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**Quelques travaux autour des  
déformations isomonodromiques**

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*À Joseph et Raphaël.*

# Contents

<b>Overview</b>	4
<b>1 Isomonodromic deformations and stability of vector bundles</b>	7
1.1 Introduction	7
1.2 Flat connections and monodromy	10
1.2.1 Connections	10
1.2.2 Monodromy	13
1.2.3 Isomonodromic deformations	14
1.3 Universal isomonodromic deformations	15
1.3.1 The Teichmüller curve	16
1.3.2 Universal isomonodromic deformations, logarithmic case	16
1.3.3 Parallel isomonodromic deformations	17
1.3.4 Universal isomonodromic deformations, meromorphic rank-two case	20
1.3.5 The parallel foliation in the general case	23
1.4 Notions of stability of vector bundles over curves	25
1.4.1 Arbitrary rank	25
1.4.2 Rank two	26
1.4.3 An elementary vector bundle lemma	28
1.5 Definition of the Atiyah bundle	30
1.5.1 Definition <i>via</i> the symbol homomorphism	30
1.5.2 The logarithmic Atiyah bundle and connections	31
1.5.3 The filtered Atiyah bundle and reducible connections	32
1.6 Some deformation theory	34
1.6.1 Deformations of curves with marked points	35
1.6.2 Deformations of vector bundles over curves	37
1.6.3 Deformations of quadruples	38
1.7 Main results	40
1.7.1 Isomonodromic deformations and stability in rank two	40
1.7.2 Isomonodromic deformations and stability in arbitrary rank	48

1.7.3	Further remarks and open questions	51
<b>2</b>	<b>Algebraic isomonodromic deformations and the mapping class group</b>	<b>54</b>
2.1	Introduction	54
2.1.1	Algebraization of universal isomonodromic deformations	54
2.1.2	Dynamical study of finite orbits in the reducible rank 2 case	55
2.2	The mapping class group	56
2.2.1	The mapping class group action on $\chi_{g,n}(G)$	57
2.2.2	The mapping class group action on $\mathcal{T}_{g,n}$	58
2.2.3	Relation to $\mathcal{M}_{g,n}$	58
2.3	Effective description of the action on $\chi_{g,n}(G)$	59
2.3.1	Presentation of the fundamental group	59
2.3.2	Mapping class group generators	60
2.4	Finite orbits of the action on $\chi_{g,n}(\text{Aff}(\mathbb{C}))$	63
2.4.1	Abelian case	64
2.4.2	Preparation lemmata	66
2.4.3	Non abelian case in genus one	68
2.4.4	Non abelian case in higher genus	70
2.5	Reducible rank two representations with finite orbit	72
2.5.1	The size of some finite orbits	72
2.5.2	Reduction to the affine case	75
2.5.3	Classification results	76
2.5.4	Further remarks and open questions	77
2.6	Algebraic logarithmic connections	78
2.6.1	Definition and relation to analytic connections	78
2.6.2	Flat connections and monodromy	79
2.6.3	Algebraic logarithmic Riemann-Hilbert correspondence	79
2.6.4	Algebraic isomonodromic deformations	81
2.7	The monodromy of the monodromy	83
2.7.1	Mapping classes of the central fiber	83
2.7.2	Splitting of the fundamental group	85
2.7.3	Splitting and the mapping class group	87
2.7.4	Extensions of representations	88
2.8	Algebraization results	90
2.8.1	Finiteness is a necessary condition	90
2.8.2	Finiteness is a sufficient condition	91
2.8.3	Proof of the main result	92

2.8.4	Proof of the universal property	93
2.8.5	Further remarks and open questions	93
<b>3</b>	<b>Flat rank 2 vector bundles over genus 2 curves</b>	<b>95</b>
3.1	Introduction	95
3.2	Parabolic bundles and their elementary transformations	98
3.2.1	Parabolic bundles and connections	98
3.2.2	Elementary transformations	99
3.2.3	Weight-stability and moduli spaces	100
3.3	Hyperelliptic descent	101
3.3.1	Topological considerations	102
3.3.2	A direct algebraic approach	106
3.3.3	Bundle-type terminology	108
3.3.4	Tyurin subbundles	110
3.4	A dictionary for hyperelliptic descent	111
3.4.1	Stable bundles (Figure 2)	112
3.4.2	Gunning planes (Figures 3 and 4)	114
3.4.3	The unipotent family and its 15 twists (Figures 5 and 6)	115
3.4.4	Generic decomposable bundles (Figure 7)	117
3.4.5	The trivial bundle and its 15 twists (Figure 8)	118
3.4.6	The 6 + 10 Gunning bundles (Figures 9 and 10)	119
3.4.7	Affine bundles	121
3.5	Classical approaches and their relations	121
3.5.1	The Narasimhan-Ramanan moduli space	122
3.5.2	Natural coordinates for $\mathcal{M}_{\text{NR}}$	124
3.5.3	Tyurin parametrization	129
3.5.4	The Bertram-Bognessi moduli space	132
3.6	A chart of the moduli stack $\mathfrak{Bun}(X)$	137
3.6.1	Semi-stable bundles and projective charts	138
3.6.2	The chart $\mathbb{P}_R^1 \times \mathbb{P}_G^1 \times \mathbb{P}_T^1$	139
3.6.3	The chart $\mathbb{P}_b^3 = \mathbb{P}_p^3$	142
3.6.4	The four democratic charts and the Geiser involution	143
3.6.5	A sufficient collection of charts	146
3.6.6	Application to the Hitchin fibration	147
3.7	The isomonodromy foliation	151
3.7.1	An atlas of $\mathfrak{Con}^*(X/\iota)$	152
3.7.2	The isomonodromy Hamiltonian system	154

3.7.3	Transversality to the locus of Gunning bundles	156
3.7.4	Projective structures and Hejhal's theorem	158
3.7.5	Further results and open questions	160

<b>References</b>		<b>161</b>
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# Overview

The aim of this manuscript is to present a selection of the works by the author, together with several collaborators, since her thesis. These works concern certain aspects of the following: the Riemann-Hilbert problem, dynamics on character varieties and moduli of connections and vector bundles over curves. What they have in common is that they all arise from questions related to isomonodromic deformations of meromorphic connections on holomorphic vector bundles over Riemann surfaces (of any genus  $g$ ), defined in the sense of Malgrange.

However, the nature of the considered questions and used techniques are quite different, and the presentation will be divided in three main chapters accordingly. Each chapter will have its own detailed introduction, followed by a mostly self-contained exposition of the results with detailed proofs. For interested readers discouraged by the resulting length, each of the brief summaries below is followed by a minimal reading guide through the corresponding chapter, covering the main results, the essential ones among the non-standard definitions, as well as further remarks and conclusions. In order to obtain simply an overview of the considered problems and some of their perspectives, it is sufficient to follow the grey columns in Tables [1](#)–[3](#).

**Chapter 1** presents in a uniform way (at the dispense of full generality), a series of generalizations of a result of Bolibruch in [\[Bol90\]](#) that were obtained in collaboration with I. Biswas and J. Hurtubise. For any linear representation of the fundamental group of a given curve minus some points, one can construct a meromorphic connection over the curve realizing the given representation as its monodromy. The connection is moreover uniquely determined if one prescribes some auxiliary data [\[Ber80\]](#). But what can be said about the vector bundle underlying the connection? This Riemann-Hilbert type question is not adressed directly. Rather, several results were achieved in the general theme of showing *via* deformation theory that along deformations of irreducible connections over curves obtained by varying the complex structure but fixing the monodromy and auxiliary data, the locus of non generic vector bundles has at least the codimension one might expect. Here non generic means for example not being a stable bundle when  $g > 1$ .

Introduction	Definitions	Main results	Further
§ <a href="#">1.1</a>	$\mathcal{T}_{\parallel}$ § <a href="#">1.3.3</a> $\kappa(E)$ Def. <a href="#">1.4.3</a> $\text{At}_D(E)$ § <a href="#">1.5</a>	rank 2 § <a href="#">1.7.1</a> (new proof) rank $r$ § <a href="#">1.7.2</a>	§ <a href="#">1.7.3</a>

Table 1: Quick reader’s guide for Chapter [1](#)

**Chapter 2** presents the collaboration [\[CH16\]](#) with G. Cousin, which consist in a generalization to higher genus of previous works [\[Cou17\]](#) [\[CM16\]](#). The germ of the universal isomonodromic deformation of a logarithmic connection on a stable  $n$ -pointed genus  $g$ -curve always exists in the analytic category. The question is under which conditions it is the analytic germification of an algebraic isomonodromic deformation. Up to some minor technical conditions, this turns out to be the case if

and only if the monodromy of the connection has finite orbit under the action of the mapping class group. A detailed study of the dynamics of this action in the particular case of reducible rank two representations and genus  $g > 0$  leads to the complete classification of the finite orbits in this case.

Introduction	Tools and definitions	Main results	Further
§ 2.1	$\Gamma_{g,n}$ -action § 2.3 Iso Def. 2.6.3 RH § 2.6 Splitting § 2.7.2, § 2.7.3	Dynamics § 2.5.3 Algebraization § 2.8.1, § 2.8.2	§ 2.5.4 § 2.8.5

Table 2: Quick reader’s guide for Chapter 2

**Chapter 3** presents results obtained in collaboration with F. Loray in [HL15]. They concern the forgetful map from the moduli space of irreducible holomorphic  $\mathfrak{sl}_2$ -connections over a curve of genus two towards the (non-separated) moduli space of underlying vector bundles (including unstable bundles). As a particularity of the genus 2 case, it can actually be studied, *via* hyperelliptic descent, from the point of view of the forgetful map from the moduli space of certain logarithmic rank two connections over the Riemann sphere towards the moduli space of underlying parabolic bundles. The latter is well-known by [AL97, LS15] and the authors establish explicit links between it and classical approaches by Narasimhan-Ramanan, Tyurin and Bertram. This provides an explanation for a certain number of known geometric phenomena of the birational models of the Kummer surface in the moduli spaces associated to the latter. Moreover, this allows to recover explicitly Bolognesi’s Poincaré family on a two-cover of the Narasimhan-Ramanan moduli space. Several applications can be deduced, such as an explicit version of the Hitchin fibration, as well as applications to unbranched (in [HL15]) and branched (in [CDHL18]) projective structures, *via* isomonodromic deformations.

Introduction	Hyperelliptic descent (main tool)	Main results	Further
§ 3.1	elm § 3.2.2 $\Phi$ § 3.3 $\phi$ § 3.4.1	$\mathbb{P}_{\text{NR}}^3$ § 3.5.2 $\mathbb{P}_B^3 \rightarrow \mathbb{P}_{\text{NR}}^3$ § 3.5.4 $\mathfrak{Bun}^*(X)$ § 3.6.2–§ 3.6.4 Hitchin § 3.6.6 Hejhal § 3.7.3, § 3.7.4	§ 3.7.5

Table 3: Quick reader’s guide for Chapter 3

**List of publications.** The complete list of publications of the author, as well as works currently under review for publication, is the following.

- [01] *Stability of rank 2 vector bundles along isomonodromic deformations*, Mathematische Annalen, 344(2):346–490, 2009. [Heu09]
- [02] *Universal isomonodromic deformations of meromorphic rank 2 connections on curves*, Annales de l’Institut Fourier, 60(2):515–549, 2010. [Heu10]
- [03] (with Biswas and Hurtubise) *Isomonodromic deformations of logarithmic connections and stability*, Mathematische Annalen, 366(1-2): 121–140, 2016. [BHH16]
- [04] (with Biswas and Hurtubise) *Isomonodromic deformations and very stable vector bundles of rank two*, Communications in Mathematical Physics, 356(2):627–640, 2017. [BHH17]



- [05] (with Biswas and Hurtubise) *Isomonodromic deformations of irregular connections and stability of bundles*, 13 pages, to appear in Communications in Analysis and Geometry, ArXiv e-prints 1608.00780. [\[BHH18b\]](#)
- [06] (with Biswas and Hurtubise) *Isomonodromic deformations of logarithmic connections and stable parabolic vector bundles*, 25 pages, submitted for publication, ArXiv e-prints 1703.07203, 2018. [\[BHH18a\]](#)
- [07] (with Cousin) *Algebraic isomonodromic deformations and the mapping class group*, 38 pages, submitted for publication, ArXiv e-prints 1612.05779, 2016. [\[CH16\]](#)
- [08] (with Loray) *Flat rank two vector bundles over genus two curves*, 90 pages, to appear in Memoirs of the AMS (2018), ArXiv e-prints 1401.2449. [\[HL15\]](#)
- [09] (with Loray) *Hitchin Hamiltonians in genus two*, Analytic and Algebraic Geometry, Hyderabad conference proceedings, editors Aryasomayajula, Biswas, Morye, Parameswaran (2017). [\[HL17\]](#)
- [10] (with Calsamiglia, Deroin and Loray) *The Riemann-Hilbert mapping for  $\mathfrak{sl}_2$ -systems over genus two curves*, 35 pages, to appear in Bulletins de la SMF (2018), ArXiv e-prints 1602.02273. [\[CDHL18\]](#)
- [11] (with Biswas) *On the logarithmic connections over curves*, Journal of the Ramanujan Mathematical Society, 28A(SPL):21–40, 2013. [\[BH13b\]](#)
- [12] (with Biswas) *Non flat extensions of flat vector bundles*, International Journal of Mathematics, 26(14): 1550114, 2015. [\[BH15\]](#)
- [13] (with Biswas) *Holomorphic Connections on Filtered Bundles over Curves*, Documenta Mathematica, 18:1473–1480, 2013. [\[BH13a\]](#)
- [14] (with Biswas) *Holomorphic connections on filtered vector bundles with given connection on the graded bundle*, Mathematical Proceedings of the Royal Irish Academy, 114(2):1–7, 2014. [\[BH14a\]](#)
- [15] (with Biswas) *Meromorphic connections on vector bundles over curves*, Proceedings Mathematical Sciences, 124: 487–496, 2014. [\[BH14b\]](#)

**Appearance of the publications in the manuscript.** The papers [01] and [02] were issued from the author’s thesis; their presentation is not the objective of the present memoir. They will however be referred to and briefly summarized when needed, mostly in the first chapter. Indeed, Chapter [1](#) presents [03]–[05] and briefly mentions [06], which all are generalizations of the main result in [01] (which uses [02]). Chapter [2](#) presents [07]. Chapter [3](#) presents [08], which was announced in [09], and briefly mentions [10] and [11].

Not presented in the main text are the collaborations [12]–[15] with I. Biswas. The construction in [12] shows that for any curve  $C$  of genus  $g \geq 2$  and any  $r \geq 3$ , there are vector bundles  $E \rightarrow C$  of rank  $r$  admitting a subbundle  $F \subset E$  such that both  $F$  and  $E/F$  are flat (admit holomorphic connections), but  $E$  is not. This might seem counter-intuitive in view of [\[Sim92, Cor. 3.10\]](#), but the key is of course to consider non-semistable bundles of degree zero. The papers [13]–[15] establish sufficient conditions for the existence of different types of connections on a given vector bundle  $E \rightarrow C$  that preserve a given filtration of  $E$ . A complete characterization in terms of necessary and sufficient conditions for the existence of filtration preserving connections remains however open. The more general question of non-obvious necessary or sufficient conditions (other than being filtration-preserving) for the existence of connections on vector bundles that are Griffiths-transversal with respect to a given filtration seems to be completely open.

# Chapter 1

## Isomonodromic deformations and stability of vector bundles

### 1.1 Introduction

Take any triple of the form  $(E \rightarrow X, D, \nabla)$ , where  $E$  is a holomorphic vector bundle over a compact Riemann surface  $X$ ,  $D \subset X$  is a finite subset and  $\nabla$  is a logarithmic connection on  $E$  with polar divisor at most  $D$ . The monodromy functor associates to it a representation

$$\rho_{\nabla} : \pi_1(X \setminus D, x_0) \longrightarrow \mathrm{GL}(E_{x_0}).$$

The monodromy functor produces an equivalence between the category of logarithmic connections  $(E, \nabla)$  on  $(X, D)$  such that the real parts of residues lie in  $[0, 1)$ , and the category of linear representations of  $\pi_1(X \setminus D, x_0)$  [Del70]. Given a representation  $\rho$ , one can consider the set of all logarithmic connections  $(E \rightarrow X, D, \nabla)$  (with no condition on the residues) that produce the same monodromy representation  $\rho_{\nabla} = \rho$ . All these connections are conjugated to each other by meromorphic gauge transformations with possible poles over  $D$ , see for example [Sab07, Sec. II.3].

The classical Riemann–Hilbert problem considers  $X = \mathbb{P}^1$  and can be formulated as follows:

*Given a representation  $\rho : \pi_1(\mathbb{P}^1 \setminus D, x_0) \rightarrow \mathrm{GL}(V)$  with  $V \simeq \mathbb{C}^r$ , is there a logarithmic connection  $(E \rightarrow \mathbb{P}^1, D, \nabla)$  such that  $\rho = \rho_{\nabla}$  and  $E$  is the trivial bundle?*

The answer to this problem is

- 1) positive if rank  $r = 2$  [Ple64], [Dek79],
- 2) negative in general ( $r \geq 3$ ) [Bol90],
- 3) positive if  $\rho$  is irreducible [Bol92], [Kos92].

On the other hand, the fundamental group  $\pi_1(\mathbb{P}^1 \setminus D, x_0)$  depends only on the topological and not the complex structure of  $\mathbb{P}^1 \setminus D$ . So given an initial connection, one can consider variations of the complex structure without changing the monodromy representation. More precisely, consider the Teichmüller space  $\mathcal{T}_{0,n}$  of the  $n$ -pointed Riemann sphere together with its universal Teichmüller curve

$$p : (\mathcal{X} = \mathbb{P}^1 \times \mathcal{T}_{0,n}, \mathcal{D}) \longrightarrow \mathcal{T}_{0,n},$$

where  $n = \deg(D)$ . Since  $\mathcal{T}_{0,n}$  is contractible, the inclusion

$$(\mathbb{P}^1, \mathcal{D}_t) := p^{-1}(t) \hookrightarrow (\mathcal{X}, \mathcal{D}), \quad t \in \mathcal{T}_{0,n},$$

induces an isomorphism  $\pi_1(\mathbb{P}^1 \setminus \mathcal{D}_t, x_t) \simeq \pi_1(\mathcal{X} \setminus \mathcal{D}, x_t)$ . By the Riemann–Hilbert correspondence and Malgrange’s Lemma, we can associate to any logarithmic connection  $(E, \nabla_0)$  on  $\mathbb{P}^1$  with polar divisor  $D$ , its *universal isomonodromic deformation*: a flat logarithmic connection  $(\mathcal{E}, \nabla)$  over  $\mathcal{X}$  with polar divisor  $\mathcal{D}$  that extends the given initial connection  $(E, \nabla_0)$ . The monodromy representation for  $\nabla|_{p^{-1}(t)}$  does not change as  $t$  moves over  $\mathcal{T}_{0,n}$  (see [Mal83a] or Section 1.3 for more details).

We are led to another Riemann–Hilbert problem:

*Given a logarithmic connection  $(E, \nabla_0)$  on  $\mathbb{P}^1$  with polar divisor  $D$  of degree  $n$ , is there a point  $t \in \mathcal{T}_{0,n}$  such that the vector bundle  $\mathcal{E}_t = \mathcal{E}|_{\mathbb{P}^1 \times \{t\}}$  underlying the universal isomonodromic deformation  $(\mathcal{E}, \nabla)$  is trivial?*

Denote  $\mathcal{Y} := \{t \in \mathcal{T}_{0,n} \mid \mathcal{E}_t \text{ is not the trivial bundle}\}$ . Under the assumption that  $\mathcal{Y} \neq \mathcal{T}_{0,n}$ , it is known that  $\mathcal{Y}$  forms a (possibly empty) divisor on  $\mathcal{T}_{0,n}$  (see e.g. [Sab07, Thm. I.5.3]). This divisor is commonly referred to as the *Malgrange  $\Theta$ -divisor* (e.g. in [BB<sup>+</sup>12, p. 118]) and corresponds to zeros of the Miwa- $\tau$ -function [Mal04] (see also [Pal99]). The assumption  $\mathcal{Y} \neq \mathcal{T}_{0,n}$  is usually achieved by requiring that the initial vector bundle  $E$  is the trivial bundle. The above Riemann-Hilbert problem asks in which cases this requirement is justified. A partial answer is given by a theorem of Bolibruch.

**Theorem 1.1.1** ([Bol90]). *Let  $(E, \nabla_0)$  be an irreducible tracefree logarithmic rank two connection with  $n \geq 4$  poles on  $\mathbb{P}^1$ . There is a proper closed analytic subset  $\mathcal{Y} \subsetneq \mathcal{T}_{0,n}$  such that for all  $t \in \mathcal{T}_{0,n} \setminus \mathcal{Y}$ , the vector bundle  $\mathcal{E}_t = \mathcal{E}|_{\mathbb{P}^1 \times \{t\}}$  underlying the universal isomonodromic deformation  $(\mathcal{E}, \nabla)$  of  $(E, \nabla_0)$  is trivial.*

From the Birkhoff–Grothendieck classification of holomorphic vector bundles on  $\mathbb{P}^1$  it follows immediately that the only semistable holomorphic vector bundle of degree 0 and rank  $r$  on  $\mathbb{P}^1$  is the trivial bundle  $\mathcal{O}_{\mathbb{P}^1}^{\oplus r}$ . This leads to the following more general question:

*Given a representation  $\rho : \pi_1(X \setminus D, x_0) \rightarrow \mathrm{GL}(V)$ , with  $V \simeq \mathbb{C}^r$ , where  $X$  is a compact connected Riemann surface, is there a logarithmic connection  $(E \rightarrow X, D, \nabla)$  such that  $\rho = \rho_\nabla$  and  $E$  is semistable of degree zero?*

The answer to this problem is still

- 1) positive if rank  $r = 2$  [EH01],
- 2) negative in general [EH01],
- 2) positive if  $\rho$  is irreducible [EV99].

On the other hand, we can ask the following: Let  $p : (\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{T}_{g,n}$  be the universal Teichmüller curve.

*Given a logarithmic connection  $(E, \nabla_0)$ , with polar divisor  $D$  of degree  $n$  on a compact connected Riemann surface  $X$  of genus  $g$ , is there an element  $t \in \mathcal{T}_{g,n}$  such that the vector bundle  $\mathcal{E}_t = \mathcal{E}|_{\mathcal{X}_t} \rightarrow \mathcal{X}_t = p^{-1}(t)$  underlying the universal isomonodromic deformation  $(\mathcal{E}, \nabla)$  of  $(E, \nabla_0)$  is semistable?*

Note that we necessarily have  $\deg(\mathcal{E}_t) = \deg(E)$ . Again, Theorem 1.1.1 can be generalized:

**Theorem 1.1.2** ([Heu09]). *Let  $(E, \nabla_0)$  be an irreducible tracefree logarithmic rank 2 connection with polar divisor  $D$  of degree  $n$  on a compact connected Riemann surface  $X$  of genus  $g$  such that  $3g - 3 + n > 0$ . Consider its universal isomonodromic deformation  $(\mathcal{E}, \nabla)$  over  $p : (\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{T}_{g,n}$ .*

*There is a proper closed analytic subset  $\mathcal{Y}^{\mathrm{nms}} \subsetneq \mathcal{T}_{g,n}$  such that for any  $t \in \mathcal{T}_{g,n} \setminus \mathcal{Y}^{\mathrm{nms}}$ , the vector bundle  $\mathcal{E}_t = \mathcal{E}|_{\mathcal{X}_t}$ , where  $(\mathcal{X}_t, \mathcal{D}_t) = p^{-1}(t)$ , is maximally stable.*

A maximally stable rank two bundle  $E \rightarrow X$  (see Section [1.4](#)) is automatically semistable if  $g \geq 1$  or  $\deg(E)$  is even. Surprisingly, Theorem [1.1.2](#) remains valid when the initial connection  $\nabla_0$  is allowed to have arbitrary non-Fuchsian singularities; in particular it is valid in the presence of irregular singularities [\[Heu09\]](#). Note however that the parameter space of the universal isomonodromic deformation then has strictly larger dimension than the Teichmüller space (see [\[Heu10\]](#), see also Section [1.3](#)). The reason, observed in [\[Heu10\]](#), behind the fact that the proof of Theorem [1.1.2](#) can be adapted rather easily for non-Fuchsian singularities is that in the  $\mathfrak{sl}_2$ -case, isomonodromic deformations of meromorphic connections are *locally constant*, i.e., near each point in  $\mathcal{X}$  there exists a choice of coordinates and a choice of frame of  $\mathcal{E}$  such that the connection matrix of  $\nabla$  with respect to that choice depends on at most one variable, given by a coordinate in the fiber over the corresponding parameter  $t \in \mathcal{T}$ . Note that a universal object for locally constant isomonodromic deformations (of a given initial connection) can easily be constructed in arbitrary rank. We will call it the *parallel isomonodromic deformation* (see § [1.3.3](#)). Under suitable conditions, it can be seen as the restriction of the universal isomonodromic deformation to a submanifold in the parameter space.

For  $g = 0$ , and irreducible tracefree rank two connections over  $\mathbb{P}^1$  with four poles (counted with multiplicity), the parameter space of universal/parallel isomonodromic deformations is one-dimensional. They are closely related to solutions of Painlevé equations [\[FN80\]](#) [\[JMU81\]](#). The locus of non-maximally stable bundles then corresponds to the locus of non-trivial bundles and forms a closed analytic subset. The result in [\[Heu09\]](#) implies that this analytic subset is proper. For the special case corresponding to the third Painlevé equation, this has been independently proven in [\[Nil09\]](#), see also [\[GH17\]](#) Chap. 4,8].

The proof in [\[Heu09\]](#) of Theorem [1.1.2](#) and its meromorphic version consists in a detailed analysis of the family of Riccati-foliations induced by the universal isomonodromic deformation and relies heavily on the fact that in rank two, a subbundle of  $E \rightarrow X$ , assuming it is neither 0 nor  $E$ , corresponds to a section  $\sigma$  of  $\mathbb{P}(E)$ , so that there is a natural isomorphism  $\sigma(X) \simeq X$ . It is not at all clear how to generalize that proof to the case of bundles of arbitrary rank.

In § [1.7.1](#) we give a new proof of Theorem [1.1.2](#) and its non-Fuchsian version, not assuming tracefreeness, but assuming for simplicity that the pair  $(X, D^{\text{red}})$  is stable. This new proof reflects a machinery in arbitrary rank developed in collaboration with I. Biswas and J. Hurtubise in [\[BHH16\]](#) and [\[BHH18b\]](#). The main idea is based on two observations.

Firstly, for various notions of stability of bundles, there exist semicontinuity results along families of bundles, *e.g.* those due to Maruyama, Shatz, Atiyah-Bott, Laumon, and Gurjat-Nitsure (see Section [1.4](#) and references therein).

Secondly, the Atiyah bundle associated to a vector bundle over  $X$  (or a general complex manifold), first introduced in [\[Ati57\]](#), is a fundamental tool in the study of holomorphic connections. It is particularly maniable in the sense that it admits a variety of equivalent definitions, and it admits certain logarithmic and meromorphic generalizations (see Section [1.5](#)). Its two predominant aspects that we are concerned with, formulated here for the holomorphic case, are the following.

- A splitting of the Atiyah exact sequence associated to  $E \rightarrow X$  is equivalent to the datum of a holomorphic connection on  $E$  (see *e.g.* § [1.5.2](#)).
- The first cohomology space of the Atiyah bundle associated to  $E \rightarrow X$  classifies deformations of the pair  $(E, X)$  to first order (see *e.g.* Section [1.6](#)).

These two aspects are intertwined when considering the deformation of the pair  $(E, X)$  underlying the universal isomonodromic deformation of an initial holomorphic connection on  $E \rightarrow X$ .

In the rank 2 case, the machinery from [\[BHH16\]](#) and [\[BHH18b\]](#) can be adapted to investigate

very stability ([Lau88], see also Section 1.4) of vector bundles along isomonodromic deformations. This has been established in [BHH17] (see also § 1.7.1).

In arbitrary rank, but  $g > 0$ , we find the following (see § 1.7.2, including moreover a version of the result in [BHH18b] concerning the general meromorphic case).

**Theorem 1.1.3** ([BHH16]). *Let  $X$  be a compact Riemann surface of genus  $g \geq 1$  and let  $D$  be a reduced divisor on  $X$  of degree  $n \geq 0$ . Let  $(E, \nabla_0)$  be an irreducible logarithmic connection of arbitrary rank over  $X$ , with polar divisor  $D$ .*

*Let  $p : (\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{T}_{g,n}$  be the universal Teichmüller curve and let  $\mathcal{E} \rightarrow \mathcal{X}$  be the vector bundle underlying the universal isomonodromic deformation of  $(E, \nabla_0)$ . Denote  $\mathcal{E}_t := \mathcal{E}|_{p^{-1}(t)}$  for any  $t \in \mathcal{T}_{g,n}$ . Define*

$$\mathcal{Y}^{\text{nss}} := \{t \in \mathcal{T}_{g,n} \mid \mathcal{E}_t \text{ not semistable}\} \quad , \quad \mathcal{Y}^{\text{ns}} := \{t \in \mathcal{T}_{g,n} \mid \mathcal{E}_t \text{ not stable}\}.$$

*Then  $\mathcal{Y}^{\text{nss}}$  and  $\mathcal{Y}^{\text{ns}}$  are closed analytic subsets of  $\mathcal{T}_{g,n}$  satisfying*

$$\text{codim}(\mathcal{Y}^{\text{nss}}, \mathcal{T}_{g,n}) \geq g, \quad \text{codim}(\mathcal{Y}^{\text{ns}}, \mathcal{T}_{g,n}) \geq g - 1.$$

*In particular,  $\mathcal{Y}^{\text{nss}}$  is proper and  $\mathcal{Y}^{\text{ns}}$  is proper if  $g > 1$ .*

In the prepublication [BHH18a], not presented here, we established a version of Theorem 1.1.3, studying parabolic (semi-) stability of vector bundles (see [MS80]), as well as parabolic very stability in rank two, along isomonodromic deformations endowed with the parabolic structure naturally induced from the connection as in [BB04]. Note that parabolic semistability of a parabolic vector bundle is not a consequence of semistability of the underlying vector bundle. However, the proof of Theorem 1.1.3 goes through after appropriate modifications.

Moreover, I only indicate that in [BHH16] we established a version of Theorem 1.1.3 studying (Ramanan-)stability along isomonodromic deformations of logarithmic connections on principal  $G$ -bundles, where  $G$  is a connected reductive complex algebraic group. We refer to [Ram75] and [Boa02] for definitions of these notions, and remark that in the case  $G = \text{GL}_r\mathbb{C}$ , they are equivalent to the corresponding notions for vector bundles of rank  $r$ .

## 1.2 Flat connections and monodromy

### 1.2.1 Connections

Let  $M$  be a complex manifold<sup>1</sup> of complex dimension  $m > 0$  and let  $\mathcal{D}^{\text{red}}$  be a (possibly empty) reduced effective normal crossing divisor on  $M$ . Denote by  $\mathcal{D}^1, \dots, \mathcal{D}^n$  the irreducible components of  $\mathcal{D}^{\text{red}}$ .

A meromorphic connection of rank  $r$  over  $M$  with polar divisor at most  $\mathcal{D} = \sum_{i=1}^n n_i \mathcal{D}^i$ , where the  $n_i$ 's are positive integers, is a pair  $(E, \nabla)$ , where  $E \rightarrow M$  is a holomorphic vector bundle of rank  $r$  over  $M$ , whose sheaf of sections we shall also denote by  $E$ , and  $\nabla$  is a  $\mathbb{C}$ -linear morphism

$$\nabla : E \rightarrow E \otimes \Omega_M^1(\mathcal{D}),$$

which satisfies the Leibniz rule

$$\nabla(f \cdot e) = f \cdot \nabla(e) + e \otimes df$$

---

<sup>1</sup>Here and throughout, by *manifold* we mean connected Hausdorff manifold, except explicit mention of the contrary. The same holds for Riemann surfaces. The latter will moreover always be compact.

for any  $f \in \mathcal{O}_X(U)$ ,  $e \in E(U)$ , where  $U \subset X$  is any open subset. We say that  $(E, \nabla)$  has polar divisor  $\mathcal{D}$  if ( $\mathcal{D}$  is empty or)  $\nabla$  is not induced by a meromorphic connection on  $E$  with polar divisor at most  $\mathcal{D} - \mathcal{D}^i$  for some  $i \in \llbracket 1, n \rrbracket$ .

If  $U$  is a connected open set of  $M$  such that  $E(U)$  admits a frame  $\underline{e}$ , then  $\nabla|_U$  is of the form

$$\underline{e} \cdot Y \mapsto \underline{e} \cdot (d + \Omega) \cdot Y$$

for a matrix-valued meromorphic one-form  $\Omega$ , the *connection matrix* of  $\nabla$  with respect to  $\underline{e}$ .

A *holomorphic connection* is a meromorphic connection with empty polar divisor and a *logarithmic connection*  $(E, \nabla)$  is a  $\mathbb{C}$ -linear morphism

$$\nabla : E \rightarrow E \otimes \Omega_M^1(\log \mathcal{D}^{\text{red}})$$

satisfying the Leibniz rule.

A connection  $(E, \nabla)$  is called *flat* if its curvature  $\nabla^2$  is zero. Note that if  $\nabla$  is given with respect to a frame  $\underline{e}$  over  $U$  by the connection matrix  $\Omega$ , then the curvature of  $\nabla$  over  $U$  is given by

$$\underline{e} \cdot Y \mapsto \underline{e} \cdot (d\Omega + \Omega \wedge \Omega) \cdot Y,$$

where we consider the natural exterior product of matrix-valued one-forms. In particular, if  $M$  is of complex dimension one, any holomorphic, logarithmic or meromorphic connection over  $M$  is automatically flat.

We will be particularly interested in connections of a certain type considered by Malgrange in [\[Mal83b\]](#) [\[Mal83a\]](#). Those cover all meromorphic connections if  $m = 1$ , but the definition is more restrictive if  $m > 1$ .

**Definition 1.2.1** (Malgrange connections). *A meromorphic connection  $(E, \nabla)$  on  $M$  with polar divisor  $\mathcal{D} = \sum_{i=1}^n n_i \mathcal{D}^i$  will be called a Malgrange connection if*

- $\mathcal{D}^{\text{red}} = \sum_{i=1}^n \mathcal{D}^i$  is smooth (non-crossing),
- $\nabla$  is induced by a  $\mathbb{C}$ -linear morphism

$$E \rightarrow E \otimes \Omega_M^1(\log \mathcal{D}^{\text{red}}) \otimes \mathcal{O}_M(\mathcal{D} - \mathcal{D}^{\text{red}}),$$

which we abusively again denote by  $\nabla$ , and

- if  $U \subset M$  is connected and open such that on  $U$  we have a frame  $\underline{e}$  of  $E(U)$ , as well as coordinates  $(z_1, \dots, z_m)$  of  $M$  with  $\mathcal{D}^{\text{red}} \cap U = \mathcal{D}^i \cap U = \{z_1 = 0\}$  for some  $i \in \llbracket 1, n \rrbracket$ , then the evaluation at any point of  $\mathcal{D}^i \cap U$  of the holomorphic matrix-valued one-form

$$\Omega \cdot z_1^{n_i},$$

where  $\Omega$  is the connection matrix with respect to  $\underline{e}$ , is non-zero.

Note that flat logarithmic connections with smooth polar divisor are automatically Malgrange connections because then the residue along the polar divisor is well defined and flat, see [\[Sab07\]](#) § 0.14.b].

**Definition 1.2.2** (Locally constant connections). *A meromorphic connection  $(E, \nabla)$  on  $M$  with polar divisor  $\mathcal{D} = \sum_{i=1}^n n_i \mathcal{D}^i$  will be called locally constant if*

- $\mathcal{D}^{\text{red}} = \sum_{i=1}^n \mathcal{D}^i$  is smooth (non-crossing),

- $\nabla$  is flat, and
- for each  $i \in \llbracket 1, n \rrbracket$ , any point of  $\mathcal{D}^i$  admits a neighborhood  $U \subset M$  such that there exists a choice of coordinates  $(z_1, \dots, z_m)$  on  $U$  which is convenient in the sense that  $\mathcal{D}^i \cap U = \{z_1 = 0\}$  and there exists a frame  $\underline{e}$  of  $E(U)$  such that the connection matrix  $\Omega$  of  $(E, \nabla)$  is, with respect to this frame and these coordinates, of the form

$$\Omega(z_1, \dots, z_m) = \frac{A(z_1)}{z_1^{n_i}} dz_1, \quad (1.1)$$

where  $A$  is a matrix-valued holomorphic function on  $U$  depending only on  $z_1$ .

Note that locally constant connections are automatically Malgrange connections. The converse does not hold in general. However, flat logarithmic connections with smooth polar divisor are automatically locally constant, due to *elimination of non-essential variables* [YT76, Thm. 5]. Moreover, Malgrange connections of rank one are automatically locally constant, as one can see for example by following the argument in the proof of Theorem 1.3.5 in §1.3.4 applied to the connection matrix  $\omega$  near a pole of such a Malgrange connection of rank one.

**Lemma 1.2.3.** *Let  $\mathbb{D} \subset \mathbb{C} \times \mathbb{C}^{m-1}$  with  $m > 1$  be a polydisc centered at 0 with coordinates  $(z, \underline{t})$ . Let  $(E, \nabla)$  be a flat meromorphic connection with polar divisor  $n_i D$ , where  $D = \{z = 0\}$  and  $n_i > 1$ . If, up to shrinking  $\mathbb{D}$  to a smaller polydisc centered at 0, there exists a set of coordinates  $(z', \underline{t}) = (f(z, \underline{t}), \underline{t})$  which is convenient in the sense of Definition 1.2.2 and which in restriction to  $\{\underline{t} = 0\}$  coincides with  $(z, \underline{t})$ , then the  $(n_i - 1)$ -jet*

$$f(z, \underline{t}) \pmod{z^{n_i}},$$

*i.e., the Taylor series up to order  $n_i - 1$  of  $f$  with respect to  $z$ , does not depend on the choice of such a  $z'$ .*

*Moreover, if in any neighborhood of 0 in  $\mathbb{D}$  there is a set of coordinates  $(\hat{z}, \underline{t}) = (g(z, \underline{t}), \underline{t})$  which in restriction to  $\{\underline{t} = 0\}$  coincides with  $(z, \underline{t})$  and such that  $g(z, \underline{t}) \equiv f(z, \underline{t}) \pmod{z^{n_i}}$ , then up to shrinking  $\mathbb{D}$ , the coordinates  $(\hat{z}, \underline{t})$  are also convenient in the sense of Definition 1.2.2.*

*Proof.* Let  $z'' = h(z', \underline{t})$  be a set of coordinates with  $h|_{\underline{t}=0} = \text{id}$ . If the  $(n_i - 1)$ -jet of  $h$  admits non-zero partial derivatives with respect to some  $t_k$  with  $k \in \{1, \dots, m-1\}$ , then for any frame, the connection matrix of  $(E, \nabla)$  with respect to  $(z'', \underline{t})$  admits a non-zero  $dt_k$ -component in its polar part. In particular, if  $(z'', \underline{t})$  is convenient, then it coincides with  $(z', \underline{t})$  up to order  $n_i - 1$ .

For the second part of the statement, note that since  $(z', \underline{t})$  is convenient and coincides with  $(\hat{z}, \underline{t})$  up to order  $n_i - 1$ , the connection matrix of  $(E, \nabla)$ , with respect to some frame  $(e_1, \dots, e_r)$  and the coordinates  $(\hat{z}, \underline{t})$ , is, up to shrinking  $\mathbb{D}$ , of the form

$$\Omega = \frac{1}{\hat{z}^{\ell}} A(\hat{z}, \underline{t}) d\hat{z} + \sum_{k=1}^{m-1} B_k(\hat{z}, \underline{t}) dt_k$$

with holomorphic  $A, B_k$ . By flatness of  $(E, \nabla)$ , it satisfies the integrability condition  $d\Omega = -\Omega \wedge \Omega$ . Hence the system of linear partial differential equations

$$\begin{cases} \frac{\partial}{\partial t_k} \psi(\hat{z}, \underline{t}) = -B_k(\hat{z}, \underline{t}) \psi(\hat{z}, \underline{t}) & , \quad k \in \{1, \dots, m\} \\ \psi(\hat{z}, \underline{0}) = I_r \end{cases}$$

is integrable and therefore admits a unique solution  $\psi \in \mathrm{GL}_r \mathcal{O}(\mathbb{D})$ . The connection matrix of  $(E, \nabla)$  with respect to the frame  $(e_1, \dots, e_r) \cdot \psi$  is given by  $\psi^{-1} \Omega \psi + \psi^{-1} d\psi$ . By construction, it possesses no  $dt_k$ -components with respect to the coordinates  $(\hat{z}, \underline{t})$ . Since it still satisfies the integrability condition, it can moreover not depend on  $\underline{t}$ . In other words, the connection matrix with respect to  $(e_1, \dots, e_r) \cdot \psi$  and the coordinates  $(\hat{z}, \underline{t})$  is of the desired form [\(1.1\)](#).  $\square$

**Definition 1.2.4** (Second fundamental form). *Let  $(E, \nabla)$  be a Malgrange connection with polar divisor  $\mathcal{D}$  on  $M$ . Let  $F \subset E$  be a subbundle. The second fundamental form of  $(E, \nabla)$  with respect to  $F$  is the element*

$$\Pi_{\nabla, F} \in \Gamma \left( \mathrm{Hom} \left( F, E/F \otimes \Omega_M^1(\log \mathcal{D}^{\mathrm{red}}) \otimes \mathcal{O}_M(\mathcal{D} - \mathcal{D}^{\mathrm{red}}) \right) \right)$$

defined by the composition

$$F \hookrightarrow E \xrightarrow{\nabla} E \otimes \Omega_M^1(\log \mathcal{D}^{\mathrm{red}}) \otimes \mathcal{O}_M(\mathcal{D} - \mathcal{D}^{\mathrm{red}}) \rightarrow E/F \otimes \Omega_M^1(\log \mathcal{D}^{\mathrm{red}}) \otimes \mathcal{O}_M(\mathcal{D} - \mathcal{D}^{\mathrm{red}}).$$

The connection  $(E, \nabla)$  is said to be reducible if there exists a subbundle  $F$  of  $E$  with  $0 < \mathrm{rank}(F) < r = \mathrm{rank}(E)$  such that  $\nabla$  induces a connection on  $F$ ; more precisely, such that  $\Pi_{\nabla, F} = 0$ . If there is no such subbundle  $F$ , then  $(E, \nabla)$  is called irreducible.

Note that although  $\nabla$  is only  $\mathbb{C}$ -linear, the second fundamental form with respect to any subbundle is  $\mathcal{O}_M$ -linear.

## 1.2.2 Monodromy

Let  $M$  be a complex manifold and let  $\mathcal{D}^{\mathrm{red}}$  be a (possibly empty) reduced effective normal crossing divisor on  $M$  as before. Let  $(E, \nabla)$  be a flat meromorphic connection of rank  $r$  over  $M$  with polar divisor at most  $\mathcal{D}$  such that the corresponding reduced divisor is  $\mathcal{D}^{\mathrm{red}}$ . Denote  $M^0 := M \setminus \mathcal{D}^{\mathrm{red}}$ . Since  $(E, \nabla)$  is flat by assumption,  $(E^0, \nabla^0) := (E, \nabla)|_{M^0}$  is integrable, i.e.,  $\mathcal{S} := \ker(\nabla^0)$  is a locally constant sheaf of rank  $r$  over  $M^0$ . For any path  $\gamma : [0, 1] \rightarrow M^0$ , the pull back  $\gamma^* \mathcal{S}$  is locally constant and thus isomorphic to a constant sheaf. Hence  $\gamma$  defines an isomorphism  $\gamma(\mathcal{S}) : \mathcal{S}_{\gamma(1)} \rightarrow \mathcal{S}_{\gamma(0)}$ . This isomorphism is invariant by homotopy relative to  $\{\gamma(0), \gamma(1)\}$ . Fixing some point  $x_0 \in M^0$ , we obtain a representation

$$\rho_{\nabla} : \pi_1(M^0, x_0) \rightarrow \mathrm{GL}(\mathcal{S}_{x_0}),$$

the *monodromy representation* of  $(E, \nabla)$  with respect to  $x_0$ . Conversely, given  $\rho$  as above, there exists a unique (up to unique isomorphism) flat holomorphic connection on  $M^0$  with monodromy representation  $\rho$  by Riemann-Hilbert correspondence (see [\[Del70\]](#), Thm. I.2.17). Moreover, assuming  $\mathcal{D}^{\mathrm{red}}$  is normal crossing, given  $\rho$  and, for each  $i \in \llbracket 1, n \rrbracket$ , a choice of branch of the complex logarithm, there exists a unique corresponding flat logarithmic *Deligne connection* on  $M$  with polar divisor at most  $\mathcal{D}^{\mathrm{red}}$  and monodromy representation  $\rho$  [\[Del70\]](#), Prop. II.5.4]. Up to some minor technical conditions, the Deligne models near the punctures may be replaced by more general so-called *compatible mild transversal models* introduced by G. Cousin [\[Cou17\]](#), Th. 6]. Under more restrictive circumstances, the local models may be chosen arbitrarily due to Malgrange's Lemma (see [§ 1.2.3](#)).

**Definition 1.2.5.** *The monodromy representation  $\rho_{\nabla}$  of  $(E, \nabla)$  is said to be reducible if there exists a sub-vector space of  $\mathcal{S}_{x_0} = E_{x_0}$  of dimension  $d$  with  $0 < d < r$ , which is stable under the action of  $\pi_1(M^0, x_0)$  induced by  $\rho_{\nabla}$ . Otherwise, the monodromy representation is said to be irreducible.*



Note that if  $(E, \nabla)$  is a reducible flat meromorphic connection, then its monodromy representation  $\rho_\nabla$  is reducible. By Riemann-Hilbert correspondence, the converse holds if  $(E, \nabla)$  is holomorphic. The converse holds more generally if  $(E, \nabla)$  is logarithmic, due to the existence of local normal forms [YT76, Thm. 3] at the poles.

### 1.2.3 Isomonodromic deformations

Let  $X_0$  be a compact Riemann surface of genus  $g$ , let  $D_0^{\text{red}}$  be a reduced effective divisor of degree  $n$  on  $X_0$  of the form  $D_0^{\text{red}} = \sum_{i=1}^n [x_i]$  and let  $(E_0, \nabla_0)$  be a meromorphic connection over  $X_0$  with polar divisor  $D_0 = \sum_{i=1}^n n_i [x_i]$  where the  $n_i$ 's are positive integers.

**Definition 1.2.6** (Isomonodromic deformations). *An isomonodromic deformation of  $(X_0, E_0, \nabla_0)$  as above consists in the following data:*

- a family  $(p : \mathcal{X} \rightarrow \mathcal{T}, \mathcal{D}^{\text{red}})$  of  $n$ -pointed Riemann surfaces with  $\mathcal{D}^{\text{red}} = \sum_{i=1}^n \mathcal{D}^i$  (i.e.,  $p$  is a proper holomorphic submersion between complex manifolds, where the fibers of  $p$  are compact Riemann surfaces,  $\mathcal{D}^{\text{red}}$  is a smooth reduced effective divisor on  $\mathcal{X}$  and there exist disjoint sections  $\sigma_i : \mathcal{T} \rightarrow \mathcal{X}$  of  $p$  such that  $\mathcal{D}^i = [\sigma_i(\mathcal{T})]$ );
- a flat Malgrange connection  $(\mathcal{E}, \nabla)$  over  $\mathcal{X}$  with polar divisor  $\mathcal{D} = \sum_{i=1}^n n_i \mathcal{D}^i$ ;
- a point  $t_0$  in  $\mathcal{T}$ ; we denote  $X := p^{-1}(\{t_0\})$  and  $D^{\text{red}} := \mathcal{D}^{\text{red}}|_X$ ; and
- an (analytic) isomorphism of curves with marked points and meromorphic connections

$$(\psi, \Psi) : ((X_0, D_0^{\text{red}}), (E_0, \nabla_0)) \xrightarrow{\sim} ((X, D^{\text{red}}), (\mathcal{E}, \nabla)|_X).$$

Such deformations are called isomonodromic because for any point  $x_1$  in the fiber of  $p$  over some point  $t_1$  in  $\mathcal{T}$  away from the punctures, the choice of a path  $\gamma$  in  $\mathcal{X} \setminus \mathcal{D}_0$  from  $\psi(x_0)$  to  $x_1$  allows to identify the monodromy representation of  $(E_0, \nabla_0)$  with respect to  $x_0$  with the monodromy representation of  $(\mathcal{E}, \nabla)|_{p^{-1}(t_1)}$  with respect to  $x_1$ . Here the corresponding fibers over  $\psi(x_0)$  and  $x_1$  are identified by holonomy along the chosen path and the fibers at  $x_0$  and  $\psi(x_0)$  of the corresponding vector bundles are identified by  $\Psi$ .

An important ingredient in the construction of isomonodromic deformations is Malgrange's Lemma, see [Sab07, p. 133], from [Mal83b]. It can be stated as follows.

**Proposition 1.2.7** (Malgrange's Lemma). *Let  $\mathcal{U}_i, \mathcal{T}$  be complex manifolds. Let  $p : \mathcal{U}_i \rightarrow \mathcal{T}$  be a surjective holomorphic submersion whose fibers are smooth, connected and of complex dimension one. Let  $\mathcal{D}^i$  be a smooth divisor on  $\mathcal{U}_i$  defined by a holomorphic section of  $p$ . Let  $\Sigma := p^{-1}(t_0)$  be a fiber and let  $(E_i, \nabla_i)$  be a logarithmic connection on  $\Sigma$  with polar divisor  $\{0\} := \Sigma \cap \mathcal{D}^i$ . Assume there exists a homeomorphism*

$$\Phi : (\Sigma \times \mathcal{T}, \{0\} \times \mathcal{T}) \rightarrow (\mathcal{U}_i, \mathcal{D}^i)$$

satisfying  $p \circ \Phi = \text{pr}_2$  and assume that  $\mathcal{T}$  is simply connected.

Then there exists a unique (up to unique isomorphism) flat logarithmic connection  $(\mathcal{E}, \nabla)$  on  $\mathcal{U}_i$  with polar divisor  $\mathcal{D}^i$ , such that

$$(\mathcal{E}, \nabla)|_\Sigma = (E_i, \nabla_i).$$

Note that in particular, in the above proposition, if  $\mathcal{U}^i = \Sigma \times \mathcal{T}$ , then, up to isomorphism, the only logarithmic connection over  $\mathcal{U}^i$  extending the logarithmic connection  $(E_i, \nabla_i)$  is the one obtained by pull-back via the projection  $\Sigma \times \mathcal{T} \rightarrow \Sigma$ . Therefore, logarithmic connections over Riemann surfaces

with marked points (corresponding to the position of the poles) are rigid: they cannot be non-trivially isomonodromically deformed whilst keeping the Riemann surface with marked points fixed. To obtain interesting isomonodromic deformations in the logarithmic case, one needs to deform the base.

### 1.3 Universal isomonodromic deformations

For the purpose of this section, let us fix non-negative integers  $g, n$  satisfying

$$2g - 2 + n > 0. \tag{1.2}$$

Let us fix a quadruple

$$(X_0, D_0^{\text{red}}, E_0, \nabla_0), \tag{1.3}$$

where  $X_0$  is a compact Riemann surface of genus  $g$ ,  $D_0^{\text{red}}$  is a reduced divisor of degree  $n$  on  $X_0$  and  $(E_0, \nabla_0)$  is a meromorphic connection of rank  $r$  over  $X_0$  with a certain polar divisor  $D_0$  of degree  $N \geq n$ , such that  $D_0^{\text{red}}$  is the reduced divisor associated to  $D_0$ . Note that condition (L.2) is equivalent to saying that the pair  $(X_0, D_0^{\text{red}})$  is *stable*. We say that  $(E_0, \nabla_0)$  is *deformable* if

$$3g - 3 + N > 0. \tag{1.4}$$

Note that the only undeformable configuration over a stable pointed curve is  $(g, n, N) = (0, 3, 3)$ . We will construct an isomonodromic deformation

$$(p : \mathcal{X} \rightarrow \mathcal{T}_{\parallel}, \mathcal{D}^{\text{red}}, \mathcal{E}, \nabla, t_0, \psi, \Psi),$$

of (L.3), which will be called the *parallel isomonodromic deformation* of (L.3), where the parameter space  $\mathcal{T}_{\parallel}$  has dimension  $3g - 3 + N$  and the connection  $(\mathcal{E}, \nabla)$  is locally constant (see Definition L.2.2), such that this isomonodromic deformation of (L.3) satisfies a strong universal property with respect to germs of locally constant isomonodromic deformations of (L.3). More precisely:

**Proposition 1.3.1.** *Let  $(p' : \mathcal{X}' \rightarrow \mathcal{T}', \mathcal{D}'^{\text{red}}, \mathcal{E}', \nabla', t'_0, \psi', \Psi')$ , be an isomonodromic deformation of (L.3) such that  $(\mathcal{E}', \nabla')$  is locally constant. Then, up to restricting to a sufficiently small neighborhood of  $t'_0$  in  $\mathcal{T}'$ , there exists a classifying map*

$$c_{\parallel} : (\mathcal{T}', t'_0) \rightarrow (\mathcal{T}_{\parallel}, t_0)$$

and a unique isomorphism

$$(\mathcal{X}', \mathcal{D}'^{\text{red}}, \mathcal{E}', \nabla') \xrightarrow{\sim} c_{\parallel}^*(\mathcal{X}, \mathcal{D}^{\text{red}}, \mathcal{E}, \nabla)$$

which over  $t'_0$  induces the identity on (L.3) via  $(\psi', \Psi')$  and  $(\psi, \Psi)$ .

This parallel isomonodromic deformation of (L.3) coincides with the universal isomonodromic deformation of (L.3) in the logarithmic case  $N = n$ , as well as in the meromorphic rank-one case.

We will show, in the meromorphic rank-two case, that under suitable conditions the parameter space of the universal isomonodromic deformation of (L.3) admits a natural foliation corresponding to parallel isomonodromic deformations. We will also indicate why this holds, up to germification and more restrictive conditions, in the case of arbitrary rank. Note however that in the non-logarithmic case of rank  $r > 2$ , the construction of the universal isomonodromic deformation requires significantly more involved methods than the elementary ones that as we will show are sufficient in the logarithmic or meromorphic rank  $r \leq 2$  case. For an overview of the construction in the general case, which because of its complexity is usually performed under various additional assumptions on (L.3) and is not always complemented by the establishment of a universal property, we refer to [Sib77, Mal79, JMU81, Mal83b, Mal86, Sab98, Pal99, Kri02, Boa02].

### 1.3.1 The Teichmüller curve

Before introducing the main construction, we need to recall a few well-known results from Teichmüller theory. For a more detailed exposition, we refer to [ACG11] chap. XIV].

We fix a compact oriented real surface  $\Sigma_g$  for reference and a subset  $Y^n \subset \Sigma_g$  of cardinality  $n$ , as well as an ordering of  $Y^n$ , i.e., a sequence  $y^n := (y_1, \dots, y_n)$  with  $\{y_i\} = Y^n$ .

As a set, the *Teichmüller space*  $\mathcal{T}_{g,n}$  of Riemann surfaces of genus  $g$  with  $n$  marked points is the set of isomorphism classes  $[X, D, \phi]$  of triples  $(X, D, \phi)$ , where  $X$  is a compact Riemann surface of genus  $g$ ,  $D = D^{\text{red}}$  is a set of  $n$  distinct points in  $X$  and  $\phi$  is a Teichmüller structure, i.e., an orientation-preserving homeomorphism  $\phi : (\Sigma_g, Y^n) \rightarrow (X, D)$ . Note that the Teichmüller structure induces an ordering of the elements of  $D$ : we have  $D = \{x_1, \dots, x_n\}$  with  $x_i = \phi(y_i)$ . Two  $n$ -pointed genus- $g$  curves with Teichmüller structure  $(X, D, \phi)$  and  $(X', D', \phi')$  are said to be isomorphic if there exists an isomorphism of pointed curves  $\psi : (X', D') \rightarrow (X, D)$  such that the automorphism  $\phi^{-1} \circ \psi \circ \phi'$  of  $(\Sigma_g, Y^n)$  respects the ordering of the elements of  $Y^n$  and is isotopic to the identity by an isotopy relative to  $Y^n$ . Under our general assumption (L.2), curves with Teichmüller structure are rigid: if  $\psi$  is an automorphism of  $(X, D)$  and  $(X, D, \phi)$  is isomorphic to  $(X, D, \psi \circ \phi)$ , then  $\psi$  is the identity.

The Teichmüller space  $\mathcal{T}_{g,n}$  has a natural structure of a complex manifold such that, for any holomorphic family  $(p : \mathcal{X} \rightarrow \mathcal{T}, \mathcal{D})$  of  $n$ -pointed genus- $g$  curves endowed with a Teichmüller structure  $\Phi : (\Sigma_g, Y^n) \times \mathcal{T} \xrightarrow{\sim} (\mathcal{X}, \mathcal{D})$  with  $p \circ \Phi = \text{pr}_2$ , the associated classifying map

$$\text{class}^+ : \mathcal{T} \rightarrow \mathcal{T}_{g,n}$$

is holomorphic. Moreover, there exists a holomorphic family

$$(p_{g,n} : \mathcal{X}_{g,n} \rightarrow \mathcal{T}_{g,n}, \mathcal{D}_{g,n}) \tag{1.5}$$

of  $n$ -pointed genus- $g$  curves parametrized by the Teichmüller space, endowed with a Teichmüller structure  $\Phi_{g,n}$ , such that the corresponding classifying map is the identity. This *universal Teichmüller curve* satisfies a strong universal property with respect to families with Teichmüller structure as above, i.e., any such family is isomorphic to the pull-back of the universal Teichmüller curve under the classifying map, and this isomorphism is unique. From the real-analytic Fricke coordinates, we know that the Teichmüller space  $\mathcal{T}_{g,n}$  is diffeomorphic to  $\mathbb{R}^{6g-6+2n}$  [FK97]. In particular,  $\mathcal{T}_{g,n}$  is a contractible topological space. From the Bers construction of the Teichmüller space, we know that  $\mathcal{T}_{g,n}$  can be holomorphically embedded into  $\mathbb{C}^{3g-3+n}$ , see [Ber61]. In particular, there exist global holomorphic coordinates  $\underline{t}$  on  $\mathcal{T}_{g,n}$ .

### 1.3.2 Universal isomonodromic deformations, logarithmic case

Let  $(X_0, D_0^{\text{red}}, E_0, \nabla_0)$  be as in L.3 and assume that  $\nabla_0$  is logarithmic, i.e. its polar divisor  $D_0$  satisfies  $D_0 = D_0^{\text{red}}$ . We still assume (L.2); i.e.,  $(X_0, D_0)$  is *stable*. The *universal isomonodromic deformation*

$$(p : \mathcal{X} \rightarrow \mathcal{T}, \mathcal{D}, \mathcal{E} \rightarrow \mathcal{X}, \nabla, \star, \psi, \Psi) \tag{1.6}$$

of  $(X_0, D_0^{\text{red}}, E_0, \nabla_0)$  is usually constructed as follows. Choose a Teichmüller structure  $\phi$  on  $(X_0, D_0)$  and denote by  $\star \in \mathcal{T} := \mathcal{T}_{g,n}$  the point corresponding to  $[X_0, D_0, \phi]$ . Denote by  $p : (\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{T}$  the universal Teichmüller curve (L.5), whose fibers shall be denoted by  $(\mathcal{X}_t, \mathcal{D}_t) := p^{-1}(t)$ . We moreover denote  $\mathcal{D} = \sum_{i=1}^n \mathcal{D}^i$  the natural decomposition. Let  $\psi : (X_0, D_0) \rightarrow (\mathcal{X}_\star, \mathcal{D}_\star)$  be the unique isomorphism compatible with the given Teichmüller structures on  $X_0$  and  $X$ . *Via* this isomorphism,

the connection  $(E_0, \nabla_0)$  on  $X_0$  induces a logarithmic connection  $(\mathcal{E}_\star, \nabla_\star)$  on  $\mathcal{X}_\star$  with polar divisor  $\mathcal{D}_\star$ . We denote by  $\Psi : E_0 \rightarrow \mathcal{E}_\star$  the natural vector bundle isomorphism projecting to  $\psi$ . From the Teichmüller structure  $\Phi_{g,n}$  on the Teichmüller curve, we get, for each  $i \in \llbracket 1, n \rrbracket$ , a topological trivialization of  $(\mathcal{U}_i, \mathcal{D}^i)$  for  $\mathcal{U}_i := (\mathcal{X} \setminus \mathcal{D}) \cup \mathcal{D}^i$  yielding, by Malgrange's Lemma, a flat logarithmic connection  $(\mathcal{E}^i, \nabla^i)$  over  $\mathcal{U}_i$  with polar divisor  $\mathcal{D}^i$  extending  $(\mathcal{E}_\star, \nabla_\star)|_{\mathcal{U}_i \cap \mathcal{X}_\star}$ . By Riemann-Hilbert correspondence, those glue to a flat logarithmic connection  $(\mathcal{E}, \nabla)$  over  $\mathcal{X}$  with polar divisor  $\mathcal{D}$ , satisfying  $(\mathcal{E}, \nabla)|_{\mathcal{X}_\star} = (\mathcal{E}_\star, \nabla_\star)$ . Again by Malgrange's Lemma and Riemann-Hilbert correspondence, we have the following strong universal property.

**Proposition 1.3.2.** *Let  $(X_0, D_0)$  be a stable  $n$ -pointed genus- $g$  curve and let  $(E, \nabla)$  be a logarithmic connection over  $X_0$  with polar divisor  $D_0$ . Let  $(p' : \mathcal{X}' \rightarrow \mathcal{T}', \mathcal{D}', \mathcal{E}', \nabla', t'_0, \psi', \Psi')$ , be an isomonodromic deformation of  $(X_0, D_0, E_0, \nabla_0)$  such that there exists a Teichmüller structure*

$$\Phi' : (\Sigma_g, Y^n) \times \mathcal{T}' \xrightarrow{\sim} (\mathcal{X}', \mathcal{D}')$$

with  $p' \circ \Phi' = \text{pr}_2$ . Consider the universal isomonodromic deformation [\(L.6\)](#) of  $(X_0, D_0, E_0, \nabla_0)$  and the Teichmüller classifying map

$$\text{class}^+ : \mathcal{T}' \rightarrow \mathcal{T} = \mathcal{T}_{g,n}.$$

Up to conveniently modifying  $\Phi'$ , we may assume  $\text{class}^+(t'_0) = \star$ . There exists unique isomorphism

$$(\mathcal{X}', \mathcal{D}', \mathcal{E}', \nabla') \xrightarrow{\sim} (\text{class}^+)^*(\mathcal{X}, \mathcal{D}, \mathcal{E}, \nabla)$$

which over  $t'_0$  induces the identity on [\(L.3\)](#) via  $(\psi', \Psi')$  and  $(\psi, \Psi)$ .

Note that in the logarithmic case, any isomonodromic deformation of  $(X_0, D_0, E_0, \nabla_0)$  satisfies the condition of Proposition [1.3.2](#) up to restricting its parameter space  $\mathcal{T}'$  to for example a contractible neighborhood of  $t'_0$ .

### 1.3.3 Parallel isomonodromic deformations

Let  $(X_0, D_0^{\text{red}}, E_0, \nabla_0)$  be as in [\(L.3\)](#). We choose a Teichmüller structure  $\phi$  on  $(X_0, D_0)$  and we denote by

$$\star \in \mathcal{T}_{g,n}$$

the point in the Teichmüller space corresponding to  $(X_0, D_0, \phi)$ . The Teichmüller structure  $\phi$  also provides an ordering of the points of  $X_0$  forming the support of  $D_0^{\text{red}}$ :

$$D_0^{\text{red}} = \sum_{i=1}^n [x_i].$$

The polar divisor  $D_0$  of  $(E_0, \nabla_0)$  is then of the form  $D_0 = \sum_{i=1}^n n_i [x_i]$ , with  $n_i > 0$ .

Consider the universal Teichmüller curve [\(L.5\)](#) together with its Teichmüller structure  $\Phi_{g,n}$ . Note that the latter provides a topological trivialization of the family [\(L.5\)](#). Since moreover the Teichmüller space  $\mathcal{T}_{g,n}$  is contractible, for each  $i \in \llbracket 1, n \rrbracket$ , the  $i$ -th irreducible component of the divisor  $\mathcal{D}_{g,n}$  on the Teichmüller curve  $\mathcal{X}_{g,n}$  admits a contractible neighborhood  $\mathcal{V}_i \subset \mathcal{X}_{g,n}$ . The second Cousin problem there has a solution, so that there exists a holomorphic function  $z^{(i)} \in \Gamma \mathcal{O}_{\mathcal{V}_i}$  with zero-divisor  $\mathcal{D}_{g,n}^i$ , see [\[GR79\]](#) Thm. 2 p. 139]. Since moreover  $\mathcal{D}_{g,n}^i$  is smooth, if  $\mathcal{V}_i$  is chosen sufficiently small, then there are global holomorphic coordinates of the form  $(z^{(i)}, \underline{t})$  on  $\mathcal{V}_i$ , such that  $p_{g,n} : (z^{(i)}, \underline{t}) \mapsto \underline{t}$  with respect to a holomorphic embedding of the Teichmüller space into  $\mathbb{C}^{3g-3+n}$ . We fix these coordinates.

For each  $i \in \{1, \dots, n\}$ , denote

$$J^{(i)} := \left\{ \begin{array}{ll} \{\varepsilon \mapsto e^{a_1^{(i)}} \varepsilon + \dots + a_{n_i-1}^{(i)} \varepsilon^{n_i-1}, a_k^{(i)} \in \mathbb{C}\} & \text{if } n_i > 1 \\ \{\varepsilon \mapsto \varepsilon\} & \text{if } n_i = 1. \end{array} \right\} \simeq \mathbb{C}^{n_i-1}$$

the universal cover of the group of  $(n_i - 1)$ -jets of biholomorphisms of the germ  $(\mathbb{C}, 0)$ .<sup>2</sup> We now set

$$\mathcal{T}_{\parallel} := \mathcal{T}_{g,n} \times J_{\parallel} \quad , \quad J_{\parallel} := \prod_{i=1}^n J^{(i)}$$

and consider

$$p : (\mathcal{X}, \mathcal{D}^{\text{red}}) \rightarrow \mathcal{T}_{\parallel}, \quad \mathcal{D}^{\text{red}} = \sum_{i=1}^n \mathcal{D}^i,$$

the pull-back of the universal Teichmüller curve with respect to the projection to the first factor of  $\mathcal{T}_{\parallel}$ . We set  $t_0 := (\star, \underline{0}) \in \mathcal{T}_{\parallel}$ .

Let  $\psi : (X_0, \mathcal{D}_0^{\text{red}}) \rightarrow (\mathcal{X}_{t_0}, \mathcal{D}_{t_0}^{\text{red}}) := p^{-1}(t_0)$  be the unique isomorphism compatible with the given Teichmüller structures. *Via* the isomorphism  $\psi$ , the connection  $(E_0, \nabla_0)$  on  $X_0$  induces a meromorphic connection on  $p^{-1}(t_0)$ , which we denote by  $(\mathcal{E}_{t_0}, \nabla_{t_0})$ , and a natural vector bundle isomorphism  $\Psi$ . Since  $\mathcal{T}_{\parallel}$  is contractible, the natural morphism

$$\pi_1(\mathcal{X}_{t_0} \setminus \mathcal{D}_{t_0}^{\text{red}}) \rightarrow \pi_1(\mathcal{X} \setminus \mathcal{D}^{\text{red}})$$

is an isomorphism. By Riemann-Hilbert correspondence, the holomorphic connection  $(\mathcal{E}_{t_0}, \nabla_{t_0})|_{\mathcal{X}_{t_0} \setminus \mathcal{D}_{t_0}^{\text{red}}}$  therefore extends to a unique flat holomorphic connection  $(\mathcal{E}^{(0)}, \nabla^{(0)})$  over  $\mathcal{X} \setminus \mathcal{D}^{\text{red}}$ .

Consider now a polar component  $\mathcal{D}^i$ . In a contractible neighborhood  $\mathcal{U}_i$  of  $\mathcal{D}^i$  in  $\mathcal{X}$ , we have global holomorphic coordinates of the form

$$(z, \underline{t}, \underline{a}^{(1)}, \dots, \underline{a}^{(n)}) \quad \text{with } \underline{t} \in \mathcal{T}_{g,n}, \quad \underline{a}^{(k)} \in J^{(k)}, \quad z = z^{(i)},$$

such that  $\mathcal{D}^i$  there is given by  $\{z = 0\}$ . We shall denote by  $\varphi_{\underline{a}^{(i)}}(\varepsilon)$  the  $(n_i - 1)$ -jet of biholomorphism associated to  $\underline{a}^{(i)}$ . Consider the meromorphic connection on the trivial bundle over  $(\mathbb{C}, 0)$  obtained by restricting  $(E_0, \nabla_0)$  to a germ of neighborhood of  $x_i$  and choosing a frame of  $E_0$  over this germ. It is written

$$Y \mapsto (d + A dx) \cdot Y \quad \text{for some } x^{n_i} A(x) \in \mathfrak{gl}_r \mathbb{C}\{x\}.$$

Assume that the map  $\psi$  is given with respect to our coordinates by  $x \mapsto (z, \underline{t}) = (\varphi(x), \star)$ . We obtain a connection

$$\nabla^{(i)} : Y \mapsto (d + \Omega_i) \cdot Y \quad , \quad \Omega_i := A(\varphi^{-1}(\varphi_{\underline{a}^{(i)}}(z))) d(\varphi^{-1}(\varphi_{\underline{a}^{(i)}}(z)))$$

on the trivial bundle over  $\mathcal{U}_i$ . If we restrict the discussion to a compact subset  $K$  of  $J_{\parallel}$ , then up to shrinking  $\mathcal{U}_i$  in the  $z$ -direction, we can make sure that

$$(\varphi^{-1}(\varphi_{\underline{a}^{(i)}}(z))), \underline{t}, \underline{a}^{(1)}, \dots, \underline{a}^{(n)}$$

---

<sup>2</sup>We made a slight abuse of notation here: for any  $\underline{a}^{(i)} \in \mathbb{C}^{n_i-1}$ , there is a natural  $(n_i - 1)$ -jet of biholomorphism  $\varphi(\varepsilon) = \varphi_{\underline{a}^{(i)}}(\varepsilon)$  as stated, together with a natural choice of logarithm for  $\varphi'(0)$ . Those together are equivalent to the datum of  $\underline{a}^{(i)}$ .

are holomorphic coordinates on  $\mathcal{U}_i$ . In particular, the connection  $\nabla^{(i)}$  there is flat, has polar divisor  $n_i \mathcal{D}^i$  and is locally constant.

By Riemann-Hilbert correspondence, we obtain a unique isomorphism between the restrictions to  $\mathcal{U}_i \setminus \mathcal{D}_0$  of  $\nabla^{(i)}$  and  $(\mathcal{E}^{(0)}, \nabla^{(0)})$  respectively, which is compatible with  $(\mathcal{E}_{t_0}, \nabla_{t_0})$ . Since moreover the compact subset  $K$  can be chosen arbitrarily large, the above construction allows to extend  $(\mathcal{E}_{t_0}, \nabla_{t_0})$  to a flat locally constant connection  $(\mathcal{E}, \nabla)$  over  $\mathcal{X}$ . We have now constructed the parallel isomonodromic deformation

$$(p : \mathcal{X} \rightarrow \mathcal{T}_{\parallel}, \mathcal{D}^{\text{red}}, \mathcal{E}, \nabla, t_0, \psi, \Psi)$$

of the initial connection (1.3). More precisely, we have seen that it is a locally constant isomonodromic deformation of (1.3), whose parameter space has dimension

$$\dim(\mathcal{T}_{\parallel}) = 3g - 3 + n + \sum_{i=1}^n (n_i - 1) = 3g - 3 + \sum_{i=1}^n n_i.$$

It remains to see that it satisfies the universal property stated in Proposition 1.3.1. Let  $(p' : \mathcal{X}' \rightarrow \mathcal{T}', \mathcal{D}'^{\text{red}}, \mathcal{E}', \nabla', t'_0, \psi', \Psi')$  be a locally constant isomonodromic deformation of (1.3). We shall construct the classifying map in two steps:

- Up to restricting the parameter space  $\mathcal{T}'$  to for example to a contractible neighborhood of  $t'_0$ , we may assume that  $(\mathcal{X}' \rightarrow \mathcal{T}', \mathcal{D}'^{\text{red}})$  admits a Teichmüller structure. Then it admits a unique Teichmüller structure compatible, *via*  $\psi'$ , with the one we have chosen on  $(X_0, \mathcal{D}_0^{\text{red}})$  in the construction of the parallel isomonodromic deformation of (1.3), so that there is a classifying map  $\text{class}^+ : (\mathcal{T}', t'_0) \rightarrow (\mathcal{T}_{g,n}, \star)$ . Without loss of generality, up to unique isomorphism, by the universal property of the Teichmüller curve, we may assume that  $(p' : \mathcal{X}' \rightarrow \mathcal{T}', \mathcal{D}'^{\text{red}}, t'_0)$  is identical to the pull-back *via*  $\text{class}^+$  of  $(p_{g,n} : \mathcal{X}_{g,n} \rightarrow \mathcal{T}_{g,n}, \mathcal{D}_{g,n}, \star)$ .
- Since  $(\mathcal{E}', \nabla')$  is locally constant, up to further shrinking  $\mathcal{T}'$ , we may assume that for each  $i \in \llbracket 1, n \rrbracket$  we have global holomorphic coordinates of the form  $(z', \underline{t}') = (f_i(z, \underline{t}'), \underline{t}')$  near the  $i$ -th irreducible component of  $\mathcal{D}'^{\text{red}}$ , which are convenient in the sense of Definition 1.2.2 and do moreover coincide with  $(z, \underline{t}')$  when restricted to  $\{\underline{t}' = \underline{t}'_0\}$ . By Lemma 1.2.3, the  $(n_i - 1)$ -jet  $f_i(\varepsilon, \underline{t}') \bmod \varepsilon^{n_i}$  is intrinsically defined. This jet yields, possibly after further shrinking  $\mathcal{T}'$ , a well-defined map  $\mathbf{c}_i : (\mathcal{T}', t'_0) \rightarrow (J^{(i)}, \underline{0})$ .

The classifying map will then be

$$\mathbf{c}_{\parallel} := (\text{class}^+, \mathbf{c}_1, \dots, \mathbf{c}_n) : (\mathcal{T}', t'_0) \rightarrow (\mathcal{T}_{\parallel}, t_0).$$

The fact that there is a unique isomorphism

$$(\mathcal{X}', \mathcal{D}'^{\text{red}}, \mathcal{E}', \nabla') \xrightarrow{\sim} \mathbf{c}_{\parallel}^*(\mathcal{X}, \mathcal{D}^{\text{red}}, \mathcal{E}, \nabla)$$

which over  $t'_0$  induces the identity on (1.3) *via*  $(\psi', \Psi')$  and  $(\psi, \Psi)$ , is now an easy combination of Riemann-Hilbert correspondence and Lemma 1.2.3.

Note that by the universal property in Proposition 1.3.2, in the logarithmic case, the parallel isomonodromic deformation of  $(X_0, \mathcal{D}_0, E_0, \nabla_0)$  is uniquely isomorphic to the universal isomonodromic deformation. Since Malgrange connections of rank one are locally constant, in the meromorphic rank-one case, by Proposition 1.3.1, the parallel isomonodromic deformation of (1.3) admits a universal property with respect to germs of isomonodromic deformations (1.3). In other words, in the meromorphic rank-one case, the parallel isomonodromic deformation of (1.3) is a universal isomonodromic deformation of (1.3).

**Remark 1.3.3.** Let  $(\mathcal{E}, \nabla)$  be the flat connection underlying the parallel isomonodromic deformation of  $(X_0, D_0^{\text{red}}, E_0, \nabla_0)$ . It follows from the construction that if  $\nabla_0$  induces a connection on a subbundle  $F_0$  of  $E_0$ , then there exists a subbundle  $\mathcal{F}$  of  $\mathcal{E}$  such that  $\nabla$  induces a flat connection on  $\mathcal{F}$ . On the other hand, the connection  $(\mathcal{E}, \nabla)$  is canonically isomorphic to the connection we obtain by applying the construction of the parallel isomonodromic deformation to  $(\mathcal{E}, \nabla)|_{p^{-1}(t_1)}$  for any  $t_1 \in \mathcal{T}_{\parallel}$ . Consequently, if  $(E_0, \nabla_0)$  is irreducible then  $(\mathcal{E}, \nabla)|_{p^{-1}(t)}$  is irreducible for any parameter  $t \in \mathcal{T}_{\parallel}$ .

### 1.3.4 Universal isomonodromic deformations, meromorphic rank-two case

It was shown in [Heu10] that the universal object for isomonodromic deformations of rank 2 connections with vanishing trace, where one imposes the trace to vanish also along the isomonodromic deformation, is precisely the parallel isomonodromic deformation. We will now adapt the  $\mathfrak{sl}_2$ -construction to the case of more general rank-two connections.

Let

$$(X_0, D_0^{\text{red}}, E_0, \nabla_0), \quad (1.7)$$

be as in (L3), with two additional assumptions:

- The rank of  $E_0$  is  $r = 2$ .
- If  $x$  is a local coordinate on  $U \subset X_0$  near a pole of order  $\ell > 1$  of  $(E_0, \nabla_0)$ , and the connection  $\nabla_0 : E_0 \rightarrow E_0 \otimes \Omega_X^1(D_0)$  is, with respect to a holomorphic frame  $\underline{e}(x)$ , of the form

$$\nabla_0 : \underline{e} \cdot Y \mapsto \underline{e} \cdot \left( d + \frac{1}{x^\ell} A(x) dx \right) \cdot Y, \quad A \in M_{2 \times 2} \mathcal{O}(U),$$

then

$$\text{tr}(A(0)) \neq 0, \quad A(0) - \frac{1}{2} \text{tr}(A(0)) \cdot I_2 \neq 0. \quad (1.8)$$

**Remark 1.3.4.** The second condition above is verified for some choice of coordinates and frame near each puncture if and only if it is verified for all such choices. It amounts to the most polar parts of non-Fuchsian singularities of the connection having two distinct, but non-opposite eigenvalues, or having two identical non-zero eigenvalues, but only one Jordan block.

The universal isomonodromic deformation

$$(p : \mathcal{X} \rightarrow \mathcal{T}, \mathcal{D}^{\text{red}}, \mathcal{E} \rightarrow \mathcal{X}, \nabla, t_0, \psi, \Psi) \quad (1.9)$$

of (L7) can be obtained by the following construction, which is very similar to the one of the parallel isomonodromic deformation of (L7). As before, we choose a Teichmüller structure on the stable curve  $(X_0, D_0^{\text{red}})$ , providing a point  $\star \in \mathcal{T}_{g,n}$  and an ordering on the poles of  $(E_0, \nabla_0)$ , such that the polar divisor writes  $D_0 = \sum_{i=1}^n n_i [x_i]$ . For each  $i \in \{1, \dots, n\}$  denote by  $J^{(i)} \simeq \mathbb{C}^{n_i-1}$  the  $(n_i - 1)$ -jet space as in § 1.3.3. Set

$$\mathcal{T} := \mathcal{T}_{g,n} \times J \quad , \quad J := \prod_{i=1}^n \left( J^{(i)} \times J^{(i)} \right)$$

and consider

$$p : (\mathcal{X}, \mathcal{D}^{\text{red}}) \rightarrow \mathcal{T}, \quad \mathcal{D}^{\text{red}} = \sum_{i=1}^n \mathcal{D}^i,$$

the pull-back of the universal Teichmüller curve (L5) with respect to the projection to the first factor of  $\mathcal{T}$ . We set  $t_0 := (\star, \underline{0}) \in \mathcal{T}$ . We choose  $\psi : (X_0, D_0) \xrightarrow{\sim} (\mathcal{X}_{t_0}, \mathcal{D}_{t_0}^{\text{red}}) := p^{-1}(t_0)$  and  $\Psi : (E_0, \nabla_0) \xrightarrow{\sim}$

$(\mathcal{E}_{t_0}, \nabla_{t_0})$  projecting to  $\psi$  as before. The meromorphic Malgrange connection  $(\mathcal{E}, \nabla)$  with polar divisor  $\mathcal{D} = \sum_{i=1}^n n_i \mathcal{D}^i$ , whose restriction to  $p^{-1}(t_0)$  coincides with  $(\mathcal{E}_{t_0}, \nabla_{t_0})$ , is constructed exactly as before, except near the polar components  $\mathcal{D}^i$  with  $n_i > 0$ . Consider such a polar component and the meromorphic connection on the trivial bundle over  $(\mathbb{C}, 0)$  obtained by restricting  $(E_0, \nabla_0)$  to a germ of neighborhood of  $x_i$  and choosing a frame there. It can be written  $\nabla_0^1 \otimes \nabla_0^2$  with

$$\begin{aligned} \nabla_0^1 : Y &\mapsto (d + A dx) \cdot Y & x^{n_i} A(x) &\in \mathfrak{sl}_2 \mathbb{C}\{x\} \\ \nabla_0^2 : y &\mapsto (d + a dx) \cdot y & x^{n_i} a(x) &\in \mathbb{C}\{x\}, \end{aligned}$$

where we moreover have  $x^{n_i} A(x)|_{x=0} \neq 0$ ,  $x^{n_i} a(x)|_{x=0} \neq 0$  by our assumption on the non-logarithmic poles. In a neighborhood  $\mathcal{U}_i$  of  $\mathcal{D}^i$  in  $\mathcal{X}$ , we have preferred coordinates of the form

$$(z, \underline{t}, \alpha_1, \dots, \alpha_n) \quad \text{with} \quad \underline{t} \in \mathcal{T}_{g,n}, \quad \alpha_k = (\underline{a}^{(k,1)}, \underline{a}^{(k,2)}) \in J^{(k)} \times J^{(k)}, \quad z = z^{(i)},$$

such that  $\mathcal{D}^i$  there is given by  $\{z = 0\}$ . Assume that the map  $\psi$  is given with respect to these coordinates by  $x \mapsto (z, \underline{t}, \underline{\alpha}) = (\varphi(x), \star, \underline{\alpha})$ . Denoting by  $\varphi_{\underline{a}^{(i,\ell)}}(\varepsilon)$  the  $(n_i - 1)$ -jet of biholomorphism associated to  $\underline{a}^{(i,\ell)}$ , we obtain connections

$$\begin{aligned} \nabla^1 : Y &\mapsto \left( d + A(\varphi^{-1}(\varphi_{\underline{a}^{(i,1)}}(z))) d(\varphi^{-1}(\varphi_{\underline{a}^{(i,1)}}(z))) \right) \cdot Y \\ \nabla^2 : y &\mapsto \left( d + \alpha(\varphi^{-1}(\varphi_{\underline{a}^{(i,2)}}(z))) d(\varphi^{-1}(\varphi_{\underline{a}^{(i,2)}}(z))) \right) \cdot y. \end{aligned}$$

By the same argument as before, germifying in the  $z$ -direction, we may assume that the connection matrices of these connections are holomorphic after multiplying by  $z^{n_i}$  and non-vanishing if further restricted to  $\mathcal{D}^i$ . Moreover, the connections  $\nabla^1$  and  $\nabla^2$  are obviously flat over their domain of definition, and so is their tensor product  $\nabla^{(i)} := \nabla^1 \otimes \nabla^2$ . We now glue the  $\nabla^{(i)}$ 's with the construction outside the  $\mathcal{D}^i$ 's to obtain  $(\mathcal{E}, \nabla)$ . Note that

$$\dim(\mathcal{T}) = 3g - 3 + n + 2 \sum_{i=1}^n (n_i - 1).$$

We have the following universal property.

**Theorem 1.3.5.** *Consider the universal isomonodromic deformation [\(1.9\)](#) of [\(1.7\)](#). Let  $(p' : \mathcal{X}' \rightarrow \mathcal{T}', \mathcal{D}'^{\text{red}}, \mathcal{E}', \nabla', t'_0, \psi', \Psi')$ , be another isomonodromic deformation of [\(1.7\)](#). Then, up to restricting to a sufficiently small neighborhood of  $t'_0$  in  $\mathcal{T}'$ , there exists a classifying map*

$$\mathfrak{c} : (\mathcal{T}', t'_0) \rightarrow (\mathcal{T}, t_0)$$

and a unique isomorphism

$$(\mathcal{X}', \mathcal{D}'^{\text{red}}, \mathcal{E}', \nabla') \xrightarrow{\sim} \mathfrak{c}^*(\mathcal{X}, \mathcal{D}^{\text{red}}, \mathcal{E}, \nabla)$$

which over  $t'_0$  induces the identity on [\(1.7\)](#) via  $(\psi', \Psi')$  and  $(\psi, \Psi)$ .

*Proof.* Let us denote by  $m - 1$  the dimension of  $\mathcal{T}'$ . We may assume  $m > 1$ , otherwise there is nothing to prove. We may also assume  $\mathcal{T}'$  to be sufficiently small. Then we may endow the family  $p' : (\mathcal{X}', \mathcal{D}'^{\text{red}}) \rightarrow \mathcal{T}'$  with a Teichmüller structure  $\Phi'$  compatible with the one on  $(X_0, D_0^{\text{red}})$  associated to the point  $\star \in \mathcal{T}_{g,n}$ . We obtain a Teichmüller classifying map  $\text{class}^+ : (\mathcal{T}', t'_0) \rightarrow (\mathcal{T}_{g,n}, \star)$ . The classifying map  $\mathfrak{c}$  will be of the form  $\mathfrak{c} = (\text{class}^+, \mathfrak{c}_1, \dots, \mathfrak{c}_n)$  with  $\mathfrak{c}_i = \left( \mathfrak{c}_i^{(1)}, \mathfrak{c}_i^{(2)} \right) : \mathcal{T}' \rightarrow J^{(i)} \times J^{(i)}$ . Only those  $\mathfrak{c}_i$  where  $n_i > 1$  need our attention. Let us consider such an index  $i$ . We may assume without loss



of generality that the family  $p'$  coincides with the pull-back, *via* class<sup>+</sup>, of the universal Teichmüller curve, so that we have preferred coordinates  $(z, \underline{t}')$  in a neighborhood of the  $i$ -th component of  $\mathcal{D}'^{\text{red}}$ . Since  $\mathcal{T}'$  is sufficiently small, we may consider this neighborhood to be topologically a small polydisc  $\mathbb{D}$  in  $\mathbb{C}^m$ . Hence the vector bundle  $\mathcal{E}'|_{\mathbb{D}}$  admits a frame  $\underline{e}'$ , and we consider the connection matrix  $\Omega$  of  $\nabla'$  with respect to this frame. We have  $\Omega = \Omega_0 + \omega \cdot I_2$  with  $\text{trace}(\Omega_0) = 0$  and  $\omega$ . We may write  $\Omega_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ . From the flatness of  $\nabla'$  we deduce  $d\Omega_0 + \Omega_0 \wedge \Omega_0 = 0$  and  $d\omega = 0$ , in other words,

$$d\alpha = \gamma \wedge \beta, \quad d\beta = 2\beta \wedge \alpha, \quad d\gamma = 2\alpha \wedge \gamma, \quad d\omega = 0. \quad (1.10)$$

The assumption (1.8) on the initial connection (1.7) implies, *via*  $(\psi', \Psi')$ , that the holomorphic one-forms  $z^{n_i}\Omega_0$  and  $z^{n_i}\omega$  are not zero when evaluated at  $(z, \underline{t}') = (0, t'_0)$ . Up to a constant gauge transformation, we may assume that this holds also for the holomorphic one-form  $z^{n_i}\beta$ . Since  $\mathbb{D}$  is sufficiently small, we may assume that the holomorphic one-forms  $z^{n_i}\beta$  and  $z^{n_i}\omega$  are non-vanishing on  $\mathbb{D}$ . By (1.10), these two one-forms define non-singular holomorphic foliations on  $\mathbb{D}$ . Hence up to shrinking, by the Frobenius integrability theorem there exist holomorphic functions  $f_1, f_2$  on  $\mathbb{D}$  and non-vanishing holomorphic functions  $\lambda_1, \lambda_2$  on  $\mathbb{D}$  such that

$$z^{n_i}\beta = \lambda_1 df_1 \quad \text{and} \quad z^{n_i}\omega = \lambda_2 df_2.$$

Since in both cases, the divisor  $\{z = 0\}$  is a leaf of the foliation and  $\frac{\partial f_j}{\partial z}$  is non-vanishing, we may assume, up to shrinking, that

- $(f_j, \underline{t}')$  is a system of coordinates on  $\mathbb{D}$  for  $j \in \{1, 2\}$ ,
- $\{z = 0\} = \{f_1 = 0\} = \{f_2 = 0\}$ ; it suffices to modify the  $f_j$ 's by some constants,
- in restriction to  $\{\underline{t}' = t'_0\}$ , both  $f_1$  and  $f_2$  are the identity; it suffices to compose  $f_j$  with  $g_j(\cdot, t'_0)$ , where  $g_j$  is defined by the coordinate change  $(z, \underline{t}') = (g_j(f_j, \underline{t}'), \underline{t}')$ ,
- the functions  $\frac{\partial f_j}{\partial z}|_{z=0}$  for  $j \in \{1, 2\}$  admit a complex logarithm that yields 0 when evaluated at  $t'_0$ ; indeed, by the previous assumption, these functions yield 1 when evaluated at  $t'_0$ .

For these convenient  $f_1, f_2$ , we now define

$$\mathbf{c}_i^{(j)}(\underline{t}') := (\varepsilon \mapsto f_j(\varepsilon, \underline{t}') \pmod{\varepsilon^{n_i}}), \quad j \in \{1, 2\}.$$

We clearly have a unique isomorphism  $(\mathcal{X}', \mathcal{D}'^{\text{red}}) \xrightarrow{\sim} \mathbf{c}^*(\mathcal{X}, \mathcal{D}^{\text{red}})$  which over  $t'_0$  induces the identity on  $(X_0, D_0^{\text{red}})$  *via*  $\psi'$  and  $\psi$ . By Riemann-Hilbert correspondence and Malgrange's Lemma, in restriction to the complement of the non-logarithmic poles, we moreover have a unique isomorphism  $(\mathcal{E}', \nabla') \xrightarrow{\sim} \mathbf{c}^*(\mathcal{E}, \nabla)$  which over  $t'_0$  induces the identity on  $(E_0, \nabla_0)$  *via*  $\Psi'$  and  $\Psi$ . It remains to prove that this isomorphism extends over the non-logarithmic poles.

Let us consider such a pole of index  $i$  and adopt the notation above. We denote  $\varphi^{(j)}(z, \underline{t}') := \mathbf{c}_i^{(j)}(\underline{t}')(z)$ . Up to shrinking, since  $\nabla'$  is a Malgrange connection and the  $f_j$ 's were convenient, with respect to these coordinates we have

$$\beta = \beta_0 \left( \varphi^{(1)}, \underline{t} \right) \frac{d\varphi^{(1)}}{(\varphi^{(1)})^{n_i}} + \sum_{k=1}^{m-1} \beta_k \left( \varphi^{(1)}, \underline{t}' \right) dt'_k, \quad \omega = \omega_0 \left( \varphi^{(2)}, \underline{t} \right) \frac{d\varphi^{(2)}}{(\varphi^{(2)})^{n_i}} + \sum_{k=1}^{m-1} \omega_k \left( \varphi^{(2)}, \underline{t}' \right) dt'_k$$

with  $\beta_0, \beta_k, \omega_0, \omega_k$  holomorphic. However, (1.10) and the fact that  $\beta_0$  is non-vanishing forces  $\alpha$  and  $\gamma$  to be of an analogous form with respect to  $(\varphi^{(1)}, \underline{t})$ . From the integrability of  $\Omega_0$  and  $\omega$  we then deduce, as in Lemma 1.2.3, that there exists  $\tilde{\psi}_0 \in \text{SL}_2(\mathcal{O}(\mathbb{D}))$  and  $\tilde{\lambda} \in \mathcal{O}^*(\mathbb{D})$  with  $\tilde{\psi}_0|_{\{\underline{t}'=t'_0\}} = I_2$  and  $\tilde{\lambda}|_{\{\underline{t}'=t'_0\}} = 1$  such that

$$\tilde{\psi}_0^{-1}\Omega_0\tilde{\psi}_0 + \tilde{\psi}_0^{-1}d\tilde{\psi}_0 = M \left( \varphi^{(1)} \right) \frac{d\varphi^{(1)}}{(\varphi^{(1)})^{n_i}}, \quad \omega + \tilde{\lambda}^{-1}d\tilde{\lambda} = \mu \left( \varphi^{(2)} \right) \frac{d\varphi^{(2)}}{(\varphi^{(2)})^{n_i}}$$

for some holomorphic  $M \in \mathfrak{sl}_2(\mathbb{C}\{\varepsilon\})$ ,  $\mu \in \mathbb{C}\{\varepsilon\}$ . In other words, with respect to the frame  $\tilde{\underline{e}}' := \underline{e}' \cdot \tilde{\psi}_0 \cdot \tilde{\lambda}$  of  $E'$  and the coordinates  $(z, \underline{t}')$ , the connection matrix of  $\nabla'$  is given by

$$M \left( \varphi^{(1)}(z, \underline{t}') \right) \frac{d \left( \varphi^{(1)}(z, \underline{t}') \right)}{\left( \varphi^{(1)}(z, \underline{t}') \right)^{n_i}} + I_2 \cdot \mu \left( \varphi^{(2)}(z, \underline{t}') \right) \frac{d \left( \varphi^{(2)}(z, \underline{t}') \right)}{\left( \varphi^{(2)}(z, \underline{t}') \right)^{n_i}}.$$

It is now immediate to check that on  $\mathbb{D}$ , the unique isomorphism  $(\mathcal{E}', \nabla') \xrightarrow{\sim} \mathfrak{c}^*(\mathcal{E}, \nabla)$  defined in the union between the fiber over  $\underline{t}'_0$  and the complement of  $\{z = 0\}$ , extends (uniquely) to  $\mathbb{D}$ .  $\square$

Consider the embedding

$$\iota : \begin{cases} J_{\parallel} = \prod_{i=1}^n J^{(i)} & \rightarrow & J = \prod_{i=1}^n (J^{(i)} \times J^{(i)}) \\ (\underline{a}^{(1)}, \dots, \underline{a}^{(n)}) & \mapsto & (\underline{a}^{(1)}, \underline{a}^{(1)}, \dots, \underline{a}^{(n)}, \underline{a}^{(n)}) . \end{cases}$$

and the corresponding embedding

$$\iota' : (\mathcal{T}_{\parallel} = \mathcal{T}_{g,n} \times J_{\parallel}, (\star, \underline{0})) \rightarrow (\mathcal{T} = \mathcal{T}_{g,n} \times J, (\star, \underline{0})).$$

The pull-back of the universal isomonodromic deformation of [\(1.7\)](#) by  $\iota'$  is by construction the parallel isomonodromic deformation of [\(1.7\)](#). Moreover, the parameter space  $\mathcal{T}$  admits a foliation by leaves isomorphic to  $\mathcal{T}_{\parallel}$ . Indeed, consider the holomorphic map

$$\pi : \begin{cases} J = \prod_{i=1}^n (J^{(i)} \times J^{(i)}) & \rightarrow & \prod_{i=1}^n J^{(i)} \\ \left( \varphi_{\underline{a}^{(1,1)}}, \varphi_{\underline{a}^{(1,2)}}, \dots, \varphi_{\underline{a}^{(n,1)}}, \varphi_{\underline{a}^{(n,2)}} \right) & \mapsto & \left( \varphi_{\underline{a}^{(1,1)}} \circ \left( \varphi_{\underline{a}^{(1,2)}} \right)^{-1}, \dots, \varphi_{\underline{a}^{(n,1)}} \circ \left( \varphi_{\underline{a}^{(n,2)}} \right)^{-1} \right) . \end{cases}$$

The kernel of the differential  $\pi_*$  defines a non-singular codimension  $\sum_{i=1}^n (n_i - 1)$ -foliation on  $J$ . Its leaves, the level sets of  $\pi$ , are isomorphic to  $J_{\parallel}$ . Pulling back this foliation to  $\mathcal{T}$ , we obtain a non-singular foliation whose leaves are isomorphic to  $\mathcal{T}_{\parallel}$ . These leaves correspond to parallel isomonodromic deformations. Indeed, if  $(\mathcal{E}, \nabla)$  denotes the connection underlying the universal isomonodromic deformation [\(1.9\)](#) of [\(1.7\)](#), and we choose a point  $t_1 \in \mathcal{T}$  in its parameter space, then by construction, we see that the restriction of  $(\mathcal{E}, \nabla)$  to the leaf containing  $t_1$  of the parallel foliation in the parameter space is canonically isomorphic to the connection underlying the parallel isomonodromic deformation of  $(\mathcal{E}, \nabla)|_{p^{-1}(t_1)}$ .

**Remark 1.3.6.** *Following the same argumentation as in Remark [1.3.3](#), if  $(E_0, \nabla_0)$  is irreducible and  $(\mathcal{E}, \nabla)$  denotes the flat connection underlying the universal isomonodromic deformation of [\(1.7\)](#), then  $(\mathcal{E}, \nabla)|_{p^{-1}(t)}$  is irreducible for any parameter  $t \in \mathcal{T}$ .*

### 1.3.5 The parallel foliation in the general case

For the construction of the germ of universal isomonodromic deformation in the non-logarithmic case of arbitrary rank, with however various additional assumptions, we refer to the literature. All we need in order to establish the relation between the germ of universal and the germ of parallel isomonodromic deformation of a given initial connection is some information concerning the parameter space  $\mathcal{T}$  of the former, as well as the corresponding classifying map. As we will now see, under suitable conditions and up to germification, this parameter space  $\mathcal{T}$  still admits a foliation by leaves isomorphic to  $\mathcal{T}_{\parallel}$ , and the leaf containing the initial parameter corresponds to the parallel isomonodromic deformation. Let

$$(X_0, D_0^{\text{red}}, E_0, \nabla_0) \tag{1.11}$$

be as in [\(1.3\)](#), with the following additional assumption:

(\*) If  $x$  is a local coordinate on  $U \subset X_0$  near a pole of order  $n_i > 1$  of  $(E_0, \nabla_0)$ , and

$$\Omega = \frac{1}{x^{n_i}} A(x) dx, \quad A \in M_{r \times r} \mathcal{O}(U)$$

is the connection matrix of  $\nabla_0$  with respect to a frame  $\underline{e}$  of  $E_0$  over  $U$ , then the matrix  $A(0)$  admits  $r$  distinct non-zero eigenvalues.

As is well known [Sib62], under this condition, for any integer  $\ell$ , up to shrinking  $U$  and conveniently choosing  $\underline{e}$ , one may assume the Taylor series expansion of  $A$  up to order  $\ell$  to take values in  $\text{Diag}_r(\mathbb{C}[x])$ , where  $\text{Diag}_r$  denotes diagonal matrices in  $M_{r \times r}$ . In particular, we may assume the non-logarithmic part of  $\Omega$ , i.e., the part corresponding to the Taylor series expansion up to order  $n_i - 2$  of  $A$ , to be diagonal for suitable frames. We may now associate to  $(X_0, D_0^{\text{red}}, E_0, \nabla_0)$  a point

$$(\star, \underline{\text{pp}}_0) \in \mathcal{T} := \mathcal{T}_{g,n} \times \prod_{i=1}^n \mathcal{PP}_r^{(i)},$$

where  $\mathcal{PP}_r^{(i)} := \{0\}$  if  $n_i = 1$  and

$$\mathcal{PP}_r^{(i)} := \left\{ \left( e^{\Lambda_0} + \sum_{k=1}^{n_i-2} \Lambda_k \varepsilon^k \right) \frac{d\varepsilon}{\varepsilon^{n_i}} \mid \Lambda_0, \Lambda_k \in \text{Diag}_r \mathbb{C} \right\} \simeq (\widetilde{\mathbb{C}}^* \times \mathbb{C}^{n_i-2})^r$$

if  $n_i > 1$ . Let us be more precise. Recall that near each irreducible component  $\mathcal{D}_{g,n}^i$  of the divisor parametrizing the marked points in the universal Teichmüller curve  $p_{g,n} : \mathcal{X}_{g,n} \rightarrow \mathcal{T}_{g,n}$ , we have preferred coordinates  $(z^{(i)}, \underline{t})$ . A point  $(t_0, \underline{\text{pp}})$  in  $\mathcal{T}$  should be read as a point  $t_0$  in the Teichmüller space and a collection of possible non-logarithmic parts at the marked points of a connection over the fiber  $p_{g,n}^{-1}(\{t_0\})$  of the universal Teichmüller curve, where for each  $i \in \{1, \dots, n\}$ , the variable  $\varepsilon$  in  $\mathcal{PP}_r^{(i)}$  is seen as the local coordinate  $z^{(i)}|_{\{\underline{t}=t_0\}}$  near the  $i$ -th marked point on the curve  $p_{g,n}^{-1}(\{t_0\})$ . Such a point  $(t_0, \underline{\text{pp}})$  in  $\mathcal{T}$  is an isomorphism class of *irregular curve* (in terms of [Boa02]), endowed with a Teichmüller structure and a choice of logarithm of the leading order term at each pole (with respect to preferred coordinates). Following [Mal83b, Boa02], up to restricting<sup>3</sup> to a sufficiently small neighborhood of  $(\star, \underline{\text{pp}}_0)$  in  $\mathcal{T}$ , there exists an isomonodromic deformation

$$(p : \mathcal{X} \rightarrow \mathcal{T}, \mathcal{D}^{\text{red}}, \mathcal{E} \rightarrow \mathcal{X}, \nabla, (\star, \underline{\text{pp}}_0), \psi, \Psi) \tag{1.12}$$

of (1.11) parametrized by  $\mathcal{T}$ , where  $(\mathcal{X}, \mathcal{D}^{\text{red}})$  is the pull-back of the universal Teichmüller curve with respect to the projection  $\mathcal{T} \rightarrow \mathcal{T}_{g,n}$  to the first factor of  $\mathcal{T}$ , such that this germ of isomonodromic deformation is *versal* with respect to the natural classifying map of underlying irregular curves with preferred coordinates at the marked points. This means that for each parameter  $(t_0, \underline{\text{pp}})$ , the connection matrix of the connection obtained from  $(\mathcal{E}, \nabla)$  by restricting to  $p^{-1}(\{(t_0, \underline{\text{pp}})\})$  has non-logarithmic polar part  $\underline{\text{pp}}$  when read with respect to  $z^{(i)}|_{\{\underline{t}=t_0\}}$  and a suitable frame. Consider the non-singular holomorphic foliation of codimension  $(r-1) \sum_{i=1}^n (n_i - 1)$  on  $\mathcal{T}$  defined by the pull-back to  $\mathcal{T}$  of the level sets of the map

$$\pi : \mathcal{PP}_r := \prod_{i=1}^n \mathcal{PP}_r^{(i)} \longrightarrow \mathcal{PP}_{r-1} := \prod_{i=1}^n \mathcal{PP}_{r-1}^{(i)}$$

---

<sup>3</sup>Germification here can actually be avoided by deleting the locus of multiple eigenvalues of  $e^{\Lambda_0}$  in each  $\mathcal{PP}_r^{(i)}$  and lifting to the universal cover of the resulting parameter space.

constructed as follows. Consider firstly the projections  $f_1 : \mathcal{PP}_r \rightarrow \mathcal{PP}_1$  and  $f_2 : \mathcal{PP}_r \rightarrow \mathcal{PP}_{r-1}$  obtained by selecting the first entry and the lower right submatrix of size  $(r-1) \times (r-1)$  respectively, of each diagonal matrix of meromorphic one-forms. There is a well-defined holomorphic map  $f : \mathcal{PP}_r \rightarrow J_{\parallel} = \prod_{i=1}^n J^{(i)}$  satisfying  $f(\underline{\text{pp}}_0) = \underline{0}$  and

$$f_1(f(\underline{\Omega})^*\underline{\Omega}) = f_1(\underline{\text{pp}}_0) \bmod \left( \frac{1}{\varepsilon} \mathbb{C}\{\varepsilon\} \right)^n .$$

Here for the pull-back we consider the action of  $J^{(i)}$  on  $\mathcal{PP}_r^{(i)}$  for each  $i$ , modulo logarithmic terms, and differentiation is to be considered with respect to the variable  $\varepsilon$  only. The existence and uniqueness of  $f$  can easily be established by considering its components  $f^{(i)}$  for  $i \in \{1, \dots, n\}$  and performing an induction on  $k \in \{2, \dots, n_i\}$ , considering  $f^{(i)} \bmod \varepsilon^k$ . The map  $\pi$  is now defined by

$$\underline{\Omega} \mapsto f_2(f(\underline{\Omega})^*\underline{\Omega}) \bmod \left( \frac{1}{\varepsilon} \mathbb{C}^{r-1}\{\varepsilon\} \right)^n .$$

Let us denote by  $\mathcal{T}_{\parallel}$  the parameter space of the parallel isomonodromic deformation of [\(1.11\)](#). As one can easily check, the composition of the classifying map  $\mathcal{T}_{\parallel} \rightarrow \mathcal{T}$  with  $\text{id}_{\mathcal{T}_{g,n}} \times f$  is simply the pull-back to  $\mathcal{T}_{\parallel}$  of the involution on  $J_{\parallel}$  which to each jet associates its inverse. In particular, the classifying map  $\mathcal{T}_{\parallel} \rightarrow \mathcal{T}$  is injective and its image parametrizes the leaf containing  $(t_0, \underline{\text{pp}})$  of the foliation given by  $\pi_*$ . It is known by [\[Mal81, Mal86\]](#) that under suitable conditions, the germ of versal isomonodromic deformation [\(1.12\)](#) of [\(1.11\)](#) satisfies a universal property with respect to germs of isomonodromic deformations of [\(1.11\)](#) and the natural classifying map. When that is the case, the universal property ensures that the connection  $(\mathcal{E}, \nabla)$  in [\(1.12\)](#), restricted to the preimage under  $p$  of the germ of leaf in  $\mathcal{T}$  containing  $(t_0, \underline{\text{pp}})$ , is uniquely isomorphic to the connection underlying the germ of parallel isomonodromic deformation of [\(1.11\)](#).

## 1.4 Notions of stability of vector bundles over curves

In this section, we recall the notion of Mumford stability of vector bundles over curves and a refinement of this notion using Segre-invariants and Higgs bundles in the rank 2 case. We also recall that these are open conditions in families due to results of Maruyama, Shatz and Laumon. Furthermore, we establish an elementary vector bundle lemma, which will be needed later on. For the moment, it may just be considered as an example for the manipulation of vector bundles with respect to questions of stability.

### 1.4.1 Arbitrary rank

**Definition 1.4.1.** *Let  $X$  be a compact Riemann surface and let  $E \rightarrow X$  be a vector bundle. Then  $E$  is called (Mumford)-semi-stable (resp. stable) if for any non-zero coherent subsheaf  $F$  of  $E$  we have*

$$\mu(F) := \frac{\deg(F)}{\text{rank}(F)} \leq \mu(E) \quad (\text{resp. } \mu(F) < \mu(E) \text{ or } F = E) .$$

Note that for any coherent subsheaf  $F$  as above there exists a unique subbundle  $F^{\text{sat}} \subset E$  such that  $\text{rank}(F) = \text{rank}(F^{\text{sat}})$  and  $F \subset F^{\text{sat}}$ . In particular,  $\mu(F^{\text{sat}}) \geq \mu(F)$ . The above definition is therefore equivalent to the one we obtain by imposing that  $F$  is (the sheaf of sections of) a subbundle of  $E$ . More details can be found for example in [\[LP97\]](#).

The notion of stable bundles has been introduced by Mumford in [Mum62], where he established that the moduli space of isomorphism classes of stable bundles with fixed rank and degree over  $X$  admits a natural structure of non-singular quasi-projective variety. Semistable bundles, or rather their  $S$ -equivalence classes, have to be taken into account when describing its closure. Stable bundles of degree 0 in genus  $g \geq 2$  are also precisely those which can be endowed with a holomorphic connection whose monodromy representation is unitary [NS65], see also [Don83].

The dual and the tensor product of semistable bundles are semistable [AB81, p. 588]. Moreover, we have the following useful general fact. If  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an exact sequence of coherent sheaves over  $X$ , then the sequence of rationals  $(\mu(A), \mu(B), \mu(C))$  is monotonous: we either have  $\mu(A) \leq \mu(B) \leq \mu(C)$  or  $\mu(A) \geq \mu(B) \geq \mu(C)$  and if one of the inequalities is strict, the other is, too.

Every vector bundle  $E \rightarrow X$  admits a canonical *Harder-Narasimhan filtration*, i.e., a filtration

$$0 = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_N = E$$

such that for all  $i \in \llbracket 1, N \rrbracket$ , the vector bundle  $V_i := F_i/F_{i-1}$  is semistable and for all  $i \in \llbracket 1, N \rrbracket$ , one has  $\mu(V_i) > \mu(V_{i+1})$  [Ses80, p. 15]. If  $N = 1$ , then  $E$  is semistable.

Every semistable vector bundle  $E \rightarrow X$  admits a *Jordan-Hölder filtration*, i.e., a filtration

$$0 = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_N = E$$

such that for all  $i \in \llbracket 1, N \rrbracket$ , the vector bundle  $V_i := F_i/F_{i-1}$  is stable of slope  $\mu(V_i) = \mu(E)$ . Moreover, the graded bundle of a Jordan-Hölder filtration up to permutation of the direct summands is canonical [Ses80, p. 18]. If  $N = 1$ , then  $E$  is stable.

**Theorem 1.4.2** (Shatz). *Let  $\mathcal{E} \rightarrow \mathcal{X} \xrightarrow{p} \mathcal{T}$  be a holomorphic family of vector bundles over compact Riemann surfaces. For any  $t \in \mathcal{T}$ , denote  $\mathcal{X}_t := p^{-1}(t)$  and  $\mathcal{E}_t := \mathcal{E}|_{\mathcal{X}_t}$ . Define*

$$\mathcal{T}^{\text{nss}} := \{t \in \mathcal{T} \mid \mathcal{E}_t \text{ not semistable}\} \quad , \quad \mathcal{T}^{\text{ns}} := \{t \in \mathcal{T} \mid \mathcal{E}_t \text{ not stable}\}.$$

*Then  $\mathcal{T}^{\text{nss}}$  and  $\mathcal{T}^{\text{ns}}$  are (possibly empty) closed analytic subsets of  $\mathcal{T}$ .*

*Moreover, for a generic point<sup>A</sup>  $t_0$  in  $\mathcal{T}^{\text{nss}}$  there exists a Euclidian neighborhood  $B$  of  $t_0$  in  $\mathcal{T}^{\text{nss}}$  and a filtration by subbundles*

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \dots \subsetneq \mathcal{F}_{N+1} = \mathcal{E}|_{p^{-1}(B)}$$

*inducing, for any  $t \in B$ , the Harder-Narasimhan filtration of  $\mathcal{E}_t$  by restriction to  $\mathcal{X}_t$ .*

*Similarly, for a generic point  $t_0 \in \mathcal{T}^{\text{ns}} \setminus \mathcal{T}^{\text{nss}}$  there exists a Euclidian neighborhood  $B$  of  $t_0$  in  $\mathcal{T}^{\text{ns}} \setminus \mathcal{T}^{\text{nss}}$  and a filtration of  $\mathcal{E}|_{p^{-1}(B)}$  by subbundles inducing, for any  $t \in B$ , a Jordan-Hölder filtration of  $\mathcal{E}_t$  by restriction to  $\mathcal{X}_t$ .*

*Proof.* One can check that Shatz's results in [Sha77], see also [AB81], [Nit86], [GN14], applied to vector bundles over curves can be transferred into the analytic setting.  $\square$

## 1.4.2 Rank two

**Definition 1.4.3.** *Let  $E \rightarrow X$  be a rank 2 vector bundle over a compact Riemann surface  $X$ . The Segre-invariant of  $E$  is defined as*

$$\kappa(E) := \min_L(\deg(E) - 2\deg(L)),$$

---

<sup>A</sup>By *generic point* we mean a point contained in  $B_1 \setminus A$ , where  $B_1$  is a neighborhood of an arbitrary non-singular point  $t_1$  and  $A$  is a proper closed analytic subset of  $B_1$ .

where the minimum is taken over all holomorphic line subbundles  $L$  of  $E$ . We say that  $E$  is maximally stable if  $\kappa(E) \geq g - 1$ .

Note that a rank two vector bundle  $E \rightarrow X$  is stable if and only if  $\kappa(E) > 0$  and semistable if and only if  $\kappa(E) \geq 0$ . The Segre-invariant  $\kappa(E)$  of a rank two vector bundle  $E \rightarrow X$  is bounded above by the genus  $g$  of  $X$  [Nag70]. One of the standard references in the study of maximally stable bundles is [LN83].

**Theorem 1.4.4** (Maruyama). *Let  $\mathcal{E} \rightarrow \mathcal{X} \xrightarrow{p} \mathcal{T}$  be a holomorphic family of rank 2 vector bundles over compact Riemann surfaces. For any  $t \in \mathcal{T}$ , denote  $\mathcal{X}_t := p^{-1}(t)$  and  $\mathcal{E}_t := \mathcal{E}|_{\mathcal{X}_t}$ . For any  $k \in \mathbb{Z}$ , define*

$$\mathcal{T}_k := \{t \in \mathcal{T} \mid \kappa(\mathcal{E}_t) \leq k\}.$$

Then for any  $k \in \mathbb{Z}$ , the set  $\mathcal{T}_k$  is a closed analytic subset of  $\mathcal{T}$ .

Moreover, for a generic point  $t_0 \in \mathcal{T}_k \setminus \mathcal{T}_{k-1}$  there exists a Euclidian neighborhood  $B$  of  $t_0$  in  $\mathcal{T}_k$  such that there exists a line subbundle  $\mathcal{L}$  of  $\mathcal{E}|_{p^{-1}(B)}$  satisfying

$$\deg(\mathcal{E}_t) - 2\deg(\mathcal{L}_t) = \kappa(\mathcal{E}_t) = k \quad \forall t \in B,$$

where  $\mathcal{L}_t := \mathcal{L}|_{\mathcal{X}_t}$ .

*Proof.* One can check that Maruyama's results in [Mar76] applied to vector bundles over curves can be transferred into the analytic setting.  $\square$

**Definition 1.4.5.** *Let  $E \rightarrow X$  be a rank 2 vector bundle over a compact Riemann surface  $X$ . The vector bundle  $E$  is said to be very stable if there exists no non-zero nilpotent Higgs field on  $E$ . If  $E$  is maximally stable but not very stable, then  $E$  is said to be wobbly.*

Here a Higgs field  $\theta$  of  $E$  is a global section of  $\text{End}(E) \otimes \Omega_X^1$ . To a Higgs field  $\theta$  one can associate, by composition, a global section  $\theta^2$  of  $\text{End}(E) \otimes \Omega_X^1 \otimes \Omega_X^1$ . The Higgs field  $\theta$  is nilpotent if  $\theta^{\text{rank}(E)} = 0$ .

Very stable bundles (in arbitrary rank) were introduced by Laumon [Lau88]. They appear as a particular case in the study of Hitchin systems. The notion of wobbly bundles has been introduced in [DP08], although we took the liberty here to modify their definition in order to include the  $g \leq 1$  case and to exclude the non-maximally stable case.

Note that a rank 2 bundle  $E$  is very stable if and only if for every line subbundle  $L$  of  $E$  one has

$$h^0(X, Q^\vee \otimes L \otimes \Omega_X^1) = 0, \tag{1.13}$$

where  $Q = E/L$ . Indeed, if  $\theta$  is a non-zero nilpotent Higgs field on  $E$  then setting  $L = \ker(\theta)$  we get a non-zero morphism  $Q \rightarrow L \otimes \Omega_X^1$ . Conversely, if we have a non-zero morphism  $Q \rightarrow L \otimes \Omega_X^1$ , then we obtain a non-zero nilpotent Higgs field by composing with  $E \rightarrow Q$  and  $L \otimes \Omega_X^1 \rightarrow E \otimes \Omega_X^1$ .

Any line bundle of degree at least  $g$  on  $X$  admits a non-zero section by Riemann-Roch. Hence for any rank-two vector bundle  $E \rightarrow X$  we have that if  $E$  is very stable, then it is maximally stable. The converse is true if  $g = 0$  and false in general if  $g > 0$ .

As for stability, very stability is an open condition [Lau88, Prop. 3.5]. We need the following slightly more precise rank-two-statement.

**Proposition 1.4.6.** *Let  $\mathcal{E} \rightarrow \mathcal{X} \xrightarrow{p} \mathcal{T}$  be a holomorphic family of rank 2 vector bundles over compact Riemann surfaces of genus  $g \geq 0$ . For any  $t \in \mathcal{T}$ , denote  $\mathcal{X}_t := p^{-1}(t)$  and  $\mathcal{E}_t := \mathcal{E}|_{\mathcal{X}_t}$ . Define*

$$\mathcal{T}^{\text{nil}} := \{t \in \mathcal{T} \mid \mathcal{E}_t \text{ not very stable}\}.$$

Then  $\mathcal{T}^{\text{nil}}$  is a closed analytic subset of  $\mathcal{T}$ .

Consider

$$\mathcal{T}^{\text{wob}} := \{t \in \mathcal{T}^{\text{nil}} \mid \mathcal{E}_t \text{ maximally stable}\},$$

which by Maruyama's Theorem [\[1.4.4\]](#) is either empty or the complement of a proper closed analytic subset in  $\mathcal{T}^{\text{nil}}$ . For a generic point  $t_0 \in \mathcal{T}^{\text{wob}}$  there exists a Euclidian neighborhood  $B$  of  $t_0$  in  $\mathcal{T}^{\text{wob}}$  such that there exists a global section  $\theta$  of  $\text{End}(\mathcal{E}) \otimes \Omega_{\mathcal{X}}^1|_{p^{-1}(B)}$  inducing a non-zero nilpotent Higgs field  $\theta_t$  on each fiber  $\mathcal{E}_t$  with  $t \in B$ .

*Proof.* In view of Theorem [\[1.4.4\]](#), it is sufficient to show that  $\mathcal{T}^{\text{wob}} = \mathcal{T}^{\text{nil}} \setminus \mathcal{T}_{g-2}$  is a closed analytic subset of  $\mathcal{T} \setminus \mathcal{T}_{g-2}$ . Let  $t_0 \in \mathcal{T}$  such that  $\mathcal{E}_{t_0}$  is maximally stable. We restrict to a sufficiently small neighborhood of  $t_0$ , which we again denote by  $\mathcal{T}$ .

Assume first  $\kappa(\mathcal{E}_{t_0}) > 0$ , which is automatically the case if  $g > 1$ . Then  $t \mapsto h^0(\text{End}(\mathcal{E}_t) \otimes \Omega_{\mathcal{X}_t}^1)$  is constant. Indeed, by Serre duality and stability of  $\mathcal{E}_t$ , we have  $h^1(\text{End}(\mathcal{E}_t) \otimes \Omega_{\mathcal{X}_t}^1) = 1$ . By Grauert's Base Change Theorem [\[Gra60\]](#), see also [\[OSSG80, p. 11\]](#), since  $\mathcal{T}$  is small, there exists  $\underline{\nu} = (\nu_1, \dots, \nu_n)$  generating  $H^0(\mathcal{X}, \text{End}(\mathcal{E}) \otimes \Omega_{\mathcal{X}}^1)$  as a  $\Gamma\mathcal{O}_{\mathcal{T}}$ -module, such that  $\underline{\nu}_t := \underline{\nu}(t)$  is a basis of  $H^0(\mathcal{X}_t, \text{End}(\mathcal{E}_t) \otimes \Omega_{\mathcal{X}_t}^1)$  for any  $t$ . Similarly, we have generating families  $\underline{\omega} = (\omega_1, \dots, \omega_g)$  of  $H^0(\mathcal{X}, \Omega_{\mathcal{X}}^1)$  and  $\underline{\omega}' = (\omega'_1, \dots, \omega'_m)$  of  $H^0(\mathcal{X}, \Omega_{\mathcal{X}}^1 \otimes \Omega_{\mathcal{X}}^1)$ . For  $i = 1, 2$  we have morphisms  $\text{Tr}^i : \text{End}(\mathcal{E}) \otimes \Omega_{\mathcal{X}}^1 \rightarrow \Omega_{\mathcal{X}}^{1 \otimes i}$  given by  $\theta \mapsto \text{trace}(\theta^i)$ . The induced morphisms of global sections are given with respect to the above bases by matrices  $A_1 \in M_{g \times n}(\Gamma\mathcal{O}_{\mathcal{T}})$  and  $A_2 \in M_{m \times n}(\Gamma\mathcal{O}_{\mathcal{T}})$ . Hence

$$\mathcal{T}^{\text{wob}} = \left\{ t \in \mathcal{T} \mid \text{rank} \begin{pmatrix} A_1(t) \\ A_2(t) \end{pmatrix} < n \right\},$$

which is clearly the simultaneous zero-locus of a finite collection of holomorphic functions in  $t$ . Moreover, in restriction to a sufficiently small neighborhood  $B_1$  in  $\mathcal{T}^{\text{wob}}$  of a smooth point  $t_1$  of  $\mathcal{T}^{\text{wob}}$ , we can find a non-zero holomorphic column vector in the kernel of  $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ . Restricting to the locus where this vector does not vanish, we find  $\theta$  as in the statement in the complement of a proper closed analytic subset of  $B_1$ .

The case  $g = 0$  is trivial because then  $\mathcal{T}^{\text{wob}} = \emptyset$ .

It remains to consider the case  $g = 1$ ,  $\kappa(\mathcal{E}_{t_0}) = 0$ . Since  $\deg(\mathcal{E}_t)$  is even for every  $t$ , and we assumed  $\mathcal{T}$  sufficiently small, there exist four distinct line bundles  $\mathcal{L}_i \rightarrow \mathcal{X}$  with  $i \in [[1, 4]]$  such that  $\mathcal{L}_i^{\otimes 2} \otimes \det(\mathcal{E}) \simeq \mathcal{O}_{\mathcal{X}}$ . We claim that

$$\mathcal{T}^{\text{wob}} = \bigcup_{i=1}^4 \{t \in \mathcal{T} \mid h^0(\mathcal{E} \otimes \mathcal{L}_i^{-1}|_{\mathcal{X}_t}) > 0\}.$$

Indeed, a non-zero section of  $\mathcal{E} \otimes \mathcal{L}_i^{-1}|_{\mathcal{X}_t}$  gives rise to an invertible subsheaf  $F$  of  $\mathcal{E} \otimes \mathcal{L}_i^{-1}|_{\mathcal{X}_t}$ . Since  $\deg(F) \geq 0$  and  $0 = \kappa(\mathcal{E}_t) = \kappa(\mathcal{E} \otimes \mathcal{L}_i^{-1}|_{\mathcal{X}_t})$ , this subsheaf  $F$  has to be a subbundle of degree 0. Since moreover  $F$  is effective, we have  $F \simeq \mathcal{O}_{\mathcal{X}_t}$ . Hence  $\mathcal{L}_i|_{\mathcal{X}_t}$  is a line subbundle of  $\mathcal{E}_t$  and gives rise to a non-zero nilpotent Higgs field by the characterization of very stable bundles in [\(1.13\)](#). Conversely, again by [\(1.13\)](#), every non-zero nilpotent Higgs field has to be of that form. Hence by Grauert's Semicontinuity Theorem, see [\[OSSG80, p. 11\]](#), see also [\[Dem95\]](#),  $\mathcal{T}^{\text{wob}}$  is a closed analytic subset and in the neighborhood of a generic point in  $\mathcal{T}^{\text{wob}}$ , we can construct  $\theta$  as in the statement.  $\square$

### 1.4.3 An elementary vector bundle lemma

**Lemma 1.4.7.** *Let  $E \rightarrow X$  be a vector bundle on a compact Riemann surface  $X$ . Let*

$$0 = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_N = E \tag{1.14}$$

be a filtration of length  $N \geq 2$  by subbundles such that for all  $i \in \llbracket 1, N \rrbracket$ , the vector bundle  $V_i := F_i/F_{i-1}$  is semistable. Define a subbundle  $\text{End}^F(E)$  of  $\text{End}(E)$  by setting

$$\text{End}^F(E)(U) := \{\phi \in \text{End}(E)(U) \mid \phi(F_i|_U) \subset F_i|_U \quad \forall i \in \llbracket 1, N \rrbracket\}.$$

Assume that the sequence of slopes  $\mu_i := \mu(V_i)$  associated to the filtration (1.14) is non-increasing:  $\mu_i \geq \mu_{i+1}$ , and denote

$$\mu_{\max} := \max\{\mu_{k+1} - \mu_k \mid k \in \llbracket 1, N-1 \rrbracket\} \leq 0.$$

Then for any coherent subsheaf  $0 \neq G \subset \text{End}(E)/\text{End}^F(E)$  we have  $\mu(G) \leq \mu_{\max}$ .

*Proof.* This proof is simply a detailed version of a remark in [AB81, p. 590]. For each  $j \in \llbracket -N, N-1 \rrbracket$ , let us define  $W_j \subset \text{End}(E)$  by setting

$$W_j(U) := \{\phi \in \text{End}(E)(U) \mid \phi(F_k|_U) \subset F_{k+j}|_U \quad \forall k \in \llbracket 1, N \rrbracket\},$$

where we denote  $F_{N'} := F_N$  if  $N' > N$ ,  $F_{-n} := F_0$  if  $n > 0$  and  $V_i = F_i/F_{i-1}$  accordingly. By definition, we have  $W_0 = \text{End}^F(E)$ , whereas  $W_{-1}$  is the subsheaf of endomorphisms that are nilpotent with respect to the given filtration of  $E$ . We obtain a filtration by subbundles

$$0 = W_{-N} \subsetneq \dots \subsetneq W_0 \subsetneq \dots \subsetneq W_{N-1} = \text{End}(E)$$

satisfying

$$W_j/W_{j-1} \simeq \bigoplus_{k=1}^N \text{Hom}(V_k, V_{k+j}),$$

which is a direct sum of semistable bundles  $V_{kj} := \text{Hom}(V_k, V_{k+j})$ . If  $V_{kj} \neq 0$ , then  $\mu(V_{kj}) = \mu_{k+j} - \mu_k$ . Now let us denote  $\overline{W}_i := W_i/W_0$  for each  $i \in \llbracket 0, N-1 \rrbracket$ . Let  $0 \neq G \subset \overline{W}_{N-1}$ . Let  $i \in \llbracket 1, N-1 \rrbracket$  and consider the following diagram with exact rows and columns.

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & G \cap \overline{W}_{i-1} & \longrightarrow & G \oplus \overline{W}_{i-1} & \longrightarrow & G + \overline{W}_{i-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & G \cap \overline{W}_i & \longrightarrow & G \oplus \overline{W}_i & \longrightarrow & G + \overline{W}_i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (G \cap \overline{W}_i)/(G \cap \overline{W}_{i-1}) & \longrightarrow & \overline{W}_i/\overline{W}_{i-1} & \longrightarrow & (G + \overline{W}_i)/(G + \overline{W}_{i-1}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

We see that  $(G \cap \overline{W}_i)/(G \cap \overline{W}_{i-1})$  is a coherent subsheaf of  $\overline{W}_i/\overline{W}_{i-1} = W_i/W_{i-1}$ . In particular, it is either zero or

$$\mu((G \cap \overline{W}_i)/(G \cap \overline{W}_{i-1})) \leq \max\{\mu_{k+i} - \mu_k \mid k \in \llbracket 1, N-i \rrbracket\} \leq \mu_{\max}.$$

By induction on  $i \in \llbracket 0, N-1 \rrbracket$ , considering the left row of the diagram above and the monotony of the corresponding slopes, we conclude that for each index  $i$ , either  $G \cap \overline{W}_i = 0$  or  $\mu(G \cap \overline{W}_i) \leq \mu_{\max}$ . For  $i = N-1$  this yields the result.  $\square$

**Remark 1.4.8.** This Lemma can also be derived from general considerations considering slope filtrations for tensor products (such as  $E^\vee \otimes E$ ), see [And09, Thm. 2.3.3].



## 1.5 Definition of the Atiyah bundle

Let  $E \rightarrow M$  be a holomorphic vector bundle over a complex manifold  $M$ . The Atiyah bundle  $\text{At}(E)$  associated to  $E \rightarrow M$ , is a certain holomorphic vector bundle on  $M$  fitting into an intrinsic short exact sequence

$$0 \longrightarrow \text{End}(E) \longrightarrow \text{At}(E) \longrightarrow \text{TM} \longrightarrow 0$$

of vector bundles over  $M$ . The equivalence class of this *Atiyah exact sequence* corresponds to an extension class

$$A(E) \in \text{H}^1(M, \text{Hom}(\text{TM}, \text{End}(E))) = \text{H}^1(M, \text{End}(E) \otimes \Omega_M^1),$$

the so-called *Atiyah class*. The Atiyah bundle is particularly maniable, as illustrated by the variety of equivalent<sup>5</sup> definitions one may find in the literature. Those include the following list of approaches.

1. The choice of a particular  $\mathcal{O}_M$ -module structure on  $\text{End}_{\mathbb{C}}(E) \oplus \text{TM}$  [Ati57, § 4].
2. The sheaf of  $\text{GL}_r \mathbb{C}$ -invariant sections of the tangent bundle of the frame bundle of  $E$  [Ati57, Thm. 1].
3. Considering the element in  $\text{Ext}^1(E, E \otimes \Omega_M^1)$  obtained from the first-order jet bundle associated to  $E$ , see for example [HL10, § 10.1].
4. Using the curvature of the Chern connection associated to a hermitian structure on  $E$  [Ati57, Prop. 4].
5. Considering the sheaf of holomorphic differential operators of order at most 1 with scalar symbol, see for example [CS16, Appendix].

Each definition has its own upshots depending on the context in which one may encounter this fundamental concept; for example Approach 2.) admits an immediate generalization to the concept of the Atiyah bundle associated to a principal  $G$ -bundle, where  $G$  is a connected complex Lie group. Approach 4.) has been used in [CS16] and [Hua95] to give a differential-geometric description of deformations of pairs  $(E, X)$  à la Kodaira-Spencer.

In this section, we are going to recall in detail the definition of the Atiyah bundle corresponding to Approach 5.), which we are going to use throughout, as well as the generalizations of the Atiyah bundle encoding logarithmic and reducible connections.

### 1.5.1 Definition via the symbol homomorphism

For  $n \in \mathbb{N}$ , let  $\text{Diff}^n(E)$  denote the *sheaf of holomorphic differential operators of order at most  $n$*  of  $E$ . We have  $\text{Diff}^0(E) = \text{End}(E)$ . A section  $P \in \text{Diff}^1(E)(U)$  is a  $\mathbb{C}$ -linear endomorphism of  $E|_U$  such that, for every holomorphic frame  $\underline{e}$  over the range of a coordinate  $\underline{z} := (z_1, \dots, z_m)$  on  $V \subset U \subset M$ ,  $P$  is of the form

$$P : \underline{e} \cdot Y \mapsto \underline{e} \cdot \left( P_0 \cdot Y + \sum_{k=1}^m P_k \cdot \frac{\partial Y}{\partial z_k} \right)$$

for some  $P_0, P_k \in M_{r \times r} \mathcal{O}(V)$ . If  $\psi : E|_U \rightarrow U \times \mathbb{C}^r$  denotes the trivialization associated to the frame  $\underline{e}$ , the *symbol* of  $P$  over  $V$  is by definition

$$\sigma_1(P) = \sum_{k=1}^m \psi^{-1} \circ P_k \circ \psi \otimes \frac{\partial}{\partial z_k} \in (\text{End}(E) \otimes \text{TM})(V).$$

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<sup>5</sup>up to possibly changing the sign of the map  $\text{End}(E) \rightarrow \text{At}(E)$

The symbol depends neither on the choice of  $\underline{z}$  nor on the choice of  $\underline{e}$  [Dem12, § VI.1] and yields a natural short exact sequence

$$0 \longrightarrow \text{Diff}^0(E) \longrightarrow \text{Diff}^1(E) \xrightarrow{\sigma_1} \text{End}(E) \otimes \text{TM} \longrightarrow 0 \quad (1.15)$$

of vector bundles over  $M$ .

**Definition 1.5.1** (The Atiyah bundle *via* the symbol homomorphism). *We define*

$$\text{At}(E) := \sigma_1^{-1}(\text{id}_E \otimes \text{TM}) .$$

From (1.15), we obtain a canonical short exact sequence

$$0 \longrightarrow \text{End}(E) \longrightarrow \text{At}(E) \xrightarrow{\sigma_1} \text{TM} \longrightarrow 0. \quad (1.16)$$

Since it is useful for the comparison with other definitions, let us calculate the extension class  $A(E) \in H^1(M, \text{End}(E) \otimes \Omega_M^1)$  corresponding to equivalence class of the *Atiyah exact sequence* (1.16). Recall that  $A(E)$  is the image of  $\text{id}_{\text{TM}}$  under the connecting homomorphism in the long exact sequence associated to the short exact sequence obtained from (1.16) by applying  $\text{Hom}(\text{TM}, \bullet)$ . Let  $(\psi_i)_i$  with  $\psi_i : E|_{U_i} \rightarrow U_i \times \mathbb{C}^r$  be an atlas of holomorphic trivialization charts of  $E$ . We denote by  $\underline{e}^{(i)} = (e_1^{(i)}, \dots, e_r^{(i)})$  with  $e_k^{(i)}(z) = \psi_i^{-1}(z, \mathbf{e}_k)$ , where  $(\mathbf{e}_1, \dots, \mathbf{e}_r)$  is the canonical basis of  $\mathbb{C}^r$ , the associated collection of holomorphic frames of  $E$  over the open sets  $U_i$ . Over  $U_i \cap U_j$  we have

$$\psi_i = \psi_{ij} \cdot \psi_j \quad ; \quad \underline{e}^{(j)} = \underline{e}^{(i)} \cdot \psi_{ij} \quad (1.17)$$

for some  $\psi_{ij} \in \text{GL}_r \mathcal{O}_M(U_i \cap U_j)$ . One easily checks that  $A(E)$  is given by the Čech-cocycle

$$\psi_j^{-1} \circ d \circ \psi_j - \psi_i^{-1} \circ d \circ \psi_i \stackrel{(1.17)}{=} -\psi_j^{-1} \circ \psi_{ij}^{-1} d\psi_{ij} \circ \psi_j .$$

## 1.5.2 The logarithmic Atiyah bundle and connections

Let  $\mathcal{D} = \mathcal{D}^{\text{red}}$  be a smooth reduced effective divisor on  $M$ . Recall that the *logarithmic tangent bundle*  $\text{TM}(-\log \mathcal{D})$ , the dual of the logarithmic cotangent bundle  $\Omega_M^1(\log \mathcal{D})$ , is defined as follows. On the complement of the support of  $\mathcal{D}$ , sections of  $\text{TM}(-\log \mathcal{D})$  are identified with sections of  $\text{TM}$ , whereas over a coordinate chart  $(z_1, \dots, z_m)$  on  $U \subset M$  in which the support of  $\mathcal{D}$  is given by  $\{z_1 = 0\}$ , we have

$$\text{TM}(-\log \mathcal{D})(U) = \left\langle z_1 \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_n} \right\rangle .$$

In other words, we have a short exact sequence of  $\mathcal{O}_M$ -modules

$$0 \longrightarrow \text{TM}(-\log \mathcal{D}) \longrightarrow \text{TM} \longrightarrow \mathcal{O}_{\mathcal{D}}(\mathcal{D}) \longrightarrow 0 .$$

In the case that  $X = M$  is of complex dimension 1 and  $D = \mathcal{D}$ , one has  $\text{TX}(-\log \mathcal{D}) = \text{TX}(-D)$ .

**Definition 1.5.2** (The logarithmic Atiyah bundle). *We define*

$$\text{At}_{\mathcal{D}}(E) := \sigma_1^{-1}(\text{TM}(-\log \mathcal{D})) ,$$

where  $\sigma_1$  is given in (1.16). We obtain a canonical short exact sequence

$$0 \longrightarrow \text{End}(E) \longrightarrow \text{At}_{\mathcal{D}}(E) \xrightarrow{\sigma_1^{\log}} \text{TM}(-\log \mathcal{D}) \longrightarrow 0 \quad (1.18)$$

fitting into the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \text{End}(E) & \longrightarrow & \text{At}_{\mathcal{D}}(E) & \xrightarrow{\sigma_1^{\log}} & \text{TM}(-\log \mathcal{D}) \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{End}(E) & \longrightarrow & \text{At}(E) & \xrightarrow{\sigma_1} & \text{TM} \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathcal{O}_{\mathcal{D}}(\mathcal{D}) & \xlongequal{\quad} & \mathcal{O}_{\mathcal{D}}(\mathcal{D}) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array} \tag{1.19}$$

**Lemma 1.5.3.** *If  $\nabla : E \rightarrow E \otimes \Omega_M^1(\log \mathcal{D})$  is a logarithmic connection on  $E \rightarrow M$  with polar divisor at most  $\mathcal{D}$ , then  $\delta : \text{TM}(-\log \mathcal{D}) \rightarrow \text{At}_{\mathcal{D}}(E)$  defined by*

$$\delta(\theta)(e) = \nabla(e) \cdot \theta, \tag{1.20}$$

where  $\nabla(e) \cdot \theta$  denotes the contraction of  $\nabla(e)$  with  $\theta$ , satisfies

$$\sigma_1^{\log} \circ \delta = \text{id}_{\text{TM}(-\log \mathcal{D})}$$

and therefore defines a holomorphic splitting of the logarithmic Atiyah exact sequence (1.18). Conversely, for a holomorphic splitting  $\delta$  there exists a unique logarithmic connection  $\nabla$  on  $E \rightarrow M$  with polar divisor at most  $\mathcal{D}$ , satisfying (1.20) for all  $U \subset M$ ,  $e \in E(U)$  and  $\theta \in \text{TM}(-\log \mathcal{D})(U)$ .

*Proof.* It suffices to notice the following. Let  $\underline{e} = \underline{e}(z)$  be a holomorphic frame of  $E|_U$ . Then  $\delta$  is given by

$$\delta(\theta) : \underline{e} \cdot Y \mapsto \underline{e} \cdot \left( v_1 P_1 + \sum_{k=2}^m v_k P_k + \theta \right) \cdot Y \quad \text{for} \quad \theta = v_1 z_1 \frac{\partial}{\partial z_1} + \sum_{k=2}^m v_k \frac{\partial}{\partial z_k}$$

for some holomorphic matrices  $P_1, P_k \in M_{r \times r} \mathcal{O}_M(U)$ . The connection matrix with respect to the holomorphic frame  $\underline{e}$  of the corresponding logarithmic connection is given by

$$\Omega = P_1 \frac{dz_1}{z_1} + \sum_{k=2}^m P_k dz_k.$$

□

Note that in particular, holomorphic connections on  $E \rightarrow M$  are in one-to-one correspondence with holomorphic splittings of the Atiyah exact sequence (1.16).

### 1.5.3 The filtered Atiyah bundle and reducible connections

Let  $L \subset E$  be a subbundle of  $E \rightarrow M$ , i.e., we have a short exact sequence of vector bundles over  $M$

$$0 \longrightarrow L \longrightarrow E \longrightarrow Q \longrightarrow 0.$$

Combining this exact sequence with its dual sequence, we obtain an exact sequence

$$0 \longrightarrow Q^\vee \otimes L \longrightarrow Q^\vee \otimes E \oplus E^\vee \otimes L \longrightarrow E^\vee \otimes E \longrightarrow L^\vee \otimes Q \longrightarrow 0,$$

factoring through

$$\text{End}^L(E) := \{\phi \in \text{End}(E) \mid \phi(L) \subset L\}$$

such that we have the following two short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q^\vee \otimes L & \longrightarrow & E^\vee \otimes L \oplus Q^\vee \otimes E & \longrightarrow & \text{End}^L(E) \longrightarrow 0 \\ & & & & & & \parallel \\ & & & & & & 0 \longrightarrow \text{End}^L(E) \longrightarrow \text{End}(E) \longrightarrow L^\vee \otimes Q \longrightarrow 0. \end{array}$$

Recall from Definition [1.5.1](#) that sections of the Atiyah bundle are sections of the sheaf of holomorphic differential operators of order at most 1.

**Definition 1.5.4** (The filtered Atiyah bundle). *We define the sheaf of sections of the Atiyah bundle (respectively logarithmic Atiyah bundle) filtering with respect to  $L \subset E$  by*

$$\begin{aligned} \text{At}^L(E)(U) &:= \{P \in \text{At}(E)(U) \mid P(L(U)) \subset L(U)\}, \\ \text{At}_{\mathcal{D}}^L(E)(U) &:= \{P \in \text{At}_{\mathcal{D}}(E)(U) \mid P(L(U)) \subset L(U)\}. \end{aligned}$$

We obtain a canonical filtered logarithmic Atiyah exact sequence

$$0 \longrightarrow \text{End}^L(E) \longrightarrow \text{At}_{\mathcal{D}}^L(E) \xrightarrow{\sigma_1^{\log, L}} \text{TM}(-\log \mathcal{D}) \longrightarrow 0, \quad (1.21)$$

and a similar canonical exact sequence with middle term  $\text{At}^L(E)$ . Considering the following diagram, we see that  $\text{At}_{\mathcal{D}}^L(E)$  is a subbundle of  $\text{At}_{\mathcal{D}}(E)$  with quotient bundle  $L^\vee \otimes Q$ .

$$\begin{array}{ccccccc} & & 0 & & 0 & & (1.22) \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{End}^L(E) & \longrightarrow & \text{At}_{\mathcal{D}}^L(E) & \xrightarrow{\sigma_1^{\log, L}} & \text{TM}(-\log \mathcal{D}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \iota & & \parallel \\ 0 & \longrightarrow & \text{End}(E) & \longrightarrow & \text{At}_{\mathcal{D}}(E) & \xrightarrow{\sigma_1^{\log}} & \text{TM}(-\log \mathcal{D}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \rho & & \\ & & L^\vee \otimes Q & \xlongequal{\quad} & L^\vee \otimes Q & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Here  $\iota$  is the natural inclusion map, and  $\rho$  is defined by the diagram. In other words, the map  $\rho$  is by definition the composition of the quotient map from  $\text{At}_{\mathcal{D}}(E)$  to  $\text{At}_{\mathcal{D}}(E)/\text{At}_{\mathcal{D}}^L(E)$  with the inverse of the natural map  $\text{End}(E)/\text{End}^L(E) \rightarrow \text{At}_{\mathcal{D}}(E)/\text{At}_{\mathcal{D}}^L(E)$ , which is an isomorphism, and the identification  $\text{End}(E)/\text{End}^L(E) = L^\vee \otimes Q$  obtained from  $\rho' : \text{End}(E) \rightarrow L^\vee \otimes Q$  given by  $\rho'(\phi) = (E \rightarrow Q) \circ \phi \circ (L \rightarrow E)$ . This forces  $\rho$  to be given, for any holomorphic section  $P$  of  $\text{At}_{\mathcal{D}}(E)$ , by

$$\rho(P) = (E \rightarrow Q) \circ P \circ (L \rightarrow E). \quad (1.23)$$

**Lemma 1.5.5.** *Let  $\nabla : E \rightarrow E \otimes \Omega_M^1(\log \mathcal{D})$  be a logarithmic connection on  $E \rightarrow M$  and let  $\delta$  be the corresponding splitting of the logarithmic Atiyah exact sequence as in Lemma 1.5.3. Let  $L$  be a subbundle of  $E$  with  $Q := E/L$ . Then with respect to the canonical isomorphism*

$$\mathrm{Hom}(\mathrm{TM}(-\log \mathcal{D}), L^\vee \otimes Q) \simeq \mathrm{Hom}(L, Q \otimes \Omega_M^1(\log \mathcal{D}))$$

*we have  $\rho \circ \delta = \mathrm{II}_{\nabla, L}$ , where  $\rho$  as in equation 1.2.3 and  $\mathrm{II}_{\nabla, L}$  is the second fundamental form of  $\nabla$  with respect to  $L$  as in Definition 1.2.4.*

*Proof.* It suffices to notice that the contraction with vector fields tangential to  $\mathcal{D}$  commutes with the projection  $E \otimes \Omega_M^1(\log \mathcal{D}) \rightarrow Q \otimes \Omega_M^1(\log \mathcal{D})$ .  $\square$

Note that in particular, holomorphic splittings of the filtered logarithmic Atiyah exact sequence (1.21) are in one-to-one correspondence with logarithmic connections  $\nabla : E \rightarrow E \otimes \Omega_M^1(\log \mathcal{D})$  satisfying  $\mathrm{II}_{\nabla, L} = 0$ .

## 1.6 Some deformation theory

The aim of this section is to introduce the Kodaira Spencer maps that we will use in the proof of the main results of this chapter. More precisely, we recall the construction of a chain level refinement of Kodaira Spencer maps associated to a germ of holomorphic family  $\mathcal{E} \rightarrow \mathcal{X} \xrightarrow{p} B$  of vector bundles over compact Riemann surfaces parametrized by  $B$ , equipped with a holomorphic subbundle  $\mathcal{L} \subset \mathcal{E}$  and a holomorphic family of marked points on the fibers  $\mathcal{X}_t = p^{-1}(t)$  given by a set of disjoint sections  $\sigma_i : B \rightarrow \mathcal{X}$  for  $i \in \{1, \dots, n\}$ , such that the corresponding divisor  $\mathcal{D} = \mathcal{D}^{\mathrm{red}} := \sum_{i=1}^n \sigma_i(B)$  is smooth. We denote by  $E \rightarrow X \rightarrow \{t_0\}$  with  $L \subset E$  and  $D \subset X$  the restriction of the picture to a central parameter  $t_0 \in B$ .

What we will obtain is the following commutative diagram, where  $\mathrm{KS}_\bullet$  denotes the Kodaira-Spencer map classifying first order deformations of “objects of type  $\bullet$ ” and “ $\circ$ ” symbolizes all the forgetful maps.

$$\begin{array}{ccccccc}
 & & & & \mathrm{KS}_{(X,D)} & & \\
 & & & & \curvearrowright & & \\
 & & & & \mathrm{KS}_{(E,X,D)} & & \\
 & & & & \curvearrowright & & \\
 \mathrm{T}_{t_0} B & \xrightarrow{\mathrm{KS}_{(E,L,X,D)}} & \mathrm{H}^1(X, \mathrm{At}_D^L(E)) & \xrightarrow{\circ_{L^*}} & \mathrm{H}^1(X, \mathrm{At}_D(E)) & \xrightarrow{\circ_{\sigma_{1*}^{\log}}} & \mathrm{H}^1(X, \mathrm{TX}(-D)) \\
 & \searrow \mathrm{KS}_{(E,L,X)} & \downarrow \circ & & \downarrow \circ & & \downarrow \tau_* \circ \\
 & & \mathrm{H}^1(X, \mathrm{At}^L(E)) & \xrightarrow{\circ} & \mathrm{H}^1(X, \mathrm{At}(E)) & \xrightarrow{\circ} & \mathrm{H}^1(X, \mathrm{TX}) \\
 & & & & & & \\
 & & & & \mathrm{KS}_{(E,X)} & & \\
 & & & & \curvearrowright & & \\
 & & & & \mathrm{KS}_X & & 
 \end{array} \tag{1.24}$$

Actually, we will only introduce those maps in Diagram (1.24) that will be relevant for us, namely  $\mathrm{KS}_X, \mathrm{KS}_{(X,D)}, \mathrm{KS}_{(E,X,D)}, \mathrm{KS}_{(E,LX,D)}$  and the corresponding forgetful maps. The construction can

however be easily generalized to obtain Diagram (1.24) in full. All these Kodaira-Spencer maps will be derived from exact sequences intrinsically associated to the family of subobjects underlying the family. In particular, if we have two families of subobjects that are isomorphic (in the obvious sense generalizing for example [ACG11, p. 172] and [CS16, Def. 4.1]), or isomorphic after possibly shrinking the parameter space, then this isomorphism restricted to the central parameter identifies the corresponding Kodaira-Spencer maps.

Note further that due to the fact that we are working with  $X$  of complex dimension one, deformations of type  $(E, X)$ ,  $(E, X, D)$  and  $(X, D)$  are unobstructed and, under the condition that  $X$ , respectively  $(X, D)$  is *stable*, there exists for each such type a *Kuranishi family* such that the corresponding Kodaira-Spencer map is an isomorphism and such that the germ of any family of the given type with same central data is isomorphic to a pull-back of the Kuranishi family [ACG11, Thm. 2.12, Cor. 4.6], [CS16, § 5].

In particular, under the condition that  $X$ , respectively  $(X, D)$ , is stable, the Kodaira-Spencer maps  $\text{KS}_X, \text{KS}_{(E, X)}$ , respectively  $\text{KS}_{(X, D)}, \text{KS}_{(E, X, D)}$ , classify not only first order deformations, but germs of families up to isomorphism.

However, even in complex dimension one, deformations of type  $(E, L, X, D)$  and  $(E, L, X)$  are obstructed in the following sense. Given  $\mathcal{E} \rightarrow (\mathcal{X}, \mathcal{D}) \rightarrow B$  deforming  $E \rightarrow (X, D) \rightarrow \{t_0\}$  and  $L \subset E$ , the subbundle  $L \subset E$  may not be induced from a subbundle  $\mathcal{L} \subset \mathcal{E}$ , even after shrinking the parameter space. We construct an obstruction map in § 1.6.3.

### 1.6.1 Deformations of curves with marked points

Let  $(p : \mathcal{X} \rightarrow B, \mathcal{D})$  be a *holomorphic family of Riemann surfaces with marked points*. This means that  $\mathcal{X}$  and  $B$  are complex manifolds and  $p$  is a proper holomorphic submersion such that the fibers  $\mathcal{X}_t := p^{-1}(t)$  for  $t \in B$  are compact Riemann surfaces. Moreover,  $\mathcal{D} = \mathcal{D}^{\text{red}}$  is a (possibly empty) smooth reduced effective divisor on  $\mathcal{X}$ , whose support is the union of the images of  $n$  disjoint sections  $\sigma_i : B \rightarrow \mathcal{X}$ :

$$\mathcal{D} = \sum_{i=1}^n \sigma_i(B).$$

We consider the *germ* of such a family at a central parameter  $t_0 \in B$ ; at each of the finitely many occasions in the following where we would need to restrict to a sufficiently small Euclidian neighborhood of  $t_0$  in  $B$ , we restrict to this neighborhood, which we again denote  $B$ . We occasionally recall that we are working over a germ of family by saying that  $B$  is *sufficiently small*. We denote  $\mathcal{X}_t$  the fiber of  $p$  over  $t$ ,  $\mathcal{D}_t := \mathcal{D} \cap \mathcal{X}_t$  and  $(X, D) := (\mathcal{X}_{t_0}, \mathcal{D}_{t_0})$ .

The differential of  $p$  induces a canonical exact sequence

$$0 \longrightarrow \text{TX} \longrightarrow \text{T}\mathcal{X}|_X \xrightarrow{p^*} p^*\text{TB}|_X \longrightarrow 0. \quad (1.25)$$

The *Kodaira-Spencer map*  $\text{KS}_X$  of the germ of family above, not taking into account the marked points, is by definition the connecting morphism in the long exact sequence associated to (1.25). Using the natural identification  $\text{T}_{t_0}B = \text{H}^0(X, p^*\text{TB}|_X)$ , we write this Kodaira-Spencer map

$$\text{T}_{t_0}B \xrightarrow{\text{KS}_{(X, D)}} \text{H}^1(X, \text{TX}) .$$

We refer to [Voi02, § 9.1] for an exposition of different deformation-theoretic interpretations of this map. A few details will however be given below, when we take into account the marked points.

From the definition of the logarithmic tangent bundle (see § 1.5.2), we have the following commutative diagram with exact rows and columns, where  $\tau$  denotes the natural inclusion map:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & (1.26) \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TX(-D) & \longrightarrow & T\mathcal{X}(-\log \mathcal{D})|_X & \longrightarrow & p^*TB|_X \longrightarrow 0 \\
& & \downarrow \tau & & \downarrow & & \parallel \\
0 & \longrightarrow & TX & \longrightarrow & T\mathcal{X}|_X & \xrightarrow{p^*} & p^*TB|_X \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & \mathcal{O}_D(D) & \xlongequal{\quad} & \mathcal{O}_D(\mathcal{D})|_X & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & .
\end{array}$$

**Definition 1.6.1.** For a germ of family as above, Diagram (1.26) yields the following short exact sequence of holomorphic vector bundles over  $X$

$$0 \longrightarrow TX(-D) \longrightarrow T\mathcal{X}(-\log \mathcal{D})|_X \longrightarrow p^*TB|_X \longrightarrow 0. \quad (1.27)$$

Using the natural identification  $T_{t_0}B = H^0(X, p^*TB|_X)$ , we denote by

$$T_{t_0}B \xrightarrow{\text{KS}_{(X,D)}} H^1(X, TX(-D))$$

the connecting morphism in the long exact sequence associated to (1.27). It is by definition the Kodaira-Spencer map associated to the germ of family  $(\mathcal{X} \xrightarrow{p} B, \mathcal{D})$  with central fiber  $(X, D) \rightarrow \{t_0\}$ .

**Remark 1.6.2.** Note that Diagram (1.26) shows that the forgetful morphism yielding the Kodaira-Spencer map for the underlying germ of family of curves without marked points is induced by  $\tau$ :

$$\begin{array}{ccc}
T_{t_0}B & \xrightarrow{\text{KS}_{(X,D)}} & H^1(X, TX(-D)) \\
& \searrow \text{KS}_X & \downarrow \tau_* \\
& & H^1(X, TX).
\end{array}$$

Alternatively, for any  $v \in T_{t_0}B$ , the element  $\text{KS}_{(X,D)}(v)$  of  $H^1(X, TX(-D))$  can be computed as the Kodaira-Spencer class of the first order deformation of  $(X, D)$  obtained by restricting the family  $(\mathcal{X}, \mathcal{D}) \xrightarrow{p} B$  to the first order neighborhood of  $t_0 \in B$  in the direction of  $v$ . Since later on we will use some results of [ACG11] formulated in these terms, we shall briefly recall this construction, following [ACG11, p. 172–176], [Dia84, p. 909].

Since  $B$  is small, we may assume without loss of generality that  $B \subset \mathbb{C}^b$  with  $b = \dim(B)$ . Since  $B$  is small and  $p : \mathcal{X} \rightarrow B$  is a holomorphic submersion,  $\mathcal{X}$  is covered by open sets  $\mathcal{U}_i$  such that  $p(\mathcal{U}_i) = B$  and on  $\mathcal{U}_i$  we have coordinates of the form  $(z_i, p) : \mathcal{U}_i \rightarrow \mathbb{C}^{b+1}$ . We write those  $(z_i, \underline{t})$ . We may assume that the covering  $(\mathcal{U}_i)_i$  is conveniently chosen such that each irreducible component of  $\mathcal{D}$  is contained in one of the  $\mathcal{U}_i$ 's and does not intersect the others. Moreover, we may assume that if an irreducible component of  $\mathcal{D}$  is contained in  $\mathcal{U}_i$ , then it is there given by  $\{z_i = 0\}$ .

Let  $\varepsilon$  be the standard coordinate on  $\mathbb{C}$  and let  $S = (0, \mathcal{O}_0)$  with  $\mathcal{O}_0 = \mathcal{O}_{\mathbb{C}}/\mathcal{O}_{\mathbb{C}}\varepsilon^2$  be the double point [GR12, p. 4]. For every  $v = \sum_{\ell=1}^b v_{\ell} \frac{\partial}{\partial t_{\ell}} \in \mathbb{T}_{t_0} B$  we have a morphism of complex spaces  $\nu : S \rightarrow B$  sending 0 to  $t_0$  and  $g = g(\underline{t}) \in \mathcal{O}_B(U)$  with  $t_0 \in U$  to  $g(t_0) + \varepsilon \sum_{\ell=1}^b v_{\ell} \frac{\partial g}{\partial t_{\ell}}(t_0) \in \mathcal{O}_0$ . Denote  $\mathcal{X}_v := \nu^* \mathcal{X}$  and  $\mathcal{D}_v := \nu^* \mathcal{D}$ . Since  $\mathcal{X}$  is covered by open sets  $\mathcal{U}_i$  (each intersecting  $X$ ) with coordinates  $(z_i, t)$ , which in restriction to  $t = t_0$  yield the coordinate  $x_i = (z_i, t_0)$  for  $X$  on  $U_i := \mathcal{U}_i \cap X$ , we obtain coordinates  $(x_i, \varepsilon)$  on  $U_i \times S \subset \mathcal{X}_v$ . From the holomorphic transition maps  $(z_i, \underline{t}) = (f_{ij}(z_j, \underline{t}), \underline{t})$  on  $\mathcal{X}$  we obtain holomorphic transition maps on  $\mathcal{X}_v$ :

$$(x_i, \varepsilon) = (f_{ij}^0(x_j) + \varepsilon b_{ij}(x_j), \varepsilon) \quad \text{with} \quad f_{ij}^0(x_j) = f_{ij}(z_j, t_0). \quad (1.28)$$

Here  $b_{ij}(x_j) = \sum_{\ell=1}^b v_{\ell} \frac{\partial f_{ij}}{\partial t_{\ell}}(z_j, t_0)$ . From the atlas of charts  $(x_i, \varepsilon)$  one can reconstruct  $\mathcal{D}_v$  from  $D$  without additional information. Indeed, from  $D$  on  $X$  we know those charts in which the irreducible components of  $\mathcal{D}_v$  are supposed to occur, and they are there given by  $\{x_i = 0\}$ . From the cocycle condition

$$f_{ij}(f_{jk}(z_k, \underline{t}), \underline{t}) = f_{ik}(z_k, \underline{t})$$

on  $\mathcal{X}$  one obtains a cocycle condition on  $\mathcal{X}_v$  which amounts to the cocycle condition  $f_{ij}^0 \circ f_{jk}^0 = f_{ik}^0$  of  $X$  and

$$b_{ij}(f_{jk}^0(x_k)) + b_{jk}(x_k) \frac{\partial f_{ij}^0}{\partial x_j}(f_{jk}^0(x_k)) = b_{ik}(x_k)$$

or, equivalently,

$$b_{ij}(x_j) \frac{\partial}{\partial x_i} + b_{jk}(x_k) \frac{\partial}{\partial x_j} = b_{ik}(x_k) \frac{\partial}{\partial x_i}.$$

In particular,  $\beta_{ij} := b_{ij}(x_j) \frac{\partial}{\partial x_i}$  defines a Čech-cocycle. On the other hand, we have a short exact sequence

$$0 \longrightarrow \mathrm{TX}(-D) \longrightarrow \mathrm{T}\mathcal{X}_v(-\log \mathcal{D}_v)|_X \xrightarrow{p^*} \mathrm{T}_0 S \otimes \mathcal{O}_X \longrightarrow 0$$

(see [Voi02, p. 212] for a detailed description of the maps in this exact sequence). Note that the cocycle  $\beta_{ij}$  corresponds to the image of  $\frac{\partial}{\partial \varepsilon} \in \mathrm{H}^0(X, \mathrm{T}_0 S \otimes \mathcal{O}_X)$  under the connecting homomorphism associated to the short exact sequence above. Indeed,  $\frac{\partial}{\partial \varepsilon}$  given with respect to the chart  $(x_j, \varepsilon)$  is given with respect to the chart  $(x_i, \varepsilon)$  by  $\frac{\partial}{\partial \varepsilon} + b_{ij}(x_j) \frac{\partial}{\partial x_i}$ . Moreover,  $\frac{\partial}{\partial \varepsilon}$  is tangential to  $\mathcal{D}_v$  in each chart containing an irreducible component of  $\mathcal{D}_v$ . The elements

$$[\beta_{ij}] \in \mathrm{H}^1(X, \mathrm{TX}(-D)) \quad [\beta_{ij}] \in \mathrm{H}^1(X, \mathrm{TX})$$

are by definition the Kodaira-Spencer classes of the first order deformations  $(\mathcal{X}_v, \mathcal{D}_v) \xrightarrow{p} S$  and  $\mathcal{X}_v \xrightarrow{p} S$  of  $(X, D)$  and  $X$  respectively. It is now immediate to check that they coincide with  $\mathrm{KS}_{(X, D)}(v)$  and  $\mathrm{KS}_X(v)$  respectively.

From any element of  $\mathrm{H}^1(X, \mathrm{TX}(-D))$  one can construct, *via* (1.28), a unique equivalence class of first order deformation  $(\mathcal{X}_{\varepsilon}, \mathcal{D}_{\varepsilon}) \xrightarrow{p} S$  [ACG11, p. 173-174]. Therefore,  $\mathrm{H}^1(X, \mathrm{TX}(-D))$  parameterizes such first order deformations up to equivalence.

## 1.6.2 Deformations of vector bundles over curves

Let  $(p : \mathcal{X} \rightarrow B, \mathcal{D})$  be a *holomorphic family of Riemann surfaces with marked points* as in § 1.6.1. Let  $\mathcal{E} \rightarrow \mathcal{X}$  be a holomorphic vector bundle. Again we restrict to a germ  $(B, t_0)$  and denote  $X := p^{-1}(t_0)$ ,  $D := \mathcal{D} \cap X$ . Moreover, we denote  $E := \mathcal{E}|_X$ . From Definition 1.5.2 of the logarithmic Atiyah bundle for  $E$  and  $\mathcal{E}$  with respect to  $D$  and  $\mathcal{D}$ , we obtain the following Diagram (1.29), where the right column is identical to the canonical exact sequence (1.27).



$$\begin{array}{ccccccc}
& & 0 & & 0 & & (1.29) \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{End}(E) & \longrightarrow & \text{At}_D(E) & \xrightarrow{\sigma_1^{\log}} & \text{TX}(-D) \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{End}(\mathcal{E})|_X & \longrightarrow & \text{At}_{\mathcal{D}}(\mathcal{E})|_X & \longrightarrow & \text{T}\mathcal{X}(-\log \mathcal{D})|_X \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & p^*\text{T}B|_X & \xlongequal{\quad} & p^*\text{T}B|_X \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}
,$$

**Definition 1.6.3.** For a germ of family of vector bundles over Riemann surfaces with marked points as above, Diagram (1.29) yields the following short exact sequence of holomorphic vector bundles over  $X$

$$0 \longrightarrow \text{At}_D(E) \longrightarrow \text{At}_{\mathcal{D}}(\mathcal{E})|_X \longrightarrow p^*\text{T}B|_X \longrightarrow 0. \quad (1.30)$$

Using the natural identification  $\text{T}_{t_0}B = \text{H}^0(X, p^*\text{T}B|_X)$ , we denote by

$$\text{T}_{t_0}B \xrightarrow{\text{KS}_{(E,X,D)}} \text{H}^1(X, \text{At}_D(E))$$

the connecting morphism in the long exact sequence associated to (1.30). It is by definition the Kodaira-Spencer map associated to the germ of family  $(\mathcal{E} \rightarrow \mathcal{X} \xrightarrow{p} B, \mathcal{D})$  with central fiber  $(E \rightarrow X, D)$  over  $\{t_0\}$ .

**Remark 1.6.4.** Note that Diagram (1.26) shows that the forgetful morphism yielding the Kodaira-Spencer map for the underlying germ of family of curves with marked points is induced by  $\sigma_1^{\log}$ :

$$\begin{array}{ccc}
\text{T}_{t_0}B & \xrightarrow{\text{KS}_{(E,X,D)}} & \text{H}^1(X, \text{At}_D(E)) \\
& \searrow \text{KS}_{(X,D)} & \downarrow \sigma_1^{\log} \\
& & \text{H}^1(X, \text{TX}(-D)).
\end{array}$$

For the case  $D = \emptyset$ , one obtains the Kodaira-Spencer map  $\text{KS}_{(E,X)}$ . We refer to [CS16], complementing [Hua95], for a detailed exposition of the differential-geometric approach à la Kodaira and Spencer to germs of families of vector bundles over curves. It allows to see  $-\text{KS}_{(E,X)}$  as a derivation of a family of complex structures. For an algebro-geometric approach considering first order deformations of the pair  $(E, X)$  we refer to [Mar09, Sec. 2.3]. Both approaches, after further taking into account the marked points, yield  $\text{KS}_{(E,X,D)}$  (up to a sign).

### 1.6.3 Deformations of quadruples

Let  $(\mathcal{L} \subset \mathcal{E} \rightarrow \mathcal{X}, p : \mathcal{X} \rightarrow B, \mathcal{D})$  be a holomorphic family of filtered vector bundles over Riemann surfaces with  $n$  marked points. This means we have a holomorphic family  $p : \mathcal{X} \rightarrow B$  of compact Riemann surfaces, with marked point given by  $\mathcal{D}$  as before, a vector bundle  $\mathcal{E} \rightarrow \mathcal{X}$  and a subbundle

$\mathcal{L} \subset \mathcal{E}$ . Again we restrict to the germ  $(B, t_0)$ ; we denote  $\mathcal{X}_t$  the fiber of  $p$  over  $t$ ,  $\mathcal{E}_t := \mathcal{E}|_{\mathcal{X}_t}$ ,  $\mathcal{L}_t := \mathcal{L}|_{\mathcal{X}_t}$ ,  $\mathcal{D}_t := \mathcal{D} \cap \mathcal{X}_t$ , and  $(E, L, X, D) := (\mathcal{E}_{t_0}, \mathcal{L}_{t_0}, \mathcal{X}_{t_0}, \mathcal{D}_{t_0})$ .

From the Definition [1.5.4](#) of the filtered logarithmic Atiyah bundle, we obtain the following diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{End}^L(E) & \longrightarrow & \text{At}_D^L(E) & \longrightarrow & \text{TX}(-D) \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{End}^{\mathcal{L}}(\mathcal{E})|_X & \longrightarrow & \text{At}_{\mathcal{D}}^{\mathcal{L}}(\mathcal{E})|_X & \longrightarrow & \text{TX}(-\log \mathcal{D})|_X \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & p^*TB|_X & \xlongequal{\quad} & p^*TB|_X \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}, \tag{1.31}$$

where the right column is identical to the canonical exact sequence [\(1.27\)](#).

**Definition 1.6.5.** For a germ of family of filtered vector bundles over Riemann surfaces with marked points as above, diagram [\(1.31\)](#) yields the following short exact sequence of holomorphic vector bundles over  $X$

$$0 \longrightarrow \text{At}_D^L(E) \longrightarrow \text{At}_{\mathcal{D}}^{\mathcal{L}}(\mathcal{E})|_X \longrightarrow p^*TB|_X \longrightarrow 0. \tag{1.32}$$

Using the natural identification  $\mathbb{T}_{t_0}B = H^0(X, p^*TB|_X)$ , we denote by

$$\mathbb{T}_{t_0}B \xrightarrow{\text{KS}_{(E,L,X,D)}} H^1(X, \text{At}_D^L(E))$$

the connecting morphism in the long exact sequence associated to [\(1.32\)](#). It is by definition the Kodaira-Spencer map associated to the germ of family  $(\mathcal{L} \subset \mathcal{E} \rightarrow \mathcal{X} \xrightarrow{p} B, \mathcal{D})$  with central fiber  $(L \subset E \rightarrow X, D)$  over  $\{t_0\}$ .

From Diagram [\(1.22\)](#), denoting  $\mathcal{Q} := \mathcal{E}/\mathcal{L}$  and  $Q := \mathcal{Q}|_X$ , we obtain the following diagram with exact rows and columns.

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{At}_D^L(E) & \xrightarrow{\iota} & \text{At}_D(E) & \xrightarrow{p} & L^\vee \otimes Q \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & \text{At}_{\mathcal{D}}^{\mathcal{L}}(\mathcal{E})|_X & \longrightarrow & \text{At}_{\mathcal{D}}(\mathcal{E})|_X & \longrightarrow & \mathcal{L}^\vee \otimes \mathcal{Q}|_X \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & p^*TB|_X & \xlongequal{\quad} & p^*TB|_X & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}. \tag{1.33}$$

**Remark 1.6.6.** Note that Diagram (1.33) shows that the forgetful morphism yielding the Kodaira-Spencer map for the underlying germ of family of unfiltered vector bundles over curves with marked points is induced by  $\iota$ :

$$\begin{array}{ccc} T_{t_0}B & \xrightarrow{\text{KS}_{(E,L,X,D)}} & H^1(X, \text{At}_D^L(E)) \\ & \searrow \text{KS}_{(E,X,D)} & \downarrow \iota_* \\ & & H^1(X, \text{At}_D(E)). \end{array}$$

One can justify the name we have given to  $\text{KS}_{(E,L,X,D)}$  by checking that  $H^1(X, \text{At}_D^L(E))$  parameterizes first order deformations of quadruples  $(E, L, X, D)$  and that  $\text{KS}_{(E,L,X,D)}$  is the corresponding classification map. This is however not essential for our purposes. All we will actually need is that Definition 1.6.5 establishes a certain map  $\text{KS}_{(E,L,X,D)}$ , which comes with the forgetful morphism  $\iota_*$  yielding  $\text{KS}_{(E,X,D)}$ . One then notices that  $\rho_* : H^1(X, \text{At}(E)) \rightarrow H^1(X, L^\vee \otimes Q)$  with  $\rho$  as in Diagram (1.33) yields an obstruction for the subbundle  $L \subset E$  to be induced from a subbundle  $\mathcal{L} \rightarrow \mathcal{E}$ . Indeed, we have the following.

**Proposition 1.6.7** (An obstruction map). *Let  $p : (\mathcal{X}, \mathcal{D}) \rightarrow B$  be a germ of family of compact Riemann surfaces with marked points, with central fiber  $(X, D) \rightarrow \{t_0\}$ . Let  $\mathcal{E} \rightarrow \mathcal{X}$  be a vector bundle. Denote  $E := \mathcal{E}|_X$  and let  $L \subset E$  be a subbundle of  $E \rightarrow X$ . If there exists a subbundle  $\mathcal{L} \subset \mathcal{E}$  such that  $\mathcal{L}|_X = L \subset E$ , then the composition*

$$T_{t_0}B \xrightarrow{\text{KS}_{(E,X,D)}} H^1(X, \text{At}_D(E)) \xrightarrow{\rho_*} H^1(X, L^\vee \otimes Q)$$

vanishes identically. Here  $Q := E/L$  and  $\rho_*$  is induced from the second fundamental form

$$\begin{aligned} \rho : \text{At}_D(E) &\rightarrow L^\vee \otimes Q \\ P &\mapsto (E \rightarrow Q) \circ P \circ (L \rightarrow E). \end{aligned}$$

*Proof.* Let  $\mathcal{L} \subset \mathcal{E}$  be a subbundle as in the statement. Recall from equation (1.23) that the map  $\rho$  in diagram (1.33) coincides with the map given in the statement. Consider the top row of diagram (1.33). Since it is exact, we have  $\rho_* \circ \iota_* = 0$ . The result follows from the fact that  $\text{KS}_{(E,X,D)} = \iota_* \circ \text{KS}_{(E,L,X,D)}$ .  $\square$

## 1.7 Main results

This section contains a new proof of the Theorem 1.1.2 from [Heu09] mentioned in the introduction, concerning rank-two connections, as well as its meromorphic version, based on the techniques developed in collaboration with I. Biswas and J. Hurtubise in [BHH16] and [BHH18b]. The statements we prove will also include the investigation of very stability from [BHH17]. We will not need the assumption of tracefreeness, but we will restrict ourselves to the case of stable pointed base curves. Then we adapt these techniques in higher rank.

### 1.7.1 Isomonodromic deformations and stability in rank two

**Theorem 1.7.1.** *Let  $X_0$  be a compact Riemann surface of genus  $g \geq 0$  and let  $D_0$  be a reduced divisor on  $X$  of degree  $n$ . Let  $(E_0, \nabla_0)$  be a logarithmic connection of rank 2 over  $X_0$ , with polar divisor  $D_0$ . Assume that  $(E_0, \nabla_0)$  deformable, i.e.,  $3g - 3 + n > 0$ , and is irreducible.*

Let  $\mathcal{E} \rightarrow \mathcal{X}$  be the vector bundle over  $p : \mathcal{X} \rightarrow \mathcal{T} = \mathcal{T}_{g,n}$  underlying the universal isomonodromic deformation of  $(E_0, \nabla_0)$  (see § [1.3.2](#)). Denote  $\mathcal{E}_t := \mathcal{E}|_{p^{-1}(t)}$  for any  $t \in \mathcal{T}$ . We consider the following closed analytic subsets of  $\mathcal{T}$ , where  $k \in \mathbb{Z}$  (see § [1.4.2](#)):

$$\mathcal{T}_k := \{t \in \mathcal{T} \mid \kappa(\mathcal{E}_t) \leq k\} \quad , \quad \mathcal{T}^{\text{nil}} := \{t \in \mathcal{T} \mid \mathcal{E}_t \text{ not very stable}\}.$$

Then

$$\text{codim}(\mathcal{T}_k, \mathcal{T}) \geq g - 1 - k, \quad \text{codim}(\mathcal{T}^{\text{nil}}, \mathcal{T}) > 0.$$

**Remark 1.7.2.** Note that

$$\mathcal{T}_{2-2g-n} \subset \dots \subset \mathcal{T}_{g-3} \subset \mathcal{T}_{g-2} \subset \mathcal{T}^{\text{nil}} \subset \mathcal{T}_g = \mathcal{T}.$$

We use the convention  $\text{codim}(A, \mathcal{T}) := \dim(\mathcal{T}) - \dim(A)$ , where  $\dim(\emptyset) = -\infty$ .

*Proof.* Denote by  $\nabla$  the flat logarithmic connection on  $\mathcal{E} \rightarrow \mathcal{X}$  with polar divisor  $\mathcal{D}$  given by the universal isomonodromic deformation.

Let  $t_0 \in \mathcal{T}$ . Denote  $X := p^{-1}(t_0)$ ,  $D := \mathcal{D} \cap X$ ,  $E := \mathcal{E}|_X$ . We **claim** that the Kodaira-Spencer map

$$\text{KS}_{(E,X,D)} : \mathbb{T}_{t_0}\mathcal{T} \rightarrow \text{H}^1(X, \text{At}_D(E))$$

classifying the germ at  $t = t_0$  of the family  $(\mathcal{E} \rightarrow \mathcal{X} \xrightarrow{p} \mathcal{T}, \mathcal{D})$  coincides with the composition

$$\mathbb{T}_{t_0}\mathcal{T} \xrightarrow{\text{KS}_{(X,D)}} \text{H}^1(X, \text{TX}(-D)) \xrightarrow{\delta_*} \text{H}^1(X, \text{At}_D(E))$$

induced by the morphism  $\delta : \text{TX}(-D) \rightarrow \text{At}_D(E)$  defined by the connection  $\nabla|_X$  as in Lemma [1.5.3](#). Indeed, denote by  $\sigma_1^{\text{log}}$  the symbol homomorphism for  $\text{At}_D(E)$  and by  $\tilde{\sigma}_1^{\text{log}}$  the restriction to  $X$  of the symbol homomorphism for  $\text{At}_{\mathcal{D}}(\mathcal{E})$  (see Definition [1.5.2](#)). The Kodaira-Spencer map  $\text{KS}_{(E,X,D)}$  as above is by definition the connecting morphism in the middle column of the following commutative diagram with exact rows and columns (see Definition [1.6.3](#)).

$$\begin{array}{ccccccc} & & 0 & & 0 & & (1.34) \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{End}(E) & \longrightarrow & \text{At}_D(E) & \xrightarrow{\sigma_1} & \text{TX}(-D) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{End}(\mathcal{E}_B)|_X & \longrightarrow & \text{At}_{\mathcal{D}}(\mathcal{E})|_X & \xrightarrow{\tilde{\sigma}_1} & \text{TX}(-\log \mathcal{D})|_X \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & p^*\mathbb{T}_{t_0}\mathcal{T} & \xlongequal{\quad} & p^*\mathbb{T}_{t_0}\mathcal{T} \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}.$$

The connecting homomorphism in the right column  $\text{KS}_{(X,D)} : \mathbb{T}_{t_0}\mathcal{T} \rightarrow \text{H}^1(X, \text{TX}(-D))$  is by definition the Kodaira-Spencer map of the family  $(\mathcal{X}, \mathcal{D})$  with central fiber  $(X, D)$  over  $t_0$  (see Definition [1.6.1](#)).

On the other hand, we have the following morphism of short exact sequences, where

$$\tilde{\delta} : \text{TX}(-\log \mathcal{D})|_X \rightarrow \text{At}_{\mathcal{D}}(\mathcal{E})|_X$$

is the restriction to  $X$  of the morphism defined by the connection  $\nabla$  as in Lemma [1.5.3](#), and where unnamed arrows coincide with the corresponding ones in Diagram [\(1.34\)](#).

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{TX}(-D) & \longrightarrow & \mathrm{T}\mathcal{X}(-\log D)|_X & \longrightarrow & p^*\mathrm{T}_{t_0}\mathcal{T} \longrightarrow 0 \\
& & \downarrow \delta & & \downarrow \tilde{\delta} & & \parallel \\
0 & \longrightarrow & \mathrm{At}_D(E) & \longrightarrow & \mathrm{At}_D(\mathcal{E})|_X & \longrightarrow & p^*\mathrm{T}_{t_0}\mathcal{T} \longrightarrow 0.
\end{array} \tag{1.35}$$

The claim follows.

Now let  $k \in \mathbb{Z}$ . Assume that  $t_0$  is a generic point in  $\mathcal{T}_k \setminus \mathcal{T}_{k-1}$ . Let  $B$  be a sufficiently small neighborhood of  $t_0$  in  $\mathcal{T}_k \setminus \mathcal{T}_{k-1}$  and denote  $\mathcal{X}_B := p^{-1}(B)$ ,  $\mathcal{D}_B := \mathcal{D} \cap \mathcal{X}_B$ ,  $\mathcal{E}_B := \mathcal{E}|_{\mathcal{X}_B}$ . According to Maruyama's Theorem [1.4.4](#), there exists a line subbundle  $\mathcal{L}$  of  $\mathcal{E}_B$  such that for  $L := \mathcal{L}|_X$  and  $Q := E/L$  we have

$$\deg(L^\vee \otimes Q) = k.$$

Since  $(X, D)$  is stable and  $(\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{T}$  is the universal Teichmüller curve with marked points, the Kodaira-Spencer map

$$\mathrm{KS}_{(X,D)} : \mathrm{T}_{t_0}\mathcal{T} \rightarrow \mathrm{H}^1(X, \mathrm{TX}(-D))$$

is an isomorphism [[ACG11](#), p. 188, p. 449]. We have the following commutative diagram, where  $\iota_*$  is induced from the inclusion map  $\iota : \mathrm{At}_D^L(E) \rightarrow \mathrm{At}_D(E)$  (see Definition [1.5.4](#)) and  $\mathrm{KS}_{(E,L,X,D)}$  is as in Definition [1.6.5](#):

$$\begin{array}{ccc}
& & \mathrm{H}^1(X, \mathrm{At}_D^L(E)) \\
& \nearrow \mathrm{KS}_{(E,L,X,D)} & \downarrow \iota_* \\
\mathrm{T}_{t_0}B & \xrightarrow{\mathrm{KS}_{(E,X,D)}} & \mathrm{H}^1(X, \mathrm{At}_D(E)) \\
\downarrow & \nearrow & \uparrow \delta_* \\
\mathrm{T}_{t_0}\mathcal{T} & \xrightarrow[\mathrm{KS}_{(X,D)}]{\sim} & \mathrm{H}^1(X, \mathrm{TX}(-D)).
\end{array}$$

Now consider  $\rho : \mathrm{At}_D(E) \rightarrow L^\vee \otimes Q$  given by  $P \mapsto (E \rightarrow Q) \circ P \circ (L \rightarrow E)$  and denote  $\mathrm{II}_\delta := \rho \circ \delta$ . Recall from Lemma [1.5.5](#) that  $\mathrm{II}_\delta$  coincides with the second fundamental form of  $\nabla|_X$  with respect to  $L$ . Since  $\nabla|_X$  is irreducible (see Remark [1.3.3](#)),

$$\mathrm{II}_\delta : \mathrm{TX}(-D) \rightarrow L^\vee \otimes Q$$

is not the zero homomorphism. Since moreover source and target of  $\mathrm{II}_\delta$  are line bundles, we have a short exact sequence of coherent sheaves over  $X$

$$0 \longrightarrow \mathrm{TX}(-D) \xrightarrow{\mathrm{II}_\delta} L^\vee \otimes Q \longrightarrow \mathbb{T} \longrightarrow 0,$$

where  $\mathbb{T}$  is a torsion sheaf. It follows that

$$\mathrm{II}_{\delta_*} \circ \mathrm{KS}_{(X,D)} : \mathrm{T}_{t_0}\mathcal{T} \rightarrow \mathrm{H}^1(X, L^\vee \otimes Q)$$

is surjective. On the other hand, the composition

$$\rho_* \circ \mathrm{KS}_{(E,X,D)} : \mathrm{T}_{t_0}B \rightarrow \mathrm{H}^1(X, L^\vee \otimes Q)$$

is the zero morphism because  $\rho_* \circ \iota_* = 0$  (see Proposition [1.6.7](#)). This composition moreover coincides with the composition  $\mathrm{II}_{\delta_*} \circ \mathrm{KS}_{(X,D)} \circ (\mathrm{T}_{t_0}B \hookrightarrow \mathrm{T}_{t_0}\mathcal{T})$ . Thus  $\dim(\mathcal{T}) - \dim(B) \geq h^1(X, L^\vee \otimes Q)$ .

Yet by Riemann-Roch, we have

$$h^1(X, L^\vee \otimes Q) = h^0(X, L^\vee \otimes Q) - \deg(L^\vee \otimes Q) + g - 1 \geq 0 - k + g - 1,$$

whence

$$\text{codim}(\mathcal{T}_k, \mathcal{T}) \geq g - 1 - k.$$

Note that in particular, if  $\mathcal{T}^{\text{nil}} = \mathcal{T}_{g-2}$ , then  $\text{codim}(\mathcal{T}^{\text{nil}}, \mathcal{T}) > 0$ . Assume  $\mathcal{T}^{\text{nil}} \neq \mathcal{T}_{g-2}$  and let  $t_0$  be a generic point in  $\mathcal{T}^{\text{wob}} := \mathcal{T}^{\text{nil}} \setminus \mathcal{T}_{g-2}$ . Let  $B$  be a sufficiently small neighborhood of  $t_0$  in  $\mathcal{T}^{\text{wob}}$ . By Proposition [1.4.6](#) and Serre duality, there exists a line subbundle  $\mathcal{L} = \ker(\theta)$  of  $\mathcal{E}_B$ , inducing a line subbundle  $L \subset E$ , such that  $h^1(L^\vee \otimes Q) \neq 0$ . By the same reasoning as above, we have

$$\text{codim}(\mathcal{T}^{\text{nil}}, \mathcal{T}) \geq h^1(L^\vee \otimes Q) > 0.$$

□

**Theorem 1.7.3.** *Let  $X_0$  be a compact Riemann surface of genus  $g \geq 0$  and let  $D_0$  be an effective divisor on  $X$ . Denote by  $D_0^{\text{red}}$  the associated reduced divisor and assume that  $(X_0, D_0^{\text{red}})$  is stable. Let  $(E_0, \nabla_0)$  be a rank 2 meromorphic connection over  $X_0$ , with polar divisor  $D_0$ . Assume that  $(E_0, \nabla_0)$  is irreducible and deformable.*

*Let  $\mathcal{E} \rightarrow \mathcal{X}$  with  $p : \mathcal{X} \rightarrow \mathcal{T}_\parallel$  be the family of vector bundles underlying the parallel isomonodromic deformation of  $(E_0, \nabla_0)$  (see § [1.3.3](#)). Denote  $\mathcal{E}_t := \mathcal{E}|_{p^{-1}(t)}$  for any  $t \in \mathcal{T}_\parallel$ . We consider the following closed analytic subsets of  $\mathcal{T}_\parallel$ , where  $k \in \mathbb{Z}$  (see § [1.4.2](#)):*

$$\mathcal{T}_k := \{t \in \mathcal{T}_\parallel \mid \kappa(\mathcal{E}_t) \leq k\} \quad , \quad \mathcal{T}^{\text{nil}} := \{t \in \mathcal{T}_\parallel \mid \mathcal{E}_t \text{ not very stable}\}.$$

Then

$$\text{codim}(\mathcal{T}_k, \mathcal{T}_\parallel) \geq g - 1 - k \quad , \quad \text{codim}(\mathcal{T}^{\text{nil}}, \mathcal{T}_\parallel) > 0.$$

*Proof.* Denote by  $\nabla$  the flat meromorphic connection on  $\mathcal{E} \rightarrow \mathcal{X}$  with polar divisor  $\mathcal{D}$  and polar locus  $\mathcal{D}^{\text{red}}$  given by the parallel isomonodromic deformation. Let  $t_0 \in \mathcal{T}_\parallel$ ; denote  $X := p^{-1}(t_0)$ ,  $D := \mathcal{D}|_X$ ,  $D^{\text{red}} := \mathcal{D}^{\text{red}}|_X$  and  $E := \mathcal{E}|_X$ . We wish to apply the idea of the proof of Theorem [1.7.1](#). The key point is to show that for any line subbundle  $L \subset E$  with quotient bundle  $Q = E/L$ , the composition

$$\rho_* \circ \text{KS}_{(E, X, D)} : \text{T}_{t_0} \mathcal{T}_\parallel \rightarrow \text{H}^1(X, L^\vee \otimes Q)$$

is surjective. Recall that  $\rho$  is defined by  $\rho(P) = (E \rightarrow Q) \circ P \circ (L \rightarrow E)$  for any section  $P$  of  $\text{At}_{D^{\text{red}}}(E)$ . The main difficulty here is that  $\nabla$  is not necessarily logarithmic and it will be less straight forward to relate the Kodaira-Spencer map  $\text{KS}_{(E, X, D)}$  and the composition  $\rho_* \circ \text{KS}_{(E, X, D)}$  to the connection and its second fundamental form.

**First step:** Relate  $\text{KS}_{(E, X, D)}$  to  $\nabla$ .

We denote  $\text{At}_{\mathcal{D}}^{\text{log}}(\mathcal{E}) := \text{At}_{\mathcal{D}^{\text{red}}}(\mathcal{E})$ . From the logarithmic Atiyah exact sequence

$$0 \longrightarrow \text{End}(\mathcal{E}) \longrightarrow \text{At}_{\mathcal{D}}^{\text{log}}(\mathcal{E}) \xrightarrow{\sigma_1^{\text{log}}} \text{T}\mathcal{X}(-\log \mathcal{D}^{\text{red}}) \longrightarrow 0$$

we obtain, by tensoring with  $\mathcal{O}_{\mathcal{X}}(\mathcal{D} - \mathcal{D}^{\text{red}})$  and considering the preimage under  $\sigma_1^{\text{log}} \otimes \text{id}$  of

$$\text{T}\mathcal{X}(-\log \mathcal{D}^{\text{red}}) \subset \text{T}\mathcal{X}(-\log \mathcal{D}^{\text{red}}) \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D} - \mathcal{D}^{\text{red}}),$$

the meromorphic Atiyah exact sequence

$$0 \longrightarrow \text{End}(\mathcal{E}) \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D} - \mathcal{D}^{\text{red}}) \longrightarrow \text{At}_{\mathcal{D}}^{\text{mer}}(\mathcal{E}) \xrightarrow{\sigma_1^{\text{mer}}} \text{TX}(-\log \mathcal{D}^{\text{red}}) \longrightarrow 0.$$

Similarly to Lemma [1.5.3](#) from  $\nabla$  we obtain a splitting

$$\tilde{\delta} : \text{TX}(-\log \mathcal{D}^{\text{red}}) \longrightarrow \text{At}_{\mathcal{D}}^{\text{mer}}(\mathcal{E}), \quad \sigma_1^{\text{mer}} \circ \tilde{\delta} = \text{id}$$

of the meromorphic Atiyah exact sequence. We obtain coherent subsheaves

$$\mathcal{F}^{\text{mer}} := \text{Im}(\tilde{\delta}) \subset \text{At}_{\mathcal{D}}^{\text{mer}}(\mathcal{E}), \quad \mathcal{F}^{\text{log}} := \mathcal{F}^{\text{mer}} \cap \text{At}_{\mathcal{D}}^{\text{log}}(\mathcal{E}) \subset \text{At}_{\mathcal{D}}^{\text{log}}(\mathcal{E}),$$

where the latter is constructed from the natural exact sequence

$$0 \longrightarrow \text{At}_{\mathcal{D}}^{\text{log}}(\mathcal{E}) \longrightarrow \text{At}_{\mathcal{D}}^{\text{mer}}(\mathcal{E}) \longrightarrow \text{End}(\mathcal{E}) \otimes \mathcal{O}_{\mathcal{D}-\mathcal{D}^{\text{red}}}(\mathcal{D} - \mathcal{D}^{\text{red}}) \longrightarrow 0.$$

Hence we have a natural exact sequence

$$0 \longrightarrow \mathcal{F}^{\text{log}} \longrightarrow \mathcal{F}^{\text{mer}} \longrightarrow \tilde{\mathbb{T}}_1 \longrightarrow 0,$$

where the quotient  $\tilde{\mathbb{T}}_1$  is a torsion sheaf isomorphic to  $\mathcal{O}_{\mathcal{D}-\mathcal{D}^{\text{red}}}(\mathcal{D} - \mathcal{D}^{\text{red}})$ .

Similarly, we define  $\text{At}_{\mathcal{D}}^{\text{log}}(E)$ ,  $\text{At}_{\mathcal{D}}^{\text{mer}}(E)$ ,

$$\delta : \text{TX}(-D_0) \longrightarrow \text{At}_{\mathcal{D}}^{\text{mer}}(E)$$

from  $\nabla|_X$  as well as  $F^{\text{log}} \subset \text{At}_{\mathcal{D}}^{\text{log}}(E)$  and  $F^{\text{mer}} \subset \text{At}_{\mathcal{D}}^{\text{mer}}(E)$ . We obtain the following diagram with exact rows and columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & (1.36) \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^{\text{log}} & \longrightarrow & \mathcal{F}^{\text{log}}|_X & \longrightarrow & p^* \text{T } \mathcal{T}_{\parallel}|_X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & F^{\text{mer}} & \longrightarrow & \mathcal{F}^{\text{mer}}|_X & \longrightarrow & p^* \text{T } \mathcal{T}_{\parallel}|_X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{T}_1 & \xlongequal{\quad} & \tilde{\mathbb{T}}_1|_X & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Moreover, we find the following.

- A natural injective morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^{\text{mer}} & \longrightarrow & \mathcal{F}^{\text{mer}}|_X & \longrightarrow & p^* \text{T } \mathcal{T}_{\parallel}|_X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{At}_{\mathcal{D}}^{\text{mer}}(E) & \longrightarrow & \text{At}_{\mathcal{D}}^{\text{mer}}(\mathcal{E})|_X & \longrightarrow & p^* \text{T } \mathcal{T}_{\parallel}|_X \longrightarrow 0, \end{array}$$

where the relevant quotient bundle is isomorphic to  $\text{End}(E)(D - D^{\text{red}})$  (because  $\delta$  is a splitting of the meromorphic Atiyah exact sequence).

- A natural injective morphism of exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{At}_D^{\log}(E) & \longrightarrow & \text{At}_D^{\log}(\mathcal{E})|_X & \longrightarrow & p^*T\mathcal{T}_{\parallel}|_X \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & \text{At}_D^{\text{mer}}(E) & \longrightarrow & \text{At}_D^{\text{mer}}(\mathcal{E})|_X & \longrightarrow & p^*T\mathcal{T}_{\parallel}|_X \longrightarrow 0,
\end{array}$$

where the relevant quotient sheaf is  $\text{End}(E) \otimes \mathcal{O}_{D-D^{\text{red}}}(D - D^{\text{red}})$  (by construction of the meromorphic Atiyah bundle).

Those, together with Diagram (1.36), yield the following injective morphism of exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & F^{\log} & \longrightarrow & \mathcal{F}^{\log}|_X & \longrightarrow & p^*T\mathcal{T}_{\parallel}|_X \longrightarrow 0 \\
& & \downarrow \mu & & \downarrow & & \parallel \\
0 & \longrightarrow & \text{At}_D^{\log}(E) & \longrightarrow & \text{At}_D^{\log}(\mathcal{E})|_X & \longrightarrow & p^*T\mathcal{T}_{\parallel}|_X \longrightarrow 0,
\end{array} \tag{1.37}$$

where  $\mu : F^{\log} \rightarrow \text{At}_D^{\log}(E)$  is the natural inclusion map. Now Diagram (1.37) encodes the Kodaira-Spencer map and its relation to  $\nabla$ . More precisely, recall from Definition 1.6.3 that  $\text{KS}_{(E,X,D)}$  is the connecting homomorphism in the long exact sequence of cohomology spaces associated to the bottom row of diagram (1.37). Denote by  $f_{\parallel}$  the connecting homomorphism associated to the short exact sequence

$$0 \longrightarrow F^{\log} \longrightarrow \mathcal{F}^{\log}|_X \longrightarrow p^*T\mathcal{T}_{\parallel}|_X \longrightarrow 0$$

given by the upper row of Diagram (1.37). It follows that the Kodaira-Spencer map

$$\text{KS}_{(E,X,D)} : T_{t_0}\mathcal{T}_{\parallel} \rightarrow H^1\left(X, \text{At}_D^{\log}(E)\right)$$

classifying the germ at  $t = t_0$  of the family  $(\mathcal{E} \rightarrow \mathcal{X} \xrightarrow{p} \mathcal{T}_{\parallel}, \mathcal{D}^{\text{red}})$  coincides with the composition

$$T_{t_0}\mathcal{T}_{\parallel} \xrightarrow{f_{\parallel}} H^1\left(X, F^{\log}\right) \xrightarrow{\mu_*} H^1\left(X, \text{At}_D^{\log}(E)\right).$$

**Second step:** Appropriately lift the second fundamental form.

The first step of our proof implies that for any line subbundle  $L$  of  $E$ , the composition  $\rho_* \circ \text{KS}_{(E,X,D)}$  equals  $\rho_*^{\log} \circ f_{\parallel}$ , where  $\rho^{\log}$  is the restriction of  $\rho$  to  $F^{\log}$ . We claim that  $\rho^{\log}$  is a non-zero morphism lifting the second fundamental form of  $\nabla$ . Indeed, define

$$\rho' : \text{At}_D^{\text{mer}}(E) \rightarrow L^{\vee} \otimes Q \otimes \mathcal{O}_X(D - D^{\text{red}}) = \text{End}(E) \otimes \mathcal{O}_X(D - D^{\text{red}}) / \text{End}^L(E) \otimes \mathcal{O}_X(D - D^{\text{red}})$$

similarly to  $\rho$ . We obtain a map

$$\rho^{\text{mer}} : F^{\text{mer}} \rightarrow L^{\vee} \otimes Q \otimes \mathcal{O}_X(D - D^{\text{red}}),$$

whose precomposition with  $\delta$  identifies with the second fundamental form  $\text{II}_{\delta}$  of  $\nabla|_X$  with respect to  $L$ , analogously to Lemma 1.5.5. Since  $\text{II}_{\delta}$  is non-zero by irreducibility of  $\nabla|_X$  (see Remark 1.3.6), the map  $\rho^{\text{mer}}$  is not the zero map. It follows that  $\rho^{\log}$  is not the zero map, because we have a commutative diagram of torsion-free, invertible coherent sheaves

$$\begin{array}{ccc}
F^{\log} & \xrightarrow{\rho^{\log}} & L^{\vee} \otimes Q \\
\downarrow & & \downarrow \\
F^{\text{mer}} & \xrightarrow{\rho^{\text{mer}}} & L^{\vee} \otimes Q \otimes \mathcal{O}_X(D - D^{\text{red}}).
\end{array}$$



**Third step:** Show that  $\rho_*^{\log}$  is surjective.

From the second step, we know that  $\rho^{\log}$  fits into an exact sequence

$$0 \longrightarrow F^{\log} \xrightarrow{\rho^{\log}} L^\vee \otimes Q \longrightarrow \mathbb{T}_2 \longrightarrow 0,$$

where  $\mathbb{T}_2$  is a torsion sheaf. In particular,  $H^1(X, \mathbb{T}_2) = 0$ . We deduce that  $\rho_*^{\log}$  is surjective:

$$\rho_*^{\log} : H^1(X, F^{\log}) \longrightarrow H^1(X, L^\vee \otimes Q).$$

**Fourth step:** Show that  $f_{\parallel}$  is surjective.

At this point, we need some explicit calculations. Let  $B_{\parallel}$  be a sufficiently small neighborhood of  $t_0$  in  $\mathcal{T}_{\parallel} = \mathcal{T}_{g,n} \times J_{\parallel}$ . We may choose coordinates  $(\underline{t}, \underline{a}^{(1)}, \dots, \underline{a}^{(n)})$  on  $B_{\parallel}$ , with  $\underline{t} \in \mathcal{T}_{g,n}$  and  $\varphi_{\underline{a}^{(i)}} : \varepsilon \mapsto \sum_{k=1}^{n_i} a_k^{(i)} \varepsilon^k$  in  $J^{(i)}$  such that  $t_0$  corresponds to  $(\underline{0}, (1, \underline{0}), \dots, (1, \underline{0}))$ . The total space  $\mathcal{X}_{B_{\parallel}}$  of the family of curves parametrized by  $B_{\parallel}$  is covered by open sets  $\mathcal{U}_i$  with coordinates of the form  $(z_i, \underline{t}, \underline{a}^{(1)}, \dots, \underline{a}^{(n)})$  with holomorphic transition maps  $z_i = f_{ij}(z_j, \underline{t})$  dictacted by the Teichmüller space such that moreover each irreducible component of  $\mathcal{D}^{\text{red}}$  intersects precisely one chart, and the intersection is there given by  $\{z_i = 0\}$ . Over these charts, we have local frames  $\underline{e}_i$  of  $\mathcal{E}_{B_{\parallel}}$  such that  $\nabla|_{\mathcal{X}_{B_{\parallel}}}$  is given by

$$\begin{cases} \underline{e}_i \cdot Y \mapsto \underline{e}_i \cdot \left( d + A^{(i)}(\varphi_{\underline{a}^{(i)}}(z_i)) d(\varphi_{\underline{a}^{(i)}}(z_i)) \right) \cdot Y & \text{if } \mathcal{D}^{\text{red}} \cap \mathcal{U}_i \neq \emptyset \\ \underline{e}_j \cdot Y \mapsto \underline{e}_j \cdot dY & \text{if } \mathcal{D}^{\text{red}} \cap \mathcal{U}_j = \emptyset \end{cases}$$

Here  $A^{(i)}$  is a two-by-two matrix whose coefficients are meromorphic functions (in one variable) with poles of order at most  $n_i$  at the origin. The transition maps between two frames away from  $\mathcal{D}^{\text{red}}$  are then constant. Otherwise, keeping the above convention for indexation by  $i$  or  $j$ , we have that  $\underline{e}_i \psi_{ij} = \underline{e}_j$  implies

$$d\psi_{ij} \cdot \psi_{ij}^{-1} + A^{(i)}(\varphi_{\underline{a}^{(i)}}(z_i)) d(\varphi_{\underline{a}^{(i)}}(z_i)) = 0. \quad (1.38)$$

Let us calculate  $f_{\parallel} \left( \frac{\partial}{\partial a_k^{(i_0)}} \right)$ . Contracting  $\nabla|_{\mathcal{X}_{B_{\parallel}}}$  with  $\frac{\partial}{\partial a_k^{(i_0)}}$  yields

$$\begin{cases} \underline{e}_i \cdot Y \mapsto \underline{e}_i \cdot \left( \frac{\partial}{\partial a_k^{(i)}} + A^{(i)}(\varphi_{\underline{a}^{(i)}}(z_i)) z_i^k \right) \cdot Y & \text{if } i = i_0 \\ \underline{e}_i \cdot Y \mapsto \underline{e}_i \cdot \frac{\partial}{\partial a_k^{(i_0)}} Y & \text{if } i \neq i_0 \\ \underline{e}_j \cdot Y \mapsto \underline{e}_j \cdot \frac{\partial}{\partial a_k^{(i_0)}} Y \end{cases}$$

These however are not all local sections of  $\mathcal{F}^{\log}$ . In order to obtain such sections, which still project to  $\frac{\partial}{\partial a_k^{(i_0)}}$ , we may add for example the contraction of  $\nabla|_{\mathcal{X}_{B_{\parallel}}}$  with

$$-\frac{z_i^k}{\varphi'_{\underline{a}^{(i)}}(z_i)} \frac{\partial}{\partial z_i} = -\frac{z_i^k}{a_1^{(i)} + 2a_2^{(i)} z_i + \dots + (n_i - 1)a_{n_i-1}^{(i)} z_i^{n_i-2}} \frac{\partial}{\partial z_i}$$

over  $\mathcal{U}_i$  with  $i = i_0$ , yielding

$$\begin{cases} \underline{e}_i \cdot Y \mapsto \underline{e}_i \cdot \left( \frac{\partial}{\partial a_k^{(i)}} - \frac{z_i^k}{\varphi'_{\underline{a}^{(i)}}(z_i)} \frac{\partial}{\partial z_i} \right) \cdot Y & \text{if } i = i_0 \\ \underline{e}_i \cdot Y \mapsto \underline{e}_i \cdot \frac{\partial}{\partial a_k^{(i_0)}} Y & \text{if } i \neq i_0 \\ \underline{e}_j \cdot Y \mapsto \underline{e}_j \cdot \frac{\partial}{\partial a_k^{(i_0)}} Y \end{cases}$$

We obtain a Čech-cocycle which is identically zero on all intersections except  $\mathcal{U}_{i_0} \cap \mathcal{U}_j$ , where it is given with respect to the coordinate on  $\mathcal{U}_i$  with  $i = i_0$  by

$$\underline{e}_i \cdot Y \mapsto \underline{e}_i \cdot \left( -\frac{\partial \psi_{ij}}{\partial a_k^{(i)}} \psi_{ij}^{-1} + \frac{z_i^k}{\varphi'_{\underline{a}^{(i)}}(z_i)} \frac{\partial}{\partial z_i} \right) \cdot Y \stackrel{\text{1.38}}{=} \underline{e}_i \cdot \left( A^{(i)}(\varphi_{\underline{a}^{(i)}}(z_i)) z_i^k + \frac{z_i^k}{\varphi'_{\underline{a}^{(i)}}(z_i)} \frac{\partial}{\partial z_i} \right) \cdot Y.$$

Restricting to  $X$  we obtain a cocycle in  $H^1(X, F^{\log})$  representing  $f_{\parallel} \left( \frac{\partial}{\partial a_k^{(i_0)}} \right)$ , namely

$$\underline{e}_{i_0} \cdot Y \mapsto \underline{e}_{i_0} \cdot z_{i_0}^k \left( \frac{\partial}{\partial z_{i_0}} + A^{(i_0)}(z_{i_0}) \right) \cdot Y.$$

This cocycle also represents the image of  $\left[ \nabla|_X \cdot \left( -z_{i_0}^k \frac{\partial}{\partial z_{i_0}} \right) \right] \in H^0(X, \mathbb{T}_1)$  under the connecting homomorphism associated to the exact sequence

$$0 \longrightarrow F^{\log} \longrightarrow F^{\text{mer}} \longrightarrow \mathbb{T}_1 \longrightarrow 0.$$

Denoting by  $K_{\parallel}$  the subspace of vectors in  $\mathbb{T}_{t_0} B_{\parallel} = \mathbb{T}_{t_0} \mathcal{T}_{\parallel}$  whose projection to  $\mathbb{T}_0 \mathcal{T}_{g,n}$  is zero, we have established by the above calculation an isomorphism  $K_{\parallel} \simeq H^0(X, \mathbb{T}_1)$  fitting into the following commutative diagram with exact columns

$$\begin{array}{ccc} K_{\parallel} & \xrightarrow{\sim} & H^0(X, \mathbb{T}_1) & (1.39) \\ \downarrow & & \downarrow & \\ \mathbb{T}_{t_0} \mathcal{T}_{\parallel} & \xrightarrow{f_{\parallel}} & H^1(X, F^{\log}) & \\ \downarrow \text{KS}_{(X,D)} & & \downarrow & \\ H^1(X, \mathbb{T}X(-D^{\text{red}})) & \xrightarrow{\delta_*} & H^1(X, F^{\text{mer}}) & \\ \downarrow & & \downarrow & \\ 0 & & 0 & \end{array}.$$

Since moreover  $\delta_*$  is surjective, it follows that  $f_{\parallel}$  is surjective.

### Conclusion

We may now conclude as in the proof of Theorem 1.7.1, briefly summarized as follows.

Combining the results in the above steps with the notation from § 1.6.3 (concerning deformations of quadruples) establishes the commutativity of the following diagram, with surjectivity of double-headed arrows, where  $B$  is a neighborhood of  $t_0$  in a submanifold of  $\mathcal{T}_{\parallel}$  in which  $L$  extends to a line subbundle of  $\mathcal{E}|_{p^{-1}(B)}$ .

$$\begin{array}{ccccc}
& & \mathbb{H}^1(X, \text{At}_{D^{\text{red}}}^L(E)) & \xrightarrow{\iota_*} & \mathbb{H}^1(X, \text{At}_D^{\text{log}}(E)) & \xrightarrow{\rho_*} & \mathbb{H}^1(X, L^\vee \otimes Q) \\
& \nearrow \text{KS}_{(E, L, X, D^{\text{red}})} & \downarrow \iota_* & \nearrow & \uparrow \mu_* & \nearrow \rho_*^{\text{log}} & \\
\mathbb{T}_{t_0} B & \xrightarrow{\text{KS}_{(E, X, D^{\text{red}})}} & \mathbb{H}^1(X, \text{At}_D^{\text{log}}(E)) & \xleftarrow{\mu_*} & \mathbb{H}^1(X, F^{\text{log}}) & & \\
\downarrow & \searrow & & \nearrow f_{\parallel} & & & \\
\mathbb{T}_{t_0} \mathcal{T}_{\parallel} & \xlongequal{\quad\quad\quad} & \mathbb{T}_{t_0} B_{\parallel} & & & & 
\end{array}$$

We have  $\rho_* \circ \iota_* = 0$  by Proposition [\[L.6.7\]](#). This implies

$$\text{codim}(B, \mathcal{T}_{\parallel}) \geq h^1(X, L^\vee \otimes Q).$$

For  $t_0$  generic in  $\mathcal{T}_k \setminus \mathcal{T}_{k-1}$ , we may choose a neighborhood  $B$  of  $t_0$  in  $\mathcal{T}_k \setminus \mathcal{T}_{k-1}$  as above and  $L$  with  $\deg(L^\vee \otimes Q) = k$  by Maruyama's Theorem [\[L.4.4\]](#). By Riemann-Roch, one then has  $h^1(X, L^\vee \otimes Q) \geq g - 1 - k$ .

For  $t_0$  generic in  $\mathcal{T}^{\text{wob}} = \mathcal{T}^{\text{nil}} \setminus \mathcal{T}_{g-2}$  we may choose a neighborhood  $B$  of  $t_0$  in  $\mathcal{T}^{\text{wob}}$  as above and  $L$  with  $h^1(X, L^\vee \otimes Q) > 0$  by Proposition [\[L.4.6\]](#). □

**Remark 1.7.4.** *The calculations and arguments provided in the first and fourth step in the proof of Theorem [\[1.7.3\]](#) did not use any assumption on the rank.*

**Remark 1.7.5.** *If  $(E_0, \nabla_0)$  in the above statement satisfies moreover [\(1.8\)](#), then the conclusion is also valid when considering the family  $\mathcal{E} \rightarrow \mathcal{X}$  with  $p : \mathcal{X} \rightarrow \mathcal{T}$  underlying the universal isomonodromic deformation of  $(E_0, \nabla_0)$  (see § [\[1.3.4\]](#)). Indeed, let  $t_0$  be a generic point in  $\mathcal{T}_k$ . Let  $\mathcal{T}_{\parallel} \subset \mathcal{T}$  the leaf of the parallel foliation containing  $t_0$  (see also § [\[1.3.4\]](#)). We may assume that  $t_0$  is not a singularity of  $\mathcal{T}_k \cap \mathcal{T}_{\parallel}$ . Theorem [\[1.7.3\]](#) yields*

$$\text{codim}(\mathcal{T}_k \cap \mathcal{T}_{\parallel}, \mathcal{T}_{\parallel}) \geq g - 1 - k,$$

whence  $\text{codim}(\mathcal{T}_k, \mathcal{T}) \geq g - 1 - k$ . The argument for  $\mathcal{T}^{\text{nil}}$  is identical.

## 1.7.2 Isomonodromic deformations and stability in arbitrary rank

We now show how to adapt the ideas from § [\[1.7.1\]](#) to the case of arbitrary rank  $r \geq 2$ . The statements and proofs we obtain can be deduced from [\[BHH16\]](#) and [\[BHH18b\]](#).

**Theorem 1.7.6.** *Let  $X_0$  be a compact Riemann surface of genus  $g \geq 1$  and let  $D_0$  be a reduced divisor on  $X_0$  of degree  $n \geq 0$ . Let  $(E_0, \nabla_0)$  be a logarithmic connection of arbitrary rank over  $X_0$ , with polar divisor  $D_0$ . Assume that  $(E_0, \nabla_0)$  is deformable, i.e.,  $3g - 3 + n > 0$ , and is irreducible.*

*Let  $\mathcal{E} \rightarrow \mathcal{X}$  with  $p : \mathcal{X} \rightarrow \mathcal{T} = \mathcal{T}_{g,n}$  be the family of vector bundles underlying the universal isomonodromic deformation of  $(E_0, \nabla_0)$ . Denote  $\mathcal{E}_t := \mathcal{E}|_{p^{-1}(t)}$  for any  $t \in \mathcal{T}$ . Define*

$$\mathcal{T}^{\text{nss}} := \{t \in \mathcal{T} \mid \mathcal{E}_t \text{ not semistable}\}, \quad \mathcal{T}^{\text{ns}} := \{t \in \mathcal{T} \mid \mathcal{E}_t \text{ not stable}\}.$$

*Then  $\mathcal{T}^{\text{nss}}$  and  $\mathcal{T}^{\text{ns}}$  are closed analytic subsets of  $\mathcal{T}$  satisfying*

$$\text{codim}(\mathcal{T}^{\text{nss}}, \mathcal{T}) \geq g, \quad \text{codim}(\mathcal{T}^{\text{ns}}, \mathcal{T}) \geq g - 1.$$

*In particular,  $\mathcal{T}^{\text{nss}}$  is proper and  $\mathcal{T}^{\text{ns}}$  is proper if  $g > 1$ .*

**Remark 1.7.7.** *Note that*

$$\mathcal{T}^{\text{NSS}} \subset \mathcal{T}^{\text{NS}} \subset \mathcal{T}.$$

*Proof.* The fact that  $\mathcal{T}^{\text{NSS}}$  and  $\mathcal{T}^{\text{NS}}$  are closed analytic subsets follows from Shatz' Theorem [1.4.2](#). Let  $t_0$  be a generic point in  $\mathcal{T}^{\text{NSS}}$  or in  $\mathcal{T}^{\text{NS}} \setminus \mathcal{T}^{\text{NSS}}$ . Again by Theorem [1.4.2](#), there exists a neighborhood  $B$  of  $t_0$  in the analytic subset under consideration such that  $\mathcal{E}_B := \mathcal{E}|_{p^{-1}(B)}$  admits a filtration of length  $N \geq 2$  by subbundles, which in restriction to  $\mathcal{X}_t := p^{-1}(t)$  with  $t \in B$  corresponds to the Harder-Narasimhan filtration, respectively a Jordan-Hölder filtration of  $\mathcal{E}_t$ . Denote  $(E, X, D) := (\mathcal{E}_{t_0}, \mathcal{X}_{t_0}, D \cap \mathcal{X}_{t_0})$  and let

$$0 = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_N = E$$

be the filtration under consideration of  $E$ . Similarly to Definition [1.5.4](#), define a filtered Atiyah bundle

$$\text{At}_D^F(E)(U) := \{P \in \text{At}_D(E)(U) \mid P(F_i(U)) \subset F_i(U) \quad \forall i \in \llbracket 1, N \rrbracket\}.$$

It fits into the following diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & (1.40) \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \text{End}^F(E) & \longrightarrow & \text{At}_D^F(E) & \longrightarrow & \text{TX}(-D) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \iota & & \parallel \\
 0 & \longrightarrow & \text{End}(E) & \longrightarrow & \text{At}_D(E) & \longrightarrow & \text{TX}(-D) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \rho & & \\
 & & \text{End}(E)/\text{End}^F(E) & \xlongequal{\quad} & \text{End}(E)/\text{End}^F(E) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & .
 \end{array}$$

with  $\text{End}^F(E)$  as in Lemma [1.4.7](#), and  $\iota$  being the natural inclusion map.

Denote by  $\delta : \text{TX}(-D) \rightarrow \text{At}_D(E)$  the morphism induced by  $\nabla|_X$  as in Lemma [1.5.3](#) and consider the map

$$\text{II}_\delta := \rho \circ \delta : \text{TX}(-D) \rightarrow \text{End}(E)/\text{End}^F(E)$$

induced by composition with  $\rho$ , where  $\rho$  is defined by Diagram [\(1.40\)](#). The map  $\text{II}_\delta$  is not the zero map by irreducibility of  $(E, \nabla|_X)$ . We therefore have a non-zero coherent subsheaf

$$G := \text{Im}(\text{II}_\delta) \subset \text{End}(E)/\text{End}^F(E).$$

Denoting

$$\text{At}_D^G(E) := \rho^{-1}(G), \quad \text{End}^G(E) := \text{End}(E) \cap \text{At}_D^G(E),$$

we obtain a new diagram with exact rows and columns given in [\(1.41\)](#).

(1.41)

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{End}^F(E) & \longrightarrow & \text{At}_D^F(E) & \longrightarrow & \text{TX}(-D) \longrightarrow 0 \\
& & \downarrow & & \downarrow \iota' & \nearrow \delta & \parallel \\
0 & \longrightarrow & \text{End}^G(E) & \longrightarrow & \text{At}_D^G(E) & \longrightarrow & \text{TX}(-D) \longrightarrow 0 \\
& & \downarrow & & \downarrow \rho' & \searrow \Pi_\delta & \\
& & G & \xlongequal{\quad} & G & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

We construct a Kodaira-Spencer map  $\text{KS}_{(E,F,X,D)} : \mathbb{T}_{t_0} B \rightarrow \text{H}^1(X, \text{At}_D^F(E))$  similarly to Definition 1.6.5. Similarly to what we have done before, one shows that it fits into the following commutative diagram.

$$\begin{array}{ccccc}
& & \text{H}^1(X, \text{At}_D^F(E)) & \xrightarrow{\iota'_*} & \text{H}^1(X, \text{At}_D^G(E)) & \xrightarrow{\rho'_*} & \text{H}^1(X, G) \\
& \nearrow \text{KS}_{(E,F,X,D)} & \downarrow \iota_* & \nwarrow \nu_* & \nearrow & \nearrow & \\
\mathbb{T}_{t_0} B & \xrightarrow{\text{KS}_{(E,X,D)}} & \text{H}^1(X, \text{At}_D(E)) & & & & \\
\downarrow & \nearrow & & \nearrow \delta_* & \nearrow \Pi_{\delta_*} & & \\
\mathbb{T}_{t_0} \mathcal{T} & \xrightarrow[\text{KS}_{(X,D)}]{\sim} & \text{H}^1(X, \text{TX}(-D)) & & & & 
\end{array}$$

Here  $\nu_*$  is induced from the natural inclusion map  $\nu : \text{At}_D^G(E) \rightarrow \text{At}_D(E)$ .

On the one hand,  $\Pi_{\delta_*}$  is surjective because  $G$  is a line bundle and  $\Pi_\delta$  is non-zero. On the other hand,  $\rho'_* \circ \iota'_*$  is the zero map by exactness in Diagram (1.41). Hence by Riemann-Roch

$$\dim(\mathcal{T}) - \dim(B) \geq h^1(X, G) \geq g - 1 - \deg(G).$$

The vector bundle Lemma 1.4.7 implies  $\deg(G) < 0$  in the case  $B \subset \mathcal{T}^{\text{nss}}$  and  $\deg(G) \leq 0$  in the case  $B \subset \mathcal{T}^{\text{ns}} \setminus \mathcal{T}^{\text{nss}}$ .  $\square$

Let us make some remarks on how the statement of Theorem 1.7.6 can obviously be sharpened in view of the above proof. If  $n = 0$ , then even without the assumption of  $(E_0, \nabla_0)$  being irreducible, the estimate

$$\text{codim}(\mathcal{T}^{\text{nss}}, \mathcal{T}) \geq g$$

still holds. Indeed, it suffices to see in the proof that the second fundamental form  $\Pi_\delta$  with respect to a Harder-Narasimhan filtration of length  $N \geq 2$  cannot be the zero map. If  $\Pi_\delta$  were zero and  $n = 0$ , then  $\nabla|_X$  would induce a holomorphic connection on each  $F_i/F_{i-1}$ , which would imply  $\mu(F_i/F_{i-1}) = 0$  for all  $i \in \llbracket 1, N \rrbracket$  [Wei38] [Ati57]. This contradicts the strict monotony of slopes in a Harder-Narasimhan filtration.

In the statement, we also excluded the case  $(g = 1, n = 0)$  because then the base curve is not stable. However, if  $(E_0, \nabla_0)$  is a holomorphic  $(n = 0)$  connection on a curve of genus  $g = 1$ , then

$E_0$  is automatically semistable of slope 0 (by Riemann-Hilbert correspondence and the Atiyah-Weil-criterion).

On the other hand, the idea of the above proof does not allow to adapt the argument concerning very stability in Theorem [1.7.1](#) to higher rank, basically because from  $h^1(X, V) > 0$  with  $V = (\ker(\theta))^\vee \otimes E/\ker(\theta)$ , where  $\theta$  is a Higgs field on  $E$ , we cannot conclude  $h^1(X, G) > 0$  for an arbitrary line subbundle  $G$  of  $V$ .

**Theorem 1.7.8.** *Let  $X_0$  be a compact Riemann surface of genus  $g \geq 1$  and let  $D_0^{\text{red}}$  be a reduced divisor on  $X$  of degree  $n > 0$ . Let  $(E_0, \nabla_0)$  be a meromorphic connection of arbitrary rank over  $X_0$ , with polar locus  $D_0^{\text{red}}$  and polar divisor  $D_0$ . Assume that  $(E_0, \nabla_0)$  is irreducible.*

*Let  $\mathcal{E} \rightarrow \mathcal{X}$  with  $p : \mathcal{X} \rightarrow \mathcal{T}_\parallel$  be the vector bundle underlying the parallel isomonodromic deformation of  $(E_0, \nabla_0)$ . With the notation of Theorem [1.4.2](#) for the non-semistable locus  $\mathcal{T}^{\text{ns}}$  and the non-stable locus  $\mathcal{T}^{\text{ns}}$ , the following estimates hold*

$$\text{codim}(\mathcal{T}^{\text{ns}}, \mathcal{T}_\parallel) \geq g, \quad \text{codim}(\mathcal{T}^{\text{ns}}, \mathcal{T}_\parallel) \geq g - 1.$$

*Proof.* The proof of this theorem consists in adapting the proof of Theorem [1.7.3](#), in a manner analogous to the way we adapted the proof of Theorem [1.7.1](#) to the case of higher rank in the proof of Theorem [1.7.6](#). More precisely, we have to consider the logarithmic Atiyah bundle  $\text{At}_{D^{\text{red}}}^F(E)$  preserving a (Harder-Narasimhan or Jordan-Hölder) filtration instead of  $\text{At}_{D^{\text{red}}}^L(E)$ , which preserves just a subbundle. Moreover, it is necessary to modify the third step because  $\text{End}(E)/\text{End}^F(E)$  is no longer necessarily a line bundle. Instead, one considers

$$G^{\text{mer}} := \text{Im}(\text{II}_\delta) \subset \text{End}(E) \otimes \mathcal{O}_X(D - D^{\text{red}}) / \text{End}^F(E) \otimes \mathcal{O}_X(D - D^{\text{red}})$$

and sets  $G := G^{\text{mer}} \cap (\text{End}(E)/\text{End}^F(E))$ . As in the second step, one obtains  $G \neq 0$ . This leads to the following commutative diagram

$$\begin{array}{ccccc}
 & & \text{H}^1(X, \text{At}_{D^{\text{red}}}^F(E)) & \xrightarrow{\iota'_*} & \text{H}^1(X, \text{At}_{D^{\text{red}}}^G(E)) & \xrightarrow{\rho'_*} & \text{H}^1(X, G) \\
 & \nearrow \text{KS}_{(E, F, X, D^{\text{red}})} & \downarrow \iota_* & \swarrow \nu_* & \uparrow \mu'_* & \nearrow \rho_*^{\text{log}'} & \\
 \text{T}_{t_0} B & & \text{H}^1(X, \text{At}_D^{\text{log}}(E)) & \xleftarrow{\mu_*} & \text{H}^1(X, F^{\text{log}}) & & \\
 \downarrow & \nearrow \text{KS}_{(E, X, D^{\text{red}})} & & \nearrow f_\parallel & & & \\
 \text{T}_{t_0} \mathcal{T}_\parallel & \xlongequal{\quad} & \text{T}_{t_0} B_\parallel & & & & 
 \end{array}$$

One then concludes as in the proof of Theorem [1.7.6](#) using  $\rho'_* \circ \iota'_* = 0$  and Shatz' Theorem.  $\square$

### 1.7.3 Further remarks and open questions

Let  $(E_0, \nabla_0)$  be a logarithmic connection of rank  $r \geq 2$ , with polar divisor  $D_0$  of degree  $n$  on a compact connected Riemann surface  $X_0$  of genus  $g$ . If  $g = 0$ , then we moreover assume  $n \geq 4$ .

Let  $(E_1 \rightarrow X_1, \nabla_1)$  be another such connection. For conciseness, in the following we will say that  $(E_0, \nabla_0)$  is *isomonodromically deformable* into  $(E_1, \nabla_1)$  if there exists an isomonodromic deformation of  $(E_0, \nabla_0)$ , given by some flat logarithmic connection  $(\mathcal{E}, \nabla)$  over a family of compact Riemann surfaces  $p : \mathcal{X} \rightarrow \mathcal{T}$  with connected parameter space  $\mathcal{T}$ , endowed with marked points, given by

$\mathcal{D} \subset \mathcal{X}$ , and an isomorphism  $(\mathcal{E}, \nabla)|_{p^{-1}(t_0)} \simeq (E_0, \nabla_0)$ , such that there exists  $t_1 \in \mathcal{T}$  and an isomorphism  $(\mathcal{E}, \nabla)|_{p^{-1}(t_1)} \simeq (E_1, \nabla_1)$ .

Recall from the introduction that the Riemann-Hilbert type problem that motivated our results is the following.

*Given a logarithmic connection  $(E_0, \nabla_0)$  as above, is it isomonodromically deformable into a connection  $(E_1, \nabla_1)$  such that  $E_1$  is semistable?*

According to our results, the answer is positive if

- i)  $n = 0$  or
- ii)  $(E_0, \nabla_0)$  is irreducible and  $g \geq 1$  or
- iii)  $(E_0, \nabla_0)$  is irreducible and  $(g, r) = (0, 2)$  and  $\deg(E_0)$  even.

As one can easily check, the irreducibility assumption in the second and third point can be weakened into  $\nabla_0$  not preserving any nonzero subbundle  $F$  such that  $\mu(F) > \mu(E_0)$ , in other words, into the pair  $(E_0, \nabla_0)$  being *semistable*. Moreover, if the pair  $(E_0, \nabla_0)$  is not semistable, then the answer is negative for purely topological reasons. What is not entirely covered by our results is the genus zero case. Consider an irreducible logarithmic connection  $(E_0, \nabla_0)$  of rank  $r \geq 2$  with  $n \geq 4$  poles over  $\mathbb{P}^1$ . In view of the proof of Theorem [1.7.6](#), up to isomonodromic deformation, we may assume that  $E_0$  is of the following type:

$$E_0 \simeq \bigoplus_{k=1}^r \mathcal{O}_{\mathbb{P}^1}(a_k) \quad \text{where } \forall k \in \llbracket 1, r-1 \rrbracket, \quad a_{k+1} = a_k \text{ or } a_{k+1} = a_k + 1.$$

When  $\deg(E_0) = 0$  and  $r = 2$ , then such a vector bundle is trivial. When  $r > 2$ , even if we assume  $\deg(E_0) = 0$ , the question remains whether  $(E_0, \nabla_0)$  can be isomonodromically deformed into a connection on the trivial bundle.

In the future, we would like to develop some applications of the Theorems [1.7.6](#) and [1.7.8](#). Note that from refinements of the  $\mathfrak{sl}_2$ -case of Theorem [1.7.1](#) several applications already appeared. For example, it is shown in [Heu09](#) that any non-trivial one-parameter isomonodromic deformation of irreducible  $\mathfrak{sl}_2$ -connection with four poles (counted with multiplicity) arises as a solution of one of the Painlevé equations  $P_{\text{II}}-P_{\text{VI}}$ . Moreover, in the  $\mathfrak{sl}_2$  and  $(g, n) = (2, 0)$  case, from refinements in [HL15](#) and [CDHL18](#) of Theorem [1.7.1](#) one deduces that the monodromy functor for irreducible holomorphic connections on unstable bundles, respectively the trivial bundle, (over varying curves) is a local diffeomorphism. This represents a new proof of Hejhal's theorem and a first step towards an open question of Ghys respectively (see Chapter [3](#)). In the case of rank  $r \geq 2$  and  $g > 1$ , Krichever [Kri02](#) obtained isomodromy equations for meromorphic connections using Tyurin's approach of rational trivialization for generic stable bundles of degree  $rg$  [Tyu65](#). Theorems [1.7.6](#) and [1.7.8](#) suggest that any universal/parallel isomonodromic deformation of irreducible meromorphic connection should appear as a solution of Krichever's isomonodromy equations.

Note that Theorems [1.7.1](#) and [1.7.6](#) provide lower bounds for the codimension of the loci of certain "non-generic" vector bundles that may occur along universal isomonodromic deformations of irreducible logarithmic connections. It would be interesting to see whether these lower bounds are sharp. Even more interesting is to estimate the dimension of these subsets of the parameter space in function of the initial connection (i.e., in function of its monodromy and residues). Already the non-emptiness of these subsets, which is a degeneration problem, is widely open. For example:

**Problem 1.** *Let  $(E_0, \nabla_0)$  be an irreducible logarithmic connection of rank  $r \geq 2$  over a compact Riemann surface of genus  $g \geq 1$  with polar divisor  $D_0$  of degree  $n \geq 0$ . Assume  $\deg(E_0) = 0$ .*

Find necessary and sufficient conditions for each of the following. The connection  $(E_0, \nabla_0)$  can be isomonodromically deformed into some connection  $(E_1, \nabla_1)$ , where  $E_1$  is

- a) the trivial bundle
- b) non-stable
- c) non-semistable.

To the best of the author's knowledge, the known (partial) answers to this problem are essentially the following. For  $g \geq 2$  and  $n = 0$ , by the Narasimhan-Seshadri Theorem [NS65], unitary monodromy is sufficient condition for Problem [1b) and non-unitary monodromy is a necessary condition for Problems [1a) and [1c). If moreover  $r = 2$ , then in view of [LM09] and the Gallo-Kapovich-Marden Theorem [GKM00], non-unitary monodromy is also sufficient for Problem [1c).

Note that Problem [1c) appears also as a degeneration problem when considering  $g = 0$  and semistable initial bundles. A conjecture that, as pointed out by Loray, already appears in the works of Poincaré, says that for  $g = 0$ ,  $n = 4$ ,  $r = 2$ , any irreducible tracefree  $(E_0, \nabla_0)$  can be isomonodromically/parallelly deformed into  $(E_1, \nabla_1)$  with  $E_1 \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ .

As mentioned in the introduction, our deformation technique allows to solve the Riemann-Hilbert problem investigating (Ramanan-)semistability along isomonodromic deformations for principal connections on  $G$ -bundles (see [BHH16]) over curves of genus  $g \geq 1$ . The discrete version of this Riemann-Hilbert problem (fixing the complex structure of the pointed base curve, but not the residues) seems to be largely open for  $G \neq \mathrm{GL}_r\mathbb{C}$ :

**Problem 2.** Let  $\rho : \pi_1(X \setminus D, x_0) \rightarrow G$  be a group homomorphism, where  $G$  is a connected reductive complex algebraic group,  $X$  is a compact Riemann surface and  $D$  is a divisor on  $X$ , such that there exist principal logarithmic connections  $(E_G \rightarrow X, D, \nabla_G)$  with monodromy conjugated to  $\rho$  (see [Boa11]). Is there one such connection with  $E_G$  trivial or semistable?



## Chapter 2

# Algebraic isomonodromic deformations and the mapping class group

### 2.1 Introduction

Let  $\Sigma_g$  be a compact oriented real surface of genus  $g$ , let  $Y^n := \{y_1, \dots, y_n\}$  be a set of  $n$  distinct points in  $\Sigma_g$  and let  $y_0 \in \Sigma_g \setminus Y^n$ . Denote

$$\Lambda_{g,n} := \pi_1(\Sigma_g \setminus Y^n, y_0).$$

The mapping class group  $\Gamma_{g,n}$  of isotopy classes of orientation preserving homeomorphisms of  $\Sigma_g$  that fix  $Y^n$  pointwise, acts on the character variety

$$\chi_{g,n}(G) := \text{Hom}(\Lambda_{g,n}, G) / G,$$

where  $G$  is an algebraic subgroup of  $\text{GL}_r\mathbb{C}$  (see Section 2.2 for more details). For a representation  $\rho \in \text{Hom}(\Lambda_{g,n}, G)$ , we denote by  $[\rho] \in \chi_{g,n}(G)$  the corresponding equivalence class.

In this chapter, we present two results about finite orbits of the mapping class group action on  $\chi_{g,n}(G)$  for  $G = \text{GL}_r\mathbb{C}$ , obtained in collaboration with G. Cousin in the prepublication [CH16]. These results and their respective proofs can be read independently. On the one hand, we relate such finite orbits to the existence of an algebraic universal isomonodromic deformation of a logarithmic connection over a curve, whose monodromy belongs to that orbit. The obtained result, stated in § 2.1.1, can be seen as criterion under which a GAGA-type theorem holds for isomonodromic deformations. On the other hand, motivated by this result, we perform a dynamical study classifying conjugacy classes of reducible rank two representations with finite orbit. This classification result will be stated in § 2.1.2.

#### 2.1.1 Algebraization of universal isomonodromic deformations

As recalled in Chapter 1, any triple  $(C, E, \nabla_0)$ , where  $C$  is a compact Riemann surface and  $(E, \nabla_0)$  is an analytic logarithmic connection over  $C$ , admits a universal analytic isomonodromic deformation, which is unique up to unique isomorphism, and whose parameter space is the Teichmüller space  $\mathcal{T}_{g,n}$ . This universal analytic isomonodromic deformation satisfies a universal property with respect to germs of analytic isomonodromic deformations of  $(C, E, \nabla_0)$ . On the other hand,  $C$  can be seen as a smooth curve (a smooth projective complex variety of dimension one). By one of Serre's GAGA

theorems [Ser56, Prop. 18], the holomorphic vector bundle  $E$  is obtained by analytification from a unique algebraic vector bundle over  $C$ . Moreover, the analytic logarithmic connection  $\nabla_0$  on  $E$  is induced by a unique algebraic logarithmic connection on this algebraic vector bundle [Del70, Prop. II.4.4.]. A universal algebraic isomonodromic deformation of  $(C, E, \nabla_0)$ , if it exists, would be an algebraic isomonodromic deformation whose analytic germification is isomorphic to the germification of the universal analytic isomonodromic deformation of  $(C, E, \nabla_0)$ . In § 2.6.4, we give an alternative definition and state the corresponding universal property. Our main result is the following.

**Theorem 2.1.1.** *Let  $C$  be an irreducible smooth projective complex curve of genus  $g$ . Let  $D$  be a set of  $n$  distinct points in  $C$  and let  $\varphi : (\Sigma_g, Y^n) \rightarrow (C, D)$  be an orientation preserving homeomorphism. Let  $(E, \nabla_0)$  be an algebraic logarithmic connection of rank  $r$  over  $C$  with polar divisor  $D$  and monodromy  $[\rho] \in \chi_{g,n}(\mathrm{GL}_r\mathbb{C})$  with respect to  $\varphi$ . Assume that  $2g - 2 + n > 0$  and that  $\nabla_0$  is mild. If  $r > 2$ , then assume further that  $\rho$  is semisimple. The following are equivalent:*

- (1) *There exists a universal algebraic isomonodromic deformation of  $(C, E, \nabla_0)$ .*
- (2) *The orbit  $\Gamma_{g,n} \cdot [\rho]$  in  $\chi_{g,n}(\mathrm{GL}_r\mathbb{C})$  is finite.*

Note that the orbit  $\Gamma_{g,n} \cdot [\rho]$  in  $\chi_{g,n}(\mathrm{GL}_r\mathbb{C})$  does not depend on the choice of  $\varphi$ . We prove Theorem 2.1.1 by adapting the proof for the special case of genus  $g = 0$ , which has been established in [Cou17]. The main ingredients of the proof of Theorem 2.1.1 are: the logarithmic Riemann-Hilbert correspondence (see Section 2.6); the introduction of a base point section for a family of punctured curves and the splitting of the fundamental group of the total space of the family, together with its relation to the mapping class group (see Section 2.7). Both implications to be proven appear as special cases of stronger results: Theorems 2.8.1 and 2.8.2 respectively. We give their statements and proofs in Section 2.8.

The statement of Theorem 2.1.1 is natural in the following sense. As we recall in Section 2.2, the (algebraic) moduli space  $\mathcal{M}_{g,n}$  of stable smooth  $n$ -pointed genus- $g$  curves is the quotient of the (analytic) Teichmüller space  $\mathcal{T}_{g,n}$  by the natural action of  $\Gamma_{g,n}$ . Intuitively, a universal algebraic isomonodromic deformation should be the quotient of the universal analytic isomonodromic deformation with respect to a sufficiently large subgroup of  $\Gamma_{g,n}$  that fixes  $[\rho]$ .

## 2.1.2 Dynamical study of finite orbits in the reducible rank 2 case

Since the pure mapping class group  $\Gamma_{g,n}$  is a finite index subgroup of the full mapping class group  $\hat{\Gamma}_{g,n}$ , for any representation  $\rho \in \mathrm{Hom}(\Lambda_{g,n}, G)$ , the conjugacy class  $[\rho] \in \chi_{g,n}(G)$  has finite orbit under  $\Gamma_{g,n}$  if and only if it has finite orbit under  $\hat{\Gamma}_{g,n}$  (see Section 2.2). Note that the size of  $\hat{\Gamma}_{g,n} \cdot [\rho]$  equals the size of the set of conjugacy classes of  $m$ -tuples

$$\left\{ (\rho'(s_1), \dots, \rho'(s_m)) \mid \rho' \in \mathrm{Hom}(\Lambda_{g,n}, G) \text{ and } [\rho'] \in \hat{\Gamma}_{g,n} \cdot [\rho] \right\} / G,$$

where  $\{s_1, \dots, s_m\}$  is a set of generators of  $\Lambda_{g,n}$ . We introduce a specific presentation

$$\Lambda_{g,n} = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_n \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_n = 1 \rangle$$

and a subgroup

$$\hat{\Gamma}_{g,n}^\circ = \langle \tau_1, \dots, \tau_{3g+n-2}, \sigma_1, \dots, \sigma_{n-1} \rangle$$

of  $\hat{\Gamma}_{g,n}^\bullet := \hat{\Gamma}_{g,n+1}$  which, as such, acts on  $\mathrm{Hom}(\Lambda_{g,n}, G)$ , and which is sufficiently large in the sense that the  $\hat{\Gamma}_{g,n}^\circ$ -orbit of  $[\rho] \in \chi_{g,n}(G)$  equals its  $\hat{\Gamma}_{g,n}$ -orbit. Moreover, the action of  $\hat{\Gamma}_{g,n}^\circ$  on  $\Lambda_{g,n}$  can be explicitly described (see Section 2.3).

We then apply this explicit description of the mapping class group action to the specific study of finite  $\Gamma_{g,n}$ -orbits on  $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$  that correspond to reducible representations, assuming  $g > 0$ . Using other techniques, the complete characterization of finite  $\Gamma_{g,n}$ -orbits on  $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$  with  $g > 0$  that correspond to irreducible representations with values in  $\mathrm{SL}_2\mathbb{C}$  is under preparation by Wang [\[Wha17\]](#). The complete characterization of finite  $\Gamma_{g,n}$ -orbits on  $\chi_{0,n}(\mathrm{GL}_2\mathbb{C})$  that correspond to reducible representations (assuming  $g = 0$ ) has been carried out by Cousin and Moussard in [\[CM16\]](#). As in our case, the study then boils down to the characterization of finite orbits in  $\chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$ , where  $\mathrm{Aff}(\mathbb{C})$  is the group of affine isomorphisms of the complex line. Moreover, as a particularity of the genus zero case, the pure mapping class group acts trivially on the linear part of affine representations  $\rho_{\mathrm{Aff}}$ , and the study can be further reduced to linear dynamics. For  $g > 0$ , a more direct approach is necessary.

In the case  $g = 1$  and  $n > 0$ , we find a particular type of representations with infinite image, whose conjugacy classes have finite orbit under  $\Gamma_{g,n}$ , namely the representations  $\rho_{\mu,\mathbf{c}} \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_2\mathbb{C})$  defined by

$$\rho_{\mu,\mathbf{c}}(\alpha_1) := \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \quad \rho_{\mu,\mathbf{c}}(\beta_1) := \begin{pmatrix} 1 & -\frac{1}{\mu-1} \\ 0 & 1 \end{pmatrix} \quad \rho_{\mu,\mathbf{c}}(\gamma_i) := \begin{pmatrix} 1 & c_i \\ 0 & 1 \end{pmatrix} \quad \forall i \in \llbracket 1, n \rrbracket$$

where  $\mu \in \mathbb{C}^* \setminus \{1\}$  is a root of unity and  $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{C}^n$  with  $\sum_{i=1}^n c_i = 1$ . Note that the condition  $\sum_{i=1}^n c_i = 1$  is necessary for  $\rho_{\mu,\mathbf{c}}$  to be well defined. The complete classification, for every  $g > 0$  and  $n \geq 0$ , of reducible rank-2 representations with finite  $\Gamma_{g,n}$ -orbit is the following.

**Theorem 2.1.2.** *Let  $g > 0, n \geq 0$ . Let  $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_2\mathbb{C})$  be a reducible representation. Consider its conjugacy class  $[\rho] \in \chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ . Then the orbit  $\Gamma_{g,n} \cdot [\rho]$  is finite if and only if one of the following conditions is satisfied.*

- (1) *The representation  $\rho$  is a direct sum of scalar representations with finite images.*
- (2) *We have  $g = 1, n > 0$ , there are a root of unity  $\mu \in \mathbb{C}^* \setminus \{1\}$ ,  $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{C}^n$  with  $\sum_{i=1}^n c_i = 1$  and a scalar representation  $\lambda$  with finite image such that*

$$[\rho] \in \Gamma_{g,n} \cdot [\lambda \otimes \rho_{\mu,\mathbf{c}}].$$

The heart of the proof of [Theorem 2.1.2](#) is the complete classification of finite  $\hat{\Gamma}_{g,n}$ -orbits in  $\chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$  under the full mapping class group (see the beginning of [Section 2.4](#) for details on how we proceed). We deduce an explicit description of the finite  $\Gamma_{g,n}$ -orbits for scalar and affine representations. The decomposition of reducible representations into a tensor product of such representations then yields the result. Moreover, the size of the finite orbits can be estimated (see [Section 2.5](#)).

## 2.2 The mapping class group

Let  $g$  and  $n$  be nonnegative integers. Let  $\Sigma_g$  be a compact oriented real surface of genus  $g$ , let  $y^n = (y_1, \dots, y_n)$  be a sequence of  $n$  distinct points in  $\Sigma_g$ . We shall denote by  $Y^n := \{y_1, \dots, y_n\}$  the corresponding (unordered) set of points. The (*pure*) *mapping class group* of  $(\Sigma_g, y^n)$  is defined to be the set of orientation preserving homeomorphisms  $h$  of  $\Sigma_g$  such that  $h(y_i) = y_i$  for all  $i \in \llbracket 1, n \rrbracket := \{k \in \mathbb{Z} \mid 1 \leq k \leq n\}$ , quotiented by isotopies:

$$\Gamma_{g,n} := \mathrm{Homeo}_+(\Sigma_g, y^n) / \{\text{isotopies relative to } Y^n\}.$$

We can also consider homeomorphisms of  $\Sigma_g$  that preserve the set  $Y^n$ , but do not necessarily preserve the labelling of the punctures. This leads to the *full mapping class group*

$$\hat{\Gamma}_{g,n} := \text{Homeo}_+(\Sigma_g, Y^n) / \{\text{isotopies relative to } Y^n\}.$$

Note that we have an exact sequence of groups

$$1 \longrightarrow \Gamma_{g,n} \longrightarrow \hat{\Gamma}_{g,n} \longrightarrow \mathfrak{S}_n \longrightarrow 1,$$

where  $\mathfrak{S}_n$  denotes the symmetric group of degree  $n$ . In particular,  $\Gamma_{g,n}$  is a subgroup of  $\hat{\Gamma}_{g,n}$  of finite index  $n!$ .

### 2.2.1 The mapping class group action on $\chi_{g,n}(G)$

Let now  $y_0 \in \Sigma_g \setminus Y^n$  be a point. We denote the fundamental group of  $\Sigma_g \setminus Y^n$  with respect to the base point  $y_0$  by

$$\Lambda_{g,n} := \pi_1(\Sigma_g \setminus Y^n, y_0). \quad (2.1)$$

The composition  $\alpha \cdot \alpha'$  of two paths  $\alpha, \alpha' \in \Lambda_{g,n}$  shall denote the usual concatenation (first  $\alpha$ , then  $\alpha'$ ). For any group  $G$ , we may consider the space  $\text{Hom}(\Lambda_{g,n}, G)$  of representations as well as the set of representations modulo conjugation, which we shall denote

$$\chi_{g,n}(G) := \text{Hom}(\Lambda_{g,n}, G) / G. \quad (2.2)$$

Define the groups of orientation preserving homeomorphisms  $h$  of  $\Sigma_g$  such that  $h(y_0) = y_0$  and  $h(Y^n) = Y^n$ , respectively  $h(Y^n) = Y^n$ , modulo isotopy:

$$\begin{aligned} \Gamma_{g,n+1} &:= \text{Homeo}_+(\Sigma_g, y^n, y_0) / \{\text{isotopies relative to } Y^n \cup \{y_0\}\}, \\ \hat{\Gamma}_{g,n}^\bullet &:= \text{Homeo}_+(\Sigma_g, Y^n, y_0) / \{\text{isotopies relative to } Y^n \cup \{y_0\}\}. \end{aligned}$$

Now  $\hat{\Gamma}_{g,n}^\bullet$  naturally acts on the fundamental group  $\Lambda_{g,n}$ : for  $h \in \hat{\Gamma}_{g,n}^\bullet$  and  $\alpha \in \Lambda_{g,n}$ , we set

$$\mathbf{a}(h)(\alpha) := h_*\alpha.$$

Via the forgetful maps  $\Gamma_{g,n+1} \rightarrow \Gamma_{g,n}$  and  $\hat{\Gamma}_{g,n}^\bullet \rightarrow \hat{\Gamma}_{g,n}$  we obtain a commutative diagram

$$\begin{array}{ccccc} \Gamma_{g,n+1} & \hookrightarrow & \hat{\Gamma}_{g,n}^\bullet & \xrightarrow{\mathbf{a}} & \text{Aut}(\Lambda_{g,n}) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_{g,n} & \hookrightarrow & \hat{\Gamma}_{g,n} & \longrightarrow & \text{Out}(\Lambda_{g,n}) := \text{Aut}(\Lambda_{g,n}) / \text{Inn}(\Lambda_{g,n}). \end{array}$$

Indeed, the isotopy class of any element  $h \in \text{Homeo}_+(\Sigma_g, y^n)$  may be lifted to the isotopy class of an element  $h_0 \in \text{Homeo}_+(\Sigma_g, y^n, y_0)$ . Let  $h_1 \in \text{Homeo}_+(\Sigma_g, y^n, y_0)$  be another representative. Then these lifts are the extremities of an isotopy  $(h_t)_{t \in [0,1]}$  relative to  $Y^n$ . We have a loop  $\gamma \in \Lambda_{g,n}$  defined by  $\gamma(t) = h_t(y_0)$ . For any  $\alpha \in \Lambda_{g,n}$ , we obtain  $\mathbf{a}(h_1)(\alpha) = \gamma^{-1} \cdot \mathbf{a}(h_0)(\alpha) \cdot \gamma$ .

In particular, for any group  $G$ , the mapping class group  $\hat{\Gamma}_{g,n}$  acts on  $\chi_{g,n}(G)$ , and this action lifts to an action of  $\hat{\Gamma}_{g,n}^\bullet$  on the space  $\text{Hom}(\Lambda_{g,n}, G)$ . More precisely, for all  $\rho \in \text{Hom}(\Lambda_{g,n}, G)$ ,  $h \in \hat{\Gamma}_{g,n}^\bullet$  and  $\alpha \in \Lambda_{g,n}$ , we define

$$([h] \cdot \rho)(\alpha) := \rho(\mathbf{a}(h^{-1})(\alpha)). \quad (2.3)$$

Recall that for  $G = \mathrm{GL}_r \mathbb{C}$ , a representation  $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_r \mathbb{C})$  is called *irreducible* if the only subvector spaces  $V \subset \mathbb{C}^r$  stable under  $\mathrm{Im}(\rho)$  are  $\{0\}$  and  $\mathbb{C}^r$ . A *semisimple* representation is a direct sum of irreducible representations. These notions are invariant under conjugation, so that we may speak of irreducible or semisimple elements of  $\chi_{g,n}(\mathrm{GL}_r \mathbb{C})$ .

### 2.2.2 The mapping class group action on $\mathcal{T}_{g,n}$

We define a *curve of genus  $g$*  to be a *smooth projective complex curve*  $C$  with  $H^1(C, \mathbb{Z}) = \mathbb{Z}^{2g}$ .

As a set, the Teichmüller space  $\mathcal{T}_{g,n}$  of  $n$ -pointed genus- $g$  curves is the set of isomorphism classes  $[C, D, \varphi]$  of triples  $(C, D, \varphi)$ , where  $C$  is a curve of genus  $g$ ,  $D = \{x_1, \dots, x_n\}$  is a set of  $n$  distinct points in  $C$  and  $\varphi$  is a *Teichmüller structure*, i.e., an orientation-preserving homeomorphism  $\varphi : (\Sigma_g, Y^n) \rightarrow (C, D)$ . Two  $n$ -pointed genus- $g$  curves with Teichmüller structure  $(C, D, \varphi)$  and  $(C', D', \varphi')$  are said to be *isomorphic* if there exists an isomorphism of pointed curves  $\psi : (C, D) \rightarrow (C', D')$  such that  $[\varphi'] = [\psi \circ \varphi]$ , where  $[\varphi]$  denotes the isotopy class of  $\varphi$ .

We have a natural action of the pure mapping class group  $\Gamma_{g,n}$  on  $\mathcal{T}_{g,n}$  given by

$$[h] \cdot [C, D, \varphi] := [C, D, \varphi \circ h^{-1}]; \quad [h] \in \Gamma_{g,n}, [C, D, \varphi] \in \mathcal{T}_{g,n}.$$

The kernel of this action is finite. More precisely, we have (see [ACG11, Prop 4.11 p. 189]):

**Lemma 2.2.1.** *Let  $g, n \geq 0$  such that  $2g - 2 + n > 0$ . If the natural morphism  $\Gamma_{g,n} \rightarrow \mathrm{Aut}(\mathcal{T}_{g,n})$  has nontrivial kernel  $K_{g,n}$ , then  $K_{g,n} \simeq \mathbb{Z}/2\mathbb{Z}$  and one of the following holds.*

- $(g, n) = (2, 0)$  and the non-trivial element of  $K_{g,n}$  is the hyperelliptic involution of  $\Sigma_2$ .
- $(g, n) = (1, 1)$  and the non-trivial element of  $K_{g,n}$  is the order 2 symmetry about the puncture, given, for  $(\Sigma_1, y_1) = (\mathbb{C}/\mathbb{Z}^2, 0)$ , by  $z \mapsto -z$ .

### 2.2.3 Relation to $\mathcal{M}_{g,n}$

Let  $g, n$  be non-negative integers satisfying

$$2g - 2 + n > 0. \tag{2.4}$$

As a set, the moduli space  $\mathcal{M}_{g,n}$  of curves of genus  $g$  with  $n$  (labeled) punctures is the set of isomorphism classes  $[C, \mathbf{x}]$  of pairs  $(C, \mathbf{x})$ , where  $C$  is a genus  $g$  curve and  $\mathbf{x} = (x_1, \dots, x_n)$  is a tuple of  $n$  distinct points in  $C$ . The isomorphisms are isomorphisms of pointed curves that *respect the labellings* of the  $n$ -tuples. Notice that a pointed curve with Teichmüller structure  $(C, D, \varphi)$  defines such a pair  $(C, \mathbf{x})$ , by setting  $\mathbf{x} := (\varphi(y_i))_{i \in [1, n]}$ . In this way, we obtain a forgetful map

$$\pi_{g,n} : \mathcal{T}_{g,n} \rightarrow \mathcal{M}_{g,n} \tag{2.5}$$

whose fibers are globally fixed by the action of  $\Gamma_{g,n}/K_{g,n}$  on  $\mathcal{T}_{g,n}$ . Denote by  $\mathcal{R}_{g,n} \subset \mathcal{T}_{g,n}$  the set consisting in points with non-trivial stabilizer for the action of  $\Gamma_{g,n}/K_{g,n}$ . The subset  $\mathcal{B}_{g,n} := \pi_{g,n}(\mathcal{R}_{g,n})$  of  $\mathcal{M}_{g,n}$  characterizes pointed curves with automorphism groups not isomorphic to  $K_{g,n}$ . We say that these curves have *exceptional automorphisms*.

Recall that  $\mathcal{T}_{g,n}$  has a natural structure of a complex analytic manifold, and  $\mathcal{M}_{g,n}$  has a natural structure of a complex quasi-projective variety. The set  $\mathcal{B}_{g,n}$  of curves with exceptional automorphisms is a Zariski closed subset of  $\mathcal{M}_{g,n}$  (see [ACG11, Rem. 5.13 p. 202 and Th. 6.5 p. 207]) which is a proper subset (see [Bai62, Mon62, Poo00, Cor08]). Moreover, the map

$$\pi_{g,n}|_{\mathcal{T}_{g,n} \setminus \mathcal{R}_{g,n}} : \mathcal{T}_{g,n} \setminus \mathcal{R}_{g,n} \rightarrow \mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}$$

is a non branched analytic cover, with Galois group isomorphic to  $\Gamma_{g,n}/K_{g,n}$ . For any point  $\hat{\star} \in \mathcal{T}_{g,n}$  projecting to  $\star \in \mathcal{M}_{g,n}$  we obtain a tautological map

$$\text{taut}_{\hat{\star}} : \pi_1(\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}, \hat{\star}) \rightarrow \Gamma_{g,n} / K_{g,n}; \quad (2.6)$$

such that for the lift  $\hat{\gamma}$  in  $\mathcal{T}_{g,n}$  with  $\hat{\gamma}(0) = \hat{\star}$  of a loop  $\gamma$  corresponding to an element of  $\pi_1(\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}, \star)$  we have  $\hat{\gamma}(1) = \text{taut}_{\hat{\star}}(\gamma) \cdot \hat{\star}$ . For another point  $\hat{\star}' = [h] \cdot \hat{\star}$  we obtain  $\text{taut}_{[h] \cdot \hat{\star}} = h \cdot \text{taut}_{\hat{\star}} \cdot h^{-1}$ , so that  $\text{taut}_{\hat{\star}}$  is a morphism of groups:

$$\text{taut}_{\hat{\star}}(\gamma_1 \cdot \gamma_2) = \text{taut}_{\hat{\star}}(\gamma_1) \cdot \text{taut}_{\hat{\star}}(\gamma_2).$$

We refer to [ACG11, chap. XIV] for a more detailed exposition on all of the above.

## 2.3 Effective description of the action on $\chi_{g,n}(G)$

In this section, we describe the action of  $\hat{\Gamma}_{g,n}$  on  $\Lambda_{g,n}$  in terms of specified generators for both groups. More precisely, for a specific presentation

$$\Lambda_{g,n} = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_n \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_n = 1 \rangle$$

with  $g > 0$  we construct a subgroup

$$\hat{\Gamma}_{g,n}^\circ = \langle \tau_1, \dots, \tau_{3g+n-2}, \sigma_1, \dots, \sigma_{n-1} \rangle$$

of  $\hat{\Gamma}_{g,n}^\bullet = \hat{\Gamma}_{g,n+1}$  such that the  $\hat{\Gamma}_{g,n}^\circ$ -orbit of  $[\rho] \in \chi_{g,n}(G)$  equals its  $\hat{\Gamma}_{g,n}$ -orbit. Table 1 summarizes the action of the generators of  $\hat{\Gamma}_{g,n}^\circ$  on those generators of  $\Lambda_{g,n}$  that are not fixed by the action of the generator of  $\hat{\Gamma}_{g,n}^\circ$  under consideration.

$\tau_{2k}$	$k \in \llbracket 1, g \rrbracket$	$\alpha_k \mapsto \alpha_k \beta_k$
$\tau_{2k-1}$	$k \in \llbracket 1, g \rrbracket$	$\beta_k \mapsto \beta_k \alpha_k$
$\tau_{2g+k}$	$k \in \llbracket 1, g-1 \rrbracket$	$\alpha_{k+1} \mapsto \Theta_k^{-1} \alpha_{k+1}$ $\alpha_k \mapsto \alpha_k \Theta_k$ ; where $\Theta_k := \alpha_{k+1} \beta_{k+1}^{-1} \alpha_{k+1}^{-1} \beta_k$ $\beta_k \mapsto \Theta_k^{-1} \beta_k \Theta_k$
$\tau_{3g-1+k}$	$k \in \llbracket 1, n-1 \rrbracket$	$\alpha_g \mapsto \alpha_g \Xi_k$ $\beta_g \mapsto \Xi_k^{-1} \beta_g \Xi_k$ ; where $\Xi_k := (\gamma_1 \dots \gamma_k)^{-1} \beta_g$ $i \in \llbracket 1, k \rrbracket$ $\gamma_i \mapsto \Xi_k^{-1} \gamma_i \Xi_k$
$\sigma_k$	$k \in \llbracket 1, n-1 \rrbracket$	$\gamma_k \mapsto \gamma_k \gamma_{k+1} \gamma_k^{-1}$ $\gamma_{k+1} \mapsto \gamma_k$

Table 1: Action of  $\hat{\Gamma}_{g,n}^\circ$  on  $\Lambda_{g,n}$ .

### 2.3.1 Presentation of the fundamental group

To give an effective description of  $\Lambda_{g,n}$  and how  $\hat{\Gamma}_{g,n}$  acts, we will assume that  $\Sigma_g$  is the subsurface of genus  $g$  of  $\mathbb{R}^3$  depicted in Figure 1. On this surface we also depicted, in gray, an embedded closed

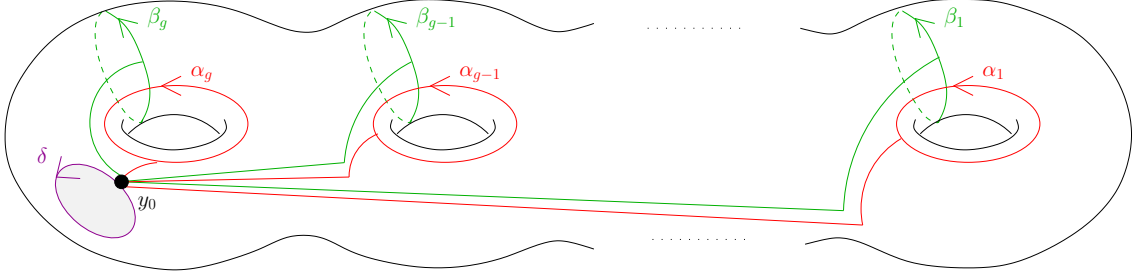


Figure 1: Preferred elements of the fundamental group, I

disk  $\bar{\Delta} \subset \Sigma_g$ , we will denote  $\Delta$  its interior. We fix  $n$  and we consider a subset  $Y^n = \{y_1, \dots, y_n\} \subset \Delta$  of cardinality  $n$ , as well as a point  $y_0 \in \bar{\Delta} \setminus \Delta$ . We have

$$\pi_1(\Sigma_g \setminus \Delta, y_0) = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \delta \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = \delta^{-1} \rangle,$$

where the mentioned generators correspond to the loops in Figure 1.

The loops in Figure 2 correspond to the following presentation.

$$\pi_1(\bar{\Delta} \setminus Y^n, y_0) = \langle \gamma_1, \dots, \gamma_n, \delta \mid \gamma_1 \cdots \gamma_n = \delta \rangle.$$

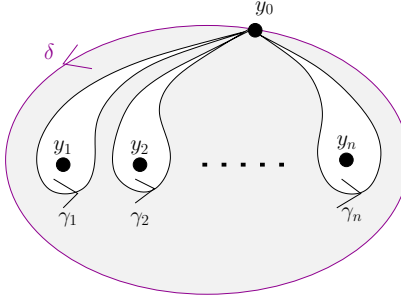


Figure 2: Preferred elements of the fundamental group, II

By the Van Kampen theorem, we have

$$\begin{aligned} \Lambda_{g,n} &= \pi_1(\Sigma \setminus \Delta, y_0) *_{\delta} \pi_1(\bar{\Delta} \setminus Y^n, y_0) \\ &= \left\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_n \mid \gamma_1 \cdots \gamma_n = ([\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g])^{-1} \right\rangle. \end{aligned}$$

**Definition 2.3.1.** *In the sequel, we will refer to the above*

$$(\alpha_i)_{i \in [1,g]}, (\beta_i)_{i \in [1,g]}, (\gamma_j)_{j \in [1,n]}$$

as the generators of  $\Lambda_{g,n}$ .

### 2.3.2 Mapping class group generators

We define  $\Gamma_g^1$  to be the mapping class group of orientation preserving homeomorphisms of  $\Sigma_g \setminus \Delta$  that restrict to the identity on the boundary  $\partial\Delta$ . Continuing such homeomorphisms by the identity on

$\Delta$ , we get a morphism

$$\varphi_g : \Gamma_g^1 \rightarrow \hat{\Gamma}_{g,n}^\bullet.$$

After Lickorish [Lic64] (see also [FM12, Th. 4.13]), the group  $\Gamma_g^1$  is generated by the (right) *Dehn-twists* along the loops  $\tau_1, \dots, \tau_{3g-1}$  represented in Figure 3.

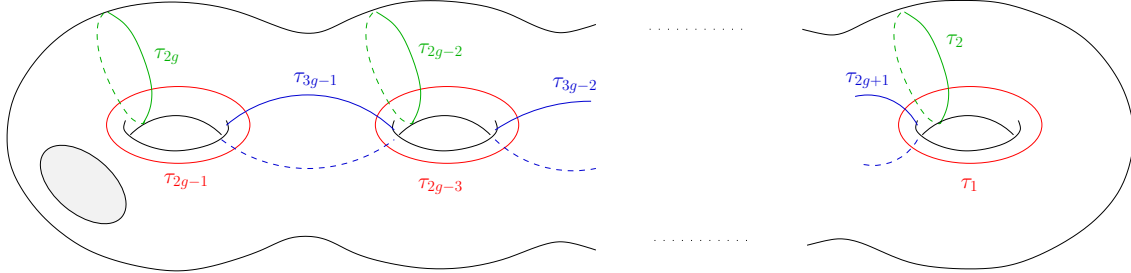


Figure 3: Dehn-twists

A *right Dehn twist* acts on paths which cross the corresponding Dehn curve as depicted in Figure 4. This action can be paraphrased as *a path crossing the Dehn curve has to turn right*. A *left Dehn twist* is the inverse of a right Dehn twist.

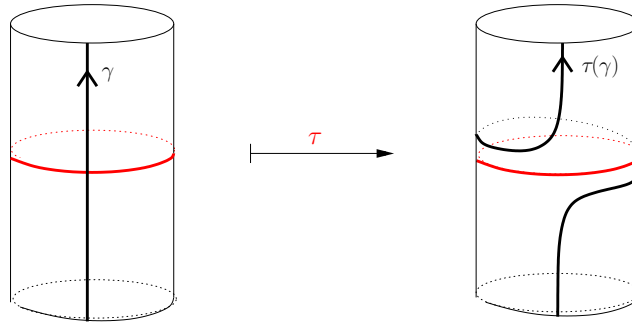


Figure 4: A Dehn-twist

One can easily check the following.

**Lemma 2.3.2** (Dehn-twists). *The action of the Dehn twists above on the fundamental group  $\pi_1(\Sigma_g \setminus \Delta, y_0)$  is given in Table 2, where we only indicate the non-trivial actions on the generators. Here for  $\tau_{2k-1}$  we give the formula for the left Dehn twist. The other generators all correspond to right Dehn twists. For  $k \in \llbracket 1, g-1 \rrbracket$ , the element  $\Theta_k$  described in Table 2 is fixed by  $\tau_{2g+k}$ .*

$\tau_{2k}$	$k \in \llbracket 1, g \rrbracket$	$\alpha_k \mapsto \alpha_k \beta_k$
$\tau_{2k-1}$	$k \in \llbracket 1, g \rrbracket$	$\beta_k \mapsto \beta_k \alpha_k$
$\tau_{2g+k}$	$k \in \llbracket 1, g-1 \rrbracket$	$\alpha_{k+1} \mapsto \Theta_k^{-1} \alpha_{k+1}$ $\alpha_k \mapsto \alpha_k \Theta_k$ ; where $\Theta_k := \alpha_{k+1} \beta_{k+1}^{-1} \alpha_{k+1}^{-1} \beta_k$ $\beta_k \mapsto \Theta_k^{-1} \beta_k \Theta_k$

Table 2: Dehn twist action



On the other hand, one can define the mapping class group of orientation preserving homeomorphisms of  $\bar{\Delta}$  that preserve the set  $Y^n$  and restrict to the identity on the boundary  $\partial\Delta$ . It is classically called the *braid group on  $n$  strands* and denoted  $B_n$ . Continuing such homeomorphisms by the identity on the complement of  $\Delta$  in  $\Sigma_g$ , we get a morphism

$$\varphi_0 : B_n \rightarrow \hat{\Gamma}_{g,n}^\bullet.$$

After Artin [Art25], the group  $B_n$  is generated by half-twists  $\sigma_1, \dots, \sigma_{n-1}$ , whose action is depicted in Figure 5.

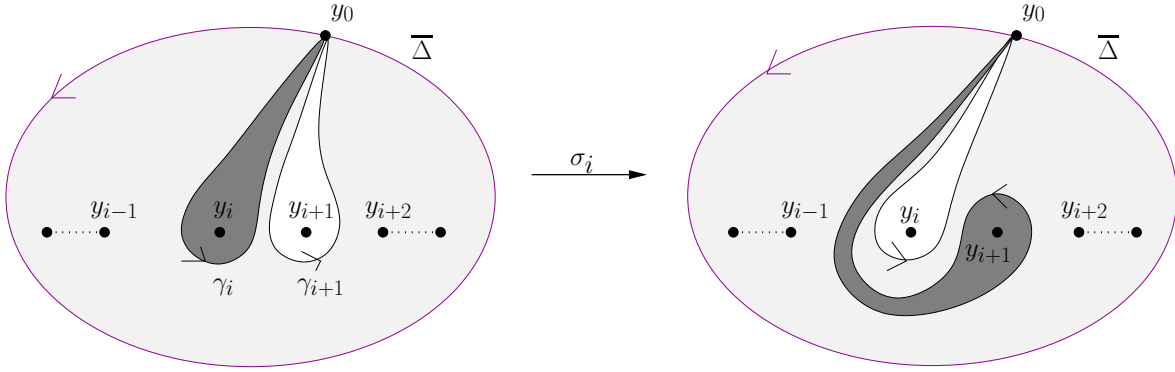


Figure 5: Half-twists

**Lemma 2.3.3** (Half-twists). *The action of the braid group  $B_n = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$  on the fundamental group  $\pi_1(\bar{\Delta} \setminus Y^n, y_0)$  is described in Table 3, where we only indicate the non-trivial actions on the generators. Moreover, Table 3 indicates the action of  $\sigma_{\text{cycl}} := \sigma_{n-1} \circ \dots \circ \sigma_1 \in B_n$  and some of its powers.*

$\sigma_k$	$k \in \llbracket 1, n-1 \rrbracket$	$\gamma_k$	$\mapsto \gamma_k \gamma_{k+1} \gamma_k^{-1}$
		$\gamma_{k+1}$	$\mapsto \gamma_k$
$\sigma_{\text{cycl}}$		$\gamma_1$	$\mapsto \delta \gamma_n \delta^{-1}$
	$i \in \llbracket 2, n \rrbracket$	$\gamma_i$	$\mapsto \gamma_{i-1}$
$\sigma_{\text{cycl}}^k$	$k \in \llbracket 1, n \rrbracket$		
	$i \in \llbracket 1, k \rrbracket$	$\gamma_i$	$\mapsto \delta \gamma_{n+i-k} \delta^{-1}$
	$j \in \llbracket k+1, n \rrbracket$	$\gamma_j$	$\mapsto \gamma_{j-k}$

Table 3: Half twist action

**Remark 2.3.4.** *Note that  $\sigma_{\text{cycl}}$  is almost a cyclic permutation of the generators of  $\pi_1(\bar{\Delta} \setminus Y^n, y_0)$ . More precisely, it acts as such on the representations  $\rho$  that satisfy  $\rho(\delta) = \text{id}$ , e.g. representations with abelian image.*

By construction, the subgroups  $\varphi_0(B_n)$  and  $\varphi_g(\Gamma_g^1)$  of  $\hat{\Gamma}_{g,n}^\bullet$  commute, and we have a morphism

$$B_n \times \Gamma_g^1 \xrightarrow{\varphi_0 \times \varphi_g} \hat{\Gamma}_{g,n}^\bullet.$$

Composing with the canonical map  $\pi : \hat{\Gamma}_{g,n}^\bullet \rightarrow \hat{\Gamma}_{g,n}$  (forgetting that  $y_0$  is fixed) yields a morphism  $B_n \times \Gamma_g^1 \rightarrow \hat{\Gamma}_{g,n}$ , which is not surjective unless  $g = 0$  or  $n \leq 1$ . In order to generate the whole

mapping class group  $\hat{\Gamma}_{g,n}$  for  $g > 0$ , it suffices to add  $\min(0, n - 1)$  Dehn twists, namely the ones corresponding to the loops  $\tau_{3g}, \dots, \tau_{3g+n-2}$  of Figure 6 (see [EM12, Sec. 4.4.4]). We call them *mixing twists*.

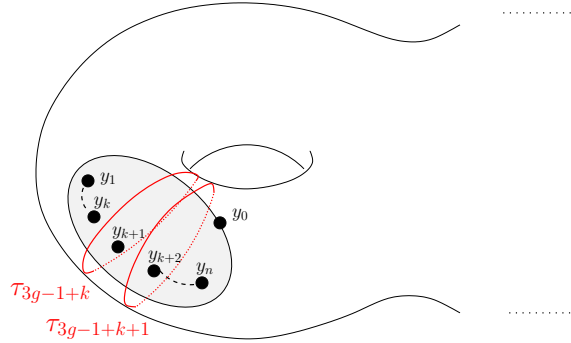


Figure 6: Mixing twists

**Lemma 2.3.5** (Mixing twists). *The action of the (right) mixing twists  $\tau_{3g}, \dots, \tau_{3g+n-2}$  on the fundamental group  $\Lambda_{g,n}$  is described in Table 4, where we only indicate the non-trivial actions on the generators. Moreover, for  $k \in \llbracket 1, n - 1 \rrbracket$ , the element  $\Xi_k$  described there is fixed by  $\tau_{3g-1+k}$ .*

$\tau_{3g-1+k}$	$k \in \llbracket 1, n - 1 \rrbracket$	$\alpha_g \mapsto \alpha_g \Xi_k$
		$\beta_g \mapsto \Xi_k^{-1} \beta_g \Xi_k$ ; where $\Xi_k := (\gamma_1 \dots \gamma_k)^{-1} \beta_g$
	$i \in \llbracket 1, k \rrbracket$	$\gamma_i \mapsto \Xi_k^{-1} \gamma_i \Xi_k$

Table 4: Mixing twist action

The twists, mixing twists and braids we introduced all fix  $y_0$ . We denote by  $\hat{\Gamma}_{g,n}^\circ$  the subgroup of  $\hat{\Gamma}_{g,n}^\bullet$  they generate. If  $g = 0$ , then we have  $\hat{\Gamma}_{g,n}^\circ = B_n$ . We are interested in the case  $g > 0$ , where we have

$$\hat{\Gamma}_{g,n}^\circ := \langle \tau_i, \sigma_j \mid i \in \llbracket 1, 3g - 1 + \min(0, n - 1) \rrbracket, j \in \llbracket 1, n - 1 \rrbracket \rangle .$$

As mentioned, the image of  $\hat{\Gamma}_{g,n}^\circ$  under  $\pi : \hat{\Gamma}_{g,n}^\bullet \rightarrow \hat{\Gamma}_{g,n}$  is  $\hat{\Gamma}_{g,n}$ .

**Remark 2.3.6.** *We did not call  $\delta = \gamma_1 \dots \gamma_n = ([\alpha_1, \beta_1] \dots [\alpha_g, \beta_g])^{-1}$  a generator of the fundamental group. It will nevertheless be useful to notice that among our preferred generators of  $\hat{\Gamma}_{g,n}^\circ$ , only the mixing twists act non trivially on  $\delta$ . More precisely, for  $k \in \llbracket 1, n - 1 \rrbracket$  we have*

$$\tau_{3g-1+k}(\delta) = [\Xi_k^{-1}, \beta_g] \delta .$$

## 2.4 Finite orbits of the action on $\chi_{g,n}(\text{Aff}(\mathbb{C}))$

In the previous section, we have established an explicit description of the full mapping class group action on  $\Lambda_{g,n}$ . This description at hand, we will now classify affine representations  $\rho \in \text{Hom}(\Lambda_{g,n}, \text{Aff}(\mathbb{C}))$  with finite orbit  $\hat{\Gamma}_{g,n} \cdot [\rho]$  in  $\chi_{g,n}(\text{Aff}(\mathbb{C}))$  for  $g > 0$ :

- We establish that for those representations  $\rho \in \text{Hom}(\Lambda_{g,n}, \text{Aff}(\mathbb{C}))$  such that the group  $\text{Im}(\rho)$  is abelian, the orbit  $\hat{\Gamma}_{g,n} \cdot [\rho]$  is finite if and only if  $\text{Im}(\rho)$  is finite (see Proposition 2.4.2).

- We then consider representations  $\rho \in \text{Hom}(\Lambda_{g,n}, \text{Aff}(\mathbb{C}))$  such that the group  $\text{Im}(\rho)$  is not abelian. We classify all finite orbits in this case in three steps.
  - We give a necessary condition for the finiteness of  $\hat{\Gamma}_{g,n} \cdot [\rho]$  in Lemma [2.4.4](#)
  - We prove that in the genus one case, this necessary condition is also sufficient (see Proposition [2.4.5](#)).
  - We prove that in the higher genus case, this necessary condition can be tightened (see Lemma [2.4.6](#)), and this tightened necessary condition cannot hold for every conjugacy class  $[\rho'] \in \hat{\Gamma}_{g,n} \cdot [\rho]$ . We conclude that in the higher genus case, there are no conjugacy classes of non abelian  $\text{Aff}(\mathbb{C})$ -representations with finite orbit under  $\hat{\Gamma}_{g,n}$  (see Proposition [2.4.7](#)).

The group  $\text{Aff}(\mathbb{C}) = \{(a_{ij}) \in \text{GL}_2(\mathbb{C}) \mid a_{21} = 0, a_{22} = 1\}$  identifies with the group  $\{z \mapsto az + b \mid a \in \mathbb{C}^*, b \in \mathbb{C}\}$  of affine transformations of the complex line  $\mathbb{C}$ . For shortness, its elements will be denoted as polynomials  $az + b$ . Our explicit calculations are easier to check with the following formulas in mind.

$$\begin{aligned} (\lambda z) \circ (az + b) \circ (\lambda z)^{-1} &= az + \lambda b \\ (z + c) \circ (az + b) \circ (z + c)^{-1} &= az + b - c(a - 1) \\ [\lambda z + c, az + b] &= z - c(a - 1) + (\lambda - 1)b \end{aligned}$$

Also, recall that by definition, for all  $\tau \in \hat{\Gamma}_{g,n}^\circ$ ,  $\rho \in \text{Hom}(\Lambda_{g,n}, \text{Aff}(\mathbb{C}))$  and  $\alpha \in \Lambda_{g,n}$ , we have

$$(\tau \cdot \rho)(\alpha) = \rho(\tau_*^{-1}\alpha).$$

### 2.4.1 Abelian case

**Lemma 2.4.1** (Finding a non trivial subgroup). *Let  $g > 0$ ,  $n \in \mathbb{N}$ . Let  $G$  be a group with identity element  $\text{id}$  and let  $\rho : \Lambda_{g,n} \rightarrow G$  be a representation. Assume that for any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ , we have*

$$\rho'(\alpha_g) = \text{id}.$$

*Then  $\rho$  is the trivial representation, i.e.,  $\text{Im}(\rho) = \{\text{id}\}$ .*

*Proof.* Note that our assumption on  $\rho$  is  $\hat{\Gamma}_{g,n}^\circ$ -invariant, so that what we prove for  $\rho$  under that assumption also holds for any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ . We denote

$$R' := \langle \rho'(\alpha_g), \rho'(\beta_g), \dots, \rho'(\alpha_1), \rho'(\beta_1) \rangle, \quad S' := \langle \rho'(\gamma_1), \dots, \rho'(\gamma_n) \rangle.$$

**First step:** *for any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ , the group  $R'$  is trivial.*

For  $k \in \llbracket 1, g \rrbracket$ , define the following property, which we shall denote  $H(k)$ :

$$\text{For any } \rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho, \text{ the group } R'_k := \langle \rho'(\alpha_g), \rho'(\beta_g), \dots, \rho'(\alpha_k), \rho'(\beta_k) \rangle \text{ is trivial.}$$

Let us first prove that our assumption implies  $H(g)$ . Consider  $\tau := \tau_{2g}^{-1}$  and  $\rho' = \tau \cdot \rho$ . Then  $\rho'(\alpha_g) = \rho(\alpha_g \beta_g) = \rho(\beta_g)$ . We have  $\rho'(\alpha_g) = \rho(\alpha_g) = \text{id}$ , hence  $\rho(\beta_g) = \rho(\alpha_g) = \text{id}$ . By  $\hat{\Gamma}_{g,n}^\circ$ -invariance, we have  $H(g)$ .

Let now  $\rho$  be a representation satisfying  $H(k)$ . In particular, we have

$$\rho(\alpha_i) = \rho(\beta_i) = \text{id} \quad \forall i \in \llbracket k, g \rrbracket.$$

For  $\rho' = \tau \cdot \rho$ , with  $\tau = \tau_{2g+k-1}^{-1}$  we have  $\rho'(\alpha_k) = \rho(\beta_{k-1}^{-1} \alpha_k \beta_k) = \rho(\beta_{k-1})^{-1}$ .

For  $\rho' = \tau \cdot \rho$ , with  $\tau = (\tau_{2k-3} \circ \tau_{2g+k-1})^{-1}$ , we have  $\rho'(\alpha_k) = \rho(\beta_{k-1} \alpha_{k-1})^{-1}$ .

Hence  $\rho$  satisfying  $H(k)$  implies  $\rho(\alpha_i) = \rho(\beta_i) = \text{id}$  for all  $i \in \llbracket k-1, g \rrbracket$ . As  $H(k)$  is  $\hat{\Gamma}_{g,n}^\circ$ -invariant, this proves  $H(k-1)$ . We conclude by noticing  $R' = R'_1$ .

**Second step:** for any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ , the group  $S'$  is trivial.

If  $n = 0$  or  $n = 1$ , there is nothing to prove. Assume  $n > 1$ . We have already proven that  $R'$  is trivial for any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ . In particular  $\rho'(\delta) = \text{id}$ . Considering, for  $i \in \llbracket 1, n \rrbracket$ , the action of  $\tau = (\sigma_{\text{cycl}}^{n-i} \circ \tau_{3g+n-2})^{-1}$  on  $\alpha_g$  then shows that for  $\rho' = \tau \cdot \rho$  we have  $\text{id} = \rho'(\alpha_g) = \rho(\gamma_i)$  (see Table [3](#) page [62](#)). Hence  $\langle \rho(\gamma_1), \dots, \rho(\gamma_n) \rangle = \{\text{id}\}$ . Since the assertion is  $\hat{\Gamma}_{g,n}^\circ$ -invariant, we have proven that  $S'$  is trivial for any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ .

We conclude that  $\text{Im}(\rho) = \text{Im}(\rho') = \langle S', R' \rangle = \{\text{id}\}$ .  $\square$

**Proposition 2.4.2** (Abelian case). *Let  $g > 0$ . Let  $\rho : \Lambda_{g,n} \rightarrow \text{Aff}(\mathbb{C})$  be a representation such that the group  $\text{Im}(\rho)$  is abelian. Then the orbit of the conjugacy class  $[\rho]$  under the action of  $\hat{\Gamma}_{g,n}$  is finite if and only if  $\text{Im}(\rho)$  is finite.*

*Proof.* If  $\text{Im}(\rho)$  is finite, then the orbit  $\hat{\Gamma}_{g,n}^\circ \cdot \rho$  is finite. *A fortiori*, the orbit  $\hat{\Gamma}_{g,n}^\circ \cdot [\rho]$  is finite.

Assume now that  $\rho$  is abelian and the orbit of  $[\rho]$  is finite. Since  $\text{Im}(\rho)$  is an abelian subgroup of  $\text{Aff}(\mathbb{C})$  it is, up to conjugation, either a non trivial subgroup of the translation group

$$\{z \mapsto z + c \mid c \in \mathbb{C}\} \subset \text{Aff}(\mathbb{C}),$$

or it is a subgroup of the linear group

$$\{z \mapsto \lambda z \mid \lambda \in \mathbb{C}^*\} \subset \text{Aff}(\mathbb{C}).$$

**Elimination of the first case:**  $\text{Im}(\rho)$  cannot be a non trivial translation group.

Indeed, if it would be the case, by Lemma [2.4.1](#), we might assume  $\rho(\alpha_g) \neq \text{id}$ . Up to conjugation, we would then have

$$\rho \begin{pmatrix} \alpha_g \\ \beta_g \end{pmatrix} = \begin{pmatrix} z + 1 \\ z + c \end{pmatrix}$$

for a certain  $c \in \mathbb{C}$ . Considering the action of  $\tau^{-m}$  with  $\tau := \tau_{2g-1}$  :

$$\tau^{-m} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \end{pmatrix} = \rho \begin{pmatrix} \alpha_g \\ \beta_g \alpha_g^m \end{pmatrix} = \begin{pmatrix} z + 1 \\ z + c + m \end{pmatrix},$$

we would deduce that, for  $m \neq m'$ , the conjugacy classes of  $\tau^m \cdot \rho$  and  $\tau^{m'} \cdot \rho$  are distinct. Hence  $\hat{\Gamma}_{g,n}^\circ \cdot [\rho]$  would be infinite, yielding a contradiction.

**Second case:** If  $\text{Im}(\rho)$  is a subgroup of the linear group, then it is finite.

Note that two distinct linear representations are not conjugated. For any  $i \in \llbracket 1, g \rrbracket$ , finiteness of the orbit under  $\langle \tau_{2i} \rangle$  yields that  $\rho(\beta_i)$  is torsion. Similarly, considering  $\langle \tau_{2i-1} \rangle$  yields that  $\rho(\alpha_i)$  is torsion for all  $i \in \llbracket 1, g \rrbracket$ . For  $j \in \llbracket 1, n-1 \rrbracket$ , finiteness of the orbit under  $\langle \tau_{3g-1+j} \rangle$  implies that  $\rho(\gamma_1 \dots \gamma_j)$  is torsion. Consequently,  $\gamma_j$  is torsion for all  $j \in \llbracket 1, n-1 \rrbracket$ . Hence

$$\text{Im}(\rho) = \langle \rho(\alpha_i), \rho(\beta_i), \rho(\gamma_j) \mid i \in \llbracket 1, g \rrbracket, j \in \llbracket 1, n-1 \rrbracket \rangle$$

is an abelian group generated by finitely many torsion elements, whence the conclusion.  $\square$

## 2.4.2 Preparation lemmata

**Lemma 2.4.3** (Finding a non abelian subgroup). *Let  $g > 0$ . Let  $\rho : \Lambda_{g,n} \rightarrow \text{Aff}(\mathbb{C})$  be a representation. Assume that for any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ , the subgroup*

$$\langle \rho'(\alpha_g), \rho'(\beta_g) \rangle$$

*of  $\text{Im}(\rho)$  is abelian. Then  $\rho$  is an abelian representation, i.e.,  $\text{Im}(\rho)$  is abelian.*

*Proof.* Denote  $R'_k := \langle \rho'(\alpha_g), \rho'(\beta_g), \dots, \rho'(\alpha_k), \rho'(\beta_k) \rangle$  and  $S' := \langle \rho'(\gamma_1), \dots, \rho'(\gamma_n) \rangle$ .

**First step:** For any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ , the group  $R'_g$  is contained in the center of  $R'_1$ . For  $k \in \llbracket 1, g \rrbracket$ , define the following property.

$$H(k) : \boxed{\text{For any } \rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho, \text{ the group } R'_g \text{ is a subgroup of the center of } R'_k.}$$

By assumption, we have  $H(g)$ . Assume now  $H(k)$  is proven. In particular,  $R_g := \langle \rho(\alpha_g), \rho(\beta_g) \rangle$  is a subgroup of the center of  $R_k := \langle \rho(\alpha_g), \rho(\beta_g), \dots, \rho(\alpha_k), \rho(\beta_k) \rangle$ . Note that  $H(k)$  is  $\hat{\Gamma}_{g,n}^\circ$ -invariant. Hence in order to prove  $H(k-1)$ , it suffices to prove that  $R_g$  is also a subgroup of the center of  $R_{k-1}$ . For  $\rho' = \tau \cdot \rho$ , with  $\tau = \tau_{2g+k-1}^{-1}$ , only one of the generators of  $R_k$  is modified, namely

$$\rho'(\alpha_k) = \rho(\beta_{k-1}^{-1} \alpha_k \beta_k) = \rho(\beta_{k-1})^{-1} \rho(\alpha_k \beta_k).$$

In particular, we have  $\rho'(\beta_g) = \rho(\beta_g)$ . Then  $H(k)$  implies that  $\rho(\beta_g)$  belongs to the center of  $\langle R_k, R'_k \rangle = \langle R_k, \rho(\beta_{k-1}) \rangle$ . For  $\rho'' = \tau' \cdot \rho$ , with  $\tau' = \tau \circ \tau_{2k-3}^{-1}$ , we have

$$\rho''(\alpha_k) = \rho(\beta_{k-1} \alpha_{k-1})^{-1} \rho(\alpha_k \beta_k).$$

Then  $H(k)$  implies that  $\rho''(\beta_g) = \rho(\beta_g)$  belongs to the center of  $\langle R_k, R''_k \rangle = \langle R_k, \rho(\beta_{k-1} \alpha_{k-1}) \rangle$ . We have now proven that for any representation  $\rho$  such that  $H(k)$  holds,  $\rho(\beta_g)$  is an element of the center of  $R_{k-1} = \langle R_k, R'_k, R''_k \rangle$ . This assertion applied to  $\tau_{2g-1}^{-1} \cdot \rho$  shows that  $\rho(\beta_g \alpha_g)$  is an element of the center of  $R_{k-1}$ . Hence  $R_g = \langle \rho(\beta_g), \rho(\beta_g \alpha_g) \rangle$  is a subgroup of the center of  $R_{k-1}$ .

**Second step:** For any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ , the group  $R' := R'_1$  is abelian.

If  $R'$  is trivial, then in particular it is abelian. If  $R'$  is non trivial, then by the first step in the proof of Lemma 2.4.1, we can find  $\tau' \in \hat{\Gamma}_{g,n}^\circ$  such that for the induced representation  $\rho'' = \tau' \cdot \rho$  we have  $R'' = R'$  and  $R''_g$  is non trivial. Hence by the first step of the current lemma,  $R'$  has a non trivial center. Yet any subgroup of  $\text{Aff}(\mathbb{C})$  with non trivial center is abelian.

**Third step:** For any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ , the group  $R'_g$  is a subgroup of the center of  $\text{Im}(\rho')$ .

We have now proven that under our assumption,  $R'$  is abelian for any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ . In particular,  $\rho(\delta) = \text{id}$ . Recall, from Remark 2.3.6, the action of the mixing twist  $\tau_{3g+n-2}$  on  $\delta$ . It is given by  $\delta \mapsto [\beta_g^{-1} \delta \gamma_n^{-1}, \beta_g] \delta$ . Hence, for  $\rho' = \tau \cdot \rho$  with  $\tau = (\sigma_{\text{cycl}}^{n-i} \circ \tau_{3g+n-2})^{-1}$ , we have

$$\rho'(\delta) = [\rho(\beta_g^{-1} \gamma_i^{-1}), \rho(\beta_g)] = [\rho(\beta_g)^{-1}, \rho(\gamma_i)^{-1}]$$

(see Table 3 page 62). Consequently,  $\rho(\beta_g)$  centralizes  $S := \langle \rho(\gamma_i) \mid i \in \llbracket 1, n \rrbracket \rangle$ . Yet we could have applied the same argument to  $\rho'' = \tau' \cdot \rho$ , where  $\tau' = \tau_{2g-1}^{-1}$  is the inverse of the Dehn-twist  $\beta_g \mapsto \beta_g \alpha_g$ , and we would have obtained that  $\rho(\beta_g \alpha_g)$  centralizes  $S$ . It follows that  $R_g$  centralizes  $S$ . By  $\hat{\Gamma}_{g,n}^\circ$ -invariance of the statement, we deduce that for any  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$ , the group  $R'_g$  centralizes  $\text{Im}(\rho') = \langle R', S' \rangle$ .

**Fourth step:**  $\text{Im}(\rho)$  is abelian.

If  $\rho$  is the trivial representation, there is nothing to prove. Otherwise, by Lemma 2.4.1, there is a representation  $\rho' \in \hat{\Gamma}_{g,n}^\circ \cdot \rho$  in the orbit of  $\rho$  such that  $R'_g = \langle \rho'(\alpha_g), \rho'(\beta_g) \rangle$  is not the trivial group. On the other hand, we have proven that  $R'_g$  is a subgroup of the center of  $\text{Im}(\rho')$ . Hence  $\text{Im}(\rho) = \text{Im}(\rho')$  is abelian.  $\square$

**Lemma 2.4.4** (Prepared form). *Let  $g > 0$ . Let  $\rho : \Lambda_{g,n} \rightarrow \text{Aff}(\mathbb{C})$  be a representation. Assume that  $\text{Im}(\rho)$  is non abelian and  $\hat{\Gamma}_{g,n} \cdot [\rho]$  is finite. Then up to the action of a certain element of the mapping class group and up to conjugation,  $\rho$  is of the following ‘‘prepared form’’*

$$\rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_i \\ \beta_i \\ \gamma_j \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ \mu^{m_i} z + a_i \\ z + b_i \\ z + c_j \end{pmatrix}, \quad (2.7)$$

for  $i \in \llbracket 1, g-1 \rrbracket$  and  $j \in \llbracket 1, n \rrbracket$ , where  $\mu \in \mathbb{C}^* \setminus \{1\}$  is a root of unity,  $m_g, m_i \in \mathbb{Z}$ ,  $a_i, b_i, c_j \in \mathbb{C}$  and  $\mu^{m_g} \neq 1$ .

*Proof.* According to Lemma 2.4.3, up to the action of an element of the mapping class group, we may assume  $\rho([\alpha_g, \beta_g]) \neq \text{id}$ . Since  $\hat{\Gamma}_{g,n} \cdot [\rho]$  is finite, the linear part  $\rho_{\text{lin}}$  of  $\rho$  also has finite orbit. After Proposition 2.4.2,  $\rho_{\text{lin}}$  takes values in a finite cyclic group  $\langle \mu \rangle \subset \mathbb{C}^*$ . Hence for each  $i \in \llbracket 1, g \rrbracket$ , we have

$$\rho(\alpha_i) = \mu^{m_i} z + a_i, \quad \rho(\beta_i) = \mu^{n_i} z + b_i$$

for integers  $m_i, n_i \in \mathbb{Z}$  and complex numbers  $a_i, b_i \in \mathbb{C}$ . Consider the actions of  $\tau_{2i}^{-1}$  and  $\tau_{2i-1}^{-1}$  on  $(m_i, n_i)$  (the other exponents are not altered) :

$\tau_{2i-1}^{-1}$	$\begin{pmatrix} m_i \\ n_i \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m_i \\ n_i \end{pmatrix}$
$\tau_{2i}^{-1}$	$\begin{pmatrix} m_i \\ n_i \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_i \\ n_i \end{pmatrix}$

These actions generate the action of  $\text{SL}_2\mathbb{Z}$  on  $(m_i, n_i) \in \mathbb{Z}^2$ . If  $(m_i, n_i) \neq (0, 0)$ , then  $\tilde{m}_i := \text{gcd}(m_i, n_i)$  is a well defined positive integer. Let  $p_i$  and  $q_i$  be integers such that  $p_i m_i + q_i n_i = \tilde{m}_i$ . The matrix

$$\begin{pmatrix} p_i & q_i \\ -\frac{n_i}{\tilde{m}_i} & \frac{m_i}{\tilde{m}_i} \end{pmatrix} \in \text{SL}_2\mathbb{Z}$$

then sends  $(m_i, n_i)$  to  $(\tilde{m}_i, 0)$ . Hence, up to the action of a word in the twists  $(\tau_{2i})_{i \in \llbracket 1, g \rrbracket}, (\tau_{2i-1})_{i \in \llbracket 1, g \rrbracket}$ , we may assume  $n_i = 0$  for each  $i \in \llbracket 1, g \rrbracket$ . The property  $\rho([\alpha_g, \beta_g]) \neq \text{id}$  is not altered by such a word, hence  $\mu^{m_g} \neq 1$ . Up to conjugation by an element of  $\text{Aff}(\mathbb{C})$ , we may moreover assume

$$\rho \begin{pmatrix} \alpha_g \\ \beta_g \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z \\ z + 1 \end{pmatrix}. \quad (2.8)$$

For  $j \in \llbracket 1, n \rrbracket$ , let  $c_j, d_j \in \mathbb{C}$  be defined by  $\rho(\gamma_j) = d_j z + c_j$ . For  $k \in \mathbb{Z}$ , consider the action of  $\tau_2^{-k}$  :

$$\tau_2^{-k} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \gamma_j \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z + k\mu^{m_g} \\ z + 1 \\ d_j z + c_j \end{pmatrix} \approx \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ d_j z + c_j - k \frac{(d_j - 1)\mu^{m_g}}{\mu^{m_g} - 1} \end{pmatrix}.$$

For these sequences of normalized triples to be finite, we must have  $d_j = 1$  for each  $j$ .  $\square$

### 2.4.3 Non abelian case in genus one

**Proposition 2.4.5** (Non abelian representations for  $g = 1$ ). *Assume  $g = 1$ . Let  $\rho : \Lambda_{g,n} \rightarrow \text{Aff}(\mathbb{C})$  be a representation with non abelian image (in particular,  $n \geq 1$ ). Then the orbit  $\hat{\Gamma}_{g,n} \cdot [\rho]$  is finite if and only if there is a root of unity  $\mu \neq 1$  and  $\mathbf{c} := (c_1, \dots, c_n) \in \mathbb{C}^n$  with  $\sum_{i=1}^n c_i = 1$  such that  $[\rho] \in \hat{\Gamma}_{g,n} \cdot [\rho_{\mu,\mathbf{c}}]$ , where  $\rho_{\mu,\mathbf{c}}$  is the representation given by*

$$\rho_{\mu,\mathbf{c}}(\alpha_1) = \mu z; \quad \rho_{\mu,\mathbf{c}}(\beta_1) = z - \frac{1}{\mu - 1}; \quad \rho_{\mu,\mathbf{c}}(\gamma_i) = z + c_i \quad \forall i \in \llbracket 1, n \rrbracket.$$

*Proof.* Recall that for  $g = 1$ , the fundamental group  $\Lambda_{g,n}$  has the following presentation

$$\Lambda_{g,n} = \langle \alpha_1, \beta_1, \gamma_1, \dots, \gamma_n \mid \gamma_1 \cdots \gamma_n = [\alpha_1, \beta_1]^{-1} \rangle,$$

and the mapping class group  $\hat{\Gamma}_{g,n}$  is generated by the elements of Table 5

$\tau_1$		$\beta_1 \mapsto \beta_1 \alpha_1$
$\tau_2$		$\alpha_1 \mapsto \alpha_1 \beta_1$
$\tilde{\tau}_{2+k} := \tau_2^{-1} \circ \tau_{2+k}$	$k \in \llbracket 1, n-1 \rrbracket$	$\alpha_1 \mapsto \alpha_1 \beta_1^{-1} \Xi_k$
	$i \in \llbracket 1, k \rrbracket$	$\beta_1 \mapsto \Xi_k^{-1} \beta_1 \Xi_k$
		$\gamma_i \mapsto \Xi_k^{-1} \gamma_i \Xi_k$
$\sigma_i$	$i \in \llbracket 1, n-1 \rrbracket$	$\gamma_i \mapsto \gamma_i \gamma_{i+1} \gamma_i^{-1}$
		$\gamma_{i+1} \mapsto \gamma_i$

Table 5: Action of the generators in genus 1

Assume  $[\rho]$  has finite orbit, then by Lemma 2.4.4, we have  $\hat{\Gamma}_{g,n} \cdot [\rho] = \hat{\Gamma}_{g,n} \cdot [\rho_{\mu,\mathbf{c}}]$  for a convenient choice of  $\mathbf{c} \in \mathbb{C}^n$  and a root of unity  $\mu \neq 1$ . Let us now prove that  $[\rho_{\mu,\mathbf{c}}]$  has finite orbit. Denote

$$N := \text{order}(\mu); \quad D_{\mathbf{c}} := \mu^{\mathbb{Z}} c_1 + \dots + \mu^{\mathbb{Z}} c_n.$$

Denote the following sets of tuples of affine transformations

$$\begin{aligned} S_{\mu,d}^1 &:= \left\{ \left( \begin{array}{c} \mu^{k_1} z \\ \mu^{k_2} z - \frac{d}{\mu^{k_1} - 1} \end{array} \right) \mid \begin{array}{l} k_1, k_2 \in \mathbb{Z}, \quad k_1 \notin N\mathbb{Z}, \\ \gcd(k_1, k_2, N) = 1 \end{array} \right\} \\ S_{\mu,d}^2 &:= \left\{ \left( \begin{array}{c} \mu^{k_1} z + \frac{d}{\mu^{k_2} - 1} \\ \mu^{k_2} z \end{array} \right) \mid \begin{array}{l} k_1, k_2 \in \mathbb{Z}, \quad k_2 \notin N\mathbb{Z}, \\ \gcd(k_1, k_2, N) = 1 \end{array} \right\} \\ R_{\mu,\mathbf{c},d} &:= \left\{ \left( \begin{array}{c} z + \tilde{c}_1 \\ \vdots \\ z + \tilde{c}_n \end{array} \right) \mid \begin{array}{l} (\hat{c}_1, \dots, \hat{c}_n) \in \mathfrak{S}_n \cdot (c_1, \dots, c_n), \\ \tilde{c}_i \in \mu^{\mathbb{Z}} \hat{c}_i \quad \forall i \in \llbracket 1, n \rrbracket, \\ d = \sum_{i=1}^n \tilde{c}_i \end{array} \right\}. \end{aligned}$$

Moreover, we set  $S_{\mu,d} := S_{\mu,d}^1 \cup S_{\mu,d}^2$ . Then by definition, we have

$$\rho_{\mu,\mathbf{c}} \left( \begin{array}{c} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \vdots \\ \gamma_n \end{array} \right) \in O_{\mu,\mathbf{c}} := \bigcup_{d \in D_{\mathbf{c}}} \left\{ \left( \begin{array}{c} \varphi_{\alpha} \\ \varphi_{\beta} \\ \varphi_1 \\ \vdots \\ \varphi_n \end{array} \right) \mid \left( \begin{array}{c} \varphi_{\alpha} \\ \varphi_{\beta} \end{array} \right) \in S_{\mu,d}, \quad \left( \begin{array}{c} \varphi_1 \\ \vdots \\ \varphi_n \end{array} \right) \in R_{\mu,\mathbf{c},d} \right\}.$$

Note that  $O_{\mu, \mathbf{c}}$  is a finite set, and we will prove that each conjugacy class in the orbit of  $\rho_{\mu, \mathbf{c}}$  under the action of the mapping class group has a representative in  $O_{\mu, \mathbf{c}}$ . We shall denote  $[O_{\mu, \mathbf{c}}]$  the image of  $O_{\mu, \mathbf{c}}$  in  $\chi_{g, n}(\text{Aff}(\mathbb{C}))$ .

- *The set  $[O_{\mu, \mathbf{c}}]$  is stable under the inverses of  $\tau_1$  and  $\tau_2$ .*

In order to prove this assertion, it is enough to prove that the sets  $S_{\mu, d}^1$  and  $S_{\mu, d}^2$  are stable under the action of  $\tau_1^{-1}$  and  $\tau_2^{-1}$  modulo conjugation by translations. Let

$$\rho \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \mu^{k_1} z \\ \mu^{k_2} z - \frac{d}{\mu^{k_1-1}} \end{pmatrix} \in S_{\mu, d}^1.$$

Then

$$\tau_1^{-1} \cdot \rho \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \mu^{k_1} z \\ \mu^{k_1+k_2} z - \frac{d}{\mu^{k_1-1}} \end{pmatrix} \in S_{\mu, d}^1$$

and

$$\tau_2^{-1} \cdot \rho \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \mu^{k_1+k_2} z - \frac{\mu^{k_1} d}{\mu^{k_1-1}} \\ \mu^{k_2} z - \frac{d}{\mu^{k_1-1}} \end{pmatrix}.$$

To see that, up to conjugation by a translation, the latter image also belongs to  $S_{\mu, d}$ , we need to distinguish two cases. Firstly, if  $k_1 + k_2 \in N\mathbb{Z}$ , then  $k_2 \notin N\mathbb{Z}$  and we obtain

$$\begin{pmatrix} \mu^{k_1+k_2} z - \frac{\mu^{k_1} d}{\mu^{k_1-1}} \\ \mu^{k_2} z - \frac{d}{\mu^{k_1-1}} \end{pmatrix} = \begin{pmatrix} z + \frac{d}{\mu^{k_2-1}} \\ \mu^{k_2} z - \frac{d}{\mu^{k_1-1}} \end{pmatrix} \approx \begin{pmatrix} z + \frac{d}{\mu^{k_2-1}} \\ \mu^{k_2} z \end{pmatrix} \in S_{\mu, d}^2.$$

Secondly, if  $k_1 + k_2 \notin N\mathbb{Z}$ , then we obtain

$$\begin{pmatrix} \mu^{k_1+k_2} z - \frac{\mu^{k_1} d}{\mu^{k_1-1}} \\ \mu^{k_2} z - \frac{d}{\mu^{k_1-1}} \end{pmatrix} \approx \begin{pmatrix} \mu^{k_1+k_2} z \\ \mu^{k_2} z - \frac{d}{\mu^{k_1+k_2-1}} \end{pmatrix} \in S_{\mu, d}^1$$

In a similar way, one can show that up to conjugation by translations, we have  $\tau_1^{-1} \cdot S_{\mu, d}^2 \subset S_{\mu, d}$  and  $\tau_2^{-1} \cdot S_{\mu, d}^2 \subset S_{\mu, d}^2$ .

- *The set  $[O_{\mu, \mathbf{c}}]$  is stable under the inverses of  $\sigma_1, \dots, \sigma_{n-1}$ .*

Indeed, for every  $\rho \in O_{\mu, \mathbf{c}}$ , the group  $\langle \rho(\gamma_1), \dots, \rho(\gamma_n) \rangle$  is a translation group. In particular, it is abelian. Hence the elements  $\sigma_i$  act as permutations. But permutations stabilize the set  $R_{\mu, \mathbf{c}, d}$ .

- *The set  $[O_{\mu, \mathbf{c}}]$  is stable under the inverse of the modified mixing twist  $\tilde{\tau}_{2+k}$ .*

Note that for every  $k \in \llbracket 1, n-1 \rrbracket$ , up to a common conjugation by  $\rho(\Xi_k)$ , the representation  $\rho' := \tilde{\tau}_{2+k}^{-1} \cdot \rho$  may be described as follows, where  $\Xi_k = (\gamma_1 \dots \gamma_k)^{-1} \beta_1$ .

$$\begin{cases} \rho'(\alpha_1) = \rho(\Xi_k \alpha_1 \beta_1^{-1}) \\ \rho'(\beta_1) = \rho(\beta_1) \\ \rho'(\gamma_i) = \rho(\gamma_i) & i \in \llbracket 1, k \rrbracket; \\ \rho'(\gamma_j) = \rho(\Xi_k \gamma_j \Xi_k^{-1}) & j \in \llbracket k+1, n \rrbracket. \end{cases}$$

In the following calculations,  $i$  represents an index less or equal to  $k$  (if such an index exists) and  $j$  represents an index greater than  $k$ .

Assume first that  $\rho(\alpha_1, \beta_1) \in S_{\mu, d}^1$ . Then

$$\rho \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_i \\ \gamma_j \end{pmatrix} = \begin{pmatrix} \mu^{k_1} z \\ \mu^{k_2} z - \frac{d}{\mu^{k_1-1}} \\ z + \tilde{c}_i \\ z + \tilde{c}_j \end{pmatrix} \quad \text{and} \quad \rho(\Xi_k) = \mu^{k_2} z - \frac{d}{\mu^{k_1-1}} - \sum_{i=1}^k \tilde{c}_i.$$



Hence

$$\rho' \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_i \\ \gamma_j \end{pmatrix} = \begin{pmatrix} \mu^{k_1} z + d - \sum_{h=1}^k \tilde{c}_h \\ \mu^{k_2} z - \frac{d}{\mu^{k_1-1}} \\ z + \tilde{c}_i \\ z + \mu^{k_2} \tilde{c}_j \end{pmatrix} \approx \begin{pmatrix} \mu^{k_1} z \\ \mu^{k_2} z - \frac{d'}{\mu^{k_1-1}} \\ z + \tilde{c}_i \\ z + \mu^{k_2} \tilde{c}_j \end{pmatrix},$$

where

$$d' = \mu^{k_2} d - (\mu^{k_2} - 1) \sum_{i=1}^k \tilde{c}_i = \sum_{i=1}^k \tilde{c}_i + \sum_{j=k+1}^n \mu^{k_2} \tilde{c}_j,$$

since  $d = \sum_{i=1}^k \tilde{c}_i + \sum_{j=k+1}^n \tilde{c}_j$ . In other words, up to conjugation by a translation, we have  $\rho' \in O_{\mu, \mathbf{c}}$ . By an almost identical argumentation, we show that if  $\rho \in O_{\mu, \mathbf{c}}$  with  $\rho(\alpha_1, \beta_1) \in S_{\mu, d}^2$ , then  $\tilde{\tau}_{2+k}^{-1} \cdot \rho$  is also in  $O_{\mu, \mathbf{c}}$  modulo conjugation.

Since every element of  $\hat{\Gamma}_{g,n}^\circ$  induces a bijection of  $\chi_{g,n}(\text{Aff}(\mathbb{C}))$  and we have proven that  $[O_{\mu, \mathbf{c}}]$  is stable under  $\tau_i^{-1}$  for every  $i \in \llbracket 1, n+1 \rrbracket$  and  $\sigma_j^{-1}$  for every  $j \in \llbracket 1, n-1 \rrbracket$ , these generators of  $\hat{\Gamma}_{g,n}^\circ$  induce bijections of

$$[O_{\mu, \mathbf{c}}] \subset \chi_{g,n}(\text{Aff}(\mathbb{C})).$$

Hence  $[O_{\mu, \mathbf{c}}]$  is also stable under  $\tau_i$  for every  $i \in \llbracket 1, n+1 \rrbracket$  and  $\sigma_j$  for every  $j \in \llbracket 1, n-1 \rrbracket$ . We conclude that the orbit  $\hat{\Gamma}_{g,n}^\circ \cdot [\rho_{\mu, \mathbf{c}}] = \hat{\Gamma}_{g,n}^\circ \cdot [\rho_{\mu, \mathbf{c}}]$  is contained in the finite set  $[O_{\mu, \mathbf{c}}]$ .  $\square$

#### 2.4.4 Non abelian case in higher genus

We are now considering the case  $g > 1$ , and arbitrary  $n \geq 0$ . Recall that  $\Lambda_{g,n}$  then contains the group

$$G := \langle \alpha_{g-1}, \beta_{g-1}, \alpha_g, \beta_g \rangle \subset \Lambda_{g,n}$$

and  $\hat{\Gamma}_{g,n}^\circ$  contains a subgroup

$$H := \langle \tau_{2g-3}, \tau_{2g-2}, \tau_{2g-1}, \tau_{2g}, \tau_{3g-1} \rangle \subset \hat{\Gamma}_{g,n}^\circ$$

acting on  $G$  as summarized by Table [6](#).

$\tau_{2k}$	$k \in \llbracket g-1, g \rrbracket$	$\alpha_k$	$\mapsto$	$\alpha_k \beta_k$
$\tau_{2k-1}$	$k \in \llbracket g-1, g \rrbracket$	$\beta_k$	$\mapsto$	$\beta_k \alpha_k$
$\tau_{3g-1}$		$\alpha_g$	$\mapsto$	$\Theta^{-1} \alpha_g$
		$\alpha_{g-1}$	$\mapsto$	$\alpha_{g-1} \Theta$
		$\beta_{g-1}$	$\mapsto$	$\Theta^{-1} \beta_{g-1} \Theta$

; where  $\Theta := \alpha_g \beta_g^{-1} \alpha_g^{-1} \beta_{g-1}$

Table 6: Action of a subgroup in genus  $g \geq 2$

**Lemma 2.4.6** (Elimination criterion). *Let  $g \geq 2$ . Let  $\rho : \Lambda_{g,n} \rightarrow \text{Aff}(\mathbb{C})$  be a representation of the following “weak prepared form”*

$$\rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ \mu^{m_{g-1}} z + a \\ z + b \end{pmatrix}, \quad (2.9)$$

where  $\mu$  is a root of unity,  $a, b \in \mathbb{C}$ ,  $m_g, m_{g-1} \in \mathbb{Z}$  and  $\mu^{m_g} \neq 1$ . If  $\hat{\Gamma}_{g,n} \cdot [\rho]$  is finite, then

$$a = 0, \quad b = 0 \quad \text{and} \quad \mu^{m_{g-1}} = \frac{1}{\mu^{m_g}}.$$

*Proof.* Note that if two representations  $\rho, \rho'$  of the form (2.9) are conjugated, then their restrictions to  $G$  are equal. Assume

$$\rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ \mu^{m_{g-1}} z + a \\ z + b \end{pmatrix}.$$

Now consider the action of  $\tau_{2g-2}^{-k}$  for  $k \in \mathbb{Z}$ :

$$\tau_{2g-2}^{-k} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ \mu^{m_{g-1}} z + a + k \cdot \mu^{m_{g-1}} b \\ z + b \end{pmatrix}.$$

Since the suborbit  $(\tau_{2g-2}^{-k} \cdot [\rho])_k$  is supposed to take finitely many values, we have  $\boxed{b = 0}$ .

Now consider the action of  $\tau_{3g-1}^{-k}$ . We have

$$\tau_{3g-1}^{-k} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z + k \mu^{m_g} \\ z + 1 \\ \mu^{m_{g-1}} z + a - k \mu^{m_g + m_{g-1}} \\ z \end{pmatrix} \approx \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ \mu^{m_{g-1}} z + a - k \cdot \frac{\mu^{2m_g + m_{g-1} - \mu^{m_g}}}{\mu^{m_g - 1}} \\ z \end{pmatrix}.$$

As the corresponding suborbit is supposed to be finite, we have  $\boxed{\mu^{m_{g-1}} = \mu^{-m_g}}$ .

In order to conclude, consider  $\tilde{\tau}_{3g-1} = \tau^{-1} \circ \tau_{3g-1} \circ \tau$ , where  $\tau := \tau_{2g-3} \circ \tau_{2g} \circ \tau_{2g-1}^{-1} \circ \tau_{2g}$ . We have

$$\tilde{\tau}_{3g-1}^k : \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\Theta}^{-k} \alpha_g \tilde{\Theta}^k \\ \beta_g \tilde{\Theta}^k \\ \alpha_{g-1} \tilde{\Theta}^k \\ \tilde{\Theta}^{-k} \beta_{g-1} \alpha_{g-1}^{-1} \tilde{\Theta}^k \alpha_{g-1} \tilde{\Theta}^k \end{pmatrix},$$

where  $\tilde{\Theta} := \tau_*^{-1} \Theta = \alpha_g^{-1} \beta_{g-1} \alpha_{g-1}^{-1}$ . Then

$$\rho(\tilde{\Theta}^k) = z - k \cdot a.$$

Hence, modulo conjugation by  $\rho(\tilde{\Theta}^k)$ , we have

$$\tilde{\tau}_{3g-1}^{-k} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} \approx \rho \begin{pmatrix} \alpha_g \\ \tilde{\Theta}^k \beta_g \\ \tilde{\Theta}^k \alpha_{g-1} \\ \beta_{g-1} \alpha_{g-1}^{-1} \tilde{\Theta}^k \alpha_{g-1} \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z \\ z + 1 - k \cdot a \\ \frac{1}{\mu^{m_g}} z + (1 - k) \cdot a \\ z - k \cdot a \mu^{m_g} \end{pmatrix}.$$

Provided  $1 - k \cdot a \neq 0$  (which is the case for an infinite number of  $k \in \mathbb{Z}$ ), we obtain

$$\tilde{\tau}_{3g-1}^{-k} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} \approx \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ \frac{1}{\mu^{m_g}} z + \frac{(1-k) \cdot a}{1-k \cdot a} \\ z - \frac{k \cdot a}{1-k \cdot a} \mu^{m_g} \end{pmatrix}.$$

Again by finiteness, we have  $\boxed{a = 0}$ . □

**Proposition 2.4.7** (Non abelian representations for  $g > 1$ ). *Assume  $g \geq 2$  and  $n \geq 0$ . Let  $\rho : \Lambda_{g,n} \rightarrow \text{Aff}(\mathbb{C})$  be a representation with non abelian image. Then the orbit  $\hat{\Gamma}_{g,n} \cdot [\rho]$  is infinite.*

*Proof.* Let  $g \geq 2$  and let  $\rho$  be a representation with finite orbit modulo conjugation. Let us assume for a contradiction that  $\rho(\Lambda_{g,n})$  is non-abelian. We may then assume that  $\rho$  is of “prepared form” as in Lemma 2.4.4. In particular, we may assume that  $\rho$  is of “weak prepared form” and hence satisfies the elimination criterion of Lemma 2.4.6. In other words, we may assume that  $\rho$  is of the following form:

$$\rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu z \\ z + 1 \\ \frac{1}{\mu} z \\ z \end{pmatrix},$$

where  $\mu \neq 1$  is a root of unity. We have

$$\tau_{3g-1}^{-1} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu z + \mu \\ z + 1 \\ \frac{1}{\mu} z - 1 \\ z \end{pmatrix}; \quad \tau_{2g} \cdot (\tau_{3g-1}^{-1} \cdot \rho) \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu z \\ z + 1 \\ \frac{1}{\mu} z - 1 \\ z \end{pmatrix}.$$

Now  $\tau_{2g} \circ \tau_{3g-1}^{-1} \cdot \rho$  is also of weak prepared form, but is not compatible with the elimination criterion of Lemma 2.4.6, whence the contradiction. □

## 2.5 Reducible rank two representations with finite orbit

Theorem 2.1.2 concerns representations  $\rho : \Lambda_{g,n} \rightarrow \text{GL}_2\mathbb{C}$  that are reducible, i.e., that globally fix a line in  $\mathbb{C}^2$ . A particular case of reducible rank 2 representations are those that are totally reducible, i.e., that globally fix two distinct lines in  $\mathbb{C}^2$ . In Proposition 2.5.8, we will prove the statement in the totally reducible case, and in Theorem 2.5.9, we will prove it in the reducible but not totally reducible case. The juxtaposition of these two results yields Theorem 2.1.2. Moreover, we will estimate the size of finite orbits of conjugacy classes of affine representation under the pure mapping class group.

### 2.5.1 The size of some finite orbits

Note that since  $\mathbb{C}^*$  is abelian, we have a natural identification between scalar representations and their conjugacy classes:  $\chi_{g,n}(\mathbb{C}^*) = \text{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ . In particular,  $\Gamma_{g,n}$  acts on  $\text{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ .

**Proposition 2.5.1.** *Let  $g > 0, n \geq 0$ . Let  $\lambda \in \text{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$  be a scalar representation with finite image. Then*

$$\text{card}(\text{Im}(\lambda))^{2g-1} \leq \text{card}(\Gamma_{g,n} \cdot \lambda) \leq \text{card}(\text{Im}(\lambda))^{2g} \tag{2.10}$$

*Proof.* Since  $\text{Im}(\lambda)$  is finite, there is a root of unity  $\mu \in \mathbb{C}^*$  such that  $\text{Im}(\lambda) = \mu^{\mathbb{Z}}$ . For each  $j \in \llbracket 1, n \rrbracket$ , choose an integer  $m_j \in \mathbb{Z}$  such that  $\lambda(\gamma_j) = \mu^{m_j}$ . Denote  $N := \text{order}(\mu)$  and

$$O_\lambda := \left\{ (\mu^{k_g}, \mu^{\ell_g}, \dots, \mu^{k_1}, \mu^{\ell_1}) \left| \begin{array}{l} \mathbf{k} := (k_g, \dots, k_1) \in \mathbb{Z}^g, \quad \ell := (\ell_g, \dots, \ell_1) \in \mathbb{Z}^g \\ \gcd(k_g, \dots, k_1, \ell_g, \dots, \ell_1, m_1, \dots, m_n, N) = 1 \end{array} \right. \right\}.$$

Note that to any element  $(\mu^{k_g}, \mu^{\ell_g}, \dots, \mu^{k_1}, \mu^{\ell_1}) \in O_\lambda$  we can associate a well defined representation  $\lambda' \in \text{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$  by setting  $\lambda'(\alpha_i) = \mu^{k_i}$ ;  $\lambda'(\beta_i) = \mu^{\ell_i}$  for all  $i \in \llbracket 1, g \rrbracket$  and  $\lambda'(\gamma_j) = \mu^{m_j}$  for all  $j \in \llbracket 1, n \rrbracket$ . In that sense, we can see  $O_\lambda$  as a subset of  $\text{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ .

We claim that  $\Gamma_{g,n} \cdot \lambda = O_\lambda$ . Notice that this claim implies (2.10). Indeed, the second inequality is obvious, and the first one follows from the fact that if we set for example  $k_g = 1$ , then we can choose all other exponents freely.

Let us now prove the claim. We clearly have  $\lambda \in O_\lambda$ . Each pure element  $\tau$  of  $\hat{\Gamma}_{g,n}^\circ$  transforms the generators  $\gamma_i$  into conjugates  $\zeta_i^{-1} \gamma_i \zeta_i$ . Since  $\mathbb{C}^*$  is abelian, this implies that for any representation  $\lambda'$  corresponding to an element of  $O_\lambda$ , we have  $(\tau \cdot \lambda')(\gamma_i) = \lambda'(\gamma_i) = \mu^{m_i}$ . Consequently,  $\Gamma_{g,n} \cdot O_\lambda = O_\lambda$  and in particular  $\Gamma_{g,n} \cdot \lambda \subset O_\lambda$ .

The orbits of  $\hat{\Gamma}_{g,n}$  on  $\chi_{g,n}(\mathbb{C}^*) = \text{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$  are the ones of  $\hat{\Gamma}_{g,n}^\circ$ . Note that the subgroup  $H := \langle \tau_i \mid i \in \llbracket 1, 3g-1 + \min(0, n-1) \rrbracket \rangle \subset \hat{\Gamma}_{g,n}^\circ$  is generated by pure elements. Translating Table 1 into an action of  $\hat{\Gamma}_{g,n}^\circ$  on the powers of  $\mu$  corresponding to the generators of  $\Lambda_{g,n}$  then yields the following.

- (a) For a given  $(\tilde{k}_g, \dots, \tilde{k}_1) \in \{1, \dots, N\}^g$  such that  $\gcd(\tilde{k}_g, \dots, \tilde{k}_1, m_1, \dots, m_n, N) = 1$ , the subgroup  $\langle \tau_{2i}, \tau_{2i-1} \mid i \in \llbracket 1, g \rrbracket \rangle \subset H$  acts transitively on those elements of  $O_\lambda$  satisfying  $\gcd(k_i, \ell_i) = k_i$  for all  $i \in \llbracket 1, g \rrbracket$  (see also the proof of Lemma 2.4.4).
- (b) For all  $\tilde{\mathbf{k}} := (\tilde{k}_g, \dots, \tilde{k}_1) \in \{1, \dots, N\}^g$  such that  $\gcd(\tilde{k}_g, \dots, \tilde{k}_1, m_1, \dots, m_n, N) = 1$ , there is an element of the subgroup  $\langle \tau_{2i}, \tau_{2i-1}, \tau_{2g+i'} \mid i \in \llbracket 1, g \rrbracket, i' \in \llbracket 1, g-1 \rrbracket \rangle \subset H$ , which sends the element of  $O_\lambda$  given by  $\mathbf{k} = (\gcd(\tilde{k}_g, \dots, \tilde{k}_1), 0, \dots, 0)$  and  $\ell = \vec{0}$  to the element of  $O_\lambda$  given by  $\mathbf{k} = \tilde{\mathbf{k}}$  and  $\ell = \vec{0}$ .
- (c) The subgroup  $\langle \tau_{3g-1+j} \mid j \in \llbracket 1, \min(0, n-1) \rrbracket \rangle \subset H$  acts transitively on those elements of  $O_\lambda$  satisfying  $\ell = \vec{0}$  and  $k_i = 0$  for all  $i \in \llbracket 1, g-1 \rrbracket$ .

Consequently, the pure subgroup  $H$  acts transitively on  $O_\lambda$ . This implies  $\Gamma_{g,n} \cdot \lambda = O_\lambda$ .  $\square$

Recall that we denote

$$\text{Aff}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \left| a, b \in \mathbb{C}, a \neq 0 \right. \right\}. \quad (2.11)$$

**Proposition 2.5.2.** *Let  $g = 1$  and let  $n > 0$ . Let  $\mu \in \mathbb{C}^*$  be a root of unity of order  $N > 1$  and let  $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{C}^n$  with  $\sum_{i=1}^n c_i = 1$ . Consider the representation  $\rho_{\mu, \mathbf{c}} \in \text{Hom}(\Lambda_{g,n}, \text{Aff}(\mathbb{C}))$  defined by*

$$\rho_{\mu, \mathbf{c}}(\alpha_1) := \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}; \quad \rho_{\mu, \mathbf{c}}(\beta_1) := \begin{pmatrix} 1 & -\frac{1}{\mu-1} \\ 0 & 1 \end{pmatrix}; \quad \rho_{\mu, \mathbf{c}}(\gamma_i) := \begin{pmatrix} 1 & c_i \\ 0 & 1 \end{pmatrix} \quad \forall i \in \llbracket 1, n \rrbracket.$$

*Its orbit  $\Gamma_{g,n} \cdot [\rho_{\mu, \mathbf{c}}]$  is finite in  $\chi_{g,n}(\text{Aff}(\mathbb{C}))$ . More precisely, we have*

$$\phi(N)(2N - \phi(N)) \cdot N^{n'-1} \leq \text{card}(\Gamma_{g,n} \cdot [\rho_{\mu, \mathbf{c}}]) \leq (N^2 - 1)N^{n'-1}, \quad (2.12)$$

*where  $n' := \text{card}\{i \in \llbracket 1, n \rrbracket \mid c_i \neq 0\}$  and  $\phi$  denotes the Euler totient function.*

**Remark 2.5.3.** Observe that the estimate (2.12) yields an equality if  $N$  is a prime number.

*Proof.* For convenience we shall represent the elements of  $\text{Aff}(\mathbb{C})$  by degree one polynomials  $az + b$ , as in page 63. Denote

$$D_{\mathbf{c}} := \mu^{\mathbb{Z}}c_1 + \dots + \mu^{\mathbb{Z}}c_n; \quad S_{\mu,d} := S_{\mu,d}^1 \cup S_{\mu,d}^2; \quad R_{\mu,\mathbf{c},d}; \quad O_{\mu,\mathbf{c}}$$

as in the proof of Proposition 2.4.5. Moreover, denote

$$R_{\mu,\mathbf{c},d}^{\text{pure}} := \left\{ \left( \begin{array}{c} z + \tilde{c}_1 \\ \vdots \\ z + \tilde{c}_n \end{array} \right) \middle| \begin{array}{l} \tilde{c}_i \in \mu^{\mathbb{Z}}c_i \quad \forall i \in \llbracket 1, n \rrbracket \\ d = \sum_{i=1}^n \tilde{c}_i \end{array} \right\}$$

$$O_{\mu,\mathbf{c}}^{\text{pure}} := \bigcup_{d \in D_{\mathbf{c}}} \left\{ (\varphi_{\alpha}, \varphi_{\beta}, \varphi_1, \dots, \varphi_n) \middle| \begin{array}{l} \left( \begin{array}{c} \varphi_{\alpha} \\ \varphi_{\beta} \end{array} \right) \in S_{\mu,d}, \quad \left( \begin{array}{c} \varphi_1 \\ \vdots \\ \varphi_n \end{array} \right) \in R_{\mu,\mathbf{c},d}^{\text{pure}} \end{array} \right\}.$$

We shall denote by  $[O_{\mu,\mathbf{c}}]$  and  $[O_{\mu,\mathbf{c}}^{\text{pure}}]$  the respective images of  $O_{\mu,\mathbf{c}}$  and  $O_{\mu,\mathbf{c}}^{\text{pure}}$  in  $\chi_{g,n}(\text{Aff}(\mathbb{C}))$ . By a slight refinement of the proof of Proposition 2.4.5, we have

$$\Gamma_{g,n} \cdot [\rho_{\mu,\mathbf{c}}] \subset [O_{\mu,\mathbf{c}}^{\text{pure}}]. \quad (2.13)$$

Indeed, recall that the pure subgroup  $\Gamma_{g,n}$  of  $\hat{\Gamma}_{g,n}$  is the subgroup that respects the labellings of the punctures. Each pure element  $\tau$  of  $\hat{\Gamma}_{g,n}^{\circ}$  transforms the generators  $\gamma_i$  into conjugates  $\zeta_i^{-1}\gamma_i\zeta_i$ . As we have  $\rho(\zeta_i) = \mu^m z + d$  for suitable  $m \in \mathbb{Z}$ ,  $d \in \mathbb{C}$ , we deduce  $(\tau \cdot \rho)(\gamma_i) = \mu^m c_i$ . This proves the inclusion (2.13). Moreover, using Table 5 page 68, we can check successively:

- (a) as observed in the proof of Lemma 2.4.4, any element  $\rho = [*_1, *_2, z + \tilde{c}_1, \dots, z + \tilde{c}_n]$  of  $[O_{\mu,\mathbf{c}}]$  can be transformed into an element  $\rho' = [z + d/(\mu - 1), \mu z, z + \tilde{c}_1, \dots, z + \tilde{c}_n]$  by an element of  $\langle \tau_1, \tau_2 \rangle$ , where  $d = \sum_{i=1}^n \tilde{c}_i$ ;
- (b) for any  $j \in \llbracket 1, n \rrbracket$ , by the action of an element of  $B_n = \langle \sigma_i \mid i \in \llbracket 1, n-1 \rrbracket \rangle$ , the element  $\rho'$  can be transformed into  $[z + d/(\mu - 1), \mu z, z + c'_1, \dots, z + c'_n]$ , where  $c'_1 = \tilde{c}_j$ ,  $c'_j = \tilde{c}_1$  and  $c'_i = \tilde{c}_i$  for  $i \neq 1, j$ ;
- (c) for any  $m_j \in \mathbb{Z}$ , using a power of  $\tilde{\tau}_3$ , we transform this latter element into  $[z + d'/(\mu - 1), \mu z, z + \mu^{m_j} c'_1, \dots, z + c'_n]$ , where  $d' = d + (\mu^{m_j} - 1)\tilde{c}_j$ ;
- (d) reusing an element of  $B_n$ , one gets  $\rho'$  altered only by replacing  $\tilde{c}_j$  by  $\mu^{m_j}\tilde{c}_j$  and  $d$  by  $d'$ .

This allows to infer that any element  $\rho = [*_1, *_2, z + \tilde{c}_1, \dots, z + \tilde{c}_n]$  of  $[O_{\mu,\mathbf{c}}]$  can be transformed into  $[z + 1/(\mu - 1), \mu z, z + c_1, \dots, z + c_n]$  by a suitable element of  $\hat{\Gamma}_{g,n}$ . Reusing (2.5.1), we deduce  $\hat{\Gamma}_{g,n} \cdot [\rho_{\mu,\mathbf{c}}] = [O_{\mu,\mathbf{c}}]$ . The conjunction of this equality and the inclusion (2.13) yields

$$\Gamma_{g,n} \cdot [\rho_{\mu,\mathbf{c}}] = [O_{\mu,\mathbf{c}}^{\text{pure}}].$$

Denote by  $[S_{\mu,d}]_t$  and  $[O_{\mu,\mathbf{c}}^{\text{pure}}]_t$  the set of equivalence classes of  $S_{\mu,d}$  and  $O_{\mu,\mathbf{c}}^{\text{pure}}$  respectively modulo conjugation by translations. For each  $d \in D_{\mathbf{c}}$ , the cardinality of  $[S_{\mu,d}]_t$  equals the cardinality of

$$K_N := \{(k_1, k_2) \in \llbracket 1, N \rrbracket^2 \mid \gcd(k_1, k_2, N) = 1\}.$$

Indeed, for  $\{i, j\} = \{1, 2\}$ , the elements of  $S_{\mu,d}^i$  that are not conjugated by a translation to an element of  $S_{\mu,d}^j$  are precisely those corresponding to  $k_j = 0$  and  $\gcd(k_i, N) = 1$ . We can estimate

$$\phi(N)(2N - \phi(N)) \leq \text{card}([S_{\mu,d}]_t) \leq N^2 - 1.$$

These inequalities are readily derived from the inclusions

$$\{(k_1, k_2) \in \llbracket 1, N \rrbracket^2 \mid \gcd(k_1, N) = 1 \text{ or } \gcd(k_2, N) = 1\} \subset K_N \subset \llbracket 1, N \rrbracket^2 \setminus \{(0, 0)\}.$$

On the other hand, conjugations by translations act trivially on  $R_{\mu, \mathbf{c}, d}^{\text{pure}}$ . By definition of  $n'$ ,

$$\text{card} \left( \bigcup_{d \in D_{\mathbf{c}}} R_{\mu, \mathbf{c}, d}^{\text{pure}} \right) = N^{n'}.$$

We deduce

$$\phi(N)(2N - \phi(N))N^{n'} \leq \text{card}[O_{\mu, \mathbf{c}}^{\text{pure}}]_{\mathfrak{t}} \leq (N^2 - 1)N^{n'}.$$

The condition  $\sum_{i=1}^n c_i = 1$  ensures  $n' > 0$ . In particular, there is an index  $i_0 \in \llbracket 1, n \rrbracket$  such that  $c_{i_0} \neq 0$ . Up to conjugation by powers of the linear transformation  $\mu z$ , we can normalize  $\tilde{c}_{i_0} = c_{i_0}$  for each element in  $[O_{\mu, \mathbf{c}}^{\text{pure}}]_{\mathfrak{t}}$ , which yields  $\text{card}[O_{\mu, \mathbf{c}}^{\text{pure}}] = \frac{1}{N} \text{card}[O_{\mu, \mathbf{c}}^{\text{pure}}]_{\mathfrak{t}}$ .  $\square$

## 2.5.2 Reduction to the affine case

Let  $A$  be a group. Consider a representation  $\rho \in \text{Hom}(A, \text{GL}_2\mathbb{C})$ , and assume it takes values in  $\text{Upp} \subset \text{GL}_2\mathbb{C}$ , where  $\text{Upp}$  is the group of invertible upper triangular matrices of rank 2. To such a representation, we may associate two other ones: the *scalar part*  $\rho_{\mathbb{C}^*} : \alpha \mapsto \rho(\alpha)_{2,2}$  and the *affine part*  $\rho_{\text{Aff}} := \rho_{\mathbb{C}^*}^{-1} \otimes \rho$ . The former takes values in  $\mathbb{C}^*$  and the latter takes values in  $\text{Aff}(\mathbb{C})$ , see (2.11).

**Lemma 2.5.4.** *Let  $\rho = \rho_{\mathbb{C}^*} \otimes \rho_{\text{Aff}}$  and  $\rho' = \rho'_{\mathbb{C}^*} \otimes \rho'_{\text{Aff}}$  be two reducible representations as above, and assume that they are not totally reducible. We have  $[\rho] = [\rho'] \in \text{Hom}(A, \text{GL}_2\mathbb{C})/\text{GL}_2\mathbb{C}$  if and only if  $\rho_{\mathbb{C}^*} = \rho'_{\mathbb{C}^*}$  and  $[\rho_{\text{Aff}}] = [\rho'_{\text{Aff}}] \in \text{Hom}(A, \text{Aff}(\mathbb{C}))/\text{Aff}(\mathbb{C})$ .*

*Proof.* The "if"-part is trivial. Assume  $[\rho] = [\rho']$ . Since they take values in  $\text{Upp}$ , both representations  $\rho$  and  $\rho'$  leave the line  $\text{span}(e_1)$  of  $\mathbb{C}^2$  invariant. Since both are not totally reducible, for each of the representations, there is no other globally invariant line. Let  $M = (m_{i,j}) \in \text{GL}_2\mathbb{C}$  conjugate both representations. Then  $M$  must leave  $\text{span}(e_1)$  invariant, i.e.  $M \in \text{Upp}$ . As the scalars are central in  $\text{GL}_2\mathbb{C}$ , the element  $M/m_{2,2} \in \text{Aff}(\mathbb{C})$  conjugates both representations. In particular  $\rho_{\mathbb{C}^*} = \rho'_{\mathbb{C}^*}$  and  $M/m_{2,2}$  conjugates  $\rho_{\text{Aff}}$  and  $\rho'_{\text{Aff}}$ .  $\square$

Lemma 2.5.4 has the following immediate consequence, where we consider the natural inclusion

$$\iota : \text{Aff}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{C}, a \neq 0 \right\} \hookrightarrow \text{GL}_2\mathbb{C}.$$

**Lemma 2.5.5.** *Let  $g > 0, n \geq 0$  and let  $\rho \in \text{Hom}(\Lambda_{g,n}, \text{GL}_2\mathbb{C})$  be a reducible but not totally reducible representation. Then there exist a unique  $\lambda \in \text{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$  and a unique conjugacy class  $[\rho_{\text{Aff}}] \in \chi_{g,n}(\text{Aff}(\mathbb{C}))$  such that  $[\rho] = [\lambda \otimes \iota_* \rho_{\text{Aff}}] \in \chi_{g,n}(\text{GL}_2\mathbb{C})$ . Moreover, we have*

$$\max\{\text{card}(\Gamma_{g,n} \cdot \lambda), \text{card}(\Gamma_{g,n} \cdot [\rho_{\text{Aff}}])\} \leq \text{card}(\Gamma_{g,n} \cdot [\rho]) \leq \text{card}(\Gamma_{g,n} \cdot \lambda) \cdot \text{card}(\Gamma_{g,n} \cdot [\rho_{\text{Aff}}]). \quad (2.14)$$

*In particular, the following are equivalent.*

- $\Gamma_{g,n} \cdot [\rho]$  is a finite subset of  $\chi_{g,n}(\text{GL}_2\mathbb{C})$ .
- $\Gamma_{g,n} \cdot \lambda$  is a finite subset of  $\text{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$  and  $\Gamma_{g,n} \cdot [\rho_{\text{Aff}}]$  is a finite subset of  $\chi_{g,n}(\text{Aff}(\mathbb{C}))$ .

Consider the natural inclusion

$$\iota : (\mathbb{C}^*)^2 \rightarrow \mathrm{GL}_2\mathbb{C}; \quad (a_1, a_2) \mapsto \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

**Lemma 2.5.6.** *Let  $g \geq 0, n \geq 0$ . Let  $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_2\mathbb{C})$  be a totally reducible representation. Then there are scalar representations  $\lambda_1, \lambda_2 \in \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$  such that  $[\rho] = [\iota_*(\lambda_1, \lambda_2)] \in \chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ . Moreover, we have*

$$\frac{1}{2} \max\{\mathrm{card}(\Gamma_{g,n} \cdot \lambda_i) \mid i \in \{1, 2\}\} \leq \mathrm{card}(\Gamma_{g,n} \cdot [\rho]) \leq \mathrm{card}(\Gamma_{g,n} \cdot \lambda_1) \cdot \mathrm{card}(\Gamma_{g,n} \cdot \lambda_2). \quad (2.15)$$

In particular, the following are equivalent:

- $\Gamma_{g,n} \cdot [\rho]$  is a finite subset of  $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ .
- $\Gamma_{g,n} \cdot \lambda_i$  is a finite subset of  $\mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$  for  $i = 1, 2$ .

*Proof.* The image of the map  $\iota_*$  from  $\mathrm{Hom}(\Lambda_{g,n}, (\mathbb{C}^*)^2)$  to  $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$  is obviously the set of conjugacy classes of totally reducible representations. By definition, the action of  $\Gamma_{g,n}$  on  $\iota_*\mathrm{Hom}(\Lambda_{g,n}, (\mathbb{C}^*)^2)$ , induced by the action on  $\mathrm{Hom}(\Lambda_{g,n}, (\mathbb{C}^*)^2)$ , coincides with the action of  $\Gamma_{g,n}$  on  $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ . Moreover,

$$\frac{1}{2} \cdot \mathrm{card}(\Gamma_{g,n} \cdot (\lambda_1, \lambda_2)) \leq \mathrm{card}(\Gamma_{g,n} \cdot [\iota_*(\lambda_1, \lambda_2)]) \leq \mathrm{card}(\Gamma_{g,n} \cdot (\lambda_1, \lambda_2)). \quad (2.16)$$

Indeed, the second inequality is obvious, and the first one follows from the fact that if  $[\iota_*(\lambda_1, \lambda_2)] = [\iota_*(\lambda'_1, \lambda'_2)]$  then either  $(\lambda_1, \lambda_2) = (\lambda'_1, \lambda'_2)$  or  $(\lambda_1, \lambda_2) = (\lambda'_2, \lambda'_1)$ . On the other hand, we can estimate

$$\max\{\mathrm{card}(\Gamma_{g,n} \cdot \lambda_i) \mid i \in \{1, 2\}\} \leq \mathrm{card}(\Gamma_{g,n} \cdot (\lambda_1, \lambda_2)) \leq \mathrm{card}(\Gamma_{g,n} \cdot \lambda_1) \cdot \mathrm{card}(\Gamma_{g,n} \cdot \lambda_2). \quad (2.17)$$

We conclude by noticing that (2.16) and (2.17) imply (2.15).  $\square$

**Remark 2.5.7.** *The equality  $[\rho] = [\iota_*(\lambda_1, \lambda_2)] \in \chi_{g,n}(\mathrm{GL}_2\mathbb{C})$  in the above Lemma is commonly written as  $\rho = \lambda_1 \oplus \lambda_2$ . We adopted this notation in the statement of Theorem 2.1.2, and we will use it in its proof.*

### 2.5.3 Classification results

**Proposition 2.5.8.** *Let  $g \geq 0, n \geq 0$ . Let  $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_2\mathbb{C})$  be totally reducible, i.e.,  $\rho = \lambda_1 \oplus \lambda_2$  is a direct sum of scalar representations. The following are equivalent:*

- the orbit  $\Gamma_{g,n} \cdot [\rho]$  in  $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$  is finite.
- the subgroup  $\mathrm{Im}(\rho)$  of  $\mathrm{GL}_2\mathbb{C}$  has finite order.

Moreover, if the orbit  $\Gamma_{g,n} \cdot [\rho]$  is finite, then its size can be estimated as follows:

$$\frac{1}{2} \max\{\mathrm{card}(\mathrm{Im}(\lambda_i))^{2g-1} \mid i \in \{1, 2\}\} \leq \mathrm{card}(\Gamma_{g,n} \cdot [\rho]) \leq \mathrm{card}(\mathrm{Im}(\rho))^{2g}. \quad (2.18)$$

*Proof.* From Lemma 2.5.6, finiteness of the orbit  $\Gamma_{g,n} \cdot [\rho]$  in  $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$  is tantamount to the finiteness of the orbits  $\Gamma_{g,n} \cdot \lambda_i \subset \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$  for  $i = 1, 2$ . Since  $[\hat{\Gamma}_{g,n} : \Gamma_{g,n}] = n!$  is finite, finiteness of  $\Gamma_{g,n} \cdot \lambda_i$  is equivalent to the finiteness of  $\hat{\Gamma}_{g,n} \cdot \lambda_i$ . Proposition 2.4.2 establishes that  $\hat{\Gamma}_{g,n} \cdot \lambda_i$  is finite if and only if  $\mathrm{Im}(\lambda_i)$  is finite. This proves the equivalence in the statement.

The left inequality in (2.18) follows from Lemma 2.5.6 and Proposition 2.5.1. Each pure element  $\tau$  of  $\hat{\Gamma}_{g,n}^\circ$  transforms the generators  $\gamma_i$  into conjugates. By abelianity, for  $\rho' = \tau \cdot \rho$  and any  $i \in \llbracket 1, n \rrbracket$ , we get  $\rho'(\gamma_i) = \rho(\gamma_i)$ . We deduce the right inequality in (2.18).  $\square$

**Theorem 2.5.9.** *Let  $g > 0, n \geq 0$  and let  $\rho \in \text{Hom}(\Lambda_{g,n}, \text{GL}_2\mathbb{C})$  be a reducible but not totally reducible representation. The following are equivalent:*

- the orbit  $\Gamma_{g,n} \cdot [\rho]$  in  $\chi_{g,n}(\text{GL}_2\mathbb{C})$  is finite.
- $g = 1, n > 0$ , there are a scalar representation  $\lambda \in \text{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$  and an affine representation  $\rho_{\mu, \mathbf{c}} \in \text{Hom}(\Lambda_{g,n}, \text{Aff}(\mathbb{C}))$  as in Proposition 2.5.2, such that

$$[\rho] \in \Gamma_{g,n} \cdot [\lambda \otimes \rho_{\mu, \mathbf{c}}].$$

Moreover, if the orbit  $\Gamma_{g,n} \cdot [\rho]$  is finite, then its size can be estimated as follows:

$$\max \left\{ N_2, \phi(N)(2N - \phi(N))N^{n'-1} \right\} \leq \text{card}(\Gamma_{g,n} \cdot [\rho]) \leq (N^2 - 1)N^{n'-1}N_2^2, \quad (2.19)$$

where  $n' := \text{card}\{i \in \llbracket 1, n \rrbracket \mid \rho(\gamma_i) \notin \mathbb{C}^*I_2\}$ ,  $N := \text{order}(\mu)$ ,  $N_2 = \text{card}(\text{Im}(\lambda))$  and  $\phi$  is the Euler totient function.

*Proof.* From Lemma 2.5.5 we know that  $[\rho]$  admits a unique decomposition  $[\rho] = [\lambda \otimes \rho_{\text{Aff}}]$ , where  $\lambda$  is a scalar representation and  $\rho_{\text{Aff}}$  is an affine representation. Moreover, since  $\rho$  is not totally reducible, the affine representation  $\rho_{\text{Aff}}$  has non abelian image. Still by Lemma 2.5.5 the orbit  $\Gamma_{g,n} \cdot [\rho] \subset \chi_{g,n}(\text{GL}_2\mathbb{C})$  is finite if and only if the orbits  $\Gamma_{g,n} \cdot \lambda \subset \text{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$  and  $\Gamma_{g,n} \cdot [\rho_{\text{Aff}}] \subset \chi_{g,n}(\text{Aff}(\mathbb{C}))$  are finite. From Proposition 2.4.2 the orbit  $\Gamma_{g,n} \cdot \lambda \subset \text{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$  is finite if and only if  $\lambda$  has finite image. Since  $[\hat{\Gamma}_{g,n} : \Gamma_{g,n}] = n!$  is finite, finiteness of the orbit  $\Gamma_{g,n} \cdot [\rho_{\text{Aff}}] \subset \chi_{g,n}(\text{Aff}(\mathbb{C}))$  is equivalent to the finiteness of the orbit  $\hat{\Gamma}_{g,n} \cdot [\rho_{\text{Aff}}] \subset \chi_{g,n}(\text{Aff}(\mathbb{C}))$ . Since  $\rho_{\text{Aff}}$  has non abelian image, by the Propositions 2.4.7 and 2.4.5 the finiteness of the latter orbit is equivalent to  $g = 1, n > 0$  and  $[\rho_{\text{Aff}}] \in \hat{\Gamma}_{g,n} \cdot [\rho_{\mu, \mathbf{c}'}]$  for a convenient choice of a non trivial root of unity  $\mu$  and  $\mathbf{c}' = (c'_1, \dots, c'_n) \in \mathbb{C}^n$  with  $\sum_{i=1}^n c'_i = 1$ . Composing with a suitable element of  $B_n$  shows this is also equivalent to  $[\rho_{\text{Aff}}] \in \Gamma_{g,n} \cdot [\rho_{\mu, \mathbf{c}}]$ , for some  $\mathbf{c} \in \mathfrak{S}_n \cdot \mathbf{c}'$ . This proves the equivalence in the statement.

The estimate (2.19) follows from Proposition 2.5.1, Proposition 2.5.2 and Lemma 2.5.5, taken into account that

$$\text{card}\{i \in \llbracket 1, n \rrbracket \mid \rho(\gamma_i) \notin \mathbb{C}^*I_2\} = \text{card}\{i \in \llbracket 1, n \rrbracket \mid \rho_{\text{Aff}}(\gamma_i) \neq \text{id}\} = \text{card}\{i \in \llbracket 1, n \rrbracket \mid c_i \neq 0\}.$$

□

## 2.5.4 Further remarks and open questions

The obvious general and largely open problem at hand is the following.

**Problem 3.** *Let  $g, n \geq 0$  be integers and let  $G$  be an algebraic subgroup of  $\text{GL}_r\mathbb{C}$  which is not finite. Characterize all finite orbits in  $\chi_{g,n}(G)$  for the action of the mapping class group  $\Gamma_{g,n}$ .*

This problem is interesting from the point of view of isomonodromic deformations as well as from the point of view of dynamics on character varieties. For some results and open problems on the latter, see for example the survey [Gol06].

A well-known question asked by M. Kisin is closely related to the question of the complete classification of finite mapping class group orbits in  $\text{Hom}(\Lambda_{g,0}, G)$ . Both have recently been answered, for  $g \geq 2$ , in [BKMS17]. The authors show that the only elements  $\rho \in \text{Hom}(\Lambda_{g,0}, G)$  with  $\Gamma_{g,0} \cdot \rho$  finite are the obvious ones: those with  $\text{Im}(\rho)$  finite. This answers Problem 3 for the case of  $n = 0, g \geq 2$  and abelian groups  $G$ .



For  $G = \text{Aff}(\mathbb{C})$ , or when considering only reducible  $[\rho] \in \chi_{g,n}(\text{GL}_2\mathbb{C})$ , Problem 3 is solved by combining [CM16] and our result.

The only other known answers to Problem 3, to the best of the author's knowledge, concern the case  $G = \text{SL}_2\mathbb{C}$ ,  $g = 0$ . Even for this special case, a complete answer is known only for  $n \leq 4$ . It has been established by Lisovyy and Tykhyy in [LT14] and turned out to provide a classification of all algebraic solutions of the sixth Painlevé equation. The latter are due to Boalch, Dubrovin, Mazzocco and others, we refer to [LT14] and the survey articles [Boa07] [Iwa13] for precise references. Some algebraic solutions of  $n$ -Garnier systems with  $n \geq 5$  corresponding to finite mapping class group orbits were found in [Tsu06], [Dia13], [Gir16] and [CM17].

## 2.6 Algebraic logarithmic connections

### 2.6.1 Definition and relation to analytic connections

Let  $M$  be a smooth quasi-projective variety<sup>1</sup> over  $\mathbb{C}$  and let  $D$  be a (possibly empty) reduced normal crossing divisor on  $M$ . Denote by  $D^1, \dots, D^n$  the irreducible components of  $D$ . By definition,  $M$  is a Zariski open subset of a projective variety  $\widehat{M}$ . By Hironaka desingularization, we can and shall moreover assume that  $\widehat{M}$  is smooth, and that  $D_\infty := \widehat{M} \setminus M$  and  $\widehat{D} := \overline{D} + D_\infty$  are normal crossing divisors on  $\widehat{M}$ .

An *algebraic logarithmic connection* of rank  $r$  over  $M$  with polar divisor at most  $D$  is a pair  $(E, \nabla)$ , where  $E \rightarrow X$  is an algebraic vector bundle of rank  $r$  over  $M$ , whose sheaf of sections we shall also denote by  $E$ , and  $\nabla$  is a  $\mathbb{C}$ -linear morphism

$$\nabla : E \rightarrow E \otimes \Omega_M^1(\log D),$$

which satisfies the Leibniz rule

$$\nabla(f \cdot e) = f \cdot \nabla(e) + e \otimes df$$

for any  $f \in \mathcal{O}_M(\Delta)$ ,  $e \in E(\Delta)$ , where  $\Delta \subset M$  is any Zariski open subset and  $\mathcal{O}_M$  denotes the sheaf of regular functions on  $M$ . We say that  $(E, \nabla)$  has *polar divisor*  $D$  if, for any  $i \in \llbracket 1, n \rrbracket$ ,  $\nabla$  does not factor through

$$E \otimes \Omega_M^1(\log(D - D^i)) \hookrightarrow E \otimes \Omega_M^1(\log D).$$

Such an algebraic logarithmic connection  $(E, \nabla)$  will be called *regular* if  $(E, \nabla)|_{M \setminus D}$  is regular in the sense of [Del70, § II.4]. We will only need the following sufficient condition: if  $(E, \nabla)$  is induced, by restriction to  $M$ , from an algebraic logarithmic connection on  $\widehat{M}$  with polar divisor at most  $\widehat{D}$ , then  $(E, \nabla)$  is regular [Del70, Thm. II.4.1]. Note however that regularity does not depend on the choice of a smooth compactification. By [Ser56, Prop. 18] and [Del70, Prop. II.4.4], any analytic logarithmic connection  $(\widehat{E}^{an}, \widehat{\nabla}^{an})$  with polar divisor at most  $\widehat{D}$  on  $\widehat{M}$  (seen as a complex manifold), as defined in § 1.2.1, is obtained by analytification from a unique (up to isomorphism) algebraic logarithmic connection  $(\widehat{E}, \widehat{\nabla})$  with polar divisor at most  $\widehat{D}$  on  $\widehat{M}$ .

An algebraic logarithmic connection  $(E, \nabla)$  is called *flat* if its curvature  $\nabla^2$  is zero.

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<sup>1</sup>Here and throughout, by *quasi-projective variety* we shall mean irreducible and separated quasi-projective variety, unless explicit mention of the contrary.

## 2.6.2 Flat connections and monodromy

Let  $(E, \nabla)$  be a flat algebraic logarithmic connection on  $M$  with polar divisor at most  $D$  as above. Let  $x_0 \in M^0 := M \setminus D$  and let  $f : E_{x_0} \xrightarrow{\sim} \mathbb{C}^r$  be an isomorphism of vector spaces. We define the *monodromy representation*

$$\rho_{\nabla} : \pi_1(M^0, x_0) \rightarrow \mathrm{GL}_r \mathbb{C}$$

of  $(E, \nabla)$  with respect to  $f$  to be the monodromy representation of its analytification  $(E^{an}, \nabla^{an})$  with respect to  $f$ , which in turn is the monodromy representation of  $(E^{an}, \nabla^{an})$  as defined in § [1.2.2](#), conjugated by  $f$ .

We shall briefly review its construction for the case where  $M = C$  is a smooth projective curve (a compact Riemann surface), while introducing the monodromy with respect to a Teichmüller structure. Later on, this will allow us to compare monodromy representations for different but homeomorphic curves. Let  $C$  be a smooth projective curve, and let  $D = \{x_1, \dots, x_n\}$  be a set of  $n \geq 0$  distinct points on  $C$ . Denote  $C^0 := C \setminus D$ . Let

$$\varphi : (\Sigma_g, Y^n) \xrightarrow{\sim} (C, D)$$

be a homeomorphism (with respect to the Euclidian topology on  $C$ ) and denote  $x_0 := \varphi(y_0)$ . Since  $C$  is of complex dimension one, any logarithmic connection over  $C$  is automatically flat. Moreover, since  $C$  is projective, any analytic logarithmic connection over  $C$  is the analytification of a unique algebraic logarithmic connection over  $C$ . Let  $(E, \nabla)$  be an algebraic logarithmic connection of rank  $r$  over  $C$  with polar divisor at most  $D$ . Since  $(E, \nabla)$  and  $(E^{an}, \nabla^{an})$  are flat,  $\mathcal{S} := \ker(\nabla^{an}|_{C^0})$  is a locally constant sheaf of rank  $r$  over  $C^0$ . For any path  $\gamma : [0, 1] \rightarrow \Sigma_g \setminus Y^n$ , the pull back  $(\varphi \circ \gamma)^* \mathcal{S}$  is locally constant and thus isomorphic to a constant sheaf. Hence  $\gamma$  defines an isomorphism  $\gamma(\mathcal{S}) : \mathcal{S}_{\gamma(1)} \rightarrow \mathcal{S}_{\gamma(0)}$ . This isomorphism is invariant by homotopy relative to  $\{\gamma(0), \gamma(1)\}$  and satisfies  $\gamma_1 \cdot \gamma_2(\mathcal{S}) = \gamma_1(\mathcal{S}) \circ \gamma_2(\mathcal{S})$  for any pair of paths  $(\gamma_1, \gamma_2)$ . Using the canonical identification  $\mathcal{S}_{x_0} = E_{x_0}$ , we may set  $\rho_{\nabla}(\gamma) = f \circ \gamma(\mathcal{S}) \circ f^{-1}$  for closed paths with end point  $x_0$ , and obtain a representation

$$\rho_{\nabla} : \Lambda_{g,n} = \pi_1(\Sigma_g \setminus Y^n, y_0) \rightarrow \mathrm{GL}_r \mathbb{C},$$

the *monodromy representation of  $(E, \nabla)$  with respect to  $\varphi$  and  $f$* . Note that its conjugacy class

$$[\rho_{\nabla}] \in \chi_{g,n}(\mathrm{GL}_r \mathbb{C}) = \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_r \mathbb{C}) / \mathrm{GL}_r \mathbb{C},$$

does not depend on the choice of  $f$ . We refer to  $[\rho_{\nabla}]$  as the *monodromy of  $(E, \nabla)$  with respect to  $\varphi$* . Note that any representative of the conjugacy class  $[\rho_{\nabla}]$  is a monodromy representation of  $(E, \nabla)$  with respect to  $\varphi$  and some choice of  $f$ .

## 2.6.3 Algebraic logarithmic Riemann-Hilbert correspondence

Let us briefly recall some notions and results from [\[Cou17\]](#), allowing to construct algebraic logarithmic connections from monodromy representations.

Denote by  $\mathbb{D}$  the unit disc around 0 in the complex line and denote by  $\mathbb{V}$  the trivial vector bundle of rank  $r$  over  $\mathbb{D}$ . Its sheaf of holomorphic sections shall be denoted by  $\mathcal{V} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{D}}$ . A (logarithmic) *transversal model* is an analytic logarithmic connection  $(\mathbb{V}, \xi)$  over  $\mathbb{D}$  with polar divisor at most  $\{0\}$ . It is called a *mild transversal model* if any automorphism of the locally constant sheaf  $\ker(\xi|_{\mathbb{D} \setminus \{0\}})$  is obtained by the restriction to  $\mathbb{D} \setminus \{0\}$  of an automorphism of the sheaf  $\mathcal{V}$ . Let us recall some examples.

- If  $(\mathbb{V}, \xi)$  is a model such that its monodromy admits only one Jordan block for each eigenvalue, then  $(\mathbb{V}, \xi)$  is mild.
- If  $(\mathbb{V}, \xi)$  is a *Deligne model*, i.e., the real parts of the eigenvalues of its residue take values in  $[0, 1)$ , then  $(\mathbb{V}, \xi)$  is mild.
- If  $(\mathbb{V}, \xi)$  is *resonant* (its residue admits two eigenvalues that differ by a non-zero integer) and has diagonalizable monodromy, then  $(\mathbb{V}, \xi)$  is not mild.

The isomorphism class of a transversal model is called a *transversal type*. Accordingly, a *mild transversal type* is the transversal type of a mild transversal model.

Let  $M$  be a smooth quasi-projective complex variety, and let  $D \subset M$  be a smooth (non-crossing) divisor on  $M$ . Denote  $(D^i)_{i \in I}$  the irreducible components of  $D$ . Let

$$\rho \in \text{Hom}(\pi_1(M \setminus D, x_0), \text{GL}_r(\mathbb{C}))$$

be a representation and  $\mathcal{L}$  be the locally constant sheaf over the complex manifold  $M \setminus D$ , which has monodromy representation  $\rho$ . For each  $i \in I$ , choose a holomorphic embedding  $f_i : \mathbb{D} \hookrightarrow M$  such that  $f_i(\mathbb{D})$  intersects  $D^i$  transversely exactly once, at  $f_i(0)$ . We say that a transversal model  $(\mathbb{V}, \xi_i)$  is *compatible* with  $\rho$  at  $D^i$  if its monodromy representation with respect to some  $\varepsilon \in \mathbb{D} \setminus \{0\}$  is isomorphic to the one of  $f_i^* \mathcal{L}$ . This is a well defined notion, independent of the choice of  $f_i$ . By isomorphism invariance, this adapts to a notion of *compatible transversal type*. Compatible mild transversal models always exist, e.g. one can choose a Deligne model.

Assume we have a flat algebraic logarithmic connection  $(E, \nabla)$  over  $M$ , with polar divisor at most  $D$ . By [Cou17, Prop. 3.2.1], the transversal type defined by  $f_i^* \nabla^{an}$  is independent of the choice of  $f_i$ , it depends only of  $D^i$  and  $(E, \nabla)$ . It is called the *transversal type of  $(E, \nabla)$  at  $D^i$* . The connection  $(E, \nabla)$  is said to be *mild* if for every component  $D^i$ , the transversal type of  $(E, \nabla)$  at  $D^i$  is mild.

**Theorem 2.6.1** (Logarithmic Riemann-Hilbert). *Let  $M$  be a smooth quasi-projective complex variety, let  $D \subset M$  be a smooth divisor and let  $\rho : \pi_1(M \setminus D, x_0) \rightarrow \text{GL}_r(\mathbb{C})$  be a representation. Denote  $(D^i)_{i \in I}$  the irreducible components of  $D$ . For each  $i \in I$ , let  $(\mathbb{V}, \xi_i)$  be a mild transversal model compatible with  $\rho$  at  $D^i$ .*

*There exists a flat algebraic logarithmic connection  $(E, \nabla)$  over  $M$  with polar divisor at most  $D$  such that*

- $(E, \nabla)$  is regular,
- the monodromy of  $(E, \nabla)$  is given by  $[\rho]$  and
- for each  $i \in I$ , the transversal type of  $\nabla$  at  $D^i$  is given by  $(\mathbb{V}, \xi_i)$ .

*Moreover, let  $(E, \nabla)$  and  $(E', \nabla')$  be two such connections. Choose isomorphisms  $f : E_{x_0} \xrightarrow{\sim} \mathbb{C}^r$  and  $f' : E'_{x_0} \xrightarrow{\sim} \mathbb{C}^r$  such that the monodromy representations of  $(E, \nabla)$  and  $(E', \nabla')$  with respect to  $f$  and  $f'$  are both given by  $\rho$ . Then there exists a unique isomorphism  $\Psi : (E', \nabla') \xrightarrow{\sim} (E, \nabla)$  such that  $f^{-1} \circ \Psi \circ f'^{-1} = I_r$ .*

*Proof.* Let  $\widehat{M}$  be smooth projective variety containing  $M$  as a Zariski open subset. Denote by  $\widehat{D}^j, j \in J$ , the irreducible components of  $\widehat{M} \setminus M$  and by  $\widehat{D}^i$  the Zariski closure of  $D^i$  in  $\widehat{M}$  for each  $i \in I$ . By Hironaka desingularization, we may suppose that  $\widehat{D} := \sum_{i \in I \cup J} \widehat{D}^i$  is a normal crossing divisor. Moreover, we may assume that  $\sum_{i \in I} \widehat{D}^i$  is non-crossing. Since  $M \setminus D = \widehat{M} \setminus \widehat{D}$ ,  $\rho$  defines  $\widehat{\rho} = \rho \in \text{Hom}(\pi_1(\widehat{X} \setminus \widehat{D}), \text{GL}_r(\mathbb{C}))$ . For each  $j \in J$ , choose a Deligne model  $(\mathbb{V}, \xi_j)$  on  $(\mathbb{D}, 0)$  compatible with  $\widehat{\rho}$  at  $\widehat{D}^j$ . According to [Cou17, §3.3, Thm. 6], there exists an analytic logarithmic connection

$(\widehat{E}^{\text{an}}, \widehat{\nabla}^{\text{an}})$  over  $\widehat{X}$  with polar divisor  $\widehat{D}$ , unique up to isomorphism, inducing  $[\widehat{\rho}]$  and all the chosen transversal models.

Since  $\widehat{M}$  is projective, by GAGA [Ser56, Prop. 18] this connection is the analytification of a unique (up to isomorphism) algebraic logarithmic connection  $(\widehat{E}, \widehat{\nabla})$  on  $\widehat{M}$ . Denote by  $(E, \nabla) := (\widehat{E}, \widehat{\nabla})|_M$  its restriction to  $M$ , which is a regular algebraic logarithmic connection. It remains to show uniqueness. Again from [Cou17, Th. 6], we know that  $(E^{\text{an}}, \nabla^{\text{an}})$  is unique up to analytic isomorphism, and that this isomorphism is moreover uniquely determined by its restriction to the fibers over  $x_0$ . Yet any analytic isomorphism between regular algebraic logarithmic connections  $(E, \nabla), (E', \nabla')$  over  $M$  is algebraic, for the isomorphism can be seen as a global horizontal section of  $(E, \nabla) \otimes (E', \nabla')^\vee$ , which is regular by [Del70, Prop. 4.6].  $\square$

## 2.6.4 Algebraic isomonodromic deformations

Let  $g, n$  be non-negative integers. Let  $(C, D)$  be a  $n$ -pointed genus- $g$  curve, i.e.,  $C$  is a smooth projective (complex, irreducible, separated) curve and  $D$  is a reduced divisor of degree  $n$  on  $C$ . Henceforth, we shall assume

$$2g - 2 + n > 0, \quad (2.20)$$

i.e.,  $(C, D)$  is *stable*.

**Definition 2.6.2** (Families of pointed curves). *An algebraic family of  $n$ -pointed genus- $g$  curves is a pair*

$$\mathcal{F} = (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}),$$

where  $\kappa : \mathcal{C} \rightarrow \mathcal{T}$  is a proper surjective smooth morphism of (complex, irreducible, separated) smooth quasi-projective varieties, whose fibers are genus- $g$  curves and  $\mathcal{D} = \sum_{i=1}^n \mathcal{D}^i$  is a smooth reduced divisor on  $\mathcal{C}$  given by pairwise disjoint sections  $\sigma_1, \dots, \sigma_n$  of  $\kappa$  with  $\sigma_i(\mathcal{T}) = \mathcal{D}^i$ . We denote, for any  $t \in \mathcal{T}$ ,  $\mathcal{C}_t := \kappa^{-1}(t)$  and  $\mathcal{D}_t := \sum_{i=1}^n \sigma_i(t)$ .

An algebraic family of  $n$ -pointed genus- $g$  curves with central fiber  $(C, D)$  is a tuple

$$\mathcal{F}_{(C,D)} = (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}, t_0, \psi),$$

where  $\kappa$  and  $\mathcal{D}$  are as above and  $\psi : (C, D) \xrightarrow{\sim} (\mathcal{C}_{t_0}, \mathcal{D}_{t_0})$  is an isomorphism of pointed curves.

Let  $(C, D)$  be as above and let  $(E, \nabla_0)$  be a logarithmic connection over  $C$  with polar divisor  $D$ .

**Definition 2.6.3** (Isomonodromic deformations). *An algebraic isomonodromic deformation of  $(C, E, \nabla_0)$  is a tuple  $\mathcal{I}_{(C,E,\nabla_0)} = (\mathcal{F}_{(C,D)}, \mathcal{E}, \nabla, \Psi)$ , where*

- $\mathcal{F}_{(C,D)} = (\kappa : X \rightarrow T, D, t_0, \psi)$  is an algebraic family of  $n$ -pointed genus- $g$  curves with central fiber  $(C, D)$  as above,
- $(\mathcal{E}, \nabla)$  is a flat regular algebraic logarithmic connection over  $\mathcal{C}$  with polar divisor  $\mathcal{D}$  and
- $\Psi : (E, \nabla_0) \rightarrow \psi^*(\mathcal{E}, \nabla)|_{\mathcal{C}_{t_0}}$  is an isomorphism of algebraic logarithmic connections over  $C$ .

If moreover  $\mathcal{F}_{(C,D)}$  is a Kuranishi family (see below), then  $\mathcal{I}_{(C,E,\nabla_0)}$  will be called *universal*.

For a detailed exposition on algebraic and analytic Kuranishi families, we refer to [ACG11, Chap. 15]. For convenience of the reader and in order to introduce some notation, we summarize some facts that will be relevant in the sequel. An algebraic *Kuranishi family* with central fiber  $(C, D)$  is by definition a certain universal object of algebraic deformations of  $(C, D)$ . Any stable  $n$ -pointed genus- $g$  curve  $(C, D)$  admits an algebraic Kuranishi family, and this remains true if one imposes certain

additional constraints on the Kuranishi family, such as, as in our case, for its parameter space to be a smooth irreducible quasi-projective variety. The universal property then reads as follows (see [\[ACG11\]](#) Rem. 6.9, p. 208]):

**Proposition 2.6.4** (Universal property of Kuranishi families). *Let  $\mathcal{F}_{(C,D)}$  and  $\mathcal{F}'_{(C,D)}$  be two algebraic families with central fiber  $(C, D)$  as in Definition [2.6.2](#). Assume that  $\mathcal{F}_{(C,D)}$  is Kuranishi. Then there exist*

- an étale base change  $p : (\mathcal{T}'', t''_0) \rightarrow (\mathcal{T}', t'_0)$ ; denote  $\mathcal{F}''_{(C,D)} := p^* \mathcal{F}'_{(C,D)}$ ;
- a morphism  $q : (\mathcal{T}'', t''_0) \rightarrow (\mathcal{T}, t_0)$  and
- a unique isomorphism  $\mathfrak{f} : \mathcal{F}''_{(C,D)} \xrightarrow{\sim} q^* \mathcal{F}_{(C,D)}$  projecting to the identity on  $\mathcal{T}''$  and inducing the identity on the central fiber.

One can, and we shall, moreover consider only Kuranishi families that are Kuranishi at any parameter, i.e., for any  $t_1 \in \mathcal{T}$ , the naturally associated family  $\mathcal{F}_{(\mathcal{C}_{t_1}, \mathcal{D}_{t_1})}$  with central fiber  $\kappa^{-1}(t_1)$  is also Kuranishi.

If  $\mathcal{F}_{(C,D)}$  is an algebraic Kuranishi family in the above sense, then the underlying analytic family  $\mathcal{F}_{(C,D)}^{an}$  of compact Riemann surfaces is an analytic Kuranishi family. Moreover, there exists a Euclidian neighborhood  $U$  of  $t_0$  in  $\mathcal{T}$  such that  $\mathcal{F}_U := \mathcal{F}^{an}|_U$  can be endowed with a Teichmüller structure

$$\Phi : (\Sigma_g, Y^n) \times B \rightarrow (\mathcal{C}_U, \mathcal{D}_U)$$

satisfying  $\kappa \circ \Phi = \text{pr}_2$  such that the Teichmüller classifying map

$$\text{class}^+(\mathcal{F}^+) : \begin{cases} U & \rightarrow & \mathcal{T}_{g,n} \\ t & \mapsto & [\mathcal{C}_t, \mathcal{D}_t, \Phi_t] \end{cases}$$

with  $\mathcal{F}^+ := (\mathcal{F}_U, \Phi)$  is a local isomorphism. The universal Teichmüller curve over  $\mathcal{T}_{g,n}$  can actually be constructed by gluing germs of analytic Kuranishi families with Teichmüller structure [\[AC09\]](#). Unless  $(g, n) = (0, 3)$ , in which case  $\mathcal{M}_{g,n}$  is reduced to a point, there is no universal curve over  $\mathcal{M}_{g,n}$ . Algebraic Kuranishi families are however a convenient substitute.

Let  $\mathcal{F}_{(C,D)}$  be an algebraic family as in Definition [2.6.2](#). Assume we have a labelling  $\mathbf{x}$  of  $D$ , i.e.,  $\mathbf{x} = (x_i)_{i \in \llbracket 1, n \rrbracket} \in C^n$  and  $D = \sum_{i=1}^n x_i$ . Then there is a well defined labelling  $\mathbf{D} = (\mathcal{D}^i)_{i \in \llbracket 1, n \rrbracket}$  of  $\mathcal{D}$  defined by  $\mathcal{D} = \sum_{i=1}^n \mathcal{D}^i$  and  $\psi(x_i) \in \mathcal{D}^i$  for all  $i \in \llbracket 1, n \rrbracket$ , yielding a well defined classifying map

$$\text{class}(\mathcal{F}) : \begin{cases} \mathcal{T} & \rightarrow & \mathcal{M}_{g,n} \\ t & \mapsto & [\mathcal{C}_t, \mathbf{D}_t] \end{cases} ,$$

which is a morphism of quasi-projective varieties with respect to the natural structure of quasi-projective variety on  $\mathcal{M}_{g,n}$ . If  $\mathcal{F}_{(C,D)}$  is Kuranishi, then for any labelling, the classifying map  $\text{class}(\mathcal{F})$  is dominant and has finite fibers.

Note that as a contrast to the analytic category, a universal algebraic isomonodromic deformation of  $(C, E, \nabla_0)$  does not need to exist; its existence is precisely the subject of Theorem [2.1.1](#). The reason is that we cannot achieve contractible parameter spaces of algebraic Kuranishi families by restricting to Zariski-open subsets, and we therefore have no canonical way of associating a (monodromy)-representation of  $\pi_1(C \setminus \mathcal{D})$  to a representation of  $\pi_1(C \setminus D)$ . When a universal algebraic isomonodromic deformation of  $(C, E, \nabla_0)$  does exist, the germ at the central parameter of its analytification is clearly uniquely isomorphic to the germ of the universal analytic isomonodromic deformation of  $(C, E, \nabla_0)^{an}$  (see [§ 1.3.2](#)). Remaining in the algebraic category, a universal property can be formulated as follows (the proof will be given in [§ 2.8.4](#)).

**Proposition 2.6.5** (Universal property of universal algebraic isomonodromic deformations). *Let  $\mathcal{I}_{(C,E,\nabla_0)}$  and  $\mathcal{I}'_{(C,E,\nabla_0)}$  be algebraic isomonodromic deformations of  $(C, E, \nabla_0)$  as above. Assume that  $\mathcal{I}_{(C,E,\nabla_0)}$  is universal, that  $(E, \nabla_0)$  is mild and the monodromy of  $(E, \nabla_0)$  is irreducible. Then*

- *up to replacing  $\mathcal{I}'_{(C,E,\nabla_0)}$  by  $p^*\mathcal{I}'_{(C,E,\nabla_0)}$ , where  $p : (\mathcal{T}'', t''_0) \rightarrow (\mathcal{T}', t'_0)$  is a convenient étale base change, and*
- *up to replacing the underlying connection  $(\mathcal{E}' \rightarrow \mathcal{C}', \nabla')$  by  $(\mathcal{E}', \nabla') \otimes \kappa'^*(\mathcal{L}, \xi)$ , where  $(\mathcal{L}, \xi)$  is a regular flat algebraic connection of rank 1 over  $\mathcal{T}'$  with empty polar divisor, and*
- *modifying  $\Psi'$  accordingly,*

*there exists a morphism  $q : (\mathcal{T}', t'_0) \rightarrow (\mathcal{T}, t_0)$  and a unique isomorphism  $\mathfrak{f} : \mathcal{I}'_{(C,E,\nabla_0)} \xrightarrow{\sim} q^*\mathcal{I}_{(C,E,\nabla_0)}$  projecting to the identity on  $\mathcal{T}'$  and inducing the identity on the central fiber.*

## 2.7 The monodromy of the monodromy

In this section, we introduce the so-called group of mapping classes of a family of curves, which is the image of a canonical morphism from the fundamental group of the parameter space of the family to the fundamental group of the central fiber. For an isomonodromic deformation, the action on the monodromy representation of the initial connection by the group of mapping classes of the underlying family of curves corresponds to the monodromy of the monodromy representation. Under suitable conditions, this group can be translated into a subgroup of  $\Gamma_{g,n}$ . We shall moreover see that up to an étale base change, any algebraic family of pointed curves can be endowed with a section avoiding the punctures. The existence of such a *base point section* allows us to decompose the fundamental group of the total space of the family of curves into an semi-direct product of the fundamental groups of the central fiber and the parameter space.

### 2.7.1 Mapping classes of the central fiber

As usual, let  $(C, D)$  be a stable  $n$ -pointed genus- $g$  curve. Let  $\mathcal{F}_{(C,D)}$  be an algebraic family as in Definition 2.6.2, with parameter space  $(\mathcal{T}, t_0)$ . Let  $\beta : [0, 1] \rightarrow \mathcal{T}$  be a closed path with end point  $t_0$ , i.e., a continuous map such that  $\beta(0) = \beta(1) = t_0$ . By Ehresmann's Theorem [Voi03, Thm. 9.3] [Hus94, Cor. 10.3], the pullback fibration  $\beta^*(\mathcal{C}, \mathcal{D}) \rightarrow [0, 1]$  possesses a topological trivialization  $\Phi : (C, D) \times [0, 1] \xrightarrow{\sim} \beta^*(\mathcal{C}, \mathcal{D})$ . For  $s \in [0, 1]$ , we denote

$$\Phi_s := \Phi|_{(C,D) \times \{s\}}$$

and deduce a homeomorphism from the central fiber seen over  $\{1\}$  to the central fiber seen over  $\{0\}$  given by

$$\psi^{-1} \circ \Phi_0 \circ \Phi_1^{-1} \circ \psi : (C, D) \xrightarrow{\sim} (C, D).$$

Its isotopy class shall be called the *mapping class associated to  $\beta$  and  $\mathcal{F}_{(C,D)}$*  and denoted

$$\text{map}_{\mathcal{F}_{(C,D)}}(\beta).$$

**Lemma 2.7.1.** *The mapping class  $\text{map}_{\mathcal{F}_{(C,D)}}(\beta)$  is well defined, i.e., it does not depend on the choice of a trivialization  $\Phi$ . Moreover,  $\text{map}_{\mathcal{F}_{(C,D)}}(\beta)$  only depends on the homotopy class of  $\beta$ .*

*Proof.* For fixed  $\beta$ , take two trivializations  $\Phi, \tilde{\Phi} : (C, D) \times [0, 1] \xrightarrow{\sim} \beta^*(\mathcal{C}, \mathcal{D})$ . The family  $\tilde{\Phi}_0 \circ \tilde{\Phi}_s^{-1} \circ \Phi_s \circ \Phi_1^{-1}$  gives an isotopy from  $\Phi_0 \circ \Phi_1^{-1}$  to  $\tilde{\Phi}_0 \circ \tilde{\Phi}_1^{-1}$ .

Consider now two paths  $\beta_1$  and  $\beta_2$  that are homotopic relative to their endpoints. By definition, there exists a continuous map  $\theta : \overline{\mathbb{D}} \rightarrow \mathcal{T}$ , where  $\overline{\mathbb{D}}$  denotes the closed unit disc, such that  $\beta_2(s) = \theta(e^{i\pi(1+s)})$  and  $\beta_1(s) = \theta(e^{i\pi(1-s)})$ . Since  $\overline{\mathbb{D}}$  is contractible, by Ehresmann's Theorem, there is a trivialization  $\Phi$  of  $\theta^*(\mathcal{C}, \mathcal{D})$ . It induces trivializations  $\Phi^i$  of  $\beta_i^*(\mathcal{C}, \mathcal{D})$  for  $i = 1, 2$ . Since they are both induced by  $\Phi$ , we have  $\Phi_0^1 = \Phi_0^2 = \Phi_{-1}$  and  $\Phi_1^1 = \Phi_1^2 = \Phi_1$ .  $\square$

**Proposition 2.7.2.** *Let  $\mathcal{F}_{(C,D)} = (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}, t_0, \psi)$  be an algebraic family as in Definition 2.6.2. Assume that none of the fibers  $(\mathcal{C}_t, \mathcal{D}_t)$  has exceptional automorphisms. Let  $\mathbf{x}$  be a labelling of  $D$  and denote  $\text{cl} : \mathcal{T} \rightarrow \mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}$  the corestriction of the induced classifying map  $\text{class}(\mathcal{F})$  (see § 2.2.3 and § 2.6.4). Let  $\varphi : (\Sigma_g, y^n) \xrightarrow{\sim} (C, \mathbf{x})$  be an orientation preserving homeomorphism and denote by  $\hat{\star} := [C, D, \varphi]$  the corresponding point in  $\mathcal{T}_{g,n}$ . Then for all  $\beta \in \pi_1(\mathcal{T}, t_0)$ , the following equation holds in  $\Gamma_{g,n}/K_{g,n}$ :*

$$\varphi^{-1} \circ \text{map}_{\mathcal{F}_{(C,D)}}(\beta) \circ \varphi = \text{taut}_{\hat{\star}}(\text{cl}_{\star}\beta),$$

where  $\text{taut}_{\hat{\star}}$  is the tautological morphism  $\text{taut}_{\hat{\star}} : \pi_1(\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}, \hat{\star}) \rightarrow \Gamma_{g,n}/K_{g,n}$  (see (2.6)) and  $\star := [C, \mathbf{x}] \in \mathcal{M}_{g,n}$ .

*Proof.* Denote  $\mathcal{F}_{g,n}^+ = (\mathcal{F}_{g,n}, \Phi_{g,n})$  the universal Teichmüller curve  $\mathcal{F}_{g,n} = (\kappa_{g,n} : \mathcal{X} \rightarrow \mathcal{T}_{g,n}, \mathcal{Y})$  endowed with the Teichmüller structure  $\Phi_{g,n} : (\Sigma_g, Y^n) \times \mathcal{T}_{g,n} \xrightarrow{\sim} (\mathcal{X}, \mathcal{Y})$ . For any point  $t \in \mathcal{T}_{g,n}$ , we shall denote

$$\Phi_t^{g,n} := \Phi_{g,n}|_{(\Sigma_g, Y^n) \times \{t\}} : (\Sigma_g, Y^n) \times \{t\} \xrightarrow{\sim} (\mathcal{X}_t, \mathcal{Y}_t).$$

Let  $p : (\tilde{\mathcal{T}}, \tilde{t}_0) \rightarrow (\mathcal{T}, t_0)$  be a universal cover and consider the pulled-back family

$$\tilde{\mathcal{F}} = (\tilde{\kappa} : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{T}}, \tilde{\mathcal{D}}) := p^*(\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}).$$

Now for any contractible analytic submanifold  $\tilde{\Delta} \subset \tilde{\mathcal{T}}$  containing  $\tilde{t}_0$ , there is a trivialization

$$\Phi : (C, D) \times \tilde{\Delta} \xrightarrow{\sim} (\tilde{\mathcal{C}}, \tilde{\mathcal{D}})|_{\tilde{\Delta}}$$

of  $\tilde{\mathcal{F}}|_{\tilde{\Delta}}$ , unique up to isotopy, such that  $\Phi_{\tilde{t}_0} = \psi$  with respect to the identification  $(\tilde{\mathcal{C}}_{\tilde{t}_0}, \tilde{\mathcal{D}}_{\tilde{t}_0}) = (\mathcal{C}_{t_0}, \mathcal{D}_{t_0})$  provided by pullback. We denote  $\tilde{\Phi} := \Phi \circ (\varphi \times \text{id})$ . Setting  $\tilde{\mathcal{F}}^+ := (\tilde{\mathcal{F}}|_{\tilde{\Delta}}, \tilde{\Phi})$  defines an analytic family of compact Riemann surfaces with marked points and Teichmüller structure. By the universal property of the Teichmüller curve, up to modifying  $\tilde{\Phi}$  by a fiber-preserving isotopy, there is a unique isomorphism  $\mathfrak{f}$  of complex manifolds fitting into the following commutative diagram:

$$\begin{array}{ccc} (\Sigma_g, Y^n) \times \tilde{\Delta} & \xlongequal{\quad} & (\Sigma_g, Y^n) \times \tilde{\Delta} \\ \tilde{\Phi} \downarrow & & \downarrow \text{class}^+(\tilde{\mathcal{F}}^+)^* \Phi_{g,n} \\ (\tilde{\mathcal{C}}, \tilde{\mathcal{D}})|_{\tilde{\Delta}} & \xrightarrow[\sim]{\mathfrak{f}} & \text{class}^+(\tilde{\mathcal{F}}^+)^*(\mathcal{X}, \mathcal{Y}) \\ \tilde{\kappa} \downarrow & & \downarrow \text{class}^+(\tilde{\mathcal{F}}^+)^* \kappa_{g,n} \\ \tilde{\Delta} & \xlongequal{\quad} & \tilde{\Delta}. \end{array}$$

Now let  $[\beta] \in \pi_1(\mathcal{T}, t_0) \setminus \{1\}$  and consider  $\tilde{\beta} : [0, 1] \rightarrow \tilde{\mathcal{T}}$ , the lift of  $\beta$  with starting point  $\tilde{t}_0$ . If the representative  $\beta$  of the homotopy class  $[\beta]$  is well chosen, then  $\tilde{\beta}$  is a  $\mathcal{C}^\infty$ -embedding. By existence of tubular neighborhoods, there is a contractible neighborhood  $\tilde{\Delta}$  of  $\tilde{t}_0$  as above, containing  $\tilde{\beta}$ . We claim that, up to isotopy,

$$\text{map}_{\mathcal{F}_{(C,D)}}(\beta) = \Phi_{\tilde{\beta}(1)}^{-1} \circ \Phi_{\tilde{t}_0}. \quad (2.21)$$

Indeed, we have  $\beta^*(\mathcal{C}, \mathcal{D}) = (p \circ \tilde{\beta})^*(\mathcal{C}, \mathcal{D}) = \tilde{\beta}^*p^*(\mathcal{C}, \mathcal{D}) = \tilde{\beta}^*(\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ . Since moreover  $\tilde{\beta}$  is an embedding, we have  $\beta^*(\mathcal{C}, \mathcal{D}) = (\tilde{\mathcal{C}}, \tilde{\mathcal{D}})|_{\tilde{\beta}([0,1])}$ . The claim then follows from the fact that  $\psi^{-1} \circ \Phi_{t_0}$  is the identity and from the definition of the mapping class.

Denote  $\hat{\beta} := \text{class}^+(\tilde{\mathcal{F}}^+)_* \tilde{\beta}$ , which is a path in  $\mathcal{T}_{g,n}$  with starting point  $\hat{\star}$ . By our definitions, the following diagram is commutative if we remove the dotted arrow.

$$\begin{array}{ccccc}
(\Sigma_g, Y^n) \times \{t_0\} & \xrightarrow[\sim]{\tilde{\Phi}_{t_0}} & (\tilde{\mathcal{C}}_{t_0}, \tilde{\mathcal{D}}_{t_0}) & \equiv & (\tilde{\mathcal{C}}_{\tilde{\beta}(1)}, \tilde{\mathcal{D}}_{\tilde{\beta}(1)}) & \xleftarrow[\sim]{\tilde{\Phi}_{\tilde{\beta}(1)}} & (\Sigma_g, Y^n) \times \{\tilde{\beta}(1)\} \\
\parallel & & \downarrow \tilde{f}_{t_0} \sim & & \downarrow \tilde{f}_{\tilde{\beta}(1)} \sim & & \parallel \\
(\Sigma_g, Y^n) \times \{\hat{\star}\} & \xrightarrow[\Phi_{\hat{\star}}^{g,n}]{\sim} & (\mathcal{X}_{\hat{\star}}, \mathcal{Y}_{\hat{\star}}) & \xrightarrow[\hat{\psi}]{\dots\dots\dots} & (\mathcal{X}_{\hat{\beta}(1)}, \mathcal{Y}_{\hat{\beta}(1)}) & \xleftarrow[\Phi_{\hat{\beta}(1)}^{g,n}]{\sim} & (\Sigma_g, Y^n) \times \{\hat{\beta}(1)\}
\end{array}$$

We define the induced isomorphism of pointed curves

$$\hat{\psi} = \tilde{f}_{\tilde{\beta}(1)} \circ (\tilde{f}_{t_0})^{-1}, \quad (2.22)$$

so that adding the dotted arrow maintains this commutativity. We have

$$\begin{cases} \Phi_{\hat{\star}}^{g,n} & = & \tilde{f}_{t_0} \circ \tilde{\Phi}_{t_0} & = & \tilde{f}_{t_0} \circ \Phi_{t_0} \circ \varphi \\ \Phi_{\hat{\beta}(1)}^{g,n} & = & \tilde{f}_{\tilde{\beta}(1)} \circ \tilde{\Phi}_{\tilde{\beta}(1)} & = & \tilde{f}_{\tilde{\beta}(1)} \circ \Phi_{\tilde{\beta}(1)} \circ \varphi. \end{cases} \quad (2.23)$$

On the other hand,  $\text{cl}_*\beta$  is a closed path in  $\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}$  with end point  $\star$ . By construction, it lifts, with respect to the forgetful map  $\pi_{g,n}$ , to  $\hat{\beta}$ , with  $\hat{\beta}(0) = \hat{\star}$ . By definition of the tautological morphism  $\text{taut}_{\hat{\star}}$ , we thus have, for  $[h] := \text{taut}_{\hat{\star}}(\text{cl}_*\beta) \in \Gamma_{g,n}/K_{g,n}$ :

$$[h] \cdot [\mathcal{X}_{\hat{\star}}, \mathcal{Y}_{\hat{\star}}, \Phi_{\hat{\star}}^{g,n}] = [\mathcal{X}_{\hat{\beta}(1)}, \mathcal{Y}_{\hat{\beta}(1)}, \Phi_{\hat{\beta}(1)}^{g,n}].$$

By the definition of the action of the mapping class group on  $\mathcal{T}_{g,n}$ , we now have

$$[h] \cdot [\mathcal{X}_{\hat{\star}}, \mathcal{Y}_{\hat{\star}}, \Phi_{\hat{\star}}^{g,n}] = [h] \cdot [\mathcal{X}_{\hat{\beta}(1)}, \mathcal{Y}_{\hat{\beta}(1)}, \hat{\psi} \circ \Phi_{\hat{\star}}^{g,n}] = [\mathcal{X}_{\hat{\beta}(1)}, \mathcal{Y}_{\hat{\beta}(1)}, \hat{\psi} \circ \Phi_{\hat{\star}}^{g,n} \circ h^{-1}].$$

Hence there is an element  $[k] \in K_{g,n}$  such that, up to isotopy,

$$\hat{\psi} \circ \Phi_{\hat{\star}}^{g,n} = \Phi_{\hat{\beta}(1)}^{g,n} \circ h \circ k.$$

Combined with (2.22) and (2.23), this implies, up to isotopy,

$$\Phi_{\tilde{\beta}(1)}^{-1} \circ \Phi_{t_0} = \varphi \circ h \circ k \circ \varphi^{-1},$$

which by (2.21) and the definitions of  $h$  and  $k$  yields the desired result.  $\square$

## 2.7.2 Splitting of the fundamental group

**Lemma 2.7.3** (Existence of a base point section). *Let  $\mathcal{F}_{(C,D)} = (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}, t_0, \psi)$  be an algebraic family of pointed curves with central fiber  $(C, D)$  as in Definition 2.6.2. Let  $x_0$  be a point in  $C \setminus D$ . Then there are*



- a Zariski open neighborhood  $U$  of  $t_0$  in  $\mathcal{T}$  and
- a finite étale cover  $p : (\mathcal{T}', t'_0) \rightarrow (U, t_0)$

such that for  $\mathcal{F}'_{(C,D)} = (\kappa' : \mathcal{C}' \rightarrow \mathcal{T}', \mathcal{D}', t'_0, \psi')$ , defined by  $\mathcal{F}'_{(C,D)} := p^* \mathcal{F}_{(C,D)}$ , there exists a section  $\sigma$  of  $\kappa'$  with values in  $\mathcal{C}' \setminus \mathcal{D}'$  such that  $\sigma(t'_0) = \psi'(x_0)$ .

*Proof.* Since  $\mathcal{C}$  is embedded in some projective space  $\mathbb{P}^N$ , by Bertini's Theorem, there exists a hyperplane  $H$  of  $\mathbb{P}^N$  which intersects  $\mathcal{C}_{t_0}$  transversely, is disjoint from  $\mathcal{D}_{t_0}$  and satisfies  $\psi(x_0) \in H$ . Since  $H$  is ample, we have  $\deg(\mathcal{C}_t \cap H) > 0$  for each  $t \in \mathcal{T}$ . In particular,  $H \cap \mathcal{C}_t \neq \emptyset$  for each parameter  $t \in \mathcal{T}$ . By irreducibility of  $\mathcal{T}$ , there exists an irreducible component  $\Delta$  of  $\mathcal{C} \cap H$  such that  $\kappa(\Delta) = \mathcal{T}$  and  $\psi(x_0) \in \Delta$ . Now  $\kappa|_{\Delta} : \Delta \rightarrow \mathcal{T}$  is a connected finite ramified covering. Denote by  $Z_1 \subset \mathcal{T}$  its branching locus. Further, denote by  $Z_2$  the adherence of  $\kappa(\Delta \cap \mathcal{D})$ . By construction,  $Z := Z_1 \cup Z_2$  is a Zariski closed proper subset of  $\mathcal{T}$  not containing  $t_0$ . Denote  $U := \mathcal{T} \setminus Z$  and

$$\mathcal{T}' := \kappa^{-1}(U) \cap \Delta.$$

We now have  $t'_0 := \psi(x_0) \in \mathcal{T}'$  and

$$p := \kappa|_{\mathcal{T}'} : (\mathcal{T}', t'_0) \rightarrow (U, t_0)$$

is a connected finite étale cover. Consider the algebraic family  $\mathcal{F}'_{(C,D)} := p^* \mathcal{F}_{(C,D)}$ . By definition of the pullback, its total space  $\mathcal{C}'$  is given by a fibered product

$$\mathcal{C}' = \{(x, t') \in \mathcal{C}|_{\kappa^{-1}(U)} \times \mathcal{T}' \mid \kappa(x) = p(t')\}$$

and we have  $\kappa' : \mathcal{C}' \rightarrow \mathcal{T}'$ ;  $(x, t') \mapsto t'$ . On the other hand,  $\mathcal{T}'$  is a subset of  $\mathcal{C}|_{\kappa^{-1}(U)}$  by construction and we can define a section  $\sigma$  of  $\kappa'$  by

$$\sigma : \mathcal{T}' \rightarrow \mathcal{C}'; t' \mapsto (t', t').$$

Since moreover  $\mathcal{T}' \cap \mathcal{D} = \emptyset$  by the choice of  $Z_2$ , we have  $\sigma(\mathcal{T}') \cap \mathcal{D}' = \emptyset$ . We conclude by noticing  $\sigma(t'_0) = (\psi(x_0), t'_0) = \psi'(x_0)$ .  $\square$

To fix notations, let us recall the definition of (inner) semi-direct products.

Let  $G$  be a group and  $A$  a subgroup. Assume we have a group  $\tilde{B}$  fitting into a split short exact sequence of groups, as follows.

$$\{1\} \longrightarrow A \longrightarrow G \overset{\sigma}{\longleftarrow} \tilde{B} \longrightarrow \{1\}$$

Assume further that the map  $A \rightarrow G$  in that sequence is defined by the inclusion map. Then  $A$  is a normal subgroup of  $G$ ; for  $B := \sigma(\tilde{B})$  we have a natural morphism  $\eta \in \text{Hom}(B, \text{Aut}(A))$  defined by  $\eta(b)(a) = b \cdot a \cdot b^{-1}$  for all  $a \in A, b \in B$ ; we have a group  $A \rtimes_{\eta} B$  defined as the set  $A \times B$  endowed with the group law

$$(a, b) \cdot (a', b') = (a \cdot \eta(b)(a'), b \cdot b'),$$

and the natural morphism  $A \rtimes_{\eta} B \rightarrow G$  defined by  $(a, b) \mapsto a \cdot b$  is bijective, allowing us to identify  $G = A \rtimes_{\eta} B$ .

**Lemma 2.7.4** (Splitting). *Let  $\mathcal{F}_{(C,D)} = (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}, t_0, \psi)$  be an algebraic family as in Definition 2.6.2. Let  $\sigma : \mathcal{T} \rightarrow \mathcal{C}$  be a section of  $\kappa$  such that  $\sigma(\mathcal{T}) \subset \mathcal{C}^0 := \mathcal{C} \setminus \mathcal{T}$ . Denote  $\mathcal{C}^0 := \mathcal{C} \setminus \mathcal{D}$  and  $x_0 := \psi^{-1}(\sigma(t_0))$ . Then*

$$\pi_1(\mathcal{C}^0, \sigma(t_0)) = \psi_* \pi_1(\mathcal{C}^0, x_0) \rtimes_{\eta} \sigma_* \pi_1(\mathcal{T}, t_0), \quad (2.24)$$

where for all  $\gamma \in \pi_1(\mathcal{C}^0, x_0)$  and  $\beta \in \pi_1(\mathcal{T}, t_0)$  we have

$$\eta(\sigma_* \beta)(\psi_* \gamma) = \sigma_* \beta \cdot \psi_* \gamma \cdot \sigma_* \beta^{-1}.$$

*Proof.* Since  $\sigma$  takes values in  $\mathcal{C}^0$ , we have a morphism of fundamental groups  $\sigma_* : \pi_1(\mathcal{T}, t_0) \rightarrow \pi_1(\mathcal{C}^0, \sigma(t_0))$ . From the embedding of the central fiber, we get the morphism  $\psi_* : \pi_1(\mathcal{C}^0, x_0) \rightarrow \pi_1(\mathcal{C}^0, \sigma(t_0))$ . Consider now the family of  $n$ -punctured curves given by  $\kappa : \mathcal{C}^0 \rightarrow \mathcal{T}$ . This family is a topologically locally trivial fibration and the fiber over  $t_0$  identifies, via  $\psi$ , with  $\mathcal{C}^0$ . Hence we have a long homotopy exact sequence

$$\cdots \longrightarrow \pi_2(\mathcal{C}^0, \sigma(t_0)) \xrightarrow{\kappa_*} \pi_2(\mathcal{T}, t_0) \longrightarrow \pi_1(\mathcal{C}^0, \sigma(t_0)) \xrightarrow{\psi_*} \pi_1(\mathcal{C}^0, \sigma(t_0)) \xrightarrow{\kappa_*} \pi_1(\mathcal{T}, t_0) \longrightarrow \{1\}.$$

The maps  $\sigma_* : \pi_j(\mathcal{T}, t_0) \rightarrow \pi_j(\mathcal{C}^0, \tau(t_0))$  are sections for the corresponding  $\kappa_*$  and we may derive the following split short exact sequence:

$$\{1\} \longrightarrow \pi_1(\mathcal{C}^0, x_0) \xrightarrow{\psi_*} \pi_1(\mathcal{C}^0, \tau(t_0)) \begin{array}{c} \xleftarrow{\sigma_*} \\ \xrightarrow{\kappa_*} \end{array} \pi_1(\mathcal{T}, t_0) \longrightarrow \{1\}. \quad \square$$

Given a decomposition (2.24), the monodromy representation of the flat connection underlying an isomonodromic deformation can be seen as an extension of the monodromy representation of the initial connection. Its existence and uniqueness will be discussed in § 2.7.4

### 2.7.3 Splitting and the mapping class group

Let  $\mathcal{F}_{(C,D)} = (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}, t_0, \psi)$  be an algebraic family of stable  $n$ -pointed genus- $g$  curves as in Definition 2.6.2. Assume there is a section  $\sigma : \mathcal{T} \rightarrow \mathcal{C}^0 := \mathcal{C} \setminus \mathcal{D}$  of  $\kappa$ . Then we can define an algebraic family of  $n + 1$ -pointed genus- $g$  curves

$$\mathcal{F}_{(C,D^\bullet)}^\bullet := (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}^\bullet, t_0, \psi)$$

by setting  $\mathcal{D}^\bullet := \mathcal{D} + \sigma(\mathcal{T})$  and  $D^\bullet := D + x_0$ , where  $x_0 := \psi^{-1}(\sigma(t_0)) \in \mathcal{C}^0 = \mathcal{C} \setminus \mathcal{D}$ . To a labelling  $\mathbf{x} = (x_1, \dots, x_n)$  of  $D$  we can associate a labelling  $\mathbf{x}^\bullet := (x_1, \dots, x_n, x_0)$  of  $D^\bullet$ . Note that if a fiber of  $\mathcal{F}^\bullet$  has exceptional automorphisms, then the corresponding fiber of  $\mathcal{F}$  also has exceptional automorphisms. If none of the fibers of  $\mathcal{F}^\bullet$  has exceptional automorphisms, we may corestrict the classifying map  $\text{class}(\mathcal{F}^\bullet)$  to obtain a morphism

$$\text{cl}^\bullet : \mathcal{T} \rightarrow \mathcal{M}_{g,n+1} \setminus \mathcal{B}_{g,n+1}.$$

Let  $\varphi : (\Sigma_g, y^n, y_0) \xrightarrow{\sim} (C, \mathbf{x}, x_0)$  be an orientation preserving homeomorphism and denote  $\hat{\star}^\bullet := [C, D^\bullet, \varphi] \in \mathcal{T}_{g,n+1}$  and  $\star^\bullet := [C, \mathbf{x}^\bullet] \in \mathcal{M}_{g,n+1}$ . Note that since we assumed  $2g - 2 + n > 0$ , we have  $K_{g,n+1} = \{1\}$  according to Lemma 2.2.1. We obtain a tautological morphism

$$\text{taut}_{\hat{\star}^\bullet} : \pi_1(\mathcal{M}_{g,n+1} \setminus \mathcal{B}_{g,n+1}, \star^\bullet) \rightarrow \Gamma_{g,n+1}$$

as in § 2.2.3

**Proposition 2.7.5.** *Let  $\mathcal{F}_{(C,D^\bullet)}^\bullet = (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}^\bullet, t_0, \psi)$  be an algebraic family of stable  $n$ -pointed genus- $g$  curves with an additional puncture as above. Assume that none of its fibers admits exceptional automorphisms. Let  $\varphi : (\Sigma_g, y^n, y_0) \xrightarrow{\sim} (C, \mathbf{x}, x_0)$  be an orientation preserving homeomorphism. Then*

$$\pi_1(C^0, \sigma(t_0)) = (\psi \circ \varphi)_* \Lambda_{g,n} \rtimes_{\eta} \sigma_* \pi_1(\mathcal{T}, t_0), \quad (2.25)$$

where for all  $\alpha \in \Lambda_{g,n}$  and  $\beta \in \pi_1(\mathcal{T}, t_0)$ , we have

$$\begin{aligned} \eta(\sigma_* \beta)((\psi \circ \varphi)_* \alpha) &= \sigma_* \beta \cdot (\psi \circ \varphi)_* \alpha \cdot \sigma_* \beta^{-1} \\ &= (\psi \circ \varphi)_* \mathbf{a}(\text{taut}_{\hat{\star}}(\text{cl}_{\star}^\bullet \beta))(\alpha). \end{aligned}$$

Here we adopt the notation above and denote  $\mathbf{a}(h)(\alpha) = h_* \alpha$  for all  $h \in \Gamma_{g,n+1}$  and  $\alpha \in \Lambda_{g,n}$  as in Section [2.2](#).

*Proof.* By Proposition [2.7.2](#), the following equation holds in  $\Gamma_{g,n+1} = \Gamma_{g,n+1}/K_{g,n+1}$  for every  $\beta \in \pi_1(\mathcal{T}, t_0)$ :

$$\varphi^{-1} \circ \text{map}_{\mathcal{F}_{(C,D^\bullet)}^\bullet}(\beta) \circ \varphi = \text{taut}_{\hat{\star}}(\text{cl}_{\star}^\bullet \beta). \quad (2.26)$$

We claim that for any  $\gamma \in \pi_1(C^0, x_0)$  and any  $\beta \in \pi_1(\mathcal{T}, t_0)$ , the following equation holds in  $\pi_1(C^0, \sigma(t_0))$ :

$$\psi_* \text{map}_{\mathcal{F}_{(C,D^\bullet)}^\bullet}(\beta)_* \gamma = \sigma_* \beta \cdot \psi_* \gamma \cdot \sigma_* \beta^{-1}. \quad (2.27)$$

Indeed, let  $\gamma : [0, 1] \rightarrow C^0$  be a closed path with end point  $x_0$ . For any  $s_0 \in [0, 1]$ , we have a closed path  $\gamma_{s_0} := \gamma \times \{s_0\}$  in the product space  $C^0 \times [0, 1]$ . We also have a path  $\theta : [0, 1] \rightarrow C^0 \times [0, 1]; s \mapsto (x_0, s)$ . The path  $\theta \cdot \gamma_1 \cdot \theta^{-1}$  is closed and homotopic to  $\gamma_0$ . Now let  $\beta \in \pi_1(\mathcal{T}, t_0)$  and let  $\Phi : (C^0, x_0) \times [0, 1] \xrightarrow{\sim} \beta^*(C^0, \sigma(\mathcal{T}))$  be a trivialization commuting with the natural projections to  $[0, 1]$ . Define the homeomorphism

$$\tilde{\Phi} := \Phi \circ ((\Phi_1^{-1} \circ \psi) \times \text{id}_{[0,1]}): (C^0, x_0) \times [0, 1] \xrightarrow{\sim} \beta^*(C^0, \sigma(\mathcal{T})),$$

which is another trivialization, satisfying  $\tilde{\Phi}_1 = \psi$  and  $\tilde{\Phi}_0 = \psi_* \text{map}_{\mathcal{F}_{(C,D^\bullet)}^\bullet}(\beta)$ . Since  $\tilde{\Phi}$  is continuous, the closed paths  $\tilde{\Phi}_* \gamma_0$  and  $\tilde{\Phi}_* \theta \cdot \tilde{\Phi}_* \gamma_1 \cdot \tilde{\Phi}_* \theta^{-1}$  are homotopic in  $\beta^*(C^0, \sigma(\mathcal{T}))$ . Considering the natural projection  $\kappa : \beta^*(C^0, \sigma(\mathcal{T})) \rightarrow (C^0, \sigma(\mathcal{T}))$ , we have  $\kappa_* \tilde{\Phi}_* \gamma_0 = \tilde{\Phi}_0_* \gamma$  and  $\kappa_* \tilde{\Phi}_* \gamma_1 = \tilde{\Phi}_1_* \gamma$ . Since moreover  $\kappa_* \tilde{\Phi}_* \theta = \sigma_* \beta$ , we have [\(2.27\)](#).

Since  $\varphi$  is a homeomorphism, the induced map  $\varphi_* : \Lambda_{g,n} \rightarrow \pi_1(C^0, x_0)$  is an isomorphism. The statement then follows from [\(2.26\)](#), [\(2.27\)](#) and the Splitting Lemma [2.7.4](#).  $\square$

## 2.7.4 Extensions of representations

We shall now consider the problem of extending a representation of the fundamental group of a fiber of a family of pointed curves to a representation for the whole family in light of Proposition [2.7.5](#).

**Lemma 2.7.6** (Extension of representations). *Let  $A, B$  be groups and let  $G = A \rtimes_{\eta} B$  be as in § [2.7.2](#). Let  $\rho_A \in \text{Hom}(A, \text{GL}_r \mathbb{C})$  be a representation.*

- *There exists a representation  $\rho \in \text{Hom}(G, \text{GL}_r \mathbb{C})$  such that  $\rho|_A = \rho_A$  if and only if there exists a representation  $\rho_B \in \text{Hom}(B, \text{GL}_r \mathbb{C})$  such that for all  $(a, b) \in A \times B$  we have*

$$\rho_A(b \cdot a \cdot b^{-1}) = \rho_B(b) \cdot \rho_A(a) \cdot \rho_B(b^{-1}).$$

- Let  $\rho, \rho' \in \text{Hom}(G, \text{GL}_r \mathbb{C})$  be representations such that  $\rho|_A = \rho'|_A = \rho_A$ . Assume that  $\rho_A$  is irreducible. Then there exists  $\lambda \in \text{Hom}(B, \mathbb{C}^*)$  such that  $\rho = \lambda \otimes \rho'$ .

The proof of this Lemma is elementary and will be left to the reader. A similar statement can be found in [Cou17] Lem. 1].

Concerning the problem of extending (monodromy)-representations, we begin with the elementary case of non semisimple rank 2 representations.

**Lemma 2.7.7.** *Let  $A, B$  be groups. Let  $\rho_A \in \text{Hom}(A, \text{GL}_2 \mathbb{C})$  be non semisimple. Let  $\theta \in \text{Hom}(B, \text{Aut}(A))$  such that for all  $h \in \text{Im}(\theta)$ , we have  $[\rho_A] = h \cdot [\rho_A] := [\rho_A \circ h^{-1}]$ . Then there exists a representation  $\rho_B \in \text{Hom}(B, \text{GL}_2 \mathbb{C})$  such that*

$$\rho_A(\theta(\beta)^{-1}(\alpha)) = \rho_B(\beta)^{-1} \rho_A(\alpha) \rho_B(\beta) \quad \forall \alpha \in A, \beta \in B.$$

*Proof.* We may assume that  $\rho_A$  takes values in  $\text{Upp}$ , the group of invertible upper triangular matrices of rank 2. By assumption, for each  $\beta \in B$ , there exists a matrix  $M_\beta \in \text{GL}_2 \mathbb{C}$  such that  $\rho_A(\theta(\beta)^{-1}(\bullet)) = M_\beta^{-1} \rho_A(\bullet) M_\beta$ . By Lemma 2.5.4, we may assume  $M_\beta \in \text{Aff}(\mathbb{C})$ , where  $\text{Aff}(\mathbb{C}) := \{(a_{i,j}) \in \text{Upp} \mid a_{2,2} = 1\}$ , which is isomorphic to the affine group of the complex line. If  $\text{Im}(\rho_A) \subset \text{Upp}$  is non abelian, then it has trivial centralizer and the matrices  $M_\beta \in \text{Aff}(\mathbb{C})$  are uniquely defined. Otherwise, we have  $\text{Im}(\rho_A) \subset \{\lambda \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{C}^*, \tau \in \mathbb{C}\}$  and the matrices  $M_\beta$  are uniquely defined if we impose  $M_\beta \in \{\begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \mid \mu \in \mathbb{C}^*\}$ . It is now straightforward to check that given these choices, the map  $\beta \mapsto M_\beta$  is well defined and a morphism of groups.  $\square$

For a similar result for semisimple representations  $\rho_A$  (of arbitrary rank), the group  $B$ , which in our case will be the fundamental group of a parameter space, might have to be modified, in order to take into account the non-unicity of the matrices  $M_\beta$  due to possible permutations of irreducible components.

**Proposition 2.7.8.** *Let  $A$  be a group. Let  $\rho_A \in \text{Hom}(A, \text{GL}_r \mathbb{C})$  be semisimple. Let  $(\mathcal{T}, t_0)$  be a smooth connected quasi-projective variety, and let  $\theta \in \text{Hom}(\pi_1(\mathcal{T}, t_0), \text{Aut}(A))$  such that  $H := \text{Im}(\theta)$  stabilizes  $[\rho_A]$ . Then there is an étale base change  $p : (\mathcal{T}', t'_0) \rightarrow (\mathcal{T}, t_0)$  and a representation  $\rho_B \in \text{Hom}(\pi_1(\mathcal{T}', t'_0), \text{GL}_r \mathbb{C})$  such that*

$$\rho_A(\theta(p_*\beta)^{-1}(\alpha)) = \rho_B(\beta)^{-1} \cdot \rho_A(\alpha) \cdot \rho_B(\beta) \quad \forall \alpha \in A, \beta \in \pi_1(\mathcal{T}', t'_0).$$

*Proof.* Let  $\rho_A = \bigoplus_{i \in I} \rho_A^i$  be a decomposition such that each  $\rho_A^i$  is irreducible. The subgroup

$$\bigcap_{i \in I} \text{Stab}_{\text{Aut}(A)}[\rho_A^i] \subset \text{Stab}_{\text{Aut}(A)}[\rho_A],$$

stabilizing the conjugacy class  $[\rho_A^i]$  for each  $i \in I$ , is of finite index (see for example [Cou17] Lemma 3]). Hence the subgroup  $\tilde{H} := H \cap_{i \in I} \text{Stab}_{\text{Aut}(A)}[\rho_A^i]$  is of finite index in  $H$ . Consider the finite connected unramified covering  $\tilde{p} : (\tilde{\mathcal{T}}, \tilde{t}_0) \rightarrow (\mathcal{T}, t_0)$  characterized by  $\tilde{p}_* \pi_1(\tilde{\mathcal{T}}, \tilde{t}_0) = \theta^{-1}(\tilde{H})$ . Note that  $\tilde{p}$  induces a structure of smooth quasi-projective variety on  $\tilde{\mathcal{T}}$ . Since  $\tilde{H}$  stabilizes  $[\rho_A^i]$ , for every  $h \in \tilde{H}$  and every  $i \in I$ , there is a matrix  $M_h^i \in \text{GL}_{r_i} \mathbb{C}$  such that

$$(M_h^i)^{-1} \cdot \rho_A^i \cdot M_h^i = [h] \cdot \rho_A^i. \tag{2.28}$$

Given  $i$  and  $h$ , the choice of  $M_h^i$  is unique up to an element of the centralizer of  $\rho_A^i$ . Since  $\rho_A^i$  is irreducible, this centralizer is given by the set of scalar matrices. Denote by  $\overline{M}_h^i \in \text{PGL}_{r_i} \mathbb{C}$  the projectivization of  $M_h^i \in \text{GL}_{r_i} \mathbb{C}$ . Then  $\overline{\rho}_B^i : \beta \mapsto \overline{M}_{\theta_* \tilde{p}_* \beta}^i$  is a well defined element of  $\text{Hom}(\pi_1(\tilde{\mathcal{T}}, \tilde{t}_0), \text{PGL}_{r_i} \mathbb{C})$ .

According to the Lifting Theorem [Cou15, Thm. 3.1], there exists a Zariski closed subset  $\tilde{Z}$  of  $\tilde{\mathcal{T}}$  not containing  $\tilde{t}_0$ , a finite morphism of smooth quasi-projective varieties

$$p' : (\tilde{\mathcal{T}}', t'_0) \rightarrow (\tilde{\mathcal{T}} \setminus \tilde{Z}, \tilde{t}_0),$$

étale in a neighborhood  $\mathcal{T}'$  of  $t'_0$ , and a representation  $\rho_B^i \in \text{Hom}(\pi_1(\tilde{\mathcal{T}}', t'_0), \text{GL}_{r_i} \mathbb{C})$  whose projectivization is  $p'^* \overline{\rho_B^i}$ . For a convenient choice of  $p'$ , this property is satisfied for all  $i \in I$  at once. We obtain a representation  $\rho_B := \bigoplus_{i \in I} \rho_B^i$  in  $\text{Hom}(\pi_1(\mathcal{T}', t'_0), \text{GL}_r \mathbb{C})$  satisfying the required properties with respect to  $p := \tilde{p} \circ p'|_{\mathcal{T}'}$ .  $\square$

## 2.8 Algebraization results

We shall now see that Theorem 2.1.1 is a corollary of the juxtaposition of Theorem 2.8.1, showing that our algebraizability criterion for germs of universal isomonodromic deformations is necessary, and Theorem 2.8.2, showing that it is also sufficient.

### 2.8.1 Finiteness is a necessary condition

**Theorem 2.8.1.** *Let  $(C, D)$  be a stable  $n$ -pointed genus  $g$ -curve. Let  $\varphi : (\Sigma_g, Y^n) \xrightarrow{\sim} (C, D)$  be an orientation preserving homeomorphism. Let  $(E, \nabla_0)$  be an algebraic logarithmic connection over  $C$  with polar divisor  $D$  and denote by  $[\rho_{\nabla_0}] \in \chi_{g,n}(\text{GL}_r \mathbb{C})$  its monodromy with respect to  $\varphi$ . Let  $\mathcal{I}_{(C,E,\nabla_0)} = (\mathcal{F}_{(C,D)}, \mathcal{E}, \nabla, \Psi)$  be an algebraic isomonodromic deformation of  $(C, E, \nabla_0)$  with parameter space  $\mathcal{T}$  as in § 2.6.4.*

*Assume that the classifying map  $\text{class}(\mathcal{F}) : \mathcal{T} \rightarrow \mathcal{M}_{g,n}$  is dominant.*

*Then the  $\Gamma_{g,n}$ -orbit of  $[\rho_{\nabla_0}]$  in  $\chi_{g,n}(\text{GL}_r \mathbb{C})$  is finite.*

*Proof.* The orbit  $\Gamma_{g,n} \cdot [\rho_{\nabla_0}]$  does not depend on the choice of  $\varphi$ . Moreover, it is canonically identified, for any  $t_1 \in \mathcal{T}$ , with the orbit  $\Gamma_{g,n} \cdot [\rho_{t_1}]$  of the monodromy of the connection  $(\mathcal{E}, \nabla)$  restricted to the fiber over  $t_1$  of the family  $\mathcal{F}$ . Since  $\text{class}(\mathcal{F})$  is dominant we may assume, without loss of generality, that  $\star := \text{class}(\mathcal{F})(t_0) \in \mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}$ . Moreover, up to restricting  $\mathcal{I}_{(C,E,\nabla_0)}$  to a Zariski open neighborhood  $U$  of  $t_0$  in  $\mathcal{T}$ , we may assume that  $\text{class}(\mathcal{F})(\mathcal{T}) \cap \mathcal{B}_{g,n} = \emptyset$ . Notice that this property, as well as the assumption of  $\text{class}(\mathcal{F})$  being dominant is not altered by finite covers and further excision of strict subvarieties not containing  $t_0$ . According to Lemma 2.7.3, up to such a manipulation, we may assume that  $\mathcal{F}_{(C,D)} = (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}, t_0, \psi)$  admits a section  $\sigma : \mathcal{T} \rightarrow \mathcal{C}$  of  $\kappa$  with values in  $\mathcal{C}^0 := \mathcal{C} \setminus \mathcal{D}$  such that  $\sigma(t_0) = \psi \circ \varphi(y_0)$ . Let  $\rho_{\nabla_0}$  be a representative of  $[\rho_{\nabla_0}]$ . Denote by  $\rho$  the monodromy representation of  $(\mathcal{E}, \nabla)$  with respect to some isomorphism  $\mathcal{E}_{x_0} \simeq \mathbb{C}^r$ , such that the restriction of  $\rho$  to the subgroup  $(\psi \circ \varphi)_* \Lambda_{g,n}$  of  $\pi_1(\mathcal{C}^0, \sigma(t_0))$ , given by the inclusion of the central fiber, is identical to  $(\psi \circ \varphi)_* \rho_{\nabla_0}$ . Such a choice of isomorphism for the fiber over  $x_0$  exists, as implies for example Theorem 2.6.1. According to Proposition 2.7.5, we then have a semi-direct product decomposition

$$\pi_1(\mathcal{C}^0, \sigma(t_0)) = (\psi \circ \varphi)_* \Lambda_{g,n} \rtimes_{\eta} \sigma_* \pi_1(\mathcal{T}, t_0),$$

where we have two different expressions for its structure morphism  $\eta$ , proving that

$$H := \text{taut}_{\star}(\text{cl}_{\star} \pi_1(\mathcal{T}, t_0)) \subset \Gamma_{g,n+1}$$

acts on  $\rho_{\nabla_0} \in \text{Hom}(\Lambda_{g,n}, \text{GL}_r \mathbb{C})$  by conjugation. More precisely, for all  $\alpha \in \Lambda_{g,n}$  and  $[h] \in \text{taut}_{\star}(\text{cl}_{\star} \beta) \in H$ , we have

$$\rho_{\nabla_0}(\mathfrak{a}(h)(\alpha)) = \rho(\sigma_* \beta) \cdot \rho_{\nabla_0}(\alpha) \cdot \rho(\sigma_* \beta^{-1})$$

and in particular  $[h^{-1}] \cdot [\rho_{\nabla_0}] = [\rho_{\nabla_0}]$ . In other words,  $H$  is a subgroup of the stabilizer of  $[\rho_{\nabla_0}]$  in  $\Gamma_{g,n+1}$ . By definition of the mapping class group action, we then have

$$\pi(H) \subset \text{Stab}_{\Gamma_{g,n}}[\rho_{\nabla_0}],$$

where  $\pi : \Gamma_{g,n+1} \rightarrow \Gamma_{g,n}$  is the projection forgetting the marking  $y_0$ . Since the size of the orbit  $\Gamma_{g,n} \cdot [\rho_{\nabla_0}]$  equals the index of  $\text{Stab}_{\Gamma_{g,n}}[\rho_{\nabla_0}]$  in  $\Gamma_{g,n}$ , it now suffices to prove that  $\pi(H)$  has finite index in  $\Gamma_{g,n}$ . Denote by  $q : \Gamma_{g,n} \rightarrow \Gamma_{g,n}/K_{g,n}$  the quotient by the normal subgroup  $K_{g,n}$ , which, by Lemma 2.2.1, has order at most 2. Hence for the indices, we have

$$[\Gamma_{g,n} : \pi(H)] \leq 2 \cdot [\Gamma_{g,n}/K_{g,n} : q(\pi(H))].$$

We have a commutative diagram

$$\begin{array}{ccc} \Gamma_{g,n+1} & \xleftarrow{\text{taut}_{\hat{x}} \circ \text{cl}^*} & \pi_1(\mathcal{T}, t_0) \\ \pi \downarrow & & \downarrow \text{taut}_{\hat{x}} \circ \text{cl}_* \\ \Gamma_{g,n} & \xrightarrow{q} & \Gamma_{g,n}/K_{g,n}, \end{array}$$

where  $\text{cl} : \mathcal{T} \rightarrow \mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}$  denotes the corestriction of  $\text{class}(\mathcal{F})$ . On the other hand, by the dominance assumption and [Deb01, Lemma 4.19], the subgroup  $\text{cl}_* \pi_1(\mathcal{T}, t_0)$  of  $\pi_1(\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}, \star)$  is of finite index. In particular, since the tautological morphism  $\text{taut}_{\hat{x}} : \pi_1(\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}, \star) \rightarrow \Gamma_{g,n}/K_{g,n}$  is onto, the subgroup  $q(\pi(H)) = \text{taut}_{\hat{x}}(\text{cl}_* \pi_1(\mathcal{T}, t_0))$  of  $\Gamma_{g,n}/K_{g,n}$  has finite index.  $\square$

## 2.8.2 Finiteness is a sufficient condition

**Theorem 2.8.2.** *Let  $\mathcal{F}_{(C,D)} = (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}, t_0, \psi)$  be an algebraic family of stable  $n$ -pointed genus- $g$  curves with central fiber  $(C, D)$  as in § 2.6.4. Let  $(E, \nabla_0)$  be an algebraic logarithmic connection over  $C$  with polar divisor  $D$  and denote by  $[\rho_{\nabla_0}] \in \chi_{g,n}(\text{GL}_r \mathbb{C})$  its monodromy with respect to an orientation preserving homeomorphism  $\varphi : (\Sigma_g, Y^n) \xrightarrow{\sim} (C, D)$ . Assume that*

- $(E, \nabla_0)$  is mild,
- $r = 2$  or  $[\rho_{\nabla_0}]$  is semisimple, and
- the  $\Gamma_{g,n}$ -orbit of  $[\rho_{\nabla_0}]$  in  $\chi_{g,n}(\text{GL}_r \mathbb{C})$  is finite.

Then there exist

- an étale base change  $p : (\mathcal{T}', t'_0) \rightarrow (\mathcal{T}, t_0)$  and
- a regular flat algebraic logarithmic connection  $(\mathcal{E}, \nabla)$  over  $p^* \mathcal{C}$  with polar divisor  $p^* \mathcal{D}$ ,

such that  $\psi^*(\mathcal{E}, \nabla)|_{\mathcal{C}'_{t'_0}}$  is isomorphic to  $(E, \nabla_0)$ .

*Proof.* Since  $(C, D)$  is stable by assumption, it only admits a finite number of automorphisms. Let  $x_0 \in C \setminus D$  be a point fixed by no automorphism other than the identity. Up to isotopy, we may assume  $\varphi(y_0) = x_0$ . Let  $\mathbf{x}^\bullet$  be the labelling of  $D^\bullet = D + x_0$  induced by  $\varphi$ . By construction, we have  $\star^\bullet := [C, \mathbf{x}^\bullet] \in \mathcal{M}_{g,n+1} \setminus \mathcal{B}_{g,n+1}$ . Up to an étale base change, we may assume, by Lemma 2.7.3, that there is a section  $\sigma : \mathcal{T} \rightarrow \mathcal{C}$  of  $\kappa$  with values in  $\mathcal{C}^0 := \mathcal{C} \setminus \mathcal{D}$  such that  $\sigma(t_0) = \psi(x_0)$ . With the notation of § 2.7.3, we may consider the family of  $n+1$ -pointed genus- $g$  curves  $\mathcal{F}_{(C,D,\bullet)}^\bullet = (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D} + \sigma(\mathcal{T}), t_0, \psi)$ . According to Proposition 2.7.5, we have a semi-direct product decomposition  $\pi_1(\mathcal{C} \setminus \mathcal{D}, \sigma(t_0)) = (\psi \circ \varphi)_* \Lambda_{g,n} \rtimes_{\eta} \sigma_* \pi_1(\mathcal{T}, t_0)$ , where

$$\eta(\sigma_* \beta)((\psi \circ \varphi)_* \alpha) = \sigma_* \beta \cdot (\psi \circ \varphi)_* \alpha \cdot \sigma_* \beta^{-1} = (\psi \circ \varphi)_* \mathbf{a}(\theta_* \beta)(\alpha)$$

and  $\theta := \text{taut}_{\star^\bullet} \circ \text{cl}^\bullet_* : \pi_1(\mathcal{T}, t_0) \rightarrow \Gamma_{g,n+1}$ .

Since the  $\Gamma_{g,n+1}$ -orbit of  $[\rho_{\nabla_0}]$  in  $\chi_{g,n}(\text{GL}_r\mathbb{C})$  is finite, the stabilizer

$$H := \text{Stab}_{\Gamma_{g,n+1}}[\rho_{\nabla_0}]$$

of the conjugacy class of  $\rho_{\nabla_0}$  under the action of  $\Gamma_{g,n+1}$  has finite index in  $\Gamma_{g,n+1}$ . Since the tautological morphism is onto, the subgroup  $\text{taut}_{\star^\bullet}^{-1}(H)$  of  $\pi_1(\mathcal{M}_{g,n+1} \setminus \mathcal{B}_{g,n+1}, \star^\bullet)$  then has also finite index. In particular, there is a finite connected étale cover  $q : (U, u_0) \rightarrow (\mathcal{M}_{g,n+1} \setminus \mathcal{B}_{g,n+1}, \star^\bullet)$  such that  $\pi_1(U, u_0) = \text{taut}_{\star^\bullet}^{-1}(H)$ . Now consider the fibered product

$$\begin{array}{ccc} (\mathcal{T}', t'_0) & \xrightarrow{p} & (\mathcal{T}, t_0) \\ \downarrow & & \downarrow \text{class}(\mathcal{F}^\bullet) \\ (U, u_0) & \xrightarrow{q} & (\mathcal{M}_{g,n+1}, \star^\bullet). \end{array}$$

We denote the pullback family of curves by  $\mathcal{F}'_{(C,D)} = (\kappa' : \mathcal{C}' \rightarrow \mathcal{T}', \mathcal{D}' + \sigma'(\mathcal{T}'), t'_0, \psi') := p^* \mathcal{F}^\bullet_{(C,D)}$ . We further denote  $\text{cl}' = \text{cl}^\bullet \circ p$ , which is the corestriction of  $\text{class}(\mathcal{F}^\bullet)$ . By construction, the morphism  $\theta' := \theta \circ p = \text{taut}_{\star^\bullet} \circ \text{cl}'_* : \pi_1(\mathcal{T}', t'_0) \rightarrow \Gamma_{g,n+1}$  takes values in  $H$ .

Again up to an étale base change of  $(\mathcal{T}', t'_0)$ , by Proposition 2.7.8 and Lemma 2.7.7, there is a representation  $\rho_B \in \text{Hom}(\pi_1(\mathcal{T}', t'_0), \text{GL}_r\mathbb{C})$  such that for all  $\beta \in \pi_1(\mathcal{T}', t'_0)$ ,  $\alpha \in \Lambda_{g,n}$ , we have

$$([\theta'_* \beta]^{-1} \cdot \rho_{\nabla_0})(\alpha) = \rho_B(\beta) \cdot \rho_{\nabla_0}(\alpha) \cdot \rho_B(\beta^{-1}).$$

Since by definition  $([\theta'_* \beta]^{-1} \cdot \rho_{\nabla_0})(\alpha) = \rho_{\nabla_0}(\mathbf{a}(\theta'_* \beta)(\alpha))$ , we obtain a well defined representation

$$\rho : \begin{cases} \pi_1(\mathcal{C}' \setminus \mathcal{D}', \sigma'(t'_0)) & \rightarrow \text{GL}_r\mathbb{C} \\ (\psi' \circ \varphi)_* \alpha \cdot \sigma'_* \beta & \mapsto \rho_{\nabla_0}(\alpha) \cdot \rho_B(\beta) \end{cases}$$

(see Lemma 2.7.6) with respect to the semi-direct product decomposition  $\pi_1(\mathcal{C}' \setminus \mathcal{D}', \sigma'(t'_0)) = (\psi' \circ \varphi)_* \Lambda_{g,n} \rtimes_{\eta} \sigma'_* \pi_1(\mathcal{T}', t'_0)$ . By construction,  $\rho$  extends  $\rho_{\nabla_0}$ . We conclude by the logarithmic Riemann-Hilbert correspondence (see Theorem 2.6.1).  $\square$

### 2.8.3 Proof of the main result

The two previous theorems imply Theorem 2.1.1 stated in the introduction:

*Proof of Theorem 2.1.1.* Let us first prove the implication (2.1.1)  $\Rightarrow$  (2.1.1). Let  $\mathcal{I}_{(C,E,\nabla_0)}$ , given by  $(\mathcal{F}_{(C,D)}, E, \nabla, \Psi)$ , be a universal algebraic isomonodromic deformation of  $(C, E, \nabla_0)$  as in § 2.6.4. Then by definition, the family  $\mathcal{F}_{(C,D)}$  is Kuranishi. In particular, the classifying map  $\text{class}(\mathcal{F}) : \mathcal{T} \rightarrow \mathcal{M}_{g,n}$  is dominant. Then by Theorem 2.8.1, the  $\Gamma_{g,n}$ -orbit of  $[\rho_{\nabla_0}]$  in  $\chi_{g,n}(\text{GL}_r\mathbb{C})$  is finite.

Let us now prove the implication (2.1.1)  $\Rightarrow$  (2.1.1). Let  $\mathcal{F}_{(C,D)} = (\kappa : \mathcal{C} \rightarrow \mathcal{T}, \mathcal{D}, t_0, \psi)$  be any algebraic Kuranishi family with central fiber  $(C, D)$  as in § 2.6.4. Note that such a family exists since  $(C, D)$  is stable, and that it remains Kuranishi after pullback *via* an étale base change. Up to such a manipulation, according to Theorem 2.8.2, the family  $\mathcal{F}_{(C,D)}$  can be endowed with a regular flat algebraic logarithmic connection  $(\mathcal{E}, \nabla)$  over  $\mathcal{C}$  with polar divisor  $\mathcal{D}$  such that there is an isomorphism  $\Psi : (E, \nabla_0) \rightarrow (\mathcal{E}, \nabla)|_{\mathcal{C}_{t_0}}$  commuting with  $\psi$  via the natural projections to  $(C, D)$  and  $(\mathcal{C}_{t_0}, \mathcal{D}_{t_0})$  respectively. Now  $\mathcal{I}_{(C,E,\nabla_0)} := (\mathcal{F}_{(C,D)}, \mathcal{E}, \nabla, \Psi)$  defines an algebraic universal isomonodromic deformation of  $(C, E, \nabla_0)$ .  $\square$

## 2.8.4 Proof of the universal property

Let us now proof the universal property of universal algebraic isomonodromic deformations stated in § [2.6.4](#)

*Proof of Proposition [2.6.5](#)* Let  $\mathcal{I}_{(C,E,\nabla_0)} = (\mathcal{F}_{(C,D)}, \mathcal{E}, \nabla, \Psi)$  be a universal algebraic isomonodromic deformation of  $(C, E, \nabla_0)$  with parameter space  $(\mathcal{T}, t_0)$  and let  $\mathcal{I}'_{(C,E,\nabla_0)}$  given by  $(\mathcal{F}'_{(C,D)}, \mathcal{E}', \nabla', \Psi')$  be an algebraic isomonodromic deformation of  $(C, E, \nabla_0)$  with parameter space  $(\mathcal{T}', t'_0)$ .

By Lemma [2.7.3](#), there is an étale base change  $\tilde{p} : (\tilde{\mathcal{T}}, \tilde{t}_0) \rightarrow (\mathcal{T}, t_0)$ , such that for  $\tilde{\mathcal{F}}_{(C,D)} := \tilde{p}^* \mathcal{F}_{(C,D)}$ , there is a section  $\sigma : \tilde{\mathcal{T}} \rightarrow \tilde{\mathcal{C}}$  avoiding the marked points. Since  $\tilde{\mathcal{F}}_{(C,D)}$  is still Kuranishi, by the universal property of Kuranishi families, we have an étale base change  $p : (\mathcal{T}'', t''_0) \rightarrow (\mathcal{T}', t'_0)$ , a morphism  $\tilde{q} : (\mathcal{T}'', t''_0) \rightarrow (\tilde{\mathcal{T}}, \tilde{t}_0)$  and a unique isomorphism

$$\tilde{f} : \mathcal{F}''_{(C,D)} := p^* \mathcal{F}'_{(C,D)} \xrightarrow{\sim} \tilde{q}^* \tilde{\mathcal{F}}_{(C,D)}.$$

In particular,  $\sigma$  lifts to a section  $\sigma'' := \tilde{f}^* \tilde{q}^* \sigma : \mathcal{T}'' \rightarrow \mathcal{C}''$  avoiding the marked points of  $\mathcal{F}''_{(C,D)}$ . Let  $\rho_{\nabla_0} \in \text{Hom}(\pi_1(C \setminus D, x_0), \text{GL}_r \mathbb{C})$  be a representative of the monodromy of  $(E, \nabla_0)$ . Denote by

$$\rho'', \tilde{\rho} \in \text{Hom}(\pi_1(\mathcal{C}'' \setminus \mathcal{D}'', \sigma''(t''_0)), \text{GL}_r \mathbb{C})$$

representatives of the monodromy of  $(\mathcal{E}'', \nabla'')$  and  $\tilde{f}^* \tilde{q}^* \tilde{p}^*(\mathcal{E}, \nabla)$  respectively, such that

$$\rho''|_{\psi''_* \pi_1(C \setminus D, x_0)} = \tilde{\rho}|_{\psi''_* \pi_1(C \setminus D, x_0)} = \psi''_* \rho_{\nabla_0}.$$

By the Splitting Lemma [2.7.4](#), we have

$$\pi_1(\mathcal{C}'' \setminus \mathcal{D}'', \sigma''(t''_0)) = \psi''_* \pi_1(C \setminus D, x_0) \rtimes_{\eta} \sigma''_* \pi_1(\mathcal{T}'', t''_0).$$

Since  $\rho_{\nabla_0}$  is irreducible, by Lemma [2.7.6](#) there is a representation  $\lambda \in \text{Hom}(\pi_1(\mathcal{T}'', t''_0), \mathbb{C}^*)$  such that  $\lambda \otimes (\sigma'')^* \rho'' = (\sigma'')^* \tilde{\rho}$ . By the Riemann-Hilbert correspondence, there is a regular flat algebraic connection  $(L, \xi)$  of rank 1 over  $\mathcal{T}''$ , without poles, whose monodromy representation is  $\lambda^{-1}$ . The monodromy representation of its lift  $\kappa''^*(L, \xi)$  is the trivial extension of  $\sigma''^* \lambda^{-1}$  to a representation  $\psi''_* \pi_1(C \setminus D, x_0) \rtimes_{\eta} \sigma''_* \pi_1(\mathcal{T}'', t''_0) \rightarrow \mathbb{C}^*$ . Now up to replacing  $(\mathcal{E}'', \nabla'')$  by  $(\mathcal{E}'', \nabla'') \otimes \kappa''^*(L, \xi)$  and choosing an appropriate modification of  $\Psi''$ , the monodromy representations of  $(\mathcal{E}'', \nabla'')$  and  $\tilde{f}^* \tilde{q}^* \tilde{p}^*(\mathcal{E}, \nabla)$  with respect to frames of the fiber over  $\sigma''(t''_0)$  induced, via  $\Psi$  and  $\Psi''$ , from a common frame of  $E_{x_0}$ , coincide. Both connections are regular, have canonically identified monodromy representations and same transversal models, given by  $(E, \nabla_0)$ . By the logarithmic Riemann-Hilbert correspondence, they are uniquely isomorphic *via* an isomorphism inducing the identity on  $(C, E, \nabla_0)$ .  $\square$

## 2.8.5 Further remarks and open questions

The proof of Theorem [2.8.1](#) asserting that finiteness of the orbit  $\Gamma_{g,n} \cdot [\rho_{\nabla_0}]$  in  $\chi_{g,n}(G)$  with  $G = \text{GL}_r \mathbb{C}$  is a necessary condition for the existence of an algebraic isomonodromic deformation of  $(C, E, \nabla_0)$  with  $\text{class}(\mathcal{F}) : \mathcal{T} \rightarrow \mathcal{M}_{g,n}$  dominant goes through word for word when considering algebraic isomonodromic deformations of principal logarithmic connections on  $G$ -bundles over curves, where  $G$  is a connected complex algebraic subgroup of  $\text{GL}_r \mathbb{C}$ . One can even drop the regularity assumption in the definition of algebraic isomonodromic deformations for that statement.

In collaboration with I. Biswas and G. Cousin, we are currently working on a generalization of the sufficient condition (Theorem [2.8.2](#)) for principal connections. This generalization is more



delicate, because we need an appropriate generalization of the algebraic logarithmic Riemann-Hilbert correspondence for  $G$ -connections. It is not known in general which elements of  $\text{Hom}(\pi_1(M \setminus D), G)$  arise as the monodromy representation of a flat principal logarithmic connection  $(E_G, \nabla)$  a complex manifold  $M$  of dimension  $m > 1$  (for  $m = 1$ , see [Boa11]). In the case of smooth divisors  $D$ , one might look, analogously to [Cou17], for  $G$ -mild models. It would be interesting to classify those groups  $G$  such that any representation  $\rho \in \text{Hom}(\pi_1(\mathbb{D} \setminus \{0\}), G)$  arises from a  $G$ -mild model. Examples of such groups are  $\text{GL}_r\mathbb{C}$ , the group of triangular matrices and the group of diagonal matrices. The group  $G = \text{SL}_r\mathbb{C}$  on the other hand, is a counter-example if  $r > 1$ . Indeed, consider for example a scalar monodromy  $\lambda \cdot I_r$  with  $\lambda^r = 1$  but  $\lambda \neq 1$ . It does not admit a mild  $\mathfrak{sl}_r\mathbb{C}$ -model. One can show that the monodromy

$$\begin{pmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{pmatrix} \in \text{SL}_4\mathbb{C},$$

where  $i = \sqrt{-1}$ , is another such example.

Even though, in order to generalize Theorem 2.8.2 for principal connections one may consider, as we did for  $G = \text{GL}_r\mathbb{C}$ , only isomonodromic deformations of  $G$ -mild initial connections  $(E_G, \nabla_0)$ , algebraization requires moreover  $G$ -mild models for the irreducible components of the divisor “at infinity”. Moreover, one has to take into account normal crossings, which boils down to the following.

**Problem 4** (Corner models). *Consider the polydisc  $M$  of radius 2 centered at the origin in  $\mathbb{C}^m$ , with  $m > 1$ , endowed with standard coordinates  $(z_1, \dots, z_m)$ . Denote by  $D \subset M$  the normal crossing divisor  $\{z_1 \cdots z_m = 0\}$ . For  $i \in \llbracket 1, m \rrbracket$ , let  $\nabla_i$  be a logarithmic connection on the trivial  $G$ -bundle over the disc  $\Sigma^i := \{z_j = 1 \ \forall j \neq i\}$  with polar divisor  $\Sigma^i \cap D$ , such that the monodromy matrices of  $\nabla_1, \dots, \nabla_m$  commute. Is there a flat principal logarithmic connection  $\nabla$  on the trivial  $G$ -bundle over  $M$ , with polar divisor  $D$ , which induces  $\nabla_i$  on  $\Sigma^i$  for  $i \in \llbracket 1, m \rrbracket$ ?*

For  $G = \text{GL}_r\mathbb{C}$ , it follows from [Cou17] that Problem 4 has a solution if the models  $\nabla_i$  are all mild and at most one of them is not a Deligne model.

Ultimately, the goal should be a classification, in terms of generalized monodromy data, of meromorphic principal  $G$ -connections over curves which admit universal algebraic isomonodromic deformations. Even up to some technical assumptions, it is not at all clear what the statement should be in the non-logarithmic case. One may start by asking the following.

**Problem 5.** *Do finite orbits of the wild mapping class group action on the wild character variety introduced in [Boa14] correspond to algebraic universal isomonodromic deformations of meromorphic principal  $G$ -connections over curves?*

# Chapter 3

## Flat rank two vector bundles on genus two curves

### 3.1 Introduction

Let  $X$  be a smooth projective curve of genus 2 over  $\mathbb{C}$ . Consider the set  $\mathfrak{Con}(X)$  of isomorphism classes of tracefree holomorphic connections  $(E, \nabla)$  of rank 2 over  $X$ . Consider moreover the set  $\mathfrak{Bun}(X)$  of isomorphism classes of rank 2 vector bundles  $E$  with trivial determinant bundle over  $X$  that are *flat* in the sense that they can be endowed with a holomorphic connection, so that the natural forgetful map

$$\text{bun} : \mathfrak{Con}(X) \rightarrow \mathfrak{Bun}(X) \tag{3.1}$$

is surjective. In collaboration with F. Loray, we provided in [HL15] an explicit description of (3.1), seen as a map of stacks, and deduced several applications. These results will be presented in this chapter.

Note that by [Nit93, IIS06a, IIS06b], the subset  $\mathfrak{Con}^{\text{irr}}(X) \subset \mathfrak{Con}(X)$  corresponding to irreducible connections has a natural structure of open subset in the smooth locus of a 6-dimensional quasi-projective variety such that the complement of  $\mathfrak{Con}^{\text{irr}}(X)$  in this variety parametrizes GIT-equivalence classes of reducible connections. A GIT-approach to the subset  $\mathfrak{Bun}^{\text{ss}}(X) \subset \mathfrak{Bun}(X)$  of semistable bundles (in the sense of Mumford) has been established in the classical work by Narasimhan and Ramanan (see [NR69] and § 3.5.1). They construct a quotient map

$$\text{NR} : \mathfrak{Bun}^{\text{ss}}(X) \rightarrow \mathcal{M}_{\text{NR}} := |2\Theta| \simeq \mathbb{P}^3 \tag{3.2}$$

onto the 3-dimensional linear system generated by twice the  $\Theta$ -divisor on  $\text{Pic}^1(X)$ . The Kummer surface  $\text{Kum}(X) = \text{Jac}(X)/_{\pm 1}$  naturally parametrizes the set of decomposable bundles in  $\mathfrak{Bun}^{\text{ss}}$ , and the classifying map NR provides an embedding  $\text{Kum}(X) \hookrightarrow \mathbb{P}_{\text{NR}}^3$  as a quartic surface with 16 ordinary double points. The map NR is one-to-one in restriction to the set  $\mathfrak{Bun}^s$  of stable bundles, which it identifies with the complement  $\mathcal{M}_{\text{NR}} \setminus \text{Kum}(X)$ . In the fiber over a smooth point of  $\text{Kum}(X)$  one finds 1 decomposable bundle and 2 *affine bundles*. Over each singular point of  $\text{Kum}(X)$  one finds 1 decomposable bundle (the trivial bundle and its *twists*) and a 1-parameter family of (*twisted*) *unipotent bundles*. Not appearing in  $\mathcal{M}_{\text{NR}}$  are the 16 *Gunning bundles* that form the complement in  $\mathfrak{Bun}(X)$  of the semistable locus. We refer to § 3.3.3 for the geometric motivation of our bundle-type terminology.

**Hyperelliptic descent** (see § 3.3). The main idea in [HL15], that already appears in works of Ramanan and Bhosle (see for example [Ram81] and [Bho84]), is to use the hyperellipticity of the

curve  $X$  to descent to the study of parabolic bundles and connections over  $\mathbb{P}^1$ . A result of Goldman combined with Riemann-Hilbert correspondence implies that any  $(E, \nabla) \in \mathfrak{Con}^{\text{irr}}(X)$  admits a lift  $h : (E, \nabla) \xrightarrow{\sim} \iota^*(E, \nabla)$  of the hyperelliptic involution  $\iota : X \rightarrow X$  and this lift is unique up to a sign (see § 3.3.1). Let us denote by  $\pi : X \rightarrow \mathbb{P}^1 \simeq X/\iota$  the natural quotient map. As an example of the more general situation studied in collaboration with I. Biswas in [BH13b], the lift  $h$  induces a natural splitting of the push-forward  $\pi_*(E, \nabla)$  into a direct sum of two logarithmic connections of rank two over  $\mathbb{P}^1$ . More precisely, each of these direct summands is an element of  $\mathfrak{Con}(X/\iota)$ , which will denote the set of isomorphism classes of pairs  $(\underline{E}, \underline{\nabla})$ , where  $\underline{E}$  is a rank two vector bundle of degree  $-3$  on  $\mathbb{P}^1$  and  $\underline{\nabla}$  is a logarithmic connection on  $E$  with poles located at the image  $\underline{W}$  under  $\pi$  of the Weierstrass points on  $X$ , such that each residue has the two eigenvalues  $0$  and  $\frac{1}{2}$ .

Conversely (see § 3.3.2), from any  $(\underline{E}, \underline{\nabla}) \in \mathfrak{Con}(X/\iota)$  one can construct a tracefree holomorphic connection on  $X$  by composing the natural lift  $\pi^*$  with positive elementary transformations at the lift of the parabolic structure on  $\underline{p}$  on  $\underline{E}$  defined by the  $\frac{1}{2}$ -eigendirections of  $\underline{\nabla}$ . In restriction to the irreducible locus, we obtain a map  $\Phi : \mathfrak{Con}(X/\iota) \rightarrow \mathfrak{Con}(X)$  whose fibers correspond to the direct summands described above. Forgetting the connections on both sides yields a well defined map  $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$  given by  $(\underline{E}, \underline{p}) \mapsto E$ , so that the following diagram commutes, where vertical arrows designate the forgetful maps:

$$\begin{array}{ccc} \mathfrak{Con}(X/\iota) & \xrightarrow[\Phi]{2:1} & \mathfrak{Con}(X) \\ \downarrow & & \downarrow \\ \mathfrak{Bun}(X/\iota) & \xrightarrow[\phi]{} & \mathfrak{Bun}(X) . \end{array} \quad (3.3)$$

**A dictionary** (see § 3.4). The map  $\phi$  turns out to be a ramified 2-cover onto its image. The image consists of the complement in  $\mathfrak{Bun}(X)$  of the affine bundles, and the ramification locus corresponds to the decomposable bundles. The fiber of  $\phi$  over a generic stable bundle  $E$  consists of two isomorphism classes of parabolic bundles  $(\underline{E}, \underline{p})$ , such that

$$\underline{E} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$$

and the parabolic structure  $\underline{p}$  is defined by the fibers over the Weierstrass points of a subbundle  $\underline{L} \simeq \mathcal{O}_{\mathbb{P}^1}(-4)$  of  $\underline{E}$ , giving rise to an anticanonical subbundle  $L \simeq \mathcal{O}_X(-K_X)$  of  $E$  which is invariant under the hyperelliptic involution. We also investigate the special cases of all other bundles in the image of  $\phi$  and describe the particularities characterizing the parabolic bundles in the associated fibers of  $\phi$ . It turns out that with the exception of those corresponding in  $\mathfrak{Bun}(X)$  to the trivial bundle and its twists, all parabolic bundles occurring in  $\mathfrak{Bun}(X/\iota)$  are indecomposable.

**Application to classical approaches** (see § 3.5). Of course  $\mathfrak{Bun}(X)$  is not a Hausdorff topological space. For example all points in a same fiber of the Narasimhan-Ramanan map (3.2) are arbitrarily close to each other. Other non-Hausdorff points in  $\mathfrak{Bun}(X)$  are the 16 (unstable) Gunning bundles. For each of them, there is a plane in  $\mathcal{M}_{\text{NR}}$  such that each point in the fiber of (3.2) over a point in this plane is arbitrarily close to this Gunning bundle. We show that these 16 *Gunning planes* are precisely the planes arising in the classical (16:6)-configuration of the Kummer surface (see [Hud90, GD94, NR69, Bol07]). With respect to a curve  $X$  with equation

$$y^2 = x(x-1)(x-r)(x-s)(x-t) \quad (3.4)$$

we exhibit in § 3.5.2, via generators of  $|\mathcal{O}(2\Theta)|$ , a natural system of coordinates on  $\mathcal{M}_{\text{NR}} \simeq \mathbb{P}^3$ . With respect to these coordinates, we compute the Narasimhan-Ramanan map for the moduli space  $\mathcal{M}_B \simeq$

$\mathbb{P}^3$  of  $\iota$ -invariant non-split extensions  $\mathcal{O}_X(-K_X) \rightarrow E \rightarrow \mathcal{O}_X(K_X)$  studied by Bertram and Bolognesi [Ber92, Bol07, Bol09]. Via Tyurin’s classical approach, moduli spaces of parabolic structures on the trivial rank two bundle over  $X$  (tensorised by  $\mathcal{O}(-K_X)$ ) supported by divisors in  $|2K_X|$  are birational models of  $\mathfrak{Bun}(X)$ . We provide such a model  $\mathcal{M}_{Tyu} \simeq \mathbb{P}^2 \times \mathbb{P}^1$ , which arises naturally from hyperelliptic descent, and which covers the trivial bundle in  $\mathfrak{Bun}(X)$ . Note however that other constructions allow to cover for example the affine bundles in  $\mathfrak{Bun}(X)$ . We construct a universal bundle on a rational 2-cover  $\mathcal{B}$  of  $\mathcal{M}_{Tyu}$ , which, via the relations we provide towards the other mentioned moduli spaces, can be seen as an explicit version of Bolognesi’s universal bundle [Bol09] over the rational two-cover  $\mathcal{M}_B$  of  $\mathcal{M}_{NR}$ . For convenience, Table 1 references where we the obtained explicit maps can be found in the manuscript. Note that by classification, each of the mentioned moduli spaces contains a birational model of the Kummer surface. We compute these (with respect to our coordinates), and are able to describe the geometry of their birational transition maps (in terms of blow ups and contractions) by consulting the special loci singled out in the dictionary.

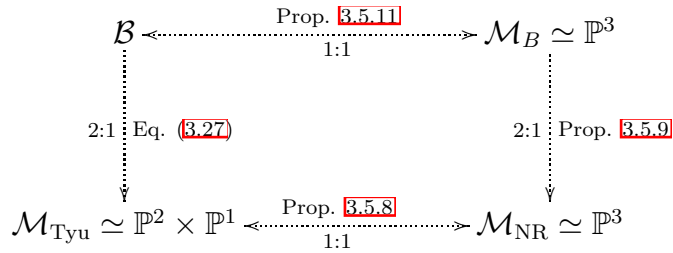


Table 1: Collection of explicit formulae.

**A chart of the stack  $\mathfrak{Bun}(X)$**  (see § 3.6). The subset  $\mathfrak{Bun}^{\text{ind}}(X/\iota)$  of indecomposable parabolic bundles in  $\mathfrak{Bun}(X/\iota)$  has been shown in the previous works [AL97, LS15] to have a natural structure of smooth non separated 3-dimensional projective scheme. More precisely, it can be constructed by patching together GIT quotients  $\text{Bun}_{\mu}^{\text{ss}}(X/\iota)$  of  $\mu$ -semi-stable parabolic bundles for a finite number of well-chosen weights  $\mu \in [0, 1]^6$ . These moduli spaces are smooth projective manifolds and they are patched together along Zariski open subsets. Note that for one of these  $\mu$ , the corresponding moduli space is naturally identified with the aforementioned Bertram-Bolognesi moduli space. We select some more particular choices of  $\mu$  and provide explicit transition maps between the corresponding  $\text{Bun}_{\mu}^{\text{ss}}(X/\iota)$ ’s. Again the special loci identified by the dictionary allow us to describe the geometry of these transition maps, as well as the Galois-involution of  $\phi$  as a composition of blow-ups, contractions and flops. This provides a geometrical and to a large extend explicit description of  $\phi : \mathfrak{Bun}^{\text{ind}}(X/\iota) \xrightarrow{2:1} \mathfrak{Bun}^*(X)$ , where the image consists of the complementary in  $\mathfrak{Bun}(X)$  of the affine bundles and (twists of) the trivial bundle.

**Application to the Hitchin fibration** (see § 3.6.6). The restriction  $\text{bun}^s : \mathcal{C}\text{on}^s(X) \rightarrow \mathfrak{Bun}^s(X)$  to the locus of stable bundles of the forgetful map (3.1) is known to be a locally trivial affine  $\mathbb{A}^3$ -bundle, whose homogeneous part can be interpreted as the moduli space  $\mathfrak{Higgs}(X)$  of tracefree holomorphic Higgs bundles  $(E, \theta)$  with  $E \in \mathfrak{Bun}^s(X)$ . This moduli space is naturally identified with the cotangent bundle  $T^{\vee}\mathfrak{Bun}^s(X)$ . We provide an explicit universal Higgs bundle for its natural counterpart  $\mathfrak{Higgs}(X/\iota)$  with respect to Diagram (3.3). This allows us, in a very direct way, to compute the Hitchin map

$$\text{Hitch} : \left\{ \begin{array}{ll} \mathfrak{Higgs}(X) & \rightarrow \text{H}^0(X, \Omega_X^1 \otimes \Omega_X^1) \\ (E, \theta) & \mapsto \det(\theta) \end{array} \right\}$$

explicitly. The result is stated in Table 11. We can relate the six Hamiltonians described by G. van

Geemen and E. Previato in [VGP96] to the three Hamiltonian coefficients of the Hitchin map. Let us mention that the map  $\text{bun}^s$  also has a natural structure of Lagrangian fibration (see *e.g.* [Sim08, § 7.2]). Using the so-called apparent map in [LS15] and Diagram (3.3), one can explicitly construct a regular rational and Lagrangian section of  $\text{bun}^s$  (see [HL15, §. 7.3], not detailed here).

**A chart of the stack  $\mathfrak{Con}(X)$**  (see § 3.7.1). We describe a collection of subsets of  $\mathfrak{Con}(X/\iota)$ , parametrized by  $\mathbb{C}^6$  and endowed with universal connections, with birational transition maps. Patching those together yields a certain subset  $\mathfrak{Con}^*(X/\iota)$  of  $\mathfrak{Con}(X/\iota)$  with a natural structure of affine (possibly non separated) scheme, which is invariant under the Galois involution of  $\Phi$ . The quotient

$$\mathfrak{Con}^*(X) := \Phi(\mathfrak{Con}^*(X/\iota)) \subset \mathfrak{Con}(X)$$

contains all irreducible connections.

**The family  $\mathfrak{M}$  and the isomonodromy foliation** (see § 3.7.2). Varying the parameter  $(r, s, t) \in \mathcal{T} := (\mathbb{C} \setminus \{0, 1\})^3 \setminus \{\text{diagonals}\}$  which encodes the complex structure of  $X$  in (3.4), this construction yields a family  $\mathfrak{M} \rightarrow \mathcal{T}$  with fibers  $\mathfrak{Con}^*(X_{(r,s,t)})$ . Note that  $\mathfrak{M}$  is actually a chart of the moduli stack of genus 2 curves endowed with tracefree rank 2 connections, i.e., isomorphism classes of triples  $(X, E, \nabla)$ . Moreover, all these triples in  $\mathfrak{M}$  arise from the hyperelliptic lift  $\Phi$ . The well known isomonodromy Hamiltonian system derived from Garnier systems over  $\mathbb{P}^1$  gives rise, in  $\mathfrak{M}$ , to a singular holomorphic foliation  $\mathcal{F}_{\text{iso}}$ . The 3-dimensional leaves of this foliation parametrize (universal) isomonodromic deformations.

**Application to projective structures** (see § 3.7.3 and § 3.7.4). In the above  $\mathfrak{M}$ , not only the isomonodromy foliation  $\mathcal{F}_{\text{iso}}$ , but also the locus of special bundles, for example the Gunning bundles, are explicit. We show that the isomonodromy foliation is transverse to the locus of Gunning bundles by direct computation in Theorem 3.7.2. As a corollary, we obtain a new proof of a result of Hejhal [Hej75], stating that the monodromy map from the space of projective structures on the genus two curves to the space of  $\text{PGL}_2$ -representations of the fundamental group is a local diffeomorphism.

## 3.2 Parabolic bundles and their elementary transformations

This section contains basic definitions that will be used in the sequel. Namely, what for practical reasons we simply call *parabolic bundles* (in the literature: quasi-parabolic bundles of rank 2), their (*semi-*)*stability* with respect to a given *weight*, *parabolic connections* and *elementary transformations* on parabolic bundles and connections.

In the following paragraphs, let  $X$  be a smooth projective curve defined over  $\mathbb{C}$  and let  $D = x_1 + \dots + x_n$  be a reduced divisor on  $X$ .

### 3.2.1 Parabolic bundles and connections

A *parabolic structure  $\mathbf{p}$*  supported by  $D$  on a given rank 2 vector bundle  $E \rightarrow X$  associates, to each  $x_i \in D$ , a 1-dimensional subspace  $p_i$  of the fiber  $E_{x_i}$ . The  $p_i$ 's will be called *parabolics*, and the pair  $(E, \mathbf{p})$  will be called a *parabolic bundle*. When, as above, we implicitly have an order on the support of  $D$ , we denote the parabolic structure  $\mathbf{p} = (p_1, \dots, p_n)$ . A trivial yet useful remark is that a parabolic structure on  $E$  supported by  $D$  can be equivalently defined by the choice of a point in each of the fibers over  $D$  of the ruled surface  $\mathbb{P}(E) \rightarrow X$ . Abusing notation, we denote these points  $p_i$  as well.

A *logarithmic connection* of rank  $r$  with polar divisor at most  $D$  is a pair  $(E, \nabla)$ , where  $E \rightarrow X$  is a vector bundle of rank  $r$  and  $\nabla : E \rightarrow E \otimes \Omega_X^1(D)$  is a  $\mathbb{C}$ -linear morphism satisfying the Leibniz

rule. At each  $x_0 \in D$ , the connection intrinsically defines an endomorphism of the fiber  $E_{x_0}$ , the *residue*  $\text{Res}_{x_0} \nabla$ . *Residual eigenvalues* and *residual eigenspaces* in  $E_{x_0}$  hence are well-defined. The connection  $\nabla$  on  $E$  induces a logarithmic connection  $\text{tr}(\nabla)$  on the determinant line bundle  $\det(E)$  over  $X$  with

$$\text{Res}_{x_0} \text{tr}(\nabla) = \text{tr}(\text{Res}_{x_0} \nabla)$$

for each  $x_0 \in D$ . By the Residue Theorem, the sum of residues of a global meromorphic 1-form on  $X$  is zero. We thereby obtain *Fuchs' relation*:

$$\deg(E) + \sum_{x_0 \in D} \text{tr}(\text{Res}_{x_0} \nabla) = 0. \quad (3.5)$$

We say that a logarithmic connection  $(E, \nabla)$  is *tracefree* when  $(\det(E), \text{tr}(\nabla))$  is isomorphic to the trivial connection  $d_X$  on the trivial line bundle. A logarithmic connection with polar divisor at most  $D$  such that all residues are zero gives rise to a *holomorphic connection*  $(E, \nabla)$  with  $\nabla : E \rightarrow E \otimes \Omega_X^1$ .

Let  $(E, \nabla)$  be a logarithmic rank  $r$  connection and let  $L$  be a  $r$ -torsion line bundle, i.e.,  $L^{\otimes r} \simeq \mathcal{O}_X$ . Then there is a unique holomorphic connection  $\zeta$  on  $L$  such that  $(L, \zeta)^{\otimes r}$  is isomorphic to the trivial connection on the trivial line bundle. We refer to the tensor product  $(E, \nabla) \otimes (L, \zeta)$  as the *twist* of  $(E, \nabla)$  by  $L$ . Note that twists preserve the isomorphism class of the trace connection.

A *parabolic connection* over  $X$  with support  $D$  will be by definition a triple  $(E, \nabla, \mathbf{p})$ , where  $E \rightarrow X$  is a vector bundle of rank  $r = 2$ ,  $\mathbf{p}$  is a parabolic structure on  $E$  supported by  $D$  and  $\nabla$  is a logarithmic connection on  $E$  with polar divisor at most  $D$ , such that moreover for each  $x_i \in D$  the corresponding parabolic  $p_i$  is an eigendirection of  $\text{Res}_{x_i} \nabla$ .

### 3.2.2 Elementary transformations

Let  $(E, p)$  be a parabolic bundle on  $X$  supported by a single point  $x_0 \in X$ . Consider the vector bundle  $E^-$  defined by the subsheaf of those sections  $s$  of  $E$  such that  $s(x_0) \in p$ . A natural parabolic direction on  $E^-$  is defined by those sections of  $E$  which are vanishing at  $x_0$  (and thus belong to  $E^-$ ). If  $x$  is a local coordinate vanishing at  $x_0$  and  $E$  is generated near  $x_0$  by  $\langle e_1, e_2 \rangle$  with  $e_1(x_0) \in p$ , then  $E^-$  is locally generated by  $\langle e_1, e'_2 \rangle$  with  $e'_2 := x e_2$  and we define  $p^- \subset E^-|_{x_0}$  to be  $\mathbb{C}e'_2(x_0)$ . By identifying the sections of  $E$  and  $E^-$  outside  $x_0$ , we obtain a natural birational morphism  $E \dashrightarrow E^-$ . We denote

$$\text{elm}_{x_0}^-(E, p) := (E^-, p^-).$$

In a similar way, we define the parabolic bundle  $(E^+, p^+)$  by the sheaf of those meromorphic sections of  $E$  having (at most) a single pole at  $x_0$ , whose residual part is an element of  $p$ . The parabolic  $p^+$  then is defined by

$$p^+ := \{s(x_0) \mid s \text{ is a holomorphic section of } E \text{ near } x_0\}.$$

In other words,  $E^+$  is generated by  $\langle e'_1, e_2 \rangle$  with  $e'_1 := \frac{1}{x}e_1$  and  $p^+ \subset E^+|_{x_0}$  defined by  $\mathbb{C}e_2$ . The natural morphism  $E \dashrightarrow E^+$  is now regular, but fails to be an isomorphism at  $x_0$ . We denote

$$\text{elm}_{x_0}^+(E, p) := (E^+, p^+).$$

These *elementary transformations* (see also [\[Mac07\]](#)) satisfy

- $\det(E^\pm) = \det(E) \otimes \mathcal{O}_X(\pm[x_0])$ ,

- $\text{elm}_{x_0}^+ \circ \text{elm}_{x_0}^- = \text{id}_{(E,p)}$  and  $\text{elm}_{x_0}^- \circ \text{elm}_{x_0}^+ = \text{id}_{(E,p)}$ ,
- $\text{elm}_{x_0}^+ = \mathcal{O}_X([x_0]) \otimes \text{elm}_{x_0}^-$ .

More generally, given a parabolic bundle  $(E, \mathbf{p})$  with support  $D$ , we define the elementary transformations  $\text{elm}_D^\pm$  as the composition of the (commuting) single elementary transformations over all points of  $D$ . We define  $\text{elm}_{D_0}^\pm$  for any subdivisor  $D_0 \subset D$  in the obvious way.

Given a parabolic connection  $(E, \nabla, \mathbf{p})$  with support  $D$ , the elementary transformations  $\text{elm}_D^\pm$  yield new parabolic connections  $(E^\pm, \nabla^\pm, \mathbf{p}^\pm)$ . In fact, the compatibility condition between  $\mathbf{p}$  and the residual eigenspaces of  $\nabla$  ensures that  $\nabla^\pm$  is still logarithmic. The monodromy is left unchanged, but the residual eigenvalues are shifted as follows: if  $\lambda_1$  and  $\lambda_2$  denote the residual eigenvalues of  $\nabla$  at  $x_0$ , with  $p$  contained in the  $\lambda_1$ -eigenspace, then

- $\text{Res}_{x_0} \nabla^+$  has eigenvalues  $(\lambda_1^+, \lambda_2^+) := (\lambda_1 - 1, \lambda_2)$ ,
- $\text{Res}_{x_0} \nabla^-$  has eigenvalues  $(\lambda_1^-, \lambda_2^-) := (\lambda_1, \lambda_2 + 1)$ ,

and  $p^\pm$  is now defined by the  $\lambda_2^\pm$ -eigenspace.

One easily verifies that elementary transformations are well-defined on isomorphism classes of parabolic bundles and parabolic connections.

Note that positive and negative elementary transformations coincide for a projectivized parabolic bundle  $(\mathbb{P}(E), p)$ . They consist, for the ruled surface  $\mathbb{P}(E) \rightarrow X$ , in composing the blowing-up of  $p$  with the contraction of the strict transform of the fiber [\[GM89\]](#). This contraction yields the new parabolic  $p^\pm$ . Elementary transformations on projectivized parabolic bundles are clearly involutive.

Note moreover that when  $\sigma \simeq X$  denotes the curve in the total space of the ruled surface  $\mathbb{P}(E) \rightarrow X$  corresponding to a line subbundle  $L \subset E$ , the self-intersection number of  $\sigma$  coincides with  $\deg(E) - 2 \deg(L)$  (see [\[Mar70\]](#)). The self-intersection number of the curve  $\hat{\sigma}$  in  $\mathbb{P}(E^\pm)$  corresponding to the line subbundle  $L^\pm$  of  $E^\pm$  determined by  $L$  satisfies

$$\#(\hat{\sigma} \cdot \hat{\sigma}) = \begin{cases} \#(\sigma \cdot \sigma) - 1 & \text{if } p \in \sigma \\ \#(\sigma \cdot \sigma) + 1 & \text{if } p \notin \sigma. \end{cases}$$

### 3.2.3 Weight-stability and moduli spaces

A *weight*  $\boldsymbol{\mu}$  supported by the divisor  $D$  associates to each  $x_i \in D$  an element  $\mu_i \in [0, 1]$ . Let  $\boldsymbol{\mu}$  be such a weight and let  $(E, \mathbf{p})$  be a parabolic bundle with parabolic structure supported by  $D$ . We refer to the triple  $(E, \mathbf{p}, \boldsymbol{\mu})$  as a *weighted parabolic bundle*. For any line subbundle  $L$  of  $E$ , we define the *stability index* of  $L$  with respect to  $\boldsymbol{\mu}$  by

$$\text{ind}_{\boldsymbol{\mu}}(L) := \deg_{\boldsymbol{\mu}}^{\text{par}}(E) - 2 \deg_{\boldsymbol{\mu}}^{\text{par}}(L),$$

where

$$\deg_{\boldsymbol{\mu}}^{\text{par}}(E) := \deg(E) + \sum_{i=1}^n \mu_i \quad \text{and} \quad \deg_{\boldsymbol{\mu}}^{\text{par}}(L) := \deg(L) + \sum_{p_i \subset L} \mu_i.$$

Here by taking the sum over  $p_i \subset L$  we mean taking the sum over those parabolics  $p_i$  that are contained in the total space of  $L \subset E$ . Note that the stability index of  $L$  with respect to  $\boldsymbol{\mu}$  writes, for the section  $\sigma$  of  $\mathbb{P}(E) \rightarrow X$  defined by  $L$ , as

$$\text{ind}_{\boldsymbol{\mu}}(L) = \text{ind}_{\boldsymbol{\mu}}(\sigma) := \#(\sigma \cdot \sigma) + \sum_{p_i \notin \sigma} \mu_i - \sum_{p_i \in \sigma} \mu_i.$$

A parabolic bundle  $(E, \mathbf{p})$  will be called  $\mu$ -semi-stable (resp.  $\mu$ -stable) if

$$\mathrm{ind}_{\mu}(L) \geq 0 \text{ (resp. } > 0) \text{ for each line subbundle } L \subset E.$$

Alternatively, we speak of (semi-)stability of weighted parabolic bundles  $(E, \mathbf{p}, \mu)$ . For vanishing weights  $\mu_1 = \dots = \mu_n = 0$ , we get the usual definition of (semi-)stability of vector bundles.

We say a bundle is *strictly semi-stable* if it is semi-stable but not stable. A bundle is called *unstable* if it is not semi-stable.

Semi-stable weighted parabolic bundles over  $X$  with fixed ordinary degree, fixed weight  $\mu$  supported by  $D$  and  $\mu$ -parabolic degree 0 admit a coarse moduli space  $\mathrm{Bun}_{\mu}^{ss}$  which is a normal projective variety; the stable locus  $\mathrm{Bun}_{\mu}^s$  is smooth [MS80] [BH95]. Note that a similar property holds for weighted parabolic connections [Nit93].

For weighted parabolic bundles  $(E, \mathbf{p}, \mu)$ , it is natural to extend the definition of elementary transformations as follows. Given a subdivisor  $D_0 \subset D$ , define

$$\mathrm{elm}_{D_0}^{\pm} : (E, \mathbf{p}, \mu) \dashrightarrow (E', \mathbf{p}', \mu')$$

by setting

$$\mu'_i = \begin{cases} 1 - \mu_i & \text{if } p_i \in D_0, \\ \mu_i & \text{if } p_i \notin D_0. \end{cases}$$

For  $L' \subset E'$  denoting the strict transform of a line subbundle  $L \subset E$ , we then have  $\mathrm{ind}_{\mu'}(L') = \mathrm{ind}_{\mu}(L)$ . Therefore,  $\mathrm{elm}_{D_0}^{\pm}$  acts as an isomorphism between the moduli spaces  $\mathrm{Bun}_{\mu}^{ss}$  and  $\mathrm{Bun}_{\mu'}^{ss}$  (resp.  $\mathrm{Bun}_{\mu}^s$  and  $\mathrm{Bun}_{\mu'}^s$ ).

### 3.3 Hyperelliptic descent

From now on, let  $X$  be the smooth complex projective curve given in an affine chart of  $\mathbb{P}^1 \times \mathbb{P}^1$  by

$$y^2 = x(x-1)(x-r)(x-s)(x-t), \quad (3.6)$$

for some pairwise distinct  $r, s, t \in \mathbb{C}^0 \setminus \{1\}$ . Denote by  $\iota : X \rightarrow X$  its hyperelliptic involution, defined in the above chart by  $(x, y) \mapsto (x, -y)$ , and denote by  $\pi : X \rightarrow \mathbb{P}^1$  the hyperelliptic projection, defined in the above chart by  $(x, y) \mapsto x$ . Denote by  $\underline{W} = \{0, 1, r, s, t, \infty\}$  the critical divisor on  $\mathbb{P}^1$  and by  $W = \{w_0, w_1, w_r, w_s, w_t, w_{\infty}\}$  the *Weierstrass divisor* on  $X$ , i.e., the branching divisor with respect to  $\pi$ .

Consider a rank 2 vector bundle  $\underline{E} \rightarrow \mathbb{P}^1$  of degree  $-3$ , endowed with a logarithmic connection  $\underline{\nabla} : \underline{E} \rightarrow \underline{E} \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$  having residual eigenvalues 0 and  $\frac{1}{2}$  at each pole. We fix the parabolic structure  $\underline{\mathbf{p}}$  defined by the  $\frac{1}{2}$ -eigenspaces over  $\underline{W}$ . After lifting the parabolic connection  $(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}})$  via  $\pi : X \rightarrow \mathbb{P}^1$ , we get a parabolic connection on  $X$

$$\left( \tilde{E} \rightarrow X, \tilde{\nabla}, \tilde{\mathbf{p}} \right) = \pi^* \left( \underline{E} \rightarrow \mathbb{P}^1, \underline{\nabla}, \underline{\mathbf{p}} \right).$$

We have  $\det(\tilde{E}) \simeq \mathcal{O}_X(-3K_X)$  and the residual eigenvalues of the connection  $\tilde{\nabla} : \tilde{E} \rightarrow \tilde{E} \otimes \Omega_X^1(W)$  are 0 and 1 at each pole, with parabolic structure  $\tilde{\mathbf{p}}$  defined by the 1-eigenspaces. After applying elementary transformations directed by  $\tilde{\mathbf{p}}$  (see § 3.2.2), we get a new parabolic connection:

$$\mathrm{elm}_W^+ : \left( \tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}} \right) \dashrightarrow (E, \nabla, \mathbf{p})$$



which is now holomorphic and tracefree.

Recall from the introduction that we denote by  $\mathfrak{Con}(X/\iota)$  the set of isomorphism classes of logarithmic rank 2 connections on  $\mathbb{P}^1$  with residual eigenvalues 0 and  $\frac{1}{2}$  at each pole in  $\underline{W}$ , and we denote by  $\mathfrak{Con}(X)$  the set of isomorphism classes of tracefree holomorphic rank 2 connections on  $X$ . Since to every element  $(\underline{E}, \underline{\nabla})$  of  $\mathfrak{Con}(X/\iota)$ , the parabolic structure  $\underline{\mathfrak{p}}$  is intrinsically defined as above, we have just defined a map

$$\Phi : \begin{cases} \mathfrak{Con}(X/\iota) & \rightarrow \mathfrak{Con}(X) \\ (\underline{E}, \underline{\nabla}, \underline{\mathfrak{p}}) & \mapsto (E, \nabla). \end{cases} \quad (3.7)$$

In § [3.3.1](#) we characterize all holomorphic and tracefree rank 2 connection  $(E, \nabla)$  on  $X$  that can be obtained as above. It turns out that irreducibility of  $\nabla$  is a sufficient condition. In § [3.3.3](#) we introduce a bundle-type terminology for flat vector bundles  $E$  and characterize those that can be obtained *via*  $\Phi$ , namely all semistable but not *affine bundles*, and also the unstable *Gunning bundles*. In § [3.3.2](#) we present a construction inverse to this  $\Phi$ , and we enrich our point of view of hyperelliptic decent by its relation to the classical approach of Tyurin [\[Tyu64\]](#) in § [3.3.4](#).

### 3.3.1 Topological considerations

By the Riemann-Hilbert correspondence,  $\mathfrak{Con}(X/\iota)$  and  $\mathfrak{Con}(X)$  are in one-to-one correspondence with spaces of representations. Let us start with  $\mathfrak{Con}(X)$  which is easier.

The monodromy of a tracefree holomorphic rank 2 connection  $(E, \nabla)$  on  $X$  gives rise to a monodromy representation, namely a homomorphism  $\rho : \pi_1(X, w) \rightarrow \mathrm{SL}_2$ . In fact, this depends on the choice of a basis on the fiber  $E_w$ . Another choice will give the conjugate representation  $M\rho M^{-1}$  for some  $M \in \mathrm{GL}_2\mathbb{C}$ . The class  $[\rho] \in \mathrm{Hom}(\pi_1(X, w), \mathrm{SL}_2) / \mathrm{PGL}_2$  however is well-defined by  $(E, \nabla)$ . Conversely, the monodromy  $[\rho]$  characterizes the connection  $(E, \nabla)$  on  $X$  modulo isomorphism, which yields a bijective correspondence

$$\mathrm{RH} : \mathfrak{Con}(X) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(X, w), \mathrm{SL}_2) / \mathrm{PGL}_2$$

which is complex analytic where it makes sense, i.e., in restriction to the subset  $\mathfrak{Con}^{\mathrm{irr}}(X)$  of irreducible connections, with its natural structure of smooth quasi-projective variety. Yet this map is highly transcendental, since we have to integrate a differential equation to compute the monodromy. Note that the space of representations only depends on the topology of  $X$ , not on the complex and algebraic structure.

In a similar way, parabolic connections in  $\mathfrak{Con}(X/\iota)$  are in one-to-one correspondence with conjugacy classes of elements  $\rho$  in  $\mathrm{Hom}^*(\pi_1^{\mathrm{orb}}(X/\iota, x), \mathrm{GL}_2\mathbb{C})$ . Here, thinking of  $\mathbb{P}^1 = X/\iota$  as the orbifold quotient of  $X$  by the hyperelliptic involution, a representation  $\rho$  of the orbifold fundamental group  $\pi_1^{\mathrm{orb}}(X/\iota)$  (killing squares of simple loops around punctures, see the proof of theorem [3.3.1](#) below) can be seen as a representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus \underline{W}, x) \rightarrow \mathrm{GL}_2$$

with 2-torsion monodromy around the punctures. Moreover, elements  $\rho \in \mathrm{Hom}^*(\pi_1^{\mathrm{orb}}(X/\iota, x), \mathrm{GL}_2\mathbb{C})$  are required to have the two eigenvalues 1 and  $-1$  around each puncture.

If  $x = \pi(w)$ , the branching cover  $\pi : X \rightarrow X/\iota$  induces a monomorphism

$$\pi_* : \pi_1(X, w) \hookrightarrow \pi_1^{\mathrm{orb}}(X/\iota, x),$$

whose image consists of words of even length in the alphabet of a system of simple generators of  $\pi_1^{\mathrm{orb}}(X/\iota, x)$ . This allows to associate, to any representation  $\rho : \pi_1^{\mathrm{orb}}(X/\iota, x) \rightarrow \mathrm{GL}_2\mathbb{C}$  as above, a

representation  $\rho \circ \pi_* : \pi_1(X, w) \rightarrow \mathrm{SL}_2$ . We have thereby defined a map  $\Phi^{\mathrm{top}}$  between corresponding representation spaces, which makes the following diagram commutative

$$\begin{array}{ccc} \mathfrak{Con}(X/\iota) & \xrightarrow[\sim]{\mathrm{RH}} & \mathrm{Hom}^*(\pi_1^{\mathrm{orb}}(X/\iota, x), \mathrm{GL}_2\mathbb{C})/\mathrm{PGL}_2 \\ \downarrow \Phi & & \downarrow \Phi^{\mathrm{top}} \\ \mathfrak{Con}(X) & \xrightarrow[\sim]{\mathrm{RH}} & \mathrm{Hom}(\pi_1(X, w), \mathrm{SL}_2)/\mathrm{PGL}_2. \end{array} \quad (3.8)$$

We now want to describe the map  $\Phi^{\mathrm{top}}$ . The quotient  $\pi_1^{\mathrm{orb}}(X/\iota, x) / \pi_*(\pi_1(X, w)) \simeq \mathbb{Z}_2$  acts (by conjugacy) as outer automorphisms of  $\pi_1(X, w)$ . It coincides with the outer action of the hyperelliptic involution  $\iota$ .

**Theorem 3.3.1.** *Given a representation  $[\rho] \in \mathrm{Hom}(\pi_1(X), \mathrm{SL}_2)/\mathrm{PGL}_2$ , the following properties are equivalent:*

- (a)  $[\rho]$  is either irreducible or abelian;
- (b)  $[\rho]$  is  $\iota$ -invariant;
- (c)  $[\rho]$  is in the image of  $\Phi^{\mathrm{top}}$ .

*If these properties are satisfied, then  $[\rho]$  has 1 or 2 preimages under  $\Phi^{\mathrm{top}}$ , depending on whether it is diagonal or not.*

*Proof.* We start making explicit the monomorphism  $\pi_*$  and the involution  $\iota$ . Let  $x \in \mathbb{P}^1 \setminus \underline{W}$  and  $w \in X$  one of the two preimages. Choose simple loops around the punctures to generate the orbifold fundamental group of  $\mathbb{P}^1 \setminus \underline{W}$  with the standard representation

$$\pi_1^{\mathrm{orb}}(X/\iota, x) = \left\langle \gamma_0, \gamma_1, \gamma_r, \gamma_s, \gamma_t, \gamma_\infty \mid \begin{array}{l} \gamma_0^2 = \gamma_1^2 = \gamma_r^2 = \gamma_s^2 = \gamma_t^2 = \gamma_\infty^2 = 1 \\ \text{and } \gamma_0\gamma_1\gamma_r\gamma_s\gamma_t\gamma_\infty = 1 \end{array} \right\rangle.$$

Even words in these generators can be lifted as loops based in  $w$  on  $X$ , generating the ordinary fundamental group of  $X$ . Using the relations, we see that  $\pi_1(X, w)$  is actually generated by the following pairs

$$\begin{cases} \alpha_1 := \gamma_0\gamma_1 & \alpha_2 := \gamma_s\gamma_t \\ \beta_1 := \gamma_r\gamma_1 & \beta_2 := \gamma_\infty\gamma_t \end{cases} \quad (3.9)$$

and they provide the standard presentation

$$\pi_1(X, w) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid [\alpha_1, \beta_1][\alpha_2, \beta_2] = 1 \rangle, \quad (3.10)$$

where  $[\alpha_i, \beta_i] = \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$  denotes the commutator. In other words, the monomorphism  $\pi_*$  is defined by  $\alpha_1 \mapsto \gamma_0\gamma_1$  *et cetera* (see Figure 1).

After moving the base point to a Weierstrass point,  $w = w_i$  say, the involution  $\iota$  acts as an involutive automorphism of  $\pi_1(X, w_i)$ : it coincides with the outer automorphism given by  $\gamma_i$ -conjugacy. For instance, for  $i = 1$ , we get

$$\begin{cases} \alpha_1 \mapsto \alpha_1^{-1} & \alpha_2 \mapsto \gamma\alpha_2^{-1}\gamma^{-1} \\ \beta_1 \mapsto \beta_1^{-1} & \beta_2 \mapsto \gamma\beta_2^{-1}\gamma^{-1} \end{cases} \quad \text{with } \gamma = \beta_1^{-1}\alpha_1^{-1}\beta_2\alpha_2.$$

Let us now prove (a) $\Leftrightarrow$ (b). That irreducible representations are  $\iota$ -invariant already appears in the last section of [\[Gol97\]](#). Let us recall the argument given there. There is a natural surjective map

$$\Psi : \mathrm{Hom}(\pi_1(X), \mathrm{SL}_2)/\mathrm{PGL}_2 \longrightarrow \mathrm{Hom}(\pi_1(X), \mathrm{SL}_2)//\mathrm{PGL}_2 =: \mathcal{X}$$

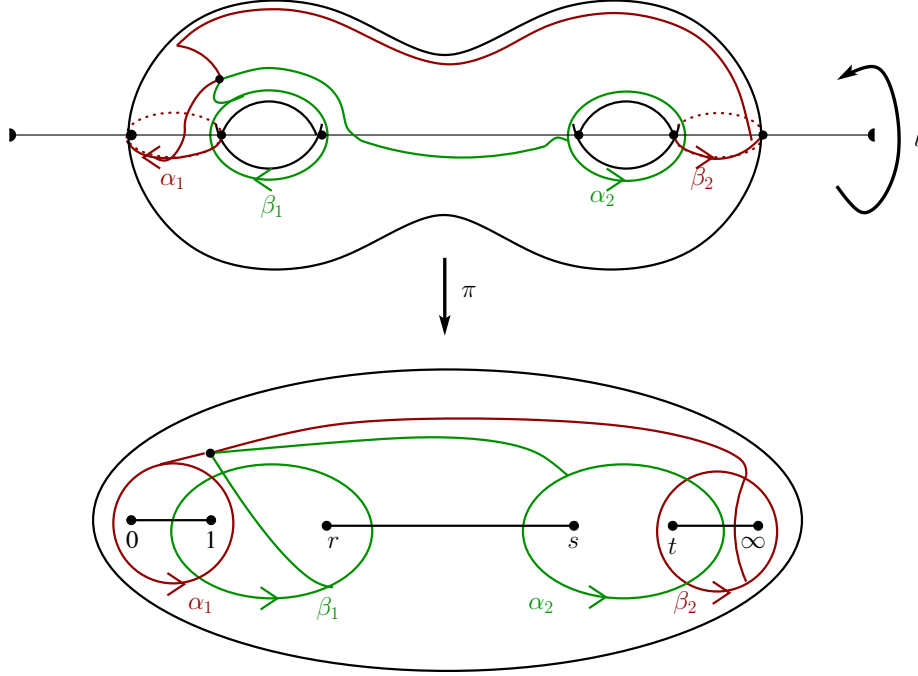


Figure 1: Elements of  $\pi_1(\mathbb{P}^1 \setminus \underline{W}, x)$  that lift as the generators of  $\pi_1(X, w)$ .

to the GIT quotient  $\chi$ , which is an affine variety. The singular locus is the image of reducible representations. There can be many different classes  $[\rho]$  over each singular point. The smooth locus of  $\chi$  however is the geometric quotient of irreducible representations, which are called stable points in this context. The above map  $\Psi$  is injective over this open subset. The involution  $\iota$  acts on  $\chi$  as a polynomial automorphism and we want to prove that this action is trivial. First note that the canonical Fuchsian representation given by the uniformisation  $\mathbb{H} \rightarrow X$  must be invariant under the hyperelliptic involution  $\iota : X \rightarrow X$ . The corresponding point in  $\chi$  therefore is fixed by  $\iota$ . On the other hand, the definition of  $\chi$  only depends on the topology of  $X$  and, considering all possible complex structures on  $X$ , we now get a large set of fixed points  $\chi_{\text{Fuchsian}} \subset \chi$ . Those Fuchsian representations actually form an open subset of  $\text{Hom}(\pi_1(X), \text{SL}_2\mathbb{R}) // \text{SL}_2\mathbb{R}$ , and thus a Zariski dense subset of  $\chi$ . It follows that the action of  $\iota$  is trivial on the whole space  $\chi$ . By injectivity of  $\Psi$ , any irreducible representation is  $\iota$ -invariant.

In other words, if an irreducible representation  $\rho$  is defined by matrices  $A_i, B_i \in \text{SL}_2$  for  $i = 1, 2$  with  $[A_1, B_1] \cdot [A_2, B_2] = I_2$ , then there exists  $M \in \text{GL}_2\mathbb{C}$  satisfying:

$$\begin{cases} M^{-1}A_1M = A_1^{-1} \\ M^{-1}B_1M = B_1^{-1} \end{cases} \quad \begin{cases} M^{-1}A_2M = CA_2^{-1}C^{-1} \\ M^{-1}B_2M = CB_2^{-1}C^{-1} \end{cases} \quad \text{with } C = B_1^{-1}A_1^{-1}B_2A_2. \quad (3.11)$$

Since the action of  $\iota$  is involutive,  $M^2$  commutes with  $\rho$  and is thus a scalar matrix. The matrix  $M$  has two opposite eigenvalues which can be normalized to  $\pm 1$  after replacing  $M$  by a scalar multiple. There are exactly two such normalizations, namely  $M$  and  $-M$ .

For an abelian representation  $\rho$  defined by matrices  $A_i, B_i \in \text{SL}_2$  for  $i = 1, 2$ , the action of  $\iota$  is simply given by

$$A_i \mapsto A_i^{-1} \quad \text{and} \quad B_i \mapsto B_i^{-1} \quad \text{for } i = 1, 2.$$

We can easily exhibit a conjugation of  $\rho$  equivalent to this action. Indeed, in the abelian case, up to conjugacy, we have:

- either  $A_1, B_1, A_2, B_2$  are diagonal and one can choose  $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,
- or  $A_1, B_1, A_2, B_2$  are upper triangular with diagonal  $\pm I_2$  and  $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  works.

It remains to consider reducible  $\iota$ -invariant representations. In the strict reducible case (i.e., reducible but not diagonal), there is a unique common eigenvector for all matrices  $A_1, B_1, A_2, B_2$ ; the representation  $\rho$  restricts to it as a representation  $\pi_1(X) \rightarrow \mathbb{C}^*$  which must be  $\iota$ -invariant. This (abelian) representation must therefore take values in  $\{\pm 1\}$ . It follows that any reducible  $\iota$ -invariant representation is abelian.

Let us now prove (b) $\Leftrightarrow$ (c). Given a representation  $[\rho] \in \text{Hom}^*(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2\mathbb{C})/\text{PGL}_2$ , its image under  $\Phi^{\text{top}}$  is  $\iota$ -invariant, because the action of  $\iota$  coincides in this case with the conjugacy by  $\rho(\gamma_1) \in \text{GL}_2\mathbb{C}$ . Conversely, let  $[\rho] \in \text{Hom}(\pi_1(X), \text{SL}_2)/\text{PGL}_2$  be  $\iota$ -invariant, i.e.,  $\iota^*\rho = M^{-1} \cdot \rho \cdot M$  for some  $M \in \text{GL}_2\mathbb{C}$  as in (3.11). From the cases discussed above, we know that  $M$  can be chosen with eigenvalues  $\pm 1$ . Then setting

$$\begin{cases} M_0 := A_1 M \\ M_1 := M \\ M_r := B_1 M \end{cases} \quad \begin{cases} M_s := B_2^{-1} A_1 B_1 M \\ M_t := A_1 B_1 M A_2 B_2 \\ M_\infty := A_1 B_1 M A_2 \end{cases}$$

we get a preimage of  $[\rho]$ , mapping  $\gamma_i$  to  $M_i$ . The preimage depends only of the choice of  $M$ . Any other choice reads  $M' := CM$  with  $C$  commuting with  $\rho$ . In the general case, i.e., when  $\rho$  is irreducible, we get two preimages given by  $M$  and  $-M$ . However, when  $\rho$  is diagonal, we get only one preimage, because the anti-diagonal matrices  $M$  and  $-M$  are conjugated by a diagonal matrix (commuting with  $\rho$ ).  $\square$

**Corollary 3.3.2.** *The Galois involution of the double cover  $\Phi^{\text{top}}$  is given by*

$$\left\{ \begin{array}{ccc} \text{Hom}^*(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2\mathbb{C})/\text{PGL}_2 & \longrightarrow & \text{Hom}^*(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2\mathbb{C})/\text{PGL}_2 \\ [\rho] & \mapsto & [-\rho] \end{array} \right\}.$$

So far, Theorem 3.3.1 provides an analytic description of the map  $\Phi$ : although  $\Phi^{\text{top}}$  is a polynomial branching cover, the Riemann-Hilbert correspondence is only analytic. In the next section, we will follow a more direct approach providing algebraic informations about  $\Phi$ . However, note that we can already deduce the following:

**Corollary 3.3.3.** *Any tracefree holomorphic connection  $(E, \nabla)$  on  $X$  which is either irreducible or totally reducible is invariant under the hyperelliptic involution: there exists a bundle isomorphism  $h: E \rightarrow \iota^*E$  conjugating  $\nabla$  with  $\iota^*\nabla$ . We can moreover assume  $h \circ \iota^*h = \text{id}_E$  and the restriction of  $h$  to the fibre  $E_w = \iota^*E_w$  over each Weierstrass point  $w = \iota(w) \in X$  is an automorphism with simple eigenvalues  $\pm 1$ . Moreover, if  $\nabla$  is irreducible, then  $h$  is unique up to a sign.*

## Symmetry group

The 16-order group of 2-torsion characters  $\text{Hom}(\pi_1(X), \{\pm 1\})$  acts on the space of representations  $\text{Hom}(\pi_1(X), \text{SL}_2)/\text{PGL}_2$  by multiplication, changing signs of the images  $A_i, B_i$  of the generators  $\alpha_i, \beta_i$ . This corresponds to the action by twist (see § 3.2.1) of 2-torsion line bundles on the moduli space  $\mathfrak{Con}(X)$ . Together with the involution of Corollary 3.3.2, we get a 32-order group acting on

$$\text{Hom}^*(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2\mathbb{C})/\text{PGL}_2,$$

changing signs of the images  $M_i$  of the generators  $\gamma_i$ . Note that only even numbers of sign changes can occur. The action of this torsion group can be described as follows (see (3.9)):

$$\begin{array}{ccc}
& (B_1, A_1, B_2, A_2) & (M_0, M_1, M_r, M_s, M_t, M_\infty) \\
\mathcal{O}_X([w_0] - [w_1]) & (-, +, +, +) & (-, -, +, +, +, +) \\
\mathcal{O}_X([w_1] - [w_r]) & (+, -, +, +) & (+, -, -, +, +, +) \\
\mathcal{O}_X([w_s] - [w_t]) & (+, +, -, +) & (+, +, +, -, -, +) \\
\mathcal{O}_X([w_t] - [w_\infty]) & (+, +, +, -) & (+, +, +, +, -, -) \\
\mathcal{O}_X & (+, +, +, +) & (-, -, -, -, -, -)
\end{array}$$

The quotient for this action identifies with one of the two connected components of

$$\mathrm{Hom}(\pi_1(X), \mathrm{PGL}_2) / \mathrm{PGL}_2,$$

namely the component of those representations that lift to  $\mathrm{SL}_2$ . We have seen in Theorem 3.3.1 that the fixed point set of the Galois involution of  $\Phi^{\mathrm{top}}$  is given by diagonal representations. We can also compute the fixed point locus of  $\mathcal{O}_X([w_0] - [w_1])$  for instance.

**Proposition 3.3.4.** *The fixed point locus of the action of  $\mathcal{O}_X([w_0] - [w_1])$  (with its unitary connection) on the space of representations  $\mathrm{Hom}(\pi_1(X), \mathrm{SL}_2) / \mathrm{PGL}_2$  is parametrized by:*

$$A_1 = \pm I, \quad B_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$$

with  $(a, b) \in \mathbb{C}^* \times \mathbb{C}^*$ .

### 3.3.2 A direct algebraic approach

Let  $(E, \nabla)$  be a holomorphic tracefree rank 2 connection on  $X$ , and let  $h$  be a  $\nabla$ -invariant lift to the vector bundle  $E$  of the action of  $\iota$  on  $X$  as in Corollary 3.3.3. Following [Bis97] and [BH13b], we can associate a parabolic logarithmic connection  $(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}})$  on  $\mathbb{P}^1$  with polar divisor  $\underline{W}$  and a natural choice of parabolic weights  $\underline{\mu}$ . Let us briefly recall this construction. The isomorphism  $h$  induces a non-trivial involutive automorphism on the rank 4 bundle  $\pi_*E$  on  $\mathbb{P}^1$ . The spectrum of such an automorphism is  $\{-1, +1\}$  with respective multiplicities 2, which yields a splitting  $\pi_*E = \underline{E} \oplus \underline{E}'$  with  $\underline{E}$  denoting the  $h$ -invariant subbundle.

In local coordinates, the automorphism  $h$  acts on  $\pi_*E$  in the following way. If  $U \subset X$  is a sufficiently small open set outside of the critical points, we have  $\Gamma(\pi(U), \pi_*E) = \Gamma(U, E) \oplus \Gamma(\iota(U), E)$  and  $h$  permutes both direct summands. Locally at a Weierstrass point with local coordinate  $y$ , one can choose sections  $e_1$  and  $e_2$  generating  $E$  such that  $h(e_1) = e_1$  and  $h(e_2) = -e_2$  (recall that  $h$  has eigenvalues  $\pm 1$  in restriction to the Weierstrass fiber). On the corresponding open set of  $\mathbb{P}^1$ , the bundle  $\pi_*E$  is generated by  $\langle e_1, e_2, ye_1, ye_2 \rangle$ , and we see that  $\langle e_1, ye_2 \rangle$  spans the  $h$ -invariant subspace. Since the connection  $\nabla$  on  $E$  is  $h$ -invariant, we can choose the sections  $e_1$  and  $e_2$  above to be horizontal for  $\nabla$ . Then considering the basis  $\underline{e}_1 = e_1$  and  $\underline{e}_2 = ye_2$  of  $\underline{E}$ , we get

$$\underline{\nabla} \underline{e}_1 = \nabla e_1 = 0 \quad \text{and} \quad \underline{\nabla} \underline{e}_2 = \nabla ye_2 = dy \otimes e_2 = \frac{dy}{y} \otimes \underline{e}_2 = \frac{1}{2} \frac{dx}{x} \otimes \underline{e}_2$$

so that  $\underline{\nabla}$  is logarithmic with eigenvalues 0 and  $\frac{1}{2}$ . To each pole in  $\underline{W}$ , we associate the parabolic  $\underline{p}_i$  defined by the eigenspace with eigenvalue  $\frac{1}{2}$ , with the natural (in the sense of [BH13b]) parabolic weight  $\underline{\mu}_i = \frac{1}{2}$ .

$$\begin{array}{ccccc}
(\tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}}) & \xleftarrow{\text{elm}_{\tilde{W}}^-} & (E, \nabla, \mathbf{p}) & \xleftrightarrow{h} & (E, \nabla) & \xleftrightarrow{-h} & (E, \nabla, \mathbf{p}') & \xrightarrow{\text{elm}_{\tilde{W}}^-} & (\tilde{E}', \tilde{\nabla}', \tilde{\mathbf{p}}') \\
\uparrow \pi^* & & & & \downarrow \pi_* & & & & \uparrow \pi^* \\
(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}}) & \xleftrightarrow{\text{eig}_{\frac{1}{2}}} & (\underline{E}, \underline{\nabla}) & \longleftrightarrow & (\underline{E}, \underline{\nabla}) \oplus (\underline{E}', \underline{\nabla}') & \longleftrightarrow & (\underline{E}', \underline{\nabla}') & \xleftrightarrow{\text{eig}_{\frac{1}{2}}} & (\underline{E}', \underline{\nabla}', \underline{\mathbf{p}}') \\
& & & & \xrightarrow{\sqrt{\text{d log}(\underline{W})} \otimes \text{elm}_{\underline{W}}^+} & & & & 
\end{array}$$

Table 2: Hyperelliptic descent, lift and involution.

However, since we consider the rank 2 case, this general construction can also be viewed in the following way (summarized in Table 2): Denote by  $\mathbf{p}$  the parabolic structure on  $E$  defined by the  $h$ -invariant directions over  $W = \{w_0, w_1, w_r, w_s, w_t, w_\infty\}$  and associate the natural homogenous weight  $\mu = 0$ . In the coordinates above, the basis  $(\underline{e}_1, \underline{e}_2)$  generates the vector bundle  $E$  after one negative elementary transformation in that direction. Now the hyperelliptic involution acts trivially on the parabolic logarithmic connection on  $X$  defined by

$$(\tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}}, \tilde{\mu}) := \text{elm}_{\tilde{W}}^-(E, \nabla, \mathbf{p}, \mu)$$

and we have

$$(\tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}}, \tilde{\mu}) = \pi^*(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}}, \underline{\mu}).$$

### Galois involution and symmetry group

With the notations above, let  $(\underline{E}', \underline{\nabla}')$  be the connection on  $\mathbb{P}^1$  we obtain for the other possibility of a lift of the hyperelliptic involution on  $(E \rightarrow X, \nabla)$ , namely for  $h' = -h$ . It is straightforward to check that the map from  $(\underline{E}', \underline{\nabla}', \underline{\mathbf{p}}')$  to  $(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}})$  and vice-versa is obtained by the elementary transformations  $\text{elm}_{\underline{W}}^+$  over  $\mathbb{P}^1$ , followed by the tensor product with a certain logarithmic rank 1 connection  $\sqrt{\text{d log}(\underline{W})}$  over  $\mathbb{P}^1$  we now define:

There is a unique rank 1 logarithmic connection  $(L, \zeta)$  on  $\mathbb{P}^1$  having polar divisor  $\underline{W}$  and eigenvalues 1; note that  $L \simeq \mathcal{O}_{\mathbb{P}^1}(-6)$ . We denote by  $\text{d log}(\underline{W})$  this connection and by  $\sqrt{\text{d log}(\underline{W})}$  its unique square root. In a similar way, define  $\sqrt{\text{d log}(\underline{D})}$  for any even order subdivisor  $\underline{D} \subset \underline{W}$ .

The Galois involution of our map  $\Phi : \mathfrak{Con}(X/\iota) \rightarrow \mathfrak{Con}(X)$  is therefore given by

$$\sqrt{\text{d log}(\underline{W})} \otimes \text{elm}_{\underline{W}}^+ : \mathfrak{Con}(X/\iota) \rightarrow \mathfrak{Con}(X/\iota).$$

There is a 16-order group of symmetries on  $\mathfrak{Con}(X)$ , consisting of twists by 2-torsion line bundles. It can be lifted as a 32-order group of symmetries on  $\mathfrak{Con}(X/\iota)$ , namely those transformations  $\sqrt{\text{d log}(\underline{D})} \otimes \text{elm}_{\underline{D}}^+$  with  $\underline{D} \subset \underline{W}$  even. For instance, if  $\underline{D} = w_i + w_j$ , then its action on  $\mathfrak{Con}(X/\iota)$  corresponds via  $\Phi$  to the twist by the 2-torsion connection on  $\mathcal{O}_X(w_i + w_j - K_X)$ . In particular, it permutes the two parabolics (of  $\mathbf{p}$  and  $\mathbf{p}'$ ) over  $w_i$  and  $w_j$ .

### Possible parabolic bundles

Note that the triples  $(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}})$  arising from hyperelliptic descent as above satisfy the following:

- $\underline{E}$  is a rank 2 vector bundle over  $\mathbb{P}^1$ ,

- $\nabla : \underline{E} \rightarrow \underline{E} \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$  is a rank 2 logarithmic connection on  $\underline{E}$  with polar divisor  $\underline{W} = [0] + [1] + [r] + [s] + [t] + [\infty]$  and residual eigenvalues 0 and  $\frac{1}{2}$  over each pole,
- $\underline{p} = (\underline{p}_0, \underline{p}_1, \underline{p}_r, \underline{p}_s, \underline{p}_t, \underline{p}_\infty)$  the parabolic structure defined by the  $\frac{1}{2}$ -eigendirections over  $x = 0, 1, r, s, t, \infty$ .

From Fuchs' relation, [Bis02], [AL97, Prop. 3], and a formula due to Brunella (see for example [Heu09, p. 464]) one can deduce the following complete characterization of possible parabolic bundles  $(\underline{E}, \underline{p})$  (see [HL15, Prop. 5.1]).

**Proposition 3.3.5.** *Given a parabolic bundle  $(\underline{E}, \underline{p})$  of rank 2 with parabolic structure supported by  $\underline{W}$ , there exists a logarithmic connection  $\nabla$  as above if and only if*

- either  $(\underline{E}, \underline{p})$  is indecomposable and then  $\underline{E} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  or  $\underline{E} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$
- or  $\underline{E} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  with 2 parabolics defined by the fibres over the Weierstrasspoints of the line subbundle  $\mathcal{O}_{\mathbb{P}^1}(-1)$ , the 4 other ones by  $\mathcal{O}_{\mathbb{P}^1}(-2)$ ,
- or  $\underline{E} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$  with all parabolics defined by  $\mathcal{O}_{\mathbb{P}^1}(-3)$ .

### 3.3.3 Bundle-type terminology

Recall the well-known flatness criterion for vector bundles over curves [Wei38, Ati57].

**Theorem 3.3.6** (Weil). *A holomorphic vector bundle on a compact Riemann surface is flat, i.e., it admits a holomorphic connection, if and only if it is the direct sum of indecomposable bundles of degree 0.*

Recall from the introduction that we denote by  $\mathfrak{Bun}(X)$  the set of isomorphism classes of vector bundles  $E \rightarrow X$  such that there exists  $(E, \nabla) \in \mathfrak{Con}(X)$ . In other words, flat rank 2 vector bundles  $E \rightarrow X$  with trivial determinant bundle  $\det(E) \simeq \mathcal{O}_X$ . From Weil's criterion, we get the following list of such bundles (up to isomorphism):

1. **flat stable bundles:** stable bundles are automatically indecomposable; therefore, all isomorphism classes of stable rank 2 bundles  $E \rightarrow X$  with trivial determinant bundle are elements of  $\mathfrak{Bun}(X)$ .
2. **flat strictly semistable bundles:**
  - (a) decomposable bundles of the form  $E = L \oplus L^{-1}$ , where  $L \in \text{Jac}(X)$  is a degree 0 line bundle. We distinguish between the following :
    - i. *generic decomposable bundles:* those satisfying  $L^{\otimes 2} \not\simeq \mathcal{O}_X$ ; they form a 2-parameter family of isomorphism classes.
    - ii. 16 *special decomposable bundles:* those satisfying  $L^{\otimes 2} \simeq \mathcal{O}_X$ , namely the trivial bundle  $E_0 = \mathcal{O}_X \oplus \mathcal{O}_X$  and its 15 *twists*  $E_0 \otimes L_0$  with  $L_0 \not\simeq L_0^{\otimes 2} \simeq \mathcal{O}_X$ ; we write those special decomposable bundles  $E_\tau = E_0 \otimes \mathcal{O}_X(\tau)$  with  $\tau = [w_i] - [w_j]$  and  $i, j \in \{0, 1, r, s, t, \infty\}$ .
  - (b) non-split extensions  $0 \rightarrow L \rightarrow E \rightarrow L^{-1} \rightarrow 0$ , where  $L \in \text{Jac}(X)$  is a degree 0 line bundle; one shows easily that such bundles are indecomposable and hence flat. We distinguish between:
    - i. *affine bundles:* those satisfying  $L^{\otimes 2} \not\simeq \mathcal{O}_X$ ; for each such  $L$ , there is (up to isomorphism) a unique corresponding affine bundle  $E_L$ ; note that the vector bundle  $E_L \otimes L$  can be endowed with (non tracefree) holomorphic connections with monodromy in the affine group  $\text{Aff}(\mathbb{C}) = \begin{pmatrix} \mathbb{C}^* & \mathbb{C} \\ 0 & 1 \end{pmatrix}$ .

- ii. *unipotent bundles*: those satisfying  $L = \mathcal{O}_X$ ; since  $\mathbb{P}H^1(X, \mathcal{O}_X) \simeq \mathbb{P}^1$ , they form a 1-parameter family of isomorphism classes of bundles; note that unipotent bundles can be endowed with tracefree holomorphic connections with unipotent monodromy.
- iii. *twists of unipotent bundles*; each unipotent bundle can be tensorised by  $\mathcal{O}_X(\tau)$  with  $\tau = [w_i] - [w_j]$  as above, preserving flatness; we call twisted unipotent bundles those obtained for the 15 non trivial  $\tau$ 's.

3. **flat unstable bundles**: we call those *Gunning bundles* in reference to [Gun67b]. In particular, a Gunning bundle over  $X$  is an unstable indecomposable rank 2 vector bundle with trivial determinant bundle. There are precisely 16 isomorphism classes of such bundles: for each of the 16 *theta characteristics*  $\vartheta \in \text{Pic}^1(X)$  such that  $\vartheta^{\otimes 2} \simeq \Omega_X^1 \simeq \mathcal{O}_X(K_X)$ , there is a unique isomorphism class of indecomposable extension  $0 \rightarrow \vartheta \rightarrow E \rightarrow \vartheta^{-1} \rightarrow 0$ ; we denote the corresponding Gunning bundle  $E = E_\vartheta$ . We distinguish between the

- (a) 6 *odd Gunning bundles*:  $\vartheta = \mathcal{O}_X([w_i])$ ,  $i \in \{0, 1, r, s, t, \infty\}$
- (b) 10 *even Gunning bundles*:  $\vartheta = \mathcal{O}_X([w_i] + [w_j] - [w_\infty])$ ,  $i \neq j \in \{0, 1, r, s, t\}$ .

For geometrical reasons, we further introduce the following terminology:

- **The Gunning planes**  $\Pi_\vartheta \subset \mathfrak{Bun}(X)$ . For any of the 16 theta characteristics  $\vartheta$ , we refer to bundles in  $\mathfrak{Bun}(X)$  that arise as a non-split extension  $0 \rightarrow \vartheta^{-1} \rightarrow E \rightarrow \vartheta \rightarrow 0$ , and that moreover are not affine bundles, as *elements of the Gunning plane*  $\Pi_\vartheta$ . As we shall see later, Gunning planes are in bijection with actual 2-planes in  $\mathbb{P}_{\text{NR}}^3$ . We speak of odd and even Gunning planes according to the type of  $\vartheta$ .
- **The unipotent family**  $\Delta \subset \mathfrak{Bun}(X)$ . We refer to the set of isomorphism classes of unipotent bundles as the *family*  $\Delta$ . Note that it is naturally parametrized by  $\mathbb{P}^1$ . When  $w$  is a Weierstrass point, we refer to an element of  $\Delta \cap \Pi_{[w]}$  as a *special unipotent bundle* associated to  $w$ . We shall see in Proposition 3.4.1 that there are (up to isomorphism) precisely six special unipotent bundles - one for each Weierstrass point. A *generic unipotent bundle* will be a unipotent bundle that is not a special unipotent bundle.

Following [Gun67a, Thm. 29] and [Mar71], the automorphism group  $\text{Aut}(E)$  and, more importantly, the projectivized automorphism group  $\mathbb{P}\text{Aut}(E) = \text{Aut}(E)/\mathbb{C}^*$  for each vector bundle  $E$  corresponding to an isomorphism class in  $\mathfrak{Bun}(X)$ , is well known. Moreover, a calculation of the dimension  $h^0(X, \mathfrak{sl}(E))$  of the affine space of connections on each fixed  $E$  and a description of the moduli space of irreducible connections up to bundle automorphism for each fixed  $E$  has been detailed in [HL15, § . 1.3]. The result is summarized in Table 3.

bundle type	$E$	$\mathbb{P}\text{Aut}(E)$	connections	moduli in $\mathfrak{Con}^{\text{irr}}(X)$
stable	$E$	1	$\mathbb{A}^3$	$\mathbb{A}^3$
decomposable	$L \oplus L^{-1}$	$(\mathbb{C}^*, \cdot)$	$\mathbb{A}^4$	$\mathbb{C}^2 \times \mathbb{C}^*$
affine	$L \rightarrow E_L \rightarrow L^{-1}$	1	$\mathbb{A}^3$	$\emptyset$
trivial+twists	$E_0, E_\tau$	$\text{PGL}_2(\mathbb{C})$	$\mathbb{A}^6$	$\mathbb{C}_{(\nu_0, \nu_1, \nu_2)}^3 \setminus \{\nu_1^2 = 4\nu_0\nu_2\}$
unipotent+twists	$\tau \rightarrow E \rightarrow \tau$	$(\mathbb{C}, +)$	$\mathbb{A}^4$	$\mathbb{C}^2 \times \mathbb{C}^*$
Gunning	$\vartheta \rightarrow E_\vartheta \rightarrow \vartheta^{-1}$	$H^0(X, \Omega_X^1)$	$\mathbb{A}^5$	$\mathbb{A}^3$

Table 3: Moduli spaces of irreducible connections for each bundle type.

We notice that a curious phenomenon occurs: *when  $E$  is an affine bundle, then all (holomorphic, tracefree) connections on  $E$  are reducible, and none of them are totally reducible*. Indeed,  $E$  has a



unique subbundle  $L$  of degree 0, which is not isomorphic to its pull-back under the hyperelliptic involution  $\iota$ . Therefore, the vector bundle  $E$  cannot admit a lift  $h$  of  $\iota$ . This implies that the monodromy of a connection  $\nabla$  on  $E$  can be neither irreducible, nor totally reducible. This phenomenon does not occur for higher genus [Hit87b, Prop. 3.3, p. 70]. Even though affine bundles do not allow hyperelliptic descent, we shall see in § 3.4.7 that they admit another natural construction by elementary transformations, due to Tyurin [Tyu64] (see also § 3.5.3).

### 3.3.4 Tyurin subbundles

Let  $E \in \mathfrak{Bun}(X)$  be a flat vector bundle. We call *Tyurin subbundle* of  $E$  any line subbundle  $L \subset E$  obtained by saturation of the inclusion of locally free sheaves  $\mathcal{O}_X(-K_X) \hookrightarrow E$  defined by a non zero element  $\varphi \in H^0(\text{Hom}(\mathcal{O}_X(-K_X), E))$ . A Tyurin subbundle  $L \subset E$  is called *degenerate* when  $L \not\subset \mathcal{O}_X(-K_X)$ .

Assume that  $E$  arises from hyperelliptic descent, i.e., that lies in the image of the composition of  $\Phi$  in (3.7) and the forgetful map. From Table 3 and the remark on affine bundles thereafter, we know from Corollary 3.3.3 that there exists a lift  $h : E \rightarrow \iota^*E$  of the hyperelliptic involution  $\iota : X \rightarrow X$ , which moreover acts non-trivially with two distinct eigenvalues on each Weierstrass fiber  $E|_w$ . The involution  $\iota$  acts linearly on  $\mathcal{O}_X(-K_X)$  and therefore  $h$  acts on  $H^0(\text{Hom}(\mathcal{O}_X(-K_X), E))$ . Since it is involutive, this action induces a splitting

$$H^0(\text{Hom}(\mathcal{O}_X(-K_X), E)) = H^+ \oplus H^-$$

into eigenspaces (relative to  $\pm 1$  eigenvalues).

**Proposition 3.3.7.** *Let  $E$ ,  $h$  and  $H^+ \oplus H^-$  be as above. Then*

$$\dim(H^+ \oplus H^-) = \begin{cases} 3 & \text{if } E \text{ is unipotent, or an odd Gunning bundle} \\ 4 & \text{if } E \text{ is the trivial bundle} \\ 2 & \text{else .} \end{cases}$$

*If  $E$  is not an even Gunning bundle, then both  $H^+$  and  $H^-$  have positive dimension, and there are two distinct Tyurin subbundles  $L^+$  and  $L^-$  of  $E$  such that all non-zero elements of  $H^\pm$  give rise to  $L^\pm$ . They are  $h$ -invariant and they are the only  $h$ -invariant Tyurin subbundles.*

*Proof.* One easily calculates  $h^0(X, E)$  for each bundle type in § 3.3.3. Serre duality, together with the fact that rank two vector bundles with trivial determinant bundle are self-dual, yields the dimension of  $H := H^+ \oplus H^-$ . Considering  $\dim(H) \geq 2$ , one checks that not all elements of  $H \setminus \{0\}$  can give rise to the same subbundle  $L \subset E$  unless  $E$  is an even Gunning bundle. Assume that  $\varphi_1$  and  $\varphi_2$  in  $H$  give rise to distinct subbundles. In the following, we consider the  $\varphi_i$ 's as maps from the total space of the line bundle  $\Omega := \mathcal{O}_X(-K_X)$  to the total space of  $E$ . When combined, their images span all fibers of  $E$ , except those over the vanishing divisor of the map  $\varphi_1 \wedge \varphi_2 : \Omega \otimes \Omega \rightarrow \det(E) = \mathcal{O}_X$ . This divisor is an element of  $|2K_X|$  and therefore of the form

$$[P] + [\iota(P)] + [Q] + [\iota(Q)]$$

for some  $P, Q \in X$ . In particular, the  $\varphi_i$ 's generate the fiber over at least one of the six Weierstrass points. But the action of  $h$  on the Weierstrass fiber has two opposite eigenvalues. Hence the  $\varphi_i$ 's cannot both belong to the same eigenspace of  $H$ . Let us assume that  $\varphi_1$  and  $\varphi_2$  yield the now well defined  $L^+$  and  $L^-$  as in the statement and let  $\varphi \in H$  correspond to a Tyurin subbundle  $L$  that is neither  $L^+$  nor  $L^-$ . We may assume  $\varphi = \varphi_1 + \varphi_2$ . Considering the distinct eigenvalues of the action of  $h$  for  $\varphi_1$  and  $\varphi_2$  over the Weierstrass points,  $L$  cannot be  $h$ -invariant.  $\square$

Let us denote, as in Table 2, by  $\mathbf{p}$  and  $\mathbf{p}'$  the parabolic structures on  $E$  determined by the  $+1$  and  $-1$ -eigendirections respectively of the action of  $h$  on the fibers over the Weierstrass points. When one of the two  $h$ -invariant Tyurin subbundles in the above proposition, say  $L^+$ , is non-degenerate, then the action of  $h$  on  $H^+$  induces an action on the global non-vanishing sections of  $E \otimes \mathcal{O}_X(K_X)$  corresponding to  $L^+ \otimes \mathcal{O}_X(K_X) \simeq \mathcal{O}_X$ . In particular, the parabolic structure on  $E$  defined by the fibers  $L^+|_w \subset E|_w$  over the Weierstrass points then coincides with  $\mathbf{p}$ . Similarly, when  $L^-$  is non-degenerate, then it determines  $\mathbf{p}'$ . So when  $L^\pm$  are both non-degenerate, then they determine both of the hyperelliptic parabolic structures  $\mathbf{p}$  and  $\mathbf{p}'$  on  $E$ . This is the case for generic stable bundles. Indeed, when  $E$  is stable and admits a degenerate  $h$ -invariant Tyurin subbundle  $L$ , then we necessarily have  $L \simeq \mathcal{O}(-[w])$ , where  $w$  is a Weierstrass point. Such a bundle is by definition an element of the odd Gunning plane  $\Pi_{[w]}$  (see §3.3.3).

A case-by-case study for each type of flat bundle of hyperelliptic parabolic structures up to bundle automorphism that are determined by the invariant Tyurin subbundles  $L^\pm$  will be included in the discussion in Section 3.4. The result is summarized in Table 4.

bundle type	degenerate invariant Tyurin subbundles	parabolic structures $\mathbf{p}, \mathbf{p}'$ (up to autom.) determined by $L^\pm$
stable off odd Gunning planes	$\emptyset$	2 out of 2
stable on $\Pi_{[w_i]}$ , off $\Pi_{[w_j]} \forall j \neq i$	$L^+ \simeq \mathcal{O}_X(-[w_i])$	1 out of 2
stable on $\Pi_{[w_i]} \cap \Pi_{[w_j]}$	$L^+ \simeq \mathcal{O}_X(-[w_i]), L^- \simeq \mathcal{O}_X(-[w_j])$	0 out of 2
generic decomposable off $\Pi_{[w]}$ 's	$\emptyset$	1 out of 1
generic decomposable on $\Pi_{[w]}$	$L^+ \simeq \mathcal{O}_X(-[w]), L^- \simeq \mathcal{O}_X(-[w])$	0 out of 1
$\tau \oplus \tau$ with $\tau^{\otimes 2} \simeq \mathcal{O}_X$	$L^+ \simeq \tau, L^- \simeq \tau$	1 out of 1
generic unipotent (off $\Pi_{[w]}$ 's)	$L^+ \simeq \mathcal{O}_X$	2 out of 2
special unipotent on $\Pi_{[w]}$	$L^+ \simeq \mathcal{O}_X, L^- \simeq \mathcal{O}_X(-[w])$	1 out of 2
twists of unipotent	$L^+ \simeq \mathcal{O}_X([w_i] - [w_j])$	1 out of 2
affine	$\emptyset$	0 out of 0
even Gunning bundle	$L^+ = L^- \simeq \vartheta$	2 out of 2
odd Gunning bundle	$L^+ \simeq \vartheta$	2 out of 2

Table 4: Invariant Tyurin subbundles for the different types of bundles. By definition non-degenerate Tyurin subbundles are isomorphic to  $\mathcal{O}_X(-K_X)$ .

### 3.4 A dictionary for hyperelliptic descent

Let us now consider the lower part of the diagram

$$\begin{array}{ccc}
 \mathfrak{Con}(X/\iota) & \xrightarrow{\Phi} & \mathfrak{Con}(X) \\
 \downarrow & & \downarrow \\
 \mathfrak{Bun}(X/\iota) & \xrightarrow{\phi} & \mathfrak{Bun}(X) .
 \end{array} \tag{3.12}$$

Recall that  $\Phi$  has been defined in (3.7), vertical arrows correspond to forgetful maps and  $\mathfrak{Bun}(X/\iota)$  denotes precisely the set of isomorphism classes of parabolic bundles  $(\underline{E}, \underline{\mathbf{p}})$  that arise from parabolic connections  $(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}})$  in  $\mathfrak{Con}(X/\iota)$  by forgetting  $\underline{\nabla}$ .

In particular, the map  $\phi$  in (3.12) is constructed as follows. Given a flat parabolic bundle  $(\underline{E}, \underline{\mathbf{p}})$  in  $\mathfrak{Bun}(X/\iota)$ , we lift it up to the curve  $X$  as  $\pi^*(\underline{E}, \underline{\mathbf{p}}) = (\tilde{E}, \tilde{\mathbf{p}})$ , then apply elementary transformations  $(E, \mathbf{p}) := \text{elm}_W^+(\tilde{E}, \tilde{\mathbf{p}})$  over the Weierstrass points and get a vector bundle  $E$  with trivial determinant bundle over  $X$ , an element of  $\mathfrak{Bun}(X)$ .

Conversely, let  $E$  be a bundle in the image of  $\phi$ . Then one can choose a lift

$$h : E \rightarrow \iota^* E \tag{3.13}$$

of the hyperelliptic involution  $\iota$  on  $X$  such that  $\iota^* h \circ h = \text{id}_E$ , and such that for each Weierstrass point  $w \in X$ , the lift  $h$  acts on  $E|_w = \iota^* E|_w$  with two opposite eigenvalues  $\pm 1$  (see Cor. 3.3.3). The  $+1$  and  $-1$  eigendirections respectively define two parabolic structures  $\mathbf{p}$  and  $\mathbf{p}'$  on  $E$ , supported by the Weierstrass divisor. The parabolic bundles

$$(E, \mathbf{p}) \quad \text{and} \quad (E, \mathbf{p}') \tag{3.14}$$

arise, via  $\phi = \text{elm}_W^+ \circ \pi^*$ , from elements  $(\underline{E}, \underline{\mathbf{p}})$  and  $(\underline{E}', \underline{\mathbf{p}'})$  in  $\mathfrak{Bun}(X/\iota)$ . The latter are moreover interchanged by the Galois involution  $\mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_W^+$  (see § 3.3.2).

We now explain which parabolic bundles in  $\mathfrak{Bun}(X/\iota)$  give rise to which types of bundles in  $\mathfrak{Bun}(X)$  and illustrate on pictures the corresponding configurations of curves and points on the ruled surfaces.

### 3.4.1 Stable bundles (Figure 2)

When  $E$  is stable, the only bundle automorphisms of  $E$  are homotheties, so that the lift  $h$  in (3.13) does not depend on the choice of a connection. It is (up to a sign) uniquely determined by the bundle  $E$ . By Proposition 3.3.7 there are precisely two  $h$ -invariant Tyurin bundles  $L$  and  $L'$  of  $E$ . When  $E$  is stable and is not a member of one of the odd Gunning planes  $\Pi_{[w_i]}$ , these Tyurin subbundles are non-degenerate:

$$L, L' \simeq \mathcal{O}_X(-K_X).$$

The two parabolic structures  $\mathbf{p}$  and  $\mathbf{p}'$  in (3.14) then are precisely the fibres over the Weierstrass points of two line subbundles  $L, L'$ . Note that the roles of  $\mathbf{p}$  and  $\mathbf{p}'$  can be interchanged by changing the sign of  $h$ , and we may focus on one of them, say  $(E, \mathbf{p})$ . After applying elementary transformations over the Weierstrass points  $(\tilde{E}, \tilde{\mathbf{p}}) := \text{elm}_W^-(E, \mathbf{p})$ , we get the lift of a unique parabolic bundle  $(\underline{E}, \underline{\mathbf{p}})$  on  $X/\iota$ ; precisely,  $\tilde{E} = \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-2K_X)$  and  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ . The two anti-canonical subbundles  $L, L' \subset E$ , being  $h$ -invariant, descend as two subbundles of  $(\underline{E}, \underline{\mathbf{p}})$ ; one easily checks that they are the destabilizing bundle  $\underline{L} = \mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \times \mathcal{O}_{\mathbb{P}^1}(-2)$  and the unique  $\underline{L}' \simeq \mathcal{O}_{\mathbb{P}^1}(-4) \subset \underline{E}$  containing all parabolics of  $\underline{\mathbf{p}}$ .

In Figure 2, we can see the projectivized total space of the parabolic bundle associated to  $E$  (a ruled surface), and its two preimages  $\underline{E}$  and  $\underline{E}'$  in  $\mathfrak{Bun}(X/\iota)$ . The anti-canonical subbundles  $L$  and  $L'$  of  $E$ , and the corresponding subbundles of  $\underline{E}$  and  $\underline{E}'$ , are the blue and red curves (sections) on the ruled surfaces. We can see the self-intersection of the curves in each case. Parabolics are just points in Weierstrass fibers; those corresponding to  $\mathbf{p}$  and  $\underline{\mathbf{p}}$  (defined by the blue curve  $L$  up-side) are the red ones and those corresponding to  $\mathbf{p}'$  and  $\underline{\mathbf{p}'}$  (defined by the red curve  $L'$  up-side) are the blue ones. We also indicate, in yellow, the intersection of the two curves in each ruled surface; it is closely related to the so-called Tyurin divisor  $D_E^T$  that will be considered in § 3.5.3. The Galois involution of  $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$  permutes the roles of  $L$  and  $L'$ ; down-side, the elementary transformation permutes the role of the two curves.

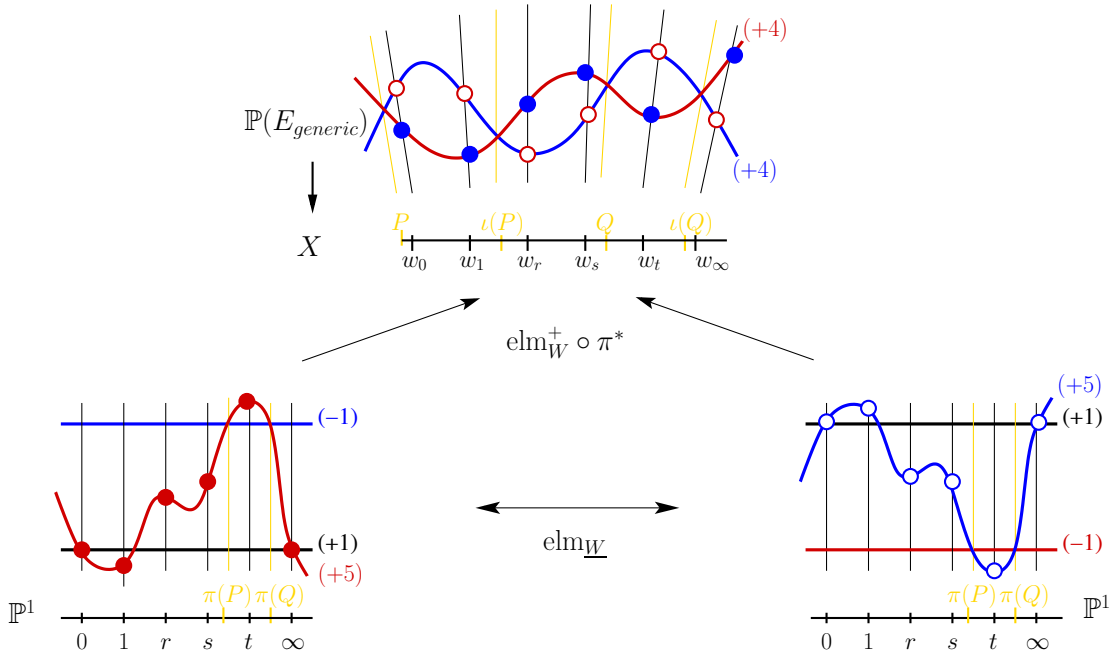


Figure 2: A generic stable bundle on  $X$ .

Consider now the case where  $E$  is stable but belongs to an odd Gunning plane  $\Pi_{[w]}$ . Then one of the two  $h$ -invariant Tyurin subbundles is degenerate, say  $L = \mathcal{O}_X(-[w])$ , and fails to determine the corresponding parabolic structure  $\mathbf{p}$  over the Weierstrass point  $w$ . When  $E \in \Pi_{[w_i]} \cap \Pi_{[w_j]}$ , then the two  $h$ -invariant Tyurin subbundles are degenerate and neither  $\mathbf{p}$ , nor  $\mathbf{p}'$  are entirely determined by these bundles. Figure 3 depicts a stable bundle on an odd Gunning plane. It can be seen as a special case of Figure 2, where one of the two  $+4$ -curves is reducible.

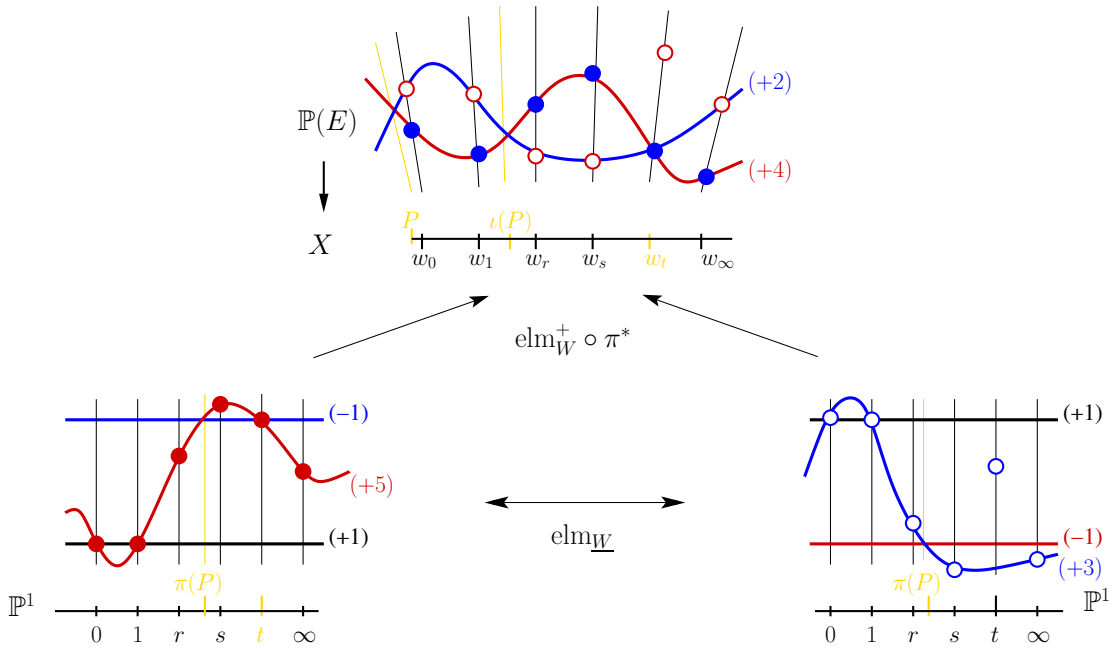


Figure 3: A stable bundle belonging to the odd Gunning plane  $\Pi_{[w_t]}$ .

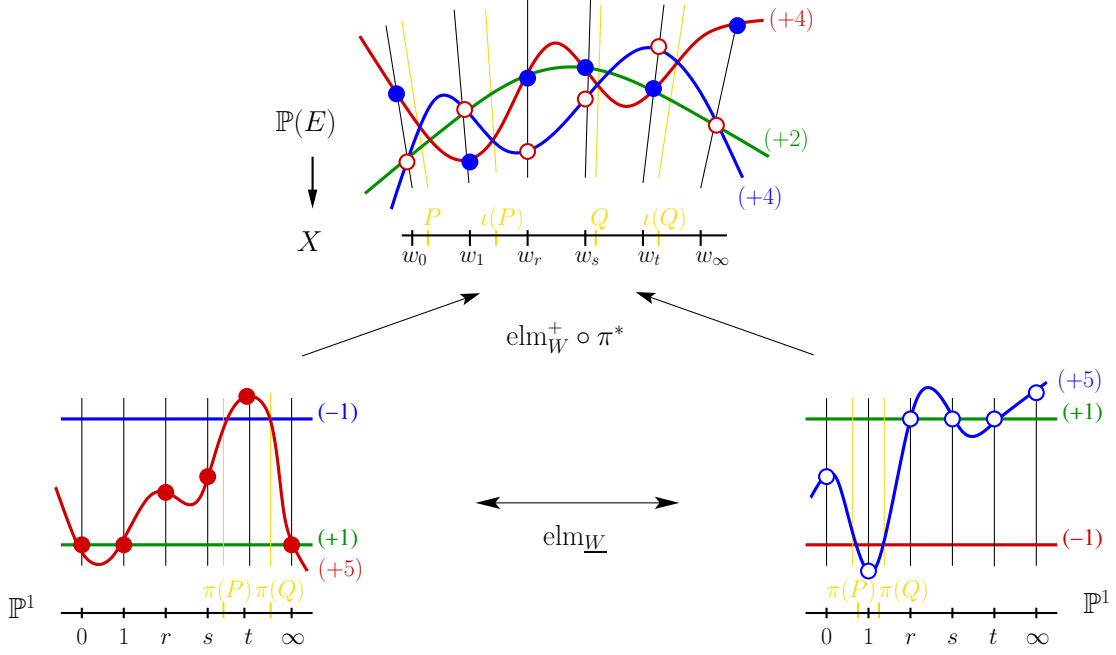


Figure 4: A stable bundle belonging to the even Gunning plane  $\Pi_{[w_r]+[w_s]-[w_t]}$ .

Another special case, drawn in Figure 4, arises when  $\mathbb{P}E$  possesses an invariant (but not anti-canonical)  $+2$ -curve (drawn here in green) containing three parabolics of each type (red and blue). This configuration corresponds to a stable bundle on an even Gunning plane.

### 3.4.2 Gunning planes (Figures 3 and 4)

Let  $\vartheta \in \text{Pic}^1(X)$  with  $\vartheta^{\otimes 2} \simeq \mathcal{O}_X(-K_X)$  be a theta-characteristic. Recall the Gunning plane  $\Pi_\vartheta \subset \mathfrak{Bun}(X)$  is by definition the set of isomorphism classes of bundles in  $\mathfrak{Bun}(X)$  that arise as non-split extensions  $\vartheta^{-1} \rightarrow E \rightarrow \vartheta$  and that admit hyperelliptic descent. Generic members of each Gunning plane are stable bundles as indicated in Figures 3 and 4. Not necessarily assuming stability, one can check that a bundle  $E$  belonging to a Gunning plane yields parabolic bundles  $(\underline{E}, \underline{p})$  and  $(\underline{E}', \underline{p}')$  characterized as follows.

**Odd Gunning planes.** For  $i \in \underline{W}$ , consider  $\vartheta \simeq \mathcal{O}_X([w_i])$ . The odd Gunning plane  $\Pi_\vartheta$  descends as

- $\underline{\Pi}_i := \{(\underline{E}, \underline{p}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_i \subset \mathcal{O}_{\mathbb{P}^1}(-1)\};$
- $\underline{\Pi}'_i := \{(\underline{E}', \underline{p}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}'_k \subset \mathcal{O}_{\mathbb{P}^1}(-3), \forall k \neq i\}.$

**Even Gunning planes.** Denote  $\underline{W} = \{i, j, k\} \cup \{l, m, n\}$  and consider  $\vartheta \simeq \mathcal{O}_X([w_i] + [w_j] - [w_k]) \simeq \mathcal{O}_X([w_l] + [w_m] - [w_n])$ . The even Gunning plane  $\Pi_\vartheta$  descends to

- $\underline{\Pi}_{i,j,k} := \{(\underline{E}, \underline{p}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_i, \underline{p}_j, \underline{p}_k \subset \mathcal{O}_{\mathbb{P}^1}(-2) \subset \underline{E}\};$
- $\underline{\Pi}_{l,m,n} := \{(\underline{E}', \underline{p}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}'_l, \underline{p}'_m, \underline{p}'_n \subset \mathcal{O}_{\mathbb{P}^1}(-2) \subset \underline{E}'\}.$

Recall from § 3.3.2 that the 16-order group of 2-torsion line bundles  $\tau$  with  $\tau^{\otimes 2} \simeq \mathcal{O}_X$  acts on  $\mathfrak{Con}(X)$  and is induced from a group action on  $\mathfrak{Con}(X/\iota)$ . With respect to the map  $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$ , this action is seen as follows. Let  $E \in \mathfrak{Bun}(X)$  be endowed with a lift  $h$  of the hyperelliptic involution as in (3.13) and let  $\tau = \mathcal{O}_X([w_i] - [w_j])$  with  $i \neq j$ . The two parabolic structures  $\underline{p}$  and  $\underline{p}'$

on  $E \otimes \tau$  are obtained from the two parabolic structures on  $E$  by interchanging there the roles of the two parabolics over  $w_i$  and  $w_j$  respectively. Equivalently, the two parabolic bundles over  $\mathbb{P}^1$  associated to  $E \otimes \tau$  by hyperelliptic descent are obtained from those for  $E$  by applying  $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \text{elm}_{[w_i]+[w_j]}^+$  on  $\underline{E}$  and  $\underline{E}'$ .

Note further that the group of 2-torsion line bundles acts on the set of theta-characteristics, and therefore on the set of Gunning planes in  $\mathfrak{Bun}(X)$ . As will shall see in the following paragraphs, the non-stable elements of each odd Gunning plane for example are precisely one special unipotent bundle and a 1-parameter family of generic decomposable bundles.

### 3.4.3 The unipotent family and its 15 twists (Figures 5 and 6)

Let  $E$  be an element of the family  $\Delta \subset \mathfrak{Bun}(X)$  of unipotent bundles. It arises as a non-split extension  $0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0$ . There are many lifts  $h$  of the hyperelliptic involution  $\iota$  since there are non-trivial automorphisms on  $E$ : one can choose  $\pm \iota^* g \circ h \circ g^{-1}$  for any  $g \in \text{Aut}(E)$ . However, since  $E$  admits a unique trivial subbundle, this subbundle must be  $h$ -invariant for any choice of  $h$ . Possibly replacing  $h$  by  $-h$ , we may assume  $L^- = \mathcal{O}_X$ . The following proposition will imply that among the 1-parameter family of unipotent bundles, there are only six special unipotent bundles, and that for those, we have  $L^- = \mathcal{O}_X$  and  $L^+ \simeq \mathcal{O}_X(-[w])$ .

**Proposition 3.4.1.** *Let  $E$  be a unipotent bundle. Then there is a regular map*

$$\varphi_1 \oplus \varphi_2 : \mathcal{O}_X \oplus \mathcal{O}_X(-K_X) \rightarrow E,$$

*invertible over the complement of a divisor  $D \in |\mathbb{K}_X|$ , and  $E$  is, up to isomorphism, uniquely determined by  $D$ .*

*Moreover, when  $w$  is a Weierstrass point and  $D = 2[w]$ , then  $E$  is the special unipotent bundle associated to  $w$  and for any lift  $h$  of the hyperelliptic involution,  $E$  admits a  $h$ -invariant Tyurin subbundle isomorphic to  $\mathcal{O}_X(-[w])$ .*

*Proof.* Let  $\varphi_1 : \mathcal{O}_X \rightarrow E$  and  $\varphi_2 : \mathcal{O}_X(-K_X) \rightarrow E$  be non-zero morphisms such that moreover the line subbundle determined by  $\varphi_2$  is not the unique trivial subbundle of  $E$ . Note that such a  $\varphi_2$  exists by Proposition 3.3.7. Then  $\varphi_1 \wedge \varphi_2 : \mathcal{O}_X(-K_X) \rightarrow \det(E) \simeq \mathcal{O}_X$  is a non-zero morphism, and  $D := \text{div}_0(\varphi_1 \wedge \varphi_2) \in |\mathbb{K}_X|$ . It follows that  $\varphi_1 \oplus \varphi_2$  is a composition of two positive elementary transformations, with support  $P$  and  $\iota(P)$  respectively, determined by  $D = [P] + [\iota(P)]$ . Conversely, let  $D \in |\mathbb{K}_X|$ . Assuming that  $D$  is reduced, one checks that there is only one isomorphism class of parabolic structure on  $\mathcal{O}_X \oplus \mathcal{O}_X(-K_X)$  such that after elementary transformation over  $D$ , one obtains an indecomposable bundle admitting a trivial subbundle. One checks that this remains true, after adapting the notion of parabolic structure, for non-reduced support  $D$ , and that one then obtains the corresponding special unipotent bundle.

Now assume  $D = 2[w]$  and let  $h$  be a lift of the hyperelliptic involution. Then  $h$  acts as a linear involution on  $H^0(X, \text{Hom}(\mathcal{O}_X(-[w]), E))$ . As in the proof of Proposition 3.3.7, one shows that there is at least one eigenvector that does not give rise to the trivial subbundle. It yields the desired subbundle.  $\square$

Consider a generic (i.e., non special) unipotent bundle  $E$ . For any lift  $h$  of the hyperelliptic involution, we then obtain  $L^- = \mathcal{O}_X$  and a non-degenerate invariant Tyurin subbundle  $L^+ \simeq \mathcal{O}_X(-K_X)$ , determining the parabolic structures  $\mathbf{p}'$  and  $\mathbf{p}$ . Note that the automorphism group of  $E$  fixes the trivial subbundle and acts transitively on the set of non-degenerate Tyurin subbundles.

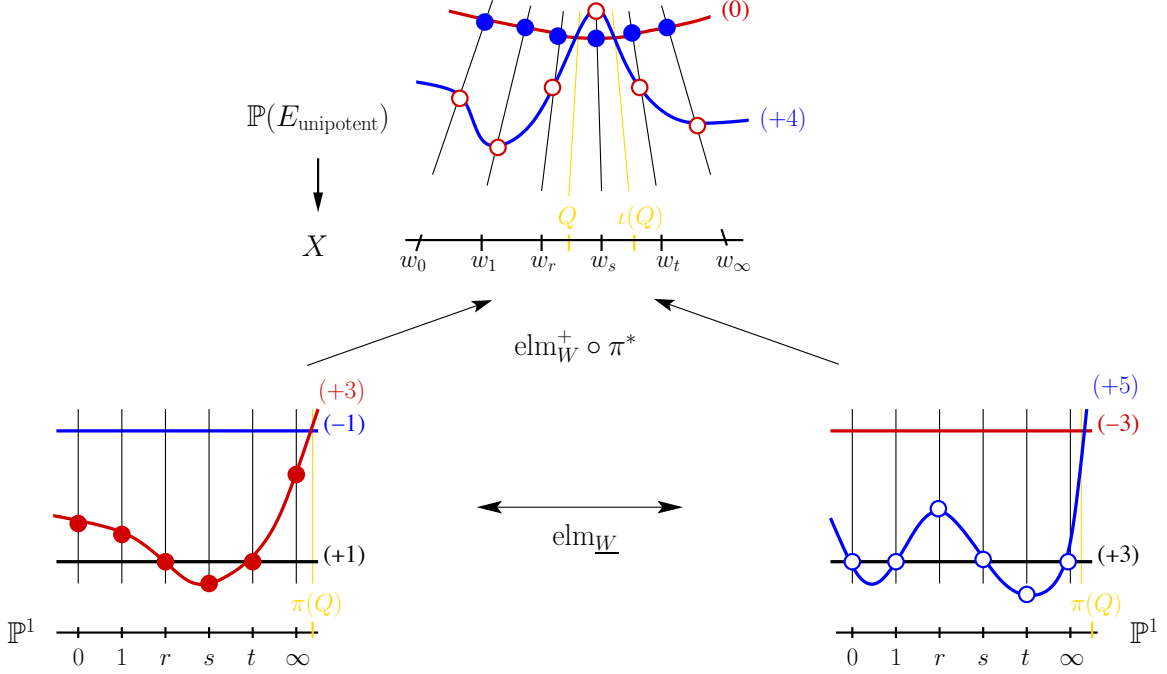


Figure 5: A generic unipotent bundle over  $X$ .

For fixed  $h$ , a generic unipotent bundle has two parabolic structures (see Figure 5):

- $\mathbf{p}$  defined by some subbundle  $\mathcal{O}_X(-K_X) \subset E$ ;
- $\mathbf{p}'$  defined by the destabilizing bundle  $\mathcal{O}_X \subset E$ .

They depend up to automorphism not on the choice of  $h$  and respectively descend to elements (determined by  $E \in \Delta$ ) of the following families

- $\underline{\Delta} := \{(\underline{E}, \underline{\mathbf{p}}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{\mathbf{p}} \subset \mathcal{O}_{\mathbb{P}^1}(-3) \subset \underline{E}\}$ ;
- $\underline{\Delta}' := \{(\underline{E}', \underline{\mathbf{p}}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3) \text{ and } \underline{\mathbf{p}}' \subset \mathcal{O}_{\mathbb{P}^1}(-4) \subset \underline{E}'\}$ .

Note that the case of special unipotent bundles, where  $L^+$  fails to determine the parabolic of  $\mathbf{p}$  over the special Weierstrass point, can be obtained as a limit case. In Figure 5, the +4 and the +5 curve become reducible and the -1 curve obtains a parabolic.

Let  $E$  be a twist of unipotent bundle, in other words an element of  $\Delta$  tensorised by a 2-torsion line bundle  $\tau = \mathcal{O}_X([w_i] - [w_j])$  with  $i \neq j$ . It arises as a non-split extension  $0 \rightarrow \tau \rightarrow E \rightarrow \tau \rightarrow 0$ . After choosing a lift  $h$  of the hyperelliptic involution, Proposition 3.3.7 yields two invariant Tyurin subbundles  $L$  and  $L'$ . One of them, say  $L'$ , is  $\tau \subset E$ , the only Tyurin subbundle of degree 0. It fails to determine the corresponding parabolic structure  $\mathbf{p}'$  over  $w_i$  and  $w_j$ . The intersection  $L \cap L'$  has to be  $[w_i] + [w_j]$  and  $L$  is therefore non-degenerate and defines the parabolic structure  $\mathbf{p}$ . The parabolic structures on  $E$  can moreover be deduced from the ones in the unipotent bundle  $E \otimes \tau$  by permuting the role of the two parabolics over  $w_i$  and  $w_j$  (see § 3.3.2, § 3.4.2).

Hence a twist of unipotent bundle has two hyperelliptic parabolic structures (see Figure 6):

- $\mathbf{p}$  with parabolics  $p_i$  and  $p_j$  on  $\mathcal{O}_X \hookrightarrow E$  and the others outside;
- $\mathbf{p}'$  with parabolics  $p_i$  and  $p_j$  outside  $\mathcal{O}_X \hookrightarrow E$  and the others on it.

They respectively descend as elements of

- $\underline{\Delta}_{i,j} := \{(\underline{E}, \underline{p}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_k \subset \mathcal{O}_{\mathbb{P}^1}(-2), \forall k \neq i, j\}$ ;
- $\underline{\Delta}'_{i,j} := \{(\underline{E}', \underline{p}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}'_i, \underline{p}'_j \subset \mathcal{O}_{\mathbb{P}^1}(-1)\}$ .

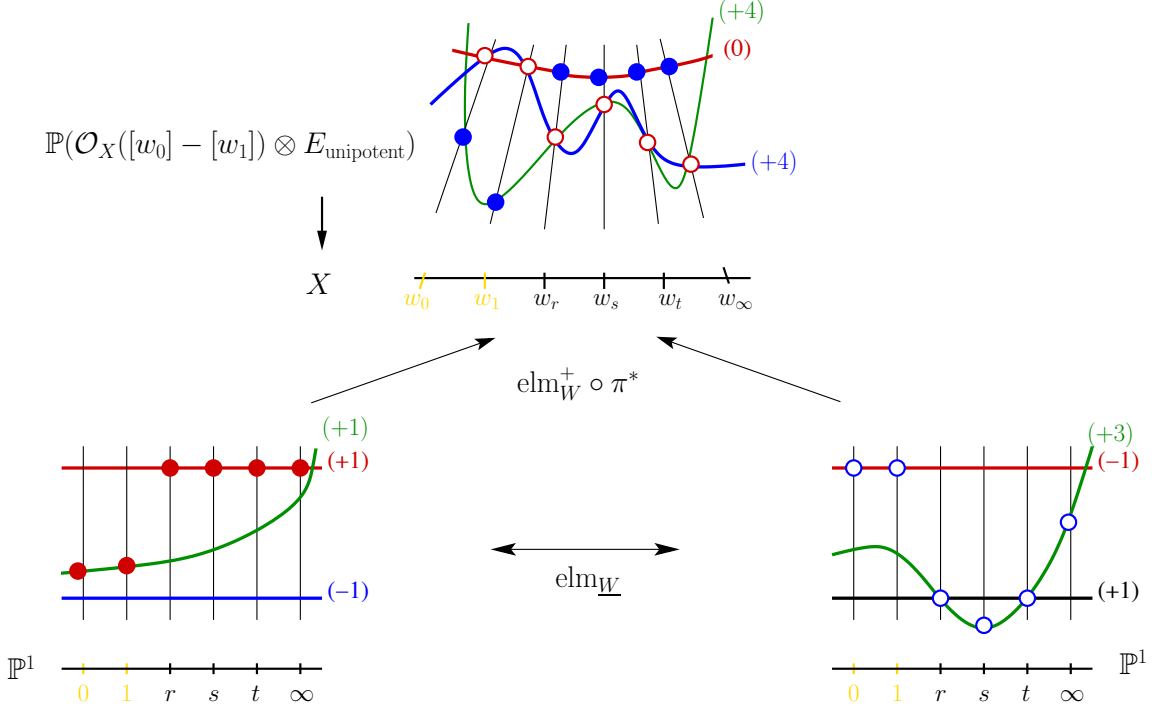


Figure 6: Twist of a generic unipotent bundle over  $X$ .

### 3.4.4 Generic decomposable bundles (Figure 7)

Let  $E = L_0 \oplus L_0^{-1}$ , where  $L_0 = \mathcal{O}_X([P] + [Q] - K_X)$  is not 2-torsion:  $L_0^2 \neq \mathcal{O}_X$ . On the projective bundle  $\mathbb{P}E$ , there are two sections  $\sigma_0, \sigma_\infty : X \rightarrow \mathbb{P}E$  coming from the two direct summands  $L_0$  and  $L_0^{-1}$  respectively, both having 0 self-intersection. They are permuted by any lift  $h$  of the hyperelliptic the involution  $\iota$  and they correspond to degenerate, but non-invariant Tyurin subbundles. There is a 1-parameter family of anticanonical embeddings  $\mathcal{O}_X(-K_X) \hookrightarrow E$  on which the automorphism group  $\text{Aut}(E)$  acts transitively. Any hyperelliptic involution  $h$  fixes two members  $L^+$  and  $L^-$  of this family. These are either both degenerate (when  $P = w$  or  $Q = w$  and thus  $E \in \Pi_{[w]}$ ), or both non-degenerate. Let us focus on the case where neither  $P$ , nor  $Q$  is a Weierstrass point. Then there are two  $h$ -invariant non-degenerate Tyurin subbundles. Note that this was also the case for generic stable bundles (see § 3.4.1). Here however, we have the particularity that up to bundle automorphism, there is only one possible hyperelliptic parabolic structure  $\underline{p}$ . It is defined by a non-degenerate Tyurin subbundle  $\mathcal{O}_X(-K_X) \hookrightarrow E$  that can be arbitrarily chosen. Choose such a subbundle and denote by  $\sigma : X \rightarrow \mathbb{P}E$  the corresponding section. The section  $\sigma$  intersects  $\sigma_0$  at  $[P] + [Q]$  and  $\sigma_\infty$  at  $[\iota(P)] + [\iota(Q)]$ . One can view  $\mathbb{P}E$  as the fiber-wise compactification of  $\mathcal{O}_X([P] + [Q] - [\iota(P)] - [\iota(Q)])$  with  $\sigma_0$  as the zero section and  $\sigma_\infty$  as the compactifying section;  $\sigma$  is a rational section with divisor  $[P] + [Q] - [\iota(P)] - [\iota(Q)]$ .

For the corresponding parabolic bundle  $(\underline{E}, \underline{p})$ , the anticanonical embedding descends as the destabilizing subbundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ . On the other hand,  $\sigma_0$  and  $\sigma_\infty$ , being permuted by the involution  $\iota$ , descend as a 2-section  $\Gamma \subset \mathbb{P}(\underline{E})$ , thus intersecting a generic



member of the ruling twice. Moreover,  $\Gamma$  intersects twice the section  $\sigma_{-1} : \mathbb{P}^1 \rightarrow \mathbb{P}(\underline{E})$  defined by the destabilizing bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \underline{E}$ , namely at  $\pi(P)$  and  $\pi(Q)$  (where  $\pi : X \rightarrow \mathbb{P}^1 = X/\iota$  is the hyperelliptic projection). The restriction of the ruling projection  $\mathbb{P}(\underline{E}) \rightarrow \mathbb{P}^1$  to the curve  $\Gamma$  is a 2-cover branching precisely over the branching divisor  $\underline{W}$  of  $\pi : X \rightarrow \mathbb{P}^1$ . The parabolic structure  $\underline{p}$  is located at the double points of  $\Gamma \subset \mathbb{P}(\underline{E})$  over  $\underline{W}$ .

Conversely, when we have a parabolic structure  $\underline{p}$  on  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  such that there is a smooth curve  $\Gamma \subset \mathbb{P}(\underline{E})$  belonging to the linear system defined by  $|2[\sigma_{-1}] + 4[f]|$  (with  $f$  any fiber of the ruling and  $\sigma_{-1}$  the negative section as before) such that  $\Gamma$  passes through all 6 parabolic points  $\underline{p}$  and is moreover vertical at these points (i.e. tangent to the ruling), then it gives rise to a generic decomposable bundle  $E$ .

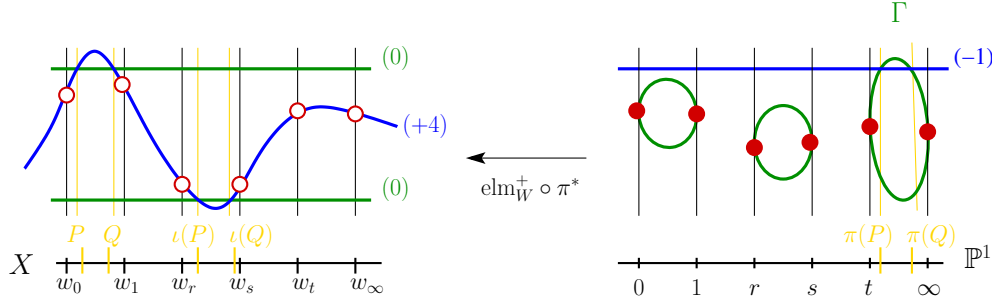


Figure 7: A generic decomposable bundle on  $X$ .

The case of  $E \in \Pi_{[w]}$  can be obtained as a limit case: in Figure 7, the +4 curve becomes reducible, and the parabolic over  $w$  on the curve  $\Gamma$  (still smooth and defined as above) now also lies on the  $-1$ -curve.

### 3.4.5 The trivial bundle and its 15 twists (Figure 8)

When  $E$  is the trivial bundle, we obtain a 1-parameter family of Tyurin subbundles formed by all embeddings  $\mathcal{O}_X \hookrightarrow E$ . Any lift  $h$  as in (3.13) fixes two of these (degenerate) Tyurin subbundles. However, up to automorphism, there is exactly one parabolic structure, which is moreover determined by the corresponding subbundle  $\mathcal{O}_X \hookrightarrow E$ . Descending to  $\mathbb{P}^1$ , we get the decomposable bundle  $\underline{E}_0 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$  with parabolic structure  $\underline{p}$  by a line subbundle  $\mathcal{O}_{\mathbb{P}^1}(-3) \subset \underline{E}_0$ . Note that the isomorphism class of  $(\underline{E}_0, \underline{p})$  does not depend on the choice of such a subbundle and is a fixed point of the Galois involution  $\mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_W^+$ .

When  $E = E_\tau = \tau \otimes E_0$  with  $\tau = \mathcal{O}_X([w_i] - [w_j])$  is a 2-torsion line bundle with  $i \neq j$ , the two parabolic structures  $\underline{p}$  and  $\underline{p}'$  can be deduced from the case of trivial bundles by permuting the role of the two parabolics over  $w_i$  and  $w_j$  with respect to  $\underline{p}$  and  $\underline{p}'$  (see § 3.3.2). In particular,  $E$  comes from the decomposable parabolic bundle  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  having parabolics  $\underline{p}_i$  and  $\underline{p}_j$  lying in the first direct summand, the other ones in the second. Note that the 16 parabolic bundles corresponding to the trivial bundle and its twists are precisely the flat decomposable bundles listed in Proposition 3.3.5.

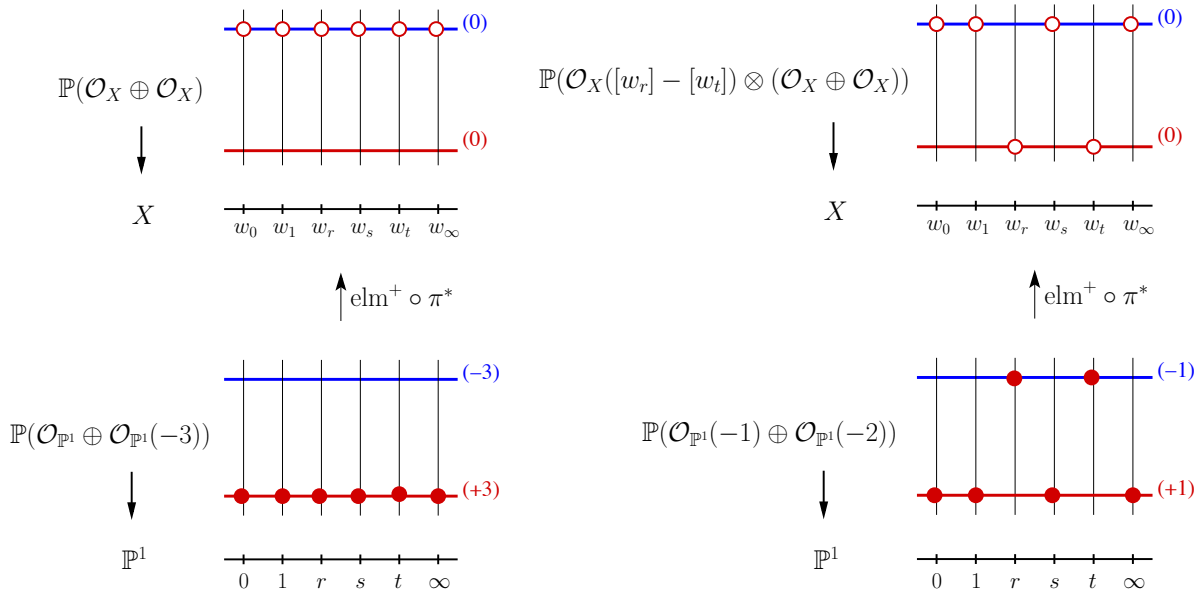


Figure 8: The trivial bundle over  $X$  and one of its twists.

### 3.4.6 The 6 + 10 Gunning bundles (Figures 9 and 10)

Let  $\vartheta$  with  $\vartheta^2 = \mathcal{O}_X(K_X)$  be a theta characteristic and  $E_\vartheta$  be the associated Gunning bundle. It arises as a non-split extension  $0 \rightarrow \vartheta \rightarrow E_\vartheta \rightarrow \vartheta^{-1} \rightarrow 0$ .

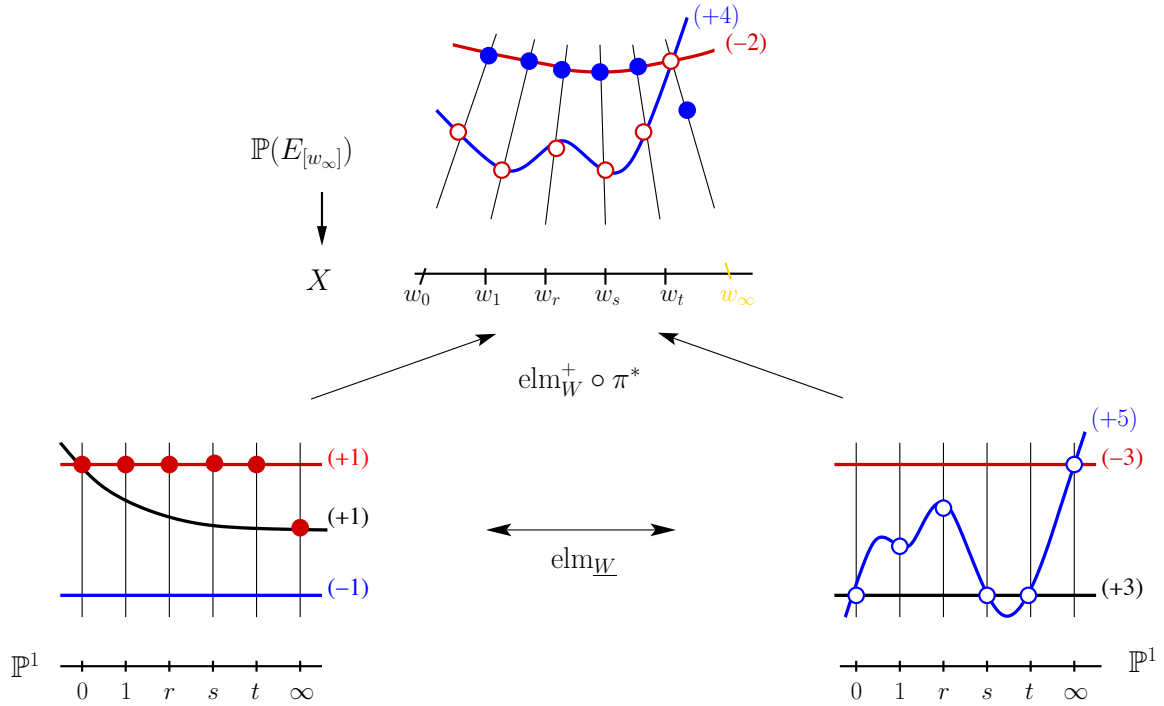


Figure 9: An odd Gunning bundle over  $X$ .

**Six odd theta characteristics.** When  $\vartheta$  is an odd theta characteristic  $\vartheta = \mathcal{O}_X([w_i])$ , the two  $h$ -invariant Tyurin subbundles  $L$  and  $L'$  are distinct and one of them, say  $L'$ , is the destabilizing subbundle  $\vartheta$ . The other one is necessarily non degenerate  $L \simeq \mathcal{O}_X(K_X)$ . The hyperelliptic parabolic  $p_i$  over  $w_i$  is defined by  $L|_{w_i} = L'|_{w_i}$  and  $p'_i$  is elsewhere. Hence each fixed  $h$  induces two non-isomorphic parabolic structures  $\mathbf{p}$  and  $\mathbf{p}'$  on  $E_\vartheta$ . Moreover, neither the isomorphism class of  $(E_\vartheta, \mathbf{p})$  nor the isomorphism class of  $(E_\vartheta, \mathbf{p}')$  depends on  $h$ . Note that indeed  $\text{Aut}(E_\vartheta)$  fixes  $\vartheta \subset E_\vartheta$  and acts transitively on the set of non-degenerate Tyurin subbundles. Hence there are exactly two isomorphism classes of hyperelliptic parabolic structures on  $E_\vartheta$ :

- $\mathbf{p}$  with parabolic  $p_i$  in  $\vartheta \hookrightarrow E$  and the others outside;
- $\mathbf{p}'$  with all parabolics in  $\vartheta \hookrightarrow E$  except  $p'_i$ .

They respectively descend as

- $Q_i : \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  and  $\underline{p}_k \subset \mathcal{O}_{\mathbb{P}^1}(-2)$ ,  $\forall k \neq i$ ;
- $Q'_i : \underline{E}' = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$  and  $\underline{p}'_i \subset \mathcal{O}_{\mathbb{P}^1}$ .

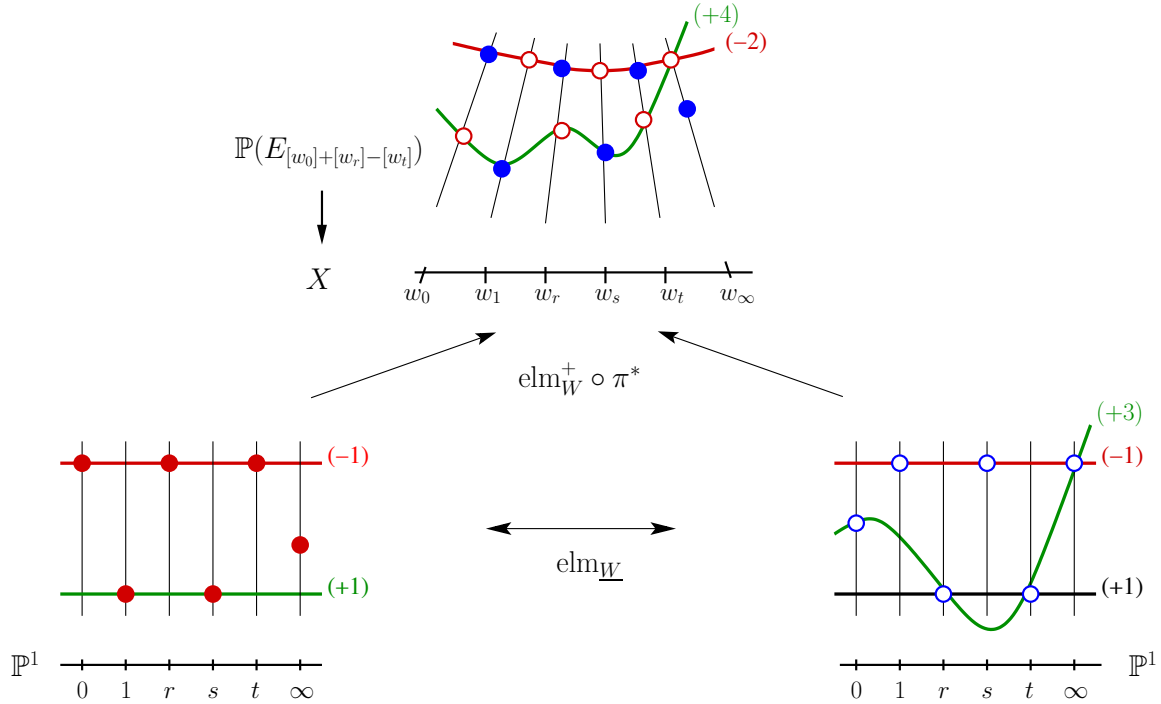


Figure 10: An even Gunning bundle over  $X$ .

**Ten even theta characteristics.** When  $\vartheta$  is an even theta characteristic  $\vartheta = \mathcal{O}_X([w_i] + [w_j] + [w_k] - K_X)$ , the space of morphisms  $\varphi : \mathcal{O}_X(-K_X) \rightarrow E_\vartheta$  is of dimension 2 and all these morphisms take values in the subbundle  $\vartheta \subset E_\vartheta$ , which is therefore the unique Tyurin bundle. We may identify the sheaves

$$\text{Hom}(\mathcal{O}_X(-K_X), \vartheta) \simeq \mathcal{O}_X([w_i] + [w_j] + [w_k]),$$

and their space of global sections is generated by the two sections

$$1, \frac{(x-x_l)(x-x_m)(x-x_n)}{y} \in H^0(X, \mathcal{O}_X([w_i] + [w_j] + [w_k])),$$

where  $\{i, j, k, l, m, n\} = \{0, 1, r, s, t, \infty\}$  and  $w_i = (x_i, 0) \in X$ . The hyperelliptic involution acts as id on the first one and  $-\text{id}$  on the second one. Hence there are two types of hyperelliptic parabolic structures on  $E_\vartheta$ :

- $\mathbf{p}$  with parabolics  $p_i, p_j$  and  $p_k$  in  $\vartheta \hookrightarrow E$  and the others outside;
- $\mathbf{p}'$  with parabolics  $p'_l, p'_m$  and  $p'_n$  in  $\vartheta \hookrightarrow E$  and the others outside.

They respectively descend as elements of

- $Q_{i,j,k} : \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  and  $\underline{p}_l, \underline{p}_m, \underline{p}_n \subset \mathcal{O}_{\mathbb{P}^1}(-1)$ ;
- $Q_{l,m,n} : \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  and  $\underline{p}'_i, \underline{p}'_j, \underline{p}'_k \subset \mathcal{O}_{\mathbb{P}^1}(-1)$ .

### 3.4.7 Affine bundles

As we have seen, affine bundles are the only ones in  $\mathfrak{Bun}(X)$  that cannot be constructed by hyperelliptic descent. They admit however a natural construction by elementary transformations given by Tyurin's approach. Indeed, even if the notion of  $h$ -invariant line subbundles does not make sense here, we can of course consider the space of Tyurin subbundles of an affine bundle. Let  $L_0 = \mathcal{O}_X([P] + [Q] - K_X) = \mathcal{O}_X(K_X - [\iota(P)] - [\iota(Q)])$  be a degree 0 line bundle such that  $L_0^{\otimes 2} \neq \mathcal{O}_X$  and let  $E$  be the unique non-trivial extension

$$0 \longrightarrow L_0 \longrightarrow E \longrightarrow L_0^{-1} \longrightarrow 0.$$

Then  $h^0(X, \text{Hom}(\mathcal{O}_X(-K_X), E)) = 2$ . Moreover, we have

$$h^0(X, \text{Hom}(\mathcal{O}_X(-K_X), L_0)) = 1.$$

In other words,  $E$  possesses a 1-parameter family of Tyurin subbundles. For  $P \neq Q$ , precisely three of them are degenerated:  $L_0$ , a unique line subbundle  $L_P \simeq \mathcal{O}_X(-P)$  of  $E$  and a unique line subbundle  $L_Q \simeq \mathcal{O}_X(-Q)$  of  $E$ . They define a parabolic structure  $\mathbf{q}$  on  $E$  over the so-called Tyurin divisor  $D = [P] + [Q] + [\iota(P)] + [\iota(Q)]$ , where the parabolics over  $\iota(P)$  and  $\iota(Q)$  are both given by  $L_0$  and the two other ones are both given by  $L_P \cap L_Q$ . Then  $\mathcal{O}_X(K_X) \otimes \text{elm}_D^-(E, \mathbf{q})$  yields the trivial bundle  $E_0$  on  $X$ . One easily checks that the thereby induced parabolic structure on  $E_0$  is isomorphic to the one given, with respect to  $\mathbb{P}(E_0) = \mathbb{P}^1 \times X$ , by  $(\lambda_P, \lambda_{\iota(P)}, \lambda_Q, \lambda_{\iota(Q)}) = (0, 1, 0, \infty) \in (\mathbb{P}^1)^4$ , inducing moreover the case  $P = Q$  as the limit.

## 3.5 Classical approaches and their relations

Let  $X$  be a smooth genus 2 curve over  $\mathbb{C}$ , together with its hyperelliptic involution  $\iota$  as at the beginning of Section 3.3. We are interested in moduli of flat rank 2 vector bundles over  $X$  with trivial determinant bundle and their geometry.

Recall from § 3.3.3 that with the exception of the 16 Gunning bundles, flat bundles as above are semistable. In § 3.5.1, we review the classical Narasimhan-Ramanan moduli space  $\mathcal{M}_{\text{NR}}$  of  $S$ -equivalence classes of semistable bundles [NR69]. We complement it by noticing that each of our 16 Gunning planes there has an alternative interpretation as sets of  $S$ -equivalence classes of semistable bundles arbitrarily close to the corresponding Gunning bundle, and that they form the 16 planes in the classical (16,6)-configuration of the Kummer surface [Hud90, GD94, NR69, Bol07]. In § 3.5.2 we compute a natural isomorphism  $\mathbb{P}_{\text{NR}}^3 \simeq \mathcal{M}_{\text{NR}}$  and the thereby arising equation of the Kummer quartic.

Tyurin’s classical approach [Tyu64, Tyu65] of minimal rational trivialization of generic stable vector bundles will be studied in its particularities for the case of our interest (rank 2, genus 2), namely in its relation to hyperelliptic descent in § 3.5.3. We obtain a natural compactification  $\mathcal{M}_{\text{Tyu}}$  of the space of Tyurin invariants, which is isomorphic to  $\mathbb{P}^2 \times \mathbb{P}^1$ . We explicitly compute the (birational) classification map towards  $\mathbb{P}_{\text{NR}}^3$ .

The Bertram [Ber92] point of view, complemented by more recent works of Bolognesi [Bol07, Bol09] (see also [Kum00]), arises from the moduli space  $\mathcal{M}_B \simeq \mathbb{P}^3$  of hyperelliptic ( $\iota$ -invariant) flags  $(E, L)$ , where  $L$  is a non-degenerate Tyurin subbundle. It is described in § 3.5.4, together with its relations to the moduli spaces mentioned before. Using Tyurin’s approach, we moreover recover an explicit version of Bolognesi’s universal bundle.

In summary, in this section we review classical approaches and we construct, using hyperelliptic descent, an explicit version of the following rational classification maps:

$$\mathcal{M}_B \xrightarrow{2:1} \mathcal{M}_{\text{NR}} \xleftarrow{1:1} \mathcal{M}_{\text{Tyu}}.$$

The geometry of the resulting birational transition maps between the three birational models of the Kummer quartic can be understood by the special loci in these spaces singled out in the dictionary from Section 3.4.

### 3.5.1 The Narasimhan-Ramanan moduli space

Two semi-stable vector bundles of same rank and degree over a curve are called *S-equivalent*, if the graded bundles associated to the Jordan-Hölder filtrations of these bundles are isomorphic. In our case, i.e., rank 2 bundles with trivial determinant bundle over  $X$ , we get that

- two stable bundles are *S*-equivalent if and only if they are isomorphic;
- two strictly semi-stable bundles are *S*-equivalent if and only if there is a line bundle  $L \in \text{Jac}(X)$  such that each of the two bundles is an extension either of  $L^{-1}$  by  $L$  or of  $L$  by  $L^{-1}$ .

To a semi-stable bundle  $E$ , we associate (following [NR69]) the set

$$C_E = \{L \in \text{Pic}^1(X) \mid h^0(X, E \otimes L) > 0\}.$$

Equivalently,  $L \in C_E$  if and only if there is a non-zero (and thus injective) morphism  $L^{-1} \rightarrow E$  of locally free sheaves. For stable bundles, the quotient  $E/L^{-1}$  then is necessarily locally free and hence defines an embedding of the total space of  $L^{-1}$  into the total space of  $E$ . The set  $C_E$  then parametrizes line subbundles of degree  $-1$ .

Narasimhan and Ramanan proved that this set  $C_E$  is the support of a uniquely defined effective divisor  $D_E$  on  $\text{Pic}^1(X)$  linearly equivalent to  $2\Theta$ , where

$$\Theta = \{[P] \mid P \in X\} \subset \text{Pic}^1(X)$$

is the locus of effective divisors of degree 1, naturally parametrized by the curve  $X$  itself. Moreover, for strictly semi-stable bundles, the divisor  $D_E$  only depends on the Jordan-Hölder filtration, i.e., on the *S*-equivalence class of  $E$ . We thus get a map

$$\text{NR} : \mathcal{M}_{\text{NR}} \rightarrow \mathbb{P}(H^0(\text{Pic}^1(X), \mathcal{O}_{\text{Pic}^1}(2\Theta))) \quad (3.15)$$

from the moduli space of *S*-equivalence classes to the linear system  $|2\Theta|$  on  $\text{Pic}^1(X)$ . Note that the Narasimhan-Ramanan classifying map is defined only for semi-stable bundles and thus not for Gunning bundles.

**Theorem 3.5.1** (Narasimhan-Ramanan). *The map NR defined above is an isomorphism. Let  $\pi : \mathcal{E} \rightarrow T$  be a smooth family of semi-stable rank 2 vector bundles with trivial determinant over  $X$ . Then the map  $\phi : T \rightarrow \mathcal{M}_{\text{NR}}$  associating to  $t \in T$  the  $S$ -equivalence class of  $E_t = \pi^{-1}(t)$  is a morphism.*

In particular, the moduli space of stable bundles naturally identifies with a Zariski open proper subset of  $\mathcal{M}_{\text{NR}} \simeq \mathbb{P}^3$ .

Let  $E = L_0 \oplus L_0^{-1}$  with  $L_0 \in \text{Jac}(X)$ . Given  $L \in \text{Pic}^1(X)$ , non-trivial sections of  $E \otimes L$  come from non-trivial sections of  $L_0 \otimes L$  or  $L_0^{-1} \otimes L$ . We promptly deduce that

$$D_E = L_0 \cdot \Theta + L_0^{-1} \cdot \Theta$$

where  $L_0 \cdot \Theta$  denotes the translation of  $\Theta$  by  $L_0$  for the group law on  $\text{Pic}(X)$ . A special case occurs for the 16 torsion points  $L_0^2 = \mathcal{O}_X$  for which  $L_0 = L_0^{-1}$  and hence  $D_E = 2(L_0 \cdot \Theta)$  is not reduced. The moduli space of semi-stable decomposable bundles naturally identifies with the *Kummer variety*

$$\text{Kum}(X) := \text{Jac}(X) / \iota,$$

the quotient of the Jacobian  $\text{Jac}(X)$  by the involution  $\iota : L_0 \mapsto \iota^* L_0 = L_0^{-1}$ . The Narasimhan-Ramanan classifying map (3.15) provides a canonical embedding

$$\text{Kum}(X) := \text{Jac}(X) / \iota \hookrightarrow \mathcal{M}_{\text{NR}}$$

and the image is a quartic surface in  $\mathcal{M}_{\text{NR}} \simeq \mathbb{P}^3$ . The moduli space of stable bundles identifies with the complement of this surface. The 16 torsion points  $L_0^2 = \mathcal{O}_X$  of the Jacobian are precisely the fixed points of the involution  $\iota$  and yield 16 conic singularities on  $\text{Kum}(X)$ .

As it shall turn out, the notion of two vector bundles being arbitrarily close in  $\mathfrak{Bun}(X)$ , is responsible for the classical geometry of the Kummer surface of strictly semi-stable bundles in  $\mathcal{M}_{\text{NR}}$ . We say two rank 2 vector bundles  $E$  and  $E'$  with trivial determinant over  $X$  are *arbitrarily close* if there are smooth families of vector bundles  $(E_t)_{t \in \mathbb{A}^1}$  and  $(E'_t)_{t \in \mathbb{A}^1}$  over  $X$  such that  $E_t \simeq E'_t$  for each  $t \neq 0$  and  $E_0 \simeq E$ ,  $E'_0 \simeq E'$ . One easily proves the following (see [HL15, Prop. 3.5]).

**Proposition 3.5.2.** *Given two extensions*

$$0 \rightarrow L \rightarrow E_0 \rightarrow L' \rightarrow 0 \quad \text{and} \quad 0 \rightarrow L' \rightarrow E'_0 \rightarrow L \rightarrow 0$$

*of the same (but permuted) line bundles, there are two deformations  $E_t$  and  $E'_t$  of these bundles (parametrized by  $\mathbb{A}^1$ ) such that  $E_t \simeq E'_t$  for  $t \neq 0$ .*

In particular, two semi-stable rank 2 bundles are arbitrarily close if and only if they are  $S$ -equivalent. Recall that for any theta-characteristic  $\vartheta$ , we introduced in § 3.3.3 the notion of Gunning plane  $\Pi_\vartheta \in \mathfrak{Bun}(X)$  (as a set), and that by § 3.4.2 it contains each  $S$ -equivalence class of semistable bundle  $E'$  arising as an extension  $0 \rightarrow \vartheta^{-1} \rightarrow E' \rightarrow \vartheta \rightarrow 0$  exactly once. Now Proposition 3.5.2 shows that all elements of the Gunning plane  $\Pi_\vartheta$  are arbitrarily close to the Gunning bundle  $E_\vartheta$ . On the other hand, the set of  $S$ -equivalence classes of semistable extensions  $E'$  as above forms a 2-plane in  $\mathcal{M}_{\text{NR}}$ ; it is given by those divisors  $D_{E'} \sim 2\Theta$  that pass through the point  $\vartheta$  on  $\text{Pic}^1(X)$ . Since they are naturally identified, we denote this 2-plane in  $\mathcal{M}_{\text{NR}}$  by the same symbol  $\Pi_\vartheta$ .

The intersection of  $\Pi_\vartheta$  with the Kummer surface is easily described as

$$\Pi_\vartheta \cap \text{Kum}(X) = \{L_0 \oplus L_0^{-1} \mid L_0 \in \vartheta^{-1} \cdot \Theta\}.$$

In fact, the 16 planes  $\Pi_\theta \subset \mathcal{M}_{\text{NR}}$  are well-known; each of them is tangent to the Kummer surface along a conic passing through 6 of the 16 nodes. The above description gives a natural parametrization of the hyperelliptic cover of this marked conic by the curve  $X$  itself (*via* the  $\Theta$  divisor). Precisely, for each  $\Pi_\theta$ , the 6 corresponding nodes are those parametrized by the 2-torsion points  $\vartheta^{-1} \otimes \mathcal{O}([w_i])$  where  $w_i$  runs over the six Weierstrass points. Conversely, through each node pass 6 of the 16 planes. This so-called (16, 6) configuration is classical (see [Hud90, GD94]) as well as the interpretation in terms of the moduli space of vector bundles (see [NR69, Bol07]). However, the interpretation of  $\Pi_\theta$  in terms of  $S$ -equivalence classes of semi-stable bundles arbitrarily close to the (unstable but flat) Gunning bundle  $E_\theta$  seems to not have been considered so far.

**Remark 3.5.3.** *In this geometric picture, the family  $\Delta \simeq \mathbb{P}^1$  of unipotent bundles can be seen as the tangent cone to the Kummer surface after blowing up the singular point corresponding to the trivial bundle. The strict transform of the  $\Pi_{[w_i]} \subset \mathcal{M}_{\text{NR}}$  then intersects this  $\mathbb{P}^1$  in a unique point which is the special unipotent bundle associated to  $[w_i]$ .*

### 3.5.2 Natural coordinates for $\mathcal{M}_{\text{NR}}$

We shall now construct two sets of coordinates on the Narasimhan-Ramanan moduli space  $\mathcal{M}_{\text{NR}} \simeq \mathbb{P}^3$ , allowing us to express explicitly the Kummer surface of strictly semi-stable bundles as well as the involutions of the moduli space given by tensor products with 2-torsion line bundles. The first set of coordinates  $(v_0 : v_1 : v_2 : v_3)$  has the advantage that they allow to easily calculate the Narasimhan-Ramanan classifying map for the families of bundles we will encounter. The second set of coordinates  $(t_0 : t_1 : t_2 : t_3)$  exploits symmetries of the Kummer surface which are useful in certain applications (see Cor. 3.5.6 and § 3.6.6).

For all computations, the curve  $X$  is the smooth compactification of the affine complex curve defined by

$$X : y^2 = x(x-1)(x-r)(x-s)(x-t)$$

where  $0, 1, r, s, t \in \mathbb{C}$  are pair-wise distinct; we denote by  $\infty$  the point at infinity. Moreover, in all of the resulting formulae, we denote

$$\sigma_1 = r + s + t, \quad \sigma_2 = rs + st + tr, \quad \sigma_3 = rst. \quad (3.16)$$

**Coordinates**  $(v_0 : v_1 : v_2 : v_3)$ . Let us first calculate a basis of  $H^0(\text{Pic}^1(X), \mathcal{O}_{\text{Pic}^1(X)}(2\Theta))$  in order to introduce explicit projective coordinates on the three-dimensional projective space

$$\mathbb{P}H^0(\text{Pic}^1(X), \mathcal{O}_{\text{Pic}^1(X)}(2\Theta)).$$

Since  $\text{Pic}^1(X)$  is birationally equivalent to the symmetric product  $X^{(2)}$ , rational functions on  $\text{Pic}^1(X)$  can be conveniently expressed as symmetric rational functions on  $X \times X$ .

$$\begin{array}{ccc} X \times X & \longrightarrow & X^{(2)} \xrightarrow{\phi^{(2)}} \text{Pic}^2(X) \xrightarrow{\sim} \text{Pic}^1(X) \\ & & \{P, Q\} \longmapsto [P] + [Q] \end{array}$$

The pull-back of the divisor  $\Theta \subset \text{Pic}^1(X)$  (resp.  $\Theta + [\infty] \subset \text{Pic}^2(X)$ ) to  $X \times X$  is  $\overline{\Delta} + \infty_1 + \infty_2$ , where  $\overline{\Delta}$  is the anti-diagonal  $\{(P, Q) \in X \times X \mid Q = \iota(P)\}$ ,  $\infty_1$  is the divisor  $\{\infty\} \times X$  and  $\infty_2$  is the divisor  $X \times \{\infty\}$ . The pull-back to  $X \times X$  of  $2\Theta$ , viewed as a divisor on  $\text{Pic}^1(X)$ , is then (see Figure 11):

$$2\overline{\Delta} + 2\infty_1 + 2\infty_2.$$

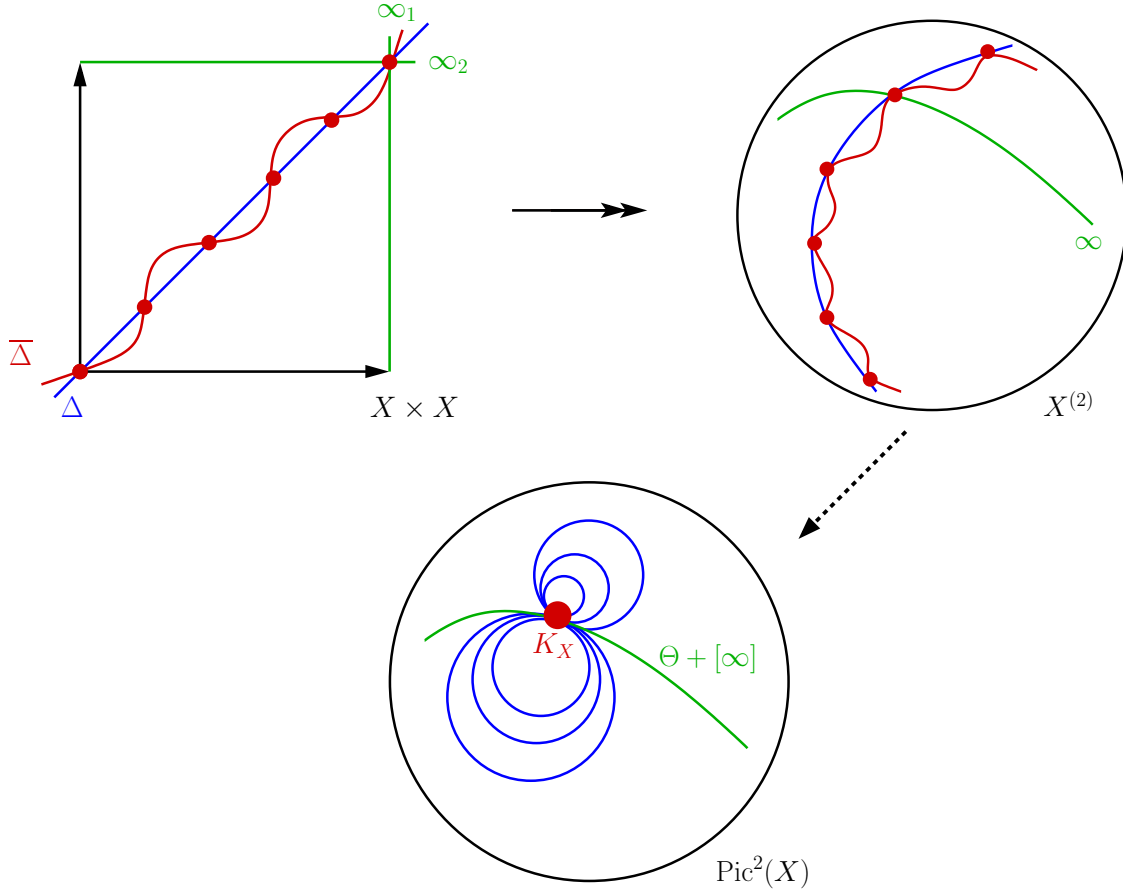


Figure 11:  $X^2$  as a rational cover of  $\text{Jac}(X)$ .

**Lemma 3.5.4.** *Let  $(P_1, P_2) = ((x_1, y_1), (x_2, y_2))$  be coordinates of  $X \times X$ . Denote by  $\mathfrak{S}_2$  the group generated by the symmetric involution  $(P_1, P_2) \mapsto (P_2, P_1)$  on  $X \times X$ . Then the space of  $\mathfrak{S}_2$ -invariant sections of the line bundle  $\mathcal{O}_{X \times X}(2\bar{\Delta} + 2\infty_1 + 2\infty_2)$  is given by*

$$H^0(X \times X, \mathcal{O}_{X \times X}(2\bar{\Delta} + 2\infty_1 + 2\infty_2))^{\mathfrak{S}_2} = \text{Vect}_{\mathbf{C}}(1, \text{Sum}, \text{Prod}, \text{Diag}),$$

where (with respect to (3.16))

$$\begin{aligned}
1 &: (P_1, P_2) \mapsto 1 \\
\text{Sum} &: (P_1, P_2) \mapsto x_1 + x_2 \\
\text{Prod} &: (P_1, P_2) \mapsto x_1 x_2, \\
\text{Diag} &: (P_1, P_2) \mapsto \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - (x_1 + x_2)^3 + (1 + \sigma_1)(x_1 + x_2)^2 + \\
&\quad + x_1 x_2 (x_1 + x_2) - (\sigma_1 + \sigma_2)(x_1 + x_2).
\end{aligned} \tag{3.17}$$

*Proof.* We have

$$\dim \left( H^0(X \times X, \mathcal{O}_{X \times X}(2\bar{\Delta} + 2\infty_1 + 2\infty_2))^{\mathfrak{S}_2} \right) = h^0 \left( \text{Pic}^1(X), \mathcal{O}_{\text{Pic}^1(X)}(2\Theta) \right) = 4$$



(see [\[NR69\]](#) or [\[Mum08\]](#)). The function  $Diag$  can be rewritten as

$$Diag = \frac{1}{(x_1 - x_2)^2} \cdot \left[ -2y_1y_2 - 2(1 + \sigma_1)x_1^2x_2^2 - (\sigma_2 + \sigma_3)(x_1^2 + x_2^2) + (x_1 + x_2) \cdot (x_1^2x_2^2 + (\sigma_1 + \sigma_2)x_1x_2 + \sigma_3) \right] \quad (3.18)$$

The expression of  $Diag$  in [\(3.17\)](#) shows that it has no poles off the anti-diagonal and the infinity (and in particular no poles on the diagonal). From the expression [\(3.18\)](#) follows easily that  $Diag$  has polar divisor  $2\bar{\Delta} + 2\infty_1 + 2\infty_2$ . Indeed, if  $u_1$  is the local parameter for  $X_1$  near  $\infty_1$  defined by  $x_1 = \frac{1}{u_1}$ , then the principal part of the generating functions is given by

$$1, \quad Sum = \frac{1}{u_1^2} + x_2, \quad Prod = \frac{x_2}{u_1^2} \quad \text{and} \quad Diag \sim \frac{x_2^2}{u_1^2} - \frac{y_2}{u_1} + \dots$$

As a section of  $H^0(\text{Pic}^1(X), \mathcal{O}_{\text{Pic}^1(X)}(2\Theta))$ , the function 1 vanishes twice along  $\Theta$  while the other ones do not vanish identically on  $\Theta$ .  $\square$

In the sequel, denote by  $(v_0 : v_1 : v_2 : v_3) \in \mathbb{P}_{\text{NR}}^3$  the projective coordinates representing the rational function

$$v_0 + v_1 \cdot Sum + v_2 \cdot Prod + v_3 \cdot Diag$$

on  $X^{(2)}$ . We use the natural isomorphism  $\mathbb{P}_{\text{NR}}^3 \simeq \mathcal{M}_{\text{NR}}$  given by identifying the zero-divisor of such a function with an element of  $|2\Theta|$  as explained above, combined with the inverse of the Narasimhan-Ramanan map [\(3.15\)](#).

**Equation of the Kummer surface.** We will now compute the equation in  $\mathbb{P}_{\text{NR}}^3$  of the Kummer surface embedded in  $\mathcal{M}_{\text{NR}}$  with respect to the isomorphism  $\mathbb{P}_{\text{NR}}^3 \simeq \mathcal{M}_{\text{NR}}$ . It is sufficient to consider decomposable semi-stable bundles. Let  $L = \mathcal{O}_X([P_1] + [P_2] - [\infty]) \in \text{Pic}^1(X)$  and denote by  $\tilde{L}$  the associated degree 0 bundle  $\tilde{L} = \mathcal{O}_X([P_1] + [P_2] - 2[\infty])$ . The corresponding Narasimhan-Ramanan divisor on  $\text{Pic}^1(X)$  is  $\tilde{L} \cdot \Theta + \tilde{L}^{-1} \cdot \Theta$ . The first component  $\tilde{L} \cdot \Theta$  is parametrized by

$$X \rightarrow \text{Pic}^1(X); \quad Q \mapsto [P_1] + [P_2] + [Q] - 2[\infty].$$

Setting  $[P_1] + [P_2] + [Q] - 2[\infty] \sim [P_1] + [P_2] - [\infty]$ , we get that  $[P_1] + [P_2] + [Q]$  belongs to the linear system  $|[P_1] + [P_2] + [\infty]|$  on  $X$ . The latter is generated by the two functions 1 and  $f(P) := \frac{y+y_1}{x-x_1} - \frac{y+y_2}{x-x_2}$  on the curve. Therefore,  $[P_1] + [P_2] - [\infty] \in \tilde{L} \cdot \Theta$  (the support of) if, and only if,  $f(P_1) = f(P_2)$ ; this gives the following equation for  $P_1 = (x_1, y_1)$  and  $P_2 = (x_2, y_2)$ :

$$\frac{y_1 + y_1}{x_1 - x_1} - \frac{y_1 + y_2}{x_1 - x_2} = \frac{y_2 + y_1}{x_2 - x_1} - \frac{y_2 + y_2}{x_2 - x_2}.$$

The equation for the other component  $\tilde{L}^{-1} \cdot \Theta$  is deduced by changing signs  $y_i \rightarrow -y_i$  for  $i = 1, 2$ . Taking into account the two equations, we get an equation for  $\tilde{L} \cdot \Theta + \tilde{L}^{-1} \cdot \Theta$ :

$$\left( \frac{y_1 + y_1}{x_1 - x_1} - \frac{y_1 + y_2}{x_1 - x_2} - \frac{y_2 + y_1}{x_2 - x_1} + \frac{y_2 + y_2}{x_2 - x_2} \right) \left( \frac{-y_1 + y_1}{x_1 - x_1} - \frac{-y_1 + y_2}{x_1 - x_2} - \frac{-y_2 + y_1}{x_2 - x_1} + \frac{-y_2 + y_2}{x_2 - x_2} \right) = 0$$

which, after reduction, reads

$$-Diag(P_1, P_2) \cdot 1 + Prod(P_1, P_2) \cdot Sum - Sum(P_1, P_2) \cdot Prod + 1 \cdot Diag = 0 \quad (3.19)$$

**Remark 3.5.5.** *The symmetric form of this equation is due to the fact that for any vector bundle  $E \in \mathcal{M}_{\text{NR}}$  and any line bundle  $L \in \text{Pic}^1(X)$  such that  $h^0(X, E \otimes L) > 0$ , the divisor  $D_E$  associated to  $E$  and the divisor  $\tilde{L} \cdot \Theta + \tilde{L}^{-1} \cdot \Theta$  associated to  $\tilde{L} \oplus \tilde{L}^{-1}$  intersect precisely in  $L$  and  $\iota(L)$  on  $\text{Pic}^1(X)$ .*

Hence, according to equation (3.19), the Kummer embedding

$$\begin{aligned} \text{Jac}(X) &\rightarrow \text{Kum}(X) \subset \mathbb{P}_{\text{NR}}^3 \\ \mathcal{O}_X([\underline{P}_1] + [\underline{P}_2] - 2[\infty]) &\mapsto (v_0 : v_1 : v_2 : v_3) \end{aligned}$$

is explicitly given by

$$(v_0 : v_1 : v_2 : v_3) = (-\text{Diag}(\underline{P}_1, \underline{P}_2) : \text{Prod}(\underline{P}_1, \underline{P}_2) : -\text{Sum}(\underline{P}_1 : \underline{P}_2) : 1) \quad (3.20)$$

One can now eliminate parameters  $\underline{P}_1$  and  $\underline{P}_2$  from (3.20) as follows: express  $\underline{y}_1 \underline{y}_2$  in terms of functions  $\underline{x}_1 + \underline{x}_2$  and  $\underline{x}_1 \underline{x}_2$  and variable  $v_0/v_3$ , so that the square can be replaced by

$$\left(\underline{y}_1 \underline{y}_2\right)^2 = \prod_{w=0,1,r,s,t} (w^2 - (\underline{x}_1 + \underline{x}_2)w + \underline{x}_1 \underline{x}_2);$$

then replace  $\underline{x}_1 \underline{x}_2$  and  $\underline{x}_1 + \underline{x}_2$  by  $v_1/v_3$  and  $-v_2/v_3$  respectively. We get

$$\begin{aligned} \text{Kum}(X) : \\ 0 = & (v_0 v_2 - v_1^2)^2 \cdot 1 \\ & - 2 [[(\sigma_1 + \sigma_2)v_1 + (\sigma_2 + \sigma_3)v_2](v_0 v_2 - v_1^2) \\ & + 2(v_0 + \sigma_1 v_1)(v_0 + v_1)v_1 + 2(\sigma_2 v_1 + \sigma_3 v_2)(v_1 + v_2)v_1] \cdot v_3 \\ & [-2\sigma_3(v_0 v_2 - v_1^2) + [(\sigma_1 + \sigma_2)^2 v_1 + (\sigma_2 + \sigma_3)^2 v_2](v_1 + v_2) \\ & - (\sigma_1 + \sigma_3)^2 v_1 v_2 + 4[(\sigma_2 + \sigma_3)v_0 - \sigma_3 v_2]v_1] \cdot v_3^2 \\ & - 2\sigma_3 [(\sigma_1 + \sigma_2)v_1 - (\sigma_2 + \sigma_3)v_2] \cdot v_3^3 \\ & + \sigma_3^2 \cdot v_3^4. \end{aligned}$$

Here, we see that  $v_3 = 0$  is a (Gunning-) plane tangent to  $\text{Kum}(X)$  along a conic.

Following formula (3.20), we can compute the locus of the trivial bundle  $E_0$  and its twists  $E_\tau := E_0 \otimes \mathcal{O}_X(\tau)$ , where  $\tau = [w_i] - [w_j]$ :

$E_\tau$	$(v_0 : v_1 : v_2 : v_3)$
$E_{[w_0]-[w_1]}$	$(rs + st + rt : 0 : -1 : 1)$
$E_{[w_0]-[w_r]}$	$(r(st + s + t) : 0 : -r : 1)$
$E_{[w_0]-[w_s]}$	$(s(rt + r + t) : 0 : -s : 1)$
$E_{[w_0]-[w_t]}$	$(t(rs + r + s) : 0 : -t : 1)$
$E_{[w_1]-[w_r]}$	$((1 + r)st : r : -1 - r : 1)$
$E_{[w_1]-[w_s]}$	$((1 + s)rt : s : -1 - s : 1)$
$E_{[w_1]-[w_t]}$	$((1 + t)rs : t : -1 - t : 1)$
$E_{[w_r]-[w_s]}$	$((r + s)t : rs : -r - s : 1)$
$E_{[w_r]-[w_t]}$	$((r + t)s : rt : -r - t : 1)$
$E_{[w_s]-[w_t]}$	$((s + t)r : st : -s - t : 1)$
$E_0$	$(1 : 0 : 0 : 0)$
$E_{[w_0]-[w_\infty]}$	$(0 : 0 : 1 : 0)$
$E_{[w_1]-[w_\infty]}$	$(1 : -1 : 1 : 0)$
$E_{[w_r]-[w_\infty]}$	$(r^2 : -r : 1 : 0)$
$E_{[w_s]-[w_\infty]}$	$(s^2 : -s : 1 : 0)$
$E_{[w_t]-[w_\infty]}$	$(t^2 : -t : 1 : 0)$

The Gunning planes  $\Pi_\vartheta$  are the planes passing through 6 of these 16 singular points. Precisely, the odd Gunning plane with  $\vartheta = [w_i]$  is passing through all  $E_\tau$  with  $\tau = [w_i] - [w_j]$  (including the trivial bundle  $E_0$  for  $i = j$ ); for an even Gunning plane with  $\vartheta = [w_i] + [w_j] - [w_k] \sim [w_l] + [w_m] - [w_n]$ , where  $\{i, j, k, l, m, n\} = \{0, 1, r, s, t, \infty\}$ , we get

$$\left. \begin{aligned} & E_{[w_i]-[w_j]}, E_{[w_j]-[w_k]}, E_{[w_i]-[w_k]} \\ & E_{[w_l]-[w_m]}, E_{[w_m]-[w_n]}, E_{[w_l]-[w_n]} \end{aligned} \right\} \in \Pi_{[w_i]+[w_j]-[w_k]} = \Pi_{[w_l]+[w_m]-[w_n]}.$$

We can derive explicit equations, for instance:

$\Pi_{[w_0]}$	$v_1 = 0$
$\Pi_{[w_1]}$	$v_1 + v_2 + v_3 = 0$
$\Pi_{[w_\infty]}$	$v_3 = 0$
$\Pi_{[w_0]+[w_1]-[w_\infty]}$	$v_0 + v_1 = (rs + st + rt)v_3$

**Coordinates**  $(t_0 : t_1 : t_2 : t_3)$ . By construction of the Narasimhan-Ramanan moduli space, the action of the group of 2-torsion line bundles by twist is linear and free on  $\mathcal{M}_{\text{NR}}$  and it preserves  $\text{Kum}(X)$ . Since we know the action on the sixteen bundles  $E_\tau$  corresponding to the singularities of the Kummer surface, the action of the group of 2-torsion line bundles on  $\mathcal{M}_{\text{NR}}$  with respect to the coordinates  $(v_0 : v_1 : v_2 : v_3)$  can easily be made explicit. It is given in [HL17, p. 158]. If we allow to choose square-roots  $\rho_0, \rho_1, \rho_r, \rho_s$  such that

$$\rho_0^2 = F'(0), \quad \rho_1^2 = -F'(1), \quad \rho_r^2 = F'(r), \quad \rho_s^2 = F'(s),$$

where  $F(x) = x(x-1)(x-r)(x-s)(x-t)$  and  $F'(x)$  is its derivative with respect to  $x$ , then we can construct an isomorphism between  $\mathbb{P}_t^3$  with coordinates  $(t_0 : t_1 : t_2 : t_3)$  and  $\mathbb{P}_{\text{NR}}^3$  (with coordinates  $(v_0 : v_1 : v_2 : v_3)$ ) such that in  $\mathbb{P}_t^3$ , the action of the group of 2-torsion line bundles is generated by double-transpositions of variables and double-changes of signs as in the table below.

$\tau$	$(t_0 : t_1 : t_2 : t_3) \otimes E_\tau$
0	$(t_0 : t_1 : t_2 : t_3)$
$[w_0] - [w_\infty]$	$(t_2 : t_3 : t_0 : t_1)$
$[w_1] - [w_\infty]$	$(t_1 : -t_0 : -t_3 : t_2)$
$[w_r] - [w_\infty]$	$(t_0 : -t_1 : -t_2 : t_3)$
$[w_s] - [w_\infty]$	$(t_1 : t_0 : -t_3 : -t_2)$
$[w_t] - [w_\infty]$	$(t_2 : t_3 : -t_0 : -t_1)$

An explicit coordinate change satisfying this property is given by

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ c & d & a & b \\ d & -c & -b & a \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & -\sigma_2 \\ 0 & \rho_1 & 0 & 0 \\ 0 & \rho_0 & \rho_0 & \rho_0 \\ 0 & 0 & 0 & \rho_0\rho_1 \end{pmatrix} \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad (3.21)$$

$$\begin{aligned} \text{where} \quad a &= rst(r-s)\rho_1 + t\rho_r\rho_s - rt(r-1)\rho_s - st\rho_1\rho_r \\ b &= -st(s-1)\rho_r + rt\rho_1\rho_s \\ c &= t(r-s)\rho_0\rho_1 - t(r-1)\rho_0\rho_s \\ d &= -t(r-1)(s-1)(r-s)\rho_0 + t(s-1)\rho_0\rho_r. \end{aligned}$$

Note that from the product formulation, this can be seen as a composition of two coordinate changes, the first one transforming the Gunning planes  $\Pi_{[w_0]}$ ,  $\Pi_{[w_1]}$ ,  $\Pi_{[w_\infty]}$  and  $\Pi_{[w_0]+[w_1]-[w_\infty]}$  into coordinate hyperplanes, the second one respecting this condition, but further normalizing the group action. Since for our new coordinates, the group action of the 2-torsion bundles is normalized to double-transpositions of variables and double-changes of signs, the equation of the Kummer surface in these coordinates must be invariant under such manipulations. According to [Hud90, § 53, p. 80-81], this forces the equation of the Kummer surface with respect to the coordinates  $(t_0 : t_1 : t_2, t_3)$  to be of the following very nice form

$$\begin{aligned} &(t_0^4 + t_1^4 + t_2^4 + t_3^4) + 2D(t_0t_1t_2t_3) \\ &+ A(t_0^2t_3^2 + t_1^2t_2^2) + B(t_1^2t_3^2 + t_0^2t_2^2) + C(t_2^2t_3^2 + t_0^2t_1^2) = 0, \end{aligned} \quad (3.22)$$

with coefficients  $A, B, C, D$  satisfying the relation

$$4 - A^2 - B^2 - C^2 + ABC + D^2 = 0.$$

Indeed, we find such an equation, and more precisely the one given by

$$\begin{aligned} A &= -2\frac{s(t-1)+(t-s)}{t(s-1)} & B &= -2\frac{r+(r-t)}{t} \\ C &= 2\frac{(r-1)+(r-s)}{s-1} & D &= -4\frac{r(s-t)+(r-s)}{t(s-1)}. \end{aligned} \tag{3.23}$$

The five  $t$ -polynomials occurring in the Kummer equation (3.22) are fundamental invariants for the action of the translation group and define a natural map  $\mathcal{M}_{\text{NR}} \simeq \mathbb{P}_{\text{NR}}^3 \simeq \mathbb{P}_t^3 \rightarrow \mathbb{P}^4$  whose image is a quartic hypersurface (see [Dol12], Proposition 10.2.7).

**Corollary 3.5.6.** *The quartic in  $\mathbb{P}^4$  defined by the natural map  $\mathcal{M}_{\text{NR}} \rightarrow \mathbb{P}^4$  is a coarse moduli space of  $S$ -equivalence classes of semi-stable  $\mathbb{P}^1$ -bundles over  $X$ .*

*Proof.* Let  $T$  be a smooth parameter space and  $\mathcal{S} \rightarrow X \times T$  a family of  $\mathbb{P}^1$ -bundles over  $X$ . Denote by  $\pi_T$  the projection  $X \times T \rightarrow T$ . The  $\mathbb{P}^1$ -bundle  $\mathcal{S}$  lifts to a rank 2 bundle  $\mathcal{E} \rightarrow X \times T$  such that  $\det(\mathcal{E}) = \pi_T^* \mathcal{O}_X$  and  $\mathbb{P}\mathcal{E} = \mathcal{S}$ . This vector bundle is unique up to tensor product with  $\pi_T^*(L)$  where  $L$  is a 2-torsion line bundle on  $X$ . According to Theorem 3.5.1, the classification map  $T \rightarrow \mathcal{M}_{\text{NR}}$  then is a morphism as is its composition with the natural map  $\mathcal{M}_{\text{NR}} \rightarrow \mathbb{P}^4$ . The resulting morphism  $T \rightarrow \mathbb{P}^4$  no longer depends on the choice of  $\mathcal{E}$ .  $\square$

### 3.5.3 Tyurin parametrization

Recall from § 3.3.4 that we defined Tyurin subbundles as subbundles obtained by anticanonical embeddings, and that they are called degenerate when they are not isomorphic to the anticanonical line bundle.

**Tyurin invariants for generic stable bundles.** Let  $E \in \mathfrak{Bun}(X)$  be a stable bundle. Recall from Proposition 3.3.7 (see also § 3.4.1) that there is a well defined decomposition

$$H^0(X, \text{Hom}(\mathcal{O}_X(-K_X), E)) = H \oplus H'$$

into one-dimensional subspaces that are  $h$ -invariant for any lift  $h : E \xrightarrow{\sim} \iota^* E$  of the hyperelliptic involution  $\iota$ , and that the subspaces  $H$  and  $H'$  define two distinct Tyurin subbundles  $L$  and  $L'$  of  $E$ . Let  $\varphi_1 \in H \setminus \{0\}$  and  $\varphi_2 \in H' \setminus \{0\}$ . The divisor

$$D_E^T := \text{div}_0(\varphi_1 \wedge \varphi_2) \in |2K_X|$$

does not depend on the choice of the  $\varphi_i$ 's. It is a well-defined invariant of the bundle  $E$ . We call it the *Tyurin divisor*.

**Proposition 3.5.7.** *Let  $E \in \mathfrak{Bun}(X)$  be a stable bundle and let  $D_E \in |2\Theta|$  be the divisor on  $\text{Pic}^1(X)$  defined by Narasimhan-Ramanan (see § 3.5.1). Then the Tyurin divisor  $D_E^T$  is the intersection between the divisor  $D_E$  and the natural embedding  $X \rightarrow \Theta; P \mapsto [P]$  on  $\text{Pic}^1(X)$ :*

$$D_E^T = D_E \cdot \Theta.$$

*For each point  $P$  in the support of  $D_E^T$ , there is one and only one subbundle  $L_P \subset E$  such that  $L_P \simeq \mathcal{O}_X(-[P])$ . These are precisely the degenerate Tyurin subbundles.*

*For any Weierstrass point  $w = \iota(w) \in X$ , the bundle  $E$  lies on the odd Gunning plane  $\Pi_{[w]}$  if and only if  $w$  is in the support of  $D_E^T$ .*

*Proof.* Let  $P$  be a point in the support of  $D_E^T$ . Then  $\iota(P)$  is also in the support, and there is a linear combination  $\lambda_1\varphi_1 + \lambda_2\varphi_2$  vanishing at  $\iota(P)$ . It gives rise to an subsheaf  $\mathcal{O}_X(-[P]) \hookrightarrow E$ , which, by stability of  $E$ , has to be a subbundle. Again by stability of  $E$ , there cannot be another subbundle isomorphic to  $\mathcal{O}_X(-[P])$ . Moreover,  $E$  cannot admit any line subbundle of the form  $\mathcal{O}_X(-[P'])$  with  $P' \in X$  other than the (at most four) degenerate Tyurin subbundles arising from  $D_E^T$ . Those however form precisely the support of  $D_E \cdot \Theta$ . Since this divisor is reduced for generic stable bundles  $E$ , we can conclude by continuity that  $D_E^T = D_E \cdot \Theta$ . By definition, an element of the Gunning plane  $\Pi_{[w]}$  contains a degenerate Tyurin subbundle isomorphic to  $\mathcal{O}_X(-[w])$ .  $\square$

Let  $E$  be as above. Its Tyurin divisor is of the form

$$D_E^T = [P] + [\iota(P)] + [Q] + [\iota(Q)]$$

for some  $P, Q \in X$ . Let us assume that the Tyurin divisor  $D_E^T$  is reduced, i.e.,  $E$  does not lie on any odd Gunning plane. Then we have a parabolic structure  $\mathbf{q}^+$  on  $E$ , defined by the fibers of the line subbundles  $L_P, L_{\iota(P)}, L_Q, L_{\iota(Q)}$  over the corresponding points. This parabolic structure then coincides with the intersection in  $E$  of the two distinguished and non-degenerate Tyurin subbundles  $L, L' \simeq \mathcal{O}_X(-K_X)$  that are  $h$ -invariant. In particular,  $\text{elm}_{D_E^T}^-(E, \mathbf{q}^+)$  is naturally isomorphic to  $\Omega := \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$ , endowed with a parabolic structure  $\mathbf{q}_E^T$ . Here the direct summands of  $\Omega$  correspond to the pair  $(L, L')$  of  $h$ -invariant Tyurin subbundles of  $E$  in a chosen order. Note that conversely,  $\text{elm}_{D_E^T}^+(\Omega, \mathbf{q}_E^T)$  is nothing but the regular rational map

$$\varphi_1 \oplus \varphi_2 : \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X) \rightarrow E.$$

The isomorphism class of  $(\Omega, \mathbf{q}_E^T)$  is a well-defined invariant for the isomorphism class of the stable bundle  $E$  off the Gunning planes.

**Tyurin's construction.** Conversely, let us consider a divisor, say reduced for simplicity:

$$D = [P_1] + [\iota(P_1)] + [P_2] + [\iota(P_2)] \in |2K_X|,$$

and a parabolic structure  $\mathbf{q}$  on  $\Omega = \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$  supported by  $D$ . Note that the datum of  $\mathbf{q}$  is equivalent to the datum of points in the fibers of the trivial  $\mathbb{P}^1$ -bundle  $\mathbb{P}\Omega \rightarrow X$  over the support of  $D$ . We are going to consider  $\mathbb{P}\Omega$  as the trivialized trivial  $\mathbb{P}^1$ -bundle, so that the parabolic structure  $\mathbf{q}$  is defined by

$$(\lambda_{P_1}, \lambda_{\iota(P_1)}, \lambda_{P_2}, \lambda_{\iota(P_2)}) \in (\mathbb{P}^1)^4.$$

From this data, one can associate a vector bundle  $E \rightarrow X$  with trivial determinant bundle by

$$\text{elm}_D^+(\Omega, \mathbf{q}) = (E, \mathbf{q}^+). \quad (3.24)$$

The list of vector bundles that can be obtained by Tyurin's construction (with possibly reduced Tyurin divisor) is given in Table [5](#).

**A Turin parameter space.** Consider the parameter space

$$(P_1, P_2, \lambda) \in X_1 \times X_2 \times \mathbb{P}^1 := X \times X \times \mathbb{P}^1.$$

We define a parabolic structure  $\mathbf{q}$  on  $\Omega$  as a function of a generic parameter  $(P_1, P_2, \lambda)$ , with ordered support and associated parabolic structure given respectively by

$$(P_1, \iota(P_1), P_2, \iota(P_2)) \quad \text{and} \quad (\lambda_{P_1}, \lambda_{\iota(P_1)}, \lambda_{P_2}, \lambda_{\iota(P_2)}) := \left( \lambda, -\lambda, \frac{1}{\lambda}, -\frac{1}{\lambda} \right). \quad (3.25)$$

Note that this particular choice of normalized parabolic structure does not allow to construct affine bundles. In order to obtain those, one could consider another normalization where for example three of the parabolics are fixed to  $(0, 1, \infty)$  and the fourth one is a free parameter. Our choice (3.25) is motivated by hyperelliptic descent. Indeed, the transformation

$$\begin{aligned} (X_1 \times X_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) &\longrightarrow (X_1 \times X_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) \\ ((P_1, P_2, \lambda), ((x, y), z)) &\longmapsto ((P_1, P_2, \lambda), ((x, -y), -z)) \end{aligned}$$

preserves the parabolic structure  $\mathbf{q}$  and induces a lift  $h$  of the hyperelliptic involution on the vector bundle  $E$  associated to  $\mathbf{q}$  as in (3.24). The two subbundles of  $\Omega$  corresponding to  $\{z = 0\}$  and  $\{z = \infty\}$  on  $\mathbb{P}\Omega = \mathbb{P}_z^1 \times X$  are fixed under this transformation and yield the two  $h$ -invariant Tyurin subbundles of  $E$ .

We now consider symmetries that yield isomorphic parabolic bundles. One can first independently permute  $P_1 \leftrightarrow \iota(P_1)$ ,  $P_2 \leftrightarrow \iota(P_2)$  and  $P_1 \leftrightarrow P_2$ : this generates a order 8 group of permutations. Moreover, even for a fixed order on the support of  $D$  there is a freedom in the choice of  $\lambda$ : our choice of normalization, characterized by

$$\lambda_{P_1} + \lambda_{\iota(P_1)} = \lambda_{P_2} + \lambda_{\iota(P_2)} = 0 \quad \text{and} \quad \lambda_{P_1} \cdot \lambda_{P_2} = 1, \quad (3.26)$$

is invariant under the Klein 4 group  $\langle z \mapsto -z, z \mapsto \frac{1}{z} \rangle$  acting on the projective variable  $z \in \mathbb{P}^1$  parametrizing fibers of  $\mathbb{P}\Omega = \mathbb{P}^1 \times X$ . The transformation group taking into account all this freedom is generated by the following 4 transformations

$$\begin{aligned} (X_1 \times X_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) &\longrightarrow (X_1 \times X_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) \\ ((P_1, P_2, \lambda), ((x, y), z)) &\begin{cases} \xrightarrow{\sigma_{12}} & ((P_2, P_1, \frac{1}{\lambda}), ((x, y), z)) \\ \xrightarrow{\sigma_\iota} & ((\iota(P_1), \iota(P_2), -\lambda), ((x, y), z)) \\ \xrightarrow{\sigma_{iz}} & ((P_1, \iota(P_2), i\lambda), ((x, y), iz)) \\ \xrightarrow{\sigma_{1/z}} & ((P_1, P_2, \frac{1}{\lambda}), ((x, y), \frac{1}{z})) \end{cases} \end{aligned}$$

(here,  $i = \sqrt{-1}$ ). The 32-order group  $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz}, \sigma_{1/z} \rangle$  acts faithfully on the parameter space  $X_1 \times X_2 \times \mathbb{P}_\lambda^1$ . We define the *Tyurin configuration space*  $\mathcal{M}_{\text{Tyu}}$  to be the GIT quotient for this action, given by:

$$\begin{aligned} X_1 \times X_2 \times \mathbb{P}_\lambda^1 &\xrightarrow{(32:1)} \mathcal{M}_{\text{Tyu}} = \mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \\ ((x_1, y_1), (x_2, y_2), \lambda) &\mapsto ((1 : -\underline{s} : \underline{p}), \boldsymbol{\lambda}) = ((1 : -x_1 - x_2 : x_1 x_2), (\lambda^2 + \frac{1}{\lambda^2}) y_1 y_2) \end{aligned} \quad (3.27)$$

Here  $P_1 = (x_1, y_1)$  and  $P_2 = (x_2, y_2)$ . Note that  $\mathbb{P}_D^2$  naturally parametrizes the linear system  $|2K_X| \simeq \mathbb{P}^2$  of possible divisors.

**Relation to  $\mathbb{P}_{\text{NR}}^3$ .** Note that by construction, the open set in  $\mathcal{M}_{\text{Tyu}}$  given by reduced divisors is the moduli space of parabolic structures on  $\Omega$  with reduced support in  $|2K_X|$  whose parabolic directions can moreover be normalized in a particular way (3.26), and that it is in bijective correspondence with isomorphism classes of certain parabolic bundles  $(E, \mathbf{q}^+)$  with reduced support in  $|2K_X|$ . One checks that our construction implies that the vector bundles  $E$  that one may obtain are all flat (and hence elements of  $\mathfrak{Bun}(X)$ ), and moreover semistable, generically stable and admitting hyperelliptic descent.

We may observe that the surface  $\{\boldsymbol{\lambda} = \infty\}$  in  $\mathcal{M}_{\text{Tyu}}$ , corresponding to  $\lambda = 0$  or  $\infty$ , is the locus of the trivial bundle  $E = E_0$ . We can also find an equation for the surface corresponding to

generic decomposable bundles  $E$ . Those are obtained from  $\lambda \in \{1, -1, i, -i\}$ , and hence correspond to  $\lambda^2 = 4(y_1 y_2)^2$  which, after expansion, reads

$$\begin{aligned} \lambda^2 = & \underline{p}(\underline{p} - \underline{s} + 1) \cdot \left( \underline{p}^3 - \sigma_1 \underline{p}^2 \underline{s} + \sigma_2 \underline{p} \underline{s}^2 - \sigma_3 \underline{s}^3 + (\sigma_1^2 - 2\sigma_2) \underline{p}^2 + (3\sigma_3 - \sigma_1 \sigma_2) \underline{p} \underline{s} \right. \\ & \left. + \sigma_1 \sigma_3 \underline{s}^2 + (\sigma_2^2 - 2\sigma_1 \sigma_3) \underline{p} - \sigma_2 \sigma_3 \underline{s} + \sigma_3^2 \right) \end{aligned}$$

Note that here and in the following, we used the notation (3.16). This surface is birationally equivalent to the Kummer surface in  $\mathcal{M}_{\text{NR}}$ , via the Narasimhan-Ramanan classification map given explicitly in the following proposition.

**Proposition 3.5.8.** *The natural classifying map  $\mathcal{M}_{\text{Tyu}} = \mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \xrightarrow{1:1} \mathcal{M}_{\text{NR}} \simeq \mathbb{P}_{\text{NR}}^3$  reads*

$$\begin{aligned} (\underline{s}, \underline{p}, \lambda) & \mapsto (v_0 : v_1 : v_2 : v_3) \\ & = \left( \frac{\lambda - \underline{s} \underline{p}^2 + 2(1 + \sigma_1) \underline{p}^2 - (\sigma_1 + \sigma_2) \underline{s} \underline{p} + (\sigma_2 + \sigma_3)(\underline{s}^2 - 2\underline{p}) - \sigma_3 \underline{s}}{\underline{s}^2 - 4\underline{p}} : \underline{p} : -\underline{s} : 1 \right). \end{aligned}$$

Note that the fibration  $\mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \rightarrow \mathbb{P}_D^2$  is send onto the pencil of lines of  $\mathbb{P}_{\text{NR}}^3$  passing through the trivial bundle  $E_0 : (1 : 0 : 0 : 0)$ .

*Sketch of proof.* The idea, detailed in [HL15, Prop. 4.10] is to consider the family of +6-curves in  $\mathbb{P}\Omega$  that contain all four Tyurin parabolics. It yields, after elementary transformations, the family of +2-curves in  $\mathbb{P}E$  forming the Narasimhan-Ramanan divisor  $D_E$  (in the generic situation, which is sufficient). One then uses the definition of the coordinates on  $\mathbb{P}_{\text{NR}}^3$ .  $\square$

### 3.5.4 The Bertram-Bolognesi moduli space

The space of non trivial extensions  $0 \rightarrow \mathcal{O}_X(-K_X) \rightarrow E \rightarrow \mathcal{O}_X(K_X) \rightarrow 0$  is

$$\mathbb{P}H^1(X, \mathcal{O}_X(-2K_X)) \simeq \mathbb{P}H^0(X, \mathcal{O}_X(3K_X))^\vee.$$

This space naturally parametrizes the moduli space of those pairs  $(E, L)$  where  $L \subset E$  is a non-degenerate Tyurin bundle. The hyperelliptic involution  $\iota$  acts naturally on  $H^0(X, \mathcal{O}_X(3K_X))$  and thus on its dual: the invariant subspace is an hyperplane that naturally parametrizes those pairs  $(E, L)$  that are invariant under the involution. We define the *Bertram-Bolognesi moduli space*  $\mathcal{M}_B := \mathbb{P}_B^3$  to be this invariant hyperplane:

$$\mathbb{P}_B^3 \subset \mathbb{P}H^0(X, \mathcal{O}_X(3K_X))^\vee =: \mathbb{P}_B^4.$$

A cubic differential  $\omega \in H^0(X, \mathcal{O}_X(3K_X))$  reads

$$\omega = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 y) \left( \frac{dx}{y} \right)^{\otimes 3}$$

uniquely so that the coefficients  $a_i$  provide a full set of coordinates. Let  $(b_0 : b_1 : b_2 : b_3 : b_4)$  be dual homogeneous coordinates for  $\mathbb{P}_B^4$ . The natural action of the hyperelliptic involution  $\iota : X \rightarrow X$  on cubic differentials induces an involution on  $\mathbb{P}_B^4$  that fixes the hyperplane  $\mathbb{P}_B^3 = \{b_4 = 0\}$  and the point  $(0 : 0 : 0 : 0 : 1)$ .

Table 5: List of vector bundles that can be obtained from Tyurin's construction.  
136

bundle type		Tyurin divisor	reduced	parabolic structure
stable	off $\Pi_{[w_i]}$	$D_E^T$	yes	generic
	on $\Pi_{[w_i]}$ , off $\Pi_{[w_j]}$	$2[w_i] + [P] + [\iota(P)]$	no	$(\lambda_{w_i}, \lambda_P, \lambda_{\iota(P)}) = (0, 1, \infty)$
	on $\Pi_{[w_i]} \cap \Pi_{[w_j]}$	$2[w_i] + 2[w_j]$	no	$\lambda_{w_i} \neq \lambda_{w_j}$
unipotent	generic	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$\lambda_P = \lambda_{\iota(P)}$
	special	$[P] + [\iota(P)] + 2[w]$	no	$\lambda_P = \lambda_{\iota(P)}$
	twisted by $\mathcal{O}_X([w_i] - [w_j])$	$2[w_i] + 2[w_j]$	no	$\lambda_{w_i} = \lambda_{w_j}$
affine $L_0 \rightarrow E \rightarrow L_0^{-1}$	$L_0^{\otimes 2} \neq \mathcal{O}_X$ $L_0 = \mathcal{O}_X([P] + [Q] - K_X)$	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$(\lambda_P, \lambda_{\iota(P)}, \lambda_Q, \lambda_{\iota(Q)})$ $= (0, 1, 0, \infty)$
semi-stable decomposable $L_0 \oplus L_0^{-1}$	generic: $L_0^{\otimes 2} \neq \mathcal{O}_X$ $L_0 = \mathcal{O}_X([P] + [Q] - K_X)$	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$\lambda_P = \lambda_Q \neq \lambda_{\iota(P)} = \lambda_{\iota(Q)}$
	trivial: $L_0 = \mathcal{O}_X$	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$\lambda_P = \lambda_{\iota(P)} \neq \lambda_Q = \lambda_{\iota(Q)}$
	twist: $L_0 = \mathcal{O}_X([w_i] - [w_j])$	$2[w_i] + 2[w_j]$	no	$\lambda_{w_i} \neq \lambda_{w_j}$
unstable decomposable $L \oplus L^{-1}$	$L = \mathcal{O}_X([P]), P \notin W$	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$\lambda_{\iota(P)} \neq \lambda_P = \lambda_Q = \lambda_{\iota(Q)}$
	$L = \mathcal{O}_X([w])$	$2[w] + [Q] + [\iota(Q)]$	no	$\lim_{P \rightarrow w}$ of the previous one
	$L = \mathcal{O}_X(K_X)$	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$\lambda_P = \lambda_{\iota(P)} = \lambda_Q = \lambda_{\iota(Q)}$
odd Gunning bundle	$E_w$	$2[w] + [P] + [\iota(P)]$ ( $P$ arbitrary)	no	$\lambda_w = \lambda_P = \lambda_{\iota(P)}$



**Some known results.** Following the introductions of [Ber92, Kum00] and § 5 of [Bol07], the locus of unstable bundles is given by the natural embedding of the curve  $X$ :

$$\beta : X \hookrightarrow \mathbb{P}_B^4; (x, y) \mapsto (1 : x : x^2 : x^3 : y).$$

The locus of strictly semi-stable bundles is given by the quartic hypersurface  $\text{Wed} \subset \mathbb{P}_B^4$  spanned by the 2-secant lines of  $X$ . The Narasimhan-Ramanan moduli map

$$\mathbb{P}_B^4 \dashrightarrow \mathbb{P}_{\text{NR}}^3$$

is given by the full linear system of quadrics that contain  $X$ ; it restricts to  $\mathbb{P}_B^3$  as the full linear system of quadrics (of  $\mathbb{P}_B^3$ ) that contain the six points  $X \cap \mathbb{P}_B^3$ . After blowing-up the locus  $X$  of unstable bundles, we get a morphism

$$\widetilde{\mathbb{P}}_B^4 \rightarrow \mathbb{P}_{\text{NR}}^3$$

namely a conic bundle; its restriction to the strict transform  $\widetilde{\mathbb{P}}_B^3$  of  $\mathbb{P}_B^3$  is generically  $2 : 1$ , ramifying over the Kummer surface  $\text{Kum} \subset \mathbb{P}_{\text{NR}}^3$ . The quartic hypersurface  $\text{Wed}$  restricts to  $\mathbb{P}_B^3$  as the (*dual*) *Weddle surface*; it is sent onto the Kummer surface. There is a Poincaré vector bundle  $\mathcal{E} \rightarrow X \times \mathbb{P}_B^4$  realizing the classifying map above. Hence by restriction, there is a Poincaré bundle  $\mathcal{E} \rightarrow X \times \mathbb{P}_B^3$  on the double cover  $\mathbb{P}_B^3$  of  $\mathbb{P}_{\text{NR}}^3$ . The projectivized Poincaré bundle  $\mathbf{P}(\mathcal{E}) \rightarrow X \times \mathbb{P}_B^3$  defines a conic bundle  $\mathcal{C} \rightarrow X \times \mathbb{P}_{\text{NR}}^3$  over the quotient  $\mathbb{P}_{\text{NR}}^3$ . For each vector bundle  $E \in \mathbb{P}_{\text{NR}}^3$ , the fibre  $\mathcal{C}_E$  of the conic bundle represents the family of Tyurin-subbundles of  $E$ . Yet the conic bundle  $\mathcal{C}$  is not a projectivized vector bundle over  $\mathbb{P}_{\text{NR}}^3$ , not even up to birational equivalency, because a Poincaré bundle over a Zariski-open set of  $\mathbb{P}_{\text{NR}}^3$  does not exist [NR68].

**Explicit classifying map towards  $\mathbb{P}_{\text{NR}}^3$ .** Note that a point in the Bertram-Bolognesi moduli space  $\mathcal{M}_B$  corresponds to a pair  $(E, L)$  with  $E \in \mathfrak{Bun}(X)$  admitting hyperelliptic descent, together with a specified choice of  $h$ -invariant non-degenerate Tyurin subbundle  $L$ . It therefore corresponds to a well-defined parabolic bundle  $(\underline{E}, \underline{\mathbf{p}}) \in \mathfrak{Bun}(X/\iota)$  such that moreover  $\underline{E} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ , where the destabilizing subbundle of  $\underline{E}$  arises from  $L$  and carries no parabolic. Since  $E$ , as a non-split extension, does not admit a subbundle isomorphic to  $L^{-1}$ , the parabolic bundle  $(\underline{E}, \underline{\mathbf{p}})$  is moreover indecomposable. Conversely, parabolic bundles arising as a non-split extensions

$$0 \rightarrow (\mathcal{O}_{\mathbb{P}^1}(-1), \emptyset) \rightarrow (\underline{E}, \underline{\mathbf{p}}) \rightarrow (\mathcal{O}_{\mathbb{P}^1}(-2), \underline{W}) \rightarrow 0$$

are elements of  $\mathbb{P}\mathbb{H}^1(\mathbb{P}^1, \text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(\underline{W}), \mathcal{O}_{\mathbb{P}^1}(-1)))$ , which by Serre duality, is identified with  $\mathbb{P}\mathbb{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^\vee$ . After lifting such parabolic bundles by  $X \rightarrow \mathbb{P}^1$ , applying elementary transformations and forgetting the parabolic structure, we precisely get those extensions

$$0 \rightarrow \mathcal{O}_X(-K_X) \rightarrow E \rightarrow \mathcal{O}_X(K_X) \rightarrow 0$$

i.e. by those points of  $\mathbb{P}_B^4 = \mathbb{P}\mathbb{H}^0(X, \mathcal{O}_X(3K_X))^\vee$ , that are  $\iota$ -invariant. Using this relation, two results (Prop. 3.6.1 and Prop. 3.6.2) that will be presented in § 3.6 yield, as an immediate corollary, the explicit Narasimhan-Ramanan classification map.

**Proposition 3.5.9.** *The natural map  $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_{\text{NR}}^3$  is generically  $2 : 1$  and given by*

$$(b_0 : b_1 : b_2 : b_3) \mapsto \begin{cases} v_0 = & b_2b_3 - (1 + \sigma_1)b_2^2 + (\sigma_1 + \sigma_2)b_1b_2 - (\sigma_2 + \sigma_3)b_0b_2 + \sigma_3b_0b_1 \\ v_1 = & b_2^2 - b_1b_3 \\ v_2 = & b_0b_3 - b_1b_2 \\ v_3 = & b_1^2 - b_0b_2 \end{cases}$$

Moreover, the (dual) Weddle surface, i.e., the lift to  $\mathbb{P}_B^3$  of the Kummer equation, reads

$$\begin{aligned} &(-b_0b_2b_3^2 + b_1^2b_3^2 + b_1b_2^2b_3 - b_2^4) + (1 + \sigma_1)(b_0b_2^2b_3 - 2b_1^2b_2b_3 + b_1b_2^3) + (\sigma_1 + \sigma_2)(-b_0b_2^3 + b_1^3b_3) \\ &+ (\sigma_2 + \sigma_3)(-b_0b_1^2b_3 + 2b_0b_1b_2^2 - b_1^3b_2) + \sigma_3(b_0^2b_1b_3 - b_0^2b_2^2 - b_0b_1^2b_2 + b_1^4) = 0. \end{aligned}$$

Note that on  $X$ , we have the following equation:

$$y^2 = x(x-1)(x-r)(x-s)(x-t) = x^5 - (1 + \sigma_1)x^4 + (\sigma_1 + \sigma_2)x^3 - (\sigma_2 + \sigma_3)x^2 + \sigma_3x.$$

It follows that the equation for  $v_0$  in Proposition 3.5.9 vanishes along the embedding  $\beta(X) \subset \mathbb{P}_B^4$ . Moreover, we have the following equations on  $\beta(X)$ :

$$b_0b_2 = b_1^2 = x^2, \quad b_0b_3 = b_1b_2 = x^3 \quad \text{and} \quad b_1b_3 = b_2^2 = x^4.$$

It follows that the components  $v_0, \dots, v_3$  of the map  $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_{\text{NR}}^3$  exactly correspond to the restriction to  $\mathbb{P}_B^3$  of the natural quadratic forms on  $\mathbb{P}_B^4$  vanishing along the embedding  $\beta : X \hookrightarrow \mathbb{P}_B^4$ . It is quite surprising that the most natural basis both appearing from the Bertram point of view, and the Narasimhan-Ramanan point of view, are so compatible. They provide the same system of coordinates on  $\mathcal{M}_{\text{NR}}$  which is however not considered in the classical theory of Kummer surfaces (see [Hud90, GD94]).

Moreover, since we know that points in  $\mathcal{M}_B$  correspond to the particular type of parabolic bundles in  $\mathfrak{Bun}(X/\iota)$  described above, the dictionary in Section 3.4 classifies all bundles  $E$  arising from points  $(E, L) \in \mathcal{M}_B$ . Again those bundles  $E$  which are special in the sense that they are neither stable bundles off the Gunning planes nor generic decomposable bundles give a bundle-type interpretation of geometrical features of the dual Weddle surface in  $\mathcal{M}_B$ . Since we know where the special bundles occur in  $\mathbb{P}_{\text{NR}}^3$ , we can explicitly compute their locus in  $\mathbb{P}_B^3$  by Proposition 3.5.9. The result is given in the following proposition and summarized in Figure 12.

**Proposition 3.5.10.** *The special bundles occurring in  $\mathbb{P}_B^3 = \mathcal{M}_B$  are precisely the following*

- **Odd Gunning bundles  $Q_i$ :** they are the 6 special points of the twisted cubic parametrized by

$$\underline{\beta} : X/\iota \rightarrow \mathbb{P}_B^3 ; x \mapsto (1 : x : x^2 : x^3),$$

namely  $Q_i$  is the image of the Weierstrass point  $w_i$ .

- **Generic unipotent bundles  $\underline{\Delta}$ :** the 1-parameter family defined by the twisted cubic  $\underline{\beta}(X/\iota)$  (minus the special points) corresponds to the set of non-special unipotent bundles.
- **Twisted unipotent bundles  $\underline{\Delta}_{i,j}$ :** lines of  $\mathbb{P}_B^3$  passing through  $Q_i$  and  $Q_j$ .
- **Even Gunning planes  $\underline{\Pi}_{i,j,k}$ :** planes of  $\mathbb{P}_B^3$  passing through  $Q_i, Q_j$  and  $Q_k$ .
- **Odd Gunning planes  $\underline{\Pi}'_i$ :** the quadric surface of  $\mathbb{P}_B^3$  with a conic singular point at  $Q_i$  that contains the 5 lines  $\underline{\Delta}_{i,j}$  and the cubic  $\underline{\Delta}$ .

The preimage under the Narasimhan-Ramanan classifying map  $\gamma : \mathbb{P}_B^3 \xrightarrow{2:1} \mathbb{P}_{\text{NR}}^3$  of the Kummer surface is the dual Weddle surface  $\text{Wed}(X)$ , which is, via  $\gamma$ , a birational model of  $\text{Kum}(X)$ . It is also a quartic surface, but with only 6 nodes (see [Hud90, GD94]). The above bundle-type interpretation describes this birational map geometrically: The 16 singular points of  $\text{Kum}(X)$  are blown-up and replaced in  $\text{Wed}(X)$  by the cubic  $\underline{\Delta}$  and the 15 lines  $\underline{\Delta}_{i,j}$ . The six new conic points arise from the contraction to the point  $Q_i$  of the lift  $\underline{\Pi}_i$  of the Gunning plane  $\Pi_i$  in  $\mathbb{P}_{\text{NR}}^3$ . One deduces that the map

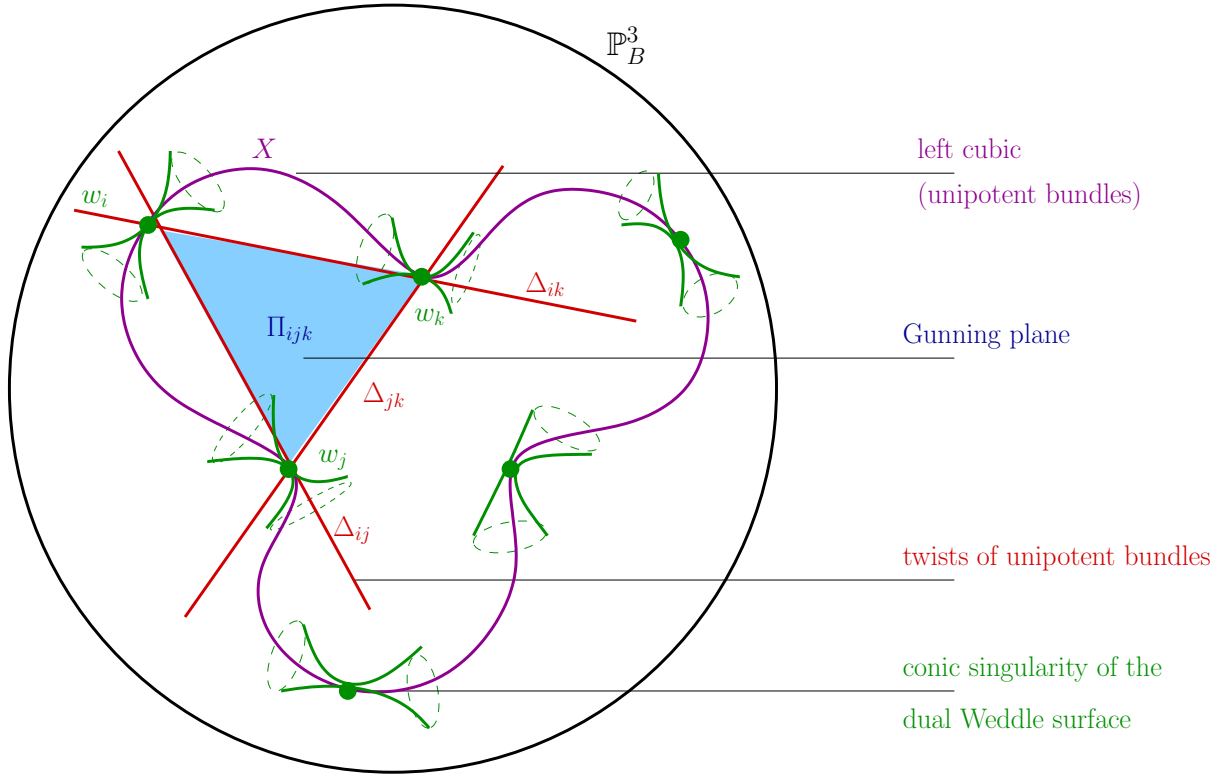


Figure 12: Special bundles in the Bertram-Bolognesi moduli space  $\mathbb{P}_B^3$ .

$\gamma$  is defined by the linear system of quadrics passing through the 6 points  $Q_i$ ; indeed, for a general plane  $\Pi \in \mathbb{P}_{\text{NR}}^3$ ,  $\gamma^*\Pi$  must intersect each contracted  $\underline{\Pi}_i$ . We thus recover the quadric system in [Dol10, § 4.6]. Those  $\Pi$  tangent to  $\text{Kum}(X)$  have a singular lift  $\underline{\Pi}$ ; when  $\Pi$  runs over the tangent planes of  $\text{Kum}(X)$ , the singular point of  $\underline{\Pi}$  runs over the (dual) Weddle surface. The 10 even Gunning planes in  $\mathbb{P}_{\text{NR}}^3$  have each two preimages, giving rise to the 20 lifted Gunning planes  $\underline{\Pi}_{i,j,k}$  (each passing through 3 of the 6 conic points). Note that the complement of the dual Weddle surface covers the open set of stable bundles in  $\mathbb{P}_{\text{NR}}^3$

$$\mathbb{P}_B^3 \setminus \text{Wed}(X) \xrightarrow{\gamma} \mathbb{P}_{\text{NR}}^3 \setminus \text{Kum}(X),$$

but that this is not a covering since over odd Gunning planes, only the lift  $\underline{\Pi}'_i$  occurs in  $\mathbb{P}_B^3$  (the other one corresponds to the indeterminacy point  $Q_i$ ).

**The universal family via Tyurin's approach.** Recall from § 3.5.7 that we have introduced a parameter space  $(X_1 \times X_2 \times \mathbb{P}_\lambda^1)$  of a family of parabolic structures  $\mathbf{q}$  on  $\Omega$  with  $\mathbb{P}\Omega = X \times \mathbb{P}_\lambda^1$  supported by elements of  $|2K_X|$  yielding, by elementary transformations, ordered triples  $(E, L, L')$  of flat bundles  $E$  with (generically) non-degenerate and  $h$ -invariant Tyurin subbundles  $L, L'$ . Considering only the data  $(E, L)$  then yields an element of  $\mathcal{M}_B$ . Using [LS15, Thm. 4.2], we showed in [HL15, Prop. 7.4] that the classifying map is explicitly described as follows.

**Proposition 3.5.11.** *The natural map  $X_1 \times X_2 \times \mathbb{P}_\lambda^1 \xrightarrow{16:1} \mathbb{P}_B^3$  is given by*

$$\begin{cases} b_0 &= \lambda y_2 - \frac{1}{\lambda} y_1 \\ b_1 &= \lambda x_1 y_2 - \frac{1}{\lambda} x_2 y_1 \\ b_2 &= \lambda x_1^2 y_2 - \frac{1}{\lambda} x_2^2 y_1 \\ b_3 &= \lambda x_1^3 y_2 - \frac{1}{\lambda} x_2^3 y_1 \end{cases}$$

Consider now the total space  $(X_1 \times X_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1)$ . It is by construction equipped with the 4 rational sections

$$(P_1, \lambda), (\iota(P_1), -\lambda), \left(P_2, \frac{1}{\lambda}\right), \left(\iota(P_2), -\frac{1}{\lambda}\right) : (X_1 \times X_2 \times \mathbb{P}_\lambda^1) \rightarrow (X \times \mathbb{P}_z^1)$$

which are globally invariant under the action of the 32-order group  $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz}, \sigma_{1/z} \rangle$ , whose orbits (for generic points) correspond to isomorphism classes of  $E$ . The quotient provides a projective Poincaré bundle, namely a (non trivial)  $\mathbb{P}^1$ -bundle over  $(\mathbb{P}_D^2 \times \mathbb{P}_\lambda^1) \times X$  (actually, over an open set of Tyurin configuration space) equipped with a universal parabolic structure. After positive elementary transformation, we get a universal  $\mathbb{P}^1$ -bundle over an open subset of  $\mathbb{P}_{\text{NR}}^3$ . However, we cannot lift the construction to a vector bundle because the action of  $\langle z \mapsto -z, z \mapsto \frac{1}{z} \rangle$  (induced by  $\langle \sigma_{iz}^2, \sigma_{1/z} \rangle$ ) does not lift to a linear  $\text{GL}_2\mathbb{C}$ -action (indeed,  $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  do not commute). This is the reason why there is no Poincaré bundle for  $\mathbb{P}_{\text{NR}}^3$ , but only a projective version of it. The ambiguity is killed-out if we do not take  $\sigma_{1/z}$  into account, meaning that we choose one of the two  $h$ -invariants Tyurin subbundles and consider isomorphism classes of  $(E, L)$ : we then obtain Bolognesi's Poincaré bundle [Bol09] mentioned earlier. It is here explicitly given as follows.

Consider the vector bundle  $\tilde{\mathcal{E}} = \pi^*(\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X))$  over  $(X_1 \times X_2 \times \mathbb{P}^1) \times X$ , where  $\pi$  denotes the projection from  $X_1 \times X_2 \times \mathbb{P}^1 \times X$  to  $X$ . We shall moreover denote  $\pi_i$  the projection to  $X_i$ , and consider the divisor  $\mathcal{D} = \{\pi = \pi_1\} + \{\pi = \iota \circ \pi_1\} + \{\pi = \pi_2\} + \{\pi = \iota \circ \pi_2\}$ . We define a parabolic structure over  $\mathcal{D}$  given by

$$(P_1, P_2, \lambda, P_1, \begin{pmatrix} \lambda \\ 1 \end{pmatrix}), (P_1, P_2, \lambda, \iota(P_1), \begin{pmatrix} -\lambda \\ 1 \end{pmatrix}), (P_1, P_2, \lambda, P_2, \begin{pmatrix} 1 \\ \lambda \end{pmatrix}), (P_1, P_2, \lambda, \iota(P_2), \begin{pmatrix} -1 \\ \lambda \end{pmatrix}).$$

The action of the group  $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz}, \sigma_{1/z} \rangle$  on the base lifts to an action on the bundle  $\tilde{\mathcal{E}}$  given by

$$(P_1, P_2, \lambda, P, Z) \begin{cases} \xrightarrow{\sigma_{12}} & (P_2, P_1, \frac{1}{\lambda}, P, Z) \\ \xrightarrow{\sigma_\iota} & (\iota(P_1), \iota(P_2), -\lambda, P, Z) \\ \xrightarrow{\sigma_{iz}} & \left( (P_1, \iota(P_2), i\lambda, P, \begin{pmatrix} \sqrt{i} & 0 \\ 0 & \frac{1}{\sqrt{i}} \end{pmatrix} Z) \right). \end{cases}$$

By [Bis97],  $\tilde{\mathcal{E}}$  is the lift of a vector bundle  $\hat{\mathcal{E}} \rightarrow X \times \mathcal{B}$  with  $\mathcal{B} = (X_1 \times X_2 \times \mathbb{P}^1) / \langle \sigma_{12}, \sigma_\iota, \sigma_{iz} \rangle$ . Moreover, the parabolic structure we gave on  $\tilde{\mathcal{E}}$  is also invariant and descends to a parabolic structure on  $\hat{\mathcal{E}}$ . At least in restriction to a codimension 2 subset of  $\mathcal{B}$ , it makes sense to perform positive elementary transformations along this (not everywhere reduced) parabolic structure. This yields a vector bundle  $\mathcal{E} \rightarrow X \times \mathcal{B}$ . After appropriate restriction and identification, it corresponds to a universal bundle on the 2-cover  $\mathcal{M}_B \dashrightarrow \mathcal{M}_{\text{NR}}$ .

### 3.6 A chart of the moduli stack $\mathfrak{Bun}(X)$

Consider the map  $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$  defined in Section 3.4. Consider moreover the subset  $\mathfrak{Bun}^{\text{ind}}(X/\iota)$  of indecomposable parabolic bundles in  $\mathfrak{Bun}(X/\iota)$ . We know from Proposition 3.3.5, § 3.4.5 and § 3.3.3 that its image  $\mathfrak{Bun}^*(X)$  under  $\phi$  consists in the complement in  $\mathfrak{Bun}(X)$  of the affine bundles, the trivial bundle and the twists of the trivial bundle. Moreover, we know from Section 3.4 that the restricted map

$$\phi : \mathfrak{Bun}^{\text{ind}}(X/\iota) \twoheadrightarrow \mathfrak{Bun}^*(X) \tag{3.28}$$

is a ramified two-cover, ramifying over the locus of decomposable bundles in  $\mathfrak{Bun}^*(X)$ . On the other hand, according to [AL97, LS15],  $\mathfrak{Bun}^{\text{ind}}(X/\iota)$  has the structure of a smooth non-separated

projective scheme. In particular, (3.28) describes a chart of  $\mathfrak{Bun}(X)$ , seen as a moduli stack. We are now going to describe  $\mathfrak{Bun}^*(X)$  by describing both  $\mathfrak{Bun}^{\text{ind}}(X/\iota)$  and the Galois-involution of  $\phi$  geometrically and, to a large extent, explicitly.

Note that  $\mathfrak{Bun}^{\text{ind}}$  can be constructed by patching spaces naturally isomorphic to  $(\mathbb{P}^1)^3$  (see [AL97]) or  $\mathbb{P}^3$  (see [LS15]) along Zariski-open subsets. We are going to present a certain number of those, which will be referred to as *charts*, with particular geometrical meaning, natural explicit coordinates and transition maps. Note that, as one can easily convince oneself, affine parts of these charts admit natural parabolic Poincaré bundles. One of those -and one of our preferred charts- corresponds to Bolognesi's universal bundle over Bertram's moduli space (see § 3.5.4). The geometry of the explicit transition maps and the Galois involution will be described by extensively using the dictionary from Section 3.4.

As an application, from another one of our preferred charts, endowed with its Narasimhan-Ramanan classification map, we deduce in § 3.6.6 an explicit expression of the Hitchin map

$$\mathfrak{Higgs}(X) \simeq T^{\vee}\mathfrak{Bun}^s(X) \rightarrow H^0(X, \Omega_X^1 \otimes \Omega_X^1).$$

### 3.6.1 Semi-stable bundles and projective charts

For any choice of  $\boldsymbol{\mu} = (\mu_0, \mu_1, \mu_r, \mu_s, \mu_t, \mu_{\infty}) \in [0, 1]^6$ , we denote by  $\text{Bun}_{\boldsymbol{\mu}}^{ss}(X/\iota) \subset \mathfrak{Bun}(X/\iota)$  the set of parabolic bundles that are semistable when endowed with the weight  $\boldsymbol{\mu}$  (see § 3.2.3). For a generic weight  $\boldsymbol{\mu}$ , semi-stable bundles are automatically stable; in this case, the moduli space  $\text{Bun}_{\boldsymbol{\mu}}^{ss}(X/\iota)$  is projective, smooth and a geometric quotient. The special weights  $\boldsymbol{\mu}$ , for which some bundles are strictly semi-stable, form a finite collection of affine planes in the weight-space  $[0, 1]^6 \ni \boldsymbol{\mu}$  called *walls*. They cut-out  $[0, 1]^6$  into finitely many *chambers*: the connected components of the complement of walls. Along walls, the moduli space is no more a geometric quotient, but a categorical quotient, identifying some semi-stable bundles together to get a (Hausdorff) projective variety, which might be singular in this case; outside of the strictly semi-stable locus,  $\text{Bun}_{\boldsymbol{\mu}}^{ss}(X/\iota)$  is still smooth and a geometric quotient. The moduli space  $\text{Bun}_{\boldsymbol{\mu}}^{ss}(X/\iota)$  is constant in a given chamber; if not empty, it has dimension three and contains as an open set the geometric quotient of those bundles  $(\underline{E}, \underline{\boldsymbol{p}})$  with  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  and parabolics  $\underline{\boldsymbol{p}}$  in general position:

- no parabolic in  $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow E$ ,
- no 3 parabolics in the same  $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow E$ ,
- no 5 parabolics in the same  $\mathcal{O}_{\mathbb{P}^1}(-3) \hookrightarrow E$ .

Between (non empty!) moduli spaces in any two chambers, we get a natural birational map

$$\text{can} : \text{Bun}_{\boldsymbol{\mu}}^{ss}(X/\iota) \xrightarrow{\sim} \text{Bun}_{\boldsymbol{\mu}'}^{ss}(X/\iota)$$

arising from the identification of the generic bundles occurring in both of them. The indeterminacy locus comes from those special parabolic bundles that are stable for  $\boldsymbol{\mu}$  but not for  $\boldsymbol{\mu}'$  and *vice-versa*; this configuration occurs each time we cross a wall. In [LS15] it is shown that a parabolic bundle  $(\underline{E}, \underline{\boldsymbol{p}})$  is indecomposable if, and only if, it is stable for a good choice of weights  $\boldsymbol{\mu} \in [0, 1]^6$ . The moduli space  $\mathfrak{Bun}^{\text{ind}}(X/\iota)$  of indecomposable bundles can thus be covered by a finite collection of such moduli spaces, by choosing one  $\boldsymbol{\mu}$  in each non empty chamber; therefore,  $\mathfrak{Bun}^{\text{ind}}(X/\iota)$  can be constructed by patching together these moduli spaces by means of canonical maps along the open set of common bundles, yielding a structure of smooth non separated scheme. In this context, two parabolic bundles  $(\underline{E}, \underline{\boldsymbol{p}})$  and  $(\underline{E}', \underline{\boldsymbol{p}'})$  are said to be *arbitrarily close* if there are families  $(\underline{E}_t, \underline{\boldsymbol{p}}_t)_{t \in \mathbb{A}^1}$  and  $(\underline{E}'_t, \underline{\boldsymbol{p}}'_t)_{t \in \mathbb{A}^1}$

such that  $(\underline{E}_t, \underline{\mathbf{p}}_t) \simeq (\underline{E}'_t, \underline{\mathbf{p}}'_t)$  for each  $t \neq 0$  but  $(\underline{E}_0, \underline{\mathbf{p}}_0) \simeq (\underline{E}, \underline{\mathbf{p}})$  and  $(\underline{E}'_0, \underline{\mathbf{p}}'_0) \simeq (\underline{E}', \underline{\mathbf{p}})$ . If two parabolic bundles over  $\mathbb{P}^1$  are arbitrarily close then of course the corresponding vector bundles over  $X$  are arbitrarily close in the sense of § 3.5.1. As shown in [LS15], for example any point of  $\underline{\Delta}$  is arbitrary close to any point of  $\underline{\Delta}'$  (see § 3.4.3). This will give rise to a flop phenomenon when we will compare certain semi-stable projective charts. The same phenomenon occurs for twisted unipotent bundles.

It is closely related to the reason why decomposable flat bundles are not taken into account in this picture. Indeed, consider for example the preimage  $(\underline{E}_0, \underline{\mathbf{p}}^0) := \phi^{-1}(E_0)$  of the trivial bundle on  $X$  (see § 3.4.5). If the bundle  $\underline{E}_0 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$  equipped with the decomposable parabolic structure  $\underline{\mathbf{p}}^0$  defined by the fibres of  $\mathcal{O}_{\mathbb{P}^1}(-3) \hookrightarrow \underline{E}_0$  is semi-stable for some choice of weights  $\underline{\mu}$ , then all other parabolic structures  $\underline{\mathbf{p}}$  on  $\underline{E}_0$  with no parabolics in the total space of  $\mathcal{O}_{\mathbb{P}^1} \subset \underline{E}_0$  are also semi-stable and arbitrarily close to  $\underline{\mathbf{p}}^0$ ; they are represented by the same point in the Hausdorff quotient  $\text{Bun}_{\underline{\mu}}^{ss}(X/\iota)$ . One can check that this point is moreover necessarily singular.

As an example that will be useful in the following, let us consider weights of the form

$$\underline{\mu} = (\mu_0, \mu_1, \mu_r, \mu_s, \mu_t, \mu_\infty) = (\mu, \mu, \lambda, \lambda, \lambda, \mu) \quad (3.29)$$

with  $\lambda, \mu \in [0, 1]$ . Considering only  $\underline{\mu}$  as in (3.29), a parabolic bundle belongs a wall if its projectivization possesses a section with self-intersection number  $k \in 2\mathbb{Z} + 1$  containing  $m$  parabolics over  $\{0, 1, \infty\}$  and  $\ell$  parabolics over  $\{r, s, t\}$  such that

$$0 = k + (3 - 2m)\mu + (3 - 2\ell)\lambda$$

for some  $\lambda, \mu \in [0, 1]$ . Table 6 lists all possible configurations. They are visualized in Figure 13.

Possible configuration	$(k, m, \ell)$	Walls
$\lambda = -\mu + \frac{5}{3}$	$(-5, 0, 0), (5, 3, 3)$	
$\lambda = -\mu + 1$	$(-3, 0, 0), (-1, 1, 1), (1, 2, 2), (3, 3, 3)$	
$\lambda = -\mu + \frac{1}{3}$	$(-1, 0, 0), (1, 3, 3)$	
$\lambda = -3\mu + 1$	$(-1, 0, 1), (1, 3, 2)$	③
$\lambda = -3\mu + 3$	$(-3, 0, 1), (3, 3, 2)$	
$\lambda = -\frac{1}{3}\mu + 1$	$(-3, 1, 0), (3, 2, 3)$	
$\lambda = -\frac{1}{3}\mu + \frac{1}{3}$	$(-1, 1, 0), (1, 2, 3)$	
$\lambda = 3\mu - 1$	$(-1, 0, 2), (1, 3, 1)$	②
$\lambda = \frac{1}{3}\mu + \frac{1}{3}$	$(-1, 2, 0), (1, 1, 3)$	
$\lambda = \mu + \frac{1}{3}$	$(-1, 3, 0), (1, 0, 3)$	
$\lambda = \mu - \frac{1}{3}$	$(-1, 0, 3), (1, 3, 0)$	①

Table 6: Possible wall-configurations for weights of the form  $\underline{\mu} = (\mu, \mu, \lambda, \lambda, \lambda, \mu)$ .

The pink line in in Figure 13 is not a wall, it will be considered later. The particular meaning of the red parts of Table 6 and Figure 13 will also be considered later. The two blue chambers are non-empty and correspond to our two main charts.

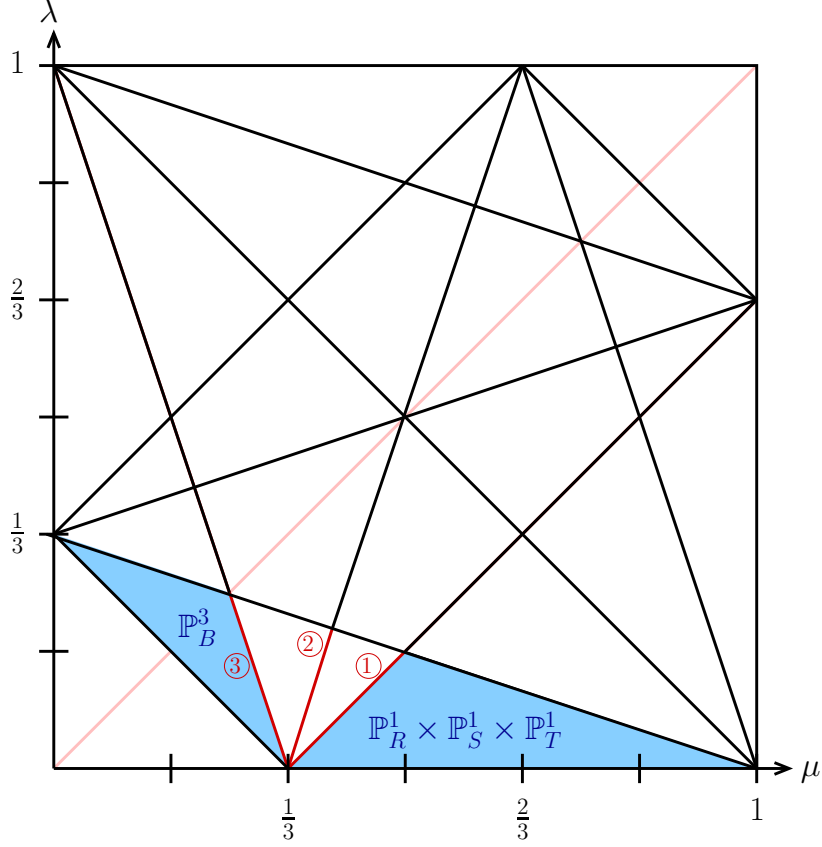


Figure 13: Chambers of moduli spaces for the weights  $\boldsymbol{\mu} = (\mu, \mu, \lambda, \lambda, \lambda, \mu)$ .

### 3.6.2 The chart $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$

Our first main chart (see [AL97] and [LS15, § 3.4]) is given by weights of the form

$$\mu_0 = \mu_1 = \mu_\infty = \frac{1}{2} \quad \text{and} \quad \mu_r = \mu_s = \mu_t = 0$$

and is isomorphic to  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ . Precisely,  $\boldsymbol{\mu}$ -stable bundles  $(\underline{E}, \underline{\mathbf{p}})$  are given by  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  with  $\underline{p}_0, \underline{p}_1, \underline{p}_\infty$  outside of  $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E}$  and not all three of them contained in the same  $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$ . Within the 2-parameter family of line subbundles isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(-2)$  we can choose one containing at least  $\underline{p}_0$  and  $\underline{p}_\infty$  say, and then choose meromorphic sections  $e_1$  and  $e_2$  of  $\mathcal{O}_{\mathbb{P}^1}(-1)$  and  $\mathcal{O}_{\mathbb{P}^1}(-2)$  (whose divisor is supported at  $x = \infty$ ) such that the parabolic structure is normalized to

$$\underline{p}_i = \lambda_i e_1 + e_2 \quad \text{with} \quad (\lambda_0, \lambda_1, \lambda_\infty) = (0, 1, 0) \quad \text{and} \quad (\lambda_r, \lambda_s, \lambda_t) = (R, S, T) \in \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1.$$

To compare to the point of view of [AL97], note that

$$\mathcal{O}_{\mathbb{P}^1}(1) \otimes \text{elm}_\infty^+(\underline{E}, \underline{\mathbf{p}}) = (\underline{E}'_0, \underline{\mathbf{p}}') \tag{3.30}$$

is the trivial bundle  $\underline{E}'_0 = \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}$  equipped with a parabolic structure having  $\underline{p}'_0, \underline{p}'_1$  and  $\underline{p}'_\infty$  pairwise distinct (with respect to the trivialization of the bundle).

**Proposition 3.6.1.** *The classifying map  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \dashrightarrow \mathbb{P}_{\text{NR}}^3$  is explicitly given by  $(R, S, T) \mapsto (v_0 : v_1 : v_2 : v_3)$  where*

$$\begin{aligned}
v_0 &= s^2 t^2 (r^2 - 1)(s - t)R - r^2 t^2 (s^2 - 1)(r - t)S + s^2 r^2 (t^2 - 1)(r - s)T + \\
&\quad + t^2 (t - 1)(r^2 - s^2)RS - s^2 (s - 1)(r^2 - t^2)RT + r^2 (r - 1)(s^2 - t^2)ST \\
v_1 &= rst [(r - 1)(s - t)R - (s - 1)(r - t)S + (t - 1)(r - s)T + \\
&\quad + (t - 1)(r - s)RS - (s - 1)(r - t)RT + (r - 1)(s - t)ST] \\
v_2 &= -st(r^2 - 1)(s - t)R + rt(s^2 - 1)(r - t)S - rs(t^2 - 1)(r - s)T - \\
&\quad - t(t - 1)(r^2 - s^2)RS + s(s - 1)(r^2 - t^2)RT - r(r - 1)(s^2 - t^2)ST \\
v_3 &= st(r - 1)(s - t)R - rt(s - 1)(r - t)S + sr(t - 1)(r - s)T + \\
&\quad + t(t - 1)(r - s)RS - s(s - 1)(r - t)RT + r(r - 1)(s - t)ST
\end{aligned}$$

The indeterminacy points

$$(R, S, T) = (0, 0, 0), \quad (1, 1, 1), \quad (\infty, \infty, \infty) \quad \text{and} \quad (r, s, t)$$

of this map correspond to the odd Gunning bundles  $E_{[w_1]}$ ,  $E_{[w_0]}$  and  $E_{[w_\infty]}$  respectively. Conversely, a generic point  $(v_0 : v_1 : v_2 : v_3) \in \mathcal{M}_{\text{NR}}$  has precisely two preimages in  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ :

$$\begin{aligned}
R &= \frac{r(t-1)(v_0+rv_1-r(s+t+st)v_3)T}{t(r-1)(v_0+tv_1-t(r+s+rs)v_3)-(r-t)(v_0+v_1-\sigma_2 v_3)T} \\
S &= \frac{s(t-1)(v_0+sv_1-s(r+t+rt)v_3)T}{t(s-1)(v_0+tv_1-t(r+s+rs)v_3)-(s-t)(v_0+v_1-\sigma_2 v_3)T},
\end{aligned}$$

where  $T$  is any solution of  $aT^2 + bT + ct^2 = 0$  with

$$\begin{aligned}
a &= (v_1 + v_2 t + v_3 t^2)(v_0 + v_1 - \sigma_2 v_3) \\
b &= -(1 + t)(v_0 v_2 + v_1^2 + tv_1 v_3) - 2(v_0 v_1 + tv_0 v_3 + tv_1 v_2) \\
&\quad + \sigma_2 (tv_1 + v_2 + tv_3)v_3 + (r + s + rs)(v_1 + t^2 v_2 + t^2 v_3)v_3 \\
c &= (v_1 + v_2 + v_3)(v_0 + tv_1 - t(r + s + rs)v_3).
\end{aligned}$$

The discriminant of this polynomial leads again to our equation of the Kummer surface in the coordinates  $(v_0 : v_1 : v_2 : v_3)$  given in Section [3.5.2](#). Here and below, we use the  $\sigma$ -notation from [\(3.16\)](#).

*Sketch of proof.* The idea of the proof of this proposition, detailed in [\[HL15, Prop. 5.2\]](#), is the following. Using [\(3.30\)](#), one may work with the trivial bundle  $\underline{E}'_0$ , and more precisely, with its trivialized lift  $E_0$  to  $X$ . One finds that in the generic situation, the family of degree  $-4$  subbundles of  $E_0$  that contain all parabolics of  $\mathbf{p}'$  yields, after application of  $\mathcal{O}_X(-3[w_\infty]) \otimes \text{elm}_W^+$ , the 1-parameter family of line subbundles of  $E$  that forms the Narasimhan-Ramanan divisor  $D_E$  (see [3.5.1](#)). Degree  $-4$  subbundles of the trivial bundle of the general form  $\mathcal{O}_X(-[P_1] - [P_2] - 2[w_\infty])$  are easily parametrized. Containing all parabolics yields a condition on  $\{P_1, P_2\}$  in terms of  $(R, S, T)$  that it suffices to translate into a coordinate  $(v_0 : \dots : v_3)$ .  $\square$

The Galois involution  $(R, S, T) \mapsto (\tilde{R}, \tilde{S}, \tilde{T})$  of the classifying map  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \dashrightarrow \mathbb{P}_{\text{NR}}^3$  is easily calculated and given by

$$\begin{aligned}
\tilde{R} &= \lambda(R, S, T) \cdot \frac{(s-t)+(t-1)S-(s-1)T}{-t(s-1)S+s(t-1)T+(s-t)ST} \\
\tilde{S} &= \lambda(R, S, T) \cdot \frac{(r-t)+(t-1)R-(r-1)T}{-t(r-1)R+r(t-1)T+(r-t)RT} \\
\tilde{T} &= \lambda(R, S, T) \cdot \frac{(r-s)+(s-1)R-(r-1)S}{-s(r-1)R+r(s-1)S+(r-s)RS}
\end{aligned}$$



$$\text{where } \lambda(R, S, T) = \frac{t(r-s)RS - s(r-t)RT + r(s-t)ST}{(s-t)R - (r-t)S + (r-s)T}.$$

Its fixed points provide the equation in coordinates  $(R, S, T)$  of the lift of the Kummer surface, namely

$$\begin{aligned} & ((s-t)R + (t-r)S + (r-s)T)RST \\ & + t((r-1)S - (s-1)R)RS + r((s-1)T - (t-1)S)ST + s((t-1)R - (r-1)T)RT \\ & - t(r-s)RS - r(s-t)ST - s(t-r)RT = 0. \end{aligned}$$

### 3.6.3 The chart $\mathbb{P}_b^3 = \mathbb{P}_B^3$

Our second main chart is defined by democratic weights

$$\frac{1}{6} < \mu_0 = \mu_1 = \mu_r = \mu_s = \mu_t = \mu_\infty < \frac{1}{4}$$

and corresponds to the moduli space of the indecomposable parabolic structures on  $\underline{E} := \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  having no parabolic in the total space of  $\mathcal{O}_{\mathbb{P}^1}(-1)$ . It corresponds to the main chart  $\mathbb{P}_b^3$  of [\[LS15\]](#) given by  $\mathbb{P}H^1(\mathbb{P}^1, \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(\underline{W}), \mathcal{O}_{\mathbb{P}^1}(-1))) \simeq \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^\vee$ . As explained in § [3.5.4](#) it coincides with the Bertram-Bognessi-moduli space  $\mathbb{P}_B^3$  given by the  $\iota$ -invariant hyperplane of  $\mathbb{P}_B^4 = \mathbb{P}H^0(X, \mathcal{O}_X(3K_X))^\vee$ . From this point of view, the projective coordinates  $\mathbf{b} = (b_0 : b_1 : b_2 : b_3) \in \mathbb{P}_b^3$  are dual to the coordinates of  $\iota$ -invariant cubic forms

$$(a_0 + a_1x + a_2x^2 + a_3x^3) \left( \frac{dx}{y} \right)^{\otimes 3}.$$

**Proposition 3.6.2.** *The natural birational map  $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  is given by*

$$(b_0 : b_1 : b_2 : b_3) \mapsto \begin{cases} R &= r \frac{b_3 - (s+t+1)b_2 + (st+s+t)b_1 - stb_0}{b_3 - \sigma_1 b_2 + \sigma_2 b_1 - \sigma_3 b_0} \\ S &= s \frac{b_3 - (r+t+1)b_2 + (rt+r+t)b_1 - rtb_0}{b_3 - \sigma_1 b_2 + \sigma_2 b_1 - \sigma_3 b_0} \\ T &= t \frac{b_3 - (r+s+1)b_2 + (rs+r+s)b_1 - rsb_0}{b_3 - \sigma_1 b_2 + \sigma_2 b_1 - \sigma_3 b_0} \end{cases}$$

The inverse map is given by  $(R, S, T) \mapsto (b_0 : b_1 : b_2 : b_3)$  with

$$\begin{cases} b_0 &= \frac{R-r}{r(r-1)(r-s)(r-t)} + \frac{S-s}{s(s-1)(s-r)(s-t)} + \frac{T-t}{t(t-1)(t-r)(t-s)} \\ b_1 &= \frac{R-r}{(r-1)(r-s)(r-t)} + \frac{S-s}{(s-1)(s-r)(s-t)} + \frac{T-t}{(t-1)(t-r)(t-s)} \\ b_2 &= \frac{r(R-r)}{(r-1)(r-s)(r-t)} + \frac{s(S-s)}{(s-1)(s-r)(s-t)} + \frac{t(T-t)}{(t-1)(t-r)(t-s)} \\ b_3 &= \frac{r^2 R}{(r-1)(r-s)(r-t)} + \frac{s^2 S}{(s-1)(s-r)(s-t)} + \frac{t^2 T}{(t-1)(t-r)(t-s)} - \frac{1}{(r-1)(s-1)(t-1)} \end{cases}$$

*Sketch of proof.* The idea is to use the identification  $\mathbb{P}_B^3 = \mathbb{P}_b^3$  and to apply a result of [\[LS15\]](#) (see also [\[LSS12\]](#)). The elements of  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  appearing in  $\mathbb{P}_b^3$  are those with  $(R, S, T)$  finite. For any  $(E, \mathbf{p}) \in \mathbb{C}_{(R,S,T)}^3$ , and any compatible apparent parabolic Higgs field (see § [3.6.6](#)), one can compute its apparent map (the second fundamental form with respect to the destabilizing bundle), which yields a global section of  $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(\underline{W})$ . For example for  $\Theta_r$  in [\(3.33\)](#) one obtains  $\frac{(R-r)x^3 + (r(1+s+t) - R\sigma_1)x^2 + (R\sigma_2 - r(s+t) - \sigma_3)x - \sigma_3(R-1)}{x(x-1)(x-r)(x-s)(x-t)} dx$ . There are two maps

$$\mathbb{C}_{(R,S,T)}^3 \rightarrow \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^\vee.$$

One consists in associating to each  $(E, \mathbf{p})$  the hyperplane locus of compatible apparent parabolic Higgs fields. The other one associates the corresponding  $(b_0 : b_1 : b_2 : b_3)$ , which are the dual coordinates for  $\frac{a_0 + a_1x + a_2x^2 + a_3x^3}{x(x-1)(x-r)(x-s)(x-t)} dx$ . According to the proof of [LS15, Thm. 4.2], these two maps coincide. In particular, our example for  $\Theta_r$  yields  $b_3(R-r) + b_2(r(1+s+t) - R\sigma_1) + b_1(R\sigma_2 - r(s+t) - \sigma_3) - b_0\sigma_3(R-1) = 0$ . Similar equations from  $\Theta_s$  and  $\Theta_t$  in (3.33) yield the result.  $\square$

Recall that the classifying map  $\mathbb{P}_b^3 = \mathbb{P}_B^3 \rightarrow \mathbb{P}_{\text{NR}}^3$ , which is now a direct consequence of Propositions 3.6.1 and 3.6.2 is stated in Proposition 3.5.9

In order to understand the geometry of the birational map  $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  explicitly given in Proposition 3.6.2 we have to consider a path in  $\mathfrak{Bun}^{\text{ind}}(X/\iota)$  linking the corresponding chambers and the wall-crossing phenomena along this path. What we will obtain is summarized in Figure 14. Since  $\mathbb{P}_B^3$  corresponds to the weight  $\boldsymbol{\mu} = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$  and  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  corresponds to the weight  $\boldsymbol{\mu} = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{1}{2})$ , we may consider only the walls between chambers of the form  $\boldsymbol{\mu} = (\mu, \mu, \lambda, \lambda, \lambda, \mu)$  with  $\lambda, \mu \in [0, 1]$  as in (3.29). First, we want to consider the crossing of the walls ①, ② and ③ in Figure 13, in order to describe the birational map  $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ . We look at Table 6 and check which configurations  $(k, m, \ell)$  (corresponding to self-intersection number  $k$ ,  $m$  parabolics over  $\{0, 1, \infty\}$  and  $\ell$  parabolics over  $\{r, s, t\}$ ) arise in these walls. The configuration  $(k, m, \ell) = (1, 3, 0)$  is not stable in  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ , but  $(k, m, \ell) = (-1, 0, 3)$  is. By the dictionary in Section 3.4, the latter corresponds to the even Gunning bundle  $E_\vartheta$  with  $\vartheta = \mathcal{O}_X([w_r] + [w_s] - [w_t])$ . The point in the moduli space  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  corresponding to this bundle is blown up when crossing the wall ① and replaced by the corresponding Gunning plane:  $(k, m, \ell) = (1, 3, 0)$ . Passing on to wall ②, the three lines  $(k, m, \ell) = (-1, 0, 2)$  in the moduli space corresponding to the unipotent bundles tensored by  $\mathcal{O}_X([w_r] - [w_s])$ ,  $\mathcal{O}_X([w_s] - [w_t])$  and  $\mathcal{O}_X([w_s] - [w_t])$  respectively are no longer stable. Here a flop phenomenon occurs: these three lines are blown up and the resulting planes are contracted to three lines  $(k, m, \ell) = (1, 3, 1)$  corresponding to the families of the same types of unipotent bundles. Passing on to wall ③, the three planes  $(k, m, \ell) = (-1, 0, 1)$  corresponding to the odd Gunning planes with characteristic  $\vartheta \in \{\mathcal{O}_X([w_r]), \mathcal{O}_X([w_s]), \mathcal{O}_X([w_t])\}$  are contracted and replaced by three points corresponding to the configurations  $(k, m, \ell) = (1, 3, 2)$ : the Gunning bundles with characteristic  $\vartheta$ .

### 3.6.4 The four democratic charts and the Geiser involution

Following the pink line in Figure 13 means studying wall-crossing phenomena for moduli spaces  $\text{Bun}_\mu^{\text{ss}}(X/\iota)$  with democratic weights  $\boldsymbol{\mu} = (\mu, \mu, \mu, \mu, \mu, \mu)$  for  $\mu \in [0, 1]$ . As we see, walls occur for  $\mu \in \{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}\}$ . Those  $\mu$  corresponding neither to a wall nor to an empty chamber are charts of  $\mathfrak{Bun}^{\text{ind}}(X/\iota)$ . We will find four of those. As before, in order to describe the geometry of their transition maps, it is however useful to consider the moduli spaces on the walls as well. We are going to do so by moving  $\mu$  from 0 to 1. For each choice of  $\mu \in [0, 1]$ , one can easily check for which types of special bundles which hyperelliptic lifts are semi-stable with respect to the corresponding  $\boldsymbol{\mu}$ ; this is summarized in Table 7 (in terms of Section 3.4). Note that (avatars in  $\mathfrak{Bun}^{\text{ind}}(X/\iota)$  of) even Gunning bundles are not semistable for any choice of democratic weights.

**For  $\mu \in [0, \frac{1}{6}]$ :** The moduli space  $\text{Bun}_\mu^{\text{ss}}(X/\iota)$  is empty since  $\mathcal{O}_{\mathbb{P}^1}(-1)$  is destabilizing the generic parabolic bundle (even if it carries no parabolic).

**For  $\mu = \frac{1}{6}$ :** The moduli space  $\text{Bun}_\mu^{\text{ss}}(X/\iota)$  reduces to a single point. Indeed, it also contains the (non flat) decomposable bundle  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  with all parabolics  $\mathbf{p}$  lying in the total space of  $\mathcal{O}_{\mathbb{P}^1}(-2)$ . But the generic parabolic bundle is arbitrarily close to this decomposable bundle so that they have to be identified in the Hausdorff quotient  $\text{Bun}_\mu^{\text{ss}}(X/\iota)$ .

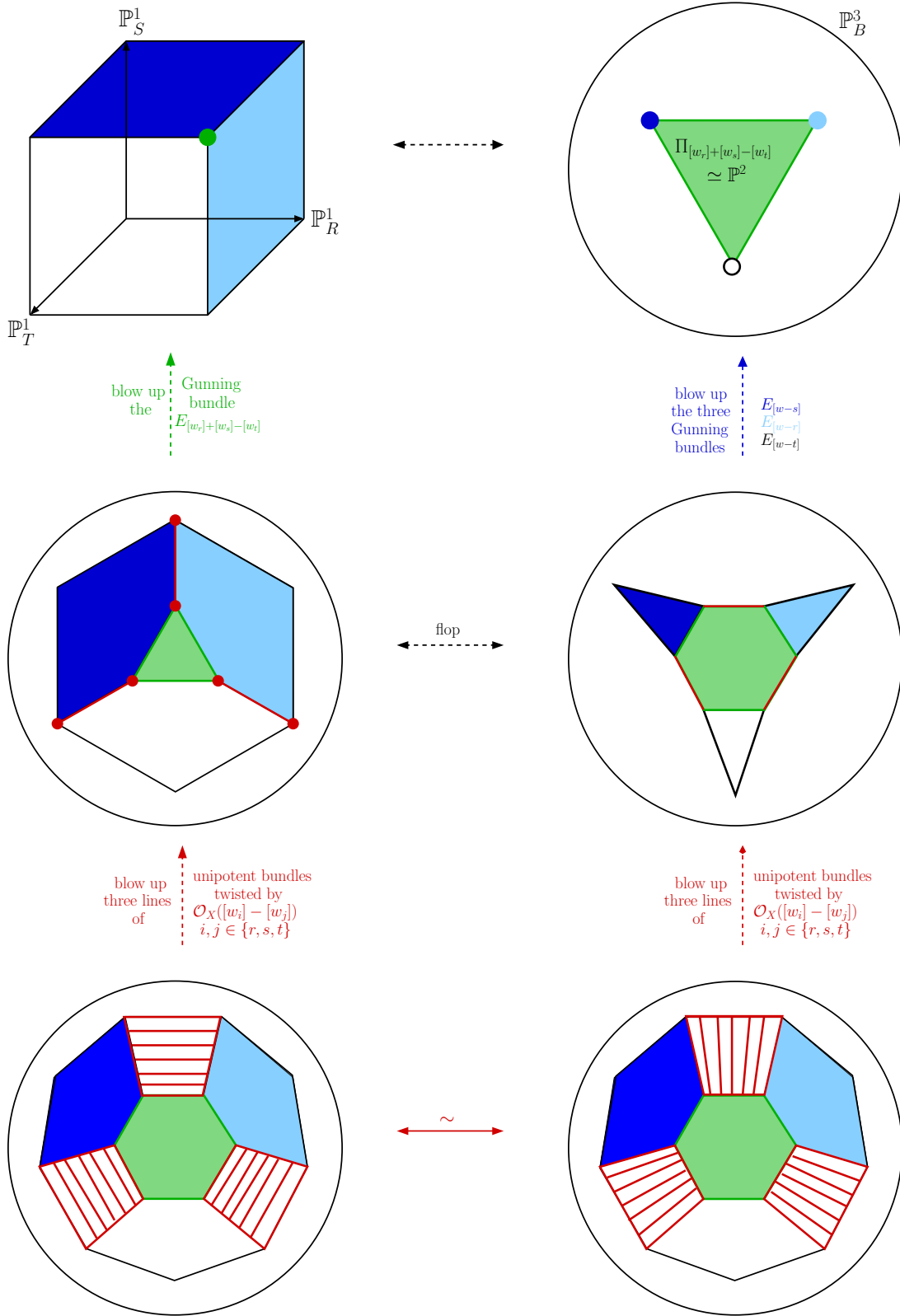


Figure 14: Geometry of the natural birational map  $\mathbb{P}^3_B \dashrightarrow \mathbb{P}^1_R \times \mathbb{P}^1_S \times \mathbb{P}^1_T$ .

$\mu$	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{6}$	1
unipotent bundles (and twists)			$\underline{\Delta}$		$\underline{\Delta}'$		
			$\underline{\Delta}_{ij}$		$\underline{\Delta}'_{ij}$		
odd Gunning bundles and planes		$Q_i$			$\underline{\Pi}_i$		
				$\underline{\Pi}'_i$		$Q'_i$	
even Gunning planes				$\underline{\Pi}_{ijk}$			

Table 7: Moving weights.

**For  $\mu \in ]\frac{1}{6}, \frac{1}{4}[$ :** Here, we recover our chart  $\mathbb{P}_B^3 = \mathbb{P}_b^3 = \text{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota)$  from § 3.5.4 and § 3.6.3 with special families  $\underline{\Delta}$ ,  $\underline{\Delta}_{ij}$ ,  $Q_i$ ,  $\underline{\Pi}'_i$  and  $\underline{\Pi}_{ijk}$ . The natural map  $\phi : \text{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota) \dashrightarrow \mathbb{P}_{\text{NR}}^3$  has indeterminacy points at all 6 points  $Q_i$ .

**For  $\mu = \frac{1}{4}$ :** Now, odd Gunning planes  $\underline{\Pi}_i$  become semi-stable, but are arbitrarily close to the corresponding point  $Q_i$ , so that they are identified in the quotient  $\text{Bun}_{\mu}^{ss}(X/\iota)$ . Therefore, the moduli space is still the same  $\mathbb{P}_b^3$ , but no longer a geometric quotient.

**For  $\mu \in ]\frac{1}{4}, \frac{1}{2}[$ :** Odd Gunning bundles  $Q_i$  are no longer semi-stable and are replaced by the corresponding Gunning planes  $\underline{\Pi}_i$ . The natural map

$$\text{can} : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \rightarrow \text{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota)$$

is the blow-up of  $\mathbb{P}_b^3$  at all 6 points  $Q_i$ , and the exceptional divisors represent the corresponding planes  $\underline{\Pi}_i$ . The classifying map  $\text{NR} \circ \phi : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \rightarrow \mathbb{P}_{\text{NR}}^3$  is a morphism.

**For  $\mu = \frac{1}{2}$ :** the trivial bundle and its 15 twists become semi-stable (and just for this special value of  $\mu$ ). In particular, unipotent families are identified with these bundles in the moduli space, which has the effect to contract the strict transforms of lines  $\underline{\Delta}_{ij}$  and the rational curve  $\underline{\Delta}$  to 16 singular points of  $\text{Bun}_{\mu}^{ss}(X/\iota)$ . This moduli space is exactly the double cover of  $\mathbb{P}_{\text{NR}}^3$  ramified along  $\text{Kum}(X)$ , therefore singular with conic points over each singular point of  $\text{Kum}(X)$ . The natural map

$$\text{can} : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \rightarrow \text{Bun}_{\frac{1}{2}}^{ss}(X/\iota)$$

is a minimal resolution.

**For  $\mu \in ]\frac{1}{2}, \frac{3}{4}[$ :** The families  $\underline{\Delta}$  and  $\underline{\Delta}_{ij}$  are no longer semi-stable, and are replaced by the families  $\underline{\Delta}'$  and  $\underline{\Delta}'_{ij}$ . But mind that the canonical map

$$\text{can} : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \dashrightarrow \text{Bun}_{\frac{1}{2}, \frac{3}{4}}^{ss}(X/\iota)$$

is not biregular: there is a flop phenomenon around each of the 16 aforementioned rational curves. Precisely, after blowing-up the 16 curves, we exactly get the resolution  $\widehat{\text{Bun}}_{\frac{1}{2}}^{ss}(X/\iota)$  of the previous moduli space by blowing-up the 16 conic points. Then, exceptional divisors are  $\simeq \mathbb{P}^1 \times \mathbb{P}^1$  and we can contract them back to rational curves by using the other ruling; this is the way the map can is constructed here. In particular, we get a second minimal resolution of  $\text{Bun}_{\frac{1}{2}}^{ss}(X/\iota)$ .

**For  $\mu \in [\frac{3}{4}, \frac{5}{6}[$ :** Here, we finally contract the strict transforms of  $\underline{\Pi}'_i$  to the points  $Q_i$ .

The Galois involution of the ramified cover  $\phi : \mathfrak{Bun}(X/\iota) \xrightarrow{2:1} \mathfrak{Bun}(X)$

$$\Upsilon := \mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_{\underline{W}}^+ : \mathfrak{Bun}(X/\iota) \xrightarrow{\sim} \mathfrak{Bun}(X/\iota)$$

induces isomorphisms between moduli spaces

$$\Upsilon : \text{Bun}_{\mu}^{ss}(X/\iota) \xrightarrow{\sim} \text{Bun}_{\mu'}^{ss}(X/\iota)$$

where  $\mu'$  is defined by  $\mu'_i = \frac{1}{2} - \mu_i$  for all  $i$ . In particular, it underlines the symmetry of our democratic family of moduli spaces around  $\mu = \frac{1}{2}$ : the Galois involution induces a biregular involution of  $\text{Bun}_{\frac{1}{2}}^{ss}(X/\iota)$ , as well as isomorphisms

$$\text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \xrightarrow{\sim} \text{Bun}_{\frac{1}{2}, \frac{3}{4}}^{ss}(X/\iota) \quad \text{and} \quad \text{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota) \xrightarrow{\sim} \text{Bun}_{\frac{3}{4}, \frac{5}{6}}^{ss}(X/\iota).$$

Considering now the composition

$$\text{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota) \xrightarrow{\text{can}} \text{Bun}_{\frac{3}{4}, \frac{5}{6}}^{ss}(X/\iota) \xrightarrow{\Upsilon} \text{Bun}_{\frac{1}{6}, \frac{1}{2}}^{ss}(X/\iota),$$

we get the (birational) Galois involution of the map  $\phi : \mathbb{P}_{\mathfrak{b}}^3 \dashrightarrow \mathbb{P}_{\text{NR}}^3$  described in Proposition [3.5.9](#). This is known as the Geiser involution (see [\[Dol10\]](#), § 4.6); it is a degree 7 birational map. The combination of all wall-crossing phenomena described in § [3.6.4](#) when  $\mu$  is varying from  $\frac{1}{6}$  to  $\frac{5}{6}$ , provides a complete decomposition of this map (see Table [8](#)):

- first blow-up 6 points (the  $Q_i$  along the embedding  $X/\iota \xrightarrow{\sim} \underline{\Delta} \subset \mathbb{P}_{\mathfrak{b}}^3$ ),
- flop 16 rational curves (the strict transforms of the twisted cubic  $\underline{\Delta}$  and all lines  $\underline{\Delta}_{ij}$ ),
- contract 6 planes (namely strict transforms of  $\Pi'_i$  onto  $Q_i$ ),
- then compose by the unique isomorphism sending  $Q'_i \rightarrow Q_i$ .

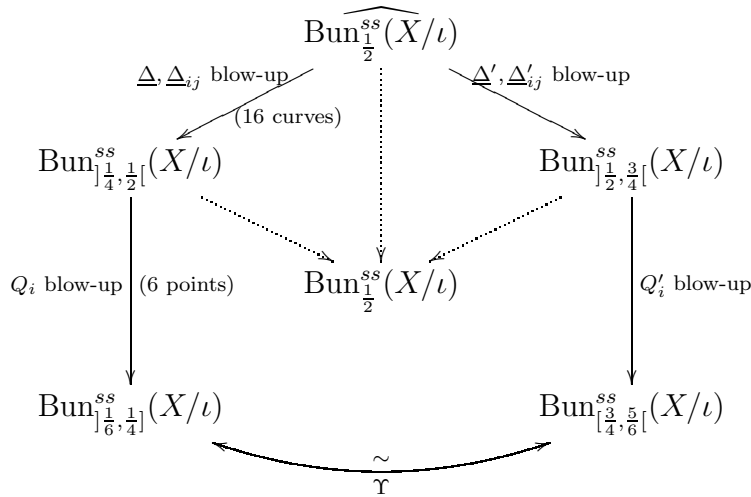


Table 8: Geometry of the Geiser involution.

### 3.6.5 A sufficient collection of charts

As a note, we now describe how to complete our preferred charts in order to cover  $\mathfrak{Bun}^{\text{ind}}(X/\iota)$ .

Consider our first main chart  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  and permutations thereof. By the latter, we mean charts  $\mathbb{P}_i^1 \times \mathbb{P}_j^1 \times \mathbb{P}_k^1$  obtained by choosing three distinct elements  $i, j, k \in \{0, 1, r, s, t, \infty\}$ , setting

$\mu_i = \mu_j = \mu_k = 0$  and all the other weights equal to  $\frac{1}{2}$ . The birational relations between those are obvious. In view of § 3.4.1, all of these together are sufficient to cover all stable bundles in  $\mathfrak{Bun}(X)$ .

Now add the two democratic charts  $\mathbb{P}_B^3 = \mathbb{P}_b^3 = \text{Bun}_{\frac{1}{5}}^{ss}(X/\iota)$  and  $\text{Bun}_{\frac{1}{3}}^{ss}(X/\iota)$ . Table 9 lists which non-stable elements of  $\mathfrak{Bun}(X)$  occur in the image under  $\phi$  of these and  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ . Here we use a checkmark sign ( $\checkmark$ ) only if, in view of Section 3.4, we can be certain that every bundle of a given type can be found, a quotient mark (?) when certain of them can possibly be found, and nothing when no bundle of this type can be found.

We conclude that all of the above are sufficient to cover  $\mathfrak{Bun}^*(X)$ . It then suffices to add for each the corresponding Galois-involution chart in order to cover  $\mathfrak{Bun}^{\text{ind}}(X/\iota)$ .

bundle type		$\mathbb{P}_b^3$	$\text{Bun}_{\frac{1}{3}}^{ss}(X/\iota)$	$\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$
unipotent	generic	$\checkmark$	$\checkmark$	?
	special		$\checkmark$	?
	twisted	$\checkmark$	$\checkmark$	?
affine				
semi-stable decomposable $L_0 \oplus L_0^{-1}$	$L_0 = \mathcal{O}_X([P] - [Q])$ with $P, Q \in X \setminus W$ and $P \neq Q$	$\checkmark$	$\checkmark$	?
	$L_0 = \mathcal{O}_X([P] - [w])$ with $P \in X \setminus W$ and $w \in W$		$\checkmark$	?
	$L_0^{\otimes 2} = \mathcal{O}_X$			
Gunning bundle	even, $\vartheta \simeq \mathcal{O}_X([w_r] + [w_s] - [w_t])$			$\checkmark$
	even, other $\vartheta$			
	odd, $\vartheta \in \{[w_1], [w_0], [w_\infty]\}$	$\checkmark$		$\checkmark$
	odd, other $\vartheta$	$\checkmark$		

Table 9: Non stable bundles that can be found in the image in  $\mathfrak{Bun}(X)$  .

### 3.6.6 Application to the Hitchin fibration

Let us denote by  $\mathfrak{Bun}^s(X) \simeq \mathcal{M}_{\text{NR}} \setminus \text{Kum}(X)$  the moduli space of stable rank two bundles with trivial determinant bundle over  $X$ . For  $E \in \mathfrak{Bun}^s(X)$ , the cotangent space of  $\mathfrak{Bun}^s(X)$  at  $E$  is isomorphic, by Serre duality, to  $H^0(X, \mathfrak{sl}(E) \otimes \Omega_X^1)$ , i.e., to the space of trace free holomorphic Higgs fields  $\theta : E \rightarrow E \otimes \Omega_X^1$  on  $E$ . In particular, we have a canonical isomorphism

$$T^\vee \mathfrak{Bun}^s(X) \simeq \mathfrak{Higgs}(X),$$

where  $\mathfrak{Higgs}(X)$  denotes the moduli space of tracefree holomorphic Higgs bundles  $(E, \theta)$  with  $E$  stable. In [Hit87a], Hitchin considered (in a more general setting) the map

$$\text{Hitch} : \left\{ \begin{array}{ll} \mathfrak{Higgs}(X) & \rightarrow H^0(X, \Omega_X^1 \otimes \Omega_X^1) \\ (E, \theta) & \mapsto \det(\theta) \end{array} \right\} \quad (3.31)$$

and established that it defines an algebraically completely integrable Hamiltonian system: the Liouville form on  $\mathcal{M}_{\text{NR}}$  induces a symplectic structure on  $\mathfrak{Higgs}(X)$  and writing a quadratic differential as  $(h_2 x^2 + h_1 x + h_0) \left(\frac{dx}{y}\right)^{\otimes 2}$ , the 3 components of Hitch

$$h_0, h_1, h_2 : \mathfrak{Higgs}(X) \rightarrow \mathbb{C}$$

are holomorphic functions commuting to each other for the Poisson structure. Moreover, fibers of the Hitchin map are open sets of 3-dimensional abelian varieties. One can also associate to  $(E, \theta)$  the spectral curve  $\text{spec}(\theta)$ , which is the double-section of the projectivized bundle  $\mathbb{P}E \rightarrow X$  defined by the eigendirections of  $\theta$ . This curve  $\text{spec}(\theta)$  is thus a two-fold ramified cover of  $X$ , ramifying at zeroes of the quadratic form  $\text{Hitch}(E, \theta)$ ; the spectral curve is constant along Hitchin fibers and its Jacobian is the compactification of the fiber (see for example [Hur96]). A broad field of applications has been deduced from the various algebraic and geometric properties of the Hitchin system and its generalizations since then. As an application of our description of the map  $\mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$ , we shall now calculate the Hitchin Hamiltonians  $h_i$  explicitly.

Denote by  $\mathfrak{Higgs}(X/\iota)$  the set of triples  $(\underline{E}, \underline{\theta}, \underline{p})$  where  $(\underline{E}, \underline{p}) \in \mathfrak{Bun}(X/\iota)$  corresponds to a stable bundle  $E \in \mathfrak{Bun}^s(X)$  and where

$$\underline{\theta} : \underline{E} \rightarrow \underline{E} \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$$

is a logarithmic Higgs field with at most apparent singularities over  $\underline{W}$ : the residue at any  $\underline{w}_i \in \underline{W}$  is either zero or nilpotent and the parabolic  $\underline{p}_{\underline{w}_i}$  is a 0-eigendirection of the residue. For fixed  $(\underline{E}, \underline{p})$ , we denote by  $\text{H}^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{p}}$  the space of such apparent logarithmic Higgs fields on  $\underline{E}$ . Note that  $\mathfrak{Higgs}(X/\iota)$  is canonically identified with the cotangent bundle of  $\mathfrak{Bun}(X/\iota)$  (restricted to the preimage of  $\mathfrak{Bun}^s(X)$ ). Indeed, firstly, we identify the tangent space of the moduli space of parabolic bundles at the point  $(\underline{E}, \underline{p})$  with  $\text{H}^1(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{p}))$  where  $\mathfrak{sl}(\underline{E}, \underline{p})$  is the sheaf of tracefree endomorphisms of  $E$  over  $\mathbb{P}_x^1$  that preserve the parabolic structure. Secondly, by Serre duality, we have a perfect pairing

$$\langle \cdot, \cdot \rangle : \begin{cases} \text{H}^1(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{p})) \times \text{H}^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{p}} & \rightarrow \text{H}^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1) \simeq \mathbb{C} \\ (\phi, \underline{\theta}) & \mapsto \sum \text{Res}(\text{trace}(\phi \cdot \underline{\theta})) \end{cases} .$$

Let  $E \in \mathfrak{Bun}^s(X)$ . Since  $E$  is stable, the hyperelliptic involution (up to a sign)  $h : E \rightarrow \iota^*E$  given by Corollary 3.3.3 does not depend on the choice of an irreducible connection on  $E$ . It follows that for any Higgs field  $\theta$  on  $E$ , the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\theta} & E \otimes \Omega_X^1 \\ \downarrow h & & \uparrow \iota^*h \\ \iota^*E & \xrightarrow{\iota^*\theta} & \iota^*E \otimes \Omega_X^1. \end{array}$$

Hyperelliptic descent of the pair  $(E, \theta)$  then produces two triples  $(\underline{E}, \underline{\theta}, \underline{p}) \in \mathfrak{Higgs}(X/\iota)$ . Let us calculate the composition

$$\mathfrak{Higgs}(X/\iota) \xrightarrow{2:1} \mathfrak{Higgs}(X) \xrightarrow{\text{Hitch}} \text{H}^0(X, \Omega_X^1 \otimes \Omega_X^1) \quad (3.32)$$

in the affine part  $\mathbb{C}_{(R,S,T)}^3$  of our first main chart of  $\mathfrak{Bun}(X/\iota)$  (see § 3.6.2). Let  $(\underline{E}, \underline{p}) \in \mathfrak{Bun}(X/\iota)$  correspond to  $(R_0, S_0, T_0) \in \mathbb{C}_{(R,S,T)}^3$ . We have  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ . We may choose a pair of meromorphic sections  $(e_1, e_2)$  of  $\underline{E}$  forming a frame over  $\mathbb{P}^1 \setminus \{\infty\}$  such that  $e_1$  is a section of the subbundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \underline{E}$  and  $e_2$  defines a subbundle  $L \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$ , such that moreover the parabolic structure  $\underline{p}$  is given by

$$\begin{array}{cccccc} x=0 & x=1 & x=r & x=s & x=t & x=\infty \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} R_0 \\ 1 \end{pmatrix} & \begin{pmatrix} S_0 \\ 1 \end{pmatrix} & \begin{pmatrix} T_0 \\ 1 \end{pmatrix} & L \end{array} ,$$

The vector field  $\frac{\partial}{\partial R} \in T_{(R_0, S_0, T_0)} \mathfrak{Bun}(X/\iota)$  for instance is given in  $H^1(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{\mathbf{p}}))$  by the cocycle

$$\phi_{01} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

over  $U_{01} = U_0 \cap U_1$  with  $U_0 := \mathbb{P}^1 \setminus \{r\}$  and  $U_1 := D_\varepsilon(r)$ . Indeed, if we glue the restrictions  $(\underline{E}, \underline{\mathbf{p}})|_{U_0}$  and  $(\underline{E}, \underline{\mathbf{p}})|_{U_1}$  by the map

$$\exp(\zeta \phi_{01}) = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} : ((\underline{E}, \underline{\mathbf{p}})|_{U_1})|_{U_{01}} \rightarrow (\underline{E}, \underline{\mathbf{p}})|_{U_0},$$

we get the new parabolic bundle defined by  $(R_0 + \zeta, S_0, T_0)$ , i.e., the point defined by the time- $\zeta$  map generated by the vector field  $\frac{\partial}{\partial R}$ . A straightforward calculation (see [HL15, Prop. 6.1]) shows that the dual basis with respect to  $\langle \cdot, \cdot \rangle$  of the basis

$$\left( \frac{\partial}{\partial R}, \frac{\partial}{\partial S}, \frac{\partial}{\partial T} \right)$$

of  $T_{(R_0, S_0, T_0)} \mathfrak{Bun}(X/\iota)$  then is  $(\Theta_r, \Theta_s, \Theta_t)|_{(R, S, T) = (R_0, S_0, T_0)}$ , given with respect to the trivial chart over  $\mathbb{P}^1$  of  $\underline{E}$  defined by the frame  $(e_1, e_2)$  by

$$\begin{aligned} \Theta_r &= \begin{pmatrix} 0 & 0 \\ 1 - R & 0 \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} R & -R \\ R & -R \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} -R & R^2 \\ -1 & R \end{pmatrix} \frac{dx}{x-r} \\ \Theta_s &= \begin{pmatrix} 0 & 0 \\ 1 - S & 0 \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} S & -S \\ S & -S \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} -S & S^2 \\ -1 & S \end{pmatrix} \frac{dx}{x-s} \\ \Theta_t &= \begin{pmatrix} 0 & 0 \\ 1 - T & 0 \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} T & -T \\ T & -T \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} -T & T^2 \\ -1 & T \end{pmatrix} \frac{dx}{x-t}. \end{aligned} \tag{3.33}$$

**Corollary 3.6.3.** *The Liouville form on  $\mathbb{C}_{(R, S, T)}^3$  given by  $dR + dS + dT$  defines a holomorphic symplectic 2-form on  $\mathfrak{Higgs}(X/\iota) \subset T^\vee \mathfrak{Bun}(X/\iota)$  given with respect to the chart  $\mathbb{C}_{(R, S, T, c_r, c_s, c_t)}^6$  by*

$$dR \wedge dc_r + dS \wedge dc_s + dT \wedge dc_t.$$

Any Higgs field  $\underline{\Theta}$  on  $\underline{E}$  respecting the parabolic structure  $\underline{\mathbf{p}}$  given by  $(R, S, T)$  is a unique linear combination of the above  $\Theta_i$ 's:

$$\underline{\Theta} = c_r \Theta_r + c_s \Theta_s + c_t \Theta_t. \tag{3.34}$$

In particular, we obtain a universal Higgs bundle over  $\mathbb{C}_{(R, S, T, c_r, c_s, c_t)}^6 \subset T^\vee \mathfrak{Bun}(X/\iota)$ . The image of such a triple  $(\underline{E}, \underline{\Theta}, \underline{\mathbf{p}})$  under the map (3.32) is simply given by  $\det(\Theta)$ , where  $(E, \Theta) := \text{elm}_W^+(\pi^*(\underline{E}, \underline{\Theta}))$  and the elementary transformations are taken with respect to the hyperelliptic lift  $\underline{\mathbf{p}}$  of the parabolic structure  $\underline{\mathbf{p}}$ . On the other hand, in affine charts of  $X$ , elementary transformations are simply meromorphic gauge transformations, acting by conjugacy on  $\Theta$ . Therefore, to get Hitchin Hamiltonians on the chart  $(R, S, T, c_r, c_s, c_t)$ , we just have to compute

$$\det(c_r \Theta_r + c_s \Theta_s + c_t \Theta_t) = (h_2 x^2 + h_1 x + h_0) \frac{(dx)^{\otimes 2}}{x(x-1)(x-r)(x-s)(x-t)}.$$

A straightforward computation yields the explicit Hitchin Hamiltonians for  $\mathfrak{Higgs}(X/\iota)$  given in Table 10.



$$\begin{aligned}
h_0 &= (c_r(R-1) + c_s(S-1) + c_t(T-1))(c_rst(R-r)R + c_srt(S-s)S + c_trs(T-t)T) \\
h_1 &= +c_r(c_r(s+t)(r+1) + c_ss(t+1) + c_tt(s+1))R^2 - c_r^2(t+s)R^3 \\
&\quad + c_s(c_s(r+t)(s+1) + c_rr(t+1) + c_tt(r+1))S^2 - c_s^2(t+r)S^3 \\
&\quad + c_t(c_t(r+s)(t+1) + c_rr(s+1) + c_ss(r+1))T^2 - c_t^2(r+s)T^3 \\
&\quad - c_rc_s(t(R-1+S-1) + r(S-s) + s(R-r))RS \\
&\quad - c_rc_t(s(R-1+T-1) + r(T-t) + t(R-r))RT \\
&\quad - c_sc_t(r(S-1+T-1) + s(T-t) + t(S-s))ST \\
&\quad - (c_tt(r+s) + c_rr(s+t) + c_ss(r+t))(c_rR + c_sS + c_tT) \\
h_2 &= (c_r(R-1)R + c_s(S-1)S + c_t(T-1)T)(c_r(R-r) + c_s(S-s) + c_t(T-t))
\end{aligned}$$

Table 10: Explicit Hitchin Hamiltonians for the chart  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  of  $\mathfrak{Bun}(X/\iota)$

It is easy to check that these functions indeed Poisson-commute: for any  $f, g \in \{h_0, h_1, h_2\}$ , we have

$$\sum_{i=r,s,t} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} = 0$$

in Darboux notation  $(p_r, p_s, p_t, q_r, q_s, q_t) := (R, S, T, c_r, c_s, c_t)$ .

Consider the natural rational map  $\phi^* : \mathbb{T}^\vee \mathbb{P}_{\text{NR}}^3 \dashrightarrow \mathbb{T}^\vee \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  induced by the explicit map  $\phi : \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \dashrightarrow \mathbb{P}_{\text{NR}}^3$  of Proposition [3.6.1](#). Then, for a general section  $\mu_0 d\left(\frac{v_0}{v_3}\right) + \mu_1 d\left(\frac{v_1}{v_3}\right) + \mu_2 d\left(\frac{v_2}{v_3}\right)$ , the Hitchin Hamiltonians with respect to these coordinates (see [§ 3.5.2](#)) are straightforward to calculate. The page-filling formula is given explicitly in [\[HL15, Table 12\]](#). From the explicit coordinate change  $(v_0 : v_1 : v_2 : v_3) \leftrightarrow (t_0 : t_1 : t_2 : t_3)$  given in [\(3.21\)](#), we know how to identify general sections

$$\eta_0 d\left(\frac{t_0}{t_3}\right) + \eta_1 d\left(\frac{t_1}{t_3}\right) + \eta_2 d\left(\frac{t_2}{t_3}\right) = \mu_0 d\left(\frac{v_0}{v_3}\right) + \mu_1 d\left(\frac{v_1}{v_3}\right) + \mu_2 d\left(\frac{v_2}{v_3}\right),$$

yielding

$$\text{Hitch} : \left\{ \begin{array}{ccc} \mathfrak{Higgs}(X) & \rightarrow & \mathbb{H}^0(X, \Omega_X^1 \otimes \Omega_X^1) \\ ((t_0 : t_1 : t_2 : t_3), \eta_0, \eta_1, \eta_2) & \mapsto & (h_2 x^2 + h_1 x + h_0) \frac{(dx)^{\otimes 2}}{x(x-1)(x-r)(x-s)(x-t)} \end{array} \right\},$$

where Hitchin Hamiltonians are given in Table [11](#).

$$\begin{aligned}
h_0 &= \frac{1}{4t_3^4} \cdot \begin{cases} rst \cdot [\eta_0(t_0^2 - t_3^2) + \eta_1(t_0t_1 + t_2t_3) + \eta_2(t_0t_2 + t_1t_3)]^2 \\ -st \cdot [\eta_0(t_0t_1 - t_2t_3) + \eta_1(t_1^2 + t_3^2) + \eta_2(t_0t_3 + t_1t_2)]^2 \\ +4rs \cdot (\eta_0t_0 + \eta_1t_1)^2 t_3^2 \\ -rt \cdot [\eta_0(t_0^2 + t_3^2) + \eta_1(t_0t_1 + t_2t_3) + \eta_2(t_0t_2 - t_1t_3)]^2 \end{cases} \\
h_1 &= \frac{1}{4t_3^4} \cdot \begin{cases} t \cdot (t_0^2 + t_1^2 + t_2^2 + t_3^2) [(\eta_0^2 + \eta_1^2 + \eta_2^2)t_3^2 + (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)^2] \\ +st \cdot (t_0^2 - t_1^2 + t_2^2 - t_3^2) [(\eta_0^2 - \eta_1^2 + \eta_2^2)t_3^2 - (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)^2] \\ +4r \cdot (t_0t_2 - t_1t_3) t_3 [\eta_0\eta_2t_3 + (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)\eta_1] \\ +4sr \cdot (t_0t_2 + t_1t_3) t_3 [\eta_0\eta_2t_3 - (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)\eta_1] \\ +4s \cdot (t_0t_3 + t_1t_2) t_3 [\eta_1\eta_2t_3 - (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)\eta_0] \\ +4rt \cdot (t_0t_1 + t_2t_3) t_3 [\eta_0\eta_1t_3 - (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)\eta_2] \end{cases} \\
h_2 &= \frac{1}{4t_3^4} \cdot \begin{cases} s \cdot [\eta_0(t_0t_2 + t_1t_3) + \eta_1(t_0t_3 + t_1t_2) + \eta_2(t_2^2 - t_3^2)]^2 \\ -1 \cdot [\eta_0(t_0t_2 - t_1t_3) + \eta_1(t_0t_3 + t_1t_2) + \eta_2(t_2^2 + t_3^2)]^2 \\ -t \cdot [\eta_0(t_0t_1 + t_3t_3) - \eta_2(t_0t_3 - t_1t_2) + \eta_1(t_2^2 + t_3^2)]^2 \\ +4r \cdot (\eta_1t_1 + \eta_2t_2)^2 t_3^2 \end{cases}
\end{aligned}$$

Table 11: Explicit Hitchin Hamiltonians for the coordinates  $(t_0 : t_1 : t_2 : t_3)$  of  $\mathcal{M}_{\text{NR}}$ .

The reason behind the fact that the explicit Hitchin Hamiltonians have a much simpler expression with respect to the coordinates  $(t_0 : t_1 : t_2 : t_3)$  than with respect to the coordinates  $(v_0 : v_1 : v_2 : v_3)$  of  $\mathcal{M}_{\text{NR}}$  is that equation of the Kummer surface with respect to the former is very symmetric. As pointed out in [VGP96], the classical line geometry for Kummer surfaces in  $\mathbb{P}^3$  is related to certain symmetries of the Hitchin Hamiltonians. Note that in [VGP96], B. van Geemen and E. Previato conjectured a projective version of explicit Hitchin Hamiltonians, which has been confirmed in [GTNB98]. Their Hamiltonians  $H_1, \dots, H_6$  are evaluations, up to functions in the base, of the explicit Hitchin map at the Weierstrass points. This can now be made precise: from the coefficients  $A, B, C, D$  of the Kummer surface in (3.22), one checks that our normalized  $\underline{W}$  corresponds to the  $\lambda_i$ 's in [VGP96] when identifying  $(\lambda_1, \lambda_2, \dots, \lambda_6) = (0, t, 1, s, r, \infty)$ . Now if we denote

$$h(x) := h_2x^2 + h_1x + h_0,$$

then

$$\begin{aligned}
H_1 &= \frac{4h(0)}{rst} & H_4 &= \frac{4h(s)}{s(s-1)(s-r)(s-t)} \\
H_2 &= -\frac{4h(t)}{t(t-1)(t-r)(t-s)} & H_5 &= \frac{4h(r)}{r(r-1)(r-s)(r-t)} \\
H_3 &= \frac{4h(1)}{(r-1)(s-1)(t-1)} & H_6 &= 0.
\end{aligned}$$

### 3.7 The isomonodromy foliation

Let  $\mathfrak{Con}(X)$  be as in Section 3.3, and denote by  $\mathfrak{Con}^*(X)$  the subset defined as the complement of the trivial connection  $d_X$  on the trivial vector bundle over  $X$  and twists thereof. Moreover, denote by  $\mathfrak{Con}^*(X/\iota) \subset \mathfrak{Con}(X/\iota)$  the set of those parabolic connections  $(\underline{E}, \underline{\nabla})$  that are mapped to an element

of  $\mathfrak{Con}^*(X)$  under the hyperelliptic lift  $\text{elm}_W^+ \circ \pi^*$ . We are going to cover  $\mathfrak{Con}^*(X/\iota)$  by an atlas of affine charts  $\mathbb{C}^6$  in § 3.7.1. Then, by allowing variations of the position of the points  $\underline{W}$  over which the parabolic structure  $\underline{p}$  given by  $\underline{\nabla}$  is defined, we obtain a space  $\underline{\mathfrak{M}}$  of triples  $(\underline{E}, \underline{\nabla}, \underline{W})$  which by hyperelliptic lift describes a space  $\mathfrak{M}$  of triples  $(X, E, \nabla)$ , where  $X$  is a curve of genus 2,  $E$  is a rank 2 vector bundle with trivial determinant bundle and  $\nabla$  a holomorphic trace free connection on  $E$  with either abelian (but not trivial or a twist thereof) or irreducible monodromy. The isomonodromy foliation  $\mathcal{F}_{\text{iso}}$  on  $\mathfrak{M}$  is induced from the isomonodromy foliation  $\underline{\mathcal{F}}_{\text{iso}}$  on  $\underline{\mathfrak{M}}$ . The latter is well-known and can be expressed explicitly as a Hamiltonian system derived from the Garnier equations (see § 3.7.2). On the other hand, we have explicit expressions of the loci in  $\underline{\mathfrak{M}}$  corresponding to special types of flat bundles in  $\mathfrak{Bun}(X)$ , for example the Gunning bundles. We prove by explicit computation that  $\mathcal{F}_{\text{iso}}$  is transversal to this locus of Gunning bundles in § 3.7.3, which provides an alternative proof of a theorem of Hejhal (see § 3.7.4).

### 3.7.1 An atlas of $\mathfrak{Con}^*(X/\iota)$

Let  $\underline{\tilde{E}} := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ , given by the affine chart  $(\mathbb{P}^1 \setminus \{\infty\}) \times \mathbb{C}^2$  with coordinates  $(x, Y)$ , together with the affine chart  $(\mathbb{P}^1 \setminus \{0\}) \times \mathbb{C}^2$  with coordinates  $(\hat{x}, \hat{Y}) = (\frac{1}{x}, (\frac{1}{0} \ 0) Y)$ . Unless otherwise specified, the formulae in the following will be with respect to the coordinates  $(x, Y)$ . We define a parabolic structure  $\underline{\tilde{p}}$  on  $\underline{\tilde{E}}$  as follows:

$$\begin{array}{cccccc} x=0 & x=1 & x=r & x=s & x=t & x=\infty \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} z_r \\ 1 \end{pmatrix} & \begin{pmatrix} z_s \\ 1 \end{pmatrix} & \begin{pmatrix} z_t \\ 1 \end{pmatrix} & \mathcal{O}_{\mathbb{P}^1}(-1). \end{array} \quad (3.35)$$

Note that the only automorphisms of  $\underline{\tilde{E}}$  fixing  $\underline{\tilde{p}}$  are the scalar ones. For any choice of  $\kappa_i \in \mathbb{C}^*$  with  $i \in \underline{W} = \{0, 1, r, s, t, \infty\}$  we define  $\rho \in \mathbb{C}$  by

$$\kappa_0 + \kappa_1 + \kappa_r + \kappa_s + \kappa_t + \kappa_\infty + 2\rho = 1.$$

We are going to consider logarithmic connections  $\underline{\tilde{\nabla}} : \underline{\tilde{E}} \rightarrow \underline{\tilde{E}} \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$  with eigenvalues

$$\begin{pmatrix} x=0 & x=1 & x=r & x=s & x=t & x=\infty \\ 0 & 0 & 0 & 0 & 0 & \rho \\ \kappa_0 & \kappa_1 & \kappa_r & \kappa_s & \kappa_t & \kappa_\infty + \rho \end{pmatrix}. \quad (3.36)$$

More precisely, we consider such connections such that moreover the parabolic structure  $\underline{\tilde{p}}$  on  $\underline{\tilde{E}}$  coincides with the parabolic structure defined by the  $\kappa_i$ -eigendirections of the residues of  $\underline{\tilde{\nabla}}$  over  $\underline{W} \setminus \{\infty\}$  and the  $(\rho + \kappa_\infty)$ -eigendirection of the residue of  $\underline{\tilde{\nabla}}$  over  $x = \infty$ . Such a connection can be written as

$$\underline{\tilde{\nabla}} = \nabla_0 + c_r \Theta_r + c_s \Theta_s + c_t \Theta_t \quad (3.37)$$

with

$$\nabla_0 := d + \begin{pmatrix} 0 & 0 \\ \rho & \kappa_0 \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} -\rho & \rho + \kappa_1 \\ -\rho & \rho + \kappa_1 \end{pmatrix} \frac{dx}{x-1} + \sum_{i \in \{r, s, t\}} \begin{pmatrix} 0 & z_i \kappa_i \\ 0 & \kappa_i \end{pmatrix} \frac{dx}{x-i}$$

and

$$\Theta_i := \begin{pmatrix} 0 & 0 \\ 1 - z_i & 0 \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} z_i & -z_i \\ z_i & -z_i \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} -z_i & z_i^2 \\ -1 & z_i \end{pmatrix} \frac{dx}{x-i},$$

where  $(z_r, z_s, z_t, c_r, c_s, c_t) \in \mathbb{C}^6$ . Note that the residue of  $\tilde{\nabla}$  at  $x = \infty$  is given with respect to the coordinate  $\tilde{Y}$  by

$$\begin{pmatrix} \rho & 0 \\ \rho + c_r(r - z_r) + c_s(s - z_s) + c_t(t - z_t) & \rho + \kappa_\infty \end{pmatrix}.$$

Moreover, for residues of  $\tilde{\nabla}$  over finite points (and coordinate  $Y$ ), the eigendirections with respect to the 0-eigenvalues are generated by

$$\begin{pmatrix} x=0 \\ -\frac{\kappa_0}{\rho + \sum_{i \in \{r,s,t\}} c_i(z_i - 1)} \\ 1 \end{pmatrix} \begin{pmatrix} x=1 \\ 1 + \frac{\kappa_1}{\rho + \sum_{i \in \{r,s,t\}} c_i z_i} \\ 1 \end{pmatrix} \begin{pmatrix} x=r \\ z_r - \frac{\kappa_r}{c_r} \\ 1 \end{pmatrix} \begin{pmatrix} x=s \\ z_s - \frac{\kappa_s}{c_s} \\ 1 \end{pmatrix} \begin{pmatrix} x=t \\ z_t - \frac{\kappa_t}{c_t} \\ 1 \end{pmatrix}.$$

It is immediate to check that for any fixed choice of the  $\kappa_i$ 's in  $\mathbb{C}^*$ , two connections  $(\tilde{\underline{E}}, \tilde{\nabla})$  constructed as above for distinct values of

$$(\mathbf{z}, \mathbf{c}) := (z_r, z_s, z_t, c_r, c_s, c_t) \in \mathbb{C}^6$$

are non isomorphic. We deduce a description of certain affine charts of  $\mathfrak{Con}(X/\iota)$  as follows.

- **The canonical chart  $U_0$ :** Set  $\kappa_i = \frac{1}{2}$  for all  $i \in \underline{W}$ . For any element of  $\mathbb{C}_{(\mathbf{z}, \mathbf{c})}^6$ , consider the corresponding connection  $(\tilde{\underline{E}}, \tilde{\nabla})$  in (3.37). Let  $(\mathcal{O}_{\mathbb{P}^1}(-1), \zeta)$  be the (unique) logarithmic rank 1 connection over  $\mathbb{P}^1$  having a single pole at infinity. Then  $(\underline{E}, \underline{\nabla}) := (\tilde{\underline{E}}, \tilde{\nabla}) \otimes (\mathcal{O}_{\mathbb{P}^1}(-1), \zeta)$  is a logarithmic connection whose eigenvalues are given by (3.36) except at  $x = \infty$ , where the eigenvalues have been shifted to  $(\rho + 1, \rho + \kappa_\infty + 1)$ . In particular, since here we have  $\kappa_\infty = \frac{1}{2}$  and thus  $\rho = -1$ , each of the residues of  $(\underline{E}, \underline{\nabla})$  has eigenvalues 0 and  $\frac{1}{2}$ . Hence  $(\underline{E}, \underline{\nabla})$  is an element of  $\mathfrak{Con}(X/\iota)$ , whose parabolic structure  $\underline{\mathbf{p}}$  (given by the  $\frac{1}{2}$ -eigendirections of  $\underline{\nabla}$ ) is the following

$$\begin{matrix} x=0 & x=1 & x=r & x=s & x=t & x=\infty \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} z_r \\ 1 \end{pmatrix} & \begin{pmatrix} z_s \\ 1 \end{pmatrix} & \begin{pmatrix} z_t \\ 1 \end{pmatrix} & \mathcal{O}_{\mathbb{P}^1}(-2). \end{matrix} \quad (3.38)$$

If we denote by  $U_0 \subset \mathfrak{Con}(X/\iota)$  the set of parabolic connections thusly constructed, we have a canonical isomorphism  $U_0 \simeq \mathbb{C}_{(\mathbf{z}, \mathbf{c})}^6$  and a universal connection over  $U_0 \times \mathbb{P}^1$  given by (3.37), tensorised by  $(\mathcal{O}_{\mathbb{P}^1}(-1), \zeta)$ . Note that the Higgs fields  $\Theta_i$  are identical to the ones in (3.33) for  $(R, S, T) = (z_r, z_s, z_t)$ . In particular,  $U_0$  is isomorphic to the restriction of  $\mathrm{T}^\vee \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  to the affine part of  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \subset \mathfrak{Bun}(X/\iota)$ .

- **The 7 switched charts  $U_J$ :** Let  $\emptyset \neq J \subset \{r, s, t\}$ . Set  $\kappa_j = -\frac{1}{2}$  for all  $j \in J$  and  $\kappa_i = \frac{1}{2}$  for all  $i \in \underline{W} \setminus J$ . For any element of  $\mathbb{C}_{(\mathbf{z}, \mathbf{c})}^6$  consider the corresponding connection  $(\tilde{\underline{E}}, \tilde{\nabla})$  in (3.37). Let  $(\mathcal{O}_{\mathbb{P}^1}, \eta)$  be the (unique) logarithmic rank 1 connection over  $\mathbb{P}^1$  of the form  $\eta : \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(J + [\infty])$  having eigenvalues  $+\frac{1}{2}$  over each element in  $J$  and eigenvalue  $1 - \frac{\#J}{2}$  over  $\infty$ . Then  $(\underline{E}, \underline{\nabla}) := (\tilde{\underline{E}}, \tilde{\nabla}) \otimes (\mathcal{O}_{\mathbb{P}^1}(-1), \eta)$  is an element of  $\mathfrak{Con}(X/\iota)$ . Denote by  $U_J$  the set of logarithmic connections thusly constructed. Again we have a canonical isomorphism  $U_J \simeq \mathbb{C}_{(\mathbf{z}, \mathbf{c})}^6$  and a universal connection over  $U_J \times \mathbb{P}^1$  given by (3.37), tensorised by  $(\mathcal{O}_{\mathbb{P}^1}, \eta)$ . Here, the parabolic structure  $\underline{\mathbf{p}}$  defined by the  $\frac{1}{2}$ -eigendirections of  $\underline{\nabla}$  is the following

$$\begin{matrix} x=0 & x=1 & x=i & x=j & x=\infty \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} z_i \\ 1 \end{pmatrix} & \begin{pmatrix} 2c_j z_j + 1 \\ c_j \end{pmatrix} & \mathcal{O}_{\mathbb{P}^1}(-2) \end{matrix} \quad \text{for } i \in \{r, s, t\} \setminus J, \quad j \in J,$$

wheras the 0-eigendirections over  $J$  are given by  ${}^\top(z_j, 1)$ .

- **The twisted chart  $U_{\text{tw}}$ :** Set  $\kappa_0 = \kappa_1 = -\frac{1}{2}$  and  $\kappa_i = \frac{1}{2}$  for all  $i \in \underline{W} \setminus \{0, 1\}$ . Denote again by  $(\tilde{\underline{E}}, \tilde{\nabla})$  the connection in (3.37) associated to a point in  $\mathbb{C}_{(z,c)}^6$  for this choice of the  $\kappa_i$ 's. Let  $(\underline{E}, \underline{\nabla})$  be the connection obtained from  $(\tilde{\underline{E}}, \tilde{\nabla})$  by applying the two negative elementary transformations in the 0-eigendirections of  $\tilde{\nabla}$  over  $x = 0$  and  $x = 1$ . Again,  $(\underline{E}, \underline{\nabla})$  is an element of  $\mathfrak{Con}(X/\iota)$  and we denote by  $U_{\text{tw}} \simeq \mathbb{C}^6$  the set of logarithmic connections thusly constructed, which comes with a universal connection over  $U_{\text{tw}} \times \mathbb{P}^1$ . Note that it may happen, for particular choices of  $(z, c)$ , that the 0-eigendirections of  $\tilde{\nabla}$  over  $x = 0$  and  $x = 1$  are both given by  $\text{Tr}(1, 0)$ . In that case, we have  $\underline{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ . Otherwise, we obtain  $\underline{E} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ .
- **The  $9 \cdot (6! - 1)$  permuted charts  $U_{0,\sigma}, U_{J,\sigma}, U_{\text{tw},\sigma}$ :** For any  $\sigma \in \mathfrak{S}(\{0, 1, r, s, t, \infty\}) \setminus \{\text{id}\}$ , instead of  $\nabla_0$  and the  $\Theta'_i$ 's in (3.37) respecting the parabolic structure  $\tilde{\underline{p}}$  in (3.36) on  $\tilde{\underline{E}}$ , we could have started with an analogous construction of connection and Higgs fields respecting the normalized parabolic structure given by

$$\begin{array}{cccccc} x = \sigma(0) & x = \sigma(1) & x = \sigma(r) & x = \sigma(s) & x = \sigma(t) & x = \infty \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} z_r \\ 1 \end{pmatrix} & \begin{pmatrix} z_s \\ 1 \end{pmatrix} & \begin{pmatrix} z_t \\ 1 \end{pmatrix} & \mathcal{O}_{\mathbb{P}^1}(-1). \end{array}$$

For that new choice, we obtain similar construction of canonical, switched and twisted chart.

- **The  $9 \cdot 6!$  Galois involution charts:** Choose any of the above charts  $U$  together with its universal family of connections. First apply positive elementary transformations in all  $\frac{1}{2}$ -eigendirections of the universal family of connections. We obtain logarithmic connections with eigenvalues 0 and  $-\frac{1}{2}$  over each Weierstrass point. Then tensorize this connection by the unique logarithmic (rank 1) connection on  $\mathcal{O}_{\mathbb{P}^1}(-3)$  having eigenvalues  $\frac{1}{2}$  over each Weierstrass point. We obtain a new chart  $U'$ , such that the images of  $U$  and  $U'$  under the hyperelliptic lift  $\mathfrak{Con}(X/\iota) \rightarrow \mathfrak{Con}(X)$  coincide.

By construction, the above are indeed all affine charts of  $\mathfrak{Con}(X/\iota)$ . Moreover, the transition maps between charts are all birational. Using Proposition 3.3.5, one easily checks that the image under  $\mathfrak{Con}(X/\iota) \rightarrow \mathfrak{Bun}(X/\iota)$  of the union of all of the above charts is surjective. Then, analysing possible connections over different types of bundles, one deduces the following (see [HL15, Prop. 7.1]).

**Proposition 3.7.1.** *The union of all of the above charts is given by the subset  $\mathfrak{Con}(X/\iota)^* \subset \mathfrak{Con}(X/\iota)$  defined as the complementary of the preimage of those elements of  $\mathfrak{Con}(X)$  that are not the trivial connection on the trivial bundle nor a twist thereof.*

### 3.7.2 The isomonodromy Hamiltonian system

Recall that our base curve  $X$  and its quotient  $X/\iota$  are defined, via  $X = X_{r,s,t} : y^2 = x(x-1)(x-r)(x-s)(x-t)$ , by a parameter  $(r, s, t)$  in

$$\mathcal{T} := \{(r, s, t) \in \mathbb{C}^3 \mid r, s, t \neq 0, 1, r \neq s, r \neq t, s \neq t\}.$$

The construction carried out in § 3.7.1 of charts of  $\mathfrak{Con}^*(X_{(r,s,t)}/\iota)$  naturally generalises to families with varying  $(r, s, t)$ , yielding families  $\mathfrak{M} \rightarrow \mathcal{T}$  and  $\underline{\mathfrak{M}} \rightarrow \mathcal{T}$  such that  $\mathfrak{M}|_{(r,s,t)} = \mathfrak{Con}^*(X_{(r,s,t)})$  and  $\underline{\mathfrak{M}}|_{(r,s,t)} = \mathfrak{Con}^*(X_{(r,s,t)}/\iota)$ . We shall denote by  $\Phi : \underline{\mathfrak{M}} \rightarrow \mathfrak{M}$  the map which for each fixed  $(r, s, t)$  corresponds to the hyperelliptic lift.

Note that a family of tracefree rank two connections defined over the total space of a family of genus 2 curves  $X_{r,s,t}$  parametrized by an open set of  $\mathcal{T}$  is isomonodromic if and only if it is isomonodromic after hyperelliptic descent. Since families of logarithmic connections are isomonodromic if

and only if the connection matrices of the family are integrable, and locally in  $\underline{\mathfrak{M}}$  we have universal families of connections, the integrability condition defines a (singular holomorphic) foliation  $\underline{\mathcal{F}}_{\text{iso}}$  on  $\underline{\mathfrak{M}}$ , inducing a foliation  $\mathcal{F}_{\text{iso}}$  on  $\mathfrak{M}$ . We refer to  $\underline{\mathcal{F}}_{\text{iso}}$  and  $\mathcal{F}_{\text{iso}}$  as the isomonodromy foliation. Note moreover that isomonodromic families of logarithmic connections over  $\mathbb{P}^1$  remain isomonodromic if we perform elementary transformations in the eigendirections of the residues and or tensorise with a flat rank 1 connection. We may thus describe the isomonodromic foliation in (lifts over  $\mathcal{T}$  of) the canonical chart, the switched charts and the twisted chart simultaneously by describing the integrability condition for families of connections  $(\underline{E}, \underline{\nabla})$  with  $\underline{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  and  $\underline{\nabla}$  given by a point in  $\mathbb{C}_{(z,c)}^6$  as at the beginning of § 3.7.1. This integrability condition is well-known (see [Oka86]) to be given by a Hamiltonian system, usually expressed with respect to the coordinates

$$(\mathbf{q}, \mathbf{p}) = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3),$$

which will also be Darboux coordinates with respect to the symplectic form  $\omega = \sum_{i \in \{r,s,t\}} dz_i \wedge dc_i$ , and are defined as follows. The vector  $e_1 = \top(1, 0)$  becomes an eigenvector of the connection matrix of  $\underline{\nabla}$  for 3 points  $x = \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$  (counted with multiplicity), namely at the zeroes of the  $(2, 1)$ -coefficient of the connection matrix:

$$-\rho + \sum_{i \in \{r,s,t\}} c_i \frac{(z_i - i)x - i(z_i - 1)}{x - i} = \left( -\rho + \sum_{i=1}^3 c_i(z_i - i) \right) \frac{\prod_{k=1}^3 (x - \mathbf{q}_k)}{\prod_{i \in \{r,s,t\}} (x - i)} \quad (3.39)$$

At each of the three solutions  $x = \mathbf{q}_k$  of (3.39), the eigenvector  $e_1 = \top(1, 0)$  is associated to the eigenvalue

$$\mathbf{p}_k := -\frac{\rho}{\mathbf{q}_k - 1} + \sum_{i \in \{r,s,t\}} c_i z_i \left( \frac{1}{\mathbf{q}_k - 1} - \frac{1}{\mathbf{q}_k - i} \right). \quad (3.40)$$

The equations (3.39) and (3.40) allow us to express our initial variables  $(z, c)$  as rational functions of the new variables  $(\mathbf{p}, \mathbf{q})$  as follows. Set  $\Delta = (\mathbf{q}_1 - \mathbf{q}_2)(\mathbf{q}_2 - \mathbf{q}_3)(\mathbf{q}_3 - \mathbf{q}_1)$  and

$$\Lambda = \rho + \sum_{\{k,l,m\}=\{1,2,3\}} \frac{\mathbf{p}_k(\mathbf{q}_k - r)(\mathbf{q}_k - s)(\mathbf{q}_k - t)}{(\mathbf{q}_k - \mathbf{q}_l)(\mathbf{q}_k - \mathbf{q}_m)}.$$

For  $i \in \{r, s, t\}$ , denote

$$\Lambda_i := \Lambda|_{i=1}$$

the rational function obtained by setting  $i = 1$  in the expression of  $\Lambda$ . Then we have, for  $\{i, j, k\} = \{r, s, t\}$

$$c_i = -\frac{(\mathbf{q}_1 - i)(\mathbf{q}_2 - i)(\mathbf{q}_3 - i)}{i(i-1)(i-j)(i-k)} \Lambda \quad \text{and} \quad z_i = i \frac{\Lambda_i}{\Lambda}. \quad (3.41)$$

The rational map

$$\Pi : \mathbb{C}_{(\mathbf{q}, \mathbf{p})}^6 \dashrightarrow \mathbb{C}_{(z, c)}^6 \quad (3.42)$$

has degree 6: the (birational) Galois group of this map is the permutation group on indices  $k = 1, 2, 3$  for pairs  $(\mathbf{q}_k, \mathbf{p}_k)$ . For  $i \in \{r, s, t\}$  define  $H_i$  by

$$i(i-1) \prod_{j \in \{r,s,t\} \setminus \{i\}} (j-i) \cdot H_i := \sum_{l=1}^3 \frac{\prod_{k \neq l} (\mathbf{q}_k - i)}{\prod_{k \neq l} (\mathbf{q}_k - \mathbf{q}_l)} F(\mathbf{q}_l) \left( \mathbf{p}_l^2 - G(\mathbf{q}_l) \mathbf{p}_l + \frac{\mathbf{p}_l}{\mathbf{q}_l - i} \right) + \rho(\rho + \kappa_\infty) \prod_{l=1}^3 (\mathbf{q}_l - i),$$

where  $F(x) = x(x-1)(x-r)(x-s)(x-t)$  and  $G(x) = \frac{\kappa_0}{x} + \frac{\kappa_1}{x-1} + \frac{\kappa_r}{x-r} + \frac{\kappa_s}{x-s} + \frac{\kappa_t}{x-t}$ . Then, assuming  $\kappa_i \notin \mathbb{Z}$  for any  $i \in \{0, 1, r, s, t, \infty\}$ , a local analytic map

$$\chi : (r, s, t) \mapsto (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$$

induces an isomonodromic deformation of the connection (3.37) if, and only if,

$$\frac{\partial \mathfrak{q}_k}{\partial i} = \frac{\partial H_i}{\partial \mathfrak{p}_k} \quad \text{and} \quad \frac{\partial \mathfrak{p}_k}{\partial i} = -\frac{\partial H_i}{\partial \mathfrak{q}_k} \quad \forall i \in \{r, s, t\}, \quad k \in \{1, 2, 3\}. \quad (3.43)$$

In other words, the isomonodromic foliation  $\underline{\mathcal{F}}_{\text{iso}}$  is given with respect to the various charts of  $\underline{\mathcal{M}}$  corresponding to families of  $\mathbb{C}_{(z, \mathbf{c})}^6$  over  $\mathcal{T}$  as the kernel of the 2-form

$$\Omega = \sum_{k=1}^3 d\mathfrak{q}_k \wedge d\mathfrak{p}_k + \sum_{i \in \{r, s, t\}} dH_i \wedge di.$$

In particular,  $\underline{\mathcal{F}}_{\text{iso}}$  is 3-dimensional, transversal to fibers of the family  $\underline{\mathcal{M}} \rightarrow \mathcal{T}$  and  $\chi$  locally parametrizes a leaf of this foliation. The tangent space to the foliation is defined by the 3 vector fields  $V_r, V_s, V_t$  given by

$$V_i := \frac{\partial}{\partial i} + \sum_{k=1}^3 \left( \frac{\partial H_i}{\partial \mathfrak{p}_k} \right) \frac{\partial}{\partial \mathfrak{q}_k} - \sum_{k=1}^3 \left( \frac{\partial H_i}{\partial \mathfrak{q}_k} \right) \frac{\partial}{\partial \mathfrak{p}_k}. \quad (3.44)$$

Note that the polar locus of these vector fields is given by  $(\mathfrak{q}_1 - \mathfrak{q}_2)(\mathfrak{q}_2 - \mathfrak{q}_3)(\mathfrak{q}_1 - \mathfrak{q}_3) = 0$ , namely the critical locus of the map (3.42).

### 3.7.3 Transversality to the locus of Gunning bundles

**Theorem 3.7.2.** *For any even theta-characteristic  $\vartheta$ , the locus  $\{(X, E_\vartheta, \nabla) \in \underline{\mathcal{M}}\}$  of connections on the Gunning bundle  $E_\vartheta$ , is transversal to the isomonodromy foliation  $\mathcal{F}_{\text{iso}}$ .*

*Proof.* Up to permuting the role of the Weierstrass points, we can assume

$$\vartheta = \mathcal{O}_X([w_r] + [w_s] - [w_t]) = \mathcal{O}_X([w_0] + [w_1] - [w_\infty]).$$

Recall from §3.4.6 that the corresponding Gunning bundle  $E_\vartheta$  has two preimages  $(\underline{E}, \underline{\mathfrak{p}}) \in \underline{\mathfrak{Bun}}(X/\iota)$ . They both satisfy  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  and the two parabolic structures are characterized by the fact that the destabilizing bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E}$  contains precisely all parabolics associated to one of the two subsets  $\{r, s, t\} \cup \{0, 1, \infty\} = \underline{W}$  (see Figure 15).

Up to a Möbius transformation in  $X/\iota$  (and thus a rational change of variables  $(r, s, t)$ ), it is sufficient to consider the configuration on the left hand side of Figure 15. Any connection in  $\underline{\mathfrak{Con}}^*(X/\iota)$  yielding this configuration under the forgetful map  $\underline{\mathfrak{Con}}^*(X/\iota) \rightarrow \underline{\mathfrak{Bun}}^*(X/\iota)$  is visible in the Switched chart  $U_J \simeq \mathbb{C}_{(z, \mathbf{c})}^6$  for  $J = \{r, s, t\}$ , and is therefore obtained (by tensor product with a certain rank 1 connection) for a particular value of  $(z, \mathbf{c})$  from  $(\underline{E}, \underline{\nabla})$  in (3.37) with the following exponents:

$$\kappa_0 = \kappa_1 = \kappa_\infty = \frac{1}{2} \quad \text{and} \quad \kappa_r = \kappa_s = \kappa_t = -\frac{1}{2} \quad (\Rightarrow \rho = \frac{1}{2})$$

(see §3.7.1). More precisely, such a point  $(z, \mathbf{c}) \in U_J$  yields a configuration corresponding to the Gunning bundle  $E_\vartheta$  as in Figure 15 if and only if the 0-eigenspaces of the residues of  $\underline{\nabla}$  over  $x = r, s, t$

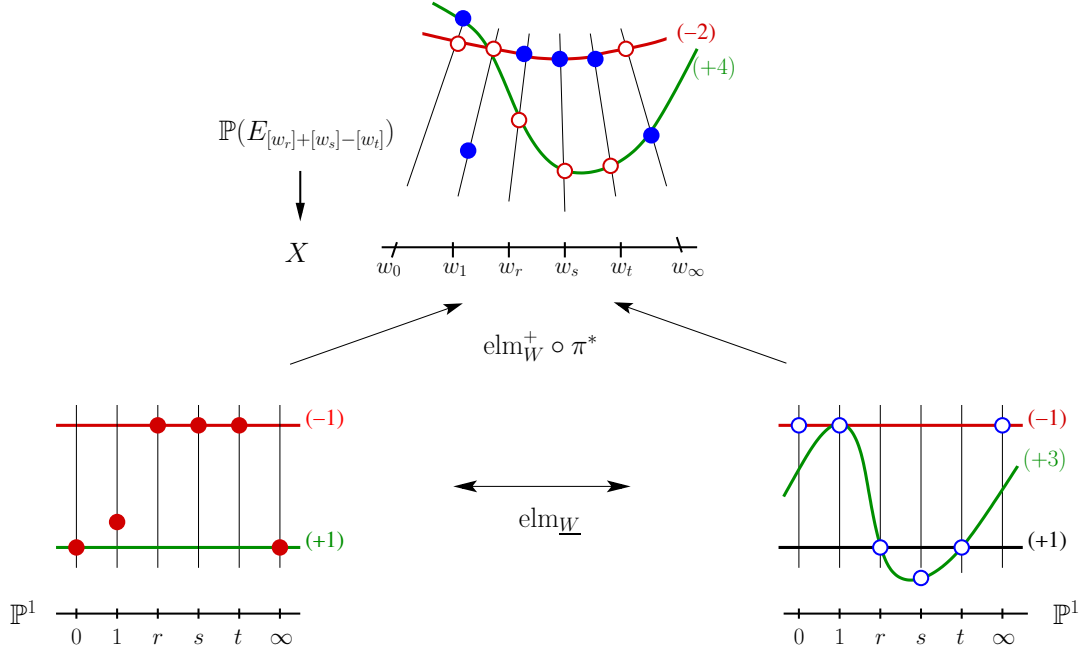


Figure 15: The two parabolic bundles corresponding to  $E_\vartheta$  with  $\vartheta = \mathcal{O}_X([w_r] + [w_s] - [w_t])$ .

coincide with the destabilizing subbundle  $\mathcal{O}_{\mathbb{P}^1} \subset \tilde{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  in the fibers over these points. From the explicit formulae for these 0-eigenspaces we deduce that the locus of the Gunning bundle  $E_\vartheta$  in  $U_J$  is given by  $\{c_r = c_s = c_t\} = 0$  and is parametrized by  $z \in \mathbb{C}^3$ . Allowing now variations the parameter  $(r, s, t)$  defining  $X$ , we are let to consider the locus in  $\mathfrak{M}$  of even Gunning bundles  $E_\vartheta$ , given by

$$\Xi := \{((r, s, t), (z, \mathbf{c})) \in \mathcal{T} \times U_J \mid c_i = 0\}.$$

Consider now the rational map

$$\begin{array}{ccc} \mathcal{T} \times \mathbb{C}_{(\mathbf{q}, \mathbf{p})}^6 & \xrightarrow{\Pi} & \mathcal{T} \times U_J \subset \mathfrak{M} \\ ((r, s, t), \mathbf{q}, \mathbf{p}) & \xrightarrow{6:1} & ((r, s, t), z, \mathbf{c}). \end{array} \quad (3.45)$$

defined in (3.42). For practical reasons, in the following we denote  $(\mathbf{q}, \mathbf{p}) = (q_r, q_s, q_t, p_r, p_s, p_t)$  instead of the indices 1, 2, 3 in (3.42). Namely, formula (3.41) shows that the Gunning locus  $\Xi$  coincides with the image under  $\Pi$  of

$$\Xi^{\text{Darb}} = \{((r, s, t), (\mathbf{q}, \mathbf{p})) \in \mathcal{T} \times \mathbb{C}_{(\mathbf{q}, \mathbf{p})}^6 \mid q_r = r, q_s = s, q_t = t\}.$$

Note that both  $\Xi$  and  $\Xi^{\text{Darb}}$  are Zariski-open subsets of six-dimensional linear subspaces of  $\mathbb{C}^9$ , parametrized by  $(r, s, t, \mathbf{p})$  and  $(r, s, t, z)$  respectively. The map  $\Pi$  induces on these linear spaces an affine transformation given by

$$z_r = r(2(r-1)p_r + 1), \quad z_s = s(2(s-1)p_s + 1), \quad z_t = t(2(t-1)p_t + 1).$$

Note that according to (3.41), the polar locus of the map  $\Pi$  is  $\{(q_r - q_s)(q_r - q_t)(q_s - q_t) = 0\}$ , which is disjoint from  $\Xi^{\text{Darb}} \subset \mathcal{T} \times \mathbb{C}_{(\mathbf{q}, \mathbf{p})}^6$ . On the other hand, locally in  $\mathcal{T}$  we can define a right inverse of  $\Pi$  by using (3.39) to define  $\mathbf{q}$  with the additional requirement that  $\Xi$  is sent to  $\Xi^{\text{Darb}}$ , and using the



following formula for  $\mathbf{p}$ :

$$\mathbf{p}_j = -\frac{\rho}{\mathbf{q}_j - 1} + \sum_{i \in \{r, s, t\}} \frac{c_i z_i}{\mathbf{q}_j - 1} + \frac{z_j(\mathbf{q}_k - j)(\mathbf{q}_\ell - j)(\rho - \sum_{i \in \{r, s, t\}} c_i(z_i - i))}{j(j-1)(j-k)(j-\ell)} - \frac{c_k z_k}{\mathbf{q}_j - k} - \frac{c_\ell z_\ell}{\mathbf{q}_j - \ell},$$

where  $\{j, k, \ell\} = \{r, s, t\}$  (instead of (3.40), which is not defined on  $\Xi$ ). This shows that there is a Euclidian neighborhood of  $\Xi^{\text{Darb}}$  such that  $\Pi$  restricted to this neighborhood is a local diffeomorphism, which moreover sends  $\Xi^{\text{Darb}}$  surjectively onto  $\Xi$ .

In order to show that the isomonodromy foliation  $\mathcal{F}_{\text{iso}}$  is transversal to  $\Xi$  it is thus sufficient to prove the transversality of  $\Xi^{\text{Darb}}$  with the vector fields  $V_i$  defined in (3.44). Modulo the vector fields

$$\frac{\partial}{\partial r} + \frac{\partial}{\partial \mathbf{q}_r}, \quad \frac{\partial}{\partial s} + \frac{\partial}{\partial \mathbf{q}_s}, \quad \frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{q}_t}, \quad \frac{\partial}{\partial \mathbf{p}_r}, \quad \frac{\partial}{\partial \mathbf{p}_s}, \quad \frac{\partial}{\partial \mathbf{p}_t},$$

that are tangent to  $\Xi^{\text{Darb}}$ , the vector fields  $V_i$  for  $i \in \{r, s, t\}$  are equivalent respectively to

$$\tilde{V}_i = -\frac{\partial}{\partial \mathbf{q}_i} + \sum_{j \in \{r, s, t\}} \left( \frac{\partial H_i}{\partial \mathbf{p}_j} \right) \frac{\partial}{\partial \mathbf{q}_j}.$$

We associate the column vector

$${}^\top(\tilde{V}_r, \tilde{V}_s, \tilde{V}_t) = \left[ \left( \frac{\partial H_i}{\partial \mathbf{p}_j} \right)_{i, j \in \{r, s, t\}} - I \right] \cdot {}^\top \left( \frac{\partial}{\partial \mathbf{q}_r}, \frac{\partial}{\partial \mathbf{q}_s}, \frac{\partial}{\partial \mathbf{q}_t} \right),$$

where  $I$  denotes the identity 3-by-3 matrix. Calculation shows

$$\left[ \left( \frac{\partial H_i}{\partial \mathbf{p}_j} \right)_{i, j \in \{r, s, t\}} - I \right] \Big|_{(\mathbf{q}_r, \mathbf{q}_s, \mathbf{q}_t) = (r, s, t)} = \frac{1}{2} \cdot I,$$

which is clearly invertible. This proves transversality of the isomonodromic foliation  $\mathcal{F}_{\text{iso}}$  with the locus  $\Xi$  of our even Gunning bundle in  $\mathfrak{M}$ . The transversality of  $\mathcal{F}_{\text{iso}}$  with the locus  $\Phi(\Xi)$  of our even Gunning bundle in  $\mathfrak{M}$  then follows from the fact that the Gunning bundle is not in the fixed point locus of the action of the Galois involution in the Switched chart  $U_J$  we considered, so that the two-fold cover  $(r, s, t, \mathbf{z}, \mathbf{c}) \xrightarrow{2:1} \Phi(r, s, t, \mathbf{z}, \mathbf{c})$  is a local diffeomorphism in a neighborhood of  $\Xi$ .  $\square$

### 3.7.4 Projective structures and Hejhal's theorem

As we shall now see, Theorem 3.7.2 provides an alternative proof, in the genus 2 case, of a theorem of Hejhal concerning projective structures. The notion of projective structures on compact Riemann surfaces goes back to the works of Schwarz on the hypergeometric equation (see [dSG10, Chap. VIII]). A *projective structure* on  $X$  is given by an atlas of charts  $f_i : U_i \subset X \rightarrow \mathbb{C} \subset \mathbb{P}^1$  (holomorphic diffeomorphisms) such that the transition maps  $\varphi_{ij} := f_j \circ f_i^{-1}$  are (restrictions to  $f_i(U_i \cap U_j)$ ) of Möbius transformations:  $\varphi_{ij} \in \text{PGL}_2(\mathbb{C})$ . As proven in [Gun67b], the set of equivalence classes of projective structures on  $X$  is in bijective correspondence with the vector space of quadratic differentials

$$(\nu_2 x^2 + \nu_1 x + \nu_0) \left( \frac{dx}{y} \right)^{\otimes 2} \in H^0(X, \Omega_X^1 \otimes \Omega_X^1)$$

with  $\nu = (\nu_0, \nu_1, \nu_2) \in \mathbb{C}^3$ . This correspondence associates to  $\nu$  an atlas of projective charts formed by local solutions  $f$  of the Schwarzian differential equation

$$\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{x^3 + \nu_2 x^2 + \nu_1 x + \nu_0}{F} - \frac{1}{2}\frac{F''}{F} + \frac{7}{8}\frac{F'F'}{F^2},$$

where  $F(x) = x(x-1)(x-r)(x-s)(x-t)$  and primes denote differentiation with respect to  $x$  (see [dSG10, § IX.3.4]). On the other hand there is a bijective correspondence between equivalence classes of projective structures on  $X$  and isomorphism classes of projectivised connections  $\mathbb{P}(E, \nabla)$ , where  $E$  is any Gunning bundle (see again [Gun67b]). Equivalently, one may consider the set of isomorphism classes of tracefree connections  $(E_\vartheta, \nabla)$  for a fixed  $\vartheta$ -characteristic  $\vartheta^{\otimes 2} \simeq \Omega_X^1$ , which we have parametrized by  $\mathbb{C}_c^3$ . Following the calculation in [LM09, Prop. 2.1], one sees that the natural identification  $\mathbb{C}_\nu^3 \simeq \mathbb{C}_c^3$  is holomorphic. The monodromy of a projective structure can be equivalently defined as the monodromy of  $\mathbb{P}(E, \nabla)$ , the Schwarzian differential equation, or of any local projective chart (it can be analytically continued along any loop). Let us denote by  $\Sigma_2$  the real oriented surface underlying  $X$ . We may consider projective structures along the family of complex structures on  $\Sigma_2$  parametrized by  $\mathcal{T} = \mathbb{C}_{(r,s,t)}^3 \setminus \{\text{diagonals}\}$ . After lifting to the Teichmüller space, namely the universal cover  $\tilde{\mathcal{T}} \rightarrow \mathcal{T}$ , the monodromy map

$$\text{Mon} : \tilde{\mathcal{T}} \times \mathbb{C}_b^3 \rightarrow \text{Hom}(\pi_1(\Sigma_2, w), \text{PGL}_2)/\text{PGL}_2. \quad (3.46)$$

is well-defined and analytic. A problem which goes back to the work of Poincaré on Fuchsian functions was to decide which kinds of conjugacy classes of representations in  $\chi := \text{Hom}(\pi_1(\Sigma_2, w), \text{PGL}_2)/\text{PGL}_2$  arise as the monodromy of a projective structure, i.e. as monodromy of  $(E_\vartheta, \nabla)$ , for a suitable complex structure on  $\Sigma_2$ . Since both source and target of Mon are 6-dimensional, one expects most elements of  $\chi$  to be realizable by a projective structure. On the other hand, we know three types of representations in  $\chi$  that are certainly not realizable: those which are not in the image in  $\chi$  of  $\text{Hom}(\pi_1(\Sigma_2, w), \text{SL}_2)$  - they do not come from tracefree connections; those which are in the image in  $\chi$  of unitary representations - they come from connections on stable bundles; and those which are in the image of reducible representations in  $\text{Hom}(\pi_1(\Sigma_2, w), \text{SL}_2)$  - they come from connections on strictly semistable bundles. It was proven in [GKM00] that any element of  $\chi$  that does not belong to one of these three sets of exceptions can be realised. Some time earlier, D. A. Hejhal proved in [Hej75] a local version:

**Theorem 3.7.3** (Hejhal). *The monodromy map (3.46) is a local diffeomorphism.*

Going back to our space  $\mathfrak{M}$  of triples  $(X, E, \nabla)$  we also have, locally in  $\mathcal{T}$  (or, alternatively, after lifting  $\mathfrak{M}$  over the universal cover of  $\mathcal{T}$ ), a monodromy map

$$\text{Mon} : \mathfrak{M} \rightarrow \text{Hom}(\pi_1(\Sigma_2, w), \text{SL}_2)/\text{PGL}_2, \quad (3.47)$$

whose fibers are the leaves of the isomonodromic foliation  $\mathcal{F}_{\text{iso}}$ . Note that in restriction to the irreducible locus,  $\mathcal{F}_{\text{iso}}$  is non-singular and Mon is a submersion. Moreover, the locus  $\Xi$  in  $\mathfrak{M}$  of Gunning bundles with fixed even  $\vartheta$ -characteristic is smooth and has dimension 6, complementary to the dimension 3 of the isomonodromy leaves. Theorem 3.7.2 is therefore equivalent to saying that the restriction  $\text{Mon}|_{\Xi}$  of the monodromy map in (3.47) to  $\Xi$  is a local diffeomorphism. After proper identifications, it is therefore equivalent to Hejhal's theorem.

**Remark 3.7.4.** *Note that transversality of two submanifolds of  $\mathfrak{M}$  of complementary dimension implies their topological transversality in the sense that their intersection is at most 0-dimensional. The topological transversality of  $\Xi$  with the isomonodromy leaves, or equivalently, the openness of the monodromy map  $\text{Mon}|_{\Xi}$ , also follows from the main result in [Heu09].*

### 3.7.5 Further results and open questions

Consider the Teichmüller space  $\mathcal{T}_{g,0}$  of curves  $C$  of genus  $g$  endowed with a Teichmüller structure  $\varphi : \Sigma_g \rightarrow X$ . Consider the space  $\text{Syst}^{\text{irr}}$  of pairs  $(C, A)$  with  $C \in \mathcal{T}_{g,0}$  and  $A \in \mathfrak{sl}_2\Omega_C^1$  corresponding to an irreducible tracefree connection  $d + A$  on the trivial bundle  $C \times \mathbb{C}^2$  (an irreducible *system*), modulo gauge equivalence.

**Theorem 3.7.5.** *When  $g = 2$ , then the monodromy map*

$$\text{Mon} : \text{Syst}^{\text{irr}} \rightarrow \text{Hom}(\pi_1(\Sigma_g), \text{SL}_2\mathbb{C}) //_{\text{SL}_2\mathbb{C}}$$

*is a local diffeomorphism.*

This theorem has two proofs, one developed with F. Loray by showing, similarly to Theorem [3.7.2](#) that the "irreducible" leaves of the isomonodromy foliation in  $\mathfrak{M}$  are transversal to the locus of the trivial bundle, and another one, developed by G. Calsamiglia and B. Deroin, using moduli spaces of branched projective structures on  $\Sigma_g$ . They appear in our common recent publication [\[CDHL18\]](#). Notably, the statement analogous to Theorem [3.7.5](#) is false in genus  $g > 2$  (see [\[CDHL18, § 6\]](#)). Theorem [3.7.5](#) should be seen as a local result in genus two towards the following question raised by E. Ghys.

**Problem 6** (Ghys). *Classify of the image of  $\text{Syst}^{\text{irr}}$  under the monodromy map for  $g \geq 2$ .*

The motivation for this problem comes from the study of quotients  $Y := \text{SL}_2/\Gamma$  by cocompact lattices  $\Gamma \subset \text{SL}_2$ . These compact complex manifolds are not Kähler. Huckleberry and Margulis proved in [\[HM83\]](#) that they admit no complex hypersurfaces (and therefore no non-constant meromorphic functions). Elliptic curves exist in such quotients, while the existence of compact curves of genus at least two remains open and is related to Ghys' question. Indeed, assuming that for a non trivial system on a curve  $C$ , its monodromy has image contained in  $\Gamma$  (up to conjugation), then the corresponding fundamental matrix induces a non trivial holomorphic map from  $C$  to  $Y$ . Reciprocally, any curve  $C$  in  $Y$  can be lifted to  $\text{SL}_2\mathbb{C}$  and gives rise to the fundamental matrix of some system on  $C$ , whose monodromy is contained in  $\Gamma$ . In fact, it is not known whether holomorphic  $\mathfrak{sl}_2$ -systems on Riemann surfaces of genus  $> 1$  give rise to representations with discrete or real image. Although Ghys' question remains open, our result shows on the one hand that we can locally realize arbitrary deformations of the monodromy representation of a given system over a genus two curve by allowing deformations of both the curve and the system, and on the other hand that, if a genus two curve exists in some  $Y$  as before, it is rigid in  $Y$  up to left translations.

The results in [\[HL15\]](#) also raise some questions which should be easier to tackle. For example to study the geometry of isomonodromic deformations in  $\mathfrak{M}$  near a point of  $\text{Syst}^{\text{irr}}$ . Note that the comparison of the two approaches in [\[CDHL18, § 11\]](#) shields some light on this question. In particular, not only in the moduli space of bundles, but also in  $\mathfrak{M}$ , and even after restricting to the locus of irreducible connections, the locus of the trivial bundle appears as a singularity in the locus of strictly semistable bundles. One might also deduce some more applications from the explicit description of the Hitchin map in Table [11](#). For example, it seems to suggest that in the spirit of Torelli type theorems on moduli spaces of vector bundles over curves [\[MN68, BG03\]](#), these explicit Hitchin Hamiltonians encode the complex structure of the base curve - namely, the equivalence class of the triple  $(r, s, t)$  in [\(3.4\)](#). Moreover, it would be interesting to compute the nilpotent cone explicitly for example. This is not a trivial question - for the naïve approach, this calculation has not manageable size.

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