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Géométrie quaternionienne en basses dimensions

par

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Introduction

Si (M, g) est une variété riemannienne connexe, le groupe d'holonomie réduit de M désigne l'ensemble des applications linéaires $T_xM \to T_xM$ obtenues par transport parallèle le long des lacets homotopes à zéro et de point base x. Lorsque la métrique g est irréductible, c'est-à-dire non localement difféomorphe à un produit riemannien, et lorsque (M, g) n'est pas localement symétrique, Berger [**Ber55**] a montré que le groupe d'holonomie réduit de g est nécessairement $SO(n)$, $U(n/2)$, $SU(n/2)$, $Sp(n/4)$, $Sp(n/4)Sp(1)$, G_2 ou $Spin(7)$. Si le groupe d'holonomie est $Sp(n/4)Sp(1)$, on dit que g est quaternion-kählerienne, la métrique est alors Einstein et son tenseur de Ricci est non nul.

Le fait que la courbure scalaire s d'une telle métrique ne soit pas réduite à zéro est un aspect important de la géométrie quaternion-kählerienne et entraîne l'apparition de comportements très différents selon que l'on soit en courbure scalaire positive ou négative ; lorsque $s > 0$, LeBrun [**LeB93**] et Salamon [**LeB-Sal94**] ont montré qu'il existe au plus un nombre fini de variétés quaternion-kähleriennes compactes de dimension n à isométrie et homothétie près ; au contraire, le cas $s < 0$ est beaucoup plus riche, même dans le cas homogène [**Ale75**]. En particulier LeBrun [**LeB91**] a construit une famille à une infinité de paramètres de métriques quaternion-kähleriennes complètes à courbure scalaire négative sur la boule unité. Ces métriques g sont analytiques et admettent un pôle d'ordre 2 au bord, de sorte que si ρ est une fonction qui s'annule à l'ordre 1 sur la sphère $\mathbb{S}$, on obtient une classe conforme de métriques $[\rho^2 g|_{T\mathbb{S}}]$ sur $\mathbb{S}$, dégénérées et dont le noyau H est appelé distribution de contact quaternion-kählerienne [**Biq00**]. On dit que la distribution H est l'infini conforme de g (voir la définition 1.0.2 du chapitre 1). Avant de donner une définition précise des structures de contact quaternioniennes, nous décrivons l'exemple fondamental que constitue le bord de la métrique quaternion hyperbolique $g_{\mathcal{H}}$.

Une structure quaternionienne sur un espace vectoriel V est la donnée d'un triplet (I_1, I_2, I_3) de structures presque complexes vérifiant les relations de commutation $I_1 I_2 = -I_2 I_1 = I_3$. Si $\mathbb{H}$ désigne le corps gauche des quaternions, une base orthonormale (i, j, k) des imaginaires purs fournit une telle structure sur l'espace vectoriel $\mathbb{H}^n \simeq \mathbb{R}^{4n}$. La métrique hyperbolique s'écrit alors

$$g_{\mathcal{H}} = \frac{4euc}{\rho} + \frac{1}{\rho^2}((d\rho)^2 + (I_1 d\rho)^2 + (I_2 d\rho)^2 + (I_3 d\rho)^2)$$

où euc est la métrique euclidienne de $\mathbb{R}^{4n}$ et $\rho = (1 - |x|^2)$. Les structures presque complexes I_i sont orthogonales pour la métrique $g_{\mathcal{H}}$ et engendrent un fibré $\mathcal{Q} \subset End(TM)$ qui est stable sous l'action de la connexion de Levi-Civita de $g_{\mathcal{H}}$. L'existence d'un fibré $\mathcal{Q}$ localement engendré par une structure quaternionienne, orthogonale et stable sous l'action de la connexion de Levi-Civita au-dessus d'une variété riemannienne caractérise les métriques quaternion-kähleriennes.

Dans le cas de la métrique hyperbolique, la distribution de contact sur le bord est

$$H^{can} = \cap_{i=1}^3 \ker I_i d\rho;$$

elle est stable sous l'action de I_1, I_2 et I_3 et on a

$$d(I_i d\rho)(X, Y) = 4euc(I_i X, Y)$$

pour des vecteurs X et Y de H. Ceci nous amène à la définition

Définition 0.1. Soit H une distribution lisse de codimension 3 sur une variété M. Si il existe une métrique g_H sur H, une structure quaternionienne locale (I_1, I_2, I_3) sur H ainsi que des 1-formes locales η_1, η_2 et η_3 telles que $\eta_1|_H = \eta_2|_H = \eta_3|_H = 0$ et pour tout i

$$d\eta_i|_H = g_H(I_i \cdot, \cdot),$$

on dit que H est une structure de contact quaternionienne.

Cette définition peut être simplifiée de manière à mieux rendre compte des spécificités de la dimension 7.

Définition 0.2. Soit H une distribution orientable, de codimension 3 sur une variété de dimension 7 et $\Lambda^2_+ H^*$ le fibré engendré par les $d\eta|_H$ où η décrit l'ensemble des 1-formes s'annulant sur H. On dit que H est une structure de contact quaternionienne si $\Lambda^2_+ H^*$ est un fibré de rang 3 et si la restriction à $\Lambda^2_+ H^*$ du produit extérieur

$$\Lambda^2 H^* \otimes \Lambda^2 H^* \to \Lambda^4 H^* \to \mathbb{R}$$

est une métrique définie positive.

Sous les conditions de la définition (0.2), l'existence d'une métrique g_H et d'une structure quaternionienne vérifiant les hypothèses de la définition (0.1) est un fait classique d'algèbre linéaire et revient à choisir une orientation sur H et une base orthonormale $(\frac{1}{\sqrt{2}}d\eta_i|_H)$ de $\Lambda^2_+H^*$.

L'objet principal de cette thèse est la question suivante :

– Etant donnée une structure de contact quaternionienne H sur une variété M de dimension 7, existe-t-il une métrique quaternion-kählerienne g, définie sur un voisinage de M, admettant un pôle d'ordre 2 le long de M et d'infini conforme H ?

Rappelons qu'en dimension $4n+3 \geq 11$, une réponse positive a été obtenue par Biquard dans [**Biq00**]. En dimension 7, à l'aide de techniques twistorielles, je montre l'existence d'une condition nécessaire et suffisante d'intégrabilité pour qu'une distribution de contact quaternionienne soit le bord d'une métrique quaternion-kählerienne.

Twisteurs

Une des propriétés remarquables de la géométrie quaternion-kählerienne est la possibilité de lui appliquer des techniques d'analyse complexe via l'espace des twisteurs. Ce dernier est une variété holomorphe, fibrée au dessus de toute variété quaternion-kählerienne, inventée de manière indépendante par Bérard Bergery [**Bér79**] et Salamon [**Sal82**] au début des années 80 et généralisant l'espace des twisteurs des variétés anti-autoduales de dimension 4 découvert par Penrose ([**Pen72**] et aussi [**At-Hi-Si78**]).

Si (M^{4n}, g) est une variété quaternion-kählerienne, l'espace des twisteurs $\pi : \mathcal{T} \to M$ est le fibré en sphères rondes S^2

$$\mathcal{T} = \{x_1 I_1 + x_2 I_2 + x_3 I_3\,,\ x_1^2 + x_2^2 + x_3^2 = 1\}$$

où (I_1, I_2, I_3) est une structure quaternionienne locale sur M qui engendre $\mathcal{Q}$. La connexion de Levi-Civita de g permet d'identifier π^*TM à une distribution D, transverse aux fibres de $\mathcal{T}$. Si $I \in \mathcal{T}$, la structure complexe $\mathcal{J}$ sur $\mathcal{T}$ est définie par $\mathcal{J}_I(X) = IX$ sur D et correspond à la structure complexe canonique des sphères sur l'espace tangent aux fibres. La distribution D est holomorphe et la projection $\Theta : T\mathcal{T} \to T\mathcal{T}/D$ vérifie

$$\Theta \wedge d\Theta^n \neq 0,$$

on dit que Θ est une structure de contact holomorphe. Enfin, l'application antipodale $I \mapsto -I$ fournit une involution anti-holomorphe de $\mathcal{T}$.

Le grand intérêt de cette construction est qu'elle peut être inversée, [**LeB89**] : soit $(\mathcal{T},\Theta)$ une variété de contact holomorphe de dimension $2n+1$, munie d'une involution anti-holomorphe $\sigma:\mathcal{T}\to\mathcal{T}$ et désignons par $M^{\mathbb{C}}$ l'ensemble des sphères holomorphes de $\mathcal{T}$, invariantes sous l'action de σ, transverses à $D=\ker\Theta$ et de fibré normal $2n\mathcal{O}(1)$. Si $M^{\mathbb{C}}$ est non vide, alors le sous-ensemble M des sphères σ-invariantes peut être muni d'une métrique pseudo-riemannienne qui est quaternion-kählerienne dans le cas où elle est définie positive. De plus, si $\mathcal{T}$ vient d'une variété quaternion-kählerienne M', alors M et M' sont isométriques.

Nous allons maintenant illustrer cette construction dans le cas hyperbolique, à l'aide d'une description plus intrinsèque de la métrique $g_{\mathcal{H}}$. L'espace hyperbolique quaternionien est

$$\mathbb{H}\mathcal{H}^n=\{[q_1,...,q_{n+1}]\,,\ \sum_{k=1}^{n}|q_k|^2<|q_{n+1}|^2\}\subset\mathbb{H}P^n,$$

que l'on identifie à $(B^{4n},g_{\mathcal{H}})$ via la restriction de l'injection $\mathbb{H}^n\hookrightarrow\mathbb{H}P^n$, $(q_1,...,q_n)\mapsto[q_1,...,q_n,1]$. L'identification

$$(z_1,...,z_{2n+2})\mapsto(z_1+jz_2,...,z_{2n+1}+jz_{2n+2})$$

entre $\mathbb{C}^{2n+2}$ et $\mathbb{H}^{n+1}$ permet de définir une projection $\pi:\mathbb{C}P^{2n+1}\to\mathbb{H}P^n$, de fibre $\mathbb{C}P^1$ et dont la restriction à

$$\mathbb{C}P^{2n+1}_{+}=\{[z_1,...,z_{2n+2}]\,,\ \sum_{k=1}^{2n}|z_k|^2<|z_{2n+1}|^2+|z_{2n+2}|^2\}$$

identifie l'espace des twisteurs de $\mathbb{H}\mathcal{H}^n$ avec $\mathbb{C}P^{2n+1}_{+}$. La structure réelle est la multiplication à droite par j, et la forme de contact à valeurs dans $\mathcal{O}(2)$ est définie sur tout $\mathbb{C}P^{2n+1}$ par

$$\Theta=\sum_{k=1}^{n}(z_{2k-1}dz_{2k}-z_{2k}dz_{2k-1})-(z_{2n+1}dz_{2n+2}-z_{2n+2}dz_{2n+1})\,.$$

Le bord de $\mathbb{C}P^{2n+1}_{+}$ est l'hypersurface

$$\mathbb{C}P^{2n+1}_{0}=\{[z_1,...,z_{2n+2}]\,,\ \sum_{k=1}^{2k}|z_k|^2=|z_{2n+1}|^2+|z_{2n+2}|^2\}\,.$$

C'est une variété CR, de forme de contact $\theta^r=\Im(\sum_{k=1}^{2n}z_kd\overline{z}_k-(z_{2n+1}d\overline{z}_{2n+1}+z_{2n+2}d\overline{z}_{2n+2}))$, fibrée en $\mathbb{C}P^1$ au-dessus du bord

$$\partial\mathbb{H}\mathcal{H}^n=\{[q_1,...,q_{n+1}]\,,\ \sum_{k=1}^{n}|q_k|^2=|q_{n+1}|^2|\}\,,$$

que l'on appelle espace des twisteurs de la structure de contact quaternionienne H^{can} sur $\partial\mathbb{H}\mathcal{H}^n$. Comme $\Theta(zj) = \sum_{k=1}^n |z_k|^2 - |z_{2n+1}|^2 - |z_{2n+2}|^2$, la variété $\mathbb{C}P_0^{2n+1}$ est exactement la réunion des fibres de π qui sont tangentes à la distribution $\ker\Theta$. La construction twistorielle inverse donnant des variétés holomorphes, nous décrivons maintenant la complexification de l'espace hyperbolique quaternionien.

On définit la forme symplectique $w = \sum_{k=1}^n dz_{2k-1} \wedge dz_{2k} - dz_{2n+1} \wedge dz_{2n+2}$. Les complexifications de $\mathbb{H}\mathcal{H}^n$ et de $\partial\mathbb{H}\mathcal{H}^n$ sont les grassmaniennes $Gr_2(\mathbb{C}^{2n+2})$ et $Gr_2^0(\mathbb{C}^{2n+2})$ respectivement, où $Gr_2^0(\mathbb{C}^{2n+2})$ désigne l'ensemble des plans complexes de $\mathbb{C}^{2n+2}$ isotropes pour w. Chaque point P de $Gr_2(\mathbb{C}^{2n+2})$ définit par projection une courbe complexe de genre nul $\mathbb{C}P_P^1$ dans $\mathbb{C}P^{2n+1}$ et $Gr_2^0(\mathbb{C}^{2n+2})$ s'identifie alors à l'ensemble des $P \in Gr_2(\mathbb{C}^{2n+2})$ tels que $\mathbb{C}P_P^1$ soit tangent au noyau de Θ. La structure réelle provient de la multiplication par j sur $\mathbb{H}^{n+1} \simeq \mathbb{C}^{2n+2}$.

Pour montrer qu'une distribution de contact quaternionienne (M, H) est l'infini conforme d'une métrique quaternion-kählerienne, on construit une variété CR intégrable $\mathcal{T} \to M$ au-dessus de M que l'on appelle espace des twisteurs de M. On complexifie M en une variété holomorphe $M^{\mathbb{C}}$; puis à l'aide de $\mathcal{T}$, on obtient une variété complexe $\mathcal{T}^c$ munie d'une structure de contact holomorphe Θ et d'une famille de sphères holomorphes $(\mathcal{C}_m)_{m\in N^{\mathbb{C}}}$ telle que $M^{\mathbb{C}}$ soit une hypersurface de $N^{\mathbb{C}}$ qui paramètre les sphères tangentes au noyau de Θ. On applique alors la construction inverse (**[LeB89]** et **[Biq00]**) pour montrer que M est le bord d'une variété quaternion-kählerienne.

Principaux résultats

Si H est une distribution de contact quaternionienne sur une variété M, la construction d'une métrique quaternion-kählerienne d'infini conforme H se réduit à la construction d'un espace des twisteurs CR intégrable. Je définis une condition d'intégrabilité pour une structure de contact quaternionienne qui permet de faire cette construction (définition 1.0.3, chapitre 1). Ceci me permet d'obtenir le résultat suivant :

THÉORÈME 0.1. *Une structure de contact quaternionienne H sur une variété de dimension 7 est l'infini conforme d'une métrique quaternion kählerienne si et seulement si H est intégrable.*

Ce résultat est démontré dans le chapitre 1 à l'aide de la construction d'une connexion adaptée aux structures de contact quaternioniennes en dimension 7.

Cette condition d'intégrabilité est vérifiée pour les bords des métriques construites par LeBrun. Galicki **[Gal91]** a construit une famille d'exemples ayant $Sp(1)$ comme sous-groupe d'isométrie par quotient de l'espace hyperbolique quaternionien. Une question naturelle est alors de chercher des structures de contact quaternioniennes sur la sphère $\mathbb{S}^7$, intégrables, et qui ont $Sp(1)$ comme sous-groupe de symétries, où $Sp(1)$ désigne ici l'action du groupe associé à la fibration de Hopf $\mathbb{S}^7 \to \mathbb{S}^4$.

THÉORÈME 0.2. *L'espace des déformations $Sp(1)$-invariantes, intégrables de H^{can} est une famille continue à 25-paramètres.*

Ce résultat est démontré dans le second chapitre et permet d'en déduire l'existence d'une famille de déformations quaternion-kähleriennes et $Sp(1)$-invariantes de la métrique hyperbolique quaternionienne.

COROLLAIRE 0.1. *Il existe une famille à 25 paramètres de déformations quaternion kähleriennes, $Sp(1)$-invariantes de l'espace hyperbolique quaternionien.*

Sur une variété M, une 4-forme Ω de stabilisateur $Sp(n)Sp(1)$ définit une métrique riemannienne g qui est quaternion-kählerienne lorsque Ω est parallèle pour la connexion de Levi-Civita de g. En dimension 8 et contrairement aux dimensions supérieures, il existe des 4-formes de stabilisateur $Sp(n)Sp(1)$ qui sont fermées mais qui ne sont pas parallèles (**[Swa89]** et **[Sal01]**).

Soit ρ une fonction strictement positive sur B^8 s'annulant à l'ordre 1 sur la sphère $\mathbb{S}^7$ et soit H une distribution de contact quaternionienne sur $\mathbb{S}^7$. La condition d'intégrabilité sur H apparaît comme une obstruction à trouver une 4-forme asymptotiquement hyperbolique quaternionienne (définition 1.2, chapitre 3), de bord H et telle que $|\nabla\Omega|_g = O(\sqrt{\rho})$ où g est la métrique définie par Ω et ∇ est la connexion de Levi-Civita de g. Le troisième chapitre est principalement dédié à la preuve du

THÉORÈME 0.3. *Si H est proche de la structure standard H^{can}, Il existe une 4-forme asymptotiquement hyperbolique quaternionienne Ω, de bord H et telle que $|d\Omega|_g = O(\rho^3)$.*

Mazzeo (**[Maz88]**) a montré que le spectre essentiel du laplacien des métriques asymptotiquement hyperboliques réelles (ou conformément compactes) est identique à celui de l'espace hyperbolique réel. Il est naturel de faire une hypothèse analogue à propos du spectre essentiel du laplacien de Hodge pour les métriques asymptotiquement hyperboliques quaternioniennes, sachant que sur les 5-formes, le spectre du laplacien pour $g_{\mathcal{H}}$ est $[1;+\infty[$. La construction formelle du théorème précédent permet alors d'obtenir le résultat suivant :

THÉORÈME 0.4. *Supposons qu'il existe un voisinage U de H^{can} dans l'espace des structures de contact quaternioniennes tel que si g est une métrique asymptotiquement hyperbolique quaternionienne de bord $H \in U$, le spectre essentiel du laplacien sur les 5-formes est contenu dans $[\frac{1}{2};+\infty[$. Alors il existe un voisinage V de H^{can} tel que toute structure de contact quaternionienne $H \in V$ peut être réalisée comme le bord d'une 4-forme fermée, asymptotiquement hyperbolique quaternionienne et de bord H.*

Enfin, le dernier chapitre donne une définition des quotients de contact quaternioniens, analogue à celle des quotients quaternioniens définis par Galicki et Lawson dans [**Gal88**].

CHAPTER 1

Quaternionic contact structures in dimension 7

1. Introduction

In this paper we solve a boundary problem for quaternionic-Kähler metrics. This problem is a degenerate version of a problem initially posed for Einstein metrics. If g is a metric on a manifold M with boundary N, and $[b]$ is a conformal class of metrics on N, $[b]$ is the conformal infinity of g if there exists a function ρ positive in M and vanishing to first order on N such that $\rho^2 g$ extends continuously on N with $\rho^2 g|_{T\mathbb{S}^3} \in [b]$. The standard example is the hyperbolic metric g_{hyp} on the ball B^{n+1} given by

$$g_{hyp} = 4\frac{euc}{\rho^2} ,$$

where euc is the Euclidean metric on $\mathbb{R}^{n+1}$ and $\rho(x) = 1 - |x|^2$. The conformal infinity of g_{hyp} is the conformal class of the round metric on $\mathbb{S}^n$.

The problem of finding complete Einstein metrics with prescribed conformal infinity on the ball was solved by Graham and Lee in [**Gra91**]. In dimension 4, one can search for selfdual Einstein metrics. LeBrun [**LeB82**] shows using twistor theoretic arguments that a conformal metric on a 3-manifold N is the conformal infinity of a selfdual Einstein metric defined near N. However, a conformal metric on the sphere $\mathbb{S}^3$ is not always the conformal infinity of a complete selfdual Einstein metric on the ball B^4, see [**Biq02**].

In the same way, the degenerate version is modeled on the quaternionic hyperbolic metric. Let $\mathbb{H}$ be the skew field of quaternions and $\mathbb{H}^n$ the n-dimensional $\mathbb{H}$-vector space. The action of the standard basis (i, j, k) of imaginary quaternions gives endomorphisms (I_1, I_2, I_3) of $\mathbb{H}^n \simeq \mathbb{R}^{4n}$. Each I_i is an almost complex structure on $\mathbb{H}^n$ and one has the commutations rules $I_1 I_2 = -I_2 I_1 = I_3$. A such triple of endomorphisms on a real vector space V is called a quaternionic structure on V. The quaternionic hyperbolic metric on the ball $B^{4n} \subset \mathbb{H}^n$ is given by

$$g_{\mathcal{H}} = \frac{4euc}{\rho} + \frac{1}{\rho^2}((d\rho)^2 + (I_1 d\rho)^2 + (I_2 d\rho)^2 + (I_3 d\rho)^2) ,$$

where $\rho = 1 - |x|^2$ and *euc* is the Euclidean metric. In this case, the function ρ is positive in B^{4n}, vanishes to first order on $\mathbb{S}^{4n-1}$, and $[\rho^2 g_{\mathcal{H}}|_{T\mathbb{S}^{4n-1}}]$ is a conformal class of degenerate metrics on $\mathbb{S}^{4n-1}$ with kernel

$$H^{can} = \cap_{i=1}^{3} \ker I_i d\rho|_{T\mathbb{S}^{4n-1}} .$$

The distribution H^{can} is a so called quaternionic contact structure ([**Biq00**] and [**Mont02**, p. 115]) whose definition in dimension 7 is:

Definition 1.1. Let H be an oriented distribution of codimension 3 on a 7-dimensional manifold N and let $\mathcal{I}$ be the set of one forms vanishing on H. The distribution H is called a quaternionic contact structure if

$$\Lambda^2_+ H^*_x = \{d\eta|_{H_x} \, , \; \eta \in \mathcal{I}\}$$

is a rank three subbundle of $\Lambda^2 H^*$ such that the restriction to $\Lambda^2_+ H^*$ of the exterior product

$$\Lambda^2 H^* \otimes \Lambda^2 H^* \to \Lambda^4 H^* \xrightarrow{\simeq} \mathbb{R}$$

gives a positive definite metric on $\Lambda^2_+ H^*$.

If H is a quaternionic contact structure in dimension 7, a classical fact in 4-dimensional linear algebra gives the existence of a unique conformal class $[g]$ of metrics on H such that $\Lambda^2_+ H^*$ coincides with the space of selfdual 2-forms with respect to $[g]$. Moreover, taking a local oriented orthonormal basis $(\frac{1}{\sqrt{2}} w_i = \frac{1}{\sqrt{2}} d\eta_i|_H)$ of $\Lambda^2_+ H^*$ with respect to a particular choice of metric g in this conformal class, one gets a quaternionic structure $(I_i)_{i=1,2,3}$ on H satisfying $w_i(\cdot,\cdot) = g(I_i \cdot, \cdot)$ and defined up to a rotation by an element of $SO(3)$.

This description shows the link with the following definition given by Biquard in [**Biq00**]: a quaternionic contact structure is a distribution H of codimension 3 on a manifold N^{4n+3}, locally given by three 1-forms (η_1, η_2, η_3) such that there exists a metric g on H and a quaternionic structure (I_i) on H satisfying the conditions $d\eta_i|_H = g(I_i \cdot, \cdot)$. The conformal class $[g]$ is uniquely determined by H.

Our definition enlights the fact that in dimension 7, quaternionic contact distributions form an open set in the set of codimension 3 distributions. This fact is no more true in higher dimensions.

Let us now come back to quaternionic-Kähler geometry. First, using the previous notations, we give the following definition:

Definition 1.2. A metric g on a manifold M with boundary N is asymptotically quaternionic hyperbolic (AQH) if one has a quaternionic contact structure H on N with compatible metric g_H on H and a function ρ, positive in M vanishing to first order on N such that on a neighbourhood $]0,a] \times N$ of N, the behaviour of g near N is given by

$$g \sim \frac{1}{\rho^2}(d\rho^2 + \eta_1^2 + \eta_2^2 + \eta_3^2) + \frac{1}{\rho} g_H \text{ when } \rho \to 0.$$

The quaternionic contact structure H is called the conformal infinity of g. In the case where g is also quaternionic-Kähler, one says that g is asymptotically hyperbolic quaternionic-Kähler (AHQK).

Biquard [**Biq00**] has shown that every quaternionic contact structure of dimension $4n+3 \geq 11$ is at least locally the conformal infinity of a unique AHQK metric. Moreover, he showed in [**Biq02**] that a quaternionic contact structure on $\mathbb{S}^{4n+3}$ with $4n+3 \geq 11$ and close to the canonical one is the conformal infinity of a AHQK complete metric on the ball B^{4n+4}. The question remains open in dimension 7.

In this paper, we answer this last question. We show that the conformal infinity of an AHQK 8-manifold must satisfy an additionnal integrability property which is empty in higher dimensions. Conversely, we prove that an integrable quaternionic contact 7-manifold is the conformal infinity of a unique AHQK manifold.

Definition 1.3. Let H be a quaternionic contact structure on a manifold N of dimension 7 and choose a compatible metric g. The quaternionic contact structure H is called integrable if for each local oriented orthonormal basis $(d\eta_i|_H)$ of $\Lambda^2_+ H^*$, there exist vector fields (R_1, R_2, R_3) satisfying

- $i_{R_i}\eta_j = \delta_{ij}$,
- $i_{R_i} d\eta_j|_H = -i_{R_j} d\eta_i|_H$.

This property does not depend on the choice of metric g inside the conformal class.

We can now give the statements of the main results.

Theorem 1.1. *Let H be a real analytic quaternionic contact structure on a manifold N^7. Then H is the conformal infinity of an AHQK metric g defined on a neighbourhood of N and admitting a real analytic extension on the boundary with pole of order 2 iff H is integrable. Moreover, the germ of g along N is uniquely determined by H.*

Using [**Biq02**] and this theorem, we can fill in the 8-ball by globally defined complete AHQK metrics whose boundaries are close to the canonical quaternionic contact structure H^{can}:

COROLLARY 1.1. *Let H be an integrable quaternionic contact structure on $\mathbb{S}^7$, close to the canonical distribution H^{can}. Then H is the conformal infinity of a complete AHQK metric on the ball B^8.*

The paper is organized as follows. In section 2, we construct a connection associated to each compatible metric. A part T^W of its torsion gives a conformal invariant named vertical torsion. The vanishing of T^W is equivalent to the integrability of H.

In the third section, we study the boundaries of AHQK manifolds and we show that they are integrable. This gives the motivation to study more carefully the torsion and the curvature of this case. In particular, the curvature on H looks like that of anti-selfdual Riemannian 4-manifolds except for an additional term coming from the Bianchi identity. The computation is done in section 4.

Still assuming the integrability condition, we construct an integrable CR-manifold, the twistor space of the quaternionic contact structure. This is done in section 5 and gives the converse statement to the third section, namely that a quaternionic contact structure with vanishing vertical torsion is the boundary of a unique AHQK manifold of dimension 8.

2. Construction of the connection

In the following, one has a smooth manifold N of dimension 7, a quaternionic contact structure H on N and g a fixed compatible metric g on H. We fixe local contact forms (η_1, η_2, η_3) and a local quaternionic structure (I_i) on H such that $d\eta_i(\cdot, \cdot) = g(I_i \cdot, \cdot)$ on H.

In the first three parts of this section, we construct an adapted connection associated to g. This connection will be used in the twistorial construction of section 5. To look at the conformal invariance of this twistorial construction, we will need to know how a conformal change of metric changes the connection. This is done in part 5 of this section.

2.1. Partial connection. If N is a manifold, E a vector bundle and D a distribution on N, a D-connection on E is a differential operator

$$\nabla : \Gamma(E) \to \Gamma(D^* \otimes E),$$

satisfying the Leibniz rule $\nabla(f\, s) = (df)|_D \otimes s + f \nabla s$ for every function f and section s of E.

LEMMA 2.1. *Assume that W is a distribution on N giving a splitting $TN = H \oplus W$. There exists a unique H-connection ∇ on H preserving the metric g and such that the torsion satisfies*

$$\forall X, Y \in H,\ (T_{X,Y})_H = 0\,,$$

where the subscript H indicates the projection on H in the direction of W.

PROOF. If ∇ is such a connection, we must have for every sections X, Y and Z of H the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X.g(Y,Z) + Y.g(Z,X) - Z.g(X,Y) \\ &\quad + g([X,Y]_H, Z) - g([X,Z]_H, Y) - g([Y,Z]_H, X)\,. \end{aligned}$$

It gives both uniqueness and existence. □

Otherwise stated, the vector fields X, Y, Z are sections of H, and given a complementary W, a vector field R is a section of W, and (R_1, R_2, R_3) is the dual basis of $(\eta_1|_W, \eta_2|_W, \eta_3|_W)$. We equip W with the metric $\sum_i \eta_i^2$.

REMARK 2.1. If W is a complement to H, the torsion of the H-connection associated to W on H satisfies

$$T_{X,Y} = -[X,Y]_W = \sum_{i=1}^{3} d\eta_i(X,Y) R_i\,.$$

2.2. Extension of the connection.

LEMMA 2.2. *Let W be a complement of H in TN. One can find a unique connection ∇^W on N such that :*

(i) *∇^W preserves the splitting $TN = H \oplus W$ and the metrics on H and W,*
(ii) *if X, $Y \in H$ and R, $R' \in W$, then $(T_{X,Y})_H = 0$ and $(T_{R,R'})_W = 0$,*
(iii) *the torsion T satisfies*

$$\forall X \in H\ ,\ T^W_X := (R \mapsto (T_{X,R})_W) \in \mathfrak{so}(W)^\perp\,, \tag{1}$$

$$\forall R \in W\ ,\ T^H_R := (X \mapsto (T_{R,X})_H) \in \mathfrak{so}(H)^\perp\,, \tag{2}$$

PROOF. Let ∇ be the partial connection on H defined by lemma 2.1. We extend it to a true connection which preserves the metric on H, still denoted by ∇. If $a \in \Gamma(W^* \otimes \mathfrak{so}(H))$, the connection $\nabla' = \nabla + a$ is metric and its torsion T' satisfies

$$T'_{R,X} = \nabla'_R X - [R,X]_H = T_{R,X} + a_R(X)\,,$$

so that there exists a unique a_R which annihilates the $\mathfrak{so}(H)$-part of $T_{R,\cdot}$. The connection on W is constructed in the same way. □

We put $\alpha_{ij}(X) = d\eta_j(R_i, X)$. One has

$$T^W_X(R_i) = \nabla^W_X(R_i) - [X, R_i]_W = \nabla^W_X R_i - \sum_{j=1}^{3} \alpha_{ij}(X) R_j\,,$$

from which we obtain

$$\nabla^W_X R_i = -\frac{1}{2}\sum_{j=1}^{3}(\alpha_{ji}(X) - \alpha_{ij}(X))R_j$$

and

$$T^W_X(R_i) = -\frac{1}{2}\sum_{j=1}^{3}(\alpha_{ji}(X) + \alpha_{ij}(X))R_j\,. \tag{3}$$

2.3. Reducing torsion. We search now a particular choice of W giving the simplest torsion. To fix the notations, we recall some basic facts about representations of $SO(4)$.

The universal covering of $SO(4)$ is $Spin(4) = Sp(1) \times Sp(1)$ where $Sp(1)$ is the group of unitary quaternions. Let S_+ and S_- be the representations of the first and the second factor respectively on $\mathbb{H} \simeq \mathbb{C}^2$. The irreducible representations of $Spin(4)$ are the $S^m_+ \otimes S^n_-$ where S^m_+ and S^n_- are the symmetric power of order m and n of S_+ and S_- respectively. The following Clebsch-Gordan formula gives the irreducible decomposition of tensorial products :

$$S^n_+ \otimes S^p_+ \simeq S^{n+p}_+ \oplus S^{n+p-2}_+ \oplus \cdots \oplus S^{n-p}_+\,,\ p \leq n\,.$$

The real irreducible representations of $SO(4)$ are the real parts of $S^n_+ \otimes S^m_-$ with $n+m$ even. We will denote them by $S^{n,m}$. In particular, we have

$$\mathbb{R}^4 \simeq S^{1,1}\,,\ \Lambda^2_+ \simeq S^{2,0}\,,\ \Lambda^2_- \simeq S^{0,2}\,.$$

We now give the explicit isomorphism $\mathbb{R}^4 \otimes Sym^2(\Lambda^2_+) \simeq S^{5,1} \oplus S^{3,1} \oplus S^{1,1}$. Let (I_1, I_2, I_3) be a quaternionic structure on $\mathbb{R}^4$ given a $SO(3)$-trivialization of Λ^2_+. Then

$$\begin{array}{rcl} S^{5,1} & \simeq & \{\sum_{i,j} a_{ij} \otimes I_i \otimes I_j\,,\ a_{ij} = a_{ji} \in \mathbb{R}^4\ \textit{and}\ \forall j\,,\ \sum_i I_i a_{ij} = 0\}\,, \\ S^{1,1} & \simeq & \{\sum_i r \otimes I_i \otimes I_i\,,\ r \in \mathbb{R}^4\}\,, \\ S^{3,1} & \simeq & \{\sum_{i,j}(I_i r_j + I_j r_i) \otimes I_i \otimes I_j\,,\ r_i \in \mathbb{R}^4\,,\ \sum_i I_i r_i = 0\}\,. \end{array}$$

In our case we have the natural identification

$$W \simeq \Lambda^2_+ H^*\,,\ R_i \mapsto d\eta_i|_H$$

so that T^W becomes a section of $H^* \otimes End(\Lambda^2_+ H^*)$. We put $w_i = d\eta_i|_H$ and w_i^* the dual basis.

REMARK 2.2. The metric g allows us to identify H^* and H and we use it throughout the text. In particular $\Lambda^2_+ H^*$ can be considered as a subspace of the space of 2-forms or that of skew-symmetric endomorphisms.

PROPOSITION 2.1. *For each choice of compatible metric g on H, there is a unique complement W^g of H such that $T^{W^g} \in \Gamma(S^{5,1})$.*

PROOF. Let W be transverse to H and (R_1, R_2, R_3) be the dual basis of (η_1, η_2, η_3) on W. We have obtained in (3)

$$T^W = -\frac{1}{2} \sum_{j=1}^{3} (\alpha_{ij} + \alpha_{ji}) \otimes w_i^* \otimes w_j \,,$$

If W' is another complementary to H spanned by the vectors $R'_i = R_i + r_i$ with $r_i \in H$, then $\alpha'_{ij} = i_{R'_i} d\eta_j|_H = \alpha_{ij} + (I_j r_i)^\flat$ ($\flat$ and $\sharp$ are the usual musical isomorphisms). With the explicit decomposition of $H^* \otimes Sym^2(\Lambda^2_+ H)$ we wrote down, the existence and the uniqueness of W follow. □

REMARK 2.3. Another choice of complementary does not change the $S^{5,1}$ part of the torsion.

2.4. Derivation of the quaternionic structure. We fix $W = W^g$ and note ∇ the corresponding connection. This connection is metric and so preserves the bundle $\Lambda^2_+ H^*$:

$$\nabla I_j|_H = \sum_{i=1}^{3} \gamma_{ij} \otimes I_i \,.$$

Here we just look at the derivation in the direction of H, i.e. $\gamma_{ij} \in H^*$.

Let $X, Y, Z \in \Gamma(H)$, and $\mathfrak{a}$ be the skew-symmetrisation in X Y and Z. One has the identity

$$d(d\eta_j)(X,Y,Z) = \mathfrak{a}(\nabla d\eta_j)(X,Y,Z) \tag{4}$$

$$+ d\eta_j(T_{X,Y}, Z) + d\eta_j(T_{Z,X}, Y) + d\eta_j(T_{Y,Z}, X) \,,$$

and can be rewritten in the following form :

$$\sum_i (g(\gamma_{ij}^\sharp + \alpha_{ij}^\sharp, X) I_i + I_i X \wedge (\gamma_{ij}^\sharp + \alpha_{ij}^\sharp)) = 0 \,.$$

Projecting on $\Lambda^2_+ H$ and $\Lambda^2_- H$ gives the equivalent condition

$$\forall j \in \{1,2,3\} \,, \; \sum_{i=1}^{3} (\alpha_{ij} + \gamma_{ij}) \circ I_i = 0 \,.$$

But our particular choice of complementary vector bundle ensures that

$$\sum_i (\alpha_{ij} + \alpha_{ji}) \circ I_i = 0$$

hence we get

$$\gamma_{ij} = -\frac{1}{2}(\alpha_{ij} - \alpha_{ji}) .$$

2.5. Conformal change. Let $\eta' = f^2\eta$ be such a conformal change, and (R_1, R_2, R_3) be the dual basis of (η_1, η_2, η_3) on W^g. We put (R'_1, R'_2, R'_3) the dual basis of $(f^2\eta_1, f^2\eta_2, f^2\eta_3)$ on W^{f^2g}.

PROPOSITION 2.2. *The conformal change of metric corresponds to the following change of basis of associated complementaries :*

$$R'_i = f^{-2}(R_i + r_i) ,$$

where $r_i^\flat = 2f^{-1}df|_H \circ I_i$ *(the musical isomorphisms* $\sharp$ *and* $\flat$ *are taken with respect to* g *on* H*, after restriction if necessary for 1-forms). Moreover, we get*

$$i_{R'_i} d\eta'_j|_H + i_{R'_j} d\eta'_i|_H = i_{R_i} d\eta_j|_H + i_{R_j} d\eta_i|_H .$$

PROOF. We put $\alpha'_{ij} = i_{R'_i} d\eta'_j|_H$. We have

$$\eta'_i(R'_j) = f^2\eta_i(R'_j) = \delta_{ij}$$

so that $R'_i = f^{-2}(R_i + r_i)$ with $r_i \in H$ and finally

$$\alpha'_{ij} + \alpha'_{ji} = \alpha_{ij} + \alpha_{ji} - (r_i^\flat \circ I_j + r_j^\flat \circ I_i) - 4\delta_{ij} f^{-1} df|_H . \tag{5}$$

The conformal change left $S^{5,1}$, $S^{3,1}$ and $S^{1,1}$ globally invariant and $(r_i^\flat \circ I_j + r_j^\flat \circ I_i) + 4\delta_{ij} f^{-1} df|_H \in S^{3,1} \oplus S^{1,1}$ therefore the conditions $\alpha'_{ij} + \alpha'_{ji} \in S^{5,1}$ and $\alpha_{ij} + \alpha_{ji} \in S^{5,1}$ imply $(r_i^\flat \circ I_j + r_j^\flat \circ I_i) + 4\delta_{ij} f^{-1} df|_H = 0$ and the lemma follows. □

COROLLARY 2.1. *The torsion* T^{W^g} *associated to the Carnot-Carathéodory metric is conformally invariant. We call it the vertical torsion and denote it by* T^{W^g} *or* T^W*.*

PROOF. If we change the metric in the conformal class, the 2-forms w_i are multiplied by the conformal factor and elements of the dual basis are multiplied by its inverse. So the only thing we must look at is the invariance of $(\alpha_{ij} + \alpha_{ji})_{i,j}$ which follows from 2.2.

□

Let us summarize the results we have obtained in the following proposition.

PROPOSITION 2.3. *Let (N, H) be a quaternionic contact structure. The integrability of H does not depend on the choice of an adapted metric on H. Moreover, if g is particular a choice of compatible metric on H, the following conditions are equivalent :*

- *The distribution H is integrable.*
- *The torsion T^{W^g} vanishes.*
- *For any choice of complementary distribution W, the $S^{5,1}$ part of the torsion vanishes.*
- *For any choice of oriented orthonormal basis $(\frac{1}{\sqrt{2}}d\eta_i|_H)$ of H^+ and any choice of vector fields (R_1, R_2, R_3) such that*

$$\eta_j(R_i) = \delta_{ij} ,$$

the $S^{5,1}$ part of $(i_{R_i}d\eta_j|_H + i_{R_j}d\eta_i|_H)_{i,j}$ vanishes.

In the study of the twistor space, we will need to know how the connection is changed when the metric is multiplied by a conformal factor. We put $\theta = f^{-1}df$. Recall that we write $\theta^\sharp$ for $(\theta|_H)^\sharp$ and that the change of complementary distribution is parametrized by $R'_i = f^{-2}(R_i - 2I_i\theta^\sharp)$. The following lemma will be useful in the twistorial construction.

LEMMA 2.3. *The connection ∇' adapted to f^2g is given by*

$$\begin{array}{rcl} \nabla'_X & = & \nabla_X + \theta(X) + \theta^\sharp \wedge X + \sum_i I_i\theta^\sharp \wedge I_iX + \sum_i \langle I_i\theta^\sharp, X\rangle I_i \\ \nabla'_{R_i} & = & \nabla_{R_i} + \theta(R_i) + 2|\theta^\sharp|^2 I_i + 2\theta^\sharp \wedge I_i\theta^\sharp - \frac{1}{2}\sum_j(\alpha^\sharp_{ij} + \alpha^\sharp_{ji}) \wedge I_j\theta^\sharp \\ & & +2(I_i\nabla\theta^\sharp)^{\mathfrak{so}(H)} \end{array}$$

where $(I_i\nabla\theta^\sharp)^{\mathfrak{so}(H)}$ means that we take the $\mathfrak{so}(H)$ part of the endomorphism $X \mapsto I_i\nabla_X\theta^\sharp$.

PROOF. We put $\nabla' = \nabla + \theta + a$ and $\nabla^1 = \nabla + \theta$. The connection ∇^1 preserves f^2g and its torsion is

$$\begin{array}{rcl} T^1_{X,Y} & = & \sum_i d\eta_i(X,Y)R_i + \theta(X)Y - \theta(Y)X \\ & = & T'_{X,Y} - \sum_i d\eta_i(X,Y,)r_i + \theta(X)Y - \theta(Y)X \end{array}$$

so that $a_XY - a_YX = \sum_i d\eta_i(X,Y)r_i - \theta(X)Y + \theta(Y)X$. The connections ∇' and ∇^1 both preserve f^2g hence a is a 1-form with values in $\mathfrak{so}(H)$. The skew-symmetrisation in the two first variables gives an isomorphism $H^* \otimes \mathfrak{so}(H) \to \Lambda^2H^* \otimes H$, with inverse b

$$\langle b(c)_XY, Z\rangle = \frac{1}{2}\left(\langle c(X,Y), Z\rangle + \langle c(Z,X), Y\rangle - \langle c(Y,Z), X\rangle\right)$$

from which we deduce the first part of the lemma.

We now look at the change of the connection in the direction of W^g. If $U \in TS$, $U_{V/W}$ is its projection on V in the direction of $W = W^g$. We have

$$\begin{aligned} a_{R_i} X &= \nabla'_{R_i} X - \nabla_{R_i} X - \theta(R_i) X \\ &= \nabla'_{R_i + r_i} X - \nabla_{R_i} X - \theta(R_i) X - \nabla'_{r_i} X \end{aligned}$$

Introducing the torsion, we obtain

$$\begin{aligned} a_{R_i} X &= (T_{R_i + r_i, X})_{V/W'} - (T_{R_i, X})_{V/W} + [R_i, X]_{V/W'} - [R_i, X]_{V/W} \\ &\quad -\theta(R_i) X - \nabla'_{r_i} X + [r_i, X]_{V/W'} \\ &= (T_{R_i + r_i, X})_{V/W'} - (T_{R_i, X})_{V/W} + \textstyle\sum_j d\eta_j(R_i, X) r_j \\ &\quad -\theta(R_i) X - \nabla'_X r_i \end{aligned}$$

But $a_{R_i} \in \mathfrak{so}(H)$, so that it suffices to compute the skew-symmetric part of the right hand term in the previous equality. The contributions of the torsions vanish by definition, that of $\sum_j d\eta_j(R_i, X) r_j$ is

$$\frac{1}{2} \sum_j \alpha_{ij}^\sharp \wedge r_j = - \sum_j \alpha_{ij}^\sharp \wedge I_j \theta^\sharp \,,$$

and that of $\nabla' r_i$ is

$$-2\theta^\sharp \wedge I_i \theta^\sharp - 2|\theta^\sharp|^2 I_i + (\nabla r_i)^{\mathfrak{so}(H)} \,.$$

Using the expression of ∇I_i obtained in 2.4, we get

$$\begin{aligned} \nabla r_i &= -2\nabla(I_i \theta^\sharp) = -2(\nabla I_i)\theta^\sharp - 2 I_i \nabla \theta^\sharp \\ &= \textstyle\sum_j (\alpha_{ji} - \alpha_{ij}) \otimes I_j \theta^\sharp - 2 I_i \nabla \theta^\sharp \,. \end{aligned}$$

Mixing all this together gives the lemma. □

2.6. Higher dimensional case. Let us do some remarks about what is going on in higher dimensions. Let H be a quaternionic contact structure on a manifold N^{4n+3} with $n > 1$ and g be a compatible metric on H. In the same way and always with the same notations, one can show that there exists a unique complementary W^g such that

$$\sum_i (\alpha_{ij} + \alpha_{ji}) \circ I_i = 0$$

for all j.

On the other hand, lemma 2.1 is always true and give a metric H-connection ∇ on H. Then, using (4) and an argument of representation theory, one can show that in fact $\alpha_{ij} + \alpha_{ji} = 0$ and that ∇ preserves not only the metric but also the $Sp(n)Sp(1)$ structure on H. Hence, there is no integrability condition. It is the reason why all quaternionic contact structures in dimension strictly greater than 7 are the boundaries of AHQK metrics.

3. Conformal infinity of AHQK manifolds

In this section, we will study the conformal infinity of an AQH quaternion-Kähler manifold. We find a particular trivialization of the quaternionic structure admitting an analytic extension to the boundary with pole of order 2. Then, we use it to show that the quaternionic contact structure on the boundary is integrable.

3.1. Twistor space and asymptotic development. The following is essentially the work of [**Biq00**, III.2] and [**LeB91**]. Let (M, g) be an AHQK manifold of dimension 8 and suppose that the metric g admits an analytic extension to the boundary N. We will apply the twistor machinery to obtain a particular choice of local trivialization of the quaternionic structure in a neighbourhood of the boundary. The twistor space [**Sal82**] of M is a 5-dimensional holomorphic manifold with the following data :

- a holomorphic contact structure η with values in a line bundle L;
- a family of dimension 8 of compact genus zero curves $(\mathcal{C}_m)_{m\in M^{\mathbb{C}}}$ with normal bundle $\mathcal{O}(1)\otimes\mathbb{C}^4$;
- an hypersurface $N^{\mathbb{C}}\subset M^{\mathbb{C}}$ of curves tangent to the contact distribution;
- a compatible real structure τ, without fixed points.

Remark 3.1. M is the real slice of $M^{\mathbb{C}}-N^{\mathbb{C}}$ and N that of $N^{\mathbb{C}}$.

On each $\mathcal{C}_m$, the line bundle L is isomorphic to $\mathcal{O}(2)$ so that $\mathcal{L}_m = H^0(\mathcal{C}_m, Hom(T\mathcal{C}_m, L))$ is a line bundle on $M^{\mathbb{C}}$. By restriction, the 1-form η gives a section Θ of $\mathcal{L}$ and $S^{\mathbb{C}}$ is the null set of Θ. We choose local square root $L^{1/2}$ of L, but the conclusions do not depend on this choice. Let us define

$$\begin{aligned} E_m &= H^0(\mathcal{C}_m, L^{-1/2}\otimes N_m)\,,\\ H_m &= H^0(\mathcal{C}_m, L^{1/2})\,,\end{aligned}$$

so that

$$T_m M^{\mathbb{C}} = E_m\otimes H_m\,.$$

For $m\notin N^{\mathbb{C}}$ and $u, v\in H_m$, the Wronskian $w(u\wedge v) = udv - vdu$ defines a two form

$$w_H : \Lambda^2 H^0(\mathcal{C}_m, L^{1/2}) \xrightarrow{w} \mathcal{L}_m \xrightarrow{\Theta^{-1}} \mathbb{C}\,, \tag{6}$$

and therefore a $SO_3(\mathbb{C})$-structure $w_H\otimes w_H$ on $H^0(\mathcal{C}_m, L)\simeq Sym^2(H_m)$.

The normal bundle N_m of a curve $\mathcal{C}_m$ has a natural identification with $\ker\eta$ if $m \notin N^{\mathbb{C}}$ so that we have a well defined 2-form

$$\Lambda^2 H^0(\mathcal{C}_m, N_m) \xrightarrow{d\eta} Sym^2(H_m)\,.$$

The choice of a $SO_3(\mathbb{C})$-trivialization on $Sym^2(H_m)$ exhibits three 2-forms w_1, w_2, w_3 giving the $Sp_2(\mathbb{C})Sp_1(\mathbb{C})$ structure. The complexified quaternionic-Kähler metric is

$$g = w_E \otimes w_H \text{ on } E_m \otimes H_m \tag{7}$$

where

$$w_E : E_m \xrightarrow{d\eta} \mathbb{C}\,.$$

We now look at the contact structure on the boundary. Let $l : \mathcal{L} \to \mathbb{C}$ be a local choice of trivialization of $\mathcal{L}$ in a neighbourhood of $s \in N^{\mathbb{C}}$ and extend it on $M^{\mathbb{C}}$. In the same way, we obtain a symplectic form

$$\hat{w}_H : \Lambda^2 H^0(\mathcal{C}_m, L^{1/2}) \xrightarrow{w} \mathcal{L}_m \xrightarrow{l} \mathbb{C}\,,$$

and thus a $SO_3(\mathbb{C})$-metric $\hat{w}_H \otimes \hat{w}_H = l^2\Theta^2 w_H \otimes w_H$. We choose a local $SO_3(\mathbb{C})$-trivialization $Sym^2(H_m) \to \mathbb{C}^3$.

If $s \in N^{\mathbb{C}}$, one has $T\mathcal{C}_s \subset \ker\eta$ hence η gives three 1-forms (η_1, η_2, η_3) along $N^{\mathbb{C}}$

$$H^0(\mathcal{C}_s, N_s) \xrightarrow{\eta} H^0(\mathcal{C}_s, L) \simeq Sym^2(H_s) \to \mathbb{C}^3\,.$$

On the other hand, on $M^{\mathbb{C}} - N^{\mathbb{C}}$ we obtain three 2-forms

$$\wedge^2 H^0(\mathcal{C}_m, N_m) \xrightarrow{d\eta} Sym^2(H_m) \to \mathbb{C}^3\,,$$

which can be written as $l^2\Theta^2 w_i$ with w_i defining the quaternionic structure of $M^{\mathbb{C}} - N^{\mathbb{C}}$.

We put $\rho = l\Theta : M^{\mathbb{C}} \to \mathbb{C}$.

LEMMA 3.1. *The forms w_i have pole of order 2 along $N^{\mathbb{C}}$. More precisely, the 2-forms $l^2\Theta^2 w_i$ are defined on $N^{\mathbb{C}}$ and satisfy*

$$l^2\Theta^2 w_i = -d\rho \wedge \eta_i + \frac{1}{2}\sum_{r,s} \varepsilon^{rsi}\eta_r \wedge \eta_s$$

on $N^{\mathbb{C}}$ where ε^{rsi} is the signature of the permutation (r, s, i) of $(1, 2, 3)$.

PROOF. Because the w_i define a quaternionic structure, we need only show that $i_{\partial/\partial\rho} l^2\Theta^2 w_i = -\eta_i$ to obtain the lemma. We take $s \in N^{\mathbb{C}}$.

There exists a section ϕ of N_s along $\mathcal{C}_s$ such that $\eta(\phi) = 0$ and $i_\phi d\eta|_{T\mathcal{C}_s} \neq 0$, cf [**Biq00**, lemma III.2.5]. We normalize ϕ in order to have $l(i_\phi d\eta|_{T\mathcal{C}_s}) = 1$. It is a vector in $T_s M^{\mathbb{C}}$ with the properties $d\rho(\phi) = 1$ and $\eta_i(\phi) = 0$. Remark that whereas the symplectic form $d\eta$ is not defined along $N^{\mathbb{C}}$, the 3-form $\eta \wedge d\eta$ admits an extension to $N^{\mathbb{C}}$. By restriction, we have

$$\Theta d\eta = \eta \wedge d\eta \in H^0(\mathcal{C}_m, T^*\mathcal{C}_m \otimes \Lambda^2 N_m^* \otimes L^2) = \mathcal{L}_m \otimes \Lambda^2 T^* M^{\mathbb{C}} \otimes H^0(\mathcal{C}_m, L)\,.$$

If u is tangent to $\mathcal{C}_s$ and $\sigma \in H^0(\mathcal{C}_s, N_s)$, then

$$\eta \wedge d\eta(u, \phi, \sigma) = \eta(\sigma) d\eta(u, \phi)\,,$$

i.e. $i_\phi l \Theta d\eta = -\eta$ and finally

$$i_\phi l^2 \Theta^2 w_i = -\eta_i\,.$$

□

The intersection of the kernels of $\rho^2 w_1$, $\rho^2 w_2$, and $\rho^2 w_3$ on $N^{\mathbb{C}}$ is

$$H^{\mathbb{C}} = H^0(\mathcal{C}_s, N_s \cap \ker \eta \cap T\mathcal{C}_s^{\perp d\eta})$$

and coincides with the contact structure of the boundary. The symplectic form w_i has well defined terms of order -1 on $H^{\mathbb{C}}$ and one can show [**Biq00**, Lemma III.2.6] that

$$w_i = \frac{1}{\rho^2} w_{i,-2} + \frac{1}{\rho} w_{i,-1} + \cdots\,,$$

with

$$w_{i,-2} = -d\rho \wedge \eta_i + \frac{1}{2} \sum_{r,s} \varepsilon^{rsi} \eta_r \wedge \eta_s\,,\ w_{i,-1}|_{H^{\mathbb{C}}} = d\eta_i|_{H^{\mathbb{C}}}\,.$$

If we put

$$\hat{E}_m = H^0(\mathcal{C}_s, (N_s \cap \ker \eta \cap T\mathcal{C}_s^{\perp d\eta}) \otimes L^{-1/2})\,,$$

we obtain by restriction a complex metric on $H^{\mathbb{C}}$

$$g_{H^{\mathbb{C}}} = d\eta|_{E_m} \otimes \hat{w}_H\,.$$

The quaternionic metric on $M^{\mathbb{C}}$ has the asymptotic development

$$g = \frac{1}{\rho^2} g_{-2} + \frac{1}{\rho} g_{-1} + \cdots$$

with

$$g_{-2} = d\rho^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 \text{ and } g_{-1}|_{H^{\mathbb{C}}} = g_{H^{\mathbb{C}}}\,.$$

Finally, we put $w_{i,-1} = d\eta_i + \gamma_i$ where $\gamma_i|_{H^{\mathbb{C}}} = 0$.

3.2. Boundary conditions. We follow the notations of the previous section and restrict ourselves to the real slice. We choose an arbitrary complementary W to H. Let (R_1, R_2, R_3) be the dual basis of (η_1, η_2, η_3) on W and let $\tilde{I}_i$ be the almost complex structures on H.

The symplectic forms w_i and the metric define almost complex structures I_i. Because of the form of the w_i, we have the analytic development

$$I_i\partial_\rho = I_{i,0}\partial_\rho + \rho I_{i,1}\partial_\rho + \cdots = R_i + \psi_i + \cdots$$

where $\psi_i \in H$ is independent of ρ and if $X \in H$,

$$I_i X = I_{i,0}X + \rho I_{i,1}X + \cdots = \tilde{I}_i X + \cdots$$

We are now in position to show the following

PROPOSITION 3.1. *The boundary of an AHQK manifold admitting is an integrable quaternionic contact structure.*

PROOF. If $X \in H$, one has

$$w_i(I_j\partial_\rho, X) + w_j(I_i\partial_\rho, X) = -2\delta_{ij}g(\partial_\rho, X)\,.$$

The order -2 terms do not give anything but from the order -1 terms we deduce the equation

$$\begin{aligned} d\eta_i(R_j, X) + d\eta_j(R_i, X) &= -\gamma_i(R_j, X) - \gamma_j(R_i, X) - 2\delta_{ij}g_{-1}(\partial_\rho, X) \\ &\quad + g_{-1}(\psi_j, \tilde{I}_i X) + g_{-1}(\psi_i, \tilde{I}_j X)\,. \end{aligned}$$

The second line gives an element in $S^{3,1} \oplus S^{1,1}$ therefore we need only to look at γ_i. We will now use the fact that the metric is quaternionic-Kähler. Indeed, there exists one forms β_{ij} such that the 2-forms (w_i) satisfy

$$dw_i = \sum_j \beta_{ji} \wedge w_j\,, \quad \beta_{ji} = -\beta_{ij}\,.$$

The application $(\Lambda^1)^3 \to \Lambda^3$

$$(a_i)_{i=1,2,3} \mapsto \sum_i a_i \wedge w_i$$

is an injection so that the β_{ij} are unique.

We have

$$\begin{aligned} dw_i &= -\tfrac{1}{\rho^3}\textstyle\sum_{r,s}\varepsilon^{rsi} d\rho \wedge \eta_r \wedge \eta_s - \tfrac{1}{\rho^2} d\rho \wedge (d\eta_i + \gamma_i) \\ &\quad + \tfrac{1}{\rho^2}\Big(d\rho \wedge d\eta_i + \tfrac{1}{2}\textstyle\sum_{r,s}\varepsilon^{rsi}(d\eta_r \wedge \eta_s - \eta_r \wedge d\eta_s)\Big) + \cdots \end{aligned}$$

and then

$$dw_i = -\frac{1}{\rho^3}\sum_{r,s}\varepsilon^{rsi} d\rho\wedge\eta_r\wedge\eta_s + \frac{1}{\rho^2}\left(\sum_{r,s}\varepsilon^{rsi} d\eta_r\wedge\eta_s - d\rho\wedge\gamma_i\right) + \cdots .$$

We have $\sum_{r,s,p,q}\varepsilon^{irs}\varepsilon^{pqr}\eta_s\wedge\eta_p\wedge\eta_q = 0$ so

$$dw_i = \sum_r\left(\frac{1}{\rho}\sum_s\varepsilon^{irs}\eta_s\right)\wedge\frac{w_{r,-2}}{\rho^2} + \frac{1}{\rho^2}\left(\sum_{r,s}\varepsilon^{rsi} d\eta_r\wedge\eta_s - d\rho\wedge\gamma_i\right) + \cdots$$

The exterior product of 1-forms with $w_{r,-2}$ is an injection, so β_{ri} is of the form

$$\beta_{ri} = \frac{1}{\rho}\beta_{ri,-1} + \beta_{ri,0} + \cdots \text{ and } \beta_{ri,-1} = \sum_s\varepsilon^{irs}\eta_s .$$

Looking at the order -2 terms with respect to ρ, one obtains the equations

$$\sum_r\beta_{ri,0}\wedge w_{r,-2} + \sum_r\beta_{ri,-1}\wedge w_{r,-1} = \sum_{r,s}\varepsilon^{rsi} d\eta_r\wedge\eta_s - d\rho\wedge\gamma_i .$$

We put $\beta_{ri,0} = \lambda_{ri} d\rho + \beta^N_{ri,0}$ and $\gamma_r = d\rho\wedge\gamma^\rho_r + \gamma^N_r$ where $\beta^N_{ri,0}\in T^*N$ and $\gamma^N_r\in\Lambda^2T^*N$.

Taking the $d\rho$ component in the previous equation, one gets

$$\gamma^N_i = \sum_r\eta_r\wedge\beta^N_{ri,0} - \frac{1}{2}\sum_{r,k,s}\lambda_{ri}\varepsilon^{ksi}\eta_k\wedge\eta_s + \sum_{r,s}\varepsilon^{irs}\eta_s\wedge\gamma^\rho_r .$$

But then $\gamma_i(R_j,X) + \gamma_j(R_i,X) = 0$ and the lemma follows. □

In the two next sections, we will look at integrable quaternionic contact structures in order to show that they are the boundaries of AHQK metrics.

4. Integrable quaternionic contact structures

Let (N,H) be a quaternionic contact structure.

In section 2, we computed the derivation of the quaternionic structure in the direction of H. On the other hand, from the identity $d(d\eta_j)(R_i,X,Y) = 0$, we obtain

$$(8)\qquad (\nabla_{R_i} d\eta_j)(X,Y) = \mathfrak{a}(\nabla\alpha_{ij})(X,Y) - \sum_k\alpha_{ik}\wedge\alpha_{kj}(X,Y) + \sum_k d\eta_j(R_i,R_k)d\eta_k(X,Y) - g(I_jT_{R_i,X} + T_{R_i,I_jX},Y)$$

where $\mathfrak{a}(\nabla\alpha_{ij})(X,Y) = (\nabla_X\alpha_{ij})(Y) - (\nabla_Y\alpha_{ij})(X)$.

From now on, we suppose that the quaternionic structure is integrable. We choose a compatible metric g on H and $W = W^g$ the associated complementary vector bundle defining the adapted connection ∇.

4.1. Torsion. The computations of section 2.4 give for any $X \in H$,

$$\nabla_X I_j = -\sum_{i=1}^{3} \alpha_{ij}(X) I_i \,. \tag{9}$$

LEMMA 4.1. *Let (M, H) be an integrable quaternionic contact structure. The tensor T^H defined in lemma 2.2 lives in the component $S^{2,2}$ of $W^* \otimes \mathfrak{so}(H)^\perp$.*

PROOF. By construction, T^H is a section of

$$\Lambda^2_+ H \otimes \mathfrak{so}(H)^\perp = S^{2,0} \oplus S^{4,2} \oplus S^{2,2} \oplus S^{0,2} \,,$$

so we can put

$$T_{R_i} = \lambda_i Id + \sum_p I_p A_{pi}$$

with $A_{pi} \in \Gamma(\Lambda^2_- H)$ (seen as skew-symmetric endomorphisms). We apply (8) with $i = j$ and obtain $\lambda_i = 0$ and $A_{ii} = 0$. Applying one more time (8), we see that A_{pi} is equal to the Λ^2_- part of $\mathfrak{a}(\alpha_{pi}) - \sum_k \alpha_{pk} \wedge \alpha_{ik}$ which is skew-symmetric in p and i.

Writing $A_i = \frac{1}{2} \sum_{r,s} \varepsilon^{rsi} A_{rs}$, we obtain T^H as the image of $\sum_i I_i \otimes A_i \in \Lambda^2_+ H \otimes \Lambda^2_- H$ by the $SO(4)$-equivariant map $I_i \otimes B \mapsto \frac{1}{2} \sum_j \eta_j \otimes [I_i, I_j] B$. $\square$

We are now able to calculate more precisely the vertical derivatives of the quaternionic structure.

LEMMA 4.2. *There exists a function λ on N such that*

$$\nabla_{R_i} I_j = \frac{1}{2} \sum_{k=1}^{3} \left(d\eta_j(R_i, R_k) + d\eta_i(R_j, R_k) - d\eta_k(R_i, R_j) \right) I_k + \lambda [I_i, I_j] \,.$$

PROOF. Symmetrizing (8) gives

$$\nabla_{R_i} I_j + \nabla_{R_j} I_i = \sum_{k=1}^{3} (d\eta_j(R_i, R_k) + d\eta_i(R_j, R_k)) I_k \,, \tag{10}$$

In particular,

$$\langle \nabla_{R_j} I_j, I_i \rangle = -\langle \nabla_{R_j} I_i, I_j \rangle = -2 d\eta_j(R_i, R_j) \,,$$

so that we know $\nabla_{R_j} I_i$ except for its component on $[I_i, I_j]$. We can put

$$\nabla_{R_i} I_j = \frac{1}{2} \sum_{k=1}^{3} \left(d\eta_j(R_i, R_k) + d\eta_i(R_j, R_k) - d\eta_k(R_i, R_j) \right) I_k + \lambda_{ij} [I_i, I_j] \,,$$

with $\lambda_{ii} = 0$. From (10), we have $\lambda_{ij} = \lambda_{ji}$. Moreover, taking for instance $i = 1$, $j = 2$ and using the skew-symmetry $\langle \nabla_{R_1} I_2, I_3 \rangle = -\langle \nabla_{R_1} I_3, I_2 \rangle$, we get $\lambda_{12} = \lambda_{13}$. The other equalities are obtained in the same way. □

4.2. The curvature tensor. We will give some results about the curvature tensor in the $T^{W^g} = 0$ case. They will be useful for the twistorial construction.

We are now interested in the curvature R of ∇, and more precisely in its horizontal part. This is a section $R \in \Gamma(\Lambda^2 H^* \otimes \mathfrak{so}(H))$. The splitting $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ allows us to decompose the curvature in $\Lambda^2_+ \otimes \Lambda^2_+$, $\Lambda^2_- \otimes \Lambda^2_+$ and $\Lambda^2_- \otimes \Lambda^2_-$ parts. Looking at its action on $\Lambda^2_+ H$, we have

$$\begin{aligned} R_{X,Y} I_i &= \nabla_X \nabla_Y I_i - \nabla_Y \nabla_X I_i - \nabla_{[X,Y]_H} I_i + \textstyle\sum_j d\eta_j(X,Y) \nabla_{R_j} I_i \\ &= \textstyle\sum_j \left(-\mathfrak{a}(\nabla \alpha_{ji}) + \sum_{k=1}^3 \alpha_{jk} \wedge \alpha_{ki} \right) (X,Y) I_j \\ &\quad + \textstyle\sum_j d\eta_j(X,Y) \nabla_{R_j} I_i \,. \end{aligned}$$

Proposition 4.1. *The $\Lambda^2_+ H \otimes \Lambda^2_+ H$ part of the curvature is scalar. More precisely, if we denote it by $\mathcal{S} \in \Gamma(End(\Lambda^2_+ H))$, we obtain with the notations of lemma 4.2 :*

$$\mathcal{S} = 2\lambda Id_{\Lambda^2_+} \,.$$

Proof. Using Lemma 4.2 and (8), one sees that

$$\begin{aligned} (-\mathfrak{a}(\nabla \alpha_{ji}) + \textstyle\sum_k \alpha_{jk} \wedge \alpha_{ki})_+ &= \lambda [I_i, I_j] \\ +\tfrac{1}{2} \textstyle\sum_k \left(d\eta_i(R_j, R_k) - d\eta_j(R_i, R_k) - d\eta_k(R_i, R_j) \right) I_k & \end{aligned}$$

where the subscript + means the selfdual part. Injecting this in the curvature formula, one easily deduces the proposition. □

We can define a Ricci tensor and a scalar curvature for the partial curvature R. As usual, we put

$$\begin{aligned} Ric(X,Y) &= tr_H(Z \mapsto R_{Z,X} Y) \\ s &= tr_H(Ric) \,, \end{aligned}$$

where the subscript H means that the trace is taken only on H. We note Ric_0 the trace-free part of the Ricci tensor. In order to obtain the exact form of the curvature, we use the first Bianchi identity

$$R_{X,Y} Z + R_{Y,Z} X + R_{Z,X} Y = (d^\nabla T)_{X,Y,Z}$$

Let X, Y, Z and R_i be parallel at the point p. Since the horizontal covariant derivatives of R_i and I_i are identical,

$$(\nabla T|_H)_p = (\nabla \sum_i w_i \otimes R_i)_p = 0\,,$$

so that at p, we have

$$\begin{array}{rcl} R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y & = & -T_{[X,Y],Z} - T_{[Y,Z],X} - T_{[Z,X],Y} \\ & = & \sum_{i=1}^{3}(d\eta_i \wedge T^H_{R_i})(X,Y,Z) \end{array}$$

The image by the Bianchi map b of the curvature R lives in the component $S^{2,2} \simeq \Lambda^2_+ H \otimes \Lambda^2_- H$ of $\Lambda^3 H^* \otimes H \simeq S^{2,0} \oplus S^{0,2} \oplus S^{0,0} \oplus S^{2,2}$.

PROPOSITION 4.2. *The horizontal part $R \in \Gamma(\Lambda^2 H^* \otimes \mathfrak{so}(H))$ of the curvature tensor seen as an endomorphism of $\Lambda^2 H = \Lambda^2_+ H \oplus \Lambda^2_- H$ has matrix*

$$R = \begin{pmatrix} \frac{s}{12} Id & Ric_0 + B \\ {}^t Ric_0 - {}^t B & \frac{s}{12} Id + W^- \end{pmatrix}$$

PROOF. Recall that the kernel of b is exactly the Riemannian curvature tensors. We have

$$\begin{array}{rcl} S^2(\Lambda^2 H^*) & = & \ker b \oplus \Lambda^4 H^* \\ \Lambda^2(\Lambda^2 H^*) & = & \Lambda^2_+ H^* \oplus \Lambda^2_- H^* \oplus \Lambda^2_+ H^* \otimes \Lambda^2_- H^* \end{array}$$

We have shown that $b(R) \in S^{2,2}$ so that R is the sum of a Riemannian tensor and an element in the unique irreducible $S^{2,2}$ component which appears in $\Lambda^2(\Lambda^2 H^*) \subset End(\Lambda^2(H^*))$. Moreover, $Ric(B) = 0$ if $B \in S^{2,2} \subset \Lambda^2(\Lambda^2 H^*)$ so that the Ricci tensor behaves like the Riemannian Ricci tensor, hence is symmetric. □

We show the following lemma which will be useful in the next section.

LEMMA 4.3. *If the vertical torsion vanishes, the curvature R of the adapted connection satisfies the following equality*

$$[R_{X,Y} + IR_{X,IY} + IR_{X,IY} - R_{IX,IY}, I] = 0\,,$$

for all $X, Y \in H$ and $I \in \Lambda^2_+ H$.

PROOF. This lemma is well known in the case of anti-selfdual Riemannian curvature in dimension 4. In our case, it is similar except for the Bianchi part B of the curvature tensor, hence we need only to show that B satisfies the previous equality. We take for instance

$$B : w \in \mathfrak{so}(H) \mapsto tr(wK)J - tr(wJ)K \in End(\mathfrak{so}(H))\,,$$

where $J \in \Lambda^2_- H$ and $K \in \Lambda^2_+ H$. We must show that

$$C = [B(w) + IB(Iw) - IB(wI) + B(IwI), I] = 0\,.$$

One has $[J, K] = 0$, so that we get

$$C = [-tr(wJ)K - tr(IwJ)IK + tr(wIJ)IK - tr(IwIJ)K, I]\ .$$

The result follows then from the two equalities

$$\begin{cases} tr(IwJ) = tr(JIw) = tr(IJw) = tr(wIJ) \\ tr(IwIJ) = tr(IwJI) = -tr(wJ) \end{cases}$$

□

5. Twistor space

In this section, we will end the proof of theorem 1.1.

5.1. Definitions. Let N^7 be a smooth manifold and H be a quaternionic contact structure on N with vanishing vertical torsion. Let g be a compatible Carnot-Carathéodory metric, W the adapted complementary distribution and ∇ the connection associated to g.

Let $\mathcal{T}$ be the set of 2-forms $w \in \Lambda^2_+ H^*$ of norm $\sqrt{2}$. This is a 2-sphere bundle on M called the twistor space of (N, H). It can be identified with the set of almost complex structures compatible with g and the orientation. Let π be the projection $\mathcal{T} \to N$ and choose a local quaternionic structure (I_1, I_2, I_3) associated to the 1-forms (η_1, η_2, η_3). At a point $I = x_1 I_1 + x_2 I_2 + x_3 I_3$, we put

$$\eta^r = x_1 \pi^* \eta_1 + x_2 \pi^* \eta_2 + x_3 \pi^* \eta_3\,.$$

It is a well defined 1-form on $\mathcal{T}$ not depending on the choice of SO_3-trivialization (I_1, I_2, I_3).

Using the connection ∇, we split the tangent bundle of $\mathcal{T}$ at $I \in \pi^{-1}(s)$ for $s \in N$:

$$T_I\mathcal{T} = T_I\mathcal{T}_s \oplus \pi^* T_s N\,.$$

Here $\mathcal{T}_s$ is the fiber above s of the fibration π. We call $Hor_I\mathcal{T} \simeq T_s N = W_s \oplus H_s$ the horizontal space. Let (R_1, R_2, R_3) be the dual basis of (η_1, η_2, η_3) on W. At I_1, we have an almost complex structure J on $\ker \eta_r \simeq \ker \eta_1 \oplus T_I\mathcal{T}_s$ satisfying

- on $\ker \eta_1$, the almost complex structure satisfies $J = I_1$ after extending I_1 to all $\ker \eta_1$ by $I_1 R_2 = R_3$ and $I_1 R_3 = -R_2$;
- on $T_{I_1}\mathcal{T}_s$ J is the natural complex structure given by the metric and the orientation on the sphere $\mathcal{T}_s$.

PROPOSITION 5.1. *Let H be an integrable quaternionic contact structure on a 7-dimensional manifold N. The almost complex structure J defined on the kernel of η^r is independent of the choice of compatible metric g on H.*

PROOF. Let $\eta' = f^2\eta$ be a conformal change, we follow the notations of 2.5. The distribution $\ker\eta_r$ on the twistor space is left unchanged. The conformal change gives a new complementary W^{f^2g} spanned by (R_1', R_2', R_3') and a new connection $\nabla' = \nabla + a$. The distribution $Hor_I'\mathcal{T}$ is the horizontal subspace on $\mathcal{T}$ corresponding to ∇', and J' is the corresponding almost complex structure.

The vertical part of J is left unchanged.

At $I_1 \in \mathcal{T}_s$, we take $U \in \ker\eta^r$, horizontal for the connection ∇, and X its projection on N. We assume here that $X \in H$. In the decomposition $T_{I_1}\mathcal{T} = Hor_{I_1}\mathcal{T} \oplus T_{I_1}\mathcal{T}_s$, we have $U = (X, 0)$ and $JU = (I_1X, 0)$. On the other hand, in the decomposition $T_{I_1}\mathcal{T} = Hor'_{I_1}\mathcal{T} \oplus T_{I_1}\mathcal{T}_s$, we have $U = (X, -a_XI_1)$, $JU = (I_1X, -a_{I_1X}I_1)$ and $J'U = (I_1X, -\frac{1}{2}[I_1, a_XI_1])$ thus J and J' coincide iff $a_{I_1X}I_1 = \frac{1}{2}[I_1, a_XI_1]$ for all $X \in \ker\eta_1$.

One has $a \in \Omega^1(\mathbb{R} \oplus \mathfrak{so}(H))$, and we decompose the $\mathfrak{so}(H)$-part in selfdual and anti-selfdual part that we write respectively a^+ and a^-. From 2.3, one gets for $X \in H$

$$a_X^+ = \sum_j \langle I_j\theta^\sharp, X\rangle I_j\,.$$

and the equality $a_{I_1X}I_1 = \frac{1}{2}[I_1, a_XI_1]$.

We must now verify the same kind of identity for $X = R_2$. This work in the same way, only that we must pay attention to the fact that the complementary spaces adapted to the choice of metric changes with the conformal change. We have the decompositions $T_{I_1}\mathcal{T} = W^g \oplus H \oplus T_{I_1}\mathcal{T}_s$ and $T_{I_1}\mathcal{T} = W^{f^2g} \oplus H \oplus T_{I_1}\mathcal{T}_s$ where W^g and H are in $Hor_{I_1}\mathcal{T}$ for the first case, and in $Hor'_{I_1}\mathcal{T}$ for the second case. Taking $U = R_2$, horizontal for ∇ and writting the vectors in the second décomposition, we obtain $U = (f^2R_2', -r_2, -a_{R_2}I_1)$, $JU = (f^2R_3', -r_3, -a_{R_3}I_1)$ and $J'U = (f^2R_3', -I_1r_2, -\frac{1}{2}[I_1, a_{R_2}I_1])$. But we have $r_i = -2I_i\theta^\sharp$, hence it suffices to verify that $a_{R_3}I_1 = \frac{1}{2}[I_1, a_{R_2}I_1]$ which is a straightforward computation, remarking that with the vanishing of the torsion, we get from lemma 2.3

$$a_{R_i} = \theta(R_i) + 2|\theta^\sharp|^2 I_i + 2\theta^\sharp \wedge I_i\theta^\sharp + 2(I_i\nabla\theta^\sharp)^{\mathfrak{so}(H)}\,,$$

so that the selfdual part is

$$a_{R_i}^+ = 3|\theta^\sharp|^2 I_i - \frac{1}{2}\sum_k tr(I_kI_i\nabla\theta^\sharp)I_k\,.$$

□

5.2. Integrability of the twistor space. This section is devoted to the proof of the following theorem :

THEOREM 5.1. *Let H be a quaternionic contact structure with vanishing vertical torsion and J be the almost complex structure on the kernel of η^r on the twistor space. Then*

- *J is adapted to the symplectic form $d\eta^r$ on $\ker \eta^r$ and gives a metric of signature $(6,2)$.*
- *J is integrable.*

PROOF. The first point is similar to [**Biq00**] and

$$d\eta^r(\cdot, J\cdot) = g_H + d\eta_1(R_2,R_3)(\eta_2^2+\eta_3^2) + \eta_3 \odot dx_2 - \eta_2 \odot dx_3\,,$$

where $\alpha \odot \beta = \frac{1}{2}(\alpha\otimes\beta + \beta\otimes\alpha)$ is the symmetric product. This is the metric of signature $(6,2)$.

We must now verify the integrability of J. This is given by the vanishing of the Nijenhuis tensor

$$N(X,Y) = [X,Y] + J[X,JY] + J[JX,Y] - [JX,JY]\,.$$

If X and Y are vertical, it follows from the fact that J is the complex structure of the 2-sphere which is integrable, and if X is horizontal and Y vertical this is similar to the proof of 14.68 in [**Bes87**].

Assume now that X and Y are horizontal. In this case the vertical part and the horizontal part of $N(X,Y)$ at $I\in\mathcal{T}$ are given by

$$\begin{aligned}(N(X,Y))_{Hor} &= T_{X,Y} + IT_{X,IY} + IT_{IX,Y} - T_{IX,IY}\\ (N(X,Y))_{Ver} &= [R_{X,Y} + IR_{X,IY} + IR_{IX,Y} - R_{IX,IY}, I]\end{aligned}$$

We look first at the horizontal part. If $X,Y\in H$, then

$$T_{X,Y} = \sum_i d\eta_i(X,Y)R_i$$

and we deduce that $(N(X,Y))_{Hor} = 0$. If $X = R_2$ and $Y = R_3$ at $I = I_1$, then $(N(X,Y))_{Hor} = T_{R_2,R_3} - T_{R_3,-R_2} = 0$ so that the only no-trivial case at I_1 is $X\in H$ and $Y = R_2$. Following the notations of 4.1, the W-part of the torsion $T_{R,X}$ vanishes and the H-part is $T_{R_i,X} = \sum_p I_pA_{pi}X$ where $A_{pi} = -A_{ip}\in\Lambda^2_-H$. Therefore, we have

$$\begin{aligned}(N(X,R_2))_{Hor} &= -\textstyle\sum_p I_pA_{p2}X - I_1\sum_p I_pA_{p3}X\\ &\quad -I_1\textstyle\sum_p I_pA_{p2}I_1X + \sum_p I_pA_{p3}I_1X\\ &= -I_3A_{32}X - I_3A_{23}X - I_1I_3A_{32}I_1X + I_2A_{23}I_1\,.\end{aligned}$$

The A_{ij} and I_k commute hence the skew-symmetry $A_{23} = -A_{32}$ gives the vanishing of $(N_{X,R_2})_{Hor}$.

We show now the vanishing of the vertical part. If $X, Y \in H$, this is just lemma 4.3. It remains to show that for $X \in H$,

$$c_{R_2,X,Y} = [R_{R_2,X} + I_1 R_{R_3,X} + I_1 R_{R_2,I_1 X} - R_{R_3,I_1 X}, I_1]\, Y = 0\,.$$

We put $I_1 R_1 = 0$ and $I_1 R_2 = R_3$, in order to have $c_{X,Y,Z}$ defined for all X, Y and Z. Because we have the same identities on the torsion, the computation is very similar to [**Biq00**, Lemma II.5.3] and one gets

$$c_{R_2,X,Y} + c_{Y,R_2,X} = 0\,, \quad \forall X, Y \in H\,.$$

$c_{R_2,X,Y}$ is in the subspace spanned by $I_2 Y$ and $I_3 Y$ therefore if the $\mathbb{C}$-subspaces spanned by Y and X for the almost complex structure I_1 are transverses, then $c_{R_2,X,Y} = 0$. We deduce that $c_{R_2,X,Y} =$ in all cases.

□

5.3. Proof of theorem 1.1. We have shown that any integrable quaternionic contact structure H admits a twistor space $\mathcal{T}$ which is CR-integrable. This is sufficient to apply the results of Biquard [**Biq00**] which give the theorem 1.1 (see part III for the twistorial construction). With the notations of 3.1, the AHQK metric is $g = w_E \otimes w_H$ and is quaternionic-Kähler, [**LeB89**].

The corollary 1.1 follows immediately from our theorem 5.1 and the theorem 0.4 of [**Biq02**].

5.4. Concluding remarks. We have shown that an integrable quaternionic contact distribution on $\mathbb{S}^7$ close to the canonical one is the conformal infinity of a quaternionic-Kähler metric on the ball B^8.

A quaternionic Kähler manifold can be defined with the help of a parallel 4-form Ω with stabilizer $Sp(n)Sp(1)$. Swann [**Swa89**] showed that in dimension greater than 8, if Ω is closed, then Ω is parallel. On the other hand, one can construct an 8-manifold with closed Ω which is not parallel, [**Sal01**]. So one can ask if a quaternionic contact structure in dimension 7 is the conformal infinity of an asymptotically hyperbolic metric associated to a closed 4-form with stabilizer $Sp(2)Sp(1)$.

CHAPTER 2

$Sp(1)$-invariant deformations of the 7-sphere

1. Introduction

This chapter is devoted to the construction of a family of integrable quaternionic contact structures on the 7-sphere. The idea is to look for $Sp(1)$-invariant distributions, so that they are connexions on the Hopf bundle $S^7 \to S^4$. The integrability condition becomes a semi-linear equation on the basis S^4, and the infinitesimal integrable deformations of the canonical quaternionic contact distribution H^{can} are parametrized by the first homology group of an elliptic complex.

Among the integrable quaternionic contact structures on $\mathbb{S}^7$, we show the existence of an interesting family of $Sp(1)$-invariant integrable quaternionic contact structures on the 7-sphere:

THEOREM 1.1. *Let H^{can} be the canonical quaternionic contact structure of $\mathbb{S}^7$. Let $\mathcal{H}$ be the set of integrable $Sp(1)$-invariant quaternionic contact structures and $\mathcal{G}$ be the group of diffeomorphisms of $\mathbb{S}^7$ commuting with the $Sp(1)$-action. There is a neighbourhood $\mathcal{V}$ of $[H^{can}]$ in $\mathcal{H}/\mathcal{G}$ which is homeomorphic to the quotient of a 35-dimensional ball B^{35} by the isotropy group $Sp(2)$ of H^{can}. One obtains a 25-parameter family of integrable quaternionic contact structures.*

Then, we can construct a family of $Sp(1)$-invariant complete quaternionic Kähler metrics on the 8-ball :

COROLLARY 1.1. *Let $g_{\mathcal{H}}$ be the quaternionic hyperbolic metric on the 8-ball. There exists a 25-parameter family of $Sp(1)$-invariant AHQK metrics with boundaries close to the boundary H^{can} of $g_{\mathcal{H}}$.*

This examples generalize a 3-parameter family constructed by Galicki in **[Gal91]**, which were obtained by quaternionic quotient of the hyperbolic quaternionic space $\mathbb{H}\mathcal{H}^3$. All these metrics have isometry group strictly greater than $Sp(1)$.

2. Deformations of contact structures

Hereafter, we assume that $N = \mathbb{S}^7$ is the 7-sphere in $\mathbb{H}^2$ where $\mathbb{H}^2$ is an $\mathbb{H}$-vector space with $\mathbb{H}$ acting on right. Let $\langle \cdot, \cdot \rangle$ be the canonical metric on $\mathbb{H}^2 \simeq \mathbb{R}^8$. Recall that we have a quaternionic contact structure on $\mathbb{S}^7$ given by $H^c_x = (x\mathbb{H})^\perp$ for $x \in \mathbb{S}^7$. The restriction to H^{can} of the round metric on $\mathbb{S}^7$ defines an adapted metric g_0. The adapted complementary is $W_x = xIm\mathbb{H}$ and is spanned by $R_1(x) = xi$, $R_2(x) = xj$ and $R_3(x) = xk$.

H^{can} is a connection on the principal $Sp(1)$-bundle $\mathbb{S}^7 \to \mathbb{S}^4$ (Hopf-bundle). We call $\eta \in \Omega^1(\mathbb{S}^7) \otimes \mathfrak{sp}(1) \simeq \Omega^1(W)$ its connection form. Let us write it $\eta = \sum_i \eta_i$ or $\sum_i \eta_i \otimes R_i$. One has $d\eta_i(W, H^{can}) = 0$ so that the the torsions $T^W = -\frac{1}{2}\sum_{i,j}(\alpha_{ij} + \alpha_{ji}) \otimes w_i^* \otimes w_j$ and T^H vanish.

Let ν be the canonical volume form of $\mathbb{S}^7$ that we decompose as $\nu = \nu^c \wedge \eta_1 \wedge \eta_2 \wedge \eta_3$ so that $\nu^c|_H$ is a volume form on H^{can}.

In this section, we compute the complex of integrable infinitesimal deformations of H^{can}.

2.1. Deformation of the integrability condition. A deformation of H^{can} is given by a 1-form θ with values in W which vanishes on W, or equivalently by a section of $End(H^{can}, W)$. The link between the new distribution and θ is given by

$$H_\theta = \{X - \theta(X), X \in H^{can}\} = \ker(\eta + \theta)\,.$$

Assume now that θ^t is a 1-parameter family of such 1-forms, each giving a vertical torsion free distribution denoted by $H_t = \ker(\eta + \theta^t)$. For small t, the forms $d(\eta_i + \theta_i^t)|_{H_t} \in \Gamma(\Lambda^2 H_t^*)$ span a space of selfdual 2-forms on H_t with respect to a metric g_t on H_t. We choose g_t such that g_0 is the restriction of the round metric on H^{can}.

In order to write the condition on the torsion, one has to take an orthonormal basis of $\Lambda^2_+ H_t^*$. We identify the functions and the 4-forms on H^{can} using ν^c. We search $a^t : \mathbb{S}^7 \to GL(3, \mathbb{R})$ such that $a^0 = Id$ and

$$\begin{aligned}[a^t \cdot d(\eta + \theta^t)]_i \wedge [a^t \cdot d(\eta + \theta^t)]_j \\ \wedge [a^t \cdot (\eta + \theta^t)]_1 \wedge [a^t \cdot (\eta + \theta^t)]_2 \wedge [a^t \cdot (\eta + \theta^t)]_3 = 2\delta_{ij}\nu\,.\end{aligned}$$

Setting $\dot\psi = \frac{d\psi^t}{dt}|_{t=0}$, one obtains

$$\dot a_{ij} + \dot a_{ji} + (d\dot\theta_i \wedge d\eta_j + d\dot\theta_j \wedge d\eta_i)|_{H^{can}} + tr(\dot a) = 0 \tag{11}$$

REMARK 2.1. We used the fact that $\alpha_{ij} = 0$. In general, one has

$$\dot{a}_{ij} + \dot{a}_{ji} + (d\dot{\theta}_i \wedge d\eta_j + d\dot{\theta}_j \wedge d\eta_i) + \sum_k (\alpha_{ki} \wedge \dot{\theta}_k \wedge d\eta_j + \alpha_{kj} \wedge \dot{\theta}_k \wedge d\eta_i) + tr(\dot{a}) = 0$$

on H.

We put $\beta^t = a^t \cdot (\eta + \theta^t)$ with dual basis (R_1^t, R_2^t, R_3^t) on W. Our choice of a^t ensures that we obtain an orthonormal direct basis in $\Lambda^2_+ H_t$ for the metric g_t. Let I_i^t be the associated quaternionic structure. By 2.3, the deformation preserves the integrability iff there exist γ_i^t such that for $X \in H^{can}$,

$$i_{R_i^t}\beta_j^t(X - \theta^t(X)) + i_{R_j^t}\beta_i^t(X - \theta^t(X)) = \gamma_i^t I_j^t(X - \theta^t(X)) + \gamma_j^t I_i^t(X - \theta^t(X)) \,.$$

The γ_i^0 vanish so that one obtains the following lemma.

LEMMA 2.1. *If θ^t is a 1-parameter smooth deformation of the quaternionic contact structure on $\mathbb{S}^7$ which preserves the integrability, we have*

$$\mathcal{A}_0(\dot{\theta}) = -d(\dot{a}_{ij} + \dot{a}_{ji})|_{H^{can}} + (i_{R_i} d\dot{\theta}_j + i_{R_j} d\dot{\theta}_i)|_{H^{can}} \in S^{3,1} \oplus S^{1,1} \,,$$

where

$$\dot{a}_{ij} + \dot{a}_{ji} + (d\dot{\theta}_i \wedge d\eta_j + d\dot{\theta}_j \wedge d\eta_i)|_{H^{can}} + tr(\dot{a}) = 0 \,.$$

REMARK 2.2. The statement has exactly the same form if one deforms Einstein selfdual Levi-Civita connections with non-zero scalar curvature (which give 3-Sasakian manifolds and so integrable quaternionic contact structures, see **[Kon75]**).

The composition of $\mathcal{A}_0$ with the projection on $S^{5,1}$ gives a differential operator $\mathcal{A} : \Gamma((H^{can})^* \otimes W) \to \Gamma(S^{5,1})$. Its kernel gives the infinitesimal deformations of H^{can} preserving the integrability. This kernel contains the image of the infinitesimal diffeomorphisms through

$$\begin{array}{rcl} \mathcal{D} : \Gamma(T\mathbb{S}^7) & \to & \Gamma((H^{can})^* \otimes W) \\ \zeta & \mapsto & \{X \in H \mapsto X.\eta(\zeta) + d\eta(\zeta, X)\} \end{array}$$

2.2. A Bianchi identity. Because of the dimensions of the different vector bundles, the previous complex cannot be elliptic, even in the direction of H^{can}. We will show now a Bianchi identity.

LEMMA 2.2. *Let (M^7, H, g) be a quaternionic contact structure where g is a particular choice of Carnot-Carathéodory metric. Let W be the adapted complementary and ∇ be the corresponding adapted connection. The vertical torsion T^W of H is a section of $S^{5,1} \subset H^* \otimes S^{4,0}$. Let $\mathcal{B}_H$ be the composition* *of* $d^\nabla : \Gamma(H^* \otimes S^{4,0}) \to \Gamma(\Lambda^2 H^* \otimes S^{4,0})$ *with the projection on* $S^{6,0}$*. Then we have*

$$\mathcal{B}_H(T^W) = 0\,.$$

REMARK 2.3. Here is a small abuse of notation. Indeed d^∇ can be applied only on true 1-forms with values in a vector bundle. Nevertheless we can give the following meaning to d^∇ : a section σ of $H^* \otimes E$ is extended in a true 1-form vanishing on W and we use then the vanishing $(T_{X,Y})_H = 0$ in order to obtain

$$\begin{aligned}(d^\nabla \sigma)(X,Y) &= \nabla_X \sigma_Y - \nabla_Y \sigma_X - \sigma_{T_{X,Y}} \\ &= (\nabla_X \sigma)_Y - (\nabla_Y \sigma)_X\,,\end{aligned}$$

for vector fields $X, Y \in H$. This kind of equalities will be used throughout the proof for every elements of $\Gamma(H^* \otimes E)$ and every vector bundle E.

Summing over cyclic permutations will be denoted by $\sum_{(\cdot,\cdot,\cdot)}$.

PROOF. Let (I_1, I_2, I_3) be a local direct orthonormal basis of $\Lambda^2_+ H^*$ corresponding to local 1-forms (η_1, η_2, η_3) defining the contact structure. Denotes by (R_1, R_2, R_3) the corresponding dual basis on W. The first Bianchi identity is

$$\mathfrak{S}_{X,Y,R_i} \left(R_{X,Y} R_i - T_{T_{X,Y},R_i} - (\nabla_X T)_{Y,R_i} \right) = 0\,,$$

for vector fields X and Y in H. Taking the W-part, we obtain

$$\begin{gathered} R_{X,Y} R_i = (T_{T_{R_i,X},Y})_W + (T_{T_{Y,R_i},X})_W \\ +((\nabla_X T)_{Y,R_i})_W + ((\nabla_{R_i} T)_{X,Y})_W + ((\nabla_Y T)_{R_i,X})_W\,. \end{gathered}$$

We calculate first $A_1(X,Y,R_i) = (T_{T_{R_i,X},Y})_W + (T_{T_{Y,R_i},X})_W$. One has

$$\begin{aligned} A_1(X,Y,R_i) = -T^W_{T^W_X(R_i)}(Y) + \sum_j \langle I_j T^H_{R_i}(X), Y\rangle R_j + T^W_{T^W_Y(R_i)}(X) \\ - \sum_j \langle I_j T^H_{R_i}(Y), X\rangle R_j\,. \end{aligned}$$

Putting $a_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji})$, one gets

$$A_1(X,Y,R_i) = -\sum_{k,j=1}^{3} a_{ji} \wedge a_{kj}(X,Y) R_k + \sum_{k=1}^{3} \langle (I_k T^H_{R_i} + T^H_{R_i} I_k)(X), Y\rangle R_k\,. \tag{12}$$

Assume now that $p \in M$, and that X, Y and R_i are parallel at p. In particular, at p, one has $\alpha_{ij} = \alpha_{ji}$ at p and

$$\Big(\sum_{(X,Y,R_i)} (\nabla_X T)_{Y,R_i}\Big)_W = -\sum_k (d^\nabla a_{ki})(X,Y) R_k - \sum_k (\nabla_{R_i} d\eta_k)(X,Y) R_k\,,$$

so that we obtain

$$\begin{aligned} R_{X,Y}R_i &= -\sum_{k,j=1}^3 a_{ji}\wedge a_{kj}(X,Y)R_k + \sum_{k=1}^3\langle (I_kT^H_{R_i}+T^H_{R_i}I_k)(X),Y\rangle R_k \\ &\quad -\sum_k (d^\nabla a_{ki})(X,Y)R_k - \sum_k(\nabla_{R_i}d\eta_k)(X,Y)R_k\,. \end{aligned}$$

From the equation $\langle R_{X,Y}R_i,R_k\rangle + \langle R_{X,Y}R_k,R_i\rangle = 0$, we deduce that

$$\begin{aligned} 2d^\nabla a_{ki}(X,Y) &= \langle (I_kT^H_{R_i}+T^H_{R_i}I_k)(X),Y\rangle + \langle (I_iT^H_{R_k}+T^H_{R_k}I_i)(X),Y\rangle \\ &\quad -\nabla_{R_i}d\eta_k(X,Y) - \nabla_{R_k}d\eta_i(X,Y)\,. \end{aligned}$$

Remark that (8) is true even if T^W does not vanish. At p, it gives

$$\begin{aligned} 4d^\nabla a_{ki} &= \sum_j (d\eta_k(R_j,R_i)+d\eta_i(R_j,R_k))\langle I_j\cdot,\cdot\rangle \\ &\quad +2\langle (I_kT^H_{R_i}+T^H_{R_i}I_k)\cdot,\cdot\rangle + \langle (I_iT^H_{R_k}+T^H_{R_k}I_i)\cdot,\cdot\rangle\,. \end{aligned}$$

This is a 2-form whose selfdual part is

$$4(d^\nabla a_{ki})_+ = \sum_j (d\eta_k(R_j,R_i)+d\eta_i(R_j,R_k))\langle I_j\cdot,\cdot\rangle$$

$$-2\,tr(T^H_{R_i})\langle I_k\cdot,\cdot\rangle - 2\,tr(T^H_{R_k})\langle I_i\cdot,\cdot\rangle\,.$$

This is an element of $S^{2,0}\otimes(S^{4,0}\oplus S^{0,0})\subset (S^{2,0})^3$. We take the projection in $Sym^3(S^{2,0})\simeq S^{6,0}\oplus S^{2,0}$ and then the $S^{6,0}$-part to obtain the lemma.

□

2.3. The complex of infinitesimal deformations. We take the infinitesimal part of the previous equation and obtain the complex of infinitesimal deformations of the 7-sphere

$$(\mathcal{C}_0)\quad \Gamma(T\mathbb{S}^7)\xrightarrow{\mathcal{D}}\Gamma(H^*\otimes W)\xrightarrow{\mathcal{A}}\Gamma(S^{5,1})\xrightarrow{\mathcal{B}_c}\Gamma(S^{6,0})\,.$$

Here $\mathcal{B}_c$ means the Bianchi operator on H^{can}.

We have the decomposition $\Gamma(T\mathbb{S}^7)=\Gamma(W)\oplus\Gamma(H^{can})$ and on the other hand $\Gamma((H^{can})^*\otimes W)=\Gamma(S^{3,1})\oplus\Gamma(S^{1,1})$ with the property that $\mathcal{A}(\Gamma(S^{1,1}))=0$. The restriction of $\mathcal{D}$ to $\Gamma(H^{can})$ is an isomorphism $\Gamma(H^{can})\to\Gamma(S^{1,1})$ so that if $\tilde{\mathcal{D}}$ is the composition of $\mathcal{D}$ restricted to $\Gamma(W)$ with the projection on $\Gamma(S^{3,1})$, we obtain an isomorphism

$$\frac{\ker\mathcal{A}}{\mathcal{D}(\Gamma(T\mathbb{S}^7))}\simeq\frac{\ker\mathcal{A}\cap\Gamma(S^{3,1})}{\tilde{\mathcal{D}}(\Gamma(W))}\,.$$

In other words, we can compute the first homology group of the complex

$$(\mathcal{C})\quad \Gamma(W)\xrightarrow{\tilde{\mathcal{D}}}\Gamma(S^{3,1})\xrightarrow{\mathcal{A}}\Gamma(S^{5,1})\xrightarrow{\mathcal{B}_c}\Gamma(S^{6,0})\,.$$

REMARK 2.4. This complex is not elliptic. Nevertheless a straightforward computation shows that $(\mathcal{C})$ is elliptic in the direction of H^{can}. This was not the case of $(\mathcal{C}_0)$.

LEMMA 2.3. *If $\xi = \xi_{H^{can}} + \xi_W \in T_x\mathbb{S}^7$, the principal symbols σ_ξ of the previous differential operators satisfy :*

- *If $\xi_{H^{can}} = 0$, then $\ker \sigma_\xi(\tilde{\mathcal{D}}) = W_x$, or else if $\xi_{H^{can}} \neq 0$, then $\ker \sigma_\xi(\tilde{\mathcal{D}}) = \{0\}$.*
- *If $\xi_{H^{can}} = 0$, then $\ker \sigma_\xi(\mathcal{A}) = S^{3,1}_x$, or else if $\xi_{H^{can}} \neq 0$, then $\ker \sigma_\xi(\mathcal{A}) = Im\, \sigma_\xi(\tilde{\mathcal{D}})$.*
- *If $\xi_{H^{can}} = 0$, then $\ker \sigma_\xi(\mathcal{B}) = S^{5,1}_x$, or else if $\xi_{H^{can}} \neq 0$, then $\ker \sigma_\xi(\mathcal{B}_c) = Im\, \sigma_\xi(\mathcal{A})$.*

3. $Sp(1)$-invariant case

We have seen in the previous section that infinitesimal deformations of the standard quaternionic contact structure on $\mathbb{S}^7$ are parametrized by the first cohomology group of the complex $(\mathcal{C})$. This complex is not elliptic and even not hypoelliptic. Indeed, **[LeB91]** ensures the existence of an infinite dimensional moduli space of integrable quaternionic contact structures on $\mathbb{S}^7$.

In order to obtain an elliptic complex, we will look at quaternionic contact structures on $\mathbb{S}^7$ admitting a free $Sp(1)$-action. Here, $Sp(1)$ is viewed as the group of unitary quaternions. There is a canonical action of $Sp(1)$ on $\mathbb{S}^7$ given by the diagonal action of $Sp(1)$ on $\mathbb{S}^7 \subset \mathbb{H}^2$. The quotient is the 4-sphere and the projection $\mathbb{S}^7 \to \mathbb{S}^4$ is the Hopf projection. Smooth deformations of this $Sp(1)$-action on $\mathbb{S}^7$ are always diffeomorphic to the canonical one. Therefore, we fix the $Sp(1)$-action to be the canonical one.

3.1. G-invariant structures. In this section, we do some general remarks about quaternionic contact structures H invariant under a free smooth G-action, where $G = SO(3)$ or $G = Sp(1)$. Let (N, H) be such a quaternionic contact structure. The action must be transverse to the contact distribution so that H is a connection on a G-principal bundle $N \xrightarrow{\pi} B$. Let (η_1, η_2, η_3) be the connection form of H with values in $\mathfrak{sp}(1)$. The symplectic forms $d\eta_i|_H$ define a unique adapted conformal class of metrics $[g]$ on H. Because of the G-invariance, the conformal class $[g]$ can be pushed down on B and gives a conformal class of Riemannian metrics $[g]$ on B. Let $E = M \times_{Ad} \mathfrak{g}$ be the adjoint bundle. The connection H gives a covariant derivative ∇^E on E, with curvature R^E. By definition of $[g]$, the curvature R^E gives an isomorphism

$$\begin{array}{rcl} R^E : \Lambda^2_+ TB & \to & E \\ \zeta & \mapsto & R_\zeta \in \mathfrak{so}(E) \simeq E \end{array}$$

Let D be a linear connection, preserving the conformal class. Every choice of D is available, but in general one chooses a metric g in the conformal class and the corresponding Levi-Civita connection.

The tensor $(R^E)^{-1}\nabla^{D,E}R^E$ is a section of $T^*B \otimes End(\Lambda^2_+ B)$ and taking the symmetric part of $End(\Lambda^2_+ B)$ with respect to any choice of metric in the conformal class $[g]$, it gives a tensor T in $\Gamma(S^{1,1} \otimes (S^{4,0} \oplus S^{0,0}))$. The $S^{5,1}$ part of T is the vertical torsion of the quaternionic contact structure H. We put $Tor(H) = T$.

3.2. Infinitesimal $Sp(1)$-invariant deformations of $\mathbb{S}^7$. We now come back to the deformations of the canonical quaternionic contact structure on $\mathbb{S}^7$. Let $\mathcal{H}$ be the set of $Sp(1)$-invariant quaternionic contact structures on the Hopf bundle $\mathbb{S}^7 \to \mathbb{S}^4$ and $\mathcal{G}$ be the group of diffeomorphisms of $\mathbb{S}^7$ commuting with the $Sp(1)$ action. Let ∇ be the Levi-Civita connection of the round metric of $\mathbb{S}^4$. In the $Sp(1)$-invariant case, the complex $(\mathcal{C})$ can be written on the basis $\mathbb{S}^4$ in the following way :

LEMMA 3.1. *The complex $(\mathcal{C})$ applied to $Sp(1)$-invariant deformations on the Hopf bundle $\mathbb{S}^7 \to \mathbb{S}^4$ can be written on the basis as*

$$(\mathcal{C}) \quad \Gamma(S^{2,0}) \xrightarrow{\tilde{\mathcal{D}}} \Gamma(S^{3,1}) \xrightarrow{\mathcal{A}} \Gamma(S^{5,1}) \xrightarrow{\mathcal{B}_c} \Gamma(S^{6,0})$$

where $\tilde{\mathcal{D}} = p^{3,1}\nabla$, $\mathcal{A} = p^{5,1}\nabla^2$ and $\mathcal{B}_c = p^{6,0}\nabla$. The homology groups H^0, H^1, H^2 and H^3 of $(\mathcal{C})$ have dimensions 10, 35, 0 *and* 0 *respectively.*

PROOF. The operator $\mathcal{A}$ is the composition of $\mathcal{A}_1 = p^{4,0}\nabla$ and $\mathcal{A}_2 = p^{5,1}\nabla$ so that the previous complex splits into

$$\begin{array}{ll} (\mathcal{C}_1) & \Gamma(S^{2,0}) \xrightarrow{\tilde{\mathcal{D}}} \Gamma(S^{3,1}) \xrightarrow{\mathcal{A}_1} \Gamma(S^{4,0})\,, \\ (\mathcal{C}_2) & \Gamma(S^{4,0}) \xrightarrow{\mathcal{A}_2} \Gamma(S^{5,1}) \xrightarrow{\mathcal{B}_c} \Gamma(S^{6,0})\,. \end{array}$$

One recognize in $(\mathcal{C}_1)$ the complex of deformations of anti-selfdual metrics. This complex is well-known and one can show that $\mathcal{A}_1$ gives an isomorphism between $\ker \tilde{\mathcal{D}}^*$ and $\Gamma(S^{4,0})$ (see for instance the proof of [**Bes87**, theorem 13.30, p. 376]). Therefore the kernel of $\mathcal{A} \oplus \mathcal{D}^*$ can be identified with the kernel of $\mathcal{A}_2$, and we are reduced to the study of $(\mathcal{C}_2)$.

First we give some Weitzenböck formula. Let $\mathcal{D}^\nabla$ be the Dirac operator on $S \otimes E$ where $S = S^{1,0} \oplus S^{0,1}$ is the spinor bundle and E can be any

$S^{n,m}$. The Dirac operator is the composition of the connection and the Clifford multiplication. The Clifford multiplication is a morphism of representation on $Spin(4)$-modules so is the identity on each irreducible component of $S \otimes E$, up to a multiplicative constant. If $E = S^{5,0}$, we see for instance that $\mathcal{D}^\nabla = b\mathcal{B}_c \oplus a\mathcal{A}_2^*$ for some constants a and b. The Weitzenböck formula is

$$\mathcal{D}^\nabla(\phi \otimes s) = \nabla^*\nabla(\phi \otimes s) + \frac{s}{4}\phi \otimes s + \sum_{e_i, e_j} e_i \cdot e_j \phi \otimes R^\nabla_{e_i, e_j} s\,,$$

and in our case, the curvature R^∇ is scalar so that the last term in the previous equality is a combination of Casimir operators. One obtains finally

$$(\mathcal{D}^\nabla)^2 = \nabla^*\nabla + \frac{s}{4}\,,$$

and so $\ker(\mathcal{B} \oplus \mathcal{A}_2^*) = 0$, that is to say the complex $(\mathcal{C})$ has no second homology group.

In the same way, regarding $\mathcal{B}_c^* : \Gamma(S^{6,0}) \to \Gamma(S^{5,1})$, it appears to be the Dirac operator on $S^{6,0} \subset S^{1,0} \otimes S^{5,0}$ (up to a multiplicative constant). One can show that

$$((\mathcal{D}^\nabla)^2 - \nabla^*\nabla)|_{S^{6,0}} = \frac{2s}{3}\,,$$

which gives $\ker(\mathcal{B}_c^*) = 0$. From this results, we deduce that $\dim \ker \mathcal{A}_2$ is exactly the index of $(\mathcal{C}_2)$ which is the index of the Dirac operator

$$\mathcal{D}^\nabla : \Gamma(S^{1,0} \otimes S^{5,0}) \to \Gamma(S^{0,1} \otimes S^{5,0})\,.$$

By the Atiyah-Singer index theorem,

$$\begin{aligned} index\, \mathcal{D}^\nabla &= \{ch(S^{5,0})\hat{A}(\mathbb{S}^4)\}[\mathbb{S}^4] \\ &= (6 + 35 ch_2(S^{1,0}))(1 - p_1/24)[\mathbb{S}^4] \\ &= 35\,. \end{aligned}$$

□

3.3. Moduli space. In this section, we will end the proof of theorem 1.1. Here we must be more precise in our notations. If g is a conformal class of metric on $\mathbb{S}^4$, there is a subbundle $S_g^{5,1}$ of $T^*\mathbb{S}^4 \otimes \Lambda^2 T^*\mathbb{S}^4 \otimes \Lambda^2 T\mathbb{S}^4$ associated to the representation $S^{5,1}$ and g. In the same way, one defines $S_g^{6,0}$ in $\Lambda^2 T^*\mathbb{S}^4 \otimes \Lambda^2 T^*\mathbb{S}^4 \otimes \Lambda^2 T\mathbb{S}^4$.

Remark 3.1. We have seen that each $H \in \mathcal{H}$ defines a conformal class of metrics on $\mathbb{S}^4$. In fact, the quaternionic contact structure H defines a true metric on $\mathbb{S}^4$. Indeed, if we come back to section 3.1, the vector bundle E is an oriented bundle which gives an volume form on $\Lambda^2_+ T\mathbb{S}^4$ such that R^E preserves the two orientations. Then, we can choose the metric on $\mathbb{S}^4$ which gives the same volume form on $\Lambda^2_+ T\mathbb{S}^4$. We obtain a well defined map

$$G : \mathcal{H} \to \mathcal{M} ,$$

where $\mathcal{M}$ is the set of smooth metrics on $\mathbb{S}^4$. The round metric on $\mathbb{S}^4$ is called g_0 and is the metric $G(H^{can})$.

With the help of the canonical structure H^{can}, we identify $\mathcal{H}$ with an open subset in $\Gamma(T^*\mathbb{S}^4 \otimes S^{2,0}_{g_0})$. Let $p^{i,j}$ be the orthogonal projection with respect to g_0 in $S^{i,j}_{g_0}$ ($S^{i,j}_{g_0}$ will appear at most one time in our vector bundles so that the $p^{i,j}$ are well defined). We restrict ourselves to a neighbourhood $\mathcal{U}$ of H^{can} in $\mathcal{H}$ where $p^{5,1}$ (resp. $p^{6,0}$) gives by restriction an isomorphism from $S^{5,1}_{G(H)}$ onto $S^{5,1}_{g_0}$ (resp. from $S^{6,0}_{G(H)}$ onto $S^{6,0}_{g_0}$). With the identifications given by the $p^{i,j}$, one gets maps

$$\mathcal{T} : \mathcal{U} \to \Gamma(S^{5,1}_{g_0}) \, , \; H \mapsto p^{5,1}(Tor(H)) \, ,$$

and

$$\mathcal{B} : \mathcal{U} \oplus \Gamma(S^{5,1}_{g_0}) \to (S^{6,0}_{g_0}) \, , \; (H,T) \mapsto p^{6,0}(\mathcal{B}_H(T)) \, .$$

Because of the Bianchi identity of lemma 2.2, we have $\mathcal{B}(H, \mathcal{T}(H)) = 0$ for $H \in \mathcal{U}$. We want to apply an implicit function theorem so we must work in Banach spaces. We assume now that our sections are $C^{k+2,\alpha}$ (Hölder-spaces). We have seen in section 6.3 that we can search a slice in $\Gamma(S^{3,1})$. We put $\mathcal{U}_1 = \mathcal{U} \cap \Gamma(S^{3,1})$. Let us define the smooth map

$$\Psi : \mathcal{U}_1 \to \operatorname{Im} \tilde{\mathcal{D}}^* \oplus \ker \mathcal{B}_c \, , \; a \mapsto (\tilde{\mathcal{D}}^*(a), p\,\mathcal{T}(a)) \, ,$$

where p is the projection on $\ker \mathcal{B}_c$ in the direction of $\operatorname{Im} \mathcal{B}^*_c$. Because of the vanishing of the second homology group of $(\mathcal{C})$, the differential $d_{H^{can}}\psi$ is surjective. Its kernel is $\ker(\tilde{\mathcal{D}}^* \oplus \mathcal{A})$ and is of finite dimension 35. Therefore, there is a submanifold $X^{35} \subset \ker \tilde{\mathcal{D}}^* \oplus \Gamma(S^{6,0})$ such that on a neighbourhood of H^{can} in $\mathcal{U}_1$, one has $\Psi(a) = 0$ iff $a \in X^{35}$. Because of the vanishing of the homology groups H^2 and H^3, we can apply the inverse function theorem with the Bianchi operator $\mathcal{B}$ at $(H^{can}, 0)$ in order to obtain that if $p\mathcal{T}(a) = 0$ then $\mathcal{T}(a) = 0$ for a sufficiently small. We obtain a neighbourhood V of H^{can} such that

$$(\tilde{\mathcal{D}}^*(a), \mathcal{T}(a)) = 0 \text{ iff } a \in M = X^{35} \cap V \, .$$

We have obtained a 35-dimensional family of integrable $C^{k+2,\alpha}$ quaternionic contact structures on $\mathbb{S}^7$. If $a \in M$, it satisfies a non-linear but elliptic equation, hence a is smooth.

The isotropy group G of H^{can} under the action of $\mathcal{G}$ is $Sp(2)$. Because $\ker \tilde{\mathcal{D}}^*$ is $Sp(2)$-invariant and $Sp(2)$ is compact, we can assume that M is stable under the action of $Sp(2)$. Hence, the manifold M is not the moduli space of integrable quaternionic contact structures. Nevertheless, the only diffeomorphisms acting on M are in $Sp(2)$. Indeed, it follows from the properness of the action of $\mathcal{G}$ on $\mathcal{H}$: an element $\phi \in \mathcal{G}$ gives a diffeomorphism ψ on $\mathbb{S}^4$ acting on the metrics $G(\mathcal{H})$. The diffeomorphism ϕ is determined up to a gauge transformation by ψ. The both nice behaviours of the action of diffeomorphisms on the metrics and of the gauge transformations on the connections give the properness of the action of $\mathcal{G}$.

Therefore there exists a neighbourhood of $[H^{can}]$ in $\mathcal{H}/\mathcal{G}$ which is homeomorphic to a neighbourhood of H^{can} in M quotiented by $Sp(2)$. It gives the theorem 1.1, and using the theorem 0.4 of [**Biq02**], one gets the corollary 1.1.

Among these, there is a family obtained as the boundary of quaternionic quotient constructed by Galicki in [**Gal91**]. Let us describe these more precisely. Choose $D \in \mathfrak{sp}(2)$ and let S^D be

$$S^D = \{x \in \mathbb{H}^k, |x|^2 + \frac{|x^* D x|^2}{4} = 1\}\,.$$

Here x^* means the adjoint of x with respect to the canonical quaternionic hermitian metric of $\mathbb{H}^2$. S^D is isomorphic to the 7-sphere and invariant under the diagonal action of $Sp(1)$ on right. One has the codimension 3-distribution

$$H_x^D = \{v \in \mathbb{H}^2, x^* v - \frac{x^* D x}{4}(x^* D v + v^* D x) = 0\} \subset T_x S^D\,.$$

This is a quaternionic contact structure which is the conformal infinity of an AQH quaternionic-Kähler metric on the interior B^D of S^D. Therefore H^D is an integrable quaternionic contact structure. Remark that H_x^D is different from the subspace of $T_x S^D \subset \mathbb{H}^2$ stable under the right-action of $\mathbb{H}$. The isotropy group of H^D is a quotient of $K \times Sp(1)$ where K is the subgroup of elements of $Sp(2)$ which commute with D.

CHAPTER 3

Quaternion-symplectic forms

1. Introduction

If (M^{4n}, g) is a quaternionic-Kähler manifold, and if (I_1, I_2, I_3) is a local quaternionic structure (I_1, I_2, I_3) defining local symplectic forms $w_i(\cdot,\cdot) = g(I_i\cdot,\cdot)$, ones obtains a well defined parallel 4-form

$$\Omega = \sum_{i=1}^{3} w_i^2$$

whose stabilizer is $Sp(n)Sp(1)$. Mutually, a 4-form Ω with stabilizer $Sp(n)Sp(1)$ on a $4n$-manifold defines a metric g which is quaternionic-Kähler iff Ω is parallel for the Levi-Civita connexion of g. In dimension $4n \geq 12$, Swann [**Swa89**] has shown that such a 4-form Ω is parallel for the Levi-Civita connection iff Ω is closed. On the contrary, in dimension 8, there exist 4-form with stabilizer $Sp(2)Sp(1)$, which are closed and not parallel, see for instance [**Sal01**].

DEFINITION 1.1. Let M be a smooth manifold of dimension 8. A closed 4-form Ω on M is quaternionic symplectic if for each point $x \in M$, the stabilizer of Ω under the action of $GL(T_xM)$ is isomophic to $Sp(2)Sp(1)$. If Ω is not closed, we say that Ω is a quaternionic 4-form.

One knows now that an integrable quaternionic contact structure closed to the canonical one on the 7-sphere is the boundary of an asymptotically hyperbolic quaternionic-Kähler 4-form. We give a general definition :

DEFINITION 1.2. Let M^8 be a manifold with boundary N and let Ω be a quaternionic 4-form on M. Assume given a quaternionic contact structure H on N. Let g_H be a compatible metric on H and let $(d\eta_i|_H)_i$ be a local $SO(3)$-trivialization of $\Lambda^2_+ H^*$. The 4-form Ω is called asymptotically hyperbolic, with boundary H, if there exists a positive function ρ, vanishing to first order on N such that $\Omega = \sum_i w_i^2$ with

$$w_i \sim \frac{1}{\rho^2}\left(-d\rho\wedge\eta_i + \frac{1}{2}\sum_{r,s}\varepsilon^{irs}\eta_r\wedge\eta_s\right) + \frac{1}{\rho}\psi_i \text{ when } \rho \to 0$$

and $\psi_i|_H = d\eta_i|_H$. In this case, the Riemannian metric defined by Ω is asymptotically hyperbolic quaternionic.

A natural question is to search if every quaternionic contact structure close to the canonical one is the conformal infinity of a closed asymptotically hyperbolic quaternionic form. First, I show that if ρ is a function vanishing up to first order on the boundary, then one can find an asymptotically hyperbolic quaternionic 4-form Ω, with metric g and such that

$$|d\Omega|_g = O(\rho^3)\,.$$

Assuming then the existence of a uniform Poincaré estimate for the Hodge laplacian of quaternionic-hyperbolic metrics with boundary H close to H^{can}, one constructs a closed asymptotically hyperbolic quaternionic form whose boundary is H.

2. Preliminaries

First, we give some notations and collect some definitions and preliminary facts which will be used throughout the chapter. More details about representations of $Sp(2)Sp(1)$ appear in Salamon's book [**Sal89**].

2.1. The group $Sp(2)Sp(1)$. Let $\mathbb{R}^8$ be equipped with an orientation and its canonical metric, and let $(e_1, \cdots, e_8)$ be an oriented orthonormal basis of $(\mathbb{R}^8)^*$. Let us define the symplectic forms

$$\begin{aligned} w_1 &= e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6 + e_7 \wedge e_8\,, \\ w_2 &= e_1 \wedge e_3 - e_2 \wedge e_4 + e_5 \wedge e_7 - e_6 \wedge e_8\,, \\ w_3 &= e_1 \wedge e_4 + e_2 \wedge e_3 + e_5 \wedge e_8 + e_6 \wedge e_7\,. \end{aligned}$$

Let ρ_0 be the representation $\Lambda^4(\mathbb{R}^8)^*$ of $GL(8,\mathbb{R})$. The subgroup of $GL(8,\mathbb{R})$ preserving the selfdual 4-form $\Omega_0 = w_1^2 + w_2^2 + w_3^2$ is $Sp(2)Sp(1)$ and the orbit of Ω_0 under the action of $GL(8,\mathbb{R})$ is called $\mathcal{O}$.

We give now some facts about representations of $Sp(2)Sp(1)$. The irreducible representation of $Sp(2)$ with highest root (n_1, n_2) is denoted by $V^{(n_1,n_2)}$. If $\sigma \simeq \mathbb{C}^2$ is the standard representation of $Sp(1)$, the irreducible representations of $Sp(1)$ are the symmetric powers $\sigma^p = Sym^p(\sigma)$. With this notations, one obtains the irreducible representations of $Sp(2)Sp(1)$ as the tensor products $V^{(n_1,n_2)} \otimes \sigma^p$ for $n_1 + n_2 + p$ even. The real irreducible representations $[V^{(n_1,n_2)} \otimes \sigma^p]$ of $Sp(2)Sp(1)$ are just the real parts of the previous ones.

Following Salamon, we put $\lambda_s^r = V^{(n_1,n_2)}$ where

$$(n_1, n_2) = (\underbrace{2,\dots,2}_{s}, \underbrace{1,\dots,1}_{r-2s}, 0,\dots,0)\,,$$

and we abbreviate the real representation $[\lambda_s^r \otimes \sigma^p]$ to $[\lambda_s^r\sigma^p]$. With this notations, one has $\mathfrak{sp}(1) \simeq [\sigma^2]$ and $\mathfrak{sp}(2) \simeq [\lambda_1^2]$. The decomposition of the exterior algebra is given by Swann in [**Swa89**]:

$$\begin{aligned}
\Lambda^1 &\simeq [\lambda_0^1\sigma^1]\,,\\
\Lambda^2 &\simeq \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus [\lambda_0^2\sigma^2]\,,\\
\Lambda^3 &\simeq \Lambda^1 \oplus [\lambda_0^1\sigma^3] \oplus [\lambda_1^3\sigma^1]\,,\\
\Lambda^4 &\simeq [\lambda_1^2\sigma^2] \oplus [\lambda_0^2] \oplus [\lambda_0^2\sigma^2] \oplus \mathbb{R} \oplus [\sigma^4] \oplus [\lambda_2^4]\,.
\end{aligned}$$

The Hodge operator gives a decomposition $\Lambda^4 = \Lambda^4_+ \oplus \Lambda^4_-$ in selfdual and antiselfdual 4-forms. From the inclusion $\mathbb{R} \oplus [\sigma^4] \subset \Lambda^4_+$ and a dimension count, one obtains

$$\Lambda^4_- \simeq [\lambda_1^2\sigma^2] \oplus [\lambda_0^2] \quad \text{and} \quad \Lambda^4_+ \simeq [\lambda_0^2\sigma^2] \oplus \mathbb{R} \oplus [\sigma^4] \oplus [\lambda_2^4]\,.$$

Remark that the image of the infinitesimal action of $\mathfrak{gl}(8,\mathbb{R})$ on Ω_0 is

$$T_{\Omega_0}\mathcal{O} \simeq \frac{\mathfrak{gl}(8,\mathbb{R})}{\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)} \simeq [\lambda_1^2\sigma^2] \oplus [\lambda_0^2] \oplus [\lambda_0^2\sigma^2] \oplus \mathbb{R}$$

so that Λ^4_- appears to be a direct summand of $T_{\Omega_0}\mathcal{O}$ and can be identified in this way with the space Sym_0^2 of traceless symmetric endomorphisms of $\mathbb{R}^8$.

2.2. First step. We start with a quaternionic-contact structure H on the 7-sphere $\mathbb{S}^7$, close to the conformal infinity of the quaternionic hyperbolic metric and with a choice of metric g_H on H, compatible with the quaternionic-contact structure. We choose locally an orthonormal basis $(\frac{1}{\sqrt{2}}d\eta_i)$ of $\Lambda^2_+ H^*$. Let $T\mathbb{S}^7 = H \oplus W$ be a splitting of $T\mathbb{S}^7$. We denote again by g_H the degenerate metric on $T\mathbb{S}^7$ which vanishes on W and coincides with g_H on H. If r denotes the Euclidean radius on $\mathbb{R}^8$, let $\rho = 1 - |r|^2 <$ be a function vanishing to first order on the sphere $\mathbb{S}^7$. With the help of ρ, one identifies a boundary U of $\mathbb{S}^7$ in B^8 with $]0,\varepsilon[\times\mathbb{S}^7$, and one puts

$$g = \frac{1}{\rho^2}(d\rho^2 + \eta_1^2 + \eta_2^2 + \eta_3^2) + \frac{1}{\rho}g_H\,,$$

so that g becomes a riemannian metric on U.

We define locally the 2-forms

$$w_i = \frac{1}{\rho^2}\left(-d\rho\wedge\eta_i + \frac{1}{2}\sum_{r,s}\varepsilon^{irs}\eta_r\wedge\eta_s\right) + \frac{1}{\rho}(d\eta_i + \gamma_i)\,.$$

where $\gamma_i|_H = 0$. and $(d\eta_i + \gamma_i)(W, T\mathbb{S}^7) = 0$. More precisely, one has

$$\gamma_i = \sum_{j=1}^{3} i_{R_j} d\eta_i \wedge \eta_j \, , \tag{13}$$

where (R_1, R_2, R_3) is the dual basis of $(\eta_1|_W, \eta_2|_W, \eta_3|_W)$.

We obtain a globally defined 4-form

$$\Omega = \sum_{i=1}^{3} w_i^2 \, ,$$

which gives a $Sp(2)Sp(1)$ structure on U. It can be extended in an $Sp(2)Sp(1)$ structure on B^8 depending smoothly on H, for instance in deforming smoothly H in H^{can} using the ρ parameter. We still call Ω the 4-form thus obtained and we assume that the choice of g_H and the construction of Ω depends smoothly on the boundary H.

One defines the metric $g_s = \eta_1^2 + \eta_2^2 + \eta_3^2 + g_H$ on $\mathbb{S}^7$.

2.3. Asymptotic behaviour of $d\Omega$. The splitting $T\mathbb{S}^7 = W \oplus H$ gives a decomposition $\Lambda^d T^*\mathbb{S}^7 = \oplus_{i=0}^{3} \Lambda_i^d$ where

$$\Lambda_i^d \simeq \Lambda^i W^* \wedge \Lambda^{d-i} H^* \, ,$$

and with projections $p_i^d : \Lambda^d \to \Lambda_i^d$.

Let $\Omega_i^d(B^8)$ be the set of sections of $\Lambda_i^d \subset \Lambda^d T^* B^8$ over B^8. If $\alpha \in \Omega_i^d(B^8)$, then one has the estimate

$$||\alpha||_g = \rho^{\frac{d+i}{2}} ||\alpha||_{g_s} \tag{14}$$

on U.

LEMMA 2.1. *When ρ goes to zero, the 5-form $d\Omega$ satisfies*

$$||d\Omega||_g = O(\sqrt{\rho}) \, ,$$

and

$$||p_1^4(i_{\rho\frac{\partial}{\partial\rho}} d\Omega) + \frac{2}{\rho^2} \sum_{i,j} (i_{R_j} d\eta_i|_H \wedge d\eta_i|_H + i_{R_i} d\eta_j|_H \wedge d\eta_i|_H) \wedge \eta_j||_g = O(\rho) \, .$$

PROOF. In a neighbourhood of $\mathbb{S}^7$, one has

$$\begin{aligned} dw_i &= -\frac{1}{\rho^3}\sum_{r,s}\varepsilon^{irs}d\rho\wedge\eta_r\wedge\eta_s+\frac{1}{\rho^2}(d\rho\wedge d\eta_i+\sum_{r,s}\varepsilon^{irs}d\eta_r\wedge\eta_s) \\ &\quad -\frac{1}{\rho^2}d\rho\wedge(d\eta_i+\gamma_i)+\frac{1}{\rho}d\gamma_i\,, \\ &= -\frac{1}{\rho^3}\sum_{r,s}\varepsilon^{irs}d\rho\wedge\eta_r\wedge\eta_s+\frac{1}{\rho^2}\sum_{r,s}\varepsilon^{irs}d\eta_r\wedge\eta_s-\frac{1}{\rho^2}d\rho\wedge\gamma_i+\frac{1}{\rho}d\gamma_i \end{aligned}$$

This gives for $d\Omega$ when ρ goes to zero,

$$\begin{aligned} \frac{1}{2}d\Omega &= -\frac{3}{2\rho^4}\sum_{i,r,s}\varepsilon^{irs}d\rho\wedge\gamma_i\wedge\eta_r\wedge\eta_s+\frac{1}{\rho^3}\sum_{i,r,s}\varepsilon^{irs}d\eta_r\wedge\eta_s\wedge\gamma_i \\ &\quad -\frac{1}{\rho^3}\sum_i d\rho\wedge\gamma_i\wedge(d\eta_i+\gamma_i) \\ &\quad -\frac{1}{\rho^3}\sum_i d\rho\wedge\eta_i\wedge d\gamma_i+\frac{1}{2\rho^3}\sum_{i,r,s}\varepsilon^{irs}d\gamma_i\wedge\eta_r\wedge\eta_s+O(\rho)\,. \end{aligned}$$

Pay attention to the fact that in this expression, the term $O(\rho)$ is taken with respect to the metric g. We use here the fact that γ_i vanishes on $H\otimes H$ hence that $||\gamma_i||_{g_\rho}=O(\rho\sqrt{\rho})$. In particular, we see that

$$||d\Omega||_g=O(\sqrt{\rho})\,,$$

which gives the first part of the lemma. Moreover, we have

$$i_{\rho\frac{\partial}{\partial\rho}}d\Omega=-\frac{3}{\rho^3}\sum_{i,r,s}\varepsilon^{irs}\gamma_i\wedge\eta_r\wedge\eta_s-\frac{2}{\rho^2}\sum_i\gamma_i\wedge(d\eta_i+\gamma_i)-\frac{2}{\rho^2}\sum_i\eta_i\wedge d\gamma_i+O(\rho)$$

where $O(\rho)$ is still taken with respect to g. The sums

$$\sum_{i,r,s}\varepsilon^{irs}\gamma_i\wedge\eta_r\wedge\eta_s \text{ and } \sum_i\gamma_i\wedge\gamma_i$$

have no components in Λ^4_1, therefore

$$p_1^4(\rho\frac{\partial}{\partial\rho}d\Omega)=\frac{2}{\rho^2}(\sum_{i,j}i_{R_j}d\eta_i\wedge\eta_j\wedge d\eta_i+\sum_i\eta_i\wedge p_0^3(d\gamma_i))+O(\rho)\,,$$

and we get the lemma using

$$\begin{aligned} d\gamma_i|_H &= d(\sum_j i_{R_j}d\eta_i\wedge\eta_j)|_H\,, \\ &= -\sum_j i_{R_j}d\eta_i|_H\wedge d\eta_j|_H\,. \end{aligned}$$

□

Recall that there exists a unique complement W^{g_H} of H such that

$$\sum_{i,j}(i_{R_j}d\eta_i|_H \wedge d\eta_i|_H + i_{R_i}d\eta_j|_H \wedge d\eta_i|_H) = 0\,.$$

The distribution W^{g_H} depends smoothly on H, the metric g_H and there first derivatives. From now on, we assume that $W = W^{g_H}$.

3. Approximated solutions

In the previous section, for a given quaternionic-contact structure H on the 7-sphere, we have obtained a quaternionic 4-form Ω with boundary H, and such that $d\Omega = O(\sqrt{\rho})$. In this section, we show the existence of a quaternionic 4-form Ω_0, with boundary H and such that $d\Omega_0 = O(\rho^3)$.

3.1. General remarks. We begin with our approximated solution Ω, and we look for w such that $d(\Omega + w) = 0$ and $\Omega + w \in \Gamma(\mathcal{O})$. Thus, we have to study a differential operator $P : \Gamma(\mathcal{O}) \to \Omega^5(B^8)$, given infinitesimaly by

$$\dot{P} : \Gamma(T_\Omega\mathcal{O}) \to \Omega^5(B^8)\,,\ w \mapsto dw\,.$$

The operator $\dot{P}$ is not elliptic, but from 2.1, we have $\Lambda^4_- \subset T_\Omega\mathcal{O}$, hence we will take the restriction of $\dot{P}$ to the sections of Λ^4_-. This restriction is elliptic and explain why we study the operator $d : \Omega^4_-(B^8) \to \Omega^5(B^8)$ in the next section.

3.2. The indicial operators. We start with a quaternionic contact structure H on $\mathbb{S}^7$ and an asymptotically hyperbolic 4-form Ω quaternionic metric g on B^8, with boundary H. There exists a function ρ vanishing to first order on $\mathbb{S}^7$ such that on a neighboorhoud $]0,a[\times\mathbb{S}^7$ of $\mathbb{S}^7$ in B^8, one has

$$g \sim \frac{1}{\rho^2}(d\rho^2 + \sum_i \eta_i^2) + \frac{1}{\rho}g_H \text{ when } \rho \to 0$$

where g_H is a metric on H compatible with the quaternionic contact structure. Let W be a supplementary vector bundle to H in $T\mathbb{S}^7$. On $\mathbb{S}^7$, we define the metrics $g_\rho = (\eta_1^2 + \eta_2^2 + \eta_3^2)/\rho^2 + g_H/\rho$ and $g_s = \eta_1^2 + \eta_2^2 + \eta_3^2 + g_H$ where by definition $g_H(W, T\mathbb{S}^7) = 0$. The Hodge operators associated to g, g_ρ and g_s are denoted respectively by $*$, $*_\rho$ and $*_s$, and ∇^7 is the connection with respect to g_s on $\mathbb{S}^7$, and d_7 is the differential on $\mathbb{S}^7$. Let $\Omega^d_\rho(\mathbb{S}^7)$ be the space of d-forms on $\mathbb{S}^7$ which depend on the parameter $\rho \in]0,a[$. Remark then that one has for $\alpha \in \Omega^d(\mathbb{S}^7)$,

$$*(-\frac{d\rho}{\rho} \wedge *_\rho \alpha) = \alpha + O(\sqrt{\rho})\alpha$$

We are looking at anti-selfdual deformations Ω. We have the isomorphism

$$\Psi_4 : \Omega^4_\rho(\mathbb{S}^7) \to \Omega^4_-(B^8)\,,\ \alpha \mapsto -\frac{d\rho}{\rho} \wedge *_\rho \alpha - *(-\frac{d\rho}{\rho} \wedge *_\rho \alpha)\,.$$

If $\alpha \in \Omega^4_\rho(\mathbb{S}^7)$, one has

$$d\Psi_4(\alpha) = \frac{d\rho}{\rho} \wedge P_1(\alpha) + P_2(\alpha) \text{ where } P_1(\alpha) \in \Omega^4_\rho(\mathbb{S}^7) \text{ and } P_2(\alpha) \in \Omega^5_\rho(\mathbb{S}^7)\,,$$

We are interested in the asymptotic developpments of the operators P_1 and P_2 with respect to the norm of $\Psi_4(\alpha)$, hence we consider now the normalised operators

$$\mathcal{P}_1(\alpha) = \sum_{i,j} \rho^{\frac{4+i}{2}} p_i^4 P_1(\rho^{-\frac{4+j}{2}} p_j^4(\alpha)) \quad \text{and} \quad \mathcal{P}_2(\alpha) = \sum_{i,j} \rho^{\frac{5+i}{2}} p_i^5 P_2(\rho^{-\frac{4+j}{2}} p_j^4(\alpha))\,;$$

and we put

$$\alpha = f *_s (\eta_1 \wedge \eta_2 \wedge \eta_3) + \sum_i \nu_i \wedge \eta_i + \frac{1}{2}\sum_{i,r,s} \varepsilon^{irs} \gamma_i \wedge \eta_r \wedge \eta_s + \xi \wedge \eta_1 \wedge \eta_2 \wedge \eta_3\,,$$

where ε^{irs} is the signature of the permutation (i,r,s) of $(1,2,3)$ and

$$\nu_i \in \Lambda^3 H^*\,,\ \gamma_i \in \Lambda^2 H^*\,,\ \xi \in H^*\,.$$

To simplify the notations, we put $\eta_{123} = \eta_1 \wedge \eta_2 \wedge \eta_3$ and $\eta_{ij} = \eta_i \wedge \eta_j$. The Hodge operator of the metric g_H on H is $*_H$.

LEMMA 3.1. *With the previous decomposition of $\alpha \in \Omega^4_\rho(\mathbb{S}^7)$, we get*

$$\begin{aligned}
\mathcal{P}_1(\alpha) = (-\rho\frac{\partial \xi}{\partial \rho} + \frac{7}{2}\xi) \wedge \eta_{123} + \sum_{i,r,s} \frac{\varepsilon^{irs}}{2}(-\rho\frac{\partial \gamma_i}{\partial \rho} + 3\gamma_i + 2f d_7\eta_i|_H) \wedge \eta_{rs} \\
+ \sum_i (-\rho\frac{\partial \nu_i}{\partial \rho} + \frac{5}{2}\nu_i + \sum_{r,s} \varepsilon^{sri} *_H \nu_s \wedge d\eta_r|_H) \wedge \eta_i \\
+(-\rho\frac{\partial f}{\partial \rho} + 2f) *_s \eta_{123} + \sum_i *_H \gamma_i \wedge d_7\eta_i|_H \\
+O(\sqrt{\rho}\,\alpha) + O(\sqrt{\rho}\,\nabla^7\alpha) + O(\rho\sqrt{\rho}\,\frac{\partial \alpha}{\partial \rho})\,.
\end{aligned}$$

Proof. From the definition of $\mathcal{P}_1$, we get

$$\mathcal{P}_1(\alpha) = \sum_{i,j} \rho^{\frac{i-j}{2}} p_i^4 d_7 *_\rho p_j^4(\alpha) + \sum_i \frac{4+i}{2} p_i^4(\alpha) - \rho \sum_i \frac{\partial}{\partial\rho} p_i^4(\alpha)$$
$$+ O(\sqrt{\rho}\,\alpha) + O(\rho\sqrt{\rho}\,\frac{\partial\alpha}{\partial\rho}),$$

so that the only term which remains to compute is

$$\sum_{i,j} \rho^{\frac{i-j}{2}} p_i^4 d_7 *_s p_j^4(\alpha) = \sum_{i,j} \rho^{\frac{i+j-2}{2}} p_i^4 d_7 *_\rho p_j^4(\alpha).$$

Because we are interested in the high order terms, we can now assume that $\xi = 0$. One has

$$*_s\alpha = f\eta_{123} - \frac{1}{2}\sum_{i,r,s} \varepsilon^{irs} *_H \nu_i \wedge \eta_{rs} + \sum_i *_H\gamma_i \wedge \eta_i .$$

Applying d_7, we get

$$\begin{aligned}\sum_{i,j} \rho^{\frac{i+j-2}{2}} p_i^4 d_7 *_s p_j^4\alpha &= \sum_{i,r,s} \varepsilon^{irs} f d_7\eta_i|_H \wedge \eta_{rs} + \sum_{i,r,s} \varepsilon^{irs} *_H \nu_i \wedge d_7\eta_r|_H \wedge \eta_s \\ &\quad + \sum_i *_H\gamma_i \wedge d_7\eta_i|_H + O(\sqrt{\rho}\,\nabla^7\alpha)\end{aligned}$$

and the lemma follows. □

In the same way, we obtain the asymptotic behaviour of $\mathcal{P}_2$.

Lemma 3.2. *The operator $\mathcal{P}_2$ satisfies*

$$\mathcal{P}_2(\alpha) = \frac{1}{2}\sum_{i,r,s} \varepsilon^{irs}(*_H I_i\xi)\wedge\eta_{rs} + \frac{1}{2}\sum_{i,r,s} \varepsilon^{irs}\gamma_i \wedge d_7\eta_r|_H \wedge \eta_s + O(\sqrt{\rho}\,\alpha) + O(\sqrt{\rho}\,\nabla^7\alpha).$$

Definition 3.1. The indicial operator of $\mathcal{P}_1$ on $\Omega^4_\rho(\mathbb{S}^7)$ is

$$I(\mathcal{P}_1) = (-\rho\frac{\partial\xi}{\partial\rho} + \frac{7}{2}\xi) \wedge \eta_{123} + \sum_{i,r,s} \frac{\varepsilon^{irs}}{2}(-\rho\frac{\partial\gamma_i}{\partial\rho} + 3\gamma_i + 2f d_7\eta_i|_H) \wedge \eta_{rs}$$
$$+ \sum_i (-\rho\frac{\partial\nu_i}{\partial\rho} + \frac{5}{2}\nu_i + \sum_{r,s} \varepsilon^{sri} *_H \nu_s \wedge d\eta_r|_H) \wedge \eta_i$$
$$+ (-\rho\frac{\partial f}{\partial\rho} + 2f) * \eta_{123} + \sum_i *_H\gamma_i \wedge d_7\eta_i|_H .$$

In the following, we compute the kernel of $I(\mathcal{P}_1)$. For this purpose, we must refine the decomposition of $\Omega_\rho(\mathbb{S}^7)$. The forms $d\eta_i|_H$ span the space $\Lambda^2_+ H^*$ of selfdual 2-forms on H with respect to the metric g_H. We can now refine the splitting of some Λ^p_i, identifying the spaces W^*, $\Lambda^2 W^*$ and $\Lambda^2_+ H^*$. In particular, we have using this identifications,

$$\Lambda^4_2 \simeq \Lambda^2 H^* \otimes \Lambda^2_+ H^* = \Lambda^4_{2,1} \oplus \Lambda^4_{2,8} \oplus \Lambda^4_{2,9}\,, \text{ where } \left\{ \begin{array}{lcl} \Lambda^4_{2,1} & \simeq & \mathbb{R}\,, \\ \Lambda^4_{2,8} & = & End_0(\Lambda^2_+ H^*)\,, \\ \Lambda^4_{2,9} & = & \Lambda^2_- H^* \otimes \Lambda^2_+ H^*\,, \end{array} \right.$$

and $End_0(\Lambda^2_+ H^*)$ denotes the trace-free endomorphisms of $\wedge^2_+ H^*$. If $(\gamma_i)_{i\in\{1,2,3\}}$ is in Λ^4_2, we put

$$\gamma_i = \gamma w_i + \gamma_i^+ + \gamma_i^-$$

with $\gamma \in \mathbb{R}$, $(\gamma_i^+)_i \in \Lambda^4_{2,8}$ and $(\gamma_i^-)_i \in \Lambda^4_{2,9}$. In the same way, we have a decomposition

$$\Lambda^4_1 = \Lambda^4_{1,4} \oplus \Lambda^4_{1,8} \text{ where } \left\{ \begin{array}{lcl} \Lambda^4_{1,4} & = & \{(\nu_i) \in \Lambda^3 H^*\,,\ \nu_i = *_H(I_i \nu)\,,\ \nu \in H^*\}\,, \\ \Lambda^4_{1,8} & = & \{(\nu_i) \in \Lambda^3 H^*\,,\ \nu_i = *_H(I_i \nu_i^0)\,,\ \sum_i I_i \nu_i = 0\}\,. \end{array} \right.$$

Proposition 3.1. *The indicial operator $I(\mathcal{P}_1)$ can be written*

$$I(\mathcal{P}_1) = -\rho \frac{\partial}{\partial \rho} + A\,,$$

where A is linear and satisfies

$$\begin{array}{lcl} A & = & \frac{7}{2} Id \text{ on } \Gamma_\rho(\Lambda^4_3 \oplus \Lambda^4_{1,8})\,, \\ A & = & \frac{1}{2} Id \text{ on } \Gamma_\rho(\Lambda^4_{1,4})\,, \\ A & = & 3Id \text{ on } \Gamma_\rho(\Lambda^4_{2,9} \oplus \Lambda^4_{2,8})\,, \\ A & = & \begin{pmatrix} 3Id & 2Id \\ 6Id & 2Id \end{pmatrix} \text{ on } \Gamma_\rho(\Lambda^4_0 \oplus \Lambda^4_{2,1})\,. \end{array}$$

Moreover, the eigenvalues of A on $\Lambda^4_0 \oplus \Lambda^4_{2,1}$ are 6 and -1.

Proof. We follow the previous notations for $\alpha \in \Omega^4_\rho(\mathbb{S}^7)$. The restriction of A to Λ^4_3 comes immediately from the definition 3.1. Now, if $\alpha \in \Gamma_\rho(\Lambda^4_{2,9} \oplus \Lambda^4_{2,8})$, then one has

$$\sum_i *_H(\gamma_i^+ + \gamma_i^-) \wedge d_7 \eta_i|_H = 0\,,$$

therefore still from 3.1, we obtain $A\alpha = 3\alpha$.

Next, if $\alpha = \sum_i *_H I_i\nu \wedge \eta_i \in \Lambda^4_{1,4}$ then one has

$$\begin{aligned} A\alpha &= \tfrac{5}{2}\alpha + \textstyle\sum_{r,s} \varepsilon^{sri} I_s\nu \wedge w_r\,, \\ &= \tfrac{5}{2}\alpha + \textstyle\sum_{r,s} \varepsilon^{sri} *_H (I_r I_s \nu)\,, \\ &= \tfrac{5}{2}\alpha - 2 *_H I_i\nu\,, \end{aligned}$$

hence $A = \frac{1}{2} Id$ on $\Lambda^4_{1,4}$. The case $\alpha \in \Lambda^4_{1,8}$ is very similar, and it remains now the case $\alpha \in \Lambda^4_0 \oplus \Lambda^4_{2,1}$ which is a completely analogous computation. □

Definition 3.2. The indicial operator of $\mathcal{P}_2$ is

$$I(\mathcal{P}_2)(\alpha) = \frac{1}{2}\sum_{i,r,s} \varepsilon^{irs}(*_H I_i\xi) \wedge \eta_{rs} + \frac{1}{2}\sum_{i,r,s} \varepsilon^{irs}\gamma_i \wedge d_7\eta_r|_H \wedge \eta_s\,.$$

We need now to compute the indicial operators of $d : \Omega^5(B^8) \to \Omega^6(B^8)$. In the same way, we identify $\gamma \in \Omega^d(B^8_*)$ with $(\alpha, \beta) \in \Omega^{d-1}_\rho(\mathbb{S}^7) \times \Omega^d_\rho(\mathbb{S}^7)$, where $\gamma = \frac{d\rho}{\rho} \wedge \alpha + \beta$. We write $d\gamma = (Q_1(\alpha,\beta), Q_2(\alpha,\beta))$ using this identification. Following what has been done with the operators P_i, we normalize the operators Q_i and write $\mathcal{Q}_i$ for the new operators. We choose the notations

$$\alpha = f *_s \eta_{123} + \sum_i \nu_i \wedge \eta_i + \frac{1}{2}\sum_{i,r,s} \varepsilon^{irs}\gamma_i \wedge \eta_{rs} + \xi \wedge \eta_{123}\,,$$

$$\beta = \sum_s h_s \wedge \eta_s + \sum_i \varepsilon^{irs}\mu_i \wedge \eta_{rs} + \zeta \wedge \eta_{123}$$

where f, ν_i, γ_i, ξ, h_s, μ_i and ζ are forms vanishing on W. A straightforward computation gives the indicial operators of $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

Proposition 3.2. *The indicial operators of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are*

$$I(\mathcal{Q}_1)(\alpha,\beta) = -\rho\frac{\partial\beta}{\partial\rho} + 3\sum_s h_s \wedge \eta_s + \frac{7}{2}\sum_{i,r,s} \varepsilon^{irs}\mu_i \wedge \eta_{rs} + 4\zeta \wedge \eta_{123}$$

$$-\sum_{i,r,s} \gamma_i \wedge d\eta_r|_H \wedge \eta_s - \frac{1}{2}\sum_{i,r,s} \varepsilon^{irs}\xi \wedge d\eta_i|_H \wedge \eta_{rs}\,,$$

and

$$I(\mathcal{Q}_2)(\alpha,\beta) = \frac{1}{2}\sum_i \varepsilon^{irs}\zeta \wedge d\eta_i|_H \wedge \eta_{rs}\,.$$

3.3. Main result of the section. Let E be a vector bundle on $\mathbb{S}^7$, giving a vector bundle on $\overline{B^8} - \{0\}$. If k is a positive integer and p is real, let $\tilde{C}^k_p(E)$ be the set of sections σ of E on B^8 such that $\rho^{-p}\sigma$ admits a C^k extension over $\overline{B^8} - \{0\}$ for any function ρ vanishing up to order 1 on the boundary $\partial B^8 = \mathbb{S}^7$. We define now $C^k_p = \tilde{C}^k_p + \tilde{C}^k_{p+\frac{1}{2}}$.

With respect to the decomposition $\Lambda^5 T^* B^8 \simeq \frac{d\rho}{\rho} \wedge \Lambda^4 T^*\mathbb{S}^7 \oplus \Lambda^5 T^*\mathbb{S}^7$ and if H is C^{k+1}, the 4-form Ω satisfies

$$\sum_i p^4_i \rho^{-\frac{4+i}{2}} d\Omega + \sum_i p^5_i \rho^{-\frac{5+i}{2}} d\Omega \in C^k_{\frac{1}{2}}(\Lambda^4 \oplus \Lambda^5)\,.$$

Moreover, from lemma 2.1, the projection $p^4_1 \rho^{-\frac{5}{2}} d\Omega$ is $C^k_1(\Lambda^4_1)$.

THEOREM 3.1. *If H is a quaternionic contact structure of regularity C^{k+6}, close to the boundary of the quaternionic hyperbolic metric, there exists a quaternionic form Ω_H of regularity C^{k+1} on B^8, with conformal infinity H and depending smoothly on H, and such that*

$$\sum_i p^4_i \rho^{-\frac{4+i}{2}} d\Omega_H + \sum_i p^5_i \rho^{-\frac{5+i}{2}} d\Omega_H \in C^k_3(\Lambda^4 \oplus \Lambda^5)$$

In particular,

$$||d\Omega_H||_g = O(\rho^3)\,.$$

We need to prove the lemma :

LEMMA 3.3. *Assume that $1/2 \leq p < 3$. Let (α, β) be in $C^k_p(\Lambda^4 T^*\mathbb{S}^7) \times C^k_p(\Lambda^5 T^*\mathbb{S}^7)$, without Λ^4_1 term if $p = 1/2$. If $\mathcal{Q}_1(\alpha,\beta)$ and $\mathcal{Q}_2(\alpha,\beta)$ are $C^{k-1}_{p+\frac{1}{2}}$ sections, then there exists a section $\alpha' \in C^k_p(\Lambda^4 T^*\mathbb{S}^7)$ such that $\mathcal{P}_1(\alpha') - \alpha \in C^{k-1}_{p+\frac{1}{2}}(\Lambda^4 T^*\mathbb{S}^7)$ and $\mathcal{P}_2(\alpha') - \beta \in C^{k-1}_{p+\frac{1}{2}}(\Lambda^5 T^*\mathbb{S}^7)$.*

PROOF. We follow the notations of the previous section, adding a $'$ on each component of α'. Moreover, we identify f and $f *_s \eta_{123}$, ν_i and $\nu_i \wedge \eta_i$, etc.

From the indicial operator $I(\mathcal{Q}_2)$ obtained in proposition 3.2.2, we obtain $\zeta \in C^k_{p+\frac{1}{2}}(H^*)$. We explain now how to remove the ξ and μ_i components of (α, β). We define $\xi'(\rho, x)$ to be $\rho^p/(7/2 - p)$ times $(\rho^{-p}\xi)(0, x)$. Because $\rho^p\xi$ is C^k up to the boundary, the section ξ' is well defined. Then, one has $\mathcal{P}_1(\xi')(\rho,x) - \xi(\rho,x) = \rho^p((\rho^{-p}\xi)(0,x) - (\rho^{-p}\xi(\rho,x)) + O(\rho^{p+\frac{1}{2}})$, hence $\mathcal{P}_1(\xi') - \xi$ is a $C^{k-1}_{p+\frac{1}{2}}$ section. Because of the indicial operator of $\mathcal{P}_2$, we must now verify the compatibility $\mu_i = \frac{1}{2} *_H I_i \xi'$, up to a section of $C^k_{p+\frac{1}{2}}$. But from the indicial operator of $\mathcal{Q}_1$, we obtain

$$-\rho\frac{\partial\mu_i}{\partial\rho}+\frac{7}{2}\mu_i-\frac{1}{2}*_H I_i\xi=-\rho\frac{\partial}{\partial\rho}(\mu_i-\frac{1}{2}*_H I_i\xi')+\frac{7}{2}(\mu_i-\frac{1}{2}*_H I_i\xi')=0$$

up to a $C^{k-1}_{p+\frac{1}{2}}$ term. This is exactly what we need to conclude that $\mu_i-\mathcal{P}_2(\xi')\in C^{k-1}_{p+\frac{1}{2}}$. The other components can be obtained in the same way, except for the Λ^4_1 component when $p=\frac{1}{2}$. □

We show now the theorem 3.3.1 :

Proof. We begin with the first approximation Ω, which depends smoothly on H and satisfies $||d\Omega||_g=O(\sqrt{\rho})$. The first step is to search for a section A of traceless symetric endomorphisms Sym^2_0 such that $\mathrm{Id}+A_1$ is inversible and $||d\rho_0(\mathrm{Id}+A_1)^{-1}\Omega||_g=O(\rho)$ where ρ_0 is the representation Λ^4 of $GL(8,\mathbb{R})$. One has $\rho_0(\mathrm{Id}+A_1)^{-1}\Omega=\Omega+w(A_1)+F(A_1)$ where $w:A\in Sym^2_0\mapsto w(A)\in\Lambda^4_-$ is linear and isometric in A, and where $F(A)$ is polynomial in A, without terms of order 0 or 1, and with constant coefficients in every orthonormal basis.

Hence we need only look at the equation $||d(\Omega+w(A_1))||_g=O(\rho)$. From the previous section, it suffices to look at the indicial operators. The first approximation Ω can be chosen such that $d\Omega$ does not have $C^{k-1}_{\frac{1}{2}}(\Lambda^4_1)$ terms. Therefore, we can apply the previous lemma to obtain $w(A_1)$. This argument can be applied recursively up to $p=3$. We obtain finally a 4-form Ω_0. Each step depends smoothly on the boundary, and if the boundary is sufficiently close to the canonical one, the endomorphism $\mathrm{Id}+A=(\mathrm{Id}+A_1)(\mathrm{Id}+A_2)\cdots$ is inversible. We obtain a quaternionic 4-form on a neighbourhood of $\mathbb{S}^7$, which can be extended to a quaternionic 4-form on the ball provided H is not to far from H^{can}. □

Remark 3.1. The volume form of g is $\rho^{-6}d\rho\wedge\nu_7$ for a volume form ν_7 on S^7, admitting a C^k extension to $\overline{B^8}$. Thus, the 5-form $d\Omega_H$ is square integrable.

4. Towards quaternionic symplectic forms

The aim of this section is to give an idea which could be used to get closed asymptotically hyperbolic quaternionic forms on the 8-ball with prescribed boundaries.

4.1. Preliminaries. Here g denotes an asymptotically hyperbolic quaternionic metric on the 8-ball B with Levi-Civita connection ∇ and volume form $d\nu_g$. In the following, let (E,h) be a metric vector bundle on B associated to a $SO(8)$-representation. The Levi-Civita connection gives a metric connection on E still denoted by ∇.

Let $H^k(E)$ be the set of sections σ of E satisfying

$$\int_{B^8} |\nabla^j \sigma|_h^2 d\nu_g < +\infty \,.$$

for $j \in \{0, \cdots, k\}$ with its usual norm $||\cdot||_{H^k}$. The set of L^2-functions on (B^8, g) with k L^2-derivatives is just denoted by H^k. We state now some analysis facts using the geometry at infinity of g.

Let H be the boundary of g, let g_H be a metric on H and ρ be a function vanishing up to order 1 on $\mathbb{S}^7$ such that

$$g \sim \frac{1}{\rho^2}(d\rho^2 + \sum_i \eta_i^2) + \frac{1}{\rho} g_H$$

when ρ goes to infinity. Here (η_1, η_2, η_3) is a $SO(3)$-choice of 1-forms vanishing on H with dual vector fields R_1, R_2, and R_3. One defines the vector fields $\xi_0 = \rho \frac{\partial}{\partial \rho}$, $\xi_i = \rho R_i$ and $\xi_j = \sqrt{\rho} X_j$ if$(X_j)_{j=4,\ldots,7}$ is an orthonormal basis of H. We assume that the asymptotic developpment of g is such that

$$g(\xi_i, \xi_j) - \delta_{ij} \in C^k_{\frac{1}{2}} \tag{15}$$

for a given integer $k > 0$. Then, one has

$$\begin{aligned}
g([\xi_0, \xi_i], \xi_j) &= g(\xi_i, \xi_j) = \delta_{ij} + O(\sqrt{\rho}) && \text{if } 1 \leq i \leq 3\,;\\
g([\xi_0, \xi_i], \xi_j) &= \tfrac{1}{2}\delta_{ij} + O(\sqrt{\rho}) && \text{if } 4 \leq i \leq 7\,;\\
g([\xi_i, \xi_j], \xi_k) &= O(\sqrt{\rho}) && \text{if } 1 \leq i \leq 3\,;\\
g([\xi_i, \xi_j], \xi_k) &= -\textstyle\sum_{r=1}^3 d\eta_r(X_i, X_j)\delta_{rk} + O(\sqrt{\rho}) && \text{if } 4 \leq i,\, j \leq 7\,.
\end{aligned}$$

We put $g([\xi_i, \xi_j], \xi_k) = \gamma_{ij}^k = \gamma_{ij,can}^k + O(\sqrt{\rho})$ where the one forms $\gamma_{ij,can}^k$ equal to $g_{\mathcal{H}}([\xi_i^{can}, \xi_j^{can}], \xi_k^{can})$ for a particular choice (ξ_i^{can}) of orthonormal frame for the hyperbolic quaternionic metric $g_{\mathcal{H}}$. The coefficients R_{ijk}^l of the curvature R of ∇ in the frame (ξ_i) admit an asymptotic developpment $R_{ijk}^l = R_{ijk,can}^l + O(\sqrt{\rho})$ where the constants $(R_{ijk,can}^l)$ are the coefficients of the curvature of $g_{\mathcal{H}}$ in the frame (ξ_i^{can}). Therefore, the curvature is bounded and the geometry of g is uniform at infinity. Analysis on complete Riemannian manifolds with bounded curvature shares a lot of the features of analysis on $\mathbb{R}^n$. In particular, the Sobolev embeddings are true in such manifolds, see [**Aub91**, chapter 2].

THEOREM 4.1. *Let (M^n, g) be a complete Riemannian manifold with bounded curvature and injectivity radius $\delta > 0$. If $k > n/2$, we have the continuous* *embedding $H^k \subset C^0_B$ where C^0_B is the set of continuous bounded functions on M^n.*

We assume now that $g(\xi_i, \xi_j) - \delta_{ij} \in C^k_{\frac{1}{2}}$ with $k \geq 5$.

COROLLARY 4.1. *One has a continuous embedding $H^5(E) \subset C^0_B(E)$ where $C^0_B(E)$ is the set of continuous bounded sections of E on (B^8, g).*

PROOF. Such a vector bundle can be emdedded in a tensorial product $T^*B^{\otimes^r} \otimes TB^{\otimes^s}$ so that it is sufficient to look at the sections of TB. For a vector field X and from the relation $dg(X, \xi_i) = g(\nabla X, \xi_i) + g(X, \nabla \xi_i)$ outside a compact set K of B, we deduce that $g(X, \xi_i)$ is a H^1 function and obtain the estimate $||g(X, \xi_i)||_{H^1(B-K)} \leq A_i ||X||_{H^1(B-K)}$. The same kind of estimate are true with higher order derivatives terms. Indeed, let us look at second covariante derivatives. One has

$$\nabla^2 \xi_j = \nabla \sum_{i,k} \gamma^k_{ij} \xi^*_i \otimes \xi_k = \sum_{i,k} d\gamma^k_{ij} \otimes \xi^*_i \otimes \xi_k + \sum_{i,k} \gamma^k_{ij} \nabla(\xi^*_i \otimes \xi_k)$$

so that it is sufficient to look at the behaviour of $d\gamma^k_{ij}$. But from (15), we obtain $\gamma^k_{ij} - \gamma^k_{ij,can} \in C^{k-1}_{\frac{1}{2}}$ so that $d\gamma^k_{ij}(\xi_p)$ is still a $C^{k-2}_{\frac{1}{2}}$ function. The higher order estimate are obtained in the same way and finally, one has $||g(X, \xi_i)||_{H^5(B-K)} \leq C_i ||X||_{H^5(B-K)}$. Therefore, the functions $g(X, \xi_i)$ are continuous and bounded. Using the Grahm-Schmidt orthonormalization process with the basis (ξ_i), we obtain the corollary. □

If Ω is an asymptotically hyperbolic quaternionic form, one defines $\Phi : \Gamma((\mathfrak{sp}(2) \oplus \mathfrak{sp}(1))^\perp) \to \Gamma(\Lambda^5 T^*B)$ be the function $A \mapsto \rho_0(\mathrm{Id} + A)(\Omega)$ when $\mathrm{Id} + A$ is inversible. All the sections A of $\mathfrak{gl}(TB)$ we shall use will be small enough to obtain automatically the inversibility of $\mathrm{Id} + A$.

If $P : E \to \mathbb{R}$ is polynomial with constant coefficient in every orthonormal basis, we obtain with the previous corollary that $P(\sigma) \in H^{10}$ for $\sigma \in H^{10}(E)$. Therefore Φ is a map from a neighbourhood of zero in $H^{10}((\mathfrak{sp}(2) \oplus \mathfrak{sp}(1))^\perp)$ to $H^9(\Lambda^5)$.

4.2. A partial result. From now on, one fixes the function $\rho = 1 - |x|^2$ on the 8-ball B. If g_H is a compatible metric on a quaternionic contact structure, we extend g_H by zero on W^{g_H}.

In this section, we assume that the following hypothesis holds :

HYPOTHESIS 4.1. *There exist constants C, ε and a neighbourhood U of H^{can} in the set of quaternionic contact structures of regularity C^1 on $\mathbb{S}^7$ such* *that if g is a metric on B of the form*

$$g = \frac{1}{\rho^2}(d\rho^2 + \sum_i \eta_i^2) + \frac{1}{\rho} g_H$$

on $\{\rho \leq \varepsilon\}$, then for all smooth forms α with compact support in $\{\rho \leq \varepsilon\}$, one has the Poincaré estimate

$$||\Delta\alpha||_{L^2} \geq C||\alpha||_{L^2} \,. \tag{16}$$

Remark 4.1. The Hodge Laplacian for the quaternion hyperbolic metric $g_{\mathcal{H}}$ on 5-forms is an isomorphism (**[Ped04]** for a study of the Hodge Laplacian on the quaternion hyperbolic space), and its spectrum is $[1; +\infty[$. The previous hypothèsis is equivalent to an hypothesis of uniform bellow boundness on the essential spectrum of quaternion hyperbolic metrics, i.e. is equivalent to the fact that the essential spectrum of such metrics is contained in $[\frac{1}{2}; +\infty[$ if the boundary is close to H^{can}.

Let us show now how this hypothesis can be used to prove the existence of asymptotically hyperbolic quaternionic symplectic forms.

Let p^4_- be the projection $\Lambda^4 \to \Lambda^4_-$. On smooth 5-forms with compact support, one has $\Delta = 2d\delta_- + \delta d$.

Remark 4.2. We will use the metrics associated to the quaternionic form Ω_H given by the theorem 3.1. This metrics can be written

$$g(H) = \frac{1}{\rho^2}(d\rho^2 + \sum_i \eta_i^2) + \frac{1}{\rho} g_H + \frac{1}{\rho^{\frac{1}{2}}} g_{-\frac{1}{2}} + \cdots$$

on $\{\rho \leq \varepsilon\}$ with a finite number of g_i depending smoothly on H. Therefore, possibly changing C and U (neighbourhood of H^{can} in C^k norm), we can assume that hypothesis 4.1 still holds for the metrics $g(H)$.

Lemma 4.1. *There exists $\varepsilon > 0$ and a constant C such that for the metrics $g(H)$ with $H \in U$, one can replace the estimate (16) by the estimate*

$$||\Delta\alpha||_{L^2} \geq C||\alpha||_{H^2}$$

when α has support in $\{\rho \leq \varepsilon\}$.

Proof. By homogeneity, in the case of the hyperbolic metric $g_{\mathcal{H}}$ one has on each ball $B(1)$ of radius 1 the elliptic inequality

$$||\alpha||_{H^2(B(1/2))} \leq c(||\alpha||_{L^2(B(1))} + ||\Delta\alpha||_{L^2(B(1))})$$

with a constant c which does not depend on the center of $B(1)$. The lemma follows for the hyperbolic metric. For a general asymptotically hyperbolic metric, the uniform geometry of $g(H)$ at infinity gives the same result, possibly shrinking the set $\{\rho \leq \varepsilon\}$. $\square$

LEMMA 4.2. *There exists a neighbourhood U of H^{can} such that if $g(H)$ is the asymptotically hyperbolic metric with boundary $H \in U$ obtained with 3.1, then the Hodge Laplacian Δ is an isomorphism $H^2(\Lambda^5) \to L^2(\Lambda^5)$ with uniformly bounded inverse.*

PROOF. Suppose that Δ is not inversible with bounded inverse for $g(H)$ near the hyperbolic metric $g_{\mathcal{H}}$. Then, there exists a sequence of quaternionic contact structures H_n which converges to H^{can} and a sequence of 5-forms α_n such that

$$||\alpha_n||_{H^2(g(H_n))} = 1 \text{ and } ||\Delta^{g(H_n)}(\alpha_n)||_{L^2(g(H_n))} \to 0\,. \tag{17}$$

We can assume that α_n is strongly H^1-convergent on every compact set towards a H^1-form α which satisfies $\Delta^{g_{\mathcal{H}}}\alpha = 0$. Let $(\chi, 1-\chi)$ be a partition of unity associated to the covering $(\{\rho < \varepsilon\}, \{\rho > \varepsilon/2\})$. From the previous lemma, we obtain

$$C||\chi\alpha_n||_{H^2(g(H_n))} \leq ||\Delta^{g(H_n)}(\chi\alpha_n)||_{L^2(g(H_n))}\,.$$

One can write $\Delta^{g(H_n)}(\chi\alpha_n) = \chi\Delta^{g(H_n)}(\alpha_n) + P_n(\alpha_n)$ where P_n is a differential operator of order 1, with support in $\{\rho \geq \varepsilon/2\}$ and which converges towards an operator P of order 1. Therefore $||P_n(\alpha_n)||_{L^2(g(H_n))}$ converges to $||P(\alpha)||_{L^2(g_{\mathcal{H}})}$. If $P(\alpha) \neq 0$, then $\alpha \neq 0$ and this contradicts the fact that $\Delta^{g_{\mathcal{H}}}$ is an isomorphism. From the inequality

$$C||\chi\alpha_n||_{H^2(g(H_n))} \leq ||\chi\Delta^{g(H_n)}(\alpha_n)||_{L^2(g(H_n))} + ||P_n(\alpha_n)||_{L^2(g(H_n))}$$

we deduce that if $P(\alpha) = 0$, then $||\chi\alpha_n||_{H^2(g(H_n))} \to 0$ and from (17), we get

$$||(1-\chi)\alpha_n||_{H^2(g(H_n))} \to 1\,.$$

One has uniform elliptic estimates on $\{\rho \geq \varepsilon/2\}$, and we obtain in the same way,

$$\begin{aligned} C'||(1-\chi)\alpha_n||_{H^2(g(H_n))} &\leq ||(1-\chi)\Delta^{g(H_n)}\alpha_n||_{L^2(g(H_n))} \\ &+||Q_n(\alpha_n)||_{L^2(g(H_n))} + ||(1-\chi)\alpha_n||_{L^2(g(H_n))} \end{aligned}$$

with $||Q_n(\alpha_n)||_{L^2(g(H_n))} \to 0$. Hence $||(1-\chi)\alpha||_{L^2(g_{\mathcal{H}})} \neq 0$ and so α is not zero. This still contradicts the fact that $\Delta^{g_{\mathcal{H}}}$ is an isomorphism. $\square$

REMARK 4.3. By uniform elliptic estimates, this is still true if we replace H^2 by H^{11} and L^2 by H^9.

On $H^9(\Lambda^5) \cap \ker d$, we define $\Psi = 2\delta_-(\Delta)^{-1}$. This a uniformly bounded inverse for the smooth map $d : H^{10}(\Lambda^4_-) \to H^9(\Lambda^5) \cap \ker d$.

LEMMA 4.3. *Let $\Phi : E \to F$ a C^1 function between Banach spaces such that $\Phi(0) = 0$ and such that there exists a right inverse Ψ to $d_0\Phi$, with $||\Psi|| \leq C$. Let $\tau > 0$ be such that for all $||y|| \leq C\tau$,*

$$||d_y\Phi - d_0\Phi|| \leq \frac{1}{2C}\,.$$

Then for all $z \in F$ such that $||z|| < \frac{\tau}{2}$, there exists $||y|| < C\tau$ solution to the equation $\Phi(y) = z$.

PROOF. Let ϕ_z be the C^1 map on F defined by $\phi_z(x) = z - \Phi \circ \Psi(x) + x$. A fixed point $||x|| < \tau$ of ϕ_z satisfies $\Phi \circ \Psi(x) = z$ and $||\Psi(x)|| < C\tau$. Therefore, it suffices to apply the fixed point theorem to ϕ_z for $||x|| < \tau$. One has

$$||d_x\phi_z|| \leq ||d_{\Psi(x)}\Phi - d_0\Phi|| \cdot ||\Psi|| \leq \frac{1}{2}$$

when $||x|| < \tau$. Moreover, we get

$$||\phi_z(x)|| \leq ||z|| + C \sup_{||y||<C\tau} ||d_y\Phi - d_0\Phi|| \cdot ||x|| < \tau$$

if $||x|| < \tau$ and $||z|| < \frac{\tau}{2}$ so that we can apply the fixed point theorem. □

THEOREM 4.2. *Assume that hypothesis 4.1 holds. Then, if H is a quaternionic contact structure of regularity C^{15} close to the boundary H^{can} of the hyperbolic quaternionic metric, then there exists a closed asymptotically hyperbolic form Ω, with regularity H^{10} and boundary H.*

PROOF. The 4-form Ω_H constructed in theorem 3.1 satisfies $d\Omega_H \in H^4(\Lambda^5)$ when H is C^{10}.

Moreover, Ω_H depends smoothly on H, and is closed when $H = H^{can}$. In order to apply the previous lemma it is sufficient to remark that $||d_A\Phi - d_0\Phi||$ is uniformly bounded for H in a neighbourhood of H^{can} and for A uniformly small. □

We discuss now some facts about the space of quaternionic symplectic forms with boundary H^{can} of Ω^{can}, the quaternionic form corresponding to the hyperbolic space. The complex of infinitesimal deformations of Ω^{can} is

$$\begin{array}{ccccc} \Gamma(\Lambda^1) & \longrightarrow & \Gamma(T_{\Omega^{can}}\mathcal{O}) & \longrightarrow & \Gamma(\Lambda^5) \\ X & \mapsto & di_X\Omega^{can} & & \\ & & w & \mapsto & dw\,. \end{array}$$

A dimension count is sufficient to see that this complex is not elliptic in the middle term. This explains why we considered only anti-selfdual forms. One puts $T_{\Omega^{can}}\mathcal{O} \cap \Lambda^2_+ = E$.

The space of infinitesimal deformations is parametrized by the 4-forms $w = w_E + w_-$, where $w_E \in E$ and $w_- \in \Lambda^4_-$, satisfying $dw_E + dw_- = 0$ and $pdw_E = 0$ where p is the projection $\Lambda^5 \to [\lambda^1_0\sigma^1]$. Because there are no harmonic 5-forms, if $pdw_E = 0$, there exists a section w_- of Λ^4_- such that $dw_- = dw_E$.

Moreover, there is no harmonic $L^2(E)$ forms ([**Ped04**]), and the kernel of the restriction of the Hodge Laplacian to Λ^4_- is made of a unique discrete series representation (π, V) contained in the sections of $[\lambda^2_1\sigma^2]$. Thus, the space of infinitesimal deformations of Ω^{can} is parametrized by the direct sum of V and $\{w_E \in L^2(E)\,,\ pdw_E = 0\}$. The vanishing of the kernel of the Laplacian on 5-forms shows that this deformations are integrable.

Finally, one can ask the following question :

- Given a quaternionic contact structure H on $\mathbb{S}^7$, can we find an asymptotically hyperbolic 4-form Ω, with boundary H, and such that the associated metric is Einstein ?

Remark that it is true if the boundary is integrable, the solution then is the quaternionic-Kähler metric.

Because of the work of Biquard [**Biq00**], the metric is uniquely determined by the boundary (at least up to small diffeomorphisms).

CHAPTER 4

Examples

1. Quaternionic-contact quotients

1.1. The main result. Let H be a quaternionic contact distribution on a smooth manifold M. Consider the Lie group

$$\mathrm{Aut}(H) = \{g \in \mathrm{Diff}(M)\,,\ g_*H = H\}$$

whose Lie algebra is

$$\mathfrak{aut}(H) = \{X \in \Gamma(TM)\,,\ \mathcal{L}_X H \subset H\}\,,$$

where $\mathcal{L}_X(H) = \{[X,Y]\,,\ Y \in \Gamma(H)\}$. Let Q be the bundle TM/H and p be the canonical projection $TM \to Q$.

Definition 1.1. The momentum map is the $\mathrm{Aut}(H)$-equivariant function $\mathcal{M} : M \to \mathfrak{aut}(H)^* \otimes Q$ defined for $x \in M$ by

$$\langle \mathcal{M}(x), X\rangle = p(X)\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $\mathfrak{aut}(H)$ and $\mathfrak{aut}(H)^*$.

If G is a subgroup of $\mathrm{Aut}(M)$ which is a Lie group of Lie algebra $\mathfrak{g}$, one obtains by restriction a map $\mathcal{M} : M \to \mathfrak{g}^* \otimes Q$. In the same way that in the quaternionic-Kähler case, the only natural subset of M which remains G-invariant is

$$\tilde{M}_0^G = \{x \in M\,,\ \mathcal{M}(x) = 0\}\,.$$

Here is the main result of the section :

Theorem 1.1. *Let H be a quaternionic contact distribution on a smooth manifold M and G be a compact subgroup of $Aut(H)$ with momentum map $\mathcal{M}$. Let M_0 be the G-invariant subset of $\tilde{M}_0^G$ where $\mathcal{M}$ intersects the zero section transversally and where G acts freely. Then M_0/G is equipped with a quaternionic contact structure H_0. Moreover, if $i : M_0 \hookrightarrow M$ is the injection and $\pi : M_0 \to M_0/G$ the projection, then H_0 is uniquely defined by $i^*H = \pi^*H_0$.*

In the case where M_0/G is of dimension 7, the distribution H_0 is integrable.

PROOF. Let g_H be a choice of adapted metric on H and $(\frac{1}{\sqrt{2}}d\eta_i)_{i=1,2,3}$ be a local choice of $SO(3)$ trivialization of $\Lambda^2_+H^*$ on a neighbourhood of a point $x_0 \in M_0$. Let (I_1, I_2, I_3) be the associated quaternionic structure on H. If $g \in G$, the restriction to H of $d\eta_i \wedge d\eta_j$ is equal to δ_{ij} times a volume form on H, so that $(dg^*\eta_i \wedge dg^*\eta_j)|_H = g^*(d\eta_i \wedge d\eta_j)|_H$ is still equal to δ_{ij} times a volume form on H. Eventually, a rescalling allows us to assume that the 1-forms η_i are G-invariant. We denote by W the complementary vector bundle associated to g_H and by (R_1, R_2, R_3) the dual basis of the restriction to W of the 1-forms η_i.

Let m be the dimension of G and $4n+3$ be the dimension of M. The submanifold M_0 is of dimension $4n+3-3m$ and the rank of $H \cap TM_0$ is at least $4n-3m$. On the other hand, putting V for the vector bundle spanned by the $X(x)$ for $X \in \mathfrak{g}$ and $x \in M_0$, one obtains $H \cap TM_0 \subset (I_1V \oplus I_2V \oplus I_3V)^\perp$, so that $i^*H = H \cap TM_0 = (I_1V \oplus I_2V \oplus I_3V)^\perp$ is a codimension 3 distribution on M_0. Indeed, if $Y \in H \cap TM_0$ and $X \in \mathfrak{g}$, X is tangent to H along M_0, therefore

$$\eta_i([X,Y]) = -d\eta_i(X,Y) = g_H(I_iX, Y) = 0\,,$$

whereas taking $Y \in \mathfrak{g}$ gives that $I_1V + I_2V + I_3V$ is a direct sum. The distribution i^*H is left invariant under the action of G on M_0 and is tangent to the orbits of the G-action so that one obtains a unique codimension 3 distribution H_0 on M_0/G such that $\pi^*H_0 = i^*H$. It remains to show that H_0 is a quaternionic contact distribution on M_0/G. Because of the G-invariance, the forms η_i can be pushed forward to give forms η'_i on M_0/G, defining H_0 and such that $\pi^*\eta'_i = i^*\eta_i$. Taking the differential, one has $\pi^*d\eta'_i = i^*d\eta_i$ and at x_0,

$$(d\eta'_i)_{\pi(x_0)}(\pi_*\cdot, \pi_*\cdot) = g_H(I_i i_*\cdot, i_*\cdot)\,,$$

so that identifying $(H_0)_{\pi(x_0)}$ with $(V \oplus I_1V \oplus I_2V \oplus I_3V)^\perp$ gives the existence of a metric g on H_0 and of a quaternionic structure (I'_1, I'_2, I'_3) such that at $\pi(x_0)$

$$d\eta'_i(\cdot,\cdot) = g(I'_i\cdot, \cdot)\,.$$

Finally, we show that H_0 is integrable. Because of the G-invariance, the vector fields R_i are tangent to M_0 and project on M_0/G giving a dual basis (R'_1, R'_2, R'_3) for the forms η'_i. If $X \in H_0$ and $\tilde{X}$ is the unique vector field in $(V \oplus I_1V \oplus I_2V \oplus I_3V)^\perp$ such that $\pi_*\tilde{X} = X$, one has

$$d\eta'(R'_j, X) = d\eta_i(R_j, \tilde{X})$$

so that the integrability $i_{R'_j} d\eta'_i|_H = i'_{R_i} d\eta'_j|_H$ follows from that of H, recalling that M is of dimension strictly greater than 7 and so that H is automatically integrable. □

REMARK 1.1. If the quotient M_0/G is 3-dimensionnal, we obtain a conformal structure $[(\eta'_1)^2 + (\eta'_2)^2 + (\eta'_3)^2]$ on M_0/G.

1.2. Link with the quaternionic-Kähler case. We recall briefly the construction of the momentum map in the quaternionic-Kähler case and do the link with the quaternionic-contact case, see **[Gal88]**. Let g be a quaternionic-Kähler metric on a manifold M, with 4-form Ω and X be a vector field such that $\mathcal{L}_X \Omega = 0$. Let us put locally $\Omega = \sum_i w_i^2$ and $\mathcal{G}$ be the bundle spanned by w_1, w_2 and w_3. To X, one associates the $\mathcal{G}$-valued 1-form

$$\Theta_X = \sum_i i_X w_i \otimes w_i\,.$$

Integration provides a unique section f_X of $\mathcal{G}$ such that $\nabla f_X = \Theta_X$, where ∇ is the Levi-Civita connexion of g. The momentum map $\mu : M \to \mathfrak{aut}(\Omega)^* \otimes \mathcal{G}$ is defined by $\langle \mu(x), X\rangle = f_X(x)$.

We see now how our momentum map on quaternionic-contact structures can be defined in the same way. Let H be a quaternionic contact structure, let g_H be a choice of metric on H and $(\frac{1}{\sqrt{2}} w_i) = (\frac{1}{\sqrt{2}} d\eta_i)$ be a local $SO(3)$ trivialization of $\Lambda^2_+ H^*$, associated to the quaternionic structure (I_1, I_2, I_3). Let (R_1, R_2, R_3) be the dual basis of (η_1, η_2, η_3) spanning W^{g_H}, and let ∇ be the associated connexion. If X is an infinitesimal automorphism of H, we put

$$X = \sum_{i=1}^{3} x_i R_i + X_H$$

where $X_H \in H$. If $Y \in H$, one has

$$\sum_{j=1}^{3} \eta_j([X,Y]) w_j = 0$$

which can be written

$$-\sum_j d\eta_j(X_H, Y) - \sum_{i,j} x_i i_{R_i} d\eta_j(Y) w_j - \sum_i Y.x_i w_i = 0\,.$$

In dimension greater than 7 **[Biq00]**, or in the integrable case in dimension 7, cf (9), one obtains

$$\nabla(\sum_i x_i w_i)|_H = -\sum_j i_{X_H} w_i \otimes w_i\,,$$

thus our momentum map is completely analogue to that in quaternionic-Kähler geometry.

1.3. Some examples. The examples we describe here are the boundaries of a family of AHQK metrics contructed by Galicki in [**Gal91**]. Let $\langle u, v\rangle = u^*v$ be the canonical quaternion-hermitian product on $\mathbb{H}^{k+1}$. The standard quaternionic contact structure on the sphere $\mathbb{S}^{4k+3}$ is the distribution

$$H_u = \{v \in \mathbb{H}^{k+1}, u^*v = 0\}\,.$$

An element $u \in \mathbb{S}^{4k+3}$ is written $u = (y, x)$ with $y \in \mathbb{H}$ and $x \in \mathbb{H}^k$. Let λ, β be two real constants and D be any $\mathfrak{sp}(k)$-matrix. We consider a non-compact action $\phi^s = (\phi^s_1, \phi^s_2)$ by $\mathbb{R} \simeq SO(1,1) \subset Sp(1, k+1)$ on the sphere $\mathbb{S}^{4k+3}$

$$\begin{aligned}
\phi^s_1(y,x) &= ((e^{i\beta s}\cosh\lambda s)y + e^{i\beta s}\sinh\lambda s)((e^{i\beta s}\sinh\lambda s)y + e^{i\beta s}\cosh\lambda s)^{-1}\,,\\
\phi^s_2(y,x) &= e^{sD}x((e^{i\beta s}\sinh\lambda s)y + e^{i\beta s}\cosh\lambda s)^{-1}\,.
\end{aligned}$$

The infinitesimal action is the vector field

$$X(y,x) = (-\lambda y^2 - \beta yi + i\beta y + \lambda, Dx - \lambda xy - \beta xi)\,.$$

and the momentum map is

$$\begin{aligned}
\mathcal{M}(y,x) &= \langle (y,x), X(y,x)\rangle\\
&= x^*Dx + \beta y^*iy + \lambda(y^* - y) - \beta i\,.
\end{aligned}$$

REMARK 1.2. The action ϕ^s can be extended in an action on the hyperbolic ball $B^{4(k+1)}$, whose momentum map is still

$$\mu(y,x) = x^*Dx + \beta y^*iy + \lambda(y^* - y) - \beta i\,.$$

Let us assume now that $\lambda > 0$. The points $(-1, 0)$ and $(1, 0)$ are left invariant under the $\mathbb{R}$ action, and $\Re\langle(1,0), X(y,x)\rangle = \lambda(1 - \Re(y^2)) > 0$ for all $y \neq 1, -1$. Therefore, the action of $\mathbb{R}$ on $\mathbb{S}^{4n+3} - \{(1,0), (-1,0)\}$ is free, the orbits are transverse to the spheres $\Re(y) = c$, and we obtain a slice

$$\mathcal{S} = \{(y,x) \in \mathbb{S}^{4n+3}, y + y^* = 0\}$$

to the $\mathbb{R}$ action. The manifold $\mathcal{M}^{(-1)}(0)/\mathbb{R}$ is diffeomorphic to $\mathcal{S}_0 = \mathcal{S} \cap \mathcal{M}^{(-1)}(0)$ and one obtains by quaternionic-contact quotient a quaternionic contact structure H_0 on $\mathcal{S}_0$. When $\beta = 0$, one can describe the quaternionic contact structure in the following way : the manifold $\mathcal{S}_0$ can be identified with

$$S^D = \{x \in \mathbb{H}^k\,,\ |x|^2 + \frac{1}{4}|x^*Dx| = 1\}$$

via $x \in S^D \mapsto (x^*Dx/2, x)$. The quaternionic contact structure on S^D is at a point $x \in S^D$,

$$H_0(x) = \{v \in \mathbb{H}^k\, ,\ x^*v - \frac{1}{4}x^*Dx(x^*Dv + v^*Dx)\}\, ,$$

and does not coincide with the $\mathbb{H}$-stable subspace of T_xS^D.

2. Levi-Civita connections

2.1. Preliminaries. In chapter 2, we constructed a family of integrable quaternionic contact structures whose automorphism group contains $Sp(1)$. This provides $Sp(1)$ connections on a family of conformal 4-manifolds, satisfying a semi-linear PDE coming from the integrability condition. One may want to specialize to the Levi-Civita connections of a Riemannian 4-manifold (M, g). The action of the Levi-Civita ∇ connection on $\Lambda^2_+T^*M$ gives a codimension-3 distribution $\mathcal{H}$ on a $SO(3)$-bundle $\pi : P^+ \to M$, and one may search for g such that the following conditions are satisfied:

(18) $\mathcal{H}$ is a quaternionic-contact distribution,

(19) $\pi^*g|_H$ is adapted to the quaternionic-contact structure,

(20) $\mathcal{H}$ is integrable.

The condition (19) forces the curvature of ∇ on $\Lambda^2_+T^*M$ to be a section of $\Lambda^2_+T^*M \otimes \Lambda^2_+T^*M$, hence the traceless Ricci tensor must vanish and g is Einstein. Let W^+ be the seldual curvature of ∇ and s be the scalar curvature. Then, the condition (18) is equivalent tothe inversability of $\frac{s}{12}\mathrm{Id} + W^+$ as an endomorphism of $\Lambda^2_+T^*M$ and the integrability condition can be written

$$(\frac{s}{12}\mathrm{Id}+W^+)^{-1}\nabla(\frac{s}{12}\mathrm{Id}+W^+)+(\nabla(\frac{s}{12}\mathrm{Id}+W^+))(\frac{s}{12}\mathrm{Id}+W^+)^{-1} \in S^{3,1}\oplus S^{1,1}$$

where the notations follow 2.3.

2.2. A rigidity result. In this section, we show that the conditions (18), (19) and (20) are quite restrictive and that they force the quaternionic contact distribution to be 3-Sasakian. More precisely, one has

THEOREM 2.1. *Let g be an Einstein metric on a 4-manifold M. If the $S^{5,1}$ part of*

$$(\frac{s}{12}Id + W^+)^{-1}(\nabla(\frac{s}{12}Id + W^+)^2)(\frac{s}{12}Id + W^+)^{-1} \tag{21}$$

vanishes, then we are in one of the following mutually exclusive cases

(i) *the distribution $\mathcal{H}$ is 3-Sasakian,*

(ii) *there exist a section of* $\Lambda^2_+T^*M$ *which is Kähler with respect to* g*, and* $\mathcal{H}$ *is not a quaternionic-contact distribution.*

Before going to the proof of this result, we recall some results and facts.

REMARK 2.1. In case (ii), the curvature R is a section of $\Lambda^{1,1}T^*M \otimes \Lambda^{1,1}T^*M$, therefore $\frac{s}{12}\mathrm{Id} + W^+$ is not inversible. It explains why (i) and (ii) cannot be both true.

THEOREM 2.2. *(Konishi,* **[Kon75]** *). Let* g *be a Riemannian metric, Einstein and anti-selfdual, with non-vanishing scalar curvature. Then, the distribution* $\mathcal{H}$ *is* 3*-Sasakian.*

PROPOSITION 2.1. *(see for instance* **[Sal84]** *) Let* g *an Einstein metric on a* 4*-manifold. Then* $\nabla W^+ \in S^{5,1}$*.*

PROOF. Let g be an Einstein metric satisfying (21). Let U be an open subset of M where W^+ admits an orthonormal basis of smooth eigenvectors (I_1, I_2, I_3) with smooth eigenvalues λ_i (this condition is true in an open dense subset of M). There exists 1-forms θ_i such that in the basis (I_1, I_2, I_3), one has

$$((\frac{s}{12}\mathrm{Id} + W^+)^{-1}(\nabla(\frac{s}{12}\mathrm{Id} + W^+)^2)(\frac{s}{12}\mathrm{Id} + W^+)^{-1})_{ij} = \theta_i \circ I_j + \theta_j \circ I_i\,.$$

Let us put

$$\nabla I_i = \sum_{j,k} \varepsilon^{ijk}\gamma_j \otimes I_k\,,$$

so that one gets

$$\nabla(\frac{s}{12}\mathrm{Id} + W^+)^2 = 2\sum_i(\frac{s}{12} + \lambda_i)d\lambda_i \otimes I_i \otimes I_i$$

$$-\sum_{i,j}\varepsilon^{ijk}((\frac{s}{12} + \lambda_i)^2 - (\frac{s}{12} + \lambda_j)^2)\gamma_k \otimes I_i \otimes I_j\,.$$

On one hand, taking $i = j$ gives $d\lambda_i = (\lambda_i + \frac{s}{12})\theta_i \circ I_i$ and on the other hand, taking $i \neq j$ gives

$$-(\lambda_i - \lambda_j)(\lambda_i + \lambda_j + \frac{s}{6})\sum_k \varepsilon^{ijk}\gamma_k = (\lambda_i + \frac{s}{12})(\lambda_j + \frac{s}{12})(\theta_i \circ I_j + \theta_j \circ I_i)$$

We use now the constraint $\nabla W^+ \in S^{5,1}$ which can be written

$$\forall j\,,\ -d\lambda_j - \sum_i \varepsilon^{ijk}(\lambda_i - \lambda_j)\gamma_k \circ I_i I_j = 0$$

and then obtain that

$$\begin{pmatrix} (\lambda_3-\lambda_2)(2\lambda_1-s/3) & (\lambda_1-\lambda_3)(\lambda_2+s/12) & (\lambda_2-\lambda_1)(\lambda_3+s/12) \\ (\lambda_3-\lambda_2)(\lambda_1+s/12) & (\lambda_1-\lambda_3)(2\lambda_2-s/3) & (\lambda_2-\lambda_1)(\lambda_3+s/12) \\ (\lambda_3-\lambda_2)(\lambda_1+s/12) & (\lambda_1-\lambda_3)(\lambda_2+s/12) & (\lambda_2-\lambda_1)(2\lambda_3-s/3) \end{pmatrix} \begin{pmatrix} \gamma_1 \circ I_1 \\ \gamma_2 \circ I_2 \\ \gamma_3 \circ I_3 \end{pmatrix}$$

vanishes. Let A be the the previous left hand side matrix, whose determinant is

$$(\lambda_3-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_1)(4\lambda_1\lambda_2\lambda_3 - \frac{7}{6}s(\lambda_1\lambda_2+\lambda_3\lambda_1+\lambda_3\lambda_2) - \frac{25}{864}s^3)\,.$$

If this determinant is not zero, then the γ_i vanish so that we are in case (ii) (even hyperkählerian). From now on, the determinant of A is assumed to be zero.

Assume now that $\lambda = \lambda_1 = \lambda_2$ and that I_3 is not Kähler. Then the determinant of a 2×2 submatrix of A must vanish, so that $\lambda = \lambda_3 = 0$ or that λ is constant. In this last case, all the eigenvalues are constant, and using $\nabla W^+ \in S^{5,1}$ one obtain $\gamma_2 = \gamma_1 = 0$, which contradicts the fact that I_3 is not Kähler.

Finally, we consider the case where all the eigenvalues λ_i are distinct. Then, one of the eigenvalues is equal to a constant times s, and so is constant. Choose this one to be λ_1. Using again $\nabla W^+ \in S^{5,1}$, one gets

$$\gamma := (\lambda_2-\lambda_1)\gamma_3 \circ I_3 = (\lambda_1-\lambda_3)\gamma_2 \circ I_2.$$

We put $\gamma' = (\lambda_3-\lambda_2)\gamma_1 \circ I_1$ and we obtain the system

$$\begin{cases} (2\lambda_1-\frac{s}{3})\gamma' + (\lambda_2+\lambda_3+\frac{s}{6})\gamma &= 0 \\ (\lambda_1+\frac{s}{12})\gamma' + (2\lambda_2+\lambda_3-\frac{s}{4})\gamma &= 0 \\ (\lambda_1+\frac{s}{12})\gamma' + (\lambda_2+2\lambda_3-\frac{s}{4})\gamma &= 0 \end{cases}$$

If γ is zero, then I_1 is Kähler, and if not, then

$$det \begin{pmatrix} \lambda_1+\frac{s}{12} & 2\lambda_2+\lambda_3-\frac{s}{4} \\ \lambda_1+\frac{s}{12} & \lambda_2+2\lambda_3-\frac{s}{4} \end{pmatrix} = 0$$

and so, because $\lambda_2 \neq \lambda_3$, we get $\lambda_1 = -s/12$ and $\lambda_2 = \lambda_3 = s/12$ and the theorem follows.

□

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