On Moduli of Pointed Real Curves of Genus Zero

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To Başak, with love and gratitude

Abstract

The aim of this thesis is to explore the moduli of pointed real curves of genus zero. We investigate the actions of a set of natural real structures

$$c_\sigma : (\Sigma; p_{s_1}, \cdots, p_{s_n}) \mapsto (\overline{\Sigma}; p_{\sigma(s_1)}, \cdots, p_{\sigma(s_n)}),$$

on the moduli space $\overline{M}_{\mathbf{S}}$ of stable $\mathbf{S}$-pointed complex curves of genus zero where σ is an involution acting on the labeling set $\mathbf{S} = \{s_1, \cdots, s_n\}$.

First, we determine the moduli functor of σ-equivariant families represented by the real variety $(\overline{M}_{\mathbf{S}}, c_\sigma)$. We introduce the fixed point set $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ of the real structure c_σ as the moduli space of σ-invariant real curves.

We introduce a natural combinatorial stratification of the real moduli space $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ through the stratification of $\overline{M}_{\mathbf{S}}$. Each stratum gives the equisingular deformations of σ-invariant real curves. We identify the strata of $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ with the products of spaces of $\mathbb{Z}_2$-equivariant point configurations in the projective line $\mathbb{CP}^1$ and the moduli spaces $\overline{M}_{\mathbf{S}'}$. The degeneration types of σ-invariant real curves are encoded by trees with corresponding decorations. We calculate the first Stiefel-Whitney class of $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ in terms of its strata. We construct the orientation double cover $\mathbb{R}\widetilde{M}^\sigma_{\mathbf{S}}$ of $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$, and show that the moduli space $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ is not orientable for $|\mathbf{S}| \geq 5$ and $\mathbf{Fix}(\sigma) \neq \emptyset$. The double covering which is constructed in this work significantly differs from the 'double covering' in the recent literature on open Gromov-Witten invariants and moduli spaces of pseudoholomorphic discs: Our double covering has no boundary which is better suited for the application of intersection theory.

We then explore the further topological properties of $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$. We construct a graph complex $\mathcal{G}_\bullet$ generated by the fundamental classes of the strata of $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$. We show that the homology of $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ is isomorphic to the homology of the graph complex $\mathcal{G}_\bullet$.

Finally, we give presentations of the fundamental groups of the real moduli space $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ and its orientation double cover $\mathbb{R}\widetilde{M}^\sigma_{\mathbf{S}}$.

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Contents

Chapter 1

Introduction

I mean ..., you know ...

Sur l'espace de modules des courbes rationnelles réelles pointées

Le but de cette thèse est d'explorer les propriétés topologiques des espaces de modules des courbes rationnelles réelles pointées.

Modules de variétés réelles: Stratégie générale. Les problèmes de modules en géométrie réelle se posent naturellement comme des versions équivariantes des problèmes de modules analogues de la géométrie complexe. Ainsi, une *variété réelle* X peut être définie comme une variété complexe munie d'une involution antiholomorphe $c_X : X \to X$ et une famille réelle de variétés comme une famille complexe $\pi : \mathcal{U}_B \to B$ munie d'une paire de structures réelles $c_{\mathcal{U}} : \mathcal{U}_B \to \mathcal{U}_B$ et $c_B : B \to B$ de sorte que le diagramme suivant commute

$$\begin{array}{ccc} \mathcal{U}_B & \xrightarrow{c_{\mathcal{U}}} & \mathcal{U}_B \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{c_B} & B, \end{array} \tag{1.1}$$

(notons que seules les fibres au-dessus des points réels de B portent une structure réelle qui est donnée par $c_{\mathcal{U}}$). Dès que l'espace de modules M du problème complexe associé est fin, le problème de modules réel se réduit à l'étude des structures réelles sur M, les espaces de points fixes des structures réelles sur M servant comme les véritables espaces de modules réels.

L'espace de modules des courbes rationnelles complexes pointées. Appelons *courbe* **S**-*pointée stable* toute courbe rationnelle complexe Σ avec des points distincts et lisses $\mathbf{p} = (p_{s_1}, \cdots, p_{s_n}) \subset \Sigma$ *marqués* par des éléments d'un ensemble fini $\mathbf{S} = \{s_1, \ldots, s_n\}$ telle que les conditions suivantes sont vérifiées :

- Σ a seulement des singularités nodales;
- le groupe d'automorphismes holomorphes $Aut(\Sigma; \mathbf{p})$ est trivial.

L'espace de modules $\overline{M}_{\mathbf{S}}$ des courbes rationnelles pointées stables a été intensivement étudié comme l'un des modèles fondamentaux des problèmes de modules en géométrie algébrique (voir [22, 26, 27, 28, 29, 32]). Pendant les deux dernières décennies, l'espace de modules $\overline{M}_{\mathbf{S}}$, aussi bien que sa strate ouverte $M_{\mathbf{S}}$ (formée par des courbes non singulières), ont joué des rôles centraux dans diverses branches des mathématiques. La représentation du groupe fondamental de $M_{\mathbf{S}}$ en termes d'intégrales réitérées de l'équation de KZ a mené à la théorie de Drinfeld des groupes quantiques (voir, par exemple [10]). Des intégrales réitérées semblables sont naturellement apparues dans la description de Kontsevich des invariants de noeuds de Vasiliev ([25]). L'espace de modules $\overline{M}_{\mathbf{S}}$ et la théorie d'intersection sur $\overline{M}_{\mathbf{S}}$ sont devenues les pierres angulaires dans la théorie des invariants de Gromov-Witten, la cohomologie quantique et la symétrie miroir (voir, par exemple [26, 27, 28, 32]).

L'espace de modules des courbes rationnelles réelles pointées. L'espace de modules $\overline{M}_{\mathbf{S}}$ porte un ensemble d'involutions antiholomorphes :

$$c_\sigma : (\Sigma; p_{s_1}, \cdots, p_{s_n}) \mapsto (\overline{\Sigma}; p_{\sigma(s_1)}, \cdots, p_{\sigma(s_n)}), \tag{1.2}$$

où $\sigma \in \mathbb{S}_n$ est une involution sur $\mathbf{S}$. Puisque l'espace de modules $\overline{M}_{\mathbf{S}}$ des courbes $\mathbf{S}$-pointées stables est fin, nous pouvons appliquer la méthode ci-dessus pour étudier l'espace de modules des courbes rationnelles réelles pointées.

Les ensembles $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ des points fixes des c_σ paramétrisent les *courbes σ-invariantes* qui sont des courbes $\mathbf{S}$-pointées stables $(\Sigma; \mathbf{p})$ avec les structures réelles $c_\Sigma : \Sigma \to \Sigma$ telles que $c_\Sigma(p_s) = p_{\sigma(s)}$.

Les espaces de modules des courbes rationnelles réelles pointées stables ont récemment attiré l'attention dans divers contextes tels que les ζ-motifs multiples [14], les représentations des groupes quantiques [9, 18, 21, 36], et les invariants de Welschinger [37, 38].

Dans cette thèse, nous explorons les propriétés topologiques des espaces de modules $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ (que nous visons à appliquer à certains des problèmes ci-dessus).

Théorème. *(a) La famille universelle de courbes $\pi : \overline{U}_{\mathbf{S}} \to \overline{M}_{\mathbf{S}}$ est une famille σ-équivariante.*

(b) Toute famille σ-équivariante de courbes rationelle **S***-pointées stables* $\pi_B : \mathcal{U}_B \to B$ *est induite par une paire unique de morphismes réels*

$$\begin{array}{ccc} \mathcal{U}_B & \xrightarrow{\hat{\kappa}} & \overline{U}_{\mathbf{S}} \\ {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\kappa} & \overline{M}_{\mathbf{S}}. \end{array}$$

(c) Soit $\mathfrak{M}_\sigma$ le foncteur contravariant qui envoie une variété réelle (B, c_B) sur la famille σ-équivariante des courbes au-dessus de B. Le foncteur de modules $\mathfrak{M}_\sigma$ est représenté par la variété réelle $(\overline{M}_{\mathbf{S}}, c_\sigma)$.

(d) Pour tous $|\mathbf{S}| \geq 3$, l'espace de modules $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ de courbes σ-invariantes est une variété lisse de dimension réelle $|\mathbf{S}| - 3$.

Les variétés quasi-projectives $D_\tau \subset \overline{M}_{\mathbf{S}}$, qui donnent des déformations équisingulières de courbes rationnelles **S**-pointées stables , sont classées par les **S**-arbres τ. L'espace de modules $\overline{M}_{\mathbf{S}}$ est stratifié par D_τ. D'autre part, la structure réelle d'une courbe σ-invariante engendre des structures additionnelles: un ordre cyclique sur les points marqués se situant dans $\mathbb{R}\Sigma$, et une partition des points marqués dans $\Sigma \setminus \mathbb{R}\Sigma$. Ces données additionnelles sont codées de façon combinatoire par *les arbres u-planaires* (τ, u). Nous obtenons une stratification de $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ semblable à celle de $\overline{M}_{\mathbf{S}}$ en employant les arbres u-planaires.

Théorème. *(a) L'espace de module $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ est stratifié par les sous-ensembles semi-algébriques $C_{(\gamma,u)}$ deux à deux disjoints.*

(b) L'adhérence de n'importe quelle strate $\overline{C}_{(\gamma,u)}$ est stratifiée par $C_{(\gamma',u')}$ où (γ, u) est obtenu en contractant un ensemble invariant d'arêtes dans (γ', u').

En employant cette stratification, nous calculons la première classe de Stiefel-Whitney de $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$. Notons $\mathbf{Fix}(\sigma) = \{s \in \mathbf{S} \mid s = \sigma(s)\}$ et $\mathbf{Perm}(\sigma) = \{s \in \mathbf{S} \mid \sigma(s) \neq s\}$. Si $|\mathbf{Fix}(\sigma)| > 0$, nous supposons que $s_n = \sigma(s_n)$ et pour tout arbre γ à deux sommets notés $\{v_e, v^e\}$, nous choisissons comme v_e tel que $\boldsymbol{\partial}_\gamma(s_n) = v_e$.

Théorème. *(a) Pour $|\mathbf{Fix}(\sigma)| > 0$, le dual de Poincaré de la première classe de Stiefel-Whitney de $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ est donné par*

$$[w_1] = \sum_{(\gamma,u)} [\overline{C}_{(\gamma,u)}] = \sum_{\gamma} [\mathbb{R}\overline{D}_\gamma] \mod 2, \tag{1.3}$$

où les deux sommes portent sur tous les arbres γ à deux sommets tels que

- $|\mathbf{F}_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$ *et* $|v^e| = 0 \mod 2$*, ou*
- $|\mathbf{F}_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| \neq 3$ *et* $|v_e|(|v^e| - 1) = 0 \mod 2$*, ou*
- $|\mathbf{F}_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| = 3$ *et* $|\mathbf{F}^{\mathbb{R}}_\gamma(v^e)| = 1$

et sur toutes les structures u-planaires sur γ (pour la première somme).

(b) Pour $|\mathbf{Fix}(\sigma)| = 0$, le dual de Poincaré de la première classe de Stiefel-Whitney de $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ s'annule.

Ce théorème montre que l'espace de modules $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ est orientable quand $|\mathbf{S}| = 4$ ou $|\mathbf{Fix}(\sigma)| = 0$. Nous donnons une construction combinatoire du revêtement double d'orientation pour le reste des cas i.e., $|\mathbf{S}| > 4$ et $|\mathbf{Fix}(\sigma)| > 0$. En remarquant la non-trivialité du revêtement double d'orientation dans ces cas là, nous montrons que $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ n'est pas orientable.

Le revêtement double d'orientation dans cette thèse diffère de manière significative du revêtement double d'orientation dans la littérature récente sur les invariants de Gromov-Witten ouverts et les espaces de modules des disques pseudoholomorphes (voir, par exemple [11, 31]). Notre revêtement double n'a pas de bord ce qui convient mieux pour les applications à la théorie d'intersection.

Toutes les applications de $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ mentionnées ci-dessus exigent des informations sur l'homologie ou le groupe fondamental de $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$. Dans cette thèse, nous présentons un 'complexe de graphes' $\mathcal{G}_\bullet$ où

$$\mathcal{G}_d := \left(\bigoplus_{(\tau,o):|\mathbf{E}_\tau|=|\mathbf{S}|-d-3} \mathbb{Z}\,[\overline{C}_{(\tau,o)}] \right) \Big/ \sim$$

sont les groupes abélien produit par des arbres décorés (τ, o) avec $|\mathbf{S}| - d - 3$ arêtes, modulo des relations naturelles additionnelles. La différentielle $\partial : \mathcal{G}_d \to \mathcal{G}_{d-1}$ est donnée par

$$\partial\,[\overline{C}_{(\tau,o)}] \;=\; \sum_{(\gamma,\delta(o))/e=(\tau,o)} \pm\,[\overline{C}_{(\gamma,\delta(o))}],$$

où $(\gamma, \delta(o))$ sont les types de dégénération des courbes σ-invariantes qui représentent les faces de codimension un de $\overline{C}_{(\tau,o)}$.

Bien que les strates $C_{(\tau,o)}$ de $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ soient topologiquement non triviales, la suite spectrale d'une filtration de $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ donnée par la stratification se comporte bien et nous permet de montrer le résultat suivant.

Théorème. *$H_*(\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}})$ est isomorphe à $H_*(\mathcal{G}_\bullet)$.*

Ceci nous donne une description combinatoire de l'homologie des espaces $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ en termes de leur stratification.

C'est un fait bien connu que $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ est un espace $K(\pi_1, 1)$ pour $\sigma = \mathbf{id}$ ([7, 5]). Nous considérons le groupoïde des chemins qui sont transversaux aux strates de codimension un de $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ (et un groupoïde semblable pour $\mathbb{R}\widetilde{M}^{g}_{\mathbf{S}}$). Nous donnons des présentations de groupes fondamentaux de $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ (et $\mathbb{R}\widetilde{M}^{g}_{\mathbf{S}}$) en termes des générateurs et relations en employant leur stratification.

On moduli of pointed real curves of genus zero

The aim of this thesis is to explore the topological properties of the moduli spaces of pointed real curves of genus zero.

Moduli problem for real varieties: General strategy. The moduli problems in real geometry naturally appear as equivariant moduli problems in complex geometry. A *real variety* X is a complex variety with an antiholomorphic involution $c_X : X \to X$ called a *real structure*, and a *real family* of variety X is a complex family $\pi_B : \mathcal{U}_B \to B$ with a pair of real structures $c_{\mathcal{U}} : \mathcal{U}_B \to \mathcal{U}_B$ and $c_B : B \to B$ which makes the following diagram commute

$$\begin{array}{ccc} \mathcal{U}_B & \xrightarrow{c_{\mathcal{U}}} & \mathcal{U}_B \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow{c_B} & B. \end{array} \tag{1.4}$$

Note that the fibers over a real point of B admit real structures which are determined by $c_{\mathcal{U}}$. If the moduli space M of the complex variety X is fine, then the moduli problem of the real variety (X, c_X) reduces to the study of real structures on M, and the fixed point sets of real structures of M give the moduli spaces of the real variety (X, c_X).

Moduli space of pointed complex curves of genus zero An $\mathbf{S}$-*pointed stable curve* $(\Sigma; \mathbf{p})$ is a connected complex algebraic curve Σ of arithmetic genus zero with distinct, smooth, *labeled points* $\mathbf{p} = (p_{s_1}, \cdots, p_{s_n}) \subset \Sigma$, satisfying the following conditions:

- Σ has only nodal singularities;
- the group of holomorphic automorphisms of Σ is trivial.

The moduli space $\overline{M}_{\mathbf{S}}$ of $\mathbf{S}$-pointed stable curves has been extensively studied as one of the fundamental models of moduli problems in algebraic geometry (see [15, 22, 29, 27, 28, 32]). During the last two decades, the moduli space $\overline{M}_{\mathbf{S}}$ as well as its open stratum $M_{\mathbf{S}}$ have played central roles in various branches of mathematics: The representation of the fundamental group of $M_{\mathbf{S}}$ in terms of iterated integrals of the KZ equation led to Drinfeld's theory of quantum groups (see, for example [10]). Similar iterated integrals have naturally appeared in Kontsevich's description of Vasiliev knot invariants (see [25]). The moduli space $\overline{M}_{\mathbf{S}}$ and its intersection theory have become the corner stone in the theory of Gromov-Witten invariants, quantum cohomology and mirror symmetry (see, for example [32]).

Moduli space of pointed real curves of genus zero The moduli space $\overline{M}_{\mathbf{S}}$ carries a set of anti-holomorphic involutions. For each involution $\sigma \in \mathbb{S}_n$ acting on the labeling set $\mathbf{S} = \{s_1, \cdots, s_n\}$, there is a real structure

$$c_\sigma : (\Sigma; p_{s_1}, \cdots, p_{s_n}) \mapsto (\overline{\Sigma}; p_{\sigma(s_1)}, \cdots, p_{\sigma(s_n)}). \tag{1.5}$$

Since $\overline{M}_{\mathbf{S}}$ is a fine moduli space, we can apply the prescription given above for moduli problem of pointed real curves of genus zero.

The fixed point set of the real structure c_σ is the moduli space $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ of *σ-invariant curves* that are $\mathbf{S}$-pointed stable curves $(\Sigma; \mathbf{p})$ with a real structure $c_\Sigma : \Sigma \to \Sigma$ satisfying $c_\Sigma(p_s) = p_{\sigma(s)}$.

The moduli space of σ-invariant curves has recently attracted attention in various contexts such as multiple ζ-motives [14], representation theory and quantum groups [9, 18, 21, 36], and Welschinger invariants [37, 38].

In this thesis, our aim is to explore the topological properties of the moduli space of σ-invariant curves.

Theorem. *(a) The universal family of curves $\pi : \overline{U}_{\mathbf{S}} \to \overline{M}_{\mathbf{S}}$ is a σ-equivariant family.*

(b) Any σ-equivariant family of $\mathbf{S}$-pointed stable curves over $\pi_B : U_B \to B$ is induced by a unique pair of real morphisms

$$\begin{array}{ccc} U_B & \xrightarrow{\hat{\kappa}} & \overline{U}_{\mathbf{S}} \\ {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\kappa} & \overline{M}_{\mathbf{S}}. \end{array}$$

(c) Let $\mathfrak{M}_\sigma$ be the contravariant functor that sends real varieties (B, c_B) to the set of all σ-equivariant family of curves over B. The moduli functor $\mathfrak{M}_\sigma$ is represented by the real variety $(\overline{M}_{\mathbf{S}}, c_\sigma)$.

(d) For $|\mathbf{S}| \geq 3$, the moduli space $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ of σ-invariant curves is a smooth projective real manifold of dimension $|\mathbf{S}| - 3$.

The equisingular deformations of $\mathbf{S}$-pointed stable curves are given by quasi-projective varieties $D_\tau \subset \overline{M}_{\mathbf{S}}$ indexed by trees τ. The moduli space $\overline{M}_{\mathbf{S}}$ is stratified by these subspaces D_τ. On the other hand, the real structure of a σ-invariant curve determines additional structures: A cyclic ordering of special points lying in $\mathbb{R}\Sigma$, a partition of special points lying in $\Sigma \setminus \mathbb{R}\Sigma$. These additional structures are encoded by *u-planar trees* (τ, u). We obtain a stratification of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ similar to $\overline{M}_{\mathbf{S}}$ by using u-planar trees.

Theorem. *(a) The moduli space $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is stratified by pairwise disjoint semi-algebraic subsets $C_{(\gamma,u)}$.*

(b) The closure of any stratum $\overline{C}_{(\gamma,u)}$ is stratified by $\{C_{(\gamma',u')} \mid (\gamma', u') < (\gamma, u)\}$.

By using this stratification, we calculate the first Stiefel-Whitney class of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$. Let $\mathbf{Fix}(\sigma) = \{s \in \mathbf{S} \mid s = \sigma(s)\}$ and $\mathbf{Perm}(\sigma) = \{s \in \mathbf{S} \mid s \neq \sigma(s)\}$. If $|\mathbf{Fix}(\sigma)| > 0$, we assume that $s_n = \sigma(s_n)$ and for all trees γ have two-vertices $\{v_e, v^e\}$ such that $\boldsymbol{\partial}_\gamma(s_n) = v_e$.

Theorem. *(a) For $|\mathbf{Fix}(\sigma)| > 0$, the Poincare dual of the first Stiefel-Whitney class of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is*

$$[w_1] \; = \; \sum_{(\gamma,u)} [\overline{C}_{(\gamma,u)}] \; = \; \sum_{\gamma} [\mathbb{R}\overline{D}_\gamma] \mod 2,$$

where the both sums are taken over all two-vertex trees such that

- $|\mathbf{F}_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$ *and* $|v^e| = 0 \mod 2$, *or*
- $|\mathbf{F}_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| \neq 3$ *and* $|v_e|(|v^e| - 1) = 0 \mod 2$, *or*
- $|\mathbf{F}_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| = 3$ *and* $|\mathbf{F}^{\mathbb{R}}_\gamma(v^e)| = 1$,

and, in the first sum, in addition over all u-planar structures on γ.

(b) For $|\mathbf{Fix}(\sigma)| = 0$, the Poincare dual of the first Stiefel-Whitney class of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ vanishes.

This theorem shows that the moduli space $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is orientable when $|\mathbf{S}| = 4$ or $|\mathbf{Fix}(\sigma)| = 0$. We give a combinatorial construction of the orientation double cover of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ for the rest of the cases i.e., $|\mathbf{S}| > 4$ and $|\mathbf{Fix}(\sigma)| > 0$. By showing the non-triviality of the orientation double covers in these cases, we prove that $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is not orientable for $|\mathbf{S}| > 4$ and $|\mathbf{Fix}(\sigma)| > 0$.

The orientation double covering in this work significantly differs from the 'double covering' in the recent literature on open Gromov-Witten invariants and moduli spaces of pseudoholomorphic discs (see [11, 31]): Our double covering has no boundary which suits better for the use of intersection theory.

All of the applications mentioned above require a description of the homology or the fundamental group of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$. In this thesis, we introduce a combinatorial graph complex $\mathcal{G}_\bullet$ where

$$\mathcal{G}_d := \left(\bigoplus_{(\tau,o):|\mathbf{E}_\tau| = |\mathbf{S}| - d - 3} \mathbb{Z} \, [\overline{C}_{(\tau,o)}] \right) \Big/ \sim$$

are the Abelian groups generated by the relative fundamental classes of the strata $\overline{C}_{(\tau,o)}$ modulo the relations induced by the gluing of codimension one faces of top-dimensional strata and some additional natural relations. The differential $\partial : \mathcal{G}_d \to \mathcal{G}_{d-1}$ is given by

$$\partial \, [\overline{C}_{(\tau,o)}] \; = \sum_{(\gamma,\delta(o))=(\tau,o)} \pm \, [\overline{C}_{(\gamma,\delta(o))}],$$

where $(\gamma, \delta(o))$ are the degeneration types of the pointed real curves lying in the codimension one faces of $\overline{C}_{(\tau,o)}$.

Although the strata $\overline{C}_{(\tau,o)}$ of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ are topologically non-trivial, the spectral sequence of a filtration of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ given by the stratification behaves nicely and allows us to prove the following theorem.

Theorem. *$H_*(\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}})$ is isomorphic to $H_*(\mathcal{G}_\bullet)$.*

This gives us a combinatorial description of the homology of the real moduli space in terms of its stratification.

It is quite well-known fact that $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is a $K(\pi_1, 1)$-space for $\sigma = \mathbf{id}$ (see, for example [5]). We consider the groupoid of paths that are transversal to the codimension one strata of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ (and a similar groupoid for $\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}}$). We give presentations of the fundamental groups of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ and $\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}}$ in terms of generators and relations by using their stratifications.

Notation/Convention

We denote the finite set $\{s_1, \cdots, s_n\}$ by $\mathbf{S}$, and the symmetric group consisting of all permutations of $\mathbf{S}$ by $\mathbb{S}_n$. For an involution $\sigma \in \mathbb{S}_n$, we denote the subsets $\{s \in \mathbf{S} \mid \bar{s} = \sigma(s)\}$ and $\{s, \bar{s} \in \mathbf{S} \mid \bar{s} \neq \sigma(s)\}$ respectively by $\mathbf{Fix}(\sigma)$ and $\mathbf{Perm}(\sigma)$. Through this paper, we only consider the involution

$$\sigma = \begin{pmatrix} s_1 & \cdots & s_k & s_{k+1} & \cdots & s_{2k} & s_{2k+1} & \cdots & s_{2k+l} \\ s_{k+1} & \cdots & s_{2k} & s_1 & \cdots & s_k & s_{2k+1} & \cdots & s_{2k+l} \end{pmatrix}, \tag{1.6}$$

where $2k + l = n$. We fix $\mathbf{Perm}(\sigma) = \{s_\alpha \mid \alpha = 1, \cdots 2k\}$, and $\mathbf{Fix}(\sigma) = \{s_{2k+i} \mid i = 1, \cdots, l\}$.

In this paper, the genus of the curves is zero except when the contrary is stated explicitly. Therefore, we omit mentioning the genus of the curves.

Chapter 2

Pointed complex curves of genus zero and their moduli

This chapter reviews the basic facts on pointed complex curves of genus zero and their moduli space. The moduli space of **S**-pointed stable curves of genus zero is stratified according to degeneration types of these curves. The degeneration types of pointed curves are encoded by trees. The combinatorial structure of the stratification and the intersection ring of the moduli space $\overline{M}_{\mathbf{S}}$ are discussed. The group of holomorphic automorphisms $Aut_{\sharp}(\overline{M}_{\mathbf{S}})$ respecting the stratification is introduced and shown that it is isomorphic to permutation group $\mathbb{S}_n$.

2.1 Pointed curves and their trees

Definition 2.1.1. An **S**-*pointed complex curve* $(\Sigma; \mathbf{p})$ is a connected complex algebraic curve Σ with distinct, smooth, *labeled points* $\mathbf{p} = (p_{s_1}, \cdots, p_{s_n}) \subset \Sigma$, satisfying the following conditions:

- Σ has only nodal singularities.
- The arithmetic genus of Σ is equal to zero.

The nodal points and labeled points are called *special* points.

A *family of* **S**-*pointed complex curves* over a complex manifold B is a proper, holomorphic map $\pi_B : U_B \to B$ with n sections $p_{s_1}, \cdots, p_{s_n}$ such that each geometric fiber $(\Sigma(b); \mathbf{p}(b))$ is an **S**-pointed curve.

Two such curves $(\Sigma; \mathbf{p})$ and $(\Sigma'; \mathbf{p}')$ are *isomorphic* if there exists a bi-holomorphic equivalence $\Phi : \Sigma \to \Sigma'$ mapping p_s to p'_s for all $s \in \mathbf{S}$.

An **S**-pointed curve is *stable* if its group of automorphisms is trivial (i.e., on each irreducible component, the number of special points is at least three).

2.1.1 Graphs

Definition 2.1.2. A *graph* Γ is a collection of finite sets of *vertices* $\mathbf{V}_\Gamma$ and *flags (or half edges)* $\mathbf{F}_\Gamma$ with a boundary map $\boldsymbol{\partial}_\Gamma : \mathbf{F}_\Gamma \to \mathbf{V}_\Gamma$ and an involution $\mathbf{j}_\Gamma : \mathbf{F}_\Gamma \to \mathbf{F}_\Gamma$ ($\mathbf{j}_\Gamma^2 = \mathbf{id}$). We call $\mathbf{E}_\Gamma = \{(f_1, f_2) \in \mathbf{F}_\Gamma^2 \mid f_1 = \mathbf{j}_\Gamma(f_2) \ \& \ f_1 \neq f_2\}$ the set of *edges*, and $\mathbf{T}_\Gamma = \{f \in \mathbf{F}_\Gamma \mid f = \mathbf{j}_\Gamma(f)\}$ the set of *tails*. For a vertex $v \in \mathbf{V}_\Gamma$, let $\mathbf{F}_\Gamma(v) = \boldsymbol{\partial}_\Gamma^{-1}(v)$ and $|v| = |\mathbf{F}_\Gamma(v)|$ be the *valency* of v.

We think of a graph Γ in terms of its following *geometric realization* $||\Gamma||$: Consider the disjoint union of closed intervals $\bigsqcup_{f_i \in \mathbf{F}_\Gamma} [0,1] \times f_i$, and identify $(0, f_i)$ with $(0, f_j)$ if $\boldsymbol{\partial}_\Gamma(f_i) = \boldsymbol{\partial}_\Gamma(f_j)$, and identify (t, f_i) with $(1-t, \mathbf{j}_\Gamma(f_i))$ for $t \in]0,1[$ and $f_i \neq f_j$. The geometric realization of Γ has a piecewise linear structure.

Definition 2.1.3. A *tree* γ is a graph whose geometric realization is connected and simply-connected. If $|v| > 2$ for all vertices, then such a tree is called *stable*.

We associate a subtree γ_v for each vertex $v \in \mathbf{V}_\gamma$ which is given by $\mathbf{V}_{\gamma_v} = \{v\}, \mathbf{F}_{\gamma_v} = \mathbf{F}_\gamma(v)$, $\mathbf{j}_{\gamma_v} = id$, and $\boldsymbol{\partial}_{\gamma_v} = \boldsymbol{\partial}_\gamma$.

Definition 2.1.4. Let γ and τ be trees with n tails. A *morphism* between these trees $\boldsymbol{\phi} : \gamma \to \tau$ is a pair of maps $\boldsymbol{\phi}_{\mathbf{F}} : \mathbf{F}_\tau \to \mathbf{F}_\gamma$ and $\boldsymbol{\phi}_{\mathbf{V}} : \mathbf{V}_\gamma \to \mathbf{V}_\tau$ satisfying the following conditions:

- $\boldsymbol{\phi}_{\mathbf{F}}$ is injective and $\boldsymbol{\phi}_{\mathbf{V}}$ is surjective.
- The following diagram commutes

$$\begin{array}{ccc} \mathbf{F}_\gamma & \xrightarrow{\boldsymbol{\partial}_\tau} & \mathbf{V}_\gamma \\ {\scriptstyle \boldsymbol{\phi}_{\mathbf{F}}}\big\uparrow & & \big\downarrow{\scriptstyle \boldsymbol{\phi}_{\mathbf{V}}} \\ \mathbf{F}_\tau & \xrightarrow{\boldsymbol{\partial}_\gamma} & \mathbf{V}_\tau . \end{array}$$

- $\boldsymbol{\phi}_{\mathbf{F}} \circ \mathbf{j}_\tau = \mathbf{j}_\gamma \circ \boldsymbol{\phi}_{\mathbf{F}}$.
- $\boldsymbol{\phi}_{\mathbf{T}} := \boldsymbol{\phi}_{\mathbf{F}}|_{\mathbf{T}}$ is a bijection.

An *isomorphism* $\boldsymbol{\phi} : \gamma \to \tau$ is a morphism where $\phi_{\mathbf{F}}$ and $\phi_{\mathbf{V}}$ are bijections. We denote the isomorphic trees by $\gamma \approx \tau$.

Each morphism $\boldsymbol{\phi} : \gamma \to \tau$ induces a piecewise linear map between the geometric realizations of γ and τ.

Lemma 2.1.5. *Let γ and τ be stable trees with n tails. Any isomorphism $\boldsymbol{\phi} : \gamma \to \tau$ is uniquely defined by its restriction on tails $\boldsymbol{\phi}_{\mathbf{T}} : \mathbf{T}_\tau \to \mathbf{T}_\gamma$.*

Proof. Let $\boldsymbol{\phi}, \boldsymbol{\varphi} : \gamma \to \tau$ be two isomorphisms such that their restrictions on tails are the same. Consider the path ${}_{f_1}P_{f_2}$ in $||\gamma||$ that connects a pair of tails f_1, f_2. The automorphism $\boldsymbol{\varphi}^{-1} \circ \boldsymbol{\phi}$ of γ maps ${}_{f_1}P_{f_2}$ to itself; otherwise, the union of the ${}_{f_1}P_{f_2}$ and its image $\boldsymbol{\varphi}^{-1} \circ \boldsymbol{\phi}({}_{f_1}P_{f_2})$ gives a loop in $||\gamma||$, which contradicts simply-connectedness. Moreover, the restriction of $\boldsymbol{\varphi}^{-1} \circ \boldsymbol{\phi}$ to the path ${}_{f_1}P_{f_2}$ is the identity map since it preserves distances of vertices to tails f_1, f_2. This follows from the compatibility of the automorphism $\boldsymbol{\varphi}^{-1} \circ \boldsymbol{\phi}$ with $\boldsymbol{\partial}_\gamma$ and $\mathbf{j}_\gamma$.

The geometric realization $||\gamma||$ of γ can be covered by paths that connects pairs of tails of γ. We conclude that the automorphism $\boldsymbol{\varphi}^{-1} \circ \boldsymbol{\phi}$ is the identity since it is the identity on every such path. □

There are only finitely many isomorphisms classes of stable trees whose set of tails are $\mathbf{S}$. We call the isomorphism classes of such trees $\mathbf{S}$-*trees*. We denote the set of all $\mathbf{S}$-trees by $\mathcal{T}ree$.

2.1.2 Dual trees of pointed curves

Let $(\Sigma; \mathbf{p})$ be an $\mathbf{S}$-pointed stable curve and $\eta : \hat{\Sigma} \to \Sigma$ be its normalization. Let $(\hat{\Sigma}_v; \hat{\mathbf{p}}_v)$ be the following pointed stable curve: $\hat{\Sigma}_v$ is a component of $\hat{\Sigma}$, and $\hat{\mathbf{p}}_v$ is the set of points consisting of the preimages of special points on $\Sigma_v := \eta(\hat{\Sigma}_v)$. The points $\hat{\mathbf{p}}_v = (\hat{p}_{f_1}, \cdots, \hat{p}_{f_{|v|}})$ on $\hat{\Sigma}_v$ are ordered by the elements f_* of a set $\mathbf{F}_\gamma(v)$.

Definition 2.1.6. The *dual tree* γ of an $\mathbf{S}$-pointed curve $(\Sigma; \mathbf{p})$ is the $\mathbf{S}$-tree consisting of the following data:

- $\mathbf{V}_\gamma$ is the set of components of $\hat{\Sigma}$.
- $\mathbf{F}_\gamma(v)$ is the set consisting of the preimages of special points in $\hat{\Sigma}_v$.
- $\boldsymbol{\partial}_\gamma : f \mapsto v$ if and only if $\hat{p}_f \in \hat{\Sigma}_v$.
- $\mathbf{j}_\gamma : f \mapsto f$ if and only if $\hat{p}_f$ is a labeled point, and $\mathbf{j}_\gamma : f_1 \mapsto f_2$ if and only if $\hat{p}_{f_1} \in \hat{\Sigma}_{v_1}$ and $\hat{p}_{f_2} \in \hat{\Sigma}_{v_2}$ are the preimages of the nodal point $\Sigma_{v_1} \cap \Sigma_{v_2}$.

Lemma 2.1.7. *Let Φ be an isomorphism between the $\mathbf{S}$-pointed stable curves $(\Sigma; \mathbf{p})$ and $(\Sigma'; \mathbf{p}')$.*

(a) Φ induces an isomorphism $\boldsymbol{\phi}$ between their dual trees γ, γ'.

(b) Φ is uniquely defined by its restriction on labeled points.

Proof. (a) The result follows from the decomposition of Φ into its restriction to each irreducible component and the Definition 2.1.4.

(b) Due to Lemma 2.1.5, the isomorphism $\boldsymbol{\phi} : \gamma \to \gamma'$ is determined by the restriction of Φ to the labeled points. The isomorphism $\boldsymbol{\phi}$ determines which component of

Figure 2.1: Dual tree of an $\mathbf{S}$-pointed curve for $|\mathbf{S}| = 5$.

Σ is mapped to which component of Σ' as well as the restriction of Φ to the special points. Each component of Σ is rational and has at least three special points. Therefore, the restriction of Φ to a component is uniquely determined by the images of the three special points. □

2.2 Deformations of pointed curves

Let γ be the dual tree of an $\mathbf{S}$-pointed curve $(\Sigma; \mathbf{p})$, and $\hat{\Sigma} \to \Sigma$ be its normalization. Let $(\hat{\Sigma}_v; \hat{\mathbf{p}}_v)$ be the following $\mathbf{F}_\gamma(v)$-pointed stable curve corresponding to the irreducible component Σ_v of $(\Sigma; \mathbf{p})$. Let Ω^1_Σ be the sheaf of Kähler differentials.

The infinitesimal deformations of a nodal curve Σ with divisor $D_\mathbf{p} = p_{s_1} + \cdots + p_{s_n}$ is canonically identified with the complex vector space

$$Ext^1_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_\mathbf{p}), \mathcal{O}_\Sigma), \tag{2.1}$$

and the obstructions lie in

$$Ext^2_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_\mathbf{p}), \mathcal{O}_\Sigma).$$

In this case, it is known that there is no obstruction (see, for example [31] or [17]).

The space of infinitesimal deformations is the tangent space of the space of deformations at $(\Sigma; \mathbf{p})$. It can be written explicitly in the following form:

$$\bigoplus_{v \in \mathbf{V}_\gamma} H^1(\hat{\Sigma}_v, T_{\hat{\Sigma}_v}(-D_{\hat{\mathbf{p}}_v})) \oplus \bigoplus_{(f_e, f^e) \in \mathbf{E}_\gamma} T_{\hat{p}_{f_e}} \hat{\Sigma} \otimes T_{\hat{p}_{f^e}} \hat{\Sigma}. \tag{2.2}$$

The first part corresponds to the equisingular deformations of Σ with the divisor $D_{\mathbf{p}_v} = \sum_{f_i \in \mathbf{T}_\gamma} p_{f_i}$, and the second part corresponds to the smoothing of nodal points p_e of the edges $e = (f_e, f^e)$ (see [17]).

2.2.1 Combinatorics of degenerations

Let γ be the dual tree of an $\mathbf{S}$-pointed curve $(\Sigma; \mathbf{p})$. Consider the deformation of a nodal point of $(\Sigma; \mathbf{p})$. Such a deformation of $(\Sigma; \mathbf{p})$ gives a *contraction* of an edge of γ: Let $e = (f_e, f^e) \in \mathbf{E}_\gamma$ be the edge corresponding to the nodal point and $\boldsymbol{\partial}_\gamma(e) = \{v_e, v^e\}$, and consider the equivalence relation $\sim$ on the set of vertices, defined by: $v \sim v$ for all $v \in \mathbf{V}_\gamma \setminus \{v_e, v^e\}$, and $v_e \sim v^e$. Then, there is an $\mathbf{S}$-tree γ/e whose vertices are $\mathbf{V}_\gamma/\sim$ and whose flags are $\mathbf{F}_\gamma \setminus \{f_e, f^e\}$. The boundary map and involution of γ/e are the restrictions of $\boldsymbol{\partial}_\gamma$ and $\mathbf{j}_\gamma$.

We use the notation $\gamma < \tau$ to indicate that τ is obtained by contracting some edges of γ.

2.3 Stratification of the moduli space $\overline{M}_\mathbf{S}$

The moduli space $\overline{M}_\mathbf{S}$ is the space of isomorphism classes of $\mathbf{S}$-pointed stable curves. This space is stratified according to degeneration types of $\mathbf{S}$-pointed stable curves which are given by $\mathbf{S}$-trees. The principal stratum $M_\mathbf{S}$ corresponds to the one-vertex $\mathbf{S}$-tree and is the quotient of the product $(\mathbb{CP}^1)^n$ minus the diagonals $\Delta = \bigcup_{k<l}\{(p_{s_1}, \cdots, p_{s_n}) | p_{s_k} = p_{s_l}\}$ by $Aut(\mathbb{CP}^1) = PSL_2(\mathbb{C})$.

Theorem(Knudsen & Keel, [29, 22])**.** *(a) For any* $|\mathbf{S}| \geq 3$*,* $\overline{M}_\mathbf{S}$ *is a smooth projective algebraic variety of (real) dimension* $2|\mathbf{S}| - 6$.

(b) Any family of $\mathbf{S}$*-pointed stable curves over* B *is induced by a unique morphism* $\kappa : B \to \overline{M}_\mathbf{S}$*. The universal family of curves* $\overline{U}_\mathbf{S}$ *of* $\overline{M}_\mathbf{S}$ *is isomorphic to* $\overline{M}_{\mathbf{S}\bigcup\{s_{n+1}\}}$.

(c) For any $\mathbf{S}$*-tree* γ*, there exists a quasi-projective subvariety* $D_\gamma \subset \overline{M}_\mathbf{S}$ *parameterizing the curves whose dual tree is given by* γ*.* D_γ *is isomorphic to* $\prod_{v\in\mathbf{V}_\gamma} M_{\mathbf{F}_\gamma(v)}$*. The (real) codimension of* D_γ *in* $\overline{M}_\mathbf{S}$ *is* $2|\mathbf{E}_\gamma|$.

(d) $\overline{M}_\mathbf{S}$ *is stratified by pairwise disjoint subvarieties* D_γ*. The closure of any stratum* D_γ *is stratified by* $\{D_{\gamma'} \mid \gamma' \leq \gamma\}$.

Example 2.3.1. (i) For $|\mathbf{S}| < 3$, $\overline{M}_\mathbf{S}$ is empty due to the definition of $\mathbf{S}$-pointed stable curves. For $|\mathbf{S}| = 3$, the moduli space $\overline{M}_\mathbf{S}$ is simply a point, and its universal curve $\overline{U}_\mathbf{S}$ is $\mathbb{CP}^1$ endowed with three labeled points.

(ii) For $|\mathbf{S}| = 4$, the moduli space $\overline{M}_\mathbf{S}$ is $\mathbb{CP}^1$ with three points. These points $D_{\gamma_1}, D_{\gamma_2}$ and D_{γ_3} correspond to the curves with two irreducible components, and the open stratum $M_\mathbf{S}$ is the complement of these three points (see Fig. 2.2). The universal family $\overline{U}_\mathbf{S}$ is a del Pezzo surface of degree five which is obtained by blowing up three points of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

(iii) For $|\mathbf{S}| = 5$, the moduli space $\overline{M}_\mathbf{S}$ is isomorphic to $\overline{U}_{\mathbf{S}\setminus\{s_5\}}$ i.e., it is a del Pezzo surface of degree five. It has ten divisors and each of these divisors contains three codimension two strata. The corresponding $\mathbf{S}$-trees are shown in Figure 2.2.

Figure 2.2: All strata of $\overline{M}_{\mathbf{S}}$ for $|\mathbf{S}| = 3, 4$, and 5.

2.4 Forgetful morphism

We say that $(\Sigma; p_{s_1}, \cdots, p_{s_{n-1}})$ is obtained by forgetting the labeled point p_{s_n} of the $\mathbf{S}$-pointed curve $(\Sigma; p_{s_1}, \cdots, p_{s_n})$. However, the resulting pointed curve may well be unstable. This happens when the component Σ_v of Σ supporting p_{s_n} has only two additional special points. In this case, we contract this component to its intersection point(s) with the components adjacent to Σ_v. With this *stabilization* we extend this map to whole space, and obtain $\pi_{\{s_n\}} : \overline{M}_{\mathbf{S}} \to \overline{M}_{\mathbf{S}'}$ where $\mathbf{S}' = \mathbf{S} \setminus \{s_n\}$. There exists a canonical isomorphism $\overline{M}_{\mathbf{S}} \to \overline{U}_{\mathbf{S}'}$ commuting with the projections to $\overline{M}_{\mathbf{S}'}$. In other words, $\pi_{\{s_n\}} : \overline{M}_{\mathbf{S}} \to \overline{M}_{\mathbf{S}'}$ can be identified with the universal family of curves $\overline{U}_{\mathbf{S}'} \to \overline{M}_{\mathbf{S}'}$.

2.5 Automorphisms of $\overline{M}_{\mathbf{S}}$

The open stratum $M_{\mathbf{S}}$ of the moduli space $\overline{M}_{\mathbf{S}}$ can be identified with the orbit space $((\mathbb{CP}^1)^n \setminus \Delta)/PSL_2(\mathbb{C})$. The latter orbit space may be viewed as the configuration space of $(n-3)$ ordered distinct points of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$:

$$\begin{aligned} M_{\mathbf{S}} \cong \{\mathbf{p} = (z_{s_1}, \cdots, z_{s_n}) \in \mathbb{C}^{n-3} \mid z_{s_i} \neq z_{s_j} \ \forall s_i \neq s_j, \\ \text{and } z_{s_{n-2}} = 0, z_{s_{n-1}} = 1, z_{s_n} = \infty\}, \end{aligned} \tag{2.3}$$

where $z_{s_i} := [z_{s_i} : 1]$ are the coordinates of labeled points p_{s_i} in an affine chart of $\mathbb{CP}^1$.

Let $\psi = (\psi_{s_1}, \ldots, \psi_{s_{n-3}}) : M_{\mathbf{S}} \to M_{\mathbf{S}}$ be a non-constant holomorphic map. In [19], Kaliman discovered the following fact:

Theorem (Kaliman, [19])**.** *For $|\mathbf{S}| \geq 4$, every non-constant holomorphic endomorphism $\psi = (\psi_{s_1}, \ldots, \psi_{s_{n-3}})$ of $M_{\mathbf{S}}$ is an automorphism and its components ψ_s are of the form*

$$\psi_s(\mathbf{p}) = \frac{z_{\varrho(s)} - z_{\varrho(s_{n-2})}}{z_{\varrho(s)} - z_{\varrho(s_n)}} \Bigg/ \frac{z_{\varrho(s_{n-2})} - z_{\varrho(s_{n-1})}}{z_{\varrho(s_n)} - z_{\varrho(s_{n-1})}}, \ s \in \{s_1, \cdots s_{n-3}\}$$

where $\varrho \in \mathbb{S}_n$ is a permutation not depending on s.

Kaliman's theorem implies the following corollary.

Theorem (Kaliman & Lin, [19, 30])**.** *Every holomorphic automorphism of* $M_{\mathbf{S}}$ *is produced by a certain permutation* $\varrho \in \mathbb{S}_n$*. Hence,* $Aut(M_{\mathbf{S}}) \cong \mathbb{S}_n$*.*

On the other hand, the permutation group $\mathbb{S}_n$ acts on the compactification $\overline{M}_{\mathbf{S}}$ of $M_{\mathbf{S}}$ via relabeling: For each $\varrho \in \mathbb{S}_n$, there is a map ψ_ϱ which is given by

$$\psi_\varrho : (\Sigma; \mathbf{p}) \mapsto (\Sigma; \varrho(\mathbf{p})) := (\Sigma; p_{\varrho(s_1)}, \cdots, p_{\varrho(s_n)}). \tag{2.4}$$

Lemma 2.5.1. *The map* ψ_ϱ *is holomorphic.*

Proof. The differentiability of ψ_ϱ follows from the Kodaira-Spencer construction of infinitesimal deformations given in Section 2.2. We need to show that the differential $d\psi_\varrho$ is linear at each $(\Sigma; \mathbf{p}) \in \overline{M}_{\mathbf{S}}$. It is sufficient to show that without taking the quotient with respect to $PSL_2(\mathbb{C})$.

The differential of the permutation maps

$$d\psi_\varrho : Ext^1_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_{\mathbf{p}}), \mathcal{O}_\Sigma) \to Ext^1_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_{\varrho(\mathbf{p})}), \mathcal{O}_\Sigma).$$

By using the explicit form of the tangent space given in (2.2), $d\psi_\varrho$ can be written explicitly as follows.

$$\begin{aligned} \bigoplus_{v \in \mathbf{V}_\tau} H^1(\hat{\Sigma}_v, T_{\hat{\Sigma}_v}(-D_{\hat{\mathbf{p}}_v})) \quad &\to \quad \bigoplus_{v \in \mathbf{V}_\tau} H^1(\hat{\Sigma}_v, T_{\hat{\Sigma}_v}(-D_{\varrho(\hat{\mathbf{p}}_v)})), \ \mathbf{u} \mapsto \mathbf{u}, \\ \bigoplus_{(f_e, f^e) \in \mathbf{E}_\tau} T_{\hat{p}_{f_e}} \hat{\Sigma} \otimes T_{\hat{p}_{f^e}} \hat{\Sigma} \quad &\to \quad \bigoplus_{(f_e, f^e) \in \mathbf{E}_\tau} T_{\hat{p}_{f_e}} \hat{\Sigma} \otimes T_{\hat{p}_{f^e}} \hat{\Sigma}, \ \mathbf{v} \mapsto \mathbf{v}. \end{aligned}$$

Hence the differential $d\psi_\varrho$ is linear. □

Therefore, the permutation group $\mathbb{S}_n$ is a subgroup of holomorphic automorphisms $Aut(\overline{M}_{\mathbf{S}})$.

Let $Aut_\sharp(\overline{M}_{\mathbf{S}})$ be the group of holomorphic automorphisms of $\overline{M}_{\mathbf{S}}$ that respect the stratification: $\psi \in Aut_\sharp(\overline{M}_{\mathbf{S}})$ maps D_τ onto D_γ where $\dim D_\tau = \dim D_\gamma$. Kaliman's theorem leads us to the following immediate corollary.

Theorem 1. *The group* $Aut_\sharp(\overline{M}_{\mathbf{S}})$ *is* $\mathbb{S}_n$*.*

Proof. Let $\psi \in Aut_\sharp(\overline{M}_{\mathbf{S}})$. The restriction of ψ to the open stratum gives the permutation action on $M_{\mathbf{S}}$ since the automorphism group of the open stratum $M_{\mathbf{S}}$ contains only permutations ψ_ϱ. The unicity theorem of holomorphic maps implies that $\psi = \psi_\varrho$ since they coincide on the open stratum $\psi|_{M_{\mathbf{S}}} = \psi_\varrho|_{M_{\mathbf{S}}}$. □

Remark 2.5.2. Note that the group of holomorphic automorphisms $Aut(\overline{M}_{\mathbf{S}})$ is not necessarily isomorphic to $\mathbb{S}_n$. For example, the group of automorphisms $Aut(\overline{M}_{\mathbf{S}})$ is $PSL_2(\mathbb{C})$ when $|\mathbf{S}| = 4$. It is a well-known fact that $Aut(\overline{M}_{\mathbf{S}}) = \mathbb{S}_5$ for $|\mathbf{S}| = 5$ (see, for example [8]). According to our knowledge, there is no systematic exposition of $Aut(\overline{M}_{\mathbf{S}})$ for $|\mathbf{S}| > 5$.

2.6 Intersection ring of $\overline{M}_{\mathbf{S}}$

In [22], Keel gave a construction of the moduli space $\overline{M}_{\mathbf{S}}$ via a sequence of blowups of $\overline{M}_{\mathbf{S}\setminus\{s_n\}} \times \mathbb{CP}^1$ along the certain codimension two subvarieties. This inductive construction of $\overline{M}_{\mathbf{S}}$ allowed him to calculate the intersection ring in terms of the Poincare duals D^γ of the divisor classes $[\overline{D}_\gamma]$. Note that the divisors D_γ parameterize curves whose dual trees have only one edge.

For $|\mathbf{S}| \geq 4$, choose $i, j, k, l \in \mathbf{S}$, and let $\gamma, \tau \in \mathcal{T}ree$ such that $\tau \not\approx \gamma$ and $|\mathbf{E}_\tau| = |\mathbf{E}_\gamma| = 1$. We write $ij\gamma kl$ if tails labeled by i, j and k, l belongs to different vertices of γ. We call γ and τ *compatible* if there is no $\{i, j, k, l\} \subset \mathbf{S}$ such that simultaneously $ij\gamma kl$ and $ik\tau jl$.

Theorem (Keel, [22]). *For* $|\mathbf{S}| \geq 3$,

$$H^*(\overline{M}_{\mathbf{S}}, \mathbb{Z}) = \mathbb{Z}[D^\gamma \mid \gamma \in \mathcal{T}ree,\ |\mathbf{E}_\gamma| = 1]/I_{\mathbf{S}}$$

is a graded polynomial ring, deg $D^\gamma = 1$*. The ideal* $I_{\mathbf{S}}$ *is generated by the following relations:*

1. *For any distinct four elements* $i, j, k, l \in \mathbf{S}$*:*

$$\sum_{ij\gamma kl} D^\gamma - \sum_{ik\tau jl} D^\tau = 0.$$

2. $D^\gamma D^\tau = 0$ *unless* γ *and* τ *are compatible.*

2.6.1 Additive and multiplicative structures of $H^*(\overline{M}_{\mathbf{S}})$

The precise description of homogeneous elements in $H^*(\overline{M}_{\mathbf{S}}, \mathbb{Z})$ is given by Kontsevich and Manin in [28]. The monomial $D^{\gamma_1} \dots D^{\gamma_d}$ is called *good*, if $|\mathbf{E}_{\gamma_i}| = 1$ for all i, and γ_i's are pairwise compatible. Consider any $\mathbf{S}$-tree γ. Any edge $e \in \mathbf{E}_\gamma$ defines an $\mathbf{S}$-tree $\gamma(e)$ which is obtained by contracting all edges of γ but e. Then, we can associate a good monomial $D^\gamma := \Pi_{e \in \mathbf{E}_\gamma} D^{\gamma(e)}$ of degree $|\mathbf{E}_\gamma|$ to γ. The map $\gamma \mapsto D^\gamma$ establishes a bijection between the good monomials of degree d in $H^*(\overline{M}_{\mathbf{S}}, \mathbb{Z})$, and $\mathbf{S}$-trees γ with $|\mathbf{E}_\gamma| = d$ (see [28]). Since boundary divisors intersect transversally, the Poincare duality maps a good monomials to the homology classes represented by the corresponding closed stratum

$$PD : D^\gamma \mapsto [\overline{D}_\gamma]. \tag{2.5}$$

Theorem (Kontsevich and Manin, [28]). *The classes of good monomials linearly generate* $H^*(\overline{M}_{\mathbf{S}}, \mathbb{Z})$.

Multiplication on $H^*(\overline{M}_{\mathbf{S}})$

Let $\tau, \gamma \in \mathcal{T}ree$ and $|\mathbf{E}_\tau| = 1$. In [28], a product formula of $D^\tau D^\gamma$ is given in three distinguished cases:

1. Suppose that there exists an $e \in \mathbf{E}_\gamma$ such that $\gamma(e)$ and τ are not compatible (i.e., $\overline{D}_\tau \cap \overline{D}_{\gamma(e)} = \emptyset$). Then $D^\tau D^\gamma = 0$.

2. Suppose that $D^\tau D^\gamma$ is a good monomial i.e., $\tau, \gamma(e)$'s are pairwise compatible for all $e \in \mathbf{E}_\gamma$. Then there exists a unique $\mathbf{S}$-tree τ' with $e' \in \mathbf{E}_{\tau'}$ such that $\tau'/e' = \gamma$, $\tau'(e) = \tau$, and $D^\tau D^\gamma = D^{\tau'}$.

3. Suppose now that there exists an $e \in \mathbf{E}_\gamma$ such that $\gamma(e) = \tau$ i.e., D^τ divides D^γ. For a given quadruple $\{i, j, k, l\}$ such that $ij\tau kl$, we have
$$\sum_{ij\tau_1 kl} D^{\tau_1}\ D^\gamma - \sum_{ik\tau_2 jl} D^{\tau_2}\ D^\gamma = 0.$$
since the elements of the second sum are not compatible with D^γ. Therefore,
$$D^\tau D^\gamma = -\sum_{\substack{\tau_1 \neq \tau \\ ij\tau_1 kl}} D^{\tau_1} D^\gamma.$$
Here, $D^{\tau_1} D^\tau$ are good monomials, so they can be computed as in case (2).

Additive relations of $H^*(\overline{M}_{\mathbf{S}})$

It remains to give the linear relations between degree d monomials. In [28], these relations are given in the following way. Consider an $\mathbf{S}$-tree γ with $|\mathbf{E}_\gamma| = d - 1$, and a vertex $v \in \mathbf{V}_\gamma$ with $|v| \geq 4$. Let $f_1, f_2, f_3, f_4 \in \mathbf{F}_\gamma(v)$ be pairwise distinct flags. Put $\mathbf{F} = \mathbf{F}_\gamma(v) \setminus \{f_1, f_2, f_3, f_4\}$ and let $\mathbf{F}_1, \mathbf{F}_2$ be two disjoint subsets of $\mathbf{F}$ such that $\mathbf{F} = \mathbf{F}_1 \bigcup \mathbf{F}_2$. We define two $\mathbf{S}$-trees γ_1, γ_2. The $\mathbf{S}$-tree γ_1 is obtained by inserting a new edge $e = (f_e, f^e)$ to γ at v with boundary $\boldsymbol{\partial}_{\gamma_1}(e) = \{v_e, v^e\}$ and flags $\mathbf{F}_{\gamma_1}(v_e) = \mathbf{F}_1 \bigcup \{f_1, f_2, f_e\}$ and $\mathbf{F}_{\gamma_1}(v^e) = \mathbf{F}_2 \bigcup \{f_3, f_4, f^e\}$. The $\mathbf{S}$-tree γ_2 is also obtained by inserting an edge e to γ at the same vertex v, but the flags are distributed differently on vertices $\boldsymbol{\partial}_{\gamma_2}(e) = \{v_e, v^e\}$: $\mathbf{F}_{\gamma_2}(v_e) = \mathbf{F}_1 \bigcup \{f_1, f_3, f_e\}$ and $\mathbf{F}_{\gamma_2}(v^e) = \mathbf{F}_2 \bigcup \{f_2, f_4, f^e\}$. Put
$$R(\gamma, v; f_1, f_2, f_3, f_4) := \sum_{\gamma_1} D^{\gamma_1} - \sum_{\gamma_2} D^{\gamma_2} \tag{2.6}$$
where summation is taken over all possible γ_1 and γ_2 given above.

Theorem(Kontsevich and Manin, [28])**.** *All linear relations between good monomials of degree d are spanned by $R(\gamma, v; f_1, f_2, f_3, f_4)$ with $|\mathbf{E}_\gamma| = d - 1$.*

For proofs and details, see [22, 29] and Chapter 3 in [32].

Chapter 3

Moduli of σ-invariant curves

In this chapter, we introduce σ-invariant curves and their families. We give the moduli spaces of σ-invariant curves as the fixed point sets of real structures c_σ of $\overline{M}_{\mathbf{S}}$.

3.1 Real structures on $\overline{M}_{\mathbf{S}}$

The moduli space $\overline{M}_{\mathbf{S}}$ comes equipped with an involution

$$c_{\mathbf{id}} : (\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \mathbf{p}). \tag{3.1}$$

Here, a complex curve Σ is regarded as a pair $\Sigma = (C, J)$, where C is the underlying two-dimensional manifold and J is a complex structure on it, and $\overline{\Sigma} = (C, -J)$ is its complex conjugated pair.[1]

Lemma 3.1.1. *The map $c_{\mathbf{id}}$ is a real structure on $\overline{M}_{\mathbf{S}}$.*

Proof. The differentiability of $c_{\mathbf{id}}$ follows form the Kodaira-Spencer construction of infinitesimal deformations. We need to show that the differential of $c_{\mathbf{id}}$ is anti-linear at each $(\Sigma; \mathbf{p}) \in \overline{M}_{\mathbf{S}}$. It is sufficient to show that it is anti-linear without taking the quotient with respect to $PSL_2(\mathbb{C})$.

The infinitesimal deformations of a nodal curve Σ with divisor $D_{\mathbf{p}}$ is canonically identified with the complex vector space $Ext^1_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_{\mathbf{p}}), \mathcal{O}_\Sigma)$, (see Section 2.2). By reversing the complex structure on Σ, we reverse the complex structure on the tangent space $Ext^1_{\mathcal{O}_\Sigma}(\Omega_\Sigma(D_{\mathbf{p}}), \mathcal{O}_\Sigma)$ at $(\Sigma; \mathbf{p})$. The differential of the map $(\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \mathbf{p})$

$$Ext^1_{\mathcal{O}_\Sigma}(\Omega_\Sigma(D_{\mathbf{p}}), \mathcal{O}_\Sigma) \to Ext^1_{\mathcal{O}_{\overline{\Sigma}}}(\Omega_{\overline{\Sigma}}(D_{\mathbf{p}}), \mathcal{O}_{\overline{\Sigma}}),\ \mathbf{v} \mapsto \mathbf{v}$$

[1] There is some notational ambiguity here. The bar over $\overline{M}_{\mathbf{S}}$ and that over $\overline{\Sigma}$ refer to two different structures on underlying manifolds: The first one refers to the compactification of $\overline{M}_{\mathbf{S}}$ and the second refer to the manifold with reverse complex structure. Both of these notations are widely used, and we use the bar for both cases. The context should make it clear which structure is referred to.

is clearly anti-linear. $\square$

The permutation group $\mathbb{S}_n$ acts on $\overline{M}_{\mathbf{S}}$ via relabeling: For each $\varrho \in \mathbb{S}_n$, there is a holomorphic map ψ_ϱ defined in (2.4). For each involution $\sigma \in \mathbb{S}_n$, we have an additional real structure

$$c_\sigma := c_{\mathbf{id}} \circ \psi_\sigma : (\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \sigma(\mathbf{p})) \tag{3.2}$$

on $\overline{M}_{\mathbf{S}}$. We denote the fixed point set of $c_\sigma : \overline{M}_{\mathbf{S}} \to \overline{M}_{\mathbf{S}}$ by $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$.

Lemma 3.1.2. *Each real structure preserving the stratification of $\overline{M}_{\mathbf{S}}$ is given by a certain involution $\sigma \in \mathbb{S}_n$ and is of the form (3.2).*

Proof. Theorem 1 implies that the set of anti-holomorphic automorphisms that respect the stratification is obtained by composing the principal real structure $c_{\mathbf{id}}$ with a permutation ψ_ϱ of $\varrho \in \mathbb{S}_n$. Therefore, the real structures preserving the stratification of $\overline{M}_{\mathbf{S}}$ are $c_{\mathbf{id}} \circ \psi_\sigma$ for $\sigma^2 = \mathbf{id}$. $\square$

3.2 σ-invariant curves and their families

An $\mathbf{S}$-pointed stable curve $(\Sigma; \mathbf{p})$ is called *σ-invariant* if it admits a real structure $c_\Sigma : \Sigma \to \Sigma$ such that $c_\Sigma(p_s) = p_{\sigma(s)}$ for all $s \in \mathbf{S}$.

A family of $\mathbf{S}$-pointed stable curves $\pi_B : U_B \to B$ is called *σ-equivariant* if there exist a pair of real structures

$$\begin{array}{llll} c_B : & b & \mapsto & c_B(b), \\ c_U : & b \times (\Sigma(b); \mathbf{p}(b)) & \mapsto & c_B(b) \times (\overline{\Sigma}(b); \sigma(\mathbf{p}(b))) \end{array}$$

of B and U_B which make the following diagram commute

$$\begin{array}{ccc} U_B & \xrightarrow{c_U} & U_B \\ {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ B & \xrightarrow{c_B} & B. \end{array}$$

Lemma 3.2.1. *If $(\Sigma; \mathbf{p})$ is isomorphic to $(\overline{\Sigma}'; \mathbf{p}')$, then there exist anti-holomorphic maps $c : \Sigma \to \Sigma'$ and $c' : \Sigma' \to \Sigma$ such that $c(p_s) = p'_{\sigma(s)}$ and $c'(p'_s) = p_{\sigma(s)}$. The maps c, c' are unique and reverse to each other.*

Proof. It direclty follows from Lemma 2.1.7. $\square$

Remark 3.2.2. If $(\Sigma; \mathbf{p})$ is σ-invariant, then the real structure $c_\Sigma : \Sigma \to \Sigma$ is uniquely determined by the permutation σ due to Lemma 2.1.7.

Let R be a real analytic manifold, and let B be a complexification of R. A *family of σ-invariant curves* over R is the restriction of a σ-equivariant family over B to its real part R.

Figure 3.1: A family of σ-invariant curves for $|\mathbf{S}| = 5$ and $|\mathbf{Fix}(\sigma)| = 1$.

3.3 The moduli space of σ-invariant curves

The real part of $(\overline{M}_{\mathbf{S}}, c_\sigma)$ gives us the moduli space of σ-invariant curves.

Theorem 2. *(a) For any $|\mathbf{S}| \geq 3$, $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is a smooth projective real manifold of dimension $|\mathbf{S}| - 3$.*

(b) The universal family of curves $\pi : \overline{U}_{\mathbf{S}} \to \overline{M}_{\mathbf{S}}$ is a σ-equivariant family.

(c) Any σ-equivariant family of $\mathbf{S}$-pointed stable curves over $\pi_B : U_B \to B$ is induced by a unique pair of real morphisms

$$\begin{array}{ccc} U_B & \xrightarrow{\hat{\kappa}} & \overline{U}_{\mathbf{S}} \\ {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\kappa} & \overline{M}_{\mathbf{S}}. \end{array}$$

(d) Let $\mathfrak{M}_\sigma$ be the contravariant functor that sends real varieties (B, c_B) to the set of isomorphism classes of σ-equivariant families over B. The moduli functor $\mathfrak{M}_\sigma$ is represented by the real variety $(\overline{M}_{\mathbf{S}}, c_\sigma)$.

(e) Let $\mathbb{R}\mathfrak{M}_\sigma$ be the contravariant functor that sends real analytic manifolds R to the set of isomorphism classes of families of σ-invariant curves over R. The moduli functor $\mathbb{R}\mathfrak{M}_\sigma$ is represented by the real part $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ of $(\overline{M}_{\mathbf{S}}, c_\sigma)$.

Proof. (a) The smoothness of the real part of c_σ is a consequence of the implicit function theorem, and $dim_{\mathbb{R}}\, \mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}} = dim_{\mathbb{C}}\, \overline{M}_{\mathbf{S}} = |\mathbf{S}| - 3$ since the real part $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is not empty.

(b) The fiber over $(\Sigma; \mathbf{p}) \in \overline{M}_{\mathbf{S}}$ is $\pi^{-1}((\Sigma; \mathbf{p})) = \Sigma$. We define real structures on $\overline{M}_{\mathbf{S}}$ and $\overline{U}_{\mathbf{S}}$ as follows:

$$\begin{array}{rlcl} c_\sigma : & (\Sigma; \mathbf{p}) & \mapsto & (\overline{\Sigma}; \sigma(\mathbf{p})), \\ \hat{c}_\sigma : & z \in \pi^{-1}((\Sigma; \mathbf{p})) & \mapsto & z \in \pi^{-1}((\overline{\Sigma}; \sigma(\mathbf{p}))). \end{array}$$

Due to Lemma 3.2.1, the maps c_σ and $\hat{c}_\sigma$ are well-defined. They clearly satisfy the conditions of σ-equivariant families in given Section 3.2.

(c) Due to Knudsen's theorem (see Section 2.3), each of the morphisms $\kappa : B \to \overline{M}_{\mathbf{S}}$ and $\hat{\kappa} : \mathcal{U}_B \to \overline{U}_{\mathbf{S}}$ are unique. Therefore, they are the same as $c_\sigma \circ \kappa \circ c_B : B \to \overline{M}_{\mathbf{S}}$ and $\hat{c}_\sigma \circ \hat{\kappa} \circ c_{\mathcal{U}} : \mathcal{U}_B \to \overline{U}_{\mathbf{S}}$. Hence, the morphisms $\kappa, \hat{\kappa}$ are real.

(d) Directly follows from the definition of $\mathfrak{M}_\sigma$ and (c).

(e) This statement is a direct corollary of (d). □

Remark 3.3.1. Let $\varrho \in \mathbb{S}_n$, and ψ_ϱ be the corresponding automorphism of $\overline{M}_{\mathbf{S}}$. The conjugation of real structure c_σ with ψ_ϱ provides a conjugate real structure $c_{\sigma'} = \psi_\varrho \circ c_\sigma \circ \psi_{\varrho^{-1}}$. The conjugacy classes of real structures are determined by the cardinalities $|\mathbf{Fix}(\sigma)| = l$ and $|\mathbf{Perm}(\sigma)| = 2k$. For this reason, we only consider c_σ where σ as in (1.6) i.e.,

$$\sigma = \begin{pmatrix} s_1 & \cdots & s_k & s_{k+1} & \cdots & s_{2k} & s_{2k+1} & \cdots & s_{2k+l} \\ s_{k+1} & \cdots & s_{2k} & s_1 & \cdots & s_k & s_{2k+1} & \cdots & s_{2k+l} \end{pmatrix}.$$

For such an involution, σ-invariant curves are called $(2k, l)$*-pointed real curves.* The fixed point set $\mathrm{Fix}(c_\sigma) = \mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is called the *moduli space of* $(2k, l)$*-pointed real curves* and denoted by $\mathbb{R}\overline{M}^{\sigma}_{(2k,l)}$.

Chapter 4

Combinatorial types of σ-invariant curves

This chapter contains some preliminary observations on σ-invariant real curves and their degeneration types. The degeneration types of σ-invariant curves are encoded by the **S**-trees with additional decorations. The dual trees of σ-invariant curves and their morphisms are introduced.

4.1 Topological types of σ-invariant curves

Definition 4.1.1. A σ-invariant curve $(\Sigma; \mathbf{p})$ is called

- of *type 1*, if $\mathbb{R}\Sigma$ is not empty set or a point,
- of *type 2*, if $\mathbb{R}\Sigma = \emptyset$,
- of *type 3*, if $\mathbb{R}\Sigma$ a solitary point.

Lemma 4.1.2. *(a) Each irreducible real component of a σ-invariant curve is isomorphic to $\mathbb{CP}^1$ with a real structure which is either $[z_1 : z_2] \mapsto [\bar{z}_1 : \bar{z}_2]$ or $[z_1 : z_2] \mapsto [-\bar{z}_2 : \bar{z}_1]$.*

(b) Each σ-invariant curve is either of type 1, type 2 or type 3.

Proof. (a) Due to its definition, each component of a σ-invariant curve is isomorphic to $\mathbb{CP}^1$. Let $conj$ be an anti-holomorphic involution on $\mathbb{CP}^1$. Choose a point p which is not invariant under $conj$, and set $conj(p) = \infty$. We then consider a meromorphic function f which has a simple zero at p and a simple pole at ∞. Let $\bar{f} = \overline{f \circ conj}$, then $\bar{f} = af^{-1}$. Since $\bar{\bar{f}} = f$, we have $a = \bar{a}$. Replacing f by bf changes a to $|b^2|a$. By choosing b, we can normalize $a = 1$ or $a = -1$. These two cases give the two real structures given in the statement.

(b)Let $(\Sigma; \mathbf{p})$ be a σ-invariant curve. The real structure c_Σ maps the components $\Sigma_v \mapsto \Sigma_{\bar{v}}$, and special points $p_f \mapsto p_{\bar{f}}$. Then, we need to consider two possibilities: (i) there exists (at least) one real component $c_\Sigma : \Sigma_v \to \Sigma_v$, (ii) there isn't any real component.

In the case (i), the real structures in (a) give the (possible) real parts of type 1 and 2 respectively.

In the case (ii), the fixed point set $\mathbb{R}\Sigma$ is finite. The induced homomorphism $(c_\Sigma)_*$ maps the fundamental classes of conjugate components $\Sigma_v, \Sigma_{\bar{v}}$ to each other i.e, $[\Sigma_v] \mapsto -[\Sigma_{\bar{v}}]$ and $[\Sigma_{\bar{v}}] \mapsto -[\Sigma_v]$. Hence, the trace of the linear map $(c_\Sigma)_* : H_2(\Sigma) \to H_2(\Sigma)$ is zero. Moreover, the trace of $(c_\Sigma)_* : H_1(\Sigma) \to H_1(\Sigma)$ is zero, since $H_1(\Sigma)$ is trivial. Due to Lefschetz fixed point theorem, the Euler characteristic of the fixed point set is one. Therefore, the real part is a solitary point. □

Remark 4.1.3. If $|\mathbf{Fix}(\sigma)| > 0$, then all σ-invariant curves are of type 1. This follows from the facts that real parts such σ-invariant curves can not be empty set and the labeled points are different than the nodal points. By contrast, σ-invariant curves can be of type 1, type 2 or type 3 when $|\mathbf{Fix}(\sigma)| = 0$.

4.2 Combinatorial types of σ-invariant curves.

The real structure of a σ-invariant curve determines additional structures. We introduce these structures for different topological types of σ-invariant curves separately.

4.2.1 Oriented combinatorial types

σ-invariant curves of type 1. Let $(\hat{\Sigma}, \hat{\mathbf{p}})$ be the normalization of a σ-invariant curve $(\Sigma; \mathbf{p})$ of type 1. By identifying $\hat{\Sigma}_v$ with $\Sigma_v \subset \Sigma$, we obtain a real structure on $\hat{\Sigma}$ for each real component Σ_v. The real part $\mathbb{R}\hat{\Sigma}_v$ of this real structure divides $\hat{\Sigma}_v$ into two halves: two 2-discs Σ_v^+ and Σ_v^- having $\mathbb{R}\hat{\Sigma}_v$ as their common boundary. Then, the set of the preimages of the special points $\hat{\mathbf{p}}_v$ admits the following structures:

- *An oriented cyclic ordering on the set of points lying in $\mathbb{R}\hat{\Sigma}_v$:* For any point $\hat{p}_{f_r} \in \hat{\mathbf{p}}_v \bigcap \mathbb{R}\hat{\Sigma}_v$, there is unique $\hat{p}_{f_{r'}} \in \hat{\mathbf{p}}_v \bigcap \mathbb{R}\hat{\Sigma}_v$ which follows the point $\hat{p}_{f_r}$ in the positive direction of $\mathbb{R}\hat{\Sigma}_v$ (the direction which is determined by the orientation induced from the complex orientation of Σ_v^+).
- *An ordered two-partition of the set of points lying in $\hat{\Sigma}_v \setminus \mathbb{R}\hat{\Sigma}_v$.* The subset $\hat{\mathbf{p}}_v \bigcap (\hat{\Sigma}_v \setminus \mathbb{R}\hat{\Sigma}_v)$ of $\hat{\mathbf{p}}_v$ admits a partition into two disjoint subsets $\{\hat{p}_{f_s} \in \Sigma_v^{\pm}\}$.

The preimages of special points $\hat{\mathbf{p}}_v$ are labelled by $\mathbf{F}_\gamma(v)$. Therefore, if we pick an element $\hat{p}_{f_n} \in \mathbb{R}\hat{\Sigma}_v$, the cyclic ordering can be seen as a linear ordering on $(\hat{\mathbf{p}}_v \bigcap \mathbb{R}\hat{\Sigma}_v) \setminus$

$\{p_{f_n}\}$. This linear ordering gives an *oriented cyclic ordering* on $\mathbf{F}^{\mathbb{R}}_{\gamma}(v)$ which we denote by $\{f_{r_1}\} < \cdots < \{f_{r_{l-1}}\} < \{f_{r_l}\}$ where $f_{r_l} = f_n$. Moreover, the partition $\{\hat{p}_{f_s} \in \Sigma^{\pm}_v\}$ gives an *ordered two-partition* $\mathbf{F}^{\pm}_{\gamma}(v) := \{f_s \mid \hat{p}_{f_s} \in \Sigma^{\pm}_v\}$ of $\mathbf{F}_{\gamma}(v) \setminus \mathbf{F}^{\mathbb{R}}_{\gamma}(v)$.

The *oriented combinatorial type* of the real component Σ_v of $(\Sigma;\mathbf{p})$ is a set of data

$$o_v := \{\text{type } 1; \text{two partition } \mathbf{F}^{\pm}_{\gamma}(v); \text{an oriented cyclic ordering on } \mathbf{F}^{\mathbb{R}}_{\gamma}(v)\}.$$

σ-invariant curves of type 2. Let $(\Sigma;\mathbf{p})$ be a σ-invariant curve of type 2. In this case, $(\Sigma;\mathbf{p})$ has a unique real component Σ_v since the real components must intersect at real points, and Σ has none. Moreover, $\mathbf{Fix}(\sigma) = \emptyset$ since $\mathbb{R}\Sigma = \emptyset$.

In this case, the *oriented combinatorial type* of the real component Σ_v of $(\Sigma;\mathbf{p})$ is a set of data

$$o_v := \{\text{type } 2; \mathbf{V}^{\mathbb{R}}_{\gamma} = \{v\}\}.$$

σ-invariant curves of type 3. Let $(\Sigma;\mathbf{p})$ be a σ-invariant curve of type 3. In this case, the real part $\mathbb{R}\Sigma$ of $(\Sigma;\mathbf{p})$ divides Σ into two connected pointed complex curves $(\Sigma^{\pm}, \mathbf{p}^{\pm})$ having $\mathbb{R}\Sigma$ as their intersection point. We denote the set of components of $(\Sigma^{\pm}, \mathbf{p}^{\pm})$ by $\mathbf{V}^{\pm}_{\gamma}$, and $\bigcup_{v \in \mathbf{V}^{\pm}_{\gamma}} \boldsymbol{\partial}^{-1}_{\gamma}(v)$ by $\mathbf{F}^{\pm}_{\gamma}$. Note that, $\mathbf{V}^{\mathbb{R}}_{\gamma} = \emptyset$ since there is no real component in this case.

The *oriented combinatorial type* of $(\Sigma;\mathbf{p})$ is a set of data

$$o := \{\text{type } 3; \mathbf{V}^{\pm}_{\gamma}; \mathbf{F}^{\pm}_{\gamma}\}.$$

4.2.2 Unoriented combinatorial types

The definition of oriented combinatorial types requires additional data on σ-invariant curves. By identifying the oriented combinatorial types for different choices, we obtain *un-oriented combinatorial types* of σ-invariant curves.

σ-invariant curves of type 1. For each real component Σ_v of a σ-invariant curve $(\Sigma;\mathbf{p})$ of type 1, there are two possible ways of choosing Σ^+_v in Σ_v. These two different choices give the *opposite* oriented combinatorial types o_v and $\overline{o}_v$. Namely, the oriented combinatorial type $\overline{o}_v$ is obtained from o_v by reversing the cyclic ordering on $\mathbf{F}^{\mathbb{R}}_{\gamma}(v)$ and swapping $\mathbf{F}^{+}_{\gamma}(v)$ and $\mathbf{F}^{-}_{\gamma}(v)$.

An *un-oriented combinatorial type* of Σ_v is a pair of opposite oriented combinatorial types $u_v := \{o_v, \bar{o}_v\}$.

σ-invariant curves of type 2. For a σ-invariant curve $(\Sigma;\mathbf{p})$ of type 2, the *un-oriented combinatorial type* is the same set of data with the oriented combinatorial type i.e., $u_v := o_v = \{\text{type } 2; \mathbf{V}^{\mathbb{R}}_{\gamma} = \{v\}\}$

σ-invariant curves of type 3. For a σ-invariant curve $(\Sigma;\mathbf{p})$ of type 3, there are two possible ways of choosing $(\Sigma^+,\mathbf{p}^+)$ in $(\Sigma;\mathbf{p})$. These two different choices give the *opposite* oriented combinatorial types o and $\bar{o}$. Namely, the oriented combinatorial type $\bar{o}$ is obtained from o by swapping $\mathbf{V}_\gamma^+$ and $\mathbf{V}_\gamma^-$, and swapping $\mathbf{F}_\gamma^+$ and $\mathbf{F}_\gamma^-$.

An *un-oriented combinatorial type* of $(\Sigma;\mathbf{p})$ is a pair of opposite oriented combinatorial types $u := \{o,\bar{o}\}$.

4.3 Dual trees of σ-invariant curves

The combinatorial types of σ-invariant curves can be encoded on their dual trees.

4.3.1 O-planar trees

Let γ be the dual tree of a σ-invariant curve $(\Sigma;\mathbf{p})$.

Definition 4.3.1. An *oriented locally planar structure* (*o-planar* for short) on γ is the set of data which encodes oriented combinatorial type of a σ-invariant curve $(\Sigma;\mathbf{p})$. The o-planar structures of different types are explicitly given as follows:

- *O-planar structure of type 1.*
  - The real structure c_Σ is of type 1 (i.e., $\mathbb{R}\Sigma$ is not empty set or a point).
  - $\mathbf{V}_\gamma^{\mathbb{R}} \subset \mathbf{V}_\gamma$ is the set of real components of Σ (i.e., the set of *real vertices*).
  - $\mathbf{F}_\gamma^{\mathbb{R}}(v) \subset \mathbf{F}_\gamma(v)$ is the set of the preimages of special points in $\mathbb{R}\Sigma_v$ (i.e., the set of *real flags* adjacent to real vertices $v \in \mathbf{V}_\gamma^{\mathbb{R}}$).
  - An oriented cyclic ordering on $\mathbf{F}_\gamma^{\mathbb{R}}(v)$ for every $v \in \mathbf{V}_\gamma^{\mathbb{R}}$.
  - A two-partition $\mathbf{F}_\gamma^{\pm}(v)$ of $\mathbf{F}_\gamma(v) \setminus \mathbf{F}_\gamma^{\mathbb{R}}(v)$ for every $v \in \mathbf{V}_\gamma^{\mathbb{R}}$.
- *O-planar structure of type 2.*
  - The real structure c_Σ is of type 2 (i.e., $\mathbb{R}\Sigma$ is empty set).
  - $\mathbf{V}_\gamma^{\mathbb{R}} = \{v_r\} \subset \mathbf{V}_\gamma$ is set of the real components of Σ (i.e., the set of *real vertex*); it contains only one element.
- *O-planar structure of type 3.*
  - The real structure c_Σ is of type 3 (i.e., $\mathbb{R}\Sigma$ is a point).
  - The *special real edge* $e = (f_e, f^e)$ is the edge corresponding to the solitary nodal point of Σ.
  - Two partitions $\mathbf{F}_\gamma^{\pm}$ and $\mathbf{V}_\gamma^{\pm}$ of $\mathbf{F}_\gamma$ and $\mathbf{V}_\gamma$ respectively.

An *o-planar tree* is an $\mathbf{S}$-tree γ with an o-planar structure. We denote o-planar trees by (γ, o).

Notations. For each vertex $v \in \mathbf{V}_\gamma^\mathbb{R}$ of (γ, o) (of type 1 or type 2), we associate a subtree (γ_v, o_v) which is given by the one-vertex tree γ_v and the o-planar structure o_v of (γ, o) assigned to vertex v.

A pair of vertices $v, \bar{v} \in \mathbf{V}_\gamma \setminus \mathbf{V}_\gamma^\mathbb{R}$ are said to be *conjugate* if $c_\Sigma(\Sigma_v) = \Sigma_{\bar{v}}$. Similarly, we call the flags $f, \bar{f} \in \mathbf{F}_\gamma \setminus \mathbf{F}_\gamma^\mathbb{R}$ *conjugate* if c_Σ swaps the corresponding special points $\eta(\hat{p}_f)$ and $\eta(\hat{p}_{\bar{f}})$. Here $\eta : \hat{\Sigma} \to \Sigma$ is the normalization.

We associate the subsets of vertices $\mathbf{V}_\gamma^\pm$ and flags $\mathbf{F}_\gamma^\pm$ to every o-planar tree (γ, o) of type 1 as follows. Let $v_1 \in \mathbf{V}_\gamma \setminus \mathbf{V}_\gamma^\mathbb{R}$ and let $v_2 \in \mathbf{V}_\gamma^\mathbb{R}$ be the closest invariant vertex to v_1 in $||\gamma||$. Let $f \in \mathbf{F}_\gamma(v_2)$ be in the shortest path connecting the vertices v_1 and v_2. The set $\mathbf{V}_\gamma^\pm$ is the subset of vertices $v_1 \in \mathbf{V}_\gamma \setminus \mathbf{V}_\gamma^\mathbb{R}$ such that the flag f (defined as above) is in $\mathbf{F}_\gamma^\pm(v_2)$. The subsets of flags $\mathbf{F}_\gamma^\pm$ are defined as $\partial_\gamma^{-1}(\mathbf{V}_\gamma^\pm)$.

4.3.2 U-planar trees

A *u-planar* structure on the dual tree of $(\Sigma; \mathbf{p})$ is the set of data encoding the unoriented combinatorial type of $(\Sigma; \mathbf{p})$. It is given by

$$u := \begin{cases} \{(\gamma_v, o_v), (\gamma_v, \bar{o}_v) \mid v \in \mathbf{V}_\gamma^\mathbb{R}\} & \text{if } (\Sigma; \mathbf{p}) \text{ is of type 1,} \\ \{(\gamma_v, o_v) \mid v \in \mathbf{V}_\gamma^\mathbb{R}\} & \text{if } (\Sigma; \mathbf{p}) \text{ is of type 2,} \\ \{\text{special real edge } e = (f_e, f^e)\} & \text{if } (\Sigma; \mathbf{p}) \text{ is of type 3.} \end{cases}$$

4.4 Contraction morphism of o-planar trees

Let $(\Sigma; \mathbf{p})$ be a σ-invariant curve with oriented combinatorial types at each real components. Consider the deformation of real or conjugate pairs of nodal points. Such deformations give contraction morphisms of o-planar trees.

Let $(\gamma, \hat{o})$ be an o-planar tree, and $\boldsymbol{\phi} : \gamma \to \tau$ be a morphism of $\mathbf{S}$-trees contracting an invariant set of edges $\mathbf{E}_{con} = \mathbf{E}_\gamma \setminus \boldsymbol{\phi}_\mathbf{E}(\mathbf{E}_\tau)$. In such a situation, we associate a particular o-planar structure o on τ, as described below in separate cases **(a)** and **(b)**, and speak of a *contraction morphism* $\boldsymbol{\varphi} : (\gamma, \hat{o}) \to (\tau, o)$. In all the cases, except **(a-2)**, the o-planar structure o is uniquely defined by $\hat{o}$.

(a) Let $\mathbf{E}_{con} = \{e = (f_e, f^e)\} \subset \mathbf{E}_\gamma^\mathbb{R}$.

1. If $\partial_\gamma(e) = \{v_e, v^e\} \subset \mathbf{V}_\gamma^\mathbb{R}$, then we convert the o-planar structures

$$\begin{aligned} \hat{o}_{v_e} &= \{\text{type } 1; \mathbf{F}_\gamma^\pm(v_e); \mathbf{F}_\gamma^\mathbb{R}(v_e) = \{\{f_{r_1}\} < \cdots < \{f_{r_m}\} < \{f_e\}\}\} \\ \hat{o}_{v^e} &= \{\text{type } 1; \mathbf{F}_\gamma^\pm(v^e); \mathbf{F}_\gamma^\mathbb{R}(v^e) = \{\{f'_{r'_1}\} < \cdots < \{f'_{r'_m}\} < \{f^e\}\}\}. \end{aligned}$$

at v_e and v^e to an o-planar structure at vertex $v = \boldsymbol{\phi}_{\mathbf{V}}(\{v_e, v^e\})$ of (τ, o) defining it by

$$\begin{aligned} o_v &= \{\text{type } 1; \mathbf{F}^{\pm}_{\tau}(v) = \mathbf{F}^{\pm}_{\gamma}(v_e) \bigcup \mathbf{F}^{\pm}_{\gamma}(v^e); \\ &\quad \mathbf{F}^{\mathbb{R}}_{\tau}(v) = \{\{f'_{r'_1}\} < \cdots < \{f'_{r'_m}\} < \{f_{r_1}\} < \cdots < \{f_{r_m}\}\}. \end{aligned}$$

The o-planar structures are kept unchanged at all other invariant vertices.

2. If e is a special real edge and $\partial_{\gamma}(e) = \{v_e, v^e\}$, then we convert the o-planar structure $\hat{o} = \{\text{type } 3; \mathbf{F}^{\pm}_{\gamma}; \mathbf{V}^{\pm}_{\gamma}\}$ of γ into an o-planar structure at the vertex $v_r = \boldsymbol{\phi}_{\mathbf{V}}(\{v_e, v^e\})$ of τ defining it by

$$o_{v_r} = \{\text{type } 1; \mathbf{F}^{+}_{\tau}(v) = \mathbf{F}^{+}_{\gamma}(v_e) \setminus \{f_e\}, \mathbf{F}^{-}_{\tau}(v) = \mathbf{F}^{-}_{\gamma}(v^e) \setminus \{f^e\}; \mathbf{F}^{\mathbb{R}}_{\tau}(v) = \emptyset\}.$$

or by

$$o_{v_r} = \{\text{type } 2\}.$$

(b) Let $\mathbf{E}_{con} = \{e_i = (f_{e_i}, f^{e_i}) \mid i = 1,2\}$ where $f_{e_i}, i = 1,2$ and $f^{e_i}, i = 1,2$ are conjugate pairs of flags.

1. If $\partial_{\gamma}(e_i) = \{\hat{v}, v^{e_i}\}$, and $\hat{v} \in \mathbf{V}^{\mathbb{R}}_{\gamma}$, $v^{e_i} \notin \mathbf{V}^{\mathbb{R}}_{\gamma}$, then we convert the o-planar structure

$$o_{\hat{v}} = \{\text{type } 1; \mathbf{F}^{\pm}_{\gamma}(\hat{v}); \mathbf{F}^{\mathbb{R}}_{\gamma}(\hat{v}) = \{\{f_{r_1}\} < \cdots < \{f_{r_m}\}\}.$$

at $\hat{v}$ to an o-planar structure at $v = \boldsymbol{\phi}_{\mathbf{V}}(\{\hat{v}, v^{e_1}, v^{e_2}\})$ of τ defining it by

$$\begin{aligned} o_v &= \{\text{type } 1; \mathbf{F}^{+}_{\tau}(v) = \mathbf{F}^{+}_{\gamma}(\hat{v}) \bigcup \mathbf{F}^{+}_{\gamma}(v^{e_1}) \setminus \{f_{e_1}, f^{e_1}\}, \\ &\quad \mathbf{F}^{-}_{\tau}(v) = \mathbf{F}^{-}_{\gamma}(\hat{v}) \bigcup \mathbf{F}^{-}_{\gamma}(v^{e_2}) \setminus \{f_{e_2}, f^{e_2}\}; \\ &\quad \mathbf{F}^{\mathbb{R}}_{\tau}(v) = \{\{f_{r_1}\} < \cdots < \{f_{r_m}\}\}. \end{aligned}$$

2. If $\mathbf{E}_{con} = \{e_i = (f_{e_i}, f^{e_i}) \mid i = 1,2\}$ and $\partial_{\gamma}(e_i) \bigcap \mathbf{V}^{\mathbb{R}}_{\gamma} = \emptyset$, then we define the o-planar structure at each real vertex v of τ to be the same as the o-planar structure at v of $(\gamma, \hat{o})$.

4.4.1 Contraction morphism of u-planar trees

Let $\pi_B : U_B \to B$ be a σ-equivariant family which is a deformation of a nodal point of the central fiber $(\Sigma(b_0), \mathbf{p}(b_0))$. Let (τ, u) and $(\gamma, \hat{u})$ be the u-planar trees associated respectively to generic fibers $(\Sigma(b), \mathbf{p}(b))$ and the central fiber $(\Sigma(b_0), \mathbf{p}(b_0))$ of B. Let e be the edge corresponding to the nodal point that is deformed. We say that (τ, u) is obtained by *contracting* the edge e of $(\gamma, \hat{u})$, and to indicate that we use the notation $(\gamma, \hat{u}) < (\tau, u)$.

It is important to note that the contraction of an edge of a u-planar tree is not a well-defined operation: For example, we can think about a deformation of a real nodal point as the family $\{x \cdot y = t \mid t \in \mathbb{R}\}$. According to the sign of the deformation parameter t, we obtain σ-invariant curves with two different u-planar structures, see Figure 4.1. Different u-planar trees (τ, u_i) obtained by contraction of edges $(\gamma, \hat{u})$ correspond to different signs of deformation parameters.

Figure 4.1: Two possible deformation of a real nodal point.

4.5 Forgetful morphism of o-planar trees

Let $\mathbf{S}' \subset \mathbf{S}$ such that $\sigma(\mathbf{S}') = \mathbf{S}'$. Denote the restriction σ on $\mathbf{S}'$ by σ'. The morphism $\pi_{\mathbf{S}\setminus\mathbf{S}'} : \overline{M}_{\mathbf{S}} \to \overline{M}_{\mathbf{S}'}$ forgetting the points labelled by $\mathbf{S} \setminus \mathbf{S}'$ is a real morphism i.e., $\pi_{\mathbf{S}\setminus\mathbf{S}'} \circ c_\sigma = c_{\sigma'} \circ \pi_{\mathbf{S}\setminus\mathbf{S}'}$. Therefore, $\pi_{\mathbf{S}\setminus\mathbf{S}'}$ maps the real part of $(\overline{M}_{\mathbf{S}}, c_\sigma)$ onto the real part of $(\overline{M}_{\mathbf{S}'}, c_{\sigma'})$.

Let (γ^*, o^*) be an o-planar representative of dual tree (γ^*, u^*) of $(\Sigma; \mathbf{p}) \in \mathbb{R}\overline{M}^{\sigma}_{S}$. Set $\overline{C}_{(\gamma^*,o^*)} = \overline{C}_{(\gamma^*,u^*)}$. Then, we say that the o-planar (γ, o) of $\pi_{\mathbf{S}\setminus\mathbf{S}'}((\Sigma; \mathbf{p}))$ is obtained by *forgeting* the tails $\mathbf{S} \setminus \mathbf{S}'$ of (γ^*, o^*).

We denote the set of o-planar trees $\{(\gamma^*, o^*)\}$ that give (γ, o) after forgetting the tails $\mathbf{S} \setminus \mathbf{S}'$ by $\mathbf{G}_{(\gamma,o)}(\mathbf{S}, \mathbf{S}')$.

Chapter 5

Stratification of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$

A stratification for $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ can be obtained by using the stratification of $\overline{M}_{\mathbf{S}}$ given in Section 2.3.

Lemma 5.0.1. *Let γ and $\overline{\gamma}$ be the dual trees of $(\Sigma; \mathbf{p})$ and $(\overline{\Sigma}; \sigma(\mathbf{p}))$ respectively.*

(a) If γ and $\overline{\gamma}$ are not isomorphic, then the restriction of c_σ on the union of complex strata $D_\gamma \bigcup D_{\overline{\gamma}}$ gives a real structure with empty real part.

(b) If γ and $\overline{\gamma}$ are isomorphic, then the restriction of c_σ on D_γ gives a real structure whose corresponding real part $\mathbb{R}D_\gamma$ is the intersection of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ with D_γ.

Proof. (a) Since γ and $\overline{\gamma}$ are not isomorphic, D_γ and $D_{\overline{\gamma}}$ are disjoint complex strata. The restriction of c_σ on $D_\gamma \bigcup D_{\overline{\gamma}}$ swaps the strata. Therefore, the real part of this real structure is empty.

(b) Since γ and $\overline{\gamma}$ are isomorphic, $\mathbf{S}$-pointed curves $(\Sigma; \mathbf{p})$ and $(\overline{\Sigma}; \sigma(\mathbf{p}))$ are in D_γ. Therefore, $D_\gamma = D_{\overline{\gamma}}$ and the restriction of c_σ on D_γ is a real structure. The real part $\mathbb{R}D_\gamma$ of the σ-equivariant family D_γ is $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}} \bigcap D_\gamma$ since $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}} = \mathrm{Fix}(c_\sigma)$. □

Definition 5.0.2. An $\mathbf{S}$-tree γ is called *σ-invariant* if it is isomorphic to $\bar{\gamma}$, and the set of σ-invariant $\mathbf{S}$-trees is denoted by $\mathcal{T}ree(\sigma)$.

Theorem 3. *The real moduli space $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is stratified by real analytic subsets $\mathbb{R}D_\tau$ where $\tau \in \mathcal{T}ree(\sigma)$.*

Proof. Due to Lemma 5.0.1, the restrictions of c_σ act as real structures on D_τ for $\tau \approx \overline{\tau}$, and $D_\tau \bigcup D_{\overline{\tau}}$ for $\tau \not\approx \overline{\tau}$. Since the real part of $(D_\tau \bigcup D_{\overline{\tau}}, c_\sigma)$ is emptyset, the real moduli space $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is the union of real parts $\mathbb{R}D_\tau$ of the pairwise disjoint strata (D_τ, c_σ) for $\tau \approx \overline{\tau}$. □

Although the notion of σ-invariant trees leads us to a combinatorial stratification of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ as given in Theorem 3, it does not give a stratification in terms of connected strata. For a σ-invariant tree γ, the real part of the stratum $\mathbb{R}D_\gamma$ has many connected subspaces. In this chapter, we refine this stratification by using the spaces of $\mathbb{Z}_2$-equivariant point configurations in the projective line $\mathbb{CP}^1$ and u-planar trees.

5.1 Spaces of $\mathbb{Z}_2$-equivariant point configurations in $\mathbb{CP}^1$

Let $z := [z : 1]$ be the affine coordinate on $\mathbb{CP}^1$. Consider the upper half-plane $\mathbb{H}^+ = \{z \in \mathbb{CP}^1 \mid \Im(z) > 0\}$ (resp. lower half plane $\mathbb{H}^- = \{z \in \mathbb{CP}^1 \mid \Im(z) < 0\}$) as a half of the $\mathbb{CP}^1$ with respect to $z \mapsto \bar{z}$, and the real part $\mathbb{RP}^1$ as its boundary. Denote by $\mathbb{H}$ the compactified disc $\mathbb{H}^+ \cup \mathbb{RP}^1$.

5.1.1 Configuration spaces

Let $\mathbf{F} = \{f_1, \cdots, f_m\}$ be a finite set. Let $\mathbb{S}_m$ be the group of permutations of $\mathbf{F}$, and $\rho \in \mathbb{S}_m$ be an involution. We denote the subsets $\{f_r \in \mathbf{F} \mid \bar{f}_r = \rho(f_r)\}$ and $\{f_s, f_{\bar{s}} \in \mathbf{F} \mid \bar{f}_s \neq \rho(f_s)\}$ respectively by $\mathbf{F}^{\mathbb{R}}(\rho)$ and $\mathbf{F}^{\mathbb{C}}(\rho)$. Without loss of generality, we choose the involution

$$\rho = \begin{pmatrix} f_1 & \cdots & f_i & f_{i+1} & \cdots & f_{2i} & f_{2i+1} & \cdots & f_{2i+j} \\ f_{i+1} & \cdots & f_{2i} & f_1 & \cdots & f_i & f_{2i+1} & \cdots & f_{2i+j} \end{pmatrix},$$

where $2i + j = m$.

Let $conj : \mathbb{CP}^1 \to \mathbb{CP}^1$ be an anti-holomorphic involution.

Definition 5.1.1. A ρ-invariant point configuration on $\mathbb{CP}^1$ is a finite set of points $\mathbf{p} = (p_{f_1}, \cdots, p_{f_m}) \subset \mathbb{CP}^1$ labelled by $\mathbf{F}$ such that $conj(p_f) = p_{\rho(f)}$.

The permutations $\varrho \in \mathbb{S}_m$ of $\mathbf{F}$ *relabel* ρ-invariant point configurations:

$$\psi_\varrho : (p_{f_1}, \cdots, p_{f_m}) \mapsto (p_{\varrho(f_1)}, \cdots, p_{\varrho(f_m)}).$$

If the image $\psi_\varrho(\mathbf{p})$ is also ρ-invariant, we call ψ_ϱ a *ρ-invariant relabelling.*

We will consider the spaces of configurations for the real structures with non-empty and empty real parts as separate cases (i.e., $conj : z \mapsto \bar{z}$ and $conj : z \mapsto -1/\bar{z}$).

Case I. *Configurations on $\mathbb{CP}^1$ with non-empty real part.* Each ρ-invariant point configuration $\mathbf{p}$ in $\mathbb{CP}^1$ with $z \mapsto \bar{z}$ inherits an o-planar structure.

(a) An oriented cyclic ordering $\{f_{r_1}\} < \cdots < \{f_{r_{l-1}}\} < \{f_{r_l} := f_m\}$ on $\mathbf{F}^{\mathbb{R}}(\rho)$.

(b) An ordered two-partition $\mathbf{F}^{\pm}(\rho) := \{f_s \mid p_{f_s} \in \mathbb{H}^{\pm}\}$ of $\mathbf{F}^{\mathbb{C}}(\rho)$.

The set of data given in **(a)** and **(b)** is called the *oriented combinatorial type* a ρ-invariant point configuration $\mathbf{p}$ on $(\mathbb{CP}^1, z \mapsto \bar{z})$. We denote an oriented combinatorial type by o.

The oriented combinatorial types of ρ-invariant point configurations on a real variety $(\mathbb{CP}^1, z \mapsto \bar{z})$ enumerate the connected components of the space $\widetilde{Conf}_{(\mathbf{F},\rho)}$ of $|\mathbf{F}^+(\rho)|$ distinct pairs of conjugate points in $\mathbb{H}^+ \bigcup \mathbb{H}^-$ and $|\mathbf{F}^{\mathbb{R}}(\rho)|$ distinct points in $\mathbb{RP}^1$:

$$\begin{aligned}\widetilde{Conf}_{(\mathbf{F},\rho)} \;&:= \{(p_{f_1}, \cdots, p_{f_{2i}}; q_{f_{2i+1}}, \cdots, q_{f_{2i+j}}) \mid p_f \in \mathbb{H}^+ \bigcup \mathbb{H}^-, \\ &\quad \text{for } f \in \mathbf{F}^{\mathbb{C}}(\rho),\ \&\ p_f = p_{f'} \Leftrightarrow f = f', p_f = \bar{p}_{f'} \Leftrightarrow f = \rho(f'), \\ &\quad \text{and } q_f \in \mathbb{RP}^1,\ q_f = q_{f'} \Leftrightarrow f = f'\}.\end{aligned}$$

The number of connected components is $2^i(j-1)!$.[1] They are all pairwise diffeomorphic; natural diffeomorphisms are ρ-invariant relabelings ψ_ϱ.

Let $z := [z : 1]$ be the affine coordinate on $\mathbb{CP}^1$, and $x := [x : 1]$ be affine coordinate on $\mathbb{RP}^1$. The action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ is given by

$$SL_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{H}, \quad (\Lambda, z) \mapsto \Lambda(z) = \frac{az+b}{cz+d}, \quad \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

in affine coordinates. It induces an isomorphism $SL_2(\mathbb{R})/\pm\mathbb{I} \to Aut(\mathbb{H})$. The automorphism group $Aut(\mathbb{H})$ acts on $\widetilde{Conf}_{(\mathbf{F},\rho)}$

$$\begin{aligned}(z_1, \cdots, z_{2i}; x_{2i+1}, \cdots, x_{2i+j}) \;&\mapsto \\ &(\Lambda(z_1), \cdots, \Lambda(z_{2i}); \Lambda(x_{2i+1}), \cdots, \Lambda(x_{2i+j})).\end{aligned}$$

It preserves each connected component of $\widetilde{Conf}_{(\mathbf{F},\rho)}$. This action is free when $|\mathbf{F}| \geq 3$, and it commutes with ρ-invariant relabelings. Therefore, the quotient space

$$\widetilde{C}_{(\mathbf{F},\rho)} := \widetilde{Conf}_{(\mathbf{F},\rho)}/Aut(\mathbb{H})$$

is a manifold of dimension $|\mathbf{F}| - 3$ whose connected components are pairwise diffeomorphic.

In addition to the automorphisms considered above, there is a diffeomorphism $-\mathbb{I}$ of $\widetilde{Conf}_{(\mathbf{F},\rho)}$ which is given in affine coordinates as follows.

$$-\mathbb{I} : (z_1, \cdots, z_{2i}; x_{2i+1}, \cdots, x_{2i+j}) \mapsto (-z_1, \cdots, -z_{2i}; -x_{2j+1}, \cdots, -x_{2i+j}). \tag{5.1}$$

Consider the quotient space $Conf_{(\mathbf{F},\rho)} = \widetilde{Conf}_{(\mathbf{F},\rho)}/(-\mathbb{I})$. Note that, $-\mathbb{I}$ swaps the components of $\widetilde{Conf}_{(\mathbf{F},\rho)}$ that have *opposite* oriented combinatorial types. Namely, the combinatorial type $\overline{o}$ of $-\mathbb{I}(\mathbf{p})$ is obtained from the combinatorial type o of $\mathbf{p}$ by reversing the cyclic ordering on $\mathbf{F}^{\mathbb{R}}(\rho)$ and swapping $\mathbf{F}^+(\rho)$ and $\mathbf{F}^-(\rho)$. The equivalence classes of oriented combinatorial types with respect to the action of $-\mathbb{I}$ are called *un-oriented combinatorial types* of ρ-invariant point configurations on $(\mathbb{CP}^1, z \mapsto \overline{z})$. The un-oriented combinatorial types enumerate the connected components of $Conf_{(\mathbf{F},\rho)}$.

[1] Here we use the convention $n! = 1$ whenever $n \leq 0$.

The diffeomorphism $-\mathbb{I}$ commutes with each ρ-invariant relabeling and normalizing action of $Aut(\mathbb{H})$. Therefore, the quotient space

$$C_{(\mathbf{F},\rho)} := Conf_{(\mathbf{F},\rho)}/Aut(\mathbb{H})$$

is a manifold of dimension $|\mathbf{F}|-3$, its connected components are diffeomorphic to the components of $\widetilde{C}_{(\mathbf{F},\rho)}$. Moreover, the quotient map $\widetilde{C}_{(\mathbf{F},\rho)} \to C_{(\mathbf{F},\rho)}$ is a trivial double covering.

Case II. *Configurations on $\mathbb{CP}^1$ with empty real part.* Let $\mathbf{p}$ be ρ-invariant point configurations in $\mathbb{CP}^1$ with $conj : z \mapsto -1/\bar{z}$.

The group of automorphisms of $\mathbb{CP}^1$ which commutes with $conj$ is

$$Aut(\mathbb{CP}^1, conj) \cong SU_2 := \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SL_2(\mathbb{C}) \right\}.$$

Thus, $Aut(\mathbb{CP}^1, conj)$ acts naturally on the space

$$Conf^{\emptyset}_{(\mathbf{F},\rho)} := \{(p_{f_1}, \cdots, p_{f_{2i}}) \mid conj(p_{f_k}) = p_{\rho(f_k)}\}$$

of ρ-invariant point configurations on $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$. For $|\mathbf{F}| \geq 4$, the action is free and the quotient $B_{(\mathbf{F},\rho)} := Conf^{\emptyset}_{(\mathbf{F},\rho)}/Aut(\mathbb{CP}^1, conj)$ is a $|\mathbf{F}|-3$ dimensional connected manifold.

The *combinatorial type* of ρ-invariant point configurations on $\mathbb{CP}^1$ with $z \mapsto -1/\bar{z}$ is unique and given by the topological type of the real structure $z \mapsto -1/\bar{z}$.

5.1.2 A normal position of ρ-invariant point configurations in $\mathbb{CP}^1$

By using the automorphisms we can choose representatives of the points in $\widetilde{C}_{(\mathbf{F},\rho)}$ and $B_{(\mathbf{F},\rho)}$.

Case I. *Configurations on $\mathbb{CP}^1$ with non-empty real part.* Every element in $\widetilde{C}_{(\mathbf{F},\rho)}$ is represented by $\mathbf{p} \in \widetilde{Conf}_{(\mathbf{F},\rho)}$. In order to calibrate the choice, we consider an isomorphism $\mathbb{CP}^1 \mapsto \mathbb{CP}^1$ which is mapping $\mathbf{p} \mapsto \mathbf{p}'$, puts the points in the following normal position $\mathbf{p}' \in \mathbb{CP}^1$.

(A) If $|\mathbf{F}^{\mathbb{R}}(\rho)| \geq 3$, then the three consecutive points $(p'_{f_{j-1}}, p'_{f_m}, p'_{f_1})$ in $\mathbb{RP}^1$ are put in the position $x'_{f_{j-1}} = 1, x'_{f_m} = \infty, x'_{f_1} = 0$. We then obtain

$$\mathbf{p}' = (z_1, \cdots, z_i, \overline{z}_1, \cdots, \overline{z}_i, x_{2i+1}, \cdots, x_{2i+j-1}, \infty).$$

(B) If $|\mathbf{F}^{\mathbb{R}}(\rho)| = j = 1, 2$, then the three points $\{p_{f_i}, p_{f_{2i}}, p_{f_m}\}$ are put in the position $\{\pm\sqrt{-1},\ \infty\}$. Then,

$$\mathbf{p}' = \begin{cases} (z_1, \cdots, z_{i-1}, \epsilon\sqrt{-1}, \overline{z}_1, \cdots, \overline{z}_{i-1}, -\epsilon\sqrt{-1}, x_{2i+1}, \infty) & \text{if } j = 2, \\ (z_1, \cdots, z_{i-1}, \epsilon\sqrt{-1}, \overline{z}_1, \cdots, \overline{z}_{i-1}, -\epsilon\sqrt{-1}, \infty) & \text{if } j = 1 \end{cases}$$

where $\epsilon = \pm$.

(C) If $|\mathbf{F}^{\mathbb{R}}(\rho)| = 0$, then the points $\{p_{f_i}, p_{f_{2i}}\}$ are fixed at $\{\pm\sqrt{-1}\}$ and p_{f_k} where $\{f_k\} = \{f_{i-1}, f_{2i-1}\} \bigcap \mathbf{F}^+(\rho)$ is placed on the interval $]0, \sqrt{-1}[\subset \mathbb{H}^+$. Then,

$$\mathbf{p}' = (z_1, \cdots, z_{i-2}, \epsilon_1\lambda\sqrt{-1}, \epsilon_2\sqrt{-1}, \overline{z}_1, \cdots, \overline{z}_{i-2}, -\epsilon_1\lambda\sqrt{-1}, -\epsilon_2\sqrt{-1}).$$

where $\lambda \in]0, 1[$ and $\epsilon_i = \pm, i = 1, 2$.

Case II. ***Configurations on $\mathbb{CP}^1$ with empty real part.*** Every element of $B_{(\mathbf{F},\rho)}, |\mathbf{F}^{\mathbb{C}}(\rho)| \geq 4$, is represented by $\mathbf{p} \in Conf^{\emptyset}_{(\mathbf{F},\rho)}$. In order to calibrate the choice by using $Aut(\mathbb{CP}^1, conj)$, we consider an isomorphism $\mathbb{CP}^1 \mapsto \mathbb{CP}^1$ which is mapping $\mathbf{p} \mapsto \mathbf{p}'$, puts the points of $\mathbf{p}$ in the following normal position $\mathbf{p}' \in \mathbb{CP}^1$.

(D)

$$\mathbf{p} = (z_1, \cdots, z_{i-2}, \lambda\sqrt{-1}, \sqrt{-1}, \frac{-1}{\overline{z}_1}, \cdots, \frac{-1}{\overline{z}_{i-2}}, -\frac{\sqrt{-1}}{\lambda}, -\sqrt{-1}),$$

where $\lambda \in]-1, 1[$.

5.2 The open moduli space $\mathbb{R}M^{\sigma}_{\mathbf{S}}$

In this section, we choose $\mathbf{F}$ to be the labeling set $\mathbf{S}$, and ρ to be the involution σ.

Every σ-invariant point configuration gives a σ-invariant irreducible real curve. Hence, we define

$$\Xi : \begin{cases} C_{(\mathbf{S},\sigma)} \to \mathbb{R}M^{\sigma}_{\mathbf{S}} & \text{when } |\mathbf{Fix}(\sigma)| > 0, \\ C_{(\mathbf{S},\sigma)} \sqcup B_{(\mathbf{S},\sigma)} \to \mathbb{R}M^{\sigma}_{\mathbf{S}} & \text{when } |\mathbf{Fix}(\sigma)| = 0, \end{cases} \tag{5.2}$$

which maps each σ-invariant point configurations to the corresponding isomorphism classes of irreducible σ-invariant curves.

Lemma 5.2.1. *The map Ξ is a diffeomorphism.*

Proof. The map Ξ is clearly smooth. It is surjective since any σ-invariant irreducible curve is isomorphic either to $(\mathbb{CP}^1, z \mapsto \bar{z})$ or $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$ with a σ-invariant point configuration $\mathbf{p}$ on it. It is injective since the group of holomorphic automorphisms commuting with the real structure $z \mapsto \bar{z}$ is generated by $Aut(\mathbb{H})$ and $-\mathbb{I}$, and the group of holomorphic automorphisms commuting with the real structure $z \mapsto -1/\bar{z}$ is $Aut(\mathbb{CP}^1, conj)$. These automorphism are taken into account during construction of the configuration spaces. □

5.2.1 Connected components of $\mathbb{R}M^{\sigma}_{\mathbf{S}}$

As shown in Section 5.1.1, each connected component of $C_{(\mathbf{S},\sigma)}$ for $|\mathbf{Fix}(\sigma)| > 0$ (resp. $C_{(\mathbf{S},\sigma)} \sqcup B_{(\mathbf{S},\sigma)}$ for $|\mathbf{Fix}(\sigma)| = 0$) is associated to a unique un-oriented combinatorial type u of σ-invariant point configurations. Note that, combinatorial types of σ-invariant point configurations and u-planar trees (τ, u) of irreducible σ-invariant curves are encoded by same set of data since τ is one-vertex $\mathbf{S}$-tree. We denote the connected components of $C_{(\mathbf{S},\sigma)}$ (resp. $C_{(\mathbf{S},\sigma)} \sqcup B_{(\mathbf{S},\sigma)}$) by $C_{(\tau,u)}$.

Similarly, each connected component of $\widetilde{C}_{(\mathbf{F},\rho)}$ is associated to a unique oriented combinatorial type o. We denote the connected components of $\widetilde{C}_{(\mathbf{F},\rho)}$ by $C_{(\tau,o)}$ where τ is the one-vertex $\mathbf{S}$-tree.

Lemma 5.2.2. *The connected component $C_{(\tau,u)}$ of $C_{(\mathbf{S},\sigma)}$ (resp. $C_{(\mathbf{S},\sigma)} \sqcup B_{(\mathbf{S},\sigma)}$) is diffeomorphic to*

- $((\mathbb{H}^+)^{|\mathbf{Perm}^+|} \setminus \Delta) \times \square^{|\mathbf{Fix}(\sigma)|-3}$ *if* $|\mathbf{Fix}(\sigma)| > 2$,
- $((\mathbb{H}^+ \setminus \{\sqrt{-1}\})^{|\mathbf{Perm}^+|-1} \setminus \Delta) \times \square^{|\mathbf{Fix}(\sigma)|-1}$ *if* $|\mathbf{Fix}(\sigma)| = 1, 2$,
- $((\mathbb{H}^+ \setminus \{\sqrt{-1}, \sqrt{-1}/2\})^{|\mathbf{Perm}^+|-2} \setminus \Delta) \times \square^1$ *if* $|\mathbf{Fix}(\sigma)| = 0$ *and the u-planar structure is of type 1,*
- $((\mathbb{CP}^1 \setminus \{-\sqrt{-1}, -\sqrt{-1}/2, 2\sqrt{-1}, \sqrt{-1}\})^{|\mathbf{Perm}^+|-2} \setminus (\Delta \cup \Delta^c)) \times \square^1$ *if* $|\mathbf{Fix}(\sigma)| = 0$ *and the u-planar structure is of type 2.*

Here, Δ is the union of all diagonals $z_i \neq z_j (i \neq j)$, Δ^c is the union of all cross-diagonals $z_i \neq -\frac{1}{\bar{z}_j} (i \neq j)$, and $\square^d$ is the d-dimensional open simplex.

Proof. As it is shown in Section 5.1.1, $C_{(\tau,u)}$ is the quotient $C_{(\tau,o)} \bigsqcup C_{(\tau,\bar{o})}/(-\mathbb{I})$ where $C_{(\tau,o)}$ and $C_{(\tau,\bar{o})}$ have opposite oriented combinatorial types, and $-\mathbb{I}$ swaps $C_{(\tau,o)}$ and $C_{(\tau,\bar{o})}$. The spaces $C_{(\tau,o)}$ and $C_{(\tau,\bar{o})}$ are clearly diffeomorphic. To replace $C_{(\tau,u)}$ by $C_{(\tau,o)}$, we choose an oriented representative for each un-oriented combinatorial type as follows:

- if $|\mathbf{Fix}(\sigma)| \geq 3$, we choose the oriented combinatorial type for which $\{2k+1\} < \{n-1\} < \{n\}$ with respect to the cyclic ordering on $\mathbf{Fix}(\sigma)$.
- if $|\mathbf{Fix}(\sigma)| = 0, 1, 2$, we choose the oriented combinatorial type such that $k \in \mathbf{Perm}^+$.

We put the σ-invariant point configurations into a normal position as in 5.1.2. For $|\mathbf{Fix}(\sigma)| > 0$, the parameterizations stated above follow from (**A**) and (**B**) in Section 5.1.2. In the case of $|\mathbf{Fix}(\sigma)| = 0$ and the u-planar structure is of type 1, according to (**C**) the configuration space $C_{(\tau,u)}$ is a locally trivial fibration over $\square^1 =]0, 1[$ whose fibers over $\lambda \in \square^1$ are $(\mathbb{H}^+ \setminus \{\sqrt{-1}, \lambda\sqrt{-1}\})^{|\mathbf{Perm}^+|-2} \setminus \Delta$. Similarly, in the case of $|\mathbf{Fix}(\sigma)| = 0$ and the u-planar structure is of type 2, according to (**D**) the configuration space $C_{(\tau,u)}$ is a locally trivial fibration over $\square^1 =]-1,1[$ whose fibers over $\lambda \in \square^1$ are $(\mathbb{CP}^1 \setminus \{\lambda\sqrt{-1}, \sqrt{-1}, -\sqrt{-1}/\lambda, -\sqrt{-1}\})^{|\mathbf{Perm}^+|-2} \setminus \Delta$. Since the bases of these locally trivial fibrations are contractible, they are trivial fibrations, and the result follows. $\square$

5.3 Stratification of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$

We associate a product of configuration spaces of ρ_v-invariant point configurations $C_{(\tau_v,o_v)}$, and moduli space of pointed complex curves $\overline{M}_{\mathbf{F}_\tau(v)}$ to each o-planar tree (τ, o):

$$C_{(\tau,o)} := \begin{cases} \prod_{v\in \mathbf{V}^{\mathbb{R}}_\tau} C_{(\tau_v,o_v)} \times \prod_{v\in \mathbf{V}^+_\tau} M_{\mathbf{F}_\tau(v)} & \text{if } o \text{ is of type 1,} \\ C_{(\tau_{v_r},o_{v_r})} \times \prod_{\{v,\bar{v}\}\subset \mathbf{V}_\tau\setminus \mathbf{V}^{\mathbb{R}}_\tau} M_{\mathbf{F}_\tau(v)} & \text{if } o \text{ is of type 2,} \\ \prod_{v\in \mathbf{V}^+_\tau} M_{\mathbf{F}_\tau(v)} & \text{if } o \text{ is of type 3,}. \end{cases} \tag{5.3}$$

In the case of type 2, the product runs over the un-ordered pairs of conjugate vertices belonging to $\mathbf{V}_\tau \setminus \mathbf{V}^{\mathbb{R}}_\tau$ i.e., $\{v, \bar{v}\} = \{\bar{v}, v\}$, and v_r is the vertex corresponding to the unique real component of σ-invariant curves.

For each u-planar (τ, u), we first choose an o-planar representative (τ, o), and then put $C_{(\tau,u)} = C_{(\tau,o)}$. Note that the so defined space $C_{(\tau,u)}$ does not depend on the o-planar representatives.

Lemma 5.3.1. *Let $\gamma \in \mathcal{T}ree(\sigma)$. The real part $\mathbb{R}D_\gamma$ is diffeomorphic to $\bigsqcup_{(\gamma,u)} C_{(\gamma,u)}$ where the disjoint union is taken over all possible u-planar structures of γ.*

Proof. The complex strata D_γ is diffeomorphic to the product $\prod_{v\in \mathbf{V}_\gamma} \overline{M}_{\mathbf{F}_\gamma(v)}$. The real structure $c_\sigma : D_\gamma \to D_\gamma$ maps the factor $\overline{M}_{\mathbf{F}_\gamma(v)}$ onto $\overline{M}_{\mathbf{F}_\gamma(\bar{v})}$ for conjugate pair of vertices v and $\bar{v}$, and maps the factor $\overline{M}_{\mathbf{F}_\gamma(v)}$ onto itself when $v \in \mathbf{V}^{\mathbb{R}}_\gamma$. Therefore, the real part $\mathbb{R}D_\gamma$ of c_σ is given by

$$\begin{aligned} &\prod_{v\in \mathbf{V}^{\mathbb{R}}_\gamma} C_{(\mathbf{F}_\gamma(v),\rho_v)} \times \prod_{\{v,\bar{v}\}\subset \mathbf{V}_\gamma\setminus \mathbf{V}^{\mathbb{R}}_\gamma} \overline{M}_{\mathbf{F}_\gamma(v)} \quad \text{when } |\mathbf{V}^{\mathbb{R}}_\gamma| > 1, \\ &(C_{(\mathbf{F}_\gamma(v),\rho_v)} \bigsqcup B_{(\mathbf{F}_\gamma(v),\rho_v)}) \times \prod_{\{v,\bar{v}\}\subset \mathbf{V}_\gamma\setminus \mathbf{V}^{\mathbb{R}}_\gamma} \overline{M}_{\mathbf{F}_\gamma(v)} \quad \text{when } |\mathbf{V}^{\mathbb{R}}_\gamma| = 1, \\ &\prod_{\{v,\bar{v}\}\subset \mathbf{V}_\gamma\setminus \mathbf{V}^{\mathbb{R}}_\gamma} \overline{M}_{\mathbf{F}_\gamma(v)} \quad \text{when } |\mathbf{V}^{\mathbb{R}}_\gamma| = 0, \end{aligned}$$

where ρ_v is the involution whose action on $\mathbf{F}_\gamma(v)$ for $v \in \mathbf{V}^{\mathbb{R}}_\gamma$ is given by the restriction of $c_\Sigma : \Sigma \to \Sigma$ to special points on Σ_v. The decompositions of the spaces $C_{(\mathbf{F}_\gamma(v),\rho_v)}$ and $C_{(\mathbf{F}_\gamma(v),\rho_v)} \bigsqcup B_{(\mathbf{F}_\gamma(v),\rho_v)}$ into their connected components are given in Lemma 5.2.1. $\square$

Intermezzo: Coordinates around the codimension one strata Let γ be a two-vertex $\mathbf{S}$-tree given by $\mathbf{V}_\gamma = \{v_e, v^e\}$, $\mathbf{F}_\gamma(v^e) = \{s_1, \cdots, s_m, f^e\}$ and $\mathbf{F}_\gamma(v_e) = \{f_e, s_{m+1}, \cdots, s_{n-1}, s_n\}$. Let $(z, w) := [z : 1] \times [w : 1]$ be affine coordinates on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Here we introduce coordinates around D_γ.

Consider a neighborhood $V \subset D_\gamma$ of a nodal $\mathbf{S}$-pointed curve $(\Sigma^o, \mathbf{p}^o) \in D_\gamma$. Any $(\Sigma; \mathbf{p}) \in V$ can be identified with a nodal curve $\{(z - z_{f_e}) \cdot (w - w_{f^e}) = 0\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ with special points $\mathbf{p}_{v_e} = (a_{f_e}, a_{s_{m+1}}, \cdots, a_{s_{n-1}}, a_{s_n}) \subset \{w - w_{f^e} = 0\}$ and $\mathbf{p}_{v^e} = (b_{f^e}, b_{s_1}, \cdots, b_{s_m}) \subset \{z - z_{f_e} = 0\}$. In order to determine a nodal curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the position of its special points uniquely defined by $(\Sigma; \mathbf{p})$, we make the following choice. Firstly, we fix three labelled points $a_{s_{m+1}}, a_{s_{n-1}}, a_{s_n}$ on the line $\{w - w_{f^e} = 0\}$ whenever $|v_e| > 3$, and three special points $a_{f_e}, a_{s_{n-1}}, a_{s_n}$ whenever $|v_e| = 3$. Secondly, we fix three special points $b_{f^e}, b_{s_1}, b_{s_m}$ on $\{z - z_{f_e} = 0\}$. Finally, we choose

$$\begin{array}{llll} a_{s_{m+1}} = (0,0), & a_{s_{n-1}} = (1,0), & a_{s_n} = (\infty, 0) & \text{for } |v_e| > 3, \\ a_{f_e} = (0,0), & a_{s_{n-1}} = (1,0), & a_{s_n} = (\infty, 0) & \text{for } |v_e| = 3; \end{array}$$

and

$$b_{s_1} = (z_{f_e}, 1), \quad b_{s_m} = (z_{f_e}, \infty), \quad b_{f^e} = (z_{f_e}, 0).$$

Then, the components z and w of the special points provide a coordinate system in V; in particular, for $|v_e| > 3$ such a coordinate system is formed by z_{f^e}, z_{i_*} with $i_* = s_{m+2}, \cdots, s_{n-2}$, and w_{j_*} with $j_* = s_2, \cdots, s_{m-1}$.

We now consider a family of $\mathbf{S}$-pointed curves over V times the ϵ-ball $B_\epsilon = \{|t| < \epsilon\}$. It is given by a family curves $\{(z - z_{f_e}) \cdot w + t = 0 \mid t \in B_\epsilon\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$. The labelled points (z_s, w_s), $s \in \mathbf{S}$ on these curves are chosen in the following way. If $|v_e| > 3$, we put

$$\begin{array}{l} (z_{s_1}, w_{s_1}) = (z_{f_e} - t, 1), (z_{s_m}, w_{s_m}) = (z_{f_e}, \infty), (z_{s_{m+1}, w_{s_{m+1}}}) = (0, t/z_{f_e}), \\ (z_{s_{n-1}}, w_{s_{n-1}}) = (1, -t/(1 - z_{f_e})) \text{ and } (z_{s_n}, w_{s_n}) = (\infty, 0). \end{array}$$

Similarly, for $|v_e| = 3$, $(z_{s_1}, w_{s_1}) = (-t, 1)$, $(z_{s_{n-2}}, w_{s_{n-2}}) = (0, \infty)$, $(z_{s_{n-1}}, w_{s_{n-1}}) = (1, -t)$ and $(z_{s_n}, w_{s_n}) = (\infty, 0)$. The other labelled points are taken in an arbitrary position. The component z of the special points and the parameter t provide a coordinate system in $V \times B_\epsilon$.

Due to Knudsen's theorem there exists a unique $\kappa : V \times B_\epsilon \to \overline{M}_{\mathbf{S}}$ which gives the family of $\mathbf{S}$-pointed curves given above.

Lemma 5.3.2. $\det(d\kappa) \neq 0$ *at* $(\Sigma^o, \mathbf{p}^o) \in D_\gamma$. *Hence,* κ *gives a local isomorphism.*

Proof. The parameter t gives a regular function on $\kappa(V \times B_\epsilon)$ which is vanishing along $D_\gamma \bigcap \kappa(V \times B_\epsilon)$. The differential $d\kappa(\vec{v}) = \vec{v}$ for $\vec{v} \in T_{(\Sigma^o, \mathbf{p}^o)} V$ since the restriction of κ on $V \times \{0\}$ is the identity map. We need to prove that $d\kappa(\partial_t) \neq 0$. In other words, the curves are non-isomorphic for different values of the parameter t. Let $(\Sigma(t_i), \mathbf{p}(t_i)) \in V \times B_\epsilon$ be two $\mathbf{S}$-pointed curves for $t_1 \neq t_2$. A biholomorphic map $\Phi : \Sigma(t_1) \to \Sigma(t_2)$ is determined by the images of $p_{s_{m+1}}, p_{s_{n-1}}, p_{s_n}$ when $|v_e| > 3$, and by the images of $p_{s_{n-2}}, p_{s_{n-1}}, p_{s_n}$ when $|v_e| = 3$. However, the biholomorphic map Φ mapping $(p_{s_{m+1}}, p_{s_{n-1}}, p_{s_n})(t_1) \mapsto (p_{s_{m+1}}, p_{s_{n-1}}, p_{s_n})(t_2)$ (resp. $(p_{s_{n-2}}, p_{s_{n-1}}, p_{s_n})(t_1) \mapsto (p_{s_{n-2}}, p_{s_{n-1}}, p_{s_n})(t_2)$) maps $p_{s_1}(t_1) = (z_{f_e} - t_1, 1)$ to $(z_{f_e} - t_1, t_2/t_1) \neq p_{s_1}(t_2)$ (resp. $p_{s_1}(t_1) = (-t_1, 1)$ to $(-t_1, t_2/t_1) \neq p_{s_1}(t_2)$), i.e., Φ can not be an isomorphism. □

Remark 5.3.3. Due to Lemma 5.3.2, the coordinates on $V \times B_\epsilon$ provide a coordinate system at $(\Sigma^o; \mathbf{p}^o) \in D_\gamma$. There is a natural coordinate projection $\rho : V \times B_\epsilon \to V$.

For a σ-invariant γ and c_σ-invariant V, the above coordinates and the local isomorphism κ are equivariant with respect to a suitable real structure ($(z, w) \mapsto (\bar{z}, \bar{w})$ when $\mathbb{R}\Sigma \neq \emptyset$, and $(z, w) \mapsto (\bar{w}, \bar{z})$ when $\mathbb{R}\Sigma = \emptyset$) on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Therefore, the real part $\mathbb{R}V \times]-\epsilon, \epsilon[$ of $V \times B_\epsilon$ provides a neighborhood for a $(\Sigma^o; \mathbf{p}^o)$ in $\mathbb{R}D_\gamma$ with a set of coordinates on it.

Boundary of strata

Proposition 5.3.4. *A stratum $C_{(\gamma,\hat{u})}$ is contained in the boundary of $\overline{C}_{(\tau,u)}$ if and only if the u-planar structures $u, \hat{u}$ can be lifted to o-planar structures $o, \hat{o}$ in such a way that (τ, o) is obtained by contracting an invariant set of edges of $(\gamma, \hat{o})$.*

Proof. We need to consider the statement only for the strata of codimension one and two. These cases correspond to the contraction morphisms from two/three-vertex o-planar (sub)trees to one-vertex o-planar (sub)trees given in **(a)** and **(b)** of Section 4.4. For a stratum of higher codimension, the statement can be proved by applying the elementary contractions **(a)** and **(b)** inductively. Here, we consider only the case **(a-1)**. The proof for other cases is the same.

We first assume that (τ, o) is obtained by contracting the edge e of $(\gamma, \hat{o})$, where $(\gamma, \hat{o})$ is an o-planar two-vertex tree with $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$. An element $(\Sigma; \mathbf{p}) \in C_{(\gamma,\hat{o})}$ can be represented by the nodal curve $\{(z - z_{f_e}) \cdot w = 0\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ with special points $a_f = (z_f, 0)$ and $b_f = (z_{f_e}, w_f)$ such that

$$\begin{array}{ll} a_f \in \{w = 0 \ \& \ \Im(z) > 0\} & \text{for } f \in \mathbf{F}_\gamma^+(v_e) \\ a_{\bar{f}} \in \{w = 0 \ \& \ \Im(z) < 0\} & \text{for } \bar{f} \in \mathbf{F}_\gamma^-(v_e) \\ \{a_{f_{r_1}} < \cdots < a_{f_{r_m}}\} \subset \{w = 0 \ \& \ \Im(z) = 0\} & \text{for } f_* \in \mathbf{F}_\gamma^{\mathbb{R}}(v_e) \end{array}$$

and

$$\begin{array}{ll} b_f \in \{z = 0 \ \& \ \Im(w) > 0\} & \text{for } f \in \mathbf{F}_\gamma^+(v^e) \\ b_{\bar{f}} \in \{z = 0 \ \& \ \Im(w) < 0\} & \text{for } \bar{f} \in \mathbf{F}_\gamma^-(v^e) \\ \{b_{f'_{r'_1}} < \cdots < b_{f'_{r'_m}}\} \subset \{z = 0 \ \& \ \Im(w) = 0\} & \text{for } f'_* \in \mathbf{F}_\gamma^{\mathbb{R}}(v^e). \end{array}$$

When we include the curve $\{(z-z_{f_e})\cdot w=0\}$ into the family $\{(z-z_{f_e})\cdot w+t=0\}$, the complex orientation defined on the irreducible components $w=0$ and $z-z_{f_e}=0$ by the halves $\Im(z)>0$ and, respectively, $\Im(w)>0$ extends continuously to a complex orientation of $\{(z-z_{f_e})\cdot w+t=0\}$ with $t\in[0,\epsilon[$ defined by, say, $\Im(z)>0$. As a result, the curves $\{(z-z_{f_e})\cdot w+t=0\}$ with $t\in[0,\epsilon[$ acquire an o-planar structure given by

$$\begin{array}{ll} (z_f,w_f)\in\{z\cdot w+t=0 \ \&\ \Im(z)>0\} & \text{for } f\in\mathbf{F}^+_\gamma(v_e)\bigcup\mathbf{F}^+_\gamma(v^e) \\ (z_{\bar f},w_{\bar f})\in\{z\cdot w+t=0 \ \&\ \Im(z)<0\} & \text{for } \bar f\in\mathbf{F}^-_\gamma(v_e)\bigcup\in\mathbf{F}^-_\gamma(v^e) \\ (z_f,w_f)\in\{z\cdot w+t=0 \ \&\ \Im(z)=0\} & \text{for } f\in\mathbf{F}^{\mathbb{R}}_\gamma(v_e)\bigcup\mathbf{F}^{\mathbb{R}}_\gamma(v^e) \end{array}$$

where the points on the real part of the curves $\{z\cdot w+t=0\}$ are cyclicly ordered by

$$z_{f'_{r'_1}}<\cdots<z_{f'_{r'_m}}<z_{f_{r_1}}<\cdots<z_{f_{r_m}}.$$

This is exactly the o-planar structure (τ,o) defined in **(a-1)** of Section 4.4.

Now assume that $C_{(\gamma,\hat u)}$, where $(\gamma,\hat u)$ is an u-planar tree with $V_\gamma=V^{\mathbb{R}}_\gamma=\{v_e,v^e\}$, is contained in the boundary of $\overline{C}_{(\tau,u)}$. There are four different o-planar representatives of $(\gamma,\hat u)$, and any pair of o-planar representatives $\hat o_1,\hat o_2$ which are not opposite to each other, provide two different o-planar structures $(\tau,o_i), i=1,2$ after contraction. By the already proved part of the statement, $C_{(\gamma,\hat u)}$ is contained in the boundary of $\overline{C}_{(\tau,o_i)}$ for each $i=1,2$. It remains to notice that any codimension one stratum is adjacent to at most two main strata. □

Remark 5.3.5. Let $(\gamma,\hat o)$ be an o-planar tree type 1, and let $\boldsymbol{\varphi}_e:(\gamma,\hat o)\to(\tau,o)$ be the contraction of an edge $e\in\mathbf{E}_\gamma$. If the o-planar tree (τ,o) and the u-planar tree $(\gamma,\hat u)$ underlying $(\gamma,\hat o)$ are given, then the o-planar structure $\hat o$ can be reconstructed. For this reason, we denote the corresponding o-planar structure $\hat o$ by $\delta(o)$.

Stratification of $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$

Theorem 4. *(a) $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ is stratified by $C_{(\gamma,u)}$.*

(b) The closure of any stratum $\overline{C}_{(\gamma,u)}$ is stratified by $\{C_{(\gamma',u')}\mid(\gamma',u')<(\gamma,u)\}$.

Proof. (a) The moduli space $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ can be stratified by $\mathbb{R}D_\gamma$ due to Theorem 3. The claim directly follows from the decomposition of open strata $\mathbb{R}D_\gamma$ into its connected components given in Lemma 5.3.1.

(b) The claim direcly follows from the part (a) and Proposition 5.3.4. □

Example 5.3.6. (i) The first nontrivial example is $\overline{M}_{\mathbf{S}}$ with $|\mathbf{S}|=4$. There are three conjugancy classes of real structures: c_{σ_1}, c_{σ_2}, c_{σ_3}, where

$$\sigma_1=\mathbf{id},\ \sigma_2=\begin{pmatrix} s_1 & s_2 & s_3 & s_4\\ s_2 & s_1 & s_3 & s_4\end{pmatrix} \text{ and } \sigma_3=\begin{pmatrix} s_1 & s_2 & s_3 & s_4\\ s_3 & s_4 & s_1 & s_2\end{pmatrix}.$$

These real structures give $\mathbb{R}\overline{M}^{\sigma}_{(2k,l)}$, where $(2k,l) = (0,4), (2,2)$, and $(4,0)$ respectively.

In the case of $\sigma = \sigma_1$, $\mathbb{R}M^{\sigma}_{(0,4)}$ is the configuration space of four distinct points on $\mathbb{RP}^1$ up to the action of $PSL_2(\mathbb{R})$. Each $\mathbf{S}$-pointed curve $(\Sigma; \mathbf{p}) \in \mathbb{R}M^{\sigma}_{(0,4)}$ can be identified with $(0, x_2, 1, \infty)$ where $x_2 \in \mathbb{RP}^1 \setminus \{0, 1, \infty\}$. Hence, $\mathbb{R}M^{\sigma}_{(0,4)} = \mathbb{RP}^1 \setminus \{0, 1, \infty\}$, and its compactification is $\mathbb{R}\overline{M}^{\sigma}_{(0,4)} = \mathbb{RP}^1$. The three intervals of $\mathbb{R}M^{\sigma}_{(0,4)}$ are the three configuration spaces $C_{(\tau, u_i)}$ and the three points are the configuration spaces $C_{(\gamma_i, \hat{u}_i)}$. The u-planar trees (τ, u_i) and $(\gamma, \hat{u}_i)$ are given in Figure 5.1.

Figure 5.1: All strata of $\mathbb{R}\overline{M}^{\sigma}_{(0,4)}$.

In the case of $\sigma = \sigma_2$, $\mathbb{R}M^{\sigma}_{(2,2)}$ is the space of distinct configurations of two points in $\mathbb{RP}^1$ and a pair of complex conjugate points in $\mathbb{CP}^1 \backslash \mathbb{RP}^1$. $(\Sigma; \mathbf{p}) \in \mathbb{R}M^{\sigma}_{(2,2)}$ is identified with $(\sqrt{-1}, -\sqrt{-1}, x_3, \infty) \in C_{(\tau,u)}, -\infty < x_3 < \infty$. Hence, $\mathbb{R}M^{\sigma}_{(2,2)} = \mathbb{RP}^1 \setminus \{\infty\}$, and its compactification is $\mathbb{R}\overline{M}^{\sigma}_{(2,2)} = \mathbb{RP}^1$. The interval $\mathbb{R}M^{\sigma}_{(2,2)}$ is $C_{(\tau,u)}$ and the point at its closure is $C_{(\gamma,\hat{u})}$.

In the case of $\sigma = \sigma_3$, the moduli space $\mathbb{R}M^{\sigma}_{(4,0)}$ has different pieces parameterizing real curves with non-empty and empty real parts: The subspace of $\mathbb{R}M^{\sigma}_{(4,0)}$ parameterizing the σ-invariant curves with $\mathbb{R}\Sigma \neq \emptyset$ is $(\lambda\sqrt{-1}, \sqrt{-1}, -\lambda\sqrt{-1}, -\sqrt{-1})$ where $\lambda \in]-1, 1[\backslash\{0\}$. The subspace of $\mathbb{R}M^{\sigma}_{(4,0)}$ parameterizing the real curves with $\mathbb{R}\Sigma = \emptyset$ is $(\lambda, 1, -\lambda, -1)$, where $\lambda \in]-1, 1[$. Note that, the pieces parameterizing $\mathbb{R}\Sigma \neq \emptyset$ and $\mathbb{R}\Sigma = \emptyset$ are joined through the boundary points corresponding to curves with isolated real singular points. The compactification $\mathbb{R}\overline{M}^{\sigma}_{(4,0)}$ is $\mathbb{RP}^1$.

(ii) For $|\mathbf{S}| = 5$, the moduli space $\overline{M}_{\mathbf{S}}$ has three different real structures $c_{\sigma_1}, c_{\sigma_2}$ and c_{σ_3} where

$$\sigma_1 = \mathbf{id},\ \sigma_2 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ s_2 & s_1 & s_3 & s_4 & s_5 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_1 & s_2 & s_5 \end{pmatrix}. \quad (5.4)$$

For $\sigma = \sigma_1$, the space $\mathbb{R}M^{\sigma}_{\mathbf{S}}$ is identified with the configuration space of five distinct points on $\mathbb{RP}^1$ modulo $PSL_2(\mathbb{R})$. It is $(\mathbb{RP}^1 \setminus \{0, 1, \infty\})^2 \setminus \Delta$, where Δ is union of all diagonals. Each connected component of $\mathbb{R}M^{\sigma}_{(0,5)}$ is isomorphic to a two dimensional simplex. The closure of each cell can be obtained by adding the boundaries described in Proposition 5.3.4; for an example see Figure 5.2a. It gives the compactification of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ which is a torus with three points blown up: the cells corresponding to u-planar trees (τ, u_1) and (τ, u_2) are glued along the face corresponding to $(\gamma, \hat{u})$ which gives $(\tau, u_i), i = 1, 2$ by contracting some edges, see Fig. 5.2b.

Figure 5.2: (a) Stratification of $C_{(\tau,u)}$. (b) The stratification of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ for $|\mathbf{S}| = 5$.

For $\sigma = \sigma_2$, the space $\mathbb{R}M_{\mathbf{S}}^{\sigma}$ is configuration space of conjugate pairs of points on $\mathbb{CP}^1$ minus three points. The automorphisms allows us to identify such configurations with $(z, \bar{z}, 0, 1, \infty)$ where $z \in \mathbb{C} \setminus \mathbb{R}$. Hence, it can be given as $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. The moduli space $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ is obtained as a sphere with three points blown up according to the stratification given in Proposition 5.3.4.

Finally, for $\sigma = \sigma_3$, the elements of $\mathbb{R}M_{\mathbf{S}}^{\sigma}$ can be identified with the point configurations $(z, \sqrt{-1}, \bar{z}, -\sqrt{-1}, \infty)$. Hence it can be identified with $\mathbb{CP}^1 \setminus (\mathbb{RP}^1 \cup \{\sqrt{-1}, -\sqrt{-1}\})$. Therefore, connected components are isomorphic to $\mathbb{H}^+ \setminus \{\sqrt{-1}\}$. The moduli space $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ is a sphere with a point blown up.

In fact, for $|\mathbf{S}| = 5$, the moduli space $\overline{M}_{\mathbf{S}}$ is a del Pezzo surface of degree five, and these are all the possible real parts of this del Pezzo surface (see [6]).

Chapter 6

First Stiefel-Whitney class of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$

In this chapter, we calculate the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ by using its stratification.

6.1 Orientations of top-dimensional strata

Let (τ, o) be a one-vertex o-planar tree. The coordinates of configuration spaces given in Section 5.1.2 determine an orientation of $C_{(\tau,o)}$. For instance, let $|\mathbf{Fix}(\sigma)| \geq 3$ and let the o-planar structure on (τ, o) be given by $\mathbf{Perm}^{\pm}$ and by a linear ordering $x_{r_1} = 0 < x_{r_2} < \cdots < x_{r_{l-1}} = 1 < x_{r_l} := x_n = \infty$ on $\mathbf{Fix}(\sigma)$. The coordinates in **(A)** of 5.1.2 generate the following top-dimensional differential form on $C_{(\tau,o)}$:

$$\omega_{(\tau,o)} := \left(\frac{\sqrt{-1}}{2}\right)^{|\mathbf{Perm}^{+}|} \bigwedge_{\alpha_* \in \mathbf{Perm}^{+}} dz_{\alpha_*} \wedge d\overline{z}_{\alpha_*} \bigwedge dx_{r_2} \wedge \cdots \wedge dx_{r_{l-2}}. \tag{6.1}$$

The multiplication of top-dimensional forms with a positive valued function $\Theta : C_{(\tau,o)} \to \mathbb{R}_{>0}$ defines an equivalence relation on sections of $\det(TC_{(\tau,o)})$. An *orientation* is an equivalence class of nowhere zero top-dimensional forms with respect to this equivalence relation. We denote the equivalence class of $\omega_{(\tau,o)}$ by $[\omega_{(\tau,o)}]$.

Similarly, we determine differential forms $\omega_{(\tau,o)}$ and orientations $[\omega_{(\tau,o)}]$ of $C_{(\tau,o)}$ for all (τ, o) with $|\mathbf{V}_\tau| = 1$ by using the coordinates given in **(B)**, **(C)** and **(D)** in Section 5.1.2 and their ordering.

6.2 Orientations of codimension one strata

Let (γ, o) be a two-vertex o-planar tree. Let $\mathbf{V}_\gamma = \{v_e, v^e\}$ and $e = (f_e, f^e)$ be the edge where $\boldsymbol{\partial}_\gamma(s_n) = \boldsymbol{\partial}_\gamma(f_e) = v_e$ and $\boldsymbol{\partial}_\gamma(f^e) = v^e$.

By choosing three flags in $\mathbf{F}_\gamma(v_e)$ and $\mathbf{F}_\gamma(v^e)$, and using the calibrations as in Section 5.1.2, we obtain a coordinate system in $C_{(\gamma_v,o_v)}$ for each $v \in \{v_e, v^e\}$. More precisely, we use the following choice.

I. Let o_v be an o-planar structure of type 1, and let $\mathbf{Fix}(\sigma) \neq \emptyset$. If $|\mathbf{F}^{\mathbb{R}}_\gamma(v_e)| \geq 3$ (resp. $|\mathbf{F}^{\mathbb{R}}_\gamma(v^e)| \geq 3$), then we specify an isomorphism $\Phi_{v_e} : \Sigma_{v_e} \to \mathbb{CP}^1$ (resp. $\Phi_{v^e} : \Sigma_{v^e} \to \mathbb{CP}^1$) by mapping three consecutive special points as follows: If $|\mathbf{F}^{\mathbb{R}}_\gamma(v_e)| > 3$ and the special points p_{f_e} and p_{s_n} are not consecutive, then $\Phi_{v_e} : (p_{r_{l-1}}, p_{s_n}, p_{r_1}) \mapsto (1, \infty, 0)$. If $|\mathbf{F}^{\mathbb{R}}_\gamma(v_e)| \geq 3$ and the special points p_{f_e} and p_{s_n} are consecutive and

$$\begin{array}{lcll} \{f_e\} < \{s_n\} < \{r_1\}, & \Longrightarrow & \Phi_{v_e} : (p_{f_e}, p_{s_n}, p_{r_1}) & \mapsto \ (1, \infty, 0), \\ \{r_{l-1}\} < \{s_n\} < \{f_e\}, & \Longrightarrow & \Phi_{v_e} : (p_{r_{l-1}}, p_{s_n}, p_{f_e}) & \mapsto \ (1, \infty, 0). \end{array}$$

For $|\mathbf{F}^{\mathbb{R}}_\gamma(v^e)| \geq 3$, $\Phi_{v^e} : (p_{r_{i+1}}, p_{r_{i+j}}, p_{f^e}) \mapsto (1, \infty, 0)$.

If $|\mathbf{F}^{\mathbb{R}}_\gamma(v_e)| < 3$ (resp. $|\mathbf{F}^{\mathbb{R}}_\gamma(v^e)| < 3$), then $p_{s_n} \mapsto \infty$ (resp. $p_{f^e} \mapsto 0$). We pick the *maximal element* $\alpha = s_i \in \mathbf{F}^+_\gamma(v_e)$ such that $i > j$ for all $s_j \in \mathbf{F}^+_\gamma(v_e)$ (resp. in $\mathbf{F}^+_\gamma(v^e)$), and map the pair of conjugate labelled points $(p_\alpha, p_{\overline{\alpha}})$ to $(\sqrt{-1}, -\sqrt{-1})$.

II. Let o_v be an o-planar structure of type 1, and let $\mathbf{Fix}(\sigma) = \emptyset$. We specify an isomorphism $\Phi_v : \Sigma_v \to \mathbb{CP}^1$ by mapping the pair of conjugate labelled points $(p_\alpha, p_{\overline{\alpha}})$ to $(\sqrt{-1}, -\sqrt{-1})$ for the maximal element α in $\mathbf{F}^+_\gamma(v)$, and $p_{f_e} \mapsto 0$ (resp. $p_{f^e} \mapsto 0$).

III. Let o be an o-planar structure of type 3. We pick a maximal element α_{k-1} in $\mathbf{F}^+_\gamma(v_e) \setminus \{n\}$ and specify isomorphisms $\Phi_{v_e} : \Sigma_{v_e} \to \mathbb{CP}^1$ and $\Phi_{v^e} : \Sigma_{v^e} \to \mathbb{CP}^1$ by mapping the special points $(p_{f_e}, p_{\alpha_{k-1}}, p_{s_n})$ to $(0, \sqrt{-1}/2, \sqrt{-1})$ and, $(p_{f^e}, p_{\overline{\alpha}_{k-1}}, p_{\overline{s_n}})$ to $(0, -\sqrt{-1}/2, -\sqrt{-1})$.

For each $v \in \{v_e, v^e\}$, we arrange the coordinates of the special points in the following order

$$(z_{\alpha_1}, \cdots, z_{\alpha_{k_v}}, x_{r_1}, \cdots, x_{r_{l_v}}),$$

by using the o-planar structure

$$o_v = \begin{cases} \{\text{type } 1; \mathbf{F}^{\pm}_\gamma(v); \mathbf{F}^{\mathbb{R}}_\gamma(v) = \{\{f_{r_1}\} < \cdots < \{f_{r_{l_v}}\}\}\} & \text{for case } \mathbf{I}, \\ \{\text{type } 1; \mathbf{F}^{\pm}_\gamma(v); \mathbf{F}^{\mathbb{R}}_\gamma(v) = \emptyset\} & \text{for case } \mathbf{II}, \end{cases}$$

of γ_v, where $\alpha_* \in \mathbf{F}^+_\gamma(v)$. We fix special points as in (**I**) and (**II**), and apply (6.1) to introduce top-dimensional differential forms $\Omega_{(\gamma_{v^e}, o_{v^e})}$ and $\Omega_{(\gamma_{v_e}, o_{v_e})}$ on $C_{(\gamma_{v^e}, o_{v^e})}$ and $C_{(\gamma_{v_e}, o_{v_e})}$ (note that the resulting forms do not depend on the order of z-coordinates).

In case (**III**), there are no real special points, so we may get a top-dimensional differential form $\Omega_{(\gamma,o)}$ on $C_{(\gamma,o)}$ via choosing the vertex $v \in \mathbf{V}_\gamma^+$ with ordering arbitrarily the z-coordinates

$$(z_{\alpha_1}, \cdots, z_{\alpha_{k_v}})$$

where $\mathbf{F}_\gamma^+ = \{\alpha_1, \cdots, \alpha_{k_v}\}$.

In such a way, we obtain well-defined orientations $[\Omega_{(\gamma_{v^e}, o_{v^e})}]$ and $[\Omega_{(\gamma_{v_e}, o_{v_e})}]$ of, respectively, $C_{(\gamma_{v^e}, o_{v^e})}$ and $C_{(\gamma_{v_e}, o_{v_e})}$, and finally get an orientation on $C_{(\gamma,o)}$ given by

$$\begin{array}{ll} [\Omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\Omega_{(\gamma_{v_e}, o_{v_e})}] & \text{when } \mathbf{V}_\gamma = \mathbf{V}_\gamma^{\mathbb{R}} \\ [\Omega_{(\gamma,o)}] & \text{when } \mathbf{V}_\gamma^{\mathbb{R}} = \emptyset, \text{ and } v \in \mathbf{V}_\gamma^+. \end{array}$$

6.2.1 Induced orientations on codimension one strata

Let $C_{(\tau,u)}$ be a top-dimensional stratum and $C_{(\gamma,\hat{u})}$ be a codimension one stratum contained in the boundary of $\overline{C}_{(\tau,u)}$. We lift the u-planar structures $u, \hat{u}$ to o-planar representatives $o, \hat{o} = \delta(o)$ such that (τ, o) is obtained by contracting the edge of $(\gamma, \delta(o))$ (see Proposition 5.3.4). Then, we pick a point $(\Sigma^o, \mathbf{p}^o) \in C_{(\gamma,\delta(o))}$ and consider a tubular neighborhood $\mathbb{R}V \times [0, \epsilon[$ of $(\Sigma^o, \mathbf{p}^o)$ in $\overline{C}_{(\tau,o)}$ as in Section 5.3.

The orientation $[\omega_{(\tau,o)}]$, introduced in Section 6.1, induces some orientation on $C_{(\gamma,\delta(o))}$: The outward normal direction of $\overline{C}_{(\tau,o)}$ on $\mathbb{R}V \times \{0\} \subset C_{(\gamma,\delta(o))}$ is $-\partial_t$, where t is the standard coordinate on $[0, \epsilon[\subset \mathbb{R}$. Therefore a differential form $\omega_{(\gamma,\delta(o))}$ defines the induced orientation, if and only if

$$-dt \wedge \omega_{(\gamma,\delta(o))} = \Theta\, \omega_{(\tau,o)} \qquad (6.2)$$

with $\Theta > 0$ at each point of $\mathbb{R}V \times]0, \epsilon[$.

In what follows we compare the induced orientation $[\omega_{(\gamma,\delta(o))}]$ with $[\Omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\Omega_{(\gamma_{v_e}, o_{v_e})}]$.

Case I: $|\mathbf{Fix}(\sigma)| = l \geq 1$.

Lemma 6.2.1. *Let $(\gamma, \delta(o))$ be an o-planar tree as above in Section 6.2, where $\mathbf{F}_\gamma^{\mathbb{R}}(v^e) = \{\{r_{i+1}\} < \cdots \{r_{i+j}\} < \{f^e\}\}$. Then,*

$$[\omega_{(\gamma,\delta(o))}] = (-1)^{\aleph}\ [\Omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\Omega_{(\gamma_{v_e}, o_{v_e})}]$$

where the values of $\aleph$ for separate cases are given in the following table.

$\aleph$	$l-j\geq 3$	$l-j=2$		$l-j=1$
		$\{r_1\}<\{f_e\}<\{s_n\}$	$\{f_e\}<\{r_{l-1}\}<\{s_n\}$	
$j\geq 2$	$(i+1)(j+1)$	0	$l+1$	$l+1$
$j=1$	1	1	1	1
$j=0$	$i+1$	0	0	0

*Here, the third and fourth columns correspond to two possible cyclic orderings of $\mathbf{F}_\gamma^{\mathbb{R}}(v_e)$ for $|\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| = 3$ in Case **I** of Section 6.2.*

Figure 6.1: Codimension 1 boundaries of $\overline{C}_{(\tau,\delta(o))}$ where $l-j \geq 3$ & $j \geq 2$.

Proof. We will prove the statement only for the special case of $l-j\geq 3$, $j \geq 2$. The calculations for other cases are almost identical.

Let $(\Sigma^o, \mathbf{p}^o) \in C_{(\gamma,\delta(o))}$. We set $\Sigma^o_{v_e}$ to be $\{w=0\}$ and $\Sigma^o_{v^e}$ to be $\{z - x_{f_e} = 0\}$. According to the convention in Section 6.2, the consecutive special points $(p_{r_{l-1}}, p_{s_n}, p_{r_1})$ (resp. $(p_{r_{i+1}}, p_{r_{i+j}}, p_{f^e})$) on the component Σ_{v_e} (resp. Σ_{v^e}) are fixed at $(1, \infty, 0)$. As shown in the proof of Proposition 5.3.4, a tubular neighborhood $\mathbb{R}V \times [0,\epsilon[$ of $(\Sigma^o, \mathbf{p}^o)$ in $\overline{C}_{(\tau,o)}$ can be given by the family $\{(z - x_{f_e})\cdot w + t = 0 \mid t \in [0,\epsilon[\}$ with labelled points $p_{r_1} = (0, t/x_{f_e})$, $p_{r_{i+1}} = (x_{f_e} - t, 1)$, $p_{r_{i+j}} = (x_{f_e}, \infty)$, $p_{r_{l-1}} = (1, -t/(1-x_{f_e}))$, $p_{s_n} = (\infty, 0)$, $p_{r_*} = (x_{r_*}, -t/(x_{r_*} - x_{f_e}))$ for $r_* \in \mathbf{F}^{\mathbb{R}}_\gamma \setminus \{r_1, r_{i+1}, r_{i+j}, r_{l-1}, s_n\}$ and $p_\alpha = (z_\alpha, -t/(z_\alpha - x_{f_e}))$ for $\alpha \in \mathbf{F}^+_\gamma$.

We first consider the following subcase: the special points p_{f_e} and p_{s_n} are not consecutive. According to the convention of Section 6.2, the differential forms $\Omega_{(\gamma_{v_e}, o_{v_e})}$ and $\Omega_{(\gamma_{v^e}, o_{v^e})}$ of this case are as follows:

$$\begin{aligned}
\Omega_{(\gamma_{v_e},o_{v_e})} &= \left(\tfrac{\sqrt{-1}}{2}\right)^{|\mathbf{F}^+_\gamma(v_e)|} \textstyle\bigwedge_{\alpha\in\mathbf{F}^+_\gamma(v_e)} dz_\alpha \wedge d\overline{z}_\alpha \wedge \\
&\quad dx_{r_2} \wedge \cdots \wedge dx_{r_i} \wedge dx_{f_e} \wedge \widehat{dx_{r_{i+1}}} \wedge \widehat{\cdots} \wedge \widehat{dx_{r_{i+j}}} \wedge dx_{r_{i+j+1}} \wedge \cdots \wedge dx_{r_{l-2}}, \\
\Omega_{(\gamma_{v^e},o_{v^e})} &= \left(\tfrac{\sqrt{-1}}{2}\right)^{|\mathbf{F}^+_\gamma(v^e)|} \textstyle\bigwedge_{\beta\in\mathbf{F}^+_\gamma(v^e)} dw_\beta \wedge d\overline{w}_\beta \bigwedge dy_{r_{i+2}} \wedge \cdots \wedge dy_{r_{i+j-1}}
\end{aligned}$$

By using the identities $w_\beta = -t/(z_\beta - x_{f_e})$ for $\beta \in \mathbf{F}^+_\gamma(v^e)$ and $y_r = -t/(x_r - x_{f_e})$ for $r \in \mathbf{F}^{\mathbb{R}}_\gamma(v^e)$, we obtain the following equalities:

$$\begin{aligned}
& dt = -dx_{r_{i+1}} + dx_{f_e}, \quad dx_{f_e} = dx_{r_{i+j}}, \\
& dw_\beta = -\tfrac{dt}{z_\beta - x_{f_e}} + \tfrac{tdz_\beta}{(z_\beta - x_{f_e})^2} - \tfrac{tdx_{f_e}}{(z_\beta - x_{f_e})^2} \quad \text{for } \beta \in F^{\pm}_\gamma(v^e), \\
& dy_r = -\tfrac{dt}{x_r - x_{f_e}} + \tfrac{tdx_r}{(x_r - x_{f_e})^2} - \tfrac{tdx_{f_e}}{(x_r - x_{f_e})^2} \quad \text{for } r = r_{i+2}, \cdots, r_{i+j-1}.
\end{aligned}$$

These identities imply that $-dt \wedge \Omega_{(\gamma_{v^e},o_{v^e})} \wedge \Omega_{(\gamma_{v_e},o_{v_e})}$ is equal to

$$(-1)^{(i-1)(j-1)} \left(\frac{\sqrt{-1}}{2}\right)^{|\mathbf{F}^+_\gamma|} \Theta \bigwedge_{\alpha \in \mathbf{F}^+_\gamma} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{r_2} \wedge \cdots \wedge dx_{r_{l-2}}$$

where $\Theta = \prod_{\beta\in\mathbf{F}^+_\gamma(v^e)} t(z_\beta - x_{f_e})^{-2} \prod_{r=r_{i+2},\cdots,r_{i+j-1}} t(x_r - x_{f_e})^{-2}$. Since $\Theta > 0$, the orientation defined by $-dt \wedge \Omega_{(\gamma_{v^e},o_{v^e})} \wedge \Omega_{(\gamma_{v^e},o_{v^e})}$ is equal to $(-1)^{\aleph}[\omega_{(\tau,o)}]$.

We now consider the cases $\{f_e\} < \{s_n\} < \{r_1\}$ (i.e, $i+j = l-1$) and $\{r_{l-1}\} < \{s_n\} < \{f_e\}$ (i.e, $i = 0$). According to convention in Section 6.2, the differential forms $\Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v_e}, o_{v_e})}$ are equal to

$$\begin{array}{ll} \left(\frac{\sqrt{-1}}{2}\right)^{|\mathbf{F}_\gamma^+|} & \left(\bigwedge_{\beta \in \mathbf{F}_\gamma^+(v^e)} dz_\beta \wedge d\overline{z}_\beta \bigwedge dy_{r_{i+2}} \wedge \cdots \wedge dy_{r_{l-2}}\right) \\ & \wedge \left(\bigwedge_{\alpha \in \mathbf{F}_\gamma^+(v_e)} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{r_2} \wedge \cdots \wedge dx_{r_i}\right) \end{array} \quad \text{when } i+j = l-1,$$

$$\begin{array}{ll} \left(\frac{\sqrt{-1}}{2}\right)^{|\mathbf{F}_\gamma^+|} & \left(\bigwedge_{\beta \in \mathbf{F}_\gamma^+(v^e)} dz_\beta \wedge d\overline{z}_\beta \bigwedge dy_{r_2} \wedge \cdots \wedge dy_{r_{i-1}}\right) \\ & \wedge \left(\bigwedge_{\alpha \in \mathbf{F}_\gamma^+(v_e)} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{r_{i+1}} \wedge \cdots \wedge dx_{r_{l-2}}\right) \end{array} \quad \text{when } i = 0.$$

The equation $(z - x_{f_e}) \cdot w + t = 0$ implies the following equalities:

$$\begin{array}{ll} dt = -dx_{r_{i+1}}, \quad dx_{f_e} = dx_{r_{l-1}} & \text{when } i+j = l-1 \\ dt = dx_{r_i}, \quad dx_{f_e} = dx_{r_{i+j}} & \text{when } q = 0, \end{array}$$

and

$$\begin{array}{ll} dw_\beta = -\frac{dt}{z_\beta - x_{f_e}} + \frac{t dz_\beta}{(z_\beta - x_{f_e})^2} - \frac{t dx_{f_e}}{(z_\beta - x_{f_e})^2} & \text{for } \beta \in \mathbf{F}_\gamma^\pm(v^e), \\ dy_r = -\frac{dt}{x_r - x_{f_e}} + \frac{t dx_r}{(x_r - x_{f_e})^2} - \frac{t dx_{f_e}}{(x_r - x_{f_e})^2} & \text{for } r = r_{i+2}, \cdots, r_{i+j-1}. \end{array}$$

By using these identities we obtain that $-dt \wedge \Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v^e}, o_{v^e})}$ is equal to

$$(-1)^{(i-1)(l-i-2)} \left(\frac{\sqrt{-1}}{2}\right)^{|\mathbf{F}_\gamma^+|} \Theta \bigwedge_{\alpha \in \mathbf{F}_\gamma^+} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{r_2} \wedge \cdots \wedge dx_{r_{l-2}}$$

when $i+j = l-1$, and

$$(-1)^{(j-1)} \left(\frac{\sqrt{-1}}{2}\right)^{|\mathbf{F}_\gamma^+|} \Theta \bigwedge_{\alpha \in \mathbf{F}_\gamma^+} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{r_2} \wedge \cdots \wedge dx_{r_{l-2}}$$

when $i = 0$. Since $\Theta = \prod_{\beta \in F_\gamma^+(v^e)} t(z_\beta - x_{f_e})^{-2} \prod_{r = r_{i+2}, \cdots, r_{i+j-1}} t(x_r - x_{f_e})^{-2} > 0$, the orientation $[\omega_{(\gamma, \delta(o))}]$ induced by $[\omega_{(\tau, o)}]$ is equal to

$$\begin{array}{ll} (-1)^{(i+1)(j-1)} \left[\Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v_e}, o_{v_e})}\right] & \text{when } i+j = l-1, \\ (-1)^{(j-1)} \left[\Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v_e}, o_{v_e})}\right] & \text{when } i = 0. \end{array}$$

□

Case II. $|\mathbf{Fix}(\sigma)| = 0$.

The different cases for boundaries of $C_{(\tau,o)}$ are treated separately.

Subcase: σ-invariant curves of type 1. Let (τ, o) be a one-vertex o-planar tree of type 1, and let $[\omega_{(\tau,o)}]$ be the orientation of $C_{(\tau,o)}$ defined by the differential form

$$\omega_{(\tau,o)} := (\frac{\sqrt{-1}}{2})^{k-2} \bigwedge_{\alpha_* \in \mathbf{Perm}^+ \setminus (\mathbf{Perm}^+ \bigcap \{s_{k-1}, s_k, s_{2k-1}, s_{2k}\})} dz_{\alpha_*} \wedge d\overline{z}_{\alpha_*} \bigwedge d\lambda \tag{6.3}$$

(which is given by the coordinates in **(C)** of 5.1.2). Here $\lambda = \Im(z_{\alpha'})$ and $\alpha' \in \{s_{k-1}, s_{2k-1}\} \bigcap \mathbf{Perm}^+$.

Lemma 6.2.2. *Let $(\gamma, \delta(o))$ be a two-vertex o-planar tree, and let the corresponding strata $C_{(\gamma,\delta(o))}$ be contained in the boundary of $\overline{C}_{(\tau,o)}$.*

(a) If $\mathbf{V}_\gamma^{\mathbb{R}} = \mathbf{V}_\gamma$, then the orientation $[\omega_{(\gamma,\delta(o))}]$ induced by the orientation $[\omega_{(\tau,o)}]$ is equal to $-[\Omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\Omega_{(\gamma_{v_e}, o_{v_e})}]$.

(b) If $\mathbf{V}_\gamma^{\mathbb{R}} = \emptyset$, then the orientation $[\omega_{(\gamma,\delta(o))}]$ induced by the orientation $[\omega_{(\tau,o)}]$ is equal to $[\Omega_{(\gamma,\delta(o))}]$.

Proof. (a) The proof of this case is the same with the proof of Lemma 6.2.1. We will not repeat it here.

(b) Let $(\Sigma^o, \mathbf{p}^o) \in C_{(\gamma,\delta(o))}$. We set $\Sigma_{v_e}^o$ to be $\{w = 0\}$ and $\Sigma_{v^e}^o$ to be $\{z = 0\}$. Let α_{k-1} in $\mathbf{F}_\gamma^+(v_e) \setminus \{s_n\}$ be the maximal element. According to the convention in Section 6.2, we specify the isomorphisms $\Phi_{v_e} : \Sigma_{v_e} \to \mathbb{CP}^1$ and $\Phi_{v^e} : \Sigma_{v^e} \to \mathbb{CP}^1$ by mapping the special points

$$\begin{aligned} (p_{f_e}, p_{\alpha_{k-1}}, p_{s_n}) &\mapsto (0, \sqrt{-1}/2, \sqrt{-1}), \\ (p_{f^e}, p_{\overline{\alpha}_{k-1}}, p_{\overline{s}_n}) &\mapsto (0, -\sqrt{-1}/2, -\sqrt{-1}). \end{aligned}$$

Due to Proposition 5.3.4, a tubular neighborhood $\mathbb{R}V \times [0, \epsilon[$ of $(\Sigma^o, \mathbf{p}^o)$ in $\overline{C}_{(\tau,o)}$ can be given by the family $\{z \cdot w = t \mid t \in [0, \epsilon[\}$ with labelled points

$$\begin{aligned} p_{\alpha_{k-1}} &= (\sqrt{-1}/2, -2t\sqrt{-1}),\ p_{s_n} = (\sqrt{-1}, -t\sqrt{-1}), \\ p_{\overline{\alpha}_{k-1}} &= (2t\sqrt{-1}, -\sqrt{-1}/2),\ p_{\overline{s}_n} = (t\sqrt{-1}, -\sqrt{-1}), \\ p_\alpha &= (z_\alpha, t/z_\alpha),\ p_{\overline{\alpha}} = (t/w_{\overline{\alpha}}, w_{\overline{\alpha}}). \end{aligned}$$

for $\alpha \in \mathbf{F}_\gamma^+$ (resp. $\overline{\alpha} \in \mathbf{F}_\gamma^-$). We use the z components of position of labelled points as coordinates on $C_{(\tau,o)}$.

The orientation of $C_{(\gamma,\delta(o))}$ is

$$\Omega_{(\gamma,\delta(o))} = A \bigwedge_{\alpha \in \mathbf{F}_\gamma^+ \setminus \{f^e, \alpha_{k-1}, s_n\}} d\, z_\alpha \wedge d\, \overline{z}_\alpha. \tag{6.4}$$

where $A = (\sqrt{-1}/2)^{|\mathbf{F}_\gamma^+|-3}$ due to Section 6.2. We put the labelled points into a normal position by using the following transformation

$$\Lambda : z_\alpha \mapsto q_\alpha = \sqrt{-1} \left(\frac{z_\alpha - \sqrt{-t}}{z_\alpha + \sqrt{-t}} \right) / \left(\frac{1 - \sqrt{t}}{1 + \sqrt{t}} \right).$$

Hence, we have

$$\begin{aligned} dq_\alpha &= \left(\frac{2t(1-t)}{\sqrt{t}(-1+\sqrt{t})^2(\sqrt{t}-z_\alpha\sqrt{-1})^2} \right) dz_\alpha \\ &+ \left(\frac{(t-z_\alpha\sqrt{-1})(-\sqrt{-1}+z_\alpha)}{\sqrt{t}(-1+\sqrt{t})^2(\sqrt{t}-z_\alpha\sqrt{-1})^2} \right) dt, \\ d(Im(q_{\alpha_{k-1}})) &= -\frac{1+2t}{\sqrt{t}(1+\sqrt{t}-2t)^2} dt, \end{aligned}$$

and

$$-dt \wedge \Omega_{(\gamma,\delta(o))} = A\ \Theta \bigwedge_{\alpha \in \mathbf{F}^+ \setminus \{f^e, \alpha_{k-1}, s_n\}} d\ q_\alpha \wedge d\ \overline{q}_\alpha \bigwedge d(Im(q_{\alpha_{k-1}}))$$

where $\Theta = \Phi \times \Psi$ and

$$\begin{aligned} \Phi &= \prod_{\alpha \in \mathbf{F}^+_\gamma \setminus \{f^e, \alpha_{k-1}, s_n\}} \left(\frac{2t(1-t)}{\sqrt{t}(-1+\sqrt{t})^2} \right)^2 \left(\frac{1}{(t+\sqrt{t}\cdot Im(z_\alpha)+|z_\alpha|^2)} \right)^2 > 0, \\ \Psi &= \left(\frac{\sqrt{t}(1+\sqrt{t}-2t)^2}{1+2t} \right) > 0 \end{aligned}$$

for $t \in]0, \epsilon[$. Note that, $\Theta > 0$.

Since the labelled points are in normal positions in coordinates q, the form

$$\omega_{(\tau,o)} = A \bigwedge_{\alpha \in \mathbf{F}^+ \setminus \{f^e, \alpha_{k-1}, s_n\}} d\ q_\alpha \wedge d\ \overline{q}_\alpha \bigwedge d(Im(q_{\alpha_{k-1}}))$$

gives the orientation of the stratum for a one-vertex (τ, o). The orientation $[\omega_{(\gamma,\delta(o))}]$ induced by $\omega_{(\tau,o)}$ is $[\Omega_{(\gamma,\delta(o))}]$ since $\Theta > 0$. □

Subcase: σ-invariant curves of type 2. Let (τ, o) be a one-vertex o-planar tree of type 2, and let $[\omega_{(\tau,o)}]$ be the orientation of $C_{(\tau,o)}$ defined by

$$\omega_{(\tau,o)} := -(\frac{\sqrt{-1}}{2})^{k-2} \bigwedge_{\alpha_* \in \{s_1, \cdots, s_{k-2}\}} dz_{\alpha_*} \wedge d\overline{z}_{\alpha_*} \bigwedge d\lambda \tag{6.5}$$

(which is given by the coordinates in **(D)** of 5.1.2). Here $\lambda = \Im(z_{s_{k-1}})$.

Lemma 6.2.3. *Let $(\gamma, \hat{o})$ be a two-vertex o-planar tree where $\mathbf{V}^{\mathbb{R}}_\gamma = \emptyset$, and let $C_{(\gamma,\hat{o})}$ be contained in the boundary of strata $\overline{C}_{(\tau,o)}$ given above. Then, the orientation $[\omega_{(\gamma,\hat{o})}]$ induced by the orientation $[\omega_{(\tau,o)}]$ is equal to $(-1)^{\aleph}[\Omega_{(\gamma,\hat{o})}]$ where $\aleph = |\{1, \cdots, k-1\} \bigcap \mathbf{F}^-_\gamma| + 1$.*

Proof. Let $(\Sigma^o, \mathbf{p}^o) \in C_{(\gamma,\hat{o})}$. We set Σ^o to be $\{z \cdot w = 0\}$ and the labelled points $\mathbf{p}^o$ to be a set points in $\{z \cdot w = 0\}$ as in proof Lemma 6.2.2. Due to Proposition 5.3.4, a tubular neighborhood $\mathbb{R}V \times [0, \epsilon[$ of $(\Sigma^o, \mathbf{p}^o)$ in $\overline{C}_{(\tau,o)}$ can be given by the family $\{z \cdot w = -t \mid t \in [0, \epsilon[\}$ with labelled points

$$
\begin{aligned}
p_{\alpha_{k-1}} &= (\sqrt{-1}/2, 2t\sqrt{-1}),\ p_{s_n} = (\sqrt{-1}, t\sqrt{-1}),\\
p_{\overline{\alpha}_{k-1}} &= (2t\sqrt{-1}, \sqrt{-1}/2),\ p_{\overline{s}_n} = (t\sqrt{-1}, \sqrt{-1}),\\
p_\alpha &= (z_\alpha, -t/z_\alpha),\ p_{\overline{\alpha}} = (-t/w_{\overline{\alpha}}, w_{\overline{\alpha}}).
\end{aligned}
$$

for $\alpha \in \mathbf{F}^+_\gamma$ (resp. $\overline{\alpha} \in \mathbf{F}^-_\gamma$). We use the z components of position of labelled points as coordinates on $C_{(\tau,o)}$.

Due to convention in Section 6.2, the orientation of $C_{(\gamma,\hat{o})}$ is given in (6.4).

We put the labelled points into a normal position by using the following transformation

$$
\Lambda : z_\alpha \mapsto q_\alpha \quad := \quad \left(\frac{Az_\alpha + B\sqrt{t}}{Bz_\alpha + A\sqrt{t}}\right)
$$

where $A = -(1 + \sqrt{t})$ and $B = \sqrt{-1}(1 - \sqrt{t})$.

Therefore, we have

$$
\begin{aligned}
dq_\alpha &= \left(\frac{2t(1+t)}{\sqrt{t}(t + \sqrt{t}(1 + z_\alpha\sqrt{-1}) - z_\alpha\sqrt{-1})^2}\right) dz_\alpha\\
&+ \left(\frac{-(t + z_\alpha\sqrt{-1})(-\sqrt{-1} + z_\alpha)}{\sqrt{t}(t + \sqrt{t}(1 + z_\alpha\sqrt{-1}) - z_\alpha\sqrt{-1})^2}\right) dt,\\
d(Im(q_{\alpha_{k-1}})) &= \frac{-1 + 2t}{\sqrt{t}(1 + \sqrt{t} + 2t)^2} dt,
\end{aligned}
$$

and

$$
\begin{aligned}
-dt \wedge \Omega_{(\gamma,o)} &= A\ \Theta \bigwedge_{\alpha \in \mathbf{F}^+ \setminus \{f^e, \alpha_{k-1}, s_n\}} d\ q_\alpha \wedge d\ \overline{q}_\alpha \bigwedge d(Im(q_{\alpha_{k-1}})),\\
&= A\ \Theta\ (-1)^{|\mathbf{F}^-\gamma \bigcap \{s_1, \cdots, s_{k-2}\}|} \bigwedge_{\alpha \in \{s_1, \cdots, s_{k-2}\}} d\ q_\alpha \wedge d\ \overline{q}_\alpha \bigwedge d(Im(q_{\alpha_{k-1}}))
\end{aligned}
$$

where $\Theta = \Phi \times \Psi > 0$ since

$$
\begin{aligned}
\Phi &= \prod_{\mathbf{F}^+_\gamma \setminus \{f^e, \alpha_{k-1}, s_n\}} \left(\frac{\sqrt{t}}{2t(1+t)}\right)^2 (|(t + \sqrt{t}) - z_\alpha\sqrt{-1}(1 - \sqrt{t})|^2)^2 > 0, \text{ and}\\
\Psi &= \left(\frac{\sqrt{t}(1 + \sqrt{t} + 2t)^2}{1 - 2t}\right) > 0
\end{aligned}
$$

for $t \in]0, \epsilon[$.

If $\alpha_{k-1} \in \mathbf{F}^+_\gamma$, then $|\mathbf{F}^-_\gamma \bigcap \{s_1, \cdots, s_{k-1}\}| = |\mathbf{F}^-_\gamma \bigcap \{s_1, \cdots, s_{k-2}\}|$ If $\alpha_{k-1} \in \mathbf{F}^-_\gamma$, then $|\mathbf{F}^-_\gamma \bigcap \{s_1, \cdots, s_{k-1}\}| = |\mathbf{F}^-_\gamma \bigcap \{s_1, \cdots, s_{k-2}\}| + 1$.

Therefore, the orientation $\Omega_{(\gamma,\hat{o})}$ is induced by

$$(-1)^{|F^-_\gamma \bigcap \{1,\cdots,k-1\}|} A \bigwedge_{\alpha \in \{s_1,\cdots,s_{k-2}\}} d\, q_\alpha \wedge d\, \overline{q}_\alpha \bigwedge d(Im(q_{k-1})) = (-1)^{\aleph}[\omega_{(\tau,o)}].$$

□

6.3 Conventions

Let $(\tau, o_\star)$ be the one-vertex o-planar tree where the o-planar structure $o_\star$ is given by

$$\left\{ \begin{array}{l} \text{type 1; } \mathbf{F}^+_\tau = \{s_1, s_2, \cdots, s_k\};\ \mathbf{F}^-_\tau = \{s_{k+1}, \cdots, s_{2k}\}; \\ \mathbf{F}^{\mathbb{R}}_\tau = \{\{s_{2k+1}\} < \{s_{2k+2}\} < \cdots < \{s_{2k+l} := s_n\}\} \end{array} \right\}$$

All the other o-planar structures of type 1 on τ are obtained as follows.

Let $\varrho \in \mathbb{S}_n$ be a permutation which commutes with σ and, if $l > 0$, preserves s_n. It determines an o-planar structure given by

$$\varrho(o_\star) = \left\{ \begin{array}{l} \text{type 1; } \mathbf{F}^+_\tau = \{\varrho(s_1), \cdots, \varrho(s_k)\};\ \mathbf{F}^-_\tau = \{\varrho(s_{k+1}), \cdots, \varrho(s_{2k})\}; \\ \mathbf{F}^{\mathbb{R}}_\tau = \{\{\varrho(s_{2k+1})\} < \cdots < \{\varrho(s_n) = s_n\}\} \end{array} \right\}$$

The parity of ϱ depends only on $o = \varrho(o_\star)$ and we call it *parity* $|o|$ of $o = \varrho(o_\star)$.

Convention of orientations

We fix an orientation for each top-dimensional stratum as follows.

a. Case type 1. First, we select o-planar representatives for each one-vertex u-planar tree of type 1 as follows:

1. If $|\mathbf{Fix}(\sigma)| \geq 3$, we choose the representative (τ, o) of (τ, u) in which $\{s_{2k+1}\} < \{s_{n-1}\} < \{s_n\}$;
2. If $|\mathbf{Fix}(\sigma)| < 3$, we choose the representative (τ, o) of (τ, u) such that $s_k \in \mathbf{Perm}^+$.

We denote the set of o-planar representatives of u-planar trees by $\mathcal{U}Tree(\sigma)$. We select the orientation for $C_{(\tau,u)} = C_{(\tau,o)}$ to be

$$(-1)^{|o|}[\omega_{(\tau,o)}], \tag{6.6}$$

where $(\tau, o) \in \mathcal{U}Tree(\sigma)$ and $\omega_{(\tau,o)}$ is the form defined according to Section 6.1 and $|o|$ is the parity introduced in Section 6.3.

b. Case type 2. Here, we choose the orientation defined by the form (6.5).

In what follows, we denote the set of flags $\{s_{2k+1}, s_{n-1}, s_n\}$ (for $|\mathbf{Fix}(\sigma)| \geq 3$ case) and $\{s_k, s_{2k}, s_n\}$ (for $|\mathbf{Fix}(\sigma)| < 3$ case) by $\mathfrak{F}$.

6.4 Adjacent top-dimensional strata of type 1

Let $C_{(\tau,u_i)}, i = 1, 2$, be a pair of adjacent top-dimensional strata of (τ, u_i) of type 1, and $C_{(\gamma,u)}$ be their common codimension one stratum. Let (τ, o_i) be the o-planar representatives of (τ, u_i) given in Section 6.3. Consider the pair of o-planar representatives $(\gamma, \delta(o_i))$ of (γ, u) which respectively give (τ, o_i) after contracting their edges.

Lemma 6.4.1. *The o-planar tree $(\gamma, \delta(o_1))$ is obtained by reversing the o-planar structure $\delta(o_2)_v$ of $(\gamma, \delta(o_1))$ at vertex v where $|\mathbf{F}_\gamma(v) \bigcap \mathfrak{F}| \leq 1$.*

Proof. Obviously, $(\gamma, \delta(o_1))$ can be obtained from $(\gamma, \delta(o_2))$ by reversing the o-planar structures at one or both its vertices v_e, v^e. If we reverse the o-planar structure of $(\gamma, \delta(o_2))$ at the vertex v such that $|\mathbf{F}_\gamma(v) \bigcap \mathfrak{F}| > 1$, or at both of its vertices v_e and v^e, then the resulting o-planar tree will not be an element of $\mathcal{U}Tree(\sigma)$ after contracting its edge: reversing the o-planar structure at the vertex v with $|\mathbf{F}_\gamma(v) \bigcap \mathfrak{F}| > 1$, or at both of the vertices reverses cyclic order of the elements $\{s_{2k+1}, s_{n-1}, s_n\}$ when $l \geq 3$, and moves s_k from $\mathbf{Perm}^+(\sigma)$ to $\mathbf{Perm}^-(\sigma)$ when $l < 3$. □

For a pair of two-vertex o-planar trees $(\gamma, \delta(o_i))$ as above, we calculate the differences of parities as follows.

Lemma 6.4.2. *Let $(\gamma, \delta(o_i)), i = 1, 2$ be a pair of o-planar trees as above. Let $\mathbf{V}_\gamma = \mathbf{V}_\gamma^{\mathbb{R}} = \{v_e, v^e\}$, and let o-planar structures at the vertices v_e and v^e be*

$$\begin{aligned}
\delta(o_1)_{v_e} &= \left\{ \begin{array}{l} \text{type } 1; \mathbf{F}_\gamma^{\pm}(v_e); \mathbf{F}_\gamma^{\mathbb{R}}(v_e) = \{\{r_1\} < \cdots < \{r_i\} < \{f_e\} < \\ < \{i_{i+j+1}\} < \cdots < \{r_{l-1}\} < \{n\}\} \end{array} \right\} \\
\delta(o_2)_{v^e} &= \{\text{type } 1; \mathbf{F}_\gamma^{\pm}(v^e); \mathbf{F}_\gamma^{\mathbb{R}}(v^e) = \{\{r_{i+1}\} < \cdots < \{r_{i+j}\} < \{f^e\}\}\}.
\end{aligned}$$

Let v be the vertex such that $|\mathbf{F}_\gamma(v) \bigcap \mathfrak{F}| \leq 1$. Then, the parity $|o_1| - |o_2|$ is equal to

$$\begin{array}{ll}
|\mathbf{F}_\gamma^+(v^e)| + \frac{j(j-1)}{2} & \text{when } v = v^e, \\
|\mathbf{F}_\gamma^+(v_e)| + ij + jm + im + \frac{m(m-1)}{2} + \frac{i(i-1)}{2} & \text{when } v = v_e \text{ and } |\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| > 3, \\
|\mathbf{F}_\gamma^+(v_e)| + |\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| - 1 & \text{when } v = v_e \text{ and } |\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| = 3, \\
|\mathbf{F}_\gamma^+(v_e)| & \text{when } v = v_e \text{ and } |\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| = 2.
\end{array}$$

Here, $j = |\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| - 1$ and $m = |\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| - i - 2$.

In Section 6.2, we have introduced differential forms $\Omega_{(\gamma_v, \delta(o_i)_v)}$ for each $v \in \mathbf{V}_\gamma$. When we reverse the o-planar structure at the vertex v, the differential forms $\Omega_{(\gamma_v, \delta(o_2)_v)}$, $\Omega_{(\gamma_v, \delta(o_1)_v)}$ become related as follows.

Lemma 6.4.3. *Let $(\gamma, \delta(o_i)), i = 1, 2$ be two-vertex o-planar trees as above. Then,*

$$\Omega_{(\gamma_v, \delta(o_1)_v)} = (-1)^{\mu(v)}\ \Omega_{(\gamma_v, \delta(o_2)_v)},$$

where

$$\mu(v) = |\mathbf{F}_\gamma^+(v)| + \frac{(|\mathbf{F}_\gamma^{\mathbb{R}}(v)| - 2)(|\mathbf{F}_\gamma^{\mathbb{R}}(v)| - 3)}{2}.$$

Lemmata 6.4.2 and 6.4.3 follow from straightforward calculations.

6.5 The First Stiefel-Whitney class

This section is devoted to the proof of the following theorem.

Theorem 5. *(a) For $|\mathbf{Fix}(\sigma)| > 0$, the Poincare dual of the first Stiefel-Whitney class of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is*

$$[w_1] \;=\; \sum_{(\gamma,u)} [\overline{C}_{(\gamma,u)}] \;=\; \sum_{\gamma} [\mathbb{R}\overline{D}_{\gamma}] \quad \mod 2,$$

where the both sums are taken over all two-vertex trees such that

- $|\mathbf{F}_{\gamma}(v^e)\bigcap\mathfrak{F}| \leq 1$ *and* $|v^e| = 0 \mod 2$, *or*
- $|\mathbf{F}_{\gamma}(v_e)\bigcap\mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_{\gamma}(v_e)| \neq 3$ *and* $|v_e|(|v^e|-1) = 0 \mod 2$, *or*
- $|\mathbf{F}_{\gamma}(v_e)\bigcap\mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_{\gamma}(v_e)| = 3$ *and* $|\mathbf{F}^{\mathbb{R}}_{\gamma}(v^e)| = 1$,

and, in the first sum, in addition over all u-planar structures on γ.

(b) For $|\mathbf{Fix}(\sigma)| = 0$, the Poincare dual of the first Stiefel-Whitney class of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ vanishes.

Proof. Fix an orientation for each top-dimensional stratum as in 6.3. The orientation $(-1)^{|o|}[\omega_{(\tau,o)}]$ of a top-dimensional stratum $C_{(\tau,o)}$ induces some orientation of each codimension one stratum $C_{(\gamma,\delta(o))}$ (and $C_{(\gamma,\hat{o})}$) contained in the boundary of $\overline{C}_{(\tau,o)}$. The induced orientations $(-1)^{|o|}[\omega_{(\gamma,\delta(o))}]$ and $(-1)^{|o|}[\omega_{(\gamma,\hat{o})}]$ are determined in Lemmata 6.2.1, 6.2.2 and 6.2.3, and they give (relative) fundamental cycles $[\overline{C}_{(\gamma,\delta(o))}]$ and $[\overline{C}_{(\gamma,\hat{o})}]$ of the codimension one strata $\overline{C}_{(\gamma,\delta(o))}$ and $\overline{C}_{(\gamma,\hat{o})}$ respectively.

The Poincare dual of the first Stiefel-Whitney class of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is given by

$$[w_1] = \begin{cases} \frac{1}{2}\sum_{(\tau,u)}\left(\sum_{(\gamma,\delta(o))}[\overline{C}_{(\gamma,\delta(o))}]\right) \mod 2, & \text{when } |\mathbf{Fix}(\sigma)| > 0, \\ \frac{1}{2}\sum_{(\tau,u)}\left(\sum_{(\gamma,\hat{o})}[\overline{C}_{(\gamma,\hat{o})}]\right) \mod 2, & \text{when } |\mathbf{Fix}(\sigma)| = 0, \end{cases} \tag{6.7}$$

where the external summation runs over all one-vertex u-planar trees (τ,u) and the internal one over all codimension one strata of $\overline{C}_{(\tau,o)}$ for the one-vertex o-planar tree (τ,o) which represents (τ,u) in accordance with 6.3. Indeed, the sum (6.7) detects where the orientation on $\mathbb{R}M^{\sigma}_{\mathbf{S}}$ can not be extended to $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$.

We prove the theorem by evaluating (6.7).

Case $|\mathbf{Fix}(\sigma)| > 0$. Let $C_{(\tau,o_i)}, i = 1,2$ be a pair of adjacent top-dimensional strata, and $C_{(\gamma,\delta(o_i))} \subset \overline{C}_{(\tau,o_i)}$ be their common codimension one boundary stratum. We calculate $\left[\overline{C}_{(\gamma,\delta(o_1))}\right] + \left[\overline{C}_{(\gamma,\delta(o_2))}\right]$ as follows. According to 6.3, the strata $C_{(\tau,o_i)}$ are oriented by $(-1)^{|o_i|}[\omega_{(\tau,o_i)}]$, and these orientations induce the orientations $(-1)^{|o_i|}[\omega_{(\gamma,\delta(o_i))}]$ on $C_{(\gamma,\delta(o_i))}$. The induced orientations $(-1)^{|o_i|}[\omega_{(\gamma,\delta(o_i))}]$ are given by $(-1)^{|o_i|+\aleph_i}[\Omega_{(\gamma_{v^e},\delta(o_i)_{v^e})} \wedge \Omega_{(\gamma_{v_e},\delta(o_i)_{v_e})}]$ in Lemmata 6.2.1 and 6.2.2 according to the convention introduced in Section 6.2. We denote by v be the vertex such that $|\mathbf{F}_\gamma(v) \bigcap \mathfrak{F}| \leq 1$ as in Section 6.4, and compare the induced orientations by calculating

$$\Pi(o_1, o_2) = (|o_1| + \aleph_1) - (|o_2| + \aleph_2) - \mu(v)$$

for each of the following three subcases.

First, assume that $|\mathbf{F}_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$. In this subcase, the o-planar structure is reversed at the vertex $v = v^e$. Therefore, $\aleph_1 = \aleph_2$ according to Lemma 6.2.1. Finally, by applying Lemmata 6.4.2 and 6.4.3 and using relation $j = |\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| - 1$ we obtain

$$\begin{aligned}
\Pi(o_1, o_2) = |o_1| - |o_2| - \mu(v^e) &= \frac{j(j-1)}{2} - \frac{(|\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| - 2)(|\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| - 3)}{2} \\
&= |\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| - 2 \\
&= |v^e| \mod 2.
\end{aligned}$$

The latter equality follows from the fact that $|\mathbf{F}_\gamma^{\mathbb{R}}(v)| = |v| \mod 2$.

Second, assume that $|\mathbf{F}_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$ and $|\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| \neq 3$. In this subcase, the o-planar structure is reversed at the vertex $v = v_e$. Since $|\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| \neq 3$, once more $\aleph_1 = \aleph_2$ according to the Lemma 6.2.1. Finally, by applying Lemmata 6.4.2 and 6.4.3 and using relation $|\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| = i + m + 2$, we obtain

$$\begin{aligned}
\Pi(o_1, o_2) &= ij + jm + im + \frac{m(m-1)}{2} + \frac{i(i-1)}{2} - \frac{(i+m)(i+m-1)}{2} \\
&= j(i+m), \\
&= (|\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| - 1)(|\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| - 2), \\
&= |v_e|(|v^e| - 1) \mod 2
\end{aligned}$$

when $|\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| > 3$, and

$$\begin{aligned}
\Pi(o_1, o_2) &= 2|\mathbf{F}_\gamma^{+}(v_e)| = 0 \mod 2 \\
&= |v_e|(|v^e| - 1) \mod 2
\end{aligned}$$

when $|\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| = 2$.

Third, we consider $|\mathbf{F}_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$ and $|\mathbf{F}_\gamma^{\mathbb{R}}(v_e)| = 3$ case. In this subcase, the o-planar structure is reversed at the vertex v_e. Hence, $\aleph_1 = \aleph_2$ whenever $|\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| = 1, 2$, and $\aleph_1 - \aleph_2$ is $\pm(l+1) = \pm(|\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| + 2)$ whenever $|\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| \geq 3$. Finally, by applying Lemmata 6.4.2 and 6.4.3, we obtain

$$\Pi(o_1, o_2) = \begin{cases} |\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| - 1 \pm (|F_\gamma^{\mathbb{R}}(v^e)| + 2) & = 1 \mod 2, \quad \text{when } |\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| \geq 3, \\ |\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| - 1 & = 1 \mod 2, \quad \text{when } |\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| = 2, \\ |\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| - 1 & = 0 \mod 2, \quad \text{when } |\mathbf{F}_\gamma^{\mathbb{R}}(v^e)| = 1, \end{cases}$$

The induced orientations $(-1)^{|o_i|}[\omega_{(\gamma,\delta(o_i))}]$ are the same if and only if $\Pi(o_1,o_2)=0$ mod 2. Hence, we have

$$\left[\overline{C}_{(\gamma,\delta(o_1))}\right]+\left[\overline{C}_{(\gamma,\delta(o_2))}\right] \;=\; \frac{1+(-1)^{\Pi(o_1,o_2)}}{2}\left[\overline{C}_{(\gamma,\delta(o_1))}\right].$$

The sum $([\overline{C}_{(\gamma,\delta(o_1))}]+[\overline{C}_{(\gamma,\delta(o_2))}])/2$ gives us the fundamental cycle $[\overline{C}_{(\gamma,\delta(o_1))}]$ when $(-1)^{|o_1|}[\omega_{(\gamma,\delta(o_1))}]=(-1)^{|o_2|}[\omega_{(\gamma,\delta(o_2))}]$, and it turns to zero otherwise. Finally, as is follows from the above case-by-case calculations of $\Pi(o_1,o_2)$, the fundamental class of a codimension one strata $\overline{C}_{(\gamma,\delta(o_1))}$ is involved in $[w_1]$ if and only if one of the three conditions given in Theorem are verified. It gives the first expression for $[w_1]$ given in Theorem. Since in this first expression the sum is taken over all u-planar structures on γ, it can be shorten to the sum of the fundamental classes of $\mathbb{R}\overline{D}_\gamma$.

Case $|\mathbf{Fix}(\sigma)|=0$. Let $C_{(\tau,u_i)}, i=1,2$, be a pair of adjacent top-dimensional strata and $C_{(\gamma,u)}$ be their common codimension one stratum. Let (τ,o_i) be the o-planar representatives of (τ,u_i) given in 6.3. Here, we have to consider two subcases: (i) $C_{(\gamma,u)}$ is a stratum of real curves with two real components (i.e, $|\mathbf{V}_\gamma|=|\mathbf{V}_\gamma^{\mathbb{R}}|=2$), and (ii) $C_{(\gamma,u)}$ is a stratum of real curves with two complex conjugated components (i.e, $|\mathbf{V}_\gamma|=2$ and $|\mathbf{V}_\gamma^{\mathbb{R}}|=0$).

(i) Consider the pair of o-planar representatives $(\gamma,\delta(o_i))$ of (γ,u) which respectively give (τ,o_i) after contracting the edges and compare their o-planar structure. Since the both tails n and $\sigma(n)$ are in $\mathbf{F}_\gamma(v_e)$, the o-planar structure is reversed at the vertex v^e. Therefore, $\aleph_1=\aleph_2$ according to the Lemma 6.2.1. Finally, by applying Lemmata 6.4.2 and 6.4.3, we obtain

$$\Pi(o_1,o_2)=2|\mathbf{F}_\gamma^+(v^e)|+1=1 \mod 2.$$

In other words, $\left[\overline{C}_{(\gamma,\delta(o_1))}\right]+\left[\overline{C}_{(\gamma,\delta(o_2))}\right]=0$ for this case.

(ii) Let $C_{(\tau,o_2)}$ be a stratum of of real curves with empty real part, and let $(\gamma,\hat{o})$ be an o-planar representative of (γ,u).

The orientations of $C_{(\gamma,u)}$ induced by the orientations $(-1)^{|o_1|}[\omega_{(\tau,o_1)}]$ and $[\omega_{(\tau,o_2)}]$ of $C_{(\tau,o_1)}$ and $C_{(\tau,o_2)}$ are given in Lemmata 6.2.2 and 6.2.3. Namely, they are respectively given by the following differential forms

$$(-1)^{|o_1|}\bigwedge_{\alpha_*\in\mathbf{F}_\gamma^+\setminus(\mathbf{F}_\gamma^+\bigcap\{k-1,k,2k-1,2k\})} dz_{\alpha_*}\wedge d\bar{z}_{\alpha_*},$$
$$(-1)^{\aleph}\bigwedge_{\alpha_*\in\mathbf{F}_\gamma^+\setminus(\mathbf{F}_\gamma^+\bigcap\{k-1,k,2k-1,2k\})} dz_{\alpha_*}\wedge d\bar{z}_{\alpha_*},$$

where $|o_1|=|\{1,\cdots,k-1\}\bigcap\mathbf{F}_\gamma^-|$ and $\aleph=|\{1,\cdots,k-1\}\bigcap\mathbf{F}_\gamma^-|+1$. Therefore, the orientations induced from different sides are opposite and the sum $(-1)^{\aleph-1}\left[\overline{C}_{(\gamma,u)}\right]+(-1)^{\aleph}\left[\overline{C}_{(\gamma,u)}\right]$ vanishes for all such $(\gamma,\hat{o})$. □

Example 6.5.1. Due to Theorem 5, the Poincare dual of the first Stiefel-Whitney class $[w_1]$ of $\mathbb{R}\overline{M}^{\sigma}_{(0,5)}$ can be represented by $\sum_{\gamma}[\mathbb{R}\overline{D}_{\gamma}] = \sum_{(\gamma,u)}[\overline{C}_{(\gamma,u)}]$ where γ are **S**-trees with a vertex v satisfying $|v| = 4$ and $|\mathbf{F}_{\gamma}(v) \bigcap \{s_1, s_4, s_5\}| = 1$. These **S**-trees are given in Figure 6.2a, and the union corresponding strata $\bigcup_{\tau} \mathbb{R}\overline{D}_{\gamma}$ is given the three exceptional divisors obtained by blowing up the three highlighted points in Figure 6.2b.

Figure 6.2: (a) σ-invariant trees contributing the Stiefel-Whitney class of $\mathbb{R}\overline{M}^{\sigma}_{(0,5)}$ due to Theorem 5, (b) The blown-up locus in $\mathbb{R}\overline{M}^{\sigma}_{(0,5)}$

Chapter 7

The orientation covering of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$

In Chapter 6, the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ is determined in terms of its strata. We have also proved that the moduli space $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ is orientable when $|\mathbf{S}| = 4$ or $|\mathbf{Fix}(\sigma)| = 0$. In this chapter, we give a combinatorial construction of orientation double covering for the rest of the cases i.e, $|\mathbf{S}| > 4$ and $|\mathbf{Fix}(\sigma)| > 0$. By observing the non-triviality of the orientation double cover in these cases, we prove that $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ is not orientable. Some other double covers of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ which appeared in the recent literature is discussed at the end of the chapter.

7.1 Construction of orientation double covering

Let $|\mathbf{S}| > 4$ and $|\mathbf{Fix}(\sigma)| > 0$. In Section 5.1.1, we have shown that the map $\widetilde{C}_{(\mathbf{S},\sigma)} \to \mathbb{R}M_{\mathbf{S}}^{\sigma}$, which is identifying the opposite o-planar structures, is a trivial double covering. The disjoint union of closed strata $\overline{C}_{(\mathbf{S},\sigma)} = \bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)}$, where $|\mathbf{V}_{\tau}| = 1$ and (τ, o) runs over all possible o-planar structures on τ, is a natural compactification of $\widetilde{C}_{(2k,l)}$.

To obtain the orientation double covering of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$, we need to get rid of the codimension one strata by pairwise gluing them. We use the following simple recipe: for each pair $(\tau, o_i), = 1, 2$, of one-vertex o-planar trees obtained by contracting the edge in a pair $(\tau, \delta(o_i)), i = 1, 2$, of two-vertex o-planar trees with the same underlying tree such that $\mathbf{V}_{\gamma} = \mathbf{V}_{\gamma}^{\mathbb{R}} = \{v_e, v^e\}$, $v_e = \boldsymbol{\partial}_{\gamma}(s_n)$, we glue $\overline{C}_{(\tau,o_i)}$ along $\overline{C}_{(\gamma,\delta(o_i))}, i = 1, 2$, if

$\mathcal{A}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v^e, $|\mathbf{F}_{\gamma}(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 1 \mod 2$,

$\mathcal{B}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v_e, $|\mathbf{F}_{\gamma}(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 0 \mod 2$,

$\mathcal{C}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v_e, $|\mathbf{F}_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|\mathbf{F}^{\mathbb{R}}_\gamma(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 1 \mod 2$,

$\mathcal{D}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v^e, $|\mathbf{F}_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|\mathbf{F}^{\mathbb{R}}_\gamma(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 0 \mod 2$,

$\mathcal{E}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v_e, $|\mathbf{F}_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|\mathbf{F}^{\mathbb{R}}_\gamma(v_e)| = 3$ and $|F^{\mathbb{R}}_\gamma(v^e)| \neq 1$,

$\mathcal{F}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v^e, $|\mathbf{F}_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|\mathbf{F}^{\mathbb{R}}_\gamma(v_e)| = 3$ and $|F^{\mathbb{R}}_\gamma(v^e)| = 1$.

We denote by $\mathbb{R}\widetilde{M}^g_{\mathbf{S}}$ the resulting factor space.

Theorem 6. *$\mathbb{R}\widetilde{M}^g_{\mathbf{S}}$ is the orientation double cover of $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$.*

Proof. Let $\widetilde{M}$ be the orientation double covering of $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$. The points of $\widetilde{M}$ can be considered as points in $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ with local orientation. On the other hand, by using opposite o-planar structures of a one-vertex τ, we can determine orientations $(-1)^{|o|}[\omega_{(\tau,o)}]$ and $(-1)^{|\bar{o}|+|\mathbf{Fix}(\sigma)|-1}[\omega_{(\tau,\bar{o})}]$ on $C_{(\tau,o)}$ and $C_{(\tau,\bar{o})}$ where $(\tau, o) \in \mathcal{U}Tree(\sigma)$. These orientations are opposite with respect to the identification of $C_{(\tau,o)}$ and $C_{(\tau,\bar{o})}$ by the canonical diffeomorphism $-\mathbb{I}$ introduced in Subsection 5.1.1. Hence, there is a natural continuous embedding $\widetilde{C}_{(\mathbf{S},\sigma)} = \bigsqcup_{(\tau,o)\in\mathcal{U}Tree(\sigma)} (C_{(\tau,o)} \sqcup C_{(\tau,\bar{o})}) \to \widetilde{M}$. It extends to a surjective continuous map $\overline{C}_{(\mathbf{S},\sigma)} = \bigsqcup_{(\tau,o)\in\mathcal{U}Tree(\sigma)} (\overline{C}_{(\tau,o)} \sqcup \overline{C}_{(\tau,\bar{o})}) \to \widetilde{M}$. Since $\overline{C}_{(\mathbf{S},\sigma)}$ is compact and $\widetilde{M}$ is Hausdorff, the orientation double covering $\widetilde{M}$ is a quotient space $\overline{C}_{(\mathbf{S},\sigma)}/R$ of $\overline{C}_{(\mathbf{S},\sigma)}$ under the equivalence relation R defined by the map $\overline{C}_{(\mathbf{S},\sigma)} \to \widetilde{M}$.

This equivalence relation is uniquely determined by its restriction to the codimension one faces of $\overline{C}_{(\mathbf{S},\sigma)}$, which cover the codimension one strata of $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ under the composed map $\overline{C}_{(\mathbf{S},\sigma)} \to \widetilde{M} \to \mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$. On the other hand, the equivalence relation on the codimension one faces is determined by the first Stiefel-Whitney class: A partial section of the induced map $\overline{C}_{(\mathbf{S},\sigma)}/R \to \mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$ given by distinguished strata $\bigsqcup_{(\tau,o)\in\mathcal{U}Tree(\sigma)} C_{(\tau,o)}$. Over a neighborhood of a codimension one stratum of $\mathbb{R}\overline{M}^\sigma_{\mathbf{S}}$, a partial section extends to a section if this codimension one strata is not involved in the expression for the first Stiefel-Whitney class given in Theorem 5, and it should not extend, otherwise. Notice that the faces $\overline{C}_{(\tau,\delta(o_i))}$ considered in relations $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{E}$ are mapped onto the strata $\overline{C}_{(\tau,\delta(u))}$ which do not contribute to the expression $[w_1]$ given in Theorem 5, and the faces $\overline{C}_{(\tau,\delta(o_i))}$ in relations $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{F}$ are mapped onto the strata $\overline{C}_{(\tau,\delta(u))}$ which contribute to the expression $[w_1]$. There four different faces $\overline{C}_{(\tau,\delta(o)_i)}, i = 1, \cdots, 4$ over each codimension one stratum $\overline{C}_{(\tau,\delta(u))}$. Lemma 6.4.1 determines the pairs $\overline{C}_{(\tau,\delta(o)_i)}$, $\overline{C}_{(\tau,\delta(o)_j)}$ to be glued to each other. □

Corollary 7. *The moduli space* $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ *is not orientable when* $|\mathbf{S}| > 4$ *and* $|\mathbf{Fix}(\sigma)| > 0$.

Proof. Let $|\mathbf{Fix}(\sigma)| \geq 3$, and (τ, o) be an o-planar structure with $\{s_{2k+1}\} < \{s_{n-1}\} < \{s_n\}$. It is clear that, we can produce any o-planar structure on τ with $\{s_{2k+1}\} < \{s_{n-1}\} < \{s_n\}$ by applying following operations consecutively:

- interchanging the order of two consecutive tails $\{r_i, r_{i+1}\}$ for $|\{r_i, r_{i+1}\} \bigcap \mathfrak{F}| \leq 1$ and $s_n \notin \{r_i, r_{i+1}\}$,
- swapping $s_j \in \mathbf{Perm}^+(\sigma)$ with $s_{\bar{j}} \in \mathbf{Perm}^-(\sigma)$ for $s_j \neq s_k, s_{2k}$.

The one-vertex o-planar trees with $\{s_{n-1}\} < \{s_{2k+1}\} < \{s_n\}$ can be produced from the o-planar tree $(\tau, \bar{o})$ via same procedure.

Let $|\mathbf{Fix}(\sigma)| = 1, 2$. Similarly, if we start with o-planar tree (τ, o) with $k \in \mathbf{Perm}^+(\sigma)$ ($k \in \mathbf{Perm}^-(\sigma)$), then we can produce any o-planar structure on τ with $k \in \mathbf{Perm}^+(\sigma)$ ($k \in \mathbf{Perm}^-(\sigma)$) by swapping $s \in \mathbf{Perm}^+(\sigma)$ with $\bar{s} \in \mathbf{Perm}^-(\sigma)$ for $s \neq s_k, s_{2k}$.

Note that, these operations correspond to passing from one top-dimensional stratum to another in $\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}}$ through the certain faces. These faces correspond to the one-edge o-planar trees $(\gamma, \delta(o)_i)$ with $|v^e| = 3$, and $|\mathbf{F}_\gamma(v^e) \cap \mathfrak{F}| \leq 1$ i.e, these are faces glued according to the relations of type $\mathcal{A}$. Any two top-dimensional strata in $\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}}$ with same cyclic ordering of $\mathfrak{F}$ (resp. with s_k is in same set $\mathbf{Perm}^{\pm}(\sigma)$) can be connected through a path passing through these codimension faces $\overline{C}_{(\gamma,\delta(o)_i)}$. The quotient space $\overline{C}_{(\mathbf{S},\sigma)}/\mathcal{A}$ has two connected components since there are two possible cyclic orderings of $\{s_{2k+1}, s_{n-1}, s_n\}$ when $|\mathbf{Fix}(\sigma)| \geq 3$ (resp. two possibilities for $|\mathbf{Fix}(\sigma)| = 1, 2$ case: $k \in \mathbf{Perm}^+(\sigma)$ and $k \in \mathbf{Perm}^-(\sigma)$).

The set of relations of type $\mathcal{B}$ is not empty when $|\mathbf{S}| > 4$ and $|\mathbf{Fix}(\sigma)| > 0$. Moreover, the relation of type $\mathcal{B}$ reverses the cyclic ordering on $\mathfrak{F}$ (resp. moves s_k from $\mathbf{Perm}^{\pm}$ to $\mathbf{Perm}^{\mp}$). Hence, the faces glued according to the relations of type $\mathcal{B}$ connect the connected components of $\overline{C}_{(\mathbf{S},\sigma)}/\mathcal{A}$. Therefore, the orientation double cover $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is nontrivial when $|\mathbf{S}| > 4$ and $|\mathbf{Fix}(\sigma)| > 0$ which simply means that the moduli space $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is not orientable in this case. □

Example 7.1.1. In Example 5.3.6, we obtained that $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ are respectively a torus with three points blown up, a sphere with three points blown up, and a sphere with one point blown up for the involutions σ given in (5.4). The double covering $\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}}$ is obtained by taking the two copies of the corresponding moduli space of real curves and replacing the blown up loci by annuli. Therefore, $\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}}$ are surfaces of genus 4, genus 2 and genus 0, respectively (see Figure 7.1 which illustrates the case $\sigma = id$).

Figure 7.1: Stratification of $\mathbb{R}\widetilde{M}^g_{\mathbf{S}}$ for $\sigma = id$

7.2 Combinatorial types of strata of $\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}}$

While constructing $\mathbb{R}\widetilde{M}^g_{\mathbf{S}}$, the closure of the each codimension one strata are glued in a consistent way. This identification of codimension strata gives an equivalence relation among the o-planar trees when $|\mathbf{Fix}(\sigma)| \neq 0$.

We define the notion of *R-equivalence* on the set of such o-planar trees by treating different cases separately. Let $(\gamma_1, o_1), (\gamma_2, o_2)$ be o-planar trees.

1. If $|\mathbf{V}^{\mathbb{R}}_{\gamma_i}| = 1$, then we say that they are R-equivalent whenever γ_1, γ_2 are isomorphic (i.e, $\gamma_1 \approx \gamma_2$) and the o-planar structures are the same.

2. If γ_i have an edge corresponding to real node (i.e. $\mathbf{E}^{\mathbb{R}}_{\gamma_i} = \{e\}$ and $\mathbf{V}^{\mathbb{R}}_{\gamma_i} = \boldsymbol{\partial}_{\gamma}(e) = \{v^e, v_e\}$), we first obtain $(\gamma_i(e), o_i(e))$ by contracting conjugate pairs of edges until there will be none. We say that (γ_1, o_1) and (γ_2, o_2) are R-equivalent whenever $\gamma_1 \approx \gamma_2$ and

   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v^e, $|\mathbf{F}_{\gamma}(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 1 \mod 2$,
   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v_e, $|\mathbf{F}_{\gamma}(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 0 \mod 2$,
   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v_e, $|\mathbf{F}_{\gamma}(v_e) \bigcap \mathfrak{F}| \leq 1$, $|\mathbf{F}^{\mathbb{R}}_{\gamma}(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 1 \mod 2$,
   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v^e, $|\mathbf{F}_{\gamma}(v_e) \bigcap \mathfrak{F}| \leq 1$, $|\mathbf{F}^{\mathbb{R}}_{\gamma}(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 0 \mod 2$,
   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v_e, $|\mathbf{F}_{\gamma}(v_e) \bigcap \mathfrak{F}| \leq 1$, $|\mathbf{F}^{\mathbb{R}}_{\gamma}(v_e)| = 3$ and $|\mathbf{F}^{\mathbb{R}}_{\gamma}(v^e)| \neq 1$,
   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v^e, $|\mathbf{F}_{\gamma}(v_e) \bigcap \mathfrak{F}| \leq 1$, $|\mathbf{F}^{\mathbb{R}}_{\gamma}(v_e)| = 3$ and $|\mathbf{F}^{\mathbb{R}}_{\gamma}(v^e)| = 1$,

3. Otherwise, if γ_i have more than one invariant edge (i.e. $|\mathbf{E}_{\gamma_i}|^{\mathbb{R}} > 1$), we say that (γ_1, o_1), (γ_2, o_2) are R-equivalent whenever $\gamma_1 \approx \gamma_2$ and there exists an edge $e \in \mathbf{E}^{\mathbb{R}}_{\gamma_i}$ such that the o-planar trees $(\gamma_i(e), o_i(e))$, which are obtained by contracting all edges but e, are R-equivalent in the sense of the Case (2).

We call the maximal set of pairwise R-equivalent o-planar trees by *R-equivalence classes* of o-planar trees.

Theorem 8. *A stratification of the orientation double cover* $\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}}$ *is given by*

$$\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}} = \bigsqcup_{\substack{\text{R-equivalance classes}\\ \text{of o-planar } (\gamma,o)}} C_{(\gamma,o)}.$$

7.3 Some other double coverings of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$

In [21], Kapranov constructed a different double covering $\widehat{\mathbb{R}M}_{\mathbf{S}}$ of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ having no boundary for $\sigma = id$. He has applied the following recipe to obtain the double covering: Let $\overline{C}_{(\mathbf{S},\sigma)}$ be the disjoint union of closed strata $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)}$ for $\sigma = id$. Let $(\gamma, \delta(o_i)), i = 1, 2$ be two-vertex o-planar trees representing the same u-planar tree (γ, u), and let (τ, o_i) be the one-vertex trees obtained by contracting the edges of $(\gamma, \delta(o_i))$. The strata $\overline{C}_{(\gamma,\delta(o_i))}, i = 1, 2$ are glued if $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at vertex $v^e \neq \boldsymbol{\partial}_{\gamma}(s_n)$. We obtain first Stiefel-Whitney class of $\widehat{\mathbb{R}M}_{\mathbf{S}}$ by using the same arguments in Theorem 5.

Proposition 7.3.1. *The Poincare dual of the first Stiefel-Whitney class of* $\widehat{\mathbb{R}M}_{\mathbf{S}}$ *is*

$$[\widehat{w_1}] = \frac{1}{2} \sum_{(\tau,o)} \sum_{(\gamma,\delta(o)):|v^e|=0 \mod 2} \left[\overline{C}_{(\gamma,\delta(o))}\right] \mod 2.$$

It is well-known that these spaces are not orientable when $l \geq 5$, see for example [7].

7.3.1 A double covering from open-closed string theory

In [11, 31], a different 'orientation double covering' is considered. It can be given as the disjoint union $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)}$ where $\mathbf{F}^{+}_{\tau} = \{s_1, \cdots, s_k\}$, and $\mathbf{F}^{\mathbb{R}}_{\tau}$ carries all possible oriented cyclic ordering. It is a disjoint union of manifolds with corners. The covering map $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)} \to \mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ is two-to-one only over a subset of the open space $\mathbb{R}M^{\sigma}_{\mathbf{S}}$. It only covers the subset $\bigsqcup_{(\tau,u)} \overline{C}_{(\tau,u)}$ of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ where u-planar trees (τ, u) have the partition $\{\{s_1, \cdots, s_k\}, \{s_{k+1}, \cdots, s_{2k}\}\}$ of $\mathbf{F}_{\tau} \setminus \mathbf{F}^{\mathbb{R}}_{\tau}$. Moreover, the covering map is not two-to-one over the strata with codimension higher than zero.

Chapter 8

Homology of the strata of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$

In this chapter, we calculate the homology of the strata of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ relative to the union their substrata of codimension one and higher.

8.1 Forgetful morphism revisited

In this section, we discuss some properties of the restriction of the forgetful map to the strata of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$.

8.1.1 Forgetting a conjugate pair of labelled points

Let $\mathbf{S} = \{s_1, \cdots, s_n\}$ and $\sigma \neq id$. Let $\{s, \bar{s} := \sigma(s)\}$ be a conjugate pair in $\mathbf{Perm}(\sigma)$ which is different than $\{s_n, s_{\bar{n}}\}$ when $\mathbf{Fix}(\sigma) = \emptyset$. Let $\mathbf{S}' = \mathbf{S} \setminus \{s, \bar{s}\}$. Denote the restriction of σ on $\mathbf{S}'$ by σ'. From now on, we denote the map $\pi_{\{s,\bar{s}\}} : \mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma} \to \mathbb{R}\overline{M}_{\mathbf{S}'}^{\sigma'}$ forgetting the labeled point $p_s, p_{\bar{s}}$ by simply π.

Let $\pi : C_{(\gamma^*,o^*)} \to C_{(\gamma,o)}$ be the restriction of the forgetful map to the stratum $C_{(\gamma^*,o^*)} \subset \mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$. Let $v_s := \boldsymbol{\partial}_{\gamma^*}(s)$. Whenever $v_s \in \mathbf{V}_{\gamma^*}^{\mathbb{R}}$ and $|v_s| = 3$, there is unique vertex in $\mathbf{V}_{\gamma^*}^{\mathbb{R}}$ next to v_s since v_s supports both s, $\bar{s}$ and a flag of an real edge connecting v_s to the rest of γ^*. We denote this closest vertex to v_s by v_c.

We will denote the fibers $\pi^{-1}(\Sigma; \mathbf{p})$ of the forgetful map π simply by F_s.

Lemma 8.1.1. *(a) Let $(\Sigma; \mathbf{p}) \in C_{(\gamma,o)}$ be a σ-invariant curve of type 1, and $s \in \mathbf{F}_{\gamma^*}^{+}$ (resp. $s \in \mathbf{F}_{\gamma^*}^{-}$), then the fiber F_s is*

1. *a two-dimensional open disc $\Sigma_{v_s}^{+}$ (resp. $\Sigma_{v_s}^{-}$) minus the special points p_f where $f \in \mathbf{F}_{\gamma^*}^{+}(v_s) \setminus \{s\}$ (resp. $f \in \mathbf{F}_{\gamma^*}^{-}(v_s) \setminus \{s\}$) if $v_s \in \mathbf{V}_{\gamma^*}^{\mathbb{R}}$ and $|v_s| \geq 5$;*
2. *a two-dimensional sphere Σ_{v_s} minus the special points $p_f \in \Sigma_{v_s}$ where $f \in \mathbf{F}_{\gamma^*}(v_s) \setminus \{s\}$ if $v_s \notin \mathbf{V}_{\gamma^*}^{\mathbb{R}}$ and $|v_s| \geq 4$;*

3. *an open interval if* $v_s \in \mathbf{V}^{\mathbb{R}}_{\gamma^*}$ *and* $|v_s| = 4$*;*

4. *an open interval if* $v_s \in \mathbf{V}^{\mathbb{R}}_{\gamma^*}$*,* $|v_s| = 3$*,* $|v_c| \geq 4$ *and* $|\mathbf{F}^{\mathbb{R}}_{\gamma^*}(v_c)| > 1$*;*

5. *a circle if* $v_s \in \mathbf{V}^{\mathbb{R}}_{\gamma^*}$*,* $|v_s| = 3$*, and* $|\mathbf{F}^{\mathbb{R}}_{\gamma^*}(v_c)| = 1$*;*

6. *a point if* $v_s \in \mathbf{V}^{\mathbb{R}}_{\gamma^*}$ *and* $|v_s| = |v_c| = 3$*;*

7. *a point if* $v_s \notin \mathbf{V}^{\mathbb{R}}_{\gamma^*}$ *and* $|v_s| = 3$*.*

(b) Let $(\Sigma; \mathbf{p}) \in C_{(\gamma,o)}$ *be a* σ*-invariant curve of type 2, then the fiber* F_s *is*

1. *a two-dimensional sphere* Σ_{v_s} *minus the special points* $p_f \in \Sigma_{v_s}$ *where* $f \in \mathbf{F}_{\gamma^*}(v_s) \setminus \{s, \bar{s}\}$ *if* $v_s \in \mathbf{V}^{\mathbb{R}}_{\gamma^*}$ *and* $|v_s| \geq 6$*;*

2. *a two-dimensional sphere* Σ_{v_s} *minus the special points* $p_f \in \Sigma_{v_s}$ *where* $f \in \mathbf{F}_{\gamma^*}(v_s) \setminus \{s\}$ *if* $v_s \notin \mathbf{V}^{\mathbb{R}}_{\gamma^*}$ *and* $|v_s| \geq 4$*;*

3. *a point if* $|v_s| = 3$*.*

(c) Let $(\Sigma; \mathbf{p}) \in C_{(\gamma,o)}$ *be a* σ*-invariant curve of type 3, then the fiber* F_s *is*

1. *a two-dimensional sphere* Σ_{v_s} *minus the special points* $p_f \in \Sigma_{v_s}$ *where* $f \in \mathbf{F}_{\gamma^*}(v_s) \setminus \{s\}$ *if* $|v_s| \geq 4$*;*

2. *a point if* $|v_s| = 3$*.*

Proof. Here, we will prove only (a). The proofs of the other cases are essentially the same.

Let (γ, o) be an o-planar tree of type 1. Pick a σ'-invariant curve $(\Sigma; \mathbf{p}) \in C_{(\gamma,o)}$. Let $(\Sigma^*, \mathbf{p}^*)$ be points in the fiber F_s. If $(\Sigma^*, \mathbf{p}^*) \in F_s$ is a σ-invariant curve which does not require the contraction of its component $\Sigma^*_{v_s}$ after forgetting the labeled points $p_s, p_{\bar{s}}$, then there are two possible subcases:

(1) $\Sigma^*_{v_s}$ is a real component and supporting five or more special points, or

(2) $\Sigma^*_{v_s}$ is not a real component and supporting four or more special points.

If $(\Sigma^*, \mathbf{p}^*) \in F_s$ is a σ-invariant curve which requires the contraction of $\Sigma^*_{v_s}$ after forgetting the labeled points $p_s, p_{\bar{s}}$, then the component $\Sigma^*_{v_s}$ supports only three or four special points including p_s and $p_{\bar{s}}$. In this case, there are three possible subcases:

(3) $\Sigma^*_{v_s}$ is a real component and supporting four special points,

(4-5-6) $\Sigma^*_{v_s}$ is a real component and supporting three special points, or

(7) $\Sigma^*_{v_s}$ is not a real component and supporting three special points.

Since we consider the fiber over a fixed $(\Sigma;\mathbf{p})$, the positions of special points of $(\Sigma^*,\mathbf{p}^*)\in F_s$ are fixed except the labeled points $p_s, p_{\bar{s}}$.

(1) If $v_s\in\mathbf{V}^{\mathbb{R}}_{\gamma^*}$ and $|v_s|\geq 5$, then $(\Sigma^*,\mathbf{p}^*)$ does not require the contraction of $\Sigma^*_{v_s}$ after forgetting the labelled points $p_s, p_{\bar{s}}$. For all such $(\Sigma^*,\mathbf{p}^*)$, $\Sigma^*=\Sigma$ so $\Sigma^*_{v_s}=\Sigma_{v_s}$. Assume that $s\in\mathbf{F}^+_{\gamma^*}(v_s)$. For $f\in\mathbf{F}^+_{\gamma^*}(v_s)\setminus\{s\}$, the special points p_f are distinct in $\Sigma^+_{v_s}$. Therefore, the elements $(\Sigma^*,\mathbf{p}^*)$ of F_s are determined by the positions of the labelled point p_s in $\Sigma^+_{v_s}$. Since all special points are distinct, p_s is in $\Sigma^+_{v_s}\setminus\{p_f\}$. Hence, the fiber is $\Sigma^+_{v_s}\setminus\{p_f\}$ where $f\in\mathbf{F}^+_{\gamma^*}(v_s)\setminus\{s\}$.

(2) If $v_s\notin\mathbf{V}^{\mathbb{R}}_{\gamma^*}$ and $|v_s|\geq 4$, then $(\Sigma^*,\mathbf{p}^*)$ does not require the contraction of $\Sigma^*_{v_s}$ after forgetting $p_s, p_{\bar{s}}$. Similar to the above case, $\Sigma^*_{v_s}=\Sigma_{v_s}$. The elements $(\Sigma^*,\mathbf{p}^*)$ in the fiber are given by the position of the point p_s in Σ_{v_s}. Hence, the fiber is $\Sigma_{v_s}\setminus\{p_f\}$ where $f\in\mathbf{F}_{\gamma^*}(v_s)\setminus\{s\}$.

(3) Since all special points but $p_s, p_{\bar{s}}$ are fixed, a fiber of π is a family of σ-invariant curves which, in this case, is the deformations of the irreducible real component $(\Sigma^*_{v_s};\mathbf{p}^*_{v_s})$ with two real special points and the conjugate pair $p_s, p_{\bar{s}}$. It clearly gives an open interval (see Example 5.3.6).

(4-5-6) In this case, $(\Sigma^*_{v_s};\mathbf{p}^*_{v_s})$ can not be deformed since $|v_s|=3$. Here, the family along fiber gives the deformation of $(\Sigma^*_{v_c};\mathbf{p}^*_{v_c})$ (instead of $(\Sigma^*_{v_s};\mathbf{p}^*_{v_s})$). The fiber F_s parameterizes the nodal point $\Sigma_{v_s}\cap\Sigma_{v_c}$ which disappears after forgetting p_s and $p_{\bar{s}}$. There three subcases here: (6) The fiber is a point when $|v_c|=3$ since $(\Sigma^*_{v_c};\mathbf{p}^*_{v_c})$ can not be deformed. (5) The fiber is a circle when $|\mathbf{F}^{\mathbb{R}}_{\gamma}(v_c)|=1$. It is given by the position of the nodal point $\Sigma_{v_s}\cap\Sigma_{v_c}$. (4) The fiber is an open interval when $|\mathbf{F}^{\mathbb{R}}_{\gamma}(v_c)|>1$. It is given by the position of the nodal point $\Sigma_{v_s}\cap\Sigma_{v_c}$ which can vary between two other special points in the real part of $\Sigma^*_{v_c}$.

(7) The point $(\Sigma^*,\mathbf{p}^*)$ in the fiber is unique since the contracted component supports only three points.

The o-planar trees associated to $(\Sigma^*,\mathbf{p}^*)$ are simply obtained by considering the cases above. □

Consider the forgetful map for the closed strata $\pi:\overline{C}_{(\gamma^*,o^*)}\to\overline{C}_{(\gamma,o)}$. In this case, we denote the fiber $\pi^{-1}(\Sigma;\mathbf{p})$ for $(\Sigma;\mathbf{p})\in C_{(\gamma,o)}$ by $\overline{F}_s$ since it is the closure of the fiber F_s of $\pi:C_{(\gamma^*,o^*)}\to C_{(\gamma,o)}$. By using the stratification of $\overline{C}_{(\gamma^*,o^*)}$, we obtain a stratification of fibers $\overline{F}_s$.

Lemma 8.1.2. *Let $\overline{F}^i_s$ be the fibers of $\pi^i:\overline{C}_{(\gamma^*_i,o^*_i)}\to\overline{C}_{(\gamma,o)}$ over $(\Sigma;\mathbf{p})\in C_{(\gamma,o)}$. Then, $\overline{F}^1_s\subset\overline{F}^2_s$ if and only if (γ^*_1,o^*_1) produces (γ^*_2,o^*_2) by contracting one of its real edges or a conjugate pair edges.*

Proof. This statement is a direct corollary of Proposition 5.3.4. □

8.1.2 Homology of the fibers of the forgetful morphisms

Let (γ^*, o^*) be a one-vertex o-planar tree, and let $\pi : C_{(\gamma^*,o^*)} \to C_{(\gamma,o)}$ be the map forgetting the special points $p_s, p_{\bar{s}}$ which is discussed in Section 8.1.1. Assume that the fibers are two-dimensional i.e., a punctured disc or a punctured sphere (see corresponding cases in Lemma 8.1.1).

Case type 1. Let (γ^*, o^*) be a one-vertex o-planar tree of type 1. Assume that $s \in \mathbf{F}^+_{\gamma^*}$ (resp. $s \in \mathbf{F}^-_{\gamma^*}$). Then, each fiber F_s of π is homotopy equivalent to a bouquet of $|\mathbf{F}^+_{\gamma^*}| - 1$ circles $S^1 \vee \cdots \vee S^1$. The cohomology of F_s is generated by the logarithmic differentials;

$$\begin{aligned} H^0(F_s) &= \mathbb{Z} \\ H^1(F_s) &= \bigoplus_f \mathbb{Z}\, \omega_{sf} \end{aligned}$$

where

$$\omega_{sf} = \frac{1}{2\pi\sqrt{-1}}\, d\log(z_s - z_f) \tag{8.1}$$

for $f \in \mathbf{F}^+_{\gamma^*} \setminus \{s\}$ (resp. $f \in \mathbf{F}^-_{\gamma^*} \setminus \{s\}$).

The homology with closed support $H^c_1(F_s)$ is isomorphic to the cohomology group $H^1(F_s)$ and generated by the duals of the logarithmic forms i.e., by the arcs connecting the punctures z_f to a point in boundary of the closure of the fiber $\overline{F}_s$ of π in $\overline{C}_{(\gamma^*,o^*)}$ (see Figure 8.1).

We denote the dual of the generator ω_{sf} by $\mathscr{T}_{sf}$. The homology group $H^c_2(F_s)$ is isomorphic to $H^0(F_s)$. Hence,

$$\begin{aligned} H^c_2(F_s) &= \mathbb{Z} \\ H^c_1(F_s) &= \bigoplus_f \mathbb{Z}\, \mathscr{T}_{sf} \end{aligned}$$

where $f \in \mathbf{F}^+_{\gamma^*} \setminus \{s\}$ (resp. $f \in \mathbf{F}^-_{\gamma^*} \setminus \{s\}$).

Case type 2. Let (γ^*, o^*) be a one-vertex o-planar tree of type 2. Then, each fiber F_s of π is homotopy equivalent to a bouquet of $|\mathbf{F}_{\gamma^*}(v_s)| - 3$ circles $S^1 \vee \cdots \vee S^1$. Therefore, the cohomology of F_s is generated by the logarithmic differentials;

$$\begin{aligned} H^0(F_s) &= \mathbb{Z} \\ H^1(F_s) &= \bigoplus_f \mathbb{Z}\, \omega_{sf} \end{aligned}$$

where

$$\omega_{sf} = \frac{1}{2\pi\sqrt{-1}}\, d\log(z_s - z_f)$$

Figure 8.1: The generators of $H_1^c(F)$.

for $f \in \mathbf{F}_{\gamma^*} \setminus \{s, \bar{s}, s_{2k}\}$.

The homology with closed support $H_1^c(F_s)$ is isomorphic to the cohomology group $H^1(F_s)$ and generated by the arcs connecting the pairs of punctures at z_{f_1}, z_{f_2}. These arcs are the duals of the cohomology classes $\omega_{sf_1} - \omega_{sf_2}$. We denote them by $\mathscr{R}_{s,f_1f_2}$. The homology group $H_2^c(F_s)$ is isomorphic to $H^0(F_s)$. Hence,

$$\begin{aligned} H_2^c(F_s) &= \mathbb{Z} \\ H_1^c(F_s) &= \left(\bigoplus_f \mathbb{Z}\, \mathscr{R}_{s,f_1f_2} \right) / \mathscr{J}_s \end{aligned}$$

where $f_i \in \mathbf{F}_{\gamma^*} \setminus \{s, \bar{s}, s_{2k}\}$, and the ideal $\mathscr{J}_s$ is generated by

$$\mathscr{R}_{s,f_1f_2} + \mathscr{R}_{s,f_2f_3} + \mathscr{R}_{s,f_3f_1}. \tag{8.2}$$

Homology of the fibers of the forgetful morphism $\pi_{\{s\}} : \overline{M}_{\mathbf{S}} \to \overline{M}_{\mathbf{S'}}$. Let $\mathbf{S}$ be a finite set of labeling with $|\mathbf{S}| \geq 4$, and let $s \in \mathbf{S}$ be different than s_n. Let $\mathbf{S'} = \mathbf{S} \setminus \{s\}$. Then, each fiber F_s of forgetful map $\pi_{\{s\}} : M_{\mathbf{S}} \to M_{\mathbf{S'}}$ is homotopy equivalent to a bouquet of $|\mathbf{S}| - 2$ circles $S^1 \vee \cdots \vee S^1$. Therefore, the cohomology of a fiber is generated by the logarithmic differentials;

$$\begin{aligned} H^0(F_s) &= \mathbb{Z} \\ H^1(F_s) &= \bigoplus_f \mathbb{Z}\, \omega_{sf} \end{aligned}$$

where

$$\omega_{sf} = \frac{1}{2\pi\sqrt{-1}}\, d\log(z_{s_n} - z_f)$$

for $f \in \mathbf{S} \setminus \{s, s_n\}$.

The homology with closed support $H_1^c(F_{s_n})$ is isomorphic to the cohomology group $H^1(F_{s_n})$ and generated by the arcs connecting the pairs of punctures at z_{f_1}, z_{f_2}. These arcs are the duals of the cohomology classes $\omega_{sf_1} - \omega_{sf_2}$. We denote them by $\mathscr{P}_{s,f_1f_2}$. The homology group $H_2^c(F_s)$ is isomorphic to $H^0(F_s)$. Hence,

$$\begin{aligned} H_2^c(F_s) &= \mathbb{Z} \\ H_1^c(F_s) &= \left(\bigoplus_f \mathbb{Z}\ \mathscr{P}_{s,f_1f_2}\right)/\mathcal{J}_s \end{aligned}$$

where $f_i \in \mathbf{S} \setminus \{s, s_n\}$ and the ideal $\mathcal{J}_s$ is generated by

$$\mathscr{P}_{s,f_1f_2} + \mathscr{P}_{s,f_2f_3} + \mathscr{P}_{s,f_3f_1}. \tag{8.3}$$

8.2 Homology of the strata

In this section, we give the generators of the homology of the strata $\overline{C}_{(\gamma,o)}$ relative to the union of their substrata $Q_{(\gamma,o)} := \bigcup_{(\tau,\hat{o})<(\gamma,o)} \overline{C}_{(\tau,\hat{o})}$.

Lemma 8.2.1. *Let $\pi : C_{(\gamma^*,o^*)} \to C_{(\gamma,o)}$ be the fibration discussed Section 8.1. Then,*

$$H_d^c(C_{(\gamma^*,o^*)};\mathbb{Z}) = \bigoplus_{p+q=d} H_p^c(C_{(\gamma,o)};\mathbb{Z}) \otimes H_q^c(F_s;\mathbb{Z})$$

Proof. We first consider the subcases where $\dim F_s = 2$. Assume that (γ^*, o^*) be of type 1. The strata $C_{(\gamma^*,o^*)}$ and $C_{(\gamma,o)}$ are given by the products

$$\begin{aligned} \prod_{v\in\mathbf{V}^{\mathbb{R}}_{\gamma^*}} C_{(\gamma^*_v,o^*_v)} &\times \prod_{v\in\mathbf{V}^{+}_{\gamma^*}} M_{\mathbf{F}_{\gamma^*}(v)}, \\ \prod_{v\in\mathbf{V}^{\mathbb{R}}_{\gamma}} C_{(\gamma_v,o_v)} &\times \prod_{v\in\mathbf{V}^{+}_{\gamma}} M_{\mathbf{F}_{\gamma}(v)} \end{aligned}$$

(see (5.3) in Section 5.3). The forgetful map π preserves the components $(\Sigma^*_v, \mathbf{p}^*_v)$ of $(\Sigma^*, \mathbf{p}^*) \in C_{(\gamma^*,o^*)}$ for $v \neq v_s$. Hence, it is the identity map on the factors

$$\begin{aligned} C_{(\gamma^*_v,o^*_v)} &\to C_{(\gamma_v,o_v)}, \\ M_{\mathbf{F}_{\gamma^*}(v)} &\to M_{\mathbf{F}_{\gamma}(v)} \end{aligned}$$

for $v \neq v_s$. On the other hand, it gives a fibration

$$\begin{aligned} \pi_{res} : C_{(\gamma^*_{v_s},o^*_{v_s})} &\to C_{(\gamma_{v_s},o_{v_s})}, \quad \text{when } v_s \in \mathbf{V}^{\mathbb{R}}_{\gamma^*}, \text{ and} \\ \pi_{res} : M_{\mathbf{F}_{\gamma^*}(v_s)} &\to M_{\mathbf{F}_{\gamma}(v_s)} \quad \text{when } v_s \notin \mathbf{V}^{\mathbb{R}}_{\gamma^*} \end{aligned} \tag{8.4}$$

with the same fibers F_s of $\pi : C_{(\gamma^*,o^*)} \to C_{(\gamma,o)}$. Therefore, we only need to consider the fibrations in (8.4) to calculate the (co)homology.

The strata $C_{(\gamma^*_{v_s}, o^*_{v_s})}$ and $C_{(\gamma_{v_s}, o_{v_s})}$ are diffeomorphic to the products of simplices with the products of upper half plane minus (usual and cross) diagonals (see Lemma 5.2.2). The map π_{res} forgets the coordinate subspace $\mathbb{H}^+$ corresponding to the labelled point p_s. For instance, when $|\mathbf{F}^{\mathbb{R}}_{\gamma^*}(v_s)| > 2$, the map $\pi_{res} : C_{(\gamma^*_{v_s}, o^*_{v_s})} \to C_{(\gamma_{v_s}, o_{v_s})}$ is

$$((\mathbb{H}^+)^{|\mathbf{F}^+_{\gamma^*}(v_s)|} \setminus \Delta^*) \times \square^{|\mathbf{F}^{\mathbb{R}}_{\gamma^*}(v_s)|-3} \to ((\mathbb{H}^+)^{|\mathbf{F}^+_{\gamma}(v_s)|} \setminus \Delta) \times \square^{|\mathbf{F}^{\mathbb{R}}_{\gamma}(v_s)|-3},$$

forgetting the coordinate subspace $\mathbb{H}^+$ of the labelled point p_s. Similarly, $\pi_{res} : M_{\mathbf{F}_{\gamma^*}(v_s)} \to M_{\mathbf{F}_{\gamma}(v_s)}$ is

$$\mathbb{C}^{|\mathbf{F}_{\gamma^*}(v_s)|-3} \setminus \Delta^* \to \mathbb{C}^{|\mathbf{F}_{\gamma}(v_s)|-3} \setminus \Delta,$$

forgetting the coordinate subspace $\mathbb{C}$ of the labelled point p_s.

The logarithmic differentials $d\log(z_s - z_j)$ for $j \in \mathbf{F}^+_{\gamma}(v_s)$ (resp. $j \in \mathbf{F}_{\gamma}(v_s)$) give global cohomology classes on $C_{(\gamma^*, o^*)}$ (resp. on $M_{\mathbf{F}_{\gamma^*}(v_s)}$). On the other hand, we have seen that the restrictions of these logarithmic forms to each fiber freely generate the cohomology of the fiber (see Section 8.1.2). By using Leray-Hirsch theorem, we obtain

$$H^d(C_{(\gamma^*, o^*)}) = \bigoplus_{p+q=d} H^p(C_{(\gamma, o)}) \otimes H^q(F_s).$$

If $\dim F_s = 1$ and the fibers are open intervals, then we directly have

$$H^p(C_{(\gamma^*, o^*)}) = H^p(C_{(\gamma, o)}; H^0(F_s)) = H^p(C_{(\gamma, o)}) \otimes H^0(F_s).$$

since the fibers are contractible.

If $\dim F_s = 1$ and the fibers are S^1, then $C_{(\gamma^*, o^*)}$ is diffeomorphic to $(\mathbb{H}^+ \setminus \{\sqrt{-1}\})^{|\mathbf{F}_{\gamma^*}(v_s)|-2} \times \square^1 \times S^1$. This is obtained by using a slightly different normalization of coordinates than the normalization used in Section 5.1.2 and Lemma 5.2.2 i.e., by mapping $(p_{f_1}, p_{\bar{f}_1})$ to $(\sqrt{-1}, -\sqrt{-1})$, and $(p_{f_2}, p_{\bar{f}_2})$ to $(\lambda\sqrt{-1}, -\lambda\sqrt{-1})$ where $f_1, f_2 \in \mathbf{F}^+_{\gamma^*}$. Then, the map π_{res} forget the special point parameterized by S^1. Hence, $C_{(\gamma^*, o^*)}$ is $C_{(\gamma, o)} \times S^1$, and claim follows form the Künneth formula.

If $\dim F_s = 0$, then each fiber is a single point and statement is obvious.

Finally, the duality between cohomology and homology with closed support gives us the isomorphisms which we need to complete the proof

$$\begin{aligned} H^d(C_{(\gamma^*, o^*)}) &= H^c_{\dim(C_{(\gamma^*, o^*)})-d}(C_{(\gamma^*, o^*)}) \\ H^p(C_{(\gamma, o)}) &= H^c_{\dim(C_{(\gamma, o)})-p}(C_{(\gamma, o)}) \\ H^p(F_s) &= H^c_{\dim(F_s)-p}(F_s). \end{aligned}$$

The same statements for type 2 and type 3 cases are proved by using the same strategy and arguments above. □

Since the strata of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ are the products given in (5.3), their homology is the product of homology of their factors. Here, we are going to give the relative homology for the strata corresponding to one-vertex trees.

Now, let (γ, o) be a one-vertex o-planar tree of type 1, and $\mathbf{F}^{+}_{\gamma} = \{s_1, \cdots, s_k\}$. Let $Q_{(\gamma,o)}$ be the union of the codimension one and higher strata of $\overline{C}_{(\gamma,o)}$.

Proposition 8.2.2. *The relative homology group* $H_{\dim(\overline{C}_{(\gamma,o)})-d}(\overline{C}_{(\gamma,o)}, Q_{(\gamma,o)}; \mathbb{Z})$ *is generated by*

$$\mathscr{T}_{s_{i_1} s_{j_1}} \otimes \cdots \otimes \mathscr{T}_{s_{i_d} s_{j_d}}$$

where $j_* < i_*$ *and* $i_1 < \cdots < i_d \leq |\mathbf{F}^{+}_{\gamma}|$. *In particular,*

$$H_{\dim(\overline{C}_{(\gamma,o)})}(\overline{C}_{(\gamma,o)}, Q_{(\gamma,o)}; \mathbb{Z}) = \mathbb{Z}\ [\overline{C}_{(\gamma,o)}].$$

Proof. Due its definition, the homology with closed support

$$H^{c}_{*}(C_{(\gamma,o)}) = \lim_{\rightarrow} H_*(C_{(\gamma,o)}, C_{(\gamma,o)} \setminus K)$$

where K ranges over all closed subsets of $C_{(\gamma,o)}$. The group $H_*(C_{(\gamma,o)}, C_{(\gamma,o)} \setminus K)$ is isomorphic $H_*(\overline{C}_{(\gamma,o)}, \overline{C}_{(\gamma,o)} \setminus \overline{K})$ where K ranges over all closed subsets of $\overline{C}_{(\gamma,o)}$ which does not intersect with $Q_{(\gamma,o)}$. In the limit, $\overline{C}_{(\gamma,o)} \setminus \overline{K}$ gives $Q_{(\gamma,o)}$. Hence, the homology with closed support is indeed isomorphic to the relative homology of $\overline{C}_{(\gamma,o)}$.

On the other hand, Lemma 8.2.1 implies that

$$H_d(\overline{C}_{(\gamma^*,o^*)}, Q_{(\gamma^*,o^*)}; \mathbb{Z}) = \bigoplus_{p+q=d} H_p(\overline{C}_{(\gamma,o)}, Q_{(\gamma,o)}; \mathbb{Z}) \otimes H^{c}_{q}(F_s; \mathbb{Z}).$$

where F_s is a fiber of the map forgetting $s, \bar{s}$.

We obtain the result by applying the maps forgetting the conjugate pairs of points successively and using the generators of $H^{c}_{*}(F_{s_i})$ given in Section 8.1.2. We forget the pairs of points the in following order

$$\begin{array}{ll} (s_k, s_{2k}), (s_{k-1}, s_{2k-1}), \cdots, (s_1, s_{k+1}), & \text{when } |\mathbf{F}^{\mathbb{R}}_{\gamma}| \geq 3, \\ (s_k, s_{2k}), (s_{k-1}, s_{2k-1}), \cdots, (s_2, s_{k+2}), & \text{when } |\mathbf{F}^{\mathbb{R}}_{\gamma}| = 1, 2, \\ (s_k, s_{2k}), (s_{k-1}, s_{2k-1}), \cdots, (s_3, s_{k+3}), & \text{when } |\mathbf{F}^{\mathbb{R}}_{\gamma}| = 0. \end{array}$$

We obtain that

$$H_d(\overline{C}_{(\gamma^*,o^*)}, Q_{(\gamma^*,o^*)}) = \begin{cases} H^{c}_{*}(F_{s_1}) \otimes \cdots \otimes H^{c}_{*}(F_{s_k}) \otimes H^{c}_{*}(C_{(\tau,o(\mathbb{R}))}) & \text{if } |\mathbf{F}^{\mathbb{R}}_{\gamma}| \geq 3, \\ H^{c}_{*}(F_{s_2}) \otimes \cdots \otimes H^{c}_{*}(F_{s_k}) & \text{if } |\mathbf{F}^{\mathbb{R}}_{\gamma}| = 2, \\ H^{c}_{*}(F_{s_2}) \otimes \cdots \otimes H^{c}_{*}(F_{s_k}) & \text{if } |\mathbf{F}^{\mathbb{R}}_{\gamma}| = 1, \\ H^{c}_{*}(F_{s_3}) \otimes \cdots \otimes H^{c}_{*}(F_{s_k}) & \text{if } |\mathbf{F}^{\mathbb{R}}_{\gamma}| = 0 \end{cases}$$

where $(\tau, o(\mathbb{R}))$ is a one-vertex tree which is obtained by forgetting all tails of (γ^*, o^*) labelled by $\mathbf{F}^{\pm}_{\gamma^*}$. The space $C_{(\tau,o(\mathbb{R}))}$ is an open simplex due to Lemma 5.2.2. This directly gives us that the relative homology is generated by the products of the generators of $H^c(F_{s_i})$ as stated above. In order to simplify the notation, we omit the factors coming from the generators $H_2^c(F_{s_i})$ and $H^c_{\dim C_{(\tau,o)}}(C_{(\tau,o)})$.

It is clear that the top dimensional relative homology is generated by the relative fundamental class $[\overline{C}_{(\gamma,o)}]$. □

Now, let (γ, o) be a one-vertex o-planar tree of type 2, and $\mathbf{F}_\gamma = \{s_1, \cdots, s_{2k}\}$. Let $Q_{(\gamma,o)}$ be the unions of the codimension one and higher strata of $\overline{C}_{(\gamma,o)}$.

Proposition 8.2.3. *The relative homology group* $H_{\dim(\overline{C}_{(\gamma,o)})-d}(\overline{C}_{(\gamma,o)}, Q_{(\gamma,o)}; \mathbb{Z})$ *is generated by*

$$\mathscr{R}_{s_{i_1}, s_{j_1} s_{k_1}} \otimes \cdots \otimes \mathscr{R}_{s_{i_d}, s_{j_d} s_{k_d}}$$

where $\sigma(s_{j_*}), \sigma(s_{k_*}) \neq s_{i_*}$, $j_*, k_* < i_*$ *and* $2 < i_1 < \cdots < i_d \leq |\mathbf{F}_\gamma|$. *In particular,*

$$H_{\dim(\overline{C}_{(\gamma,o)})}(\overline{C}_{(\gamma,o)}, Q_{(\gamma,o)}; \mathbb{Z}) = \mathbb{Z}\ [\overline{C}_{(\gamma,o)}].$$

We also calculate the homology of $\overline{M}_{\mathbf{S}}$ relative its strata for $\mathbf{S} = \{s_1, \cdots, s_n\}$. We denote the union of strata of $\overline{M}_{\mathbf{S}}$ of codimension one or higher by $W_{\mathbf{S}}$.

Proposition 8.2.4. *The relative homology group* $H_{\dim(\overline{M}_{\mathbf{S}})-d}(\overline{M}_{\mathbf{S}}, W_{\mathbf{S}}; \mathbb{Z})$ *is generated by*

$$\mathscr{P}_{s_{i_1}, s_{j_1} s_{k_1}} \otimes \cdots \otimes \mathscr{P}_{s_{i_d}, s_{j_d} s_{k_d}}$$

where $j_*, k_* < i_*$ *and* $i_1 < \cdots < i_d \leq |\mathbf{S}|$. *In particular,*

$$H_{\dim(\overline{M}_{\mathbf{S}})}(\overline{M}_{\mathbf{S}}, W_{\mathbf{S}}; \mathbb{Z}) = \mathbb{Z}\ [\overline{M}_{\mathbf{S}}]$$

where $[\overline{M}_{\mathbf{S}}]$ *is the fundamental class of* $\overline{M}_{\mathbf{S}}$.

The proofs of Proposition 8.2.3 and 8.2.4 are essentially the same with Proposition 8.2.2. We will not repeat it.

Remark 8.2.5. The open strata considered in Proposition 8.2.2 and 8.2.4 are topologically same with the braid spaces, and the strata in Proposition 8.2.3 have very similar topological properties with braid spaces. For that reason, the proof of Lemma 8.2.1 uses essentially the same arguments in [1].

Chapter 9

Graph homology of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$

In this chapter, we give a combinatorial graph complex whose homology is the homology of the moduli space of σ-invariant curves $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$.

9.1 The graph complex of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$

We define a graded group

$$\mathcal{G}_d := \left(\bigoplus_{(\gamma,o)\in\mathcal{T}ree(\sigma):|\mathbf{E}_\gamma|=|\mathbf{S}|-d-3} H_{\dim(\overline{C}_{(\gamma,o)})}(\overline{C}_{(\gamma,o)}, Q_{(\gamma,o)}; \mathbb{Z}) \right) / I_d. \tag{9.1}$$

The homology group $H_{\dim(\overline{C}_{(\gamma,o)})}(\overline{C}_{(\gamma,o)}, Q_{(\gamma,o)})$ is of rank one and generated by the relative fundamental cycle $[\overline{C}_{(\gamma,o)}]$ of the strata $\overline{C}_{(\gamma,o)}$ (see Section A for orientations convention).

The ideal I_d (of degree d) is generated by the following elements.

The generators of the ideal of graph complex. Case $|\mathbf{Fix}(\sigma)| > 0$.

$\mathfrak{R}$-1. Degeneration at a real vertex of type 1 o-planar trees. Consider an o-planar tree (γ, o) such that $|\mathbf{E}_\gamma| = d-2$, and one of its vertex $v \in \mathbf{V}_\gamma^{\mathbb{R}}$ with $|v| \geq 5$ and $|\mathbf{F}_\gamma^{+}(v)| \geq 2$. Let $f_i, \bar{f}_i \in \mathbf{F}_\gamma \setminus \mathbf{F}_\gamma^{\mathbb{R}}$ be conjugate pairs of flags for $i = 1, 2$, and let $f_3 \in \mathbf{F}_\gamma^{\mathbb{R}}$. Put $\mathbf{F} = \mathbf{F}_\gamma(v) \setminus \{f_1, f_2, \bar{f}_1, \bar{f}_2, f_3\}$.

We define two o-planar trees $(\gamma_1, o_1), (\gamma_2, o_2)$.

The first one (γ_1, o_1) is obtained by inserting a pair of conjugate edges $e = (f_e, f^e), \bar{e} = (f_{\bar{e}}, f^{\bar{e}})$ to (γ, o) at v with boundaries $\boldsymbol{\partial}_{\gamma_1}(e) = \{\tilde{v}, v^e\}$, $\boldsymbol{\partial}_{\gamma_1}(\bar{e}) = \{\tilde{v}, v^{\bar{e}}\}$. The distribution of flags is given by $\mathbf{F}_{\gamma_1}(\tilde{v}) = \mathbf{F}_1 \bigcup \{f_3, f_e, f_{\bar{e}}\}$, $\mathbf{F}_{\gamma_1}(v^e) = \mathbf{F}_2 \bigcup \{f_1, f_2, f^e\}$ and $\mathbf{F}_{\gamma_1}(v^{\bar{e}}) = \overline{\mathbf{F}}_2 \bigcup \{\bar{f}_1, \bar{f}_2, f^{\bar{e}}\}$, where $\mathbf{F}$ is the disjoint union of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\overline{\mathbf{F}}_2$. The set $\mathbf{F}_2$ contains the flags conjugate to the ones in $\overline{\mathbf{F}}_2$.

The second one (γ_2, o_2) is obtained by inserting a pair of real edges $e_1 = (f_{e_1}, f^{e_1})$, $e_2 = (f_{e_2}, f^{e_2})$ to (γ, o) at v with boundaries $\boldsymbol{\partial}_{\gamma_2}(e_1) = \{\tilde{v}, v^{e_1}\}$, $\boldsymbol{\partial}_{\gamma_2}(e_2) = \{\tilde{v}, v^{e_2}\}$. The sets of flags are given by $\mathbf{F}_{\gamma_2}(\tilde{v}) = \mathbf{F}_1 \bigcup \{f_3, f_{e_1}, f_{e_2}\}$, $\mathbf{F}_{\gamma_2}(v^{e_1}) = \mathbf{F}_2 \bigcup \{f_1, \bar{f}_1, f^{e_1}\}$ and $\mathbf{F}_{\gamma_2}(v^{e_2}) = \mathbf{F}_3 \bigcup \{f_2, \bar{f}_2, f^{e_2}\}$ where $\mathbf{F}$ is the disjoint union of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$. Here, the flags f_3, f_{e_1}, f_{e_2} are ordered $\{f_3\} < \{f_{e_1}\} < \{f_{e_2}\}$ according to the cyclic ordering of $\mathbf{F}_{\gamma_2}(\tilde{v})$. See Figure 9.1 for $|v| = 5$.

Then, we define

$$\mathcal{R}(\gamma, o; v, f_1, f_2, f_3) := \sum_{(\gamma_1, o_1)} [\overline{C}_{(\gamma_1, o_1)}] - \sum_{(\gamma_2, o_2)} [\overline{C}_{(\gamma_2, o_2)}]. \tag{9.2}$$

Here summation is taken over all possible $(\gamma_i, o_i), i = 1, 2$ defined above.

Figure 9.1: The o-planar tree (γ, o) with $|v| = 5$ and o-planar trees $(\gamma_i, o_i), i = 1, 2$ which appear in the sum $\mathcal{R}(\gamma, o; v, f_1, f_2, f_3)$.

$\mathfrak{R}$-2. Degeneration at a conjugate pair of vertices. Consider an o-planar tree (γ, o) of type 1 such that $|\mathbf{E}_\gamma| = d - 2$, and a pair of its conjugate vertices $\{v, \bar{v}\} \subset \mathbf{V}_\gamma \setminus \mathbf{V}_\gamma^{\mathbb{R}}$ such that $|v| = |\bar{v}| \geq 4$. Let $\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4 \in \mathbf{F}_\gamma(\bar{v})$ be the flags conjugate to $f_1, f_2, f_3, f_4 \in \mathbf{F}_\gamma(v)$. Put $\mathbf{F} = \mathbf{F}_\gamma(v) \setminus \{f_1, f_2, f_3, f_4\}$. Let $\mathbf{F}_1, \mathbf{F}_2$ be two disjoint subsets of $\mathbf{F}$ such that $\mathbf{F} = \mathbf{F}_1 \bigcup \mathbf{F}_2$. Let $\overline{\mathbf{F}}_1, \overline{\mathbf{F}}_2$ be the sets of flags conjugate to the flags in $\mathbf{F}_1, \mathbf{F}_2$ respectively.

We define two o-planar trees $(\gamma_1, o_1), (\gamma_2, o_2)$.

The first one (γ_1, o_1) is obtained by inserting a pair of conjugate edges $e = (f_e, f^e), \bar{e} = (f_{\bar{e}}, f^{\bar{e}})$ to (γ, o) respectively at $v, \bar{v}$ such that $\boldsymbol{\partial}_{\gamma_1}(e) = \{v_e, v^e\}$, $\boldsymbol{\partial}_{\gamma_1}(\bar{e}) = \{v_{\bar{e}}, v^{\bar{e}}\}$. The set of flags are $\mathbf{F}_{\gamma_1}(v_e) = \mathbf{F}_1 \bigcup \{f_1, f_2, f_e\}$, $\mathbf{F}_{\gamma_1}(v^e) = \mathbf{F}_2 \bigcup \{f_3, f_4, f^e\}$ and $\mathbf{F}_{\gamma_1}(v_{\bar{e}}) = \overline{\mathbf{F}}_1 \bigcup \{\bar{f}_1, \bar{f}_2, f_{\bar{e}}\}$, $\mathbf{F}_{\gamma_1}(v^{\bar{e}}) = \overline{\mathbf{F}}_2 \bigcup \{\bar{f}_3, \bar{f}_4, f^{\bar{e}}\}$.

The second one (γ_2, o_2) is also obtained by inserting a pair of conjugate edges to (γ, o) at the same vertices $v, \bar{v}$, but the flags are distributed differently on vertices $\boldsymbol{\partial}_{\gamma_2}(e) = \{v_e, v^e\}, \boldsymbol{\partial}_{\gamma_2}(\bar{e}) = \{v_{\bar{e}}, v^{\bar{e}}\}$: $\mathbf{F}_{\gamma_2}(v_e) = \mathbf{F}_1 \bigcup \{f_1, f_3, f_e\}$, $\mathbf{F}_{\gamma_2}(v^e) = \mathbf{F}_2 \bigcup \{f_2, f_4, f^e\}$ and $\mathbf{F}_{\gamma_2}(v_{\bar{e}}) = \overline{\mathbf{F}}_1 \bigcup \{\bar{f}_1, \bar{f}_3, f_{\bar{e}}\}$, $\mathbf{F}_{\gamma_2}(v^{\bar{e}}) = \overline{\mathbf{F}}_2 \bigcup \{\bar{f}_2, \bar{f}_4, f^{\bar{e}}\}$.

Then, we define

$$\mathcal{R}(\gamma, o; v, f_1, f_2, f_3, f_4) := \sum_{(\gamma_1, o_1)} [\overline{C}_{(\gamma_1, o_1)}] - \sum_{(\gamma_2, o_2)} [\overline{C}_{(\gamma_2, o_2)}]. \tag{9.3}$$

Here summation is taken over all possible $(\gamma_i, o_i), i = 1, 2$ defined above.

$\mathfrak{R}$**-3. Reversing o-planar structure at a real vertex.** Let (γ, o_1) be an o-planar tree of type 1 such that $|\mathbf{E}_\gamma| = d$, and $v \in \mathbf{V}_\gamma^{\mathbb{R}}$. Let $\mathbf{V}_\gamma(v)$ be the set of all vertices in $\mathbf{V}_\gamma^+$ such that the closest real vertex to its elements is v. Let (γ, o_2) be the o-planar tree which produces (γ, o_1) by reversing the o-planar structure o_v at the vertex v.

Then, we define

$$\mathcal{R}(\gamma, o_1; v) = [\overline{C}_{(\gamma,o_1)}] - (-1)^\mu [\overline{C}_{(\gamma,o_2)}] \tag{9.4}$$

where

$$\mu = |\mathbf{F}_\gamma^+(v)| + \frac{(|\mathbf{F}_\gamma^{\mathbb{R}}(v) - 2|)(|\mathbf{F}_\gamma^{\mathbb{R}}(v) - 3|)}{2} + \sum_{v_j \in \mathbf{V}_\gamma(v)} (|v_j| - 3). \tag{9.5}$$

Remark 9.1.1. Obviously, $(\gamma, o_1), (\gamma, o_2)$ are o-planar representatives of same u-planar tree (γ, u). We identify $C_{(\gamma,o_i)}$ with $C_{(\gamma,u)}$ in $\mathbb{R}\overline{M}_{\mathbf{S}}^\sigma$, and pick the coordinates defined in Appendix A by using o-planar structures $o_i, i = 1, 2$. These coordinates can be transformed to each other by

$$C_{(\gamma_v,o_v)} \times \prod_{v_j \in \mathbf{V}_\gamma(v)} \overline{M}_{\mathbf{F}_\gamma(v_j)} \to C_{(\gamma_v,\bar{o}_v)} \times \prod_{v_j \in \mathbf{V}_\gamma(v)} \overline{M}_{\mathbf{F}_\gamma(\bar{v}_j)}$$

where the map $C_{(\gamma_v,o_v)} \to C_{(\gamma_v,\bar{o}_v)}$ is $-\mathbb{I}$ given in (5.1).

Let (z_α, x_i) and (w_α, y_i) be coordinates in $C_{(\gamma_v,o_v)}$ and $C_{(\gamma_v,\bar{o}_v)}$ respectively. We have $dz_\alpha = -dw_\alpha$ and $dx_i = -dy_i$. If $|\mathbf{F}_\gamma^{\mathbb{R}}(v)| \geq 3$, then we obtain

$$\begin{aligned}
\Omega_{(\gamma_v,o_v)} &= \bigwedge_{\alpha \in \mathbf{F}_\gamma^+(v)} dz_\alpha \wedge d\bar{z}_\alpha \bigwedge dx_{f_{r_2}} \wedge \cdots \wedge dx_{f_{r_{l-2}}}, \\
&= (-1)^{\mu_1} \bigwedge_{\bar{\alpha} \in \mathbf{F}_\gamma^-(v)} dw_{\bar{\alpha}} \wedge d\bar{w}_{\bar{\alpha}} \bigwedge dy_{f_{r_{l-2}}} \wedge \cdots \wedge dy_{f_{r_2}}, \\
&= (-1)^{\mu_1} \Omega_{(\gamma_v,\bar{o}_v)}
\end{aligned}$$

where $\mu_1 = (-1)^{|\mathbf{F}_\gamma^+(v)| + (|\mathbf{F}_\gamma^{\mathbb{R}}(v) - 2|)(|\mathbf{F}_\gamma^{\mathbb{R}}(v) - 3|)/2}$. We obtain the special cases of this formula when $|\mathbf{F}_\gamma^{\mathbb{R}}| = 1, 2$ by similar calculations.

The transformation from $\overline{M}_{\mathbf{F}_\gamma(v_j)}$ to $\overline{M}_{\mathbf{F}_\gamma(\bar{v}_j)}$ gives $\Omega_{\gamma_{v_j}} = (-1)^{\mu_2} \Omega_{\gamma_{\bar{v}_j}}$ where $\mu_2 = |v_j| - 3$. This follows from a direct calculation similar to above.

Therefore, the difference $\mathcal{R}(\gamma, o_1; v) = [\overline{C}_{(\gamma,o_1)}] - (-1)^\mu [\overline{C}_{(\gamma,o_2)}]$ is indeed zero when $\mu = \mu_1 + \mu_2$ as given in (9.5).

The generators of the ideal of graph complex. Case $|\mathbf{Fix}(\sigma)| = 0$.

$\mathfrak{S}$**-1. Degeneration at the real vertex of type 1 o-planar trees.** Consider an o-planar tree (γ, o) of type 1 such that $|\mathbf{E}_\gamma| = d - 2$ and $|\mathbf{F}_\gamma^{\mathbb{R}}| = 0$. Let v be its real vertex, and assume that $|v| \geq 6$. Let $f_i \in \mathbf{F}_\gamma^+, \bar{f}_i \in \mathbf{F}_\gamma^-$ be conjugate pairs of flags for $i = 1, 2, 3$. Put $\mathbf{F} = \mathbf{F}_\gamma(v) \setminus \{f_1, f_2, f_3, \bar{f}_1, \bar{f}_2, \bar{f}_3\}$.

We define two o-planar trees $(\gamma_1, o_1), (\gamma_2, o_2)$.

The first one (γ_1, o_1) is obtained by inserting a pair of conjugate edges $e = (f_e, f^e), \bar{e} = (f_{\bar{e}}, f^{\bar{e}})$ to (γ, o) at v with boundaries $\boldsymbol{\partial}_{\gamma_1}(e) = \{\tilde{v}, v^e\}$, $\boldsymbol{\partial}_{\gamma_1}(\bar{e}) = \{\tilde{v}, v^{\bar{e}}\}$. The set of flags are $\mathbf{F}_{\gamma_1}(\tilde{v}) = \mathbf{F}_1 \bigcup \{f_1, \bar{f}_1, f_e, f_{\bar{e}}\}$, $\mathbf{F}_{\gamma_1}(v^e) = \mathbf{F}_2 \bigcup \{f_2, f_3, f^e\}$ and $\mathbf{F}_{\gamma_1}(v^{\bar{e}}) = \overline{\mathbf{F}}_2 \bigcup \{\bar{f}_2, \bar{f}_3, f^{\bar{e}}\}$, where $\mathbf{F}$ is the disjoint union of $\mathbf{F}_1, \mathbf{F}_2$ and $\overline{\mathbf{F}}_2$. The set $\mathbf{F}_2$ contains the flags that are conjugate to flags in $\overline{\mathbf{F}}_2$ and vice versa.

The second one (γ_2, o_2) is obtained in a similar way. First, we swap f_1 and $\bar{f}_1$ (i.e, put f_1 in $\mathbf{F}_\gamma^-$ and $\bar{f}_1$ in $\mathbf{F}_\gamma^+$). Then, we obtain (γ_2, o_2) by inserting a pair of conjugate edges at the vertex v same way, but the flags are distributed differently on vertices $\mathbf{F}_{\gamma_2}(\tilde{v}) = \mathbf{F}_1 \bigcup \{f_3, \bar{f}_3, f_e, f_{\bar{e}}\}$, $\mathbf{F}_{\gamma_2}(v^e) = \mathbf{F}_2 \bigcup \{\bar{f}_1, f_2, f^e\}$ and $\mathbf{F}_{\gamma_2}(v^{\bar{e}}) = \overline{\mathbf{F}}_2 \bigcup \{f_1, \bar{f}_2, f^{\bar{e}}\}$. See Figure 9.2 for $|v| = 6$.

Then, we define

$$\mathcal{R}(\gamma, o; v, f_1, f_2, f_3) := \sum_{(\gamma_1, o_1)} [\overline{C}_{(\gamma_1, o_1)}] - \sum_{(\gamma_2, o_2)} [\overline{C}_{(\gamma_2, o_2)}]. \tag{9.6}$$

Here summation is taken over all possible $(\gamma_i, o_i), i = 1, 2$ defined above.

Figure 9.2: The o-planar tree (γ, o) with $|v| = 6$ and o-planar trees $(\gamma_i, o_i), i = 1, 2$ which appear in $\mathcal{R}(\gamma, o; v, f_1, f_2, f_3)$ given in (9.6).

$\mathfrak{S}$-2. Degeneration at the real vertex of type 2 o-planar trees. Consider an o-planar tree (γ, o) of type 2 such that $|\mathbf{E}_\gamma| = d - 2$. Let v be its real vertex, and assume that $|v| \geq 6$. Let $f_i \bar{f}_i \in \mathbf{F}_\gamma(v)$ be conjugate pairs of flags for $i = 1, 2, 3$. Put $\mathbf{F} = \mathbf{F}_\gamma(v) \setminus \{f_1, f_2, f_3, \bar{f}_1, \bar{f}_2, \bar{f}_3\}$.

We define two o-planar trees $(\gamma_1, o_1), (\gamma_2, o_2)$ as follows.

The first one (γ_1, o_1) is obtained by inserting a pair of conjugate edges $e = (f_e, f^e), \bar{e} = (f_{\bar{e}}, f^{\bar{e}})$ to (γ, o) at v with boundaries $\boldsymbol{\partial}_{\gamma_1}(e) = \{\tilde{v}, v^e\}$, $\boldsymbol{\partial}_{\gamma_1}(\bar{e}) = \{\tilde{v}, v^{\bar{e}}\}$ The set flags are given by $\mathbf{F}_{\gamma_1}(\tilde{v}) = \mathbf{F}_1 \bigcup \{f_1, \bar{f}_1, f_e, f_{\bar{e}}\}$, $\mathbf{F}_{\gamma_1}(v^e) = \mathbf{F}_2 \bigcup \{f_2, f_3, f^e\}$ and $\mathbf{F}_{\gamma_1}(v^{\bar{e}}) = \overline{\mathbf{F}}_2 \bigcup \{\bar{f}_2, \bar{f}_3, f^{\bar{e}}\}$, where $\mathbf{F}$ is a disjoint union of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\overline{\mathbf{F}}_2$. The $\mathbf{F}_2$ contains the flags that are conjugated that are conjugate to flags in $\overline{\mathbf{F}}_2$ and vice versa.

The second one (γ_2, o_2) is also obtained by inserting a pair of conjugate edges at the vertex v same way, but the flags are distributed differently on vertices $\mathbf{F}_{\gamma_2}(\tilde{v}) = \mathbf{F}_2 \bigcup \{f_3, \bar{f}_3, f_e, f_{\bar{e}}\}$, $\mathbf{F}_{\gamma_2}(v^e) = \mathbf{F}_1 \bigcup \{f_1, f_2, f^e\}$ and $\mathbf{F}_{\gamma_2}(v^{\bar{e}}) = \overline{\mathbf{F}}_2 \bigcup \{\bar{f}_1, \bar{f}_2, f^{\bar{e}}\}$.

Then, we define

$$\mathcal{R}(\gamma, o; v, f_1, f_2, f_3) := \sum_{(\gamma_1, o_1)} [\overline{C}_{(\gamma_1, o_1)}] - \sum_{(\gamma_2, o_2)} [\overline{C}_{(\gamma_2, o_2)}], \tag{9.7}$$

Here summation is taken over all possible $(\gamma_i, o_i), i = 1, 2$ defined above.

$\mathfrak{S}$-3. Degeneration at a conjugate pair of vertices. Consider an o-planar tree (γ, o) of type 2 or type 3 such that $|\mathbf{E}_\gamma| = d - 2$, and a pair of its conjugate vertices $\{v, \bar{v}\} \subset \mathbf{V}_\gamma \setminus \mathbf{V}_\gamma^{\mathbb{R}}$ with $|v| = |\bar{v}| \geq 4$. Let $\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4 \in \mathbf{F}_\gamma(\bar{v})$ be the flags conjugate to $f_1, f_2, f_3, f_4 \in \mathbf{F}_\gamma(v)$. Put $\mathbf{F} = \mathbf{F}_\gamma(v) \setminus \{f_1, f_2, f_3, f_4\}$. Let $\mathbf{F}_1, \mathbf{F}_2$ be two disjoint subsets of $\mathbf{F}$ such that $\mathbf{F} = \mathbf{F}_1 \bigcup \mathbf{F}_2$. Let $\overline{\mathbf{F}}_1, \overline{\mathbf{F}}_2$ be the sets of flags conjugate to the flags in $\mathbf{F}_1, \mathbf{F}_2$ respectively.

We define two o-planar trees $(\gamma_1, o_1), (\gamma_2, o_2)$.

The first one (γ_1, o_1) is obtained by inserting a pair of conjugate edges $e = (f_e, f^e), \bar{e} = (f_{\bar{e}}, f^{\bar{e}})$ to (γ, o) respectively at $v, \bar{v}$ such that $\boldsymbol{\partial}_{\gamma_1}(e) = \{v_e, v^e\}$, $\boldsymbol{\partial}_{\gamma_1}(\bar{e}) = \{v_{\bar{e}}, v^{\bar{e}}\}$. The set of flags are $\mathbf{F}_{\gamma_1}(v_e) = \mathbf{F}_1 \bigcup \{f_1, f_2, f_e\}$, $\mathbf{F}_{\gamma_1}(v^e) = \mathbf{F}_2 \bigcup \{f_3, f_4, f^e\}$ and $\mathbf{F}_{\gamma_1}(v_{\bar{e}}) = \overline{\mathbf{F}}_1 \bigcup \{\bar{f}_1, \bar{f}_2, f_{\bar{e}}\}$, $\mathbf{F}_{\gamma_1}(v^{\bar{e}}) = \overline{\mathbf{F}}_2 \bigcup \{\bar{f}_3, \bar{f}_4, f^{\bar{e}}\}$.

The second one (γ_2, o_2) is also obtained by inserting a pair of conjugate edges to (γ, o) at the same vertices $v, \bar{v}$, but the flags are distributed differently on vertices $\boldsymbol{\partial}_{\gamma_2}(e) = \{v_e, v^e\}, \boldsymbol{\partial}_{\gamma_2}(\bar{e}) = \{v_{\bar{e}}, v^{\bar{e}}\}$: $\mathbf{F}_{\gamma_2}(v_e) = \mathbf{F}_1 \bigcup \{f_1, f_3, f_e\}$, $\mathbf{F}_{\gamma_2}(v^e) = \mathbf{F}_2 \bigcup \{f_2, f_4, f^e\}$ and $\mathbf{F}_{\gamma_2}(v_{\bar{e}}) = \overline{\mathbf{F}}_1 \bigcup \{\bar{f}_1, \bar{f}_3, f_{\bar{e}}\}$, $\mathbf{F}_{\gamma_2}(v^{\bar{e}}) = \overline{\mathbf{F}}_2 \bigcup \{\bar{f}_2, \bar{f}_4, f^{\bar{e}}\}$.

Then, we define

$$\mathcal{R}(\gamma, o; v, f_1, f_2, f_3, f_4) := \sum_{(\gamma_1, o_1)} [\overline{C}_{(\gamma_1, o_1)}] - \sum_{(\gamma_2, o_2)} [\overline{C}_{(\gamma_2, o_2)}]. \tag{9.8}$$

Here summation is taken over all possible $(\gamma_i, o_i), i = 1, 2$ defined above.

$\mathfrak{S}$-4. Reversing o-planar structure at a vertex Let (γ, o) be an o-planar tree of type 3, and $(\gamma, \bar{o})$ be the o-planar tree with opposite o-planar structure.

Then, we define

$$\mathcal{R}(\gamma, o; v) = [\overline{C}_{(\gamma, o)}] - (-1)^\mu [\overline{C}_{(\gamma, \bar{o})}] \tag{9.9}$$

where

$$\mu = |\mathbf{F}_\gamma^+(v)| + \sum_{v_j \in \mathbf{V}_\gamma^+} (|v_j| - 3). \tag{9.10}$$

Remark 9.1.2. We identify $C_{(\gamma,o)}$ and $C_{(\gamma,\bar{o})}$ with $C_{(\gamma,u)}$ in $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$, and pick the coordinates defined in Appendix A by using o-planar structures $o_i, i = 1, 2$. These coordinates can be transformed to each other by

$$\prod_{v_j \in \mathbf{V}_\gamma^+} \overline{M}_{\mathbf{F}_\gamma(v_j)} \to \prod_{v_j \in \mathbf{V}_\gamma^+} \overline{M}_{\mathbf{F}_\gamma(\bar{v}_j)}.$$

The transformation from $\overline{M}_{\mathbf{F}_\gamma(v_j)}$ to $\overline{M}_{\mathbf{F}_\gamma(\bar{v}_j)}$ gives $\Omega_{\gamma_{v_j}} = (-1)^{\mu_2}\Omega_{\gamma_{\bar{v}_j}}$ where $\mu_2 = |v_j| - 3$. This follows from a direct calculation similar to $\mathfrak{R}$-3.

Therefore, the difference $\mathcal{R}(\gamma, o_1; v) = [\overline{C}_{(\gamma,o_1)}] - (-1)^{\mu}[\overline{C}_{(\gamma,o_2)}]$ is indeed zero when $\mu = \mu_1 + \mu_2$ as given in (9.10).

9.1.1 The boundary homomorphism of the graph complex

We define the *graph complex* $\mathcal{G}_\bullet$ *of the moduli space* $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ by introducing the boundary map $\partial : \mathcal{G}_d \to \mathcal{G}_{d-1}$:

$$\partial : [\overline{C}_{(\tau,o)}] \mapsto \sum_{(\gamma,\hat{o})<(\tau,o)} \pm\, [\overline{C}_{(\gamma,\hat{o})}]. \tag{9.11}$$

Here, summation is taken over all o-planar trees $(\gamma, \hat{o})$ that give (τ, o) after contracting one of their real edges. .

9.2 Homology of the graph complex

Theorem 9. *The homology of the graph complex* $\mathcal{G}_\bullet$ *is isomorphic to the singular homology of* $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$.

Proof. First, we note that the statement directly follows when $\sigma = \mathbf{id}$. In this case, each stratum $\overline{C}_{(\gamma,u)}$ is a disc and attached to finitely many lower dimensional strata i.e, the stratification of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ is a cell decomposition. The codimension of a stratum $\overline{C}_{(\gamma,u)}$ is equal to $|\mathbf{E}_\gamma|$. Therefore, the d^{th}-skeleton of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ contains the strata $\overline{C}_{(\gamma,u)}$ where $|\mathbf{E}_\gamma| = |\mathbf{S}| - d - 3$. Alternatively, we take the fundamental classes $[\overline{C}_{(\gamma,o)}]$ for all possible o-planar representatives of u-planar trees (γ, u), and identify them according to the relations given in $\mathfrak{R}$-3. The differential of this cell complex is clearly given by (9.11) since a stratum $\overline{C}_{(\gamma,\hat{o})}$ is in the codimension one boundary of another stratum $\overline{C}_{(\tau,o)}$ if and only if $(\gamma, \hat{o})$ produces (τ, o) by contracting one its edges (see Proposition 5.3.4). Therefore, the graph complex $\mathcal{G}_\bullet$ of $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ for $\sigma = \mathbf{id}$ is a cell complex of this moduli space.

Similarly, the stratifications of the moduli spaces $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ are cell decompositions for any involution σ when $|\mathbf{S}| = 4$ (see Example 5.3.6). By using the way, we obtain that the graph complexes are cell complexes for these cases.

We prove the statement for $\sigma \neq \mathbf{id}$ by induction on the cardinality of $\mathbf{Perm}(\sigma)$.

Let $\pi : \mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}} \to \mathbb{R}\overline{M}^{\sigma'}_{\mathbf{S}'}$ be the map forgeting $s, \bar{s} \in \mathbf{Perm}(\sigma)$. Here, we use the notations introduced in Section 8.1.

Let B_d denote the union of d-dimensional strata of $\mathbb{R}\overline{M}^{\sigma'}_{\mathbf{S}'}$ (i.e, $\bigcup \overline{C}_{(\gamma,u)}$ where $|\mathbf{E}_\gamma| = |\mathbf{S}'| - d - 3$). Let $\mathbb{R}\overline{M}^{\sigma'}_{\mathbf{S}'}$ be filtered by

$$\emptyset = B_{-1} \subset B_0 \subset \cdots \subset B_{(|\mathbf{S}'|-4)} \subset B_{(|\mathbf{S}'|-3)} = \mathbb{R}\overline{M}^{\sigma'}_{\mathbf{S}'}.$$

The forgetful map π induces a filtration of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$:

$$\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_{(|\mathbf{S}'|-4)} \subset E_{(|\mathbf{S}'|-3)} = \mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$$

where $E_d = \pi^{-1}(B_d)$. Then, the spectral sequence of double complex gives us

$$\mathbb{E}^1_{p,q} = H_{p+q}(E_p, E_{p-1}) \Longrightarrow H_{p+q}(\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}; \mathbb{Z}). \tag{9.12}$$

We prove the theorem by writing down this spectral sequence explicitly. As a first step, we calculate the homology groups $H_{p+q}(E_p, E_{p-1})$.

From now on, we assume that the statement of the theorem holds for $\mathbb{R}\overline{M}^{\sigma'}_{\mathbf{S}'}$.

Step 1. We can write homology of (E_p, E_{p-1}) as a direct sum of the homology of its pieces:

$$H_{p+q}(E_p, E_{p-1}) = \bigoplus_{(\gamma,u):|\mathbf{E}_\gamma|=|\mathbf{S}'|-d-3} H_{p+q}(\pi^{-1}(\overline{C}_{(\gamma,u)}), \pi^{-1}(Q_{(\gamma,u)})).$$

Consider the following filtration of $\pi^{-1}(\overline{C}_{(\gamma,u)})$:

$$\emptyset \subset Y_0 \subset Y_1 \subset Y_2 = \pi^{-1}(\overline{C}_{(\gamma,u)})$$

where Y_j's are the unions of strata

$$Y_0 = \bigcup_{(\gamma^*_m,u^*_m)} \overline{C}_{(\gamma^*_m,u^*_m)}, \quad Y_1 = \bigcup_{(\zeta^*_l,u^*_l)} \overline{C}_{(\zeta^*_l,u^*_l)}, \quad Y_2 = \bigcup_{(\tau^*_k,u^*_k)} \overline{C}_{(\tau^*_k,u^*_k)}$$

such that π maps each stratum onto $\overline{C}_{(\gamma,u)}$ and the dimension of the fibers of π is j:

By using this filtration, we obtain the following spectral sequence.

$$\mathbb{Y}^1_{i,j} = H_{i+j}(Y_i, Y_{i-1} \bigcup (Y_i \cap \pi^{-1}(Q_{(\gamma,u)}))) \Longrightarrow H_{i+j}(E_p, E_{p-1}).$$

Clearly, Y_i contains strata of dimension $p+i$, and $Y_i \cap \pi^{-1}(Q_{(\gamma,u)})$ contains the substrata that maps to B_{p-1} (i.e, substrata of codimension one or higher in $\overline{C}_{(\gamma^*,u^*)}$).

Hence, we have

$$\begin{aligned}
\mathbb{Y}^1_{0,j} &= \bigoplus_{(\gamma^*_m, u^*_m)} H_j(\overline{C}_{(\gamma^*_m,u^*_m)}, Q_{(\gamma^*_m,u^*_m)}),\\
\mathbb{Y}^1_{1,j} &= \bigoplus_{(\zeta^*_l, u^*_l)} H_{j+1}(\overline{C}_{(\zeta^*_l,u^*_l)}, Q_{(\zeta^*_l,u^*_l)}),\\
\mathbb{Y}^1_{2,j} &= \bigoplus_{(\tau^*_k, u^*_k)} H_{j+2}(\overline{C}_{(\tau^*_k,u^*_k)}, Q_{(\tau^*_k,u^*_k)}).
\end{aligned}$$

By using Lemma 8.2.1 (and isomorphism between relative homology and homology with closed support), we can have the groups $\mathbb{Y}^1_{i,j}$ as products of homology groups of base and fiber. The dimension of the fibers $\overline{F}_s(u^*_m)$ of $\pi : \overline{C}_{(\gamma^*_m,u^*_m)} \to \overline{C}_{(\gamma,u)}$ is zero, hence

$$\mathbb{Y}^1_{0,j} = \bigoplus_{(\gamma^*_m, u^*_m)} H_j(\overline{C}_{(\gamma,u)}, Q_{(\gamma,u)}) \otimes H^c_0(F_s(u^*_m)).$$

The dimension of the fibers $\overline{F}_s(u^*_l)$ of $\pi : \overline{C}_{(\zeta^*_l,u^*_l)} \to \overline{C}_{(\gamma,u)}$ is one, hence

$$\mathbb{Y}^1_{1,j} = \bigoplus_{(\zeta^*_l, u^*_l)} H_j(\overline{C}_{(\gamma,u)}, Q_{(\gamma,u)}) \otimes H^c_1(\overline{F}_s(u^*_l)).$$

Finally, the dimension of the fibers $\overline{F}_s(u^*_k)$ of $\pi : \overline{C}_{(\tau^*_0,u^*_k)} \to \overline{C}_{(\gamma,u)}$ is zero, hence

$$\begin{aligned}
\mathbb{Y}^1_{2,j} &= \bigoplus_{(\tau^*_k, u^*_k)} H_j(\overline{C}_{(\gamma,u)}, Q_{(\gamma,u)}) \otimes H^c_2(F_s(u^*_k)),\\
\mathbb{Y}^1_{2,j-1} &= \bigoplus_{(\tau^*_k, u^*_k)} H_j(\overline{C}_{(\gamma,u)}, Q_{(\gamma,u)}) \otimes H^c_1(F_s(u^*_k)),
\end{aligned}$$

Then, the differential $d_1 : \mathbb{Y}^1_{2,j} \to \mathbb{Y}^1_{1,j}$ and $d_1 : \mathbb{Y}^1_{1,j} \to \mathbb{Y}^1_{0,j}$ are respectively given by the differentials

$$\begin{aligned}
\partial_* :\ & H^c_2(F_s(u^*_k)) \to H^c_1(F_s(u^*_l)),\\
\partial_* :\ & H^c_1(F_s(u^*_1)) \to H_0(F_s(u^*_m)).
\end{aligned} \tag{9.13}$$

The differential d_1 maps the fundamental class $[\overline{C}_{(\gamma^*/e,o^*)}]$ to $\pm[\overline{C}_{(\gamma^*,\delta(o^*))}]$ (see Lemma 8.1.2).

Finally, the differential $d_2 : \mathbb{Y}^1_{2,j} \to \mathbb{Y}_{0,j+1}$ is given by the differentials

$$\partial_* :\ H^c_1(F_s(u^*_k)) \to H_0(F_s(u^*_m)). \tag{9.14}$$

For each pair of points in $\overline{F}_s(u^*_m)$, there is a generator in $H^c_1(F_s(u^*_k))$ whose image under ∂_* gives the difference of these points (see Section 8.1.2). Therefore, each pair of strata $\overline{C}_{(\gamma^*_{m_1},u^*_{m_1})}$ $\overline{C}_{(\gamma^*_{m_2},u^*_{m_2})}$ that are zero dimensional fibrations over $C_{(\gamma,u)}$ are homologous relative to $\pi^{-1}Q_{(\gamma,u)}$ i.e.,

$$[\overline{C}_{(\gamma^*_{m_1},u^*_{m_1})}]-[\overline{C}_{(\gamma^*_{m_2},u^*_{m_2})}]=0. \tag{9.15}$$

It is important to note that, the kernel of the differential d_2 is trivial. This follows from the fact that same is true for ∂_* given in (9.14). It is a consequence of the relations of the homology of the fibers given in (8.2) and (8.3). Therefore, the homology $H_*(E_p,E_{p-1})$ is given by the total homology of the spectral sequence $(\mathbb{Y}_{i,j}/I_0,d_1)$ where the ideal is generated by the relations (9.15).

Instead of considering u-planar trees, from now on, we pick all o-planar representatives of u-planar trees and impose the relation $\mathfrak{R}$-3 given in (9.4) and $\mathfrak{S}$-4 given in (9.9) arising from reversing o-planar structures.

Step 2. The calculations in Step 1 imply that the $\mathbb{E}^1_{**}$ is generated by the relative fundamental classes of the strata. Moreover, it admits the relations that are imposed in the definition of $\mathcal{G}_\bullet$:

The chains defined in $\mathfrak{R}$-1 and $\mathfrak{R}$-2 (resp. $\mathfrak{S}$-1, $\mathfrak{S}$-2 and $\mathfrak{S}$-3) with $f_i\neq s$ are mapped onto the relation between the strata of $\mathbb{R}\overline{M}^{\sigma'}_{\mathbf{S}'}$ of same type.

On the other hand, the relations with $f_1=s$ come as consequence of the calculation of Step 1. For each relation (9.15) in relative homology $H_*(E_p,E_{p-1})$, there is a relation in $H_*(\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}})$. The sums cycles (9.2), (9.3), (9.6), (9.7) and (9.8) in $H_*(E_p)$ are mapped onto the difference given by (9.15) in $H_*(E_p,E_{p-1})$ since that the valency of the vertex v_s supporting s must be three. If otherwise, forgetting the tail doesn't require contraction and the o-planar trees obtained by forgetting $s,\bar{s}$ have two additional edges. Therefore, they are codimension one or higher, and lie in E_{p-1}.

We need to confirm that the sums defined in $\mathfrak{R}$-1 and $\mathfrak{R}$-2 (resp. $\mathfrak{S}$-1, $\mathfrak{S}$-2 and $\mathfrak{S}$-3) are indeed homologous to zero. We can show this by using certain forgetful maps. Here, we are going to show it for $\mathfrak{R}$-1 and $\mathfrak{R}$-2. The other cases are essentially the same.

The relations of type $\mathfrak{R}$-1. Consider the composition of projection $\overline{C}_{(\gamma^*,o^*)}\to\overline{C}_{(\gamma^*_v,o^*_v)}$ onto the factor corresponding to vertex v, and forgetful maps $\overline{C}_{(\gamma^*_v,o^*_v)}\to\overline{C}_{(\tau^*_v,o(v^*))}$ where $(\tau^*_v,o(v^*))$ is an one-vertex o-planar tree with $\mathbf{F}^+_{\tau^*}=\{f_1,f_2\}$ and $\mathbf{F}^{\mathbb{R}}_{\tau^*}=\{f_3\}$ which is obtained by forgetting all tails but $f_1,\bar{f}_1,f_2,\bar{f}_2,f_3$.

The space $\overline{C}_{(\tau^*_v,o(v^*))}$ is a two-dimensional disc with a puncture, and it is stratified as in Figure 9.3.

If a codimension two stratum $\overline{C}_{(\gamma^*_1,o^*_1)}$ of $\overline{C}_{(\gamma^*,o^*)}$ is in the fiber over a codimension two stratum lying in boundary of $\overline{C}_{(\tau^*_v,o(v^*))}$ (see, Figure 9.3), then (γ^*_1,o^*_1) is obtained from (γ^*,o^*) by inserting a pair of real edges at vertex v. Similarly, if a codimension two stratum $\overline{C}_{(\gamma^*_2,o^*_2)}$ of $\overline{C}_{(\gamma^*,o^*)}$ is in the fiber over a codimension two stratum lying

Figure 9.3: The strata of $\overline{C}_{(\tau_v^*, o_v^*)}$.

inside of $\overline{C}_{(\tau_v^*, o(v^*))}$ (see, Figure 9.3), then (γ_2^*, o_2^*) is obtained from (γ^*, o^*) by inserting a pair of conjugate edges at vertex v as above. Since $\overline{C}_{(\tau_v^*, o(v^*))}$ is a punctured disc, the fibers of forgetful map over any two points of $\overline{C}_{(\tau_v^*, o(v^*))}$ are homologous i.e. $\mathcal{R}(\gamma, o; v, f_1, f_2, f_3)$ is homologous to zero.

The relations of type $\mathfrak{R}$-2. The relations in (9.3) are obtain in a similar way with $\mathfrak{R}$-1. Here, we use a projection map $\overline{C}_{(\gamma,o)} \to \overline{M}_{\mathbf{F}_\gamma(v)}$. The relations in the complex moduli space $\overline{M}_{\mathbf{F}_\gamma(v)}$ are given by Kontsevich and Manin (see, Section 2.6.1) give the relations in above.

Step 3. We have a complete description of generators and relations in $\mathbb{E}^1$. We need to calculate the differentials.

The first differential $d_1 : \mathbb{E}^1_{p,q} \to \mathbb{E}^1_{p-1,q}$ is given by

$$d_1 : [\overline{C}_{(\gamma^*,\hat{o}^*)}] \mapsto \sum_{\substack{(\tau^*,o^*)\in \mathbf{G}_{(\tau,o)};\gamma/e=\tau \text{ for } e\in\mathbf{E}^{\mathbb{R}}_{\gamma} \\ \gamma^*/e^*=\tau^* \text{ for } e^*\in\mathbf{E}^{\mathbb{R}}_{\gamma^*}}} \pm[\overline{C}_{(\tau^*,o^*)}]. \tag{9.16}$$

In order to complete the proof, we only need to show that the higher differentials d_2 and d_3 of $\mathbb{E}_{**}$ vanish.

Consider the strata of type 1 o-planar trees. Due dimensional reasons,the differential d_2 is zero except $d_2 : \mathbb{E}^1_{p,1} \to \mathbb{E}^1_{p-2,2}$ and $d_2 : \mathbb{E}^1_{p,0} \to \mathbb{E}^1_{p-2,1}$. Due to Lemma 8.1.1,

I if $[\overline{C}_{(\zeta_l^*,o_l^*)}] \in \mathbb{E}_{p,1}$, then $v_s \in \mathbf{V}^{\mathbb{R}}_{\zeta_l^*}$ and either $|v_s| = 4$, or $|v_s| = 3$ and $|v_c| \geq 4$;

II if $[\overline{C}_{(\gamma_m^*,o_m^*)}] \in \mathbb{E}_{p,0}$, then either $v_s \in \mathbf{V}^{\mathbb{R}}_{\gamma_m^*}$ and $|v_s| = |v_c| = 3$, or $v_s \notin \mathbf{V}^{\mathbb{R}}_{\gamma_m^*}$ and $|v_s| = 3$.

Assume that

$$d_2([\overline{C}_{(\zeta_l^*, o_l^*)}]) = \sum \pm[\overline{C}_{(\tau_k^*, o_k^*)}]$$

for $[\overline{C}_{(\zeta_l^*, o_l^*)}] \in \mathbb{E}^1_{p,1}$. Then, each o-planar trees (τ_k^*, o_k^*) must produce (ζ_l^*, o_l^*) by contracting one of its real edges due to Theorem 4 and Proposition 5.3.4. On the other hand, the vertex $v_s \in \mathbf{V}_{\tau_k^*}$ must be $|v_s| \geq 5$ if it is a real vertex, and $|v_s| \geq 4$ if it is not a real vertex (see Lemma 8.1.1). The contraction of an real edge of (τ_k^*, o_k^*) increases or preserves the valency of the vertex v_s. This contradicts with condition I above. Hence, $d_2 : \mathbb{E}^1_{p,1} \to \mathbb{E}^1_{p-2,2}$ must be zero.

Assume that

$$d_2([\overline{C}_{(\gamma_m^*, o_m^*)}]) = \sum \pm[\overline{C}_{(\zeta_l^*, o_l^*)}]$$

for $[\overline{C}_{(\gamma_m^*, o_m^*)}] \in \mathbb{E}^1_{p,0}$. Then, each o-planar tree (ζ_l^*, o_l^*) must produce (γ_m^*, o_m^*) by contracting one of its real edges due to Theorem 4 and Proposition 5.3.4. On the other hand, the vertex $v_s \in \mathbf{V}_{\tau_k^*}$ must be a real vertex and $|v_s| = 4$ or $|v_s| = 3$ and $|v_c| > 3$. The contraction of an real edge of (ζ_l^*, o_l^*) increases or preserves the valencies of the vertices v_s and v_c. This contradicts with condition II above. Hence, $d_2 : \mathbb{E}^1_{p,0} \to \mathbb{E}^1_{p-2,1}$ must be zero.

It remains to check the differential $d_3 : \mathbb{E}^1_{p,0} \to \mathbb{E}^1_{p-3,2}$.

Assume that

$$d_3([\overline{C}_{(\gamma_m^*, o_m^*)}]) = \sum \pm[\overline{C}_{(\tau_k^*, o_k^*)}]$$

for $[\overline{C}_{(\gamma_m^*, o_m^*)}] \in \mathbb{E}^1_{p,0}$. Then, the o-planar tree (τ_k^*, o_k^*) must produce (γ_m^*, o_m^*) by contracting one of its real edges due to Theorem 4 and Proposition 5.3.4. On the other hand, the vertex $v_s \in \mathbf{V}_{\tau_k^*}$ must be $|v_s| \geq 5$ if it is a real vertex, and $|v_s| \geq 4$ if it is not a real vertex. The contraction of an real edge of (τ_k^*, o_k^*) increases or preserves the valency of the vertex v_s. This contradicts with condition I above. Hence, $d_3 : \mathbb{E}^1_{p,0} \to \mathbb{E}^2_{p-3,1}$ must be zero.

The images of the strata of type 2 and type 3 o-planar trees can be check in a same way. The same arguments show that the differentials d_2 and d_3 are zero. □

Chapter 10

Fundamental groups of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ and $\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}}$

In this chapter, we give presentations of the fundamental groups of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ and $\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}}$ by using the groupoid of paths transversal to codimension one strata. This idea has been used by Kamnitzer and Henriques in [18] to calculate the fundamental group of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ for $\sigma = \mathbf{id}$. This section extends their description to $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ and $\mathbb{R}\widetilde{M}^{\sigma}_{\mathbf{S}}$ for any σ.

10.1 Fundamental groups of open parts of strata

Here, we consider a particular subset of the set of o-planar trees. Let $\mathcal{R}Tree(\sigma)$ be the set of o-planar trees having no conjugate pair of edges. If $(\gamma, o) \in \mathcal{R}Tree(\sigma)$ is of type 1, then $\mathbf{V}_{\gamma} = \mathbf{V}^{\mathbb{R}}_{\gamma}$, if $(\gamma, o) \in \mathcal{R}Tree(\sigma)$ is of type 2, then $|\mathbf{V}_{\gamma}| = 1$, and if $(\gamma, o) \in \mathcal{R}Tree(\sigma)$ is of type 3, then $|\mathbf{V}_{\gamma}| = 2$.

For an o-planar tree $(\gamma, o) \in \mathcal{R}Tree(\sigma)$, the *open part* $O_{(\gamma,o)}$ of the stratum is the closed stratum $\overline{C}_{(\gamma,o)}$ minus the union of the closure of its codimension one strata.

Proposition 10.1.1. *For $(\gamma, o) \in \mathcal{R}Tree(\sigma)$, the open part of the stratum $\overline{C}_{(\gamma,o)}$ is simply connected.*

Proof. For $(\gamma^*, o^*) \in \mathcal{R}Tree(\sigma)$, we have

$$\overline{C}_{(\gamma^*,o^*)} = \begin{cases} \prod_{v \in \mathbf{V}_{\gamma^*}} \overline{C}_{(\gamma^*_v, o^*_v)} & \text{if } o^* \text{ is of type 1,} \\ \overline{C}_{(\gamma^*_{v_r}, o^*_{v_r})} & \text{if } o^* \text{ is of type 2,} \\ \overline{M}_{\mathbf{F}_{\gamma}(v)} & \text{if } o^* \text{ is of type 3} \end{cases}$$

(see Section 5.3). The open part $O_{(\gamma^*,o^*)}$ of $\overline{C}_{(\gamma^*,o^*)}$ is the product of the open parts of its factors given above. Hence, we only need to consider the factors that are corresponding to the one-vertex trees.

We prove the statement by induction on the cardinality of $\mathbf{Perm}(\sigma)$. First, we note that the open part of the strata of $\mathbb{R}\overline{M}_{\mathbf{S}'}^{\sigma'}$ are contractible for $\sigma(\mathbf{S}') = \mathbf{S}'$, $|\mathbf{S}'| = 3$ and $|\mathbf{Fix}(\sigma)| = 1$, $|\mathbf{S}'| = 4$ and $|\mathbf{Fix}(\sigma)| = 2$, and $|\mathbf{S}'| = 4$ and $|\mathbf{Fix}(\sigma)| = 0$. In these cases, the stratification is a cell decomposition, and the open parts of the strata are open discs (see [7, 21] and Example 5.3.6).

Let (γ^*, o^*) be a one-vertex o-planar tree of type 1. Let $|\mathbf{Perm}^+| > 0$, and $\pi : \overline{C}_{(\gamma^*,o^*)} \to \overline{C}_{(\gamma,o)}$ be the map forgetting map the conjugate pairs of points $s, \bar{s}$. Let O be a subset of the fiber $\overline{F}_s = \pi^{-1}(\Sigma; \mathbf{p})$ such that $(\Sigma^*, \mathbf{p}^*) \in O$ does not require any stabilization forgetting $s, \bar{s}$. For $(\Sigma^*, \mathbf{p}^*) \in O$, $\Sigma^* = \Sigma$. Since all special points are fixed in Σ^*, the different points of O are given by the position of the labelled point s. The labelled point s is in $(\Sigma \setminus (\{\text{special points}\} \bigcup \mathbb{R}\Sigma))/c_\Sigma$. This follows from the fact that all special points must be distinct (hence, we need to remove special points and $\mathbb{R}\Sigma$ where s and $\bar{s}$ collide and give a real node) and s in either $\mathbf{F}^+_{\gamma^*}$ or $\mathbf{F}^-_{\gamma^*}$ (so that, we need to take the quotient with respect to the real structure $c_\Sigma : \Sigma \to \Sigma$).

The degenerations of the curves $(\Sigma^*, \mathbf{p}^*) \in O$, which are obtain from limit s goes to a special point in $\Sigma \setminus \mathbb{R}\Sigma$, give us points in $O_{(\gamma^*,o^*)}$ since the limit elements have an additional conjugate pair of edges. On the other hand, the degeneration of the curves $(\Sigma^*, \mathbf{p}^*)$ obtain from limit s goes to a point in $\mathbb{R}\Sigma$ gives a curve with a real node i.e, the limits does not lie in $O_{(\gamma^*,o^*)}$. Therefore, the restriction of the forgetful map $\pi : O_{(\gamma^*,o^*)} \to O_{(\gamma,o)}$ has a fiber $(\Sigma \setminus \mathbb{R}\Sigma)/c_\Sigma$ over $(\Sigma; \mathbf{p}) \in O_{(\gamma,o)}$. It is clearly that the fiber is simply connected.

If we assume simply connectedness of $O_{(\gamma,o)}$, then $O_{(\gamma^*,o^*)}$ is clearly simply connected. We prove the statement by induction on the cardinality of labeling set $\mathbf{Perm}(\sigma)$.

The proofs for o-planar trees $(\gamma, o) \in \mathcal{R}Tree(\sigma)$ of type 2 and type 3 are the same with type 1 case. The fiber of the forgetful map $\pi : O_{(\gamma^*,o^*)} \to O_{(\gamma,o)}$ over $(\Sigma; \mathbf{p})$ is Σ when (γ^*, o^*) is of type 2, and Σ/c_Σ when (γ^*, o^*) is of type 3. In both cases, the fibers are simply connected. □

Let (γ, o) be an o-planar representative of the u-planar tree (γ, u). We define the open part of a stratum $\overline{C}_{(\gamma,u)}$ to be $O_{(\gamma,u)} := O_{(\gamma,o)}$. Note that so defined space $O_{(\gamma,u)}$ does not depend on the o-planar representative.

Proposition 10.1.2. *The moduli space $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ is stratified by simply connected subspaces $O_{(\gamma,u)}$.*

Proof. We only need to prove that the open parts $O_{(\gamma,u)}$ of the strata $\overline{C}_{(\gamma,u)}$ are pairwise disjoint.

First, we note that, if an u-planar tree $(\tau, \hat{u})$ produces (γ, u) be contracting only conjugate pairs of edges, then $C_{(\tau,\hat{u})}$ is contained in $O_{(\gamma,u)}$. This follows from Theorem 4 (b) and the definition of the open part of a stratum.

Let $(\gamma_1, u_1), (\gamma_1, u_1) \in \mathcal{R}Tree(\sigma)$. If two closed strata $\overline{C}_{(\gamma_i,u_i)}$ intersect, they intersect along the union of strata $\bigcup_{(\tau,\hat{u})} \overline{C}_{(\tau,\hat{u})}$ such that two o-planar trees $(\tau, \hat{o})$ gives (γ_i, o_i) by contracting a set of real edges (since by contracting conjugate pairs of edges we obtain strata contained in the same open part). However, the strata $\overline{C}_{(\tau,\hat{u})}$ are not contained in open part $O_{(\gamma_i,o_i)}$ since $(\tau, \hat{o})$ must have additional real edges and $\overline{C}_{(\tau,\hat{o})}$ contained in the union of codimension one strata. □

10.2 Groupoid of paths in $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$

Let's consider the following groupoid $\mathcal{P}$ of paths in $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$. Choose a point $(\Sigma(u), \mathbf{p}(u))$ in every connected component $C_{(\tau,u)}$ of $\mathbb{R}M_{\mathbf{S}}^{g}$. The objects Ob$(\mathcal{P})$ are these elements $(\Sigma(u), \mathbf{p}(u))$. The morphisms $\langle\gamma, \hat{u}\rangle_{u_1}^{u_2}$ in Hom$(\mathcal{P})$ are the homotopy classes of paths in $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ that connect the point $(\Sigma(u_1), \mathbf{p}(u_1)), (\Sigma(u_2), \mathbf{p}(u_2))$ through the common codimension 1 boundary $C_{(\gamma,\hat{u})}$ of the strata $\overline{C}_{(\tau,u_i)}, i = 1, 2$. Notice that such paths connecting $(\Sigma(u_1), \mathbf{p}(u_1)), (\Sigma(u_2), \mathbf{p}(u_2))$ are homotopic to each other since the open parts of $\overline{C}_{(\tau,u_i)}$'s are simply connected (see Proposition 10.1.1). The homotopy classes of paths that intersect with only codimension 1 strata are given by concatenations of paths $\langle\gamma, \hat{u}_i\rangle_{u_i}^{u_{i+1}}$.

Theorem 10. *The fundamental group $\pi_1(\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma})$ is presented by the loops*

$$\langle\gamma, \hat{u}_1\rangle_{u_1}^{u_2}\langle\gamma, \hat{u}_2\rangle_{u_2}^{u_3}\cdots\langle\gamma, \hat{u}_{n-1}\rangle_{u_{n-1}}^{u_n}\langle\gamma, \hat{u}_n\rangle_{u_n}^{u_1} \tag{10.1}$$

subject to the following relations relations:

- *For each $(\gamma, \hat{u})$ with $|\mathbf{E}_\gamma| = 1$,*

$$\langle\gamma, \hat{u}\rangle_{u_1}^{u_2}\langle\gamma, \hat{u}\rangle_{u_2}^{u_1} = 1. \tag{10.2}$$

- *For each (γ', u) with $|\mathbf{E}_{\gamma'}| = 2$*

$$\begin{aligned} &\langle\gamma, \hat{u}_1\rangle_{u_1}^{u_2}\ \langle\gamma, \hat{u}_2\rangle_{u_2}^{u_1} = 1 \\ &\langle\gamma, \hat{u}_1\rangle_{u_1}^{u_2}\ \langle\gamma, \hat{u}_2\rangle_{u_2}^{u_3}\ \langle\gamma, \hat{u}_3\rangle_{u_3}^{u_4}\ \langle\gamma, \hat{u}_4\rangle_{u_4}^{u_1} = 1 \end{aligned} \tag{10.3}$$

  where $(\gamma, \hat{u}_i)$ are the u-planar trees that are obtained by contracting an edge of (γ', u),

Proof. Every loop in $\mathbb{R}\overline{M}_{\mathbf{S}}^{\sigma}$ is homotopic a loop which is transversal to codimension one faces. Such transversal loops can be obtained by perturbing the original loops. Hence, we can choose the loops given in (10.1) as representatives of homotopy classes of loops.

These loops are subject to the following relations. The concatenation

$$\langle\gamma, \hat{u}\rangle_{u_1}^{u_2}\langle\gamma, \hat{u}\rangle_{u_2}^{u_1}$$

of $\langle\gamma, \hat{u}\rangle_{u_1}^{u_2}$ with the reverse path $\langle\gamma, \hat{u}\rangle_{u_2}^{u_1}$ is obviously homotopic to a point (see Fig. 10.1a) and gives the relation (10.2).

If two paths in $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$ are homotopic, then they are homotopic by a homotopy of paths that are transversal to the codimension one strata. Therefore, the homotopy relations arise from the passing the paths through codimension 2 strata: Let (γ', u) be a u-planar tree corresponding to a codimension two stratum of $\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}}$. The stratum $\overline{C}_{(\gamma',u)}$ is contained in two or four codimension one strata, since we can obtain two or four u-planar trees by contracting one of the two edges of (γ', u). Let $\overline{C}_{(\tau,u_i)}, i \in I$ intersect along codimension two stratum $\overline{C}_{(\gamma',u)}$. Therefore, the loops

$$\begin{array}{ll} \langle\gamma, \hat{u}_1\rangle_{u_1}^{u_2}\ \langle\gamma, \hat{u}_2\rangle_{u_2}^{u_1} & \text{if } I = \{1, 2\} \\ \langle\gamma, \hat{u}_1\rangle_{u_1}^{u_2}\ \langle\gamma, \hat{u}_2\rangle_{u_2}^{u_3}\ \langle\gamma, \hat{u}_3\rangle_{u_3}^{u_4}\ \langle\gamma, \hat{u}_4\rangle_{u_4}^{u_1} & \text{if } I = \{1, \cdots, 4\} \end{array} \tag{10.4}$$

around $\overline{C}_{(\gamma',u)}$ is contractible (see Fig. 10.1b and 10.1c) and give the relations (10.3).

Figure 10.1: (a) Concatenation of a path with its inverse, (b) and (c) Concatenations of paths around a codimension two stratum.

□

10.3 Groupoid of paths in $\mathbb{R}\widetilde{M}^{g}_{\mathbf{S}}$

The group $\pi_1(\mathbb{R}\widetilde{M}^{g}_{\mathbf{S}})$ can be given in a similar way to $\pi_1(\mathbb{R}\overline{M}^{\sigma}_{\mathbf{S}})$. Let $\tilde{\mathcal{P}}$ be the groupoid of paths in $\mathbb{R}\widetilde{M}^{g}_{\mathbf{S}}$ given as follows. Choose a point $(\Sigma(o), \mathbf{p}(o))$ in every top dimensional strata $C_{(\tau,o)} \subset \mathbb{R}\widetilde{M}^{g}_{\mathbf{S}}$ (i.e, $|\mathbf{V}_\tau| = 1$). The set of objects $\mathrm{Ob}(\tilde{\mathcal{P}})$ is $\{(\Sigma(o), \mathbf{p}(o)) \in C_{(\tau,o)} \mid |\mathbf{V}_\tau| = 1\}$. The morphisms $\langle\gamma, \hat{o}\rangle_{o_1}^{o_2}$ in $\mathrm{Hom}(\mathcal{P})$ are the homotopy classes of paths in $\mathbb{R}\widetilde{M}^{g}_{\mathbf{S}}$ that connect the point $(\Sigma(o_1), \mathbf{p}(o_1)), (\Sigma(o_2), \mathbf{p}(o_2))$ through the common codimension one boundary $C_{(\gamma,\hat{o})}$ of the strata $\overline{C}_{(\tau,o_i)}, i = 1, 2$. The concatenations of paths $\langle\gamma, \hat{o}_i\rangle_{o_i}^{o_{i+1}}$ give the homotopy classes of paths that meet only with codimension one strata.

Theorem 11. *The fundamental group $\pi_1(\widetilde{\mathbb{R}M^g_{\mathbf{S}}})$ is presented by the loops*

$$\langle\gamma,\hat{o}_1\rangle_{o_1}^{o_2}\langle\gamma,\hat{o}_2\rangle_{o_2}^{o_3}\cdots\langle\gamma,\hat{o}_{n-1}\rangle_{o_{n-1}}^{o_n}\langle\gamma,\hat{o}_n\rangle_{o_n}^{o_1} \tag{10.5}$$

subject to the following relations relations:

- *For each $(\gamma,\hat{o})$ with $|\mathbf{E}_\gamma|=1$,*

$$\langle\gamma,\hat{o}\rangle_{o_1}^{o_2}\langle\gamma,\hat{o}\rangle_{o_2}^{o_1}=1. \tag{10.6}$$

- *For each (γ',o) with $|\mathbf{E}_{\gamma'}|=2$*

$$\begin{aligned}&\langle\gamma,\hat{o}_1\rangle_{o_1}^{o_2}\ \langle\gamma,\hat{o}_2\rangle_{o_2}^{o_1}=1\\&\langle\gamma,\hat{o}_1\rangle_{o_1}^{o_2}\ \langle\gamma,\hat{o}_2\rangle_{o_2}^{o_3}\ \langle\gamma,\hat{o}_3\rangle_{o_3}^{o_4}\ \langle\gamma,\hat{o}_4\rangle_{o_4}^{o_1}=1\end{aligned} \tag{10.7}$$

  where $(\gamma,\hat{o}_i)$ are the o-planar trees that are obtained by contracting an edge of the all o-planar trees in R-equivalence class of (γ',o).

The proof this theorem is exactly the same with the proof of the Theorem 10. We will not repeat it here.

Appendix A

Orientations of the strata

Let (γ, o) be an o-planar tree. By choosing three flags in $\mathbf{F}_\gamma(v)$, and using the calibrations as in Section 5.1.2, we obtain a coordinate system in $C_{(\gamma_v, o_v)}$ for each $v \in \mathbf{V}_\gamma^{\mathbb{R}}$. More precisely, we use the following choice.

- Let v be a real vertex, o_v be a o-planar structure of type 1, and $\mathbf{Fix}(\sigma) \neq \emptyset$. In this case, there is a unique real flag f_m lying in the shortest path between vertices v and $\boldsymbol{\partial}_\gamma(s_n)$ since real locus $\mathbb{R}\Sigma$ is connected.
  - If $|\mathbf{F}_\gamma^{\mathbb{R}}(v)| \geq 3$, then we specify an isomorphism $\Phi_v : \Sigma_v \to \mathbb{CP}^1$ by mapping three consecutive special points $(p_{f_{j-1}}, p_{f_m}, p_{f_1})$ to $(1, \infty, 0)$.
  - If $|\mathbf{F}_\gamma^{\mathbb{R}}(v)| = 1, 2$, then we specify an isomorphism $\Phi_v : \Sigma_v \to \mathbb{CP}^1$ by mapping three special points $(p_{f_m}, p_{f_\alpha}, p_{f_{\bar{\alpha}}})$ to $(\infty, \sqrt{-1}, -\sqrt{-1})$ for an arbitrary $f_\alpha \in \mathbf{F}_\gamma^+(v)$.
- Let v be a real vertex, o_v be a o-planar structure of type 1, and $\mathbf{Fix}(\sigma) = \emptyset$.
  - If $|\mathbf{F}_\gamma^{\mathbb{R}}(v)| \geq 3$, first we pick an arbitrary real flag f_m. Then, we specify an isomorphism $\Phi_v : \Sigma_v \to \mathbb{CP}^1$ by mapping three consecutive special points $(p_{f_{j-1}}, p_{f_m}, p_{f_1})$ to $(1, \infty, 0)$.
  - If $|\mathbf{F}_\gamma^{\mathbb{R}}(v)| = 1, 2$, first we pick an arbitrary real flag f_m. Then, we specify an isomorphism $\Phi_v : \Sigma_v \to \mathbb{CP}^1$ by mapping three special points $(p_{f_m}, p_{f_\alpha}, p_{f_{\bar{\alpha}}})$ to $(\infty, \sqrt{-1}, -\sqrt{-1})$ for an arbitrary $f_\alpha \in \mathbf{F}_\gamma^+(v)$.
  - If $|\mathbf{F}_\gamma^{\mathbb{R}}(v)| = 0$ and $s_n \in \mathbf{F}_\gamma^+$ (resp. $\overline{s}_n \in \mathbf{F}_\gamma^+$), then there is a unique flag $f_\alpha \in \mathbf{F}_\gamma^+(v)$ lying in the shortest path to vertices v and $\boldsymbol{\partial}_\gamma(s_n)$ (resp. $\boldsymbol{\partial}_\gamma(\overline{s}_n)$). We specify an isomorphism $\Phi_v : \Sigma_v \to \mathbb{CP}^1$ by mapping four special points $(p_{f_\beta}, p_{f_\alpha}, p_{f_{\bar{\beta}}}, p_{\bar{f}_\beta})$ to $(\lambda\sqrt{-1}, -\lambda\sqrt{-1}, -\sqrt{-1})$ for an arbitrary $f_\alpha \in \mathbf{F}_\gamma^+(v)$.

- Let v_r be the real vertex, o_{v_r} be the o-planar structure of type 2. Then, there is a unique flag $f_\alpha \in \mathbf{F}^+_\gamma(v)$ lying in the shortest path to vertices v and $\boldsymbol{\partial}_\gamma(s_n)$ We specify an isomorphism $\Phi_v : \Sigma_v \to \mathbb{CP}^1$ by mapping four special points $(p_{f_\beta}, p_{f_\alpha}, p_{f_{\bar\beta}}, p_{\bar f_\beta})$ to $(\lambda\sqrt{-1}, -\lambda\sqrt{-1}, -\sqrt{-1})$ for an arbitrary $f_\alpha \in \mathbf{F}^+_\gamma(v)$.

Similarly, by choosing three flags in $\mathbf{F}_\gamma(v)$ for $v \notin \mathbf{V}^{\mathbb{R}}_\gamma$, we obtain a coordinate system in $\overline{M}_{\mathbf{F}_\gamma(v)}$. We use the following choice.

- Let $v, \bar v$ are a pair of conjugate vertices. Then, we specify isomorphisms $\Phi_v : \Sigma_v \to \mathbb{CP}^1$ and $\Phi_{\bar v} : \Sigma_v \to \mathbb{CP}^1$ by mapping three special points $(p_{f_{\alpha_1}}, p_{f_{\alpha_2}}, p_{f_{\alpha_2}})$ of Σ_v, and their conjugates $(p_{f_{\overline{\alpha}_1}}, p_{f_{\overline{\alpha}_2}}, p_{f_{\overline{\alpha}_2}})$ to $(0, 1, \infty)$.

Remark A.0.1. It is important to note that the choices above and in Section 6.2 give different normalized coordinates on codimension one strata. They can be transformed to each other by rational transformation.

In such a way, we obtain orientations $[\Omega_{(\gamma_v, o_v)}]$ for $v \in \mathbf{V}^{\mathbb{R}}_\gamma$, and $[\Omega_{\gamma_v}]$ for $v \notin \mathbf{V}^{\mathbb{R}}_\gamma$.

The product

$$[\Omega_{(\gamma,o)}] = \bigwedge_{v \in \mathbf{V}^{\mathbb{R}}_\gamma} [\Omega_{(\gamma_v, o_v)}] \wedge \bigwedge_{v \in \mathbf{V}^{+}_\gamma} [\Omega_{\gamma_v}]$$

gives an orientation of $C_{(\gamma,o)}$ i.e, determines the relative fundamental cycles $[\overline{C}_{(\gamma,o)}]$.

A.1 Boundary homomorphism

Let e be a real edge of an o-planar tree $(\gamma, \hat o)$, and $\boldsymbol{\partial}_\gamma(e) = \{v_1, v_2\}$. Let $\gamma \to \tau$ be the contraction of the edge e and v be image of the vertives v_1, v_2 under the contraction.

The orientation $[\Omega_{(\tau_v, o_v)}]$ induces an orientation

$$\begin{array}{ll} [\Omega_{(\gamma_{v_1}, o_{v_1})}] \wedge [\Omega_{(\gamma_{v_2}, o_{v_2})}] & \text{when } v_1, v_2 \in \mathbf{V}^{\mathbb{R}}_\gamma, \\ \left[\Omega_{\gamma_{v_1}}\right] & \text{when } v_1, v_2 \notin \mathbf{V}^{\mathbb{R}}_\gamma, \end{array}$$

on the boundary $C_{(\gamma_{v_1}, o_{v_1})} \times C_{(\gamma_{v_2}, o_{v_2})}$: Pick a point $(\Sigma^0, \mathbf{p}^0) \in C_{(\gamma_{v_1}, o_{v_1})} \times C_{(\gamma_{v_2}, o_{v_2})}$, and consider a tubular neighborhood $V \times [0, \epsilon[$ of $(\Sigma^0, \mathbf{p}^0)$ in $\overline{C}_{(\tau_v, o_v)}$ as in Section 5.3. The outward normal direction of $(\Sigma^0, \mathbf{p}^0) \in C_{(\gamma_{v_1}, o_{v_1})} \times C_{(\gamma_{v_2}, o_{v_2})}$ in $C_{(\tau_v, o_v)}$ is $-\partial_t$ where t is the standart coordinate on $[0, \epsilon[\subset \mathbb{R}$. Therefore, we define homomorphism

$$[\Omega_{(\tau_v, o_v)}] \mapsto \left\{ \begin{array}{l} \pm[\Omega_{(\gamma_{v_1}, o_{v_1})}] \wedge [\Omega_{(\gamma_{v_2}, o_{v_2})}] \\ \pm \left[\Omega_{\gamma_{v_1}}\right] \end{array} \right.$$

where the differential forms satisfy

$$\Omega_{(\tau_v o_v)} = -\Theta \; dt \wedge \Omega_{(\gamma_{v_1} o_{v_1})} \wedge \Omega_{(\gamma_{v_2} o_{v_2})} \tag{A.1}$$

for $\Theta > 0$ at all points of $V \times [0, \epsilon[$.

In order to determine the sign of $[\Omega_{(\tau_v, o_v)}]$, first we apply rational transformations to $C_{(\gamma_{v_i}, o_{v_i})}$ and put the point on Σ_{v_i} in a normalization position as in Section 6.2. Then, by appying the formulas for induced orientations in Section 6.2.1 we compare the signs of the orientations.

This gives us coboundary homomorphism of cochains of strata

$$\partial : [\overline{C}_{(\tau,o)}] \mapsto \sum_{(\gamma,\hat{o})<(\tau,o)} \pm [\overline{C}_{(\gamma,\hat{o})}] \tag{A.2}$$

where the signs are determined by (A.1).

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