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State estimation for a class of non-linear systems

by

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Notations

Sets

$I\!R$	Set of real numbers
$I\!R_+$	Set of positive real numbers, *i.e.* $I\!R_+ = \left\{x \in I\!R \mid x \geq 0\right\}$
$I\!R^n$	Set of n-dimensional real vectors
$I\!R^{n\times m}$	Set of n×m-dimensional real matrices
$\mathbb{Z}_+$	Set of positive integers
$\mathbb{S}^n$	Set of symmetric matrices in $I\!R^{n\times n}$, *i.e.* $\mathbb{S}^n = \left\{X \in I\!R^{n\times n} \mid X = X^T\right\}$
E_N	Set of indices of subsystems for switched system, $E_N = \{1, ..., N\}$
E^S	Set of tuples indicating the possible mode changes of the switched system, $E_S \subset E_N \times E_N$

Matrices and Operators

$A > 0$ $(A \geq 0)$	Real symmetric (semi)positive-definite matrix A
$A < 0$ $(A \leq 0)$	Real symmetric (semi)negative-definite matrix A
I_n	Identity matrix of dimension $n \times n$
A^{-1}	Inverse of matrix $A \in I\!R^{n\times n}$, $\det A \neq 0$
A^T	Transpose of matrix A
$A^\dagger$	Moore-Penrose pseudo-inverse of matrix A

$(\star)$	Block induced by symmetry
$\det A$	Determinant of matrix $A \in I\!R^{n\times n}$
$\operatorname{rank} A$	Rank of matrix $A \in I\!R^{n\times m}$
$\lambda(A)$	Set of eigenvalues of matrix $A \in I\!R^{n\times n}$
$\|A\|$	Induced Euclidian norm of matrix $A \in I\!R^{n\times n}$
$\mathrm{Co}(x, y)$	Convex hull of the set x, y, $\mathrm{Co}(x, y) = \{\lambda\, x + (1-\lambda)\, y, 0 \leq \lambda \leq 1\}$

Additional Notations

θ	Switching signal
σ	Switching sequence
$\sum(C_i, A_i, B_i)_M$	Switched linear system with subsystems (C_i, A_i, B_i), $i \in M$
$\phi(t, t_0, x_0, u, \theta)$	State trajectory of the switched system
$UO(C_i, A_i, B_i)_M$	Unobservable set of the switched system $\sum(C_i, A_i, B_i)_M$, $i \in M$

Abbreviations

EFS	Externally forced switching
IFS	Internally forced switching
LPV	Linear Parameter Varying
$DMVT$	Differential Mean Value Theorem

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Chapter 1

General introduction

In the theory of systems, the internal state of a dynamic process is characterized by a vector quantity. This notion is involved in the areas such as control, diagnosis and surveillance. In most of the real-world applications, the measurement of the physical state of the process by direct observation can be difficult when possible. There are almost two reasons for that: difficulties from technical aspects (necessary sensors are not available, insufficient accuracy, ...) and economic considerations (choice of a minimum number of sensors to reduce the costs of calibration and maintenance). In the absence of direct measurement, the value of the internal state can be reconstructed from the measurement of different inputs and outputs of the process. The auxiliary dynamic system which rebuilds the internal state of a process from the record of the signal measurement, is called *state observer* or *state estimator*. This is usually a mathematical model which can be implemented by computer. All the models of process cannot determine their internal state from the measurements. When this happens, the process is said to be *observable*.

The stability of system is an extremely important concept in the study of dynamical systems. The Lyapunov approach is one of the more general paradigms for the study of stability. It relies on an energy function also called Lyapunov function. The stability is proved as soon as a decreasing Lyapunov function is found. When considering estimation, it is required that the estimate converges to the original value of the state. In a sense, the observer should synchronize to the original system. The synchronization issue can be turned into a stabilization issue by considering a Lyapunov function based on the estimation error. In this thesis, the Lyapunov approach is used to solve to the state observation problem.

A hybrid system is a dynamic system that combines events in continuous-time and events in discrete-time. A switched system is a special case of hybrid system. It consists of several subsystems and a switching law that select at any moment, which system is active. The multi-model systems are similar to switched systems: they are composed of a series of subsystems (linear or nonlinear) and an interpolation function which allows to mix the different models. Switched systems are multimodel systems in which the law of mixture is piecewise constant.

Due to the fact that many processes and systems in real world applications can be modeled as switched systems and/or multi-model systems, the synthesis of observers for these classes of systems has received a growing interest in the last decades. A second reason that justifies the interest for this research area comes from the fact that it can be applied to data encryption/decrytion for telecommunication applications. In this case, the message is encrypted by mixing it to a chaotic dynamic process. At the reception side, the signal is reconstructed by synchronizing a chaotic system of the same nature, using the technique of state estimation.

In this thesis, we propose some methods for synthesizing state observers for switched systems and multi-model systems. By using new Lyapunov functions, these methods reduce the conservatism of the current approaches available in the literature.

In chapter 2 of this manuscript, the state of the art of switched systems is presented. After a simple introduction on hybrid systems and particularly on switched systems, we present the modeling and stability analysis for switched systems. The stability in presence of an arbitrary signal, stability in presence of a control signal and stabilization are introduced successively.

Chapter 3 presents the principle of state estimation, some definitions on the concept of observability, and the state of the art on the various techniques of the observer synthesis for switched systems and multi-model systems.

In chapter 4, the problem of the observer design for a class of non-linear switched system is studied. The developed solution is based on the assumption that the active mode of the switched system is unknown. In particular, the observer updates the estimated state at every commutation instant. Our approach relies on the Mean Value Theorem, which reduces the problem of state estimation for a nonlinear dynamic system to a stability problem for a linear parameter varying (LPV) system. By using a multi-quadratic Lyapunov function, we offer sufficient conditions for the observer synthesis to guarantee an upper bound on the dynamics of the estimation error. By determining the value of a scalar parameter, the synthesis problem is transformed into a linear matrix inequalities (LMI) problem, which is easily solvable numerically by convex optimization techniques.

Chapter 5 presents the observer synthesis for a class of discrete-time multi-model systems. The multi-model system is composed of two non-linear Lipschitz systems. By using a particular Lyapunov function with a non-linear term, sufficient conditions for the observer synthesis are given. This result is applicable to the systems with Lipschitz constants whose values are less than one.

In chapter 6, we study the observer synthesis for a class of discrete-time multi-model systems with unknown input. In order to ensure the synchronization between the transmitter system and receiver system, we introduce a particular Lyapunov function including a non-linear term. This leads to less restrictive synthesis conditions which can be expressed as a LMI feasibility problem.

In the last chapter, we give the conclusion of this manuscript and discuss the perspective of our research.

Chapter 2

Introduction to switched systems

2.1 Hybrid systems and switched systems

A *hybrid system* is a dynamical system that contains interacting continuous dynamics and discrete events. The former are associated with physical first principles, the latter are associated with logic devices, such as switches, digital circuitry and software code.

Since many complex systems, such as chemical processes, automotive systems, computer-controlled systems, etc., are hybrid in nature, hybrid systems researches has been extensively investigated from both the academia and the industry in the last decade.

There are two categories of approaches for hybrid systems studies. The first approach is taken by the computer scientists, which takes more attention to the discrete events. The problems that have been studied in these domains include verification, safety analysis, etc. The hybrid automata theory and logic are largely applied in the studies of such problems. The second approach is taken by the control engineers and takes more attention to the continuous dynamics. Development of classic control theory have been tempted to study continuous systems with discrete mode changes. The problems that have been studied in these domains include stability, optimal control, stabilization, etc. More details of these approaches can be found in a lot of publications as follows.

Branicky proposed to model hybrid systems in the form of a dynamic system, embedding the discrete part into the continuous parts [1]. Based on the classification of different hybrid phenomena, this model defines very clearly a hybrid system as a dynamic system and specifies how the change of models are involved. Another modeling of hybrid system is obtained by the extension of the finite state automation theory in considering a differential equation for each discrete arrivals [2] [3]. An interest of this modeling in the form of hybrid automation is the modularity of this approach. There are other approaches for the modeling of hybrid system, which can be find in [4] [5] [6] [7] [8].

Some very common systems have hybrid nature, like as a room heating system, or computer-controlled system, etc.. The bouncing ball is a typical example of hybrid system [9]. It consists of a ball dropped from an initial height and bouncing on the ground, dissipating its energy with each bounce. The ball exhibits continuous dynamics between each bounce; however, when the ball impacts the ground, its velocity undergoes a discrete change modeled after an inelastic collision. Let x_1 be the height of the ball and x_2 be the velocity of the ball, then the mathematical description of bouncing ball can be given as follows:

- When $x \in C = \{x_1 > 0\}$, flow is governed by $\dot{x_1} = x_2, \dot{x_2} = -g$, where g is the acceleration due to gravity. These equations state that when the ball is above ground, it is being drawn to the ground by gravity.
- When $x \in D = \{x_1 = 0\}$, jumps are governed by $x_1^+ = x_1, x_2^+ = -\gamma x_2$, where $0 < \gamma < 1$ is a dissipation factor. That means that when the height of the ball is zero (it touches the ground), the velocity of the ball is reversed and decreased by a factor of γ. This describes the nature of the inelastic collision.

More examples of hybrid systems can be found in [10] [11] [12] [13] [14].

In the following, we focus on the presentation of a particular class of hybrid systems, named *switched systems*. A switched system consists of several subsystems and a switched law indicating the active subsystem at each instant of time. Switched systems can exhibit jumps, particularly at the switching instants.

The motivation for studying switched systems comes from many aspect. It is known that many practical systems are inherently multimodal in the sense that their behavior may depend on various environmental factors. Since these systems are essentially switched systems, powerful analysis or design results of switched systems are helpful dealing with real systems. Another important observation is that switching among a set of controllers for a specified system can be considered as a

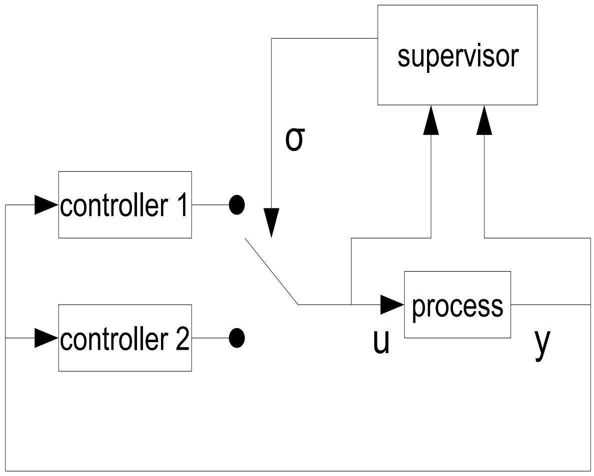

Figure 2.1: An example of switched system

switched system. That switching has been used in adaptive control, to ensure stability in situations where stability cannot be proved otherwise, or to improve transient response of adaptive control systems. Also, the methods of intelligent control design are based on the idea of switching among different controllers. Therefore, the study of switched systems contributes greatly in switching controller and intelligent controller design.

Let us consider a simple example of switched system which is applicable on adaptive supervisory control. The graphical representation of this system is shown in Figure 2.1. The process can either be (2.1) or (2.2) as follows:

$$\begin{cases} \dot{x} = A_2 x + B_2 u \\ y = C_2 x \end{cases} \tag{2.1}$$

$$\begin{cases} \dot{x} = A_1 x + B_1 u \\ y = C_1 x \end{cases} \tag{2.2}$$

The supervisor determines which is the correct process model by observing u and y, then select the appropriate controller.

2.2 Classes of switched systems

Switched systems can be classified by different ways. Usually, switching events in switched systems can be *state-dependent* or *time-dependent*, and it can be *autonomous* or *controlled* [15].

2.2.1 General formulation

Let consider the continuous-time case and consider a switched system as follows:

$$\dot{x} = f_\theta(x), \quad \theta \in E_N \tag{2.3}$$

where f_p, $p \in E_N$ is a family of sufficiently regular functions from $I\!R^n$ to $I\!R^n$. It is parameterized by some index θ in some index set E_N. $E_N = \{1, ..., N\}$ is the set of indices for subsystems.

A particular case is when the switched system is linear:

$$\dot{x} = A_\theta x, \quad \theta \in E_N. \tag{2.4}$$

The notion of *switching sequence* is necessary for the following parts.

Definition 2.2.1 *(Switching Sequence) [16]*
For a switched system, a switching sequence σ in $[t_0, t_f]$ is defined as:

$$\sigma = ((t_0, i_0), (t_1, e_1), (t_2, e_2), ..., (t_K, e_K)), \tag{2.5}$$

with $0 \leq K \leq \infty$, $t_0 \leq t_1 \leq t_2 \leq ... \leq t_K \leq t_f$, and $i_0 \in E_N$, and $e_k = (i_{k-1}, i_k) \in E_S$ for $k = 1, 2, ..., K$ (if $K = 0$, $\sigma = ((t_0, i_0)$. E_S is the set of tuples indicating the possible mode changes of the switched system and $E_S \subset E_N \times E_N$, it means that $e_k = (i_{k-1}, i_k)$ corresponds to a transition from mode $k-1$ to k. K is the total number of switchings.

Usually, a switching sequence is generated by a *switching law*, which is defined as follows.

Definition 2.2.2 *(Switching Law) [16]*
For a switched system, a switching law s is defined to be a mapping $s : I\!R^n \times I\!R \rightarrow \bigcup_{t_0} \sum_{[t_0,\infty)}$ which specifies a switching sequence $\sigma = \sigma(x_0, t_0)$ for any initial point x_0 and any initial time t_0.

Moreover, the following assumptions on the switchings must be distinguished:

1. *The known and the unknown modes cases:*

   If the active mode of the switched system is known, it allows to activate the corresponding mode in the observer (see an example in [17]).

2. *The autonomous and the non-autonomous cases:*

   The mode can be a map of the state (autonomous) or can be an arbitrary input of the system (non-autonomous).

In the unknown autonomous case, the observability problem is more complex to resolve, as the current mode must be estimated (see an example in [18]). This case is considered in the contributions given in the following chapters.

2.2.2 State-dependent switching

The active mode can be considered as a function of the state, i.e, $\theta = \theta(x)$.

Consider that the state space is partitioned into a finite or infinite number of *regions*, separated by a family of *switching surfaces*. In every region, a continuous-time dynamical system is given. When the trajectory intersects the switching surface, a state jump can occur instantaneously. Then, a switched system with *state-dependent switching* can be described by:

- a family of regions and corresponding switching surfaces,
- a family of continuous-time systems, one for each region,
- the jump function.

An example of state partition is shown in Figure 2.2, where the thin plain lines represent the state trajectory, the plain lines represent the switching surfaces which separate the regions.

Remark 2.2.1 *Note that the instantaneous state jump at the switching moment is equivalent to an impulse effect. If the impulse effect is absent, then the state trajectory is continuous. In this dissertation, we focus on switched systems without impulse effects.*

Remark 2.2.2 *A system with sliding mode controller can be considered as a switched system. That means that when the state trajectory hits the switching surface, it slides on the switching surface (see more details in [15]).*

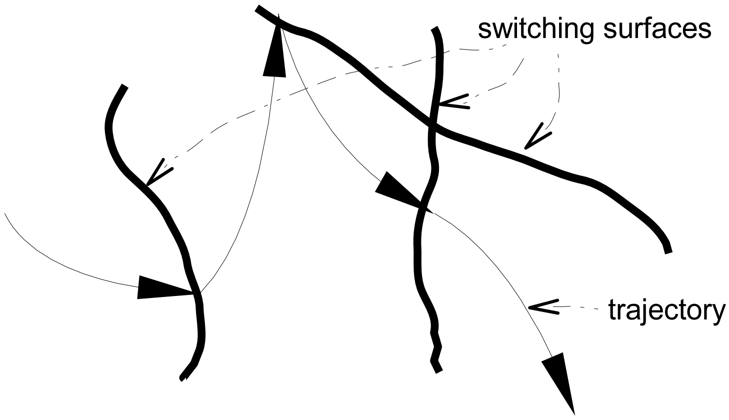

Figure 2.2: State-dependent switching

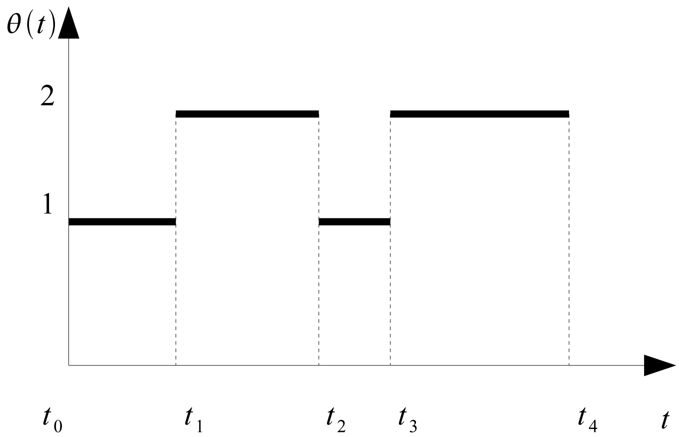

Figure 2.3: Time-dependent switching

2.2.3 Time-dependent switching

The active mode can be also considered as a function of the time, i.e, $\theta = \theta(t)$. Then the sequence is time-scheduled.

In Figure 2.3, a simple example of time-dependent switching is shown, with the set $E_N = \{1, 2\}$. When the time $t \in [t_o, t_1)$ or $t \in [t_2, t_3)$, the active mode of switched system is 1; when the time $t \in [t_1, t_2)$ or $t \in [t_3, t_4)$, the active mode of switched system is 2.

2.2.4 Autonomous switching and controlled switching

If the switching events are triggered by an internal mechanism over which we do not have direct control, then the resulting systems is said to be a switched system with *autonomous switching*, or an *autonomous switched system*.

Otherwise, if we have direct control over the switching mechanism to achieve a desired behavior of the resulting system, the systems is said to be a switched system with *controlled switching*.

Notice that the state-dependent and time-dependent switching systems can be considered as autonomous.

2.3 Multimodel systems

Multiple model representation consists in constructing a nonlinear dynamic system by mixing the behavior of several nonlinear time invariant subsystems. Consider the following discrete-time multimodel representation:

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{N} \mu_i(y_k) f_i(k, x_k) \\ y_k &= C x_k \end{aligned}$$

where $x_k \in I\!R^n$ is the state vector, function $f_i(k, x_k) : I\!R^n \to I\!R^n$ is the i^{th} subsystem which is nonlinear. The interpolation functions $\mu_i(y_k)$ satisfy the following properties:

$$\begin{aligned} &\sum_{i=1}^{N} \mu_i(y_k) = 1, \\ &0 \leq \mu_i(y_k) \leq 1, \forall i = 1, ..., N. \end{aligned}$$

At each time, the function $\mu_i(y_k)$ quantifies the relative contribution of each local model.

If all the subsystems are linear, this becomes a linear multimodel system:

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{N} \mu_i(y_k) A_i x_k \\ y_k &= C x_k \end{aligned}$$

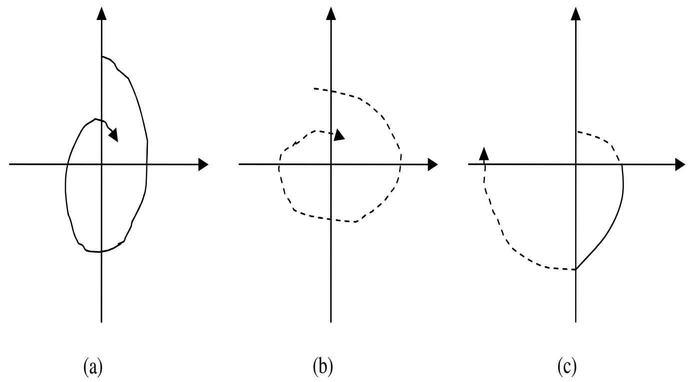

Figure 2.4: Possible trajectories of switched systems

where A_i is constant matrix of appropriate dimensions.

For the particular case, when $\mu_i(y_k) = 1$ and $\mu_j(y_k) = 0, j \neq i$, only the i^{th} subsystem is active, the system becomes a discrete-time switched system.

2.4 Stability analysis for switched systems

In this section, several stability issues for switched systems are briefly introduced. The kind of stability considered herein is the asymptotic stability.

2.4.1 Stability under arbitrary switching signal

Consider the switched system (2.3). Obviously, a necessary condition for asymptotic stability under arbitrary switching signal is that all the subsystems are asymptotically stable [15]. But this condition is not sufficient. This can be seen on an Figure 2.4: consider two second-order asymptotically stable systems whose trajectories are shown in (a), (b) of Figure 2.4. With a particular switching signal, the trajectory of the switched system might be instable, as it has been shown on Figure 2.4(c).

In order to guarantee the stability of the switched system, one feasible approach consists in looking for a common Lyapunov function for all subsystems [15]. For the system (2.3), the existence of a common Lyapunov function implies the asymptotic stability. However, this stability condition may be difficult to check and approaches that are less conservative have been developed. They rely on several Lyapunov functions, one for each mode. These approaches are used in this thesis and will be introduced in the following.

2.4.2 Stability with multiple Lyapunov functions

Let us assume here that the intervals between consecutive switching times are large enough, which is universal in the literature (see more details in [19]). With this slow switching assumption, an extension of the Lyapunov's second method, named *multiple Lyapunov functions approach*, has been developed in [20].

Definition 2.4.1 *(Lyapunov-like function) [16]*
Assume that we are given a switching law s. A smooth real-valued function $V_i(x)$ is called a Lyapunov-like function for the subsystem $\dot{x} = f_i(x)$ if it satisfies the following conditions

- *$V_i(x)$ is positive definite and $V_i(0) = 0$,*
- *$\dot{V}_i(x) = \frac{\partial V_i(x)}{\partial x} \leq 0$.*

With the Lyapunov-like functions, the following results for linear switched systems where $\dot{x} = A_i x, i \in E_N$ was developed by Peleties and DeCarlo in [20] [21].

Theorem 2.4.2 *[20] [21]*
Consider a switching sequence $\sigma(x)$ is generated by a given switching law s. Assume there exists a Lyapunov-like function $V_i(x)$ for each subsystem i in $I\!R^n$. If for any sequence $\sigma(x)$, for any $(i, j) \in E_N$ and for all the transitions from mode i to mode j, we have:

$$V_i(x(t_b)) - V_i(x(t_a)) < 0 \tag{2.6}$$

where t_a, t_b, $(t_b > t_a)$ are two switching instants from mode i to mode j, then the switched system is stable.

A simple example of the evolution of these Lyapunov-like functions is shown in Figure 2.5. Notice that V_i is nonincreasing in the intervals where subsystem i is active. For instance, for mode $i = 2$ to mode $j = 1$, two switching instants $t_a = t_2$ and $t_b = t_4$, we have $V_2(t_4) < V_2(t_2)$.

An alternative function was developed by Branicky [22] [23].

Theorem 2.4.3 *[22] [23]*
Consider a switching sequence $\sigma(x)$ is generated by a given switching law s for a

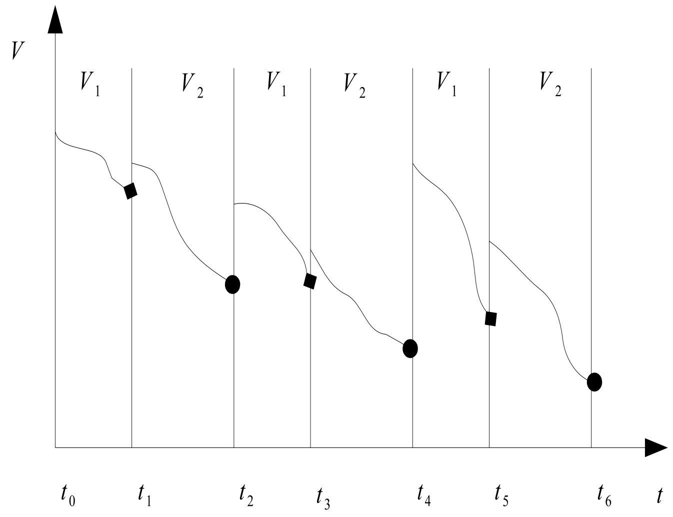

Figure 2.5: Evolution of the Lyapunov-like functions for Theorem 2.4.2

nonlinear switched system. Assume there exists a Lyapunov-like function $V_i(x)$ *for each subsystem i. If for any switching sequence* $\sigma(x)$*, for any* $(i, j) \in E_N$ *and for all the transitions from mode i to mode j, we have:*

$$V_j(x(t_b)) - V_j(x(t_a)) < 0 \tag{2.7}$$

where t_a*,* t_b*,* $(t_b > t_a)$ *are two switching instants, from mode i to mode j, then the switched system is stable.*

A simple example of the evolution of these Lyapunov-like functions is shown in Figure 2.6, notice that V_i is nonincreasing in the intervals where subsystem i is active. For instance, for mode $i = 2$ to mode $j = 1$, two switching instants $t_a = t_2$ and $t_b = t_4$, we have $V_1(t_4) < V_1(t_2)$.

Notice that a more general result for multiple Lyapunov functions approach is proposed by Ye et al.. This result does not require V_i to be nonincreasing in the intervals where subsystem i is active (see more details in [24] [25]).

2.4.3 Stabilization of switched systems

Due to the possible instability of switched system for certain switching signals, the important question is how to find a switching signal for which the switched system

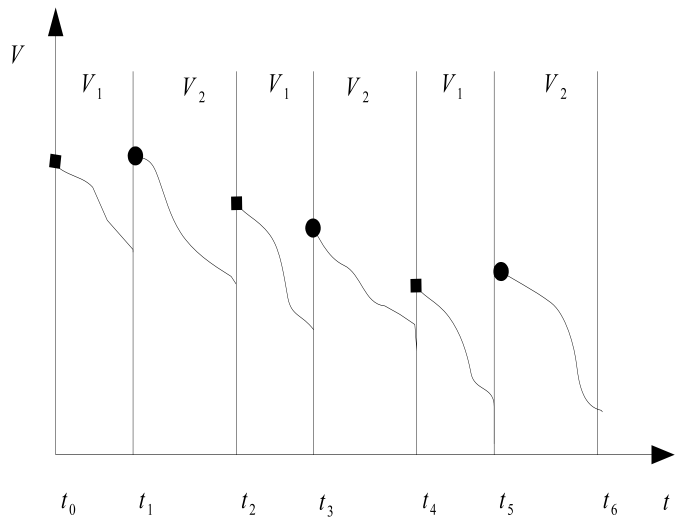

Figure 2.6: Evolution of the Lyapunov-like functions for Theorem 2.4.3

is asymptotically stable. With the assumption that none of the individual subsystems are asymptotically stable, the problem consists in constructing a stabilizing switching signal. There are many results in the literature dealing with this issue. They are based on the stability analysis results presented in the precious section. In order to design a stabilizing switching law, the methods are based on a common Lyapunov function or multiple Lyapunov functions and LMIs have been introduced [26] [27] [28] [29] [30]. These methods are applicable for asymptotic stability, for linear case or non-linear case. Notice these methods are based on sufficient conditions, which means that if the method fails, one cannot conclude that the system is not stabilizable.

Chapter 3

State of the art for observers of switched systems

Due to the fact the many real-world processes and systems can be modeled as switched systems, the observer synthesis for switched systems has known a growing interest over the last decades. In this chapter, the observability of switched systems and the existent approaches for the observation of switched systems is presented.

This chapter is structured as follows. In section 3.1, we present the principle of the state estimation. In section 3.2, some definitions on the notion of the observability is presented. In section 3.3, we introduce the state of the art of the different techniques for observer design of switched system.

3.1 Principle of state estimation

An observer is a dynamic system (O), whose inputs are the inputs/outputs of a dynamic system (S) whose state is to be estimated. The observer outputs give an estimation of the state of the system (S). The principle of the state observation can be shown in Figure 3.1:

As this dissertation concerns both the continuous and the discrete-time case, we use the following unique notation to describe a dynamic system (S):

$$\rho_x = f(x, u), \tag{3.1a}$$

$$y = h(x, u), \tag{3.1b}$$

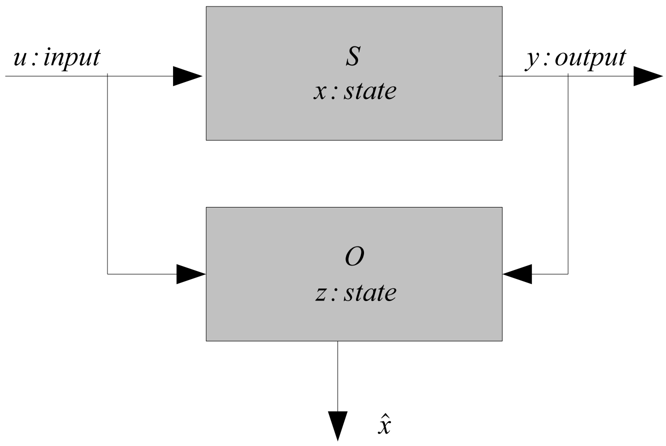

Figure 3.1: Principle of state observation

where

$$\rho_x = \begin{cases} \dot{x}, & \text{in the continuous-time case} \\ x_{k+1}, \quad k \in \mathbb{Z}_+ & \text{in the discrete-time case} \end{cases}$$

Definition 3.1.1 *The dynamic system (O) described by the equations:*

$$\rho_z = \Phi(z, u, y), \tag{3.2a}$$

$$\hat{x} = \Psi(z, u, y), \tag{3.2b}$$

$z \in I\!R^s$, is a local asymptotic observer for the system (S) if the following conditions are satisfied:

1. $x_0 = \hat{x}_0 \quad \Rightarrow \quad x_t = \hat{x}_t \quad \forall t \geq 0$;
2. *There exist an open neighborhood of the origin $\Omega \subseteq I\!R^n$ such that:*

$$x_0 - \hat{x}_0 \in \Omega \Rightarrow \|x_t - \hat{x}_t\| \to 0 \quad \textit{when} \quad t \to +\infty.$$

If $\|x_t - \hat{x}_t\|$ tends exponentially towards zero, the system (O) is called an exponential observer of the system (S).

When $\Omega = I\!R^n$, the system (O) is called a global observer of the system (S).

The second condition of Definition 3.1.1 indicates that the estimation error is asymptotically stable. When the system (O) is linear, if there exists an observer of the form (3.2) such that the second condition is satisfied, then the system is said to be detectable. For the nonlinear case, the additional conditions are necessary for the detectability: the nonlinear functions must satisfy Lipschitz condition (see more details in [31]).

The condition 1 means that if the observer (O) and the system (S) both have the same initial state, then the estimated state (O) equals the actual state of the system (S) at any time.

Let restrict to the case when $z = \hat{x}$ (i.e. $\Psi(z, u, y) = z$). Then (3.2b) is equivalent to:

$$\rho_{\hat{x}} = \Phi(\hat{x}, u, y) \tag{3.3}$$

The condition 1 can be written as:

$$\hat{x} = x \quad \Rightarrow \quad \rho_x = \rho_{\hat{x}}$$

which leads to:

$$\hat{x} = x \quad \Rightarrow \quad \Phi(\hat{x}, u, y) = f(\hat{x}, u).$$

Then, without loss of generality, (3.3) can be rewritten as follows:

$$\rho_{\hat{x}} = f(\hat{x}, u) + \kappa(\hat{x}, u, y)$$

with

$$\hat{x} = x \quad \Rightarrow \quad \kappa(\hat{x}, u, y) = 0. \tag{3.4}$$

A function κ that would be proportional to $x - \hat{x}$ would satisfy (3.4), but since x cannot be measured, this is not possible. However, $\hat{x} = x \Rightarrow h(\hat{x}, u) = h(x, u) = y$, so we can take a function κ with the following form:

$$\kappa(\hat{x}, u, y) = K(\hat{x}, u, y)\,(y - \hat{y})$$

where

$$\hat{y} = h(\hat{x}, u).$$

Then the observer can be described as follows:

$$\rho_{\hat{x}} = f(\hat{x}, u) + K(\hat{x}, u, y)(y - \hat{y}), \tag{3.5a}$$

$$\hat{y} = h(\hat{x}, u). \tag{3.5b}$$

Notice that in the present case, we focus on observation of switched systems and multimodel systems. Nevertheless, observer can be used with others classes of systems. Let mention for instance the works on a class of nonlinear systems with unknown inputs [32]. An unknown input observer is proposed, which is applicable to Fuel Cell Stacks.

3.2 Observability of switched systems

In this section, some concepts for observability of switched systems are presented (see [33] and [34]).

Consider the general model of switched linear systems as follows:

$$\begin{aligned} \rho_x &= A_\theta x + B_\theta u, \\ y &= C_\theta x, \end{aligned} \tag{3.6}$$

where $x(t) \in I\!R^n$, $u(t) \in I\!R^m$, $y(t) \in I\!R^p$ are the state vector, the input vector and the output vector, respectively. $\theta \in E_N$ is the switching signal, and ρ_x is the derivative operator in continuous-time and shift forward operator in discrete-time system case.

Let $\phi(t, t_0, x_0, u, \theta)$ denote the state trajectory of switched system (3.6), starting from $x(t_0) = x_0$ with input u and switching path θ, at time t.

Definition 3.2.1 *State x is said to be unobservable, if for any switching path θ, there exists an input u such that*

$$C_\theta \phi(t, t_0, x, u, \theta) = C_\theta \phi(t, t_0, 0, u, \theta), \quad \forall t \geq t_0.$$

The unobservable set of system (3.6), denoted by $UO(C_i, A_i, B_i)_M$ or UO in short, is the set of states which are unobservable.

Definition 3.2.2 *System (3.6) is said to be (completely) observable, if its unobservable set is null.*

Definition 3.2.3 *State $x \in I\!R^n$ is unobservable via θ, if there exists an input u such that*

$$C_\theta \phi(t, t_0, x, u, \theta) = C_\theta \phi(t, t_0, 0, u, \theta), \quad \forall t \geq t_0.$$

The unobservable set of system (3.6)*, denoted by $UO_\theta(C_i, A_i, B_i)_M$ or UO_θ in short, is the set of states which are unobservable via θ.*

Observability of discrete-time switched linear systems

In the following, we introduce some recent definitions and concepts for the observability of discrete-time switched linear systems, which are referred to the thesis of Mohamed Babaali [34].

Consider the general model of discrete-time switched linear systems given by:

$$x_{k+1} = A_{\theta(k)} x_k + B_{\theta(k)} u_k, \tag{3.7a}$$

$$y_k = C_{\theta(k)} x_k, \tag{3.7b}$$

where $x_k \in I\!R^n$, $u_k \in I\!R^m$, $y_k \in I\!R^p$ are the state vector, the input vector and the output vector, respectively. $A(.)$, $B(.)$ and $C(.)$ are real matrices of appropriate dimensions. $\theta(k)$ is the mode at time k and takes values in the finite set $\{1, ..., s\}$.

In order to lighter the notations, we write $\theta(k)$ to recall that θ may vary from a sample to another, even if θ can be a map of x_k in the case of state-dependent switching, i.e. $\theta(k) = \theta(x_k)$.

Pathwise observability (under known mode case)

The observability of discrete-time switched linear systems under known modes is called *pathwise observability*.

Without lost of generality, consider the autonomous switched linear systems as follows:

$$x_{k+1} = A_{\theta(k)} x_k, \tag{3.8a}$$

$$y_k = C_{\theta(k)} x_k, \tag{3.8b}$$

For a path $\{\theta_1 \theta_2 ... \theta_N\}$ of length N, let define the observability matrix as:

$$\Gamma(\theta) \triangleq \begin{bmatrix} C_{\theta_1} \\ C_{\theta_2} A_{\theta_1} \\ \vdots \\ C_{\theta_N} A_{\theta_{N-1}} ... A_{\theta_1} \end{bmatrix},$$

Moreover, a path θ is observable if and only if its observability matrix $\Gamma(\theta)$ has full rank n. Then, the definition is given as follows:

Definition 3.2.4 *(Pathwise Observability [34])*
The set of pairs $\{(A_1, C_1), \dots, (A_s, C_s)\}$ is pathwise observable if and only if there exists an integer N such that all paths of length N are observable. We refer to the smallest such integer as the index of pathwise observability.

Further results on pathwise observability can be found in chapter 2 and chapter 3 of [34].

Several observability concepts (under unknown mode case)

When the active mode of a switched system is unknown, consider the autonomous system as follows:

$$x_{k+1} = A_{\theta(k)} x_k, \tag{3.9a}$$

$$y_k = C_{\theta(k)} x_k. \tag{3.9b}$$

We define a path θ as a finite sequence of modes $\theta = \theta_1\theta_2...\theta_N$, where N is the path length denoted by $|\theta|$. We define Θ_N as the set of all paths of length N. Moreover, we denote by $\theta_{[i,j]}$ the product of θ between i and j, i.e. $\theta_{[i,j]} = \theta_i\theta_{i+1}...\theta_j$. We use $\theta\theta'$ to indicate the concatenation of θ with θ', and we let $\phi(\theta) \triangleq A(\theta_N)...A(\theta_1)$ denote the transition matrix of the path θ. We define the observability matrix $\Gamma(\theta)$ of a path θ as:

$$\Gamma(\theta) \triangleq \begin{bmatrix} C_{\theta_1} \\ C_{\theta_2} A_{\theta_1} \\ \vdots \\ C_{\theta_N} A_{\theta_{N-1}} ... A_{\theta_1} \end{bmatrix}.$$

Now, we define:

$$Y(\theta, x) \triangleq \Gamma(\theta)x, \tag{3.10}$$

if $x = x_1$ and $\theta = \theta_1\theta_2...\theta_N$ in (3.9), then $Y(\theta, x) = [\, y_1^T ... y_N^T]^T$.

Definition 3.2.5 *(Mode Observability [34])*
The switched linear system (3.9) *is mode observable at N if there exists an integer N' such that* $\forall\, x \in I\!R^n$ *and* $\forall\, \theta \in \Theta_{N+N'}$,

$$\theta_{[1,N]} \neq \theta'_{[1,N]} \Rightarrow Y(\theta, x) \neq Y(\theta', x') \quad \forall x' \in I\!R^n. \tag{3.11}$$

The index of mode observability is the smallest such N'.

Definition 3.2.6 *(State Observability [34])*
The switched linear system (3.9) *is state observable if there exists an integer N (the smallest being the index) such that* $\forall\, x \in I\!R^n$ *and* $\forall\, \theta \in \Theta_N$,

$$x \neq x' \Rightarrow Y(\theta, x) \neq Y(\theta', x') \quad \forall \theta' \in \theta_N. \tag{3.12}$$

This means that a system is state observable if any N consecutive measurements $Y(\theta, x)$ yield x uniquely without knowledge of θ, i.e. if the map $(x, \theta) \to Y(\theta, x)$ is injective in its first coordinate.

let us mention [35] in which a algebraic condition is given for a class of hybrid systems.

3.3 Different types of observers for switched systems

In general, there are two kinds of approaches for the observer synthesis problem. In the first kind, the active mode of the switched system is assumed to be known which allows the mode of the observer to be changed accordingly. There are many results in the literature dealing with this issue [36] [37] [38] [39]. In this context, Alessandri *et al.* [36] proposed a method for the design of a Luenberger-like asymptotic observer. Their method uses a common quadratic Lyapunov function that guarantees the stability of the estimation error. In addition to the analysis of the observability of linear switched systems presented in [37], a moving-horizon estimation technique has been employed for the state estimation [38]. Moreover, for a particular class of linear switched systems, a switched state jump observer was developed in [39] using multiple Lyapunov functions.

In the second kind of approaches, which is of interest in this paper, the active mode of the switched system is unknown. There are some results in the literature dealing with this issue [18] [40] [41] [42]. In [18], a Luenberger-type observer for continuous-time bi-modal piecewise affine systems has been proposed. The observer synthesis is based on a common quadratic Lyapunov function. The discrete-time counterpart of [18] was developed in [40]. In [41], a switched state jump observer was synthesized for continuous-time linear switched systems with multiple modes, by using multiple quadratic Lyapunov functions. A moving horizon observer for mode and continuous state estimation of nonlinear switched systems is developed in [42], where mode and continuous state estimation is expressed as an optimization problem which is solved using the Gauss-Newton algorithm. In [43], a hybrid observer for a class of switched systems is proposed. With the association of a discrete state detection method and a piecewise-linear switched observer, the observation error is guaranteed to converge toward zero or to be bounded.

For a linear system, if the pair (A, C) is detectable (which means that there exist $L \in \mathbb{R}^{n\times p}$ such that the eigenvalues of matrix $(A - LC)$ lies strictly in the left complex half-plane), then we can guarantee that the estimation error exponentially converges to zero. However, for a switched linear system, even if all the pairs (A_i, C_i) are detectable, the convergence of the estimation error will not be guaranteed.

Alessandri *et al.* proposed a method for the design of a Luenberger-like asymptotic observer in [36]. Their method uses a common quadratic Lyapunov function that guarantees the asymptotic stability of the estimation error.

For the general model of continuous-time switched linear systems given by:

$$\dot{x} = A_{\theta(t)}x + B_{\theta(t)}u, \tag{3.13a}$$

$$y = C_{\theta(t)}x, \tag{3.13b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are the state vector, the input vector and the output vector, respectively. $A(.)$, $B(.)$ and $C(.)$ are real matrices of appropriate dimensions. $\theta(t)$ is the mode at time t and takes values in the finite set $\{1, ..., s\}$. The mode $\{\theta(t)\}_{t=1}^{\infty}$ is assumed to be an exogenous variable that is governed by some external process, which could be controller itself, so the mode sequence $\{\theta(t)\}_{t=1}^{\infty}$ is assumed to be arbitrary.

Consider the Luenberger type observer as follows:

$$\dot{\hat{x}} = A_{\theta(t)}\hat{x} + B_{\theta(t)}u + L_{\theta(t)}(y - \hat{y}), \tag{3.14a}$$
$$\hat{y} = C_{\theta(t)}\hat{x}, \tag{3.14b}$$

where $L_{\theta(t)}$ is the observer gain at instant t. The following theorem has been introduced by Alessandri [36].

Theorem 3.3.1 *[36] Consider the system* (3.13) *and assume that the pairs (A_i, C_i), $i = 1, 2, ..., n$, are detectable. If there exists a symmetric positive definite matrix P as the solution of the algebraic Lyapunov inequalities:*

$$(A_i - L_iC_i)^T P + P(A_i - L_iC_i) < 0, \quad i = 1, 2, ..., n \tag{3.15}$$

then the observer (3.14) *has an estimation error that is exponentially convergent towards zero.*

Let us consider (3.7), which is the discrete-time case of (3.13), the proposed Luenberger type observer is as follows:

$$\hat{x}_{k+1} = A_{\theta(k)}\hat{x}_k + B_{\theta(k)}u_k + L_{\theta(k)}(y_k - \hat{y}_k), \tag{3.16a}$$
$$\hat{y}_k = C_{\theta(k)}\hat{x}_k, \tag{3.16b}$$

where $L_{\theta(t)}$ is the observer gain at instant t. The following theorem has been introduced. Then the following theorem is given by Alessandri [36].

Theorem 3.3.2 *[36] Consider the system* (3.7) *and assume that the pairs (A_i, C_i), $i = 1, 2, ..., n$, are detectable. If there exists a symmetric positive definite matrix P as the solution of the algebraic Lyapunov inequalities*

$$(A_i - L_iC_i)^T P(A_i - L_iC_i) - P < 0, \quad i = 1, 2, ..., n \tag{3.17}$$

then the observer (3.16) *involves an estimation error exponentially convergent to zero.*

Pettersson proposed an observer for a class of switched linear systems in [39]. By updating properly the estimated state at each switching instant, the convergence of the estimation error can be guaranteed exponentially. When the mode changes, a jump of the estimated state is computed which only depend on the observer states and the measured output. This jump allows to update the estimate at the switching instants and allows better convergence.

Chapter 4

Switched observers with jump for nonlinear switched systems

4.1 Introduction

In this chapter, the problem of observer design is addressed for a class of switched systems, which is continuous-time system including a non-linear term. The developed solution is based on the assumption that the active mode of the switched system is unknown. In particular, the observer updates the estimated state at each switching instant. This work can be seen as an extension to nonlinear switched systems of the result developed in [41] for continuous-time switched linear system with multiple modes. Unlike the approach in [41], our approach uses the Differential Mean Value Theorem (DMVT) which allows to write the dynamics of the estimation error as a LPV system. Using multiple quadratic Lyapunov functions, we propose sufficient conditions for the observer synthesis guaranteeing an upper bound on the estimation error. For a fixed value of a scalar parameter, the problem is brought to one of solving LMIs that are easily tractable by optimization techniques.

This chapter is structured as follows. In section 4.2, the considered class of nonlinear switched systems and the structure of the state jump observer is presented. In section 4.3, the Differential Mean Value Theorem approach is preseted. In section 4.4, we present our synthesis method of the switched state jump observer. In section 4.5, the applicability of the proposed approach is demonstrated on a numerical example.

4.2 Problem formulation

Consider the following class of switched systems with a nonlinear term:

$$\begin{cases} \dot{x} = A_{q(t)}x + Bf(x), \\ y = Cx, \end{cases} \tag{4.1}$$

where $x \in I\!R^n$, $y \in I\!R^p$ are the state vector and the output vector, respectively. $A_{q(t)}$,B and C are constant matrices of appropriate dimensions. The function $f : I\!R^n \to I\!R$ is nonlinear and assumed to be differentiable. $q_{(t)}$ is the switching function:

$$q_{(t)} : \mathbb{Z}_+ \to E_N = \{1, \ldots, N\}$$

indicating which of the modes of the switched system is active at a certain time. In the following, we consider that $\{\Omega_j \mid j \in E_N\}$ is a collection of polyhedral subsets of $I\!R^n$ with mutually disjointed interiors and $\cup_j \Omega_j = I\!R^n$. Moreover, the boundary between the regions Ω_i and Ω_j is defined by the set S_{ij}, which is described by:

$$S_{ij} = \{x \in I\!R^n \mid s_{ij}^T x = 0\}, \quad \forall (i,j) \in E_S$$

where $E_S \subset E_N \times E_N$ is a set of tuples indicating the possible mode changes of the switched system and $s_{ij} \in I\!R^{1\times 1}$.

Moreover, for a hyperplane, a region containing the origin can be described by $x^T Q_i x \geq 0$ for every mode i of the switched system (1) where $Q_i \in I\!R^{n\times n}$ (see [44] for more details). For example, if the region Q_1 is given by two set of states restricted by two half-planes $s_{12}^T x \geq 0$ and $s_{21}^T x \geq 0$, then $Q_1 = s_{12}s_{21}^T + s_{21}s_{12}^T$, which is shown in Figure 4.1.

The following assumption is made on the nonlinear part of the system (4.1).

Assumption. We assume that the functions:

$$h_i(t) = \frac{\partial f}{\partial x_i}(x(t)), \forall i \in \{1, \ldots, n\}$$

are bounded, then the function $h(t)$ evolves in a bounded domain H_n of which 2^n vertices are defined by:

$$\Lambda_{H_n} = \{\gamma = (\gamma_1, \ldots, \gamma_n) \mid \gamma_i \in \{\underline{h}_i, \overline{h}_i\}\}$$

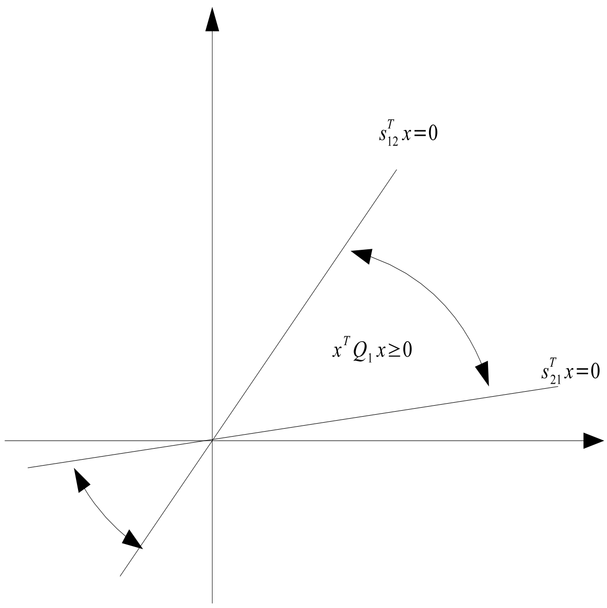

Figure 4.1: State spaces satisfying $x^T Q_1 x \geq 0$

where $\underline{h}_i = \min(h_i(t)), \overline{h}_i = \max(h_i(t))$.

Now we consider the following observer described by:

$$\begin{cases} \dot{\hat{x}} = A_{r(t)}\hat{x} + Bf(\hat{x}) + K_{r(t)}(y - \hat{y}), \\ \hat{y} = C\hat{x}, \end{cases} \tag{4.2}$$

where $\hat{x} \in I\!R^n$ is the estimate of the state vector x and $K_r \in I\!R^{n \times p}$ for $r \in E_N$ are the observer gains to be designed. The switching function $r_{(t)} : \mathbb{Z}_+ \to E_N$ indicates which of the observer modes is active at a certain instant.

The switching function $r_{(t)}$ changes its value according to the polyhedral subsets $\hat{\Omega}_j$ which boundaries are defined by the sets

$$\hat{S}_{ij} = \{\hat{x} \in I\!R^n \mid s_{ij}\hat{x} = 0\}, \quad \forall (i,j) \in E_s.$$

Moreover, when the observer is in the mode i and its state $\hat{x}$ reaches the boundary between the region Ω_i and the region Ω_j, the estimated state is updated according to

$$\hat{x}^+ = T_{1i}\hat{x} + T_{2i}y \tag{4.3}$$

where T_{1i}, T_{2i} are two matrices to be designed. This equation shows that the state of the observer jumps at the switching instants. For this reason, the observer is called a jump observer.

Our objective is to guarantee that the state observation error is bounded. The tuning parameters K_i, T_{1i}, T_{2i} with $i \in E_N$ will be designed accordingly.

4.3 Differential Mean Value Theorem

In this section, the differential mean value theorem (DMVT) approach for vector functions is presented. This allows to write the dynamics of the estimation error as a LPV system, which will be used in the following section.

The following definition is necessary:

Definition 4.3.1 *Let x, y be two elements in $\mathbb{R}^n$, we define by $Co(x, y)$ the convex hull of the set x, y, i.e.: $Co(x, y) = \lambda x + (1 - \lambda)y, \lambda \in [0, 1]$.*

Let

$$M_s = \{e_s(i) \mid e_s(i) = (0, ..., 0, 1, 0, ...0)^T, i = 1, ..., s.\}$$

be the canonical basis of the vectorial space $\mathbb{R}^s$ for all $s \geq 1$. Let: $f : \mathbb{R}^n \to \mathbb{R}^q$ be a vector function. Then we have: $f(x) = [f_1(x), ..., f_q(x)]^T$ where f_i: $\mathbb{R}^n \to \mathbb{R}$ is the i^{th} component of f.

The vectorial space $\mathbb{R}^q$ is generated by the canonical basis M_q, so we can write:

$$f(x) = \sum_{i=1}^{q} e_q(i) f_i(x).$$

Now we give the following theorem which was presented in [45].

Theorem 4.3.2 *(DMVT for vector functions) [45]*
Let f: $\mathbb{R}^n \to \mathbb{R}^q$. Let $a, b \in \mathbb{R}^n$. We assume that f is differentiable on $Co(a, b)$. Then, there exist constant vectors $c_1, ..., c_q \in Co(a, b), c_i \neq a, c_i \neq b$ for $i = 1, ..., q$ such that:

$$f(a) - f(b) = \sum_{i,j=1}^{q,n} e_q(i) e_n^T(j) \frac{\partial f_i}{\partial x_j}(c_i)(a - b).$$

4.4 Main result

Our contribution, presented in this subsection, consists in sufficient conditions for the synthesis of observer (4.2).

The dynamic of the estimation error is given by:

$$\begin{aligned}\dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= (A_r - K_r C)e + (A_q - A_r)x + B(f(x) - f(\hat{x}))\end{aligned}$$

Using the the differential mean value theorem presented before, it can be shown that there exist $z(t) \in Co(x, \hat{x})$ such that:

$$f(x) - f(\hat{x}) = \frac{\partial f}{\partial x}(z)(x - \hat{x}) \tag{4.4}$$

with

$$\frac{\partial f}{\partial x}(z) = \sum_{k=1}^{n} e_{n(k)}^T \frac{\partial f}{\partial x_k}(z).$$

Using the notation $h_k = \frac{\partial f}{\partial x_k}(z)$, we get:

$$f(x) - f(\hat{x}) = \sum_{k=1}^{n} e_{n(k)}^T h_k e. \tag{4.5}$$

This allows to write the dynamics of estimation error as a LPV system.

In order to study the stability of the estimation error, let us consider multiple Lyapunov functions, one for each observer mode i:

$$V_i(e) = e^T P_i e$$

where $P_i \in I\!R^{n \times n}$ are symmetric positive definite matrices.

For given modes $j = q_{(t)}$ of the system and $i = r_{(t)}$ of the observer,

$$\dot{e} = (A_i - K_i C)e + (A_j - A_i)x + B(f(x) - f(\hat{x})),$$

then we can get the time derivative of the energy of the estimation error:

$$\begin{aligned}\dot{V}_i(e) &= \dot{e}^T P_i e + e^T P_i \dot{e} \\ &= e^T([A_i - K_iC]^T P_i + P_i[A_i - K_iC])e \\ &\quad + e^T P_i(A_j - A_i)x + x^T(A_j - A_i)^T P_i e \\ &\quad + e^T P_i B(f(x) - f(\hat{x})) + B^T P_i e(f(x) - f(\hat{x})).\end{aligned} \tag{4.6}$$

Owing (4.5) and (4.6), we have:

$$\begin{aligned}\dot{V}_i(e) &= e^T([A_i - K_iC]^T P_i + P_i[A_i - K_iC])e \\ &\quad + e^T P_i(A_j - A_i)x + x^T(A_j - A_i)^T P_i e + 2e^T P_i B \sum_{k=1}^{n} e_{n(k)}^T h_k e.\end{aligned} \tag{4.7}$$

The following theorem states sufficient conditions for the observer synthesis.

Theorem 4.4.1 *There exists an observer* (4.2) *for the switched system* (4.1) *if there exist positive-definite matrices P_i, $i \in E_N$, scalars $\epsilon > 0$, $\alpha > 0$, $\beta > 0$, $\mu_{ij} > 0$, $\nu_{ij} > 0$ and matrices D_{ij}, for all $(i,j) \in E_S$, such that the following conditions are satisfied:*

$$\alpha I \leq P_i \leq \beta I, i \in E_N \tag{4.8a}$$

$$\Gamma_{i,j} = \begin{bmatrix} \Gamma_{i,j}^{11} & \Gamma_{i,j}^{12} \\ (\Gamma_{i,j}^{12})^T & \Gamma_{i,j}^{22} \end{bmatrix} \leq 0, (i,j) \in E_S \tag{4.8b}$$

$$P_j = P_i + D_{ij}^T C + C^T D_{ij}, (i,j) \in E_S \tag{4.8c}$$

where

$$\begin{aligned}\Gamma_{i,j}^{11} &= (A_i - K_iC)^T P_i + P_i(A_i - K_iC) + I \\ &\qquad + \nu_{ij} I + 2P_i B \sum_{k=1}^{n} e_{n(k)}^T \gamma_k, \forall \gamma_k \in \{\underline{h}_k, \overline{h}_k\} \\ \Gamma_{i,j}^{12} &= P_i(A_j - A_i), \\ \Gamma_{i,j}^{22} &= \mu_{ij} Q_j - \epsilon^2 \nu_{ij} I.\end{aligned}$$

When the observer switches from mode i to mode j, the state of the observer is *updated as following:*

$$\hat{x}^+ = (I - R_i^{-1}(CR_i^{-1})^\dagger C)\hat{x} + R_i^{-1}(CR_i^{-1})^\dagger y, \\ \forall \hat{x} \in S_{i,j}, (i,j) \in E_S, \tag{4.9}$$

and $R_i \in I\!R^{n\times n}$ *is a symmetric positive-definite matrix such that* $P_i = R_i^T R_i$.

Then if for some $T_0 > 0$

$$\sup_{t > T_0} \|x(t)\| \leq x_{\max},$$

we have:

$$\lim_{t\to\infty} \sup \|e(t)\| \leq \sqrt{\frac{\bar{\nu}}{\bar{\nu}+1}}\sqrt{\frac{\beta}{\alpha}}\epsilon x_{\max}$$

where $\bar{\nu}$ *is the largest value of* ν_{ij}, $(i,j) \in E_S$.

Proof. In order to prove that the overall energy of the estimation error is upper bounded, we show the following sufficient conditions:

- the energy decreases at the switching instants when the observer changes its mode,
- the energy in every mode is upper bounded by a constant.

First, we need to prove that, when the observer switches from mode i to mode j, the energy decreases at the switching instant *i.e.* $V_j(e^+) \leq V_i(e)$, which is equivalent to

$$(x - \hat{x}^+)^T P_j (x - \hat{x}^+) \leq (x - \hat{x})^T P_i (x - \hat{x}). \tag{4.10}$$

Let $\hat{x}^+$ be an arbitrary updated state estimate satisfying the measurement equation $y = C\hat{x}^+$. Then, we have $C(\hat{x}^+ - x) = y - y = 0$ which implies that for any matrix $D_{i,j}$, the following relation is satisfied:

$$(x - \hat{x}^+)^T (D_{i,j}^T C + C^T D_{i,j})(x - \hat{x}^+) = 0.$$

According to (4.8c), we have

$$(x - \hat{x}^+)^T (P_j - P_i)(x - \hat{x}^+) = 0.$$

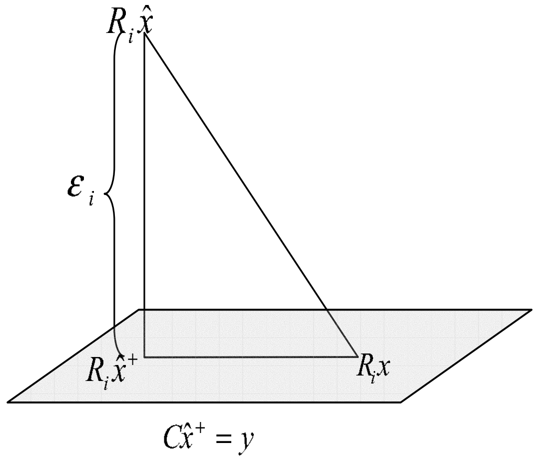

Figure 4.2: Projection of $R_i\hat{x}$ onto the plan $y = C\hat{x}^+$

Hence, (4.10) is equivalent to

$$(x - \hat{x}^+)^T P_i (x - \hat{x}^+) \leq (x - \hat{x})^T P_i (x - \hat{x}). \tag{4.11}$$

Since R_i is the factorization of P_i, (4.11) is equivalent to:

$$\|R_i(x - \hat{x}^+)\| \leq \|R_i(x - \hat{x})\|. \tag{4.12}$$

Now let's compute the updated estimated state $\hat{x}^+$ in the hyperplane $y = C\hat{x}^+$ such that the distance $\|R_i(\hat{x}^+ - \hat{x})\|$ is minimized. This leads to the following optimization problem:

$$\begin{aligned} &\min \|R_i(\hat{x}^+ - \hat{x})\|, \\ &\text{subject to} : y = C\hat{x}^+. \end{aligned} \tag{4.13}$$

This problem can be described geometrically in Figure 4.2.

Let's introduce new variables $\epsilon_i = R_i(\hat{x}^+ - \hat{x})$, then we have $\hat{x}^+ = R_i^{-1}(\epsilon_i + R_i\hat{x})$, the optimization problem (4.13) can be replaced as:

$$\begin{aligned} &\min \|\epsilon_i\|, \\ &\text{subject to} : CR_i^{-1}\epsilon_i = y - C\hat{x}. \end{aligned}$$

The solution of this problem is $R_i(\hat{x}^+ - \hat{x}) = (CR_i^{-1})^\dagger(y - C\hat{x})$, which implies

(4.9) of the main theorem.

Moreover, the vectors $R_i(\hat{x}^+ - \hat{x})$ and $R_i(x - \hat{x}^+)$ are orthogonal in the plane $y = C\hat{x}^+$, this implies that (4.12) is verified, herein (4.10) is satisfied, ending the proof of the first part. Note that this part of the proof is same as in [41].

To prove the second part of the theorem, we introduce a new variable $\bar{g} = [e^T x^T]\Gamma_{i,j}[e^T x^T]^T$. From (4.8b), we have

$$\begin{aligned}
\bar{g} &= e^T\Gamma_{i,j}^{11}e + x^T\Gamma_{i,j}^{21}e + e^T\Gamma_{i,j}^{12}x + x^T\Gamma_{i,j}^{22}x \\
&= e^T(A_i - K_iC)^T P_i e + e^T P_i(A_i - K_iC)e + e^T e + \nu_{ij}e^T e \\
&\quad + x^T(A_j - A_i)^T P_i e + e^T P_i(A_j - A_i)x + \mu_{ij}x^T Q_i x - \epsilon^2\nu_{ij}x^T x \\
&\quad + 2e^T P_i B \sum_{k=1}^{n} e_{n(k)}^T \gamma_k e. \qquad (4.14)
\end{aligned}$$

Let's introduce another variable g, so that the difference between $\bar{g}$ and g is that γ_k is replaced by h_k:

$$\begin{aligned}
g &= e^T(A_i - K_iC)^T P_i e + e^T P_i(A_i - K_iC)e + e^T e + \nu_{ij}e^T e \\
&\quad + x^T(A_j - A_i)^T P_i e + e^T P_i(A_j - A_i)x + \mu_{ij}x^T Q_i x - \epsilon^2\nu_{ij}x^T x \\
&\quad + 2e^T P_i B \sum_{k=1}^{n} e_{n(k)}^T h_k e. \qquad (4.15)
\end{aligned}$$

Due to the fact that γ_k are the vertices of h_k and g is affine according to h_k, $g \le 0$ is equivalent to $\bar{g} \le 0$. Hence, if the conditions of the theorem are satisfied, we have $g \le 0$.

By subtracting (4.15) from (4.7), we have

$$\dot{V}_i(e) = g - \nu_{ij}e^T e - \mu_{ij}x^T Q_i x - e^T e + \epsilon^2\nu_{ij}x^T x.$$

With $g \le 0$, this implies

$$\dot{V}_i(e) \le -\nu_{ij}e^T e - \mu_{ij}x^T Q_i x - e^T e + \epsilon^2\nu_{ij}x^T x.$$

Now, let set $G = \frac{-\nu_{ij}-1}{\beta}$ and $H = \nu_{ij}\epsilon^2 x_{\max}^2$, then we obtain

$$\dot{V}_i(e) \le GV_i(e) + H.$$

By integrating this first order differential equation, we obtain:

$$\begin{aligned} V_i(e) &\leq e^{G(t-t_0)}V(e(t_0)) - \frac{H}{G}(1 - e^{G(t-t_0)}) \\ &\leq e^{\frac{-\nu-1}{\beta}(t-t_0)}V(e(t_0)) + \frac{\bar{\nu}\epsilon^2 x_{\max}^2 \beta}{\bar{\nu}+1}(1 - e^{\frac{-\nu-1}{\beta}(t-t_0)}). \end{aligned}$$

For $t \to \infty$, this inequality simplifies yields:

$$V_i(e) \leq \frac{\bar{\nu}\beta}{\bar{\nu}+1}\epsilon^2 x_{\max}^2$$

which implies

$$\lim_{t\to\infty} \sup \|e(t)\| \leq \sqrt{\frac{\bar{\nu}}{\bar{\nu}+1}}\sqrt{\frac{\beta}{\alpha}}\epsilon x_{\max}.$$

It means that the energy in every mode is upper bounded by a constant. This ends the proof.

Remark 4.4.1 *Note that inequality* (4.8b) *is nonlinear in the unknown variables* P_i, K_i, ϵ *and* ν_{ij}. *By introducing the new variables* $W_i = P_i K_i$ *with* $W_i \in I\!R^{n\times p}$, *we get*

$$\Gamma_{i,j}^{11} = A_i^T P_i - C^T W_i^T + P_i A_i - W_i C + I + \nu_{ij} I + 2P_i B \sum_{k=1}^{n} e_{n(k)}^T \gamma_k.$$

This inequality becomes linear in the unknown variables by fixing ϵ. *Hence, it can be easily solved by using numerical methods. If the conditions of Theorem 4.4.1 admit a solution, the gain matrices* K_i *can be calculated as* $K_i = P_i^{-1} W_i$ *because* P_i *are positive-definite. In practice, we proceed by varying the parameter* ϵ *until a solution to the LMIs is met.*

Remark 4.4.2 *To prevent the observer gains* K_i *to be too large which will make the observer dynamics sensitive to measurement noise, it is necessary to restrict the value of* W_i. *For instance, we can restrict* W_i *using* $W_i^T W_i \leq \lambda^2 I$, *which can be formulated as*

$$\begin{bmatrix} W_i & \lambda_i I_{n\times n} \\ \lambda_i I_{p\times p} & W_i^T \end{bmatrix} \leq 0$$

where λ_i *are design parameters.*

Remark 4.4.3 *When $f(x) = 0$, theorem 1 is equivalent to the main theorem in [41], which is the linear case. Our contribution is an extension of this approach to the case.*

4.5 Numerical example

In this section, we present a numerical example in order to show the applicability of our result.

Consider the switched system (4.1) with two modes where A_i, B and C are given by:
$A_1 = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ -2.4 \end{bmatrix}^T$, $f(x) = a\cos(x_1)$ with $a = 0.01$ and $s_{12} = [1.56 \quad 1]$, $s_{21} = [1 \quad -1.56]$.

The design parameters are chosen as $\lambda_1 = \lambda_2 = 5$, $\epsilon = 4.8$. By solving the LMI conditions of Theorem 4.4.1 with the SeDuMi1.1 solver, we obtain the following observer gains:

$$K_1 = [1.8561 \quad -2.9405]^T$$
$$K_2 = [-5.7495 \quad -7.7089]^T$$

and $\alpha = 0.4$, $\beta = 18.8$, $\bar{\nu} = 5.4$. With the processor Inter Pentium M, 1400 Mhz, the CPU execution time is s = 813 ms. According to Theorem 4.4.1, we obtain the bound $\| e_k \| \leq 31.4\ x_{\max}$.

With the initial conditions $x_{(0)} = [0 \quad 1]^T$ (in region Ω_1) and $\hat{x}_{(0)} = [1 \quad 0]^T$ (in region Ω_2), the trajectories of the system and the observer are shown in Figure 4.3 and Figure 4.4. Figure 4.5 shows the evolution of the energy function. The evolution of the active modes of the system and of the observer are shown in Figure 4.6. The simulation of this example shows the convergence of the estimation error in 2.5 seconds, which is better than the upper bound of the main theorem.

4.6 Conclusion

In this chapter, new sufficient conditions for the observer synthesis of nonlinear switched system are proposed. By using multiple quadratic Lyapunov functions, the main advantage of the proposed observer lies in the fact that the knowledge of the

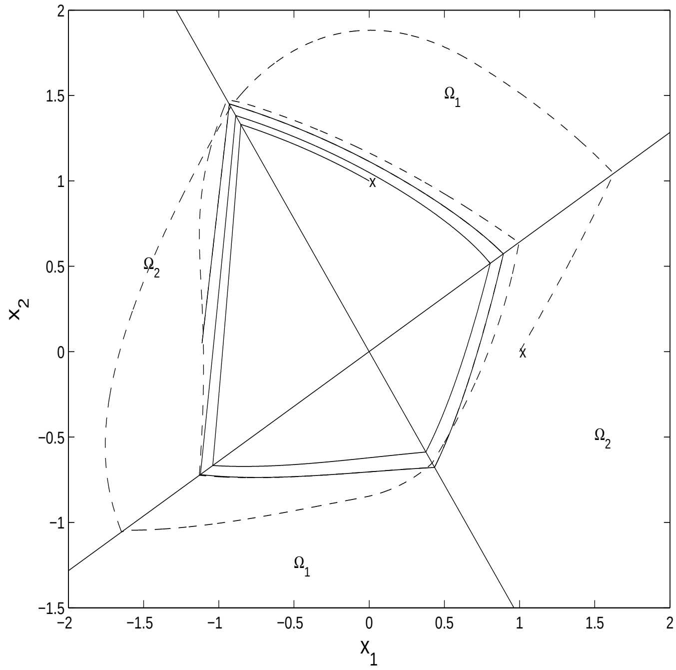

Figure 4.3: Trajectory of the system (solid line) and the observer (dash-dotted)

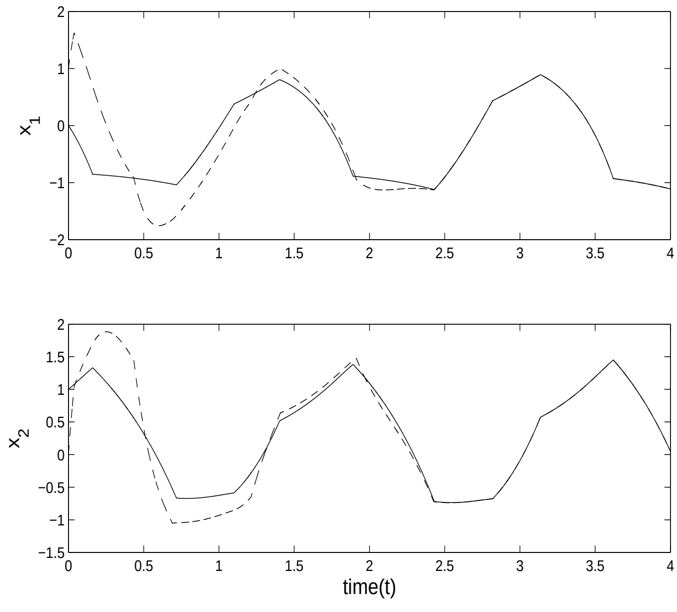

Figure 4.4: Trajectory of the system (solid line) and the observer (dash-dotted) with respect to time

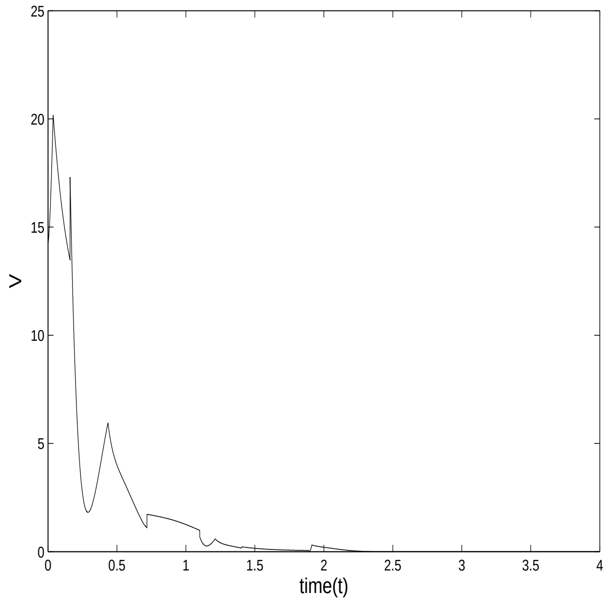

Figure 4.5: Energy function

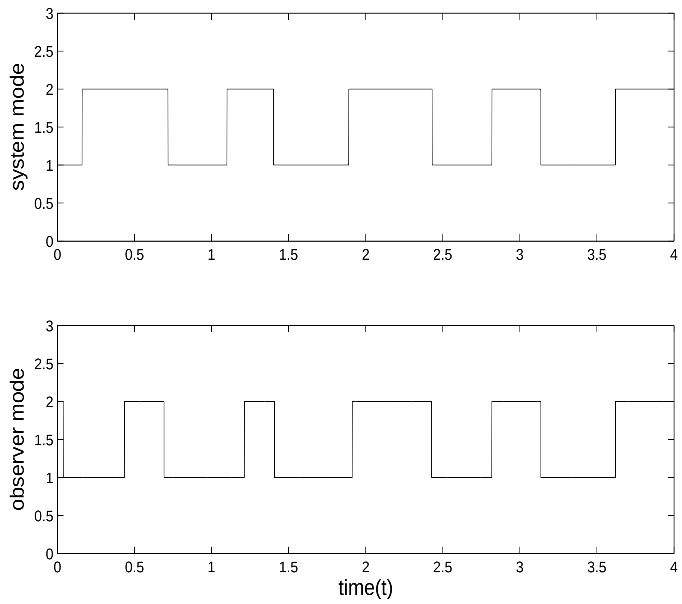

Figure 4.6: Evolution of the active mode of the system (first subfigure) and of the observer (second subfigure)

active mode of the switched system is not required. In order to improve the bound of the estimation error, the estimated state is updated at each switching instant. When a scale parameter is fixed a priori, these conditions can be expressed as a LMI feasibility problem that is easily tractable by convex optimization techniques. The interest of the proposed conditions have been validated on a numerical example.

Chapter 5

Observer design for nonlinear multimodel systems

5.1 Introduction

In this chapter, we address the problem of observer synthesis for a class of discrete-time multimodel systems. The multimodel system is composed of two Lipschitz nonlinear systems. By using a particular Lyapunov function, we give sufficient conditions for the observer synthesis. The synthesis problem is brought to a Linear Matrix Inequality (LMI) feasibility problem which is easily tractable by optimization techniques.

This chapter is structured as follows. In section 5.2, we introduce the considered class of multimodel systems and the structure of the observer. In section 5.3, we present our synthesis method of the observer. In section 5.4, the applicability of the proposed approach is demonstrated on a numerical example.

5.2 Problem formulation

In this section, a class of multimodel systems, based on two discrete-time nonlinear chaotic system, is introduced.

Consider two discrete-time systems with nonlinear terms:

$$\begin{cases} x_{k+1} = A_1 x_k + f_1(x_k) \\ y_k = C x_k \end{cases} \tag{5.1}$$

and:

$$\begin{cases} x_{k+1} = A_2 x_k + f_2(x_k) \\ y_k = C x_k \end{cases} \tag{5.2}$$

where $x \in I\!R^n$, $y \in I\!R^p$ are the state vector and the output vector, respectively. Variables A_1, A_2 and C represent constant matrices of appropriate dimensions. Functions f_1 and $f_2 : I\!R^n \to I\!R^n$ are nonlinear and assumed to be Lipschitz with respect to x_k, i.e $\exists k_i > 0$ such that:

$$\| f_i(x) - f_i(y) \|_2 \leq k_i \| x - y \|_2 \tag{5.3}$$

The $k_i, i = 1, 2$ are the Lipschitz constants.

Owing (1) and (2), we get the following multimodel nonlinear system:

$$\begin{cases} x_{k+1} = \mu(y_k)(A_1 x_k + f_1(x_k)) \\ \qquad\qquad +(1 - \mu(y_k))(A_2 x_k + f_2(x_k)) \\ y_k = C x_k \end{cases} \tag{5.4}$$

in which $\mu(y_k)$ is the activation function depending of the value of y_k and $0 \leq \mu(y_k) \leq 1$. Notice that $\mu(y_k)$ is not required to be Lipschitz.

We consider the following observer described by:

$$\begin{cases} \hat{x}_{k+1} = \mu(y_k)(A_1 \hat{x}_k + f_1(\hat{x}_k) + K_1(y_k - C\hat{x}_k)) \\ \qquad\qquad +(1 - \mu(y_k))(A_2 \hat{x}_k + f_2(\hat{x}_k) + K_2(y_k - C\hat{x}_k)) \\ \hat{y}_k = C\hat{x}_k \end{cases} \tag{5.5}$$

where $\hat{x} \in I\!R^n$ is the estimate of the state vector x and $K_i \in I\!R^{n \times p}$ for $i = 1, 2$ are the observer gains to be determined.

Denoting the estimation error by $e_k = x_k - \hat{x}_k$, its dynamic writes:

$$\begin{aligned} e_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\ &= \mu(y_k)((A_1 - K_1 C)e_k + \Delta f_{1k}) \\ &\quad +(1 - \mu(y_k))((A_2 - K_2 C)e_k + \Delta f_{2k}) \end{aligned} \tag{5.6}$$

where $\Delta f_{ik} = f_i(x_k) - f_i(\hat{x}_k), \forall i = 1, 2.$

5.3 Main result

In this section, we introduce the main contribution of our paper which consists in sufficient conditions for the synthesis of the observer (5.5).

Theorem 5.3.1 *There exists an observer* (5.5) *for the nonlinear multimodel system* (5.4) *if there exist positive-definite matrices* $P = P^T > cI$ $(P \in I\!R^{n\times n})$*,* $Q = Q^T > 0 (Q \in I\!R^{2n\times 2n})$*,* $R_i (R_i \in I\!R^{p\times n})$ *for all* $i = 1, 2$ *and scalars* $a > 0$*,* $b > 0$*,* $c > 0$*, such that the conditions* (5.7a) (5.7b) (5.7c) *are satisfied,*

$$\begin{bmatrix} (ak_1^2+bk_2^2)I-P & (\star) & 0 & (\star) \\ (d+1)(PA_1-R_1^TC) & (d+1)P-aI-Q_{11} & (\star) & 0 \\ 0 & -Q_{21} & -bI-Q_{22} & 0 \\ PA_1-R_1^TC & 0 & 0 & -P/(d+1) \end{bmatrix} < 0 \quad (5.7a)$$

$$\begin{bmatrix} (ak_1^2+bk_2^2)I-P & 0 & (\star) & (\star) \\ 0 & -aI-Q_{11} & (\star) & 0 \\ (d+1)(PA_2-R_2^TC) & -Q_{21} & (d+1)P-bI-Q_{22} & 0 \\ PA_2-R_2^TC & 0 & 0 & -P/(d+1) \end{bmatrix} < 0 \quad (5.7b)$$

$$Q - cI < 0 \quad (5.7c)$$

with:

$$d = k_1^2 + k_2^2,$$
$$Q = \begin{bmatrix} Q_{11} & Q_{21}^T \\ Q_{21} & Q_{22} \end{bmatrix} (Q_{ij} \in I\!R^{n\times n}, i, j = 1, 2).$$

The gains of the observer are given by $K_i = P^{-1}R_i^T$*.*

Proof. Assume that all the conditions of the theorem are satisfied. The proof of the theorem simply consists of showing that the observer is convergent in the paradigm of the Lyapunov stability. Let us consider the following Lyapunov function:

$$V_k = e_k^T P e_k + \Delta f_k^T Q \Delta f_k \quad (5.8)$$

with $\Delta f_k = [\Delta f_{1k}^T \;\; \Delta f_{2k}^T]^T$.

We have:

$$\begin{aligned}
V_{k+1} &= e_{k+1}^T P e_{k+1} + \Delta f_{k+1}^T Q \Delta f_{k+1} \\
&= e_k^T \{\mu^2 \tilde{A}_1^T P \tilde{A}_1 + \mu(1-\mu)\tilde{A}_1^T P \tilde{A}_2 + \mu(1-\mu)\tilde{A}_2^T P \tilde{A}_1 \\
&+ (1-\mu)^2 \tilde{A}_2^T P \tilde{A}_2\} e_k + e_k^T \{\mu^2 \tilde{A}_1^T P \Delta f_{1k} + \mu(1-\mu)\tilde{A}_1^T P \Delta f_{2k} \\
&+ \mu(1-\mu)\tilde{A}_2^T P \Delta f_{1k} + (1-\mu)^2 \tilde{A}_2^T P \Delta f_{2k}\} \\
&+ \{\mu^2 \Delta f_{1k}^T P \tilde{A}_1 + \mu(1-\mu)\Delta f_{1k}^T P \tilde{A}_2 + \mu(1-\mu)\Delta f_{2k}^T P \tilde{A}_1 \\
&+ (1-\mu)^2 \Delta f_{2k}^T P \tilde{A}_2\} e_k + \mu^2 \Delta f_{1k}^T P \Delta f_{1k} + \mu(1-\mu)\Delta f_{1k}^T P \Delta f_{2k} \\
&+ \mu(1-\mu)\Delta f_{2k}^T P \Delta f_{1k} + (1-\mu)^2 \Delta f_{2k}^T P \Delta f_{2k} + \Delta f_{k+1}^T Q \Delta f_{k+1}
\end{aligned}$$

where $\tilde{A}_i = A_i - K_i C, i = 1, 2$.

Now, let us compute $\Delta V = V_{k+1} - V_k$ which can be rewritten as (5.9)

$$\Delta V = \xi_k^T \begin{bmatrix} D - P & (\star) & (\star) & 0 & 0 \\ \mu^2 P \tilde{A}_1 + \mu(1-\mu) P \tilde{A}_2 & \mu^2 P - Q_{11} & (\star) & 0 & 0 \\ \mu(1-\mu) P \tilde{A}_1 + (1-\mu)^2 P \tilde{A}_2 & \mu(1-\mu) P - Q_{21} & (1-\mu)^2 P - Q_{22} & 0 & 0 \\ 0 & 0 & 0 & Q_{11} & Q_{12} \\ 0 & 0 & 0 & Q_{12} & Q_{22} \end{bmatrix} \xi_k \tag{5.9}$$

where:

$$\begin{aligned}
D &= \mu^2 \tilde{A}_1^T P \tilde{A}_1 + \mu(1-\mu)\tilde{A}_1^T P \tilde{A}_2 \\
&\quad + \mu(1-\mu)\tilde{A}_2^T P \tilde{A}_1 + (1-\mu)^2 \tilde{A}_2^T P \tilde{A}_2, \\
\xi_k^T &= \begin{bmatrix} e_k^T & \Delta f_k^T & \Delta f_{k+1}^T \end{bmatrix}.
\end{aligned}$$

The Lipschitz condition (5.3) implies that, for any positive scalars a and b, the following inequality holds:

$$(ak_1^2 + bk_2^2) e_k^T e_k - a\Delta f_{1k}^T \Delta f_{1k} - b\Delta f_{2k}^T \Delta f_{2k} \geq 0, \forall a, b > 0.$$

Denoting

$$L_1 = \begin{bmatrix} (ak_1^2+bk_2^2)I & 0 & 0 & 0 & 0 \\ 0 & -aI & 0 & 0 & 0 \\ 0 & 0 & -bI & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the inequality can be written:

$$\xi_k^T L_1 \xi_k \geq 0.$$

From the Lipschitz condition (5.3), it is possible to derive that the following condition holds:

$$(k_1^2+k_2^2)e_{k+1}^T P e_{k+1} - \Delta f_{k+1}^T P \Delta f_{k+1} \geq 0.$$

The condition $P > cI$ of theorem ($\forall c > 0$) implies:

$$\Delta f_{k+1}^T P \Delta f_{k+1} - c\Delta f_{k+1}^T \Delta f_{k+1} > 0.$$

In addition to the previous inequality, we have then:

$$(k_1^2+k_2^2)e_{k+1}^T P e_{k+1} - c\Delta f_{k+1}^T \Delta f_{k+1} > 0. \tag{5.10}$$

From the condition (5.6), the inequality (5.10) can be rewritten as:

$$\xi_k^T L_2 \xi_k \geq 0$$

$$L_2 = \begin{bmatrix} dD & (\star) & (\star) & 0 & 0 \\ d(\mu^2\tilde{A}_1^T P + \mu(1-\mu)\tilde{A}_2^T P) & d\mu^2 P & (\star) & 0 & 0 \\ d(\mu(1-\mu)\tilde{A}_1^T P + (1-\mu)^2\tilde{A}_2^T P) & d\mu(1-\mu)P & d(1-\mu)^2 P & 0 & 0 \\ 0 & 0 & 0 & -cI & 0 \\ 0 & 0 & 0 & 0 & -cI \end{bmatrix} \tag{5.11}$$

where L_2 is given in (5.11).

Let us introduce an additional matrix M given in (5.12)

$$M = \begin{bmatrix} M^{11} & (\star) & (\star) & 0 & 0 \\ M^{21} & (d+1)\mu^2 P - aI - Q_{11} & (\star) & 0 & 0 \\ M^{31} & (d+1)\mu(1-\mu)P - Q_{21} & M^{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{11} - cI & Q_{21}^T \\ 0 & 0 & 0 & Q_{21} & Q_{22} - cI \end{bmatrix} \tag{5.12}$$

with

$$\begin{aligned} d &= k_1^2 + k_2^2, \\ M^{11} &= (d+1)D + (ak_1^2 + bk_2^2)I - P, \\ M^{21} &= (d+1)(\mu^2 P\tilde{A}_1 + \mu(1-\mu)P\tilde{A}_2), \\ M^{31} &= (d+1)(\mu(1-\mu)P\tilde{A}_1 + (1-\mu)^2 P\tilde{A}_2), \\ M^{33} &= (d+1)(1-\mu)^2 P - bI - Q_{22}. \end{aligned}$$

It is easy to check that $\xi_k^T M \xi_k = \Delta V + \xi_k^T L_1 \xi_k + \xi_k^T L_2 \xi_k$. Therefore, as $\xi_k^T L_1 \xi_k \geq 0$ and $\xi_k^T L_2 \xi_k \geq 0$, the matrix inequality $M < 0$ implies $\Delta V < 0$.

Due to its block-diagonal structure, the inequality $M < 0$ is equivalent to the inequalities:

$$Q < cI \tag{5.13}$$

and

$$F < 0 \tag{5.14}$$

where F is given by (5.15).

$$F = \begin{bmatrix} (d+1)D + (ak_1^2 + bk_2^2)I - P & (\star) & \dots \\ (d+1)(\mu^2 P\tilde{A}_1 + \mu(1-\mu)P\tilde{A}_2) & (d+1)\mu^2 P - aI - Q_{11} & \dots \\ (d+1)(\mu(1-\mu)P\tilde{A}_1 + (1-\mu)^2 P\tilde{A}_2) & (d+1)\mu(1-\mu)P - Q_{21} & \dots \end{bmatrix}$$

$$\begin{matrix} \dots & (\star) \\ \dots & (\star) \\ \dots & (d+1)(1-\mu)^2 P - bI - Q_{22} \end{matrix} \Bigg] \tag{5.15}$$

Matrix F depends quadratically on μ. Based on the convexity principle, sufficient conditions are derived in order to assure that $F < 0$ holds for any value of $\mu \in [0,1]$: $F(\mu) < 0$ with $\mu = \{0,1\}$ and $\dfrac{\partial^2 F}{\partial \mu^2} \geq 0$, providing the inequalities (5.16), (5.17) and (5.18).

$$F_{(1)} = \begin{bmatrix} (d+1)\tilde{A}_1^T P \tilde{A}_1 + (ak_1^2 + bk_2^2)I - P & (\star) & 0 \\ (d+1)P\tilde{A}_1 & (d+1)P - aI - Q_{11} & (\star) \\ 0 & -Q_{21} & -bI - Q_{22} \end{bmatrix} < 0 \tag{5.16}$$

$$F_{(0)} = \begin{bmatrix} (d+1)\tilde{A}_2^T P \tilde{A}_2 + (ak_1^2 + bk_2^2)I - P & 0 & (\star) \\ 0 & -aI - Q_{11} & (\star) \\ (d+1)P\tilde{A}_2 & -Q_{21} & (d+1)P - bI - Q_{22} \end{bmatrix} < 0 \tag{5.17}$$

$$(d+1) \begin{bmatrix} \tilde{A}_1^T P \tilde{A}_1 - \tilde{A}_1^T P \tilde{A}_2 - \tilde{A}_2^T P \tilde{A}_1 + \tilde{A}_2^T P \tilde{A}_2 & (\star) & (\star) \\ P\tilde{A}_1 - P\tilde{A}_2 & P & (\star) \\ -P\tilde{A}_1 + P\tilde{A}_2 & -P & P \end{bmatrix} \geq 0 \tag{5.18}$$

By using the Schur complement and the notation $R_i = K_i^T P$, the inequalities (5.16) and (5.17) are respectively equivalent to (5.7a) and (5.7b) of the main theorem, and the inequality (5.13) is equivalent to (5.7c). By using the Schur complement for the non-strict inequalities [46], the inequality (5.18) is equivalent to the following two inequalities:

$$P \geq 0,$$
$$\begin{bmatrix} \Delta\tilde{A}^T P \Delta\tilde{A} & \Delta\tilde{A}^T P \\ P\Delta\tilde{A} & P \end{bmatrix} - \begin{bmatrix} -\Delta\tilde{A}^T P \\ -P \end{bmatrix} P^{-1} \begin{bmatrix} -P\Delta\tilde{A} & -P \end{bmatrix} \geq 0$$

where $\tilde{A} = \tilde{A}_1 - \tilde{A}_2$.

Because of $\begin{bmatrix} -\Delta\tilde{A}^T P \\ -P \end{bmatrix} (I - PP^{-1}) = 0$, we deduce that these conditions are always satisfied.

Remark 5.3.1 *Note that for a negative-definite matrix, each (i,i) bloc of the diagonal is negative-definite. Therefore, if the LMI conditions of the main theorem are* *verified then the blocs (1,1), (2,2) of* (5.7a) *and (3,3) of* (5.7b) *are negative-definite which implies that the Lipschitz constants* k_1, k_2 *are less than one. Thus, the developed approach will not be efficient with nonlinear models that do not satisfy this constraint.*

Remark 5.3.2 *This Lyapunov function* (5.8) *method has some conservatism because of the diagonal structure of the matrices* P *and* Q*. This conservatism could be eliminated by using the following Lyapunov function:*

$$V_k = \begin{bmatrix} e_k \\ \Delta f_k \end{bmatrix}^T \begin{bmatrix} P & S^T \\ S & Q \end{bmatrix} \begin{bmatrix} e_k \\ \Delta f_k \end{bmatrix}.$$

5.4 Numerical example

In this section, we present a numerical example in order to show the applicability of our method.

Consider the following nonlinear system, which is described under the form (5.1) (5.2) with the following parameters:

$$A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0.199 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0.199 & 0 \\ 3 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T,$$
$$f_1(x_k) = \begin{bmatrix} 0.02e^{-x_{1k}^2} \\ 0 \\ -5.7 - 0.02e^{-x_{3k}^2} \end{bmatrix}, f_2(x_k) = \begin{bmatrix} 0.01e^{-x_{1k}^2} \\ 0 \\ -5.7 - 0.02e^{-x_{3k}^2} \end{bmatrix}.$$

Now we establish the multimodel system by using the function $\mu(y_k) = (1 + \tanh(\epsilon y_k))/2$. With initial condition $x_{(0)} = [0 \quad 0 \quad 0]^T$, the evolution of $\mu(y_k)$, the trajectories of nonlinear systems and the trajectory of the multimodel system (5.4) are shown respectively in Figure 5.1, Figure 5.2, Figure 5.3 and Figure 5.4.

By solving the LMI conditions of Theorem 5.3.1 with the SeDuMi1.1 solver, we obtain the following observer gains:

$$K_1 = [0.1501 \;\; 0.9702 \;\; 1.000]^T,$$
$$K_2 = [0.1503 \;\; 0.9701 \;\; 3.000]^T,$$

with the processor Inter Pentium M, 1400 Mhz, the CPU execution time is 344 ms.

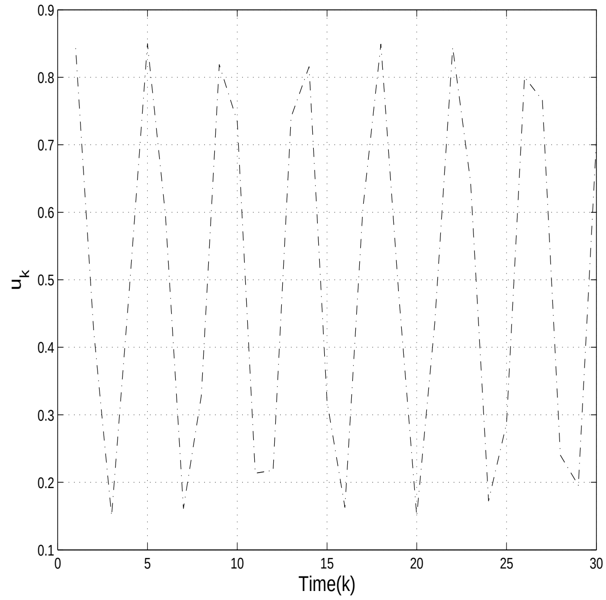

Figure 5.1: Evolution of the active function μ

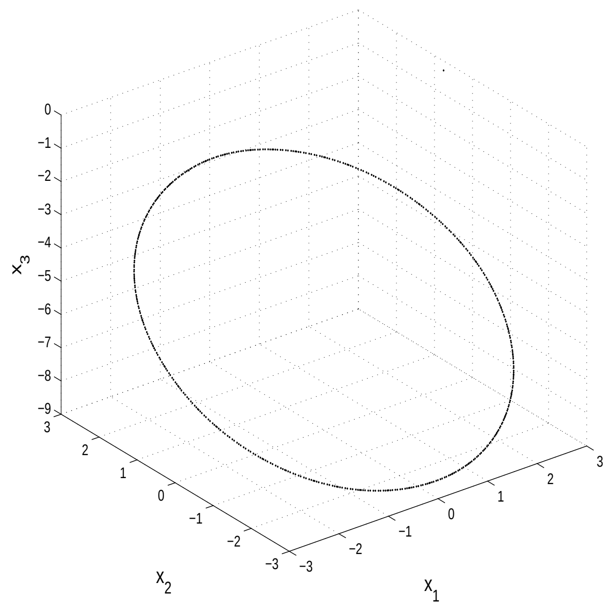

Figure 5.2: Trajectory of nonlinear system 1

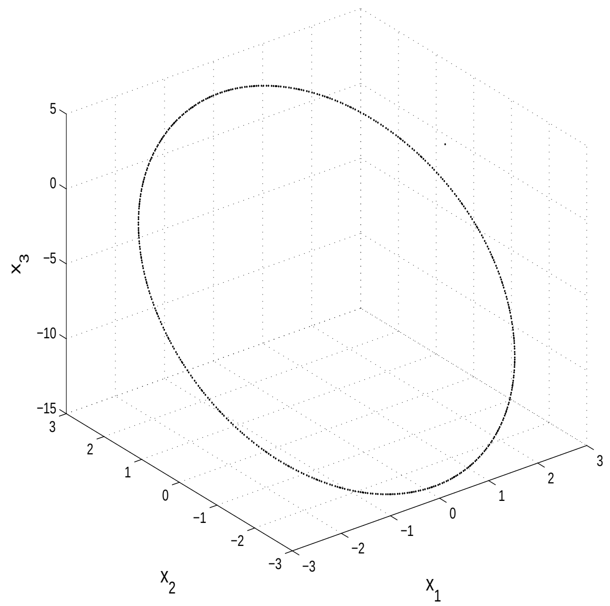

Figure 5.3: Trajectory of nonlinear system 2

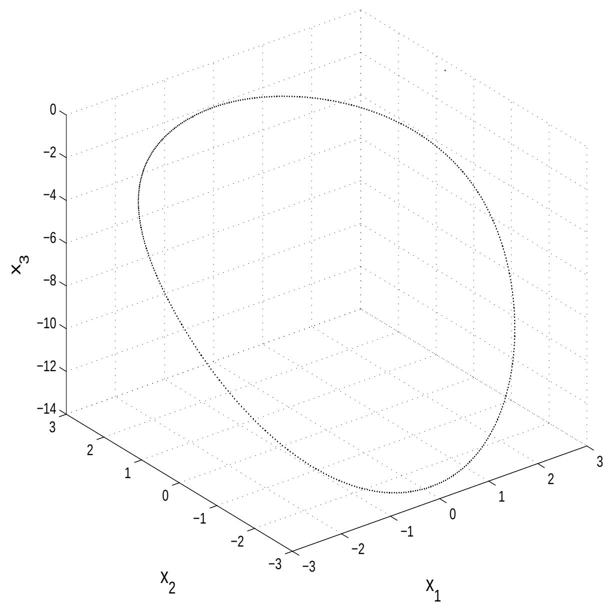

Figure 5.4: Trajectory of multimodel system

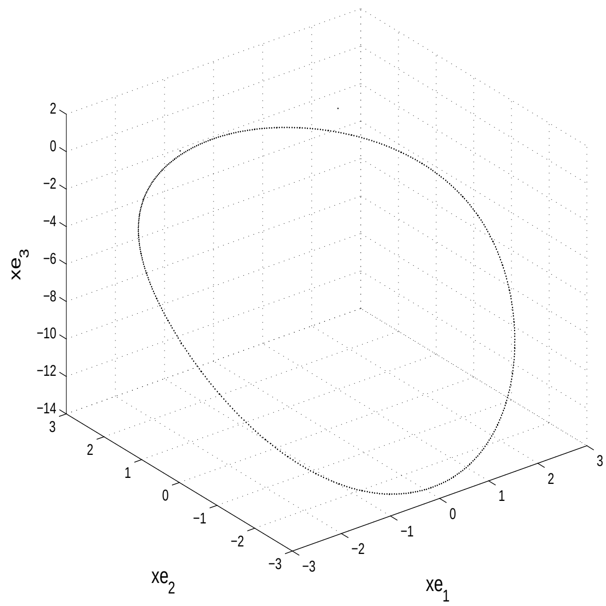

Figure 5.5: Trajectory of multimodel observer

With initial condition $\hat{x}_{(0)} = [1 \quad 1 \quad 1]^T$, the observer trajectory is shown in Figure 5.5. Figure 5.6 shows the evolution of the estimation error. This shows that the estimation error converges after two iterations.

5.5 Conclusion

In this chapter, the problem of observer synthesis is addressed for a class of discrete-time multimodel systems. The multimodel system is established by two discrete-time Lipschitz systems. By using a particular Lyapunov function, sufficient conditions for the synthesis of the observer is proposed. The applicability of this condition is confirmed by a numerical example.

Notice that the result is only applicable to systems with Lipschitz constants whose values are less than one. Future work on this topic could be dedicated to the suppression of this limitation.

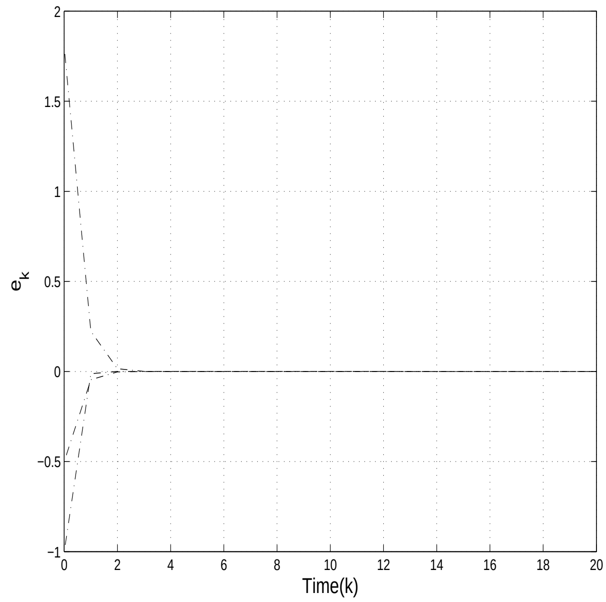

Figure 5.6: Evolution of the estimation error

Chapter 6

Unknown-input observer design for nonlinear multimodel systems

6.1 Introduction

Over the past decade, the synchronization problem between a transmitter and a receiver, in the context of secure communication, became a challenging issue. Let us notice that one of the most attractive and efficient synchronization techniques, largely investigated during the last decade, is based on state observers [47], [48], [49], [50], [51], [52]. If continuous-time nonlinear systems benefit from a well developed control theory, analysis and synthesis of discrete-time systems remain a complex and difficult problem in particular state observers design as can be shown in [53], [54]. However, communication systems are generally digital and require specific approaches.

Nonlinear Multi-Models were recently introduced to generate a new class of systems with complex behaviors that may be used for data encryption in communication systems. Note that, there are few results for the estimation of multiple model system with unknown input. In [55], [56], a multiple observer with sliding mode for a multimodel system with unknown input is proposed. In [57], the synthesis of a Luenberger type observer is proposed, the considered system is represented by a discrete-time multimodel linear system with unknown input.

In this chapter, we investigate observer design for a nonlinear multimodel system to assure transmitter-receiver synchronization. To match this goal, we introduce a simple and useful Lyapunov function, with a nonlinearity term, that leads to non conservative and convex conditions for convergence. Indeed, through some artefact we provide sufficient and Linear Matrix Inequalities to assure synchronization. After that, at the receiver, we easily deduce the encrypted signals. A diagram illustrating synchronization and input recovery for our approach is shown in Figure 6.1.

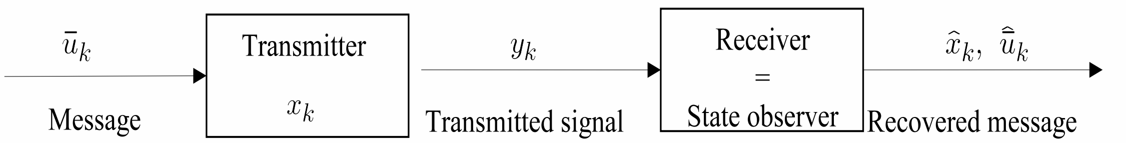

Figure 6.1: Synchronization based observer approach

This chapter is structured as follows. In Section 6.2, we introduce the considered class of multimodel systems, the structure of the observer and the estimation, then the input recovery equation is given. In Section 6.3, we present our synthesis method of the observer design. In section 6.4, the applicability of the proposed approach is demonstrated on a numerical example. Finally, concluding remarks are made in the last section.

6.2 Problem formulation

In this section, the observer problem for a class of multimodel systems with unknown input, which is based on two discrete-time nonlinear systems, is introduced.

Consider the following multimodel nonlinear system:

$$\begin{cases} x_{k+1} = \mu(A_1 x_k + f_1(x_k)) + (1-\mu)(A_2 x_k + f_2(x_k)) + D\bar{u}_k \\ y_k \quad\;\; = C x_k \end{cases} \tag{6.1}$$

where $x \in I\!R^n$, $y \in I\!R^p$, $\bar{u} \in I\!R^m$ are the state vector, the output vector and the unknown input respectively. A_1, A_2, C and D are constant matrices of appropriate dimensions. Without loss of generality, we assume that D is of full column rank. The functions f_1 and f_2: $I\!R^n \to I\!R^n$ are nonlinear and assumed to be Lipschitz with respect to x_k, i.e

$$\| f_i(x_1) - f_i(x_2) \|_2 \leq k_i \| x_1 - x_2 \|_2 \tag{6.2}$$

where $k_i > 0$ is the Lipschitz constant for $i = 1, 2$. $\mu(y_k)$ is the activation function according to the value of y_k and $0 \leq \mu(y_k) \leq 1$.

Inspired by the full order unknown input observer presented in [58], we consider the following observer:

$$\begin{cases} z_{k+1} = \mu(N_1 z_k + L_1 y_k + M(\mu) f_1(\hat{x}_k)) \\ \qquad\qquad +(1-\mu)(N_2 z_k + L_2 y_k + M(\mu) f_2(\hat{x}_k)) \\ \hat{x}_{k+1} = z_{k+1} - \mu E_1 y_{k+1} - (1-\mu) E_2 y_{k+1} \end{cases} \tag{6.3}$$

where $\hat{x} \in I\!R^n$ is the estimate of the state x and $L_i \in I\!R^{n\times p}$ is the observer gain. The matrix functions N_i, L_i and $M(\mu)$ are defined by:

$$\begin{aligned} &N_i = M(\mu)A_i - K_i C, \ \forall i = 1, 2, \\ &L_i = K_i(I + CE_i) - M(\mu)A_i E_i, \ \forall i = 1, 2, \\ &M(\mu) = \sum_{i=1}^{2} M_i \text{ where } M_1 = \mu(I + E_1 C) \text{ and } M_2 = (1-\mu)(I + E_2 C), \end{aligned} \tag{6.4}$$

E_i,K_i for $i = 1, 2$ are the matrices to be find.

The estimation error is defined as:

$$\begin{aligned} e_k &= x_k - \hat{x}_k \\ &= x_k - z_k + \mu E_1 y_k + (1-\mu) E_2 y_k \\ &= (I + \mu E_1 C + (1-\mu) E_2 C) x_k - z_k \\ &= M x_k - z_k \end{aligned}$$

and its dynamics is written as:

$$\begin{aligned} e_{k+1} &= M x_{k+1} - z_{k+1} \\ &= \mu\{(MA_1 - N_1 M - L_1 C) x_k + M(f_1(x_k) - f_1(\hat{x}_k)) + N_1 e_k\} \\ &\quad +(1-\mu)\{(MA_2 - N_2 M - L_2 C) x_k \\ &\quad +M(f_2(x_k) - f_2(\hat{x}_k)) + N_2 e_k\} + MD\bar{u}_k. \end{aligned} \tag{6.5}$$

Let's assume that $(I + E_i C)D = 0$ for all $i = 1, 2$, which implies

$$MD = 0. \tag{6.6}$$

From (6.4) and (6.6), we can deduce that

$$MA_i - N_i M - L_i C = 0 \ \forall i = 1, 2. \tag{6.7}$$

Then (6.5) becomes:

$$e_{k+1} = \mu(N_1 e_k + M\Delta f_{1k}) + (1-\mu)(N_2 e_k + M\Delta f_{2k}). \tag{6.8}$$

where

$$\Delta f_{ik} = f_i(x_k) - f_i(\hat{x}_k) \ \forall i = 1, 2.$$

If CD is of full column rank, all possible solutions of $(I + E_i C)D = 0$ have the following form:

$$E_i = -D(CD)^\dagger + Y_i(I - (CD)(CD)^\dagger)$$

where $(CD)^\dagger = ((CD)^T(CD))^{-1}(CD)^T$ and Y_i can be chosen arbitrarily. This can be rewritten as:

$$E_i = U + Y_i V \tag{6.9}$$

with $U = -D(CD)^\dagger$ and $V = I - (CD)(CD)^\dagger$.

Using (6.4) and (6.9), matrices M and N_i write:

$$\begin{aligned} M &= I + \mu(UC + Y_1 VC) + (1-\mu)(UC + Y_2 VC), \\ N_i &= A_i + \mu(UC + Y_1 VC)A_i \\ &\quad +(1-\mu)(UC + Y_2 VC)A_i - K_i C. \end{aligned} \tag{6.10}$$

If we obtain the matrices gains Y_i,K_i for $i = 1, 2$, then the observer (6.3) can be established.

Note that when both vectors x_k and $\hat{x}_k$ are synchronized, the information $\bar{u}_k$ can be recoverable. In fact, the estimation of the error converges implies

$$\hat{x}_{k+1} = \mu(A_1 \hat{x}_k + f_1(\hat{x}_k) + (1-\mu)(A_2 \hat{x}_k + f_2(\hat{x}_k)) + D\hat{\bar{u}}_k$$

where $\hat{\bar{u}}_k$ is the estimation of unknown input.

Using the least square method, we can deduce the recovered massage signal

$$\begin{aligned}\hat{u}_k &= (D^T D)^{-1} D^T \{\hat{x}_{k+1} - \mu(A_1 \hat{x}_k + f_1(\hat{x}_k)) \\ &\quad -(1-\mu)(A_2 \hat{x}_k + f_2(\hat{x}_k))\}\end{aligned}$$

where $\hat{x}_{k+1}$ is given by observer (6.3).

6.3 Observer design

In this section, we introduce the main contribution of our paper which consists in sufficient conditions for the synthesis of the observer (6.3) to guarantee the asymptotic stability of the estimation.

Theorem 6.3.1 *Assume that CD of the nonlinear multimodel system* (6.1) *is of full column rank, then there exists an observer* (6.3) *if there exist positive-definite matrices $P = P^T > cI$ ($P \in \mathbb{R}^{n\times n}$), $Q = Q^T > 0$ ($Q \in \mathbb{R}^{2n\times 2n}$), $\bar{Y}_i$ ($\bar{Y}_i \in \mathbb{R}^{n\times p}$), $\bar{K}_i$ ($\bar{K}_i \in \mathbb{R}^{n\times p}$) for all $i = 1, 2$ and scalars $a > 0$, $b > 0$, $c > 0$, such that the conditions* (6.11a) (6.11b) (6.11c) *are satisfied,*

$$\begin{bmatrix} (ak_1^2 + bk_2^2)I - P & 0 & 0 & (\star) \\ 0 & -aI - Q_{11} & (\star) & (\star) \\ 0 & -Q_{21} & -bI - Q_{22} & 0 \\ P(A_1 + UCA_1) + \bar{Y}_1 VCA_1 - \bar{K}_1 C & P(I + UC) + \bar{Y}_1 VC & 0 & -P/(d+1) \end{bmatrix} < 0 \tag{6.11a}$$

$$\begin{bmatrix} (ak_1^2 + bk_2^2)I - P & 0 & 0 & (\star) \\ 0 & -aI - Q_{11} & (\star) & 0 \\ 0 & -Q_{21} & -bI - Q_{22} & (\star) \\ P(A_2 + UCA_2) + \bar{Y}_2 VCA_2 - \bar{K}_2 C & 0 & P(I + UC) + \bar{Y}_2 VC & -P/(d+1) \end{bmatrix} < 0 \tag{6.11b}$$

$$Q - cI < 0 \tag{6.11c}$$

with:

$$d = k_1^2 + k_2^2,$$
$$Q = \begin{bmatrix} Q_{11} & Q_{21}^T \\ Q_{21} & Q_{22} \end{bmatrix} (Q_{ij} \in \mathbb{R}^{n\times n}, i, j = 1, 2).$$

$$\Delta V = \xi_k^T \begin{bmatrix} \Gamma - P & (\star) & (\star) & (\star) & 0 & 0 \\ \mu^2 M^T P N_1 + \mu(1-\mu) M^T P N_2 & \mu^2 M^T P M - Q_{11} & \dots & (\star) & 0 & 0 \\ \mu(1-\mu) M^T P N_1 + (1-\mu)^2 M^T P N_2 & \mu(1-\mu) M^T P M - Q_{21} & \dots & (1-\mu)^2 M^T P M - Q_{22} & 0 & 0 \\ 0 & 0 & \dots & 0 & Q_{11} & Q_{12} \\ 0 & 0 & \dots & 0 & Q_{12} & Q_{22} \end{bmatrix} \xi_k \tag{6.13}$$

Proof. Let us consider the following Lyapunov function:

$$V_k = e_k^T P e_k + \Delta f_k^T Q \Delta f_k$$

with $\Delta f_k = [\Delta f_{1k}^T \;\; \Delta f_{2k}^T]^T$.

Then we have

$$\begin{aligned} V_{k+1} &= e_{k+1}^T P e_{k+1} + \Delta f_{k+1}^T Q \Delta f_{k+1} \\ &= e_k^T \Gamma e_k + e_k^T \mu^2 N_1^T P M_1 \Delta f_{1k} + e_k^T \mu(1-\mu) N_1^T P M_2 \Delta f_{2k} \\ &\quad + e_k^T \mu(1-\mu) N_2^T P M_1 \Delta f_{1k} + e_k^T (1-\mu)^2 N_2^T P M_2 \Delta f_{2k} \\ &\quad + \mu^2 \Delta f_{1k}^T M_1^T P N_1 e_k + \mu(1-\mu) \Delta f_{1k}^T M_1^T P N_2 e_k \\ &\quad + \mu(1-\mu) \Delta f_{2k}^T M_2^T P N_1 e_k + (1-\mu)^2 \Delta f_{2k}^T M_2^T P N_2 e_k \\ &\quad + \mu^2 \Delta f_{1k}^T M_1^T P M_1 \Delta f_{1k} + \mu(1-\mu) \Delta f_{1k}^T M_1^T P M_2 \Delta f_{2k} \\ &\quad + \mu(1-\mu) \Delta f_{2k}^T M_2^T P M_1 \Delta f_{1k} + (1-\mu)^2 \Delta f_{2k}^T M_2^T P M_2 \Delta f_{2k} \\ &\quad + \Delta f_{k+1}^T Q \Delta f_{k+1} \end{aligned} \tag{6.12}$$

where

$$\Gamma = \mu^2 N_1^T P N_1 + \mu(1-\mu) N_1^T P N_2 + \mu(1-\mu) N_2^T P N_1 + (1-\mu)^2 N_2^T P N_2.$$

Now, let us compute $\Delta V = V_{k+1} - V_k$ which can be rewritten in (6.13) where

$$\xi_k^T = \begin{bmatrix} e_k^T & \Delta f_k^T & \Delta f_{k+1}^T \end{bmatrix}.$$

The Lipschitz condition (6.2) implies the following inequality:

$$(ak_1^2 + bk_2^2)\, e_k^T e_k - a\,\Delta f_{1k}^T\,\Delta f_{1k} - b\,\Delta f_{2k}^T\,\Delta f_{2k} \geq 0, \forall a, b > 0.$$

This inequality is equivalent to:

$$\xi_k^T\, L_1\, \xi_k \geq 0, \forall a, b > 0$$

where

$$L_1 = \begin{bmatrix} (ak_1^2 + bk_2^2)I & 0 & 0 & 0 & 0 \\ 0 & -aI & 0 & 0 & 0 \\ 0 & 0 & -bI & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By the Lipschitz condition (6.2) and $P > cI$, we obtain:

$$(k_1^2 + k_2^2)e_{k+1}^T P e_{k+1} - c\,\Delta f_{k+1}^T \Delta f_{k+1} > 0, \quad \forall c > 0. \tag{6.14}$$

By substituting e_{k+1} given by (6.5), inequality (6.14) can be rewritten as:

$$\xi_k^T\, L_2\, \xi_k \geq 0, \quad \forall a, b > 0$$

where L_2 is given by:

$$L_2 = \begin{bmatrix} d\Gamma & (\star) & \dots \\ d(\mu^2 M^T P N_1 + \mu(1-\mu) M^T P N_2) & d\mu^2 M^T P M & \dots \\ d(\mu(1-\mu) M^T P N_1 + (1-\mu)^2 M^T P N_2) & d\mu(1-\mu) M^T P M & \dots \\ 0 & 0 & \dots \\ 0 & 0 & \dots \end{bmatrix}$$

$$\begin{matrix} \dots & (\star) & 0 & 0 \\ \dots & (\star) & 0 & 0 \\ \dots & d(1-\mu)^2 M^T P M & 0 & 0 \\ \dots & 0 & -cI & 0 \\ \dots & 0 & 0 & -cI \end{matrix}\Bigg] \tag{6.15}$$

Let's introduce matrix Ω given by

$$\Omega = \begin{bmatrix} \Omega^{11} & (\star) & (\star) & 0 & 0 \\ \Omega^{21} & (d+1)\mu^2 M^T P M - aI - Q_{11} & (\star) & 0 & 0 \\ \Omega^{31} & (d+1)\mu(1-\mu) M^T P M - Q_{21} & \Omega^{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{11} - cI & Q_{21}^T \\ 0 & 0 & 0 & Q_{21} & Q_{22} - cI \end{bmatrix} \tag{6.16}$$

with

$$\begin{aligned} d &= k_1^2 + k_2^2, \\ \Omega^{11} &= (d+1)\Gamma + (ak_1^2 + bk_2^2)I - P, \\ \Omega^{21} &= (d+1)(\mu^2 M^T P N_1 + \mu(1-\mu) M^T P N_2), \\ \Omega^{31} &= (d+1)(\mu(1-\mu) M^T P N_1 + (1-\mu)^2 M^T P N_2), \\ \Omega^{33} &= (d+1)(1-\mu)^2 M^T P M - bI - Q_{22}. \end{aligned}$$

It is easy to derive $\xi_k^T \Omega \xi_k = \Delta V + \xi_k^T L_1 \xi_k + \xi_k^T L_2 \xi_k$. From the inequality $\Omega < 0$, we can deduce $\Delta V < 0$.

Due to its block-diagonal structure, the inequality $\Omega < 0$ is equivalent to the inequalities $Q < cI$ and $F < 0$ where F is given by

$$F = \begin{bmatrix} (d+1)\Gamma + (ak_1^2 + bk_2^2)I - P & (\star) & (\star) \\ (d+1)(\mu^2 M^T P N_1 + \mu(1-\mu) M^T P N_2) & (d+1)\mu^2 M^T P M - aI - Q_{11} & (\star) \\ (d+1)(\mu(1-\mu) M^T P N_1 + (1-\mu)^2 M^T P N_2) & (d+1)\mu(1-\mu) M^T P M - Q_{21} & F^{33} \end{bmatrix} \tag{6.17}$$

with $F^{33} = (d+1)(1-\mu)^2 M^T P M - bI - Q_{22}$.

Since F is a matrix function depending quadratically on μ, using the convexity principle, we deduce that $F < 0$ for any $\mu \in [0,1]$ if $F(\mu) < 0$ with $\mu = \{0,1\}$ and

$\dfrac{\partial^2 F}{\partial \mu^2} \geq 0$, providing equations (6.18), (6.19) and (6.20)

$$F_{(1)} = \begin{bmatrix} (d+1)N_{1(1)}^T P N_{1(1)} + (ak_1^2 + bk_2^2)I - P & (\star) & 0 \\ (d+1)M_{(1)}^T P N_{1(1)} & (d+1)M_{(1)}^T P M_{(1)} - aI - Q_{11} & (\star) \\ 0 & -Q_{21} & -bI - Q_{22} \end{bmatrix} < 0 \qquad (6.18)$$

$$F_{(0)} = \begin{bmatrix} (d+1)N_{2(0)}^T P N_{2(0)} + (ak_1^2 + bk_2^2)I - P & 0 & (\star) \\ 0 & -aI - Q_{11} & (\star) \\ (d+1)M_{(0)}^T P N_{2(0)} & -Q_{21} & (d+1)M_{(0)}^T P M_{(0)} - bI - Q_{22} \end{bmatrix} < 0 \qquad (6.19)$$

$$(d+1)\begin{bmatrix} N_1^T P N_1 - N_1^T P N_2 - N_2^T P N_1 + N_2^T P N_2 & (\star) & (\star) \\ M^T P N_1 - M^T P N_2 & M^T P M & (\star) \\ -M^T P N_1 + M^T P N_2 & -M^T P M & -M^T P M \end{bmatrix} \geq 0 \qquad (6.20)$$

with:

$$\begin{aligned} M_{(1)} &= M(\mu = 1) = I + UC + Y_1 VC, \\ M_{(0)} &= M(\mu = 0) = I + UC + Y_2 VC, \\ N_{1(1)} &= N_1(\mu = 1) = (I + UC + Y_1 VC)A_1 - K_1 C, \\ N_{2(0)} &= N_2(\mu = 0) = (I + UC + Y_2 VC)A_2 - K_2 C. \end{aligned} \qquad (6.21)$$

By using the Schur complement for the non-strict inequalities in [46], we can deduce that (6.20) is always verified. By using the Schur complement, (6.18) and

(6.19) are equivalent to the following inequalities:

$$\begin{bmatrix} \Theta & 0 & 0 & (\star) \\ 0 & -aI - Q_{11} & (\star) & (\star) \\ 0 & -Q_{21} & -bI - Q_{22} & 0 \\ PN_{1(1)} & PM_{(1)} & 0 & \frac{-P}{d+1} \end{bmatrix} < 0 \tag{6.22}$$

and

$$\begin{bmatrix} \Theta & 0 & 0 & (\star) \\ 0 & -aI - Q_{11} & (\star) & 0 \\ 0 & -Q_{21} & -bI - Q_{22} & (\star) \\ PN_{2(0)} & 0 & PM_{(0)} & \frac{-P}{d+1} \end{bmatrix} < 0 \tag{6.23}$$

with $\Theta = (ak_1^2 + bk_2^2)I - P$.

Now let's define $Y_i = P^{-1}\bar{Y}_i$ and $K_i = P^{-1}\bar{K}_i$ for all $i = 1, 2$. Substituting M_1, M_2, $N_{1(1)}$, $N_{2(0)}$ given by (6.21) into (6.22) and (6.23) and substituting Y_1, Y_2, K_1, K_2 by $\bar{Y}_1$, $\bar{Y}_2$,$\bar{K}_1$, $\bar{K}_2$, the problem of finding $M_{(1)}$, $M_{(0)}$, $N_{1(1)}$, $N_{2(0)}$, P in (6.22) and (6.23) is equivalent to the problem of finding $\bar{Y}_1$, $\bar{Y}_2$,$\bar{K}_1$, $\bar{K}_2$, P in (6.11a) and (6.11b) of the main theorem.

Based the main theorem, the unknown input observer design algorithm can be proposed as follows:

a) Calculate $U = -D(CD)^\dagger$ and $V = I - (CD)(CD)^\dagger$.

b) Solve the LMIs of the main theorem for P, Q, $a > 0$, $b > 0$, $c > 0$, and $\bar{Y}_i$, $\bar{K}_i$ for all $i = 1, 2$.

c) Deduce $Y_i = P^{-1}\bar{Y}_i$ and $K_i = P^{-1}\bar{K}_i$ for all $i = 1, 2$.

d) Deduce the observer gains for all $i = 1, 2$ as follows:

$$\begin{aligned} E_i &= U + Y_i V, \\ N_i &= MA_i - K_i C, \\ M &= I + \mu E_1 C + (1 - \mu) E_2 C, \\ L_i &= K_i(I + CE_i) - MA_i E_i. \end{aligned}$$

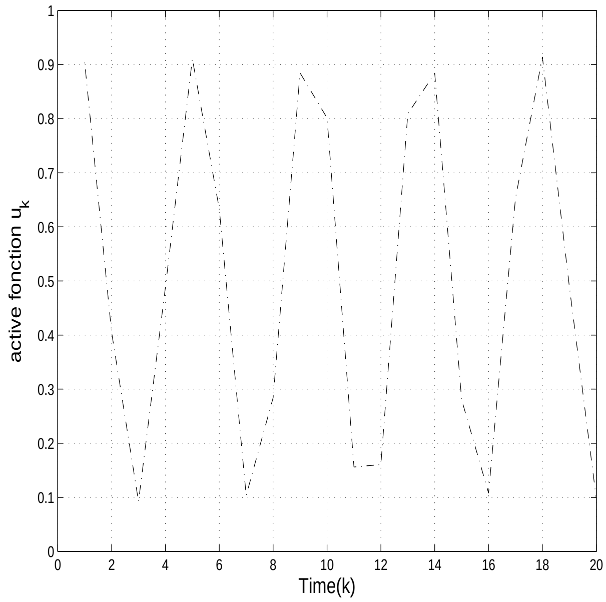

Figure 6.2: Active function μ

6.4 Numerical example

In this section, we present a numerical example in order to show the applicability of our method.

Consider the multimodel nonlinear system (6.1), which is described by the following parameters:

$$A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0.199 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0.199 & 0 \\ 3 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T, D = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T,$$

$$f_1(x_k) = \begin{bmatrix} 0.02\exp(-x_{1k}^2) \\ 0 \\ -5.7 - 0.02\exp(-x_{3k}^2) \end{bmatrix}, f_2(x_k) = \begin{bmatrix} 0.01\exp(-x_{1k}^2) \\ 0 \\ -5.7 - 0.02\exp(-x_{3k}^2) \end{bmatrix}.$$

The active function is $\mu(y_k) = (1 + \tanh(\epsilon y_k))/2$ with the active value $\epsilon = 0.4$. The message is $\bar{u}_k$ =sin$(0.01k)$. For initial condition $x_{(0)} = [2.8 \quad 0.5 \quad 0]^T$, the evolution of $\mu(y_k)$ and the trajectory of the multimodel system (6.1) are shown respectively in Figure 6.2 and Figure 6.3. Figure 6.4 shows the output trajectory y_k, which can transmit the message to the observer.

By solving the LMI conditions of Theorem 6.3.1 with the SeDuMi1.1 solver, we

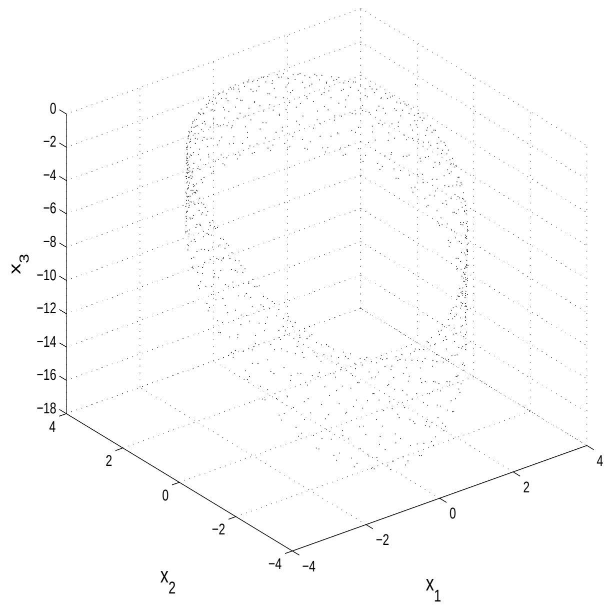

Figure 6.3: Trajectory of multimodel system

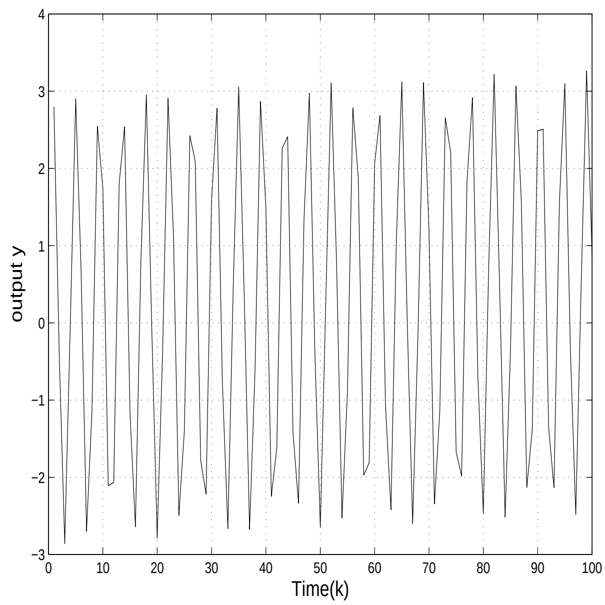

Figure 6.4: Output of multimodel system y

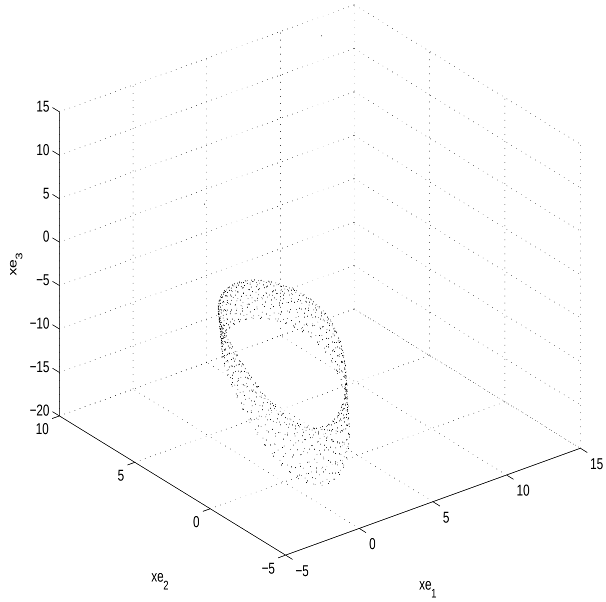

Figure 6.5: Trajectory of multimodel observer

obtain the following observer gains:

$$K_1 = [0 \;\; 1.000 \;\; 1.000]^T,$$
$$K_2 = [0 \;\; 1.000 \;\; 3.000]^T,$$

with the CPU execution time s = 605 ms (the processor Inter Pentium M, 1400 Mhz), and we obtain by (6.9):

$$E_1 = [-1 \;\; 0 \;\; -1]^T,$$
$$E_2 = [-1 \;\; 0 \;\; -1]^T.$$

With initial condition $\hat{x}_{(0)} = [10 \;\; 10 \;\; 10]^T$, the observer trajectory is shown in Figure 6.5; and the evolution of the estimation error of state x is shown in Figure 6.6. This shows that the estimation error converges in 4 discrete time. The message trajectory $\bar{u}_k$ and estimation trajectory of message $\hat{\bar{u}}_k$ is shown in Figure 6.7, and the error estimation trajectory of message is shown in Figure 6.8.

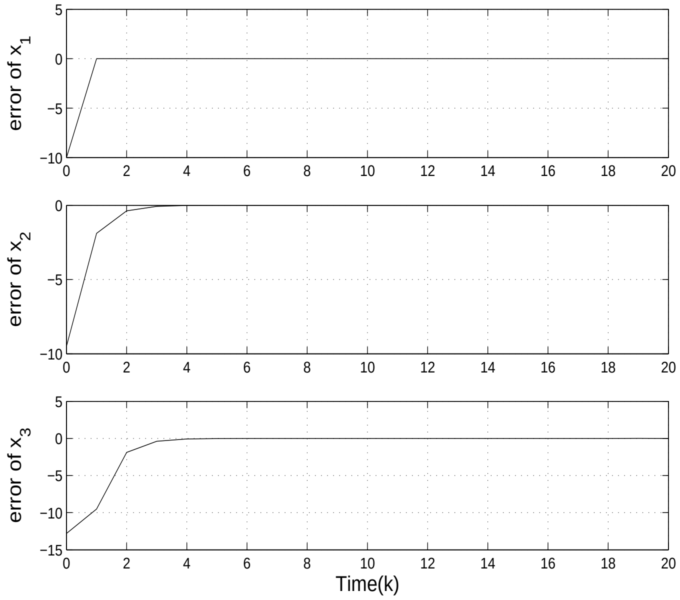

Figure 6.6: Estimation error of x

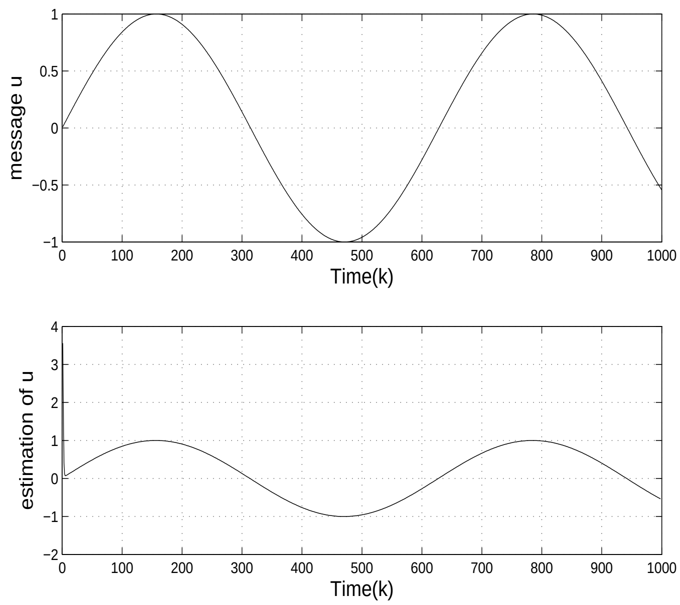

Figure 6.7: Message $\bar{u}$ and estimation of message $\hat{\bar{u}}$

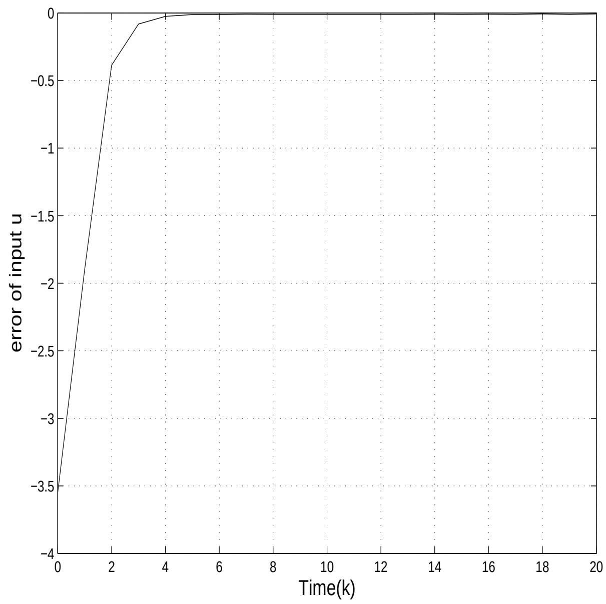

Figure 6.8: Estimation error of message $\bar{u}$

6.5 Conclusion

In this chapter, the problem of unknown input observer synthesis is addressed for a class of discrete-time multimodel systems. The multimodel system is established by two discrete-time Lipschitz systems. By using a particular Lyapunov function, a sufficient condition for the observer of the multimodel system is proposed. These conditions can be expressed as a LMI feasibility problem, which are easily tractable by convex optimization techniques. The proposed conditions have been tested on a numerical example. A particular, the proposed method can be applied to design the communication systems, the objective is to recover a message imbedded in a signal generated by a multimodel nonlinear system.

Chapter 7

General conclusion

In this thesis, several approaches for observer synthesis for a particular class of hybrid systems were proposed. We focused on the observation with/without unknown inputs for switched systems and multimodel systems. Our main contributions were given in chapters 4, 5 and 6. By using Lyapunov functions, sufficient conditions for observer synthesis were proposed. These conditions is expressed as a LMI feasibility problem that is easily solvable numerically by convex optimization techniques.

Our first approach was presented in chapter 4. In this chapter, an observer synthesis method was presented for a nonlinear switched system. The switching law of the switched system was assumed to be unknown. By using the Differential Mean Value Theorem, the dynamic of the estimation error was transformed into a LPV system. Then, by using multiple Lyapunov functions, sufficient conditions for the observer synthesis which guarantee an upper bound on estimation error were developed.

Notice that the estimation error is upper bounded by a constant in this approach. On several numerical examples, the results showed that the computed upper bound on the estimation error is large, despite of the fact that the simulation results are good. This means that the proposed method may be pessimistic. Hence, as a future work, it would be interesting to find a method which can reduce the pessimism, for instance, by looking for other kinds of Lyapunov functions.

The proposed approach could be promising for data encryption / decryption for telecommunication purposes. In order to do so, it is necessary to add an unknown input to the switched system. Extending the proposed method to the problem of observer synthesis with unknown input is an interesting and challenging issue.

Our second contribution was presented in chapter 5 which deals with observer synthesis for multimodel systems. The multimodel system is composed of two Lipschitz nonlinear systems. By using a Lyapunov function with an additional quadratic term based on the nonlinear part of the system, we proposed sufficient conditions for the observer synthesis.

Notice that this approach was developed only for multimodel systems composed of two subsystems. For the case of more than two subsystems, it is necessary to change the theorem.

Our approach is applicable to nonlinear multimodel systems where the Lipschitz parameters are less than one. This is due to the fact that the Lyapunov function method has some conservatism. In our case the bloc diagonal structures of the matrices P and Q were used. Resulting conservatism could be eliminated by using more general Lyapunov functions.

Chapter 6 presented an extension of the method developed in the previous chapter to the observer synthesis problem for multimodel system with unknown inputs. The system is composed of two nonlinear Lipschitz systems and an unknown input. By using the same Lyapunov function as in Chapter 5, sufficient conditions for observer synthesis were proposed.

As in the previous case, the main limitation of this approach comes from the fact that it is applicable only to the case of two subsystems. For the case of more than two subsystems, it is necessary to change the theorem.

The application to data encryption requires that the subsystems of the multimodel system are chaotic. But our approach is not applicable to chaotic systems at the moment because of the conservatism. How to reduce the conservatism of this approach remains a challenging problem.

Notice that observer synthesis of hybrid system is still a subject of research and there are a lot of interesting problems to explore. A direction is to look for new Lyapunov functions, which allows to reduce the conservatism of current approaches available in the literature. Another direction is to look for new structures of observers, which are adapted to new classes of systems, or reduce the conservatism of available approaches. For solving matrix inequality, if the problem is not LMI, solver for the Bilinear Matrix Inequalities (BMI) could be used. However, in this case, the solution obtained is not proved to be globally optimal.

Appendix A. Fundamental elements

Schur Complement

Lemma 7.0.1 *(Schur Complement) [46]*
Given the matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{n\times m}$ *and the matrix bloc* $M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$, *then the following representations are equivalents:*

1. M *is negative-definite.*
2. $R < 0$ *and* $Q - SR^{-1}S^T < 0$.
3. $Q < 0$ *and* $R - S^TQ^{-1}S < 0$.

Lemma 7.0.2 *(Schur complement for the non-strict inequalities) [46]*
Given the matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{n\times m}$ *and the matrix bloc* $M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$, *then the following representations are equivalents:*

1. M *is negative semi-definite.*
2. $R \leq 0$, $Q - SR^{-1}S^T \leq 0$ *and* $S(I - RR^{-1}) = 0$.
3. $Q \leq 0$, $R - S^TQ^{-1}S \leq 0$ *and* $(I - Q^{-1}Q)S = 0$.

Some elements on the convexity

In this section, we present a few definitions and properties on the convex sets, convex functions and the principle of convexity.

Definition 7.0.3 *(convex set)*
A set E is said convex if:

$$\lambda x_1 + (1-\lambda)x_2 \in E$$

for all $x_1, x_2 \in E$ *and for all* $0 \leq \lambda \leq 1$.

Geometrically, this means that every segment between any two points belonging to a convex set is included in this set.

Definition 7.0.4 *(convex function)*
A function $\varphi : I\!R^n \rightarrow I\!R$ *is said convex if:*

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)$$

for all $x_1, x_2 \in I\!R^n$ *et for all* $0 \leq \lambda \leq 1$.

The function φ is *strictly convex* if and only if:

$$\varphi(\lambda x_1 + (1-\lambda)x_2) < \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)$$

for all $x_1 \neq x_2$ et for all $0 < \lambda < 1$.

Stability of dynamical systems

In this section, we present several notions for the stability of dynamical systems (continuous-time and discrete-time). The stability is an important propriety which characterizes the comportment of dynamical systems. The Lyapunov theorem approach is the more general approach which allows us to study the stability of dynamical systems.

Several stability definitions

Consider an autonomous nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \quad \forall t \geq t_0 \tag{7.1}$$

where $x(t) \in I\!R^n$ is the state vector, $f : I\!R^n \rightarrow I\!R^n$ is a continuous locally Lipschitz function, $x(0) = x_0$. Without loss of generality, we assume the the origin is an equilibrium point of the system (7.1).

Suppose the initial time $t_0 = 0$, then

- the origin of the above system is said to be *Lyapunov stable*, if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| < \epsilon$, for every $t \geq 0$.

- the origin of the above system is said to be *asymptotically stable*, if it is Lyapunov stable and if there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\lim_{t\to\infty} x(t) = 0$.

- the origin of the above system is said to be *exponentially stable*, if it is asymptotically stable and if there exist $\alpha, \beta, \delta > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| \leq \alpha\|x(0)\|e^{-\beta t}$, for $t \geq 0$.

The definitions for discrete-time systems is almost identical to that for continuous-time systems.

Lyapunov second theorem on stability

Lyapunov stability approach is the most general approach to study the stability of dynamic systems, and there are a lot of results in this domain, for example [59]. Notice that Lyapunov stability theorems give only sufficient condition.

Consider a function $V(x) : I\!R^n \to I\!R$ such that:

- $V(x) \geq 0$ with equality if and only if $x = 0$ (positive definite),

- $\dot{V}(x(t)) < 0$ (negative definite),

then V(x) is called a Lyapunov function and the system is *asymptotically stable* in the sense of Lyapunov.

Appendix B. Publication list

Personal publication list in international conferences:

1. Yunjie Hua, Gabriela Iuliana Bara, Mohamed Boutayeb, Edouard Laroche, "Switched jump observer for nonlinear switched systems", *In Proceedings of the 3rd IFAC Symposium on System, Structure and Control, Brazil*, 2007.

2. Yunjie Hua, Gabriela Iuliana Bara, Mohamed Boutayeb, Edouard Laroche, "Observer design for a class of discrete-time multimodel nonlinear systems", *In Proceedings of the 3rd IFAC Symposium on System, Structure and Control, Brazil*, 2007.

3. Yunjie Hua, Mohamed Boutayeb, Gabriela Iuliana Bara, Edouard Laroche, "Input Recovery for a Class of Nonlinear Multimodel System: An LMI Approach", *In Proceedings of the 16th IEEE/Mediterranean Conference on Control and Automation, Ajaccio, Corsica, France*, 2008.

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