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Dissertation

On spacetime structure and symmetries in a strong gravitational field and in quantum field theory

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Contents

1. Introduction

1.1. Motivation

In the theory of quantum fields Lorentz invariance and causality are usually assumed and the discussion is carried out in the framework of Minkowski spacetime. Gravity breaks this spacetime symmetry and also the issue of causality may become non-trivial in the (strong) gravitational field. In a curved spacetime field theories require a different treatment than in a flat spacetime. There is a fundamental principle, Equivalence Principle, which realizes the concept of locality in gravity in such a way that the effects of gravity can be eliminated locally. In general relativity spacetime can be identified as a manifold, i.e. in general it looks locally like an R^N space. At every spacetime point one can always choose such a coordinate chart that the spacetime is flat and the Christoffel symbols vanish. Locally it is always possible to erect Minkowski spacetime where the laws of special relativity are obeyed. Thus, in this local coordinate system the relativistic Poincaré symmetry is expected to be satisfied. The Principle of Equivalence allows us to render gravitational interaction non-existent locally. The question is whether we can always see the same thing in this local inertial frame? Is the revealed structure of spacetime unique? There are two different points of view, two different frameworks where one could study the structure of spacetime and its symmetries: the curved spacetime, involving point-like particles and their motion, and flat Minkowski spacetime, incorporating the symmetries of quantum field theories and verifying if the presence of fields can affect the spacetime symmetries. The general idea is such that, according to the Principle of Equivalence, once the gravity is turned off, we should end up with Minkowski spacetime exhibiting Poincaré invariance. As we conduct a study solely in flat spacetime, we do not even deal with gravity. The gravitational interaction is absent. However, quantum field theory may display a spontaneous breakdown of Lorentz symmetry. The vacuum of the theory would then violate Lorentz invariance. Such a model, treated in the curved spacetime could give rise to the gravitational analogy of Higgs mechanism, the gravitons would become short-ranged. Applying the Equivalence Principle at a spacetime point in this quantum field theory would not lead to the emergence of local flat spacetime where special relativity is valid.

On the other hand, one can also analyze the motion of particles in curved spacetime. The simplest non-trivial case is Schwarzschild geometry [1]. According to the Birkhoff's theorem Schwarzschild spacetime is the most general static spherically symmetric solution. We could gain insight in the structure of this spacetime by studying the motion of free particles. The properties of geodesics in Schwarzschild geometry have been studied extensively (see [2]), however, there are a few particular characteristics that deserve attention. In the strong gravitational field, in the vicinity of the event horizon that corresponds to Schwarzschild radius, certain phenomena occur. There exists a so-called photon sphere, outside a black hole, which is a point of no return for massive geodesics. Lots of interesting properties may be observed in the vicinity of the photon sphere [3]. One of the intriguing features is superluminal, as it appears, speed of massive particles in the photon sphere interior. We found it tempting to check if similar properties have more general meaning, being present in the more general class of static, spherically symmetric spacetimes. The Schwarzschild-like spacetimes which are investigated originate in Hořava-Lifshitz gravity [4], thus a possible candidate for the UV completion of general relativity. The regulator in Hořava-Lifshitz theory breaks Lorentz invariance. In this case that is the price of arriving at a renormalizable theory of gravitation. The black-hole spacetime in Hořava-Lifshitz gravity is asymptotically flat and in a certain limit reproduces Schwarzschild geometry. Although the form of the metric in Schwarzschild-like spacetime is more complicated than in the Schwarzschild solution, there also exists a single photon sphere. We consider this and some related questions in chapter 2.

Extreme properties of strong gravitational field are manifested in the proximity of event horizon in Schwarzschild spacetime. It is convenient to use in this region the extension of Schwarzschild coordinate system - the Kruskal-Szekeres coordinates [5]. Studying even the simplest kind of motion - the radial fall one finds curious observations. In our analysis, the “observers” falling into the black hole exchange the “signals” represented by massless particles’ geodesics. The diagrams in Kruskal-Szekeres coordinate chart offer a remarkably clear view of the world lines. In the case of outgoing signals, the horizon represents a singularity; it is not possible to establish, by any observation carried out in the exterior of the black hole, what is actually the spacetime trajectory of a massless particle beneath the horizon. And the world line of the particle determines the orientation of the arrow of time. What is actually the time arrow inside the black hole? The study of infalling massive and ingoing massless geodesics cannot give a definite answer - they display the monotonic change for time together with the decrease for radial coordinate of Schwarzschild spacetime. It is the case of outgoing null geodesics that is ambiguous. One choice of the time arrow corresponds to preservation of the causal structure from above the horizon. The other possible choice for the arrow of time corresponds to the motion “forward” in the time coordinate while the radial coordinate can either decrease or increase. Causal structure is thus changed after plunging into the black hole. There is no straightforward way to show which possibility is realized by the null geodesics. Both scenarios seem valid, depending on which property one takes into account, whether the regular structure of spacetime, or the continuous parametrization of geodesics. Hence, the strong gravitational field, as in the vicinity of the event horizon, may lead to the question of the status of Equivalence Principle. If the causal structure of the spacetime is changed, then how does the local inertial frame look like? This problem is discussed in chapter 3.

In general, the quantum field theories are required to satisfy Lorentz invariance and causality so that the laws of special relativity are obeyed. Nevertheless, one can construct a model where the spontaneous breakdown of Lorentz symmetry takes place (see [6]). The starting point is the definition of the action which is manifestly Lorentz invariant, although the special choice of the terms in the action leads to the ground state - the vacuum state of the theory which is not invariant under Lorentz transformations anymore. Thus, the spontaneous symmetry breaking occurs. This phenomenon can take place in the case of $U(1)$ and $SU(2)$ gauge symmetry. The non-zero vacuum expectation value for the scalar field in the theory, originating in the special "φ^4" potential, generates the symmetry breaking. In the case of Lorentz symmetry, this role has to be played by a gauge field. To induce the vector field potential that creates the non-vanishing vacuum expectation value for this field one can employ higher orders of covariant derivatives. These terms may arise naturally in effective theories, valid only below a certain energy scale. The action for the light particles, belonging to this range of energies, can be obtained by integrating out the heavy degrees of freedom in the path integral. The elimination of a propagating mode yields long-range correlations in the remaining dynamics. Hence, the higher order derivatives emerge in the action. The expansion usually has to be truncated and the reduction of the number of terms, may cause inconsistencies in the theory. The stability and unitarity of the effective theory are thus endangered. The negative norm states as well as runaway modes may appear. The most important issue is the identification of the states with negative norm and elimination of exponentially growing amplitudes. It can be shown that negative norm states are created by skew-adjoint operators, while the positive norm states – by self adjoint operators. Moreover, the norm of the state may be determined from the properties of the operator under the time reversal transformation. Positive time-inversion parity, or the combined space- and time-reversal parity in the case of gauge fields, ensures the positive norm of the state created by a field operator. The unitary of the time evolution in the subspace of states with positive norm can be ensured by Osterwalder-Schrader reflection positivity [7] in Euclidean spacetime. The framework in which the study of the property of reflection positivity is particularly advantageous is the lattice regularization [8]–[10]. In the lattice action new variables can be introduced, corresponding to the derivatives with respect to time of all orders but the highest one, just like in the classical procedure [11]. If we treat the original fields as "coordinates", with positive time-inversion parity, the variables representing the odd-order derivatives have odd parity. The reflection positivity is satisfied by the time-reversal invariant functionals of fields acting on the time-reversal invariant vacuum. Also, the boundary conditions in the time direction have to be fulfilled – the periodic and antiperiodic trajectories in the path integral correspond to time reversal even and odd variables, respectively. The physical subspace of the Fock space is spanned by the states with positive norm, created by the time-reversal invariant functionals of fields. This should guarantee the unitary time evolution when the theory is analytically continued to real time.

However, in the case of the theory which involves the spontaneous breakdown of Lorentz symmetry one cannot truly rely on the reflection positivity argument. This property is useful to trace the states with the negative norm. To assure the unitarity in the physical subspace of the Fock space, it is convenient to first establish the unitarity and stability of the whole Fock space. When one studies the case of scalar quantum electrodynamics, with higher order covariant derivatives in the kinetic energy of the scalar field, the key point is to ensure that all the poles in the propagator correspond to real frequencies. Also, the form of action should be such that the vacuum is stable. Then, it is necessary to recognize the physical fields, represented by self-adjoint operators, with the even PT parity, and analyze the quasi-particle spectrum. Taking into account the simplest possible action, with maximum fourth order of covariant derivatives, which leads to the vacuum that violates Lorentz invariance, one finds that the spontaneous breakdown of relativistic symmetry cannot be seen in the quadratic part of the gauge field action. The electromagnetic, four-vector field A_μ reveals two transverse, massless components, just like in the usual case of electrodynamics. The Maxwell's equations are satisfied. The influence of the breakdown of Lorentz symmetry should be detected in radiative corrections due to the charged scalar field.

1.2. Organization

The thesis deals with the symmetries and conservation laws in selected models of spacetime. It comprises the methods of determining the geodesics (timelike and null geodesics) as well as the description of generalized Doppler effect arising in case of communication in curved spacetime by means of electromagnetic signals. Special attention is paid to the phenomena that occur in the strong gravitational field, in the vicinity of the event horizon of the black hole. As far as flat spacetime is concerned, the thesis involves the generalization of the idea of spontaneous symmetry breaking to relativistic Poincaré symmetry. To achieve the violation of Lorentz invariance in this manner it is necessary to introduce higher orders of covariant derivatives in the theory. It is believed that the theories with higher orders of time derivatives are plagued with inconsistencies, instabilities and negative norm states. The study of the property of reflection positivity in the Euclidean theory helps to elucidate these issues. Finally, the extended model of scalar QED is discussed where vacuum state breaks Lorentz invariance.

The thesis is organized as follows. The first part, concerning the symmetries and structure of curved spacetime, is divided into two chapters. In chapter 2 the main features of circular geodesic motion in Schwarzschild-like spacetime of Hořava-Lifshitz gravity are described. We begin with the basic notions of Schwarzschild spacetime and the circular orbits of free particles therein. Then, we focus on Schwarzschild-like spacetimes and the circular geodesics of test particles, on the photon sphere. Chapter 3 is devoted to the discussion of the communication in Schwarzschild spacetime, in the proximity to event horizon. The two-way communication between observers above and below horizon is explored. In the next chapters we follow with the considerations regarding the quantum field theory in flat spacetime. The aim of this part of the thesis is the construction of a viable quantum field theory where the spontaneous breakdown of Lorentz symmetry arises. In the chapter 4 the main features of the field theory in path integral quantization and effective theories with higher orders of derivatives are presented. Chapter 5 contains a study of an effective theory with higher order derivative terms in the action. First, we characterize linear spaces with indefinite norm. Then we introduce the Euclidean field theory in lattice regularization. The lattice model including the higher orders of derivatives is established. Finally, the property of reflection positivity is investigated. In the chapter 6 we propose an extended version of scalar QED, containing higher orders of derivatives which induce the spontaneous Lorentz symmetry breaking. The vacuum of the theory is determined. Afterwards, we assure the unitary time evolution within the subspace of physical states. Eventually, the particle content of the theory is considered. In the last chapter we present the conclusions.

The original results from chapter 2 were reported in [12]. The main content of chapter 5 was published [13] and presented on a conference. Three more manuscripts, concerning the issues discussed in chapters 3 and 6, have already been submitted for publication.

Part I

Spacetime structure and symmetries in strong gravitational fields

This part of the dissertation contains a study of selected phenomena and effects occurring in the strong gravitational field as test particles moving in curved spacetime are considered. The symmetries and corresponding conservation laws are applied to determine in the straightforward manner the timelike and null geodesics in a few models of spacetime. We start from Schwarzschild spacetime. Although it has been a subject of extensive analysis ever since the Schwarzschild's 1916 paper [1] was published one can still uncover interesting and even intriguing aspects of this geometry. The particles' geodesics in strong gravitational field, in close proximity to the event horizon, turn out to display non-trivial, curious properties, in the case of both circular and radial geodesics. We will focus on certain unusual features of circular orbits in Schwarzschild-like spacetimes, arising from Hořava-Lifshitz gravity, a theory proposed recently [4] as a UV-completion of general relativity. One of the main subjects of our investigations is the problem of communication and interaction in the vicinity of the event horizon. Determining the behavior of radial light signals in this region, by employing the Kruskal-Szekeres coordinate system, one finds unexpected characteristic features resulting from the presence of the event horizon.

2. Circular geodesics in Schwarzschild-like spacetimes

2.1. Introduction

In this chapter we will analyze the circular motion of free test particles in the Schwarzschild-like spacetime, especially in the vicinity of the so-called photon sphere. We begin with the description of basic properties of Schwarzschild geometry itself and with the determination of particle's geodesics in this spacetime (see also [14]–[16]). Then we pay attention to circular orbits in general, characterized by non-zero acceleration, which leads to the discovery of interesting properties of the photon sphere. This part is based on [3, 17]. We continue with the study of circular geodesics in Schwarzschild and Schwarzschild-like Hořava-Lifshitz spacetime. The latter case is addressed in [18]–[23] as well. We determine that the photon sphere can be reached by massive geodesics in the asymptotic way. Moreover, the photon sphere radius turns out to be the turning point for circular geodesics - the particles that cross this point fall inevitably into the black hole. No circular orbits, corresponding either to massive or massless objects moving along geodesics, can be arranged at a radius smaller than the radius of the photon sphere. The similar conclusions hold in both Schwarzschild spacetime and its counterpart in Hořava-Lifshitz gravity. Our results are presented in [12].

2.2. Schwarzschild spacetime basics

The Schwarzschild geometry is described by means of a metric

$$\begin{aligned} ds^2 &= \left(1-\frac{r_S}{r}\right)dt^2 - \left(1-\frac{r_S}{r}\right)^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2 \\ &\equiv g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \end{aligned} \tag{2.1}$$

where t is the time coordinate, r is the radial coordinate related to the spatial distance from the center, θ and ϕ are the usual spherical angles. Schwarzschild radius r_S is described by the mass that is the source of gravitational field

$$r_S = 2M \ , \tag{2.2}$$

where the system of units used is $c = G = 1$. Geodesics $x^\mu(\lambda)$,λ being a parameter along the path, satisfies the equation

$$a^\mu = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \ . \tag{2.3}$$

Here a^μ denotes the acceleration, defined by

$$a^\mu = \frac{dx^\nu}{d\lambda}\nabla_\nu \frac{dx^\mu}{d\lambda} \ , \tag{2.4}$$

and $\frac{dx^\mu}{d\lambda}$ is the vector tangent to the geodesics. To write down the covariant derivative ∇_μ one uses the Christoffel symbol

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\alpha}(\partial_\rho g_{\sigma\alpha} + \partial_\sigma g_{\alpha\rho} - \partial_\alpha g_{\rho\sigma}) \ . \tag{2.5}$$

Solving the set of four coupled equations (2.3) is not a trivial task. Still, it is not the only possible way to determine the geodesics. In an alternative approach one employs the spacetime symmetries. Static, spherically symmetric geometry is equipped with four Killing vectors – three of them originate from spherical symmetry and one is associated with time translations. Since for a Killing vector K^μ we have

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{constant}, \tag{2.6}$$

each of them has to refer to a different constant of motion for a free particle. Moreover, we can benefit from another property of geodesics – metric compatibility which allows us to treat

$$\xi = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{2.7}$$

as a constant.

The Killing vectors in Schwarzschild spacetime are [24]

$$\begin{aligned} \eta^\mu &= \delta^\mu_t \ , \\ \zeta^\mu &= \delta^\mu_\phi \ , \\ \gamma^\mu &= \delta^\mu_\theta \sin\phi + \delta^\mu_\phi \cot\theta\cos\phi \ , \\ \kappa^\mu &= -\delta^\mu_\theta \cos\phi + \delta^\mu_\phi \cot\theta \sin\phi \ , \end{aligned} \tag{2.8}$$

what can be checked by substituting them in the Killing equation

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \ . \tag{2.9}$$

These four Killing vectors are also found in flat spacetime[1] and the corresponding symmetries lead to the conservation of energy and three components of angular momentum[2]. The same quantities are conserved in the case of Schwarzschild metric. We will utilize these conservation laws in order to determine the geodesic motion of particles. The direction of angular momentum is conserved therefore the particle moves in a plane[3]. The spherical symmetry allows us to choose, without loss of generality, the equatorial plane where $\theta = \frac{\pi}{2}$. Now we can use Eq. (2.6) to obtain the relations

$$\begin{aligned} g_{tt}\frac{dt}{d\lambda} &= \epsilon \ , \\ g_{\phi\phi}\frac{d\phi}{d\lambda} &= L \ , \end{aligned} \tag{2.10}$$

which give

$$\begin{aligned} \frac{dt}{d\lambda} &= \frac{\epsilon}{1-\frac{r_S}{r}} \ , \\ \frac{d\phi}{d\lambda} &= -\frac{L}{r^2} \ . \end{aligned} \tag{2.11}$$

We have to make a distinction between massless and massive particles. For massless particles, ϵ and L are actual values of energy and angular momentum, respectively. In the case of massive particles, they stand for energy and angular momentum per unit mass and the parameter λ equals proper time. The quantities $dx^\mu/d\lambda$ are wave vectors, k^μ, of massless and four-velocities, U^μ, of massive particles. Combining equations (2.11) with (2.7) we arrive at

$$\frac{1}{1-\frac{r_S}{r}}\left[\epsilon^2 - \left(\frac{dr}{d\lambda}\right)^2\right] - \frac{L^2}{r^2} = \xi \tag{2.12}$$

where the constant ξ is zero for massless and $+1$ for massive particles. From this equation one can extract the necessary information concerning geodesic motion of particles.

2.3. Circular geodesics in Schwarzschild spacetime

All the circular geodesics of massless particles belong to the so-called photon sphere, corresponding to radius $r_{ph} = 3/2 r_S$ in Schwarzschild spacetime (see below). As in our considerations we restrict ourselves to the motion in equatorial plane, we will use the term "photon sphere" to describe the circular orbit of radius $r = r_{ph}$.

It is a well-known property of the Schwarzschild geometry that orbiting along the photon sphere uncovers many interesting features. It was indicated by Abramowicz and Lasota

[1] It can be seen in the most straightforward manner when using spherical spatial coordinates, the line element is then $ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$.

[2] One component of angular momentum determines its magnitude and two components – its direction.

[3] Also, the particle in a plane travels either in a "clockwise" or "counterclockwise direction". Thus, the conservation of two components of the angular momentum restricts the geodesic motion.

[17] that the acceleration on the circle of radius equal to the radius of photon sphere is independent of the velocity. As was emphasized recently by Abramowicz et al. [3] this property leads to a new type of twin paradox: two observers travelling with different speeds (and the same acceleration) will meet repeatedly identifying different time periods. In this context the Authors in ref. [3] indicated another interesting effect. Let us give some details of this derivation.

To describe the uniform circular motion in Schwarzschild geometry, $\frac{dr}{d\lambda} = 0$, one require two-component vector tangent to the geodesics:

$$\frac{dx^\mu}{d\lambda} = \left(\frac{dt}{d\lambda}, 0, 0, \frac{d\phi}{d\lambda} \right) . \tag{2.13}$$

The acceleration in the radial direction[4] is non-vanishing in this case (see Eq. (2.3))

$$a_r = -\frac{1}{2} \left(\frac{dt}{d\lambda} \right)^2 \frac{d}{dr} g_{tt} - \frac{1}{2} \left(\frac{d\phi}{d\lambda} \right)^2 \frac{d}{dr} g_{\phi\phi} . \tag{2.14}$$

This expression takes the same form for massive and massless particles. However, if we write it explicitly for these two types of objects (see Eq.(2.7)) we will find out that acceleration of massive particles is composed of two terms, corresponding to "gravitational" and "centrifugal" contributions

$$a_{r,m} = -\frac{1}{2} \frac{d}{dr} \ln g_{tt} - \frac{1}{2} g_{\phi\phi} \left(U^\phi \right)^2 \frac{d}{dr} \ln \left(\frac{r^2}{g_{tt}} \right) \tag{2.15}$$

whereas only "centrifugal" term is present the acceleration of massless particles:

$$a_{r,ph} = -\frac{1}{2} g_{\phi\phi} \left(k^\phi \right)^2 \frac{d}{dr} \ln \left(\frac{r^2}{g_{tt}} \right) . \tag{2.16}$$

The latter can be expressed in terms of a derivative of R^2, "effective radius" squared,

$$R^2 = \frac{r^2}{g_{tt}} . \tag{2.17}$$

This derivative vanishes for $r = \frac{3}{2} r_S$, i.e. on the photon sphere,

$$\frac{d}{dr} R^2 |_{r=3/2 r_S} = 0 . \tag{2.18}$$

On the other hand, it is possible to rewrite the acceleration (2.15) in terms of velocity with respect to the static observer. The observer is called static if he/she is in spatial rest with respect to the source of gravitational field – the central mass. His/her four-velocity is purely

[4] We have to point out that the components a_r refer to the dual vector a_μ that we keep calling, after the Authors in ref. [3], acceleration.

timelike and fulfills the normalization condition $N^\mu N_\mu = 1$:

$$N^\mu = \left((\sqrt{g_{tt}})^{-1}, 0, 0, 0 \right) \ . \tag{2.19}$$

The static observer at a certain point in space measures the energy of a moving object, passing this point, as a scalar product of the object's momentum

$$p^\mu = m_0 U^\mu \tag{2.20}$$

and his/her own velocity:

$$E = g_{\mu\nu} p^\mu N^\nu = \sqrt{g_{tt}} m_0 U^t \ , \tag{2.21}$$

where m_0 is the rest mass of the object. According to Equivalence Principle, at each space-time point one is able to choose a local inertial frame. In this frame the energy of an object is expressed as

$$E = \frac{m_0}{\sqrt{1 - v^2}} \ . \tag{2.22}$$

Comparing the quantities from (2.21) and (2.22) one can write down the relation for the speed of the object v

$$v^2 = 1 - \frac{1}{g_{tt} {U^t}^2} = -\frac{g_{\phi\phi} {U^\phi}^2}{g_{tt} {U^t}^2} \ , \tag{2.23}$$

where the equation (2.13) was used. The acceleration of a massive particle (2.15) in terms of the particle's speed reads:

$$a_{r,m} = -\frac{1}{2} \frac{d}{dr} \ln g_{tt} + \frac{1}{2} \frac{v^2}{1 - v^2} \frac{d}{dr} \ln R^2 \ . \tag{2.24}$$

In the case of a geodesic, acceleration (2.24) vanishes and velocity as a function of the circle's radius can be derived:

$$v^2 = \frac{1}{2} \frac{r_S}{r - r_S} \ . \tag{2.25}$$

As argued by Abramowicz at al. [3], this leads to the final conclusion: velocity on the photon sphere, $r = \frac{3}{2} r_S$, is equal to the speed of light,

$$v = 1 \ . \tag{2.26}$$

This appears to be a rather disturbing result: considering geodesics of massive objects (Eqs. (2.24), (2.25)), eventually one finds a null, massless-type geodesic. Following this line of reasoning, for orbits of radius within still allowed range $r_S < r < \frac{3}{2} r_S$, corresponding velocities should be even larger than 1. It would appear that massive particles moving along geodesics could reach superluminal speeds. However, this is not the case. The circular geodesics cannot be arranged arbitrarily close to the event horizon.

In order to clarify the issue of the speed of massive objects on circular geodesics we will analyze this case from another perspective (see also [14]–[16]). In Schwarzschild geometry the radial component of the vector tangent to the geodesics satisfies the following relation (see Eq. (2.12)):

$$\epsilon^2 - \left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{L^2}{r^2} + \xi\right)\left(1 - \frac{r_S}{r}\right) . \tag{2.27}$$

In circular motion this component vanishes

$$\frac{dr}{d\lambda} = 0 . \tag{2.28}$$

One may treat the term on the right hand side of the equation (2.27) as an effective potential (see [14])

$$V_\xi^{\text{eff}}(r) = \left(\frac{L^2}{r^2} + \xi\right)\left(1 - \frac{r_S}{r}\right) . \tag{2.29}$$

Circular orbits are found from the condition of vanishing derivative of this effective potential [14]–[16]:

$$\frac{d}{dr}V_\xi^{\text{eff}}(r) = 0 . \tag{2.30}$$

The solution of Eq. (2.30) takes the form

$$r_m = \frac{L^2}{r_S}\left(1 \pm \sqrt{1 - \frac{3r_S^2}{L^2}}\right) \tag{2.31}$$

in the case of massive particles, $\xi = 1$. Here the "+" corresponds to stable and "-" to unstable circular geodesics, which are described by the minima and maxima of the effective potential, respectively. One can see that the angular momentum cannot be arbitrarily small. A particle can follow circular geodesics only if its squared angular momentum is large enough

$$L^2 > 3r_S^2 . \tag{2.32}$$

For massless particles we reproduce the result

$$r_{ph} = \frac{3}{2}r_S . \tag{2.33}$$

In this case there is one possible circular orbit, the photon sphere, which corresponds to the maximum of the effective potential, hence it is unstable.

One can express the condition (2.30) as follows:

$$\frac{L^2}{L^2 + \xi r^2} = \frac{1}{2}\frac{r_S}{r - r_S} . \tag{2.34}$$

In the case of massive objects, we recognize the term on the right hand side of Eq. (2.34) as the squared speed on the geodesic 2.31 (see Eq. (2.25)). Thus the relationship between particle's speed and angular momentum is

$$v^2 = \frac{L^2}{L^2 + r_m^2} \ . \tag{2.35}$$

This relation, together with equation (2.31) implies that circular geodesics of massive particles, in the limit of large angular momentum $L \to \infty$, approach the photon sphere (2.33), whereas the corresponding speed tends to the speed of light $v^2 \to 1$. The radius of the photon sphere turns out to be a *point of no return* (see [25]) for massive particle geodesics.

It is quite surprising that the massive geodesic in the asymptotic case of infinitely large angular momentum turns out to be the massless geodesic. Furthermore, the interior of the photon sphere, $r < 3r_S/2$, while still an allowed region in spacetime, would admit only spacelike circular geodesics. To facilitate our understanding of this result we will continue our considerations in the more general case of Schwarzschild-like spacetimes.

2.4. Circular geodesics in Schwarzschild-like spacetimes

One can generalize our considerations and investigate Schwarzschild-like spacetimes which emerge in Hořava-Lifshitz gravity [4]. Hořava has recently proposed a four-dimensional renormalizable theory of gravity which admits the Lifshitz scale invariance in space and time. It is thought of as a possible candidate for UV completion of general relativity, in the case of large distances the relativistic limit is recovered [19, 20]. Hořava-Lifshitz theory has attracted a lot of attention. The different aspects of the model have been thoroughly investigated. In particular, an equivalent of Schwarzschild spacetime – a static, spherically symmetric solution has been found by Kehagias and Sfetsos [26]. The description of particle motion and geodesics in this spacetime is contained in [19, 20, 23, 27].

We will focus on the analysis of circular geodesics of massive and massless particles. We will show that the existence of a single photon sphere is the common feature of Schwarzschild-like spacetimes derived from Hořava-Lifshitz theory of gravity. Special attention will be paid to the relations between massive particles' constants of motion, location in space and their velocities, particularly in the vicinity of the photon sphere. The effects that occur in this region, characterized by strong gravitational field, are similar to ones that can be inferred from our discussion concerning Schwarzschild geometry.

2.4.1. Schwarzschild-like spacetimes

In Hořava-Lifshitz theory of gravity one can find a particular solution described by a class of static spacetimes embodying spherical symmetry [26]. The line element of this asymptotically flat black-hole spacetime reads:

$$ds^2 = g_{tt}dt^2 - \frac{1}{g_{tt}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{2.36}$$

where

$$g_{tt} = 1 + \omega r^2 - \sqrt{\omega^2 r^4 + 4M\omega r} \ . \tag{2.37}$$

The quantity ω is related to the coupling constants in the action of the theory and M is the integration constant that can be thought of as a mass of the black hole. If we recall that the Schwarzschild radius $r_S = 2M$, we recover for the spacetime (2.36) the usual behavior of a Schwarzschild metric when $r \gg (M/\omega)^{1/3}$:

$$g_{tt} = 1 - \frac{2M}{r} + \mathcal{O}\left(r^{-4}\right) \ . \tag{2.38}$$

The event horizon is defined by the radial coordinate where

$$g_{tt}(r_h) = 0 \ . \tag{2.39}$$

In the case of Hořava-Lifshitz metric (2.37) one may find more than one event horizon:

$$r_{h\pm} = M\left(1 \pm \sqrt{1 - \frac{1}{2\omega M^2}}\right) \ . \tag{2.40}$$

The signs $(+)$ and $(-)$ correspond to outer and inner horizon, respectively. Since the Ricci scalar diverges as $1/r^{3/2}$ [26] the metric is singular at the origin, $r = 0$. In order to avoid the naked singularity one has to require $\omega M^2 \geqslant 1/2$ [19, 20, 26].

The Schwarzschild-like spacetime described by (2.36) and (2.37) is static and exhibits spherical symmetry. These properties allowed us to determine the geodesics in Schwarzschild spacetime and the same arguments apply in this case. The particles' motion can be restricted to equatorial plane, $\theta = \pi/2$. The relations as in (2.10) are satisfied. We can rewrite the equation (2.12) in Hořava-Lifshitz spacetime

$$\frac{1}{g_{tt}}\left[\epsilon^2 - \left(\frac{dr}{d\lambda}\right)^2\right] = \frac{L^2}{r^2} + \xi \tag{2.41}$$

and extract from it the effective potential (see Refs. [19, 20])

$$V_\xi^{eff}(r) = g_{tt}\left(\frac{L^2}{r^2} + \xi\right) \ . \tag{2.42}$$

The shape of this potential, similarly as in the case of Schwarzschild geometry, depends on the magnitude of angular momentum, what is depicted in Fig. 2.1.

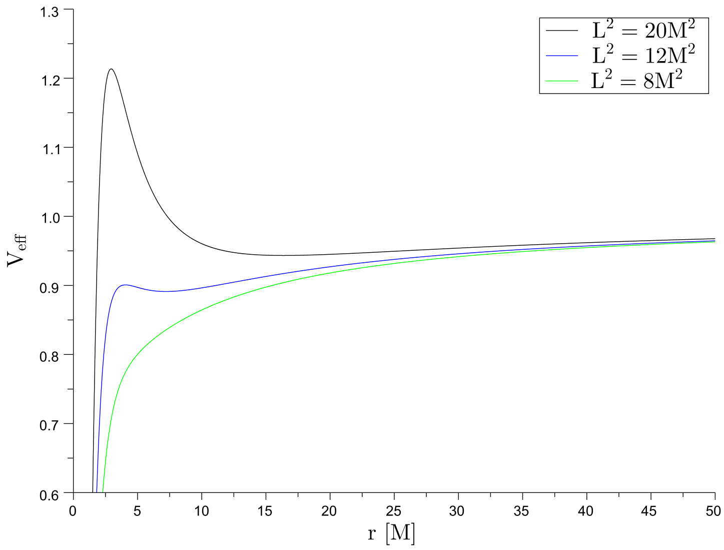

Figure 2.1. Effective potential in the case of a massive particle, for different values of L^2, where $\omega M^2 = 1/2$.

2.4.2. Circular geodesics

Due to the additional parameter ω in the metric of the Hořava-Lifshitz spacetime the equations describing circular geodesics may be more complicated than in the Schwarzschild spacetime and the solutions should differ. However, as we are going to find out, the main features of the particle's geodesic motion, like the number of possible orbits, remain unchanged.

The extrema of the effective potential correspond to the radii of circular geodesics [19, 20]. They satisfy

$$L^2(g'_{tt}r - 2g_{tt}) + g'_{tt}r^3\xi = 0 \ , \tag{2.43}$$

where $g'_{tt} = \frac{d}{dr}g_{tt}$. In the case of massive particles, $\xi = 1$, the parameter of motion which determines the existence of circular geodesics for given g^{HL}_{tt} is angular momentum. From (2.43) we can infer that the magnitude of angular momentum should exceed a certain threshold value

$$L^2 > L_0^2 = \min \frac{g'_{tt}r^3}{2g_{tt} - g'_{tt}r} \ . \tag{2.44}$$

If this condition is fulfilled, to each value of L^2 corresponds a set of two orbits: stable and unstable, described by the minimum and maximum of the effective potential, respectively. Choosing $\omega M^2 = 1/2$ we obtain, numerically, the minimal value of the squared angular momentum $L_0^2 = 11.2M^2$. The corresponding radial coordinate, $r_0 = 5.23655M$, signifies the saddle point of the effective potential. The larger the angular momentum, the more prominent the separation between inner and outer circular geodesics. The radius of stable (outer) orbit increases with L^2 while the radius of unstable orbit diminishes. On the other hand, with increasing ω the minimal: L^2 and r also increase, up to their respective values of $12M^2$ and $6M$. These values of L_0^2 and r_0 match the results that can be derived in Schwarzschild spacetime, from Eqs. (2.31) and (2.32).

Let us continue with the circular geodesics of massless particles. As in the case of

Schwarzschild geometry, the radius of the photon sphere r_{ph} does not depend on the angular momentum (see (2.43)) and satisfies the equation

$$g'_{tt}r - 2g_{tt}\Big|_{r=r_{ph}} = 0 \ . \tag{2.45}$$

By inserting the expression for the g_{tt} element of the metric in Hořava-Lifshitz spacetime (2.37) we arrive at the equality

$$r_{ph}^3 - 9M^2 r_{ph} + 4\frac{M}{\omega} = 0 \ . \tag{2.46}$$

There is always one possible circular orbit situated beyond the outer horizon, like in Schwarzschild geometry. The minimal radius of the photon sphere is $r_{ph} = (\sqrt{33}-1)M/2$ and it corresponds to $\omega M^2 = 1/2$. Increase of ω causes the massless geodesics to move further from the black hole what is presented in Fig. 2.2 and the Schwarzschild spacetime value $r_{ph} = 3M$ is reached for $\omega M^2 \gg 1$.

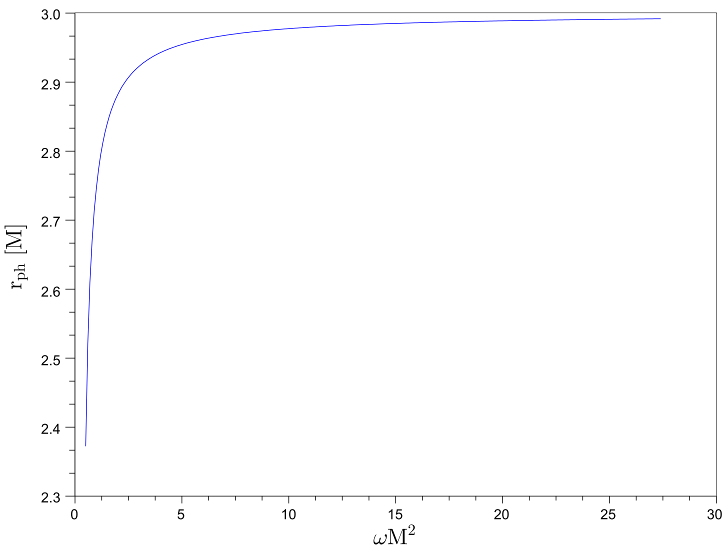

Figure 2.2. Photon sphere radius r_{ph} as a function of ωM^2.

From the Eq. (2.43) one can retrieve the relation between angular momentum of a massive particle and circular geodesics radius

$$L^2 = \frac{g'_{tt}r^3}{2g_{tt} - g'_{tt}r} \ . \tag{2.47}$$

Consequently, we infer that photon sphere can be reached in the asymptotic limit, $L^2 \to \infty$, by massive circular geodesics (see Eq. (2.45)). The radius of the photon sphere is thus *a point of no return* [25] in the Schwarzschild-like spacetimes defined by (2.36), (2.37).

Finally, one can derive the expressions that relate the massive particle's speed on circular geodesics with its angular momentum and energy as well as with the radius of the orbit.

Speed, as measured by a static local observer satisfies Eq. (2.23) which gives

$$v^2 = \frac{L^2}{L^2 + r^2} , \tag{2.48}$$

the same result as in Schwarzschild spacetime. We have a relationship between energy and angular momentum

$$\epsilon^2 = g_{tt} \left(\frac{L^2}{r^2} + \xi \right) , \tag{2.49}$$

derived from Eq. (2.41). Now we need to use the angular momentum-radius dependence (2.47) to obtain the relation between speed of the object and the radius of its orbit

$$v^2 = \frac{g_{tt}' r}{2 g_{tt}} . \tag{2.50}$$

From this equality, together with (2.45), one would obtain the speed of a massive particle equal to 1 on the photon sphere and $v^2 > 1$ for $r < r_{ph}$. However, one has to keep in mind that $r = r_{ph}$ can only be reached by massive particles in the asymptotic limit, of infinitely large momentum $L^2 \to \infty$ and energy $\epsilon^2 \to \infty$ (see Eq. (2.49)). The interior of the photon sphere is inaccessible for massive as well as massless particles' circular geodesics. It is restricted to spacelike circular orbits, $\xi = -1$. One could formally consider this region, $r_{h+} < r < r_{ph}$, as tachyonic sector of unstable orbits, characterized by positive angular momentum and energy (see Eqs. (2.43) and (2.49)) that become singular as $r \to {r_{ph}}^-$.

2.5. Summary

The static spherically symmetric spacetime of Hořava-Lifshitz gravity in a particular limit reproduces Schwarzschild spacetime. It is interesting that even in the more general case of Hořava-Lifshitz black hole there exists a single photon sphere, corresponding to the maximum of the effective potential as a function of the radial coordinate. This massless circular geodesic is *a point of no return*. The circular orbits can be arranged in three non-overlapping sectors around the central mass. Massive particles' geodesics, for large enough angular momentum $L^2 > L_0^2$, occur in pairs. There are stable (outer) and unstable (inner) circular geodesics. The latter, in the asymptotic limit of $L^2 \to \infty$, approach the photon sphere, $r \to r_{ph}^+$. The corresponding speed of the massive particle tends to the speed of light $v^2 \to 1$. The next sector, which is impenetrable for massive objects, is the photon sphere. It is the unstable geodesics of radius $r = r_{ph}$, the only circular orbit of massless particles. The region in spacetime that would correspond to spacelike circular geodesics lies within the photon sphere. These orbits cannot be arranged for $r \geqslant r_{ph}$. The sector $r < r_{ph}$ is inaccessible for massive as well as massless circular geodesics. It is worth pointing out that the massive particle's velocity-radius dependence somehow reflects this division of spacetime in the strong gravitational field. According to this relationship, a massive object would have the speed smaller, equal to or larger than the speed of light outside, on and inside the photon sphere, respectively. Naturally, one cannot draw conclusions solely from this relation, since the radius of an orbit depends on the value of angular momentum. Massive, massless and spacelike circular geodesics belong to the sectors: $r > r_{ph}$, $r = r_{ph}$ and $r < r_{ph}$, respectively. We arrived at the same conclusions in Hořava-Lifshitz gravity as in Schwarzschild spacetime of general relativity.

3. Communication in Schwarzschild spacetime

3.1. Introduction

In this chapter we will analyze the exchange of electromagnetic signals between radially falling observers (treated as test particles) in the vicinity of the event horizon in Schwarzschild spacetime. Our study is based on [28], [29]–[33]. We will use the extension of the Schwarzschild coordinates that was put forward independently by Kruskal and Szekeres [5]. This coordinate systems enables us to study geodesics crossing the horizon. Our considerations are limited to the geodesic motion in radial direction.

The notion of time in general relativity is closely connected with the observer. Each observer is equipped with a "coordinate clock" that enables him/her to measure the *proper time*. In Schwarzschild spacetime (2.1) the time coordinate t can actually be treated as proper time measured by a "master clock" of the static observer near spatial infinity, $r \to \infty$. In general, all the observers at rest could adjust and synchronize their "coordinate clocks" in such a way that they would give the correct values of coordinate time (see [14]). The direction of the flow of time, the *time arrow*, in a given geometry can be derived from the world lines of test particles. The situation is more complicated in the strong gravitational field, within the event horizon. The element g_{tt} of the Schwarzschild metric vanishes for the critical radius $r = r_S$ and becomes negative for $r < r_S$. Moreover, no static observers exist in that region of spacetime, every object is forced to move in the spatial direction. By investigating the motion of test particles in the vicinity of the horizon one can realize that the choice of time arrow and time coordinate is a truly non-trivial matter.

The event horizon as a boundary in spacetime has raised a lot of interest over the years. Its specific properties, like particle creation [34] or thermodynamics [35] were studied extensively. The phenomena associated with the communication of the observers by exchange of light signals near the horizon were also analyzed. One should mention the issues of speed of a massive particle reaching the critical radius [32], information carried away by an object [30] or the "infinite future" seen by an observer falling beneath the horizon [31].

We will consider the motion of radially falling observers and their ability to communicate with each other as well as with distant static observers. In this case the most interesting effects occur in the vicinity of the event horizon. We will study the situation that allows the maximum possible simplicity of necessary calculations but the essential features that are revealed in this approach would apply in more general cases, too. The free fall in the radial direction of two observers: Alice and Bob, starting from the same point in space, and the exchange of the electromagnetic signals between them and their "mother station" will be examined. No event that takes place below the horizon can be detected from the outside of the black hole. However, the observer that has crossed the critical radius still receives the signals originating above the horizon. We will study the frequency ratios of signals sent and received in the exterior of the black hole. Then we will take a closer look at the effects occurring in the vicinity of the horizon, especially the behavior of light signals exchanged between two infalling observers, one chasing the other. In order to describe the geodesics inside and outside the black hole in the consistent manner we will use Kruskal-Szekeres coordinates. The two-way communication is broken at the horizon. However, in the interior of the black hole one is faced with two distinct scenarios: one presuming the restoration of the communication between two observers and the other, where signals cannot be sent in both directions anymore. The choice of the scenario is related to the choice of orientation of the time arrow, time arrows are different in these two cases. It may be pointed out that it is impossible to determine by observation performed in the outer region which scenario actually occurs. The most intriguing question is whether both scenarios could arise.

First, we will describe the main features of Kruskal-Szekeres coordinate system. In this framework the geodesics of massive and massless test particles, corresponding to radial free fall will be determined. It turns out that the speed of a moving object is a useful parameter in the description of the communication process. The frequencies of signals sent and received by two infalling observers and one in spatial rest will be derived. We will follow by the study of geodesics below the event horizon and accessible communication channels. The two scenarios will be discussed, especially the one that assumes broken two-way communication below the horizon.

3.2. Exchange of signals above event horizon

We have already introduced the basic tools that can be used in Schwarzschild spacetime to determine the geodesic motion of massive or massless objects. It is necessary to extend these methods for the case of Kruskal-Szekeres coordinate system which does not display the singularity at the event horizon. Kruskal-Szekeres coordinates [5] are defined through the transformation $(t, r) \to (v, u)$

$$\begin{aligned} u &= \sqrt{\left|\frac{r}{2M} - 1\right|} e^{\frac{r}{4M}} \begin{cases} \cosh \frac{t}{4M} & r > 2M \\ \sinh \frac{t}{4M} & r < 2M \end{cases}, \\ v &= \sqrt{\left|\frac{r}{2M} - 1\right|} e^{\frac{r}{4M}} \begin{cases} \sinh \frac{t}{4M} & r > 2M \\ \cosh \frac{t}{4M} & r < 2M \end{cases} \end{aligned} \tag{3.1}$$

for $u+v>0$. In Kruskal-Szekeres coordinates the line element is described by

$$ds^2 = K(dv^2 - du^2) - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.2}$$

where $K = 32M^3\exp(-r/2M)/r$. The metric in Kruskal-Szekeres coordinates does not reveal any singularities at the event horizon $r = 2M$, what corresponds to $v = u$. We can also see that here r is no longer treated like a coordinate, it is a parameter. The inverse transformations $(v,u) \to (t,r)$ are given by

$$\begin{aligned} \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} &= u^2 - v^2, \\ \operatorname{atanh}\frac{t}{4M} &= \begin{cases} \frac{v}{u} & r > 2M \\ \frac{u}{v} & r < 2M \end{cases}. \end{aligned} \tag{3.3}$$

We have to emphasize that we are interested only in the part $u+v>0$, corresponding to Schwarzschild spacetime, above and below the event horizon. The Kruskal-Szekeres coordinates are particularly useful because the trajectories can be presented graphically in a very simple way. In the (u,v) plane (see Fig. 3.1) the lines of constant r are depicted as hyperbolae and of constant t as straight lines going through the origin. The line $v = u$ corresponds to the event horizon.

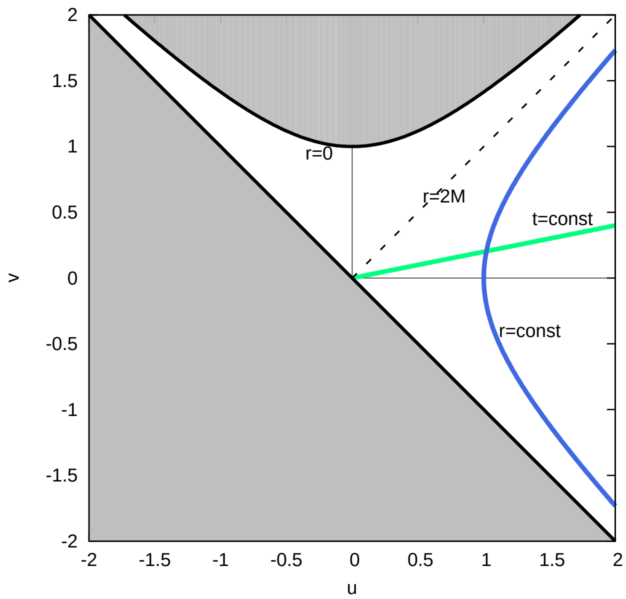

Figure 3.1. Kruskal-Szekeres coordinate system. The grey area is not covered by Schwarzschild spacetime.

In order to determine the vectors tangent to the geodesics in Kruskal-Szekeres coordinates we can use the results from section 2.2 and the following relation

$$P^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} P^{\mu} \tag{3.4}$$

for the components of vectors $P^{\mu'}$ in the transformed coordinate system $x^{\mu'}$ and P^{μ} in the original system x^{μ}. In the case of transformation $(t, r) \rightarrow (v, u)$ we have

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{u}{4M} \ , \\ \frac{\partial u}{\partial t} &= \frac{v}{4M} \ , \\ \frac{\partial v}{\partial r} &= \frac{v}{4M}\frac{r}{r-2M} \ , \\ \frac{\partial u}{\partial r} &= \frac{u}{4M}\frac{r}{r-2M} \ . \end{aligned} \tag{3.5}$$

Hence, the Killing vectors calculated in Schwarzschild spacetime (2.6), associated with the conservation of energy and magnitude of angular momentum, in Kruskal-Szekeres coordinates read:

$$\begin{aligned} \eta^{\mu} &= \frac{1}{4M}\left(u\delta^{v}_{\mu} + v\delta^{u}_{\mu}\right) \ , \\ \zeta^{\nu} &= \delta^{\phi}_{\nu} \ . \end{aligned} \tag{3.6}$$

Utilizing the results (2.7) and (2.10) we arrive at the four-velocity of the massive particles, $\xi = 1$, expressed in Schwarzschild coordinates in the form

$$U^{\mu} = (U^{t}, U^{r}, 0, 0) \ , \tag{3.7}$$

with

$$\begin{aligned} U^{t} &= \frac{\epsilon}{g_{tt}} \ , \\ U^{r} &= -\sqrt{\epsilon^{2} - g_{tt}} \ , \end{aligned} \tag{3.8}$$

where we assumed the motion to be radial free fall – hence the sign "$-$" in the formula for U^{r}. Again, ϵ is the particle's specific energy. We can write it explicitly in this case. For the particle starting at $t = t_0$ from $r = r_0$ with $U^{\mu}(t_0, r_0) = 0$ one finds the constant of motion ϵ to be

$$\epsilon = \sqrt{1 - \frac{2M}{r_0}} \ . \tag{3.9}$$

The velocity vector components U^{v} and U^{u} read

$$\begin{aligned} U^{v} &= \frac{r}{4M(r-2M)}\left(u\epsilon - v\sqrt{\epsilon^{2} - \frac{r-2M}{r}}\right) \ , \\ U^{u} &= \frac{r}{4M(r-2M)}\left(v\epsilon - u\sqrt{\epsilon^{2} - \frac{r-2M}{r}}\right) \ . \end{aligned} \tag{3.10}$$

It is useful to rewrite them using exclusively Kruskal-Szekeres coordinates:

$$\begin{aligned} U^v &= \frac{4M}{K(u^2-v^2)}\left[u\epsilon - v\sqrt{\epsilon^2 - \frac{K}{16M^2}(u^2-v^2)}\right] , \\ U^u &= \frac{4M}{K(u^2-v^2)}\left[v\epsilon - u\sqrt{\epsilon^2 - \frac{K}{16M^2}(u^2-v^2)}\right] . \end{aligned} \tag{3.11}$$

Let us introduce a function

$$h(v,u) = \frac{\sqrt{K(u^2-v^2)}}{4M} , \tag{3.12}$$

which is constant for $r = \text{const}$. It is easy to see that it vanishes on the horizon $v = u$. Using this function one can write the parameter ϵ as

$$\epsilon = h(v_0, u_0) , \tag{3.13}$$

where (v_0, u_0) are the initial conditions. Putting $\epsilon = 1$ in Eq. (3.11), which corresponds to fall from infinity, one reproduces the solution of Ref. [28].

The vector tangent to the massless geodesic k^μ, the wave vector, satisfies

$$k^\mu k_\mu = 0 \tag{3.14}$$

and its components in Schwarzschild spacetime read

$$\begin{aligned} k^t &= \frac{\Omega_\pm}{g_{tt}} , \\ k^r &= \pm\Omega_\pm . \end{aligned} \tag{3.15}$$

where the radial direction of motion is assumed. The signs "+" and "−" correspond to the ingoing and outgoing signals, respectively. The parameter $\Omega_\pm$ can be interpreted as the frequency measured by a hypothetical static observer near spatial infinity if the light signal comes from (+) or goes to (−) infinity. Let us write down the components of the wave vector in Kruskal-Szekeres coordinates

$$k^v_\pm = \mp k^u_\pm = \frac{4M\Omega_\pm}{K(u \pm v)} . \tag{3.16}$$

These null geodesics are straight lines at an angle of $\pi/4$ with respect to u and v axes.

The static observer, related to the absolute rest frame of reference, is characterized by the velocity vector at (u, v)

$$N^\mu = \frac{u\delta^\mu_v + v\delta^\mu_u}{4M\epsilon} , \ \epsilon = h(v,u) \tag{3.17}$$

Since the parameter ϵ is imaginary below the horizon, $v > u$, one can infer that no static observer can exist within the black hole. The energy and speed of a radially falling object as measured by the static observer, in accordance with Eqs. (2.21) and (2.22), is

$$v_{object}^2 = 1 - \frac{1}{(N^\mu U_\mu)^2} = 1 - \frac{h(v,u)^2}{h(v_0,u_0)^2} \ , \tag{3.18}$$

in the case where the fall started at (v_0, u_0).

Every observer, either static or moving one, determines the frequency of an electromagnetic signal as a product of his/her velocity and a wave vector

$$\omega = U^\mu k_\mu \ . \tag{3.19}$$

We will investigate communication by the exchange of electromagnetic signals between three observers: static observer, called "mother station" (ms), and Alice and Bob that move towards the black hole. Alice is assumed to begin the radial fall before Bob, so Bob chases Alice. They both start their radial free fall from the same point in space, where the "mother station" resides. The first issue that we address is the communication between Alice and ms. We denote the ratio of the frequency of a received signal to the frequency of a sent signal by f. Signals emitted (e) from ms which are received (r) by Alice reveal redshift, $f < 1$. Their frequency ratio decreases as Alice approaches the horizon. This ratio may be expressed in terms of Alice's speed, v_A, measured by a local static observer (see Eq. (3.18))

$$\begin{aligned} f(ms \to A) &= \frac{\omega^r(A)}{\omega^e(ms)} \\ &= \frac{\Omega_+ \left(h_{ms} - \sqrt{{h_{ms}}^2 - {h_A}^2}\right)}{{h_A}^2} \cdot \left[\frac{\Omega_+}{h_{ms}}\right]^{-1} \\ &= \frac{{h_{ms}}^2}{{h_A}^2}\left(1 - \sqrt{1 - \frac{{h_A}^2}{{h_{ms}}^2}}\right) \\ &= \frac{1}{1 - v_A^2}(1 - v_A) \\ &= \frac{1}{1 + v_A} \ . \end{aligned} \tag{3.20}$$

A simple calculation, similar to the one above (3.20), reveals that the signals sent by Alice and recorded by ms are also redshifted

$$f(A \to ms) = \frac{\omega^r(ms)}{\omega^e(A)} = 1 - v_A \ . \tag{3.21}$$

The frequency ratios may be used by both observers to infer Alice's speed [33], v_A, at the moment of receiving or emitting the signal. It turns out that v_A approaches the speed of light and, consequently, $f(ms \to A) \to 1/2$ as Alice reaches the event horizon. On the other hand, the ratio $f(A \to ms)$ vanishes as $v_A \to 1$, the signals are critically redshifted. Alice seems to disappear from ms's screens as a faint object. At this point we have to mention that it takes infinite coordinate time for arbitrary object, massive or massless, to reach Schwarzschild radius. Using the relations (3.8) one can calculate the coordinate time that is needed to reach the arbitrary point r from the point $r_0 > r$ in space as the integral (see also [31])

$$\begin{aligned} t &= t_0 - \int_{r_0}^{r} \frac{\epsilon}{\sqrt{\epsilon^2 - g_{tt}(r')}} dr' \\ &= t_0 + r_S \left\{ \sqrt{\frac{r_0}{r_S} - 1} \left[\left(2 + \frac{r_0}{r_S} \right) \arctan \sqrt{\frac{r_0}{r} - 1} + \sqrt{\frac{r}{r_S} \left(\frac{r_0}{r_S} - \frac{r}{r_S} \right)} \right] + \right. \\ & \left. \quad +2 \log \left(\sqrt{\frac{r}{r_S} - \frac{r}{r_0}} + \sqrt{1 - \frac{r}{r_0}} \right) - \log \left| \frac{r}{r_S} - 1 \right| \right\} \end{aligned} \tag{3.22}$$

for massive geodesics and

$$\begin{aligned} t &= t_0 - \int_{r_0}^{r} \frac{1}{g_{tt}(r')} dr' \\ &= t_0 + r_0 - r + r_S \log \left| \frac{r_0 - r_S}{r - r_S} \right| \end{aligned} \tag{3.23}$$

for massless geodesics, where t_0 is the time at the start. One can see at once that in the case of $r \to r_S$ both expressions reveal logarithmic divergences. On the other hand, it takes finite *proper* time for an observer to reach horizon [14, 31],

$$\begin{aligned} \tau &= \tau_0 - \int_{r_0}^{r} \frac{dr'}{\sqrt{\epsilon^2 - g_{tt}(r')}} \\ &= r_0 \sqrt{\frac{r_0}{r_S}} \left(\arctan \sqrt{\frac{r_0}{r} - 1} + \sqrt{\frac{r}{r_0} - \frac{r^2}{r_0^2}} \right) . \end{aligned} \tag{3.24}$$

The notion of time for static and infalling observers in the strong gravitational field is strikingly different. This asymmetry raises the question concerning the exchange of signals between two infalling observers. How is their communication broken or delayed if one of them and then the other plunges into the black hole?

Alice and Bob fall radially and Bob is following Alice, they both exchange electromagnetic signals. The signals sent by Alice are outgoing and the ones sent by Bob are ingoing. The frequency ratios have a particularly simple form when written in terms of the observers' speeds v_A and v_B (see also Eqs. (3.20) and (3.21)):

$$\begin{aligned} f(B \to A) &= \frac{\omega^r(A)}{\omega^e(B)} = \frac{1 + v_B}{1 + v_A}, \\ f(A \to B) &= \frac{\omega^r(B)}{\omega^e(A)} = \frac{1 - v_A}{1 - v_B}. \end{aligned} \tag{3.25}$$

Both these functions indicate redshift, their values never exceed 1. The first equation shows that function $f(B \to A)$ is well-behaved at the horizon, where $v_A \to 1$ and $v_B \to 1$, and could possibly be extended to the black hole interior, i.e. Alice would still receive Bob's signals after crossing the horizon. In the case of the ratio $f(A \to B)$ both its numerator and denominator vanish at the horizon.

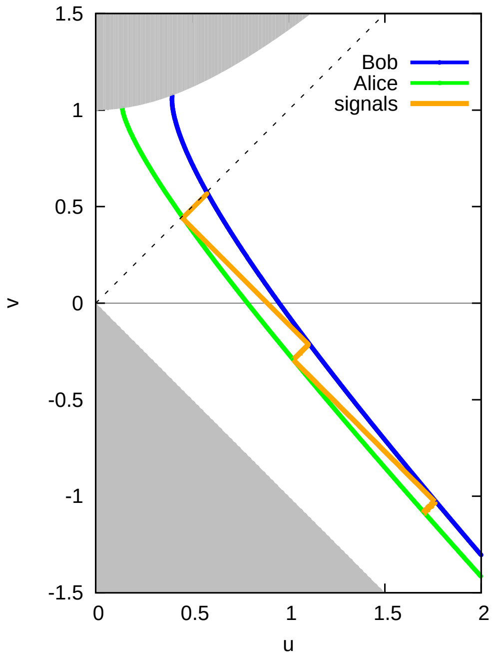

Figure 3.2. Exchange of radial signals above horizon: Bob is recording signals from Alice until crossing the event horizon (a dashed line at an angle $\pi/4$) himself.

The signals sent by Alice above the horizon will always reach Bob, no matter how much later, $t_B > t_A$, he started his free fall. Thus Bob keeps recording signals emitted by Alice until he crosses the event horizon himself. This way it may seem to him that he would collide with Alice on the horizon. What he actually detects is like an image formed by Alice's signals.

3.3. Communication below event horizon

Let us consider first solely the geodesics of infalling observers and ingoing electromagnetic signals. The corresponding tangent four-vectors, given by Eqs. (3.11) and (3.16), are well-behaved at the horizon (see also [28]). They are continuous, smooth functions of the coordinates v and u what can be seen from the explicit calculation of the limits $u \to v$ (see

Eqs. (3.11)):

$$\begin{aligned}
\lim_{u\to v} U^v &= \lim_{u\to v} \frac{4M}{K(u^2-v^2)} \left[u\epsilon - v\sqrt{\epsilon^2 - \frac{K}{16M^2}(u^2-v^2)} \right] \\
&= \lim_{u\to v} \frac{4M}{K(u^2-v^2)} \frac{u^2\epsilon^2 - v^2\left[\epsilon^2 - \frac{K}{16M^2}(u^2-v^2)\right]}{u\epsilon + v\sqrt{\epsilon^2 - \frac{K}{16M^2}(u^2-v^2)}} \\
&= \lim_{u\to v} \frac{4M}{K(u^2-v^2)} \frac{\left(\epsilon^2 + \frac{Kv^2}{16M^2}\right)(u^2-v^2)}{(u+v)\epsilon} \\
&= \lim_{u\to v} \frac{4M}{K} \frac{\epsilon^2 + \frac{Kv^2}{16M^2}}{(u+v)\epsilon} \\
&= \frac{e}{4M} \frac{\epsilon^2 + \frac{v^2}{e}}{2v\epsilon} \\
&= \frac{1}{8M}\left(\frac{\epsilon e}{v} + \frac{v}{\epsilon}\right) , \qquad (3.26)
\end{aligned}$$

and similarly

$$\begin{aligned}
\lim_{u\to v} U^u &= \lim_{u\to v} \frac{4M}{K(u^2-v^2)} \left[v\epsilon - u\sqrt{\epsilon^2 - \frac{K}{16M^2}(u^2-v^2)} \right] \\
&= \frac{1}{8M}\left(\frac{v}{\epsilon} - \frac{\epsilon e}{v}\right) . \qquad (3.27)
\end{aligned}$$

Hence, one can infer that these geodesics are not affected by the presence of the horizon. The ingoing signals can be recorded by infalling observers so Alice receives the signals from ms below the horizon. The frequency ratio reads (see Eq. (3.20))

$$f(ms \to A) = \frac{\omega^r(A)}{\omega^e(ms)} = \frac{1}{1+\sqrt{1-\frac{h_A^2}{h_{ms}^2}}} . \qquad (3.28)$$

One cannot use the notion of velocity in the interior of the black hole since it is determined with respect to the static observer and there are no such observers beneath the horizon. The frequency ratio, equal to 1/2 at the horizon, decreases monotonically to zero as $r_A \to 0$, ${h_A}^2 \to -\infty$. Ingoing signals emitted by Bob, above or below the horizon, are recorded by Alice as redshifted

$$f(B \to A) = \frac{\omega^r(A)}{\omega^e(B)} = \frac{1+\sqrt{1-\frac{h_B^2}{h_{ms}^2}}}{1+\sqrt{1-\frac{h_A^2}{h_{ms}^2}}} . \qquad (3.29)$$

This function tends to zero as $r_A \to 0$.

The case of geodesics directed "inwards" is thus pretty straightforward whereas the issue of outgoing signals is highly non-trivial when one considers the interior of the black hole. In general, the direction of the flow of time along the world line is indicated by the time arrow [29]. The value of the coordinate identified with time should either increase or decrease monotonically in this direction. It is obvious that above the horizon the coordinate that we describe as t represents time. Within the black hole the situation is more complicated due to the behavior of null outgoing geodesics in the vicinity of the horizon. Although the event horizon in Schwarzschild spacetime is not supposed to be distinguishable from other "points" along the geodesics (c.f. [36]) for the falling objects, it is still a separatrix. There are no world lines going through the horizon in the outward direction. One can see from Eq. (3.16) that the wave vector of massless outgoing geodesics is singular at $v = u$. The arrow of time as extracted from this geodesics is ill-defined at the horizon. One cannot exclude that its direction is changed in the interior of the black hole. Moreover, there is no way to determine its orientation below the horizon by any observation performed outside the black hole.

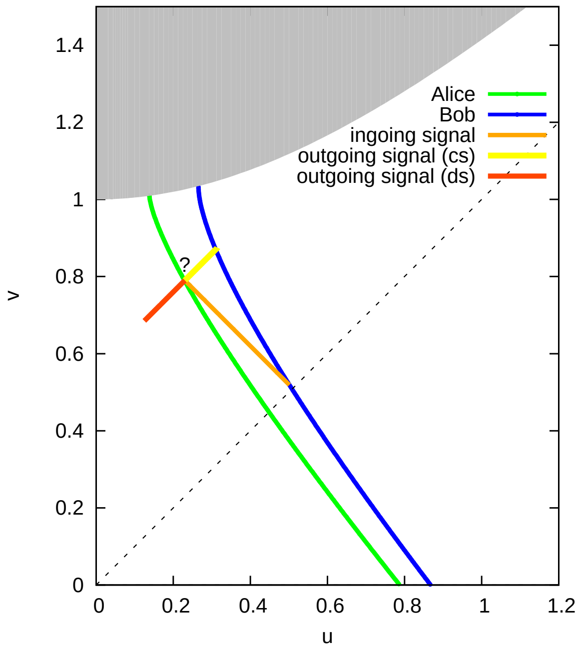

Figure 3.3. Exchange of radial signals below the horizon: Bob keeps sending signals to Alice but her outgoing signals can go in two directions, depending on the scenario ("cs" – continuous scenario, "ds" – discontinuous scenario).

One finds two distinct scenarios that may be realized below the event horizon. The difference between them is based on the choice of the direction of time arrow, whether the orientation above and below are parallel or antiparallel. The scenario that may be called *continuous* corresponds to the continuous causal structure at the horizon. In this case the Schwarzschild coordinate r is decreasing along the path for all geodesics below the horizon, therefore it is *de facto* time. Thus the metric, which depends on r, is no longer static. The directions of time arrow for outgoing null geodesics agree above and below horizon. The outgoing signal inside the black hole is actually ingoing because every object moving forward in time, $r \searrow$, is travelling in the direction towards the singularity. Furthermore, the parametrization of the outgoing massless geodesics "flips" on the horizon because

$$\Omega_{-}^{below} = -\Omega_{-}^{above} \; . \tag{3.30}$$

Obviously, Ω_- cannot be any longer interpreted as the frequency measured by the observer at spatial infinity. In this scenario both Alice and Bob can receive as well as send electromagnetic signals to each other. The two-way communication is preserved.

The *discontinuous* scenario assumes that the time arrow changes its orientation below the horizon. The time is represented by the Schwarzschild coordinate t as one can check in Fig. 3.3 – the geodesics are oriented in the direction of the flow of t (anti-clockwise direction). The geometry remains time independent in the interior of the black hole. The outgoing massless geodesic inside is antiparallel to the one outside the horizon. The parametrization of the outgoing signals geodesics is continuous

$$\Omega_-^{below} = \Omega_-^{above} \tag{3.31}$$

through the horizon, they move away from the singularity and virtually are *outgoing*. As for the exchange of signals between two observers, Bob can still send signals to Alice, just using two different channels, but Alice is unable to respond. She cannot emit any signals toward Bob. The two-way communication is broken in this case.

3.4. Causal structure of the discontinuous scenario

Let us consider the discontinuous scenario in the case of a free radial fall of the rigid body. The gravitational tidal forces are regular in an extended object moving through the event horizon in Schwarzschild spacetime. However, the causal structure in this scenario is not preserved when crossing the horizon. Consequently, it is not continuous within the body whose constituents are inside as well as outside the black hole. If the communication by means of electromagnetic signals does not survive the crossing of the horizon, the same applies to the interactions between different parts within the extended object. The components of the body which are separated by the horizon cannot maintain the equilibrium position where action and reaction forces coincide.

It is useful to find a quantity that would allow us to evaluate the extent of the rupture in communication throughout the extended body. As a first step, we introduce the measure of communication c_{ab} between particles a and b. The value of c_{ab} is one if the two-way communication of the particles is maintained and zero if it is broken. The communication weight Q_a is the number of particles of the body which are in two-way communication with a divided by the number of particles in the body

$$Q(a) = \frac{\sum_b c(a,b)}{\sum_b 1} \ . \tag{3.32}$$

It basically describes the fraction of the object that still interacts with the particle a. One can also use the average communication weight that takes into account communication between all pairs in the body

$$\bar{Q} = \frac{\sum_{ab} c(a,b)}{\sum_{ab} 1} \ . \tag{3.33}$$

Let us choose the simplest extended object that can be analyzed in a radial free fall – a one-dimensional rod. We assume that its constituents are point particles, interacting weakly with each other so that their fall can be treated as geodesic motion. The rod is arranged in the radial direction. At the same time t different particles occupy points with different values of r and start they independent free falls. A particle loses the two-way communication with another particle when it sends a signal that will not be answered, that is the signal that reaches the other particle at the event horizon.

The average communication weight, $\bar{Q}$, as a function of the radial coordinate and proper time of the particle at the outer end of the rod is depicted in Figs. 3.4 and 3.5. The way

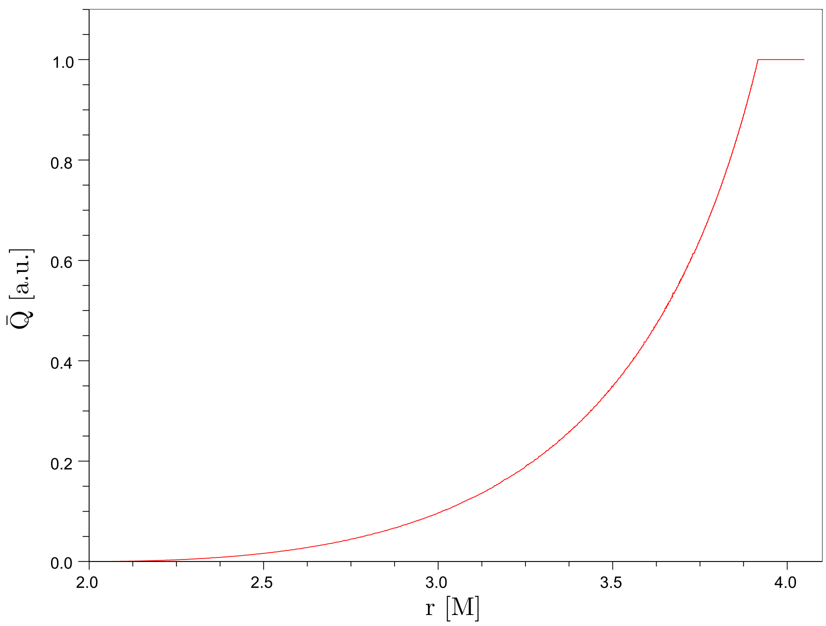

Figure 3.4. Average communication weight as a function of the radial coordinate r of the outer end of the rod during crossing the horizon.

this function is constructed is as follows. Particle N at the far end of the rod sends the signal at τ_1^N that reaches particle 1 at the horizon (at 1's proper time τ_1^1). All the other observers, numbered by $i = 2, 3, \ldots, N-1$, are above the horizon when they receive this retarded signal from N, at their respective proper times τ_1^i and at these times they maintain communication with each other and with N but not with observer 1. Thus we say that for τ_1^N the communication with observer 1 is lost for all the observers. Analogically, if at τ_2^N B sends the signal that observer 2 receives on the horizon we say that all the observers $i = 3, 4, \ldots, N-1$, and N lose communication with observer 2. So, for τ_2^N the communication

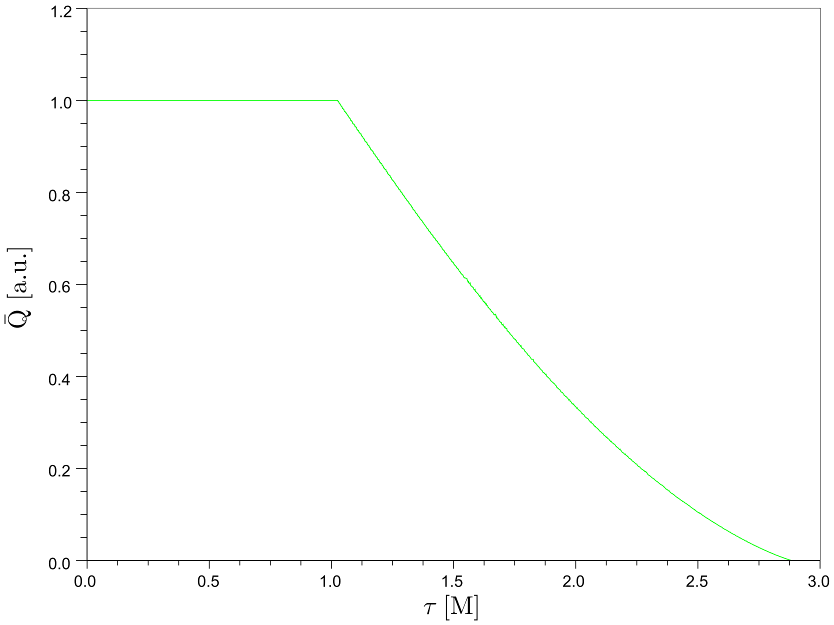

Figure 3.5. Average communication weight as a function of the proper time of the outer end of the rod during crossing the horizon.

is kept between all the observers above horizon, $3, 4, \ldots, N-1$ and N. For any bound object to exist all its constituents need to maintain two-way communication with each other. In the discontinuous scenario the part of the body which enters the black hole disintegrates. The plots present how the two-way communication is gradually lost throughout the extended object while crossing the horizon. This amounts to the body falling apart into elementary constituents. Consequently, no composite object could survive plunging into the black hole if the discontinuous scenario is realized.

3.5. Summary

The study of the exchange of electromagnetic signals in the strong gravitational field, in Schwarzschild spacetime, leads us to believe that the choice of time arrow is a non-trivial issue in the close proximity to the event horizon. In order to achieve the coherent description of physical phenomena above and below the horizon we have employed Kruskal-Szekeres coordinates. The communication between observers in the radial free fall and also with their mother station has been analyzed.

The frequency ratio of the signals emitted by mother station and recorded by Alice indicates redshift. This ratio decreases with increasing Alice's speed. It is worth noting that Alice is able to identify exactly the moment of crossing the horizon since it corresponds to the frequency ratio 1/2. There are no problems with receiving ingoing signals below the horizon, the frequency ratio keeps decreasing until it reaches 0 when Alice hits the singularity $r = 0$.

In the case of two infalling observers the frequency ratio of the exchanged signals may also be described in terms of the observers' speeds above the horizon. Both ratios correspond to redshift. It is a non-trivial observation that Bob, who chases Alice, receives her signals (sent above the horizon) until the very last instant of crossing the horizon. It is as if he falls into the "image" yielded by Alice's signals. Inside the black hole Bob can still send ingoing signals to Alice, however, the case of Alice's outgoing signals is not so simple. The null geodesic corresponding to the outgoing signal reveals singularity at the horizon.

In fact, there are two possible scenarios of null outgoing geodesics behavior below the horizon. It is not possible to distinguish which one occurs by any observation carried out in the outer region. One scenario assumes the continuous causal structure, the same orientation of the time arrow below and above the horizon. The outgoing massless geodesics are parallel in the internal and external region hence the two-way communication is restored inside the black hole. However, the parametrization of the geodesics needs to be adjusted, there is a "flip" $\Omega^{below} = -\Omega^{above}$ in the sign of frequency parameter. Also, the outgoing signals move towards the singularity, they cannot travel in the direction away from it.

In the discontinuous scenario the parametrization of null outgoing geodesics is preserved but its direction and the orientation of the time arrow below horizon are antiparallel to the direction above horizon. In this scenario Alice cannot send signals to Bob, the two way communication is broken. This is an unusual situation. It has been shown that in discontinuous scenario all the compound objects should fall apart into elementary constituents. The interactions that hold the extended body together, just like the two-way communication by means of electromagnetic signals, are lost after crossing the horizon. The balance of the action and reaction forces is spoilt and Newton's third law of motion is violated. The part of the body that enters the black hole disintegrates. No compound objects are bound to exist below the horizon.

Part II

Towards spontaneous Lorentz symmetry breaking

In the second part of the thesis we study the case of flat spacetime. Our main goal is the generalization of the idea of spontaneous symmetry breaking to relativistic symmetries. Lorentz invariance violation can be achieved through the non-vanishing expectation value for the vector field. It may be induced by the presence of the covariant derivatives of higher order in the model. One should expect the higher order derivative terms in effective theories, obtained after the elimination of the heavy degrees of freedom. The higher order derivatives are usually connected with problems with unitarity and stability. We present how the consistency of effective theories can be established, using the property of reflection positivity in lattice regularization. Then, we construct an extended model of scalar QED, with higher orders of derivatives, where the vacuum violates Lorentz symmetry. The leading order of the saddle-point expansion is considered. The study of the quasi-particle content of the theory yields quite surprising results.

4. Higher orders of derivatives in effective theories

4.1. Introduction

Most of the theories encountered in physics should be treated as effective theories. Finding the ultimate “theory of everything”, which would provide a complete description of fundamental interactions up to arbitrarily high energies (or short distances), is an almost impossible task to accomplish. On the other hand, even if this theory was constructed, it would probably prove impractical in most applications. In physics one usually is interested in the phenomena occurring at a certain scale. Too much information concerning the effects that take place far above that energy level could actually obscure the picture and impede the understanding of the problem at hand. Hence, effective theories, where only the appropriate, important features of the interesting system are considered, can be thought of as a reasonable and realistic alternative. These theories are powerful tools that enable us to describe physics properly at the given energy scale, below the threshold value.

The way to obtain effective theory from an underlying, more microscopic one is to eliminate the degrees of freedom belonging to short distances, not resolved by the model under consideration. The only variables that are taken explicitly into account correspond to energies lower than the cutoff Λ. The heavy particles, whose $M \gg \Lambda$, are integrated out from the action, however the information about them is restored in the couplings of the remaining effective Lagrangian. Due to the elimination of a propagating degree of freedom from a local theory the long range correlations exist in the resulting low-energy dynamics. These non-local features appear as higher order derivatives in the effective action, what we will show in the realm of Euclidean quantum field theory, with imaginary time.

In general, effective theories contain infinite number of terms. If the energy scale of the system considered is sufficiently low, the gradient expansion can be truncated. Then one is left with the finite number of terms that represent the crucial ingredients of the theory. However, the truncation of the effective dynamics could have undesirable consequences. Even if the high-energy theory is consistent, follows a unitary time evolution, the effective model may exhibit inconsistencies. The theory based on the truncated gradient expansion could reveal the non-unitary time evolution for the light particles.

In this chapter we will describe some rudiments of quantum field theory: Green's functions, path integral formalism and generating functionals (see also [37]-[39]). The basic features of Euclidean field theory will be introduced (see [39]). We will also show explicitly how an effective theory with higher orders of derivatives can emerge from the more microscopic model. The calculations are performed in Minkowski spacetime, with the signature $(+,-,-,-)$, unless stated otherwise. We also assume $c = \hbar = 1$.

4.2. Quantum field theory basics

4.2.1. Observables and Green's functions

In high energy physics the processes that occur are extremely fast. Due to the uncertainty of determining the frequency of an oscillation observed for a short time the possibilities of measurement of coordinates and momenta are severely limited. In consequence, it is necessary to focus on measuring scattering cross-sections and decay rates which remain well defined at arbitrarily high energies. The major breakthrough in the development of quantum field theory was the reduction formula obtained by Lehmann, Symanzik and Zimmermann [40]. It gives the general relation between the scattering amplitudes and correlation functions. The LSZ reduction formula states that the transition probability amplitudes are proportional to the residues of the connected Green functions arising from the mass-shell singularities.

The Green's function (or correlation function) of order n is defined by the expectation value

$$iG(n)(x_1, \ldots, x_n) = \langle 0|T[\phi_H(x_1)\ldots\phi_H(x_n)]|0\rangle_H \tag{4.1}$$

in the Heisenberg representation. As their values are related to measurable quantities, the calculation of the Green's functions became the main objective in quantum field theory.

The indispensable theoretical tool that makes it possible to evaluate the outcome of the experiments is the scattering matrix, usually referred to as S-matrix. It is related to the evolution of the system in time – it connects the asymptotic, initial and final, particle states

$$|f\rangle = S|i\rangle \tag{4.2}$$

It is worth noting that the S-matrix has to be a unitary matrix, this is the requirement of the unitarity of the quantum field theory. The S-matrix as an operator S is defined through its relation to the probability amplitude $\mathcal{A}$ that the initial state $|n\rangle$ results, after the interaction, in the final state $|m\rangle$ [38]:

$$\mathcal{A}_{m\leftarrow n} = \langle m|S|n\rangle \ . \tag{4.3}$$

For the set of orthonormal and complete physical states $|n\rangle$ the probability that the system ends up in any of these states has to be unity

$$\sum_m |\langle m|S|n\rangle|^2 = 1 \ . \tag{4.4}$$

Therefore, we can write

$$\sum_m |\langle m|S|n\rangle|^2 = \sum_m \langle m|S|n\rangle^* \langle m|S|n\rangle = \sum_m \langle n|S^\dagger|m\rangle\langle m|S|n\rangle = \langle n|S^\dagger S|n\rangle \tag{4.5}$$

and since this quantity equals one for arbitrary state $|n\rangle$, it is necessary that

$$S^\dagger S = SS^\dagger = \mathbb{1}. \tag{4.6}$$

The unitarity of the S-matrix represents the conservation of probabilities and thus is a very important ingredient in theories of quantized fields.

4.2.2. Perturbation expansion – path integral

Path integral was established as a formulation of Quantum Mechanics alternative to canonical approach. The idea of using the formalism based on Lagrangian rather than Hamiltonian in the description of quantum systems, introduced by Dirac [41], was later developed by Feynman [42] into a complete method of quantization. Especially in the case of quantum fields, Feynman path integral provides a very elegant and relatively simple treatment.

Path integral formalism in quantum field theory, also referred to as functional formalism, introduces the transition amplitude for a field ϕ in the form:

$$\langle \phi_f|\phi_i\rangle = \int D[\phi] e^{iS[\phi]} \ . \tag{4.7}$$

On the right hand side of the equation we have the integral over all possible field configurations and $S[\phi]$ is the classical action. The functional formalism is extremely convenient because it makes it possible to handle infinitely many Green's functions at the same time. The vacuum-to-vacuum transition amplitude in the presence of an arbitrary external source $j(x)$ that couples linearly to the field operator in the action,

$$S[\phi] \to S[\phi] + \int d^4x j(x)\phi(x) = S[\phi] + j\cdot\phi \ , \tag{4.8}$$

can be written down using the path integral formulation as

$$Z[j] = \langle 0|S|0\rangle_j = \int D[\phi] e^{iS[\phi]+ij\cdot\phi} \ , \tag{4.9}$$

where the scalar product denotes integration over spacetime coordinates. From the functional Taylor expansion of $Z[j]$

$$Z[j] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n G(x_1, \dots, x_n) j(x_1) \dots j(x_n) \ , \tag{4.10}$$

containing the interacting Green's functions, one can infer that it is the generating functional for the Green's functions,

$$G(x_1, \dots, x_n) = \frac{1}{i^n} \frac{\delta}{\delta j(x_1)} \dots \frac{\delta}{\delta j(x_n)} Z[j]|_{j=0} \tag{4.11}$$

It is straightforward to obtain $Z[j]$ for a free system characterized by the action

$$S_0[\phi] = -\frac{1}{2} \int d^4x \phi(x)(\Box + m^2)\phi(x) = \frac{1}{2}\phi \cdot G_0^{-1} \cdot \phi \ , \tag{4.12}$$

where the Feynman $i\varepsilon$ prescription is omitted. In this case

$$Z_0[j] = \int D[\phi] e^{\frac{i}{2}\phi \cdot G_0^{-1} \cdot \phi + ij \cdot \phi} \ . \tag{4.13}$$

One can change the variables

$$\phi' = \phi + G_0 j \ , \tag{4.14}$$

in order to complete the square in the exponent. Then the generating functional reads

$$Z_0[j] = \int D[\phi'] e^{\frac{i}{2}\phi' \cdot G_0^{-1} \cdot \phi' - \frac{i}{2} j \cdot G_0 \cdot j} = e^{-\frac{i}{2} j \cdot G_0 \cdot j} \ , \tag{4.15}$$

with the normalization $Z_0[0] = 1$.

The perturbation series for the generating functional can be found by splitting the action into the free part and interaction part,

$$S[\phi] = S_0[\phi] + S_i[\phi] \ , \tag{4.16}$$

and then expanding in the latter,

$$\begin{aligned} Z[j] &= \int D[\phi] e^{\frac{i}{2}\phi \cdot G_0^{-1} \cdot \phi + S_i[\phi] + ij \cdot \phi} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int D[\phi] e^{\frac{i}{2}\phi \cdot G_0^{-1} \cdot \phi + ij \cdot \phi} S_i^n[\phi] \ . \end{aligned} \tag{4.17}$$

The last term can be factorized out of the path integral

$$\begin{aligned} Z[j] &= \sum_{n=0}^{\infty} \frac{i^n}{n!} S_i^n \left[\frac{1}{i}\frac{\delta}{\delta j}\right] \int D[\phi] e^{\frac{i}{2}\phi\cdot G_0^{-1}\cdot\phi + ij\cdot\phi} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} S_i^n \left[\frac{1}{i}\frac{\delta}{\delta j}\right] Z_0[j] \ , \end{aligned} \tag{4.18}$$

where the generating functional of free Green's functions was used. The resummation of the series leads to the final result

$$Z[j] = e^{iS_i^n\left[\frac{1}{i}\frac{\delta}{\delta j}\right]} Z_0[j] = e^{iS_i^n\left[\frac{1}{i}\frac{\delta}{\delta j}\right]} e^{-\frac{i}{2} j\cdot G_0\cdot j} \ . \tag{4.19}$$

4.2.3. Connected Green's functions

The perturbation series of the generating functional represents a sum over all diagrams: connected, where each part is connected to the rest of the diagram, and disconnected, which consist of parts that are not linked together. Only the connected Feynman graphs contribute to the non-trivial part of the S-matrix. We define the functional $W[j]$ by

$$Z[j] = e^{iW[j]} \ , \tag{4.20}$$

where the initial condition $W[0] = 0$ is satisfied. The functional $W[j]$ generates the connected Green's functions G_c

$$W[j] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n G_c(x_1, \dots, x_n) j(x_1) \dots j(x_n) \ . \tag{4.21}$$

The usual, full Green's functions can always be expressed in terms of connected Green's functions, as a sum of their products.

4.2.4. One-particle irreducible Green's functions

A one-particle irreducible (1PI) Feynman graph is a connected graph that cannot be made disconnected by cutting a single one of its internal lines. The 1PI diagrams are usually evaluated with their external legs removed (truncated) what amounts to dividing out the propagator at each external line of the original diagrams. Thus the mass-shell singularity is eliminated and the value of the resulting graph represents a contribution to a scattering amplitude.

The 1PI Green's functions, $\Gamma(x_1, \dots, x_n)$, are generated by $\Gamma[\Phi]$, called the effective action,

$$\Gamma[\Phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \Gamma(x, x_1 \dots x_n) \Phi(x_1) \dots \Phi(x_n) \ . \tag{4.22}$$

It is defined by

$$e^{i\Gamma[\Phi]+i\Phi\cdot j} = \int D[\phi] e^{iS[\phi]+i\phi\cdot j} \tag{4.23}$$

where

$$\Phi(x) \equiv \langle 0|\phi(x)|0\rangle_j = \frac{1}{iZ[j]}\frac{\delta Z[j]}{\delta j(x)} = \frac{\delta W[j]}{\delta j(x)} \tag{4.24}$$

is the vacuum expectation value of the field when an external source is present. The effective action contributes to the phase of vacuum-to-vacuum transition amplitude as a function of the field expectation value, after the source term has been separated. Taking into account (4.20) and (4.23) one sees immediately that

$$\Gamma[\Phi] + \Phi \cdot j = W[j] \ , \tag{4.25}$$

the effective action is related to the connected generating functional by a functional Legendre transform. Performing the functional differentiation with respect to Φ,

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} + j(x) + \int d^4y \Phi(y)\frac{\delta j(y)}{\delta\Phi(x)} = \int d^4y \frac{\delta W[j]}{\delta j(y)}\frac{\delta j(y)}{\delta\Phi(x)} \ , \tag{4.26}$$

gives the inverse Legendre transform by

$$j(x) = -\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} \tag{4.27}$$

together with Eq. (4.24). From the effective action as the generating functional of one-particle-irreducible correlation functions all the substantial information on the quantum field theory and its predictions can be extracted.

One should distinguish between the effective action, the generating functional for 1PI Green's functions, and the action in the effective theories, obtained by eliminating heavy degrees of freedom from the path integral. In the latter case, one can obtain the expression for the effective action in the following manner (see also [43]). Let us assume the action in the theory as $S[\phi^H, \phi^L]$ where ϕ^H are the "heavy" and ϕ^L are the "light" field components. The heavy fields are the ones that cannot be observed at a certain energy scale. Only the light fields are directly observable and their dynamics is influenced by the heavy field components. In order to focus attention on the light components we write

$$\begin{aligned} Z[j] &= \int D[\phi^H]D[\phi^L]e^{iS[\phi^H,\phi^L]} \\ &= \int D[\phi^L]e^{iS_{\text{eff}}[\phi^L]} \ , \end{aligned} \tag{4.28}$$

where the effective action $S_{\text{eff}}[\phi^L]$ is defined through

$$e^{iS_{\text{eff}}[\phi^L]} = \int D[\phi^H]e^{iS[\phi^H,\phi^L]} \ . \tag{4.29}$$

If the heavy field components are eliminated exactly, then the dynamics of the light fields can be fully determined. However, the resulting expression for the effective action usually contains infinitely many terms and has to be truncated.

4.3. Action in the effective theory

4.3.1. Path integral in Euclidean spacetime

In many calculations it is very convenient to use the path integral in the framework of Euclidean quantum field theory (see also [39]). In order to perform the analytic continuation from Minkowski spacetime one needs to introduce imaginary time:

$$x_{0E} = ix_{0M} \tag{4.30}$$

which corresponds to Euclidean metric

$$x_E^2 = -x_M^2 = x_{0E}^2 + x_{1E}^2 + x_{2E}^2 + x_{3E}^2 \ . \tag{4.31}$$

The spatial coordinates are kept unchanged $x_{iE} = x_{iM}$. This is the so-called Wick rotation. One can write in Euclidean spacetime:

$$\begin{aligned} \int d^4x_E &= i\int d^4x_M \ , \\ \partial_{0E} &= -i\partial_{0M} \ , \\ \Box_E &= -\Box_M \ . \end{aligned} \tag{4.32}$$

Thus the exponent in the path integral, in the case of real scalar field, assumes the form:

$$\begin{aligned} e^{i\int d^4x_M \frac{1}{2}(\partial_{\mu M}\phi\partial^{\mu}{}_M\phi + m^2\phi^2)} &\longrightarrow e^{-\int d^4x_E \frac{1}{2}(\partial_{\mu E}\phi\partial^{\mu}{}_E\phi - m^2\phi^2)}, \\ e^{iS_M} &= e^{-S_E} \ . \end{aligned} \tag{4.33}$$

The general properties of quantum field theory for a real scalar field in Minkowski spacetime were formulated as the so-called Wightman axioms [44] in such a way that they give a realistic description of naturally occurring phenomena. In this approach the n-point correlation functions of the scalar field

$$\mathcal{W}(x_1, \ldots, x_n) = \langle 0|\phi(x_1)\ldots\phi(x_n)|0\rangle \tag{4.34}$$

are Wightman distributions which characterize the theory completely. The Hilbert space of physical states as well as the field can be reconstructed from them [44]. In Euclidean spacetime the Schwinger functions, or Euclidean Green's functions, are put forward. They correspond to Wightman distributions in real time and satisfy properties listed by Osterwalder and Schrader [7]. The Schwinger functions are analytic and covariant under Euclidean transformations

$$\mathcal{S}(x_1,\ldots,x_n)=\mathcal{S}(\Lambda x_1+a,\ldots,\Lambda x_n+a), \tag{4.35}$$

where $\Lambda\in \mathrm{SO}(4)$ is a rotation and a is a translation. They are also symmetric in their arguments and possess reflection positivity. This last property will be especially interesting for us in the later considerations. The Schwinger functions can be used to reconstruct the Wightman distributions and consequently the full quantum field theory when analytically continued to real time. Thus, the quantum fields can be studied in the frame of Euclidean theory though one has to keep in mind that real physics and the world we live in corresponds to Minkowski spacetime.

4.3.2. Effective model with higher orders of derivatives

Let us consider a model involving the complex scalar field ϕ and an Abelian gauge field A in the presence of heavy neutral scalar particle σ. We write down the action of the microscopic theory after Wick rotation as a sum of three terms:

$$S[A,\phi^*,\phi,\sigma]=S_A[A]+S_\phi[A,\phi^*,\phi]+S_\sigma[A,\phi^*,\phi,\sigma] \tag{4.36}$$

that can be evaluated as:

$$\begin{aligned}
S_A[A] &= -\frac{1}{4}\int d^4x_E F_{E\mu\nu}F_E^{\mu\nu},\\
S_\phi[A,\phi^*,\phi] &= \int d^4x_E[(D^{(e)}_{E\mu}\phi)^* D_E^{(e)\mu}\phi+U(\phi^*\phi)],\\
S_\sigma[A,\phi^*,\phi,\sigma] &= \int d^4x_E\left[\frac{1}{2}\partial_{E\mu}\sigma_x\partial_E^\mu\sigma_x+\frac{M^2}{2}\sigma_x^2+\frac{1}{2}\sigma_x^2[\gamma(D_E^{(e)2})\phi]_x^*\gamma(D_E^{(e)2})\phi_x\right]
\end{aligned} \tag{4.37}$$

Here the complex scalar field potential is denoted by $U(\phi^*\phi)$. The gauge-covariant derivative is constructed according to minimal coupling principle and its square $D_E^{(e)2}$ reads

$$D_E^{(e)2}=(\partial_E-ieA)^2=\Box_E-2ieA\partial_E-e^2A^2-ie\partial_E A. \tag{4.38}$$

The form factor in (4.37), $\gamma(z)=\sum_{i=0}^N\gamma_i z^i$ is the N-th order polynomial. Thus, the heavy scalar particle can be coupled to other degrees of freedom in a non-trivial manner.
Elimination of the heavy particle whose mass $M\gg\Lambda$, Λ being the cutoff, will allow us to express the total action only in terms of fields A, ϕ^* and ϕ:

$$S[A,\phi^*,\phi]=S_A[A]+S_\phi[A,\phi^*,\phi]+S_{\text{eff}}[A,\phi^*,\phi], \tag{4.39}$$

where $S_{\rm eff}[A,\phi^*,\phi]$ is determined from the integral

$$
\begin{aligned}
e^{-S_{\rm eff}[A,\phi^*,\phi]} &= \int D[\sigma] e^{-\frac{1}{2}\sigma D^{-1}_{M_E}\cdot\sigma - \frac{1}{2}\sigma^2\cdot\{[\gamma(D_E^2)\phi]^*\gamma(D_E^2)\phi\}} \\
&= \int D[\sigma] e^{-\frac{1}{2}\sigma\{D^{-1}_{M_E}+[\gamma(D_E^2)\phi]^*\gamma(D_E^2)\phi\}\cdot\sigma} \qquad (4.40)
\end{aligned}
$$

and we have put $D^{-1}_{M_E} = -\Box_E + M^2$.

The exponent in the above integral is quadratic in fields σ and we can consider it in analogy with the general Gaussian integral (see [37])

$$
\int D[\xi]\exp(-\xi B\xi) \to \left(\int\prod_k d\xi_k\right)\exp\left(-\xi_i B_{ij}\xi_j\right), \qquad (4.41)
$$

where B_{ij} are the elements of a symmetric matrix with eigenvalues b_i. In order to evaluate this integral we put $\xi_i = O_{ij}x_j$ where O is the orthogonal matrix of eigenvectors that diagonalizes B. Changing variables from ξ_i to x_i, we obtain

$$
\begin{aligned}
\left(\prod_k\int d\xi_k\right)\exp\left[-\xi_i B_{ij}\xi_j\right] &= \left(\prod_k\int dx_k\right)\exp\left[-\sum_i b_i x_i^2\right] \\
&= \prod_i\int dx_i\exp\left[-b_i x_i^2\right] \\
&= \prod_i\sqrt{\frac{\pi}{b_i}} \\
&= \text{const}\times[\det B]^{-1/2}, \qquad (4.42)
\end{aligned}
$$

which is a functional determinant. We write the determinant of matrix B as

$$
\det B = \Pi_i b_i = \exp\left[\sum_i \ln b_i\right] = \exp\left[\mathrm{Tr}(\log B)\right], \qquad (4.43)
$$

where the logarithm of a matrix is defined by its power series.

Now we can calculate the effective action. Omitting E in the notation, we can write

$$
S_{\rm eff}[A,\phi^*,\phi] = \frac{1}{2}\mathrm{Tr}\log\left[\frac{D_M^{-1}+([\gamma(D^2)\phi]^*\gamma(D^2)\phi)}{D_M^{-1}}\right] \qquad (4.44)
$$

where the normalization factor D_M^{-1} was inserted in the logarithm. The expansion of the logarithm gives

$$
\begin{aligned}
S_{\text{eff}}[A,\phi^*,\phi] &= \frac{1}{2}\mathrm{Tr}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(FD_M)^n \\
&= \frac{1}{2M^2}\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{n}\mathrm{Tr}\left[\left(\frac{\Box_x}{M^2}\right)^{m-1}F\right]^n
\end{aligned}
\tag{4.45}
$$

where

$$
F(x,y) = \delta(x-y)[\gamma(D^2)\phi(x)]^*\gamma(D^2)\phi(x)\ . \tag{4.46}
$$

The obtained action of the effective theory contains higher orders of derivatives of the remaining fields A and ϕ, ϕ^*.

5. Higher orders of derivatives in the action – problems and solutions

5.1. Introduction

It has been shown that eliminating a heavy particle from the theory leads to the appearance of higher order derivative terms in the action. In general, the issue of higher orders of derivatives has been present in the theoretical considerations for a very long time. The procedure for determining canonical variables in the case of Lagrangian with higher order time derivatives was given over one and a half century ago by M. Ostrogradski in [45]. In quantum field theory the higher order derivatives were employed as regulator terms [46]–[51]. It was shown explicitly in [52] that they generate states with indefinite norm.

It was assumed that the presence of a "ghost", particle with complex energy and negative norm, would not necessarily lead to instability and loss of unitarity. The argument was that within the Hilbert space of asymptotic states, of finite energy, the mass of the ghost particle diverges with the cut-off Λ [53]. It is the so-called Lee-Wick model. In the context of the effective models the heavy particle energy scale plays the role of the cut-off. Hence, the problem of negative norm states was solvable. However, the linear combinations of states with indefinite norm would give rise to zero norm states. These states' amplitudes may grow exponentially with time and lead to reappearance of instability in the model. The proposal, presented in Refs. [53] consisted of elimination of the unstable modes, by means of boundary conditions. It was required that the growing modes not appear. This approach is similar to the method utilized in [54] to exclude the runaway solutions of the radiation reaction problem in classical electrodynamics. Such boundary conditions, imposed in the future induce acausal effects. Furthermore, the unitarity of the theory is endangered due to the absence of *any* exponentially growing amplitudes, which could be produced as a result of the interactions between normal (observed) particles. The perturbative procedure that can be applied to fix this problem by the alteration of the absorptive part of the Feynman diagrams ran into difficulties, while the non-perturbative functional integral approach remained elusive [55].

The higher order derivatives have been considered, mostly as higher-order corrections, in many theories of physics. For instance, in general relativity these terms appear as higher-order curvature corrections in the Einstein-Hilbert action [56]–[61]. An extension of the Standard Model introducing higher derivative terms in the kinetic energy of the Higgs field was proposed [62] as well. There are candidates for theories of everything that involve higher derivatives [63, 64]. Also, the higher orders of derivatives can be employed in the modifications of gravity in order to avoid the necessity for dark energy [65, 66]. Classical and quantum mechanical examples of Lagrangians with higher orders of time derivatives have been extensively studied [52, 63, 65, 66, 67, 68], too.

It is a general belief that theories with higher orders of derivatives in the action are inherently flawed because of the negative norm states that endanger unitarity. One can see this problem clearly when considering a simple model

$$S[\phi] = \int d^4x \left[\phi(x) \left(\sum_{n=1}^{n_d} c_n \Box^n \right) \phi(x) - \phi(x) m^2 \phi(x) - V(\phi(x)) \right] , \tag{5.1}$$

where the scalar field action contains the derivatives up to order n_d, which we take to be an odd integer. The potential $V(\phi(x))$ is a polynomial in ϕ of degree at least 3. We assume that the model obeys time reversal invariance, the action is real and the coefficient c_{n_d} satisfies

$$(-1)^{n_d} c_{n_d} > 0 \ . \tag{5.2}$$

The generating functional for the Green's functions in this case reads

$$\int D[\phi] e^{iS[\phi]+i\int d^4x j(x)\phi(x)} = e^{-i\int d^4x V\left(\frac{\delta}{i\delta j(x)}\right)} e^{-\frac{i}{2}\int d^4x d^4y j(x) D(x-y) j(y)} \tag{5.3}$$

with the free propagator expressed in momentum space as

$$D(p) = \left(\sum_{n=1}^{n_d} (-1)^n c_n (p^2)^n - m^2 \right)^{-1} \ . \tag{5.4}$$

The Feynman $i\epsilon$ prescription has been suppressed here. Writing down the free propagator as a sum of partial fractions we obtain

$$D(p) = \sum_{j=1}^{n_d} \frac{Z_j}{p^2 - m_j^2} \ . \tag{5.5}$$

As the sum of Z_j factors in the series equals zero, at least one of them is negative. This appears as a negative contribution in the free Green's function generating functional (see Eq. (5.3)) which may represent a state with negative norm. Complex energy states, $m_j^2 < 0$, may arise as well. In this case the unitarity and stability of the theory could be compromised. The time evolution should be unitary with the subspace of physical states, with positive norm and real energy.

If the underlying microscopic theory is unitary and the heavy particle modes are eliminated exactly, the effective theory describing light particles is consistent. However, this full model would contain an infinite number of terms. In order to make it treatable one has to truncate the gradient expansion. Then, the theory could be subject to loss of unitarity, the time evolution for the light particles might turn out to be non-unitary. Furthermore, a well-defined effective model requires the specification of the initial conditions for the eliminated degrees of freedom. The initial and final states are determined through the boundary conditions in time in the path integral formulation. Identification of the states of the effective theory is a non-trivial issue. We will show that an appropriate subspace of the states in the effective model ensures a unitary time evolution and consistent dynamics.

We will demonstrate that Ostrogradski's treatment of higher order time derivatives in classical theory [45] can be adapted for the case of quantum fields. The norm of the states, its relation to self- and skew-adjoint operators (see [53, 55, 69, 70]) and the properties of the operators with respect to time reversal will be discussed first. We will proceed with the description of quantum fields on a lattice, based on [39, 71, 72]. The Yang-Mills-Higgs model with higher order derivatives will be investigated in lattice regularization since it provides a particularly adequate framework for establishing reflection positivity. This property in Euclidean spacetime assures the unitary time evolution for the real-time theory. Thus, the consistency of effective theories can be demonstrated. The major part of this chapter is contained in our article [13].

5.2. Quantum mechanics on spaces with indefinite norm

Quantum mechanics is usually constructed on linear space with positive definite norm. However, if one allows the presence of higher orders of derivatives in the theory it is useful to introduce space with indefinite norm. The non-definite metric spaces have been described in [69, 70, 53, 55]. We will present their main features, especially in connection with adjoints of operators, commutation relations and path integral formulation. As it turns out, there are relatively simple methods of distinguishing the positive definite part of linear space and the corresponding operators. One has to keep in mind that only positive norm states are physical. Negative or zero norm states may exist at intermediate times but not as asymptotic states.

5.2.1. Hilbert space with non-definite metric

Let us consider a linear space $\mathcal{H}$, consisting of the elements $|u\rangle, |v\rangle, \ldots$. It is a Hilbert space with positive definite metric if a (complex) number $\langle u|v\rangle$, corresponding to each $|u\rangle$ and $|v\rangle$, called scalar product, satisfies a set of conditions:

$$\begin{aligned} &\text{(i)} && \langle u|v\rangle = \langle v|u\rangle^* \ , \\ &\text{(ii)} && \langle u|(a|v\rangle + b|w\rangle) = a\langle u|v\rangle + b\langle u|w\rangle \ , \\ &\text{(iii)} && \langle u|u\rangle \geqslant 0 \ , \\ &\text{(iv)} && \langle u|u\rangle = 0 \text{ if, and only if } |u\rangle = 0 \ . \end{aligned}$$

In the linear space with semidefinite metric properties (iii) and (iv) are replaced by:

(iii′) always either $\langle u|u\rangle \geqslant 0$ or $\langle u|u\rangle \leqslant 0$,

(iv′) there is at least one $|u\rangle \neq 0$ with $\langle u|u\rangle = 0$.

Our interests are restricted to decomposable spaces with non-definite metric where the conditions (i), (ii) and the following:

(iii″) $H = H_+ + H_-$ where $H_\pm = \{|u\rangle | \langle u|u\rangle \gtrless 0\}$ with $\langle H_+|H_-\rangle = 0$,

(iv″) each vector $|u\rangle$ can be written as $|u\rangle = |u_+\rangle + |u_-\rangle$, $\langle u_\pm|u_\pm\rangle \gtrless 0$, in a unique manner,

are fulfilled [70]. The requirement of decomposability, (iv”), excludes zero norm vectors which are orthogonal to the whole space. Thus the space is equipped with non-degenerate metric. It is also assumed that the decomposable linear space H can be made complete with respect to the scalar product $\langle u|v\rangle' = \langle u_+|v_+\rangle - \langle u_-|v_-\rangle$ and consequently it can be promoted to a Hilbert space.

Let us introduce the metric η in our space as a non-singular Hermitian matrix, $\eta^\dagger = \eta$. It is convenient to use a basis $\{|n\rangle\}$ where the non-definite metric $\eta_{mn} = \langle m|n\rangle$ is diagonal

$$\eta_{mn} = \pm\delta_{mn} \tag{5.6}$$

and $\eta^2 = \mathbb{1}$. The matrix element of an operator A in this basis is defined by

$$\langle m|A|n\rangle = \sum_k \eta_{mk} A_{kn} \ . \tag{5.7}$$

It is preserved under multiplication

$$\eta_{mk}(AB)_{kn} = \langle m|AB|n\rangle = \sum_k \langle m|A|k\rangle\langle k|B|n\rangle = \sum_{k,\ell,j} \eta_{mk} A_{k\ell} \eta_{\ell j} B_{jn} \ . \tag{5.8}$$

The characteristic feature of the non-definite metric space is that one has to distinguish between an adjoint and Hermitian adjoint of an operator. The adjoint $\bar{A}$ is related to the operator A in the following way

$$\langle m|\bar{A}|n\rangle = \langle n|A|m\rangle^* \ . \tag{5.9}$$

It implies $\eta\bar{A} = (\eta A)^\dagger = A^\dagger\eta$ and thus

$$\bar{A} = \eta^{-1} A^\dagger \eta \ . \tag{5.10}$$

The self-adjoint operator is Hermitian for positive definite metric and may be anti-Hermitian for non-definite metric. The self- and skew-adjoint operators satisfy

$$\bar{A} = \sigma_A A \ , \tag{5.11}$$

where $\sigma_A = +1$ corresponds to self-adjoint and $\sigma_A = -1$ to skew-adjoint operators. If we consider two eigenvectors of the operator A:

$$\begin{aligned} A|u\rangle &= a_u|u\rangle \ , \\ A|v\rangle &= a_v|v\rangle \ , \end{aligned}$$

then

$$\langle u|A|v\rangle = a_v\langle u|v\rangle = \langle v|\sigma_A A|u\rangle^* = \sigma_A a_u^*\langle u|v\rangle \ . \tag{5.12}$$

Finally we obtain

$$(a_v - \sigma_A a_u^*)\langle u|v\rangle = 0 \ . \tag{5.13}$$

The spectrum is real or imaginary for self- or skew-adjoint operators, respectively, assuming that the subspace considered is the subspace of orthogonal eigenvectors with non-vanishing norm. The scalar product of two eigenvectors is non-vanishing if there is a relationship between their eigenvalues $a_u = \sigma_A a_v^*$. This way the eigenvalues of skew-adjoint operators are real in case of non-orthogonal eigenvectors.

It is worth mentioning here that the concept of unitarity is also defined with respect to the metric. However, the operators which are called unitary are the ones which preserve the usual positive definite metric. In the case of negative norm states these operators may no longer be unitary. The operators which leave the norms of all vectors invariant are referred to as *pseudo-unitary* [69]. As for the S-matrix of the effective theory (see Eq. (4.6)), it is crucial that its unitarity is maintained in the subspace of physical, positive norm states.

Let us consider a single degree of freedom represented by the pair of (Hermitian) operators $(\hat{q}_\sigma, \hat{p}_\sigma)$ satisfying the canonical commutation relations

$$[\hat{q}_\sigma, \hat{p}_\sigma] = i \ . \tag{5.14}$$

These operators are either normal, self-adjoint operators with real eigenvalues or skew-adjoint, as the negative metric modes. Demanding real spectrum in the skew-adjoint case, one has to accept non-orthogonality of the eigenstates. Thus, the metric can be expressed as

$$\eta(q, q') \equiv \langle q|q'\rangle = \delta(q - \sigma q') \tag{5.15}$$

in accordance with Eq. (5.13). In the case of the coordinate eigenstates we obtain for the self-adjoint operator:

$$\langle q|\hat{q}|q'\rangle = \eta(q,q')q' = \langle q'|\hat{q}|q\rangle^* = q^*\eta(q',q)^* = q\eta(q,q'), \quad \eta(q,q') = \delta(q-q') \; ; \tag{5.16}$$

and for the skew-adjoint operator

$$\langle q|\hat{q}|q'\rangle = \eta(q,q')q' = -\langle q'|\hat{q}|q\rangle^* = -q^*\eta(q',q)^* = -q\eta(q,q'), \quad \eta(q,q') = \delta(q+q') \tag{5.17}$$

where the eigenvalues are taken to be real. The closing relation in coordinate basis reads

$$\mathbb{1} = \int dq |\sigma q\rangle\langle q| \; . \tag{5.18}$$

The commutation relations (5.14) yield

$$e^{i\hat{p}q'}\hat{q}e^{-i\hat{p}q'} = \hat{q} + q' \; , \tag{5.19}$$

hence

$$\langle q|p\rangle = e^{ipq}/\sqrt{2\pi} \tag{5.20}$$

and then

$$\eta(p,p') = \langle p|p'\rangle = \int \frac{dq}{2\pi} e^{-iq(p-\sigma p')} = \delta(p-\sigma p') \; . \tag{5.21}$$

We arrive at the alternative closing relation

$$\mathbb{1} = \int dp |\sigma p\rangle\langle p| \; , \tag{5.22}$$

in momentum space.

It is useful to study the case of a harmonic oscillator which is a quantum mechanical analog of a quantum field theory (see also [46, 48, 53]). We consider a Hamiltonian

$$\hat{H}_\sigma = \frac{\sigma}{2}(\hat{p}_\sigma^2 + \hat{q}_\sigma^2) = \bar{a}_\sigma a_\sigma \; . \tag{5.23}$$

The operators $a_\sigma = (\hat{q}_\sigma + i\hat{p}_\sigma)/\sqrt{2}$ obey the commutation relation

$$[a_\sigma, \bar{a}_\sigma] = \frac{1}{2}[q_\sigma + ip_\sigma, \bar{q}_\sigma - i\bar{p}_\sigma] = \frac{\sigma}{2}[q_\sigma + ip_\sigma, q_\sigma - ip_\sigma] = \sigma \; . \tag{5.24}$$

We assume the Hilbert space where the set of operators a_σ and $\bar{a}_\sigma$ should act in an irreducible manner. We will analyze solely the case where $\sigma = +1$. It is obvious that the other linear space can be described by the same expressions but with interchanged a_σ and $\bar{a}_\sigma$, $a \leftrightarrow \bar{a}$. Let us introduce the operators $b = a_+$ and $\bar{b} = \bar{a}_+$. The operator $\bar{b}b$ is self-adjoint and for its eigenvector $|\lambda\rangle$ we have

$$\bar{b}b|\lambda\rangle = \lambda|\lambda\rangle \ . \tag{5.25}$$

Since

$$\begin{aligned} \bar{b}bb|\lambda\rangle &= (b\bar{b}-1)b|\lambda\rangle = (\lambda-1)b|\lambda\rangle \ , \\ \bar{b}b\bar{b}|\lambda\rangle &= \bar{b}(\bar{b}b+1)|\lambda\rangle = (\lambda+1)\bar{b}|\lambda\rangle \ , \end{aligned} \tag{5.26}$$

the states

$$\cdots, b^2|\lambda\rangle, b|\lambda\rangle, |\lambda\rangle, \bar{b}|\lambda\rangle, \bar{b}^2|\lambda\rangle, \cdots \tag{5.27}$$

have corresponding eigenvalues

$$\cdots, \lambda-2, \lambda-1, \lambda, \lambda+1, \lambda+2, \cdots \tag{5.28}$$

for $\bar{b}b$. For arbitrary λ, a real number but not an integer, this series extends to infinity in both directions so the Hamiltonian (5.23) is unbounded. In order to have a bounded Hamiltonian we require that the series stops, $\lambda = 0$. The equations

$$\begin{aligned} \langle\lambda|\bar{b}b|\lambda\rangle &= \lambda\langle\lambda|\lambda\rangle \ , \\ \langle\lambda|b\bar{b}|\lambda\rangle &= (\lambda+1)\langle\lambda|\lambda\rangle \end{aligned} \tag{5.29}$$

imply that λ ought to be an integer and the series stops at the left if $\lambda \geqslant 0$, or or at the right for $\lambda \leqslant -1$. The definite norm of the eigenstates is realized in the former case,

$$\mathrm{sign}(\langle\lambda+1|\lambda+1\rangle) = \mathrm{sign}(\langle\lambda|\lambda\rangle) \ , \tag{5.30}$$

while the indefinite norm is found in the latter,

$$\mathrm{sign}(\langle\lambda-1|\lambda-1\rangle) = -\mathrm{sign}(\langle\lambda|\lambda\rangle) \ . \tag{5.31}$$

Consequently, we can associate self- or skew-adjoint operators with the linear space with definite or non-definite metric, respectively. In both cases there exists a ground state of the Hamiltonian.

Furthermore, we can consider the path integral formulation of the time evolution amplitude for a system which includes a self-adjoint $\hat{q}$ and a skew-adjoint $\hat{q}'$ coordinate (see also [55]),

$$\langle q_f, -q'_f|e^{-itH}|q_i, q'_i\rangle = \int D[p]D[p']D[q]D[q']e^{i\int dt[p\dot{q}+p'\dot{q}'-H(q,q',p,p')]} \ , \tag{5.32}$$

where

$$q(t_i) = q_i, \quad q'(t_i) = q'_i, \quad q(t_f) = q_f, \quad q'(t_f) = q'_f \ . \tag{5.33}$$

Note that q'_f is the negative of the eigenvalue of $\hat{q}'$ at t_f, not the eigenvalue itself. Also, the function

$$H(q, q', p, p') = \langle q, q'|\hat{H}|p, p'\rangle / \langle q, q'|p, p'\rangle \qquad (5.34)$$

need not be real for self-adjoint Hamilton operator $\hat{H}$,

$$H^*(q, q', p, p') = \langle p, -p'|H|q, q'\rangle / \langle p, -p'|q, q'\rangle = H(q, -q', p, -p') \ . \qquad (5.35)$$

If we consider the Hamiltonian that is even in $\bar{q}$ and $\bar{p}$ the function $H(q, q', p, p')$ is real. The usual $i\epsilon$ prescription renders the phase space path integral convergent. Supposing the Hamiltonian is of the type

$$\hat{H} = (\hat{p}^2 - \hat{p}'^2)/2 + U(\hat{q}, \hat{q}') \qquad (5.36)$$

one can easily perform the integration over the momentum trajectories. The degree of freedom represented by the skew-adjoint operators displays an unusual sign in the kinetic energy, originating from the factor σ in the Hamiltonian (5.23). Finally, we obtain the path integral in the coordinate space in the form

$$\langle q_f, -q'_f|e^{-itH}|q_i, q'_i\rangle = \int D[q]D[q']e^{i\int dt[\frac{1}{2}\dot{q}^2 - \frac{1}{2}\dot{q}'^2 - U(q,q')]} \ . \qquad (5.37)$$

The notion of time-reversal parity can also prove useful in the determination of the signature of the metric in the linear space of states. The time reversal Θ is an anti-unitary transformation. In quantum mechanics, the state described by a wave function $\psi(t, \boldsymbol{r})$ is transformed into "time-reversed" one corresponding to the function $\psi^*(-t, \boldsymbol{r})$. In the Schrödinger representation the action of time reversal on the operators amounts to

$$\Theta : \ A \to \bar{A} \ . \qquad (5.38)$$

The operators with well defined time-reversal parity satisfy

$$\Theta A = \tau_A A \ , \qquad (5.39)$$

with $\tau_A = \pm 1$. The time reversal transformation preserves canonical commutation relations only for the variables with opposite time reversal parities $\tau_q = -\tau_p$. In the usual case $\tau_q = 1$. In the Heisenberg representation the time reversal acts on the operators as follows

$$\Theta : \ A(t) \to \bar{A}(-t) \ . \qquad (5.40)$$

In particular, the time derivatives give the contribution to the time-reversal parity with alternating signs

$$\tau_{\partial_0^n A} = (-1)^n \tau_A \ . \qquad (5.41)$$

As

$$\Theta q = \tau_q q = \bar{q} = \sigma_q q \tag{5.42}$$

we obtain a relation between the time-reversal parity of the canonical operator q and the signature of linear space $\tau_q = \sigma_q$. The positive definite norm states correspond to time-reversal invariant operators. We need to point out here that the relation (5.42) is valid in the case of scalar quantum fields. Vector fields, such as gauge fields yield a slightly different, more general relationship

$$\sigma_A = \tau_A \pi_A \tag{5.43}$$

where π_A is the space-inversion parity of the field operator A. Space reflection acts as

$$\mathcal{P} A(t, \boldsymbol{x}) = \pi_A A(t, -\boldsymbol{x}) \tag{5.44}$$

on the operators with well-defined space-inversion parity. The scalar fields also satisfy the equality (5.43). An operator with identical time- and space-inversion parities generates states of positive norm. The relation between the sign of the norm and the space- and time-reflection parity in the case of Abelian $U(1)$ gauge field is studied in more detail in the next chapter.

5.3. Quantum field theory – lattice

The lattice regularization of quantum fields is described in [39, 71, 72]. The discussion of transfer matrix and reflection positivity is based on [39].

5.3.1. Quantum fields on a lattice

In the lattice formulation of quantum field theory we consider a discretized Euclidean spacetime. Introducing the hypercubic lattice in four dimensions, we restrict the spacetime coordinates to

$$x_\mu = a n_\mu \ , \tag{5.45}$$

where a is the lattice spacing and n_μ has four integer components. Scalar fields $\phi(x)$ are defined on lattice sites

$$\phi(x) \rightarrow \phi(n) \tag{5.46}$$

numbered by n. Analogically,

$$\phi^\dagger(x) \rightarrow \phi^\dagger(n) \ . \tag{5.47}$$

The action, as a four-dimensional integral over the Lagrangian density, is replaced by a sum

$$S = \int d^4x L(\phi(x)) \rightarrow a^4 \sum_n L(\phi(n)) \tag{5.48}$$

over lattice sites. In the generating functional for Euclidean Green's functions we could define the integral over all field configurations as an ordinary integral over all fields on the lattice

$$Z = \int D[\phi]e^{-S} = \int \left(\prod_n d\phi(n) \right) e^{-S} \ . \tag{5.49}$$

The case of gauge fields cannot be treated in such a straightforward manner. They are equipped with spacetime as well as internal symmetry indices, $A_\mu = A^a_\mu(x)\tau^a$, where τ^a denotes the generators of the gauge group. Gauge fields depend on the trajectory in spacetime. Let us consider a path $\gamma_\mu(s)$, where $\gamma_\mu(0) = x^i_\mu$ and $\gamma_\mu(1) = x^f_\mu$ are initial and final points, respectively. A particle following this contour in spacetime acquires a phase factor

$$\psi \to \psi \exp \left(ig \int_\gamma A_\mu dx_\mu \right) = U_\gamma \psi \tag{5.50}$$

where the integral is calculated along the path. We can write U_γ as

$$U_\gamma = P \left[\exp \left(ig \int_0^1 ds \frac{d\gamma_\mu}{ds} A_\mu(\gamma(s)) \right) \right] \ . \tag{5.51}$$

where P stands for path ordering that arranges $A_\mu(\gamma(s))$ in such a way that ones with the larger values of s stand to the left of those with smaller s. Under a gauge transformation ω, the factor U_γ is rotated only at the endpoints of the trajectory:

$$U_\gamma \to \omega[x_\mu(\gamma(1))] U_\gamma \omega^\dagger[x_\mu(\gamma(0))] \ . \tag{5.52}$$

This property was used by Wilson [73] in the lattice formulation for gauge fields. In his approach, the fundamental variables $U_\mu(n)$ are elements of the gauge group G which are associated with each pair of the neighboring lattice sites denoted by n and $n + \hat{\mu}$, $\hat{\mu}$ being the unit vector of direction μ. They are the so-called link variables and satisfy

$$U_\mu(n) = U^\dagger_{-\mu}(n + \hat{\mu}) = \exp(igaA_\mu(n)) \ . \tag{5.53}$$

The gauge transformation of link variables yields

$$U_\mu(n) \to \omega(n + \hat{\mu}) U_\mu(n) \omega^\dagger(n) \tag{5.54}$$

while site variables transform as

$$\phi(n) \to \omega(n)\phi(n) \ , \ \phi^\dagger(n) \to \phi^\dagger(n)\omega^\dagger(n) \ . \tag{5.55}$$

In order to achieve the gauge invariant action we need to find gauge invariant quantities. The possible choices are given by scalar, matter fields connected by oriented product of link variables

$$\phi^\dagger(n)U_\mu(n)U_\nu(n+\hat{\mu})\dots U_\rho(m)\phi(m) \ , \tag{5.56}$$

or closed oriented loops described in terms of U's

$$\mathrm{Tr}\left[U_\mu(n)U_\nu(n+\hat{\mu})\dots U_\rho^\dagger(n)\right] \ . \tag{5.57}$$

The smallest closed loop on the lattice is a *plaquette* and the corresponding *plaquette variable* is

$$U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n) \ . \tag{5.58}$$

The lattice action should reproduce the classical Yang-Mills action in the continuum limit. Since the field strength tensor $F_{\mu\nu}$ is a generalized curl of fields A_μ, it would be appropriate to use integrals over small closed contours. The action proposed by Wilson reads

$$S = \sum_p S_p \ , \tag{5.59}$$

where the plaquette terms are

$$S_p = \beta\left[1 - \frac{1}{N}\mathrm{ReTr}\big(U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n)\big)\right] \ , \tag{5.60}$$

in the case of $SU(N)$ gauge group. The normalization factor is $\beta = 2N/g^2$.

Let us see that this action, with slightly redefined coupling, results in the standard action for $U(1)$ Abelian gauge field when the continuum limit is taken. The action (5.59) can be rewritten as

$$S = \frac{1}{g^2}\sum_n\sum_{\mu>\nu}\mathrm{Re}\Big\{1 - \exp\Big[iga\left(A_\mu(n) + A_\nu(n+\hat{\mu}) - A_\mu(n+\hat{\nu}) - A_\nu(n)\right)\Big]\Big\} \ , \tag{5.61}$$

where the relation (5.53) was used. The sum involves the contributions from all distinct plaquettes of the lattice. The naive continuum limit in this case is equivalent to assuming the small lattice spacing a and perform the Taylor expansion

$$A_\mu(n+\hat{\nu}) = A_\mu(n) + a\partial_\nu A_\mu(n) + \dots \ , \tag{5.62}$$

so one obtains

$$\begin{aligned} S &= \frac{1}{g^2}\sum_n\sum_{\mu>\nu}\left\{1 - \mathrm{Re}\exp\left[iga(a(\partial_\nu A_\mu(n) - \partial_\mu A_\nu(n)) + \mathcal{O}\left(a^2\right))\right]\right\} \\ &= \frac{1}{4g^2}a^4\sum_n\sum_{\mu\nu} g^2 F_{\mu\nu}^2 + \dots \\ &= \frac{1}{4}\int d^4x F_{\mu\nu}^2 \ , \end{aligned} \tag{5.63}$$

where the sum over all the plaquettes was replaced by a spacetime integral.

The continuum limit of quantum field theory is more involved due to the presence of logarithmic and power divergences in $1/a$. It is usually studied by means of perturbation expansion for asymptotically free theories and numerically when the high energy modes are strongly coupled.

5.3.2. Transfer matrix

In Euclidean spacetime transfer matrix describes the evolution of a system in an infinitesimal time interval. Its definition in the framework of lattice field theory is given quite naturally. For simplicity, in most of our derivations we will use lattice spacing $a = 1$. The time coordinate on the lattice is n^0. Let us denote it by t: $(n) = (t, \boldsymbol{n})$, with $\boldsymbol{n}$ as spatial three-vector components, keeping in mind that the coordinates are integer numbers. The lattice action can be decomposed in the following way

$$\begin{aligned} S[\phi] &= \sum_t \mathcal{L}[\phi(t+1), \phi(t)], \\ \mathcal{L}[\phi(t+1), \phi(t)] &= \sum_{\boldsymbol{n}} \frac{1}{2}\mathcal{L}_k[\phi(t+1, \boldsymbol{n}), \phi(t, \boldsymbol{n})] + \frac{1}{2}\mathcal{L}_s[\phi(t, \boldsymbol{n})] + \frac{1}{2}\mathcal{L}_s[\phi(t+1, \boldsymbol{n})] \end{aligned} \tag{5.64}$$

where $\mathcal{L}_s[\phi(t, \boldsymbol{n})]$ depends solely on the fields $\phi(t, \boldsymbol{n})$. Now let the field configuration on a timeslice t be denoted as

$$\phi(t) \equiv \phi(t, \boldsymbol{n}) \ . \tag{5.65}$$

The transfer matrix is then defined via

$$\mathrm{T}[\phi(t+1), \phi(t)] = \exp(-\mathcal{L}[\phi(t+1), \phi(t)]) \ . \tag{5.66}$$

It can be treated as the kernel of an operator T, usually called a transfer matrix as well.The wave functions which depend on the fields $\phi(t)$ at fixed times can be denoted as

$$\psi_t = \psi(\phi(t)) \ . \tag{5.67}$$

The operator T acts on the wave functions as

$$|\psi_{t+1}\rangle = \mathrm{T}|\psi_t\rangle \ . \tag{5.68}$$

Then the lattice Hamiltonian H is defined by

$$\mathrm{T} = e^{-Ha} \ , \tag{5.69}$$

where its eigenvector corresponding to the lowest eigenvalue E_0 is the vacuum $|0\rangle$.

5.3.3. Reflection positivity

The transfer matrix operator is a bounded, symmetric and positive operator. Its positivity assures the existence of a self-adjoint Hamiltonian H. It is convenient to provide a description of transfer matrix in terms of expectation values, thereby we look into reflection positivity in the lattice formulation. The property of reflection positivity for lattice models has been explored in [8, 9, 10, 39].

One kind of reflection on the lattice with site-reflection, that is the reflection with respect to a hyperplane of lattice sites with $t = 0$. It is realized by

$$\theta_s : \; t \to -t, \; \theta_s(n) = (-t, \boldsymbol{n},) \; . \tag{5.70}$$

On the other hand, we can also reflect with respect to a hyperplane placed between the lattice sites corresponding to $t = 1/2$. This is the link-reflection which acts as

$$\theta_l : \; t \to 1 - t, \; \theta_l(n) = (1 - t, \boldsymbol{n}) \; . \tag{5.71}$$

Let us begin with site-reflection positivity. We denote by F any functional of the field variables $\phi(t, \boldsymbol{x})$ taken for positive times only, $t > 0$. The Euclidean time inversion Θ of the functional F is defined in the following way:

$$\begin{aligned} \Theta(\lambda F) &= \lambda^* \Theta \; , F \\ \Theta(FG) &= \Theta F \Theta G \; , \end{aligned} \tag{5.72}$$

while

$$\Theta(\phi(n)) = (\phi(\theta_s n))^* \; . \tag{5.73}$$

It is clear that ΘF depends on the fields at negative times. Site-reflection positivity is fulfilled for

$$\langle 0 | F(\Theta F) | 0 \rangle \geqslant 0 \; . \tag{5.74}$$

Taking into account the relationship between expectation values and the states of the Hilbert space [7, 39], if the relation (5.74) is obeyed then

$$\langle \psi | \psi \rangle \geqslant 0 \; , \tag{5.75}$$

where $|\psi\rangle$ is a wave function depending on the fields at time $t = 0$. Shifting F by n lattice-spacings in the positive time direction, we arrive at the function F' of fields corresponding to positive times only, too. The separation between function F' and $\Theta F'$ sums up to $2n$ lattice spacings. Consequently, from the positivity of $\langle (\Theta F') F' \rangle$ one can infer that

$$\langle \psi | \mathrm{T}^{2n} | \psi \rangle \geqslant 0 \; . \tag{5.76}$$

Hence, we obtain positive T^2. It enables us to describe a Hamiltonian by means of

$$\mathrm{T}^2 = e^{-2Ha} \ . \tag{5.77}$$

Site-reflection positivity ensures the positive scalar product in Hilbert space. The operator T, generating time shifts by two units, is also positive and there exists a self-adjoint Hamiltonian. The complete construction of the Hilbert space and Hamiltonian from Euclidean expectation values can be found in [7, 10].

Let us show by an explicit calculation how site-reflection positivity can be demonstrated in the case of scalar field theory. The action is split according to

$$S = S_+ + S_- + S_0 \tag{5.78}$$

where S_0 depends only on the fields at $t = 0$, S_+ on the fields at positive times, $t > 0$, and $S_- = \Theta S_+$. Now we can write, be means of functional integral,

$$\langle 0|F(\Theta F)|0\rangle = \int D[\phi] e^{-S_0} F e^{-S_+} \Theta \left(F e^{-S_+} \right) \ , \tag{5.79}$$

where the logarithm of the vacuum wave function is included in $S_\pm$. If we utilize

$$\mathcal{F}[\phi] = \int D_{t>0}[\phi] F e^{-S_+} \tag{5.80}$$

the expectation value (5.79) can be expressed as

$$\int D_{t=0}[\phi] e^{-S_0} |\mathcal{F}[\phi]|^2 \ , \tag{5.81}$$

which is manifestly positive.

In the case of link-reflection we use the same definition of the operator Θ, except that θ_l replaces θ_s. The inequality of the same form as (5.74) has to be satisfied in order for the link-reflection positivity to hold. Now, however, the functions F depend on the fields at strictly positive times described by $t \geqslant 1$. The distance in the time direction between F and ΘF corresponds to an odd number of lattice spacings. By a shift of one lattice spacing forward in time in case of F increases the separation by two units. In consequence, the operator T^2, generating translation in time by two lattice spacings, is proved to be positive. Thus one is able to define the Hamiltonian and construct a Hilbert space of states with positive norm.

5.3.4. Higher orders of derivatives

Now we will consider the effective lattice model with higher orders of time derivatives. The main idea is to find the conditions sufficient for the property of reflection positivity to be satisfied [11, 13]. The action of the Yang-Mills-Higgs theory in Euclidean spacetime is

$$S[\phi,\phi^\dagger,A]=\int d^dx\left[K(D)-\phi^\dagger L(D^2)D^2\phi+V(\phi^\dagger\phi)\right]\ . \tag{5.82}$$

The gauge field $A_\mu = A^a_\mu\tau^a$, τ^a being the generators of the gauge group, appears in covariant derivatives $D_\mu = \partial_\mu - iA_\mu$. The kinetic energy of the scalar matter field ϕ includes higher orders of derivatives, as well as the pure gauge field part. The polynomials K and L of covariant derivatives are assumed to be at most of order n_d and n_d-2, respectively. The UV cutoff Λ is imposed. The vacuum of the theory may contain time reversal invariant condensates.

The Ostrogradski's method (see appendix A), based on canonical formalism, is widely used when dealing with classical systems with higher orders of time derivatives. In the field theory it assumes the introduction of new coordinates – field variables, together with their canonical pairs, corresponding to each time derivative apart from the last one. This procedure yields, for higher order time derivatives up to order n_d,

$$\begin{aligned} A_{j\mu}(x) &= D_0^jA_\mu(x)\ ,\\ \phi_j(x) &= \partial_0^j\phi(x)\ , \end{aligned} \tag{5.83}$$

where $j=0,\dots,n_d-1$. This treatment is valid also in the case of quantum fields what will be demonstrated by means of functional integral.

Before we proceed with the path integral formulation of the theory described by (5.82), let us focus attention on the time-reversal properties of the field variables. Since the action contains higher orders of time derivatives there might exist states with negative norm in the Fock space. As was mentioned in section 5.2, the time-reversal parity of the operator is associated with the signature of the norm. The variables $A_{j\mu}$ and ϕ_j reveal the following properties under time reversal:

$$\begin{aligned} \Theta A_{j\mu}(t,\boldsymbol{x}) &= (-1)^{j+\delta_{\mu,0}+1}A_{j\mu}(-t,\boldsymbol{x})\ ,\\ \Theta\phi_j(t,\boldsymbol{x}) &= (-1)^j\phi_j(-t,\boldsymbol{x})\ . \end{aligned} \tag{5.84}$$

The time-reversal invariant functional of the fields acting on the time-reversal invariant vacuum should span a space with positive definite metric. The equivalent statement is that the unitarity within the subspace of physical states with positive norm can be established by means of reflection positivity in imaginary time.

We will consider our theory with higher order derivatives in lattice regularization. The field variables appearing in the action (5.82) have their lattice analogues (see Eqs.(5.46), (5.47), (5.53)), with the behavior under gauge transformations (see Eqs. (5.54),(5.55)), as described in section 5.3. The lattice spacing is taken as $a=1$. The covariant derivative on the lattice is defined by the finite difference

$$D_\mu \phi(n) = U_\mu^\dagger(n)\phi(n+\hat{\mu}) - \phi(n) \tag{5.85}$$

and its square can be written as

$$D^2\phi(n) = \sum_\mu [U_\mu^\dagger(n)\phi(n+\hat{\mu}) + U_\mu(n-\hat{\mu})\phi(n-\hat{\mu}) - 2\phi(n)] \ . \tag{5.86}$$

The generating functional for the discretized, bare Euclidean theory is the partition function

$$Z = \int D[U]D[\phi^\dagger]D[\phi]e^{-S_L} \ . \tag{5.87}$$

The lattice action S_L reads

$$S_L = \sum_n \sum_{\gamma'} a_{\gamma'} \operatorname{Tr} U_{\gamma'}(n) + \sum_n \phi^\dagger(n) \sum_\gamma U_\gamma^\dagger(n)\phi(n+\gamma) b_\gamma + \sum_n V(\phi^\dagger(n)\phi(n)) \ . \tag{5.88}$$

Here γ' and γ are closed and open paths, respectively, of the maximum length n_d. The lattice site denoted $n+\gamma$ is the lattice site where the path γ ends, the starting point being on site n. $U_\gamma(n)$ stands for the oriented product of the link variables along this path. We need the time-reversal invariant dynamics, thus for each $\Theta\gamma$, which is the time-reversed path γ, the coefficients in the action satisfy

$$\begin{aligned} a_{\Theta\gamma'} &= a_{\gamma'}^* \\ b_{\Theta\gamma} &= b_\gamma^* \ . \end{aligned} \tag{5.89}$$

In consequence, the action S_L is rendered real. To simplify the following discussion of reflection positivity we impose the static temporal gauge so that the component U_0 is time-independent and all its time derivatives vanish.

Let us introduce now the lattice variables corresponding to time derivatives of fields. We construct them in the following way [11, 13]. Taking n_d consecutive spacelike hyperplanes, corresponding to time slices $t,\ t+1,\ \ldots,\ t+n_d-1$, we obtain blocked time slices. The new time coordinate t labels the blocked time slices and we have

$$t^{(old)} \rightarrow n_d t^{(new)} + j \ , \tag{5.90}$$

where $j = 1, \cdots, n_d$. We can express the new field variables as

$$\begin{aligned} \phi_j(t,\boldsymbol{n}) &= \phi(n_d t + j, \boldsymbol{n}) \ , \\ U_{j,\mu}(t,\boldsymbol{n}) &= U_\mu(n_d t + j, \boldsymbol{n}) \ , \quad \mu = 1,2,3 \ . \end{aligned} \tag{5.91}$$

The lattice action can be decomposed into

$$S_L = \sum_t [\mathcal{L}_s(t) + \mathcal{L}_{kg}(t) + \mathcal{L}_{km}(t)] \tag{5.92}$$

where

$$\begin{aligned}
\mathcal{L}_s(t) &= \mathcal{L}_s[\phi(t), \phi^\dagger(t), U(t)] \ , \\
\mathcal{L}_{kg}(t) &= \mathcal{L}_{kg}[U(t+1), U(t)] \ , \\
\mathcal{L}_{km}(t) &= \sum_{t,\boldsymbol{m},\boldsymbol{n}} \phi_j^\dagger(t+1, \boldsymbol{m}) \Delta_{j,k}^{-1}(\boldsymbol{m}, \boldsymbol{n}; U(t+1), U(t)) \phi_k(t, \boldsymbol{n}) + c.c.
\end{aligned} \tag{5.93}$$

As usual, *c.c.* means complex conjugate. The field variables within a single blocked time slice are gathered in $\mathcal{L}_s[U, \phi^\dagger, \phi]$ whereas the product of link variables U and U' corresponding to two successive blocked time slices is included in $\mathcal{L}_{kg}[U', U]$. The last term in (5.92), $\mathcal{L}_{km}(t)$, characterizes the coupling between scalar field variables belonging to consecutive blocked time slices which involves also link variables U and U' of these time slices.

Let us express the operator of transfer matrix T as

$$\begin{aligned}
\langle \phi', \phi'^\dagger, U' | \mathrm{T} | \phi, \phi^\dagger, U \rangle &= \exp \Bigg\{ -\frac{1}{2} \mathcal{L}_s[\phi, \phi^\dagger, U] - \frac{1}{2} \mathcal{L}_s[\phi', \phi'^\dagger, U'] - \mathcal{L}_{kg}[U', U] + \\
& - \left[\sum_{\boldsymbol{m},\boldsymbol{n}} \phi_j'^\dagger(\boldsymbol{m}) \Delta_{j,k}(\boldsymbol{m}, \boldsymbol{n}; U', U) \phi_k(\boldsymbol{n}) + c.c \right] \Bigg\} \ .
\end{aligned} \tag{5.94}$$

It is required that this operator be positive in the physical subspace of the Fock space. This condition is fulfilled if

$$\langle 0 | F \Theta F | 0 \rangle \geqslant 0 \tag{5.95}$$

is obeyed for any local functional F depending solely on fields at positive times. It is clear from (5.95) that the states with negative norm are excluded from the subspace of physical states.

The time reversal transformation of the fields yields the following relations

$$\begin{aligned}
\Theta \phi_j(n) &= \phi_{\Theta j}^\dagger(\theta n) \ , \\
\Theta \phi_j^\dagger(n) &= \phi_{\Theta j}(\theta n) \ , \\
\Theta U_{j,\mu}(n) &= U_{\Theta j,\mu}^\dagger(\theta n) \ , \quad \mu = 1, 2, 3,
\end{aligned} \tag{5.96}$$

where $\Theta j = n_d + 1 - j$. On the original lattice the reflection can be realized with respect to a lattice site or halfway between two sites. In the case of blocked time slices one has to use site-reflection for odd and link-reflection for even n_d, respectively. We will consider the site-reflection positivity, corresponding to odd n_d, thus time reversal acts on spacetime coordinates $n = (t, \boldsymbol{n})$ as

$$\theta_s(t, \boldsymbol{n}) = (-t, \boldsymbol{n}) \ . \tag{5.97}$$

We need to seek the functionals with well-defined time-reversal parity so that

$$\Theta F[\phi(n), \phi^\dagger(n), U(n)] = \tau_F F[\Theta\phi, \Theta\phi^\dagger, \Theta U] = \tau_F F[\phi(\theta n), \phi^\dagger(\theta n), U(\theta n)] \ , \tag{5.98}$$

where the internal time-reversal parity of the observable F is τ_F. The functionals which can be used include the combinations of the local fields,

$$\begin{aligned}
\phi_{\tau,j}(t, \boldsymbol{n}) &= \frac{1}{2}[\phi_j(t, \boldsymbol{n}) + \tau\phi^\dagger_{\Theta j}(t, \boldsymbol{n})] \ , \\
\phi^\dagger_{\tau,j}(t, \boldsymbol{n}) &= \frac{1}{2}[\phi^\dagger_j(t, \boldsymbol{n}) + \tau\phi_{\Theta j}(t, \boldsymbol{n})] \ , \\
U_{\tau,j,\mu}(t, \boldsymbol{n}) &= \frac{1}{2}[U_{j,\mu}(t, \boldsymbol{n}) + \tau U^\dagger_{\Theta j,\mu}(t, \boldsymbol{n})] \ , \quad \mu = 1, 2, 3
\end{aligned} \tag{5.99}$$

where $j = 1, \ldots, (n_d - 1)/2$. They display the well-defined time-inversion parity denoted as τ. The functional F may consist of products of fields provided that the product of their parities equals τ_F.

Let us follow the procedure analogical to one described in 5.3.3. We will use $\Psi = (U, \phi^\dagger, \phi)$ for the fields what simplifies the notation . We split the action (5.92) into three parts

$$S_L = S_0 + S_- + S_+ \ , \tag{5.100}$$

where

$$\begin{aligned}
S_0 &= \mathcal{L}_s(0) \ , \\
S_+ &= \sum_{t \geqslant 0}[\mathcal{L}_{kg}(t) + \mathcal{L}_{km}(t)] + \sum_{t>0} \mathcal{L}_s(t) \ , \\
S_- &= \sum_{t<0}[\mathcal{L}_s(t) + \mathcal{L}_{kg}(t) + \mathcal{L}_{km}(t)] \ .
\end{aligned} \tag{5.101}$$

One notices immediately that

$$S_\pm[\Psi(t)] = \Theta\left[S_\mp[\Psi(t)]\right] = S_\mp[\Psi(\theta t)] \tag{5.102}$$

and

$$S_0[\Psi] = \Theta S_0[\Psi] = S_0[\Psi(\theta t)] \tag{5.103}$$

due to the time-reversal invariance of the microscopic dynamics. The vacuum expectation value on the left hand side of the inequality (5.95) can be expressed as

$$\langle 0|F\Theta F|0\rangle = \int D[\Psi] e^{-S_0[\Psi]} e^{-S_+[\Psi]} F[\Psi] e^{-S_-[\Psi]} \Theta[F[\Psi]] \ . \tag{5.104}$$

We assume here that the logarithm of the wave functional of the vacuum state was inserted in the actions $S_\pm[\Psi]$. Since the vacuum state is invariant under time inversion,

$$\Theta|0\rangle = |0\rangle \ , \tag{5.105}$$

and relation (5.102) is satisfied we can rewrite the equation (5.104) as follows

$$\langle 0|F\Theta F|0\rangle = \int D[\Psi]e^{-S_0[\Psi]}e^{-S_+[\Psi]}F[\Psi]\Theta\left[e^{-S_+[\Psi]}F[\Psi]\right] \ . \tag{5.106}$$

The time-reversal invariance of $S_0[\Psi]$ (5.103) allows us to continue with

$$\begin{aligned}\langle 0|F\Theta F|0\rangle \ &= \ \int D_{t=0}[\Psi]\int D_{t>0}[\Psi(t)]e^{-\frac{1}{2}S_0[\Psi]-S_+[\Psi(t)]}F[\Psi(t)] \\ &\quad \times\Theta[F[\Psi(t)]]\int D_{\Theta t>0}[\Psi(\Theta t)]e^{-\frac{1}{2}S_0[\Theta\Psi]-S_+[\Psi(\Theta t)]} \ .\end{aligned} \tag{5.107}$$

We can now eliminate Θ from the integral and arrive at

$$\langle 0|F\Theta F|0\rangle = \tau_F\int D_{t=0}[\Psi]\left(\int D_{t>0}[\Psi]e^{-\frac{1}{2}S_0[\Psi]-S_+[\Psi]}F[\Psi]\right)^2 \tag{5.108}$$

where the time-reversal parity of the functional F is τ_F. Our expectation value is positive for $\tau_F = 1$. Thus, the reflection positivity is guaranteed for the time-reversal invariant functionals F. In the subspace of the Fock space span by the states generated by the action of the operator F on the time-reversal invariant vacuum the time evolution is unitary. However, it is also necessary to demand that the relations (5.102) and (5.103) be obeyed for every trajectory in the path integral. Therefore we impose the boundary conditions in time

$$\Psi(t_f) = \tau_\Psi\Psi(t_i) \ . \tag{5.109}$$

The time-reversal even and odd variables satisfy periodic and antiperiodic boundary conditions, respectively. This way the non-unitary runaway solutions are eliminated. Full Euclidean invariance requires similar boundary conditions in all the directions in spacetime.

5.4. Summary

The effective, low energy theories, obtained from the more microscopic theories after the elimination of heavy particle modes, contain higher orders of derivatives in the action. If one allows, in order to have a treatable model, only a finite number of terms in the expansion then identifying the states and unitarity of the effective theory become non-trivial issues. The presence of higher orders of derivatives in the truncated action implies the appearance of new degrees of freedom. We have shown that they can be included in the path integral by applying the treatment similar to Ostrogradski's classical method. We have considered the Yang-Mills-Higgs model with higher order derivatives in lattice regularization. If the periodic or antiperiodic boundary conditions in time are imposed on the new variables in the functional integral, then the reflection positivity of the Euclidean theory is established. This property leads to unitary time revolution in the subspace of physical, positive norm states of the theory and assures the existence of bounded Hamiltonian. The Euclidean model can be extended to real time in a consistent manner.

6. Spontaneous Lorentz symmetry breaking

6.1. Introduction

Over the years, there have been a certain number of proposals concerning the violation of Lorentz symmetry. It has been considered within the scheme of emerging photons [74] or the effective bumblebee models [6] with vector bosons that could be included in the Standard-Model Extension. It has been observed that Lorentz symmetry breaking should occur at high energies, the Lorentz violating terms being suppressed in the low-momentum limit [75, 76]. On the other hand, the Lorentz symmetry breaking terms can be used as regulators in the quantum field theory [4].

In the previous chapter we have studied the consistency of effective theories with higher orders of derivatives. Now we will focus attention on the effects these terms can generate as far as the symmetry and quasi-particle content of the model are concerned. The higher order derivatives may be responsible for the emergence of a vacuum that contains a condensate. If the particles which condense have non-vanishing momentum the spacetime symmetry is broken globally, the vacuum is inhomogeneous. A kind of relativistic "band structure" appears, similar to the one in solid state physics [77]. The analysis of this model might be extremely complicated. Fortunately, if we have the gauge symmetry at our disposal, the value of the covariant derivative, which is fixed by energetical consideration, can be saturated by a homogeneous gauge field. The scalar field condensate remains uniform as well, the inhomogeneity is gauged away. On the other hand, the vacuum expectation value for the gauge field is non-vanishing and spontaneous Lorentz symmetry violation takes place. One arrives at a kind of extended Higgs mechanism where some components of the gauge field are treated as Goldstone modes.

In this chapter we will study scalar quantum electrodynamics with higher order derivatives which are included in the kinetic energy of the complex scalar field. It is an attempt to extend the model of spontaneous symmetry breaking for gauge theories, to achieve Lorentz symmetry violation. The presence of higher derivative terms may be interpreted in two ways. One possibility is that they are introduced as smooth cutoff ensuring the ultraviolet finiteness of the microscopic theory. The other is that the higher order derivatives originate from the elimination of high energy particle modes so that the model may represent an effective theory. As the covariant derivatives appear as higher order terms there is a possibility of inducing the local potential for the gauge field. In the leading order of saddle point expansion the vacuum with the scalar field condensate as well as non-zero expectation value for the vector field is assumed. The spontaneous Lorentz and gauge symmetry breaking occurs. Due to the presence of higher order derivative terms, it will be necessary to establish the unitarity of time evolution in the physical subspace of states with positive norm. Finally, we will study the quasi-particle spectrum of the theory. It turns out that the standard Maxwell equations remain valid in this case. The Goldstone theorem prevents the components of electromagnetic field from gaining masses.

6.2. Scalar electrodynamics – extended model

Our model of interest is the extension of scalar QED defined by the action

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} + \int d^4x[\phi^* L(-D^2)\phi - V(\phi^*\phi)], \tag{6.1}$$

with the gauge field strength tensor

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{6.2}$$

and the covariant derivative

$$D_\mu = \partial_\mu - ieB_\mu \ , \tag{6.3}$$

which appears in the polynomial of finite order $L(z)$. The theory is time- and space-reversal invariant.

The case with global $U(1)$ symmetry is recovered for $e = 0$. If $L(p^2) = p^2$ and the local potential is of the appropriate form the vacuum reveals the homogeneous condensate. Rendering the symmetry local requires introduction of the gauge field what allows us to describe the usual Higgs mechanism, with massive photons. By including the higher orders of covariant derivatives in the polynomial L we may obtain a non-trivial local potential for the gauge field. We will use the term *condensates* to refer to the non-vanishing vacuum expectation values although one has to keep in mind that for the gauge field it is really just a coherent state. The Bose-Einstein condensation requires the conserved number of particles and this prerequisite is not satisfied in case of photons.

Our study will be carried out in the spirit of saddle point approximation. We write the fields in the action as sums

$$\begin{aligned} \phi &= \bar{\phi} + \chi \ , \\ B_\mu &= \bar{A}_\mu + A_\mu \ , \end{aligned} \tag{6.4}$$

where the terms $\bar{\phi}$, $\bar{A}$ represent the saddle point configurations ("classical fields") and the other ones correspond to quantum fluctuations. If the kinetic energy of charged scalar particles is such that the vacuum state is described by the non-vanishing value of the covariant derivative

$$-D^2\bar{\phi}(x) = k^2\bar{\phi}(x) \tag{6.5}$$

one can always, by means of a gauge transformation, adjust the contributions from the partial derivative and the compensating field. When the eigenvalue k^2 arises solely from the partial derivative we have

$$\begin{aligned} \bar{\phi}(x) &= \bar{\phi}e^{-ikx} , \\ \bar{A}_\mu &= 0 . \end{aligned} \tag{6.6}$$

After performing an appropriately chosen gauge transformation we arrive at the semi-classical vacuum where

$$\begin{aligned} \bar{\phi}(x) &= \bar{\phi} , \\ e\bar{A}_\mu &= k_\mu , \end{aligned} \tag{6.7}$$

so the condensate is homogene. Moreover, we impose the orthogonality condition between the saddle point and the fluctuations

$$\int d^4x\chi(x) = \int d^4xA_\mu(x) = 0 . \tag{6.8}$$

In our case the resulting vacuum breaks spontaneously not only global gauge invariance but also spacetime symmetry. We assume the timelike gauge field condensate so that $e\bar{A}_\mu = g_{\mu 0}k > 0$. Hence, we may obtain up to four Goldstone modes for $\bar{\phi}, \bar{A}_\mu \neq 0$, which correspond to $U(1)$ gauge rotations and Lorentz boosts [78]. It is convenient to have real $\bar{\phi}$ which can be provided by a global gauge transformation.

Let us consider the components of the gauge field in view of establishing the unitarity of our theory. In the usual case of a photon field, in relativistically covariant canonical quantization procedure the gauge fixing term is added to the Lagrangian

$$-\frac{1}{4}F^2 \to -\frac{1}{4}F^2 - \xi(\partial A)^2/2 \tag{6.9}$$

and it is assumed that the commutation relations

$$[A_\mu(t,\boldsymbol{x}), \Pi_\nu(t,\boldsymbol{y})] = -ig_{\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{6.10}$$

are obeyed. The canonical momentum reads

$$\Pi^\mu = \partial L/\partial\partial_0 A_\mu . \tag{6.11}$$

In the case of Minkowski spacetime, $g_{\mu\nu} = (1, -1, -1, -1)$, the expression on the right hand side of equation (6.10) has the wrong sign for $\mu = \nu = 0$ which corresponds to the negative norm of the A_0 state. The problem of unitarity in the physical subspace of states with positive norm is solved in standard scalar QED either by the Gupta-Bleuler quantization (see [79, 80]) or BRST symmetry (see [81]).

The wrong sign of the norm for the temporal photon states is a consequence of the fact that the operator A_0 is chosen to be skew-adjoint [13]. If this field was represented by a self-adjoint operator the corresponding eigenstates would be non orthogonal. Considering our model, in order to employ the usual path integral for real fields $A_\mu(x)$ we can treat A_0 as an auxiliary variable, with no dynamics, either in static temporal or Coulomb gauge. The former is really convenient when one needs to guarantee the unitary time evolution for the physical states and also to highlight the structure of the theory with spontaneously broken symmetry. On the other hand, the dynamical degrees of freedom are easier to follow in the Coulomb gauge so it should be used to describe the particle content of the theory.

In the static temporal gauge the field A_0 is rendered non-dynamical

$$\partial_0 A_0(x) = 0 \; . \tag{6.12}$$

The temporal gauge $A_0 = 0$ is not a good choice if the boundary conditions in time are applied. It can be seen from the Polyakov line

$$\Omega(\boldsymbol{x}) = e^{-ie\int_{t_i}^{t_f} dt A_0(t,\boldsymbol{x})} \tag{6.13}$$

which is a gauge invariant degree of freedom when some (gauge invariant) boundary conditions are imposed at the initial and final time, t_i and t_f, respectively.

6.2.1. Semi-classical vacuum

The establishment of reflection positivity for theories containing higher orders of derivatives, which is described in the previous chapter, leads to unitarity and consistency of the model provided that the Euclidean path integral is convergent and the analytic continuation to real time is possible. Thus we have to restrict the values of the poles of the propagator (c.f. Eq. (5.5)), let us denote them m_n^2. The convergence of the Euclidean path integral is ensured for $\Re m_n^2 \geqslant 0$. The Wick rotation back to real time is well-defined if, while performing the rotation of the frequency contour in the loop integrals, one avoids the singularities. Hence, the poles have to satisfy $\Im m_n^2 \cdot \Re m_n^2 \leqslant 0$. However, since the poles appear in complex conjugate pairs, leaving just one of the pair would break time-reversal invariance and induce acausal effects. Consequently, we should allow only positive real poles in the propagator of our theory. Then the unitarity shall be maintained in the subspace of states with positive norm. The frequencies of the small plane wave perturbations around the homogeneous vacuum are determined as the values of p^0 at the poles. Due to the absence of complex poles, the time dependence of the perturbative Green's functions is restricted to oscillating terms $e^{i\omega t}$, the exponentially growing terms $e^{\omega t}$ being excluded. The classical homogeneous vacuum should be stable with respect to small fluctuations, what settles the case of the homogeneous sector of the theory. The stability of the fluctuations around the vacuum will be checked through the analysis of the spectrum of elementary excitations in the quantum theory.

The energy density corresponding to $e\bar{A}_\mu$ and $\bar{\phi}$ which characterize the semi-classical vacuum is obtained from the Lagrangian by changing the sign,

$$U(e^2\bar{A}^2, \bar{\phi}^2) = -\bar{\phi}^2 L(e^2\bar{A}^2) + V(\bar{\phi}^2) \ . \tag{6.14}$$

In this case the polynomial $L(p^2)$ is supposed to be bounded from above, assuming the maximal value at $p^2 = k^2$. We minimize $U(e^2\bar{A}^2, \bar{\phi}^2)$ with respect to $\bar{A}^2$

$$0 = \frac{\partial U(e^2\bar{A}^2, \bar{\phi}^2)}{\partial e^2\bar{A}^2} \ . = -\bar{\phi}^2 L'(e^2\bar{A}^2) \tag{6.15}$$

The above relation is satisfied for $e^2\bar{A}^2 = k^2$. As mentioned before, we have $e\bar{A}_\mu = g_{\mu,0}k$. It is interesting that the four-momentum with time-like condensate can never be obtained in imaginary time.

As for the scalar field, its kinetic energy $L(p^2)$ and the local potential $V(\phi^2)$ are not separated in the unique manner. The action (6.1) is invariant under the transformation

$$L(p^2) \to L(p^2) + \Delta L, \ V(\phi^2) \to V(\phi^2) - \Delta L \phi^2 \ . \tag{6.16}$$

It implies that we have the freedom to set $L(k^2) = 0$. Now the lowest order polynomial that fulfills our requirements reads

$$L(p^2) = -\frac{1}{k^2}(p^2 - k^2)^2 \ . \tag{6.17}$$

The value of the scalar field condensate $\bar{\phi}$ can be found from the condition on the minimum of $U(k^2, \bar{\phi}^2)$

$$0 = \frac{\partial U(k^2, \bar{\phi}^2)}{\partial \bar{\phi}^2} = V'(\bar{\phi}^2) - L(k^2) \ . \tag{6.18}$$

Additionally, we assume that the first non-vanishing derivative of the potential $V(\bar{\phi}^2)$ taken at the vacuum is positive.

Let us consider now the fluctuations around homogeneous vacuum. We will focus attention on the quadratic part of the action. Writing the momentum four-vector as $p = (\omega, \boldsymbol{p})$ and the electromagnetic potential as $A = (A_0, \boldsymbol{A})$, we can express the fields as

$$\begin{aligned} \chi &= \chi_1 + i\chi_2 \ , \\ \boldsymbol{A} &= \boldsymbol{n}A_L + \boldsymbol{A}_T \ , \end{aligned} \tag{6.19}$$

where $\boldsymbol{n} = \boldsymbol{p}/|\boldsymbol{p}|$. Then, we separate their static components $\tilde{\chi}_a$, $\tilde{A}_L$ in the following way

$$\begin{aligned} \chi_a &\to \chi_a + \tilde{\chi}_a \ , \\ A_L &\to A_L + \tilde{A}_L \ , \\ \boldsymbol{A}_T &\to \boldsymbol{A}_T + \tilde{\boldsymbol{A}}_T \ . \end{aligned} \tag{6.20}$$

The quadratic action is decomposed into two parts, $S^{(2)} = S^{(2)} + \tilde{S}^{(2)}$, with $\tilde{S}^{(2)}$ collecting solely the contributions of the static parts of the fields

$$\begin{aligned} S^{(2)} &= \frac{1}{2}\int d^4x(\chi_1, \chi_2, A_L, \boldsymbol{A}_T) \begin{pmatrix} K_{11} & K_{12} & K_{1L} & 0 \\ K_{21} & K_{22} & K_{2L} & 0 \\ K_{L1} & K_{L2} & K_{LL} & 0 \\ 0 & 0 & 0 & K_{TT} \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ A_L \\ \boldsymbol{A}_T \end{pmatrix} \\ \tilde{S}^{(2)} &= \frac{t_f - t_i}{2}\int d^3x(\tilde{\chi}_1, \tilde{\chi}_2, \tilde{A}_0, \tilde{A}_L, \tilde{\boldsymbol{A}}_T) \begin{pmatrix} \tilde{K}_{11} & 0 & \tilde{K}_{10} & \tilde{K}_{1L} & 0 \\ 0 & \tilde{K}_{22} & 0 & \tilde{K}_{2L} & 0 \\ \tilde{K}_{01} & 0 & \tilde{K}_{00} & 0 & 0 \\ \tilde{K}_{L1} & \tilde{K}_{L2} & 0 & \tilde{K}_{LL} & 0 \\ 0 & 0 & 0 & 0 & \tilde{K}_{TT} \end{pmatrix} \begin{pmatrix} \tilde{\chi}_1 \\ \tilde{\chi}_2 \\ \tilde{A}_0 \\ \tilde{A}_L \\ \tilde{\boldsymbol{A}}_T \end{pmatrix} . \end{aligned} \tag{6.21}$$

The derivation of quadratic form K in momentum space,

$$K(p) = \int d^4x e^{ip(x-y)} K(x, y) \ , \tag{6.22}$$

is rather long and can be found in Appendix B. The obtained components of K are

$$\begin{aligned} K_{11} &= L_d^+(p) - 4V''\bar{\phi}^2 \\ K_{22} &= L_d^+(p) \\ K_{12} &= iL_d^-(p) = -K_{21} \\ K_{1L} &= -|\boldsymbol{p}|[z(p)L_d(p)]^- = K_{L1} \\ K_{2L} &= i|\boldsymbol{p}|[z(p)L_d(p)]^+ = -K_{L2} \\ K_{LL} &= \boldsymbol{p}^2[z^2(p)L_d(p)]^+ + \omega^2, \\ K_{TT} &= \omega^2 - \boldsymbol{p}^2, \end{aligned} \tag{6.23}$$

where the notation $f^{\pm}(p) = f(p) \pm f(-p)$ has been introduced. Also, we denote

$$L_d(p) = L((p + e\bar{A})^2) - L(k^2) \ , \tag{6.24}$$

and

$$z(p) = e\bar{\phi}/(p^2 + 2\omega k) \tag{6.25}$$

for simplicity. The four-momentum is written as $p = (\omega, \boldsymbol{p})$. The quadratic form $\tilde{K}$, for static fields, is characterized by

$$\tilde{K}(\boldsymbol{p}) = K(p)_{|\omega=0} \ , \tag{6.26}$$

extracted from Eqs. (6.23), and additional terms

$$\begin{aligned}
\tilde{K}_{10} &= -\frac{4e\bar{\phi}k}{\boldsymbol{p}^2} L_d(p)_{|\omega=0} = \tilde{K}_{01} \ , \\
\tilde{K}_{00} &= \frac{8e^2\bar{\phi}^2k^2}{(\boldsymbol{p}^2)^2} L_d(p)_{|\omega=0} + \boldsymbol{p}^2 \ .
\end{aligned} \tag{6.27}$$

6.2.2. Unitarity within the subspace of physical states

The time evolution has to be unitary within the positive norm subspace of the Fock space. This condition is satisfied as the perturbative equivalent of the lattice proof of reflection positivity (see [13]) is valid for truncated theories where the poles of the propagator are real and the Euclidean theory continued analytically to real time exhibits a manifestly unitary perturbation expansion.

The theory is not Lorentz invariant thus the partial fraction decomposition of the propagator should be given in terms of ω^2 rather than p^2. The particles' energies have to be real so that the unitarity and stability of the perturbative model is assured within the Fock space with non-definite norm. This condition amounts to the realness and positivity of all the roots of the equation for ω^2

$$\det K(\omega, \boldsymbol{p}) = 0 \ , \tag{6.28}$$

obeyed by the quadratic form K of Eq. (6.21). The determinant of K reads, in the case of the simplest kinetic term (6.17)

$$\begin{aligned}
\det K(p) \ = \ & \frac{4}{k^4}(\omega^2 - \boldsymbol{p}^2)^2\omega^2\{\omega^8 - 4\omega^6(\boldsymbol{p}^2 + 2k^2) + \omega^4[16k^4 + 16k^2\boldsymbol{p}^2 + 6(\boldsymbol{p}^2)^2 + 2V''\bar{\phi}^2k^2] \\
& -\omega^2[4V''\bar{\phi}^2(k^2\boldsymbol{p}^2 - 2k^4) + 4(\boldsymbol{p}^2)^3 + 8k^2(\boldsymbol{p}^2)^2] + 2V''\bar{\phi}^2[k^2(\boldsymbol{p}^2)^2 - 8e^2\bar{\phi}^2k^2\boldsymbol{p}^2] \\
& +(\boldsymbol{p}^2)^4\} \ .
\end{aligned} \tag{6.29}$$

It is quite obvious that the above expression, when equal to zero, corresponds to negative as well as complex values of ω^2. The unstable modes are included in the poles corresponding to scalar field and longitudinal gauge field. We can solve this particular problem by demanding

$$V''(\bar{\phi}^2) = 0 \ . \tag{6.30}$$

This choice is not natural. Such relation requires an adjustment, fine-tuning of the parameters of the theory. On the other hand, the determinant (6.29) is now reduced to

$$\det K(p) = \frac{4}{k^4}(\omega^2 - \boldsymbol{p}^2)^2\omega^2(\omega^2 - 2k\omega - \boldsymbol{p}^2)^2(\omega^2 + 2k\omega - \boldsymbol{p}^2)^2 \ . \tag{6.31}$$

Here one can infer the real quasi-particle spectrum, since $\omega = \pm|\boldsymbol{p}|$, for transverse gauge field, $\omega = 0$ and

$$\omega = \sigma_1 k + \sigma_2\sqrt{k^2 + \boldsymbol{p}^2}\ , \tag{6.32}$$

where $\sigma_1, \sigma_2 = \pm 1$, for longitudinal component of gauge field and scalar field. Taking into account the spontaneously broken symmetries, we could expect four Goldstone modes. The two of them which allow the radiation field to remain massless are associated with the violation of the invariance under Lorentz boosts. The other gapless modes may arise for $\omega = 0$ and $\sigma_1\sigma_2 = -1$ in Eq. (6.32). They can be constructed in the form of linear combinations of the scalar particle and the longitudinal component of the gauge field.

The unitarity is thus proved to be maintained in the Fock space with indefinite norm. However, we need unitary time evolution in the physical subspace of states with positive norm. The property of reflection positivity, discussed in the previous chapter, demonstrated in Euclidean spacetime in case of gauge fields requires a little more detailed explanation. For the spatial component of gauge field $\boldsymbol{A}$ the time- and space-reversal parities agree and the signature of the norm of the space $\sigma_{\boldsymbol{A}}$ is positive (see section 5.2). It is a self-adjoint operator, both in Euclidean and Minkowski spacetime. However, the Wick rotation affects the temporal component of gauge field, it changes the signature of the norm of the states created by A_0. Thus, self-adjoint operator in Euclidean spacetime becomes skew-adjoint in Minkowski case, in order that its eigenvectors stay orthogonal. The field A_0 is imaginary so it cannot be treated as a Hermitian quantum field. In consequence, A_0 can represent a non-dynamical, auxiliary variable only. This is the case in the static temporal gauge (6.12). The equation of motion for A_0, corresponding to Gauss's law, is used as a constraint, so that the dynamics is restricted to the subspace where this law is satisfied.

6.2.3. Quasi-particles

We have showed that our model follows a unitary time evolution within the subspace of physical, positive norm states. Hence, we can continue with the physical interpretation of the theory, based on the obtained quasi-particle spectrum.

The double poles of (6.17), in the kinetic energy of the scalar field may suggest that these modes are not particle-like. This kind of excitations may correspond to the scattering amplitude wave packets that vanish, in accordance with the reduction formulae. It is thus sensible to treat the scalar field as describing a yet unobserved excitation. Let us focus now solely on the dynamics of the gauge field, in such a manner that we disregard radiative corrections related to the charged scalar. We will look into the propagating, dynamical degrees of freedom and their dispersion relations. We have a freedom to choose a different gauge since the unitarity has already been established in the static temporal gauge. The particularly convenient choice is the Coulomb gauge.

If we consider the action (6.1) in the Coulomb gauge, where $A_L = 0$, we can distinguish two separate parts. One is the transverse part and corresponds to standard Maxwell action containing the square of the field strength tensor. Thus the transverse component of gauge field $\boldsymbol{A}_T$ decouples from the rest. The second part can be written as

$$S_{0m}^{(2)} = \frac{1}{2}\int d^4x(\chi_1, \chi_2, A_0) K_C \begin{pmatrix} \chi_1 \\ \chi_2 \\ A_0 \end{pmatrix} , \tag{6.33}$$

where the quadratic form in momentum space reads

$$K_C = \begin{pmatrix} L_d^+(p) & iL_d^-(p) & [(p^0+2k)z(p)L_d(p)]^+ \\ -iL_d^-(p) & L_d^+(p) & -i[(p^0+2k)z(p)L_d(p)]^- \\ [(p^0+2k)z(p)L_d(p)]^+ & i[(p^0+2k)z(p)L_d(p)]^- & [(2k+p^0)^2z^2(p)L_d(p)]^+ + \boldsymbol{p}^2 \end{pmatrix} . \tag{6.34}$$

The dispersion relation is derived from the determinant of K_C

$$\det K_C(p) = \frac{4}{k^4}\boldsymbol{p}^2(\omega^2 - 2k\omega - \boldsymbol{p}^2)^2(\omega^2 + 2k\omega - \boldsymbol{p}^2)^2 . \tag{6.35}$$

Naturally, the two massless modes that appeared in the determinant corresponding to static temporal gauge (6.31) are absent here, due to decoupling of transverse gauge field components. On the other hand, here we notice $\boldsymbol{p}^2$ instead of ω^2, there is no momentum-independent frequency, $\omega = 0$, solution. The inverse matrix that could be obtained from K_C is rather complicated but there is one significant, simple term on the diagonal

$$(K_C^{-1})_{00} = \frac{1}{\boldsymbol{p}^2}. \tag{6.36}$$

associated with A_0. Hence, the $\boldsymbol{p}^2$ in the determinant (6.35) refers to the unaltered Coulomb law. Roots that correspond to the other factors are given by Eq. (6.32) and characterize the dispersion relation of the scalar field. These roots appear as double poles in the elements of the matrix K_C^{-1} for matter fields. It suggests that the non-trivial dispersion relation for the longitudinal photon field, as observed in the static temporal gauge, can be considered a gauge artifact. It is quite surprising that the electromagnetic field displays a usual dispersion relation, it is not modified by the presence of the higher order derivatives in the action. On the other hand, the higher order covariant derivative terms imply the gauge field components in the direction of the condensate and the four-momentum only. The electromagnetic field, constructed from transverse modes and A_0, retains its usual properties. The influence of the higher orders of derivatives can only be seen in the radiative corrections due to the charged scalar field dynamics.

When looking into the matrix K_C (6.34) it is not obvious that one may arrive at the very simple term (6.36) in the inverse. It can be seen in the more straightforward manner by making an explicit calculation of the A_0 propagator, via elimination of the scalar fields.

In the complex χ basis the action can be expressed as

$$S^{(2)} = \frac{1}{2}\int d^4x(\chi^*, \chi, A_0)\begin{pmatrix} K_- & 0 & K_0 \\ 0 & 0 & 0 \\ K_0 & 0 & K_{00} \end{pmatrix}\begin{pmatrix} \chi \\ \chi^* \\ A_0 \end{pmatrix} \tag{6.37}$$

with

$$\begin{aligned} K_- &= -2(\Box - 2ik\partial_0)^2/k^2 \ , \\ K_0 &= 2e\bar{\phi}(2k + i\partial_0)(\Box - 2ik\partial_0)/k^2 \ , \\ K_{00} &= 2e^2\bar{\phi}^2(\partial_0^2 - 4k^2)/k^2 - \Delta \ . \end{aligned} \tag{6.38}$$

The equations of motion for χ^* and χ yield

$$\begin{aligned} \chi^* : \quad 0 &= K_-\chi + K_0 A_0 \\ \chi &= -K_-^{-1}K_0 A_0 \ , \\ \chi : \quad 0 &= \chi^* K_- + A_0 K_0 \\ \chi^* &= -A_0 K_0 K_-^{-1} \ , \end{aligned} \tag{6.39}$$

and thus

$$\begin{aligned} S^{(2)} &= \frac{1}{2}\int d^4x A_0(K_0K_-^{-1}K_-K_-^{-1}K_0 - K_0K_-^{-1}K_0 - K_0K_-^{-1}K_0 + K_{00})A_0 \\ &= \frac{1}{2}\int d^4x A_0[K_{00} - \frac{1}{2}(K_0K_-^{-1}K_0 + K_0^*K_-^{-1*}K_0^*)]A_0 \\ &= \frac{1}{2}\int d^4x A_0 D_{00}^{-1} A_0 \ , \end{aligned} \tag{6.40}$$

where the complex conjugate terms assure the realness of the action. It turns out that

$$D_{00}^{-1} = \boldsymbol{p}^2 \tag{6.41}$$

in momentum space. Eventually, the obtained spectrum reproduces the usual electrodynamics, comprising the massless photons with transverse polarization and the Green's function for the static field A_0 reads

$$\tilde{K}_{00}^{-1}(\boldsymbol{p}) = \frac{1}{\boldsymbol{p}^2} \ . \tag{6.42}$$

6.3. Summary

A spontaneous breakdown of gauge symmetry as well as spacetime symmetries was studied within a special model of scalar QED. The Lorentz symmetry violation occurred due to the presence of higher orders of covariant derivatives in the action. The non-vanishing vacuum expectation value was found for the gauge field. We admitted only the vacuum expectation value for the temporal component of the gauge field. Thus the spatial rotation invariance was maintained. The gauge field condensate plays the role of specific chemical potential for charged scalar particles. The scalar condensate is electrically neutral.

It was shown that the theory is stable and exhibits unitary time evolution in the physical, positive norm subspace of the Fock space, provided that certain fine-tuning is applied. If these consistency requirements are fulfilled, the model reveals the particle content that is surprisingly similar to the one that emerges in the usual electrodynamics. The gauge field components are massless what is compliant with Goldstone theorem. Moreover, the obtained linearized equations of motion reproduce Maxwell equations. Thus, the spontaneous breakdown of Lorentz symmetry in the vacuum is not reflected in the behaviour of quasi-particles. The quadratic part of the gauge field action, in the appropriately chosen gauge, is recovered as the usual case of QED. The effects of Lorentz-symmetry violation are to be seen in the photon dynamics originating in radiative corrections.

Our study may be a base for many various extensions. The introduction of non-Abelian gauge fields should result in the appearance of massive gauge bosons as it would not be possible to exclude them using Goldstone theorem. Furthermore, the theory including fermions might not need the fine-tuning, so the natural model could be constructed. Eventually, the extension that incorporates gravity could be a candidate for a model of gravitational Higgs mechanism since the spontaneous breakdown of relativistic symmetry suggests the possible appearance of massive gravitons.

7. Conclusions

The studies of spacetime structure and symmetries reveal certain interesting and even surprising features. In curved spacetime, we focused on the motion of test particles and communication by means of electromagnetic signals in strong gravitational field, in Schwarzschild and Schwarzschild-like spacetimes. The corresponding conservation laws allowed us to determine in a straightforward manner massive and massless geodesics. We studied two types of structures in strong gravitational fields: photon sphere and event horizon. Correspondingly two different types of motion, circular and radial, respectively, were thoroughly discussed. The former is known to reveal a specific relationship between the velocity of a massive particle and the radius of a circular geodesic in Schwarzschild geometry. We verified that a similar relation is satisfied in the case of Schwarzschild-like spacetime arising in Hořava-Lifshitz theory of gravity. We found that there exist three non-overlapping sectors for circular geodesics. Massless particles can only travel along a circular orbit on the photon sphere. Massive circular geodesics belong to the outer region, of larger radii, and approach the photon sphere in the asymptotic limit of infinite angular momentum and energy. The spacelike, tachyonic circular orbits reside solely in the interior of the photon sphere. The speed of a massive particle, as a function only of the radius of the orbit, would be greater than or equal to the speed of light inside and on the photon sphere, respectively. Massive particles, with large enough angular momentum, can move only along circular orbits outside the photon sphere, where their speed never exceed speed of light.

In Schwarzschild spacetime, it is a well-known fact that for a freely falling observer it takes a finite *proper* time to reach the event horizon. The amount of time as measured by any static observer is infinite. This discrepancy in the notion of time in the strong gravitational field raises the question concerning the structure of the spacetime itself in this region, in the vicinity of the event horizon. Hence, employing the Kruskal-Szekeres coordinate system, we studied the motion and communication by means of light signals between an observer falling into a black hole, Alice, her Mother Station, and an observer chasing her in the radial fall, Bob. We determined the vectors tangent to the geodesics corresponding to all the observers as well as ingoing signals, in the direction of the singularity, and outgoing signals, moving away from the black hole. This allowed us to investigate the communication in the vicinity of the horizon. Above the event horizon, the exchange of signals does not suffer any disruption – Alice and Bob keep sending and receiving signals to and from each other until they reach the Schwarzschild radius. The situation in the interior of the black hole becomes complex in a rather unexpected way. Namely, there are two possible scenarios: one preserves the causal structure and the communication between Alice and Bob, broken at the horizon, is restored beneath horizon. However, this scenario requires that the frequency parameter of the outgoing light signals be flipped. The parametrization is continuous in the other scenario which corresponds to the different causal structure above and below the horizon and an irreversible breakdown in two-way communication between the observers. For the extended object, such as a radially arranged rod composed of weakly interacting particles, crossing the horizon, the lack of the interchange of signals between its constituents would signify its disintegration. The extended body would fall apart. It is impossible to establish, by means of any observation made outside the black hole, which scenario is realized beneath the horizon. This ambiguity suggests that the recognition of what the event horizon actually represents is a non-trivial issue. It might not only be a separatrix but also a region where the structure of spacetime itself is changed in a remarkable way, with astonishing consequences. To facilitate our understanding of the choice between two possible scenarios in the interior of the black hole one should investigate the non-radial motion in the vicinity of the event horizon in Schwarzschild geometry. Our study would be extended as well by considering other types of black holes characterized, apart from the mass, by charge and angular momentum.

The case of global Poincaré symmetry was analyzed in the framework of quantum field theory, in flat spacetime. The quantum fields are supposed to be described by Lorentz-invariant theories. However, it may be possible for the Lorentz symmetry to be broken spontaneously. Our aim was to study the effects of spontaneous breakdown of relativistic symmetries in field theory. Lorentz symmetry breaking can be the consequence of the presence of vector field vacuum expectation value that arises due to the higher order covariant derivatives in the action. These terms appear in effective theories which are derived from underlying, more microscopic models by elimination of heavy particle modes. The light particles' dynamics in the effective theory is affected by the heavier particles degrees of freedom, this influence can be seen as the presence of higher order derivatives. One cannot exclude these terms completely, they are bound to appear, with small coefficients, in all realistic models describing quantum fields. However, the higher order derivative terms may lead to inconsistencies, they are associated with the emergence of additional degrees of freedom. It is possible that "ghosts" appear, particles with complex energies and negative norm. They could destroy the stability and unitarity of the theory. It was worthwhile to investigate the consistency of effective theories with higher orders of derivatives. We established in the non-perturbative manner, by means of lattice regularization, that the well-behaved models could be constructed. The theory of fields on the lattice, the Euclidean theory, should satisfy, among other requirements, Osterwalder-Schrader reflection positivity. This property assures the unitary time evolution within the subspace of the physical states of the theory when the analytical continuation to real time is performed. We showed how the theory admitting higher orders of derivatives can be described in terms of lattice variables and demonstrated that if the appropriate boundary conditions are applied, the property of reflection positivity is maintained in the Euclidean effective model. When this model is analytically continued to real time, the stable, unitary theory can be achieved in Minkowski spacetime. Hence, effective field theories described by the Lagrangians containing higher order derivatives are consistent provided that certain requirements are fulfilled. This result paves the way for the extension of existing theories by adding higher order derivative terms.

In the chapter 6 we considered an Abelian gauge model with higher order covariant derivatives. The theory was based on the modified Lagrangian of scalar quantum electrodynamics. We assumed the type of scalar field potential which leads to the presence of a scalar condensate and that contributes to spontaneous U(1) symmetry breaking. This is reminiscent of the Higgs mechanism in the Abelian gauge field case. On the other hand, the non-vanishing expectation value for the gauge field, in the temporal direction, appeared due to the higher order derivative terms in the action. This in turn indicates the spontaneous breakdown of Lorentz symmetry. The vacuum of the theory exhibited the violation of Lorentz invariance. The issue of the unitarity of the theory was thoroughly investigated. In this particular case the previously studied property of Osterwalder-Schrader positivity proved more like a guideline than an actual solution to the problem of consistency. However, we succeeded in rendering our theory stable and unitary by imposing certain conditions. The most important requirements were the appropriate fine-tuning of the parameters in the scalar field part of the action and a specific choice of gauge, the static temporal gauge. The latter amounts to treating the temporal component of the gauge field as a non-dynamical variable and the Gauss's law represents a constraint in the path integral. The particle spectrum of the theory turned out to be rather surprising, strikingly different from the one obtained for the standard Abelian Higgs phenomenon. The scalar field revealed double poles in the propagator what suggests that its normal modes are not particle-like. The physical components of the gauge field remained massless. The gauge field part in the propagator was the same as in the usual case of electrodynamics. Although the vacuum expectation value of the gauge field violates Lorentz invariance, the spontaneous breakdown of symmetry can only be seen in the radiative corrections.

Our analysis of the models with the higher order derivative terms would lead to a multitude of extensions. The study of the reflection positivity could apply to a wider range of models, containing the higher order derivatives in the terms with the higher than quadratic order of the matter fields. Moreover, the whole argument could be repeated for fermions. The spontaneous breakdown of relativistic symmetries could be investigated for the equivalent of our scalar QED model in the case of Yang-Mills fields where one should expect the appearance of massive gauge bosons. Furthermore, if one includes the charged fermions in the model with higher order covariant derivatives, it is possible that the parameters in the action will not require fine-tuning and thus one will arrive at the *natural* theory. Obviously, the further extension of the model for gravity would be of major interest since it could lead to gravitational Higgs mechanism and massive gravitons.

The study of spacetime symmetries and structure has resulted in unexpected findings. Even the well-known Schwarzschild geometry has revealed some hidden characteristics. The influence of the strong gravitational field might appear more significant and event horizon of the black hole a much more profound entity than was thought before. On the other hand, the analysis of quantum fields in flat spacetime in the case of broken relativistic symmetries touches so many fundamental and interesting aspects of quantum field theory and, if extended, may lead to such substantial consequences, that it proves genuinely worthwhile.

A. Ostrogradski's theorem

Let us consider a system described by a Lagrangian $L\left(q, \dot{q}, \ddot{q}, \ldots, q^{(n)}\right)$ which depends on the coordinate $q(t)$ and its first n time derivatives. Applying the variational principle we arrive at the Euler-Lagrange equation

$$\sum_{i=0}^{n}(-1)^i \left(\frac{d^i}{dt^i}\frac{\partial L}{\partial q^{(i)}}\right) = 0 \ , \tag{A.1}$$

which is of the order at most $2n$ so the canonical phase space has at most $2n$ variables. We define the new coordinates and their respective momenta as

$$q_i \equiv q^{(i)} \qquad \text{and} \qquad p_i \equiv \sum_{j=i}^{n-1}\Big(-\frac{d^{j-i}}{dt^{j-i}}\Big)\frac{\partial L}{\partial q^{(j)}} \ , \tag{A.2}$$

where $i = 0, 1, \ldots, n-1$. Then the Lagrangian depends on the n coordinates q_i and only the first time derivative

$$\dot{q}_{n-1} \equiv q^{(n)} \ . \tag{A.3}$$

Using the second equation in (A.2), assuming the Lagrangian depends non-degenerately on $q^{(n)}$, we can find such a function

$$a = a(q_1, \ldots, q_{n-1}, p_{n-1}) \tag{A.4}$$

that

$$\left.\frac{\partial L}{\partial q^{(n)}}\right|_{\substack{q^{(i)}=q_i \\ q^{(n)}=a}} = p_{n-1} \ . \tag{A.5}$$

For general n we write the Ostrogradski's Hamiltonian in the form,

$$\begin{aligned} H &\equiv \sum_{i=0}^{n-1} p_i q^{(i)} - L \\ &= p_0 q_1 + p_1 q_2 + \ldots + p_{n-2} q_{n-1} + p_{n-1} a - L\Big(q_0, \ldots, q_{n-1}, a(q_1, \ldots, q_{n-1}, p_{n-1})\Big) \ . \end{aligned} \tag{A.6}$$

The canonical variables satisfy the canonical equations of motion

$$\dot{q}_i \equiv \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i \equiv -\frac{\partial H}{\partial q_i} \ , \tag{A.7}$$

so the Hamiltonian (A.6) generates time evolution. Being linear in p_0, p_1, $\ldots$, p_{n-2} the Hamiltonian cannot be bounded from below. This represents an instability of the model. Ostrogradski's theorem states that a Hamiltonian is unstable if it is associated with the Lagrangian depending on higher than first order time derivatives where the dependence is non-degenerate, so it cannot be eliminated by partial integration.

B. Quadratic action in momentum space

We want to evaluate the general quadratic form K in terms of the momentum p, without fixing the gauge. To this end we will insert the test functions

$$\begin{aligned} \chi(x) &= \chi' e^{-ipx} \ , \\ A_\mu(x) &= A'_\mu e^{-ipx} \end{aligned} \tag{B.1}$$

in the quadratic action (6.21). It is straightforward to derive, from Eq. (6.18), the quadratic form of the $\mathcal{O}(\chi^*\chi)$ part which reads

$$K_{\chi^*\chi} = 2L_d(p) - 4V''\bar{\phi}^2 \ , \tag{B.2}$$

where $L_d(p)$ is described by Eq. (6.24). The other terms in $K(p)$ require the substitution for the kinetic term of the scalar field $L(-D^2)$ with the polynomial of the form

$$L(z) = \sum_{n=0}^{n_d} c_n z^n \ . \tag{B.3}$$

The block in the K matrix that represents the mixing between the scalar and the gauge field can be obtained from

$$\chi^* L(-D^2)\bar{\phi} = \chi^* \sum_{n=0}^{n_d} c_n [-(\partial - ie\bar{A} - ieA)^2]^n \bar{\phi} \ . \tag{B.4}$$

The following terms of $\mathcal{O}(A\chi)$ contribute

$$\frac{1}{2}\int d^4x d^4y \chi^*(x) K_{\chi^* A}(x,y) A(y) = ie \int d^4x \chi^*(x) \sum_{n=0}^{n_d} c_n \sum_{\ell=1}^{n} (-\bar{\Box}) \cdots (2A(x)\bar{\partial} + \partial A(x)) \cdots (-\bar{\Box})\bar{\phi} \tag{B.5}$$

where $\bar{\partial}_\mu = \partial_0 - ie\bar{A}_\mu$, $\bar{\Box} = \bar{\partial}_\mu \bar{\partial}^\mu$, and the ℓ-th factor on the right hand side is the $\mathcal{O}(A)$ term of $-D^2$. Utilizing the p-dependence (B.1) we can write

$$\begin{aligned} K_{\chi^* A}(p)A' &= 2ie \sum_{n=0}^{n_d} c_n \sum_{\ell=1}^{n} (p + e\bar{A})^2 \cdots (-2iA'_0 k - ipA')k^2 \cdots \bar{\phi} \\ &= 2e(2A'_0 k + pA')k^{-2} \sum_{n=0}^{n_d} c_n k^{2n} \sum_{\ell=0}^{n-1} \left(1 + \frac{p^2 + 2p^0 k}{k^2}\right)^\ell \bar{\phi} \ . \end{aligned} \tag{B.6}$$

The summation of the geometric series gives

$$K_{\chi^* A}(p)A' = 2e\frac{2A'_0 k + pA'}{p^2 + 2p^0 k}\sum_{n=0}^{n_d} c_n k^{2n}\left[(1+\frac{p^2+2p^0k}{k^2})^n - 1\right]\bar{\phi}\ , \qquad \text{(B.7)}$$

and finally we arrive at

$$K_{\chi^* A_\mu}(p) = 2e\bar{\phi}\left[L_d(p)\frac{2g^{\mu 0}k + p^\mu}{p^2 + 2p^0 k}\right]\ . \qquad \text{(B.8)}$$

The $\mathcal{O}(A^2)$ part of the quadratic form $K(p)$ is actually a sum of a few terms. From the pure gauge field part of the quadratic action we obtain the standard Maxwell contribution which reads

$$K^{(1)}_{A_\mu A_\nu}(p) = -T^{\mu\nu}p^2\ , \qquad \text{(B.9)}$$

where

$$T^{\mu\nu} = g^{\mu\nu} - p^\mu p^\nu / p^2\ . \qquad \text{(B.10)}$$

The next term is the $\mathcal{O}(A^2)$ part of

$$\bar{\phi}L(-D^2)\bar{\phi} = \sum_n c_n\bar{\phi}[-(\partial - ie\bar{A} - ieA)^2]^n\bar{\phi}\ . \qquad \text{(B.11)}$$

In this case it is useful to write the quadratic part as another sum $K^{(2)}_{AA}(p) + K^{(3)}_{AA}(p)$. The first term gathers the A^2 contributions from the factors $-D^2$. It can be written as

$$\begin{aligned} A'K^{(2)}_{AA}A' &= 2e^2\sum_{n=0}^{n_d} c_n \sum_{\ell=1}^{n}\bar{\phi}(-\Box')\cdots A^2(x)\cdots(-\Box')\bar{\phi} \\ &= 2A'^2\bar{\phi}^2 e^2 L'(k^2) \end{aligned} \qquad \text{(B.12)}$$

which vanishes on account of (6.15),

$$K^{(2)}_{A_\mu A_\nu}(p) = 0\ . \qquad \text{(B.13)}$$

The second term corresponds to the product of two $\mathcal{O}(A)$ terms from $-D^2$ factors and can be calculated as follows

$$\begin{aligned} A'K^{(3)}_{AA}(p)A' &= -2e^2\sum_{n=0}^{n_d} c_n\sum_{\ell=1}^{n-1}\sum_{\ell'=\ell+1}^{n}\bar{\phi}(-\Box')\cdots(2A\partial' + \partial A)\cdots(2A\partial' + \partial A)\cdots(-\Box')\bar{\phi} \\ &= 2e^2\sum_{n=0}^{n_d} c_n\sum_{\ell=1}^{n-1}\sum_{\ell'=\ell+1}^{n}\bar{\phi}k^2\cdots(2A_0k + pA)\cdots(k^2 + p^2 + 2p^0k)\cdots(2A_0k + pA)\cdots k^2\bar{\phi} \\ &= 2e^2\bar{\phi}^2(2A_0k + pA)^2 k^{-4}\sum_{n=0}^{n_d} c_n k^{2n}\sum_{\ell=1}^{n-1}\sum_{\ell'=\ell+1}^{n}\left(1 + \frac{p^2 + 2p^0k}{k^2}\right)^{\ell' - \ell - 1}\ . \end{aligned} \qquad \text{(B.14)}$$

After summing up the terms in one series we can write

$$A'K^{(3)}_{AA}(p)A' = 2e^2\bar{\phi}^2\frac{(2A_0k+pA)^2}{p^2+2p^0k}k^{-2}\sum_{n=0}^{n_d}c_nk^{2n}\left[\sum_{\ell=1}^{n-1}\left(1+\frac{p^2+2p^0k}{k^2}\right)^{n-\ell}-n+1\right] . \tag{B.15}$$

Performing the two subsequent summations of the geometric series we obtain

$$\begin{aligned} A'K^{(3)}_{AA}(p)A' &= 2e^2\bar{\phi}^2\frac{(2A_0k+pA)^2}{(p^2+2p^0k)^2}\sum_{n=0}^{n_d}c_nk^{2n}\left[\left(1+\frac{p^2+2p^0k}{k^2}\right)^{n}-1-n\frac{p^2+2p^0k}{k^2}\right] \\ &= 2e^2\bar{\phi}^2L_d(p)\frac{(2A_0k+pA)^2}{(p^2+2p^0k)^2} . \end{aligned} \tag{B.16}$$

Since the field A is real we need to symmetrize the whole $K_{AA}(p)$ contribution which yields eventually

$$K_{A_\mu A_\nu}(p) = -T^{\mu\nu}p^2+e^2\bar{\phi}^2L_d(p)\frac{(2g^{\mu 0}k+p^\mu)(2g^{\nu 0}k+p^\nu)}{(p^2+2p^0k)^2}+e^2\bar{\phi}^2L_d(-p)\frac{(2g^{\mu 0}k-p^\mu)(2g^{\nu 0}k-p^\nu)}{(p^2-2p^0k)^2} . \tag{B.17}$$

Publications

Published papers

1. A. Radosz, A. Siwek, J. Polonyi, K. Ostasiewicz
   Circular geodesics in Schwarzschild-like spacetimes
   Mod. Phys. Lett. A **26** (7), 473 (2011).
2. J. Polonyi, A. Siwek
   Boundary conditions and consistency of effective theories
   Phys. Rev. D **81**, 085040 (2010).

Submitted manuscripts

1. M. Gawełczyk, J. Polonyi, A. Radosz and A. Siwek
   Exchange of signals around the event horizon in Schwarzschild space-time.
2. A. T. Augousti, M. Gawełczyk, J. Polonyi, A. Siwek and A. Radosz
   Colliding with ghosts: observing free fall from an infalling frame of reference into a Schwarzschild black hole.
3. J. Polonyi, A. Siwek
   Spontaneous breakdown of Lorentz symmetry in scalar QED with higher order derivatives.

J. Polonyi, A. Radosz, A. Siwek, K. Ostasiewicz
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