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Des coordonnées de décalage sur le super espace de Teichmüller

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Shear coordinates on the super-Teichmüller space

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À Maeva,

On ne peut jamais tourner une page de sa vie
sans que s'y accroche une certaine nostalgie.
Ève Bélisle

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Selon Pierre Dac,

> Il faut une infinie patience pour attendre toujours ce qui n'arrive jamais.

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Contents

Introduction

Contexte

Le problème de la paramétrisation des structures complexes sur une surface donnée remonte à l'époque de Riemann. Ce dernier détermina de manière informelle le nombre de paramètres nécessaire à la description des classes de surfaces de Riemann à biholomorphismes près. Teichmüller démontra, quatre-vingts ans plus tard, que ces paramètres pouvaient être utilisés en tant que coordonnées sur une cellule, appelée *espace de Teichmüller*, dont le quotient par une action naturelle du groupe d'homéotopie est l'espace des modules de Riemann.

L'espace de Teichmüller, défini à partir des surfaces marquées, apparaît implicitement dans les travaux de Klein et Poincaré sur les groupes fuchsiens. Le théorème d'uniformisation de Klein, Poincaré et Koebe fournit une bijection entre les classes d'isomorphisme de surfaces de Riemann homéomorphes à S_g et les classes d'isométrie de surfaces hyperboliques homéomorphes à S_g, où S_g est une surface compacte de genre $g \geq 2$. À l'aide de méthodes basées sur ce théorème, Fricke construit un système de coordonnées sur l'espace de Teichmüller T_g de S_g et montre qu'il s'agit d'une cellule de dimension réelle $6g - 6$.

Il existe d'autres paramétrisations de l'espace de Teichmüller et Thurston [32] démontre, au milieu des années 80, que toute structure hyperbolique sur une surface pointée peut être décrite par l'affectation d'un nombre réel strictement positif à chaque arête d'une triangulation idéale de la surface (c'est-à-dire une décomposition de la surface en triangles de sorte que les sommets de la triangulation sont exactement les pointes ; un exemple est donné Chapitre 1, Figure 1.1). Ces paramètres de *décalage* encodent la façon de recoller deux triangles idéaux de la triangulation. Les coordonnées de décalage ont été développées par Bonahon [4] et Fock [13] qui montrent notamment comment exprimer différentes structures sur l'espace de Teichmüller à l'aide de ces dernières. Les coordonnées dépendent de la triangulation idéale, mais lorsqu'on change de triangulation par un mouvement élémentaire, appelé *flip*, les coordonnées changent de façon bien contrôlée. Un théorème dû indépendamment à Harer [17], Strebel et Penner [28] assure que toute triangulation idéale peut être obtenue à partir d'une autre en appliquant une séquence de ces mouvements qui satisfont trois types de relation (involution, commutation lointaine et l'importante relation du pentagone). En particulier les coordonnées de Thurston-Bonahon-Fock-Penner permettent d'encoder la structure de Poisson sur l'espace de Teichmüller de façon agréable et les changements de coordonnées préservent cette structure.

Ces coordonnées se révèlent être particulièrement utiles au développement de la recherche sur les espaces de Teichmüller et leurs analogues en rang supérieur :

1. La quantification des espaces de Teichmüller a été construite en déformant l'algèbre engendrée par les coordoonnées de décalage dans la direction de la structure de Poisson [6].

2. L'étude des espaces des modules des représentations du groupe fondamental d'une surface dans d'autres groupes classiques comme $\mathrm{SL}_n(\mathbb{R})$ a été développée par Fock et Goncharov [12] qui ont défini des coordonnées sur ces espaces (judicieusement décorés) et établi quelles étaient leurs transformations lorsqu'un flip était appliqué. Cela leur a permis d'identifier combinatoirement une composante "positive" correspondant à celle de Hitchin.

Le concept de *supervariété*, généralisant celui de variété classique en ajoutant des coordonnées anticommutatives aux cartes locales, a été introduit par les physiciens étudiant la supersymétrie. Il existe différentes façons de définir les supervariétés. Nous nous intéressons dans notre étude à l'approche de DeWitt [10], appélée *approche concrète* par Rogers [30, 31]. Grosso modo, dans l'approche de DeWitt, une supervariété est une variété qui est localement isomorphe à un superespace noté $\mathbb{R}^{n|m}$, dont les coordonnées sont à valeurs dans une algèbre supercommutative, *algèbre grassmannienne* (cf. Definition 2.1.2). En particulier, cela conduit à la définition des super surfaces de Riemann, qui sont fondamentales en théorie des cordes supersymétriques : elles sont les surfaces d'univers de la théorie. Dans le formalisme de Poliakov de la théorie de cordes bosoniques (cf. [29, 9]), le calcul de la fonction de partition d'une corde peut être effectué en intégrant sur l'espace des modules de surfaces de Riemann. Dans le cas supersymétrique, cet espace des modules devrait être remplacé par son superanalogue.

Une super surface de Riemann peut être vue comme une classe de conjugaison de morphismes de son groupe fondamental dans un groupe de matrices, noté $\mathrm{SpO}(2|1)$, dont les coefficients appartiennent à l'algèbre grassmannienne. Ce groupe se projette sur $\mathrm{SL}(2,\mathbb{R})$, on peut par conséquent lui associer sa *réduction* représentée par une surface de Riemann classique avec un relevé de l'holonomie de $\mathrm{PSL}(2,\mathbb{R})$ à $\mathrm{SL}(2,\mathbb{R})$: une *structure spin*. Le super espace de Teichmüller est une supervariété paramétrant les super surfaces de Riemann ayant la même topologie. Cette théorie des super espaces de Teichmüller a été développée dans le cas des surfaces fermées de genre g [9, 18]. Le besoin d'une structure spin fait que le super espace de Teichmüller admette plusieurs composantes connexes. Crane et Rabin [9] and Hodgkin [18] ont démontré que chaque composante était une "super boule" de dimension $(6g-6|4g-4)$.

Dans cette thèse, nous avons pour but de construire des coordonnées de décalage pour les super surfaces de Riemann pointées munies d'une triangulation idéale ainsi que de définir une super structure de Poisson sur cet espace à l'aide de ces coordonnées. L'un des produits issu de ces constructions est la loi de transformation des coordonnées lorsqu'on modifie la triangulation par un flip qui vérifie toutes les relations naturelles que nous avons rappelées ci-dessus, notamment la relation de super pentagone. Une des nouveautés dans ce système de coordonnées est l'apparition de coordonnées "impaires" associées à chaque triangle de triangulation, en plus des coordonnées "paires" associées à chaque arête, déjà existantes.

Structure du travail et résultats

Le premier chapitre de cette thèse est consacré à des rappels sur les surfaces de Riemann et les espaces de Teichmüller. On considère une surface S de type (g, k, m) c'est-à-dire une surface de Riemann de genre g avec deux types de composantes de bord : un ensemble de k composantes, appelées *trous*, et un ensemble de m composantes, appelées *pointes*. Soient $\mathbb{H}$ le demi-plan de Poincaré et $\mathrm{PSL}(2, \mathbb{R})$ son groupe d'automorphismes. On définit l'espace de Teichmüller de S comme l'espace des monomorphismes ρ du groupe fondamental $\pi_1(S)$ dans $\mathrm{PSL}(2, \mathbb{R})$ tels que le quotient $\mathbb{H}/\mathrm{Im}(\rho)$ est une surface du même type que S, modulo l'action de $\mathrm{PSL}(2, \mathbb{R})$ par conjugaison : on demande que chaque pointe (resp. trou) corresponde à un cusp parabolique (resp. hyperbolique). On rappelle la construction de Fock des coordonnées de décalage pour les surfaces à trous. Ces coordonnées dépendent du choix d'une triangulation idéale, ainsi un changement de triangulation conduit à un changement de coordonnées. Ainsi on rappelle ces transformations qui sont données par une formule explicite (1.2).

Dans le Chapitre 2, on rappelle les définitions et les résultats concernant les supervariétés nécessaires à notre étude. Il existe tout d'abord une super surface de Riemann (cf. [15, 9]) généralisant le demi-plan supérieur $\mathbb{H}$, noté $\mathbb{H}^S$ (cf. [9]), et le superanalogue du bord de $\mathbb{H}$ est noté $\mathbb{P}^{1|1}$. Ce super demi-plan supérieur admet un groupe de transformations (cf. [15, 9, 5]) qui est lui même une supervariété et généralise le groupe classique $\mathrm{PSL}(2, \mathbb{R})$. Il est noté $\mathrm{SpO}(2|1)$ et il agit sur $\mathbb{H}^S$ et $\mathbb{P}^{1|1}$ comme décrit en (2.3). De plus, il se projette canoniquement sur $\mathrm{SL}(2, \mathbb{R})$ puis sur $\mathrm{PSL}(2, \mathbb{R})$; si Γ est un sous-groupe de $\mathrm{SpO}(2|1)$, on note $\Gamma^\sharp$ son image par la projection sur $\mathrm{PSL}(2, \mathbb{R})$. L'existence de ces objets conduit à la définition du super espace de Teichmüller (la définition par groupe de Bryant et Hodgkin [5]) : le super espace de Teichmüller d'une surface de type (g, k, m) est l'ensemble des monomorphismes $\psi : \pi_1(S) \to \mathrm{SpO}(2|1)$ tels que le quotient $\mathbb{H}/(\psi(\pi_1(S)))^\sharp$ est une surface de type (g, k, m) modulo l'action de $\mathrm{SpO}(2|1)$ par conjugaison.

Le super espace de Teichmüller d'une surface S admet plusieurs composantes connexes indexées par l'ensemble des structures spin sur la surface. C'est pourquoi, nous rappelons dans le Chapitre 3 la définition d'une structure spin sur S et différentes caractérisations. Nous introduisons alors notre outil principal pour la construction des coordonnées sur le super espace de Teichmüller : les *orientations de Kasteleyn* sur les graphes plongés dans S. Un résultat de Cimasoni et Reshetikhin [7, 8] affirme que l'ensemble des structures spin sur S est en bijection avec l'ensemble des classes d'équivalence d'orientations de Kasteleyn, via une bijection dépendant d'une donnée combinatoire appelée *configuration de dimère*. Une structure spin σ divise l'ensemble des pointes de S en celles de type *Neveu-Schwartz* et celles de type *Ramond*. La composante connexe du super espace de Teichmüller indexée par σ est de dimension $(6g - 6 + 3k + 2m|4g - 4 + 2k + 2m - R_\sigma)$, où R_σ est le nombre de pointes Ramond pour la structure spin σ.

Le Chapitre 4 est le cœur de ce travail. Le but dans ce chapitre est de construire un ensemble de coordonnées sur le super espace de Teichmüller comme suit.

Soit S une surface de type (g, k, m) et on note par $p_1, \ldots, p_k$ l'ensemble des trous et par $p_{k+1}, \ldots, p_{k+m}$ l'ensemble des pointes.

Définition. *Le super espace de Teichmüller de type X*, noté ST, est l'ensemble des classes d'équivalence de triplets

$$\left(\rho, \{o_i\}_{i=1,\ldots,k}, \{F_j\}_{j=1,\ldots,m}\right),$$

où $\rho \in Hom\left(\pi_1(S), \mathrm{SpO}(2|1)\right)$ et $(\mathrm{Im}\rho)^\sharp$ est fuchsien, o_i est le choix d'une orientation du trou p_i et F_j est un ensemble $\pi_1(S)-$équivariant de points fixes dans $\mathbb{P}^{1|1}$ de $\rho(gm_{k+j}g^{-1})$ pour tout $g \in \pi(S)$ et m_{k+j} est un lacet simple autour de la pointe p_{k+j} et où l'on définit deux triplets $\left(\rho, \{o_i\}_{i=1,\ldots,k}, \{F_j\}_{j=1,\ldots,m}\right)$ et $\left(\rho', \{o_i'\}_{i=1,\ldots,k}, \{F_j'\}_{j=1,\ldots,m}\right)$ comme équivalents si et seulement si il existe $B \in \mathrm{SpO}(2|1)$ tel que

$$\rho' = B\rho B^{-1}, \quad o_i' = o_i \quad \text{et} \quad F_j' = B \cdot F_j.$$

Le choix de points fixes pour les pointes de type Ramond augmente la dimension du super espace de Teichmüller d'un paramètre impair et chaque composante est isomorphe à une “super boule” de dimension $6g-6+3k+2m|4g-4+2k+2m$. Nous adoptons la même approche que dans la cas classique afin d'associer des coordonnées à une triangulation idéale mais nous devons gérer trois difficultés principales.

1. Nous devons encoder une structure spin combinatoirement : pour ce faire, nous utilisons le résultat de Cimasoni et Reshetikhin sur les orientations de Kasteleyn.
2. Les coefficients des matrices que nous considérons appartiennent à une algèbre non commutative : ceci fait que beaucoup d'attention doit être portée à l'exécution des calculs.
3. De nouvelles coordonnées impaires, associées à chaque triangle, font leur apparition.

Fixons une triangulation idéale Λ de S. En associant, comme dans le cas classique, une coordonnée paire x_α (un superanalogue du birapport) à chaque arête α de Λ et une coordonnée impaire ζ_i à chaque face T_i de Λ, nous construisons une super surface de Riemann. Nous obtenons donc des coordonnées à valeurs dans $\mathbb{R}^{E(\Lambda)|T(\Lambda)}$, où $E(\Lambda)$ et $T(\Lambda)$ désignent respectivement l'ensemble des arêtes et des triangles de Λ. Nous démontrons le résultat suivant :

Théorème (cf. Theorem 4.3.10)**.** *La collection $\{x_\alpha, \zeta_i\}$ fournit un système de coordonnées sur le super espace de Teichmüller de type X de S, modulo l'action diagonale de $\mathbb{Z}_2$ par multiplication des ζ_i par -1.*

Dans le Chapitre 5 nous exposons certaines propriétés de nos coordonnées. Chaque choix de triangulation idéale Λ conduit à un système de coordonnées différent. Nous montrons alors comment ces coordonnées changent lorsqu'on modifie Λ par un flip. On obtient alors des lois de transformation semblables à celles de la Figure 1(cf. Theorem 5.1) qui se réduisent aux lois classiques décrites dans la Figure 1.6.

On démontre également que ces changements de coordonnées satisfont les relations naturelles d'involution et une super version de la relation du pentagone (cf. Definition 5.2).

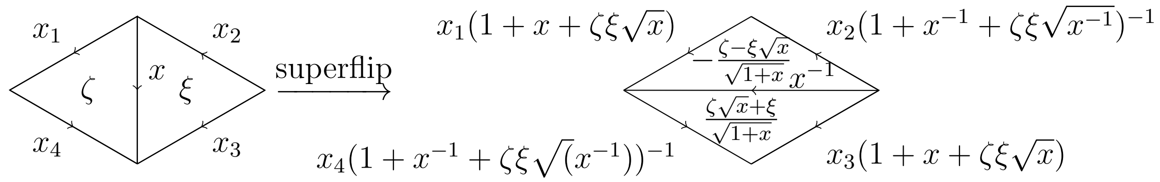


Figure 1

Théorème (cf. Theorem 5.3)**.**

1. *Le superflip est une involution*
2. *Le superflip satisfait la relation de super pentagone*

Dans le Chapitre 6, nous construisons explicitement une structure de Poisson canonique sur le super espace de Teichmüller de type X. On définit tout d'abord deux opérateurs de dérivations $\frac{\overrightarrow{\partial}}{\partial\zeta_i}$ et $\frac{\overleftarrow{\partial}}{\partial\zeta_i}$ (cf. Definition 6.2.1). Nous démontrons le résultat suivant en vérifiant que chaque flip induit une application super-Poisson.

Théorème (cf. Theorem 6.2.2)**.** *L'expression suivante définit une structure de super-Poisson paire sur le super espace de Teichmüller de type X :*

$$\{,\}_{ST} = \sum_{i,j\in E(\Lambda_\bullet)} \varepsilon^{ij} x_i x_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{1}{2}\sum_{k\in F(\Lambda_\bullet)} \frac{\overleftarrow{\partial}}{\partial\zeta_k}\frac{\overrightarrow{\partial}}{\partial\zeta_k}$$

Encore une fois, ce crochet de Poisson se réduit à celui donné dans le cas classique par l'expression (6.1).

Conclusions

Les sujets abordés dans ce manuscrit s'intègrent naturellement dans une liste de problèmes pertinents. Dans notre étude, nous construisons un système de coordonnées sur le super espace de Teichmüller des super surface de Riemann de dimension $(2|1)$. Une super surface de Riemann de dimension $(2|2)$ peut être vue comme une classe de conjugaison de morphismes de groupe fondamental dans $\mathrm{SpO}(2|2)$. Une première étape dans une étude plus approfondie des super espaces de Teichmüller serait la construction de coordonnées de décalage sur le super espace de Teichmüller des super surfaces de Riemann de dimension $(2|2)$ et plus généralement $(2|N)$.

Une autre direction de recherche pourrait être l'étude des super espaces de Teichmüller de rang supérieur, c'est-à-dire des espaces de monomorphismes du groupe fondamental d'une surface dans des super groupes qui se réduisent en des groupes de Lie classiques comme $\mathrm{SL}(n,\mathbb{R})$ par exemple.

Nous aimerions aussi tirer profit du crochet de Poisson que nous avons construit. Il serait sans doute d'une grande utilité de définir une quantification du super espace de Teichmüller en calquant la construction de la quantification de l'espace de Teichmüller classique. De plus notre crochet est pair et la construction d'un crochet de Poisson impair sur le super espace de Teichmüller, s'il existe, pourrait créer un intéressant lien avec le formalisme de Batalin-Vilkovisky utilisé en théories de jauge.

Dans notre construction, nous établissons des formules explicites pour les changements de coordonnées par application d'un flip. Ces changements pourraient être vus comme une généralisation des mutations en théorie des variétés amassées et nous l'espérons conduiraient à une version super de ces objets.

Enfin, nous rappelons que notre approche des super espaces de Teichmüller est basée sur la définition des supervariétés par DeWitt. Une autre définition des supervariétés, dite algébro-géométrique, est basée sur la théorie des faisceaux : au lieu de déformer la variété, on déforme son algèbre de fonctions. Il serait intéressant de traduire notre travail en ces termes.

English introduction

Context

The problem of the parametrization of complex structures on a given surface dates back to Riemann, who counted the number of parameters of classes of Riemann surfaces up to biholomorphic equivalence. Eighty years later, Teichmüller showed that these parameters may be used as real coordinates on a cell, called *Teichmüller space* whose quotient by a natural action of the mapping class group is Riemann's *moduli space*.

The Teichmüller space, defined in terms of marked surfaces, appeared implicitly in the study of Fuchsian groups by Klein and Poincaré. The Uniformization theorem of Riemann surfaces due to Klein, Poincaré and Koebe gives a bijective correspondence between isomorphism classes of Riemann surfaces homeomorphic to S_g and isometry classes of hyperbolic surfaces homeomorphic to S_g, where S_g is a compact surface of genus $g \geq 2$. Using methods based on this theorem, Fricke constructed a system of coordinates on the Teichmüller space T_g of S_g and showed that it is a cell of real dimension $6g - 6$.

There exist other parametrizations of the Teichmüller space and Thurston [32], in the middle of the eighties, showed that each hyperbolic structure on a punctured surface can be described by associating a positive real number to each edge of an *ideal triangulation* of the surface (i.e. a decomposition of the surface into triangles such that the vertices of the triangulation are exactly the punctures; for an example see Chapter 1, Figure 1.1): these *shear* parameters encode how to glue together two ideal triangles of the triangulation. The shear coordinates have been developed independently by Bonahon [4] and Fock [13] who showed how to express several structures on the Teichmüller space through it. The coordinates depend on the ideal triangulation, but changing the triangulation by an elementary move, called *flip* (cf. Figure 1.2), leads to a change of coordinates which is well controlled. A theorem due independently to Harer [17], Strebel and Penner [28] ensures that each ideal triangulation can be obtained from any other one by applying sequences of these moves, and that these moves satisfy three kinds of relations (involution, distant commutation and most importantly the pentagon relation). In particular the Thurston-Bonahon-Fock-Penner shear coordinates allow to encode the Poisson structure of the Teichmüller space in a convenient way and the changes of coordinates preserve the structure. The shear coordinates turned out to be particularly useful to further develop the research on the Teichmüller space and its higher rank analogues:

1. The quantization of Teichmüller spaces was achieved by deforming the algebra generated by the coordinates in the direction of the Poisson structure [6].

2. The study of moduli spaces of representations of the fundamental group of the surface in other classical groups as $SL_n(\mathbb{R})$ was developed by Fock and Goncharov [12] who defined coordinates on these moduli spaces (suitably decorated) and established how the coordinates change while applying a flip. This allowed them to identify combinatorially a "positive" component which corresponds to Hitchin's.

The concept of *supermanifold*, a generalization of the concept of classical manifold by adding some anticommuting coordinates to the local charts, was introduced by the physicists studying supersymmetry. There exist different ways to define supermanifolds. We focus in our study on the DeWitt [10] approach, also called the *concrete one* by Rogers [30, 31]. Roughly, in the DeWitt approach, a supermanifold is a manifold which is locally modeled on a superspace denoted by $\mathbb{R}^{m|n}$ whose coordinates live in super commutative algebras, called *Grassmann algebras* (cf. Definition 2.1.2). In particular, this leads to the definition of super Riemann surfaces. In supersymmetric string theory, the super Riemann surfaces are fundamental: they are the *worldsheets* of the theory. In Polyakov's formalism (cf. [29, 9]) of bosonic string theory, the computation of the partition function of a string can be performed by integrating over the moduli space of Riemann surfaces. In the supersymmetric approach to string theory, this moduli space should be replaced by its super-analog.

A super Riemann surface can be seen as a conjugacy class of morphisms of its fundamental group in a group of matrices, with coefficients in the Grassmann algebra, denoted by $\mathrm{SpO}(2|1)$. This group projects on $\mathrm{SL}(2,\mathbb{R})$ thus one can associate to each super Riemann surface its *reduction* represented by a classical Riemann surface together with a lift of the holonomy from $\mathrm{PSL}(2,\mathbb{R})$ to $\mathrm{SL}(2,\mathbb{R})$: a *spin structure*. The super Teichmüller space is a super manifold parametrizing the super Riemann surfaces carrying the same topology. This super-Teichmüller theory was developed in the case of closed surfaces of genus g [9, 18]. Because of the necessity of a spin structure, the super Teichmüller space admits several connected components. Crane and Rabin [9] and Hodgkin [18] showed that each component is a "super ball" of dimension $(6g-6|4g-4)$.

The goal of this thesis is to construct shear coordinates for punctured super Riemann surfaces equipped with an ideal triangulation and define a super Poisson structure on this space using these coordinates. One of the products of these constructions is the computation of the coordinate changes associated to the flips which satisfy all the natural relations, superanalogue of the above listed relations, notably the super pentagons. One of the new features of these coordinates is that in addition to"even" coordinates, on each edge of the triangulation, there are "odd" coordinates for each triangle.

Structure of the work and results

In the first chapter of this thesis, we recall some basic facts about Riemann surfaces and the Teichmüller spaces. Let S be an oriented surface of type (g,k,m) i.e a Riemann surface of genus g with two kinds of boundary components : a set of k components called *holes* and a set of m components called *punctures*. Let $\mathbb{H}$ be the Poincaré half plane and $\mathrm{PSL}(2,\mathbb{R})$ its automorphism group. Define the Teichmüller space as the space of monomorphisms ρ from the fundamental group $\pi_1(S)$ to $\mathrm{PSL}(2,\mathbb{R})$ such that the quotient space $\mathbb{H}/\operatorname{Im}(\rho)$ is a surface of the same kind as S up to the action of $\mathrm{PSL}(2,\mathbb{R})$ by conjugation: we ask that each punture (resp. hole) corresponds to a parabolic (resp. hyperbolic) cusp. We recall Fock's construction of the shear coordinates on a surface with holes. These coordinates depend on the choice of an ideal triangulation, so a change of triangulation leads to a change of coordinates. We also recall these transformations which are given by the explicit formula (1.2).

In Chapter 2 we recall the definitions and the results about supermanifolds needed in our study. There exists a canonical super-Riemann surface (cf. [15, 9]) generalizing the upper half-plane $\mathbb{H}$, denoted by $\mathbb{H}^S$ (cf. [9]), and the super analog of the boundary of $\mathbb{H}$ is denoted by $\mathbb{P}^{1|1}$. This super upper half-plane admits a group of transformations (cf. [15, 9, 5]) which is itself a supermanifold and generalizes the classical group $\mathrm{PSL}(2,\mathbb{R})$. It is denoted by $\mathrm{SpO}(2|1)$ and it acts on $\mathbb{H}^S$ and $\mathbb{P}^{1|1}$ by the operation described in (2.3). Moreover it projects canonically to $\mathrm{SL}(2,\mathbb{R})$ and then to $\mathrm{PSL}(2,\mathbb{R})$; if Γ is a subgroup of $\mathrm{SpO}(2|1)$, one denotes by $\Gamma^\sharp$ its image by the projection on $\mathrm{PSL}(2,\mathbb{R})$. The existence of these objects leads to the definition of the super-Teichmüller space (the group definition by Bryant and Hodgkin [5]) : the super Teichmüller space of a surface of type (g,k,m) is the set of monomorphisms $\psi : \pi_1(S) \to \mathrm{SpO}(2|1)$ such that the quotient space $\mathbb{H}/\left(\psi(\pi_1(S))\right)^\sharp$ is a surface of type (g,k,m) modulo the action of $\mathrm{SpO}(2|1)$ by conjugation.

The super-Teichmüller space of a surface S admits several connected components, indexed by the set of spin structures on the surface. Therefore we recall in Chapter 3 the definition of a spin structure on S and different characterizations of it. We then introduce our key tool in the construction of coordinates on the super-Teichmüller space: *Kasteleyn orientations* on graphs embedded in S. A result of Cimasoni and Reshetikhin [7, 8] states that the set of spin structures on S is in bijection with the set of equivalence classes of Kasteleyn orientations, via a bijection depending on a combinatorial datum called *dimer configuration*. A spin structure σ divides the set of punctures on S into those of type *Neuveu-Schwarz* and those of type *Ramond*. The component of the super-Teichmüller space indexed by σ has dimension $(6g-6+3k+2m|4g-4+2k+2m-R_\sigma)$ where R_σ is the number of Ramond punctures for the spin structure σ.

Chapter 4 is the core of this work. The aim of this chapter is to construct a set of coordinates on the super-Teichmüller X-space as follows.

Let S be a surface of type (g,k,m) and denote by $p_1,\dots,p_k$ the set of holes and by $p_{k+1},\dots,p_{k+m}$ the set of punctures.

Definition. *The super-Teichmüller X−space*, denoted by ST, is the set of equivalence classes of triples

$$\left(\rho, \{o_i\}_{i=1\dots k}, \{F_j\}_{j=1..m}\right),$$

where $\rho \in Hom\left(\pi_1(S), \mathrm{SpO}(2|1)\right)$ and $(\operatorname{Im}\rho)^\sharp$ is a Fuchsian group, o_i is the choice of an orientation of the hole p_i and F_j is a set of $\pi_1(S)-$equivariant fixed points in $\mathbb{P}^{1|1}$ of

$\rho(gm_{k+j}g^{-1})$ for all $g \in \pi_1(S)$ and m_{k+j} is a simple loop surrounding the puncture p_{k+j} and where we say that two triples, $\left(\rho, \{o_i\}_{i=1...k}, \{F_j\}_{j=1..m}\right)$ and $\left(\rho', \{o'_i\}_{i=1...k}, \{F'_j\}_{j=1..m}\right)$, are equivalent if and only if there exists $B \in \mathrm{SpO}(2|1)$ such that

$$\rho' = B\rho B^{-1}, \quad o'_i = o_i \quad \text{and} \quad F'_j = B \cdot F_j.$$

The choice of fixed points of Ramond punctures increases the dimension of the Teichmüller space by one odd parameter and each component is isomorphic to a "super ball" of dimension $6g - 6 + 3k + 2m|4g - 4 + 2k + 2m$. We follow the same approach as in the classical case to associate coordinates to an ideal triangulation but we have to handle three main difficulties.

1. We need to encode a spin structure combinatorially : to do so, we use Cimasoni and Reshetikhin's result about Kasteleyn orientations.
2. The coefficients of the matrices we are dealing with live in a non commutative algebra : this causes that some more care has to be taken when computing.
3. New coordinates of odd type appear, associated to the triangles.

Fix an ideal triangulation Λ of S. By associating, as in the classical case, one even coordinate x_α (a super-analog of the cross-ratio) to each edge α of Λ and one odd coordinate ζ_i to each face T_i of Λ, we construct a super-Riemann surface. Hence we get coordinates which live in $\mathbb{R}^{E(\Lambda)|T(\Lambda)}$, where $E(\Lambda)$ and $T(\Lambda)$ denote respectively the set of edges and triangles of Λ. We prove the following:

Theorem (see Theorem 4.3.10)**.** *The collection of numbers $\{x_\alpha, \zeta_i\}$ provides a system of coordinates on the super-Teichmüller X-space of S, up to the diagonal action of $\mathbb{Z}_2$ by multiplication of the ζ_i by -1.*

In Chapter 5 we give some properties of our coordinates. Each choice of an ideal triangulation Λ of the surface leads to a different system of coordinates. We show how these change when applying a flip on Λ. We get transformation laws as those in Figure 2 (cf. Theorem 5.1) which reduce to the classical ones described in Figure 1.6.

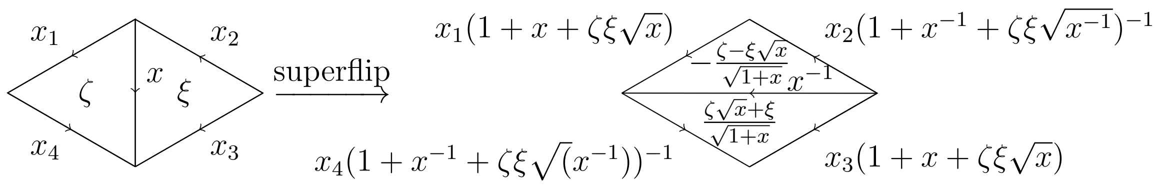


Figure 2

We also show that these coordinate changes satisfy the natural relations of involution and a super version of the pentagon relation (cf. Definition 5.2) :

Theorem (see Theorem 5.3)**.**

1. *The superflip is an involution.*

2. *The superflip satisfies the superpentagon relation.*

In Chapter 6 we construct an explicit canonical Poisson structure on the super-Teichmüller X-space. We first define two derivation operators $\frac{\overrightarrow{\partial}}{\partial \zeta_i}$ and $\frac{\overleftarrow{\partial}}{\partial \zeta_i}$ (cf. Definition 6.2.1). Then we prove the following by checking that each flip induces a *super-Poisson map.*

Theorem (see Theorem 6.2.2)**.** *The following formula defines an even super Poisson structure on the super Teichmüller X−space :*

$$\{,\}_{ST} = \sum_{i,j \in E(\Lambda_\bullet)} \varepsilon^{ij} x_i x_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{1}{2} \sum_{k \in F(\Lambda_\bullet)} \frac{\overleftarrow{\partial}}{\partial \zeta_k} \frac{\overrightarrow{\partial}}{\partial \zeta_k}$$

and this Poisson bracket does not depend on the particular triangulation.

Once again this Poisson bracket reduces to the classical one given by the formula (6.1).

Conclusions

The topics we dealt with in the present work embed naturally in a list of natural and relevant problems. In our study we construct a system of coordinates on the super Teichmüller space of super Riemann surfaces of dimension (2|1). A super Riemann surface of dimension (2|2) can be seen as a conjugacy class of morphisms of its fundamental group in SpO(2|2) (cf.[24, 27]). A first step in further investigations would be the construction of shear coordinates on the super Teichmüller space of super Riemann surfaces of dimension (2|2) and more generally $(2|N)$.

Another interesting direction of research could be the study of super Teichmüller spaces of higher rank, i.e. spaces of monomorphisms of the fundamental group of a surface to super groups which reduce to classical Lie groups, as $\mathrm{SL}(n, \mathbb{R})$ for example.

We also would like to make use of our Poisson bracket. It could be useful to define a quantization of the super Teichmüller space following the construction of the quantization of the Teichmüller space. Moreover our bracket is even and the construction of an odd Poisson bracket on the super Teichmüller space, if it exists, could make an interesting link with the Batalin-Vilkovisky formalism used in gauge theories.

In our construction, we establish explicit formulae for a change of coordinates by applying a flip. These changes could be seen as a generalization of the mutations in the theory of cluster varieties and hopefully lead to a super version of these objects.

Finally we recall that our approach to super Teichmüller spaces was based on DeWitt's supermanifold definition. Another definition of supermanifolds uses a sheaf theoretical approach: instead deforming the manifold, one deforms its algebra of functions. It will be interesting to translate our work in these terms.

CHAPTER 1

Teichmüller spaces

1.1 The Teichmüller space of Riemann surfaces of type (g, k, m)

Let S be a connected Hausdorff space with a collection $\{(V_j, z_j)\}_j$ satisfying the three following conditions :

1. Every V_j is an open subset of S, and the collection $\{V_j\}_j$ is a cover of S : $S = \bigcup_j V_j$.
2. Every z_j is a homeomorphism of V_j onto an open subset in the complex plane.
3. If $V_j \cap V_k \neq \emptyset$, the transition mapping

$$z_{kj} := z_k \circ z_j^{-1} : z_j(V_j \cap V_k) \to z_k(V_j \cap V_k)$$

   is a biholomorphism.

Definition 1.1.1. 1. The collection $\{(V_j, z_j)\}_j$ is called a system of coordinate neighborhoods on S. We say that this system defines a one-dimensional complex structure on S.

2. A Riemann surface is a connected Hausdorff space with a one-dimensional complex structure.

Local analysis on a Riemann surface S is reduced to analysis on domains in the complex plane.

Definition 1.1.2. 1. A holomorphic function on a Riemann surface S is a function f from S to $\mathbb{C}$ such that $f \circ z^{-1}$ is holomorphic on $z(V)$ for any coordinate neighborhood (V, z) of S.

2. A mapping f of S into a Riemann surface R such that $w \circ f \circ z^{-1}$ is holomorphic for all coordinate neighborhoods (V, z) of S and (W, w) of R with $f(V) \subset W$ is said to be a holomorphic mapping.

3. A holomorphic mapping $f : S \to R$ such that the inverse mapping $f^{-1} : R \to S$ exists and is holomorphic, is called a biholomorphic mapping.

Two Riemann surfaces S and R are biholomorphically equivalent if there exists a biholomorphic mapping $f : S \to R$. We say that S and R have the same complex structure.

We now come to the definition of the Teichmüller space. Let S be a closed Riemann surface of genus $g \geq 1$. The surfaces we are dealing with are endowed with two sets of points called holes and punctures. This distinction will become clear in what follows. We assume that there are k holes and m punctures on S such that $6g - 6 + 3k + 2m$ is positive. We consider triples (S, f, R) where R is a Riemann surface and $f : S \to R$ is an orientation preserving diffeomorphism. Two triples (S, f_1, R_1) and (S, f_2, R_2) are said to be equivalent if $f_2 \circ f_1^{-1}$ is homotopic to the identity.

Definition 1.1.3. The set of all equivalence classes $[S, f, R]$ of triples (S, f, R) is denoted $\mathcal{T}(S)$ and is called the Teichmüller space based on S.

The Teichmüller space may be regarded as the space of complex structures on S modulo the diffeomorphisms isotopic to the identity. In dimension 2 one can easily establish the equivalence of conformal and complex structures.

Hyperbolic structures and Fuchsian groups

Definition 1.1.4. A hyperbolic structure on S is a complete Riemannian metric of constant Gauss curvature -1.

Let $\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$ be the upper half-plane. The uniformization theorem of Klein, Koebe and Poincaré states that every Riemann surface S of negative Euler characteristic is biholomorphic to a certain quotient of the upper half-plane $\mathbb{H}$ by a well chosen subgroup of its automorphisms. This theorem gives the equivalence between conformal and hyperbolic structures.

Fuchsian groups. The group of automorphisms of $\mathbb{H}$ is the group $\mathrm{PSL}(2, \mathbb{R})$. There are three kinds of elements in $\mathrm{PSL}(2, \mathbb{R})$:

1. elliptic elements : they have exactly one fixed point in $\mathbb{H}$;

2. parabolic elements : they have exactly one fixed point on $\partial\mathbb{H}$, the boundary of $\mathbb{H}$;

3. hyperbolic elements : they have exactly two fixed points on $\partial\mathbb{H}$.

In what follows we will not deal with elliptic elements.

Definition 1.1.5. A Fuchsian group Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ having no elliptic elements. It is called *geometrically finite* if there exists a convex fundamental region for Γ with finitely many sides.

In the next section we will look at homomorphisms from the fundamental group of a surface of finite type into $\mathrm{PSL}(2,\mathbb{R})$ such that its image is a Fuchsian group. The so obtained Fuchsian group will be finitely generated. We have the following theorem about finitely generated Fuchsian groups.

Theorem 1.1.6. *If Γ is a finitely generated Fuchsian group, it is geometrically finite.*

For a proof of this statement we refer to [22].

The Teichmüller-Fricke space. We consider a Riemann surface S of genus g with two sets of marked points. The first is the set of *holes* and the second the set of *punctures*. Topologically a neighborhood of such a marked point is an annulus. As a complex surface a neighborhood of a hole is isomorphic to an annulus and a neighborhood of a puncture is isomorphic to a punctured disk. *A surface of type* (g, k, m) is a Riemann surface of genus g with k holes and m punctures. We denote by $T_{g,k,m}$ the set of monomorphisms $\psi : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ such that the quotient $\mathbb{H}/\mathrm{Im}\psi$ is a surface of type (g, k, m). The group of automorphisms of $\mathbb{H}$, $\mathrm{PSL}(2,\mathbb{R})$ acts on $T_{g,k,m}$ by conjugation.

Definition 1.1.7. The space $\mathfrak{T}_{g,k,m}$ defined as the quotient by this action $T_{g,k,m}/\mathrm{PSL}(2,\mathbb{R})$ is called the Teichmüller space based on S.

Theorem 1.1.8. *The Teichmüller space $\mathfrak{T}_{g,k,m}$ is homeomorphic to $\mathbb{R}^{6g-6+3k+2m}$.*

For a proof of the theorem in the case of closed surfaces we refer to [20, Chapter 2]. A constructive proof of this is given by Natanzon in [27, p. 28, Theorem 4.1].

1.2 Two extensions to open ciliated surfaces

The definition of the Teichmüller space of surfaces was extended by Fock [13] to surfaces with boundary components with some marked points on it. We recall here some of these results which are also exposed in [14].

1.2.1 Ciliated surfaces

Definition 1.2.1.
- A ciliated surface is a compact oriented surface with boundary and with a finite set of marked points on the boundary called *cilia*.
- A boundary component without cilia is either a *hole* or a *puncture*.
- A triangulation Λ of a ciliated surface is a decomposition of the surface with contracted holes into triangles such that every vertex is either a cilium or a contracted hole.

We will make the distinction between the edges of the triangulation belonging to the boundary and the others. The edges of the first kind will be said to be external and the edges of the second kind internal. We denote by $T(\Lambda), E(\Lambda), E_0(\Lambda), V(\Lambda)$ the sets of triangles, edges, external edges and vertices of the triangulation, respectively. The topology of a ciliated surface is determined by its genus g and a finite collection of integers

$P = (p_1, \ldots, p_s)$, where s is the number of boundary components and p_i is the number of cilia on the $i-$th component. We denote by h the number of holes, by c the number of cilia an by n the number of internal edges. Topology gives us the following relations :

1. $\sharp V(\Lambda) = h + c$,
2. $\sharp E_0(\Lambda) = c$,
3. $\sharp E(\Lambda) = 6g - 6 + 3s + 2c$,
4. $n = 6g - 6 + 3s + c$,
5. $\sharp T(\Lambda) = 4g - 4 + 2s + c$.

The topology of the triangulation Λ encodes a skew-symmetric matrix of size $\sharp E(\Lambda)$, $\varepsilon^{\alpha\beta}$, where $\alpha, \beta \in E(\Lambda)$. Consider $\alpha, \beta \in E(\Lambda)$ and $i \in T(\Lambda)$ and define $\langle \alpha, i, \beta \rangle$ equals 1 (resp. -1) if α and β are sides of the triangle i and α is in the clockwise (resp. counterclockwise) direction from β with respect to their common vertex. Otherwise it equals zero. The matrix $\varepsilon^{\alpha\beta}$ is then given by

$$\varepsilon^{\alpha\beta} = \sum_{i \in T(\Lambda)} \langle \alpha, i, \beta \rangle \,. \tag{1.1}$$

Remark 1.2.2. The entries of the matrix $\varepsilon^{\alpha\beta}$ have values in $\{0, \pm 1 \pm 2\}$.

Example 1.2.3. We consider the surface given by $g = 1, P = (0)$. We have the triangulation of the Figure 1.1. If we consider the ordered set $\{\alpha, \beta, \gamma\}$, corresponding to the edges of the triangulation we have for the matrix

$$\varepsilon = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix} .$$

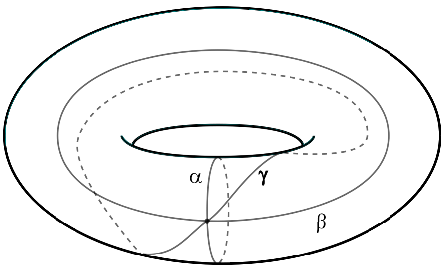


Figure 1.1: A triangulation of the torus with one contracted hole

An important result due independently to Strebel, Harer [17] and Penner [28] states that any ideal triangulation can be obtained from another one by a sequence of moves called *flips* or *Whitehead moves* (cf. Figure 1.2) and that there are three kinds of relations:

1. The square of a flip is the identity.
2. Flips in disjoint edges commute.
3. Five consecutive flips in edges having one common vertex is the identity (Pentagon relation cf. Figure 1.3).

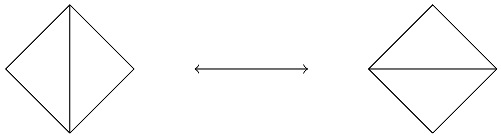


Figure 1.2: A flip.

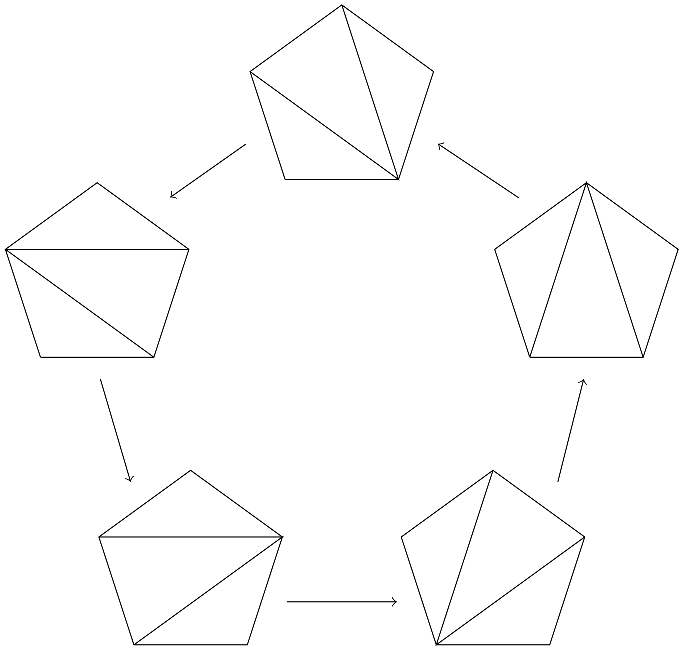


Figure 1.3: Pentagon relation

1.2.2 Teichmüller spaces of surfaces with holes.

The Teichmüller $X-$space

In this section we describe a first extension of the Teichmüller space to the case of open surfaces. We focus on surfaces of genus g with s boundary components without cilia.

Definition 1.2.4. The Teichmüller $X-$space of a surface with holes S is the space of complex structures on S with an orientation of all holes up to the diffeomorphisms homotopic to the identity. We denote it by $\mathcal{T}^X(S)$.

Remark 1.2.5. The Teichmüller $X-$space is a 2^k-cover of the Teichmüller space, where k denotes the number of holes.

If the surface has several boundary components $\partial_1, \ldots, \partial_l$, $l \geq 1$, with a non-empty set of cilia C_i on ∂_i, the definition is more or less the same but special attention has to be payed to the treatment of these marked points (cf [14]).

We now recall the definition of the Thurston-Bonahon coordinates given by the assignment of a positive real number to each internal edge of a triangulation Λ of the surface S. This collection of numbers gives a parametrization of $\mathcal{T}^X(S)$.

First of all we explain how one can lift the ideal triangulation of S to the upper half plane. We here give the construction introduced by Fock in [13]. We first consider the case where all boundary components are holes. Draw a geodesic around each hole and cut out the arising half cylinders : considering an hyperbolic element $\gamma \in \mathrm{PSL}(2,\mathbb{R})$, the quotient $\mathbb{H}/ < \gamma >$ is a hyperbolic cylinder and the axis of γ corresponds to a single closed geodesic on the cylinder. We get a surface with geodesic boundary. Cut the surface by the edges of graph of Λ into hexagons. Then take an edge and two hexagons sharing this edge and lift the resulting octagon to the upper half plane $\mathbb{H}$. The octagon has four geodesic sides corresponding to the holes. Continue this geodesics to the real axis. The orientations of the holes now induce orientations of the geodesics. Using these orientations we choose one of the two infinities of each geodesic (which correspond to the dot in Figure 1.4). These points of $\mathbb{R}P^1$ will be the vertices of the lift of our triangulation. If we have punctures instead of some holes, some edges of the octagon shrink to a point and no orientation is necessary.

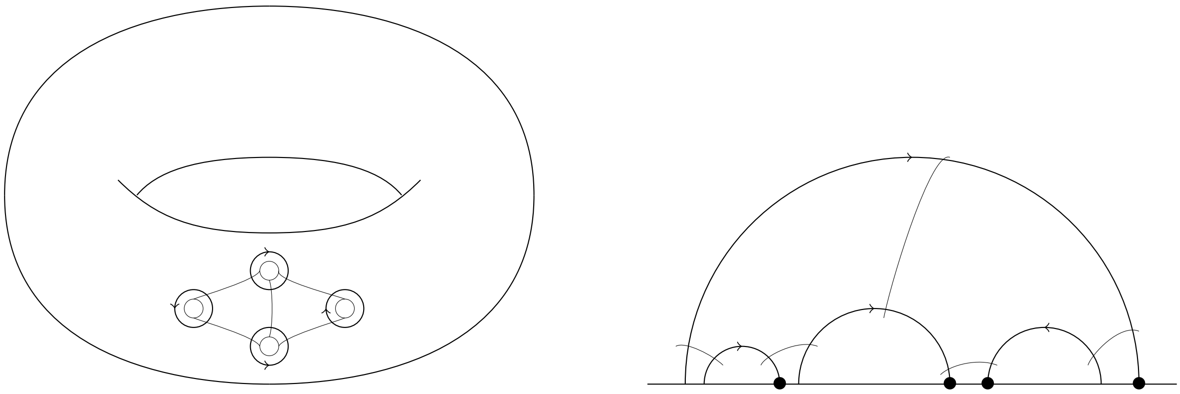


Figure 1.4: Lifting of the triangulation

Construction of coordinates After lifting Λ to the upper half plane, consider an edge α together with two adjacent ideal triangles forming a quadrilateral. The cross-ratio x, of the four vertices of this quadrilateral is invariant under the action of $\mathrm{PSL}(2,\mathbb{R})$. It is convenient to suppose that the coordinates of the ends of the edge α are 0 and ∞ and that the coordinate at the third vertex is -1 (cf Figure 1.5). Then the value of the fourth coordinate will be x.

Theorem 1.2.6 (Fock [13])**.** *The collection of positive numbers $\{x_\alpha\}_{\alpha\in E\setminus E_0(\Lambda)}$ gives a global parametrization of $\mathcal{T}^X(S)$ and the orientation of a hole γ is given by the sign of $\prod x_\alpha - 1$, where the product is taken over all the edges α incident to γ.*

To prove the theorem, Fock reconstructs a discrete monodromy group starting from an ideal triangulation of a ciliated surface S with real positive numbers on the internal edges.

Properties of the coordinates If we change the triangulation by a flip in the edge α, the coordinates change following the rule given by :

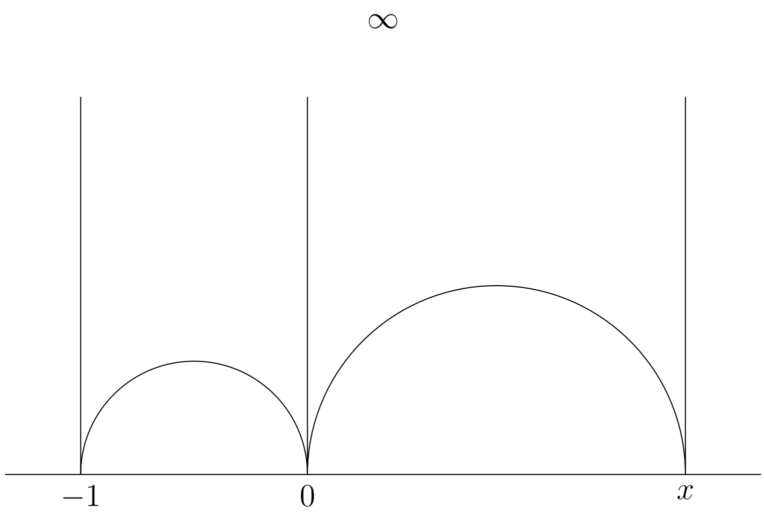


Figure 1.5: The cross-ratio

$$x'_{\beta'} = \begin{cases} x_\alpha^{-1} & \text{if } \beta = \alpha \\ x_\beta (1 + x_\alpha)^{\varepsilon^{\alpha\beta}} & \text{if } \varepsilon^{\alpha\beta} \geq 0 \\ x_\beta \left(1 + (x_\alpha)^{-1}\right)^{\varepsilon^{\alpha\beta}} & \text{if } \varepsilon^{\alpha\beta} \leq 0 \end{cases} \tag{1.2}$$

If all the edges of the quadrilateral concerned by the flip are different, the change of coordinate can be summarized in Figure 1.6.

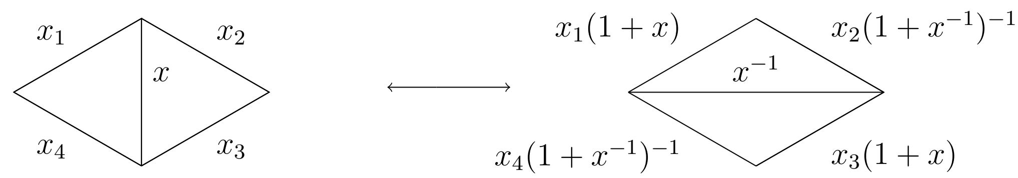


Figure 1.6

CHAPTER 2

Supermanifolds

The concept of supermanifolds is a generalization of the concept of classical manifolds including a notion of anticommuting coordinates. There exist different inequivalent ways to define supermanifolds. Roughly, in the *DeWitt approach*, a supermanifold is a manifold which is locally modeled on a superspace denoted by $\mathbb{R}^{m|n}$ defined in Section 2.1.2. In the *algebro-geometric approach*, one extends the sheaf of functions on a manifold and not the manifold itself. To study supermanifolds, one replaces the real or complex variables with elements of a super commutative algebra. In this chapter we first recall all the notions of superalgebra needed to define supermanifolds, and then we recall the approaches to supermanifolds. All the ideas developed here can be found in [31, 10, 27].

2.1 Super algebras

2.1.1 Super vector spaces and super commutative algebras

The first concept introduced in the theory of super algebras is that of *super vector space*, which is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. The elements of V_0 are said to be *even* and those of V_1 *odd*. The *parity* of an homogeneous element $v \in V_i$ is defined to be $|v| = i$.

Definition 2.1.1. 1. Let A be an algebra over $\mathbb{R}$ or $\mathbb{C}$. Then A is said to be a *super algebra* if A is a super vector space $A = A_0 \oplus A_1$ and the multiplication satisfies $A_i A_j \subset A_{i+j}$, where i and j are taken modulo 2.

2. A super algebra A is said to be *super commutative* if, for all homogeneous elements a and b of A, $ab = (-1)^{|a||b|} ba$. In particular the square of an odd element is 0.

For us, the most important examples of super commutative algebras are those of Grassmann algebras used in the definition of (concrete) supermanifolds.

Definition 2.1.2. 1. For each positive integer L, let $G_L(\mathbb{R})$ be the *real Grassmann algebra* over L generators that is

$$G_L(\mathbb{R}) = \langle 1, \alpha_1, \dots, \alpha_L | \forall i, j, 1 \leq i, j \leq L, \alpha_i = 1.\alpha_i = \alpha_i.1, \alpha_i\alpha_j = -\alpha_j\alpha_i \rangle .$$

2. The real Grassmann algebra with infinitely many generators $G(\mathbb{R})$ is defined in the same way :

$$G(\mathbb{R}) = \langle 1, \alpha_1, \alpha_2, \ldots | \forall i, j, \alpha_i = 1\alpha_i = \alpha_i 1, \alpha_i\alpha_j = -\alpha_j\alpha_i \rangle \,.$$

For $p \in \mathbb{N}^* \cup \{\infty\}$, let I_p denote the set of all multi indices $\underline{\lambda} = \lambda_1 \ldots \lambda_k$ with $1 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq p$ and including the empty index $\emptyset$. Set $I_{p,i}, i = 0, 1$ the set of multi indices in I_p which contain a number of indices of parity i.

For $L \in \mathbb{N}^* \cup \{\infty\}$, the super commutative algebra $G_L(\mathbb{R})$ splits in

$$G_L(\mathbb{R}) = G_{L,0}(\mathbb{R}) \oplus G_{L,1}(\mathbb{R}),$$

where, as a vector space,

$$G_{L,k}(\mathbb{R}) = \langle \alpha_{i_1} \ldots \alpha_{i_n} | \underline{i} = i_1 \ldots i_n \in M_{L,k} \rangle$$

Remark 2.1.3.

1. An element $a \in G(\mathbb{K})$ is invertible if and only if $a^\sharp \neq 0$.

2. If an element $a \in G(\mathbb{R})$ is such that $a^\sharp > 0$ then it admits a unique square root $\sqrt{a}$ determined by $(\sqrt{a})^2 = a$ and $(\sqrt{a})^\sharp > 0$.

2.1.2 Modules of super algebras and superspaces

In this section, the notion of super module over a super algebra is introduced and the most important example of superspace for the construction of supermanifolds is given. We first recall the definition of homomorphism of super algebras and then conclude by the matrix representation of homomorphisms of free super modules.

Definition 2.1.4. 1. Let V and W be two super vector spaces. If f is a linear map of V to W, then f is said to be a *super vector space homomorphism.*

2. A super vector space homomorphism f is said to be *even* (resp. *odd*) if for all $v \in V$, $|f(v)| = |v| \mod 2$ (resp. $|f(v)| = |v| + 1$) and the parity of f is denoted by $|f|$.

3. Let A and B be super algebras over $\mathbb{R}$ or $\mathbb{C}$. Let $f : A \to B$ be a super vector space homomorphism of a given parity, then f is said to be a *super algebra homomorphism* if

$$\forall a_1, a_2 \in A, f(a_1 a_2) = (-1)^{|f||a_1|} f(a_1) f(a_2).$$

The definition of super modules is analogous to the definition of a classical module, but a compatibility of the parities is required.

Definition 2.1.5. 1. Let $V = V_0 \oplus V_1$ be a super vector space and A a super commutative algebra. Then V is said to be a *left super A-module* if there exists a map

$$\begin{aligned} A \times V &\to V \\ (a, v) &\mapsto av \end{aligned}$$

such that

$$\forall (a_1, a_2) \in A^2, \forall v \in V, \left\{ \begin{array}{rcl} |a_1 v| & = & |a_1| + |v| \\ a_1(a_2 v) & = & (a_1 a_2) v \end{array} \right. .$$

2. Assume that there exist n elements $B_1, \ldots, B_n$ of V_0 and m elements $B_{n+1}, \ldots, B_{n+m}$ of V_1 such that each element $v \in V$ can be decomposed uniquely in

$$v = \sum_{i=1}^{n+m} a_i B_i,$$

for $a_i \in A$. In this case V is said to be a *free super A-module of dimension* (n, m), and the set $\{B_1, \ldots, B_{n+m}\}$ is said to be a (n, m) *super basis.*

Remark 2.1.6. If v is even then it may be expressed as $(a_1, \ldots, a_{n+m})$ which is an element of *the superspace* corresponding to A, $A^{n|m}$, defined as

$$A^{n|m} = (A_0)^n \times (A_1)^m .$$

The superspace which plays a particular role in the concrete construction of supermanifolds is the $(n|m)$-dimensional superspace corresponding to $G_L(\mathbb{R})$, where $L \in \mathbb{N}^* \cup \{\infty\}$, denoted by

$$\mathbb{R}_L^{n|m} = (G_{L,0}(\mathbb{R}))^n \times (G_{L,1}(\mathbb{R}))^m .$$

An element of $\mathbb{R}_L^{n|m}$ will be denoted by $(x|\zeta) = (x_1, \ldots, x_n|\zeta_1, \ldots, \zeta_m)$.

Definition 2.1.7. Let A be a super commutative algebra and let V and W be two super A-modules. Then a map $f : V \to W$ is said to be a *homomorphism of super A-modules* if f is a super vector space homomorphism and

$$\forall a \in A, \forall v \in V, f(av) = (-1)^{|a||f|} a f(v).$$

Even homomorphisms of super A-modules (in terms of particular bases) can be represented by super matrices.

Definition 2.1.8. A $(p, q) \times (r, s)$ *super matrix* over a super commutative algebra A is a $(p+q) \times (r+s)$ matrix M whose entries are elements of A, and which can be represented by blocks

$$M = \begin{array}{c} \begin{array}{cc} r \text{ columns} & s \text{ columns} \end{array} \\ \left(\begin{array}{cc} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{array} \right) \begin{array}{l} p \text{ lines} \\ q \text{ lines} \end{array} \end{array}$$

where the entries of $M_{i,j}$ are elements of A_{i+j}.

The sum and the product of super matrices are defined in the same way as in the classical case, but with the requirement that the resulting matrices always are super matrices.

2.2 Supersmooth functions on $\mathbb{R}^{n|m}$

In this section we are interested in the particular case where the super commutative algebra A is a Grassmann algebra. We focus more specifically to $G(\mathbb{R})$ and its corresponding superspace $\mathbb{R}^{n|m}$. The most important topology on $\mathbb{R}^{n|m}$ (L is infinite) is the DeWitt topology and it will now be defined. The super space $\mathbb{R}^{n|m}$ plays the role of $\mathbb{R}^n$ in the definition of supermanifolds, we then need functions which play the role of $\mathcal{C}^\infty$ functions.

2.2.1 The DeWitt topology

Let x be an element of $G_L(\mathbb{R})$ for $L \in \mathbb{N}^* \cup \{\infty\}$. There is a unique algebra homomorphism $\sharp : G_L(\mathbb{R}) \to \mathbb{R}$ which sends 1 to 1 and for all i, α_i to 0; the image $\sharp(x) = x^\sharp$ is called the *reduction* of x. The map $\sharp$ can be extended to $\mathbb{R}_L^{n|m}$ by

$$\begin{aligned} \sharp^{n|m} : \mathbb{R}_L^{n|m} &\longrightarrow \mathbb{R}^n \\ (x_1, \ldots, x_n|\zeta_1, \ldots, \zeta_m) &\longmapsto (x_1^\sharp, \ldots, x_n^\sharp). \end{aligned}$$

The image $(x_1^\sharp, \ldots, x_n^\sharp)$ of an element of $\mathbb{R}_L^{n|m}$ will be called its *reduction*.

Definition 2.2.1. A subset $U \subset \mathbb{R}^{n|m}$ is said to be open in the *DeWitt topology* if and only if there exists an open subset V of $\mathbb{R}^n$ such that $U = \left(\sharp^{n|m}\right)^{-1}(V)$.

Remark 2.2.2. The DeWitt topology is not Hausdorff.

2.2.2 Supersmooth functions on $\mathbb{R}^{n|m}$

The Grassmann analytic continuation is the key in the definition of supersmooth functions. For each positive integer L let $p_L : G(\mathbb{R}) \to G_L(\mathbb{R})$ be the projection which sends the generators α_i of $G(\mathbb{R})$ to 0 for $i > L$.

Definition 2.2.3. Let $V \subset \mathbb{R}^n$ be open and let $f : V \to G(\mathbb{R})$.

1. The function f is said to be of class $\mathcal{C}^\infty$ if for each positive integer L the function $p_L \circ f : V \to G_L(\mathbb{R})$ is $\mathcal{C}^\infty$. The set of these functions is denoted by $\mathcal{C}^\infty(V, G(\mathbb{R}))$.

2. If f belongs to $\mathcal{C}^\infty(V, G(\mathbb{R}))$ then define the function $\widehat{f} : (\sharp^{n|0})^{-1}(V) \to G(\mathbb{R})$ by

$$\widehat{f}(x) = \sum_{i_1=0,\ldots,i_n=0}^{\infty} \frac{1}{i_1! \ldots i_n!} \partial_1^{i_1} \ldots \partial_n^{i_n} f(\sharp^{n|m}(x)) s(x_1)^{i_1} \ldots s(x_n)^{i_n}.$$

Definition 2.2.4. Let $U \subset \mathbb{R}^{n|m}$ be open. The function $f : U \to G(\mathbb{R})$ is said to be of class G^∞ if and only if there exists a collection $\{f_{\underline{\lambda}} | \underline{\lambda} \in I_m, f_{\underline{\lambda}} \in \mathcal{C}^\infty(\sharp^{n|m}(U), G(\mathbb{R}))\}$ such that

$$\forall (x, \zeta) \in U, f(x|\zeta) = \sum_{\underline{\lambda} \in I_m} \widehat{f_{\underline{\lambda}}}(x) \zeta_{\underline{\lambda}}.$$

The so obtained function on U is called the *Grassmann analytic continuation* of f.

A more restricted class of supersmooth functions which makes the link between the two approaches to supermanifolds can be defined.

Definition 2.2.5. Let $U \subset \mathbb{R}^{n|m}$ be open. The function $f : U \to G(\mathbb{R})$ is said to be of class H^∞ if and only if there exists a collection $\{f_{\underline{\lambda}} | \underline{\lambda} \in I_m, f_{\underline{\lambda}} \in \mathcal{C}^\infty(\sharp^{n|m}(U), \mathbb{R})\}$ such that

$$\forall (x, \zeta) \in U, f(x|\zeta) = \sum_{\underline{\lambda} \in I_m} \widehat{f_{\underline{\lambda}}}(x) \zeta_{\underline{\lambda}}.$$

Remark 2.2.6. 1. The difference between the two definitions lies in the fact that in the case of G^∞-functions, the functions $f_{\underline{\lambda}}$ take their values in $G(\mathbb{R})$ and in the case of H^∞-functions, it is in $\mathbb{R}$.

2. Any H^∞-function is also a G^∞-function, but the converse is not true.

2.3 The definition of DeWitt supermanifolds

The construction of DeWitt supermanifolds is analogous to the construction of manifolds. The DeWitt supermanifolds are modeled on the superspace $\mathbb{R}^{n|m}$ and the transition functions will be supersmooth functions of a given class K, where $K = G^\infty$ or H^∞.

Definition 2.3.1. Let M be a set, and n and m two positive integers.

1. An $(n|m) - K$ *chart* on M is a pair (V, φ), where V is a subset of M and φ is a bijective map from V to $U \subset \mathbb{R}^{n|m}$, U open for the DeWitt topology.
2. An $(n|m) - K$ *atlas* on M is a collection of charts $\{(V_j, \varphi_j) | j \in J\}$ such that
   (a) $\bigcup_{j \in J} V_j = M$
   (b) for $i, j \in J$ such that $V_i \cap V_j \neq \emptyset$,the sets $\varphi_i(V_i \cap V_j)$ and $\varphi_j(V_i \cap V_j)$ are open and the map
   $$\varphi_j \circ \varphi_i^{-1} : \varphi_i(V_i \cap V_j) \to \varphi_j(V_i \cap V_j)$$
   is of class K.
3. An $(n|m) - K$ *DeWitt super-premanifold* is a set M together with a maximal $(n|m) - K$ atlas on M.

Two standard examples will now be given.

Example 2.3.2. The superspace $\mathbb{R}^{n|m}$ can be endowed with the structure of G^∞ or H^∞ super-premanifold. Indeed $(\mathbb{R}^{n|m}, \mathrm{Id})$ is a $(n|m)$ chart on $\mathbb{R}^{n|m}$ and $\{(\mathbb{R}^{n|m}, \mathrm{Id})\}$ is an $(n|m)$ atlas on M.

Example 2.3.3. In the same way if V is an open subset of $\mathbb{R}^{(n|m)}$, then $(V, \imath)$ (where $\imath : V \hookrightarrow \mathbb{R}^{(n|m)}$ is the inclusion) is an $(n|m)$ chart on V and $\{(V, \imath)\}$ is an $(n|m)$ atlas on V.

2.3.1 The topology of a DeWitt super-premanifold

The structure of super-premanifold gives a natural way to define a topology on M. Indeed, if M is a $(n|m)-K$ DeWitt super-premanifold together with a maximal atlas $\{(V_j,\varphi_j)|j\in J\}$, let τ_{DeWitt} be the collection of subsets $U\subset M$ such that $\forall j\in J, \varphi_j(U\cap V_j)$ is open in $\mathbb{R}^{n|m}$. Then τ_{DeWitt} is a (non-Hausdorff) topology on M. This construction is analogous to the construction in the case of classical manifolds.

2.3.2 The body of a DeWitt super-premanifold

A DeWitt super-premanifold has a naturally underlying classical topological space of dimension n. We recall the construction due to DeWitt [10] and Batchelor [2] of this space in the following theorem.

Theorem 2.3.4. *Let M be a $(n|m)-K$ DeWitt super-premanifold (with the DeWitt topology). Let $\{(V_j,\varphi_j)|j\in J\}$. Then the following holds.*

1. *the relation $\sim$ generated on M by*

$$(p\sim q)\Leftrightarrow \big(\exists j\in J|\, p,q\in V_j \quad \textit{and} \quad \sharp^{n|m}(\varphi_j(p))=\sharp^{n|m}(\varphi_j(q))\big)$$

   is an equivalence relation.

2. *The space $M^\sharp = M/\sim$ is locally diffeomorphic to $\mathbb{R}^n$, with atlas $\{(V_j^\sharp,\varphi_j^\sharp)|j\in J\}$,where*

$$\begin{aligned} V_j^\sharp &= \{[p], p\in V_j\}\\ \varphi_j^\sharp : V_j^\sharp &\rightarrow \mathbb{R}^n\\ [p] &\mapsto \sharp^{n|m}\circ\varphi_j(p). \end{aligned}$$

Definition 2.3.5. The space $M/\sim$ is called *the body* of M and denoted $M^\sharp$. The canonical projection of M to $M^\sharp$ is denoted by $\sharp$.

Definition 2.3.6. A *DeWitt supermanifold* is a super-premanifold M whose body $M^\sharp$ is a classical manifold.

In what follows we only consider supermanifolds.

2.3.3 The algebro-geometric approach to supermanifolds

In the algebro-geometric approach it is not the manifold which is extended but a sheaf of functions. Here we just recall the definition of supermanifold which was given by Leĭtes [23], and conclude by recalling the link between algebro-geometric and H^∞-DeWitt supermanifolds. We will say no more about this approach because, in what follows, we are only interested in the concrete one. Our results are based on the theory developed in Natanzon's book [27] which uses the DeWitt approach to supermanifolds.

Definition 2.3.7. A *smooth real algebro-geometric supermanifold* of dimension $(n|m)$ is a pair (M,A) where M is a real $n-$dimensional manifold and A is a sheaf of super commutative algebras over M such that

1. there exists an open cover of M, $\{(U_j, \varphi_j) | j \in J\}$ where

$$\forall j \in J, A(U_j) \cong C^\infty(U_j) \otimes \Lambda(\mathbb{R}^m),$$

2. if $\mathcal{N}$ is the sheaf of nilpotents in A, then $(M, A/\mathcal{N})$ is isomorphic to $(M, \mathcal{C}^\infty)$.

The link between the two approaches

Rogers [31] showed the existence of a unique algebro-geometric supermanifold corresponding to a given H^∞ DeWitt supermanifold.

Theorem 2.3.8. *Let M be an H^∞ DeWitt supermanifold of dimension $(n|m)$, and let A be the sheaf of super algebras on $M^\sharp$ given by $A(V) = H^\infty(\sharp^{-1}(V))$. Then $\big(M^\sharp, A\big)$ is an algebro-geometric supermanifold of dimension $(n|m)$.*

Conversely, starting with an algebro-geometric super manifold (X, A), one can construct a DeWitt supermanifold $M(X, A)$ such that $M(X, A)^\sharp$ is X and the algebro-geometric supermanifold corresponding to the sheaf of H^∞ functions on $M(X, A)$ is isomorphic to (X, A). Batchelor [3] shows that the so constructed correspondence between the two approaches is bijective.

2.4 Further examples

2.4.1 Real super projective spaces

Let $(\mathbb{R}^{n+1|m})^* \subset \mathbb{R}^{n+1|m}$ be the set $(\sharp^{n+1|m})^{-1}(\mathbb{R}^{n+1} - \{0\})$. Two elements $(x|\zeta)$ and $(x'|\zeta')$ of $(\mathbb{R}^{n+1|m})^*$ are said to be equivalent if there exists an invertible element $\ell \in G_0(\mathbb{R})$ such that

$$\begin{aligned} x_i &= \ell x_i' \quad i = 1, \dots, n+1 \\ \zeta_j &= \ell \zeta_j' \quad j = 1, \dots, m. \end{aligned}$$

Let $\sim$ denote the equivalence relation defined above and let $[(x, \zeta)]$ denote the class of (x, ζ). Then $\mathbb{P}^{n|m} = (\mathbb{R}^{n+1|m})^* / \sim$ can be endowed with a structure of DeWitt supermanifold by defining the following atlas. For all $i = 1, \dots, n+1$ let

$$V_i = \Big\{ [(x|\zeta)], (x|\zeta) \in (\mathbb{R}^{n+1|m})^*, x_i^\sharp \neq 0 \Big\}$$

$$\begin{aligned} &\varphi_i : V_i \to \mathbb{R}^{(n|m)} \\ &[(x|\zeta)] \mapsto \left(\frac{x_1}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_{n+1}}{x_i} \,\middle|\, \frac{\zeta_1}{x_i}, \dots, \frac{\zeta_m}{x_i} \right). \end{aligned}$$

2.4.2 The super upper half-plane

Before developing this example we just stress that all the definitions we recall about real Grassmann algebras and real superspaces can be given in an analogous way in the complex case. A complex DeWitt supermanifold will be modeled on $\mathbb{C}^{n|m}$ and the transition functions will be superholomorphic.

Definition 2.4.1. Let $U \subset \mathbb{C}^{n|m}$ be open. The function $f : U \to G(\mathbb{C})$ is said to be *superholomorphic* if and only if there exists a collection $\{f_{\underline{\lambda}} | \underline{\lambda} \in M_m\}$ of functions taking their values in $G(\mathbb{C})$ and being holomorphic on $\sharp^{n|m}(U)$ such that

$$\forall (z, \zeta) \in U, f(z|\zeta) = \sum_{\underline{\lambda} \in M_m} \hat{f}_{\underline{\lambda}}(z)\zeta_{\underline{\lambda}}.$$

The so obtained function on U is called the *Grassmann analytic continuation* of f.

The space $\mathbb{H}^S$ will play the role of the upper-half plane $\mathbb{H}$. It is defined by

$$\mathbb{H}^S = \{(z|\zeta) \in \mathbb{C}^{(1|1)}; \Im(z^\sharp) > 0\},$$

where $\Im(z^\sharp)$ denotes the imaginary part of $z^\sharp$. As in the classical case, there exists a notion of boundary of $\mathbb{H}^S$ which can be seen as the union of $\mathbb{R}^{1|1}$ and of points at infinity.

Definition 2.4.2. The boundary of $\mathbb{H}^S$ is defined to be $\mathbb{P}^{1|1}$

As in the classical case, $\mathbb{P}^{1|1}$ admits a covering with two charts. We will often use the following notation: given $Z \in \mathbb{P}^{1|1}$, choose a representative of Z in $\mathbb{R}^{2|1^*}$, say $\hat{Z} = \begin{pmatrix} z_1 \\ z_2 \\ \eta \end{pmatrix}$ and set $z = \frac{z_1}{z_2}$, $\varphi = \frac{\eta}{z_2}$. We then write Z as $\begin{pmatrix} z \\ \varphi \end{pmatrix}$.

Notation. If a representative $\begin{pmatrix} z_1 \\ z_2 \\ \eta \end{pmatrix}$ of an element $Z \in \mathbb{P}^{1|1}$ is such that z_2 is not invertible, then Z lies in the second chart of $\mathbb{P}^{1|1}$. In the special case where the representative has the form $\begin{pmatrix} x \\ 0 \\ \zeta \end{pmatrix}$, Z will be denoted by $\begin{pmatrix} \infty \\ \frac{\zeta}{x} \end{pmatrix}$.

2.4.3 The group $\mathrm{SpO}(2|1)(\mathbb{R})$

Consider first a matrix with coefficient in $G(\mathbb{K})$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, represented by blocks, where A and D are respectively of size 2×2 and 1×1 and have even entries, and, B and C have odd ones. Such a matrix is said to be *even*.

Definition 2.4.3. *The supertranspose* (cf. [25]) of a matrix is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{st}} = \begin{pmatrix} A^{\mathrm{t}} & C^{\mathrm{t}} \\ -B^{\mathrm{t}} & D^{\mathrm{t}} \end{pmatrix}.$$

Definition 2.4.4. *The superdeterminant* or *Berezinian* (cf. [25]) of M is defined if and only if M is square, A and D are invertible and we have

$$\operatorname{Ber} M = \det(A - BD^{-1}C)\det(D)^{-1}.$$

Definition and properties

Definition 2.4.5. The group SpO(2|1) is the group of all square even matrices B, with $\operatorname{Ber}(B) = 1$, which satisfy the relation

$$B^{\mathrm{st}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.1}$$

An element B in SpO(2|1) is a matrix of the form $B = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix}$, where

$$\begin{cases} a, b, c, d, e \in G_0(\mathbb{R}), \\ \alpha, \beta, \gamma, \delta \in G_1(\mathbb{R}), \\ ad - bc - \alpha\beta = e^2 + 2\gamma\delta = 1, \\ a\delta - c\gamma - e\alpha = b\delta - d\gamma - e\beta = 0, \\ \operatorname{Ber}(B) = 1. \end{cases} \tag{2.2}$$

These relations are equivalent to (2.1). The group SpO(2|1) is the group of automorphism of $\mathbb{H}^S$ (cf. [5]).

One defines an epimorphism $\sharp$ from SpO(2|1) to PSL$(2,\mathbb{R})$ sending $B = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix}$ to $\sharp(B) = z \mapsto B^\sharp(z) = \frac{a^\sharp z + b^\sharp}{c^\sharp z + d^\sharp}$.

Definition 2.4.6. The homography $\sharp(B)$ is called *the reduction* of B.

The action of $B = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix}$ on $\mathbb{H}^S$ and on $\mathbb{P}^{1|1}$ is given by $B \cdot Z = Z'$, where :

$$z' = \frac{az + b + \gamma\varphi}{cz + d + \delta\varphi} \quad , \quad \varphi' = \frac{\alpha z + \beta + e\varphi}{cz + d + \delta\varphi}. \tag{2.3}$$

Lemma 2.4.7. *1. For any triple* $\begin{pmatrix} z_1 \\ \theta_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ \theta_2 \end{pmatrix}, \begin{pmatrix} z_3 \\ \theta_3 \end{pmatrix}$ *in* $\mathbb{P}^{1|1}$ *with distinct bodies, there exists an element* $B \in PSL(2,\mathbb{R})$ *such that*

$$B \cdot \begin{pmatrix} z_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} \infty \\ 0 \end{pmatrix}, \quad B \cdot \begin{pmatrix} z_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} -1 \\ \theta \end{pmatrix}, \quad B \cdot \begin{pmatrix} z_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

2. *The only automorphism (which is not the identity) preserving* $\begin{pmatrix} \infty \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *and sending the points of the form* $\begin{pmatrix} -1 \\ \theta \end{pmatrix}$ *to the points of the same form, i.e.* $\begin{pmatrix} -1 \\ \zeta \end{pmatrix}$, *is the map represented by the matrix* $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Proof. If $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \infty \\ 0 \end{pmatrix}$ are fixed points of B, with the notations introduced above for the coefficients of B, we get $b = c = 0$ and $\alpha = \beta = 0$, then it follows from the equations defining B that $\gamma = \delta = 0, e^2 = ad$. The equation $\mathrm{Ber}(B) = 1$ implies $e = 1$. Then putting $a = d^{-1} = m$ we get

$$z' = m^2 z \quad \zeta' = m\zeta.$$

Choosing $m^2 = z^{-1}$ we get the proof of the second part of the lemma.

For the first part, just write the equations corresponding to $B \cdot \begin{pmatrix} z_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} \infty \\ 0 \end{pmatrix}$ and $B \cdot \begin{pmatrix} z_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and consider a as a free parameter: it can be checked that all the entries of B are uniquely determined. □

Remark 2.4.8. Observe that $\mathrm{SL}_2(\mathbb{R})$ can be embedded in $\mathrm{SpO}(2|1)$ in a canonical way:

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R}) &\longrightarrow \mathrm{SpO}(2|1) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The super-Lie group SpO(2|1)

More structure can be given to SpO(2|1). Indeed, the space of super matrices has a natural structure of DeWitt supermanifold of dimension given by the number of even and odd entries. The equations (2.2) defining an element of SpO(2|1) are polynomial in the entries of the matrix. Moreover the two group operations

$$\begin{aligned} \mathrm{SpO}(2|1) \times \mathrm{SpO}(2|1) &\to \mathrm{SpO}(2|1) \\ (A, B) &\mapsto AB \end{aligned} \qquad \begin{aligned} \mathrm{SpO}(2|1) &\to \mathrm{SpO}(2|1) \\ A &\mapsto A^{-1} \end{aligned}$$

are polynomial in the entries of the matrices. The group SpO(2|1) inherits in this way the structure of super-Lie group (for a more general definition of super-Lie groups we refer to [31, Chapter 9]). The entries of an element of SpO(2|1) are completely determined by the choice of three free even parameters and two free odd parameters, so the group SpO(2|1) is a DeWitt super-Lie group of dimension (3|2).

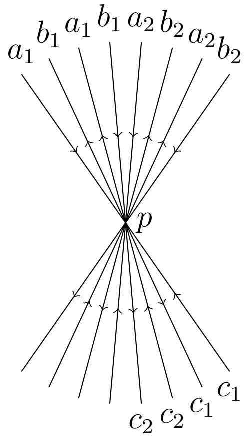


Figure 2.1

2.5 The super-Teichmüller space

The aim of this section is to define the super-Teichmüller space following [27]. For this one first introduces the analogs of Fuchsian groups which will be called super-Fuchsian groups and then using super-Fuchsian models coordinates on the super-Teichmüller space analogous to the Fricke coordinates are recalled.

2.5.1 Canonical system of generators of $\pi_1(S)$ and super-Fuchsian models

Let S be a surface of type (g, k, m). We assume implicitly that we have fixed a based point p for the fundamental group of S.

Definition 2.5.1. We say that a system of generators

$$\{A_i, B_i(i = 1 \dots g), C_i(i = g+1, \dots n)\}$$

of $\pi_1(S)$ is *standard* if it is subject to only one relation given by

$$\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} \prod_{i=g+1}^{g+k+m} C_i = 1$$

and the generators A_i, B_i can be represented by simple closed curves in S and each generator C_i can be represented by a simple closed curve surrounding a single hole or a single puncture in S. Moreover, in a neighborhood of p, the positions of the curves representing the generators are like in Figure 2.1.

Definition 2.5.2. A *super-Fuchsian* model for S is the image Γ of a monomorphism $\rho : \pi_1(S) \to \mathrm{SpO}(2|1)$ such that the reduction $\Gamma^\sharp$ is a Fuchsian model for $\pi_1(S)$.

2.5.2 Definition of the super-Teichmüller space

Consider a standard system of generators

$$\{A_i, B_i(i = 1 \dots g), C_i(i = g + 1, \dots n)\}$$

of $\pi_1(S)$. Under the isomorphism between $\pi_1(S)$ and a super-Fuchsian model Γ, denote by α_i and β_i the elements of Γ associated to A_i and B_i, respectively for $i = 1, \dots, g$. A super-Fuchsian model has now the ambiguity caused by inner automorphisms of $\mathrm{PSL}(2, \mathbb{R})$: for each $B \in \mathrm{SpO}(2|1)$, the group $\Gamma' = B\Gamma B^{-1}$ is still a Fuchsian model of S. Because of Lemma 2.4.7(1) it is possible to impose the following normalization in order to assign uniquely a super-Fuchsian group to S :

1. the element α_1 has $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \infty \\ 0 \end{pmatrix}$ as fixed points,
2. the element of α_2 has a fixed point with even part 1.

Now, as in the classical case, explicit super matrices in $\mathrm{SpO}(2|1)$ can be constructed providing the super-Fuchsian model. This construction is done by Natanzon [27]. The entries of those matrices define coordinates (the analogs of the Fricke coordinates) on the super Teichmüller space which is defined as follows.

Definition 2.5.3. Denote by $ST_{g,k,m}$ the set of monomorphisms $\psi : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ such that $(\psi(\pi_1(S)))^\sharp$ is a Fuchsian group. The group $\mathrm{SpO}(2|1)$ acts, as said before, on $ST_{g,k,m}$. The *super-Teichmüller space* of S is defined as the quotient

$$S\mathfrak{T}_{g,k,m} = ST_{g,k,m}/\,\mathrm{SpO}(2|1).$$

Remark 2.5.4. The topology on the super-Teichmüller space is induced by the topology on the space of super matrices : this topology is DeWitt.

One ambiguity given by Lemma 2.4.7(2) still remains. And we have the following theorem proved by Natanzon [27] and Hodgkin [19].

Theorem 2.5.5.
1. *The super-Teichmüller space $S\mathfrak{T}_{g,k,m}$ is a supermanifold $\widehat{S\mathfrak{T}}_{g,k,m}$ up to the diagonal action of $\mathbb{Z}_2$ on its odd part.*
2. *The space $\widehat{S\mathfrak{T}}_{g,k,m}$ has several connected component $\widehat{S\mathfrak{T}}^{\sigma}_{g,k,m}$ indexed by the spin structures on S.*
3. *For each spin structure σ, $\widehat{S\mathfrak{T}}^{\sigma}_{g,k,m}$ is diffeomorphic to $\mathbb{R}^{6g-6+3k+2m|4g-g+2k+2m-m'_\sigma}$, where m'_σ is the number of Ramond punctures for σ.*

A spin structure can be seen as a lifting of a Fuchsian group to a group of $\mathrm{SL}_2(\mathbb{R})$. Remark 2.4.8 throws light on the second point of Theorem 2.5.5. There are different ways to characterize a spin structure on a surface. The aim of the next section is to recall the characterizations needed for our study.

CHAPTER 3

Spin structures, quadratic forms and Kasteleyn orientations

We first recall the definition of spin structure, then we recall several characterizations of spin structures and we conclude by giving a combinatorial description of them using Kasteleyn orientations.

3.1 Spin structures and quadratic forms

Notation.

- We denote respectively H_1 and H^1 the two groups $H_1(S;\mathbb{Z}_2)$ and $H^1(S;\mathbb{Z}_2)$.
- The unit tangent bundle of S will be denoted by US and $\tilde{H}_1$ and $\tilde{H}^1$ are $H_1(US;\mathbb{Z}_2)$ and $H^1(US;\mathbb{Z}_2)$ respectively.
- For $\alpha \in H^1$ and for $a \in H_1$, the dual pairing is denoted by $\langle \alpha, a \rangle$.
- We write the intersection form on H_1 as a dot product.

Definition 3.1.1 (spin structure)**.** A *spin structure* on S is a principal SO_2-bundle $P \to S$ together with a 2-fold covering map $P \to US$ which restricts to the canonical covering map $SO_2 \to SO_2$ on each fiber.

Definition 3.1.2 (quadratic form)**.** We say that $\omega : H_1 \to \mathbb{Z}_2$ is a *quadratic form* over the intersection index if for all a and b in H_1 we have $\omega(a+b) = \omega(a) + \omega(b) + a \cdot b$. We denote by Ω the set of quadratic forms over the intersection index on H_1.

Proposition 3.1.3. *The following propositions are equivalent.*

1. *A spin structure on S is a class $\sigma \in \tilde{H}^1$ such that $\langle \sigma, z \rangle = 1$, where z is the generator of the homology of the fiber.*

2. *A spin structure on S is a quadratic form $\omega : H_1 \to \mathbb{Z}_2$.*

3. *If S is a hyperbolic surface, a spin structure on S is the choice of a lifting to $SL(2,\mathbb{R})$ of the holonomy of the hyperbolic structure.*

The equivalence between Definiton 3.1.1 and the point (1) is given in [1, p. 55]. This characterization via cohomology classes is also the definition chosen by Milnor [26]. The equivalence between the propositions (1) and (2) has been shown by Johnson [21, Theorem 3A]. He proves this statement in the case of a closed genus g surface, but actually all his results and all the proofs extend to the case of surfaces of type (g,k,m) without modification, the only difference being that the intersection form can be degenerate. A proof of the equivalence between the last two points of Proposition 3.1.3 is given by Natanzon [27, p.39, Theorem 7.2], who associates to each lift $\Gamma^* \subset \mathrm{SL}(2,\mathbb{R})$ of a finitely generated Fuchsian group Γ an explicit quadratic form $\omega_{\Gamma^*} : H_1 \to \mathbb{Z}_2$.

Definition 3.1.4 (Ramond and Neveu-Schwarz points)**.** Let ω be a quadratic function on $H_1(S,\mathbb{Z}_2)$ and let $\tilde{c}$ be a simple closed curve surrounding a puncture p. Let c be the homology class of $\tilde{c}$. We say that p is a Ramond point if $\omega(c)=1$ and a Neveu-Schwartz point if $\omega(c)=0$.

3.2 Kasteleyn orientations, dimer configurations and spin structures

We recall here the notion of Kasteleyn orientation on a graph. Following [7] and [8] we recall how to construct an isomorphism between the set of equivalence classes of Kasteleyn orientation and the set of Spin structures on S using dimer configurations.

Let S be a surface with holes and punctures $p_1, \dots p_{k+m}$. We also assume that S is endowed with an orientation O. Let Λ be an ideal triangulation of S. Starting from S we construct a bordered surface by cutting out for each p_i an open neighborhood D_i. We get one boundary component for each vertex of Λ. We denote by S_b the bordered surface obtained this way.

Definition 3.2.1. 1. A *graph with boundary* is a finite graph $\mathcal{G}$ with a set $\partial\mathcal{G}$ of one valent vertices called *boundary vertices.*

2. A *surface graph with boundary* on S_b is a graph with boundary $\mathcal{G}$ embedded in S_b, such that $\mathcal{G} \cap \partial S_b = \partial\mathcal{G}$ and the complement of $\mathcal{G}\backslash\partial\mathcal{G}$ in $S_b\backslash\partial S_b$ consists of open $2-$cells. We denote by X the corresponding cellular decomposition of S_b.

Definition 3.2.2 (Kasteleyn orientation)**.** A *Kasteleyn orientation* K on a surface graph with boundary on S_b is an orientation of the edges of $\mathcal{G} \cup \partial S_b$ such that for each face f of X, we have

$$\mathrm{Card}\{e \subset \partial f, K_{|e} = -O^f_{|e}\} = 1 \bmod 2,$$

where O^f is the orientation induced by O on f.

Changing the orientations of a Kasteleyn orientation at all the edges sharing a same vertex produces a new Kasteleyn orientation:

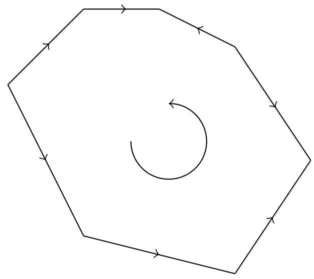


Figure 3.1: A Kasteleyn orientation on the boundary of a face.

Definition 3.2.3 (equivalence of Kasteleyn orientations)**.** Two Kasteleyn orientations are said to be *equivalent* if and only if they can be connected by a sequence of operations consisting in changing the orientations of all the edges sharing a same vertex.

Let K be a Kasteleyn orientation on a surface graph $\mathcal{G}$ in S_b. Let C be an oriented simple closed curve in $\mathcal{G} \cup \partial S_b$. We denote $\kappa_K(C)$ the number of edges of $\mathcal{G} \cup \partial S_b$, such that the orientation induced on it by C is opposed to K. We have the following result due to D. Cimasoni and N. Reshetikhin ([8]) :

Proposition 3.2.4. *Let $\mathcal{G} \subset S_b$ be a connected surface graph with boundary. Let $C_1, \dots, C_p$ be the boundary components of S_b with induced orientations. Let $n_i \in \{0, 1\}, i \in \{1, \dots, p\}$ then there exists a Kasteleyn orientation K on $\mathcal{G}$ such that $1 + \kappa_K(C_i) = n_i \bmod 2$ if and only if $n_1 + \cdots + n_p \equiv V \bmod 2$, where V is the number of vertices of $\mathcal{G}$.*

Starting from (S, Λ), we now construct a surface graph with boundary on S_b. First truncate the vertices of each triangle of Λ to obtain a hexagon.

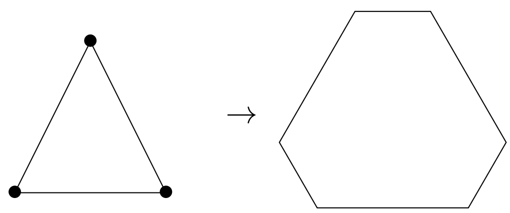


Figure 3.2: From triangles to hexagons.

Definition 3.2.5 (long and short edges)**.** The edges arising from the triangle are called *long edges*. We denote by $\mathcal{L}(\Lambda)$ the set of all long edges. The other edges obtained by truncation are said to be short. We denote the set of all short edges by $\mathcal{S}(\Lambda)$.

We now attach to each vertex of a long edge an edge with a one valent vertex on the boundary of S_b and we denote by $\mathcal{E}_d(\Lambda)$ the set of all such edges (cf. Figure 3.3).

We denote by H_Λ the obtained surface graph with boundary and we call it *hexagonalisation associated to* Λ.

We now give the relation between spin structures and Kasteleyn orientations. To do this we recall the notion of dimer configuration.

Definition 3.2.6. A *dimer configuration* D on a surface graph with boundary is a choice of edges of the underlying graph with boundary $\mathcal{G}$, called *dimers*, such that each vertex that is not a boundary vertex is adjacent to exactly one dimer.

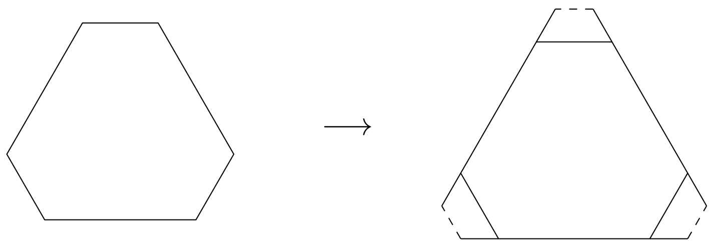


Figure 3.3: Construction of the surface graph starting from hexagons (The dashed lines represent the boundary components)

Remark 3.2.7. Let Λ be an ideal triangulation of S. Considering H_Λ, there exists a canonical dimer configuration D on H_Λ given by $D = \bigcup_{e \in \mathcal{E}_d(\Lambda)} e$.

Applying Theorem 1 of ([8]) to the case of H_Λ, we get :

Theorem 3.2.8. *Let D be the canonical dimer configuration on $H_\Lambda \subset S_b$. Given a class $\alpha \in H_1(S_b, \mathbb{Z}_2)$, represent it by oriented simple closed curves $C_1, \ldots, C_m$ in H_Λ. If K is a Kasteleyn orientation on H_Λ then the function $q_D^K : H_1(S_b, \mathbb{Z}_2) \to \mathbb{Z}_2$ given by*

$$q_D^K(\alpha) = \sum_{i<j} (C_i \cdot C_j) + \sum_{i=1}^{m} (1 + \kappa_K(C_i) + \ell_D(C_i)) \mod 2,$$

where $l_D(C)$ denotes the number of vertices in C whose adjacent dimer of D sticks out to the left of C in S_b, is a well-defined quadratic form on $H_1(S_b, \mathbb{Z}_2)$.
Moreover there is an isomorphism of affine $H^1(S_b, \mathbb{Z}_2)$-spaces from the set of equivalence classes of Kasteleyn orientations on $H_\Lambda \subset S_b$ onto the set of spin structures on S_b.

It is more convenient to work with triangles, so we introduce a new notation. Consider a hexagon F and a Kasteleyn orientation o on it. Each time the orientation of a small edge of F is opposed to the orientation induced by O, the small edge is replaced by a vertex with a dot (cf. Figure 3.4). This way we obtained a new graph on S called *dotted triangulation* and we denote it by $\Lambda_\bullet$.

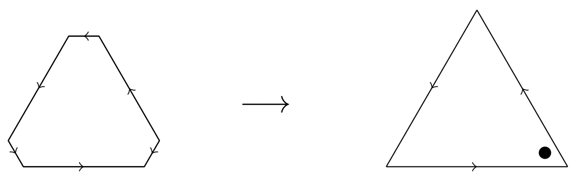
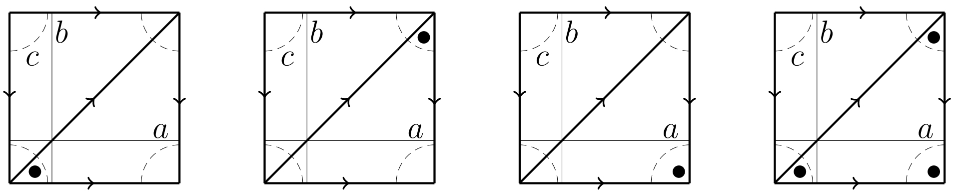


Figure 3.4: From hexagons to dotted triangles.

Remark 3.2.9. Let $\alpha \in H_1(S)$ be represented by a simple closed curve surrounding a hole. Then, by Theorem 3.2.8, $q_D^K = 0$ if and only if there is an odd number of dots surrounding the corresponding vertex of $\Lambda_\bullet$.

The example of the one-punctured torus $\mathbb{T}_1^1$

The one-punctured torus T_1^1 admits four inequivalent spin structures. They can be encoded, as seen before, by four inequivalent dotted Kasteleyn orientations on an ideal triangulation Λ and consider the set of standard generators a, b, c of $\pi_1(\mathbb{T}_1^1)$ as shown in Figure 3.5.



(a) (b) (c) (d)

Figure 3.5: The four dotted Kasteleyn orientations on $\mathbb{T}_1^1$

Let us compute the Cimasoni-Reshetikhin quadratic form in the four cases.

$$\begin{array}{llll} \text{Case (a):} & q(a) = 0 & q(b) = 0 & q(c) = 0, \\ \text{Case (b):} & q(a) = 0 & q(b) = 1 & q(c) = 0, \\ \text{Case (c):} & q(a) = 1 & q(b) = 1 & q(c) = 0, \\ \text{Case (d):} & q(a) = 1 & q(b) = 0 & q(c) = 0. \end{array}$$

In the four cases we remark that the point c is a Neuveu-Schwarz point. This is always true if the surface has only one puncture.

CHAPTER 4

Construction of coordinates on the super-Teichmüller X−space

Let S be a surface of type (g, k, m) with $k + m > 0$. Let $\{p_1, \ldots, p_{k+m}\}$ denote the set of punctures and holes. The aim of this section is to construct a set of coordinates on the super-Teichmüller space of S also called the *super-Teichmüller X-space* of S, which are the super-analogs of the shear coordinates. To this purpose we will need to encode combinatorially the datum of a spin structure on a ideally triangulated surface: for this we shall use Kasteleyin orientations.

4.1 Invariant and pseudo-invariant

In this section we recall the construction of two invariants following [24]. The first one is an even invariant of four points in $\mathbb{P}^{1|1}$ having distinct reductions and the second one an odd invariant for three points in $\mathbb{P}^{1|1}$ with distinct reductions up to the diagonal action of SpO(2|1) on 4-uples and triples of points in $\mathbb{P}^{1|1}$.

Definition 4.1.1. 1. We define an n−gon as an n−uple $(A_1, \ldots, A_n)$ of linearly ordered points in $\mathbb{P}^{1|1}$ such that $\forall i \neq j, \sharp A_i \neq \sharp A_j$.

2. An n−gon $(A_1, \ldots, A_n)$ is said to be positive if the linear order agrees with the orientation induced by the orientation of $\mathbb{P}^1 = \mathbb{P}^{1|1^\sharp}$. **In what follows we assume that all the considered n−gons are positive.**

3. We say that two n−gons $P = (P_1, \ldots, P_n)$ and $Q = (Q_1, \ldots, Q_n)$ are equivalent under the action of SpO(2|1) if there exists a matrix $B \in$ SpO(2|1) such that $\forall i, BP_i = Q_i$.

4.1.1 The super cross-ratio, the even invariant

Consider $A, A' \in \mathbb{R}^{2|1}$ and define the form $\langle A, A' \rangle = A^{\mathrm{st}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} A'$. We have $\langle A, A' \rangle = -\langle A', A \rangle$ and a straightforward computation shows that for $B \in \mathrm{SpO}(2|1)$,

$$\langle BA, BA' \rangle = \langle A, A' \rangle .$$

Given four points (Z, Z', W, W') in $\mathbb{P}^{1|1}$ with distinct reductions (bodies), we define the function

$$\chi(Z, Z', W, W') = -\frac{\langle W', W \rangle \langle Z', Z \rangle}{\langle W', Z \rangle \langle Z', W \rangle},$$

Remark 4.1.2. The function χ is well defined: the denominator is non zero because the points have distinct reductions.

The invariance of the form $\langle , \rangle$ under the diagonal action of $\mathrm{SpO}(2|1)$ implies the following:

Proposition 4.1.3. *The function χ is invariant under the diagonal action of* $\mathrm{SpO}(2|1)$ *on* $\mathbb{P}^{1|1}$.

4.1.2 The odd pseudo-invariant

Considering Lemma 2.4.7 we obviously get:

Lemma 4.1.4. *A triangle Z_1, Z_2, Z_3 is equivalent to a triangle of the form*

$$\left(\begin{pmatrix} \infty \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ \zeta \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right),$$

where the odd number $\pm\zeta$ depends only on the equivalence class of Z_1, Z_2, Z_3.

This way Lemma 4.1.4 allows to define a numeric invariant under the action of $\mathrm{SpO}(2|1)$ as

$$[Z_1 \widehat{: Z_2 :} Z_3] = (\pm\zeta) \in G_1(\mathbb{R}). \tag{4.1}$$

This invariant is defined up to a sign and is called *pseudo-invariant* by Manin [24].

4.2 Refinement of the odd invariant

Using Kasteleyn orientations on triangles we now refine the odd invariant (4.1). Let $S^{1|1} := \mathbb{R}^{2|1^*}/G_0(\mathbb{R})^*_+$. $S^{1|1}$ is a two-fold cover of $\mathbb{P}^{1|1}$ and we denote by π the projection $\pi : S^{1|1} \to \mathbb{P}^{1|1}$.

Consider an ideal triangle $T = (A_1, A_2, A_3)$ in $\mathbb{H}^S$ linearly ordered and such that this order agrees with the orientation induced on T by the orientation of $\mathbb{H}^S$. Let o be a Kasteleyn orientation on its edges.

Definition 4.2.1. A *lift* of T is the choice of points $\tilde{A}_i \in S^{1|1}$ such that $\pi(\tilde{A}_i) = A_i, \forall i$. Two lifts $\tilde{P}$ and $\tilde{P}'$ are "$\pm$ *equivalent*" if there exists $g \in \mathrm{SpO}(2|1)$ such that

$$\forall i, g(\tilde{A}_i) = \tilde{A}'_i \text{ or } \forall i, g(\tilde{A}_i) = -\tilde{A}'_i.$$

We lift T to $\tilde{T} = (\tilde{A}, \tilde{B}, \tilde{C})$ in the following way: lift A_1 to $\tilde{A}_1 = \begin{pmatrix} a_1 \\ b_1 \\ \zeta_1 \end{pmatrix}$. If the edge $e = [A_1, A_i]$ of T is oriented from A_1 to A_i lift A_i to $\tilde{A}_i = \begin{pmatrix} a_i \\ b_i \\ \zeta_i \end{pmatrix}$ such that $\operatorname{sign}\left(\det\begin{pmatrix} a_1^\sharp & a_i^\sharp \\ b_1^\sharp & b_i^\sharp \end{pmatrix}\right) = -1.$

Definition 4.2.2.
1. We say that two triples T_1 and T_2 of cyclically ordered points (triangles) in $S^{1|1}$ are $\pm$ *equivalent* if there exists $B \in \mathrm{SpO}(2|1)$ which sends T_1 to T_2 or $-T_2$.
2. The triangles T_1 and T_2 are said to be *equivalent* if there exists $B \in \mathrm{SpO}(2|1)$ which sends T_1 to T_2. We denote this equivalence by $\approx$.

Proposition 4.2.3. *Let T be an ideal triangle in H^S with vertices A, B, C. The construction above provides a bijection between the set of Kasteleyn orientations on T and the set of $\pm$ equivalence classes of lifts of A, B, C.*

We will prove later a more general result given by Proposition 4.3.4.

Lemma 4.2.4. *Each triangle A, B, C with a Kasteleyn orientation o on its edges in $S^{1|1}$ is equivalent to a triangle whose vertices are (possibly cyclically permuted)*

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ \mp 1 \\ \pm\zeta \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

where $\pm\zeta$ is determined by o. We will denote the triangle A, B, C by $[ABC]$.

Remark 4.2.5. In Lemma 4.2.4 we don't claim that $A \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $B \mapsto \begin{pmatrix} \pm 1 \\ \mp 1 \\ \pm\zeta \end{pmatrix}$ and $C \mapsto \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$.

Proof. At least one edge of the triangle A, B, C, is oriented against the orientation induced by the orientation on $\mathbb{H}$. Assume that it is AC. Now using the Lemma 2.4.7 and Proposition 4.2.3, we can send A and C respectively to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, now B is sent to a point of the desired form, which is uniquely determined. $\square$

Definition 4.2.6. Let $[ABC]$ be a triangle in $S^{1|1}$ equipped with a Kasteleyn orientation o. We associate an odd number to $([ABC], o)$ given by ζ if $[ABC] \approx \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & \zeta & 0 \end{bmatrix}$ and $-\zeta$ if $[ABC] \approx \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -\zeta & 0 \end{bmatrix}$.

Definition 4.2.7. Using the bijection between Kasteleyn orientation and $\pm$ equivalence classes of lifts, we can interpret the change of Kasteleyn orientation given by the inversion of the orientations of all the edges sharing a same vertex v as a change of leaf in $S^{1|1}$ (cf. Figure 4.1), we will call this change the *switch in* v.

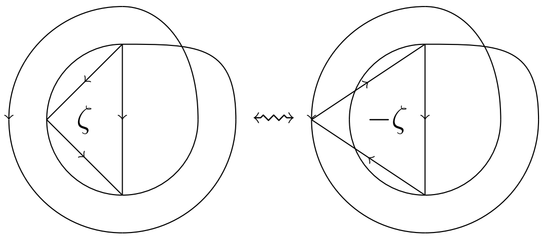


Figure 4.1: Effects of a switch. S^1 is seen as the double cover of $\mathbb{P}^1$ and so visualized as a circle running twice around the unit circle.

Lemma 4.2.8. *If we change the Kasteleyn orientation of a triangle by a switch its odd invariant gets multiplied by* -1.

Proof. Consider for example a triangle which is equivalent to $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & \zeta & 0 \end{bmatrix}$.

By the action of $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & -\zeta \\ 0 & \zeta & 1 \end{pmatrix} \in \mathrm{SpO}(2|1)$ we obtain $\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & -\zeta \end{bmatrix}$, which has by definition invariant $-\zeta$. All the possible cases can be treated in the same way. □

Lemma 4.2.9. *The following holds:*

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & \zeta & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -\zeta \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ -\zeta & 0 & 0 \end{bmatrix}.$$

Proof. The two equivalences are respectively obtained by multiplication of the first triple by the matrices of $\mathrm{SpO}(2|1)$,

$$\begin{pmatrix} 1 & 0 & -\zeta \\ 0 & 1 & 0 \\ 0 & \zeta & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\zeta \\ -\zeta & 0 & 1 \end{pmatrix}.$$

□

4.3 The super-Teichmüller X-space of a surface

In this section we define the super-Teichmüller space of surfaces with holes or super-Teichmüller X-space. We construct explicit coordinates on this space. These coordinates are the super analogs of the shear coordinates on the Teichmüller space.

4.3.1 Definition

Let S be a surface of type (g, k, m) and denote by $p_1, \ldots, p_k$ the set of holes and by $p_{k+1}, \ldots, p_{k+m}$ the set of punctures.

Definition 4.3.1. *The super-Teichmüller $X-$space*, denoted by ST, is the set of equivalence classes of triples

$$\left(\rho, \{o_i\}_{i=1\ldots k}, \{F_j\}_{j=1..m}\right),$$

where $\rho \in Hom\left(\pi_1(S), \mathrm{SpO}(2|1)\right)$ and $(\mathrm{Im}\rho)^\sharp$ is a Fuchsian group, o_i is the choice of an orientation of the hole p_i and F_j is a set of $\pi_1(S)-$equivariant fixed points in $\mathbb{P}^{1|1}$ of $\rho(gm_{k+j}g^{-1})$ for all $g \in \pi_1(S)$ and m_{k+j} is a simple loop surrounding the puncture p_{k+j} and where we say that two triples, $\left(\rho, \{o_i\}_{i=1\ldots k}, \{F_j\}_{j=1..m}\right)$ and $\left(\rho', \{o'_i\}_{i=1\ldots k}, \{F'_j\}_{j=1..m}\right)$, are equivalent if and only if there exists $B \in \mathrm{SpO}(2|1)$ such that

$$\rho' = B\rho B^{-1}, \quad o'_i = o_i \quad \text{and} \quad F'_j = B \cdot F_j.$$

This way ST is only a set, but as we will see it later, it can be endowed with a structure of H^∞ DeWitt supermanifold, by the constructions of charts on ST and transition maps between them.

4.3.2 Lifting ideal triangulations

Let S be an ideal triangulated surface with punctures and oriented holes, $p_1, \ldots, p_{k+m}$. Let Λ be its triangulation. Let $\rho : \pi_1(S) \to \Gamma \subset \mathrm{SpO}(2|1)$ be a super-Fuchsian representation of $\pi_1(S)$. To lift the triangulation to $\mathbb{H}^S$, we use the underlying hyperbolic structure, considering the morphism $\rho^\sharp : \pi_1(S) \to \Gamma^\sharp$. First lift the triangulation to $\mathbb{H}$ as explained in Subsection 1.2.2. Now using this construction, we just choose points on $\mathbb{P}^{1|1}$ which reduce to these on $\mathbb{R}P^1$ and corresponding to fixed points of the elements of Γ associated to the loops surrounding the punctures and the holes. We thus obtain a $\pi_1(S)-$covariant lift of Λ to $\mathbb{H}^S$ denoted by $\tilde{\Lambda}$.

We now modify Λ and $\tilde{\Lambda}$ to obtain their hexagonalisations H_Λ and $\tilde{H}_\Lambda$. Each vertex v of Λ leads to a boundary component with k_v vertices on it where k_v is the number of edges containing v. Each vertex $\tilde{v}$ of $\tilde{\Lambda}$, representative of a vertex v of Λ leads to k_v vertices of $\tilde{H}_\Lambda$. For the moment, by construction, these vertices are all the same point of $\mathbb{P}^{1|1}$.

Now fix a Kasteleyn orientation o on H_Λ and pull it back to $\tilde{H}_\Lambda$ we get a $\pi_1(S)-$covariant Kasteleyn orientation on $\tilde{H}_\Lambda$. Use o to lift the vertices in $\mathbb{P}^{1|1}$ to vertices in $S^{1|1}$. To do this we first explain how we deal with the vertices which are the same points in $\mathbb{P}^{1|1}$.

Definition 4.3.2. A *lift* of $\tilde{H}_\Lambda$ is the choice of points $\tilde{A}_i \in S^{1|1}$ for each vertex A_i of such $\tilde{H}_\Lambda$ that

1. $\forall i, \pi(\tilde{A}_i) = A_i$
2. if $\tilde{A}_k$ and $\tilde{A}_j$ are the ends of the same short edge we have $\pi(\tilde{A}_j) = \pi(\tilde{A}_k) = A_j = A_k$.
3. if $A_k = B \cdot A_i$ where $B = \rho(\gamma)$ for some $\gamma \in \pi_1(S)$, $\tilde{A}_k = B \cdot \tilde{A}_i$.

Two lifts $\tilde{H}_\Lambda$ and $\tilde{H}'_\Lambda$ are $\pm$ *equivalent* if there exists $g \in \mathrm{SpO}(2|1)$ such that

$$\forall i, g(\tilde{A}_i) = \tilde{A}'_i \text{ or } \forall i, g(\tilde{A}_i) = -\tilde{A}'_i.$$

Definition 4.3.3. 1. We define an $n-$gon as an $n-$uple $(A_1, \dots, A_n)$ of linearly ordered points in $\mathbb{P}^1$.

2. An $n-$gon $(A_1, \dots, A_n)$ is said to be positive if the linear order agrees with the orientation induced by the orientation of $\mathbb{P}^1$. **In what follows we assume that all the considered $n-$gons are positive.**

Proposition 4.3.4. *Let S be a triangulated hyperbolic surface, let Λ be its triangulation and let $\tilde{H}_\Lambda$ be a lift of H_Λ in $\mathbb{H}^S$ with vertices $\{A_1, \dots\}$. There exists a bijection between the set of Kasteleyn orientations on $\tilde{H}_\Lambda$ and the set of $\pm$ equivalence classes of lifts of $\tilde{H}_\Lambda$.*

Proof. Using the definition of a lift we just have to prove the proposition for a fundamental domain $\mathcal{D}$ in $\mathbb{H}^S$. We associate to each lift (up to equivalence) a Kasteleyn orientation o by the following rule. Let $\tilde{A}_i := \begin{pmatrix} a_i \\ b_i \\ \zeta_i \end{pmatrix} \in S^{1|1}$ be a given lift. Let $e_{k,l} := [A_k, A_l]$ be a long edge of H_Λ and compute the determinant $D_{k,l}$ of the matrix given by $\begin{pmatrix} a_k^\sharp & a_l^\sharp \\ b_k^\sharp & b_l^\sharp \end{pmatrix}$. We orient $e_{k,l}$ from $\tilde{A}_k$ to $\tilde{A}_l$ if $D_{k,l}$ is negative and from $\tilde{A}_l$ to $\tilde{A}_k$ otherwise, for the short edges the orientation coincides with O if and only if the two end points are the same.

Start from a vertex on a short edge of a hexagon F and follow its boundary according to the orientation induced on it by the orientation on $\mathbb{H}$, and assume first that, using the positivity, its vertices are

$$\begin{pmatrix} e_1 \\ 1 \\ \zeta_1 \end{pmatrix}, \begin{pmatrix} e_1 \\ 1 \\ \zeta_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ 1 \\ \zeta_2 \end{pmatrix}, \begin{pmatrix} e_2 \\ 1 \\ \zeta_2 \end{pmatrix}, \begin{pmatrix} e_3 \\ 1 \\ \zeta_3 \end{pmatrix}, \begin{pmatrix} e_3 \\ 1 \\ \zeta_3 \end{pmatrix}, \text{ with } e_1^\sharp < e_2^\sharp < e_3^\sharp \in \mathbb{R}.$$

The three determinants we are dealing with are $e_1^\sharp - e_2^\sharp < 0$, $e_2^\sharp - e_3^\sharp < 0$ and $e_3\sharp - e_1^\sharp > 0$. Thus the above rule ensures that exactly one edge of F will be oriented against the orientation of ∂F induced by the orientation of $\mathbb{H}$. So it is a Kasteleyn orientation on F. If we are not in the above situation, then we can reduce to it by applying switches on some vertices. Indeed by a switch at one vertex of F lying on the other leaf of $S^{1|1}$, we change the sign of one of the determinants and the extremity of a vertex of a short edge. Thus we have to change the orientation of two edges and that does not change the parity of the number of edges oriented against the orientation induced by O. Finally we get a Kasteleyn orientation on F. This holds on each face of $\mathcal{D}$, thus we get a Kasteleyn orientation on $\mathcal{D}$.

Conversely starting from a hexagonalisation of S with triangulation Λ and a Kasteleyn orientation o on H_Λ we construct a lift of H_Λ following the rule:

1. fix a vertex A_i of H_Λ and lift it in $S^{1|1}$ to $\tilde{A}_i = \begin{pmatrix} a_i \\ b_i \\ \zeta_i \end{pmatrix}$.

2. given a lift $\tilde{A}_j = \begin{pmatrix} a_j \\ b_j \\ \zeta_j \end{pmatrix}$ of a point A_j and a point A_k on an long edge A_jA_k, choose a lift $\tilde{A}_k$ of A_k by $\tilde{A}_k := \begin{pmatrix} a_k \\ b_k \\ \zeta_k \end{pmatrix}$ in such a way that
$$\operatorname{sign}\left(\det\begin{pmatrix} a_j^\sharp & a_k^\sharp \\ b_j^\sharp & b_k^\sharp \end{pmatrix}\right) = \begin{cases} 1 & \text{if } A_j \leftarrow A_k \\ -1 & \text{if } A_j \rightarrow A_k \end{cases}.$$

3. for the short edges lift the two extremities in the same point of $S^{1|1}$ if and only if the orientation o coincides with the orientation induced by O.

By construction the two maps defined by these rules are inverses to each other. □

4.3.3 Coordinates on the super-Teichmüller X-space

We are now going to assign to each edge of Λ an element of $G_0(\mathbb{R})$ and to each triangle of Λ an element of $G_1(\mathbb{R})$. The collection of all these elements will be a parametrization of the super-Teichmüller X-space of S.

First of all we give a definition of the odd invariant of a hexagon F following the ideas given in Section 4.2. Consider a hexagon arising from a triangle like in Figure 3.2 with a Kasteleyn orientation on it. The orientation O induces an orientation on the edges of F. We consider the triangle T_F formed by the left endpoints of the small edges (regardless of their Kasteleyn orientation). There is a unique orientation on the edges of T_F which, together with the initial Kasteleyn orientation on the hexagon produces a Kasteleyn orientation on the new cellularization of the hexagon (see Figure 4.2).

Definition 4.3.5. The *odd invariant of a hexagon* F is defined to be the odd invariant of the underlying triangle T_F.

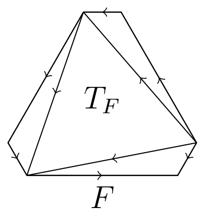


Figure 4.2: A hexagon F and its underlying triangle T_F.

Remark 4.3.6. The choice we made is arbitrary, we could have chosen the triangle in F with vertices given by the right endpoints of the small edges and all the following results would still hold (with some modifications on the signs of the odd parts). The only important point is to choose left endpoints on all the triangles of Λ to form T_F.

We now consider the dotted triangulation $\Lambda_\bullet$ of S obtained from Λ.

We recall that two Kasteleyn orientations on a graph are said to be *equivalent* if and only if they can be connected by a sequence of operations consisting in changing the orientations of all the edges sharing a same vertex. We recall also that changing the orientations of all the edges sharing a same vertex is equivalent to a switch. We now translate these operations in the notation of dotted triangulations.

Definition 4.3.7. Let T be an oriented dotted triangle and consider an angle containing a dot $\bullet$. Assuming that the edge e on the left of $\bullet$ (with respect to the orientation of T) is not identified with an edge of a triangle T' different from T, we define the left push-out of $\bullet$ to be the operation consisting in pushing the dot in the angle of T' adjacent to e and on the same side of it and reversing the orientation of e: this corresponds to applying a switch at the left extremity of the short edge of the hexagon associated to T encoded by $\bullet$. We define the right push-out of a dot in the same way. We summarize these operations in Figure 4.3.

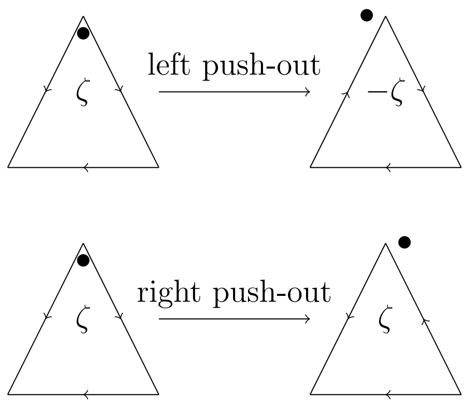


Figure 4.3: Left and right push-outs.

Remark 4.3.8. Observe that the right push-outs do not affect the value of the odd invariant of T_f because they correspond to switching one of the vertices of F which is not in T_F.

Lemma 4.3.9. *Let F be a hexagon such that each of its edges is not identified with another one. Let T be its underlying dotted triangle.*

1. *Applying left push-outs on each dot in T we obtain a new triangle T_l with a Kasteleyn orientation on it.*

2. *If ζ is the odd invariant of F, we have $\zeta = (-1)^{\mathrm{Card}\{\bullet \text{ in } F\}}[T_l]$, where $[T_l]$ is the odd invariant of T_l defined in Section 4.2.*

Proof.

1. Assume first that the orientation on T is not a Kasteleyn orientation. This means that there is an odd number of dots in T. Left push-outs correspond to changing all the orientations of the edges of F sharing the left extremity of a short edge which is oriented against O. So we change the orientation on an odd number of long edges of F. The corresponding dotted triangle T_l does not contain dots and the orientation on its edges is a Kasteleyn orientation. If the orientation on T is a Kasteleyn orientation, there is an odd number of dots in T. Left push-outs lead to the change of the orientations of an even number of long edges of F so the corresponding dotted triangle T_l does not contain dots and the orientation on its edges is a Kasteleyn orientation.

2. Applying a left push-out corresponds to applying a switch at one vertex of T_F (cf.Definition 4.3.5 and Figure 4.2).

□

Using the Kasteleyn orientation we lift the vertices of $\tilde{\Lambda}$ to $S^{1|1}$ and we associate to each edge an even number x_i and to each oriented triangle an odd number ζ_j respectively given by the super-cross-ratio and the odd invariant of the oriented triangle.

It is clear that the coordinates we obtain this way do not depend on the choice of a representative of the triple given by a morphism from $\pi_1(S)$ to $\mathrm{SpO}(2|1)$, the fixed points of the transformations representing the Ramond points and the fixed points of the transformations representing the Neveu-Schwartz points up to the diagonal action of $\mathrm{SpO}(2|1)$. We then show that given an hexagonalization of a surface equipped with a Kasteleyn orientation K and a point in ST, we can define "coordinates" for it. We claim:

Theorem 4.3.10. *The so obtained collection* $(x_1, \dots, x_n, \zeta_1, \dots, \zeta_p) \in \left(G_0(\mathbb{R})^*_+\right)^n \times G_1(\mathbb{R})^p$ *provides a global parametrization of the connected component of the super-Teichmüller* $X-$*space indexed by* K *up to the diagonal action on the odd part of* $\mathbb{Z}_2$ *by multiplication by* -1.

4.4 Proof of Theorem 4.3.10: the reconstruction of the morphism.

The aim in this section is to prove Theorem 4.3.10 that is to reconstruct a conjugacy class of a triple given by morphism from $\pi_1(S)$ to $\mathrm{SpO}(2|1)$, the fixed points of the transformations representing the Ramond points and the fixed points of the transformations representing the Neveu-Schwartz points starting from an ideal triangulation together with a dotted Kasteleyn orientation, $\Lambda_\bullet$ of an open surface with even positive numbers $\{x_i\}$ assigned to the edges and odd numbers $\{\zeta_j\}$ assigned to the dotted triangles. Like in the classical case, the orientation of a boundary component corresponding to a hole is given by the sign of the real number $\prod x_i^\sharp - 1$. Using the Kasteleyn orientation and the coordinates, after lifting the vertices of a triangle of the triangulation to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ and to a third point of even part -1, all the fixed points can be reconstructed in an unique way. To achieve the construction we consider a graph obtained from $\Lambda_\bullet$ in the following way (cf. Figure 4.4):

1. Let $\Lambda_\bullet^*$ be the dual graph of $\tilde{H}_\Lambda$.
2. Replace each three valent vertex of $\Lambda_\bullet^*$ by a triangle.
3. Orient the edges corresponding to the triangles in the negative direction with respect to the orientation induced by the triangle.

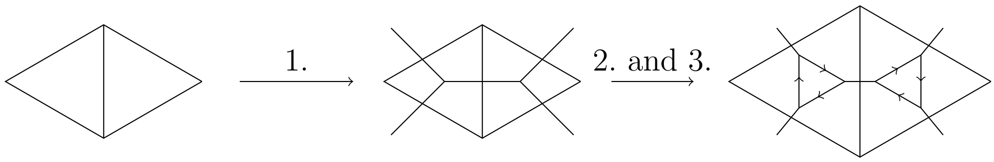


Figure 4.4: Construction of the graph for computing the morphism

On the graph $\Lambda_\bullet^*$ constructed above, there are two kinds of edges:

1. the edges belonging to a triangle, which are oriented,
2. the edges intersecting an edge of $\Lambda_\bullet$, which are not oriented.

We now associate to each edge of $\Lambda_\bullet^*$ a matrix of SpO(2|1) which depends on the dotted Kasteleyn orientation, in the following way:

1. consider a dotted triangle T_j of $\Lambda_\bullet$. Let ζ_j be the odd number associated to T_j. We associate to each edge λ of T_j a number $a_\lambda \in \{0,1\}$ encoding its orientation:

$$a_\lambda = \begin{cases} 0 & \text{if the orientation of } \lambda \text{ agrees with the orientation induced by } O \\ 1 & \text{otherwise} \end{cases}.$$

   To each vertex v of T_j we associate the number A_v of dots at v (cf. Figure 4.5).

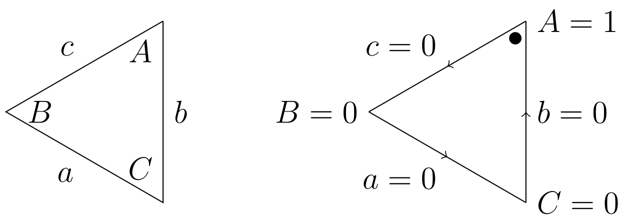


Figure 4.5

2. to each oriented edge contained in T_j we associate a matrix

$$U_{(a,b,c,A,B,C)}(\zeta_j) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{b+C} \begin{pmatrix} 1 & 1 & (-1)^{1+a+A+C}\zeta_j \\ -1 & 0 & 0 \\ (-1)^{1+a+A+C}\zeta_j & 0 & 1 \end{pmatrix}.$$

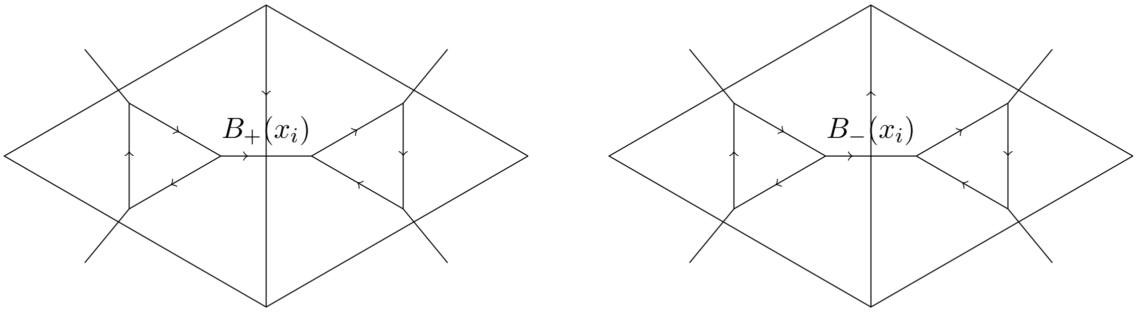


Figure 4.6

3. orient arbitrarily each non oriented edge e and associate a matrix

$$B_\pm(x_i) = \begin{pmatrix} 0 & \mp x^{\frac{1}{2}} & 0 \\ \pm x^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to the edge λ_i labeled by x_i. The Kasteleyn orientation on e and λ_i (cf. Figure 4.6) provides the sign $\pm$ (Note that $B_+(x_i) = B_-(x_i)^{-1}$).

4. Now for any oriented simple closed path in $\Lambda^*_\bullet$ we can associate a matrix of $\mathrm{SpO}(2|1)$ by multiplying on the left on the all the matrices met along it, taking

$$U_{(a,b,c,A,B,C)}(\zeta_j)^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & (-1)^{a+A+C}\zeta_j \\ 0 & (-1)^{1+a+A+C}\zeta_j & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{b+C}$$

instead of $U_{(a,b,c,A,B,C)}(\zeta_j)$ and $B_\mp(x_i)$ instead of $B_\pm(x_i)$ each time the orientation of the path disagrees with the orientation of the edge.

Lemma 4.4.1. *Let p_0 be a vertex of $\Lambda^*_\bullet$. For each loop ℓ based in p_0 let $\rho(l)$ be the matrix of $\mathrm{SpO}(2|1)$ associated to ℓ by the construction of point (4) above. The application ρ defines a homomorphism from $\pi_1(S, p_0)$ to $\mathrm{SpO}(2|1)$.*

Before proving this result we introduce

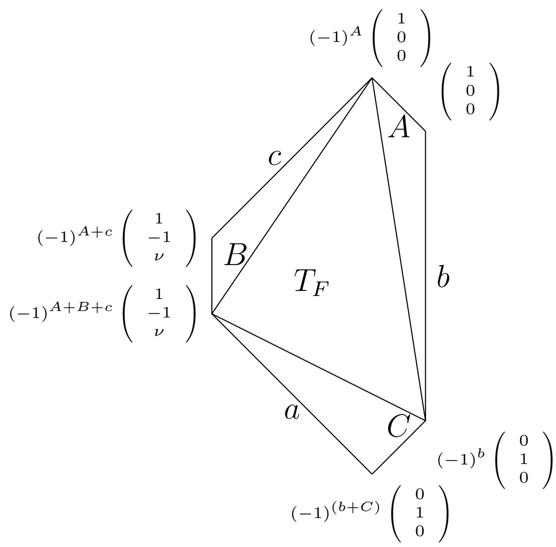
Definition 4.4.2. A hexagon whose vertices are lifts of the points

$$\begin{pmatrix} \infty \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ \theta \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is said to be in *canonical position* if the right extremity of one of its small edges is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

A dotted triangle is said to be in *canonical position* if its underlying hexagon is.

Proof. We want to prove that taking a closed path corresponding to follow the three edges of a triangle in $\Lambda^*_\bullet$, the product of the corresponding matrices is the identity.



Consider a dotted triangle in canonical position. We encode the orientation on its edges by (a, b, c) and the number of dots in a vertex by (A, B, C) as in Figure 4.5. Looking at the underlying hexagon we obtain the following.

If the odd number associated to the considered dotted triangle T is ζ, then it holds that ν must be such that the odd invariant of the triangle

$$\begin{bmatrix} (-1)^A & (-1)^{A+B+c} & 0 \\ 0 & (-1)^{1+A+B+c} & (-1)^b \\ 0 & (-1)^{A+B+c}\nu & 0 \end{bmatrix}$$

equals ζ. So we get

$$\zeta = (-1)^A(-1)^{A+B+c}(-1)^{b+1}\nu = (-1)^{B+b+c+1}\nu$$

and it holds that the odd invariant of T is ζ if and only if $\nu = (-1)^{a+A+C}\zeta$. The matrix U acts by permuting cyclically in the counterclockwise direction the vertices of the considered dotted triangle in canonical position and sends the triangle to a triangle in canonical position, indeed we have :

$$U \cdot (-1)^{b+C}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$U \cdot (-1)^{A+c}\begin{pmatrix} 1 \\ -1 \\ \nu \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon \\ 0 \end{pmatrix}$$

$$U \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \varepsilon'\begin{pmatrix} 1 \\ -1 \\ \kappa \end{pmatrix}$$

where $\varepsilon, \varepsilon' \in \{-1, 1\}$. The uniqueness of the canonical position and Proposition 4.3.4 ensure that the matrix product obtained by following the three edges of a triangle in $\Lambda^*_\bullet$ is the identity. An example is given in Figure 4.7. □

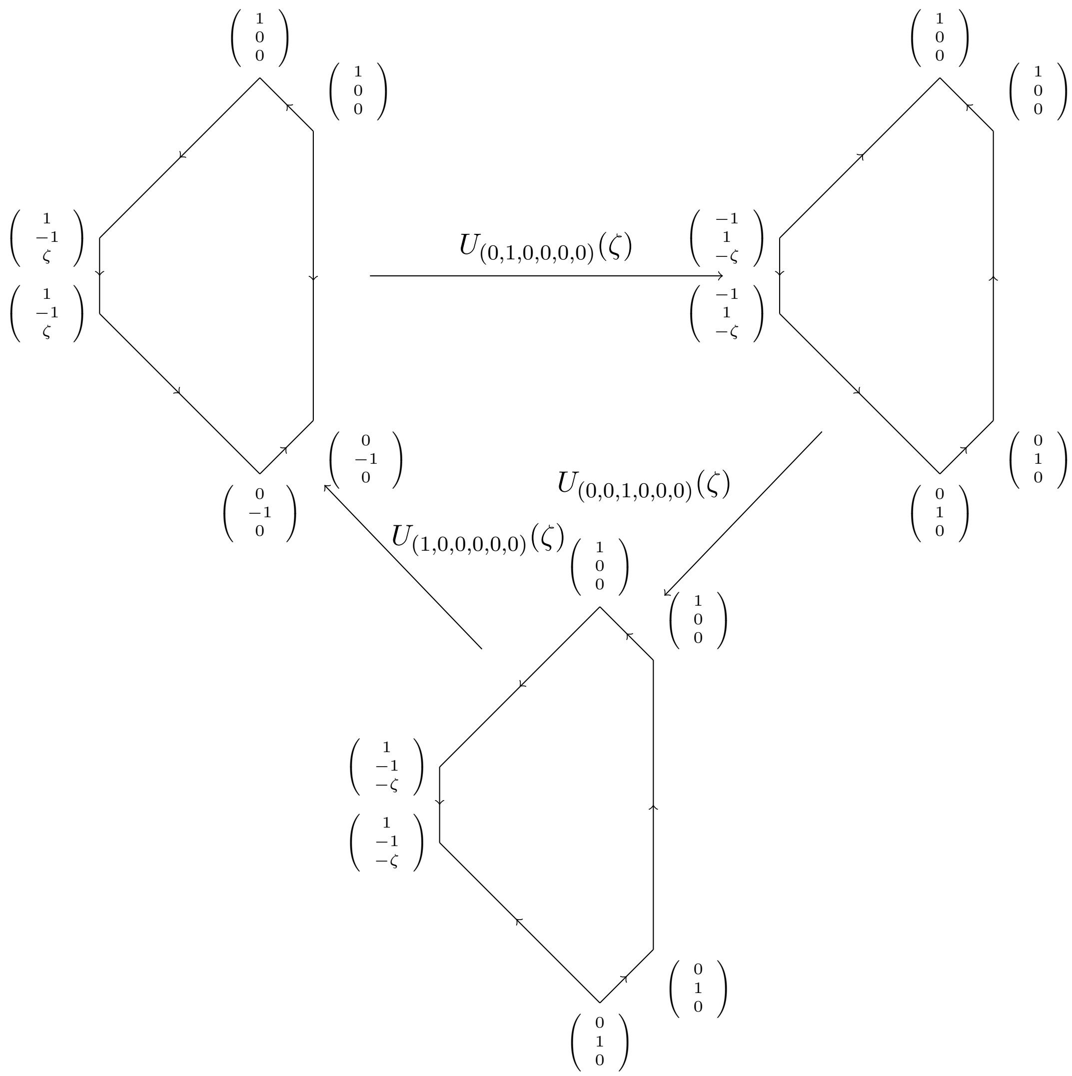


Figure 4.7: An illustration of the actions of the matrices $U_{(a,b,c)}$

Remark 4.4.3. Consider two adjacent hexagons: let us say that they are in canonical position if the vertices on the right of the short edges of one of them are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ \mp 1 \\ \mu \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}$$

and the vertices on the left of the short edges of the other are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm x^{\frac{1}{2}} \\ \pm x^{-\frac{1}{2}} \\ \mu' \end{pmatrix},$$

where μ is such that the invariant to the first hexagon is ζ. The transformation sending the second hexagon to a hexagon of the first kind is given by the matrix $B_{\pm}(x_i)$. An example is given in Figure 4.8.

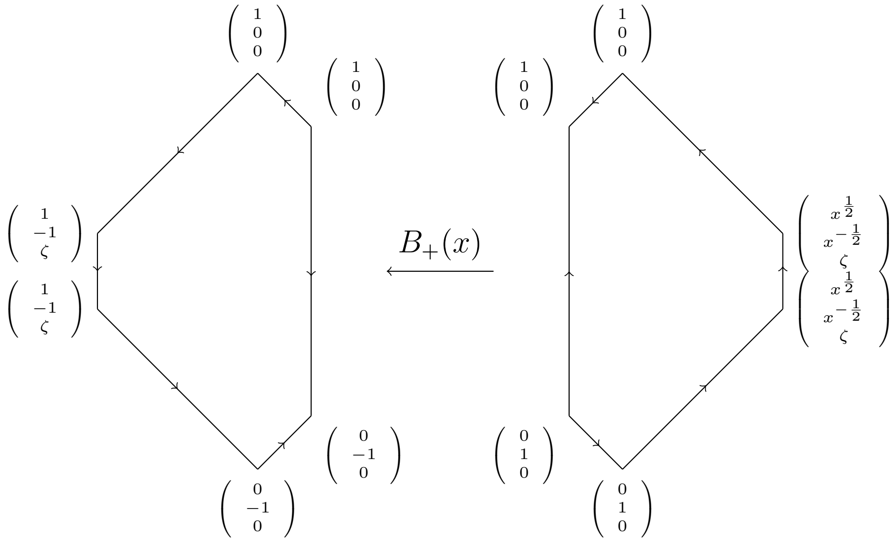


Figure 4.8: An illustration of the action of the matrix $B_\pm$

The image of ρ is a super-Fuchsian group because its reduction is a Fuchsian group. We now have to check that the constructed fixed points are the fixed points of the transformation associated to a loop surrounding a hole or a puncture. As we are reconstructing an equivalence class, it is sufficient to prove the result for the point $\begin{pmatrix} \infty \\ 0 \end{pmatrix}$.

The transformation associated to the loop γ in Figure 4.9 is given by a product

$$M = B_\pm(x_k)U^{-1}(\zeta_k)\cdots B_\pm(x_2)U^{-1}(\zeta_2)B_\pm(x_1)U^{-1}(\zeta_1). \tag{4.2}$$

Multiplying each $U^{-1}(\zeta_i)$ from the left by $J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and each $B_\pm(x_k)$ from the right by J^{-1} does not change the product (4.2) and it becomes

$$M = H_\pm(x_k)F(\zeta_k)\cdots H_\pm(x_2)F(\zeta_2)H_\pm(x_1)U^{-1}(\zeta_1),$$

where

$$\begin{cases} H_\pm(x) = \begin{pmatrix} \pm x^{\frac{1}{2}} & 0 & 0 \\ 0 & \pm x^{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \\ F(\zeta) \text{ is of the form } \begin{pmatrix} 1 & 1 & \pm\zeta \\ 0 & 1 & 0 \\ 0 & \mp\zeta & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\varepsilon}, \varepsilon \in \{0,1\}. \end{cases}$$

Each of the matrices taking part in the product sends $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ to $\pm\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, so $\begin{pmatrix} \infty \\ 0 \end{pmatrix}$ is the fixed point of the transformation associating to γ. That completes the proof of Theorem 4.3.10.

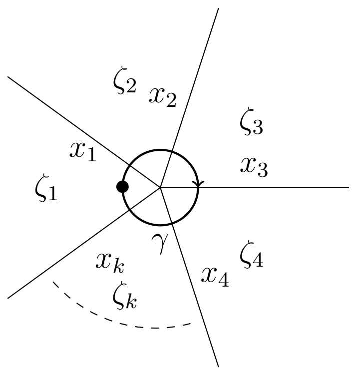


Figure 4.9

CHAPTER 5

Superflips and superpentagons

In this chapter we give explicit formulas for a change of (dotted) triangulation by a flip. We then show that these coordinate changes are involutions and that they satisfy a super version of the pentagon relation.

Theorem 5.1.

1. *Let Q be a quadrilateral in $\Lambda_\bullet$ containing no dots, we then have the following propositions.*

   (a) *If the all edges of Q are distinct, for each flip of Λ producing a triangulation Λ' there exists a unique Kasteleyn orientation o' on $H_{\Lambda'}$ coinciding with the orientation o of the edges of H_Λ not involved in the flip. Taking into account the orientations of the edges, there are four versions of a flip and their corresponding coordinate changes are given by Figure 5.1.*

   (b) *If two opposite edges are identified in Q, then the coordinates and their transformation under a flip are of the form given in Figure 5.2.*

2. *Let Q be a quadrilateral in $\Lambda_\bullet$ containing dots.*

   (a) *If the all edges of Q are distinct, we reduce to Case (1a) by pushing out the dots. Then apply the superflip and push in the dots : the coordinates and their transformation under a flip are of the form given in Figure 5.3.*

   (b) *If two opposite edges are identified in the triangulation, then the same operations as in Case (2a) lead to a change of coordinates of the form depicted in Figure 5.4.*

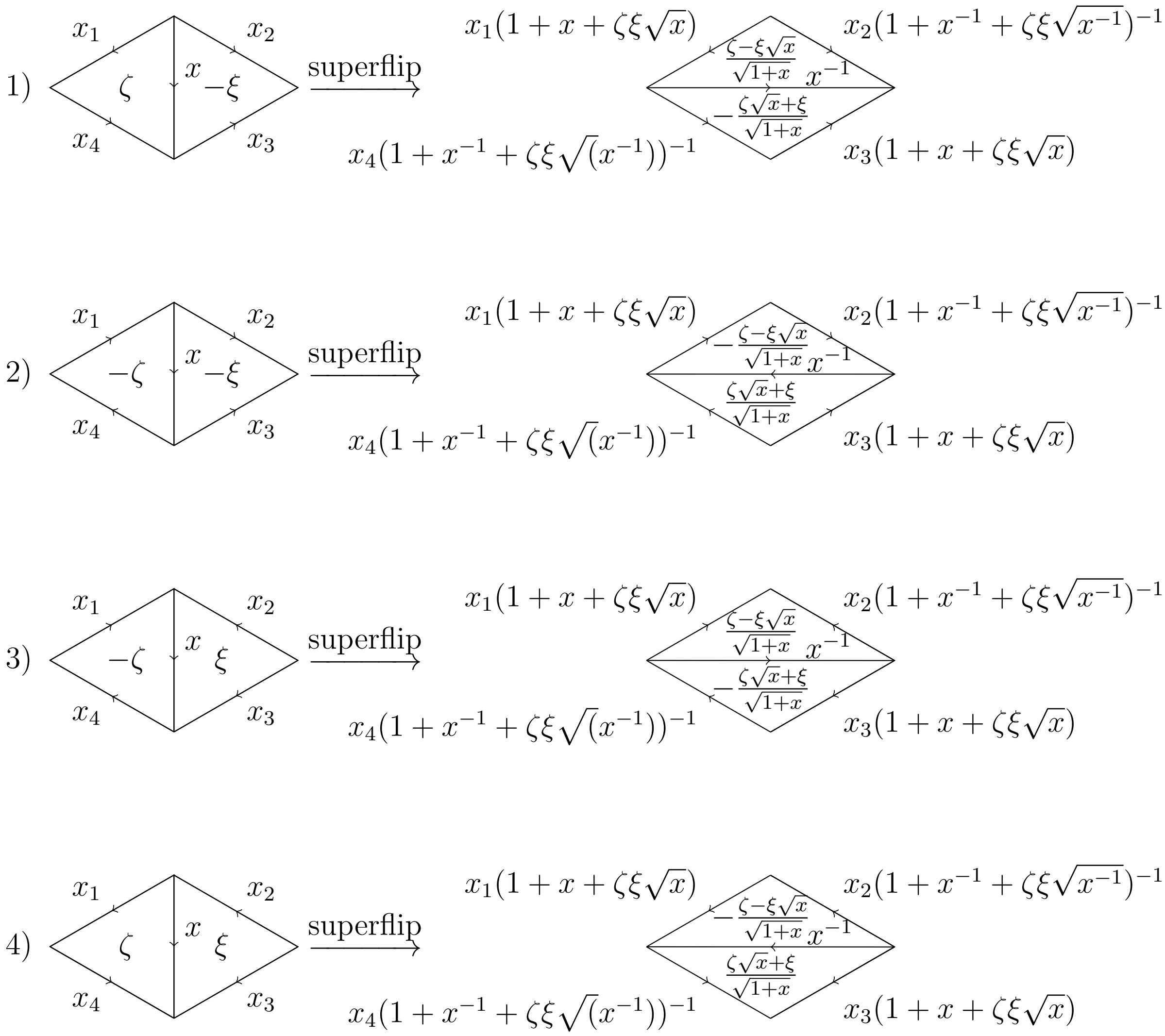


Figure 5.1: Superflips

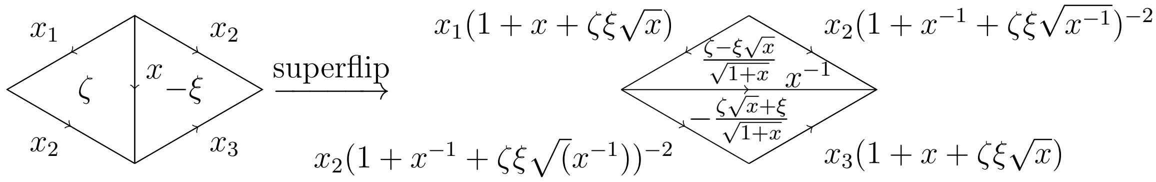


Figure 5.2: Superflip in a quadrilateral with identified opposite edges.

Proof. (1a) We prove the result in the first case of Figure 5.1. Let $[Z_1, Z_2, Z_3, Z_4]$ be a quadrilateral in $S^{1|1}$ equipped with the Kasteleyn orientation in the first picture of Figure 5.1. Up to acting with an element B of $\mathrm{SpO}(2|1)$ we can assume that the

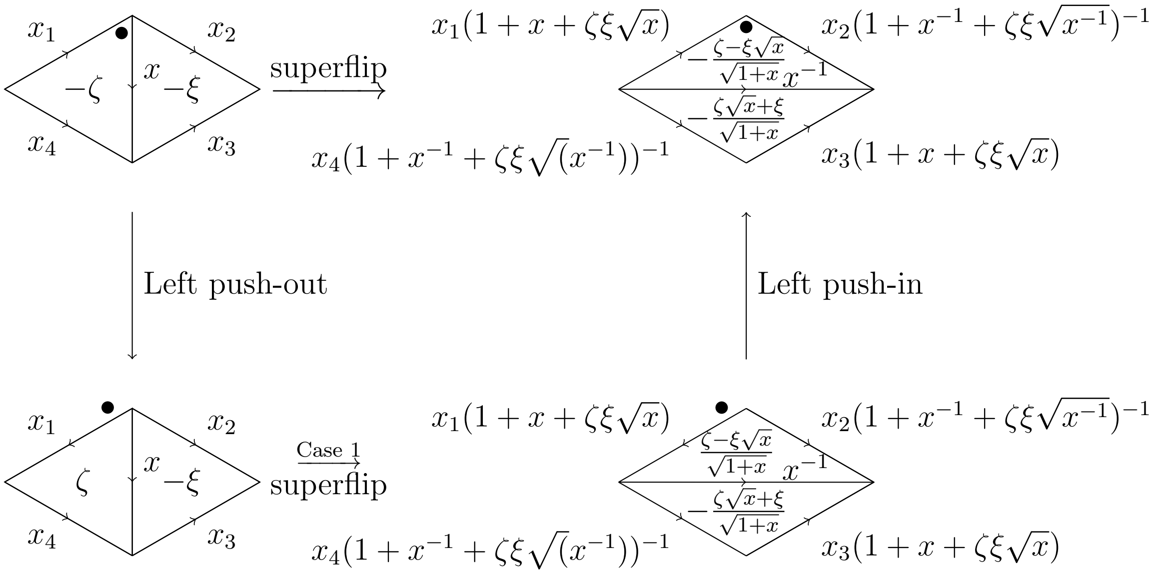


Figure 5.3: Decomposition of a superflip using push-outs

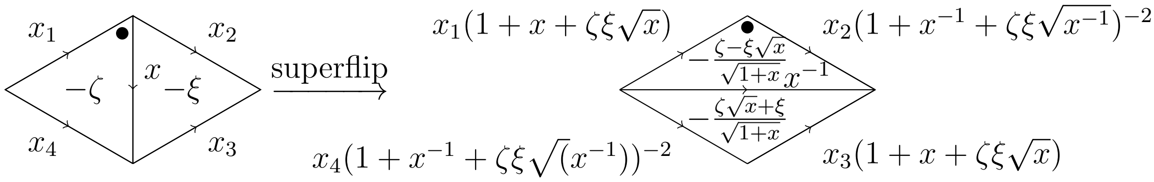


Figure 5.4: Superflip in a quadrilateral with identified opposite edges containing a dot.

vertices of the quadrilateral are the columns of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & -x \\ 0 & -1 & -1 & -1 \\ 0 & \zeta & 0 & \sqrt{x}\xi \end{bmatrix}.$$

Let's compute the odd invariants corresponding to the two oriented triangles in the picture on the right of Figure 5.1.

For convenience, we first set $y = \sqrt{\frac{1+x}{x}}\left(1 + \frac{1}{2}\frac{\zeta\xi\sqrt{x}}{1+x}\right)$. The triple $\begin{bmatrix} 1 & 0 & -x \\ -1 & -1 & -1 \\ \zeta & 0 & \sqrt{x}\xi \end{bmatrix}$ is equivalent to the product:

$$\begin{pmatrix} y^{-1} & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -\zeta \\ -1 & 0 & 0 \\ -\zeta & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 & -x \\ -1 & -1 & -1 \\ \zeta & 0 & \sqrt{x}\xi \end{bmatrix},$$

which is equal to

$$\begin{bmatrix} 0 & -y^{-1} & -\sqrt{x}\sqrt{1+x}\left(1+\frac{1}{2}\frac{\zeta\xi\sqrt{x}}{1+x}\right) \\ -y & 0 & \sqrt{x}\sqrt{1+x}\left(1+\frac{1}{2}\frac{\zeta\xi\sqrt{x}}{1+x}\right) \\ 0 & 0 & \zeta x+\xi\sqrt{x} \end{bmatrix}.$$

Up to rescaling the columns respectively by $y^{-1}, y, \frac{1}{\sqrt{x(1+x)}}\left(1-\frac{1}{2}\frac{\zeta\xi\sqrt{x}}{1+x}\right)$, the triangle is equivalent to:

$$\begin{bmatrix} 1 & 0 & -x \\ -1 & -1 & -1 \\ \zeta & 0 & \sqrt{x}\xi \end{bmatrix} \approx \begin{bmatrix} 0 & -y^{-1} & -\sqrt{x}\sqrt{1+x}\left(1+\frac{1}{2}\frac{\zeta\xi\sqrt{x}}{1+x}\right) \\ -y & 0 & \sqrt{x}\sqrt{1+x}\left(1+\frac{1}{2}\frac{\zeta\xi\sqrt{x}}{1+x}\right) \\ 0 & 0 & \zeta x+\xi\sqrt{x} \end{bmatrix}$$

$$\approx \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & \frac{\zeta\sqrt{x}+\xi}{\sqrt{1+x}} \end{bmatrix}.$$

The triple is then equivalent to the following one:

$$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & \frac{\zeta\sqrt{x}+\xi}{\sqrt{1+x}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & \frac{\zeta\sqrt{x}+\xi}{\sqrt{1+x}} \end{bmatrix}.$$

Thus by Lemma 4.2.9 the odd invariant $[Z_1, Z_2, Z_4]$ is $-\frac{\zeta\sqrt{x}+\xi}{\sqrt{1+x}}$.
The same kind of method leads us to the last invariant:

$$\begin{bmatrix} 1 & 1 & -x \\ 0 & -1 & -1 \\ 0 & \zeta & \xi\sqrt{x} \end{bmatrix} \approx \begin{bmatrix} \frac{1}{\sqrt{1+x}} & \sqrt{1+x} & 0 \\ 0 & -\sqrt{1+x} & -\sqrt{1+x} \\ 0 & \zeta & \xi\sqrt{x} \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & \frac{\zeta}{\sqrt{1+x}} & \frac{\xi\sqrt{x}}{\sqrt{1+x}} \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ -\frac{\zeta-\xi\sqrt{x}}{\sqrt{1+x}} & 0 & 0 \end{bmatrix}.$$

Thus the odd invariant $[Z_2, Z_3, Z_4]$ is $\frac{\zeta-\xi\sqrt{x}}{\sqrt{1+x}}$.

We now compute how the even coordinates change. We first look at the diagonal of the quadrilateral. We have

$$\begin{aligned} \chi(Z_2, Z_3, Z_4, Z_1) &= -\frac{\langle Z_1, Z_4\rangle \langle Z_3, Z_2\rangle}{\langle Z_1, Z_2\rangle \langle Z_3, Z_4\rangle} \\ &= \left(-\frac{\langle Z_4, Z_3\rangle \langle Z_2, Z_1\rangle}{\langle Z_4, Z_1\rangle \langle Z_2, Z_3\rangle}\right)^{-1} \\ &= (\chi(Z_1, Z_2, Z_3, Z_4))^{-1} \\ &= x^{-1}. \end{aligned}$$

Now consider the edge $e = [Z_1, Z_2]$. We assume that e is the diagonal of the quadrilateral Z_1, Y, Z_2, Z_3, where $Y \in \mathbb{P}^{1|1}$. By definition we have $x_1 = \chi(Z_1, Y, Z_2, Z_3)$. Let x_1' be the even number associated to the edge e after the flip given by Figure 5.1. We get

$$\begin{aligned} x_1' =& \chi(Z_1, Y, Z_2, Z_4) \\ =& -\frac{\langle Z_4, Z_2\rangle \langle Y, Z_1\rangle}{\langle Z_4, Z_1\rangle \langle Y, Z_2\rangle} \\ =& -\frac{\langle Z_3, Z_2\rangle \langle Y, Z_1\rangle}{\langle Z_3, Z_1\rangle \langle Y, Z_2\rangle} \frac{\langle Z_3, Z_1\rangle \langle Z_4, Z_2\rangle}{\langle Z_3, Z_2\rangle \langle Z_4, Z_1\rangle} \\ =& x_1 \frac{1 \cdot (1 + x + \zeta\xi\sqrt{x})}{1 \cdot 1}. \end{aligned}$$

The same kind of considerations provide the coordinates associated to each edge.

(1b) We now consider the opposite identified edges. Let us set $Z_\infty = \begin{pmatrix} \infty \\ 0 \end{pmatrix}, Z_{-1} = \begin{pmatrix} -1 \\ -\zeta \end{pmatrix}, Z_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, Z_x = \begin{pmatrix} x \\ -\sqrt{x}\xi \end{pmatrix}$. Let $B \in \Gamma$ be such that $[Z_{-1}, Z_0]$ and $[Z_\infty, Z_x]$ are identified by B. The edge $e = [Z_\infty, Z_x]$ is the diagonal of the quadrilateral $Z_\infty = B.Z_{-1}, Z_0, Z_x = B.Z_0, B.Z_\infty$. We have by definition

$$\begin{aligned} x_2 &= \chi\left(Z_\infty, Z_0, Z_x, B.Z_\infty\right) \\ &= -\frac{\langle B.Z_\infty, Z_x\rangle \langle Z_0, Z_\infty\rangle}{\langle B.Z_\infty, Z_\infty\rangle \langle Z_0, Z_x\rangle} \\ &= -\frac{\langle B.Z_\infty, B.Z_0\rangle \langle Z_0, Z_\infty\rangle}{\langle B.Z_\infty, B.Z_{-1}\rangle \langle Z_0, Z_x\rangle}. \end{aligned}$$

Let x_2' be the even number associated to e after the flip of Figure 5.2. The edge e is the diagonal of the quadrilateral with vertices $Z_\infty = B.Z_{-1}, Z_{-1}, Z_x = B.Z_0, B.Z_x$. We get :

$$\begin{aligned} x_2' &= \chi\left(Z_\infty, Z_{-1}, Z_x, B.Z_x\right) \\ &= -\frac{\langle B.Z_x, Z_x\rangle \langle Z_{-1}, Z_\infty\rangle}{\langle B.Z_x, Z_\infty\rangle \langle Z_{-1}, Z_x\rangle} \\ &= -\frac{\langle B.Z_x, B.Z_0\rangle \langle Z_{-1}, Z_\infty\rangle}{\langle B.Z_x, B.Z_{-1}\rangle \langle Z_{-1}, Z_x\rangle} \\ &= x_2 \frac{\langle B.Z_x, B.Z_0\rangle \langle B.Z_\infty, B.Z_{-1}\rangle}{\langle B.Z_x, B.Z_{-1}\rangle \langle B.Z_\infty, B.Z_0\rangle} \frac{\langle Z_0, Z_x\rangle \langle Z_{-1}, Z_\infty\rangle}{\langle Z_0, Z_\infty\rangle \langle Z_{-1}, Z_x\rangle} \\ &= x_2 \left(\frac{\langle Z_0, Z_x\rangle \langle Z_{-1}, Z_\infty\rangle}{\langle Z_0, Z_\infty\rangle \langle Z_{-1}, Z_x\rangle}\right)^2 \quad (\chi \text{ is SpO}(2|1)\text{−invariant}) \\ &= x_2 \left(1 + x^{-1} + \zeta\xi\sqrt{x^{-1}}\right)^{-2}. \end{aligned}$$

□

Definition 5.2. Given a triple (S, Λ, o), a *superpentagon relation* is a sequence of five superflips such that their composition is the identity and its projection to (S, Λ) (forgetting o) is a standard pentagon relation (see Figure 5.8).

Theorem 5.3.

1. *The superflip is an involution.*

2. *The superflip satisfies the superpentagon relation.*

Proof. We prove the result for the first superflip in Figure 5.1. The other results can be obtained by first pushing out the dots, then operating switches, applying that case and finally switching back the lifts.

We have the sequence of moves given by Figure 5.5.

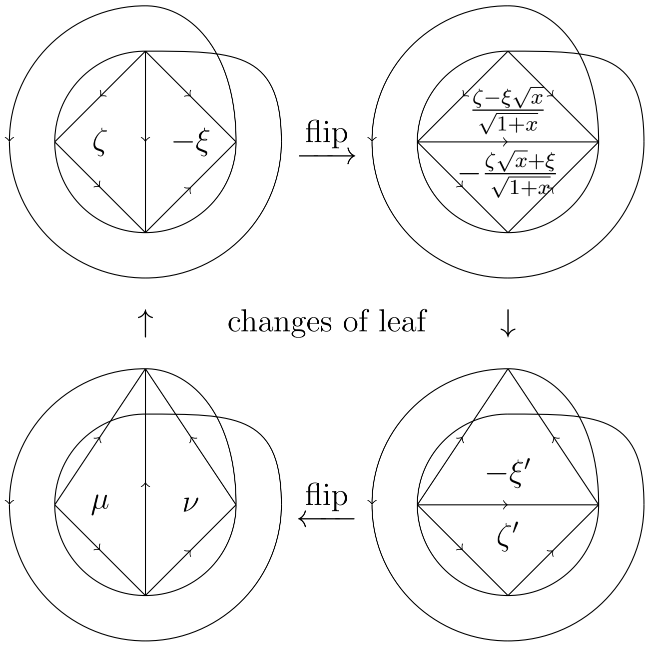


Figure 5.5: Decomposition of the effect of two successive flips in the same quadrilateral.

- Using Proposition 5.1 and Proposition 4.2.7 we have:

$$\zeta' = -\frac{\zeta\sqrt{x}+\xi}{\sqrt{1+x}},$$
$$\xi' = \frac{\zeta-\xi\sqrt{x}}{\sqrt{1+x}},$$
$$\mu = \frac{\zeta'-\xi'\sqrt{x^{-1}}}{\sqrt{1+x^{-1}}} = -\zeta,$$
$$\nu = -\frac{\zeta'\sqrt{x^{-1}}+\xi'}{\sqrt{1+x^{-1}}} = \xi.$$

  Then a last application of Proposition 4.2.7 by changing the point of leaf gives us the desired result for the odd invariants.

- For the even coordinates, it is sufficient to remark that $(x^{-1})^{-1} = x$ and

$$-\frac{\zeta\sqrt{x}+\xi}{\sqrt{1+x}}\frac{\zeta-\xi\sqrt{x}}{\sqrt{1+x}} = \zeta\xi.$$

We now prove that the pentagon relation is satisfied. We prove the result for a pentagon with distinct edges and without dots. We first decompose the superflip into the two moves showed in Figure 5.6. We have the following relations:

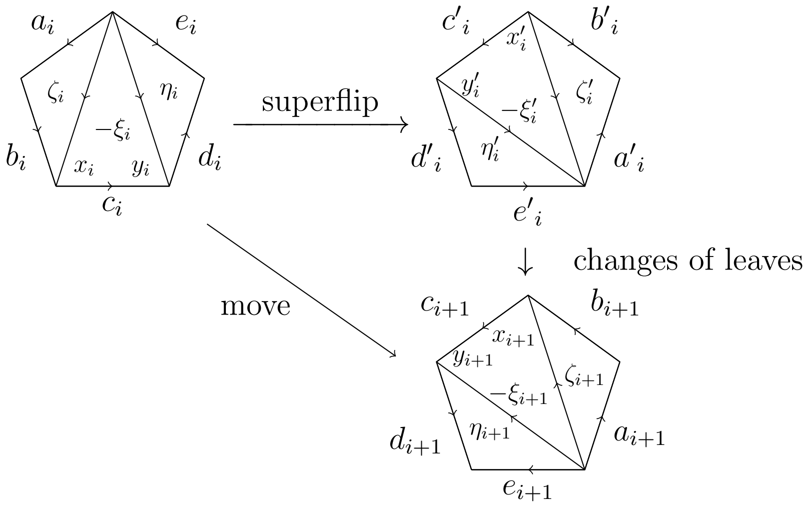


Figure 5.6

$$\begin{aligned}
a_{i+1} &= d_i, \\
b_{i+1} &= e_i, \\
c_{i+1} &= a_i\left(1 + x_i + \zeta_i\xi_i\sqrt{x_i}\right), \\
d_{i+1} &= b_i\left(1 + {x_i}^{-1} + \zeta_i\xi_i\sqrt{{x_i}^{-1}}\right)^{-1}, \\
e_{i+1} &= c_i\left(1 + x_i + \zeta_i\xi_i\sqrt{x_i}\right), \\
x_{i+1} &= y_i\left(1 + {x_i}^{-1} + \zeta_i\xi_i\sqrt{{x_i}^{-1}}\right)^{-1}, && (5.1)\\
y_{i+1} &= x_i^{-1}, && (5.2)\\
\zeta_{i+1} &= -(\eta_i), && (5.3)\\
\xi_{i+1} &= -(\zeta_i - \xi_i\sqrt{x_i})(1+x_i)^{-\frac{1}{2}}, && (5.4)\\
\eta_{i+1} &= -(\zeta_i\sqrt{x_i} + \xi_i)(1+x_i)^{-\frac{1}{2}}. && (5.5)
\end{aligned}$$

We will use that if $\alpha^2 = 0$ we have $(1+\alpha)^t = 1 + t\alpha$.

- We first prove that $y_5 = y_0$ and simultaneously $x_5 = x_0$. Using the relations and the fact above we get:

$$\begin{aligned} y_5 = x_4^{-1} = y_3^{-1}\left(1+x_3^{-1}+\zeta_3\xi_3\sqrt{x_3^{-1}}\right) \\ &= x_2\left[1+y_2^{-1}\left(1+x_2^{-1}+\zeta_2\xi_2\sqrt{x_2^{-1}}\right)+\eta_2\left(\zeta_2\sqrt{x_2^{-1}}-\xi_2\right)\sqrt{y_2^{-1}}\right) \\ &= x_1\left(1+y_1-\eta_1\xi_1\sqrt{y_1}\right) \\ &= y_0. \end{aligned}$$

Equation (5.2) gives $x_5 = x_0$.

- We now prove that $a_5 = a_0$, the same kind of considerations provide the result for the other coordinates.

$$\begin{aligned} a_5 = d_4 = b_3\left(1+x_3^{-1}+\zeta_3\xi_3\sqrt{x_3^{-1}}\right) \\ &= e_2\left(1+y_2^{-1}\left(1+x_2^{-1}+\zeta_2\xi_2\sqrt{x_2^{-1}}\right)+\left(\eta_2\zeta_2\sqrt{x_2^{-1}}-\eta_2\xi_2\right)\sqrt{y_2^{-1}}\right)^{-1} \\ &= c_1\left(1+x_1+\zeta_1\xi_1\sqrt{x_1}\right)\left[1+x_1\left\{1+y_1^{-1}\left(1+x_1^{-1}+\zeta_1\xi_1\sqrt{x_1^{-1}}\right)-\eta_1\xi_1\sqrt{y_1^{-1}}\right\}\right. \\ &\qquad \left.+\left\{\xi_1\eta_1\sqrt{y_1^{-1}}-\zeta_1\xi_1\sqrt{x_1}\right\}\right]^{-1} \\ &= c_1\left[1+y_1^{-1}+\xi_1\eta_1 y_1^{-1}\right]^{-1} \\ &= a_0. \end{aligned}$$

- We now prove that $\zeta_5 = \zeta_0$ and simultaneously $\eta_5 = \eta_0$. Using the relations and the fact above we get:

$$\begin{aligned} \zeta_5 = -\eta_4 = \left(\zeta_3\sqrt{x_3}+\xi_3\right)\left(1+x_3\right)^{-\frac{1}{2}} \\ &= -\left[1+x_2\left(1+y_2+\zeta_2\xi_2\sqrt{x_2^{-1}}\right)\right]^{-\frac{1}{2}}\left[\zeta_2-\left(\xi_2-\eta_2\sqrt{y_2}\right)\sqrt{x_2}\,\right]. \end{aligned}$$

We now want to express ζ_0 with respect to ζ_2. Therefore we have to reverse the superflip. We now consider the moves in Figure 5.7.

We have the following relations:

$$x_{i-1} = y_i^{-1}, \tag{5.6}$$

$$y_{i-1} = x_i\left(1+y_i+\xi_i\eta_i\sqrt{y_i}\right), \tag{5.7}$$

$$\eta_{i-1} = \zeta_i, \tag{5.8}$$

$$\xi_{i-1} = \left(-\xi_i+\eta_i\sqrt{y_i}\right)\left(1+y_i\right)^{-\frac{1}{2}}, \tag{5.9}$$

$$\zeta_{i-1} = \left(\xi_i\sqrt{y_i}+\eta_i\right)\left(1+y_i\right)^{-\frac{1}{2}}. \tag{5.10}$$

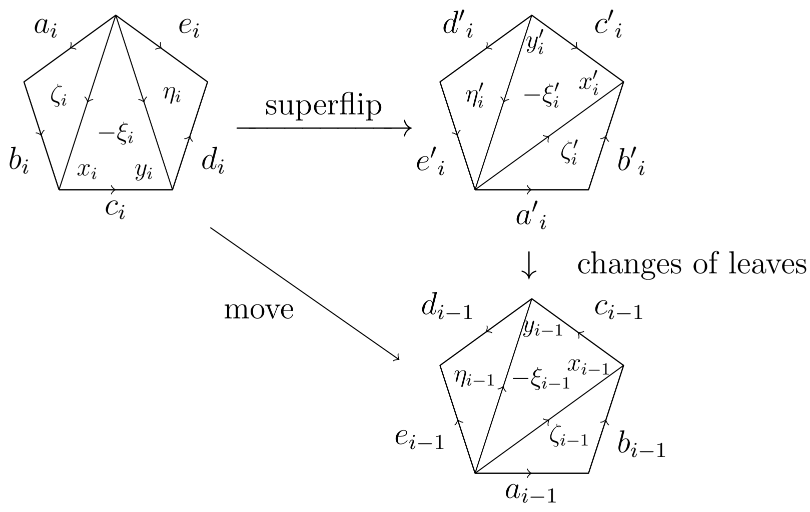


Figure 5.7

We get:

$$\begin{aligned}\zeta_0 &= \left(\xi_1\sqrt{y_1}+\eta_1\right)\left(1+y_1\right)^{-\frac{1}{2}}\\ &= \left[\left(-\xi_2+\eta_2\sqrt{y_2}\right)\left(1+y_2\right)^{-\frac{1}{2}}\sqrt{y_1}+\zeta_2\right]\left(1+y_1\right)^{-\frac{1}{2}}\\ &= \left[1+x_2\left(1+y_2+\xi_2\eta_2\sqrt{y_2}\right)\right]^{-\frac{1}{2}}\left[\zeta_2-\left(\xi_2-\eta_2\sqrt{y_2}\right)\sqrt{x_2}\right].\end{aligned}$$

Expanding the expressions of ζ_5 and ζ_0 we obtained in function of ζ_2 we remark that the two expressions are of opposite signs. But all the vertices of the obtained pentagons are on opposite leaves hence the equality. We summarize this in Figure 5.8.

□

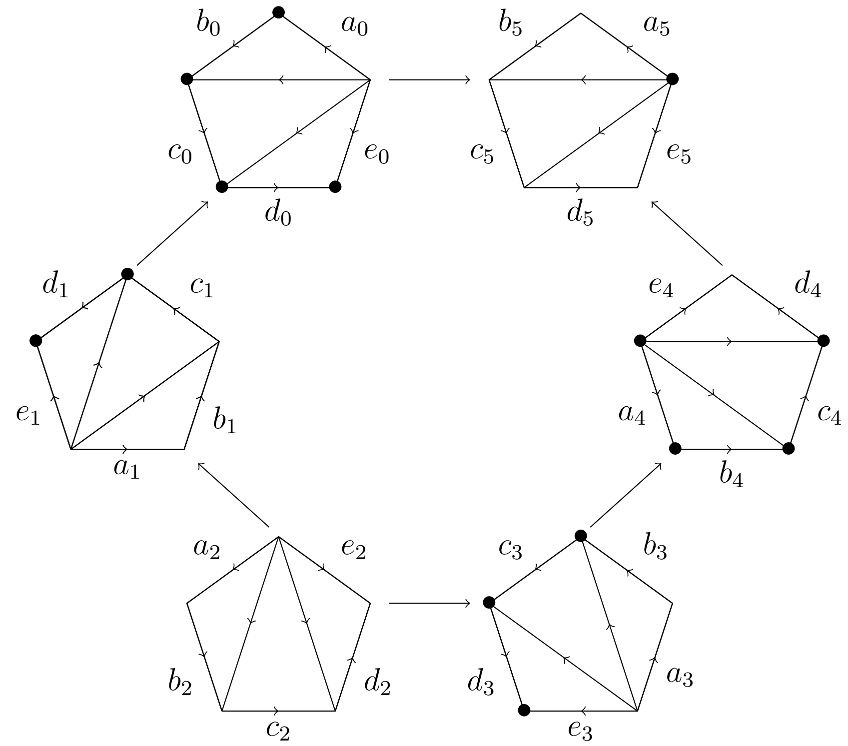


Figure 5.8: The pentagon relation (The dots correspond to the points which are on the other leaf)

CHAPTER 6

Poisson structures

6.1 Poisson manifolds and Poisson supermanifolds

6.1.1 Poisson manifolds

Definition 6.1.1. A *Poisson bracket* on a smooth manifold M is a bilinear map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),$$

such that for all f, g and h in $C^\infty(M)$ the three following properties are satisfied :

1. $\{f, g\} = -\{g, f\}$ (skew-symmetry),
2. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity),
3. $\{fg, h\} = f\{g, h\} + g\{f, h\}$ (Leibniz's rule).

Definition 6.1.2. Let $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ be two Poisson manifolds and let $f : M \to N$ be a smooth map. The map ψ is said to be a *Poisson map* if for all $f_1, f_2 \in C^\infty(N)$

$$\{f_1, f_2\}_N \circ \psi = \{f_1 \circ \psi, f_2 \circ \psi\}_M .$$

Definition 6.1.3. A *Poisson-Lie group* is a Lie group equipped with a Poisson bracket for which the multiplication $m : G \times G \to G$ is a Poisson map.

6.1.2 Poisson supermanifolds

Definition 6.1.4. A *Poisson superalgebra* is a superalgebra A over $\mathbb{R}$ with a *Poisson superbracket* $[\cdot,\cdot] : A \times A \to A$ which satisfies :

1. $[x, y] = -(-1)^{|x||y|}[y, x]$ (super skew-symmetry),
2. $[x, [y, z]] + (-1)^{|x|(|y|+|z|)}[y, [z, x]] + (-1)^{(|z|(|x|+|y|)}[z, [x, y]] = 0$ (super Jacobi identity).

3. $[x, yz] = [x, y]\, z + (-1)^{|x||y|} y\, [x, z]$ (super Leibniz's rule).

Definition 6.1.5. A *Poisson supermanifold* is a K-supermanifold ($K = G^\infty$ or H^∞) such that the superalgebra of supersmooth functions over it is equipped with a Poisson superbracket

In this chapter we give an explicit formula of a Poisson bracket $\{,\}_{ST}$ on the super-Teichmüller $X-$space but first of all we recall the definition of such a bracket in the classical case.

6.2 The Weil-Petersson Poisson structure on $\mathcal{T}^X$

There are two ways of defining the Weil-Petersson Poisson bracket on $\mathcal{T}^X$. Following [11], one knows that the space of homomorphisms of the fundamental group of a surface S into a reductive group G up to conjugation by G admits a canonical Poisson structure. For $G = \mathrm{PSL}(2, \mathbb{R})$, $\mathcal{T}^X$ can be mapped to this space and this map induces the Weil-Petersson Poisson structure on $\mathcal{T}^X$. The other definition was given by Goldman [16] in terms absolute values of traces of elements of $\mathrm{PSL}(2, \mathbb{R})$ corresponding to closed loops on the surface S. He showed that the Poisson bracket between trace functions is a linear combination of trace functions. Finally, an explicit formula of the Weil-Petersson bracket $\{,\}_{\mathcal{T}^X}$ in the term of shear coordinates (cf. [14]) is given by

$$\{,\}_{\mathcal{T}^X} = \sum_{i,j \in E(\Lambda)} \varepsilon^{ij} x_i x_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}, \tag{6.1}$$

where ε^{ij} is given by Equation (1.1).

Although the coordinates on $\mathcal{T}^X(S)$ correspond to a choice of a triangulation of S the Poisson bracket $\{,\}_{\mathcal{T}^X}$ has the interesting property to be independent of this particular triangulation. This can be checked by substitution of the change of coordinates rules (1.2) in (6.1). We do not give the computation here since it will be done in the section about the super-Poisson bracket on the super-Teichmüller $X-$space.

Definition 6.2.1. Consider the space of regular functions on ST. One defines two odd derivations, $\frac{\overrightarrow{\partial}}{\partial \zeta_i}$ and $\frac{\overleftarrow{\partial}}{\partial \zeta_i}$, which are operators acting respectively on the right and on the left (see for example [10]).

1. The action on the coordinate functions is given by:
$$\frac{\overrightarrow{\partial}\zeta_j}{\partial \zeta_i} = \frac{\overleftarrow{\partial}\zeta_j}{\partial \zeta_i} = \delta_{ij} \quad \text{and} \quad \frac{\overrightarrow{\partial}x_j}{\partial \zeta_i} = \frac{\overleftarrow{\partial}x_j}{\partial \zeta_i} = 0.$$

2. The action on a monomial $\zeta_1 \dots \zeta_i \dots \zeta_k$ is given by:
$$\frac{\overrightarrow{\partial}}{\partial \zeta_i}(\zeta_1 \dots \zeta_i \dots \zeta_k) = (-1)^{i-1}\zeta_1 \dots \hat{\zeta}_i \dots \zeta_k,$$
$$\frac{\overleftarrow{\partial}}{\partial \zeta_i}(\zeta_1 \dots \zeta_i \dots \zeta_k) = (-1)^{k-i}\zeta_1 \dots \hat{\zeta}_i \dots \zeta_k.$$

Theorem 6.2.2. *The following formula defines a super Poisson structure on the super Teichmüller $X-$space :*

$$\{,\}_{ST} = \sum_{i,j\in E(\Lambda_\bullet)} \varepsilon^{ij} x_i x_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{1}{2}\sum_{k\in F(\Lambda_\bullet)} \frac{\overleftarrow{\partial}}{\partial \zeta_k}\frac{\overrightarrow{\partial}}{\partial \zeta_k} \tag{6.2}$$

and this Poisson bracket does not depend on the particular triangulation.

Proof. Let f, g and h be three functions of given parity $|f|, |g|$ and $|h|$.

1. The bracket $\{,\}_{ST}$ is super skew-symmetric :

$$\begin{aligned}
\{f,g\}_{ST} &= \sum_{i,j\in E(\Lambda_\bullet)} \varepsilon^{ij} x_i x_j \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} + \frac{1}{2}\sum_{k\in F(\Lambda_\bullet)} \frac{\overleftarrow{\partial} f}{\partial \zeta_k}\frac{\overrightarrow{\partial} g}{\partial \zeta_k} \\
&= \sum_{i,j\in E(\Lambda_\bullet)} -\varepsilon^{ji} x_i x_j \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} + \frac{1}{2}\sum_{k\in F(\Lambda_\bullet)} (-1)^{|f|+1}\frac{\overrightarrow{\partial} f}{\partial \zeta_k}(-1)^{|g|+1}\frac{\overleftarrow{\partial} g}{\partial \zeta_k} \\
&= \sum_{i,j\in E(\Lambda_\bullet)} -\varepsilon^{ji} x_j x_i (-1)^{|f||g|}\frac{\partial g}{\partial x_j}\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{k\in F(\Lambda_\bullet)} (-1)^{|f|+|g|}\frac{\overrightarrow{\partial} f}{\partial \zeta_k}\frac{\overleftarrow{\partial} g}{\partial \zeta_k} \\
&= -(-1)^{|f||g|} \sum_{i,j\in E(\Lambda_\bullet)} \varepsilon^{ji} x_j x_i \frac{\partial g}{\partial x_j}\frac{\partial f}{\partial x_i} \\
&\qquad + \frac{1}{2}(-1)^{|f|+|g|}\sum_{k\in F(\Lambda_\bullet)} (-1)^{(|f|+1)(|g|+1)}\frac{\overleftarrow{\partial} g}{\partial \zeta_k}\frac{\overrightarrow{\partial} f}{\partial \zeta_k} \\
&= -(-1)^{|f||g|}\{g,f\}_{ST}.
\end{aligned}$$

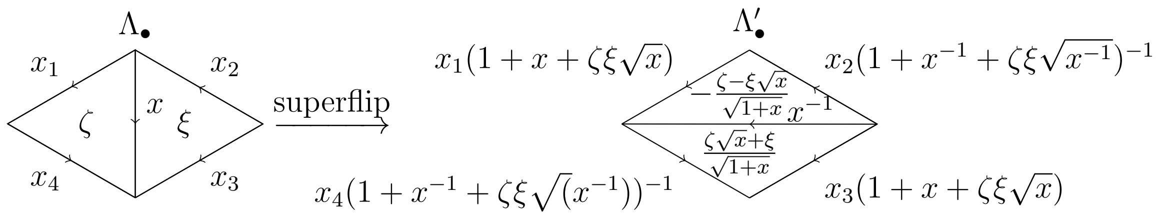
2. The bracket $\{,\}_{ST}$ satisfies the super Leibniz's rule :

$$\begin{aligned}
\{f,gh\}_{ST} &= \sum_{i,j\in E(\Lambda_\bullet)} \varepsilon^{ij} x_i x_j \frac{\partial f}{\partial x_i}\frac{\partial gh}{\partial x_j} + \frac{1}{2}\sum_{k\in F(\Lambda_\bullet)} \frac{\overleftarrow{\partial} f}{\partial \zeta_k}\frac{\overrightarrow{\partial} gh}{\partial \zeta_k} \\
&= \sum_{i,j\in E(\Lambda_\bullet)} \varepsilon^{ij} x_i x_j \frac{\partial f}{\partial x_i}\left(\frac{\partial g}{\partial x_j}h + g\frac{\partial h}{\partial x_j}\right) \\
&\qquad + \frac{1}{2}\sum_{k\in F(\Lambda_\bullet)} \frac{\overleftarrow{\partial} f}{\partial \zeta_k}\left(\frac{\overrightarrow{\partial} g}{\partial \zeta_k}h + (-1)^{|g|}g\frac{\overrightarrow{\partial} h}{\partial \zeta_k}\right) \\
&= \{f,g\}_{ST}\,h + \sum_{i,j\in E(\Lambda_\bullet)} \varepsilon^{ij} x_i x_j \frac{\partial f}{\partial x_i} g \frac{\partial h}{\partial x_j} + \frac{1}{2}(-1)^{|g|}\sum_{k\in F(\Lambda_\bullet)} \frac{\overleftarrow{\partial} f}{\partial \zeta_k} g \frac{\overrightarrow{\partial} h}{\partial \zeta_k} \\
&= \{f,g\}_{ST}\,h + (-1)^{|f||g|} g \sum_{i,j\in E(\Lambda_\bullet)} \varepsilon^{ij} x_i x_j \frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_j} \\
&\qquad + \frac{1}{2}(-1)^{|g|}(-1)^{|g|(|f|-1)} g \sum_{k\in F(\Lambda_\bullet)} \frac{\overleftarrow{\partial} f}{\partial \zeta_k}\frac{\overrightarrow{\partial} h}{\partial \zeta_k} \\
&= \{f,g\}_{ST}\,h + (-1)^{|f||g|} g\,\{f,h\}_{ST}.
\end{aligned}$$

3. The bracket $\{,\}_{ST}$ satisfies the super Jacobi identity :

   - First of all it is sufficient to show that the identity is satisfied for monomials in the element x_i and ζ_k, because of the linearity of the bracket.
   - Consider now four generators $a, b, c, d \in \{x_1, \ldots, x_E, \zeta_1, \ldots, \zeta_T\}$, where E and T are the numbers of edges and triangles in $\Lambda_\bullet$ such that the super Jacoby identity is satisfied for a, b, c and a, b, d. Using the super Leibniz's rule and the super skew-symmetry, a straightforward computation shows that the identity is satisfied for a, b, cd. So it is sufficient to show that the identity is satisfied for the coordinate functions.
   - Assume that one of the functions is ζ_i, then all the terms of the left hand side equal 0. So the identity is obviously satisfied. If all the functions are even coordinate functions then the computations reduce to the classical one, so the identity is satisfied.

We now check that the superflips and the switches are Poisson maps. The case of a switch is obvious because the change of coordinates is given by a change of sign of some odd coordinates. Consider now the following situation :



We set :

$$
\begin{aligned}
x'_1 &= x_1\left(1+x+\zeta\xi\sqrt{x}\right) & \zeta' &= -\frac{\zeta-\xi\sqrt{x}}{\sqrt{1+x}}\\
x'_2 &= x_2\left(1+x^{-1}+\zeta\xi\sqrt{x^{-1}}\right)^{-1} & \xi' &= \frac{\zeta\sqrt{x}+\xi}{\sqrt{1+x}}\\
x'_3 &= x_3\left(1+x+\zeta\xi\sqrt{x}\right)\\
x'_4 &= x_4\left(1+x^{-1}+\zeta\xi\sqrt{x^{-1}}\right)^{-1}\\
x' &= x^{-1}
\end{aligned}
$$

and we consider the Poisson bracket

$$
\{,\}'_{ST} = \sum_{i,j\in E(\Lambda'_\bullet)} (\varepsilon')^{ij} x'_i x'_j \frac{\partial}{\partial x'_i}\frac{\partial}{\partial x'_j} + \frac{1}{2}\sum_{k\in F(\Lambda'_\bullet)} \frac{\overleftarrow{\partial}}{\partial \zeta'_k}\frac{\overrightarrow{\partial}}{\partial \zeta'_k}.
$$

We get on one side

$$\{x', x'_1\}'_{ST} = -x'x'_1, \quad \{x', x'_2\}'_{ST} = x'x'_2, \quad \{x', x'_3\}'_{ST} = -x'x'_3, \quad \{x', x'_4\}'_{ST} = x'x'_4,$$
$$\{x'_1, x'_2\}'_{ST} = -x'_1x'_2, \quad \{x'_3, x'_4\}'_{ST} = -x'_3x'_4,$$

$$\{x'_1, x'_4\}'_{ST} = \{x'_1, x'_3\}'_{ST} = \{x'_2, x'_4\}'_{ST} = \{x'_2, x'_3\}'_{ST} = 0,$$

$$\{\zeta', \zeta'\}'_{ST} = \{\xi', \xi'\}'_{ST} = \tfrac{1}{2}, \quad \{\zeta', \xi'\}'_{ST} = 0,$$
$$\{\zeta', x'_i\}'_{ST} = \{\xi', x'_i\}'_{ST} = 0.$$

and on the other side :

$$\begin{aligned}\{x', x'_1\}_{ST} &= xx_1\frac{\partial x'}{\partial x}\frac{\partial x'_1}{\partial x_1} = xx_1\frac{-1}{x^2}(1 + x + \zeta\xi\sqrt{x}) \\ &= -x^{-1}x_1(1 + x + \zeta\xi\sqrt{x}) = -x'x'_1.\end{aligned}$$

$$\begin{aligned}\{x', x'_2\}_{ST} &= -xx_2\frac{\partial x'}{\partial x}\frac{\partial x'_2}{\partial x_2} = -xx_2\frac{-1}{x^2}\left(1 + x^{-1} + \zeta\xi\sqrt{x^{-1}}\right) \\ &= x^{-1}x_2\left(1 + x^{-1} + \zeta\xi\sqrt{x^{-1}}\right) = x'x'_2.\end{aligned}$$

$$\begin{aligned}\{x'_1, x'_2\}_{ST} &= -x_1x\frac{\partial x'_1}{\partial x_1}\frac{\partial x'_2}{\partial x} - xx_2\frac{\partial x'_1}{\partial x}\frac{\partial x'_2}{\partial x_2} \\ &= -x_1x(1 + x + \zeta\xi\sqrt{x})x_2\frac{1 + \frac{1}{2}\zeta\xi\sqrt{x}}{(1 + x + \zeta\xi\sqrt{x})^2} - xx_1x_2\frac{1 + \frac{1}{2}\zeta\xi\sqrt{x^{-1}}}{\left(1 + x^{-1} + \zeta\xi\sqrt{x^{-1}}\right)} \\ &= -x_1x_2\frac{1 + \frac{1}{2}\zeta\xi\sqrt{x}}{\left(1 + x^{-1} + \zeta\xi\sqrt{x^{-1}}\right)} - xx_1x_2\frac{1 + \frac{1}{2}\zeta\xi\sqrt{x^{-1}}}{\left(1 + x^{-1} + \zeta\xi\sqrt{x^{-1}}\right)} \\ &= -x_1x_2\left(1 + x^{-1} + \zeta\xi\sqrt{x^{-1}}\right)^{-1}\left(1 + x + \zeta\xi\sqrt{x}\right) = -x'_1x'_2.\end{aligned}$$

The same kind of computations works for the others pairs $\left\{x'_i, x'_j\right\}_{ST}$ and for the odd functions we get :

$$\begin{aligned}\{\zeta', \zeta'\}_{ST} &= \frac{1}{2}\left(\frac{-1}{\sqrt{1+x}}\right)^2 + \frac{1}{2}\left(\frac{\sqrt{x}}{\sqrt{1+x}}\right)^2 = \frac{1}{2}. \\ \{\xi', \xi'\}_{ST} &= \frac{1}{2}\left(\frac{\sqrt{x}}{\sqrt{1+x}}\right)^2 + \frac{1}{2}\left(\frac{1}{\sqrt{1+x}}\right)^2 = \frac{1}{2}. \\ \{\zeta', \xi'\}_{ST} &= \frac{1}{2}\frac{-1}{\sqrt{1+x}}\frac{\sqrt{x}}{\sqrt{1+x}} + \frac{1}{2}\frac{\sqrt{x}}{\sqrt{1+x}}\frac{1}{\sqrt{1+x}} = 0.\end{aligned}$$

And finally :

$$\{\zeta', x'\}_{ST} = 0$$

$$\begin{aligned}
\{x_1', \zeta'\}_{ST} &= -x_1 x \frac{\partial x_1'}{\partial x_1}\frac{\partial \zeta'}{\partial x} + \frac{1}{2}\frac{\overleftarrow{\partial}\, x_1'}{\partial \zeta}\frac{\overrightarrow{\partial}\,\zeta'}{\partial \zeta} + \frac{1}{2}\frac{\overleftarrow{\partial}\, x_1'}{\partial \xi}\frac{\overrightarrow{\partial}\,\zeta'}{\partial \xi} \\
&= -x_1 x \left(1 + x + \zeta\xi\sqrt{x}\right) \frac{\frac{1}{2}\xi\sqrt{x^{-1}}\sqrt{1+x} + (\zeta - \xi\sqrt{x})\frac{1}{2}\sqrt{(1+x)^{-1}}}{1+x} \\
&\quad + \frac{1}{2}\frac{x_1\xi\sqrt{x}}{\sqrt{1+x}} + \frac{1}{2}\frac{x_1\zeta\sqrt{x}\sqrt{x}}{\sqrt{1+x}} \\
&= -x_1 x \frac{1}{2\sqrt{1+x}}\left(\xi\sqrt{x^{-1}} + \zeta\right) + \frac{x_1\left(\xi\sqrt{x} + \zeta x\right)}{2\sqrt{1+x}} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\{x_2', \zeta'\}_{ST} &= x_2 x \frac{\partial x_2'}{\partial x_2}\frac{\partial \zeta'}{\partial x} + \frac{1}{2}\frac{\overleftarrow{\partial}\, x_2'}{\partial \zeta}\frac{\overrightarrow{\partial}\,\zeta'}{\partial \zeta} + \frac{1}{2}\frac{\overleftarrow{\partial}\, x_2'}{\partial \xi}\frac{\overrightarrow{\partial}\,\zeta'}{\partial \xi} \\
&= x_2 x \left(1 + x^{-1} + \zeta\xi\sqrt{x^{-1}}\right)^{-1} \frac{\frac{1}{2}\xi\sqrt{x^{-1}}\sqrt{1+x} + (\zeta - \xi\sqrt{x})\frac{1}{2}\sqrt{(1+x)^{-1}}}{1+x} \\
&\quad + \frac{1}{2}(1+x^{-1})^{-2}\frac{-x_2\xi\sqrt{x^{-1}}}{\sqrt{1+x}} + \frac{1}{2}(1+x^{-1})^{-2}\frac{-x_2\zeta\sqrt{x^{-1}}\sqrt{x}}{\sqrt{1+x}} \\
&= x_2 x (1+x^{-1})^{-1}\frac{\sqrt{(1+x)^{-1}}}{2(1+x)}\left(\xi\sqrt{x^{-1}} + \zeta\right) - \frac{x_2(1+x^{-1})^{-2}\left(\xi\sqrt{x^{-1}} + \zeta\right)}{2\sqrt{1+x}} \\
&= 0.
\end{aligned}$$

□

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Fabien BOUSCHBACHER

Des coordonnées de décalage sur le super espace de Teichmüller

Résumé

Dans cette thèse nous étudions un super-analogue de l'espace de Teichmüller des surfaces à trous. Le but de notre étude est la construction sur cet espace de coordonnées analogues aux coordonnées de décalage de Thurston-Bonahon-Fock-Penner. Ces coordonnées dépendent du choix d'une triangulation idéale de la surface de départ. Nous étudions les changements de coordonnées lorsque l'on change cette triangulation de la surface. Nous démontrons également que cet espace possède une structure de Poisson canonique et que cette structure est indépendante du choix de la triangulation.

Mots-clés : espace de Teichmüller, coordonnées de décalage, supervariété, structure spin, structure de Poisson.

Résumé en anglais

In this thesis we study a superanalogue of the Teichmüller space of surfaces with holes. The aim of our study is the construction of coordinates on this space which are analogous to the Thurston-Bonahon-Fock-Penner shear coordinates. These coordinates depend on a choice of an ideal triangulation of the given surface. We study the changes of coordinates when we modify the triangulation by elementary moves. We also show that this space admits a canonical Poisson structure which is independent of the choice of a triangulation.

Keywords: Teichmüller space, shear coordinates, supermanifold, spin structure, Poisson structure.