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Stabilisation rapide et observation en plusieurs instants de systèmes oscillants

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Résumé

Ce travail s'articule autour de deux concepts importants de la théorie du contrôle : la *stabilisation* et l'*observation*. Il est composé de deux parties indépendantes.

1. Stabilisation rapide. Considérons un système dont l'état x est solution du problème abstrait

$$(\mathcal{P}) \qquad \begin{cases} x'(t) = Ax(t) + Bu(t), \\ x(0) = x_0. \end{cases}$$

En pratique A est un opérateur différentiel linéaire (générateur d'un semi-groupe) qui modélise la dynamique du système et B est un opérateur de contrôle permettant d'agir sur ce système grâce au contrôle u.

Stabiliser le système régi par $(\mathcal{P})$ consiste à trouver un contrôle sous la forme d'un *feedback* (c'est-à-dire que le contrôle dépend de l'état du système à chaque instant)

$$(\mathcal{F}) \qquad u(t) = Fx(t)$$

tel que les solutions du problème en boucle fermée $(\mathcal{P})$-$(\mathcal{F})$ décroissent vers zéro lorsque t tend vers $+\infty$.

À la fin des années 1960, Lukes et Kleinman ont donné une construction systématique d'un tel feedback pour des systèmes en dimension finie. Celle-ci repose sur un Gramien de contrôlabilité et présuppose la contrôlabilité exacte de $(\mathcal{P})$. En 1974, Slemrod a étendu cette construction à des systèmes de dimension infinie avec un opérateur de contrôle *borné* et l'a améliorée : en ajoutant une fonction poids dépendant d'un paramètre ω à l'intérieur du Gramien, son feedback permet d'obtenir une décroissance exponentielle des

solutions à un taux au moins égal à ω, c'est-à-dire qu'on peut trouver une constante positive c telle que pour toute donnée initiale x_0

$$\|x(t)\| \leq ce^{-\omega t}\|x_0\|, \qquad t \geq 0.$$

Le fait que l'opérateur B soit borné limite néanmoins le champ d'application de cette méthode à des contrôles distribués excluant ainsi le cas d'un *contrôle frontière*, très important en pratique.

Sous l'impulsion de Lions [**45**], notamment avec sa méthode d'unicité hilbertienne (HUM), la théorie du contrôle des équations aux dérivées partielles a pris un nouvel essor à la fin des années 1980. En ce qui concerne le problème de la stabilisation frontière, essentiellement deux stratégies dominent alors.

D'une part, des *feedbacks explicites et relativement "simples"* à formuler permettent d'atteindre une stabilité exponentielle avec des estimations très fines du taux de décroissance (Quinn-Russell, Chen, Lasiecka-Triggiani, Komornik...). Néanmoins, avec cette classe de feedback, le taux de décroissance maximal est limité (Koch-Tataru).

Une autre approche consiste à utiliser la théorie du *contrôle optimal* [**24**]. Il s'agit de minimiser une fonctionnelle de coût et à caractériser l'optimum de ce problème de minimisation. On obtient ainsi une stabilité exponentielle (Lions, Flandoli, Lasiecka, Triggiani) mais en général ni le feedback ni le taux de décroissance ne sont explicites et le feedback dépend de la résolution d'équations de Riccati en dimension infinie, cette dernière tâche pouvant s'avérer difficile.

En 1997, Komornik [**32**] a réussi à adapter le feedback explicite construit à l'aide d'un Gramien pondéré au cas d'un opérateur de contrôle *non-borné*, incluant ainsi le cas du contrôle frontière. Le Gramien utilisé peut s'écrire formellement

$$(\mathcal{G}) \qquad \Lambda_\omega := \int_0^T e_\omega(t) e^{-tA} BB^* e^{-tA^*}\, \mathrm{d}t.$$

Sous certaines hypothèse, on peut s'assurer que Λ_ω est inversible et le feedback s'écrit alors

$$(\mathcal{F}_\omega) \qquad F = -B^* \Lambda_\omega^{-1}.$$

Remarquons que la fonction poids e_ω est choisie de telle manière à ce que l'opérateur Λ_ω soit la solution d'une équation de Riccati de la forme

$$A\Lambda_\omega + \Lambda_\omega A^* + \Lambda_\omega C^* C \Lambda_\omega - BB^* = 0$$

où l'opérateur C peut lui aussi être donné explicitement. Il y a donc un problème de contrôle optimal sous-jacent à cette méthode mais l'une des difficultés principales de la méthode de contrôle optimal – la résolution d'une équation de Riccati – est évitée. En adaptant des résultats de Flandoli [**23**] sur les équations de Riccati, Komornik a ainsi pu démontrer la stabilité exponentielle des solutions du problème $(\mathcal{P})$-$(\mathcal{F}_\omega)$ à un taux au moins égal à ω.

Il restait néanmoins des questions ouvertes concernant le caractère bien posé du problème en boucle fermée $(\mathcal{P})$-$(\mathcal{F}_\omega)$ ainsi que la stabilité des solutions :

(a) *Quelle est la notion de solution la mieux adaptée à ce problème ?*
(b) *L'opérateur $A - BB^*\Lambda_\omega^{-1}$ est-il le générateur d'un semi-groupe ?*
(c) *Le taux de décroissance en ω des solutions est-t-il optimal ?*

En utilisant l'équation de Riccati vérifiée par Λ_ω, il est possible de faire apparaître un problème dual "naturellement" associé à $(\mathcal{P})$-$(\mathcal{F}_\omega)$. On obtient alors un opérateur conjugué à $A - BB^*\Lambda_\omega^{-1}$ (au moins formellement) et dont on sait qu'il est le générateur d'un semi-groupe. L'idée est alors de définir la solution de $(\mathcal{P})$-$(\mathcal{F}_\omega)$ à partir du conjugué de ce semi-groupe. En établissant des formules analogues à celles de Flandoli, on démontre que cette notion de solution est cohérente car elle vérifie une formule de variation de la constante liée au problème. D'autre part, le générateur du semi-groupe en question est

$$\widetilde{A} - BB^*\Lambda_\omega^{-1}$$

où $\widetilde{A}$ est une extension de A et son domaine est donné par $\Lambda_\omega\mathcal{D}(A^*)$. On fait aussi apparaître une différence essentielle entre le cas d'un opérateur de contrôle B borné et celui d'un opérateur de contrôle non borné. Dans le premier cas l'extension de A peut être omise puisque $\Lambda_\omega\mathcal{D}(A^*) = \mathcal{D}(A)$. Au contraire, dans le cas non-borné, on démontre à l'aide d'exemples que l'extension est nécessaire puisque l'égalité des domaines n'est pas vraie en général.

La justification du caractère bien posé à travers l'utilisation d'un problème dual permet de donner une autre démonstration de la décroissance exponentielle des solutions à un taux supérieur à ω et aussi d'étudier plus finement le taux de décroissance effectif. Ce point est motivé par des expériences numériques ainsi que des tests mécaniques sur des poutres [**6**] réalisés à la fin des années 1990 par Bourquin et ses collaborateurs (Briffaut, Ratier, Urquiza...). Ceux-ci, outre le fait de prouver l'efficacité du feedback de Komornik, ont découvert un phénomène intéressant : le taux de décroissance effectif semble se situer autour de 2ω. En analysant plus en détail le rôle du paramètre T dans le Gramien modifié $(\mathcal{G})$, on donne une justification de ce taux "double".

2. Observation en plusieurs instants. Considérons $y(t)$ la position d'un système oscillant – par exemple une corde vibrante – à l'instant t. Motivée

par un article de Szijártó et Hegedűs [**58**], la question posée par l'observation en plusieurs instants est de savoir si à partir de la donnée des positions de la corde à certains instants $t_1, t_2, \ldots$ il est possible de déterminer complètement les données initiales c'est-à-dire, dans le cas de la corde vibrante, sa position et sa vitesse initiales $y(0)$ et $y'(0)$.

Cela revient à démontrer des estimations de la forme

$$\|y(0)\|_g + \|y'(0)\|_{g'} \leq c\big(\|y(t_1)\|_d + \|y(t_2)\|_d + \ldots\big)$$

où c est une constante positive indépendante de y.

L'existence de telles estimations dépend de certaines propriétés arithmétiques des quantités $t_i - t_j$ ainsi que des normes $\| \cdot \|_*$ choisies dans les membres de gauche et de droite de l'estimation ci-dessus.

En utilisant le développement des solutions en *série de Fourier* ainsi que des résultats classiques d'*approximation diophantienne*, on donne, dans le cas de la corde vibrante, les hypothèses minimales sur les instants d'observation et sur les normes dans les deux membres pour obtenir de telles estimations. On donne aussi une mesure de l'abondance de tels instants et on montre qu'en augmentant le nombre d'observations, on peut relaxer les normes dans les deux membres.

Dans le cas de la corde vibrante avec l'ajout d'un potentiel, il est possible d'adapter la méthode utilisée pour la corde vibrante "classique". On obtient des résultats complémentaires à ceux de Szijártó et Hegedűs. On étend aussi ces résultats à d'autres systèmes oscillants : poutre vibrante, plaque vibrante de forme rectangulaire etc.

Enfin, on utilise la méthode d'unicité hilbertienne (HUM) pour démontrer la contrôlabilité exacte d'un problème associé en donnant une caractérisation des états contrôlables.

Notations

$\|\cdot\|_X$ $\langle\cdot,\cdot\rangle_{X',X}$ $(\cdot,\cdot)_X$ $\|\cdot\|$	Given a normed vector space X, $\|.\|_X$ denotes its norm, $\langle .,.\rangle_{X',X}$ denotes the duality pairing between X and its dual X'; $(.,.)_X$ represents a scalar product. *Sometimes we will omit the name of the spaces below the norm, brackets, parentheses in order to simplify the notations.* Depending on the context, $\|.\|$ can have different meanings: • if x belongs to a normed vector space X, $\|x\|$ denotes its norm; • if T is a bounded linear operator between two vector spaces, $\|T\|$ denotes its operator norm; • if x is a real number, $\|x\|$ denotes the distance to the nearest integer (only in Part 2). We will always identify a Hilbert space with its double dual.
$y(t)$	In the applications, when a function y depends on two variables x (space) and t (time), we will often denote by $y(t)$ the map $x \mapsto y(t,x)$.
y' (resp. y_x)	denotes the time (resp. space) derivative.
$\mathcal{C}(I;X)$ (resp. $\mathcal{C}^k(I,X)$)	denotes the space of continuous (resp. k-times continuously differentiable) functions from the interval I into the normed vector space X.

$\mathcal{L}(X, Y)$	denotes the space of bounded linear applications from a normed vector space X into a normed vector space Y.
$\rho(A)$	denotes the resolvent set of the linear operator A.
$A \asymp B$	means that there are two positive constants c_1 and c_2 such that $$c_1 B \leq A \leq c_2 B.$$
semigroup (resp. *group*)	means *strongly continuous semigroup* (resp. *strongly continuous group*).
$\mathbb{1}_E$	denotes the characteristic function of the set E.
$\lambda(E)$	denotes the Lebesgue measure of a set $E \subset \mathbb{R}^n$.
$\dim_H(E)$	denotes the Hausdorff dimension of a set $E \subset \mathbb{R}^n$.
positive	> 0
nonnegative	≥ 0
negative	< 0
nonpositive	≤ 0

Part 1

Rapid stabilization

Chapter 1

Introduction

This part deals with the stabilization of a class of *infinite-dimensional*, *linear* and *time-reversible* systems; they may model oscillating mechanisms like a vibrating membrane or plate.

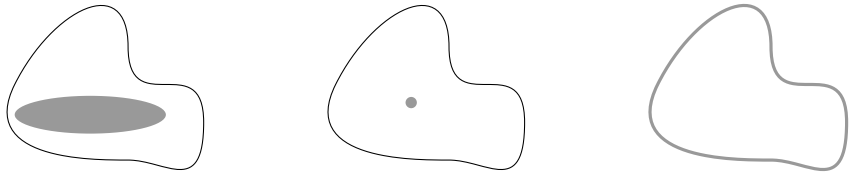
The prototype of such a system is the linear wave equation on a bounded domain, modelling for instance in dimension two the small vibrations of a membrane. An important feature is that one may act on the system by means of a *control*. In the case of the membrane, one can act on it by different ways (see Figure 1), for example using a control

(a) distributed in an *open subset* of the membrane;

(b) localized at a *point* of the membrane;

(c) restricted to the *boundary* of the membrane.

In the engineering applications, the last two possibilities are often more convenient to implement; from a theoretical point of view they are also more difficult to study, as we shall see later. We will focus on the last possibility, i.e., on the *boundary control*.



(a) distributed **(b)** pointwise **(c)** boundary

Figure 1. Three ways of acting on a membrane: the grey parts indicate the control regions.

Let Ω be an open, bounded subset of $\mathbb{R}^n$ and Γ denote its boundary. For convenience we assume that the boundary is sufficiently smooth, for example of class $\mathcal{C}^2$. The wave equation with a Dirichlet boundary control comes down to

$$\begin{cases} y'' - \Delta y = 0 & \text{in } (0,\infty)\times\Omega; \\ y = u & \text{in } (0,\infty)\times\Gamma; \\ y(0) = y_0, \quad y'(0) = y_1 & \text{in } \Omega. \end{cases} \tag{1}$$

The *stabilization problem* consists in finding a control in the form of a *feedback* (i.e., depending on the state of the system)

$$u = F(y, y') \tag{2}$$

such that the *closed-loop system* (1)-(2) is well-posed and its solution decreases to zero as t tends to infinity.

For infinite-dimensional systems, several notions of stability coexist; we will always look for *uniform exponential stability* i.e., we want the solution of the closed-loop problem to satisfy the estimation

$$\|(y(t), y'(t)\| \le ce^{-\omega t}\|(y_0, y_1)\|, \qquad t \ge 0 \tag{3}$$

for a suitable norm $\|\cdot\|$ and suitable positive constants c and ω independent of the initial data y_0 and y_1.

Since the late 1970's, many works were devoted to the obtaining of *explicit boundary feedbacks* that stabilize the wave equation; see the works by Quinn and Russell [**50**], Chen [**15**], Lagnese [**36**], Lasiecka and Triggiani [**38**], Komornik [**30, 31**]. More precisely, with feedbacks of the form

$$u = ay + by' + c\partial_\nu y \qquad \text{in } (0,\infty)\times\Gamma, \tag{4}$$

where a, b and c are suitable functions, it is possible to obtain an exponential decay of the solutions with sharp estimations of the decay rate ω. For example, from [**31**, Theorem 8.6 p. 106], we can extract the following result [1] :

Theorem 1.1 (Komornik)**.** *Let $n \ge 3$. Assume that there is a point $x_0 \in \mathbb{R}^n$ such that* [2] $(x - x_0)\cdot\nu(x) \ge 0$ *on* Γ*. Setting* $R := \sup\{|x - x_0|, x \in \Omega\}$ *and*

$$a(x) := 1 + \frac{(n-1)(x-x_0)\cdot\nu(x)}{2R^2}, \quad b(x) := \frac{(x-x_0)\cdot\nu(x)}{R}, \quad c(x) := 1$$

in (4)*, the closed-loop problem* (1)-(4) *is well-posed in* $H := H^1(\Omega)\times L^2(\Omega)$ *and there is a positive constant c such that for every initial data* $(y_0, y_1) \in H$*, the solution satisfies the estimation* (3) *with* $\omega = 1/4R$*.*

[1] For the boundary stabilization (although not necessarily exponential) of the plate equation in a bounded domain with an explicit feedback, we refer e.g., to [**51**].

[2] ν denotes the outward unit vector to the boundary Γ.

Unfortunately, a result by Koch and Tataru [**29**] states that the decay rate achievable with feedbacks of the type (4) is limited by a quantity depending on the geometry of the domain Ω.

Another efficient way to find stabilizing feedbacks is the use of *optimal control theory*. It consists of minimizing a cost functional that penalizes the control u and the state (y, y') of the corresponding solution. For instance, the cost functional could be

$$J(u) := \int_0^\infty \|u(t)\|^2_{L^2(\Gamma)}\, \mathrm{d}t + \int_0^\infty \|y(t)\|^2_{L^2(\Omega)}\, \mathrm{d}t.$$

An important but non-trivial point is to check that the functional J takes some finite values [1]. Then, convex optimization techniques ensure that there exists a unique control u_* minimizing the functional J over $L^2(0, \infty; L^2(\Gamma))$. Denoting by (y_*, y'_*) the associated optimal trajectory, it is possible to prove that the optimal control can be expressed as a feedback:

$$u_* = \partial_\nu (P y_* + Q y'_*),$$

where P and Q are bounded operators related to the solution of an infinite-dimensional Riccati equation. Eventually, the exponential decay follows using a result of Datko (see e.g., [**49**, Theorem 4.1 p. 116]). From the review paper by Lions [**45**], we can extract the following

Theorem 1.2 (Lions; Lasiecka and Triggiani)**.** *There exist two operators*

$$P : H^{-1}(\Omega) \to H^1_0(\Omega) \qquad \text{and} \qquad Q : L^2(\Omega) \to H^1_0(\Omega)$$

and two positive constants c and ω such that, setting

$$u = \partial_\nu (P y' + Q y), \tag{5}$$

the closed-loop problem (1)-(5) *is well-posed in $L^2(\Omega) \times H^{-1}(\Omega)$ and satisfies the estimation* (3)*.*

For a general description of this method of optimal control and other results, we refer to the paper by Flandoli, Lasiecka and Triggiani [**24**]. Although this method may be applied to many cost functionals, we can state two drawbacks:

- to obtain the operators P and Q, we need to solve an infinite-dimensional Riccati equation, which may be a difficult task (in general, these operators are not explicit);
- in general, the exponential decay rate ω (in the estimation (3)) is not explicit.

[1] A sufficient condition is the exact controllability of the problem (1): there is a time T such that for any initial data (y_0, y_1) and any final data (y_0^T, y_1^T), we can find a control u steering the system from the initial given state to the final given state. For this point, we refer to [**45**] or [**31**].

In the 1990's, trying to remove the drawbacks of these two methods and inspired by a finite-dimensional method, Komornik [**32**] introduced a systematic method to stabilize infinite-dimensional systems, covering in particular the case of the wave equation with a boundary control. Since this method has its origin in the stabilization of finite-dimensional systems and since it may be applied for a large class of systems, it is more convenient to describe it in an *abstract framework*.

Let us consider an abstract system which state x satisfies the problem

$$\begin{cases} x' = Ax + Bu; \\ x(0) = x_0, \end{cases} \tag{6}$$

where A is a linear operator that models the dynamics of the system and B is a control operator that allows us to act on the system through a control u.

The *stabilization problem* consists of finding a feedback operator F such that with the control

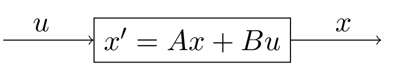
$$u = Fx \tag{7}$$

the solution of the *closed-loop problem* (6)-(7) tends to zero as t tends to $+\infty$. As already stated for the wave equation, we are interested in exponential stability [1] i.e., we want to prove the existence of two positive constants ω and c such that for each initial datum, the following estimation holds:

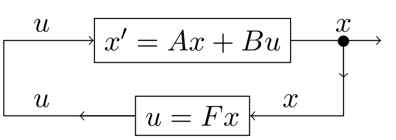
$$\|x(t)\| \leq ce^{-\omega t}\|x_0\|, \qquad t \geq 0, \tag{8}$$

for a suitable norm $\|\cdot\|$ on the state space.

Remark 1.3. The system (6) is called an *open-loop* system, while the system (6)-(7) is called a *closed-loop* system. The terminology comes from the two following diagrams. If a control $u(t)$ is defined externally, we can compute $x(t)$ through (6): the loop is open as it may be represented by the diagram [2]



If the control $u(t)$ is constructed from $x(t)$ through (7), the loop is closed and we have to modify the above diagram to



[1] Though for finite-dimensional systems, a solution tends to zero if and only if it tends to zero exponentially (see e.g., [**63**, Theorem 2.3 p. 30]), for infinite-dimensional systems several non-equivalent notions of stability coexist (see e.g., [**63**, section 3.1], [**2**, section 4.11.1]).

[2] The diagrams are taken from [**54**, p. 113].

To the problem (6) we associate the *dual problem*

$$\begin{cases} \varphi' = -A^*\varphi; \\ \varphi(0) = \varphi_0. \end{cases} \tag{9}$$

Now we recall some points of terminology in the

Definition 1.4. Let $T > 0$.

The system (6) (equivalently the pair (A, B)) is *exactly controllable in time* T if for any pair of data x_0, x_1 there is a control u such that the solution of (6) satisfies $x(T) = x_1$.

The system (9) (equivalently the pair $(-A^*, B^*)$) is *observable in time* T *via the operator* B^* if the function $B^*\varphi(\cdot)$ vanishes on $[0, T]$ implies that $\varphi_0 = 0$.

The system (6) is *stabilizable* if there exists an operator F and positive constants ω and c such that the solutions of the closed-loop problem (6)-(7) satisfy the estimation (8).

The system (6) is *completely stabilizable* if for any positive constant ω, there exist an operator $F = F(\omega)$ and a positive constant c such that the solutions of the closed-loop problem (6)-(7) satisfy the estimation (8).

Remark 1.5. For finite-dimensional systems, it is well-known that the pair (A, B) is exactly controllable if and only if the pair $(-A^*, B^*)$ is observable and that the time T has no importance in that case (see [**41**], [**54**]).

In finite dimension, a linear system is stable if and only if all its eigenvalues have a negative real part. The *pole assignment theorem* (see e.g, [**63**, Theorem 2.9 p. 44]) states that if a system is exactly controllable, then we can impose the spectrum of the closed-loop system through a suitable feedback operator F. In particular, a controllable system is completely stabilizable. But there is no systematic method to build this feedback. Nevertheless, in the late 1960's, Lukes [**47**] and Kleinman [**28**] (see also the book by Russell [**54**, pp. 112–117]) gave independently a systematic approach to stabilize the abstract system (6). It relies on an explicit feedback constructed with the operator

$$\Lambda := \int_0^T e^{-tA}BB^*e^{-tA^*}\,\mathrm{d}t.$$

Remark 1.6. This operator is also called the *controllability Gramian.* This name comes from the fact that the pair (A, B) is exactly controllable (equivalently $(-A^*, B^*)$ is observable) if and only if Λ is positive definite.

If the pair (A, B) is exactly controllable, then Λ is invertible and the feedback

$$F := -B^*\Lambda^{-1}$$

stabilizes the system.

A few years later, adding a suitable weight-function inside the Gramian operator, Slemrod [**56**] adapted and improved this result to the case of infinite-dimensional systems with *bounded* control operators. More precisely, his feedback depends on a tuning parameter $\omega > 0$ that ensures a prescribed exponential decay rate of the solutions. The weighted Gramian

$$\Lambda_\omega := \int_0^T e^{-2\omega t} e^{-tA} BB^* e^{-tA^*}\, \mathrm{d}t \tag{10}$$

is positive definite if (A, B) is exactly controllable in time T (equivalently $(-A^*, B^*)$ is exactly observable in time T). In that case, provided with the feedback

$$F = -B^* \Lambda_\omega^{-1} \tag{11}$$

the solutions of the closed-loop problem (6)-(11) satisfy the estimation (8) for a suitable constant c.

Slemrod's feedback law is well-adapted for infinite-dimensional systems with a bounded control operator B but it is not clear whether or not it works for *unbounded* [1] control operator, technical difficulties appearing in that case. In the 1990's, replacing $e^{-2\omega t}$ by a slightly different weight function in the operator Λ_ω defined above, Komornik [**32**] proved that in the general (unbounded controlled) case, the feedback (11) leads to arbitrarily large decay rates. His result can be summed up [2] as follows.

Theorem 1.7 (Komornik)**.** *Provided that the pair* $(-A^*, B^*)$ *is observable in time* T*, the closed-loop problem* (6)-(11) *is well-posed in a weak sense and for each initial datum, the solution satisfies the estimation* (8)*.*

Komornik's approach does not use the theory of optimal control through the minimization of a cost functional: an advantage is that one do not need strong existence and uniqueness results for infinite-dimensional Riccati equations. In fact, the weight function inside the Gramian has been chosen, on a suggestion by Bourquin (see [**32**, p. 1611]), in such a way that Λ_ω is the solution of an algebraic Riccati equation of the form [3]

$$A\Lambda_\omega + \Lambda_\omega A^* + \Lambda_\omega CC^* \Lambda_\omega - BB^* = 0.$$

Then, he adapted representation formulae of the the solutions of a differential Riccati equation from Flandoli [**23**] to prove the exponential decay of the solutions. Two points remained to be clarified, though.

First point (chapter 3). It deals with the *well-posedness of the closed-loop problem* (6)-(11). It is not clear how to chose the right notion of solution and

[1] As we may see later, this means that the operator B can take its values in a larger space than the usual state space. This appears naturally when the control acts on the boundary as in (1).

[2] It will be stated precisely later.

[3] A definition of the operator C and the rigorous meaning of this equation will be given later.

the right functional spaces in order to have a well-posed problem. Moreover, we can wonder if the closed-loop operator

$$A - BB^*\Lambda_\omega^{-1} \tag{12}$$

is the generator of a semigroup and in the affirmative case, how to determine its domain.

In section 1, using the Riccati equation satisfied by Λ_ω and introducing a dual problem, we show that the closed-loop operator (12) (up to replacing A by an extension) is the generator of a group on the "natural" state space and we determine its domain. While in the case of a bounded control operator, the extension of A is non-necessary [1] , we show that we must use it in the unbounded case.

In section 2, we prove a variation of constant formula for the solutions of the closed-loop problem, using a representation formula for Λ_ω inspired by Flandoli.

The main result of this chapter can be summed up as follows.

Result 1. *There is an extension $\widetilde{A}$ of A such that the operator $\widetilde{A} - BB^*\Lambda_\omega^{-1}$ is the generator of a group. Moreover the solution satisfies a variation of constant formula only involving A and B^*.*

Second point (chapter 4). It concerns the optimality of the decay rate of the solutions. Indeed, Komornik's result states that the decay rate of the solutions of the closed-loop problem is *at least* ω.

In section 1, using the well-posedness result of chapter 3, we give a short proof of the exponential decay of the solutions.

In section 2, we justify a representation formula for Λ_ω^{-1} used by Komornik in [**32**] to prove the exponential decay and we briefly recall his proof.

In section 3 we study the optimality question. It is motivated by an interesting phenomenon observed by Bourquin and his collaborators (Briffaut, Collet, Joly, Ratier, Urquiza) and for which we refer to [**6, 8**]: in some numerical and mechanical experiments on beams, the effective decay rate appears to be approximately twice bigger than expected, i.e., 2ω instead of ω. The method that we used to prove the well-posedness in chapter 3 enables us to study more precisely the decay rate. Modifying the parameter T in the definition of Λ_ω (see (19)) it is possible to obtain a more accurate lower bound of the decay rate. In particular, for dissipative systems, the actual decay rate may be close to 2ω provided that T is large enough.

The main result of this chapter can be summed up as follows:

[1] Indeed, in that case the perturbation $-BB^*\Lambda_\omega^{-1}$ is bounded and we can apply a classical perturbation result for semigroups.

Result 2. *For a dissipative system, the exponential decay rate of the solutions of the closed-loop system is bounded from below by the quantity*

$$\max\left\{\omega, 2\omega - \beta e^{-2\omega T}\right\},$$

where β is a positive constant, independent of T.

Remark 1.8.

- For a presentation of Komornik's stabilization method we can also refer to the books by Komornik and Loreti [**34**, section 2.4] and by Coron [**16**, section 13.1].
- An application of this method to partial stabilization can be found in [**34**, section 2.5]. Concrete examples to the boundary stabilization of wave and plates are given in [**32**]. This method can also be used to stabilize electromagnetic (Maxwell's equations) [**33**] and elastodynamic systems [**1**].
- Using the *backstepping method*, Smyshlyaev, Guo and Krstic [**57**] gave another explicit feedback stabilizing a vibrating beam with arbitrarily large decay rates.

Outline of the first part:

Chapter 2. Description of the hypotheses, construction of the feedback law and reminder of some useful results on abstract problems.

Chapter 3. Study of the wellposedness of the closed-loop problem.

Chapter 4. Study of the decay rate.

Chapter 2

Hypotheses and problem setting

1. Hypotheses

Two Hilbert spaces will play a leading role:

$$H = \textit{state space};$$
$$U = \textit{control space}.$$

In some applications, it will be useful not to identify them with their duals H' and U'; we denote by

$J : U' \to U$ the canonical isomorphism between U' and U;
$\widetilde{J} : H \to H'$ the canonical isomorphism between H and H'.

Unless the contrary is mentioned, in the rest of the first part, we assume that the following hypotheses are satisfied.

(H1) The operator $A : \mathcal{D}(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous group e^{tA} on H. [1]

(H2) $B \in \mathcal{L}(U, \mathcal{D}(A^*)')$.

(H3) For all $T > 0$ there exists a positive constant $c_1(T)$ such that

$$\int_0^T \|B^* e^{-tA^*} x\|_{U'}^2 \,\mathrm{d}x \leq c_1(T) \|x\|_{H'}^2$$

for all $x \in \mathcal{D}(A^*)$.

[1] Thus, its adjoint $A^* : \mathcal{D}(A^*) \subset H' \to H'$ generates a group $e^{tA^*} = (e^{tA})^*$ on H'.

(H4) There exists a number $T_0 > 0$ and a positive constant $c_2(T_0)$ such that

$$c_2(T_0)\|x\|_{H'}^2 \leq \int_0^{T_0} \|B^* e^{-tA^*} x\|_{U'}^2 \, \mathrm{d}t$$

for all $x \in \mathcal{D}(A^*)$.

Remark 2.1.

- Provided with the norm $\|x\|^2_{\mathcal{D}(A^*)} := \|x\|^2_{H'} + \|A^* x\|^2_{H'}$, $\mathcal{D}(A^*)$ is a Hilbert space. Moreover,

$$\mathcal{D}(A^*) \subset H' \qquad \Rightarrow \qquad H \subset \mathcal{D}(A^*)'$$

  with dense and continuous embeddings.

  We denote by $B^* \in \mathcal{L}(\mathcal{D}(A^*), U')$ the adjoint of B with respect to the above topology on $\mathcal{D}(A^*)$. This regularity for B^* is equivalent to a *factorization property* of B^*: more precisely, it is equivalent to the existence of a complex number λ and a bounded operator $E \in \mathcal{L}(U, H)$ such that

$$B^* = E^*(A + \lambda I)^*. \tag{13}$$

  Indeed, assume that $B^* \in \mathcal{L}(\mathcal{D}(A^*), U')$. Choosing any complex number λ in the resolvent set of $-A$ so that $\bar{\lambda}$ lies in the resolvent set of $-A^*$ and $((A + \lambda I)^*)^{-1} \in \mathcal{L}(H', \mathcal{D}(A^*))$, we can set [1]

$$\begin{aligned} & E^* := B^*((A + \lambda I)^*)^{-1} \in \mathcal{L}(H', U') \\ \Rightarrow \quad & B^* = E^*(A + \lambda I)^*. \end{aligned}$$

  The converse property is immediate.

  If $B \in \mathcal{L}(U, H)$, we say that B is *bounded*. This is the case with a distributed control (see Example 1). Otherwise, we say that B is *unbounded*, which covers the case of a boundary control (see Examples 2 and 3).

- In the applications, the inequality in (H3) represents a trace regularity result (see [**37**] and Example 3). It is usually called the *direct inequality*. Thanks to the assumptions (H1) and (H2), if this inequality satisfied for one $T > 0$, then it is satisfied for all $T > 0$ (up to a change of the positive constant). Moreover, the estimation remains true if we integrate on $(-T, T)$ instead of $(0, T)$. The corresponding inequality can be extended to all $x \in H'$ by a density argument. Hence,

  for all $x \in H'$ the function $t \mapsto B^* e^{-tA^*} x$ can be seen as an element of $L^2_{loc}(\mathbb{R}; U')$.

[1] In particular, the factorization is not unique, depending on the number λ chosen in the resolvent set of $-A$.

- The inequality of (H4) is usually called the *inverse* or *observability inequality*. Obviously it remains true if we integrate on $(0, T_1)$ with $T_1 > T_0$ but need not be true if $0 < T_1 < T_0$ (cf. Example 2).

Example 1 (transport equation with a distributed control).

$$\begin{cases} y'(t,x) = y_x(t,x) + u(t,x) & \text{in } \mathbb{R} \times [0, 2\pi]; \\ y(t, 2\pi) = y(t, 0) & \text{in } \mathbb{R}. \end{cases} \tag{14}$$

The control u acts on the whole domain $[0, 2\pi]$. We set

- *state space*:
$$H = H' := \{f : \mathbb{R} \to \mathbb{R},\ 2\pi\text{-periodic} : f \in L^2(0, 2\pi)\};$$
- *control space*:
$$U = U' = H;$$
- *dynamics*: we define the unbounded operator $A : \mathcal{D}(A) \subset H \to H$ as
$$\mathcal{D}(A) := \{f \in H : f_x \in H \text{ and } f(0) = f(2\pi)\}$$
and
$$\forall\, f \in \mathcal{D}(A), \qquad Af = f_x.$$
The operator A satisfies $A^* = -A$ and is the infinitesimal generator of a strongly continuous group in H (see e.g., [**5**, pp. 466–467]): more precisely, given $f \in H$,
$$e^{tA} f(\cdot) = f(\cdot + t), \qquad t \in \mathbb{R}.$$
In particular, the hypothesis (H1) is satisfied.
- *control operator*:
$$B = B^* = \mathrm{Id} \in \mathcal{L}(U, H).$$
Thus the hypothesis (H2) holds. We can remark that the control operator B is bounded (in the sense of Remark 2.1), which is a feature of a distributed control.

Let $T > 0$ and $\varphi \in \mathcal{D}(A^*)$. Then,

$$\begin{aligned} \int_0^T \|B^* e^{-tA^*} \varphi\|^2_{U'}\, \mathrm{d}t &= \int_0^T \Big(\int_0^{2\pi} |\varphi(x+t)|^2\, \mathrm{d}x \Big)\, \mathrm{d}t \\ &= T \int_0^{2\pi} |\varphi(x)|^2\, \mathrm{d}x \qquad (2\pi\text{-periodicity}) \\ &= T \|\varphi\|^2_H \end{aligned}$$

so that the hypotheses (H3) and (H4) hold.

Example 2 (transport equation with a periodic boundary control)**.**

$$\begin{cases} y'(t,x) = y_x(t,x) & \text{in } \mathbb{R}\times[0,2\pi]; \\ y(0,x) = y_0(x) & \text{in } [0,2\pi]; \\ y(t,2\pi) - y(t,0) = u(t) & \text{in } \mathbb{R}. \end{cases} \tag{15}$$

The control u acts only on the boundary of the domain, that is at its two ends 0 and 2π. The state space and the dynamics are the same as in the previous example; in particular (H1) is satisfied. Nevertheless we must modify the control space and operator to

- *control space:*
$$U = U' = \mathbb{R};$$
- *control operator:*
$$\forall\, f \in \mathcal{D}(A^*) = \mathcal{D}(A), \qquad B^* f = f(\text{bdry}) := f(0) = f(2\pi).$$

  From the usual trace theorem, B^* defines a bounded operator from $\mathcal{D}(A^*)$ into U'. We can specify $B \in \mathcal{L}(U, \mathcal{D}(A^*)'$: given $\alpha \in \mathbb{R} = U = U'$,
$$\forall \varphi \in \mathcal{D}(A^*), \qquad \langle B\alpha, \varphi\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)} = \alpha\varphi(\text{bdry}).$$

  Thus, the hypothesis (H2) holds. Contrary to the previous example, the control operator is unbounded (in the sense of Remark 2.1) which is the feature of a boundary control.

Let us explain briefly how we can identify the operator B^*. If we assume that y is a smooth solution of (15) and that $\varphi \in \mathcal{D}(A^*)$, an integration by parts yields

$$\begin{aligned} (y'(t),\varphi)_H &= \int_0^{2\pi} y'(t,x)\varphi(x)\,\mathrm{d}x \\ &= \int_0^{2\pi} y_x(t,x)\varphi(x)\,\mathrm{d}x \\ &= -\int_0^{2\pi} y(t,x)\varphi_x(x)\,\mathrm{d}x + y(t,2\pi)\varphi(2\pi) - y(t,0)\varphi(0) \\ &= -\int_0^{2\pi} y(t,x)\varphi_x(x)\,\mathrm{d}x + \varphi(\text{bdry})(y(t,2\pi) - y(t,0)) \\ &= (y(t), A^*\varphi)_H + \varphi(\text{bdry})u(t) \\ &= (y(t), A^*\varphi)_H + (u(t), B^*\varphi)_U. \end{aligned}$$

Let $T > 0$ and $\varphi \in \mathcal{D}(A^*)$. Then,

$$\int_0^T \|B^* e^{-tA^*}\varphi\|_{U'}^2\,\mathrm{d}t = \int_0^T |\varphi(t)|^2\,\mathrm{d}t.$$

Thus, the estimation of (H3) is satisfied whatever the choice of $T > 0$ and the estimation of (H4) is satisfied if and only if $T \geq 2\pi$.

Example 3 (wave equation with a boundary control)**.** We consider the wave equation on a bounded open set $\Omega \subset \mathbb{R}^n$ having a sufficiently smooth boundary Γ; we can act on the system through a Dirichlet control on the entire boundary:

$$\begin{cases} y''(t,x) - \Delta y(t,x) = 0 & \text{in } \mathbb{R} \times \Omega; \\ y(t,x) = u(t,x) & \text{in } \mathbb{R} \times \Gamma; \\ y(0,x) = y_0(x), \quad y'(0,x) = y_1(x) & \text{in } \Omega. \end{cases} \tag{16}$$

- *state space* [1] *:*
$$H := H^{-1}(\Omega) \times L^2(\Omega);$$
$$H' := H_0^1(\Omega) \times L^2(\Omega);$$
- *dynamics* [2] *:*
$$-A^* = \begin{pmatrix} 0 & \Delta \\ \mathrm{Id} & 0 \end{pmatrix} = \text{ wave operator in } H'$$
with
$$\mathcal{D}(A^*) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega).$$
Moreover,
$$A := (A^*)^* \approx \text{ wave operator in } H$$
with
$$\mathcal{D}(A) = L^2(\Omega) \times H_0^1(\Omega).$$
This operator is the generator of a group in H so that (H1) is satisfied.
- *control space:*
$$U = U' = L^2(\Gamma);$$
- *control operator:*
$$\forall\, (\eta_0, \eta_1) \in \mathcal{D}(A^*), \qquad B^*(\eta_0, \eta_1) := \partial_\nu \eta_0.$$
A usual trace result ensures that $B^* \in \mathcal{L}(\mathcal{D}(A^*); U')$ so that (H2) is satisfied. Moreover the factorization
$$B^* = E^* A^*$$
holds (see [**32**, p. 1602]), where $E \in L(U, H)$ is defined for $u \in L^2(\Gamma)$ by
$$Eu := (0, Du),$$

---

[1] In H, the first coordinate represents the velocity and the second coordinate represents the position (except the sign). This is the converse for the space H'.

[2] For more details on the wave operator in H' or in H, we refer to [**13**, pp. 29–31].

$D \in \mathcal{L}(L^2(\Gamma), H^{1/2}(\Omega))$ denoting the Dirichlet map defined by

$$\begin{cases} -\Delta Du = 0 & \text{in } \Omega; \\ Du = u & \text{in } \Gamma. \end{cases}$$

If $T > 0$ is large enough [1], there are positive constants $c_1(T)$ and $c_2(T)$ such that for all initial data $(\varphi_0, \varphi_1) \in \mathcal{D}(A^*)$,

$$c_1(T)\|(\varphi_0, \varphi_1)\|_{H'}^2 \leq \int_0^T \int_\Gamma |\partial_\nu \varphi(t,x)|^2 \,\mathrm{d}\Gamma \,\mathrm{d}t \leq c_2(T)\|(\varphi_0, \varphi_1)\|_{H'}^2,$$

where φ is the associated solution of the homogeneous wave equation. Therefore, the hypotheses (H3) and (H4) are satisfied.

In general, these estimations are non-trivial: they can be proved by using, for instance, the multipliers method (see [**45, 31**]) or non-harmonic Fourier series (see [**34**]). The left inequality is due to Lions [**43**] and Lasiecka and Triggiani [**37**]; the left inequality is due to Ho [**27**].

We end this section by recalling a regularity result that can be seen as an extension of (H3). It concerns the solutions of the following inhomogeneous problem in the dual space H':

$$\begin{cases} y'(t) = -A^* y(t) + g(t), & t \in \mathbb{R}, \\ y(0) = y_0, \end{cases} \tag{17}$$

where $g \in L^1_{\text{loc}}(\mathbb{R}; H')$. Here [2] , the source term does not involve any unbounded operator and the mild solution of (17) is defined by the "standard" variation of constants formula (see [**49**, p. 107])

$$y(t) = e^{-tA^*} y_0 + \int_0^t e^{-(t-r)A^*} g(r) dr, \tag{18}$$

which is a continuous function from $\mathbb{R}$ to H'. Thanks to the direct inequality stated in (H3), we can apply the operator B^* to the solution of the homogeneous problem associated to (17) (put $g = 0$ in (17)) and see this new function as an element of $L^2_{\text{loc}}(\mathbb{R}; U')$. Actually, this operation can be generalized to the solutions of the inhomogeneous problem (g can be $\neq 0$). We recall this result [3] in the

[1] larger than a constant depending on the geometry of the domain Ω

[2] contrary to the open-loop problem (see section 3)

[3] This result was firstly stated in [**37**] in the case of hyperbolic equations with Dirichlet boundary conditions.

Proposition 2.2 ([**23**, pp. 92–93], [**40**, p. 648])**.** *Fix $T > 0$. There exists a constant $c > 0$ such that for all $y_0 \in \mathcal{D}(A^*)$ and all $g \in L^1(-T, T; \mathcal{D}(A^*))$ we have the estimation*

$$\int_{-T}^{T} \|B^* y(t)\|_{U'}^2 \,\mathrm{d}t \le c\big(\|y_0\|_{H'}^2 + \|g\|_{L^1(-T,T;U')}^2\big),$$

where y is defined by (18)*. By density, we can extend this estimation for all initial data $y_0 \in H'$ and all source terms $g \in L^1(-T, T; H')$.*

We refer to Appendix A for a proof of this result.

2. Construction of the feedback

2.1. The operator Λ_ω. We assume that the hypotheses (H1)-(H4) hold true, the number $T_0 > 0$ giving the observability inequality in (H4), and we recall the construction of the feedback introduced by Komornik in [**32**]. The starting point is to define a modified, weighted Gramian. Let us fix a number $\omega > 0$, set

$$T_\omega := T_0 + \frac{1}{2\omega},$$

and define a *weight function* on the interval $[0, T_\omega]$:

$$e_\omega(s) := \begin{cases} e^{-2\omega s} & \text{if} \quad 0 \le s \le T_0; \\ 2\omega e^{-2\omega T}(T_\omega - s) & \text{if} \quad T_0 \le s \le T_\omega. \end{cases}$$

This function is *exponential* on $[0, T_0]$ and *affine* on $[T_0, T_\omega]$ (see Figure 1).

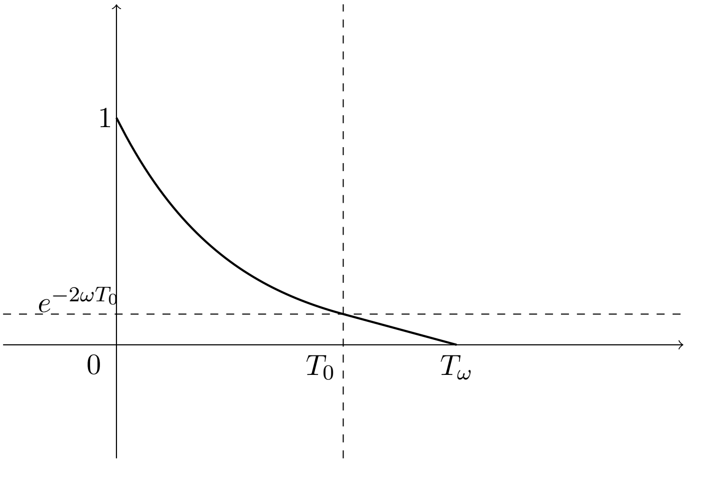


Figure 1. Weight function e_ω

The relation

$$\langle \Lambda_\omega x, y\rangle_{H,H'} := \int_0^{T_\omega} e_\omega(s)\langle JB^* e^{-sA^*} x, B^* e^{-sA^*} y\rangle_{U,U'} \,\mathrm{d}s \tag{19}$$

defines a bounded (thanks to (H3)) operator $\Lambda_\omega \in \mathcal{L}(H', H)$, self-adjoint and bounded from below [1] (thanks to (H4)). Hence [2], Λ_ω is invertible and we denote by $\Lambda_\omega^{-1} \in \mathcal{L}(H, H')$ its inverse.

Remark 2.3. The weight function e_ω has been chosen, on a suggestion by Bourquin (see [**32**, Note p. 1611]), in such a way that the operator Λ_ω is the solution of an algebraic Riccati equation.

2.2. An algebraic Riccati equation. The aim of this paragraph is to show that the operator Λ_ω satisfies an operator equation, namely an algebraic Riccati equation. This property may seem artificial but it will be crucial in the study of the well-posedness of the closed-loop problem and of the decay rate of the solutions.

Let $x, y \in \mathcal{D}((A^*)^2)$. We compute the integral

$$\int_0^{T_\omega} \frac{\mathrm{d}}{\mathrm{d}s}\left[e_\omega(s)\langle JB^* e^{-sA^*}x, B^* e^{-sA^*}y\rangle_{U,U'}\right]\mathrm{d}s \tag{20}$$

in two different ways. We can notice that the quantity between the brackets is differentiable with respect to the variable s thanks to the regularity of x and y, and the hypothesis (H2).

- On the one hand, as $e_\omega(T_\omega) = 0$ and $e_\omega(0) = 1$, the integral (20) equals
$$-\langle JB^*x, B^*y\rangle_{U,U'}.$$
- On the other hand, by differentiating inside the integral, we obtain
$$\begin{aligned}&\int_0^{T_\omega} e_\omega'(s)\langle JB^* e^{-sA^*}x, B^* e^{-sA^*}y\rangle_{U,U'}\,\mathrm{d}s\\ -&\int_0^{T_\omega} e_\omega(s)\langle JB^* e^{-sA^*}A^*x, B^* e^{-sA^*}y\rangle_{U,U'}\,\mathrm{d}s\\ -&\int_0^{T_\omega} e_\omega(s)\langle JB^* e^{-sA^*}x, B^* e^{-sA^*}A^*y\rangle_{U,U'}\,\mathrm{d}s.\end{aligned}$$

The formula

$$(Lx, y)_H := -\int_0^{T_\omega} e_\omega'(s)\langle JB^* e^{-sA^*}\Lambda_\omega^{-1}x, B^* e^{-sA^*}\Lambda_\omega^{-1}y\rangle_{U,U'}\,\mathrm{d}s$$

defines a bounded operator $L \in \mathcal{L}(H)$, self-adjoint and bounded from below because

$$-e_\omega'(s) \geq 2\omega e_\omega(s), \qquad 0 \leq s \leq T_\omega.$$

[1] This means that there is a constant $c > 0$ such that for all $x \in H'$, $\langle \Lambda_\omega x, x\rangle \geq c\|x\|^2$.

[2] As Λ_ω is bounded from below, it is one-to-one. Using its boundedness, we can also deduce that there exists a constant $c' > 0$ such that for all $x \in H'$, $\|\Lambda_\omega x\| \geq c'\|x\|$. This implies that Λ_ω has a closed range, hence $R(\Lambda_\omega) = \overline{R(\Lambda_\omega)} = N(\Lambda_\omega^*)^\perp = \{0\}^\perp = H$.

We set

$$C := \sqrt{L} \in \mathcal{L}(H).$$

For $x, y \in H$, we have

$$\begin{aligned}(Lx, y)_H &= (Cx, Cy)_H \\ &= \langle Cx, \widetilde{J}Cy\rangle_{H,H'} \\ &= \langle x, C^*\widetilde{J}Cy\rangle_{H,H'}\end{aligned}$$

where $C^* \in \mathcal{L}(H')$ denotes the adjoint of C. We can also remark the following relation between C and Λ_ω^{-1} :

$$C^*\widetilde{J}C \geq 2\omega\Lambda_\omega^{-1}. \tag{21}$$

This estimation will be important in the proof of the exponential decay of the solutions.

The second computation of the integral (20) can be written

$$\begin{aligned}&-(L\Lambda_\omega x, \Lambda_\omega y)_H - \langle \Lambda_\omega A^* x, y\rangle_{H,H'} - \langle \Lambda_\omega x, A^* y\rangle_{H,H'} \\ =&-\langle C\Lambda_\omega x, \widetilde{J}C\Lambda_\omega y\rangle_{H,H'} - \langle \Lambda_\omega A^* x, y\rangle_{H,H'} - \langle \Lambda_\omega x, A^* y\rangle_{H,H'}.\end{aligned}$$

Gathering the two computations, we obtain the following *algebraic Riccati equation* satisfied by Λ_ω :

$$\begin{aligned}&\langle \Lambda_\omega A^* x, y\rangle_{H,H'} + \langle \Lambda_\omega x, A^* y\rangle_{H,H'} \\ &\quad + \langle C\Lambda_\omega x, \widetilde{J}C\Lambda_\omega y\rangle_{H,H'} - \langle JB^* x, B^* y\rangle_{U,U'} = 0,\end{aligned} \tag{22}$$

first for $x, y \in \mathcal{D}((A^*)^2)$ and then for $x, y \in \mathcal{D}(A^*)$ by density of $\mathcal{D}((A^*)^2)$ in $\mathcal{D}(A^*)$ for the norm $\|\cdot\|_{\mathcal{D}(A^*)}$ (see the Proposition A.2).

Remark 2.4. A formal version of the Riccati equation (22), correct in finite dimension, would be

$$\Lambda_\omega A^* + A\Lambda_\omega + \Lambda_\omega C^*\widetilde{J}C\Lambda_\omega - BJB^* = 0. \tag{23}$$

Multiplying the above equation on the left and on the right by Λ_ω^{-1}, we obtain at least formally (although rigorously in finite dimension)

$$A^*\Lambda_\omega^{-1} + \Lambda_\omega^{-1}A + C^*\widetilde{J}C - \Lambda_\omega^{-1}BJB^*\Lambda_\omega^{-1} = 0. \tag{24}$$

2.3. An integral form of the algebraic Riccati equation. We rewrite the Riccati equation (22) in an integral form, satisfied for $x, y \in H$ instead of $x, y \in \mathcal{D}(A^*)$.

Setting $x, y \in \mathcal{D}(A^*)$, the Riccati equation (22) applied to $e^{-sA^*}x, e^{-sA^*}y \in \mathcal{D}(A^*)$ reads

$$\begin{aligned}(25)\qquad &\langle \Lambda_\omega A^* e^{-sA^*}x, e^{-sA^*}y\rangle_{H,H'}\\ &+ \langle \Lambda_\omega e^{-sA^*}x, A^* e^{-sA^*}y\rangle_{H,H'}\\ &+ \langle C\Lambda_\omega e^{-sA^*}x, \widetilde{J} C\Lambda_\omega e^{-sA^*}y\rangle_{H,H'}\\ &- \langle JB^* e^{-sA^*}x, B^* e^{-sA^*}y\rangle_{U,U'} = 0.\end{aligned}$$

Integrating (25) between 0 and t, we obtain the following *integral form* of the Riccati equation (22) :

$$\begin{aligned}(26)\qquad \langle \Lambda_\omega x, y\rangle_{H,H'} =& \langle \Lambda_\omega e^{-tA^*}x, e^{-tA^*}y\rangle_{H,H'}\\ &- \int_0^t \langle C\Lambda_\omega e^{-sA^*}x, \widetilde{J} C\Lambda_\omega e^{-sA^*}y\rangle_{H,H'}\,\mathrm{d}s\\ &+ \int_0^t \langle JB^* e^{-sA^*}x, B^* e^{-sA^*}y\rangle_{U,U'}\,\mathrm{d}s.\end{aligned}$$

This relation remains true for $x, y \in H'$ by density of $\mathcal{D}(A^*)$ in H' for the norm $\|\cdot\|_{H'}$.

3. The open-loop problem

We assume that the hypotheses (H1), (H2) and (H3) are satisfied and that a number $T > 0$ is fixed. The aim of this section is to recall how a solution of the *open-loop* problem

$$(27)\qquad \begin{cases} x'(t) = Ax(t) + Bu(t) & 0 \le t \le T,\\ x(0) = x_0. \end{cases}$$

can be defined. The difficulty comes from the fact that the control operator B may be unbounded in the sense that it may take its values in the larger space $\mathcal{D}(A^*)'$ (see Remark 2.1). In this section, we

- define a solution relying on a *transposition method*
- define the *mild solution* through a *variation of constants formula*
- define a notion of *weak solution*

and we show that these three definitions are in fact equivalent.

A transposition method. Let us begin with a *formal* computation. We assume that $x : [0, T] \to H$ is a regular solution of (27) and that $\varphi : [0, T] \to H'$ is a sufficiently smooth function. For the sake of simplicity, we also assume that B takes its values in H. Multiplying the first line of (27) by φ and integrating between 0 and t (where $0 \le t \le T$), we get

$$\int_0^t \langle x'(s), \varphi(s)\rangle\,\mathrm{d}s = \int_0^t \langle Ax(s), \varphi(s)\rangle\,\mathrm{d}s + \int_0^t \langle Bu(s), \varphi(s)\rangle\,\mathrm{d}s$$

Integrating by parts in the left hand side and using the adjoint of B, we get

$$\langle x(t), \varphi(t)\rangle = \langle x(0), \varphi(0)\rangle + \int_0^t \langle x(s), A^*\varphi(s) + \varphi'(s)\rangle \,\mathrm{d}s + \int_0^t \langle u(s), B^*\varphi(s)\rangle \,\mathrm{d}s.$$

In the above relation, the first integral vanishes if we assume that φ is the solution of the following homogeneous problem in H', that we will call the *dual problem* :

$$\begin{cases} \varphi'(t) = -A^*\varphi(t) & 0 \le t \le T, \\ \varphi(0) = \varphi_0. \end{cases} \tag{28}$$

Definition 2.5. Let $x_0 \in H$ and $u \in L^2(0,T;U)$. The function $x \in \mathcal{C}([0,T];H)$ is a *solution defined by transposition* of (27) if it satisfies the relation

$$\langle x(t), \varphi(t)\rangle_{H,H'} = \langle x_0, \varphi_0\rangle_{H,H'} + \int_0^t \langle u(s), B^*\varphi(s)\rangle_{U,U'} \,\mathrm{d}s \tag{29}$$

for all $0 \le t \le T$ and $\varphi_0 \in \mathcal{D}(A^*)$, φ being the corresponding solution of (28).

Proposition 2.6 ([**16**, pp. 53–54])**.** *Let $x_0 \in H$ and $u \in L^2(0,T;U)$. The open-loop problem* (27) *has a unique solution defined by transposition. Moreover, there exists a positive constant c, independent of x_0 and u, such that*

$$\|x(t)\|_H \le c\Big(\|x_0\|_H + \|u\|_{L^2(0,T;U)}\Big), \qquad 0 \le t \le T.$$

Proof. For the uniqueness, let us assume that there are two solutions defined by transposition. For a fixed t, their difference $\xi(t)$ satisfies (cf. (29))

$$\begin{aligned} &\forall \varphi_0 \in \mathcal{D}(A^*), \quad \langle \xi(t), e^{-tA^*}\varphi_0\rangle_{H,H'} = 0 \\ \Rightarrow\ &\forall \varphi_0 \in \mathcal{D}(A^*), \quad \langle \xi(t), \varphi_0\rangle_{H,H'} = 0, \end{aligned}$$

hence $\xi(t) = 0$.

We postpone the proof of the existence to the proof of Proposition 2.9 where we will see that the continuous function defined by (30) is a solution defined by transposition.

The continuity with respect to x_0 and u is a consequence of the assertion (b) of the Proposition 2.7. □

The mild solution. Now we try to obtain a variation of constants formula (the “mild solution” in the terminology of [**2**]) i.e. a solution of the form

$$e^{tA} + \int_0^t e^{(t-s)A} Bu(s)\,\mathrm{d}s.$$

Taking into account the unboundedness of B, we may use the factorization property of B^* (cf. Remark 2.1) which becomes, formally, for B,

$$B \approx (A + \lambda I)E.$$

Proposition 2.7 (see [**5**, pp. 459–460], [**40**, p. 648])**.** *Let* $u \in L^2(0,T;U)$. *Setting*

$$z(t) := \int_0^t e^{(t-s)A} Eu(s)ds, \qquad 0 \le t \le T,$$

the following results hold:

(a) $z(t) \in \mathcal{D}(A)$ *for all* $0 \le t \le T$*;*

(b) *There exists a positive constant* k*, independent of* u *such that*

$$\|(A+\lambda I)z(t)\|_H \le k\|u\|_{L^2(0,T;U)}, \qquad 0 \le t \le T;$$

(c) $(A+\lambda I)z \in \mathcal{C}([0,T];H)$*.*

This result is due to Lasiecka and Triggiani [**37**] who first proved it in the case of hyperbolic equations with Dirichlet boundary conditions. A proof is recalled in the Appendix A. Thanks to the above result, we can state the

Definition 2.8. Let $x_0 \in H$ and $u \in L^2(0,T;U)$. The *mild solution* of (27) is the continuous function with values in H defined by

$$x(t) = e^{tA}x_0 + (A+\lambda I)\int_0^t e^{(t-s)A}Eu(s)\,\mathrm{d}s, \qquad 0 \le t \le T. \tag{30}$$

Proposition 2.9 (mild=transposition)**.** *Let* $x_0 \in H$ *and* $u \in L^2(0,T;U)$. *The mild solution of* (27) *is a solution defined by transposition.*

Proof. Let $\zeta(t)$ denote the right hand side of (30). Fixing $0 \le t \le T$, for all $\varphi_0 \in \mathcal{D}(A^*)$ we have

$$\begin{aligned}
&\langle \zeta(t), \varphi(t)\rangle_{H,H'} \\
=&\langle e^{tA}x_0 + (A+\lambda I)\int_0^t e^{(t-s)A}Eu(s)\,\mathrm{d}s, \varphi(t)\rangle_{H,H'} \\
=&\langle e^{tA}x_0, e^{-tA^*}\varphi_0\rangle_{H,H'} + \langle (A+\lambda I)\int_0^t e^{(t-s)A}Eu(s)\,\mathrm{d}s, e^{-tA^*}\varphi_0\rangle_{H,H'} \\
=&\langle x_0, \varphi_0\rangle_{H,H'} + \int_0^t \langle u(s), B^*\varphi(s)\rangle_{U,U'}\,\mathrm{d}s.
\end{aligned}$$

Hence, $\zeta(\cdot)$ satisfies (29) and is a solution defined by transposition of (27). □

Remark 2.10. If B is bounded (i.e. $B \in \mathcal{L}(U,H)$), then the relation (30) reduces to the classical variation of constants formula

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)\,\mathrm{d}s.$$

Indeed, using $E = (A+\lambda I)^{-1}B \in \mathcal{L}(U,H)$ in (30), we can take the operator $(A+\lambda I)^{-1}$ out of the integral because it commutes with e^{tA} for all t and it is bounded.

Remark 2.11. The relation (29) is equivalent to

$$\langle x(t), \varphi_0\rangle_{H,H'} = \langle x_0, e^{tA^*}\varphi_0\rangle_{H,H'} + \int_0^t \langle u(s), B^* e^{(t-s)A^*}\varphi_0\rangle_{U,U'}\,\mathrm{d}s$$

for all $0 \le t \le T$ and $\varphi_0 \in \mathcal{D}(A^*)$. Hence, assuming that $\varphi_0 \in \mathcal{D}((A^*)^2)$ and recalling that $B^* = E^*(A+\lambda I)^*$ so that the differentiation with respect to t is possible, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), \varphi_0\rangle_{H,H'} = \langle x(t), A^*\varphi_0\rangle_{H,H'} + \langle u(t), B^*\varphi_0\rangle_{U,U'} \tag{31}$$

for almost all $0 \le t \le T$.

Weak solutions. The last remark leads us to another possibility to define the solutions of the open-loop problem (27).

Definition 2.12. Let $x_0 \in H$ and $u \in L^2(0,T;U)$. A function $x \in \mathcal{C}([0,T],H)$ such that

- $x(0) = x_0$,
- for all $\varphi_0 \in \mathcal{D}((A^*)^2)$, $\langle x(\cdot), \varphi_0\rangle$ is absolutely continuous on $(0,T)$ and satisfies (31),

is called a *weak solution* of (27). [1]

Proposition 2.13 (transposition=weak)**.** *Given $x_0 \in H$ and $u \in L^2(0,T;U)$, the open-loop problem* (27) *has a unique weak solution. It corresponds to the solution defined by transposition.*

Proof. The *existence* comes directly from the remark 2.11, where we have seen that the solution defined by transposition satisfies (31).

Concerning the *uniqueness*, we can use the uniqueness of the classical solution of the associated homogeneous problem (as in [**3**, p. 372]). Assuming that there are two weak solutions, we denote by $\xi(t)$ their difference. Let $y \in \mathcal{D}((A^*)^2)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi(t), y\rangle_{H,H'} = \langle \xi(t), A^* y\rangle_{H,H'}, \qquad \text{a.e. } 0 \le t \le T.$$

Integrating this relation, we get ($\xi(0)$=0)

$$\langle \xi(t), y\rangle_{H,H'} = \int_0^t \langle \xi(s), A^* y\rangle_{H,H'}\,\mathrm{d}s = \langle \int_0^t \xi(s)\,\mathrm{d}s, A^* y\rangle_{H,H'}, \qquad 0 \le t \le T.$$

Since this relation is true for all $y \in \mathcal{D}((A^*)^2)$, it remains true for all $y \in \mathcal{D}(A^*)$ by density of $\mathcal{D}((A^*)^2)$ in $\mathcal{D}(A^*)$ for the norm $\|\cdot\|_{\mathcal{D}(A^*)}$. This implies that

[1] This definition is close to the definition of an inhomogeneous problem given by Ball in [**3**] and Balakrishnan in [**2**].

$z(t) := \int_0^t \xi(s)\,\mathrm{d}s$ lies in the domain of A, $z(\cdot)$ is continuously differentiable (because $\xi(\cdot)$ is continuous) and satisfies

$$z'(t) = Az(t), \qquad 0 \le t \le T \qquad \text{and} \qquad z(0) = z_0.$$

By uniqueness of the classical solutions of the homogeneous problem, $z(t) = 0$ for all t, hence $\xi(t) = 0$ for all t. □

The three notions of solution that we have defined coincide :

$$\boxed{\text{transposition solution}} = \boxed{\text{mild solution}} = \boxed{\text{weak solution}}$$

Remark 2.14. All these constructions can be generalized to define a solution to the open-loop problem on the interval $[-T, T]$ for all $T > 0$.

4. The feedback law

With respect to the open-loop problem (27), the feedback law introduced by Komornik is

$$u(t) = -JB^*\Lambda_\omega^{-1}x(t). \tag{32}$$

We recall the main result of [**32**] in the following

Theorem 2.15 (Komornik)**.** *Assume (H1)-(H4). The operator $A - BJB^*\Lambda_\omega^{-1}$ generates a group in H in a weak sense. Moreover, the solutions of the closed-loop problem* (27)-(32) *satisfy the following estimation: there exists a positive constant c such that for all initial data x_0,*

$$\|x(t)\|_H \le ce^{-\omega t}\|x_0\|_H, \qquad t \ge 0.$$

In the next two chapters, we will analyze the expression "generates a group in H in a weak sense" and specify the exponential decay.

Chapter 3

Well-posedness of the closed-loop problem

In this chapter, we study the well-posedness of the *closed-loop problem*

$$\begin{cases} x' = (A - BJB^*\Lambda_\omega^{-1})x; \\ x(0) = x_0. \end{cases} \tag{33}$$

on the state space H or possibly on the bigger space $\mathcal{D}(A^*)'$.

If B is bounded, so is the operator $-BJB^*\Lambda_\omega^{-1}$ and a classical perturbation result for semigroups (see Proposition A.3) ensures that the closed-loop operator $A - BJB^*\Lambda_\omega^{-1}$ with domain $\mathcal{D}(A)$ remains the generator of a group in H.

In the more general case of an unbounded [1] control operator B, the situation is more intricate; it is not even clear on which subspace the closed-loop operator is well-defined:

$$\underbrace{A}_{\substack{\text{well-defined on } \mathcal{D}(A) \\ \text{with values in } H}} \quad - \quad \overbrace{BJB^*\Lambda_\omega^{-1}}^{\substack{\text{well-defined on } \Lambda_\omega\mathcal{D}(A^*) \\ \text{with values in } \mathcal{D}(A^*)'}}.$$

The idea is to use the Riccati equation satisfied by Λ_ω in order to come down to a bounded perturbation of the adjoint oparator. Multiplying on the right by Λ_ω^{-1} the Riccati equation (23) *formally* satisfied by Λ_ω, we obtain

$$A - BJB^*\Lambda_\omega^{-1} = \Lambda_\omega(-A^* - C^*\widetilde{J}C\Lambda_\omega)\Lambda_\omega^{-1}.$$

[1] Recall (H2).

In other words, the two operators

$$A - \underbrace{BJB^*\Lambda_\omega^{-1}}_{\substack{\text{unbounded}\\ \text{perturbation}}} \qquad \text{and} \qquad -A^* - \underbrace{C^*\widetilde{J}C\Lambda_\omega}_{\substack{\text{bounded}\\ \text{perturbation}}}$$

are *formally* conjugated. The operator on the right-hand side with domain $\mathcal{D}(A^*)$ is the generator of group in H' since $-A^*$ does so and the perturbation $-C^*\widetilde{J}C\Lambda_\omega$ of $-A^*$ is bounded.

The aim of the first section is to make the above formal operation rigorous in the general case, in particular when the control operator B is unbounded. This can be done by using a suitable extension $\widetilde{A}$ of A : we show that the operator $\widetilde{A} - BJB^*\Lambda_\omega^{-1}$ is the generator of a group and we give its domain. This strongly continuous group yields a natural definition of the solutions of the closed-loop problem.

In the second section, we prove a variation of constants formula for the solutions of (33), similar to the formula (30) for the solutions of the open-loop problem.

In the last section, we show how to use results from optimal control theory to study the well-posedness.

1. Generation of a group

1.1. The closed-loop operator "generates" a group. We show that by replacing A by a suitable extension $\widetilde{A}$, the closed-loop operator $\widetilde{A} - BJB^*\Lambda_\omega^{-1}$ generates a group.

At first, let us recall a classical extension result for the unbounded operator A to a bounded operator on H with values in the larger space $\mathcal{D}(A^*)'$ (see e.g., [**39**, pp. 6–7] and [**13**, pp. 21–22]).

Lemma 3.1. *The operator $A : \mathcal{D}(A) \subset H \to H$ admits a unique extension to an operator $\widetilde{A} \in \mathcal{L}(H, \mathcal{D}(A^*)')$. Moreover this extension satisfies the relation*

$$\langle \widetilde{A}x, y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)} = \langle x, A^*y\rangle_{H,H'}. \tag{34}$$

for all $x \in H$ and $y \in \mathcal{D}(A^*)$.

Proof. The *uniqueness* of such an extension is the consequence of the density of $\mathcal{D}(A)$ in H.

For the *existence* we recall that, provided with the norm $\|\cdot\|_{\mathcal{D}(A^*)}$, $\mathcal{D}(A^*)$ is a Hilbert space and $A^* \in \mathcal{L}(\mathcal{D}(A^*), H')$. We denote by $\widetilde{A}$ the adjoint [1] of A^* seen as a bounded operator between the Banach spaces $\mathcal{D}(A^*)$ and H'. Hence

$$\widetilde{A} \in \mathcal{L}(H, \mathcal{D}(A^*)')$$

[1] Banach-adjoint

and for all $x \in H$ and $y \in \mathcal{D}(A^*)$,

$$\langle \widetilde{A}x, y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)} = \langle x, A^*y\rangle_{H,H'}$$

i.e. relation (34) is true. Moreover this new operator $\widetilde{A}$ defines an extension of A, that is the two operators coincide on $\mathcal{D}(A)$. Indeed, from the above relation specialized to $x \in \mathcal{D}(A) \subset H$, we get

$$\forall y \in \mathcal{D}(A^*), \quad \langle \widetilde{A}x, y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)} = \langle Ax, y\rangle_{H,H'} \quad \Rightarrow \quad Ax = \widetilde{A}x \in H. \quad \square$$

Theorem 3.2. *The operator*

$$A_U := \widetilde{A} - BJB^*\Lambda_\omega^{-1}, \qquad \mathcal{D}(A_U) := \Lambda_\omega \mathcal{D}(A^*)$$

is the infinitesimal generator of a strongly continuous group $U(t)$ in H.

Proof. The operator

$$A_V := -A^* - C^*\widetilde{J}C\Lambda_\omega, \qquad \mathcal{D}(A_V) := \mathcal{D}(A^*)$$

generates a strongly continuous group $V(t)$ in H'. Indeed, this is a classical perturbation result for the generator of semigroups (see Proposition A.3) since the perturbation $-C^*\widetilde{J}C\Lambda_\omega$ is a *bounded* operator in H'. By another classical result on semigroups (see PropositionA.4), the *conjugated* operator

$$\Lambda_\omega A_V \Lambda_\omega^{-1}, \qquad \mathcal{D}(\Lambda_\omega A_V \Lambda_\omega^{-1}) = \Lambda_\omega \mathcal{D}(A^*)$$

is the generator of a strongly continuous group on H defined by $\Lambda_\omega V(t)\Lambda_\omega^{-1}$.

Let $z \in \Lambda_\omega \mathcal{D}(A^*)$ (i.e., $\Lambda_\omega^{-1}z \in \mathcal{D}(A^*)$) and $y \in \mathcal{D}(A^*)$. From the Riccati equation (22),

$$\langle z, A^*y\rangle_{H,H'} - \langle B^*\Lambda_\omega^{-1}z, B^*y\rangle_{H,H'} = \langle \Lambda_\omega(-A^* - C^*\widetilde{J}C\Lambda_\omega)\Lambda_\omega^{-1}z, y\rangle_{H,H'}.$$

The definition of the extension $\widetilde{A}$ and the hypothesis (H2) on B yield

$$\langle (\widetilde{A} - BJB^*\Lambda_\omega^{-1})z, y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)} = \langle \Lambda_\omega(-A^* - C^*\widetilde{J}C\Lambda_\omega)\Lambda_\omega^{-1}z, y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)}.$$

Since this identity is true for all $y \in \mathcal{D}(A^*)$,

$$A_U z = \Lambda_\omega A_V \Lambda_\omega^{-1} z \in \mathcal{D}(A^*)'.$$

In fact, this identity is true in H since the right member is an element of H.

Hence A_U and $\Lambda_\omega A_V \Lambda_\omega^{-1}$ coincides on $\Lambda_\omega \mathcal{D}(A^*)$, i.e., setting

$$U(t) := \Lambda_\omega V(t)\Lambda_\omega^{-1},$$

the operator A_U with domain $\mathcal{D}(A_U) = \Lambda_\omega \mathcal{D}(A^*)$ is the generator of $U(t)$. $\square$

Now, we can give a natural notion of solution to the closed-loop problem.

Definition 3.3. Let $x_0 \in H$. We define the corresponding *solution of the closed-loop problem* (33) by

$$x(t) := U(t)x_0 \in \mathcal{C}(\mathbb{R}; H).$$

1.2. Domain of the generator. We have seen that the domain of A_U is defined by

$$\Lambda_\omega \mathcal{D}(A^*).$$

Is it possible to link this abstract space to $\mathcal{D}(A)$ or more generally to $\mathcal{D}(A^k)$, where k is a positive integer?

The answer relies on the nature of the control operator B. More precisely, if the control is bounded then the domain of A_U is exactly $\mathcal{D}(A)$. This is coherent since the construction of the above paragraph is not "necessary" in that case, the perturbation $-BJB^*\Lambda_\omega^{-1}$ being a bounded operator in H. The situation is more complicated with an unbounded control operator : we will see through examples that *in general* $\Lambda_\omega \mathcal{D}(A^*)$ is not included in $\mathcal{D}(A)$ and $\mathcal{D}(A)$ is not included in $\Lambda_\omega \mathcal{D}(A^*)$.

Bounded control operators

Proposition 3.4. *Assume that $B \in \mathcal{L}(U, H)$. Then*

$$\mathcal{D}(A_U) = \Lambda_\omega \mathcal{D}(A^*) = \mathcal{D}(A).$$

Proof. The inclusion $\Lambda_\omega \mathcal{D}(A^*) \subset \mathcal{D}(A)$ is a consequence of the Riccati equation. Indeed, let $z \in \Lambda_\omega \mathcal{D}(A^*)$, i.e. $z = \Lambda_\omega x$ for some $x \in \mathcal{D}(A^*)$. Then, for all $y \in \mathcal{D}(A^*)$, the equation (22) implies that

$$\begin{aligned} \langle z, A^* y\rangle &= -\langle A^*\Lambda_\omega^{-1} z, \Lambda_\omega y\rangle - \langle Cz, \widetilde{J}C\Lambda_\omega y\rangle + \langle JB^*\Lambda_\omega^{-1} z, B^* y\rangle \\ &= -\langle (\Lambda_\omega A^* \Lambda_\omega^{-1} + \Lambda_\omega C^* \widetilde{J} C - BJB^*\Lambda_\omega^{-1}) z, y\rangle. \end{aligned}$$

From the definition of the adjoint operator, $z \in \mathcal{D}(A)$ and

$$Az = -(\Lambda_\omega A^* \Lambda_\omega^{-1} + \Lambda_\omega C^* \widetilde{J} C - BJB^*\Lambda_\omega^{-1}) z.$$

For the inclusion $\mathcal{D}(A) \subset \Lambda_\omega \mathcal{D}(A^*)$, we use a method of Zwart [**65**]. Set

$$A_1 := A + \Lambda_\omega C^* \widetilde{J} C - BJB^* \Lambda_\omega^{-1}, \qquad \mathcal{D}(A_1) = \mathcal{D}(A).$$

Then,

$$A_1^* = A^* + C^* \widetilde{J} C \Lambda_\omega - \Lambda_\omega^{-1} BJB^*, \qquad \mathcal{D}(A_1^*) = \mathcal{D}(A^*)$$

We can rewrite the Riccati equation (22) as

$$\langle A^* x, \Lambda_\omega y\rangle + \langle \Lambda_\omega x, A_1^* y\rangle = 0,$$

for all $x, y \in \mathcal{D}(A^*)$.
We can find a complex number s such that

$$s \in \rho(-A) \cap \rho(A_1^*).$$

Indeed, the operators $-A$ and A_1^* are the generators of strongly continuous groups. From the theorem of Hille-Yosida, their resolvent sets are unions of

two disjoint half-planes.
From the first inclusion, for each $x \in \mathcal{D}(A^*)$, $\Lambda_\omega x \in \mathcal{D}(A)$ and

$$\begin{aligned} A\Lambda_\omega x &= -(\Lambda_\omega A^* \Lambda_\omega^{-1} + \Lambda_\omega C^* \widetilde{J} C - BJB^* \Lambda_\omega^{-1})\Lambda_\omega x \\ &= -\Lambda_\omega (A^* + C^* \widetilde{J} C \Lambda_\omega - \Lambda_\omega^{-1} BJB^*)x \\ &= -\Lambda_\omega A_1^* x, \end{aligned}$$

which implies that

$$(sI + A)\Lambda_\omega x = \Lambda_\omega (sI - A_1^*)x.$$

As $s \in \rho(A_1^*) \cap \rho(-A)$, the operators $(sI + A)$ and $(sI - A_1^*)$ are invertible. Multiplying the above relation on the left by $(sI + A)^{-1}$ and on the right by $(sI - A_1^*)^{-1}$, we get

$$\Lambda_\omega (sI - A_1^*)^{-1} = (sI + A)^{-1} \Lambda_\omega \qquad \text{on } H.$$

On $\mathcal{D}(A)$, we have

$$\begin{aligned} \Lambda_\omega^{-1} &= \Lambda_\omega^{-1}(sI + A)^{-1}\Lambda_\omega \Lambda_\omega^{-1}(sI + A) \\ &= \Lambda_\omega^{-1}\Lambda_\omega (sI - A_1^*)^{-1}\Lambda_\omega^{-1}(sI + A) \\ &= (sI - A_1^*)^{-1}\Lambda_\omega^{-1}(sI + A). \end{aligned}$$

Thus, $\Lambda_\omega^{-1}\mathcal{D}(A) \subset \mathcal{D}(A_1^*) = \mathcal{D}(A^*)$ i.e. $\mathcal{D}(A) \subset \Lambda_\omega \mathcal{D}(A^*)$. □

Unbounded control operators

We return to the more general case of an unbounded control operator i.e.

$$B \in \mathcal{L}(U, \mathcal{D}(A^*)').$$

Example 4 (transport equation with a periodic boundary control, see also Example 2)**.** Let us prove that in this case

$$\Lambda_\omega \mathcal{D}(A^*) \not\subset \mathcal{D}(A) \qquad \text{and} \qquad \mathcal{D}(A) \not\subset \mathcal{D}(A^*).$$

For simplicity, we take $T = 2\pi$ and choose ω such that $1/2\omega = 2\pi$. Given φ and ψ in $\mathcal{D}(A^*) = \mathcal{D}(A)$, we have

$$\begin{aligned} (\Lambda_\omega \varphi, \psi)_H &= \int_0^{2\pi} e^{-2\omega t}\varphi(t)\psi(t)\,\mathrm{d}t + 2\omega e^{-1}\int_{2\pi}^{4\pi}(4\pi - t)\varphi(t)\psi(t)\,\mathrm{d}t \\ &= \int_0^{2\pi} e^{-2\omega t}\varphi(t)\psi(t)\,\mathrm{d}t + 2\omega e^{-1}\int_0^{2\pi}(2\pi - t)\varphi(t + 2\pi)\psi(t + 2\pi)\,\mathrm{d}t \\ &= \int_0^{2\pi}\Big(e^{-2\omega t} + e^{-1}\big(1 - \frac{t}{2\pi}\big)\Big)\varphi(t)\psi(t)\,\mathrm{d}t. \end{aligned}$$

Therefore,

$$(\Lambda_\omega \varphi)(t) = \Big(e^{-t/2\pi} + e^{-1}\big(1 - \frac{t}{2\pi}\big)\Big)\varphi(t), \qquad 0 \le t \le 2\pi.$$

For example, taking $\varphi = 1 \in \mathcal{D}(A^*)$, the constant function equal to 1, we have

$$(\Lambda_\omega 1)(t) = e^{-t/2\pi} + e^{-1}\big(1 - \frac{t}{2\pi}\big), \qquad 0 \le t \le 2\pi;$$
$$(\Lambda_\omega^{-1} 1)(t) = \Big(e^{-t/2\pi} + e^{-1}\big(1 - \frac{t}{2\pi}\big)\Big)^{-1}, \qquad 0 \le t \le 2\pi.$$

But

$$(\Lambda_\omega 1)(0) = 1 + e^{-1} \quad \neq \quad e^{-1} = (\Lambda_\omega 1)(2\pi);$$
$$(\Lambda_\omega^{-1} 1)(0) = (1 + e^{-1})^{-1} \quad \neq \quad e = (\Lambda_\omega^{-1} 1)(2\pi),$$

so that

$$\Lambda_\omega 1 \notin \mathcal{D}(A) \qquad \text{and} \qquad \Lambda_\omega^{-1} 1 \notin \mathcal{D}(A^*)$$

because the periodic boundary conditions are not satisfied.

Set $\xi_0 \in \mathcal{D}(A^*)$. Let us explain how to compute $\Lambda_\omega \xi_0$ in general. We follow the method described in [**32**, p. 1603] in the case of the wave equation with a Dirichlet boundary control and write it in an abstract framework. We can notice the similarity with the computation of the control in the Hilbert Uniqueness Method [**45**].

Step 1. We solve the homogeneous problem

$$\begin{cases} \xi'(t) = -A^*\xi(t), & 0 \le t \le T_\omega, \\ \xi(0) = \xi_0. \end{cases}$$

The solution $\xi(t)$ (which is continuously differentiable because $\xi_0 \in \mathcal{D}(A^*)$) is given by

$$\xi(t) = e^{-tA^*}\xi_0.$$

Step 2. We consider the control

$$\begin{aligned} u(t) &:= e_\omega(t) J B^* \xi(t) \\ &= e_\omega(t) J E^* (A + \lambda I)^* \xi(t) \in \mathcal{C}([0, T_\omega]; U). \end{aligned}$$

and we solve the inhomogeneous backward problem

$$\begin{cases} y'(t) = Ay(t) + Bu(t), & 0 \le t \le T_\omega, \\ y(T_\omega) = 0, \end{cases}$$

whose mild solution, analogously to the mild solution of the open-loop problem (30)), is given by

$$y(t) = -(A + \lambda I) \int_t^{T_\omega} e^{(t-s)A} E u(s) \, \mathrm{d}s, \qquad 0 \le t \le T_\omega.$$

This function is continuous on $[0, T_\omega]$ with values in H.

Step 3. We set

$$\Lambda_\omega \xi_0 = -y(0).$$

Indeed, for $\varphi_0 \in \mathcal{D}(A^*)$ we have

$$\begin{aligned}
\langle y(0), \varphi_0 \rangle &= -\int_0^{T_\omega} \langle u(s), B^* e^{-sA^*} \varphi_0 \rangle \,\mathrm{d}s \\
&= -\int_0^{T_\omega} e_\omega(s) \langle J B^* e^{-sA^*} \xi_0, B^* e^{-sA^*} \varphi_0 \rangle \,\mathrm{d}s \\
&= -\langle \Lambda_\omega \xi_0, \varphi_0 \rangle.
\end{aligned}$$

The conclusion is a consequence of the density of $\mathcal{D}(A^*)$ in H'.

Now, let us analyze how the regularity of $\Lambda_\omega \xi_0 = -y(0)$ depends on the regularity of ξ_0. Assume that $\xi_0 \in \mathcal{D}((A^*)^2)$ in order to have $u(t) \in \mathcal{C}^1([0, T_\omega]; U)$. We set

$$\begin{aligned}
z_\lambda(t) &:= e^{\lambda t} \int_t^{T_\omega} e^{(t-s)A} E u(s) \,\mathrm{d}s \\
&= \int_t^{T_\omega} e^{(t-s)(A+\lambda I)} E \widetilde{u}(s) \,\mathrm{d}s,
\end{aligned}$$

where

$$\widetilde{u}(s) := e^{\lambda s} u(s).$$

We recall that λ lies in the resolvent set of $-A$ and remark that $\widetilde{u}(T_\omega) = 0$. An integration by parts yields

$$\begin{aligned}
z_\lambda(t) &= \int_t^{T_\omega} e^{(t-s)(A+\lambda I)} E \widetilde{u}(s) \,\mathrm{d}s \\
&= \int_t^{T_\omega} (A + \lambda I) e^{(t-s)(A+\lambda I)} (A + \lambda I)^{-1} E \widetilde{u}(s) \,\mathrm{d}s \\
&= (A + \lambda I)^{-1} \int_t^{T_\omega} e^{(t-s)(A+\lambda I)} E \widetilde{u}'(s) \,\mathrm{d}s + (A + \lambda I)^{-1} E \widetilde{u}(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
y(0) &= -(A + \lambda I) z_\lambda(0) \\
&= \underbrace{-\int_0^{T_\omega} e^{-s(A+\lambda I)} E \widetilde{u}'(s) \,\mathrm{d}s}_{\in \mathcal{D}(A)} \overbrace{-E\widetilde{u}(0)}^{\in \mathcal{D}(A)?}.
\end{aligned}$$

The first term on the right side of the above identity lies in $\mathcal{D}(A)$ (see the assertion (a) of Proposition 2.7). Hence, $y(0)$ belongs to $\mathcal{D}(A)$ if and only if the second term does so. Let us take a look at this last term

$$E\widetilde{u}(0) = Eu(0) = EJB^* \xi_0 = EJE^* (A + \lambda I)^* \xi_0.$$

through an example.

Example 5 (wave equation with Dirichlet boundary conditions, see also Example 3)**.** Assume that $(\xi_0, \xi_1) \in \mathcal{D}(A^*)$. Then

$$EJB^*(\xi_0, \xi_1) = (0, -D\partial_\nu \xi_0) \in H.$$

We seek a condition on ξ_0 in order to have $(0, D\partial_\nu \xi_0) \in \mathcal{D}(A)$ i.e.

$$D\partial_\nu \xi_0 \in H_0^1(\Omega).$$

If $\xi_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then $\partial_\nu \xi_0 \in H^{1/2}(\Gamma)$ and $D\partial_\nu \xi_0 \in H^1(\Omega)$. By definition of D, the trace of $D\partial_\nu \xi_0$ on the boundary is

$$D\partial_\nu \xi_0|_\Gamma = \partial_\nu \xi_0.$$

That is why, in order to have $D\partial_\nu \xi_0 \in H_0^1(\Omega)$ it is necessary and sufficient that $\partial_\nu \xi_0 = 0$. But (see e.g., [**7**, p. 335–340])

$$\mathcal{D}((A^*)^2) = \Big\{(\xi, \psi) \in H^3(\Omega) \times H^2(\Omega) : \xi = \Delta\xi = \psi = 0 \quad \text{on } \Gamma\Big\}.$$

Hence, if $(\xi_0, \xi_1) \in \mathcal{D}((A^*)^2)$, the normal derivative of ξ_0 on Γ does not necessarily vanish. Finally, given $(\xi_0, \xi_1) \in \mathcal{D}((A^*)^2) \subset \mathcal{D}(A^*)$,

$$\Lambda_\omega(\xi_0, \xi_1) \in \mathcal{D}(A) \iff \partial_\nu \xi_0 = 0.$$

In this example of boundary control,

$$\Lambda_\omega \mathcal{D}(A^*) \not\subset \mathcal{D}(A).$$

Remark 3.5. In the case of a vibrating string, if $\Omega = (0, \pi)$, the eigenfunctions of the Laplacian with Dirichlet boundary conditions are the functions $\sin(nx)$, where $n = 1, 2, \dots$ In particular they are infinitely differentiable. Even if ξ_0 and ξ_1 are linear combinations of these functions, $\Lambda_\omega(\xi_0, \xi_1)$ does not necessarily belong to $\mathcal{D}(A)$ because the normal derivative of ξ_0 does not necessarily vanish at points 0 and π.

Remark 3.6. Concerning the converse inclusion for the general abstract problem, if the weight function e_ω is *replaced* by a continuously differentiable bump function that vanishes in a neighborhood of 0 and T_ω, then it is possible to use a regularity result of Ervedoza and Zuazua [**21**, Theorem 1.4] to prove that

$$\mathcal{D}(A) \subset \Lambda_\omega \mathcal{D}(A^*).$$

But such modification of the weight function implies the loss of the essential property (21) in order to prove the decay rate of the solutions.

2. A variation of constants formula

In this paragraph, we prove a *variation of constants formula* for the solution of the closed-loop problem (33). We recall that

$$U(t) := \Lambda_\omega V(t) \Lambda_\omega^{-1},$$

and that $V(t)$ satisfies the variation of constants formula

$$V(t)y_0 = e^{-tA^*}y_0 - \int_0^t e^{(t-r)A^*}C^*\widetilde{J}C\Lambda_\omega V(r)\,\mathrm{d}r. \tag{35}$$

Theorem 3.7. *The group $U(t)$ satisfies a variation of constants formula: for all $x_0 \in H$ and all $t \in \mathbb{R}$*

$$U(t)x_0 = e^{tA}x_0 - (A+\lambda I)\int_0^t e^{(t-r)A}EJB^*\Lambda_\omega^{-1}U(r)x_0\,\mathrm{d}r. \tag{36}$$

To prove the Theorem 3.7 we will use a representation formula for the operator Λ_ω, contained in the

Lemma 3.8. *Set $x, y \in H'$ and $t \in \mathbb{R}$. Then*

$$\begin{aligned}\langle \Lambda_\omega x, y\rangle_{H,H'} =& \langle \Lambda_\omega V(t)x, e^{-tA^*}y\rangle_{H,H'} \\ &+ \int_0^t \langle JB^*V(s)x, B^*e^{-sA^*}y\rangle_{U,U'}\,\mathrm{d}s.\end{aligned} \tag{37}$$

Remark 3.9. The integral in the above formula is meaningful. Indeed the first part of the bracket defines an element of $L^2_{\mathrm{loc}}(\mathbb{R};U)$ because of (35) and the extended regularity result stated in Proposition 2.2. The second part of the bracket defines an element of $L^2_{\mathrm{loc}}(\mathbb{R};U')$ thanks to the direct inequality stated in (H3).

Proof of Theorem 3.7. Set $x_0 \in H$, $u \in H'$ and $t \in \mathbb{R}$. In (37), replacing x by $\Lambda_\omega^{-1}x_0$ and y by $e^{tA^*}y$ we obtain

$$\langle U(t)x_0, y\rangle_{H,H'} = \langle e^{tA}x_0, y\rangle_{H,H'} - \int_0^t \langle JB^*\Lambda_\omega^{-1}U(s)x_0, B^*e^{(t-s)A^*}y\rangle_{U,U'}\,\mathrm{d}s.$$

Hence, (36) follows. □

Proof of Lemma 3.8. Flandoli [**23**] has proved a similar relation for the solution of a differential Riccati equation. We adapt his proof to the case of an algebraic Riccati equation. The proof contains two steps : at first, we use the integral form of the Riccati equation (26) and the variation of constants formula (35) to prove relation (37) modulo a rest. Then, we show that this rest vanishes. [1]

[1] In order to simplify the notations, we will omit the name of the spaces under the duality brackets in this proof.

First step. Fix $x, y \in H'$ and $t \in \mathbb{R}$. From (26) and (35) we have

$$
\begin{aligned}
&\langle \Lambda_\omega x, y\rangle \\
=&\langle \Lambda_\omega \Big[e^{-tA^*}x\Big], e^{-tA^*}y\rangle - \int_0^t \langle C\Lambda_\omega \Big[e^{-sA^*}x\Big], \widetilde{J}C\Lambda_\omega e^{-sA^*}y\rangle \,\mathrm{d}s \\
&+ \int_0^t \langle JB^* \Big[e^{-sA^*}x\Big], B^* e^{-sA^*}y\rangle \,\mathrm{d}s \\
=&\langle \Lambda_\omega \Big[V(t) + \int_0^t e^{-(t-r)A^*}C^*\widetilde{J}C\Lambda_\omega V(r)\,\mathrm{d}r\Big]x, e^{-tA^*}y\rangle \\
&- \int_0^t \langle C\Lambda_\omega \Big[V(s) + \int_0^s e^{-(s-r)A^*}C^*\widetilde{J}C\Lambda_\omega V(r)\,\mathrm{d}r\Big]x, \widetilde{J}C\Lambda_\omega e^{-sA^*}y\rangle \,\mathrm{d}s \\
&+ \int_0^t \langle JB^* \Big[V(s) + \int_0^s e^{-(s-r)A^*}C^*\widetilde{J}C\Lambda_\omega V(r)dr\Big]x, B^* e^{-sA^*}y\rangle \,\mathrm{d}s \\
=&\langle \Lambda_\omega V(t)x, e^{-tA^*}y\rangle + \int_0^t \langle JB^*V(s)x, B^*e^{-sA^*}y\rangle \,\mathrm{d}s + R.
\end{aligned}
$$

Second step. To obtain relation (37), we have to prove that the rest R vanishes. To lighten the writing, we set

$$g(r) := C^*\widetilde{J}C\Lambda_\omega V(r)x \in \mathcal{C}(\mathbb{R}; H').$$

Let us rewrite the rest :

$$
\begin{aligned}
R =&\langle \Lambda_\omega \int_0^t e^{-(t-r)A^*}g(r)\,\mathrm{d}r, e^{-tA^*}y\rangle \\
&- \int_0^t \langle C\Lambda_\omega V(s)x, \widetilde{J}C\Lambda_\omega e^{-sA^*}y\rangle \,\mathrm{d}s \\
&- \int_0^t \langle C\Lambda_\omega \int_0^s e^{-(s-r)A^*}g(r)\,\mathrm{d}r, \widetilde{J}C\Lambda_\omega e^{-sA^*}y\rangle \,\mathrm{d}s \\
&+ \int_0^t \langle JB^* \int_0^s e^{-(s-r)A^*}g(r)dr, B^*e^{-sA^*}y\rangle \,\mathrm{d}s. \\
=:&R_1 - R_2 - R_3 + R_4.
\end{aligned}
$$

- We can also write R_1 as

$$R_1 = \int_0^t \langle \Lambda_\omega e^{-(t-r)A^*}g(r), e^{-(t-r)A^*}e^{-rA^*}y\rangle \,\mathrm{d}r.$$

The integrand of the above integral corresponds to the first term in the right hand side of (26) by replacing x by $C^*\widetilde{J}C\Lambda_\omega V(r)x = g(r)$, y by $e^{-rA^*}y$ and

t by $t-r$. Hence

$$\begin{aligned}
R_1 =& \int_0^t \langle \Lambda_\omega g(r), e^{-rA^*} y\rangle \,\mathrm{d}r \\
&+ \int_0^t \Big[\int_0^{t-r} \langle C\Lambda_\omega e^{-sA^*} g(r), \widetilde{J} C\Lambda_\omega e^{-sA^*} e^{-rA^*} y\rangle ds \Big] \,\mathrm{d}r \\
&- \int_0^t \Big[\int_0^{t-r} \langle JB^* e^{-sA^*} g(r), B^* e^{-sA^*} e^{-rA^*} y\rangle ds \Big] \,\mathrm{d}r \\
=:& R_1' + R_2' - R_3'.
\end{aligned}$$

- We have

$$R_1' = R_2.$$

The change of variable $\sigma := s + r$ and Fubini's theorem give

$$\begin{aligned}
R_2' =& \int_0^t \int_r^t \langle C\Lambda_\omega e^{-(\sigma-r)A^*} g(r), \widetilde{J} C\Lambda_\omega e^{-\sigma A^*} y\rangle \,\mathrm{d}\sigma \,\mathrm{d}r \\
=& \int_0^t \int_0^\sigma \langle C\Lambda_\omega e^{-(\sigma-r)A^*} g(r), \widetilde{J} C\Lambda_\omega e^{-\sigma A^*} y\rangle \,\mathrm{d}r \,\mathrm{d}\sigma \\
=& R_3.
\end{aligned}$$

- It remains to prove that $R_3' = R_4$. Difficulties arise since the operator B^* is unbounded. The idea is to construct two approximations $R_3'(n)$ and $R_4(n)$ for R_3' and R_4. We prove that $R_3'(n) = R_4(n)$ and that $R_3'(n)$ and $R_4(n)$ converge respectively to R_3' and R_4.

We recall A^* is the infinitesimal generator of a group in H'. Hence for sufficiently large $n \in \mathbb{N}$, n lies in the resolvent set of A^*. We set

$$I_n := n(nI - A^*)^{-1} \in \mathcal{L}(H').$$

Then for all $x \in H'$, $I_n x \in \mathcal{D}(A^*)$ and $I_n x \to x$ as $n \to \infty$ (see [**49**, Lemma 3.2. p. 9]). Moreover, the sequence $\|I_n\|$ is bounded from above independently of n. Indeed, as A^* is the generator of a group, it results from Hille-Yosida theorem ([**49**, Theorem 6.3 p. 23]) that for sufficiently large $n \in \mathbb{N}$,

$$\|I_n\| = \|n(nI - A^*)^{-1}\| \le \frac{n\alpha}{n-\beta},$$

where α and β are two positive constants.

- For n sufficiently large, we set

$$R_3'(n) := \int_0^t \Big[\int_0^{t-r} \langle JB^* e^{-sA^*} I_n g(r), B^* e^{-sA^*} e^{-rA^*} y\rangle \,\mathrm{d}s \Big] \,\mathrm{d}r.$$

The application inside the duality brackets is measurable on the product space $(0,t)\times(0,t)$. [1] Moreover

$$
\begin{aligned}
&\int_0^t \int_0^{t-r} \left|\langle JB^* e^{-sA^*} I_n g(r), B^* e^{-sA^*} e^{-rA^*} y\rangle\right| \mathrm{d}s\,\mathrm{d}r \\
\leq & \int_0^t \int_0^{t} \left|\langle JB^* e^{-sA^*} I_n g(r), B^* e^{-sA^*} e^{-rA^*} y\rangle\right| \mathrm{d}s\,\mathrm{d}r \\
= & \int_0^t \Big[\int_0^{t} \left|\langle JB^* e^{-sA^*} I_n g(r), B^* e^{-sA^*} e^{-rA^*} y\rangle\right| \mathrm{d}s\Big] \mathrm{d}r \qquad \text{(Fubini-Tonelli)} \\
\leq & c\int_0^t \|g(r)\|_{H'} \|e^{-rA^*}\|_{H'}\,\mathrm{d}r \qquad \text{(Cauchy-Schwarz and direct inequalities)} \\
< & \infty.
\end{aligned}
$$

Hence we can invert the order of the integrals in $R_3'(n)$. We get (first by doing the change of variable $\sigma := s + r$) :

$$
\begin{aligned}
R_3'(n) &= \int_0^t \int_r^t \langle JB^* e^{-(\sigma-r)A^*} I_n g(r), B^* e^{-\sigma A^*} y\rangle \,\mathrm{d}\sigma\,\mathrm{d}r \\
&= \int_0^t \int_0^\sigma \langle JB^* e^{-(\sigma-r)A^*} I_n g(r), B^* e^{-\sigma A^*} y\rangle \,\mathrm{d}r\,\mathrm{d}\sigma.
\end{aligned}
$$

Finally, $R_3'(n) = \int_0^t \varphi_n(r)\,\mathrm{d}r$ and $R_3' = \int_0^t \varphi(r)\,\mathrm{d}r$ with the evident notations. For all $0 \leq r \leq t$, we have

$$
\begin{aligned}
|\varphi_n(r) - \varphi(r)| =& \left| \int_0^{t-r} \langle JB^* e^{-sA^*} [I_n g(r) - g(r)], B^* e^{-sA^*} e^{-rA^*} y\rangle \,\mathrm{d}s \right| \\
\leq & \int_0^t \left|\langle JB^* e^{-sA^*} [I_n g(r) - g(r)], B^* e^{-sA^*} e^{-rA^*} y\rangle\right| \mathrm{d}s \\
\leq & c\|I_n g(r) - g(r)\|_{H'} \|e^{-rA^*} y\|_{H'} \\
& \text{(Cauchy-Schwarz and direct inequalities).}
\end{aligned}
$$

Hence $\varphi_n(r) \to \varphi(r)$ as $n \to \infty$. Thanks to Cauchy-Schwarz, the direct inequality and because $\|I_n\|$ is bounded from above, we have

$$
|\varphi_n(r)| \leq c\|I_n g(r)\|_{H'} \|e^{-rA^*} y\|_{H'} \leq c' \|g(r)\|_{H'} \|e^{-rA^*} y\|_{H'}.
$$

We can apply the dominated convergence theorem : $R_3'(n) \to R_3$.

- For sufficiently large n, we set

$$
R_4(n) := \int_0^t \langle JB^* \int_0^s e^{-(s-r)A^*} I_n g(r) dr, B^* e^{-sA^*} y\rangle \,\mathrm{d}s.
$$

[1] The right side is measurable because it is the composition of two measurable functions (we recall that $B^* e^{-tA^*}$ is well-defined in $L^2_{\mathrm{loc}}(\mathbb{R}; U')$). In the left side we can replace B^* by $B_k^* := E^*(A_k^* + \bar{\lambda} I)$ where $A_k^* \in \mathcal{L}(H')$ is the *Yosida approximation* of A^* (see [**49**]). For all $x \in \mathcal{D}(A^*)$, $B_k^* x \to B^* x$ as $k \to \infty$ and $B_k^* \in \mathcal{L}(H', U')$. Hence, the left-hand side of the duality bracket is measurable as a simple limit of continuous (hence measurable) functions on $(0,t)\times(0,t)$.

But I_n and $e^{-(s-r)}A^*$ commute and

$$\begin{aligned} B^* I_n &= E^*(A+\lambda I)^* n(nI - A^*)^{-1} \\ &= -nE^* + (n^2 + n\lambda)E^*(nI - A^*)^{-1} \in \mathcal{L}(H'). \end{aligned}$$

Hence (see Proposition A.1 for inverting B^* and the integral sign)

$$B^* \int_0^s e^{-(s-r)A^*} I_n g(r) dr = \int_0^s B^* I_n e^{-(s-r)A^*} g(r)\, \mathrm{d}r$$

and

$$R_4(n) = \int_0^t \int_0^s \langle JB^* I_n e^{-(s-r)A^*} g(r), B^* e^{-sA^*} y\rangle\, \mathrm{d}r\, \mathrm{d}s = R_3'(n).$$

Finally, for all $0 \leq r \leq t$, $I_n g(r) \to g(r)$ and $\|I_n g(r)\| \leq c\|g(r)\|$, the right hand sign being integrable on $(0,t)$. Thanks to the dominated convergence theorem, $I_n g \to g$ in $L^1(0,t;H')$. The estimation of proposition 2.2 gives

$$B^* \int_0^s e^{-(s-r)A^*} I_n g(r)\, \mathrm{d}r \to B^* \int_0^s e^{-(s-r)A^*} g(r)\, \mathrm{d}r$$

in $L^2(0,t;U')$. Hence $R_4(n) \to R_4$ and by uniqueness of the limit, $R_3' = R_4$. □

Remark 3.10. Starting with the variation of constant formula (36) it is possible to recover the operator $\widetilde{A} - BJB^*\Lambda_\omega^{-1}$ as the generator of the group $U(t)$.

We know that for $x_0 \in \Lambda_\omega \mathcal{D}(A^*) = \mathcal{D}(A_U)$, the map

$$t \mapsto U(t)x_0$$

is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t} U(t)x_0 = A_U U(t) x_0.$$

In particular if $y \in H'$, then [1]

$$\langle \frac{\mathrm{d}}{\mathrm{d}t} U(t)x_0, y\rangle = \langle A_U U(t)x_0, y\rangle.$$

Differentiating (36) with respect to t, we want to link the generator A_U and the operator $A - BJB^*\Lambda_\omega^{-1}$ (a priori with values in $\mathcal{D}(A^*)'$). We remark that defining the domain of the latter operator is not clear. Let $x_0 \in \Lambda_\omega \mathcal{D}(A^*)$ and $y \in \mathcal{D}((A^*)^2)$.

First step. The map

$$r \mapsto B^*\Lambda_\omega^{-1} U(r)x_0$$

[1] Again, when the name of spaces under the duality brackets are unnecessary, we omit them.

is continuous from $\mathbb{R}$ to U'. Indeed, setting $y_0 := \Lambda_\omega^{-1} x_0 \in D(A^*)$, we have [1]

$$\begin{aligned}
B^*\Lambda_\omega^{-1}U(r)x_0 &= B^*\Lambda_\omega^{-1}\Lambda_\omega V(r)\Lambda_\omega^{-1}x_0\\
&= B^*V(r)y_0\\
&= E^*(A^*+\bar\lambda I)V(r)y_0\\
&= E^*(A^*+C^*\widetilde{J}C\Lambda_\omega - C^*\widetilde{J}C\Lambda_\omega+\bar\lambda I)V(r)y_0\\
&= -E^*(-A^*-C^*\widetilde{J}C\Lambda_\omega)V(r)y_0 + E^*(-C^*\widetilde{J}C\Lambda_\omega+\bar\lambda I)V(r)y_0\\
&= -E^*V(r)(-A^*-C^*\widetilde{J}C\Lambda_\omega)y_0 + E^*(-C^*\widetilde{J}C\Lambda_\omega+\bar\lambda I)V(r)y_0,
\end{aligned}$$

the latter expression being continuous in r.

Second step. The map

$$s \mapsto B^* e^{sA^*} y$$

is differentiable on $\mathbb{R}$ with values in U'. Indeed, as $y \in \mathcal{D}((A^*)^2)$, we have $(A^*+\bar\lambda I)y \in \mathcal{D}(A^*)$ and

$$B^*e^{sA^*}y = E^*e^{sA^*}(A^*+\bar\lambda I)y.$$

The latter expression is differentiable with respect to s and its derivative is $B^*e^{sA^*}A^*y$.

Third step. We deduce from the two previous steps that the map

$$t \mapsto \int_0^t \langle JB^*\Lambda_\omega^{-1}U(r)x_0, B^*e^{(t-r)A^*}y\rangle \,\mathrm{d}r$$

is differentiable on $\mathbb{R}$ and its derivative is the map

$$t \mapsto \int_0^t \langle JB^*\Lambda_\omega^{-1}U(r)x_0, B^*e^{(t-r)A^*}A^*y\rangle \,\mathrm{d}r + \langle JB^*\Lambda_\omega^{-1}U(t)x_0, B^*y\rangle.$$

It results that given two (regular) data $x_0 \in \Lambda_\omega\mathcal{D}(A^*)$ and $y \in \mathcal{D}((A^*)^2)$, we can differentiate $\langle U(t)x_0, y\rangle$ with respect to t and get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle U(t)x_0,y\rangle =& \langle e^{At}x_0, A^*y\rangle - \int_0^t \langle JB^*\Lambda_\omega^{-1}U(r)x_0, B^*e^{(t-r)A^*}A^*y\rangle\,\mathrm{d}r\\
&- \langle JB^*\Lambda_\omega^{-1}U(t)x_0, B^*y\rangle.
\end{aligned}$$

Using the relation (37) against A^*y via the duality brackets and re-injecting it in the above relation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle U(t)x_0,y\rangle = \langle U(t)x_0, A^*y\rangle - \langle JB^*\Lambda_\omega^{-1}U(t)x_0, B^*y\rangle. \tag{38}$$

With the same regularity as above for x_0 and y, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle U(t)x_0,y\rangle_{H,H'} = \langle A_U U(t)x_0,y\rangle_{H,H'} = \langle A_U U(t)x_0,y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)},$$

[1] On $\mathcal{D}(A^*) = \mathcal{D}(-A^*-C^*\widetilde{J}C\Lambda_\omega)$, the operators $V(r)$ and $-A^*-C^*\widetilde{J}C\Lambda_\omega$ (generator of $V(r)$) commute (this is a general fact about semigroups) but a priori $V(r)$ and A^* do not commute.

where A_U is the infinitesimal generator of $U(t)$. We recall from Lemma 3.1 that A admits a unique extension $\widetilde{A} \in \mathcal{L}(H, \mathcal{D}(A^*)')$. Thanks to this extension we can link A_U and $A - BJB^*\Lambda_\omega^{-1}$. From (38) and (34) we have, for $x_0 \in \Lambda_\omega \mathcal{D}(A^*)$ and $y \in \mathcal{D}((A^*)^2)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle U(t)x_0, y\rangle_{H,H'} &= \langle U(t)x_0, A^*y\rangle_{H,H'} - \langle JB^*\Lambda_\omega^{-1}U(t)x_0, B^*y\rangle_{U,U'}\\ &= \langle \widetilde{A}U(t)x_0, y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)}\\ &\qquad - \langle BJB^*\Lambda_\omega^{-1}U(t)x_0, y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)}\\ &= \langle (\widetilde{A} - BJB^*\Lambda_\omega^{-1})U(t)x_0, y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)}\\ &= \langle A_U U(t)x_0, y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)}\end{aligned}$$

In particular, the latter equality is true for $t = 0$. Hence, given a fixed $x_0 \in \Lambda_\omega \mathcal{D}(A^*)$, we have

$$\langle (\widetilde{A} - BJB^*\Lambda_\omega^{-1})x_0, y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)} = \langle A_U x_0, y\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)},$$

for all $y \in \mathcal{D}((A^*)^2)$. This relation remains true for all $y \in \mathcal{D}(A^*)$ by density of $\mathcal{D}((A^*)^2)$ in $\mathcal{D}(A^*)$ (for the norm $\|\cdot\|_{\mathcal{D}(A^*)}$). Finally,

$$\forall x_0 \in \Lambda_\omega \mathcal{D}(A^*) = \mathcal{D}(A_U), \qquad (\widetilde{A} - BJB^*\Lambda_\omega^{-1})x_0 = A_U x_0 \in H.$$

3. Another method using optimal control theory

In this paragraph, we explain how the closed-loop problem (33) is linked to an *optimal control* problem. This connection was already mentioned in [**32**, pp. 1600–1601] and we will only present an outline of this method that uses essentially results of Flandoli, Lasiecka and Triggiani [**24**].

We introduce the *cost functional*

$$J(u) := \int_0^\infty \|Cx(t)\|_H^2 + \|u(t)\|_U^2 \,\mathrm{d}t \in [0, +\infty]$$

on the space $L^2(0, \infty; U)$ where x denotes the associated solution of the open-loop problem (27).

> *The method consists of the minimization of the functional J and the characterization of the input u that realizes the minimum.*

Step 1. The functional J takes finite values [1] i.e. there exists an input $u \in L^2(0, \infty; U)$ such that $J(u) < +\infty$.

This is a consequence of the exact controllability of the pair (A, B) which is a consequence [2] of the hypotheses (H1)-(H4), in fact essentially of the

[1] In the terminology of [**24**], J satisfies the "finite cost condition".

[2] We refer to [**32**, pp. 1595–1596] and [**45**] for a proof of the implication "observability $\Rightarrow$ controllability."

observability inequality contained in (H4). Indeed, if the system (A, B) is exactly controllable in time T, is suffices to take a control u that steers the system to zero at time T and to set $u(t) = 0$ if $t > T$.

Then, a classical (convex) optimization result ensures that *there exists a unique input u^* that realizes the minimum of J* i.e.

$$J(u^*) = \min \big\{ J(u),\ u \in L^2(0, \infty; U) \big\}.$$

We denote by x^* the trajectory associated to u^*.

Step 2. Two operator equations plays an important role in the characterization of the optimal control. These are the two algebraic Riccati equations stated formally as

$$XA + A^*X + C^*\widetilde{J}C - XB^*JBX = 0; \tag{39}$$

$$AX + XA^* + XC^*\widetilde{J}CX - BJB^* = 0. \tag{40}$$

In the terminology of [**24**], the first equation is the "Algebraic Riccati Equation" while the second is the "Dual Algebraic Riccati Equation".

A result of [**24**, Theorem 2.2. pp.317–318] states that u^* can be expressed as a feedback, namely

$$u^*(\cdot) = -JB^*P_\infty x^*(\cdot),$$

where $P_\infty \in \mathcal{L}(H, H')$ satisfies the algebraic Riccati equation (39). Moreover, the closed-loop operator $A_F := A - BJB^*P_\infty$ with a domain denoted by $\mathcal{D}(A_F)$ is the infinitesimal generator of a semigroup in H. [1]

Step 3. Now it remains to *identify* P_∞ and Λ_ω^{-1}. This can be done by using further results from [**24**]. The fact that A generates a *group* and that C has "good" properties are the two essential tools.

- The operator P_∞ is the *unique* solution of the Riccati equation (39). Indeed, as C is positive definite, we can apply the Theorem 2.3 of [**24**, p. 319].
- The operator P_∞ is *invertible* because the pair (A^*, C^*) is exactly controllable (see [**24**, Theorem 2.4 pp. 319–320]).

  Let us prove that the pair (A^*, C^*) is exactly controllable in time $T > 0$ for any choice of $T > 0$. We recall that A^* generates a group and that C is bounded from below, hence invertible and so is C^*. Given $y_0, y_T \in H'$ we seek a control $u \in L^2(0, T; U)$ such that the solution of

  $$y' = -A^*y + C^*u, \qquad y(0) = y_0$$

  satisfies $y(T) = y_T$. This solution can be written

  $$y(t) = e^{-tA^*} + \int_0^t e^{-(t-s)A^*} C^*u(s)\,\mathrm{d}s, \qquad 0 \le t \le T.$$

[1] Implicitly, A has to be replaced by its extension $\widetilde{A}$ in the definition of A_F.

We can check that the control

$$u(t) = \frac{1}{T}(C^*)^{-1}e^{(t-T)A^*}(y_T - e^{TA^*}y_0), \qquad 0 \leq t \leq T$$

steers the system from y_0 at time 0 to y_T at time T.

- The (dual) Riccati equation(40) (defined on $\mathcal{D}(A^*)$) has a unique solution denoted by Q_∞. Indeed, this is a consequence of the Theorem 2.6 on [**24**, pp. 324–325]. One can apply this result since the pair (A, B) is exactly controllable. But we already know that the operator Λ_ω satisfies the Riccati equation (40) on $\mathcal{D}(A^*)$. Thus,

$$Q_\infty = \Lambda_\omega$$

- We know that A generates a group, the pairs (A, B) and (A^*, C^*) are exactly controllable. Hence, from [**24**, Theorem 2.7. p.326], we can assert that Q_∞ and P_∞^{-1} coincide. In other words

$$P_\infty = \Lambda_\omega^{-1}.$$

Step 4. Concerning the domain of the generator $\mathcal{D}(A_F)$, under the above assumptions, we can use a result of Triggiani [**60**] that asserts that

$$\mathcal{D}(A_F) = \Lambda_\omega \mathcal{D}(A^*).$$

Chapter 4

On the decay rate of the solutions

The aim of this chapter is to analyze the stability of the solutions of the closed-loop problem (33). More precisely we would like to

give a new proof of the exponential decay of the solutions and obtain a lower bound of the decay rate.

Komornik [**32**] already proved that the exponential decay rate is at least ω. Let us recall how we can obtain this result for finite-dimensional systems, so that we do note have to care about the domains of definition of the operators and the sense in which the system is well-posed. The idea is to obtain a Gronwall-type inequality for a quantity equivalent to the energy of the system.

Denoting by x the solution of (33), we consider the function

$$\langle \Lambda_\omega^{-1} x(t), x(t)\rangle \qquad \big(\asymp \|x(t)\|^2\big).$$

Differentiating it with respect to t, we obtain [1]

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\langle \Lambda_\omega^{-1} x(t), x(t)\rangle\\ =&\langle [A^*\Lambda_\omega^{-1} + \Lambda_\omega^{-1}A - 2\Lambda_\omega^{-1}BJB^*\Lambda_\omega^{-1}]x(t), x(t)\rangle\\ =&-\langle C^*\widetilde{J}Cx(t), x(t)\rangle - \langle \Lambda_\omega^{-1}BJB^*\Lambda_\omega^{-1}x(t), x(t)\rangle\\ \le&-2\omega\langle \Lambda_\omega^{-1}x(t), x(t)\rangle.\end{aligned}$$

This inequality yields

$$\langle \Lambda_\omega^{-1} x(t), x(t)\rangle \le e^{-2\omega t}\langle \Lambda_\omega^{-1} x_0, x_0\rangle, \qquad t \ge 0.$$

[1] The second equality is a consequence of the Riccati equation (24) satisfied by Λ_ω^{-1}; the inequality is a consequence of (21) and and the positiveness of $\langle \Lambda_\omega^{-1}BJB^*\Lambda_\omega^{-1}x(t), x(t)\rangle$.

and the latter relation ensures the existence of a positive constant c such that

$$\|x(t)\| \leq ce^{-\omega t}\|x_0\|, \qquad t \geq 0.$$

Now we can specify the outline of this chapter.

On the one hand, we justify the same estimation as above for infinite-dimensional systems and especially systems with an unbounded control operator. The first proof (cf. section 1) is similar to the proof given above for finite-dimensional systems: it relies on the fact that the operator $\widetilde{A} - BJB^*\Lambda_\omega^{-1}$ generates a group. The second proof (cf. section 2) details the one given by Komornik in [**32**]: in particular we justify an integral representation formula for Λ_ω^{-1}.

On the other hand, we ask the question of the *optimality of the decay rate*: is ω the effective decay rate of the stabilized system or do the solutions decrease faster? This question is the object of section 3.

In the last section, we recall briefly another stabilization method (related to the one in question here) that also leads to arbitrarily large decay rates.

1. Exponential decay

Let us give a proof of the exponential decay of the solutions of (33). The proof is different from the one given in [**32**, pp. 1598–1599]: we do not use an integral representation formula fo Λ_ω^{-1}. It is closer to the finite dimensional case (see [**32**, p. 1597] and the introduction of this chapter).

We recall that $U(t)$ denotes the group generated by $A_U = \widetilde{A} - BJB^*\Lambda_\omega^{-1}$. Its domain is $\mathcal{D}(A_U) = \Lambda_\omega \mathcal{D}(A^*)$.

Proposition 4.1 (exponential decay)**.** *There exists a positive constant c such that for each initial datum x_0,*

$$\|U(t)x_0\|_H \leq ce^{-\omega t}\|x_0\|_H, \qquad t \geq 0.$$

Proof. At first, let $x_0 \in \mathcal{D}(A_U) = \Lambda_\omega \mathcal{D}(A^*)$ and set $x(t) := U(t)x_0$. With this regularity for the initial datum, $x(t)$ is differentiable on $\mathbb{R}$ and

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \Lambda_\omega^{-1}x(t), x(t)\rangle_{H',H} &= \langle \Lambda_\omega^{-1}x(t), x'(t)\rangle_{H',H} \\
&= \langle \Lambda_\omega^{-1}x(t), A_U x(t)\rangle_{H',H} \\
&= \langle \underbrace{\Lambda_\omega^{-1}x(t)}_{\in \mathcal{D}(A^*)}, \underbrace{(\widetilde{A} - BJB^*\Lambda_\omega^{-1})x(t)}_{\in H}\rangle_{H',H} \\
&= \langle \Lambda_\omega^{-1}x(t), \widetilde{A}x(t)\rangle_{\mathcal{D}(A^*),\mathcal{D}(A^*)'} \\
&\qquad - \langle \Lambda_\omega^{-1}x(t), BJB^*\Lambda_\omega^{-1}x(t)\rangle_{\mathcal{D}(A^*),\mathcal{D}(A^*)'} \\
&= \dots
\end{aligned}$$

$$\ldots = \langle A^*\Lambda_\omega^{-1}x(t), x(t)\rangle_{H',H} \\ - \langle JB^*\Lambda_\omega^{-1}x(t), B^*\Lambda_\omega^{-1}x(t)\rangle_{U,U'}.$$

From the Riccati equation (22), we obtain for all $x, y \in \mathcal{D}(A_U)$,

$$\langle A^*\Lambda_\omega^{-1}x, y\rangle_{H',H} + \langle x, A^*\Lambda_\omega^{-1}y\rangle_{H,H'} \\ + \langle \widetilde{J}Cx, Cy\rangle_{H',H} - \langle JB^*\Lambda_\omega^{-1}x, B^*\Lambda_\omega^{-1}y\rangle_{U,U'} = 0.$$

Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \Lambda_\omega^{-1}x(t), x(t)\rangle_{H',H} &= -\langle \widetilde{J}Cx(t), Cx(t)\rangle_{H',H} \\ &\quad - \langle JB^*\Lambda_\omega^{-1}x(t), B^*\Lambda_\omega^{-1}x(t)\rangle_{U,U'} \\ &\le -\langle \widetilde{J}Cx(t), Cx(t)\rangle_{H',H} \\ &\le -2\omega\langle \Lambda_\omega^{-1}x(t), x(t)\rangle_{H',H}\end{aligned}$$

the last inequality being a consequence of (21).

Finally the above estimations yield

$$\langle \Lambda_\omega^{-1}x(t), x(t)\rangle_{H',H} \le e^{-2\omega t}\langle \Lambda_\omega^{-1}x_0, x_0\rangle_{H',H}, \qquad t \ge 0.$$

This estimation remains true for $x_0 \in H$ by density of $\mathcal{D}(A_U)$ in H. We conclude by noticing that, thanks to (H3) and (H4),

$$\langle \Lambda_\omega^{-1}x, x\rangle_{H',H} \asymp \|x\|_H. \qquad \square$$

2. Original proof via a representation formula for Λ_ω^{-1}

In this section, we give a justification to a representation formula for Λ_ω^{-1} involving the group $U(t)$. This corresponds to the formula (3.11) in [**32**]. We recall it as it is written in this paper: for all $s, t \in \mathbb{R}$,

$$\Lambda_\omega^{-1} = U(t-s)^*\Lambda_\omega^{-1}U(t-s) \\ + \int_s^t U(\tau - s)^*(C^*\widetilde{J}C + \Lambda_\omega^{-1}BJB^*\Lambda_\omega^{-1})U(\tau - s)\,\mathrm{d}\tau.$$

This formula is used in [**32**] to prove the exponential decay of the solutions of the closed-loop system. Flandoli [**23**] derived an analogous formula in the case of differential Riccati equations. We adapt his proof to the case of algebraic Riccati equations. Then, we recall Komornik's proof of the exponential decay of the solutions of the closed-loop system (33).

We first prove a similar representation formula for Λ_ω.

Proposition 4.2. *For all $x, y \in H'$ and $t \in \mathbb{R}$*

$$\langle \Lambda_\omega x, y\rangle_{H,H'} = \langle \Lambda_\omega V(t)x, V(t)y\rangle_{H,H'} \\ + \int_0^t \langle JB^*V(s)x, B^*V(s)y\rangle_{U,U'}\,\mathrm{d}s + \int_0^t \langle C\Lambda_\omega V(s)x, \widetilde{J}C\Lambda_\omega V(s)y\rangle_{H,H'}\,\mathrm{d}s. \tag{41}$$

Proof. It relies on the representation formula (37) for Λ_ω: for $x, y \in H'$,

$$\langle \Lambda_\omega x, y\rangle = \langle \Lambda_\omega V(t)x, [e^{-tA^*}y]\rangle + \int_0^t \langle JB^*V(s)x, B^*[e^{-sA^*}y]\rangle \,\mathrm{d}s.$$

In the right hand side of the above relation, we replace $e^{-tA^*}y$ and $e^{-sA^*}y$ by using the variation of constants formula (35) for V :

$$\begin{aligned}\langle \Lambda_\omega x, y\rangle =&\langle \Lambda_\omega V(t)x, V(t)y\rangle \\ &+ \langle \Lambda_\omega V(t)x, \int_0^t e^{-(t-s)A^*}C^*\widetilde{J}C\Lambda_\omega V(s)y \,\mathrm{d}s\rangle \\ &+ \int_0^t \langle JB^*V(s)x, B^*V(s)y\rangle \,\mathrm{d}s \\ &+ \int_0^t \langle JB^*V(s)x, B^* \int_0^s e^{-(s-r)A^*}C^*\widetilde{J}C\Lambda_\omega V(r)ydr\rangle \,\mathrm{d}s \\ =:&T_1 + T_2 + T_3 + T_4.\end{aligned}$$

But

$$T_2 := \int_0^t \langle \Lambda_\omega V(t-s)V(s)x, e^{-(t-s)A^*}C^*\widetilde{J}C\Lambda_\omega V(s)y\rangle \,\mathrm{d}s.$$

Thanks to (37), applied to $V(s)x$ instead of x, $C^*C\Lambda_\omega V(s)y$ instead of y and $t-s$ instead of t, we have

$$\begin{aligned}T_2 =& \int_0^t \langle \Lambda_\omega V(s)x, C^*\widetilde{J}C\Lambda_\omega V(s)y\rangle \,\mathrm{d}s \\ &- \int_0^t \int_0^{t-s} \langle JB^*V(r)V(s)x, B^*e^{-rA^*}C^*\widetilde{J}C\Lambda_\omega V(s)y\rangle \,\mathrm{d}r \,\mathrm{d}s \\ =& \int_0^t \langle C\Lambda_\omega V(s)x, \widetilde{J}C\Lambda_\omega V(s)y\rangle \,\mathrm{d}s \\ &- \int_0^t \int_0^{t-s} \langle JB^*V(r+s)x, B^*e^{-rA^*}C^*\widetilde{J}C\Lambda_\omega V(s)y\rangle \,\mathrm{d}r \,\mathrm{d}s.\end{aligned}$$

The change of variable $\sigma := r+s$ in the last term gives

$$\begin{aligned}T_2 =& \int_0^t \langle C\Lambda_\omega V(s)x, \widetilde{J}C\Lambda_\omega V(s)y\rangle \,\mathrm{d}s \\ &- \int_0^t \int_s^t \langle JB^*V(\sigma)x, B^*e^{-(\sigma-s)A^*}C^*\widetilde{J}C\Lambda_\omega V(s)y\rangle \,\mathrm{d}\sigma \,\mathrm{d}s \\ =& \int_0^t \langle C\Lambda_\omega V(s)x, \widetilde{J}C\Lambda_\omega V(s)y\rangle \,\mathrm{d}s \\ &- \int_0^t \int_0^\sigma \langle JB^*V(\sigma)x, B^*e^{-(\sigma-s)A^*}C^*\widetilde{J}C\Lambda_\omega V(s)y\rangle \,\mathrm{d}s \,\mathrm{d}\sigma.\end{aligned}$$

Hence we have shown that

$$\begin{aligned}\langle \Lambda_\omega x, y\rangle =& \langle \Lambda_\omega V(t)x, V(t)y\rangle \\ &+ \int_0^t \langle JB^*V(s)x, B^*V(s)y\rangle \,\mathrm{d}s \\ &+ \int_0^t \langle C\Lambda_\omega V(s)x, \widetilde{J}C\Lambda_\omega V(s)y\rangle \,\mathrm{d}s \\ &+ \int_0^t \langle JB^*V(s)x, B^* \int_0^s e^{-(s-r)A^*} C^*\widetilde{J}C\Lambda_\omega V(r)y \,\mathrm{d}r\rangle \,\mathrm{d}s \\ &- \int_0^t \int_0^s \langle JB^*V(s)x, B^*e^{-(s-r)A^*} C^*\widetilde{J}C\Lambda_\omega V(r)y\rangle \,\mathrm{d}r\,\mathrm{d}s.\end{aligned}$$

We have already seen in the proof of Lemma 3.8 that the two last terms in the above relation cancel each other. Hence the relation is proved. □

Proposition 4.3. *For all $x, y \in H$ and $t \in \mathbb{R}$*

(42)
$$\langle \Lambda_\omega^{-1} x, y\rangle_{H',H} = \langle \Lambda_\omega^{-1} U(t)x, U(t)y\rangle_{H',H}$$
$$+ \int_0^t \langle \widetilde{J}CU(s)x, CU(s)y\rangle_{H',H} \,\mathrm{d}s + \int_0^t \langle JB^*\Lambda_\omega^{-1}U(s)x, B^*\Lambda_\omega^{-1}U(s)y\rangle_{U,U'} \,\mathrm{d}s.$$

Proof. We replace x by $\Lambda_\omega^{-1}x$ and y by $\Lambda_\omega^{-1}y$ in the relation given by the Proposition 4.2 :

$$\begin{aligned}\langle x, \Lambda_\omega^{-1}y\rangle =& \langle \Lambda_\omega V(t)\Lambda_\omega^{-1}x, V(t)\Lambda_\omega^{-1}y\rangle + \int_0^t \langle JB^*V(s)\Lambda_\omega^{-1}x, B^*V(s)\Lambda_\omega^{-1}y\rangle \,\mathrm{d}s \\ &+ \int_0^t \langle \widetilde{J}C\Lambda_\omega V(s)\Lambda_\omega^{-1}x, C\Lambda_\omega V(s)\Lambda_\omega^{-1}y\rangle \,\mathrm{d}s.\end{aligned}$$

Then, by definition of U,

$$\langle \Lambda_\omega^{-1}x, y\rangle = \langle \Lambda_\omega^{-1}U(t)x, U(t)y\rangle$$
$$+ \int_0^t \langle JB^*\Lambda_\omega^{-1}U(s)x, B^*\Lambda_\omega^{-1}U(s)y\rangle \,\mathrm{d}s + \int_0^t \langle \widetilde{J}CU(s)x, CU(s)y\rangle \,\mathrm{d}s. \qquad \square$$

Remark 4.4. A simple change of variable in (4.3) implies that for all $s, t \in \mathbb{R}$ and all $x, y \in H$,

(43)
$$\begin{aligned}\langle \Lambda_\omega^{-1}x, y\rangle =& \langle \Lambda_\omega^{-1}U(t-s)x, U(t-s)y\rangle \\ &+ \int_s^t \langle JB^*\Lambda_\omega^{-1}U(\tau-s)x, B^*\Lambda_\omega^{-1}U(\tau-s)y\rangle \,\mathrm{d}\tau \\ &+ \int_s^t \langle \widetilde{J}CU(\tau-s)x, CU(\tau-s)y\rangle \,\mathrm{d}\tau.\end{aligned}$$

Now, we can recall the original proof of the exponential decay rate as it is stated in [**32**].

Original proof of Proposition 4.1. We denote by $x(t)$ the mild solution of (33) i.e. $x(t) = U(t)x_0$. Using the relation (43) with $x = y = U(s)x_0 = x(s)$, we have

$$\langle \Lambda_\omega^{-1} x(s), x(s)\rangle = \langle \Lambda_\omega^{-1} x(t), x(t)\rangle$$
$$+ \int_s^t \langle JB^*\Lambda_\omega^{-1} x(\tau), B^*\Lambda_\omega^{-1} x(\tau)\rangle \,\mathrm{d}\tau + \int_s^t \langle \widetilde{J} Cx(\tau), Cx(\tau)\rangle \,\mathrm{d}\tau.$$

Let $0 \le s \le t$. The estimation (21) between C and Λ_ω^{-1} and the positiveness of the second term of the right hand side in the above relation yield

$$\langle \Lambda_\omega^{-1} x(s), x(s)\rangle \ge \langle \Lambda_\omega^{-1} x(t), x(t)\rangle + 2\omega \int_s^t \langle \Lambda_\omega^{-1} x(\tau), x(\tau)\rangle \,\mathrm{d}\tau.$$

Let us assume that $x_0 \in \mathcal{D}(A_U)$ so that $x(\cdot)$ is continuously differentiable and so is the function

$$f(t) := \langle \Lambda_\omega^{-1} x(t), x(t)\rangle, \qquad t \ge 0.$$

Hence, fixing $t > 0$, we obtain for all $0 \le s < t$,

$$\frac{f(t) - f(s)}{t - s} \le -\frac{2\omega}{t - s} \int_s^t f(\tau)\,\mathrm{d}\tau = -2\omega \int_0^1 f(t - h(t - s))\,\mathrm{d}h,$$

and taking the limit as s tends to t,

$$f'(t) \le -2\omega f(t) \quad \Rightarrow \quad [f(t)e^{2\omega t}]' \le 0 \quad \Rightarrow \quad f(t) \le f(0)e^{-2\omega t}, \qquad t \ge 0.$$

The density of $\mathcal{D}(A_U)$ in H and the strong continuity of $U(t)$ imply that for all $x_0 \in H$,

$$\langle \Lambda_\omega^{-1} x(t), x(t)\rangle \le e^{-2\omega t} \langle \Lambda_\omega^{-1} x_0, x_0\rangle$$

The conclusion follows from the equivalence of $\langle \Lambda_\omega^{-1} x, x\rangle$ and $\|x\|_H^2$ in H. □

3. A better decay rate

In this paragraph we analyze the possibility of a decay rate greater than ω. We begin by stating three signs that make us expect a bigger decay rate.

A neglected term. If we return to the proof of Proposition 4.1 and have a look at the inequalities (precisely the first inequality), we notice that one term has not been taken into account: we have just dropped off the term

$$-\langle JB^*\Lambda_\omega^{-1} x(t), B^*\Lambda_\omega^{-1} x(t)\rangle.$$

If an estimation of the form

$$\|B^*\Lambda_\omega^{-1} x(t)\|^2 \ge c\langle \Lambda_\omega^{-1} x(t), x(t)\rangle, \qquad t \ge 0$$

appeared to be true (for a positive constant c), then we would obtain a decay rate bigger than ω. Unfortunately such an estimation seems difficult to reach.

A finite-dimensional example. Let us explain on a finite-dimensional example why we can expect a better decay rate than ω. We consider the the system (harmonic oscillator) governed by

$$\begin{cases} y''(t) + y(t) = u(t), & t \geq 0 \\ y(0) = y_0, \quad y'(0) = y_1. \end{cases}$$

This system is equivalent to the abstract open-loop problem (27) by setting

$$x(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}, \quad x_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hypotheses (H1) to (H3) are obviously satisfied while hypothesis (H4) follows from the observability of the pair $(-A^*, B^*)$ that one can check via the rank condition.
Now we compute the feedback operator (19) replacing Λ_ω by the slightly different operator [1]

$$\begin{aligned} \widetilde{\Lambda_\omega} &:= \int_0^{T_0} e^{-2\omega t} e^{-tA^*} BB^* e^{-tA} \,\mathrm{d}t \\ &= \begin{pmatrix} \int_0^{T_0} e^{-2\omega t} \sin^2 t \,\mathrm{d}t & -\int_0^{T_0} e^{-2\omega t} \sin t \cos t \,\mathrm{d}t \\ -\int_0^{T_0} e^{-2\omega t} \sin t \cos t \,\mathrm{d}t & \int_0^{T_0} e^{-2\omega t} \cos^2 t \,\mathrm{d}t \end{pmatrix}. \end{aligned}$$

For some particular values of T_0, the coefficients of the above matrix are particularly simple and so are the coefficients of the closed loop operator. Choosing $T_0 = k\pi$, where k is a positive integer, we obtain

$$A_U = A - BB^* \widetilde{\Lambda_\omega}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 - \frac{4\omega^2}{1-e^{-2\omega k\pi}} & \frac{-4\omega}{1-e^{-2\omega k\pi}} \end{pmatrix}.$$

We recall that for finite-dimensional systems,

$$\text{growth bound of } A_U \;=\; \max\{\mathrm{Re}(\lambda),\, \lambda \text{ eigenvalue of } A_U\}.$$

The eigenvalues of A_U are complex and conjugated since this matrix is real and the discriminant of the characteristic polynomial is negative :

$$(\mathrm{tr} A_U)^2 - 4\det A_U = \frac{8e^{-2\omega k\pi}}{(1 - e^{-2\omega k\pi})^2}\Big((1 + \frac{(2\omega)^2}{2} - \mathrm{ch}(2\omega k\pi)\Big) < 0.$$

Indeed, $\mathrm{ch}(k\pi x) > \mathrm{ch}(x) > 1 + \frac{x^2}{2}$ for all $x > 0$. Hence, denoting by λ and $\bar{\lambda}$ the eigenvalues, we obtain

$$\mathrm{Re}\lambda = \mathrm{Re}\bar{\lambda} = \frac{1}{2}\mathrm{tr} A_U = \frac{-2\omega}{1 - e^{-2\omega k\pi}} < -2\omega < -\omega.$$

Therefore, at least some choices of T_0 yield a decay rate that is at least twice better than the one obtain in Theorem 4.1. In fact for this finite-dimensional

[1] We use this operator, leading to Slemrod's feedback, just in order to make the computations easier.

system, we can prove that the decay rate is *always* bigger than 2ω, whatever the choice of T_0 (we refer to Appendix B for the details).

Some numerical and mechanical experiments. The conjecture of a bigger decay rate with this explicit feedback law was made by Bourquin and his collaborators (Briffaut, Collet, Joly, Ratier, Urquiza). They made both *numerical simulations* [**8, 61**] and *mechanical experiments* on beams [**6, 52**] and observed that the exponential decay rate of the energy for those systems was approximately twice bigger than ω. We refer for instance to Briffaut's thesis [**8**, p. 108] for a graph representing the decay of the energy of closed-loop system.

We return to the general case. The aim of the following paragraph is to show that by replacing T_0 by $T \geq T_0$ in the definition of Λ_ω (19), it is possible to have a larger decay rate of the solutions. Moreover, for dissipative systems, this decay rate approaches "quickly" the value -2ω as T increases. This may explain the larger decay rate observed in some numerical and physical experiments.

Let $c \geq 1$ and $\gamma \in \mathbb{R}$ be two constants such that

$$\forall t \geq 0, \qquad \|e^{-tA^*}\| \leq ce^{\gamma t}.$$

In the sequel, the value $\omega > 0$ is fixed and we denote by $\Lambda_{\omega,T}$ the operator obtained in (19) by replacing T_0 by $T \geq T_0$ (we will also write $e_{\omega,T}$ for the corresponding weight function). Thanks to hypotheses (H3) and (H4), this operator has the same properties as Λ_ω. In particular it is invertible. Note that we also have to replace C by an operator C_T (see the definition of C in chapter 2). We can repeat the method of chapter 3 to prove the well-posedness of the closed-loop problem with the feedback $F = -JB^*\Lambda_{\omega,T}^{-1}$ in the framework of semigroups. In this respect we set

$$U_T(t) := \text{ group generated by } \widetilde{A} - BJB^*\Lambda_{\omega,T}^{-1}.$$

Theorem 4.5. *Let $T \geq T_0$. Then,*

$$\|U_T(t)\| \leq c' \exp\big((-2\omega + \gamma + \alpha\varphi(T))t\big), \qquad t \geq 0,$$

where

$$\varphi(T) := \exp\big(\gamma T - 2\omega(T - T_0)\big), \qquad c' := c\|\Lambda_{\omega,T}\|\|\Lambda_{\omega,T}^{-1}\|$$

and α is a positive constant that depends only on T_0 and ω.

Remark 4.6. This estimation of the decay rate of the solutions of the closed-loop problem may be worst than ω but in the case of conservative or dissipative systems, we have $\gamma = 0$. Consequently, if $T - T_0$ is sufficiently large,

$$\gamma - 2\omega + \alpha\varphi(T) \approx -2\omega,$$

i.e. the decay rate of the solutions is approximately -2ω.

Proof of Theorem 4.5. Instead of working with the semigroup $U_T(t)$, we work with one of its conjugates (whose generator is easier to manipulate, see the proof of Theorem 3.2)

$$V_T(t) := \Lambda_{\omega,T}^{-1} U_T(t) \Lambda_{\omega,T}.$$

Its generator is

$$A_{V_T} = -A^* - C_T^* \widetilde{J} C_T \Lambda_{\omega,T}, \qquad \mathcal{D}(A_{V_T}) = \mathcal{D}(A^*).$$

We recall from the definition of the operator C_T that

$$C_T^* \widetilde{J} C_T \Lambda_{\omega,T} = \Lambda_{\omega,T}^{-1} \Lambda'_{\omega,T} \in \mathcal{L}(H'),$$

where $\Lambda'_{\omega,T} \in \mathcal{L}(H', H)$ is the self-adjoint, positive definite operator defined by

$$\begin{aligned}
\langle \Lambda'_{\omega,T} x, y \rangle := -& \int_0^{T+1/2\omega} e'_{\omega,T}(s) \langle JB^* e^{-sA^*} x, B^* e^{-sA^*} y \rangle \,\mathrm{d}s \\
=& 2\omega \int_0^T e^{-2\omega s} \langle JB^* e^{-sA^*} x, B^* e^{-sA^*} y \rangle \,\mathrm{d}s \\
&+ 2\omega \int_T^{T+1/2\omega} e_{\omega,T}(s) \langle JB^* e^{-sA^*} x, B^* e^{-sA^*} y \rangle \,\mathrm{d}s \\
&- 2\omega \int_T^{T+1/2\omega} e_{\omega,T}(s) \langle JB^* e^{-sA^*} x, B^* e^{-sA^*} y \rangle \,\mathrm{d}s \\
&+ 2\omega e^{-2\omega T} \int_T^{T+1/2\omega} \langle JB^* e^{-sA^*} x, B^* e^{-sA^*} y \rangle \,\mathrm{d}s \\
=:& \, 2\omega \langle \Lambda_{\omega,T} x, y \rangle + \langle R_{\omega,T} x, y \rangle,
\end{aligned}$$

with $R_{\omega,T} \in \mathcal{L}(H', H)$ self-adjoint and positive. Hence

$$A_{V_T} = -A^* - 2\omega I - \Lambda_{\omega,T}^{-1} R_{\omega,T}.$$

The operator $-A^* - 2\omega I$ with domain $\mathcal{D}(A^*)$ is the generator of a semigroup and we have the following estimation :

$$\| e^{t(-A^* - 2\omega I)} \| = e^{-2\omega t} \| e^{-tA^*} \| \leq c e^{(\gamma - 2\omega)t}, \qquad t \geq 0.$$

In order to have an estimation for the semigroup $V_T(t)$, we are going to apply a classical (bounded) perturbation result (see the Proposition A.3). The idea is that the growth of the semigroup generated by a perturbed operator can be expressed in term of the norm of the perturbation. Let us estimate the norm of the bounded perturbation $\Lambda_{\omega,T}^{-1} R_{\omega,T}$.

For all $x \in H'$,

$$c_2(T_0) e^{-2\omega T_0} \|x\|_{H'}^2 \leq \langle \Lambda_{\omega,T_0} x, x \rangle_{H,H'} \leq \langle \Lambda_{\omega,T} x, x \rangle_{H,H'},$$

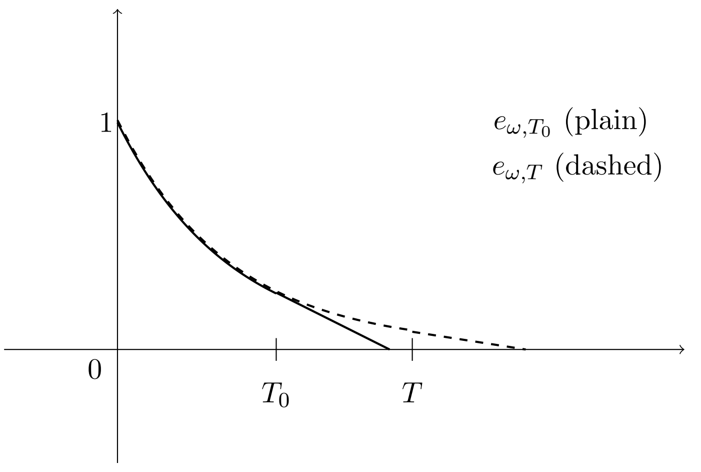


Figure 1. Changing the final time: $T \geq T_0 \Rightarrow e_{\omega,T} \geq e_{\omega,T_0}$

where $c_2(T_0)$ is the positive number given by (H4); see the Figure 1 for the second inequality. Hence, applying the estimation of Lemma 4.8 below to the operator $\widetilde{J}\Lambda_{\omega,T} \in \mathcal{L}(H')$, we get

$$\|\Lambda_{\omega,T}^{-1}\| \leq \frac{e^{2\omega T_0}}{c_2(T_0)}.$$

We remark that the weight function in the operator $R_{\omega,T}$ satisfies

$$0 \leq 2\omega(e^{-2\omega T} - e_{\omega,T}(s)) = 2\omega e^{-2\omega T}(1 - 2\omega(T-s)) \leq 2\omega e^{-2\omega T}$$

for $T \leq s \leq T + 1/2\omega$. Thus,

$$\|R_{\omega,T}\| \leq c_1(1/2\omega) c e^{\gamma T} 2\omega e^{-2\omega T},$$

where $c_1(1/2\omega)$ is a positive constant that depends only on ω (see (H3) and the Remark 2.1). [1]

Applying the perturbation result, we obtain

$$\|V_T(t)\| \leq c \exp\Big(\big(-2\omega + \gamma + 2\omega c^2 c_1(1/2\omega) c_2(T_0)^{-1} e^{\gamma T + 2\omega(T_0 - T)}\big) t\Big), \quad t \geq 0.$$

The estimation on $U_T(t)$ is a direct consequence of its relationship with $V_T(t)$. □

Remark 4.7. The relationship of the norm of the solution of a Riccati equation and the decay rate of the solutions has been studied in the framework of optimal control theory by Benabdallah and Lenczner in [**4**].

[1] The term "$ce^{\gamma T}$" appears when we transpose the direct inequality from the interval $(0, 1/2\omega)$ to $(T, T + 1/2\omega)$.

Lemma 4.8. *Let X be an Hilbert space and $T \in \mathcal{L}(X)$ be self-adjoint and positive. Assume that there exists two positive constants c_1 and c_2 such that for all $x \in X$,*

$$c_1\|x\|_X^2 \leq (Tx, x) \leq c_2\|x\|_X^2. \tag{44}$$

Then T is invertible and

$$\frac{1}{c_2} \leq \|T^{-1}\| \leq \frac{1}{c_1}.$$

Proof. The norm of T is given by

$$\|T\| = \sup_{x\neq 0} \frac{(Tx, x)}{(x, x)}.$$

Hence, the estimation (44) implies that $c_1 \leq \|T\| \leq c_2$ and that T is invertible.

Given $x \in X$, $x \neq 0$,

$$\|x\|_X = \|TT^{-1}x\|_X \leq \|T\|\|T^{-1}\|\|x\|_X$$
$$\Rightarrow \quad \|T^{-1}\| \geq \frac{1}{\|T\|} \geq \frac{1}{c_2}.$$

Given $x \in X$,

$$(Tx, x) \geq c_1\|x\|_X^2 \quad \iff \quad \|\sqrt{T}x\|_X^2 \geq c_1\|x\|_X^2.$$

Replacing x by $(\sqrt{T})^{-1}x = \sqrt{T^{-1}}x$ in the last inequality, we obtain

$$\|x\|_X^2 \geq c_1\|\sqrt{T^{-1}}x\|_X^2 \qquad \Rightarrow \qquad \|\sqrt{T^{-1}}\| \leq \frac{1}{\sqrt{c_1}}$$

and

$$\|T^{-1}\| = \|\sqrt{T^{-1}}\sqrt{T^{-1}}\| \leq \|\sqrt{T^{-1}}\|^2 \leq \frac{1}{c_1}.$$ □

4. An infinite-horizon Gramian

We end this chapter by describing briefly another explicit feedback law that may be seen as a *limit case* of the one that we have described until now. Rather than using the "finite-horizon" Gramian [1]

$$\Lambda_{\omega,T} = \int_0^T e_{\omega,T}(t)e^{-tA}BJB^*e^{-tA^*}\,\mathrm{d}t,$$

we consider the "infinite -horizon" Gramian

$$\Lambda_{\omega,\infty} := \int_0^\infty e^{-2\omega t}e^{-tA}BJB^*e^{-tA^*}\,\mathrm{d}t. \tag{45}$$

This idea is due to Bass for finite-dimensional systems (see [**54**, pp. 117–119]) and the reference therein). Later, it has been used for infinite-dimensional

[1] In this paragraph, we use the "matrix notation" in order to simplify the writing. It is correct for finite-dimensional systems or for infinite-dimensional ones with a bounded control operator. To be rigorous, one should use the "duality pairing notation" in the unbounded case.

systems with bounded control operators by Dusser and Rabah [**18**] and in the unbounded case by Urquiza [**62**].

Assuming (H1)-(H4) and also that ω is large enough, namely that [1]

$$\omega > \text{growth bound of } -A^*,$$

then, the operator $\Lambda_{\omega,\infty}$ defined by (45) belongs to $\mathcal{L}(H', H)$ and is bounded from below, hence boundedly invertible.

In the same way that we derived the Riccati equation (22) satisfied by Λ_ω in chapter 2, we can prove that $\Lambda_{\omega,\infty}$ satisfies

$$\Lambda_{\omega,\infty}A^* + A\Lambda_{\omega,\infty} + 2\omega\Lambda_{\omega,\infty} - BJB^* = 0$$

sot that (at least formally)

$$\Lambda_{\omega,\infty}^{-1}(A - BJB^*\Lambda_{\omega,\infty}^{-1})\Lambda_{\omega,\infty} = -A^* - 2\omega I. \tag{46}$$

In other words, the closed-loop operator $A - BJB^*\Lambda_{\omega,\infty}^{-1}$ is conjugated to the operator $-A^* - 2\omega I$. At this point, we could reproduce the method used in chapter 3 to analyze the well-posedness of (33) and prove that the operator

$$\widetilde{A} - BJB^*\Lambda_{\omega,\infty}^{-1}, \qquad \text{domain} = \Lambda_{\omega,\infty}\mathcal{D}(A^*),$$

is the generator of a group in H. Let us notice that Urquiza uses the optimal control theory (cf. section 3 of the previous chapter) to prove the well-posedness of the closed-loop problem in [**62**].

Moreover, the relation (46) ensures that the group generated by $\widetilde{A} - BJB^*\Lambda_{\omega,\infty}^{-1}$ is conjugated to the group generated by $-A^* - 2\omega I$ i.e. to

$$e^{-tA^*}e^{-2\omega t}.$$

The growth bound of this group is shifted to the left at a distance of -2ω with respect to the growth bound of $-A^*$. In particular, in the dissipative case (i.e. if $g(-A^*) \leq 0$) which covers the examples of the wave equation or the plate equation, the decay rate of the stabilized system turns out to be greater than 2ω.

---

[1] If A is the generator of a semigroup e^{tA}, the *growth bound* or the *type* of A (or of e^{tA}) is defined by (see e.g., [**2**, pp. 174–175])

$$g(A) = \inf_{t>0} \frac{\ln \|e^{tA}\|}{t} \in \mathbb{R} \cup \{-\infty\}.$$

This number corresponds to the infimum of the numbers $\gamma \in \mathbb{R}$ satisfying the following property: there exists a positive constant $c(\gamma)$ such that $\forall t \geq 0, \quad \|e^{tA}\| \leq c(\gamma)e^{\gamma t}$.

Hence, the phenomenon of the twice bigger decay rate observed in some experiments with the finite horizon Gramian could be explained as follow : the parameter T_0 in the Gramian was large enough so that $\Lambda_{\omega,\infty}$ was a good approximation of Λ_{ω,T_0}. Hence, the effective decay rate was close to the decay rate obtained with the infinite horizon Gramian:

$$g(-A^*) - 2\omega.$$

Remark 4.9. For an application of this stabilization method to the Korteweg-de Vries equation as for numerical simulations, we refer to [**14**].

Part 2

Observation at different time instants

Introduction

This part deals with the observation of some linear and time-reversible system whose prototype will be the vibrating string.

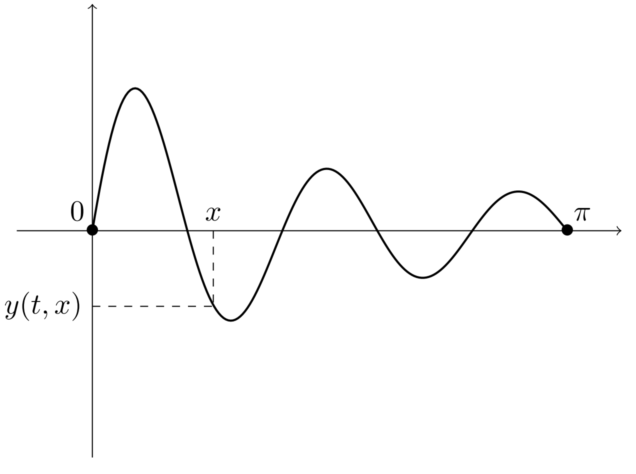


Figure 1. The vibrating string at time t

Let q be a nonnegative number. The small transversal vibrations of a string of length π fixed at its two ends satisfy [1]

$$(47)\qquad \begin{cases} y'' - y_{xx} + qy = 0 & \text{in } \mathbb{R} \times (0, \pi), \\ y = 0 & \text{in } \mathbb{R} \times \{0, \pi\}, \\ y(0) = y_0, \quad y'(0) = y_1 & \text{in } (0, \pi). \end{cases}$$

[1]The quantity $y = y(t, x)$ is the height of the string at time t and abscissa x while $y(t)$ stands for the map $y(t, \cdot)$; see also Figure 1. The choice of π for the length of the string is made in order to simplify the writing in the expansion of the solutions in Fourier series.

This system will be called the *classical string* if $q = 0$ and the *string with a potential* [1] if $q > 0$.

Obtaining observability inequalities for the vibrating string and for oscillating systems in general has been the object of many works. Indeed, observability being dual to controllability (cf. Russell [**53**]), it is often a starting point to obtain controllability results (see e.g., Lions [**45**], Haraux [**26**]). A useful tool to obtain such inequalities is the Fourier series expansion of the solutions (cf. Komornik and Loreti [**34**]).

Among all the different ways to observe the system (47), *pointwise observation* has been widely studied (see e.g., Lions [**46**], Haraux [**25**]). It consists in getting estimations of the form [2]

$$\|(y_0, y_1)\|_{\mathcal{I}} \le c\|y(\cdot, \xi)\|_{\mathcal{O}}.$$

The main difficulties are the choice of the norms for the initial data $\|\cdot\|_{\mathcal{I}}$ as for the observation $\|\cdot\|_{\mathcal{O}}$ and the choice of a *strategic point* ξ in the domain. These particular points can be characterized by some of their arithmetical properties (see e.g., Butkovskiy [**10, 11**], Komornik and Loreti [**35**]).

Following a recent paper by Szijártó and Hegedűs [**58**], we focus on a *pointwise-in-time observation*. Such type of observation seems to have been studied at first by Egorov [**19**] and Znamenskaya [**64**]. Given two norms, one for the initial data $\|\cdot\|_{\mathcal{I}}$ and one for the observation $\|\cdot\|_{\mathcal{O}}$, the objective is to find two times t_0 and t_1 such that

$$\|(y_0, y_1)\|_{\mathcal{I}} \le c(\|y(t_0)\|_{\mathcal{O}} + \|y(t_1)\|_{\mathcal{O}}). \tag{48}$$

From a practical point of view, such an inequality means that only knowing the position of the whole system at two different instants, we are able to recover the initial position y_0 and the initial velocity y_1.

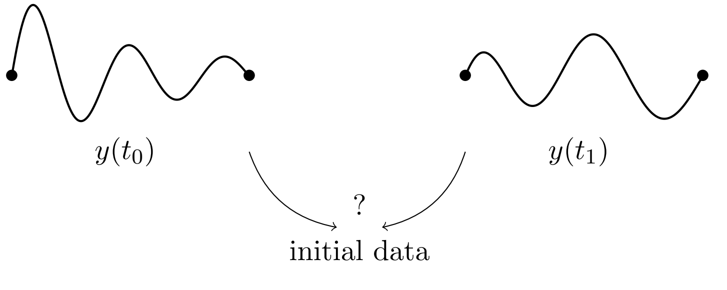


Figure 2. Do the positions of the string at two time instants uniquely determine the initial data?

[1] In quantum mechanics it is called the Klein-Gordon equation. From a mechanical point of view, it may model a "flexible string with additional stiffness forces provided by the medium surrounding the string" [**48**, pp. 138–140].

[2] Here, c denotes a positive constant, independent of the initial data y_0 and y_1.

Definition 5.1. A pair of real numbers (t_0, t_1) such that the observability inequality (48) holds is called a *strategic pair (for the inequality* (48)*).* [1]

The main idea is the following : depending on how the quantity

$$\frac{t_0 - t_1}{\pi} \tag{49}$$

is approximable by rational numbers, such pointwise-in-time observability inequalities hold. The main tools are the explicit expansion of the solutions in *Fourier series* and classical results of *Diophantine approximation.* [2]

Example 6. Let us give an evidence that arithmetical properties of the quantity (49) arise naturally in this observation problem. We know that the solution of the vibrating string equation (47) can be expressed as a Fourier series expansion. Thus, the solution can also be seen as a superposition of *harmonic oscillators* of different frequencies, each frequency corresponding to one mode. That is why we may analyze the observation problem on one harmonic oscillator, the latter representing for instance the small oscillations of a pendulum.

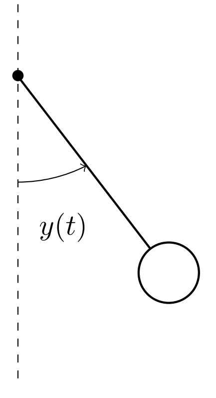


Figure 3. Angular displacement $y(t)$ of a pendulum

The angular displacement y of a pendulum having small oscillations (see Figure 3) satisfies the differential equation

$$y''(t) + \omega^2 y(t) = 0, \qquad t \geq 0, \tag{50}$$

where $\omega > 0$. [3] The initial angular displacement and the initial angular velocity are denoted by

$$y(0) = y_0 \qquad \text{and} \qquad y'(0) = v_0. \tag{51}$$

[1] In particular, this notion depends on the the norms in the left and right members.

[2] Diophantine approximation means the approximation of real numbers by rational numbers.

[3] More precisely $\omega = \sqrt{g/l}$, where g denotes the acceleration due to the gravity and l denotes the length of the pendulum.

The solution of (50)-(51) is given by the function [1]

$$y(t) = y_0 \cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t), \qquad t \geq 0. \tag{52}$$

Knowing the position $y(t_0)$ and the velocity $y'(t_0)$ of the pendulum at time $t_0 > 0$, we can recover the initial data. Indeed, the linear system

$$\begin{aligned} y_0 \cos(\omega t_0) + \frac{v_0}{\omega}\sin(\omega t_0) &= y(t_0) \\ -y_0\,\omega \sin(\omega t_0) + v_0 \cos(\omega t_0) &= y'(t_0) \end{aligned}$$

with unknowns y_0 and v_0 has a unique solution since its determinant does not vanish (it is equal to 1).

Now, we consider a slightly different problem: assume that we know the positions $y(t_0)$ and $y(t_1)$ of the pendulum at two distinct times $t_0 > 0$ and $t_1 > 0$.

Is it possible to reconstruct the initial data only from these two information?

The movement of the pendulum is periodic with period $2\pi/\omega$ (see (52)). Moreover, the position of the pendulum at time t is exactly the opposite of its position half a period later (indeed, $y(t+\pi/\omega) = -y(t)$). Hence, if the instants of observation t_0 and t_1 are separated by half a period or an integer multiple of this quantity, then, in fact, we only have one information and this is not sufficient to recover the initial data (see Figure 4, left). However, if $t_1 - t_0 \notin (\pi/\omega)\mathbb{Z}$, then the answer is positive (see Figure 4, right). Let us see this analytically.

Again, this problem is equivalent to a linear system with two unknowns y_0 and v_0 :

$$\begin{aligned} y_0 \cos(\omega t_0) + \frac{v_0}{\omega}\sin(\omega t_0) &= y(t_0) \\ y_0 \cos(\omega t_1) + \frac{v_0}{\omega}\sin(\omega t_1) &= y(t_1). \end{aligned}$$

The determinant of this system is $(1/\omega)(\cos(\omega t_0)\sin(\omega t_1) - \cos(\omega t_1)\sin(\omega t_0)) = (1/\omega)\sin(\omega(t_1 - t_0))$ and this quantity does not vanish if and only if

$$t_1 - t_0 \notin \frac{\pi}{\omega}\mathbb{Z}. \tag{53}$$

We can notice that if ω is an integer, then a sufficient condition for (53) to hold is that the quantity defined by (49) is an irrational number.

[1] The solution can also be expressed in term of complex exponentials :

$$y(t) = \alpha e^{i\omega t} + \beta e^{-i\omega t}, \qquad t \geq 0,$$

with $\alpha := (1/2)(y_0 - iv_0/\omega)$ and $\beta := (1/2)(y_0 + iv_0/\omega)$.

As a conclusion, we can recover the initial data from the observation of the positions of the pendulum at two distinct times t_0 and t_1 if and only if the condition (53) holds.

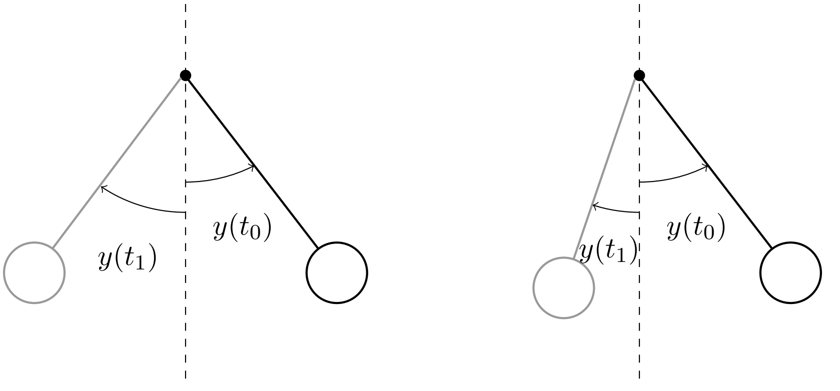


Figure 4. On the left, t_0 and t_1 are separated by half a period so that $y(t_0) = -y(t_1)$; on the right $t_1 - t_0 \notin (\pi/\omega)\mathbb{Z}$ so that $|y(t_0)| \neq |y(t_1)|$.

Let us describe the organization of this part and state (informally) the main results.

In chapter 6, after recalling the definition of adapted functional spaces to study the well-posedness of (47), we reformulate the observation problem in this setting. These spaces, denoted by D^s ($s \in \mathbb{R}$), correspond essentially to the domain of $-\Delta^{s/2}$; we may chose two real numbers r and s such that $\| \cdot \|_{\mathcal{I}} = \| \cdot \|_{D^s}$ and $\| \cdot \|_{\mathcal{O}} = \| \cdot \|_{D^r}$. Then, we investigate the observation of the vibrating string i.e. of the system (47). In section 1, we focus on of the *classical string* (i.e. $q = 0$). We prove (see Theorem 6.4) the following

Result 3. *Assume that $r - s \geq 1$. Then, there exist strategic pairs. Moreover, if $r - s > 1$, then almost all pairs are strategic. This result is optimal in the sense that there cannot be any strategic pair if $r - s < 1$.*

In section 2, we prove that the difference $r - s$ (see Theorem 6.7) can be reduced by adding further observations.

In section 3, we focus on the *string with a potential* (i.e. $q > 0$). First we recall the main result of [**58**] in Theorem 6.10, which states essentially that if $(t_0 - t_1)/\pi$ is a *rational* number along with another hypothesis, then (t_0, t_1) is a strategic pair with $r - s = 1$. After analyzing the occurrence of such pairs under the above hypotheses in Proposition 6.11, we use another method to obtain new observability inequalities. We can state the following result (see Theorem 6.12):

Result 4. *Assume that* $r - s = 1$*. If* (t_0, t_1) *is a strategic pair for the classical string, then it is also a strategic pair for the string with a potential provided that* q *is sufficiently small.*

In chapter 7, we extend our method to the vibrating string with a non-constant potential, the vibrating beam and rectangular plates. The novelty here with respect to chapter 6 is that we can prove observability results for some vibrating systems in dimension 2.

Finally, in chapter 8, applying the Hilbert Uniqueness Method, we prove an exact controllability result for the classical string ($q = 0$).

Result 5. *Let* $T > 0$*. We assume that* (t_0, t_1) *is a strategic pair. If the initial data are sufficiently smooth, then the system can be steered to rest at time* T *by mean of two impulsions at times* t_0 *and* t_1*.*

Outline of the second part:

Chapter 6. Observation of the vibrating string.

Chapter 7. Extension to other systems.

Chapter 8. An exact controllability result.

Chapter 6

Observation of the vibrating string

Let us recall [1] the construction of some useful functional spaces related to the problem (47). The functions $\sin(kx)$, $k = 1, 2, \dots$ form an orthogonal and dense system in $L^2(0, \pi)$. We denote by D the vector space spanned by these functions and for $s \in \mathbb{R}$, we define an euclidean norm on D by setting

$$\Big\| \sum_{k=1}^{\infty} c_k \sin(kx) \Big\|_s^2 := \sum_{k=1}^{\infty} k^{2s} |c_k|^2.$$

The space D^s is defined as the completion of D for the norm $\|.\|_s$. Then, D^0 coincide with $L^2(0, \pi)$ with equivalent norms and more generally, it is possible to prove that for $s > 0$,

$$D^s = \Big\{ f \in H^s(0, \pi) : f^{(2j)}(0) = f^{(2j)}(\pi) = 0, \quad \forall\, 0 \leq j \leq \Big[\frac{s-1}{2}\Big] \Big\}.$$

Identifying D^0 with its own dual, D^{-s} is the dual of D^s. For example,

$$D^0 = L^2(0, \pi), \quad D^1 = H_0^1(0, \pi) \quad \text{and} \quad D^{-1} = H^{-1}(0, \pi)$$

with equivalent norms.

Now, we recall a well-posedness result for the problem (47) via an expansion of the solutions in Fourier series. We set

$$\omega_k := \sqrt{k^2 + q}, \qquad k = 1, 2, \dots$$

Proposition 6.1 (well-posedness)**.** *Let $s \in \mathbb{R}$. For all initial data $y_0 \in D^s$ and $y_1 \in D^{s-1}$, the problem* (47) *admits a unique solution $y \in C(\mathbb{R}, D^s) \cap$*

[1] We refer to [**31**, pp. 7–11] and [**7**, pp. 335–340] for more details.

$C^1(\mathbb{R}, D^{s-1}) \cap C^2(\mathbb{R}, D^{s-2})$ *given by*

$$y(t,x) = \sum_{k=1}^{\infty}(a_k e^{i\omega_k t} + b_k e^{-i\omega_k t}) \sin kx, \tag{54}$$

where the complex coefficients a_k and b_k satisfy [1]

$$\|y_0\|_s^2 + \|y_1\|_{s-1}^2 \asymp \sum_{k=1}^{\infty} k^{2s}(|a_k|^2 + |b_k|^2). \tag{55}$$

The observability problem that we are going to investigate in this chapter is the following: given two real numbers r and s such that $s \le r$, we ask whether or not there are two times t_0 and t_1 such that

$$\|y_0\|_s + \|y_1\|_{s-1} \le c(\|y(t_0)\|_r + \|y(t_1)\|_r) \tag{56}$$

for a positive constant c, independent of the initial data $(y_0, y_1) \in D^r \times D^{r-1}$.

1. Observation of the classical string $(q = 0)$

In this paragraph, we assume that $q = 0$ in the problem (47). The following statement transforms the observation inequality (56) to a problem of Diophantine approximation.

Proposition 6.2. *Let t_0 and t_1 be real numbers. The pair (t_0, t_1) is strategic if and only if there is a positive constant c such that* [2]

$$\Big\| \frac{k(t_0 - t_1)}{\pi} \Big\| \ge \frac{c}{k^{r-s}}, \qquad k = 1, 2, \dots \tag{57}$$

For the proof, we need the following

Lemma 6.3. *Set $x \in \mathbb{R}$. We have*

$$|\sin kx| \asymp \Big\| \frac{kx}{\pi} \Big\|, \qquad k = 1, 2, \dots$$

Proof of Lemma 6.3. We follow the proof of [**35**, Lemma 2.3]. Denoting by m the nearest integer from kx/π,

$$|\sin kx| = |\sin(kx - m\pi)| = \Big|\sin\Big(\frac{kx}{\pi} - m\Big)\pi\Big|.$$

We notice that $|kx/\pi - m|\pi \le \pi/2$. Hence, using the estimations $(2/\pi)|t| \le |\sin t| \le |t|$ which hold for $|t| \le \pi/2$, we have

$$\frac{2}{\pi}\Big|\frac{kx}{\pi} - m\Big|\pi \le \Big|\sin\Big(\frac{kx}{\pi} - m\Big)\pi\Big| \le \pi\Big|\frac{kx}{\pi} - m\Big|,$$

i.e.

$$2\Big\|\frac{kx}{\pi}\Big\| \le |\sin kx| \le \pi\Big\|\frac{kx}{\pi}\Big\|. \qquad \square$$

[1] $A \asymp B$ means that there are two positive constants c_1 and c_2 such that $c_1 B \le A \le c_2 B$.

[2] If x is a real number, $\|x\|$ denotes the distance between x and the nearest integer.

Proof of Proposition 6.2. Using the Fourier series expansion (54) of the solutions of (47) and the estimation (55), we remark that the square of the left-hand side in (56) is equivalent [1] to

$$\sum_{k=1}^{\infty} k^{2s}(|a_k|^2 + |b_k|^2)$$

and the square of the right-hand side is equivalent to

$$\sum_{k=1}^{\infty} k^{2r}(|a_k e^{ikt_0} + b_k e^{-ikt_0}|^2 + |a_k e^{ikt_1} + b_k e^{-ikt_1}|^2).$$

Therefore, the observability inequality (56) holds if and only if there exists a positive constant c' such that for all $k = 1, 2, \ldots$ and all complex numbers a and b,

$$k^{2s}(|a|^2 + |b|^2) \leq c' k^{2r}(|ae^{ikt_0} + be^{-ikt_0}|^2 + |ae^{ikt_1} + be^{-ikt_1}|^2). \tag{58}$$

Now, for all k, we consider the linear maps T_k in $\mathbb{C} \times \mathbb{C}$ (endowed with its usual euclidean norm) defined by

$$T_k(a, b) := (ae^{ikt_0} + be^{-ikt_0}, ae^{ikt_1} + be^{-ikt_1}).$$

Hence, the estimation (58) holds for all k if and only if all the T_k are invertible and there exists a positive constant c'' independent of k such that

$$\frac{1}{\|T_k^{-1}\|} \geq \frac{c''}{k^{r-s}}.$$

The determinant of T_k equalling $2i \sin k(t_0 - t_1)$, we deduce that all the T_k are invertible if and only if $(t_0 - t_1)/\pi$ is irrational. In that case, their inverses are given by

$$T_k^{-1}(a, b) = \frac{1}{2i \sin k(t_0 - t_1)}(e^{-ikt_1}a - e^{-ikt_0}b, -e^{ikt_1}a + e^{ikt_0}b)$$

and a computation of their norms yields

$$\|T_k^{-1}\| = \frac{\sqrt{1 + |\cos k(t_0 - t_1)|}}{\sqrt{2}|\sin k(t_0 - t_1)|}$$

Thus,

$$\frac{1}{\|T_k^{-1}\|} \asymp |\sin k(t_0 - t_1)| \asymp \left\|\frac{k(t_0 - t_1)}{\pi}\right\|.$$

The first estimation follows from the expression of $\|T_k^{-1}\|$ while the second estimation is a consequence of the Lemma 6.3. But if (57) holds, then $(t_0 - t_1)/\pi$ must be irrational and that ensures that all the T_k are invertible. The proof is complete. □

Let us study the occurrence of such strategic pairs.

[1] in the sense of the symbol $\asymp$

Theorem 6.4.

(a) *If $r - s < 1$, the set of strategic pairs is empty.*

(b) *If $r - s = 1$, the set of strategic pairs has zero Lebesgue measure and full Hausdorff dimension in $\mathbb{R}^2$.*

(c) *If $r - s > 1$, the set of strategic pairs has full Lebesgue measure in $\mathbb{R}^2$.*

In the following lemma, we gather some classical results of (metric) Diophantine approximation [1] . For a real number α, we set

$$E_\alpha := \{x \in \mathbb{R} : \exists\, c > 0 : \; \|kx\| \geq ck^{-\alpha},\ k = 1, 2, \ldots\}.$$

Lemma 6.5 ([**12**, pp. 120–121], [**9**, p. 104], [**22**, p. 142])**.**

(a) *If $\alpha = 1$, then E_α has zero Lebesgue measure and full Hausdorff dimension* [2] *in $\mathbb{R}$.*

(b) *If $\alpha > 1$, then E_α has full Lebesgue measure in $\mathbb{R}$.*

Proof of Theorem 6.4. Set $r - s = \alpha$. As a consequence of Proposition 6.2, the set of strategic pairs coincides with the set

$$\mathcal{E}_\alpha := \left\{ (t_0, t_1) \in \mathbb{R} \times \mathbb{R} : \frac{t_0 - t_1}{\pi} \in E_\alpha \right\}$$

If $\alpha < 1$ then the set E_1 (hence $\mathcal{E}_1$) defined above is empty. Indeed, if we suppose that $x \in E_\alpha$, then for sufficiently large k, $\|kx\| \geq 1/k$ which is in contradiction with a theorem of Dirichlet (see [**12**, p.4]) that asserts that if x is irrational, then there are infinitely many positive integer k satisfying the inequality $\|kx\| < 1/k$.

If $\alpha \geq 1$, then we use the Lemma 6.5. One can notice that $\mathcal{E}_\alpha$ has full (resp. zero) Lebesgue measure in $\mathbb{R}^2$ if E_α has full (resp. zero) Lebesgue measure in $\mathbb{R}$. The same result holds for a full Hausdorff dimension. Assume for instance, that E_α as zero Lebesgue measure in $\mathbb{R}$. Then, applying Fubini's theorem, we can compute the Lebesgue measure of $\mathcal{E}_\alpha$ in $\mathbb{R}^2$:

$$\lambda(\mathcal{E}_\alpha) = \int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathcal{E}_\alpha} \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}} (\int_{\mathbb{R}} \mathbb{1}_{\pi F_\alpha + y} \,\mathrm{d}x)\,\mathrm{d}y = \int_{\mathbb{R}} 0\,\mathrm{d}y = 0.$$

The same result holds, considering the complementary set, for a full Lebesgue measure. Assume that E_α as full Hausdorff dimension in $\mathbb{R}$. Then, using the map $(x, y) \mapsto ((x - y)/\pi, y)$, bi-Lipschitz from $\mathcal{E}_\alpha$ onto $E_\alpha \times \mathbb{R}$, we obtain (see the properties (f) and (g) of Appendix C)

$$\dim_H E_\alpha = \dim_H E_\alpha + 1 = 2.$$

□

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[1] The results concerning the Lebesgue measure are due to Khinchin and the result concerning the Hausdorff dimension is due to Jarník

[2] For a reminder of the Hausdorff dimension, we refer to the Appendix C.

Remark 6.6.

- The assertion (a) of Theorem 6.4 can be seen as an *optimality result*. Indeed, it means that with only two observations, the difference $r - s$ between the orders of the Sobolev norms in the inequality (56) must be at least 1.
- One cannot obtain such estimations with only one observation. Indeed, let $t_0 \in \mathbb{R}$. Then, the function $y(t,x) = \sin(t - t_0)\sin(x)$ is a solution to (47) with $y(0) \neq 0$ or $y'(0) \neq 0$, but $y(t_0) = 0$.
- It is possible to give the Hausdorff dimension in $\mathbb{R}^2$ of the set of strategic pairs and its complementary in any case, using further results of Jarník (see e.g., [**9**, p. 104], [**22**, p. 142]):

$$\alpha = 1 \quad \Rightarrow \quad \dim_H \mathcal{E}_\alpha = \dim_H \mathbb{R}^2 \setminus \mathcal{E}_\alpha = 2;$$

$$\alpha > 1 \quad \Rightarrow \quad \dim_H \mathcal{E}_\alpha = 2 \quad \text{and} \quad \dim_H \mathbb{R}^2 \setminus \mathcal{E}_\alpha = \frac{2}{1+\alpha} + 1.$$

These values are represented on Figure 1.

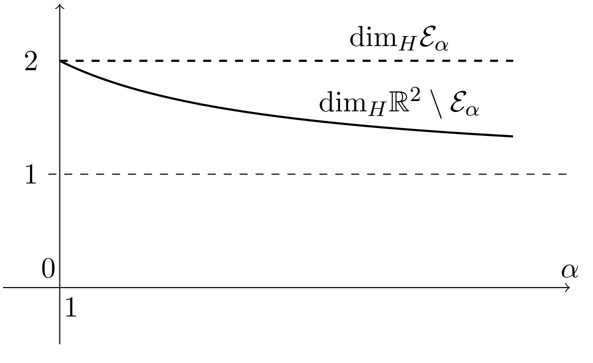

Figure 1. Hausdorff dimension of the sets of strategic and non-strategic pairs

- If the pair (t_0, t_1) is strategic, then, having only access to the two observations i.e. the position of the string at times t_0 and t_1, we can recover the initial data y_0 and y_1 using the expansion in Fourier series of $y(t_0)$ and $y(t_1)$ and the applications T_k^{-1}. Moreover, the observability inequality ensures a "continuity property" in this reconstruction process. Indeed, if two sets of observations are close, then the two sets of associated initial data must be close too.
- In the same way, if $r - s \geq 1$, we can obtain estimations of the form

$$\|y_0\|_s + \|y_1\|_{s-1} \leq c(\|y'(t_0)\|_{r-1} + \|y(t_1)\|_r),$$
$$\|y_0\|_s + \|y_1\|_{s-1} \leq c(\|y(t_0)\|_r + \|y'(t_1)\|_{r-1}),$$
$$\|y_0\|_s + \|y_1\|_{s-1} \leq c(\|y'(t_0)\|_{r-1} + \|y'(t_1)\|_{r-1}).$$

2. With more observations

We still assume that $q = 0$ in (47). In the previous section, we have seen that with only two observations, it is necessary that $r - s \geq 1$ in order to obtain the estimation (56). In this paragraph, we show that adding other observations allows to reduce the difference $r - s$.

Theorem 6.7. *Let $t_1, t_2, \ldots, t_n \in \mathbb{R}$ with $n \geq 2$, $r \in \mathbb{R}$ and set $s := r - 1/(n-1)$. Assume that among the $(t_i - t_j)/\pi$, $1 \leq i, j \leq n$, we can extract $n-1$ elements $\tau_1, \ldots, \tau_{n-1}$ that belong to a real algebraic extension of $\mathbb{Q}$ of degree n and such that $1, \tau_1, \ldots, \tau_{n-1}$ are linearly independent over $\mathbb{Q}$. Then, there exists a positive constant c such that*

$$\|y_0\|_s + \|y_1\|_{s-1} \leq c(\|y(t_1)\|_r + \ldots + \|y(t_n)\|_r)$$

for all initial data $(y_0, y_1) \in D^r \times D^{r-1}$.

The proof relies on the following

Lemma 6.8 ([**12**, p. 79]). *Let $x_1, \ldots, x_n$ be numbers that belong to a real algebraic extension of $\mathbb{Q}$ of degree $n+1$ such that $1, x_1, \ldots, x_n$ are linearly independent over $\mathbb{Q}$. Then, there exists a positive constant c, only depending on $x_1, \ldots, x_n$, such that*

$$\max \|kx_j\| \geq ck^{-1/n}, \quad k = 1, 2, \ldots$$

and [1]

$$\|k_1x_1 + k_2x_2 + \ldots k_nx_n\| \geq c(\max |k_j|)^{-n}, \quad (k_1, \ldots, k_n) \in \mathbb{Z}^n \setminus \{(0, \ldots 0)\}.$$

Proof of Theorem 6.7. Adapting the method described in the proof of Theorem 6.2, it is sufficient to obtain the estimation

$$\sum_{p=1}^{n} |ae^{ikt_p} + be^{-ikt_p}|^2 \geq ck^{-\frac{2}{n-1}}(|a|^2 + |b|^2),$$

where c is a positive constant, independent of $a, b \in \mathbb{C}$ and $k \in \mathbb{N}^*$. With no loss of generality, we can assume that $\tau_p = (t_1 - t_{p+1})/\pi$ for $p = 1, \ldots, n-1$.

[1] This estimation will be used in chapter 7.

We have

$$\begin{aligned}
\sum_{p=1}^{n} |ae^{ikt_p} + be^{-ikt_p}|^2 &= \sum_{p=2}^{n} (\frac{1}{n-1} |ae^{ikt_1} + be^{-ikt_1}|^2 + |ae^{ikt_p} + be^{-ikt_p}|^2) \\
&\geq c_1 \sum_{p=2}^{n} (|ae^{ikt_1} + be^{-ikt_1}|^2 + |ae^{ikt_p} + be^{-ikt_p}|^2) \\
&\geq c_2 \Big(\sum_{p=2}^{n} |\sin k(t_1 - t_p|^2 \Big) (|a|^2 + |b|^2) \\
&\geq c_3 \Big(\sum_{p=2}^{n} \Big\| \frac{k(t_1 - t_p)}{\pi} \Big\|^2 \Big) (|a|^2 + |b|^2) \\
&\geq c_3 \max \Big\| \frac{k(t_1 - t_p)}{\pi} \Big\|^2 (|a|^2 + |b|^2) \\
&\geq c_4 k^{-2/(n-1)} (|a|^2 + |b|^2)
\end{aligned}$$

for all $k = 1, 2, \ldots$, with positive constants c_1, c_2, c_3, c_4 independent of $a, b \in \mathbb{C}$. The numbers $1, (t_1 - t_2)/\pi, \ldots, (t_1 - t_n)/\pi$ are independent over $\mathbb{Q}$. In particular the numbers $(t_1 - t_p)/\pi$, $p = 1, \ldots, n$ are irrational. This ensures that some corresponding linear transformations on $\mathbb{C} \times \mathbb{C}$ (see the proof of Theorem 6.2) are invertible and implies the second inequality. The third inequality is a consequence of Lemma 6.3 while the last inequality results from Lemma 6.8. □

Remark 6.9. Formally, letting the number of observations tend to $+\infty$, setting $r = 0$ and $T > 0$, we recover an internal observability result [1] :

$$\|y_0\|_0^2 + \|y_1\|_{-1}^2 \leq c \int_0^T \int_0^\pi |y(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

3. Observation of the string with a potential $(q > 0)$

In this paragraph, we assume that $q > 0$ in (47) and that r and s are two real numbers such that $r - s = 1$. First, let us recall the

Theorem 6.10 (Szijártó and Hegedűs, [**58**, Theorem 1 p.4])**.** *Let t_0 and t_1 be real numbers such that*

(59) $$\frac{t_0 - t_1}{\pi} \in \mathbb{Q}$$

and

(60) $$\sin((t_0 - t_1)\sqrt{k^2 + q}) \neq 0, \qquad k = 1, 2, \ldots$$

Then, (t_0, t_1) is an strategic pair (for the string with a potential).

[1] This observability inequality remains true for the wave equation on a (sufficiently smooth) bounded domain $\Omega \subset \mathbb{R}^n$ (see [**44**, chapter 7]).

Are such hypotheses easily satisfied? We can answer this question with the following

Proposition 6.11. *The set of pairs* (t_0, t_1) *satisfying the hypotheses* (59) *and* (60) *is dense in* $\mathbb{R}^2$.

Proof. It is sufficient to prove that for each real number τ and each $\delta > 0$, there exists a real number τ' satisfying the three conditions : $|\tau - \tau'| < \delta$, $\tau' \in \pi\mathbb{Q}$ and $\sin\left(\tau'\sqrt{k^2+q}\right) \neq 0$ for all $k = 1, 2, \ldots$

First, we notice that $\sin(\zeta\sqrt{k^2+q}) = 0$ if and only if $\zeta\sqrt{k^2+q} \in \pi\mathbb{Z}$. Now, we distinguish three cases.

1. *If q is an irrational number.* The set $\pi\mathbb{Q}$ being dense in $\mathbb{R}$, there exists a number $\tau' \in \pi\mathbb{Q}$ such that $|\tau - \tau'| \leq \delta$. Moreover, τ' can be written $\tau' = (a/b)\pi$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N}^*$ relatively primes. Assume that there exist $k \in \mathbb{N}^*$ and $n \in \mathbb{Z}$ such that

$$\tau'\sqrt{k^2+q} = n\pi \qquad \Longleftrightarrow \qquad \frac{a}{b}\sqrt{k^2+q} = n.$$

Then,

$$q = \frac{n^2 b^2}{a^2} - k^2 \in \mathbb{Q},$$

which is in contradiction with our assumption on q.

2. *If q is an integer.* If $(a/b)\pi \in \pi\mathbb{Q}$, then, $\sin((a/b)\pi\sqrt{k^2+q}) = 0$ if and only if $(a/b)\sqrt{k^2+q} \in \mathbb{Z}$. Moreover, the quantity $\sqrt{k^2+q}$ is either an integer or an irrational number (depending on the fact that k^2+q is a square or not). For sufficiently large k, $\sqrt{k^2+q}$ cannot be an integer. Indeed,

$$\sqrt{k^2+q} = k\sqrt{1+\frac{q}{k^2}} = k\left(1+\frac{q}{2k^2}+o\big(\frac{1}{k^2}\big)\right) = k + \frac{q}{2k} + o\big(\frac{1}{k}\big)$$

and this is not an integer for sufficiently large k. Hence, for such k, it is an irrational number and so is $(a/b)\sqrt{k^2+q}$. Now, let $\tau'' := (a/b)\pi \in \pi\mathbb{Q}$ such that

$$|\tau'' - \tau| < \frac{\delta}{2}.$$

We are going to perturb a little bit the rational number (a/b) in order to construct a number τ' such that the sine neither vanish. From the above discussion, the quantity $\sqrt{k^2+q}$ can take at most a finite number of integer values when k varies. We denote them by $x_1, \ldots, x_N$ (if it does not take any integer value, then it is always an irrational number and we can take $\tau' = \tau''$). Let p be a prime number that is not a divisor of any of the numbers $x_1, \ldots, x_N$. For sufficiently large n,

$$\left|\pi\frac{(p^n-1)a}{p^n b} - \tau\right| < \delta.$$

and p^n does not divide a. Now, two cases are possible. If $\sqrt{k^2+q}$ is not an integer, then it is an irrational number and

$$\frac{(p^n-1)a}{p^n b}\sqrt{k^2+q} \notin \mathbb{Z}.$$

On the other hand, if $\sqrt{k^2+q}$ is an integer, then $\sqrt{k^2+q} = x_l$ for one $l \in \{1,\dots,N\}$ and

$$\frac{(p^n-1)a}{p^n b}\sqrt{k^2+q} = \frac{(p^n-1)ax_l}{p^n b} \notin \mathbb{Z}.$$

because p^n does not divide $(p^n-1)ax_l$. Finally,

$$\tau' := \frac{(p^n-1)a}{p^n b}\pi$$

satisfies the three expected conditions.

3. *If q is a rational number but not an integer.* Then, we can write $q = c/d$, where c and d are integers. Hence,

$$\tau\sqrt{k^2+q} = \tau\sqrt{k^2+\frac{c}{d}} = \tau\sqrt{k^2+\frac{cd}{d^2}} = \frac{\tau}{d}\sqrt{k^2d^2+cd}$$

and we are lead back to the case where q is an integer. □

Let us give another method, relying on the observability of the classical string, in order to obtain an observability result for the string with a potential.

Theorem 6.12. *Let (t_0, t_1) be a strategic pair for the classical string i.e.*

$$|\sin(k(t_0-t_1)| \geq \frac{c}{k}, \qquad k = 1,2,\dots \tag{61}$$

for a suitable positive constant c. Then, it is also a strategic pair for the string with a potential, provided that q is sufficiently small.

Remark 6.13. This result can be viewed as complementary to the result of Szijártó and Hegedűs (Theorem 6.10) since the hypothesis (61) implies that $(t_0-t_1)/\pi$ is irrational; hence (59) cannot hold.

Proof. Applying the method described in the proof of Proposition 6.2, a necessary and sufficient condition for estimation (56) to hold true is

$$|\sin(\omega_k(t_0-t_1))| = |\sin(\sqrt{q+k^2}(t_0-t_1))| \geq \frac{c'}{k}, \qquad k = 1,2,\dots, \tag{62}$$

where c' is a positive constant, independent of k.

Comparing the quantities $|\sin\omega_k(t_0-t_1)|$ and $|\sin(k(t_0-t_1))|$, we look for a sufficient condition that implies (62). Let us estimate the difference

$$|\sin(\sqrt{q+k^2}(t_0-t_1)) - \sin(k(t_0-t_1))|.$$

For a fixed $k \in \mathbb{N}^*$, we consider the application f_k, defined for $x \geq 0$ by

$$f_k(x) := \sin(\sqrt{k^2 + x}(t_0 - t_1)).$$

We have

$$\begin{aligned} |f_k'(x)| &= \frac{|\cos(\sqrt{k^2+x}(t_0 - t_1))||t_0 - t_1|}{2\sqrt{k^2+x}} \\ &\leq \frac{|t_0 - t_1|}{2k}. \end{aligned}$$

From the triangle inequality and the mean value theorem,

$$|f_k(0)| - |f_k(q)| \leq |f_k(q) - f_k(0)| \leq \frac{|t_0 - t_1|q}{2k}.$$

Hence [1],

$$\begin{aligned} |\sin(\sqrt{q + k^2}(t_0 - t_1))| &\geq |\sin(k(t_0 - t_1))| - \frac{|t_0 - t_1|q}{2k} \\ &\geq \frac{c}{k} - \frac{|t_0 - t_1|q}{2k} \end{aligned}$$

and these estimations are satisfied for all $k = 1, 2, \ldots$ Thus, if the quantity

$$c' := c - \frac{|t_0 - t_1|q}{2} \tag{63}$$

is positive, the estimation (56) is true. A sufficient condition is

$$q < \frac{2c}{|t_0 - t_1|}. \tag{64}$$

□

Remark 6.14.

- The inequality (64) can be rewritten more precisely as [2]

  $$q < \frac{4}{|t_0 - t_1|(K((t_0 - t_1)/\pi) + 2)}.$$

  Indeed, from the proof of Lemma 6.3 and classical results of Diophantine approximation (see [**55**]), the hypothesis (61) holds if and only if the the number $(t_0 - t_1)/\pi$ is "badly approximable by rational numbers" so that its partial quotients are bounded i.e. $K((t_0 - t_1)/\pi)$ is finite. Moreover,

  $$|\sin k(t_0 - t_1)| \geq 2\left\| k\frac{t_0 - t_1}{\pi} \right\| \geq \frac{2}{(K((t_0 - t_1)/\pi) + 2)k}.$$

[1] the constant c depends on t_0 and t_1.

[2] $K(x)$ denotes the largest partial quotient in the continued fraction of x, i.e. if the development in continued fraction of x is given by $x = [a_0; a_1, a_2, \ldots]$, then $K(x) := \sup_{k \geq 1} a_k$.

- It is possible to avoid a restriction on the size of the potential q. Set $\xi := (t_0 - t_1)/\pi \in \mathbb{R} \setminus \mathbb{Q}$ and

$$\nu(\xi) := \liminf_{k \to +\infty} k \|k\xi\|.$$

If ξ is badly approximable, then $\nu(\xi) > 0$. Moreover, if ξ' is an irrational number such that its partial quotients coincide with those of ξ from a certain rank, then $\nu(\xi') = \nu(\xi)$ (see [**12**, p. 11]). Let us construct a strictly decreasing sequence of irrational numbers by setting $\xi_0 = \xi$ and

$$\xi_{n+1} = \frac{\xi_n}{1 + \xi_n} = \frac{1}{1 + 1/\xi_n}.$$

We can assume that $0 < \xi < 1$ so that its development in continued fraction has the form

$$\xi = \xi_0 = [0; a_1, a_2, a_3, \ldots].$$

Therefore, $1/\xi_0 = [a_1; a_2, a_3, \ldots]$ and $1 + 1/\xi_0 = [1 + a_1; a_2, a_3, \ldots]$, whence $\xi_1 = [0; 1 + a_1, a_2, a_3, \ldots]$ and by recurrence

$$\xi_n = [0; n + a_1, a_2, a_3, \ldots].$$

Thus, for all n, $\nu(\xi_n) = \nu(\xi) > 0$ and the sequence (ξ_n) converges to zero. Now, from the definition of $\nu(\xi)$ and the Lemma 6.3, we obtain, for k *sufficiently large*,

$$|\sin k\pi\xi_n| \geq 2\|k\xi_n\| \geq 2\frac{\nu(\xi)}{k}.$$

Hence, going back to the relation (63), if we choose n sufficiently large so that

$$2\nu(\xi) - \frac{\xi_n \pi q}{2} > 0$$

and if we have

$$\sin(\omega_k \pi \xi_n) \neq 0, \qquad k = 1, 2, \ldots,$$

then, choosing t_0 and t_1 such that $t_0 - t_1 = \pi\xi_n$, the observability inequality holds.

Chapter 7

Extension to other systems

1. Vibrating string with a bounded potential

In this section, we generalize the observation of a vibrating string with a constant potential (see section 3 of the previous chapter) to the case of a non-constant potential: assuming that

$$q \in L^\infty(0,\pi) \qquad \text{and} \qquad q \geq 0, \tag{65}$$

we consider the problem

$$\begin{cases} y'' - y_{xx} + q(x)y = 0 & \text{in } \mathbb{R}\times(0,\pi),\\ y = 0 & \text{in } \mathbb{R}\times\{0,\pi\},\\ y(0) = y_0,\ y'(0) = y_1 & \text{in } (0,\pi). \end{cases} \tag{66}$$

Let us recall some classical facts about the well-posedness of (66) and the expression of its solutions in Fourier series. The adapted functional spaces are constructed on the same model as the spaces D^s defined in the previous chapter, *thus we will still denote them by D^s.* Let (e_k) be the Hilbert basis of $L^2(0,\pi)$ formed by the eigenfunctions (associated to the eigenvalues λ_k) of the operator $-\Delta + q(x)\mathrm{Id}$ in $H_0^1(0,\pi)$, this latter space being endowed with the norm $\|f\|^2 = \int_0^\pi (|f_x|^2 + q(x)|f|^2)\,\mathrm{d}x$ (equivalent to the usual norm). Given a real number s, we define the space D^s as the completion of the vector space spanned by these eigenfunctions for the euclidean norm

$$\left\| \sum_{k=1}^\infty c_k e_k \right\|_s^2 := \sum_{k=1}^\infty \lambda_k^s |c_k|^2.$$

In that framework, the above problem is well-posed thanks to the

Proposition 7.1 (well-posedness). *Assume that q satisfies* (65). *Let $y_0 \in D^s$ and $y_1 \in D^{s-1}$. Then,* (66) *has a unique solution $y \in \mathcal{C}(\mathbb{R}, D^s) \cap \mathcal{C}^1(\mathbb{R}, D^{s-1}) \cap \mathcal{C}^2(\mathbb{R}, D^{s-2})$ given by*

$$y(t,x) = \sum_{k=1}^{\infty} (a_k e^{i\sqrt{\lambda_k}t} + b_k e^{-i\sqrt{\lambda_k}t}) e_k(x),$$

where the complex numbers a_k and b_k are determined by the initial data y_0 and y_1. Moreover, the following estimation holds:

$$\|y_0\|_s^2 + \|y_1\|_{s-1}^2 \asymp \sum_{k=1}^{\infty} \lambda_k^s (|a_k|^2 + |b_k|^2).$$

In order to obtain an observability inequality of the type (56), we proceed analogously to the classical string (see Proposition 6.2). It yields to an estimation of the quantity

$$|\sin \omega_k \zeta|, \qquad k = 1, 2, \dots$$

for some fixed real number ζ (depending on the observation times), where we have set

$$\omega_k := \sqrt{\lambda_k}, \qquad k = 1, 2, \dots$$

We adapt both methods described in section 3 of the previous chapter; they relied on a comparison between between k and $\sqrt{k^2 + q}$ (where q was a positive real number), so that in the present case we shall compare ω_k and k. We recall (see [**17**, pp. 414–415]) that

$$\lambda_k = k^2 + \mathcal{O}(1) \qquad \text{when } k \to \infty,$$

whence

$$\omega_k = k + \mathcal{O}\Big(\frac{1}{k}\Big) \qquad \text{when } k \to \infty.$$

In order to have an explicit estimation of the reminder $\mathcal{O}(1/k)$, we assume that there exist two positive numbers a and b such that

$$0 < a \le q \le b \qquad \text{in } (0, \pi). \tag{67}$$

The eigenvalues λ_k can be computed as the min-max (taken on convenient subspaces) of the quantity[1]

$$\frac{\int_0^\pi |f_x|^2 \,\mathrm{d}x + \int_0^\pi q(x)|f|^2 \,\mathrm{d}x}{\int_0^\pi |f|^2 \,\mathrm{d}x}. \tag{68}$$

From (67) and (68), we obtain

$$a + k^2 \le \lambda_k \le b + k^2, \qquad k = 1, 2, \dots$$

Thus,

$$a \le \lambda_k - k^2 = (\omega_k - k)(\omega_k + k)$$

[1] Rayleigh quotient, see [**17**].

and

$$\omega_k - k \geq \frac{a}{\omega_k + k} \geq \frac{a}{\sqrt{b+k^2}+k} \geq \frac{a}{k(1+\sqrt{1+b/k^2})} \geq \frac{a}{k(1+\sqrt{1+b})}.$$

Similarly,

$$(\omega_k - k)(\omega_k + k) \leq b,$$

whence

$$\omega_k - k \leq \frac{b}{\omega_k + k} \leq \frac{b}{\sqrt{a+k^2}+k} \leq \frac{b}{2k}.$$

As a conclusion, there exist two positive constants α and β such that

$$\frac{\alpha}{k} \leq \omega_k - k \leq \frac{\beta}{k}, \qquad k = 1, 2, \dots \tag{69}$$

First method. Applying the mean value theorem and using (69), we get

$$|\sin(\omega_k \zeta) - \sin(k\zeta)| \leq |\omega_k - k||\zeta| \leq \frac{\beta|\zeta|}{k}.$$

If ζ is badly approximable by rational numbers,

$$\begin{aligned} |\sin(\omega_k \zeta)| &\geq |\sin(k\zeta)| - \frac{\beta|\zeta|}{k} \\ &\geq \frac{c(\zeta)}{k} - \frac{\beta|\zeta|}{k} \end{aligned}$$

for a suitable positive constant $c(\zeta)$ independent of k, thus proving the

Theorem 7.2. *Assume that q satisfies (65) and (67) and let r and s be real numbers such that $r - s = 1$. Assuming that (t_0, t_1) is a strategic pair for the classical string* [1] *and that*

$$b < \frac{2\pi c((t_0 - t_1)/\pi)}{|t_0 - t_1|} \qquad \textit{with the above notations,}$$

the pair (t_0, t_1) is strategic for the problem (66).

Second method. We adapt the method of Szijártó and Hegedűs. The following lemma is a direct consequence of [**58**, Lemma 1].

Lemma 7.3. *Assume that $\tau \in \pi\mathbb{Q}$ ($\tau \neq 0$). There exists a positive constant c such that for sufficiently large k,*

$$|\sin(\omega_k \tau)| \geq \frac{c}{k}.$$

Proof. The aim is to estimate

$$\sin(\tau\omega_k) = \sin(\tau k + \tau(\omega_k - k)).$$

Setting $\tau = (m/n)\pi$, where m and n are relatively primes integer, the quantity $\tau k = (m/n)k\pi$ takes at most n distinct values modulo π as k varies so that $|\sin(\tau k)|$ takes at most n distinct values.

[1] In other words $(t_0 - t_1)/\pi$ is badly approximable by rational numbers.

Let us assume that $\sin(\tau k)$ does not always vanish as k varies (otherwise we can skip directly to the second point below). Setting

$$\mu := \min_{\sin(\tau k)\neq 0} |\sin(\tau k)| \quad \in]0,1], \tag{70}$$

it is possible to find a real number x_μ such that

$$\sin(x_\mu) = \mu \qquad \text{and} \qquad 0 < x_\mu \leq \frac{\pi}{2}. \tag{71}$$

The estimation (69) and the fact that $\tau \neq 0$ ensure that we can find a positive constant c such that *for sufficiently large* k

$$\frac{\pi c}{2k} \leq |\tau(\omega_k - k)| \leq \frac{x_\mu}{2} \qquad \text{and} \qquad \frac{c}{k} \leq \sin\left(\frac{x_\mu}{2}\right). \tag{72}$$

We distinguish two cases.

- If $\sin(\tau k) \neq 0$, then from (70), (71) and (72), we obtain

$$|\sin(\tau\omega_k)| = |\sin(\tau k + \tau(\omega_k - k))| \geq \left|\sin\left(x_\mu - \frac{x_\mu}{2}\right)\right| = \left|\sin\left(\frac{x_\mu}{2}\right)\right| \geq \frac{c}{k}$$

  for sufficiently large k.
- If $\sin(\tau k) = 0$, then from (72), we get

$$|\sin(\tau\omega_k)| = |\sin(\tau k + \tau(\omega_k - k))| = |\sin(\tau(\omega_k - k))| \geq \frac{2}{\pi}\Big|\tau(\omega_k - k)\Big| \geq \frac{c}{k}$$

  for sufficiently large k. Indeed, $|\tau(\omega_k - k)| \leq x_\mu/2 \leq \pi/2$ so that we can use the inequality $|\sin t| \geq (2/\pi)t$, which holds for $|t| \leq \pi/2$. □

Theorem 7.4. *Assume that q satisfies* (65) *and* (67) *and let r and s be real numbers such that $r - s = 1$. Let t_0 and t_1 be real numbers such that $(t_0 - t_1) \in \pi\mathbb{Q} \setminus \{0\}$ i.e. there exist (non-vanishing) relatively prime integers m and n such that $(t_0 - t_1) = (m/n)\pi$. Assuming furthermore that*

$$b \leq \frac{2}{|m|}, \tag{73}$$

the pair (t_0, t_1) is strategic for the problem (66)*.*

Proof. [1] It sufficient to prove that there exists a positive constant c such that

$$|\sin(\omega_k(t_0 - t_1))| \geq \frac{c}{k}, \qquad k = 1, 2, \ldots$$

The Lemma 7.3 ensures that such an estimation is satisfied, at least for sufficiently large k. That is why, it is sufficient to prove that $\sin(\omega_k(t_0 - t_1))$ never vanishes. We have

$$\sin(\omega_k(t_0-t_1)) = \sin((t_0-t_1)k+(t_0-t_1)(\omega_k-k)) = \sin\left(k\frac{m\pi}{n}+k\frac{m\pi}{n}(\omega_k-k)\right).$$

[1] The proof is similar to [**58**, Remark 2, p. 9].

From (69) and (73),

$$k\frac{|m|\pi}{|n|}(\omega_k - k) \leq k\frac{|m|\pi}{|n|}\frac{b}{2k} = \frac{b}{2}\frac{|m|\pi}{|n|} < \frac{\pi}{|n|}.$$

Considering the expression,

$$\sin\left(k\frac{m\pi}{n} + k\frac{m\pi}{n}(\omega_k - k)\right),$$

the first term in the argument is either in $\pi\mathbb{Z}$ or at a distance greater than (or equal to) $(\pi/|n|)$ of $\pi\mathbb{Z}$. Moreover, the absolute value of the second term in the argument is strictly smaller than $(\pi/|n|)$. We conclude that the quantity $\sin(\omega_k(t_0 - t_1))$ never vanishes. □

Remark 7.5. It is possible to generalize the results of this this paragraph to a potential

$$q \in L^\infty(0, \pi) \tag{74}$$

with *no further assumption on its sign.*

In that case, the operator $-\Delta + q(x)\mathrm{Id}$ has only *a finite number* of non-positive eigenvalues in $H_0^1(0, \pi)$. More precisely, these eigenvalues form an increasing sequence $\lambda_1 \leq \lambda_2 \leq \dots$ of real numbers such that

$$\lambda_k = k^2 + \mathcal{O}(1) \qquad \text{as } k \to \infty.$$

and the associated (normalized) eigenfunctions e_k form an Hilbert basis of $L^2(0, \pi)$.

Then, we have to change slightly the definition of the adapted functional spaces: for a fixed real number s, we define the space D^s as the completion of the vector space spanned by the eigenfunctions e_k for the euclidean norm

$$\left\| \sum_{k=1}^{\infty} c_k e_k \right\|_s^2 := \sum_{k=1}^{\infty} (1 + |\lambda_k|)^s |c_k|^2.$$

The problem (66) is well-posed in this framework: given initial data $y_0 \in D^s$ and $y_1 \in D^{s-1}$ the problem (66) with assumption (74) has a unique solution $y \in \mathcal{C}(\mathbb{R}, D^s) \cap \mathcal{C}^1(\mathbb{R}, D^{s-1}) \cap \mathcal{C}^2(\mathbb{R}, D^{s-2})$ given by

$$y(t, x) = \sum_{k=1}^{\infty} y_k(t) e_k(x)$$

where

$$y_k(t) := \begin{cases} a_k e^{i\sqrt{\lambda_k} t} + b_k e^{-i\sqrt{\lambda_k} t} & \text{if } \lambda_k > 0; \\ a_k t + b_k & \text{if } \lambda_k = 0; \\ a_k e^{\sqrt{-\lambda_k} t} + b_k e^{-\sqrt{-\lambda_k} t} & \text{if } \lambda_k < 0. \end{cases}$$

and complex coefficients a_k and b_k only depending on the initial data. Moreover, the following estimation holds:

$$\|y_0\|_s^2 + \|y_1\|_{s-1}^2 \asymp \sum_{k=1}^{\infty}(1+|\lambda_k|)^s(|a_k|^2+|b_k|^2).$$

In order to prove an observability inequality of the form (56), it is sufficient to have

$$(1+|\lambda_k|)^{s-r}(|a_k|^2+|b_k|^2) \leq c(|y_k(t_0)|^2+|y_k(t_1)|^2), \tag{75}$$

where c is a positive constant, independent of $k = 1, 2, \ldots$ Let us notice that we can replace $(1+|\lambda_k|)^{s-r}$ by $k^{2(s-r)}$ in (75) because $1+|\lambda_k| \sim k^2$ when $k \to \infty$.

If $\lambda_k > 0$, we consider the application T_k as in the proof of Proposition 6.2. In the other cases we introduce others definitions for the applications T_k from $\mathbb{C}\times\mathbb{C}$ into $\mathbb{C}\times\mathbb{C}$, namely

$$\begin{aligned} T_k(a,b) &:= (ae^{\sqrt{-\lambda_k}t_0} + b^{-\sqrt{-\lambda_k}t_0}, ae^{\sqrt{-\lambda_k}t_1} + b^{-\sqrt{-\lambda_k}t_1}) && \text{if } \lambda_k < 0,\\ T_k(a,b) &:= (at_0+b, at_1+b) && \text{if } \lambda_k = 0. \end{aligned}$$

In these last two cases, T_k is invertible if and only if $t_0 \neq t_1$. Therefore, it is enough to estimate the quantity $\|T_k^{-1}\|$ for the positive eigenvalues λ_k as it was done in the case of a nonnegative potential. Then, it may be necessary to change the constant c to get the estimation (75) for all k, but this is possible since there is only a finite number of non-positive eigenvalues.

2. Hinged beam

We consider the small transversal vibrations of a homogeneous beam of length π with supported ends (hinged beam, see Figure 1).

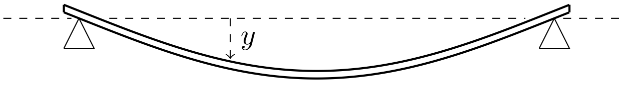

Figure 1. A beam supported at both ends

The transversal displacement y satisfies the vibrating beam equation with hinged boundary conditions [1] :

$$\begin{cases} y'' + y_{xxxx} = 0 & \text{in } \mathbb{R}\times(0,\pi),\\ y = y_{xx} = 0 & \text{in } \mathbb{R}\times\{0,\pi\},\\ y(0) = y_0, \quad y'(0) = y_1 & \text{in } (0,\pi). \end{cases} \tag{76}$$

[1] For more details about this model and especially the boundary conditions, we refer to [**17**, pp. 295–297] and [**59**, p. 341].

Using the *same spaces* D^s *as for the vibrating string* (see the first paragraph of the previous chapter), we recall the

Proposition 7.6 (well-posedness)**.** *Let* $s \in \mathbb{R}$*. For all initial data* $y_0 \in D^s$ *and* $y_1 \in D^{s-2}$*, the problem* (76) *admits a unique solution* $y \in \mathcal{C}(\mathbb{R}, D^s) \cap \mathcal{C}^1(\mathbb{R}, D^{s-2}) \cap \mathcal{C}^2(\mathbb{R}, D^{s-4})$ *given by*

$$y(t,x) = \sum_{k=1}^{\infty} (a_k e^{ik^2 t} + b_k e^{-ik^2 t}) \sin kx, \tag{77}$$

where the complex coefficients a_k *and* b_k *satisfy*

$$\|y_0\|_s^2 + \|y_1\|_{s-2}^2 \asymp \sum_{k=1}^{\infty} k^{2s}(|a_k|^2 + |b_k|^2). \tag{78}$$

In this case, the observability problem turns to the following one: given two real numbers r and s such that $s \leq r$, we are looking for two times t_0 and t_1 such that

$$\|y_0\|_s + \|y_1\|_{s-2} \leq c(\|y(t_0)\|_r + \|y(t_1)\|_r) \tag{79}$$

for a positive constant c, independent of the initial data $(y_0, y_1) \in D^r \times D^{r-1}$. Again, such a pair (t_0, t_1) will be called a *strategic pair*.

Similarly to the Proposition 6.2, we can give a characterization of the strategic pairs for the vibrating beam.

Proposition 7.7. *The pair* (t_0, t_1) *is strategic if and only if there is a positive constant* c *such that*

$$\left\| \frac{k^2(t_0 - t_1)}{\pi} \right\| \geq \frac{c}{k^{r-s}}, \qquad k = 1, 2, \ldots$$

Using exactly the same results of Diophantine approximation as in chapter 6 we can state the following result giving the occurrence of such strategic pairs.

Theorem 7.8.

(a) *If* $r - s = 2$*, the set of strategic pairs contains a subset of zero Lebesgue measure and full Hausdorff dimension in* $\mathbb{R}^2$*.*

(b) *If* $r - s > 2$*, the set of strategic pairs has full Lebesgue measure in* $\mathbb{R}^2$*.*

Remark 7.9. Due to the term k^2 (instead of k for the string) in the left-hand side of the above inequality, the Theorem 7.8 is a little less accurate than the Theorem 6.4 for the classical string. Nevertheless, an advantage of the system (76) is that one can extend the observability results to higher dimensions as we will see in the next paragraphs.

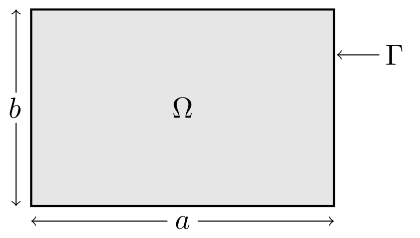

Figure 2. The domain Ω

3. Hinged rectangular plate

Let a and b be positive real numbers and $\Omega = (0,a) \times (0,b) \subset \mathbb{R}^2$ the rectangular domain whose boundary is denoted by Γ (see Figure 2). The small transversal vibrations of a hinged plate whose shape is delimited by Ω satisfy

$$\begin{cases} y'' + \Delta^2 y = 0 & \text{in } \mathbb{R} \times \Omega, \\ y = \Delta y = 0 & \text{in } \mathbb{R} \times \Gamma, \\ y(0) = y_0, \quad y'(0) = y_1 & \text{in } \Omega. \end{cases} \tag{80}$$

The eigenvalues of the operator $-\Delta$ with Dirichlet boundary conditions are (see e.g., [**17**])

$$\lambda_{m,n} = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}, \qquad m,n = 1,2,\dots$$

with associated eigenfunctions

$$e_{m,n}(x,y) = \sin\frac{mx\pi}{a}\sin\frac{ny\pi}{b}, \qquad m,n = 1,2,\dots$$

These functions form an orthogonal and dense system in $L^2(\Omega)$. For $s \in \mathbb{R}$, we define D^s as the completion of the vector space spanned by the functions $e_{m,n}$ for the euclidean norm

$$\left\| \sum_{m,n=1}^{\infty} c_{m,n} e_{m,n} \right\|_s^2 := \sum_{m,n=1}^{\infty} \lambda_{m,n}^s |c_{m,n}|^2.$$

Proposition 7.10 (well-posedness)**.** *Given $y_0 \in D^s$ and $y_1 \in D^{s-2}$, the problem* (80) *has a unique solution $y \in \mathcal{C}(\mathbb{R}, D^s) \cap \mathcal{C}^1(\mathbb{R}, D^{s-2}) \cap \mathcal{C}^2(\mathbb{R}, D^{s-4})$, whose expansion in Fourier series is*

$$y(t,x) = \sum_{m,n=1}^{\infty} (a_{m,n} e^{i\lambda_{m,n}t} + b_{m,n} e^{-i\lambda_{m,n}t}) e_{m,n}(x,y),$$

where the complex coefficients $a_{m,n}$ and $b_{m,n}$ satisfy

$$\|y_0\|_s^2 + \|y_1\|_{s-2}^2 \asymp \sum_{m,n=1}^{\infty} \lambda_{m,n}^s (|a_{m,n}|^2 + |b_{m,n}|^2).$$

The observability problem can be stated exactly as in the previous paragraph: we are looking for pairs (t_0, t_1) satisfying the estimation (79) and such pairs are called *strategic pairs.*

From the expression of the eigenvalues,

$$\lambda_{m,n} \asymp m^2 + n^2$$

and an adaptation of Lemma 6.3 yields

$$|\sin \lambda_{m,n}(t_0 - t_1)| \asymp \left\| \frac{\lambda_{m,n}(t_0 - t_1)}{\pi} \right\|.$$

Hence, setting

$$\theta_1 := (\pi(t_0 - t_1))/a^2, \qquad \theta_2 := (\pi(t_0 - t_1))/b^2$$

and applying the same method as for the vibrating string, we get the

Proposition 7.11. *The pair (t_0, t_1) is strategic if and only if there is a positive constant c such that*

$$\|m^2\theta_1 + n^2\theta_2\| \geq \frac{c}{(m^2 + n^2)^{(r-s)/2}}, \qquad m, n = 1, 2, \ldots \tag{81}$$

Let us give sufficient conditions for (81) to hold.

First case: particular domains. We assume that there exists a positive integer N such that $\theta_1 = N\theta_2$ or equivalently

$$b^2 = Na^2.$$

Therefore, setting $\theta := \theta_2$, the estimation (81) simplifies in

$$\|(Nm^2 + n^2)\theta\| \geq \frac{c}{(m^2 + n^2)^{(r-s)/2}}, \qquad m, n = 1, 2, \ldots$$

We have already seen that if $r - s \geq 2$, the above estimation holds for some choices of t_0 and t_1. More precisely *the Theorem 7.8 remains true in this case.*

Second (general) case. It is not always possible to uncouple the expression $m^2\theta_1 + n^2\theta_2$ as we did in the first case. Nevertheless, we can use some results on the approximation of linear forms by rationals.

Theorem 7.12.

(a) *Assume that $r - s = 4$. If t_0 and t_1 are real numbers such that θ_1 and θ_2 belong to a real algebraic extension of $\mathbb{Q}$ of degree 3 and $1, \theta_1, \theta_2$ are linearly independent over the rationals, then (t_0, t_1) is a strategic pair.*

(b) *Assume that $r - s > 4$. Then, almost all (in the sense of the Lebesgue measure) couples (t_0, t_1) are strategic.*

Proof. The assertion (a) is a direct consequence of the Theorem 7.11 and the second estimation of the Lemma 6.8. The assertion (b) is a consequence of the Theorem 7.11 and of a generalization of the Lemma (6.5) (see [**9**, p. 24]). □

4. Vibrating sphere

We consider the small vibrations of a rigid sphere. Denoting by S the unit sphere of $\mathbb{R}^{N+1}$ ($N \in \mathbb{N}^*$), the small vibrations of S can be modeled by the system

$$\begin{cases} y'' + \Delta^2 y = 0 & \text{in } S \times \mathbb{R}, \\ y(0) = y_0, \quad y'(0) = y_1 & \text{on } S. \end{cases} \tag{82}$$

Before giving a well-posedness result, we recall some facts and definitions about the eigenvalues and the eigenfunctions of $-\Delta$ on S (for more details, see e.g., [**34**, chapter 7]).

- For $m = 0, 1, \ldots$ we denote by $\mathcal{H}_m$ the subspace of $L^2(S)$ formed by the spherical harmonics of degree m. The space $\mathcal{H}_m$ has finite dimension d_m and $d_0 = 1$.
- If $p \neq q$, then the spaces $\mathcal{H}_p$ and $\mathcal{H}_q$ are orthogonal in $L^2(S)$.
- The spherical harmonics are the eigenfunctions of $-\Delta$ on the sphere: if $h \in \mathcal{H}_m$, then
  $$-\Delta h = \gamma_m h \qquad \text{with } \gamma_m = m(m + N - 1).$$
  Remark that $\gamma_0 = 0$ and $\forall m \geq 1$, $\gamma_m > 0$.
- There exists a Hilbert basis of $L^2(S)$ formed with spherical harmonics: $\{h_1^m, h_2^m, \ldots, h_{d_m}^m\}$ being an orthonormal basis of $\mathcal{H}_m$ for $m = 1, 2, \ldots$,
  $$\bigcup_{m=0}^{\infty} \{h_1^m, h_2^m, \ldots, h_{d_m}^m\}$$
  is an Hilbert basis of $L^2(S)$.
- We denote by D the vector space spanned by the above defined spherical harmonics $h_1^m, h_2^m, \ldots, h_{d_m}^m$ ($m = 0, 1, \ldots$) so that each f in D can be uniquely written
  $$f = \sum_{m=0}^{\infty} \sum_{k=1}^{d_m} c_{m,k} h_k^m$$
  with a finite number of non-vanishing complex coefficients $c_{m,k}$. Given $s \in \mathbb{R}$, we set
  $$\|f\|_s^2 := \left\| \sum_{m=0}^{\infty} \sum_{k=1}^{d_m} c_{m,k} h_k^m \right\|_s^2 := \sum_{m=0}^{\infty} \sum_{k=1}^{d_m} (1 + \gamma_m)^s |c_{m,k}|^2.$$

Then, we denote by D^s the completion of the space D for the euclidean norm $\|.\|_s$. In particular, $D^0 = L^2(S)$.

Now, we can state a existence and uniqueness result for the problem (82).

Proposition 7.13 (well-posedness)**.** *Given initial data $y_0 \in D^s$ and $y_1 \in D^{s-2}$, the problem* (82) *has a unique solution $y \in C(\mathbb{R}, D^s) \cap C^1(\mathbb{R}, D^{s-2}) \cap C^2(\mathbb{R}, D^{s-2})$. This solution can be written as the Fourier series*

$$y(t,x) = (a_{0,1} + b_{0,1}t)h_1^0(x) + \sum_{m=0}^{\infty}\sum_{k=1}^{d_m}(a_{m,k}e^{i\gamma_m t} + b_{m,k}e^{-i\gamma_m}t)h_k^m(x),$$

where the complex coefficients $a_{m,k}$ and $b_{m,k}$ only depends on the initial data y_0 and y_1. Moreover, we have the estimation

$$\|y_0\|_s^2 + \|y_1\|_{s-2}^2 \asymp \sum_{m=0}^{\infty}\sum_{k=1}^{d_m}(1+\gamma_m)^s(|a_{m,k}|^2 + |b_{m,k}|^2).$$

In order to obtain an estimation of the type (79), it is sufficient to establish the estimation

$$\left\|\gamma_m \frac{t_0 - t_1}{\pi}\right\| \geq \frac{c}{\gamma_m^{\alpha/2}}, \qquad m = 1, 2, \ldots,$$

where $\alpha = r - s$ and c is a positive constant, independent of m. Hence, *the Theorem 7.8 remains true for the vibrating sphere.*

Chapter 8

An exact controllability result

Let t_0, t_1 and T be fixed real numbers such that

$$0 < t_0 < t_1 < T.$$

We consider the non-homogeneous boundary problem

$$(83)\qquad \begin{cases} y'' - y_{xx} = \delta(t-t_0)v + \delta(t-t_1)w & \text{in } (0,T)\times(0,\pi),\\ y = 0 & \text{on } (0,T)\times\{0,\pi\},\\ y(0) = y_0, \quad y'(0) = y_1 & \text{in } (0,\pi). \end{cases}$$

Here, v and w are two vectors in a Hilbert space H that will be specified later. The right-hand side in the above partial differential equation is a vector valued distribution i.e. as an element of $\mathcal{D}'(\mathbb{R}, H)$, δ denoting the Dirac delta function.

In this section, we investigate the following exact controllability problem.

Can we find controls v and w such that

$$y(T) = y'(T) = 0 \quad ?$$

In other words, we want to steer the string to a rest at time T by mean of two "impulsions" at times t_0 and t_1.

1. Weak solutions

The aim of this section is twofold: at first we define a notion of weak solution to the problem (83) via the method of transposition-duality (see [**42**]). Then, we prove that the problem is well-posed with respect to this notion of weak solution and we also obtain a Fourier series representation of the solutions.

In order to simplify the problem, we consider the case of only one control:

$$(84)\qquad \begin{cases} y'' - y_{xx} = \delta(t-t_0)v & \text{in } (0,T)\times(0,\pi),\\ y = 0 & \text{on } (0,T)\times\{0,\pi\},\\ y(0) = y_0, \quad y'(0) = y_1 & \text{in } (0,\pi). \end{cases}$$

Eventually, we will define the weak solution to (83) by the superposition principle.

Le us begin with a formal computation: we consider the solution φ to the homogeneous problem

$$(85)\qquad \begin{cases} \varphi'' - \varphi_{xx} = 0 & \text{in } (0,T)\times(0,\pi),\\ \varphi = 0 & \text{on } (0,T)\times\{0,\pi\},\\ \varphi(0) = \varphi_0, \quad \varphi'(0) = \varphi_1 & \text{in } (0,\pi). \end{cases}$$

Let y be the solution to (84) and fix $S \in [0,T]$. We assume that y and φ are sufficiently regular in order to make integrations by parts. Then, multiplying the first equation of (85) by y and integrating over $(0,S)\times(0,\pi)$, we obtain (at least formally)

$$\begin{aligned} 0 &= \int_0^S \int_0^\pi y(\varphi'' - \varphi_{xx})\,\mathrm{d}x\,\mathrm{d}t\\ &= \Big[\int_0^\pi (y\varphi' - y'\varphi)\,\mathrm{d}x\Big]_0^S + \int_0^S\int_0^\pi \varphi(y''-y_{xx})\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^\pi (y(S)\varphi'(S) - y'(S)\varphi(S))\,\mathrm{d}x - \int_0^\pi (y_0\varphi_1 - y_1\varphi_0)\,\mathrm{d}x\\ &\qquad + \int_0^S\int_0^\pi \varphi\delta(t-t_0)v\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^\pi (y(S)\varphi'(S) - y'(S)\varphi(S))\,\mathrm{d}x - \int_0^\pi (y_0\varphi_1 - y_1\varphi_0)\,\mathrm{d}x\\ &\qquad H(S-t_0)\int_0^\pi \varphi(t_0)v\,\mathrm{d}x, \end{aligned}$$

where H is the Heaviside function [1] . The appearance of the Heaviside function comes from the fact that if $S < t_0$, there is no contribution of the Dirac function. If the data are sufficiently regular, we can rewrite the last relation in the abstract form

$$(86)\qquad \Big\langle (y'(S), -y(S)), (\varphi(S), \varphi'(S))\Big\rangle_{D^{-1}\times D^0, D^1\times D^0} = \Big\langle (y_1,-y_0),(\varphi_0,\varphi_1)\Big\rangle_{D^{-1}\times D^0, D^1\times D^0} + H(S-t_0)\big(v,\varphi(t_0)\big)_{D^0}.$$

This lead to the

[1] $H(s) = 0$ if $s < 0$, $H(s) = 1$ if $s \geq 1$ and $H' = \delta$ in the sense of distributions.

Definition 8.1. Let $(y_0, y_1) \in D^0 \times D^{-1}$ and $v \in D^0$. We say that $y \in \mathcal{C}([0,T]; D^0) \cap \mathcal{C}^1([0,t_0[; D^{-1}) \cap \mathcal{C}^1([t_0,T]; D^{-1})$ is a *weak solution* to (84) and the relation (86) is satisfied for all $S \in [0,T]$ and all $(\varphi_0, \varphi_1) \in D^1 \times D^0$.

Remark 8.2. In the relation (86), if $S = t_0$, $y'(t_0)$ denotes the right derivative of y at time t_0 since, in general, y' has a discontinuity at t_0 (see the proof of Proposition 8.3 below).

Proposition 8.3. *Let $(y_0, y_1) \in D^0 \times D^{-1}$ and $v \in D^0$. The problem* (83) *has a unique weak solution y. Moreover, the solutions are continuous with respect to the initial data and the control:*

$$\max_{t\in[0,T]} \|y(t)\|_0 + \sup_{t\in[0,t_0[\cup[t_0,T]} \|y'(t)\|_{-1} \le c(\|y_0\|_0 + \|y_1\|_{-1} + \|v\|_0), \tag{87}$$

where c is a positive constant, independent of y_0, y_1 and v.

Proof. We begin with the *uniqueness* of a weak solution. Assume that there are two weak solutions y and $\widetilde{y}$ and fix $S \in [0,T]$. From the relation (86), we have

$$\Big\langle (y'(S) - \widetilde{y}'(S), \widetilde{y}(S) - y(S)), (\varphi(S), \varphi'(S)) \Big\rangle_{D^{-1}\times D^0, D^1\times D^0} = 0$$

for all $(\varphi_0, \varphi_1) \in D^1 \times D^0$ (we recall that φ denotes the associated solution to the homogeneous problem (85)). But the map $(\varphi_0, \varphi_1) \mapsto (\varphi(S), \varphi'(S))$ is an isomorphism in $D^1 \times D^0$. Hence, $y(S) = \widetilde{y}(S)$. As S is arbitrary, we conclude that $y = \widetilde{y}$.

To prove the *existence* of a weak solution, we proceed in three steps.

First step. We look for a Fourier series representing the weak solution:

$$y(t,x) = \sum_{k=1}^{\infty} y_k(t) \sin(kx). \tag{88}$$

Developing the initial data and the control term in Fourier series, we have

$$y_0 = \sum_{k=1}^{\infty} a_k \sin(kx), \qquad y_1 = \sum_{k=1}^{\infty} b_k \sin(kx), \qquad v = \sum_{k=1}^{\infty} v_k \sin(kx),$$

where a_k, b_k and v_k are complex numbers. Injecting the above expressions in (84), this yields, for each $k = 1, 2, \ldots$, to the ordinary differential equation

$$y_k''(t) + k^2 y_k(t) = \delta(t - t_0) v_k \tag{89}$$

with initial data

$$y_k(0) = a_k, \quad y_k'(0) = b_k. \tag{90}$$

Remark that the function

$$\psi_k(t) := H(t - t_0) \sin(k(t - t_0)) \frac{v_k}{k} \tag{91}$$

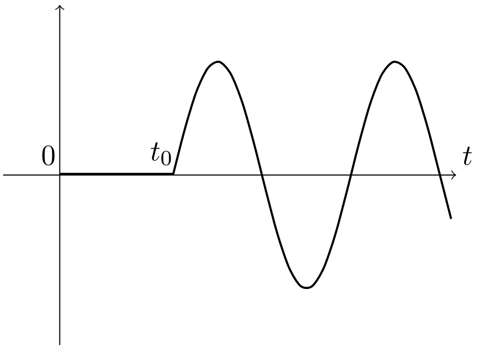

Figure 1. Shape of the map $t \mapsto \psi_k(t)$

is a particular solution to (89) in the sense of distributions. Indeed,

$$\begin{aligned}\psi_k'(t) &= \delta(t-t_0)\sin(k(t-t_0))\frac{v_k}{k} + H(t-t_0)\cos(k(t-t_0))v_k \\ &= H(t-t_0)\cos(k(t-t_0))v_k\end{aligned}$$

and

$$\begin{aligned}\psi_k''(t) &= \delta(t-t_0)\cos(k(t-t_0))v_k - kH(t-t_0)\sin(k(t-t_0))v_k \\ &= \delta(t-t_0)v_k - kH(t-t_0)\sin(k(t-t_0))v_k.\end{aligned}$$

Moreover, ψ_k is continuous, continuously differentiable except at $t = t_0$ (see Figure 1) and

$$\psi_k(0) = \psi_k'(0) = 0.$$

Then, the solution to (89)-(90) in the sense of distributions is

$$y_k(t) = c_k e^{ikt} + d_k e^{-ikt} + \psi_k(t),$$

with

$$c_k := \frac{1}{2}\Big(a_k + \frac{1}{ik}b_k\Big), \qquad d_k := \frac{1}{2}\Big(a_k - \frac{1}{ik}b_k\Big).$$

It seems natural to set

$$y_k(t) := c_k e^{ikt} + d_k e^{-ikt} + \psi_k(t). \tag{92}$$

in the Fourier series representation of y in (88).

Second step. Now, let us prove that y, defined by (88)-(92), has the expected regularity. [1]. Let $t \in [0, T]$.

$$\begin{aligned} |y_k(t)|^2 &= \Big| c_k e^{ikt} + d_k e^{-ikt} + H(t-t_0)\sin(k(t-t_0))\frac{v_k}{k}\Big|^2 \\ &\leq 2|c_k e^{ikt} + d_k e^{-ikt}|^2 + 2\Big| H(t-t_0)\sin(k(t-t_0))\frac{v_k}{k}\Big|^2 \\ &\leq 4|c_k|^2 + 4|d_k|^2 + 2|v_k|^2 \end{aligned}$$

From the regularity of y_0, y_1 and v, the right-hand side is the general term of a convergent series. Hence, $y(t) \in D^0$. Moreover,

$$\begin{aligned} k^{-2}|y_k'(t)|^2 &= k^{-2}|ik(c_k e^{ikt} + d_k e^{-ikt}) + H(t-t_0)\cos(k(t-t_0))v_k|^2 \\ &\leq 2|c_k e^{ikt} + d_k e^{-ikt}|^2 + 2k^{-2}|v_k|^2 \\ &\leq 4|c_k|^2 + 4|d_k|^2 + 2|v_k|^2. \end{aligned}$$

Again, the right-hand side is the general term of a convergent series so that

$$z(t,x) := \sum_{k=1}^{\infty} y_k'(t)\sin(kx)$$

defines an element of D^{-1}.

The above inequalities are independent of $t \in [0, T]$. This implies that the series $\sum_{k=1}^{\infty} y_k(t)\sin(kx)$ is uniformly convergent in D^0 on the interval $[0, T]$. Moreover, the series $\sum_{k=1}^{\infty} y_k'(t)\sin(kx)$ is uniformly convergent in D^{-1} on the intervals $[0, t_0[$ and $[t_0, T]$. Hence, y is continuous on $[0, T]$ and continuously differentiable on the intervals $[0, t_0[$ and $[t_0, T]$, and $y' = z$. The same inequalities yield the relation (87).

Third step. Le us prove that that y satisfies the relation (86). Fix $S \in [0, T]$. The solution to the homogeneous problem (85) can be written as a Fourier series

$$\varphi(t,x) = \sum_{k=1}^{\infty} (f_k e^{ikt} + g_k e^{-ikt})\sin(kx),$$

[1] We recall that given $s \in \mathbb{R}$,

$$\sum_{k=1}^{\infty} f_k \sin(kx) \in D^s \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} k^{2s}|f_k|^2 < \infty.$$

where the complex coefficients f_k and g_k only depend on the initial data φ_0 and φ_1. Then, recalling that $\psi_k(0) = \psi_k'(0) = 0$, we set

$$\begin{aligned}
P_1 &:= \Big\langle y'(S), \varphi(S) \Big\rangle_{D^{-1},D^1} \\
&= \sum_{k=1}^{\infty} ik(c_k e^{ikS} - d_k e^{-ikS})(f_k e^{ikS} + g_k e^{-ikS}) \\
&\quad + \psi_k'(S)(f_k e^{ikS} + g_k e^{-ikS}); \\
P_2 &:= \Big\langle y(S), \varphi'(S) \Big\rangle_{D^0,D^0} \\
&= \sum_{k=1}^{\infty} ik(c_k e^{ikS} + d_k e^{-ikS})(f_k e^{ikS} - g_k e^{-ikS}) \\
&\quad + \psi_k(S) ik(f_k e^{ikS} - g_k e^{-ikS}); \\
P_3 &:= \Big\langle y_1, \varphi_0 \Big\rangle_{D^{-1},D^1} = \sum_{k=1}^{\infty} ik(c_k - d_k)(f_k + g_k); \\
P_4 &:= \Big\langle y_0, \varphi_1 \Big\rangle_{D^0,D^0} = \sum_{k=1}^{\infty} ik(c_k + d_k)(f_k - g_k).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\phi_k'(S)(f_k e^{ikS} + g_k e^{-ikS}) - \phi_k(S) ik(f_k e^{ikS} - g_k e^{-ikS}) \\
&= H(S - t_0) v_k \Big[\cos(k(S - t_0))(f_k e^{ikS} + g_k e^{-ikS}) \\
&\qquad - i \sin(k(S - t_0))(f_k e^{ikS} - g_k e^{-ikS}) \Big] \\
&= \frac{H(S - t_0) v_k}{2} \Big[(e^{ik(S-t_0)} + e^{-ik(S-t_0)})(f_k e^{ikS} + g_k e^{-ikS}) \\
&\qquad - (e^{ik(S-t_0)} - e^{-ik(S-t_0)})(f_k e^{ikS} - g_k e^{-ikS}) \Big] \\
&= H(S - t_0)(f_k e^{ikt_0} + g_k e^{-ikt_0}) v_k.
\end{aligned}$$

Therefore,

$$P_1 - P_2 = P_3 - P_4 + H(S - t_0) \Big\langle y(t_0), v \Big\rangle_{D^0,D^0}$$

i.e., the relation (86) is satisfied. □

Remarks 8.4.

- The weak solution y to (84) given by the Proposition 8.3 has in particular the following regularity :

$$y \in L^2(0, T; D^0) \cap H^1(0, T; D^{-1}).$$

- We could define similarly the weak solution and give a well-posedness result for data $y_0, v \in D^\alpha$ and $y_1 \in D^{\alpha-1}$, where α is a real number. In that case, the weak solution would be continuous with values in D^α and continuously differentiable (except maybe at t_0) with values in $D^{\alpha-1}$.

Now, there is a natural way to define the *weak solution* to the initial non-homogeneous problem (83). Given $y_0, v, w \in D^0$ and $y_1 \in D^{-1}$, we define a weak solution to (83) as a function y continuous in $[0,T]$ with values in D^0, continuously differentiable on $[0,t_0[$, $[t_0,t_1[$ and $[t_1,T]$ with values in D^{-1} and satisfying, for all $S \in [0,T]$ and all $(\varphi_0,\varphi_1) \in D^1 \times D^0$ the formula

$$\begin{aligned}(93)\quad \Big\langle (y'(S), -y(S)), (\varphi(S), \varphi'(S)) \Big\rangle_{D^{-1}\times D^0, D^1\times D^0} &= \Big\langle (y_1, -y_0), (\varphi_0, \varphi_1) \Big\rangle_{D^{-1}\times D^0, D^1\times D^0} \\ &\quad + H(S-t_0)\big(v, \varphi(t_0)\big)_{D^0} + H(S-t_1)\big(w, \varphi(t_1)\big)_{D^0}.\end{aligned}$$

The uniqueness of a weak solution can be proved as in the proof of Proposition 8.3. We can prove the existence of a weak solution for example by superposing z and $\widetilde{z}$, the weak solutions to, respectively,

$$\begin{cases} z'' - z_{xx} = \delta(t-t_0)v & \text{in } (0,T)\times(0,\pi), \\ z = 0 & \text{on } (0,T)\times\{0,\pi\}, \\ z(0) = y_0/2, \quad z'(0) = z_1/2 & \text{in } (0,\pi). \end{cases}$$

and

$$\begin{cases} \widetilde{z}'' - \widetilde{z}_{xx} = \delta(t-t_1)w & \text{in } (0,T)\times(0,\pi), \\ \widetilde{z} = 0 & \text{on } (0,T)\times\{0,\pi\}, \\ \widetilde{z}(0) = y_0/2, \quad \widetilde{z}'(0) = y_1/2 & \text{in } (0,\pi). \end{cases}$$

2. Exact controllability via the Hilbert Uniqueness Method

In this paragraph, we prove an exact controllability result. An essential assumption is the observability of the homogeneous problem (85) in the sense defined in chapter 6. More precisely, we may assume that there exists a positive constant c such that for all $(\varphi_0,\varphi_1) \in D^0 \times D^{-1}$,

$$(94)\qquad \|\varphi_0\|_{-1}^2 + \|\varphi_1\|_{-2}^2 \le c(\|\varphi(t_0)\|_0^2 + \|\varphi(t_1)\|_0^2),$$

φ denoting the corresponding solution to (85).

Theorem 8.5 (Null controllability for smooth initial data)**.** *Let $T > 0$ be fixed. Assume that t_0 and t_1 are real numbers such that $0 < t_0 < t_1 < T$ and the observability inequality* (94) *holds. Then, for all initial data $(y_0, y_1) \in$*

$D^2 \times D^1$, *there exist control vectors* $v, w \in D^0$ *such that the weak solution* y *to the associated non-homogeneous problem* (83) *is at rest at time* T *i.e.*

$$y(T) = y'(T) = 0$$

Proof. We follow the Hilbert Uniqueness Method of Lions [**44, 45**] by looking for suitable control vectors v and w of the form

$$v = \varphi(t_0) \quad \text{and} \quad w = \varphi(t_1), \tag{95}$$

where φ is a solution to the associated homogeneous problem.

Main idea. At first, we solve the homogeneous problem

$$\begin{cases} \varphi'' - \varphi_{xx} = 0 & \text{in } (0,T) \times (0,\pi), \\ \varphi = 0 & \text{on } (0,T) \times \{0,\pi\}, \\ \varphi(0) = \varphi_0, \quad \varphi'(0) = \varphi_1 & \text{in } (0,\pi). \end{cases} \tag{96}$$

Then, we solve the backward inhomogeneous problem

$$\begin{cases} y'' - y_{xx} = \delta(t - t_0)\varphi(t_0) + \delta(t - t_1)\varphi(t_1) & \text{in } (0,T) \times (0,\pi), \\ y = 0 & \text{on } (0,T) \times \{0,\pi\}, \\ y(T) = 0, \quad y'(T) = 0 & \text{in } (0,\pi). \end{cases} \tag{97}$$

Considering the linear map

$$\Lambda : \quad (\varphi_0, \varphi_1) \mapsto (-y'(0), y(0)),$$

knowing the controllable states with the particular controls (95) amounts to determining the range of Λ. We will see that the observability inequality will be crucial for this task.

First attempt. In order to have a well-defined solution to (97) (see the Proposition 8.3 [1]), we have to ensure that the controls v and w defined by (95) lie in D^0. A sufficient assumption is that (φ_0, φ_1) belongs to $D^1 \times D^0$. In that case, Λ maps $D^1 \times D^0$ into $D^{-1} \times D^0$ and using the relation (93) (taken at $S = T$), we obtain

$$\Big\langle (\Lambda(\varphi_0, \varphi_1), (\psi_0, \psi_1) \Big\rangle_{D^{-1} \times D^0, D^1 \times D^0} = (\varphi(t_0), \psi(t_0))_{D^0} + (\varphi(t_1), \psi(t_1))_{D^0} \tag{98}$$

for all $(\psi_0, \psi_1) \in D^1 \times D^0$.

The above expression defines a continuous bilinear functional on $(D^1 \times D^0)^2$. If this functional was coercive, we could apply the Lax-Milgram theorem and conclude that Λ is an isomorphism from $D^1 \times D^0$ onto $D^{-1} \times D^0$. This means that all initial states in $D^0 \times D^{-1}$ could be steered to zero at time T. Unfortunately , the coercivity with respect to the space $(D^1 \times D^0)^2$ is not true (see the estimation (94)) and we have to introduce other functional spaces.

[1] whose an analogous statement holds for backward problems

Second attempt. From the definitions of the spaces D^s, the well-posedness of the homogeneous problem (96) and the observability inequality (94), we have the following inequalities for all $(\varphi_0, \varphi_1) \in D \times D$:

(99) $$c_1(\|\varphi_0\|_2^2 + \|\varphi_1\|_1^2) \geq \|\varphi(t_0)\|_0^2 + \|\varphi(t_1)\|_0^2 \geq c_1(\|\varphi_0\|_{-1}^2 + \|\varphi_1\|_{-2}^2)$$

with suitable positive constants c_1 and c_2.

From the right inequality, the quantity

$$\|(\varphi_0, \varphi_1)\|_F^2 := \|\varphi(t_0\|_0^2 + \|\varphi(t_1)\|_0^2$$

defines a euclidean norm on $D \times D$. Now, we define the new space

$$F := \text{ completion of } D \times D \text{ for the norm } \|.\|_F.$$

From the definition of the spaces D^s, F and the estimations (99) we deduce the following topological and algebraical inclusions :

(100) $$D^2 \times D^1 \subset F \subset D^{-1} \times D^{-2}.$$

For all (φ_0, φ_1), $(\psi_0, \psi_1) \in D \times D$, the relation (98) can be rewritten as [1]

(101) $$\Big\langle (\Lambda(\varphi_0, \varphi_1), (\psi_0, \psi_1) \Big\rangle_{D^{-2} \times D^{-1}, D^2 \times D^1} = \Big((\varphi_0, \varphi_1), (\psi_0, \psi_1) \Big)_F.$$

Hence,

$$\Lambda \in \mathcal{L}(D \times D, D^{-2} \times D^{-1}),$$

where $D \times D$ is endowed with the norm $\|.\|_F$. Indeed,

$$\begin{aligned} \left| \Big\langle (\Lambda(\varphi_0, \varphi_1), (\psi_0, \psi_1) \Big\rangle_{D^{-2} \times D^{-1}, D^2 \times D^1} \right| &\leq \|(\varphi_0, \varphi_1)\|_F \|(\psi_0, \psi_1)\|_F \\ &\leq c_3 \|(\varphi_0, \varphi_1)\|_F \|(\psi_0, \psi_1)\|_{D^2 \times D^1}, \end{aligned}$$

whence

$$\|\Lambda(\varphi_0, \varphi_1)\|_{D^{-2} \times D^{-1}} \leq c_3 \|(\varphi_0, \varphi_1)\|_F.$$

By density, we can extend (in a unique way) Λ to a bounded linear map (still denoted by Λ) from F into $D^{-2} \times D^{-1}$, i.e.

$$\Lambda \in \mathcal{L}(F, D^2 \times D^{-1}).$$

Given $(\varphi_0, \varphi_1) \in F$, the relation (101) remains true for all $(\psi_0, \psi_1) \in D \times D$. Hence, $\Lambda(\varphi_0, \varphi_1)$ defines a continuous linear functional on $D \times D$ endowed with the norm $\|.\|_F$. By density, we extend it (and identify it) to a linear functional on F i.e. $\Lambda(\varphi_0, \varphi_1) \in F'$ and for all (φ_0, φ_1), $(\psi_0, \psi_1) \in F$ we have

$$\Big\langle (\Lambda(\varphi_0, \varphi_1), (\psi_0, \psi_1) \Big\rangle_{F', F} = \Big((\varphi_0, \varphi_1), (\psi_0, \psi_1) \Big)_F.$$

Therefore

$$\|\Lambda(\varphi_0, \varphi_1)\|_{F'} = \|(\varphi_0, \varphi_1)\|_F$$

---

[1]Note the changes in the spaces below the duality brackets.

and

$$\Lambda \in \mathcal{L}(F, F').$$

Using the three last relations and the Lax-Milgram theorem [1] , we conclude that Λ is an isomorphism fro F onto F' so that all the states in the “abstract” space F' can be steered to zero at time T with a control of the form (95). From the right inclusion in (100), we get

$$D^1 \times D^2 \subset F',$$

so that, in particular, all initial states in $D^2 \times D^1$ are null-controllable in time T. □

Remarks 8.6.

- The above proof is similar to the proof of the exact controllability of the wave equation with a Neumann boundary control (see e.g.,[**31**, pp. 61-62], [**45**, pp. 19-22]).
- From the above result, we can deduce that any initial state $(y_0, y_1) \in D^2 \times D^1$ can be steered to any final state $(y_T, y'_T) \in D^2 \times D^1$ at time T using control vectors $v, w \in D^0$. Indeed, let z be a weak solution to the backward problem

$$\begin{cases} z'' - z_{xx} = 0 & \text{in } \mathbb{R} \times (0,\pi), \\ z = 0 & \text{on } \mathbb{R} \times \{0,\pi\}, \\ z(T) = y_T, \quad z'(T) = y'_T & \text{in } (0,\pi). \end{cases}$$

Let ζ be a weak solution to the problem

$$\begin{cases} \zeta'' - \zeta_{xx} = \delta(t-t_0)v + \delta(t-t_1)w & \text{in } (0,T) \times (0,\pi), \\ \zeta = 0 & \text{on } (0,T) \times \{0,\pi\}, \\ \zeta(0) = y_0 - z(0), \quad \zeta'(0) = y_1 - z'(0) & \text{in } (0,\pi), \end{cases}$$

where $v, w \in D^0$ are control vectors (given by the Theorem 8.5) such that $\zeta(T) = \zeta'(T) = 0$. Then, setting

$$y := z + \zeta,$$

this function satisfies (in a weak sense)

$$\begin{cases} y'' - y_{xx} = \delta(t-t_0)v + \delta(t-t_1)w & \text{in } (0,T) \times (0,\pi), \\ y = 0 & \text{on } (0,T) \times \{0,\pi\}, \\ y(0) = y_0, \quad y'(0) = y_1 & \text{in } (0,\pi), \\ y(T) = y_T, \quad y'(T) = y'_T & \text{in } (0,\pi). \end{cases}$$

[1] Instead, we could use the Riesz representation theorem.

- With an observability inequality with weaker norms
$$\|\varphi(t_0)\|_0^2 + \|\varphi(t_1)\|_0^2 \geq c(\|\varphi_0\|_{-\alpha}^2 + \|\varphi_1\|_{-\alpha-1}^2),$$
where $\alpha > 1$, it is also possible to obtain an exact controllability result. In this case, we can find controllable states in the smaller space $D^{\alpha+1} \times D^{\alpha}$.
- Using the Fourier series representation of the solutions to the homogeneous problem, we can give a more precise description of the space F. This space is isometric to the space of the sequences $(a_k, b_k)_{k\geq 1} \subset \mathbb{C} \times \mathbb{C}$ such that
$$\sum_{k=1}^{\infty} \left| a_k \cos(kt_0) + \frac{b_k}{k} \sin(kt_0) \right|^2 + \left| a_k \cos(kt_1) + \frac{b_k}{k} \sin(kt_1) \right|^2 < \infty,$$
the latter expression defining the square of the norm (the fact that this is a norm is a consequence of the observability inequality).

Some useful results and two proofs

1. Useful results on operators and semigroups

Proposition A.1 (an operator-integral inversion result)**.** [1] *Let H_1 and H_2 be two Hilbert spaces and C be a closed operator from H_1 into H_2 with a dense domain. Given $u \in L^2(a, b; H_1)$, we assume that $\int_a^b \|Cu(t)\|_{H_2} \,\mathrm{d}t < \infty$. Then,*

$$\int_a^b u(t)\,\mathrm{d}t \in \mathcal{D}(C) \qquad \text{and} \qquad C\int_a^b u(t)\,\mathrm{d}t = \int_a^b Cu(t)\,\mathrm{d}t.$$

Proposition A.2 (a density result)**.** *Let H be a Hilbert space and A be the generator of a semigroup on H. Then, $\mathcal{D}(A^2)$ is dense in $\mathcal{D}(A)$ for the graph norm.*

Proof. Let $x \in \mathcal{D}(A)$. We look for a sequence $(x_n) \in \mathcal{D}(A^2)$ such that

$$\begin{aligned} x_n &\to x \quad \text{in } H; \\ Ax_n &\to Ax \quad \text{in } H. \end{aligned}$$

We set $x_n = n\int_0^{1/n} e^{tA}x\,\mathrm{d}t$ so that for $h > 0$

$$\frac{1}{h}(e^{hA}x_n - x_n) \to n(e^{(1/n)A}x - x) \qquad \text{as } h \to 0.$$

Therefore, $x_n \in \mathcal{D}(A)$ and $Ax_n = n(e^{(1/n)A}x - x) \in \mathcal{D}(A)$ i.e. $x_n \in \mathcal{D}(A^2)$. Moreover, $x_n \to x$ and $Ax_n \to Ax$ in H. □

[1] For a proof, see [**2**, p. 139].

Proposition A.3 (bounded perturbation). [1] *Let A be the generator of a semigroup on a Hilbert space H, satisfying $\|e^{tA}\| \le Me^{\gamma t}$. If B is a bounded operator on H, then $A_P := A + B$ is the generator of a semigroup on H, satisfying*

$$\|e^{tA_P}\| \le Me^{(\gamma+M\|B\|)t}, \qquad t \ge 0,$$

and for all $x \in H$,

$$e^{tA_P}x = e^{tA}x + \int_0^t e^{(t-s)A}Be^{sA_P}x\,\mathrm{d}s, \qquad t \ge 0.$$

Proposition A.4 (conjugated semigroups). [2] *Let H_1 and H_2 be two Hilbert spaces, A be the generator of a semigroup (resp. a group) on H_1 and $P \in \mathcal{L}(H_1, H_2)$ be an isomorphism. Then, $Pe^{tA}P^{-1}$ is a semigroup (resp. a group) on H_2 and its generator is given by PAP^{-1} with domain $P\mathcal{D}(A)$.*

2. Proofs of some results for the open-loop problem

In this paragraph, we recall the proofs of two results from section 3 of chapter 2 concerning the open-loop problem.

For the proof of Proposition 2.2, we follow [**23**]; we need the

Lemma A.5. *Let $g \in L^1(0,T;\mathcal{D}(A^*))$. Then, for all $-T \le t \le T$,*

$$\int_0^t e^{-(t-r)A^*}g(r)\,\mathrm{d}r \in \mathcal{D}(A^*)$$

and

$$A^*\int_0^t e^{-(t-r)A^*}g(r)\,\mathrm{d}r = \int_0^t A^*e^{-(t-r)A^*}g(r)\,\mathrm{d}r.$$

Proof of Lemma A.5. The integrand $e^{-(t-r)A^*}g(r)$ lies in the domain of A^* for all r between 0 and t. Using the Proposition A.1, it is sufficient to prove that

$$\int_0^t \|A^*e^{-(t-r)A^*}g(r)\|\,\mathrm{d}r < \infty.$$

[1] This result is due to Philipps. For a proof, we refer to [**49**, p. 76–77].
[2] cf. [**20**, p. 59]

This integral is finite since

$$
\begin{aligned}
&\int_{-T}^{T}\int_{0}^{t} \|A^* e^{-(t-r)A^*} g(r)\|_{H'} \,\mathrm{d}r\,\mathrm{d}t \\
\leq &\int_{-T}^{T}\int_{-T}^{T} \|A^* e^{-(t-r)A^*} g(r)\|_{H'} \,\mathrm{d}r\,\mathrm{d}t \\
= &\int_{-T}^{T}\int_{-T}^{T} \|A^* e^{-(t-r)A^*} g(r)\|_{H'} \,\mathrm{d}t\,\mathrm{d}r \qquad \text{(Fubini-Tonelli)} \\
\leq &\int_{-T}^{T} \sqrt{2T}\Big(\int_{0}^{T} \|A^* e^{-tA^*} e^{rA^*} g(r)\|_{H'}^2 \,\mathrm{d}t\Big)^{1/2} \mathrm{d}r \qquad \text{(Cauchy-Schwarz)} \\
\leq &c\int_{-T}^{T} \|e^{rA^*} g(r)\|_{H'} \,\mathrm{d}r \qquad \text{(from (H3))} \\
\leq &c'\int_{-T}^{T} \|g(r)\|_{H'} \,\mathrm{d}r \\
< &\infty,
\end{aligned}
$$

where c and c' are positive constants independent of g. □

Proof of Proposition 2.2. Let $u \in L^2(-T,T;U)$. From the factorization property of B^* (13), the Lemma A.5 and (H3), we have

$$
\begin{aligned}
&\Big|\int_{-T}^{T}\langle B^*\int_{0}^{t} e^{-(t-r)A^*} g(r)dr, u(t)\rangle_{U',U}\,\mathrm{d}t\Big| \\
=&\Big|\int_{-T}^{T}\int_{0}^{t}\langle B^* e^{-(t-r)A^*} g(r), u(t)\rangle_{U',U}\,\mathrm{d}r\,\mathrm{d}t\Big| \\
\leq&\Big|\int_{0}^{T}\int_{r}^{T}\langle B^* e^{-(t-r)A^*} g(r), u(t)\rangle_{U',U}\,\mathrm{d}t\,\mathrm{d}r\Big| \\
&\qquad + \Big|\int_{-T}^{0}\int_{-T}^{r}\langle B^* e^{-(t-r)A^*} g(r), u(t)\rangle_{U',U}\,\mathrm{d}t\,\mathrm{d}r\Big| \\
\leq&\int_{-T}^{T} c_1\|e^{rA^*} g(r)\|_{H'}\,\mathrm{d}r\|u\|_{L^2(-T,T;U)} \\
\leq&c\|g\|_{L^1(-T,T;H')}\|u\|_{L^2(-T,T;U)},
\end{aligned}
$$

where c_1 and c are positive constants independent of g and u.
Therefore, $B^*\int_0^t e^{-(t-r)A^*} g(r)\,\mathrm{d}r \in L^2(-T,T;U')$ and

$$
\|B^*\int_{0}^{t} e^{-(t-r)A^*} g(r)\,\mathrm{d}r\|_{L^2(-T,T;U')} \leq c\|g\|_{L^1(-T,T;\mathcal{D}(A^*))}.
$$

□

Proof of Proposition 2.7. We follow the proof given in [**5**].

(a) Let $0 \leq t \leq T$ and $y \in \mathcal{D}(A^*)$.

$$\begin{aligned}\langle z(t), A^*y\rangle_{H,H'} =& \langle \int_0^t e^{(t-s)A} Eu(s)\,\mathrm{d}s, A^*y\rangle_{H,H'} \\ =& \int_0^t \langle u(s), B^* e^{(t-s)A^*} y\rangle_{U,U'}\,\mathrm{d}s - \int_0^t \langle u(s), \bar{\lambda} E^* e^{(t-s)A^*} y\rangle_{U,U'}\,\mathrm{d}s\end{aligned}$$

because $B^* = E^*(A+\lambda I)^* = E^*A^* + \bar{\lambda}E^*$. Thus, the triangle and Cauchy-Schwarz inequalities imply that

$$\begin{aligned}|\langle z(t), A^*y\rangle_{H,H'}| \leq& \Big| \int_0^t \langle u(s), B^* e^{(t-s)A^*} y\rangle_{U,U'}\,\mathrm{d}s \Big| \\ &+ \Big| \int_0^t \langle u(s), \bar{\lambda} E^* e^{(t-s)A^*} y\rangle_{U,U'}\,\mathrm{d}s \Big| \\ \leq& \|u\|_{L^2(0,T;U)} \|B^* e^{-sA^*} e^{tA^*} y\|_{L^2(0,T;U')} \\ &+ \|u\|_{L^2(0,T;U)} \|\bar{\lambda} E^* e^{-sA^*} e^{tA^*} y\|_{L^2(0,T;U')} \\ \leq& k_1 \|u\|_{L^2(0,T;U)} \|y\|_{H'},\end{aligned}$$

k_1 being a positive constant, independent of $y \in H'$. In the last inequality, we used the direct inequality (H3) and the (classical) estimation of the norm of e^{tA^*} on a bounded interval. Hence, $z(t) \in \mathcal{D}(A)$.

(b) It results from (a) that for all $y \in \mathcal{D}(A^*)$,

$$|\langle Az(t), y\rangle_{H,H'}| \leq k_1 \|u\|_{L^2(0,T;U)} \|y\|_{H'}.$$

By density of $\mathcal{D}(A^*)$ in H', this estimation remains true for all $y \in H'$. Hence,

$$\|Az(t)\|_H \leq k_1 \|u\|_{L^2(0,T;U)}.$$

Moreover, there exists a positive constant k_2 independent of u, such that

$$\|\lambda I z(t)\|_H \leq k_2 \|u\|_{L^2(0,T;U)}.$$

Setting $k = k_1 + k_2$, we conclude that assertion (b) holds.

(c) The function $z(\cdot)$ is continuous on $[0,T]$. We prove that $Az(\cdot)$ is also continuous on this interval. At first, we assume that the input u is smooth, namely that $u \in \mathcal{C}^1([0,T];U)$. From (a), we know that $z(t) \in \mathcal{D}(A)$ for all $0 \leq t \leq T$. In particular,

$$Az(t) = \lim_{h\to 0} \frac{e^{hA} - I}{h} z(t) \qquad \text{in } H.$$

For $h \neq 0$,

$$\begin{aligned}
\frac{e^{hA}-I}{h} z(t) =& \frac{1}{h}\Big(\int_0^t e^{(t+h-s)A}Eu(s)\,\mathrm{d}s - \int_0^t e^{(t-s)A}Eu(s)\,\mathrm{d}s\Big)\\
=& \frac{1}{h}\Big(\int_{-h}^{t-h} e^{(t-s)A}Eu(s+h)\,\mathrm{d}s - \int_0^t e^{(t-s)A}Eu(s)\,\mathrm{d}s\Big)\\
=& \int_0^{t-h} e^{(t-s)A}E\frac{u(s+h)-u(s)}{h}\,\mathrm{d}s\\
&+ \frac{1}{h}\int_{-h}^{0} e^{(t-s)A}Eu(s+h)\,\mathrm{d}s - \frac{1}{h}\int_{t-h}^{t} e^{(t-s)A}Eu(s)\,\mathrm{d}s,
\end{aligned}$$

Now, we let h tend to zero. The first integral in the right hand side of the last equation equals

$$\int_0^t \mathbb{1}_{[0,t-h]}e^{(t-s)A)}E\frac{u(s+h)-u(s)}{h}\,\mathrm{d}s \longrightarrow \int_0^t e^{(t-s)A)}Eu'(s)\,\mathrm{d}s \quad \text{when } h \to 0,$$

Indeed, we can use the dominated convergence theorem, remarking that (for $s \leq t-h$)

$$\Big\|\frac{u(s+h)-u(s)}{h}\Big\|_U = \Big\|\frac{1}{h}\int_s^{s+h} u'(\sigma)\,\mathrm{d}\sigma\Big\|_U \leq \max_{0\leq s\leq T}\|u'(s)\|_U.$$

For the second integral,

$$\frac{1}{h}\int_{-h}^{0} e^{(t-s)A}Eu(s+h)\,\mathrm{d}s = \frac{1}{h}\int_0^h e^{(t-s+h)A}Eu(s)\,\mathrm{d}s = \int_0^1 e^{(t-hs+h)A}Eu(hs)\,\mathrm{d}s,$$

this expression tending to $e^{tA}Eu(0)$ in H as h tends to zero.
Similarly, for the third integral,

$$\frac{1}{h}\int_{t-h}^{t} e^{(t-s)A}Eu(s)\,\mathrm{d}s = \int_0^1 e^{-hsA}Eu(t-hs)\,\mathrm{d}s,$$

and this expression tends to $Eu(t)$ in H as h tends to zero.
Therefore,

$$Az(t) = \int_0^t e^{(t-s)A}Eu'(s)\,\mathrm{d}s + e^{tA}Eu(0) - Eu(t) \in \mathcal{C}([0,T];H).$$

Now, if $u \in L^2(0,T;U)$, then we can find a sequence of functions $u_n \in \mathcal{C}^1([0,T];U)$ tending to u in $L^2(0,T;U)$. Setting $z_n(t) := \int_0^t e^{(t-s)A}Eu_n(s)\,\mathrm{d}s$, thanks to (b) we have, for all $0 \leq t \leq T$,

$$\|(A+\lambda I)(z_n(t)-z(t))\|_H \leq K\|u_n-u\|_{L^2(0,T;U)}.$$

thus, $(A+\lambda I)z_n$ converges uniformly to $(A+\lambda I)z$ on $[0,T]$, proving the continuity of the latter function. □

Decay rate in finite-dimension

1. Stabilization of the harmonic oscillator

We prove on a simple finite-dimensional system that Slemrod's feedback law yields a decay rate of the solution that may be better than expected. More precisely, we prove that in the case of the harmonic oscillator, the decay rate is at least 2ω, where ω is a parameter inside the feedback operator.

The system that we want to stabilize is

$$\begin{cases} y''(t) + y(t) = u(t), \\ y(0) = y_0. \end{cases} \tag{102}$$

Setting

$$x(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}, \qquad x_0 = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the system (102) is equivalent to the first-order system

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \\ x(0) = x_0. \end{cases} \tag{103}$$

1.1. Construction of the feedback. Slemrod's feedback is defined by

$$F := -B^* \Lambda_\omega^{-1},$$

where

$$\Lambda_\omega := \int_0^T e^{-2\omega t} e^{-tA} B B^* e^{-tA^*} \, dt.$$

Let us compute F explicitly for the system (102). The matrix A is skew-adjoint and

$$e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Moreover, we have

$$e^{-tA}BB^*e^{-tA^*} = \begin{pmatrix} \sin^2 t & -\sin t\cos t \\ -\sin t\cos t & \cos^2 t \end{pmatrix}.$$

Setting

$$\begin{aligned} k &:= 2\omega, \\ a &:= \int_0^T e^{-kt}\sin^2 t dt, \\ b &:= \int_0^T e^{-kt}\cos^2 t dt, \\ c &:= \int_0^T e^{-kt}\sin t\cos t dt, \end{aligned}$$

we obtain

$$\Lambda_\omega = \begin{pmatrix} a & -c \\ -c & b \end{pmatrix}.$$

The pair (A, B) is controllable. Indeed (Kalman rank condition),

$$[B, AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \operatorname{rank}[B, AB] = 2.$$

Hence, the symmetric matrix Λ_ω is positive definite and therefore invertible. With the above notations, its inverse is given by

$$\Lambda_\omega^{-1} = \frac{1}{ab - c^2}\begin{pmatrix} b & c \\ c & a \end{pmatrix}.$$

Finally, the closed-loop operator is

$$A_F := A - BB^*\Lambda_\omega^{-1} = \begin{pmatrix} 0 & 1 \\ -1 - \frac{c}{ab-c^2} & -\frac{a}{ab-c^2} \end{pmatrix}.$$

We can compute the coefficients of the matrix Λ_ω :

$$\begin{aligned} a =& \frac{k^2e^{-kT}(\cos(2T) - 1) + 4(1 - e^{-kT}) - 2ke^{-kT}\sin(2T)}{2k(k^2+4)}; \\ b =& \frac{2k^2 + 4(1 - e^{-kT}) - k^2e^{-kT}(1 + \cos(2T)) + 2ke^{-kT}\sin(2T)}{2k(k^2+4)}; \\ c =& \frac{-ke^{-kT}\sin(2T) - 2e^{-kT}\cos(2T) + 2}{2(k^2+4)}. \end{aligned}$$

Moreover,

$$\det(\Lambda_\omega) = ab - c^2 = \frac{2(1-e^{-kT})^2 + k^2 e^{-kT}(\cos(2T)-1)}{2k^2(k^2+4)} > 0, \tag{104}$$

the matrix Λ_ω being positive definite.

1.2. Stability. In this paragraph, we focus on the decay rate of the closed-loop problem. Here, the problem being finite-dimensional, it is equivalent to estimate the real parts of the eigenvalues of A_F. More precisely, we would like to prove that each one of the two eigenvalues of A_F as a real part that is less than $-k = -2\omega$.

Proposition B.1. *Each eigenvalue of A_F has a real part less than or equal to*

$$-k = -2\omega.$$

Proof. This is equivalent to prove that each eigenvalue of the matrix $A_F + k\mathrm{Id}$ has a nonpositive real part.

$$A_F + k\mathrm{Id} = \begin{pmatrix} k & 1 \\ -1 - \frac{c}{ab-c^2} & -\frac{a}{ab-c^2} + k \end{pmatrix}$$

The product of the two eigenvalues equals $\det(A_F + k\mathrm{Id})$ and their sum equals $\mathrm{tr}(A_F + k\mathrm{Id})$. Hence, each eigenvalue of $A_F + k\mathrm{Id}$ has a nonpositive real part if and only if the two following inequalities are satisfied :

$$\mathrm{tr}(A_F + k\mathrm{Id}) \leq 0; \tag{105}$$

$$\det(A_F + k\mathrm{Id}) \geq 0. \tag{106}$$

Now, the inequality (105) is proved in Lemma B.2 and the inequality (106) is proved in Lemma B.3 below. □

Lemma B.2. $\mathrm{tr}(A_F + kId) \leq 0$.

Proof. With the above notations, we have

$$\begin{aligned} \mathrm{tr}(A_F + k\mathrm{Id}) \leq 0 &\iff 2k - \frac{a}{ab-c^2} \leq 0 \\ &\iff a - 2k(ab-c^2) \geq 0. \end{aligned}$$

and

$$a - 2k(ab-c^2) = \frac{e^{-kT}(k^2 + 4 - k^2\cos(2T) - 2k\sin(2T) - 4e^{-kT})}{2k(k^2+4)}.$$

It is necessary and sufficient to study the sign of the map

$$\varphi(T) := k^2 + 4 - k^2\cos(2T) - 2k\sin(2T) - 4e^{-kT}.$$

First step : when T is "small". Let us compute some derivatives of φ :

$$\begin{aligned}\varphi'(T) &= 2k^2\sin(2T) - 4k\cos(2T) + 4ke^{-kT};\\ \varphi''(T) &= 4k^2\cos(2T) + 8k\sin(2T) - 4k^2e^{-kT};\\ \varphi'''(T) &= -8k^2\sin(2T) + 16k\cos(2T) + 4k^3e^{-kT}\\ &= -8k\sqrt{k^2+4}\sin(2T - \arctan(2/k)) + 4k^3e^{-kT}.\end{aligned}$$

Therefore, $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$ and $\varphi'''(0) = 16k + 4k^3 > 0$, which implies that φ''' is nonnegative in a neighborhood of zero. Hence the same property holds for φ'', φ' and φ. Thanks to the second expression of φ''', we can be more precise : $\varphi''' \geq 0$ in the interval $\left[0, \frac{1}{2}\arctan(\frac{2}{k})\right]$. We conclude that

$$0 \leq T \leq \frac{1}{2}\arctan\left(\frac{2}{k}\right) \quad \Rightarrow \quad \varphi(T) \geq 0.$$

Second step : when T is "large". We can rewrite $\varphi(T)$ as

$$\varphi(T) = \sqrt{k^2+4}(\sqrt{k^2+4} - k\cos(2T - \arctan(2/k))) - 4e^{-kT}.$$

We recall that for all $x \geq 0$, $1 + x \leq e^x$ so that $\frac{1}{1+x} \geq e^{-x}$ and

$$\varphi(T) \geq \sqrt{k^2+4}(\sqrt{k^2+4} - k) - \frac{4}{1+kT} = k^2 + 4 - k\sqrt{k^2+4} - \frac{4}{1+kT}.$$

The right member is nonnegative if and only if

$$\begin{aligned} & \frac{4}{1+kT} \leq k^2 + 4 - k\sqrt{k^2+4}\\ \iff \quad & 1 + kT \geq \frac{4}{k^2 + 4 - k\sqrt{k^2+4}}\\ \iff \quad & kT \geq \frac{-k^2 + k\sqrt{k^2+4}}{k^2 + 4 - k\sqrt{k^2+4}}\\ \iff \quad & T \geq \frac{\sqrt{k^2+4} - k}{k^2 + 4 - k\sqrt{k^2+4}}\\ \iff \quad & T \geq \frac{1}{\sqrt{k^2+4}}.\end{aligned}$$

We have

$$T \geq \frac{1}{\sqrt{k^2+4}} \quad \Rightarrow \quad \varphi(T) \geq 0.$$

But $\frac{1}{\sqrt{k^2+4}} \leq \frac{1}{2}\arctan\left(\frac{2}{k}\right)$. Indeed, setting

$$f(k) := \frac{1}{2}\arctan\left(\frac{2}{k}\right) - \frac{1}{\sqrt{k^2+4}},$$

we have

$$\begin{aligned} f'(k) &= -\frac{1}{k^2}\frac{1}{1+\frac{4}{k^2}} + \frac{2k}{2(k^2+4)\sqrt{k^2+4}} \\ &= -\frac{1}{k^2+4} + \frac{k}{(k^2+4)\sqrt{k^2+4}} \\ &= \frac{1}{k^2+4}\left(\frac{k}{\sqrt{k^2+4}} - 1\right) \\ &< 0. \end{aligned}$$

Thus, f is strictly decreasing and f tends to zero as k tends to $+\infty$ so that $f > 0$. We conclude that

$$\forall T > 0, \quad \varphi(T) \geq 0. \qquad \square$$

Lemma B.3. $\det(A_F + kId) \geq 0$.

Proof.

$$\begin{aligned} &\det(A_F + k\mathrm{Id}) \geq 0 \\ \Longleftrightarrow\ & k^2 - \frac{ak}{ab-c^2} + 1 + \frac{c}{ab-c^2} \geq 0 \\ \Longleftrightarrow\ & \frac{k^2(ab-c^2) - ak + ab - c^2 + c}{ab-c^2} \geq 0 \\ \Longleftrightarrow\ & k^2(ab-c^2) - ak + ab - c^2 + c \geq 0 \end{aligned}$$

because $ab - c^2 > 0$. The last quantity equals

$$\frac{2k^2e^{-2kT} - k^2e^{-kT} + k^3e^{-kT}\sin(2T) - k^2e^{-kT}\cos(2T) + 2 + 2e^{-2kT} - 4e^{-kT}}{2k^2(k^2+4)}.$$

It is necessary and sufficient to prove that the numerator is nonnegative.

First step. Using the inequality (106), we have

$$\begin{aligned} &2k^2e^{-2kT} - k^2e^{-kT} + k^3e^{-kT}\sin(2T) - k^2e^{-kT}\cos(2T) + 2(1-e^{-kT})^2 \\ \geq& 2e^{-2kT}k^2 + k^3e^{-kT}\sin(2T) - 2k^2e^{-kT}\cos(2T) \\ =& k^2e^{-kT}\big(2e^{-kT} + k\sin(2T) - 2\cos(2T)\big). \end{aligned}$$

Setting

$$\psi(T) := 2e^{-kT} + k\sin(2T) - 2\cos(2T),$$

we have

$$\begin{aligned} \psi'(T) &= -2ke^{-kT} + 2k\cos(2T) + 4\sin(2T); \\ \psi''(T) &= 2k^2e^{-kT} - 4k\sin(2T) + 8\cos(2T). \end{aligned}$$

But $\psi(0) = \psi'(0) = 0$ and $\psi''(0) = 2k^2 + 8 > 0$, so that $\psi \geq 0$ in a neighborhood of 0. More precisely, rewriting $\psi''(T)$ as

$$\psi''(T) = 2k^2 e^{-kT} + 4\sqrt{4+k^2}\cos(2T + \arctan(k/2)),$$

if $2T + \arctan(k/2) \leq \pi/2$ that is $T \leq (1/2)\arctan(2/k)$, then ψ'' and ψ are nonnegative. Conclusion :

$$T \leq \frac{1}{2}\arctan\left(\frac{2}{k}\right) \quad \Rightarrow \quad \det(A_F + k\mathrm{Id}) \geq 0$$

Second step. With the notations of the first step, we have

$$\begin{aligned} \psi(T) &:= 2e^{-kT} + k\sin(2T) - 2\cos(2T) \\ &= 2e^{-kT} - \sqrt{k^2+4}\cos(2T + \arctan(\frac{k}{2})). \end{aligned}$$

If $\frac{\pi}{2} \leq 2T + \arctan(\frac{k}{2}) \leq \frac{3\pi}{2}$, then the cosine in the last expression is nonpositive and $\psi(T) \geq 0$. But

$$\begin{aligned} & \frac{\pi}{2} \leq 2T + \arctan(\frac{k}{2}) \leq \frac{3\pi}{2} \\ \Longleftrightarrow\ & \frac{\pi}{2} - \arctan(\frac{k}{2}) \leq 2T \leq \pi + \frac{\pi}{2} - \arctan(\frac{k}{2}) \\ \Longleftrightarrow\ & \arctan(\frac{2}{k}) \leq 2T \leq \pi + \arctan(\frac{2}{k}) \\ \Longleftrightarrow\ & \frac{1}{2}\arctan(\frac{2}{k}) \leq T \leq \frac{1}{2}\arctan(\frac{2}{k}) + \frac{\pi}{2}. \end{aligned}$$

Conclusion :

$$\frac{1}{2}\arctan(\frac{2}{k}) \leq T \leq \frac{1}{2}\arctan(\frac{2}{k}) + \frac{\pi}{2} \quad \Rightarrow \quad \det(A_F + k\mathrm{Id}) \geq 0$$

Third step. In the first and second steps, we have seen that $\det(A_F + k\mathrm{Id}) \geq 0$ if $0 < T \leq \frac{1}{2}\arctan(\frac{2}{k}) + \frac{\pi}{2}$ and in particular that $\det(A_F + k\mathrm{Id}) \geq 0$ if $0 < T \leq \frac{3}{2}$. In this last step, we prove the inequality on the determinant for $T \geq \frac{3}{2}$. We want to prove that the quantity

$$\xi(T) := 2k^2e^{-2kT} - k^2e^{-kT} + k^3e^{-kT}\sin(2T) - k^2e^{-kT}\cos(2T) + 2(1-e^{-kT})^2$$

is nonnegative. We rewrite it as

$$\begin{aligned} \xi(T) &= 2k^2e^{-2kT} - k^2e^{-kT} + 2(1-e^{-kT})^2 - k^2e^{-kT}\sqrt{k^2+1}\cos(2T + \arctan(k)) \\ &\geq 2k^2e^{-2kT} - k^2e^{-kT} + 2(1-e^{-kT})^2 - k^2e^{-kT}\sqrt{k^2+1}. \end{aligned}$$

But [1]

$$\begin{aligned}
&2k^2e^{-2kT}-k^2e^{-kT}+2(1-e^{-kT})^2-k^2e^{-kT}\sqrt{k^2+1}\\
=&e^{-kT}(2k^2e^{-kT}-k^2-k^2\sqrt{k^2+1}+2e^{kT}+2e^{-kT}-4)\\
=&e^{-kT}(2k^2e^{-kT}+4\mathrm{ch}(kT)-4-k^2-k^2\sqrt{k^2+1})\\
\geq&e^{-kT}(4+4\frac{k^2T^2}{2}+4\frac{k^4T^4}{24}-4-k^2-k^3-k^2)\\
=&e^{-kT}(2k^2(T^2-1)+k^4\frac{T^4}{6}-k^3)\\
=&k^2e^{-kT}(2(T^2-1)+k^2\frac{T^4}{6}-k)\\
=&k^2e^{-kT}(2(T^2-1)+k(\frac{kT^4}{6}-1))
\end{aligned}$$

If $T\geq 1.5$, then ($1.5^2=2.25>2$, $1.5^4\simeq 5.06>5$) the right member is greater than

$$k^2e^{-kT}(2+k(k\frac{5}{6}-1))\geq 0$$

Indeed, $k|k\frac{5}{6}-1|\leq\frac{6}{5}<2$ if $k\leq\frac{6}{5}$. Otherwise, $k(k\frac{6}{5}-1)\geq 0$. Conclusion :

$$T\geq\frac{3}{2}\quad\Rightarrow\quad\det(A_F+k\mathrm{Id})\geq 0. \qquad\square$$

2. Some numerical simulations

In section 3 of chapter 4, we have seen that choosing $T=k\pi$ ($k\in\mathbb{N}^*$) in the construction of Slemrod's feedback law for the harmonic oscillator (102), the decay rate of the solutions is exactly

$$\frac{2\omega}{1-e^{-2\omega k\pi}}\qquad\big(:=-\max\{\mathrm{Re}(x),x\text{ eigenvalue of }A_F\}\big),$$

the latter quantity being strictly greater than ω. This is consistent with the result of the previous paragraph.

Let us represent (see Figure 1 and Figure 2) for two different values of ω, thanks to numerical simulations, the variations of the maximum of the real parts of the eigenvalues of A_F with respect to T (red graph) and the function $T\mapsto(2\omega)/(1-e^{-2\omega T})$ (green graph).

For the last simulation, the coefficients of an antisymmetric matrix A of size 5×5 have been picked (by Matlab) in the interval $[-100,100]$, as for the coefficients of a vector B of size 5×1. Then, we have checked through the rank condition that the pair (A,B) is controllable and built an approximation of the Gramian operator Λ_ω for different values of T and a fixed $\omega=2$. The Figure 3 represents the variations of the maximum of the real parts of the eigenvalues

[1] $\forall a,b,\geq 0,\quad\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$ and $\forall x\in\mathbb{R},\quad\mathrm{ch}(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots$

of A_F with respect to T. Again, we see that the represented quantity is always lower (resp. the decay rate is always bigger) than $-2\omega = -4$ (resp. 2ω).

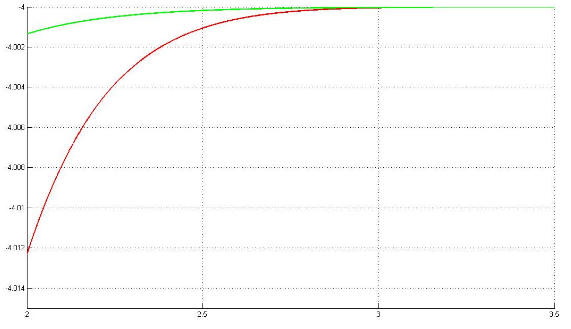

Figure 1. Harmonic oscillator, $\omega = 2$.

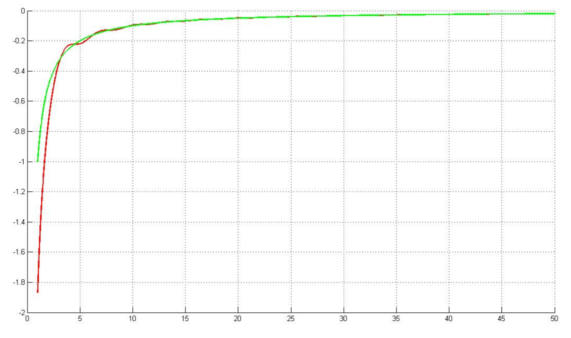

Figure 2. Harmonic oscillator, $\omega = 10^{-6}$.

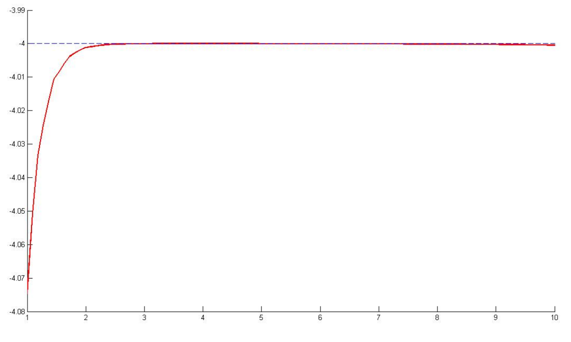

Figure 3. $A =$ antisymmetric 5×5 matrix, $\omega = 2$.

Hausdorff dimension

In this paragraph, we recall the definition of the Hausdorff dimension of a set and some of its main properties. We follow the book by Falconer [**22**] (see also [**9**, pp. 90–93] and [**34**, pp. 204–205]).

Let F be a subset of $\mathbb{R}^n$ and s be a non-negative number. For any $\delta > 0$, we define

$$\mathcal{H}^s_\delta(F) := \inf \sum_{i=1}^{\infty} (\text{diam } U_i)^s, \tag{107}$$

where the infimum is taken over all the countable δ-covers $\{U_i\}$ of F, i.e. [1]

$$F \subset \bigcup_{i=1}^{\infty} U_i \qquad \text{and} \qquad \forall i, \quad 0 < \text{diam } U_i \leq \delta,$$

From the above definition, $\mathcal{H}^s_\delta(F)$ increases as δ tends to zero. Therefore, we define a new quantity, called the *s-dimensional Hausdorff measure of F*, by setting

$$\mathcal{H}^s(F) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(F) \in [0, +\infty]. \tag{108}$$

From (107), if $\delta < 1$, then $\mathcal{H}^s_\delta(F)$ is decreasing with respect to s. Thus, from (108), $\mathcal{H}^s(F)$ is also decreasing with respect to s. Moreover, it is possible

[1] $\text{diam } U_i := \sup\{\|x - y\|,\ x, y \in U_i\}$ is the diameter of U_i.

to prove [1] that there is a critical value s_0 such that (see Figure 1)

$$\forall s < s_0, \quad \mathcal{H}^s(F) = +\infty \qquad \text{and} \qquad \forall s > s_0, \quad \mathcal{H}^s(F) = 0.$$

This value s_0 is called the *Hausdorff dimension of* F:

$$\dim_H F := \sup\{s : \mathcal{H}^s(F) = +\infty\} = \inf\{s : \mathcal{H}^s(F) = 0\}.$$

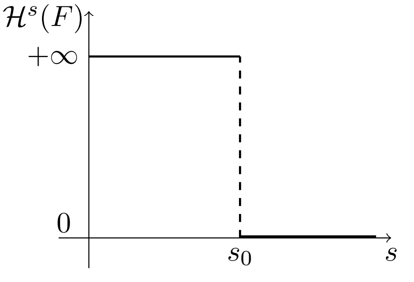

Figure 1. The critical value s_0

Next, we gather some useful properties of this quantity. Let $F, F_1, F_2, \ldots$ be (measurable if necessary) subsets of $\mathbb{R}^n$ and G be a subset of $\mathbb{R}^m$.

(a) $\dim_H F \le n$;

(b) $\lambda(F) > 0 \Rightarrow \dim_H F = n$ (λ denoting the Lebesgue measure in $\mathbb{R}^n$);

(c) $F_1 \subset F_2 \Rightarrow \dim_H F_1 \le \dim_H F_2$;

(d) $\dim_H \bigcup_{i=1}^{\infty} F_j = \sup_{j\ge 1} \dim_H F_j$;

(e) If F is finite or countable, then $\dim_H F = 0$;

(f) In general, $\dim_H F \times G \ge \dim_H F + \dim_H G$. If G is sufficiently "regular" (for example, G is a smooth submanifold of $\mathbb{R}^m$), then $\dim_H F \times G = \dim_H F + \dim_H G$;

(g) If $f : F \to \mathbb{R}^m$ is a bi-Lipschitz transformation i.e. for $x, y \in F$, $c_1|x-y| \le |f(x)-f(y)| \le c_2|x-y|$ $(0 \le c_1 \le c_2)$, then $\dim_H f(F) = \dim_H F$.

[1] If $t > s$ and $\{U_i\}$ is a δ-cover of F, then

$$\sum_{i=1}^{\infty} \frac{(\text{diam } U_i)^t}{\delta^t} \le \sum_{i=1}^{\infty} \frac{(\text{diam } U_i)^s}{\delta^s} \quad \Rightarrow \quad \sum_{i=1}^{\infty} (\text{diam } U_i)^t \le \delta^{t-s} \sum_{i=1}^{\infty} (\text{diam } U_i)^s.$$

Hence,

$$\mathcal{H}^t_\delta(F) \le \delta^{t-s} \mathcal{H}^s_\delta(F)$$

and letting $\delta \to 0$, we observe that if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$ for $t > s$.

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Ce travail est constitué de deux parties indépendantes traitant chacune d'un problème issu de la théorie du contrôle des équations aux dérivées partielles. La première partie est consacrée à l'étude d'un feedback explicite et déjà connu, s'appliquant à des systèmes linéaires, réversibles en temps et éventuellement munis d'un opérateur de contrôle non-borné. On justifie le caractère bien posé du problème en boucle fermée via la théorie des semi-groupes puis on étudie le taux de décroissance des solutions du système régulé. La seconde partie concerne un problème d'observation pour la corde vibrante : on détermine comment choisir des instants d'observation pour que la position de la corde à ces instants permette de retrouver les conditions initiales tout en préservant une certaine régularité. La méthode, qui repose sur des résultats de théorie des nombres, est ensuite étendue à d'autres systèmes. En utilisant une méthode de dualité on démontre aussi un résultat de contrôlabilité exacte.

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