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Deux aspects de la géométrie birationnelle des variétés algébriques : la formule du fibré canonique et la décomposition de Zariski

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DEUX ASPECTS DE LA GÉOMÉTRIE BIRATIONNELLE DES VARIÉTÉS ALGÉBRIQUES : LA FORMULE DU FIBRÉ CANONIQUE ET LA DÉCOMPOSITION DE ZARISKI

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La risposta è dentro di te, epperò è sbagliata.
(C. Guzzanti)

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Introduction

Ce travail de thèse se divise en deux parties. Le premier chapitre est consacré à la partie modulaire dans la formule du fibré canonique. Les résultats originaux qui y sont inclus proviennent de [11] et [12]. Plus précisément, les Sections 1.2 et 1.4 contiennent définitions et résultats généraux sur la formule du fibré canonique. Les Sections 1.3, 1.5, 1.6 et 1.7 coïncident avec les Sections 4,3,5 et 6 de [12]. Subsection 1.8.1, 1.8.2 et 1.8.3 sont exactement Subsection 2.2 et Sections 3 et 4 de [11]. Le deuxième chapitre concerne la décomposition de Zariski sur les variétés lisses de dimension 3.

La formule du fibré canonique

La formule du fibré canonique est un outil important en géométrie algébrique complexe. Elle a été developpée et améliorée suivant les utilisations qui en ont été faites. Dans son article *On compact analytic surfaces: II* [28] de 1963, K. Kodaira s'intéresse à la classification des surfaces algébriques : il y étudie certaines surfaces algébriques S dotées d'une fibration $f\colon S \to C$ dont la base C est une courbe lisse et dont la fibre générique est une courbe elliptique. Le fibré canonique d'une telle surface est donc trivial sur les fibres de f. De plus, l'application f a un nombre fini de fibres singulières. Kodaira classifie les fibres singulières et démontre, avec Ueno [43], que, si f n'a pas de fibres multiples, alors

$$12K_{S/C} \sim f^*(R + j^*\mathcal{O}(1)),$$

où R est un diviseur sur C ayant comme support le lieu singulier de f et dont les coefficients ont une interprétation en termes de la classification des fibres singulières. L'application j est l'application induite par f sur l'espace de modules des courbes elliptiques. Dans son article *Zariski decomposition and canonical rings of elliptic threefolds* [21] Fujita généralise la formule au cas où f peut avoir des fibres multiples. Kodaira était intéressé par l'étude des surfaces, tandis que Ueno et Fujita consacrent leur attention à des variétés de dimension trois dotées d'une fibration $f\colon X \to Z$ dont les fibres sont des courbes elliptiques. La méthode pour étudier de telles variétés est la même : exprimer le fibré canonique comme tiré en arrière d'une somme de diviseurs sur la base ayant des propriétés spécifiques.

La classification birationnelle des variétés est un des problèmes dont la solution est le but ultime de la géometrie algébrique complexe. Fujita et Ueno suivaient la voie indiquée par Iitaka et qui fait toujours partie de la stratégie pour résoudre le problème. L'idée de Iitaka était de regarder les systèmes pluricanoniques $|mK_X|$ et l'ordre de croissance de leur dimension quand m tend vers l'infini. Cet ordre de croissance est appelé *dimension de Kodaira* et a été introduit par Iitaka dans [22]. La dimension de Kodaira d'une variété X peut prendre les valeurs $\{-\infty, 0, \ldots, \dim X\}$. Iitaka démontre que, si la dimension de Kodaira de X est positive, alors il existe un modèle birationnel X^* de X qui admet une fibration $f\colon X^* \to Z$ telle que la dimension de Z est la dimension de Kodaira de X et la fibre générale de f a dimension de Kodaira zéro (cf. [32, Definition 2.1.36]).

Donc l'étude des variétés algébriques peut être divisée en

1. une étude des variétés X de dimension de Kodaira $-\infty, 0$ ou $\dim X$;
2. une étude des variétés qui sont dotées d'une fibration dont la fibre générale a dimension de Kodaira zéro.

Les techniques qu'on utilise en géométrie birationnelle et le besoin de considérer aussi des variétés singulières ou quasi-projectives ont déterminé le passage de variétés à *paires* (X, B), qui sont la donnée d'une variété normale X et d'un $\mathbb{R}$-diviseur B tels que $K_X + B$ est un diviseur $\mathbb{R}$-Cartier. La première distinction parmi les paires est faite par l'ordre de croissance des sections de $\lfloor m(K_X + B) \rfloor$, la *log-dimension de Kodaira*. On dispose aussi de notions de régularité pour les paires, comme *lc* et *klt*, pour la définition desquelles on renvoie à la Definition 1.2.4.
Si on a une fibration $f \colon X \to Z$ comme dans le point 2. à fibre générale de dimension de Kodaira 0, alors il existent un diviseur $\mathbb{Q}$-Cartier D sur Z, un diviseur $\mathbb{Q}$-Cartier E sur X et un nombre entier $r > 0$ tels que

$$rK_X = f^*D + E$$

(cf. [18]). Si on pose $B = -E/r$, alors

$$f \colon (X, B) \to Z$$

est un *fibration lc-triviale*, c'est-à-dire la donnée d'une paire lc (X, B) et d'une fibration f telles que

$$K_X + B \sim_{\mathbb{Q}} f^*D$$

et qui satisfait certaines hypothèses techniques (cf. Definition 1.2.8). L'outil principal pour étudier les fibrations lc-triviales est la formule du fibré canonique.

La formule du fibré canonique a atteint sa formulation actuelle avec Kawamata qui l'utilise pour traiter le *problème de sous-adjonction*. Il est bien connu que, si $Y \subseteq X$ est une hypersurface lisse, alors

$$K_Y \sim (K_X + Y)|_Y$$

et cette formule est appelée *formule d'adjonction*. Le *problème de sous-adjonction* consiste à trouver une formule similaire pour Y singulier ou de codimension plus grande que 1. La formule de sous-adjonction est un ingrédient fondamental dans les preuves par récurrence sur la dimension. Kawamata démontre dans [27] qu'une telle formule existe quitte à rajouter à $K_X + B$ un diviseur ample arbitrairement petit. Un outil fondamental pour ce résultat est une formule du fibré canonique écrite d'une façon significative du point de vue des singularités de la paire (X, B) et de la fibration f. Kawamata observe que, étant donné

$$f \colon (X, B) \to Z$$

telle que $(K_X + B)|_F \sim_{\mathbb{Q}} 0$, où F est la fibre générale de f, on peut écrire $K_X + B$ comme tiré en arrière

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z).$$

Le diviseur B_Z est appelé *discriminant* et il est défini par $B_Z = \sum(1 - \gamma_P)P$ où P varie parmi les diviseurs premiers de Z et

$$\gamma_P = \sup\{t \in \mathbb{R} |\, (X, B + tf^*(P)) \text{ est lc sur } P\}.$$

Pour une définition précise de lc sur une sous-variété de Z on renvoie à la Definition 1.2.5. Le *seuil log canonique* γ_P est une mesure de la singularité de la paire $(f^{-1}P, B|_{f^{-1}P})$. Il vaut 1 si la paire est lc, il est plus petit si elle est plus singulière. Donc SuppB_Z peut être interprété comme le lieu singulier de f vue comme une application *d'une paire* vers une variété. Dans la formule de Kodaira, le diviseur B_Z coïncide avec R. Le diviseur M_Z, appelé *partie modulaire*, est $\mathbb{Q}$-Cartier et nef sur un modèle birationnel de Z par [27, Theorem 2]. Il contient des informations sur la variation birationnelle des fibres. D'autres résultats importants dans la théorie et ses applications ont été démontrés autour des années 2000 avec les travaux de F. Ambro et avec l'article [18] de O. Fujino et S. Mori. Le premier auteur traite le problème de la sous-adjontion et démontre [1, Proposition 3.4] que les singularités de la paire (Z, B_Z) sont les mêmes que celles de (X, B). Si (X, B) est klt (cf. Definition 1.2.4), il prouve [3, Theorem 0.2] qu'il y a un diviseur Δ_Z sur Z tel que (Z, Δ_Z) est klt et $K_X + B \sim_{\mathbb{Q}} f^*(K_Z + \Delta_Z)$. Dans [18] les auteurs démontrent le résultat suivant.

Théorème 0.0.1 (Theorem 5.2 [18])**.** *Soit (X, Δ) une paire klt avec $\kappa(X, \Delta) = l \geq 0$. Alors il existe une paire klt (C', Δ') de dimension l et avec $\kappa(C', \Delta') = l$, deux $e', e \in \mathbb{Z}_{>0}$ et un isomorphisme d'anneaux gradués*

$$R(X, K_X + \Delta)^{(e)} \cong R(C', K_{C'} + \Delta')^{(e')}$$

où

$$R(X, K_X + \Delta)^{(e)} = \bigoplus_{m \in \mathbb{Z}} H^0(X, \lfloor meK_X + \Delta \rfloor).$$

De nos jours, le problème qui guide la recherche sur la formule du fibré canonique est la conjecture suivante.

EbS(k) 0.0.1 (Effective b-Semiampleness, Semiamplitude birationnelle effective, Conjecture 7.13.3, [38])**.** *Il existe un nombre entier $m = m(d, r)$ tel que pour toute fibration lc-triviale $f\colon (X, B) \to Z$ avec d la dimension de la fibre générale, k la dimension de Z et r l'indice de Cartier de la fibre générale $(F, B|_F)$ il existe un morphisme birationnel $\nu\colon Z' \to Z$ tel que $mM_{Z'}$ est sans point base.*

Un diviseur de Cartier D sur une variété Z est *birationnellement semiample* ou *b-semiample*, s'il existe un morphisme birationnel $\mu\colon Z' \to Z$ et un diviseur semiample D' sur Z' tel que $\mu_* D' = D$. On a des réponses partielles à la conjecture **EbS**, notamment s'il existe un espace de modules pour les fibres de f (cf. [38, Theorem 8.1], [15] et [16]). Mais même la version faible, où l'on conjecture juste la "b-semiamplitude" sans aucune condition sur l'entier qui rend la partie modulaire sans point base, n'a de solution que lorsque M_Z est numériquement triviale ([3, Theorem 3.5] et [12, Theorem 1.3]). La difficulté de la conjecture **EbS** est bien illustrée par un résultat de X. Jiang, qui a démontré en [24] que la conjecture **EbS** implique un énoncé d'uniformité de la fibration de Iitaka pour *toute* variété avec dimension de Kodaira positive sous l'hypothèse que les fibres ont un bon modèle minimal.

Dans ce travail de thèse on démontre plusieurs résultats relatifs à la conjecture **EbS**. Le premier concerne le cas où les fibres sont de dimension $d = 1$ et donne une description des dénominateurs de M_Z.

Théorème 0.0.2 (Theorem 1.6, [11])**.** *1. Il ne peut pas exister une borne polynomiale en r sur les dénominateurs de M_Z. Précisément, pour tout N, il existe une fibration lc-triviale*

$$f\colon (X, B) \to Z$$

telle que $\dim Z = 1$ *et pour laquelle* $V \geq r^{N+1}$ *pour tout* V *tel que* VM_Z *à coefficients entiers.*

2. *Soit* $f\colon (X,B) \to Z$ *une fibration lc-triviale dont la fibre générale est une courbe rationnelle. Alors il existe un entier* $N(r)$ *qui ne dépend que de* r*, tel que* $N(r)M_Z$ *à coefficients entiers. Plus précisément* $N(r) = r\,\mathrm{ppcm}\{l \mid l \leq 2r\}$.

Théorème 0.0.2 coïncide avec Theorem 1.1.3 et il est démontré en Subsection 1.8.3.
En outre on donne, pour tout r impair, un exemple de fibration lc-triviale tel que si V est le plus petit entier naturel pour lequel VM_Z à coefficients entiers, alors $V = N(r)/r$. Dans [38, Remark 8.2] les auteurs conjecturaient que l'entier qui rend la partie modulaire sans point base est $12r$. Le théorème 0.0.2 donne un contrexemple à cette conjecture. D'autre part, les dénominateurs de M_Z sont importants dans les applications quand il faut obtenir des résultats effectifs pour les applications pluricanoniques (cf. [41], [42]).

On démontre aussi le théorème suivant qui affirme que la conjecture **EbS** peut être réduite au cas où la base est une courbe.

Théorème 0.0.3 (Theorem 1.2, [12])**.** ***EbS(1)*** *implique* ***EbS(k)****.*

Théorème 0.0.3 est Theorem 1.1.4 et sa preuve se trouve dans Section 1.5.
Une approche par récurrence sur la dimension de la base comme celle de théorème 0.0.3 donne aussi un résultat de semiamplitude effective dans le cas $M_Z \equiv 0$. En effet, on démontre une version effective de [3, Theorem 3.5].

Théorème 0.0.4 (Theorem 1.3, [12])**.** *Il existe un entier* $m = m(b)$ *tel que pour toute fibration klt-triviale* $f\colon (X,B) \to Z$ *avec*

- Z *lisse ;*
- $M_Z \equiv 0$ *;*
- $\mathrm{Betti}_{\dim E'}(E') = b$ *où* E' *est un modèle lisse du revêtement* $E \to F$ *associé à l'unique élément de* $|r(K_F + B|_F)|$

on a $mM_Z \sim 0$.

Si la paire (X,B) est lc mais non klt sur le point générique de la base on démontre le théorème suivant qui généralise [3, Theorem 3.5].

Théorème 0.0.5 (Theorem 1.4, [12])**.** *Soit* $f\colon (X,B) \to Z$ *une fibration lc-triviale avec* Z *lisse et* $M_Z \equiv 0$*. Alors* M_Z *est un diviseur de torsion.*

Théorème 0.0.4 et théorème 0.0.5 sont Theorem 1.1.5 and Theorem 1.1.6 et sont démontrés en Subsection 1.7.1 et Subsection 1.7.2 respectivement.

La décomposition de Zariski

Soit S une surface projective lisse définie sur $\mathbb{C}$. Soit D un diviseur effectif sur S. En 1962, O. Zariski démontre (cf. [46]) l'existence de diviseurs P, N tels que

1. $N = \sum a_i N_i$ est effectif, P est nef et $D = P + N$;
2. soit $N = 0$, soit la matrice $(N_i \cdot N_j)$ est définie négative ;
3. $(P \cdot N_i) = 0$ pour tout i.

Une telle décomposition est unique et est appelée la *décomposition de Zariski de D*.
Fujita en [20] généralise l'énoncé aux diviseurs pseudoeffectifs. De plus, il remarque en [21] que le diviseur P est l'unique diviseur qui satisfait la propriété suivante :

(α) pour tout modèle birationnel $f\colon X' \to X$ et pour tout diviseur nef L sur X' tel que $f_*L \leq D$ on a $f_*L \leq P$.

À cause de l'importance de la décomposition de Zariski sur les surfaces, plusieurs généralisations aux variétés de dimension supérieure ont été étudiées. La propriété (α) donne lieu à la généralisation suivante.

Définition 0.0.2 (Definition 6.1, [39]). *Soit X une variété projective lisse et D un diviseur pseudoeffectif. Une décomposition $D = P_f + N_f$ est appelée une décomposition de Zariski au sens de Fujita (où décomposition de Fujita-Zariski) si*

1. $N_f \geq 0$ *;*
2. P_f *est nef ;*
3. *pour tout modèle birationnel $\mu\colon X' \to X$ et pour tout diviseur nef L sur X' tel que $\mu_*L \leq D$ on a $\mu_*L \leq P_f$.*

Il découle de la définition que, s'il existe une décomposition de Fujita-Zariski, alors elle est unique. L'importance de la décomposition de Fujita-Zariski est bien mise en évidence par les resultats de Birkar [4] et Birkar-Hu [5] qui ont démontré l'équivalence entre l'existence des modèles log minimaux des paires et l'existence de la décomposition de Fujita-Zariski pour les diviseurs log canoniques.

Dans [35] Nakayama démontre plusieurs résultats partiels, concernant la décomposition de Zariski en dimension 3, qui mettent en relation l'existence d'une décomposition *sur une courbe* Σ (cf. [35, III.4] pour une définition complète) avec les propriétés de stabilité du fibré conormal de Σ. Plus précisément, soit D un diviseur pseudoeffectif sur X et Σ une courbe lisse telle que $D \cdot \Sigma < 0$. Soit I_Σ l'idéal de Σ dans X. Si le fibré conormal I_Σ/I_Σ^2 est semistable, alors il y a une décomposition $\varphi^*D = P + N$ telle que $N \geq 0$ et le diviseur P a intersection positive avec toute courbe incluse dans le diviseur exceptionnel de

$$\varphi\colon \mathrm{Bl}_\Sigma X \to X$$

(cf. [35, Lemma III.4.5]). Si I_Σ/I_Σ^2 est instable alors il existe une suite exacte courte (cf. Lemma 2.2.10)

$$0 \to \mathcal{L} \to I_\Sigma/I_\Sigma^2 \to \mathcal{M} \to 0$$

telle que $\deg \mathcal{L} > \deg \mathcal{M}$. Par [35, Lemma III.4.6], si le fibré conormal n'est pas "trop instable", notamment si $2 \deg \mathcal{M} \geq \deg \mathcal{L}$, alors il existe un modèle birationnel $\varphi\colon X' \to X$ tel que φ^*D a une décomposition de Fujita-Zariski sur Σ. Le théorème suivant pourrait constituer une étape technique vers un résultat d'existence de la décomposition de Zariski en dimension 3.

Théorème 0.0.6. *Soit X une variété projective lisse de dimension 3. Soit $\Sigma \subseteq X$ une courbe et supposons que le fibré conormal*

$$I_\Sigma/I_\Sigma^2$$

n'est pas semistable comme fibré vectoriel de rang 2 sur Σ. Alors il existe une suite d'éclatements $\varphi\colon \hat{X} \to X$ de courbes lisses non contenues dans Σ telle que, si $\hat{\Sigma}$ est la transformée stricte de Σ dans $\hat{X}$, alors $I_{\hat{\Sigma}}/I_{\hat{\Sigma}}^2$ est semistable.

En réalité, on démontre un énoncé qui est beaucoup plus précis que celui du théorème 0.0.6, c'est-à-dire théorème 2.3.3, qui affirme que l'on peut aussi contrôler le degré du fibré conormal.

Chapter 1

The moduli part in the canonical bundle formula

1.1 Introduction

The canonical bundle formula is an important tool in complex algebraic geometry. It was developed and refined through time following its uses and applications. In one of his fundamental papers on the classification of compact complex surfaces K. Kodaira studied algebraic surfaces S endowed with a morphism $f\colon S \to C$ whose base C is a smooth curve and whose generic fiber is an elliptic curve. The canonical bundle of S is thus trivial on the fibers of f. The map f has a finite number of singular fibers which are classified by Kodaira. If there are no multiple fibers, Kodaira and Ueno [43] proved that

$$12K_{S/C} \sim f^*(R + j^*\mathcal{O}(1))$$

where R is a divisor on C supported on the singular locus of f and whose coefficients have an interpretation in terms of the classification of the singular fibers. The map j is the map induced by f on the moduli space of elliptic curves. Fujita generalized in [21] the formula to the case where f can have multiple fibers. Although Kodaira was interested in studying surfaces, whereas Ueno and Fujita focused their attention to some special threefolds, their method for studying these varieties is the same: find a fiber space structure for which the canonical bundle of the ambient space is trivial on the fibers.

The birational classification of complex algebraic varieties is one of the big problems whose solution is the ultimate goal of complex algebraic geometry. Fujita and Ueno followed the philosophy outlined by Iitaka which is still nowadays part of the main strategy for solving the problem. Iitaka's idea was to look at the pluricanonical systems $|mK_X|$ and the order of growth of their dimensions when m goes to infinity. The invariant that carries the information of this order of growth is called *Kodaira dimension* and was introduced by Iitaka in [22]. The Kodaira dimension of a variety X can take the values $\{-\infty, 0, \ldots, \dim X\}$. Iitaka proved that, if the Kodaira dimension of X is non-negative, then there exists a birational model X^* of X that has a structure of fiber space $f\colon X^* \to Z$ such that the dimension of Z is the Kodaira dimension of X and the general fiber has Kodaira dimension zero (see e.g. [32, Definition 2.1.36]). Thus the study of algebraic varieties is roughly divided into two parts:

1. the study of varieties of Kodaira dimension $-\infty, 0$ or equal to the dimension;

2. the study of varieties with a structure of fiber space and whose generic fibers have Kodaira dimension zero.

The techniques used in birational geometry and the need to consider singular or quasi projective varieties led to enlarge the category of varieties in order to consider pairs. A pair (X, B) is the data of a normal variety X and an $\mathbb{R}$-divisor B such that $K_X + B$ is an $\mathbb{R}$-Cartier divisor. Thus the first distinction between pairs is the order of growth of the sections of $\lfloor m(K_X + B)\rfloor$, the *log-Kodaira dimension*. There are also notions of regularity for pairs, such as *lc* or *klt*, for which we refer to Definition 1.2.4.

If we have a fibration $f\colon X \to Z$ as in point 2. whose general fiber has Kodaira dimension 0, then there exist a $\mathbb{Q}$-Cartier divisor D on Z, a $\mathbb{Q}$-Cartier divisor E on X and an integer $r > 0$ such that

$$rK_X = f^*D + E.$$

If we set $B = -E/r$ then $f\colon (X, B) \to Z$ is an *lc-trivial fibration*, that is, the data of an lc pair (X, B) and of a fibration f such that

$$K_X + B \sim_{\mathbb{Q}} f^*D$$

satisfying some technical hypotheses (cf. Definition 1.2.8).
The main tool for studying lc-trivial fibrations is the canonical bundle formula.

The canonical bundle formula as it stands is due to Kawamata who used it in order to treat the *subadjunction problem*. It is well known that, if $Y \subseteq X$ is a smooth hypersurface, then

$$K_Y \sim (K_X + Y)|_Y$$

and the formula is called *adjunction formula*. The *subadjunction problem* consists in finding a similar formula in the case where Y is singular or has codimension greater than one. The subadjunction is a key tool whenever one tries to argue by induction on the dimension. Kawamata proved in [27] the existence of such formula up to adding to $K_X + B$ a small ample divisor. A fundamental technique for this result is the canonical bundle formula written in a form that is meaningful in view of the singularities of the pair (X, B) and the fibration f. Indeed, let

$$f\colon (X, B) \to Z$$

be such that $(K_X + B)|_F \sim_{\mathbb{Q}} 0$. Kawamata observed that

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z),$$

where the divisor B_Z is called the *discriminant* and it is defined by $B_Z = \sum(1 - \gamma_P)P$, the sum being taken over the prime divisors P of X and

$$\gamma_P = \sup\{t \in \mathbb{R} |\, (X, B + tf^*(P)) \text{ is lc over } P\}.$$

For the precise definition of lc over P see Definition 1.2.5. The *log canonical threshold* γ_P is a measure of the singularity of the pair $(f^{-1}P, B|_{f^{-1}P})$. It is 1 when the pair is lc, it smaller when it is more singular. Thus SuppB_Z can be roughly interpreted as the singular locus of f seen as a map whose source is a pair. In the case of Kodaira's formula, the divisor B_Z coincides with R. The divisor M_Z, called the *moduli part*, is a $\mathbb{Q}$-Cartier divisor and it is nef on some birational modification of Z by [27, Theorem 2]. It carries informations on the birational variation of the fibers.

A great progress in the theory and its applications was accomplished around 2000 with the works of F. Ambro, and the paper [18] by O. Fujino and S. Mori. The first author treated the subadjunction problem and proved [1, Proposition 3.4] that the singularities of the pair (Z, B_Z) are the same as the singularities of (X, B). If (X, B) is klt (cf. Definition 1.2.4 for the definition of klt), he proved [3, Theorem 0.2] that there exists a divisor Δ_Z on Z such that (Z, Δ_Z) is klt and $K_X + B \sim_{\mathbb{Q}} f^*(K_Z + \Delta_Z)$. In [18] the authors proved a result that is in line with Iitaka's program.

Theorem 1.1.1 (Theorem 5.2. [18])**.** *Let (X, Δ) be a proper klt pair with $\kappa(X, \Delta) = l \geq 0$. Then there exist an l-dimensional klt pair (C', Δ') with $\kappa(C', \Delta') = l$, two integers $e', e \in \mathbb{Z}_{>0}$ and an isomorphism of graded rings*

$$R(X, K_X + \Delta)^{(e)} \cong R(C', K_{C'} + \Delta')^{(e')}$$

where

$$R(X, K_X + \Delta)^{(e)} = \bigoplus_{m \in \mathbb{Z}} H^0(X, \lfloor meK_X + \Delta \rfloor).$$

The problem that guides the research on the canonical bundle formula is the following difficult conjecture.

EbS(k) 1.1.2 (Effective b-Semiampleness, Conjecture 7.13.3, [38])**.** *There exists an integer $m = m(d, r)$ such that for any lc-trivial fibration $f : (X, B) \to Z$ with dimension of the generic fiber F equal to d, dimension of Z equal to k and Cartier index of $(F, B|_F)$ equal to r there exists a birational morphism $\nu : Z' \to Z$ such that $mM_{Z'}$ is base-point-free.*

The initials **EbS** stand for Effective b-Semiampleness. A Cartier divisor D on a variety Z is *birationally semiample*, or *b-semiample*, if there exists a birational morphism $\mu : Z' \to Z$ and a semiample divisor D' on Z' such that $\mu_* D' = D$. Conjecture **EbS** is far from being proved. Even the weaker version, which predicts that M_Z is b-semiample, without any condition on the integer that makes it base-point-free, has just a partial solution, namely when M_Z is numerically trivial ([3, Theorem 3.5] and [12, Theorem 1.3] for the lc case). There are partial results (see [38, Theorem 8.1], [15] and [16]) when there exists a moduli space for the fibers of f. The difficulty of Conjecture **EbS** is well illustrated by a result due by X. Jiang [24] who proved that Conjecture **EbS** implies a uniformity statement for the Iitaka fibration of *any* variety of positive Kodaira dimension under the assumption that the fibers have a good minimal model.

In this thesis we present several results on the canonical bundle formula. The first deals with fibration whose fibers are curves and is a study of the denominators of M_Z.

Theorem 1.1.3 (Theorem 1.6, [11])**.** *1. A polynomial global bound on the denominators of M_Z cannot exist. Precisely for any N there exists an lc-trivial fibration*

$$f : (X, B) \to Z$$

such that $\dim Z = 1$ and whose generic fiber is a rational curve such that if V is the smallest integer such that VM_Z has integral coefficients then

$$V \geq r^{N+1}.$$

2. *Let $f\colon (X,B) \to Z$ be an lc-trivial fibration whose generic fiber is a rational curve. Then there exists an integer $N(r)$ which depends only on r such that $N(r)M_Z$ has integral coefficients. More precisely $N(r) = r\mathrm{lcm}\{l | l \leq 2r\}$.*

Theorem 1.1.3 is proved in Subsection 1.8.3.
Moreover for any r odd we give an example such that if V is the smallest integer such that VM_Z has integral coefficients then $V = N(r)/r$. In [38, Remark 8.2] it was conjectured that the effective constant that makes the moduli part base-point-free is $12r$. Theorem 1.1.3 gives a counterexample to this conjecture. The denominators of M_Z are very important in applications in order to obtain effective results for the pluri-log-canonical maps of pairs with positive Kodaira dimension (see for instance [41], [42]).

The second result proves that Conjecture **EbS** can be reduced to the case where the base is a curve.

Theorem 1.1.4 (Theorem 1.2, [12]). ***EbS(1)*** *implies* ***EbS(k)****.*

The proof of Theorem 1.1.4 the object of Section 1.5. An inductive approach on the dimension of the base, as in Theorem 1.1.4, allows us to prove a result of effective semiampleness in the case $M_Z \equiv 0$. Indeed we are able to prove an effective version of [3, Theorem 3.5].

Theorem 1.1.5 (Theorem 1.3, [12]). *There exists an integer $m = m(b)$ such that for any klt-trivial fibration $f\colon (X,B) \to Z$ with*

- *Z smooth;*
- *$M_Z \equiv 0$;*
- $\mathrm{Betti}_{\dim E'}(E') = b$ *where E' is a non-singular model of the cover $E \to F$ associated to the unique element of $|r(K_F + B|_F)|$*

we have $mM_Z \sim 0$.

Moreover for the case where the pair (X,B) is lc but not klt on the generic point of the base we have the following that generalizes [3, Theorem 3.5].

Theorem 1.1.6 (Theorem 1.4, [12]). *Let $f\colon (X,B) \to Z$ be an lc-trivial fibration with Z smooth and $M_Z \equiv 0$. Then M_Z is torsion.*

The proofs of Theorem 1.1.5 and Theorem 1.1.6 are in Subsection 1.7.1 and Subsection 1.7.2 respectively.

This chapter is organized as follows: Section 1.2 contains som basic notation about the canonical bundle formula; in Section 1.4 we present a proof of Theorem 1.2.15 (see [8, Chapter 8]) that makes only use of the theory of variations of Hodge structure instead of variations of mixed Hodge structure. Sections 1.3, 1.5, 1.6 and 1.7 coincide with Sections 4, 3, 5 and 6 of [12]. Subsections 1.8.1, 1.8.2 and 1.8.3 are Subsection 2.2 and Sections 3 and 4 of [11].

1.2 Notation, definitions and known results

We will work over $\mathbb{C}$. In the following $\equiv$, $\sim$ and $\sim_{\mathbb{Q}}$ will respectively indicate numerical, linear and $\mathbb{Q}$-linear equivalence of divisors. The following definitions are taken from [31].

Definition 1.2.1. *A pair (X, B) is the data of a normal variety X and a $\mathbb{Q}$-Weil divisor B such that $K_X + B$ is $\mathbb{Q}$-Cartier.*

Definition 1.2.2. *Let (X, B) be a pair and write $B = \sum b_i B_i$. Let $\nu \colon Y \to X$ be a birational morphism, Y smooth. We can write*

$$K_Y \equiv \nu^*(K_X + B) + \sum a(E_i, X, B)E_i.$$

where $E_i \subseteq Y$ are distinct prime divisors and $a(E_i, X, B) \in \mathbb{R}$. Furthermore we adopt the convention that a nonexceptional divisor E appears in the sum if and only if $E = \nu_*^{-1} B_i$ for some i and then with coefficient $a(E, X, B) = -b_i$.
The $a(E_i, X, B)$ are called discrepancies.

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The $a(E_i, X, B)$ are called discrepancies.

A divisor E is exceptional over X if there exists a birational morphism $\nu \colon Y \to X$ such that $E \subseteq Y$ is ν-exceptional.

Definition 1.2.3. *Let (X, B) be a pair and $f \colon X \to Z$ be a morphism. Let $W \in Z$ be an irreducible subvariety. A log resolution of (X, B) over W is a birational morphism $\nu \colon X' \to X$ such that for any $x \in f^{-1}W$ the divisor $\nu^*(K_X + B)$ has simple normal crossings at x.*

Definition 1.2.4. *We set*

$$\operatorname{discrep}(X, B) = \inf\{a(E, X, B) \mid E \text{ exceptional prime divisor over } X\}.$$

A pair (X, B) is defined to be

- *klt (kawamata log terminal) if* $\operatorname{discrep}(\mathrm{X}, \mathrm{B}) > -1$ *and* $\lfloor B \rfloor \leq 0$,
- *plt (purely log terminal) if* $\operatorname{discrep}(\mathrm{X}, \mathrm{B}) > -1$,
- *lc (log canonical) if* $\operatorname{discrep}(\mathrm{X}, \mathrm{B}) \geq -1$.

Definition 1.2.5. *Let $f \colon (X, B) \to Z$ be a morphism and $W \subseteq Z$ an irreducible subvariety. For an exceptional divisor E over X we let $c(E)$ be its image in X. We set*

$$\operatorname{discrep}_W(X, B) = \inf\{a(E, X, B) \mid E \text{ prime divisor over } X,\ f(c(E)) = W\}.$$

A pair (X, B) is defined to be

- *klt over W (kawamata log terminal) if* $\operatorname{discrep}_{\mathrm{W}}(\mathrm{X}, \mathrm{B}) > -1$,
- *lc over W (log canonical) if* $\operatorname{discrep}_{\mathrm{W}}(\mathrm{X}, \mathrm{B}) \geq -1$.

Definition 1.2.6. *Let (X, B) be a pair. A place for (X, B) is a prime divisor on some birational model $\nu\colon Y \to X$ of X such that $a(E, X, B) = -1$. The image of E in X is called a center.*

Definition 1.2.7. *Let (X, B) be a pair and $\nu\colon X' \to X$ a log resolution of the pair. We set*

$$A(X, B) = K_{X'} - \nu^*(K_X + B)$$

and

$$A^*(X, B) = A(X, B) + \sum_{a(E,X,B)=1} E.$$

Definition 1.2.8. *A klt-trivial (resp. lc-trivial) fibration $f\colon (X, B) \to Z$ consists of a surjective morphism with connected fibers of normal varieties $f\colon X \to Z$ and of a log pair (X, B) satisfying the following properties:*

1. *(X, B) has klt (resp. lc) singularities over the generic point of Z;*
2. $\operatorname{rank} f'_* \mathcal{O}_X(\lceil A(X, B)\rceil) = 1$ *(resp.* $\operatorname{rank} f'_* \mathcal{O}_X(\lceil A^*(X, B)\rceil) = 1$*) where $f' = f \circ \nu$ and ν is a given log resolution of the pair (X, B);*
3. *there exists a positive integer r, a rational function $\varphi \in \mathbb{C}(X)$ and a $\mathbb{Q}$-Cartier divisor D on Z such that*
$$K_X + B + \frac{1}{r}(\varphi) = f^*D.$$

Remark 1.2.9. Condition (2) does not depend on the choice of the log resolution ν. It is verified for instance if B is effective because

$$\lceil A^*(X, B)\rceil = \lceil K_{X'} - \nu^*(K_X + B) + \sum_{a(E,X,B)=1} E\rceil$$

is exceptional over X.

Remark 1.2.10. Let $f\colon (X, B) \to Z$ be an lc-trivial fibration. Let $\mu\colon \hat{X} \to X$ be a birational morphism. Let $\hat{f} = f \circ \nu$ and let $\hat{B}$ be the divisor defined by

$$K_{\hat{X}} + \hat{B} = K_X + B.$$

Then $\hat{f}\colon (\hat{X}, \hat{B}) \to Z$ is again an lc-trivial fibration. Indeed the singularities of $(\hat{X}, \hat{B})$ are the same as the singularities of (X, B) and condition (1) in Definition 1.2.8 is verified. A log resolution for $(\hat{X}, \hat{B})$ is also a log resolution for (X, B), thus condition (2) is verified. Finally

$$K_{\hat{X}} + \hat{B} + \frac{1}{r}(\varphi \circ \mu) = \mu^*(K_X + B + \frac{1}{r}(\varphi)) = \hat{f}^*D$$

and we are done.

Remark 1.2.11. The smallest possible r that can appear in Definition 1.2.8 is the minimum of the set

$$\{m \in \mathbb{N} |\, m(K_X + B)|_F \sim 0\}$$

that is the Cartier index of the fiber. We will always assume that the r that appears in the formula is the smallest one.

Definition 1.2.12. *Let $P \subseteq Z$ be a prime Weil divisor. The log canonical threshold γ_P of $f^*(P)$ with respect to the pair (X, B) is defined as follows. Let $\bar{Z} \to Z$ be a resolution of Z. Let $\mu \colon \bar{X} \to X$ be the birational morphism obtained as a desingularization of the main component of $X \times_Z \bar{Z}$. Let $\bar{f} \colon \bar{X} \to \bar{Z}$. Let $\bar{B}$ be the divisor defined by the relation*

$$K_{\bar{X}} + \bar{B} = \mu^*(K_X + B).$$

Let $\bar{P}$ be the strict transform of P in $\bar{Z}$. Set

$$\gamma_P = \sup\{t \in \mathbb{Q} |\, (\bar{X}, \bar{B} + t\bar{f}^*(\bar{P})) \text{ is lc over } \bar{P}\}.$$

We define the discriminant *of $f \colon (X, B) \to Z$ as*

$$B_Z \quad = \quad \sum_P (1 - \gamma_P) P. \tag{1.2.1}$$

Remark 1.2.13 ([27], p.14 [1])**.** The log canonical threshold γ_P is a rational number. Indeed, assume that Z is smooth and let $\mu \colon X' \to X$ be a log resolution of the pair $(X, B + f^*(P))$. Let B' be the divisor defined by $K_{X'} + B' = \mu^*(K_X + B)$. Then $f \circ \mu \colon (X', B') \to Z$ is an lc-trivial fibration.

Let P be a prime Weil divisor of Y and set $f^*P = \sum w_j Q_j$. Set $b_j = \mathrm{mult}_{Q_j} B$. Then

$$\gamma_P = \min_j \frac{1 - b_j}{w_j}.$$

In particular γ_P is a rational number. We notice also that $-B' + \mu^* f^* B_Z$ is effective over codimension-one points: with the same notation as above, over P we have

$$-B' + \mu^* f^* B_Z = \sum_j [(1 - \gamma_P) w_j - b_j] Q_j.$$

Since $\gamma_P \leq (1 - b_j)/w_j$ for any j, we have

$$w_j - \gamma_P w_j - b_j \geq w_j - 1 \geq 0.$$

We remark that, since the above sum is finite, B_Z is a $\mathbb{Q}$-Weil divisor.

Definition 1.2.14. *Fix $\varphi \in \mathbb{C}(X)$ such that $K_X + B + \frac{1}{r}(\varphi) = f^*D$. Then there exists a unique divisor M_Z such that we have*

$$K_X + B + \frac{1}{r}(\varphi) \quad = \quad f^*(K_Z + B_Z + M_Z) \tag{1.2.2}$$

where B_Z is as in (1.2.1). The $\mathbb{Q}$-Weil divisor M_Z is called the moduli part*.*

We have the two following results.

Theorem 1.2.15 (Theorem 0.2 [2], [8])**.** *Let $f \colon (X, B) \to Z$ be an lc-trivial fibration. Then there exists a proper birational morphism $Z' \to Z$ with the following properties:*

(i) *$K_{Z'} + B_{Z'}$ is a $\mathbb{Q}$-Cartier divisor, and for every proper birational morphism $\nu\colon Z'' \to Z'$*

$$\nu^*(K_{Z'} + B_{Z'}) = K_{Z''} + B_{Z''}.$$

(ii) *$M_{Z'}$ is a nef $\mathbb{Q}$-Cartier divisor and for every proper birational morphism $\nu\colon Z'' \to Z'$*

$$\nu^*(M_{Z'}) = M_{Z''}.$$

Proposition 1.2.16 (Proposition 5.5 [2]). *Let $f\colon (X,B) \to Z$ be an lc-trivial fibration. Let $\tau\colon Z' \to Z$ be a generically finite projective morphism from a non-singular variety Z'. Assume there exists a simple normal crossing divisor $\Sigma_{Z'}$ on Z' which contains $\tau^{-1}\Sigma_Z$ and the locus where τ is not étale. Let $M_{Z'}$ be the moduli part of the induced lc-trivial fibration $f'\colon (X', B') \to Z'$. Then $M_{Z'} = \tau^* M_Z$.*

Theorem 1.2.17 (Inverse of adjunction, Proposition 3.4, [1], see also Theorem 4.5 [18]). *Let $f\colon (X,B) \to Z$ be an lc-trivial fibration. Then (Z, B_Z) has klt (lc) singularities in a neighborhood of a point $p \in Z$ if and only if (X,B) has klt (lc) singularities in a neighborhood of $f^{-1}p$.*

The Formula (1.2.2) is called the *canonical bundle formula.*

1.3 Variation of Hodge structure and covering tricks

1.3.1 Variation of Hodge structure

Let $\mathcal{S}$ be $\mathbb{C}^*$ viewed as an $\mathbb{R}$-algebra.

Definition 1.3.1 (2.1.4 [10]). *A real Hodge structure is a real vector space V of finite dimension together with an action of $\mathcal{S}$.*
The representation of $\mathcal{S}$ on V induces a bigraduation on V, such that $\overline{V^{pq}} = V^{qp}$. We say that V has weight n if $V^{pq} = 0$ whenever $p + q \neq n$.

Definition 1.3.2 (2.1.10 [10]). *A Hodge structure H of weight n is*

- *a $\mathbb{Z}$-module of finite type $H_{\mathbb{Z}}$;*
- *a real Hodge structure of weight n on $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.*

Definition 1.3.3. *Let S be a topological space. A local system on S is a sheaf $\mathbb{V}$ of $\mathbb{Q}$-vector spaces on S.*

Let now S be a complex manifold.

Definition 1.3.4. *Let $\mathcal{V} \to S$ be a vector bundle. A connection is a morphism*

$$\nabla\colon \mathcal{V} \to \Omega^1_S \otimes \mathcal{V}$$

that satisfies the Leibniz rule.
The curvature of a connection is $\nabla \circ \nabla\colon \mathcal{V} \to \Omega^2_S \otimes \mathcal{V}$.
A connection is said to be integrable if $\nabla \circ \nabla = 0$.

By [9, Proposition 2.16] the data of a local system $\mathbb{V}$ is equivalent to the data of a vector bundle $\mathcal{V} \to S$ together with an integrable connection ∇ and the correspondence is given by associating to $\mathbb{V}$ the vector bundle

$$\mathcal{V} = \mathbb{V} \otimes \mathcal{O}.$$

Definition 1.3.5. *A flat subsystem of a local system $\mathbb{V}$ is a sub-local system $\mathbb{W}$ of $\mathbb{V}$ or equivalently a subbundle $\mathcal{W}$ of $\mathcal{V}$ on which the curvature of the connection is zero.*

Definition 1.3.6 ((3.1) [40]). *A variation of Hodge structure of weight m on S is:*

- *a local system $\mathbb{V}$ on S;*
- *a flat bilinear form*

$$Q\colon \mathcal{V} \times \mathcal{V} \to \mathbb{C}$$

  which is a bilinear form rational on $\mathbb{V}$, where $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_S$;
- *a Hodge filtration $\{\mathcal{F}^p\}$, that is a decreasing filtration of $\mathcal{V}$ by holomorphic subbundles such that for any p we have $\nabla(\mathcal{F}^p) \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}$.*

Definition 1.3.7 ((3.4) [40]). *A variation of mixed Hodge structure on S is:*

- *a local system $\mathbb{V}$ on S;*
- *a Hodge filtration $\{\mathcal{F}^p\}$ that is a decreasing filtration of $\mathcal{V}$ by holomorphic subbundles such that for any p we have $\nabla(\mathcal{F}^p) \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}$;*
- *a Weight filtration $\{\mathcal{W}_k\}$ that is an increasing filtration of $\mathcal{V}$ by local subsystems, or equivalently, the subsheaf $\mathcal{W}_k$ is defined over $\mathbb{Q}$ for every k;*

Moreover we require that the filtration induced by $\{\mathcal{F}^p\}$ on $\mathcal{W}_k/\mathcal{W}_{k-1}$ determines a variation of Hodge structure of weight k.

From now on we will be interested in variations of Hodge structure and of mixed Hodge structure defined on a Zariski open subset Z_0 of a projective variety Z. We assume moreover that $\Sigma_Z = Z \backslash Z_0$ is a simple normal crossing divisor.

The following is a fundamental result about the behavior of a variation of Hodge structures on Z_0 near Σ_Z. For the definition of *monodromy* and *unipotent monodromy* of variations of Hodge structure and residue of a connection see [37, Definition 10.16, section 11.1.1]

Proposition 1.3.8 (Proposition 5.2(d), [9]). *Let $\mathcal{V}$ be a variation of Hodge structure on Z_0 that has unipotent monodromies around Σ_Z. Let z be a local variable with center in Σ_Z. Then*

a *There exists a unique extension $\tilde{\mathcal{V}}$ of $\mathcal{V}$ on Z such that*

i *every horizontal section of $\mathcal{V}$ as a section of $\tilde{\mathcal{V}}$ on Z_0 grows at most as*

$$O(\log \parallel z \parallel^k)$$

for some integer k near Σ_Z;

ii *let $\mathcal{V}^*$ be the dual of $\mathcal{V}$. Every horizontal section of $\mathcal{V}^*$ grows at most as*

$$O(\log \parallel z \parallel^k)$$

for some integer k near Σ_Z.

b *Conditions (i) and (ii) are equivalent respectively to conditions (iii) and (iv).*

iii *The matrix of the connection on $\mathcal{V}$ on a local frame for $\tilde{\mathcal{V}}$ has logarithmic poles near Σ_Z.*

iv *Each residue of the connection along each irreducible component of Σ_Z is nilpotent.*

c *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be variations of Hodge structure on Z_0 that have unipotent monodromies around Σ_Z. Every morphism $f\colon \mathcal{V}_1 \to \mathcal{V}_2$ extends to a morphism $\tilde{\mathcal{V}}_1 \to \tilde{\mathcal{V}}_2$. Moreover the functor $\mathcal{V} \mapsto \tilde{\mathcal{V}}$ is exact and commutes with $\otimes$, $\wedge$,* Hom.

The extension $\tilde{\mathcal{V}}$ is called ***the canonical extension****.*

Remark 1.3.9. In the situation of Proposition 1.3.8 the matrix Γ of the connection on $\mathcal{V}$ has the following form

$$\Gamma = \sum U_i \frac{dz_i}{z_i}$$

where U_i is the matrix that represents the nilpotent part of the monodromy around the component Σ_i of Σ_Z.

Let V, Z be smooth projective varieties and $h\colon V \to Z$ a surjective morphism with connected fibers. Let $Z_0 \subseteq Z$ be the largest Zariski open set where h is smooth and $V_0 = h^{-1}(Z_0)$. Assume that $\Sigma_Z = Z \backslash Z_0$ and $\Sigma_V = V \backslash V_0$ are simple normal crossing divisors on Z and V. Set $d = \dim V - \dim Z$. Consider $\mathcal{H}_{\mathbb{C}} = (R^d h_* \mathbb{C}_{V_0})_{\text{prim}}$, where the subscript prim stands for the primitive part of the cohomology. Set $\mathcal{H}_0 = \mathcal{H}_{\mathbb{C}} \otimes \mathcal{O}_{Z_0}$, $\mathcal{F} = h_* \omega_{V/Z}$ and $\mathcal{F}_0 = \mathcal{F} \otimes \mathcal{O}_{Z_0}$. Then $\mathcal{H}_{\mathbb{C}}$ is a local system over Z_0. Moreover $\mathcal{H}_0$ has a descending filtration $\{\mathcal{F}^p\}_{0 \leq p \leq d}$, the *Hodge filtration* and $\mathcal{F}_0 = \mathcal{F}^d$.
There is a canonical way to extend $\mathcal{H}_0$ and $\mathcal{F}_0$ to locally free sheaves on Z:

Theorem 1.3.10 (Proposition 5.4 [9], Theorem 2.6 [29], [19])**.** *1. $\mathcal{H}_0$ has a canonical extension to a locally free sheaf on Z.*

2. $h_* \omega_{V/Z}$ coincides with the canonical extension of the bottom piece of the Hodge filtration.

Let $h_0\colon V_0 \to Z_0$ be as before. Let $D \subseteq V$ be a simple normal crossing divisor such that the restriction $h_0|_D$ is flat. Assume that $D + \Sigma_V$ has simple normal crossings. Let us denote the restriction by

$$h_0\colon V_0 \backslash D \to Z_0.$$

Thus $R^d (h_0)_* \mathbb{C}_{V_0 \backslash D}$ is a local system on Z_0 by [40, section 5.2]. Let $\{\mathcal{F}^p\}$ be the Hodge filtration and let

$$W_k \quad = (h_0)_* \Omega^k_{V_0/Z_0}(\log D) \wedge \Omega^{\bullet - k}_{V_0/Z_0} \tag{1.3.1}$$

be the weight filtration of the complex $(h_0)_*\Omega^\bullet_{V_0/Z_0}(\log D)$. In particular W_k is a complex. We will adopt the following notation

$$W_k((h_0)_*\Omega^s_{V_0/Z_0}(\log D)) = (h_0)_*\Omega^k_{V_0/Z_0}(\log D) \wedge \Omega^{s-k}_{V_0/Z_0}.$$

Let $h\colon V \to Z$ be a morphism such that Σ_Z is a simple normal crossing divisor and let $\tau\colon Z' \to Z$ be a morphism from a smooth variety Z' such that $\tau^{-1}(\Sigma_Z)$ is a simple normal crossing divisor. Let V' be a desingularization of the component of $V \times_Z Z'$ that dominates Z'.

$$\begin{array}{ccc} V' & \longrightarrow & V \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ Z' & \xrightarrow[\tau]{} & Z. \end{array}$$

Assume now that h' and h are such that $R^d h_*\mathbb{C}_{V_0}$ and $R^d h'_*\mathbb{C}_{V'_0}$ have unipotent monodromies. By Proposition 1.3.8[c] we have a commutative diagram of sheaves on Z'

$$\begin{array}{ccc} (h')_*\omega_{V'/Z'} & \xrightarrow{\sim} & (i')_*(\mathcal{F}'_0) \cap \mathcal{H}' \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ \tau^*(h)_*\omega_{V/Z} & \xrightarrow{\sim} & \tau^*(i)_*(\mathcal{F}_0) \cap \mathcal{H} \end{array}$$

where $i\colon Z_0 \to Z$, $i'\colon Z'_0 \to Z'$ are inclusions, $\mathcal{H}$ (resp. $\mathcal{H}'$) is the canonical extension of $R^d h_*\mathbb{C}_{V_0}$ (resp. $R^d h'_*\mathbb{C}_{V'_0}$), $\mathcal{F}_0 = h_*\omega_{V/Z|_{Z_0}}$ (resp. $\mathcal{F}'_0 = h'_*\omega_{V'/Z'|_{Z'_0}}$) and α, β are the pullbacks by τ. If we have an isomorphism

$$\alpha\colon (h')_*\omega_{V'/Z'} \to \tau^*(h)_*\omega_{V/Z}$$

then for any $p \in Z'$ we have an isomorphism of $\mathbb{C}$-vector spaces

$$\alpha_p\colon ((h')_*\omega_{V'/Z'})_p \to (h_*\omega_{V/Z})_{\tau(p)}.$$

If τ is a birational automorphism of Z that fixes p, then α_p is an element of the linear group of $((h)_*\omega_{V/Z})_p$. In particular we have the following:

Proposition 1.3.11. *Let $h\colon V \to Z$ be a fibration such that $R^d h_{0*}\mathbb{C}_{V_0}$ has unipotent monodromies. Assume that we have a birational action of a group G on Z given by a homomorphism*

$$G \to \mathrm{Bir}(Z) = \{\nu\colon Z \dashrightarrow Z \,|\, \nu \,\text{is birational}\}.$$

Let G_p be the stabilizer of $p \in Z$. Then we have an induced action of G_p on $\mathcal{H}_p$ and on $(h_*\omega_{V/Z})_p$ and these actions commute with the inclusion $(h_*\omega_{V/Z})_p \subseteq \mathcal{H}_p$.

By [25, Theorem 17] we can assume that $R^d h_{0*}\mathbb{C}_{V_0}$ (or, more generally, a local system that has quasi-unipotent monodromies) has unipotent monodromies modulo a finite base change by a Galois morphism. We restate Kawamata's result in a more precise way that is useful for our purposes.

Theorem 1.3.12 (Theorem 17, Corollary 18 [25]). *Let $h\colon V \to Z$ be an algebraic fiber space. Let $Z_0 \subseteq Z$ be the largest Zariski open set where h is smooth and $V_0 = h^{-1}(Z_0)$. Assume that $\Sigma_Z = Z \backslash Z_0$ and $\Sigma_V = V \backslash V_0$ are simple normal crossing divisors on Z and V. Set $d = \dim V - \dim Z$.*
Then there exists a finite surjective morphism $\tau\colon Z' \to Z$ from a smooth projective algebraic variety Z' such that for a desingularization V' of $V \times_Z Z'$ the morphism $h'\colon V' \to Z'$ induced from h is such that $R^d h'_{0*}\mathbb{C}_{V'_0}$ has unipotent monodromies.

Moreover τ is a composition of cyclic coverings τ_j

$$\tau\colon Z' = Z_{k+1} \xrightarrow{\tau_k} Z_k \cdots \to Z_2 \xrightarrow{\tau_1} Z_1 = Z$$

where τ_j is defined by the building data

$$\delta_j A_j \sim H_j$$

where A_j is very ample on Z_j and H_j has simple normal crossings.

We have the following results by Deligne. We state them in our situation, but they hold in a more general setting.

Theorem 1.3.13 (Theorem 4.2.6 [10]). *The representation of the fundamental group $\pi_1(Z_0, z)$ on the fiber $\mathcal{H}_{\mathbb{Q},z}$ is semisimple.*

Here we prove a slight modification of [10, Corollary 4.2.8(ii)(b)].

Corollary 1.3.14. *Let W be a local subsystem of $\mathcal{H}_{\mathbb{C}}$ of rank one. Let $b = \dim \mathcal{H}$. Let*

$$m(x) = \mathrm{lcm}\{k \mid \phi(k) \leq x\}$$

where ϕ is the Euler function. Then $W^{\otimes m(b)}$ is a trivial local system.

Proof. By Theorem 1.3.13 we can write

$$\mathcal{H}_{\mathbb{C},z} = \bigoplus_{i=1}^{r} \mathcal{H}_i$$

where the $\mathcal{H}_i$ are the isotypic components (that is, the components that are direct sum of simple representations of the same weight). Since $\dim W = 1$, the subspace W_z is contained in one isotypic component, say $\mathcal{H}_1$. We have

$$\mathcal{H}_1 = \mathcal{H}_\lambda^{\oplus k},$$

where $\mathcal{H}_\lambda$ is a simple component of weight λ. Since W_z is simple, it identifies with one of the factors $\mathcal{H}_\lambda$'s and then

$$\bigwedge^k \mathcal{H}_1 \cong W_z^{\otimes k}.$$

If χ is the character that determines W_z as a representation then the character χ^k determines $\bigwedge^k \mathcal{H}_1$. Let $\mathcal{S}$ be the real algebraic group $\mathbb{C}^*$. We have an action of $\mathcal{S}$ on $\mathcal{H}_1$. Indeed by [10, Corollary 4.2.8(ii)(a)] for any $t \in \mathcal{S}$ we have $t\mathcal{H}_1 \cong \mathcal{H}_1$. In particular $t\mathcal{H}_1$ and $\mathcal{H}_1$ have the same weight. But since $\mathcal{H}_1$ is isotypic, we have $t\mathcal{H}_1 = \mathcal{H}_1$. The vector space $\hat{\mathcal{H}}_1 = \mathcal{H}_1 + \bar{\mathcal{H}}_1$ is real and $\mathcal{S}$-invariant, thus it is defined by a real Hodge substructure of $\mathcal{H}_{\mathbb{R},z}$. Thus a polarization on $\mathcal{H}$ induces a non-degenerate bilinear form on $\hat{\mathcal{H}}_1$ that is invariant under the action of $\pi_1(Z_0, z)$.
Then, if we set $e = \dim \hat{\mathcal{H}}_1$, we have that $(\bigwedge^e \hat{\mathcal{H}}_1)^{\otimes 2}$ is trivial. Since $\hat{\mathcal{H}}_1 = \mathcal{H}_1 + \bar{\mathcal{H}}_1$, there are two possibilities:

(a) if $\mathcal{H}_1$ is real, the character χ^{2k} is trivial,

(b) or else $\chi^{2k}\bar{\chi}^{2k}$ is trivial.

In any case $|\chi| = 1$.
The representation of $\pi_1(Z_0, z)$ on $\mathcal{H}_{\mathbb{C},z}$ comes from a representation on $\mathcal{H}_{\mathbb{Q},z}$ and all the conjugate representations of W_z appear in $\mathcal{H}_{\mathbb{C},z}$. Thus we have at most $b = \dim \mathcal{H}$ conjugate representations of W_z.

We proved that for any $\gamma \in \pi_1(Z_0, z)$ the number $\chi(\gamma)$ is a complex number of modulus one and with at most b complex conjugates. Thus $\chi(\gamma)$ is a k-th root of unity, with $k \leq b$. If we define

$$m(b) = \mathrm{lcm}\{k \mid \phi(k) \leq b\}$$

where ϕ is the Euler function, then $\chi^{m(b)}$ is trivial. □

1.3.2 Covering tricks

In order to give an interpretation of the moduli part in terms of variation of Hodge structure we need to consider an auxiliary log pair (V, B_V) with a fibration $h\colon V \to Z$.

Let $f\colon X \to Z$ be an lc-trivial fibration. Set $\Sigma_Z = \mathrm{Supp} B_Z$ and we assume that Σ_Z is a simple normal crossing divisor. Set $\Sigma_X = \mathrm{Supp} f^*\Sigma_Z$ and assume that $B + \Sigma_X$ has simple normal crossing support. We define $g\colon V \to X$ as the desingularization of the covering induced by the field extension

$$\mathbb{C}(X) \subseteq \mathbb{C}(X)(\sqrt[r]{\varphi}), \tag{1.3.2}$$

that is, the desingularization of the normalization of X in $\mathbb{C}(X)(\sqrt[r]{\varphi})$ where φ is as in (1.2.2). Let B_V be the divisor defined by the equality $K_V + B_V = g^*(K_X + B)$. Set $h = f \circ g\colon V \to Z$. Then h and f induce the same discriminant and moduli part. Let Σ_V be the support of $h^*\Sigma_Z$ and assume that $\Sigma_V + B_V$ has simple normal crossing support.

The Galois group of (1.3.2) is cyclic of order r, then we have an action of

$$\mu_r = \{x \in \mathbb{C} \mid x^r = 1\}$$

on $g_*\mathcal{O}_V$. Then we have also an action of μ_r on $h_*\omega_{V/Z}$ and on $h_*\omega_{V/Z}(P_V)$ where P_V are the horizontal places of the pair (V, B_V).

Proposition 1.3.15 (Claim 8.4.5.5, Section 8.10.3 [8])**.** *Let $f\colon X \to Z$ and $V \to X$ be as above. The decomposition in eigensheaves is*

$$h_*\omega_{V/Z} = \bigoplus_{i=0}^{r-1} f_*\mathcal{O}_X(\lceil (1-i)K_{X/Z} - iB + if^*B_Z + if^*M_Z \rceil).$$

Let P_V be the places of (V, B_V) and P the places of (X, B). Then we have

$$h_*\omega_{V/Z}(P_V) = \bigoplus_{i=0}^{r-1} f_*\mathcal{O}_X(\lceil (1-i)K_{X/Z} - iB + P + if^*B_Z + if^*M_Z \rceil)$$

and the right-hand side is the eigensheaf decomposition of the left-hand side with respect to the action of μ_r.

Proposition 1.3.16 (Proposition 5.2, [2]). *Assume that $R^d h_{0*}\mathbb{C}_{V_0}$ has unipotent monodromies. Then M_Z is an integral divisor and $\mathcal{O}_Z(M_Z)$ is isomorphic to the eigensheaf $f_*\mathcal{O}_X(\lceil -B+P+f^*B_Z+f^*M_Z\rceil)$ corresponding to a fixed primitive rth root of unity.*

1.4 Nefness: generalization to the lc case

This section will be devoted to a proof of Theorem 1.2.15 for a fibration that is lc and not klt over the generic point of the base. This result is stated in [8, §8]. The key result for the proof is Theorem 1.4.5 that is implied by the very deep results in [15] about variation of mixed Hodge structure (see also [17, Theorem 3.6]). Here we present a proof that makes use only of the theory of variation of Hodge structure and follows Ambro's proof of [2, Theorem 0.2].

Remark 1.4.1. If $B' = B + f^*\Delta$, then $f\colon (X, B') \to Z$ is an lc-trivial fibration and its discriminant is $B_Z + \Delta$.

Let X be a smooth variety, L a line bundle on X and D an integral and not necessarily effective divisor such that $L^r \cong \mathcal{O}_X(D)$. In [8, §8.10.3] it is explained that with this data it is possible to define a covering of X. Take s a rational section of L and 1_D the constant rational section of $\mathcal{O}_X(D)$. Then we can define $\pi\colon \tilde{X} \to X$ as the normalization of X in $\mathbb{C}(X)(\sqrt[r]{1_D/s^r})$. Moreover we have

$$\begin{aligned} \pi_*\mathcal{O}_{\tilde{X}} &= \bigoplus_{i=0}^{r} L^{-i}(\lfloor iD/r\rfloor); \\ \pi_*\omega_{\tilde{X}} &= \bigoplus_{i=0}^{r} \omega_X \otimes L^i(-\lfloor iD/r\rfloor). \end{aligned}$$

The Galois group of the extension $\mathbb{C}(X) \subseteq \mathbb{C}(X)(\sqrt[r]{1_D/s^r})$ acts on $\pi_*\mathcal{O}_{\tilde{X}}$ by $\sqrt[r]{1_D/s^r} \mapsto \zeta\cdot\sqrt[r]{1_D/s^r}$ where ζ is a primitive r-th root of unity. The eighensheaf corresponding to ζ is $L^{-1}(\lfloor D/r\rfloor)$.

Now we take as building data of the covering

$$\begin{aligned} L &= \mathcal{O}_X, \\ D &= -(\varphi) = r(K_{X/Z} + B - f^*(B_Z + M_Z)), \\ L^r &= \mathcal{O}_X(D). \end{aligned}$$

Now let B be a divisor such that (X, B) is lc over the generic point of Z. In particular we have

$$\begin{aligned} \pi_*\omega_{\tilde{X}/Z} &= \bigoplus_{i=0}^{r-1} \omega_{X/Z}\otimes\mathcal{O}_X(-\lfloor i(K_{X/Z}+B-f^*(B_Z+M_Z))\rfloor) && (1.4.1) \\ &= \bigoplus_{i=0}^{r-1} \mathcal{O}_X(K_{X/Z}+\lceil -i(K_{X/Z}+B-f^*(B_Z+M_Z))\rceil) && (1.4.2) \\ &= \bigoplus_{i=0}^{r-1} \mathcal{O}_X(\lceil (1-i)K_{X/Z}-iB+if^*B_Z+if^*M_Z)\rceil). && (1.4.3) \end{aligned}$$

We assume that $K_X + B$ has simple normal crossing support and call E the sum of all the lc-centers of (X, B) that dominate Z. Set $\tilde{E} = \pi^* E$, then

$$\pi_*(\omega_{\tilde{X}/Z} \otimes \mathcal{O}_{\tilde{X}}(\tilde{E})) = \bigoplus_{i=0}^{r-1} \mathcal{O}_X(\lceil (1-i)K_{X/Z} - iB + E + if^*B_Z + if^*M_Z) \rceil).$$

The eigensheaf of ζ in $\pi_*(\omega_{\tilde{X}/Z} \otimes \mathcal{O}_{\tilde{X}}(\tilde{E}))$ is

$$\mathcal{O}_X(\lceil -B + E + f^*B_Z + f^*M_Z) \rceil).$$

Let V be a non-singular model of $\tilde{X}$. We have a diagram

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & \tilde{X} & \longleftarrow & V \\ {\scriptstyle f}\downarrow & \swarrow_{\tilde{f}} & & \swarrow_{h} & \\ Z. & & & & \end{array}$$

Set $g \colon V \to X$ and $B_V = g^*(K_X + B) - K_V$. In [2, p. 245] are stated the following properties for $h \colon (V, B_V) \to Z$:

- The field extension $\mathbb{C}(V)/\mathbb{C}(X)$ is Galois and its Galois group G is cyclic of order r. There exists $\psi \in \mathbb{C}(V)$ such that $\psi^r = \varphi$. A generator of G acts by $\psi \mapsto \zeta\psi$, where ζ is a fixed primitive r-th root of unity.
- The relative log pair $h \colon (V, B_V) \to Z$ satisfies all properties of an lc-trivial fibration, except that the rank of $h_*\mathcal{O}_X(\lceil A^*(V, B_V) \rceil)$ might be bigger than one.
- Both $f \colon (X, B) \to Z$ and $h \colon (V, B_V) \to Z$ induce the same discriminant and moduli part on Z.

The canonical bundle formula for $h \colon (V, B_V) \to Z$ is

$$K_V + B_V + (\psi) \quad = \quad h^*(K_Z + B_Z + M_Z) \tag{1.4.4}$$

Let E_V be the sum of all the centers of (V, B_V).

SNC setting 1.4.2. By [25, pp. 262-263] and [45, p. 334], in order to prove the nefness of the moduli part, we can suppose the following (cf. [2, p. 245]):

i the varieties X, V, Z are non-singular quasi-projective and there exist simple normal crossing divisors Σ_X, Σ_V, Σ_Z on X, V and Z respectively such that the morphisms f and h are smooth over $Z \backslash \Sigma_Z$ and the divisors Σ_X^h/Z and Σ_V^h/Z have relative simple normal crossings over $Z \backslash \Sigma_Z$;;

ii the morphisms f and h are projective;

iii we have $f^{-1}(\Sigma_Z) \subseteq \Sigma_X$, $f(\Sigma_X^v) \subseteq \Sigma_Z$ and $h^{-1}(\Sigma_Z) \subseteq \Sigma_V$, $h(\Sigma_V^v) \subseteq \Sigma_Z$.

iv the divisors B, B_V and B_Z ,M_Z are supported by Σ_X, Σ_V and Σ_Z, respectively.

Lemma 1.4.3. *The following properties hold for the above set-up:*

1. *The group G acts naturally on $h_*\mathcal{O}_V(K_{V/Z}+E_V)$. The eigensheaf corresponding to the eigenvalue ζ is*
$$\tilde{\mathcal{L}} := f_*\mathcal{O}_X(-B+E+f^*B_Z+f^*M_Z).$$
2. *Assume that $h\colon V\to Z$ is semistable in codimension one. Then M_Z is an integral divisor, $\tilde{\mathcal{L}}$ is semipositive and $\tilde{\mathcal{L}}=\mathcal{O}_Z(M_Z)\cdot\psi$.*

Proof. Since (φ) has SNC support, the variety $\tilde{X}$ has canonical singularities and
$$h_*\mathcal{O}_V(K_{V/Z}+E_V)=f_*\pi_*(\omega_{\tilde{X}/Z}\otimes\mathcal{O}_{\tilde{X}}(\tilde{E})).$$
The action on $h_*\mathcal{O}_V(K_{V/Z}+E_V)$ is induced by the one on $\pi_*(\omega_{\tilde{X}/Z}\otimes\mathcal{O}_{\tilde{X}}(\tilde{E}))$, thus the eigensheaf of ζ is
$$\tilde{\mathcal{L}}=f_*\mathcal{O}_X(\lceil -B+E+f^*B_Z+f^*M_Z)\rceil).$$
This completes the proof of item (1).
We claim that there exists an open set $Z^\dagger\subseteq Z$ such that the codimension of $Z\backslash Z^\dagger$ is at least two, such that $(-B_V+h^*B_Z)|_{h^{-1}Z^\dagger}$ supports no fibers and $(-B_V+E_V+h^*B_Z)|_{h^{-1}Z^\dagger}$ is effective and supports no fibers. Indeed, since h is semistable, using the same notation as in the Remark 1.2.13, there exists j_0 such that $\gamma_p=1-b_{j_0}$ (here $w_j=1$ for any j).
Then $1-\gamma_p-b_{j_0}=0$ and $-B_V+h^*B_Z$ does not contain the fiber of p. Since E_V is horizontal, the same reasoning holds for $-B_V+E_V+h^*B_Z$.
For the effectivity, from Formula (1.4.4) we deduce that the coefficients of $(B_V)^h$ are integral, thus they are either 1 or negative. Then $(-B_V+E_V+h^*B_Z)^h=(-B_V+E_V)^h$ is effective. The effectivity of $(-B_V+E_V+h^*B_Z)^v=(-B_V+h^*B_Z)^v=$ follows from [27], [1, p. 14]. Let H be a general fiber of h. By restricting Formula (1.4.4) to H we get
$$(\psi|_H)+K_H+E_V|_H=-(B_V-E_V)|_H\geq 0.$$
This implies that there exists an open subset $U\subseteq Z$ such that $((\psi)+K_{V/Z}+E_V)|_U\geq 0$ and ψ is a rational section of $h_*\mathcal{O}(K_{V/Z}+E_V)$. Moreover, since by the action of G we have $\psi\mapsto\zeta\psi$, the function ψ is a rational section of $\tilde{\mathcal{L}}$ the eigensheaf of ζ. The sheaf $\tilde{\mathcal{L}}$ has rank one because for general $y\in Z$ we have $\tilde{\mathcal{L}}_y\cong H^0(F,\lceil -B+E+f^*B_Z+f^*M_Z)\rceil|_F)=H^0(F,\lceil -B+E\rceil|_F)$ and the last one is a rank one $\mathbb{C}$-vector space by the hypothesis (2) in the definition of lc-trivial fibration. Thus we can consider $\tilde{\mathcal{L}}$ as a subsheaf of $\mathbb{C}(X)\psi$.
We prove now that $\tilde{\mathcal{L}}|_{Z_\dagger}=\mathcal{O}_Z(M_Z)\psi|_{Z_\dagger}$.
Since $(-B_V+E_V+h^*B_Z)|_{h^{-1}(Z^\dagger)}$ is effective and $h^*M_Z-B_V+E_V+h^*B_Z=K_{V/Z}+E_V$ we have
$$h^*\mathcal{O}_V(M_Z)|_{h^{-1}(Z^\dagger)}\subseteq\mathcal{O}_V(K_{V/Z}+E_V)|_{h^{-1}(Z^\dagger)}$$
and
$$h_*h^*\mathcal{O}_V(M_Z)|_{Z^\dagger}\subseteq h_*\mathcal{O}_V(K_{V/Z}+E_V)|_{Z^\dagger}.$$
Now let $a\in k(Z)$ such that $h^*a+K_{V/Z}+E_V\geq 0$. Since $(-B_V+E_V+h^*B_Z)|_{h^{-1}(Z^\dagger)}$ contains no fibers we have $h^*a+h^*M_Z\geq 0$, thus $h_*\mathcal{O}_V(K_{V/Z}+E_V)|_{Z^\dagger}\subseteq h_*h^*\mathcal{O}_V(M_Z)|_{Z^\dagger}$. Then
$$h_*\mathcal{O}_V(K_{V/Z}+E_V)|_{Z^\dagger}=h_*h^*\mathcal{O}_V(M_Z)|_{Z^\dagger}.$$

By considering the action of G, we obtain the equality between the eigensheaves of ζ.
Since $Z_\dagger \subseteq Z$ is such that $Z\backslash Z_\dagger$ has codimension at least two, we have $\tilde{\mathcal{L}}^{**} = \mathcal{O}_Z(M_Z)\psi$. The sheaf $h_*\mathcal{O}_V(K_{V/Z}+E_V)$ is locally free and by the Theorem 4.16 in [7] it is semipositive. Since $\tilde{\mathcal{L}}$ is a direct summand of $h_*\mathcal{O}_V(K_{V/Z}+E_V)$ it is also locally free and semipositive. From the local freeness of $\tilde{\mathcal{L}}$ follows the equality $\tilde{\mathcal{L}} = \mathcal{O}_Z(M_Z)\psi$.
Now we prove that M_Z is an integral divisor.
Since $(-B_V + h^*B_Z)|_{h^{-1}(Z^\dagger)}$ contains no fibers and $Z^\dagger$ is big, $(-B_V + h^*B_Z)$ contains no fibers over codimension-one points. Hence for any prime Weil divisor $P \subseteq Z$ there exists a prime Weil divisor $Q \subseteq X$ such that $h(Q) = P$ and $\mathrm{mult}_Q(-B_V + h^*B_Z) = 0$. From the canonical bundle formula 1.4.4 we have

$$\mathrm{mult}_Q h^*M_Z = \mathrm{mult}_Q(K_{V/Z} + (\psi)) \in \mathbb{Z}.$$

Moreover we have $\mathrm{mult}_P M_Z \cdot \mathrm{mult}_Q h^*P = \mathrm{mult}_Q h^*M_Z$. The fact that the morphism h is semistable in codimension one implies that $\mathrm{mult}_Q h^*P = 1$, thus $\mathrm{mult}_P M_Z = \mathrm{mult}_Q h^*M_Z \in \mathbb{Z}$. □

Lemma 1.4.4 (Theorem 4.3 [2]). *We use the same notation as 1.4.2. There exists a finite Galois cover $\tau\colon Z' \to Z$ from a non-singular variety Z' which admits a simple normal crossing divisor supporting $\tau^{-1}(\Sigma_Z)$ and the locus where τ is not étale, and such that $h'\colon V' \to Z'$ is semistable in codimension one for some set-up $(V', B_{V'}) \to (X', B_{X'}) \to Z'$ induced by base change.*

The following theorem is a generalization of Theorem 4.4 in [2]. It was proven in [14] by using variation of mixed Hodge structure. Here we give a proof based on variation of Hodge structure.

Theorem 1.4.5. *Let $f\colon X \to Z$ be a surjective morhpism, let D_i be a reduced and irreducible divisor such that $f(D_i) = Z$. Set $D = \sum_{i=1}^N D_i$. Assume that*

- *we are in the SNC setting 1.4.2;*
- *the monodromies of $R^i f_{0*}\mathbb{C}_{X_0\backslash D_0}$ are unipotent for any i where $Z_0 = Z\backslash\Sigma_Z$, $X_0 = f^{-1}Z_0$, $D_0 = D \cap X_0$, $f_0 = f|_{X_0\backslash D_0}$.*

Let $\rho\colon Z' \to Z$ be a projective morphism from a non-singular variety Z' such that $\rho^{-1}\Sigma_Z$ is a simple normal crossings divisor. Let $X' \to (X \times Z')_{\mathrm{main}}$ be a resolution of the component of $X \times Z'$ which dominates Z', and let $f'\colon X' \to Z'$ be the induced fiber space:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow[\rho]{} & Z. \end{array}$$

Then for any $i \geq 0$ there exists a natural isomorphism $\rho^* R^i f_*\omega_{X/Z}(D) \cong R^i f'_*\omega_{X'/Z'}(D')$, where D' is the strict transform of D, which extends the base change isomorphism over $Z\backslash\Sigma_Z$.

First we have to state and prove some preliminary result.

Proposition 1.4.6. *Let $f\colon X \to Z$ be a surjective morphism. Assume that we are in the SNC setting 1.4.2. Let Z_0 be $Z\backslash\Sigma_Z$, let X_0 be $f^{-1}Z_0$ and $f = f|_{X_0}$. Assume that the local systems*

$R^i f_{0*}\mathbb{C}_{X^0}$ have unipotent monodromies around Σ_Z for any i. Let $\rho\colon Z' \to Z$ and X' be a projective morphism from a non-singular variety Z' such that $\rho^{-1}\Sigma_Z$ is a simple normal crossings divisor. Let $X' \to (X \times Z')_{main}$ be a resolution of the component of $X \times Z'$ which dominates Z', and let $f'\colon X' \to Z'$ be the induced fiber space. Then for any $i \geq 0$ there exists a natural isomorphism $\rho^* R^i f_* \omega_{X/Z} \cong R^i f'_* \omega_{X'/Z'}$.

Proof. Set $\Sigma_{Z'} = \rho^{-1}\Sigma_Z$, $Z'_0 = Z' \backslash \Sigma_{Z'}$, $X'_0 = f'^{-1}Z'_0$ and $f'_0 = f'|_{X'_0}$. The locally free sheaves $H_0^{(i)} = R^{m+i} f_{0*}\mathbb{C}_{X_0}$ and $H_0^{'(i)} = R^{m+i} f'_{0*}\mathbb{C}_{X'_0}$ are the underlying spaces of variation of Hodge structure of weight $m-i$. In [29, Thm 2.6, p. 176], is proven that

$$R^i f_* \omega_{X/Z} \cong {}^u\mathcal{F}^b(R^{m+i} f_* \mathbb{C}_{X_0}) \ \ \forall i \geq 0$$

$$R^i f'_* \omega_{X'/Z'} \cong {}^u\mathcal{F}^b(R^{m+i} f'_* \mathbb{C}_{X'_0}) \ \ \forall i \geq 0$$

where ${}^u\mathcal{F}^b$ denotes the upper canonical extension of the bottom part of the Hodge filtration. Since $H_0^{(i)}$ has unipotent local monodromies, the upper canonical extensions coincide with the canonical extensions. Moreover, by the unipotent monodromies assumption, the canonical extension is compatible with base change by [26, Prop 1, p. 4]. Hence by unicity of the extension the isomorphism $\rho^* R^i f_{0*} \omega_{X_0/Z_0} \cong R^i f'_{0*} \omega_{X'_0/Z'_0}$ induces an isomorphism $\rho^* R^i f_* \omega_{X/Z} \cong R^i f'_* \omega_{X'/Z'}$.

□

Proof of Theorem 1.4.5. Let N be the number of irreducible components of D. We prove the statement by double induction on N and on the dimension d of the fiber.

If $N = 0$ or $d = 0$ the result follows from Proposition 1.4.6. Suppose $N > 0$ and consider the exact sequence

$$0 \to \mathcal{O}_X(D_2) \to \mathcal{O}_X(D) \to \mathcal{O}_{D_1}(D) \to 0 \tag{1.4.5}$$

where $\tilde{D} = \sum_{i=2}^N D_i$. Set $\tilde{D}' = \sum_{i=2}^N D'_i$ and

$$A_i = \rho^* R^i f_* \omega_{X/Z}(\tilde{D}) \quad B_i = \rho^* R^i f_* \omega_{X/Z}(D) \quad C_i = \rho^* R^i f_* \omega_{D_1/Z}(\tilde{D})$$

$$A'_i = R^i f'_* \omega_{X'/Z'}(\tilde{D}') \quad B'_i = R^i f'_* \omega_{X'/Z'}(D') \quad C'_i = R^i f'_* \omega_{D'_1/Z'}(\tilde{D}').$$

We have a commutative diagram with exact lines

$$\begin{array}{ccccccccc} C_{i-1} & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & A_{i+1} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ C'_{i-1} & \longrightarrow & A'_i & \longrightarrow & B'_i & \longrightarrow & C'_i & \longrightarrow & A'_{i+1}. \end{array}$$

The morphisms β and ε are isomorphisms by the inductive hypothesis on N. The morphisms α and δ are isomorphisms by the inductive hypothesis on d. Then, by the snake lemma, also γ is also an isomorphism. □

Lemma 1.4.7. *Let $\gamma\colon Z' \to Z$ be a generically finite projective morphism from a non-singular variety Z'. Assume there exists a simple normal crossing divisor $\Sigma_{Z'}$ on Z' which contains $\gamma^{-1}\Sigma_Z$ and the locus where γ is not étale. Let $M_{Z'}$ be the moduli part of the induced set-up $(V', B_{V'}) \to (X', B_{X'}) \to Z'$. Then $\gamma^*(M_Z) = M_{Z'}$.*

Proof. The proof is exactly the same as that of [2, p. 248]. We just replace $\gamma^* h_* \mathcal{O}_V(K_{V/Z})$ with $\gamma^* h_* \mathcal{O}_V(K_{V/Z}+E)$ and $h'_* \mathcal{O}_{V'}(K_{V'/Z'})$ with $h'_* \mathcal{O}_{V'}(K_{V'/Z'}+E')$.
Step 1 Assume V/Z and V'/Z' are semistable in codimension one. In particular, M_Z and $M_{Z'}$ are integral divisors. Since h is semistable in codimension one, Theorem 1.4.5 implies

$$\gamma^* h_* \mathcal{O}_V(K_{V/Z}+E) \cong h'_* \mathcal{O}_{V'}(K_{V'/Z'}+E')$$

This isomorphism is natural, hence compatible with the action of the Galois group G. We have an induced isomorphism of eigensheaves corresponding to ζ, $\gamma^*\mathcal{O}_Z(M_Z) \cong \mathcal{O}_{Z'}(M_{Z'})$. Therefore $\gamma^* M_Z - M_{Z'}$ is linearly trivial, and is exceptional over Z . Thus $\gamma^* M_Z = M_{Z'}$.
Step 2 By [2, Theorem 4.3, p. 240] and [2, Theorem 4.1, p. 242], we can construct a commutative diagram

$$\begin{array}{ccc} \bar{Z} & \xrightarrow{\gamma'} & \bar{Z}' \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau'} \\ Z & \xrightarrow[\gamma]{} & Z'. \end{array}$$

as in [2, Remark 4.2, p. 241], so that $\bar{V}/\bar{Z}$ is semistable in codimension one for an induced set-up $(\bar{V}, B_{\bar{V}}) \dashrightarrow (\bar{X}, B_{\bar{X}}) \dashrightarrow \bar{Z}$. By [2, Theorem 4.3, p. 240] and [2, Theorem 4.1, p. 242], we replace Z' by a finite covering so that $\bar{V}'/\bar{Z}'$ is semistable in codimension one for an induced set-up $(\bar{V}', B_{\bar{V}'}) \dashrightarrow (\bar{X}', B_{\bar{X}'}) \dashrightarrow \bar{Z}'$. By Step 1, we have $M_{\bar{Z}'} = \gamma'^*(M_{\bar{Z}'})$. Since τ and τ' are finite coverings, Lemma 1.2.16 implies $\tau^*(M_Z) = M_{\bar{Z}}$ and $\tau'^*(M_{Z'}) = M_{\bar{Z}'}$. Therefore $\tau'^*(M_{Z'} - \gamma^*(M_Z)) = 0$, which implies $M_{Z'} = \gamma^*(M_Z)$.

□

Theorem 1.4.8. *Let $f\colon (X,B) \to Z$ be an lc-trivial fibration. Then there exists a proper birational morphism $Z' \to Z$ with the following properties:*

(i) *$K_{Z'} + B_{Z'}$ is a $\mathbb{Q}$-Cartier divisor;*

(ii) *$M_{Z'}$ is a nef $\mathbb{Q}$-Cartier divisor and for every proper birational morphism $\nu\colon Z'' \to Z'$*

$$\nu^*(M_{Z'}) = M_{Z''}$$

where $B_{Z'}$, $M_{Z'}$ and $M_{Z''}$ are the discriminant and the moduli parts of the lc-trivial fibrations induced by the base change

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ Z'' & \xrightarrow[\nu]{} & Z' & \longrightarrow & Z. \end{array}$$

Proof. The proof follows the same lines as in [2, p. 249].
We can suppose that we are in a SNC setting,

$$(V', B_V) \to (X', B) \to Z'.$$

- We prove that for any birational morphism $\mu\colon Z' \to Z$ we have $\mu^* M_{Z'} = M_{Z''}$ (we use Lemma 1.4.7). This proves the first point of the theorem.
- By Lemma 1.4.4 there exists a finite morphism $\tau\colon \bar{Z}' \to Z'$ such that $\bar{h}'\colon \bar{V}' \to \bar{Z}'$ is semistable in codimension one.
- By Lemma 1.4.3, the divisor $M_{\bar{Z}'}$ is integral and semipositive.
- Since τ is finite we can apply Lemma 1.2.16 and have $\tau^* M_{Z'} = M_{\bar{Z}'}$.
- Again since τ is finite and $M_{\bar{Z}'}$ is nef, $M_{Z'}$ is also nef.

□

1.5 Reduction theorems

This section is devoted to the proof of Theorem 1.1.4. Throughout this part we will assume that the bases of the lc-trivial fibrations are smooth projective varieties.

Lemma 1.5.1. *Let $f\colon (X,B) \to Z$ be an lc-trivial fibration. Then there exists a hyperplane section $H \subseteq Z$ such that, if $f|_{X_H}\colon (X_H, B|_{X_H}) \to H$ is the induced lc-trivial fibration, where $X_H = f^{-1}H$, we have $B_Z|_H = B_H$ and $M_Z|_H = M_H$.*

Proof. By the Bertini theorem, since Z is smooth, we can find a smooth hyperplane section $H \subseteq Z$ such that the pair $(X, B + f^{-1}(H) + \gamma_P f^* P)$ is lc for any prime Weil divisor $P \subseteq Z$ and $(X, B + f^{-1}(H) + t f^* P)$ is plt for any $P \subseteq Z$ prime divisor and for any $t < \gamma_P$. Let P be any prime Weil divisor in Z. Set

$$X_H = f^{-1}(H);\ \ B_{X_H} = B|_{X_H};\ \ P_H = P \cap H.$$

The restriction $f_H\colon (X_H, B_{X_H}) \to H$ is again an lc-trivial fibration. The canonical bundle formula for f_H is

$$K_{X_H} + B_{X_H} + \frac{1}{r}(\psi) = f_H^*(K_H + B_H + M_H).$$

By [30, Theorem 7.5] the pair $(X_H, B_{X_H} + \gamma_P f_H^* P_H)$ is lc for any $P_H \subseteq H$ prime divisor and $(X_H, B_{X_H} + t f_H^* P_H)$ is klt for any $P_H \subseteq H$ prime divisor and for any $t < \gamma_P$. Moreover let P such that P_H is a component of $P|_H$. Assume that we can compute the log canonical threshold over P on a component Q_{i_0} of f^*P. Then the coefficient of $B_{X_H} + \gamma_P f_H^* P_H$ along the components of $Q_{i_0}|_{X_H}$ is 1. Thus the log canonical threshold of $f_H^* P_H$ with respect to (X_H, B_H) is equal to the log canonical threshold of f^*P with respect to (X, B) and we have $B_Z|_H = B_H$.

If we write the canonical bundle formula for f, we have

$$K_X + B + \frac{1}{r}(\varphi) = f^*(K_Z + B_Z + M_Z).$$

If we sum f^*H on both sides of the equality, restrict to $f^{-1}H = X_H$ and apply the adjunction formula, we obtain

$$K_{X_H} + B_{X_H} + \frac{1}{r}(\varphi|_{X_H}) = f_H^*(K_H + B_Z|_H + M_Z|_H).$$

Since we have $B_Z|_H = B_H$, we must also have $M_Z|_H = M_H$. □

Lemma 1.5.1 is the main tool in order to prove by induction Theorem 1.1.4.

Proposition 1.5.2. *Conjecture **EbS(1)** implies that for any lc-trivial fibration $f\colon X \to Z$ we have*

$$\operatorname{codim}(\mathrm{Bs}|mM_Z|) \geq 2$$

*where $m = m(d, r)$ is as in Conjecture **EbS(1)**.*

Proof. We prove the statement by induction on $k = \dim Z$. The case $\dim Z = 1$ follows from **EbS(1)**. Suppose then that the statement is true for an lc-trivial fibration whose base has dimension $k - 1$ and let $f\colon X \to Z$ be an lc-trivial fibration with $\dim Z = k > 1$. Then we have

$$|mM_Z| = |M| + \mathrm{Fix}$$

where Fix is the fixed part of the linear system and $\operatorname{codim}(\mathrm{Bs}|M|) \geq 2$. Let H be a hyperplane section as in Lemma 1.5.1, such that $H - mM_Z$ is ample. By the Kodaira vanishing theorem

$$h^0(Z, mM_Z - H) = h^1(Z, mM_Z - H) = 0$$

and the restriction induces an isomorphism

$$H^0(Z, mM_Z) \cong H^0(H, mM_Z|_H) \cong H^0(H, mM_H).$$

Then if we write

$$\begin{aligned} |mM_Z|_{|H} &= |M|_{|H} + \mathrm{Fix}|_H \\ |mM_H| &= |L| + \mathrm{fix} \end{aligned}$$

where fix is the fixed component of the linear system $|mM_H|$, we have $\mathrm{fix} \supseteq \mathrm{Fix}|_H$. And since by inductive hypothesis $\mathrm{fix} = 0$ also $\mathrm{Fix}|_H = 0$ and then $\mathrm{Fix} = 0$. □

Corollary 1.5.3. *Conjecture **EbS(1)** implies that for any lc-trivial fibration $f\colon X \to Z$ we have $h^0(Z, mM_Z) \geq 2$, unless M_Z is torsion, where m is as in **EbS(1)**.*

Proof. By Proposition 1.5.2 there must be at least two sections, unless M_Z is torsion. □

Proof of Theorem 1.1.4. <u>We treat first the torsion case</u>. We prove by induction on the dimension of the base of the lc-trivial fibration that there exists a nonzero integer $m = m(d, r)$ such that $mM_Z \cong \mathcal{O}_Z$. If the dimension of the base equals one then it follows from Conjecture **EbS(1)**. Assume then that $f\colon X \to Z$ is an lc-trivial fibration with $\dim Z = k > 1$ and M_Z is torsion, that is, there exists an integer a such that $aM_Z \sim 0$.

Let H be a hyperplane section such that $M_Z|_H = M_H$, as in Lemma 1.5.1, and such that $H - mM_Z$ is an ample divisor. Since $M_Z|_H = M_H$, also M_H is torsion because

$$\mathcal{O}_H \cong \mathcal{O}_Z|_H \cong \mathcal{O}_H(aM_Z) \cong \mathcal{O}_H(aM_H).$$

By the Kodaira vanishing theorem, since $H - mM_Z$ is an ample divisor and $\dim Z = k > 1$, we have

$$H^0(Z, mM_Z) \cong H^0(H, mM_H).$$

By the inductive hypothesis mM_H is trivial, hence $h^0(H, mM_H) = 1$. Thus also $h^0(Z, mM_Z) = 1$ and $mM_Z \sim 0$.
Then we assume that M_Z is not torsion and we prove the statement by induction on the dimension of the base of the lc-trivial fibration. The one-dimensional case is exactly Conjecture **EbS(1)**. Suppose then that the statement is true for all the lc-trivial fibrations whose base has dimension $k-1$ and let $f\colon X \to Z$ be an lc-trivial fibration with $\dim Z = k$. Let $Z' \to Z$ be the birational model given by Theorem 1.2.15**(ii)**. We prove that $mM_{Z'}$ is base-point-free. Let $\nu\colon \hat{Z} \to Z'$ be a resolution of the linear system $|mM_{Z'}|$. Then $\nu^*|mM_{Z'}| = |\mathrm{Mob}| + \mathrm{Fix}$ where $|\mathrm{Mob}|$ is a base-point-free linear system and Fix is the fixed part. We have

$$\nu^*|mM_{Z'}| = |\nu^*(mM_{Z'})| = |mM_{\hat{Z}}|.$$

Since by Proposition 1.5.2 we have $\operatorname{codim}(\mathrm{Bs}|mM_{\hat{Z}}|) \geq 2$, it follows that $\mathrm{Fix} = 0$ and $|mM_{Z'}|$ is base-point-free. □

Remark 1.5.4. By considering as in Proposition 1.5.2 the long exact sequence associated with

$$0 \to \mathcal{O}_Z(mM_Z - H) \to \mathcal{O}_Z(mM_Z) \to \mathcal{O}_H(mM_Z|_H) \to 0$$

for a hyperplane section H as in Lemma 1.5.1, it is possible to also prove an inductive result on **effective non-vanishing**. That is, the existence of an integer $m = m(d, r)$ such that $H^0(Z, mM_Z) \neq 0$ for all lc-trivial fibrations $f\colon (X, B) \to Z$ with $\dim Z = 1$ implies the same result for lc-fibrations with $\dim Z = k \geq 1$ (and with same dimension of the fibers and Cartier index).

1.6 Bounding the denominators of the moduli part

Conjecture **EbS** 1.1.2 implies in particular the existence of an integer $N = N(d, r)$ such that for any $f\colon (X, B) \to Z$ lc-trivial fibration with fibers of dimension d and Cartier index of $(F, B|_F)$ equal to r the divisor NM_Z has integral coefficients. The result was proved in [41, Theorem 3.2] when the fiber is a rational curve. In Section 1.8 we find, by a different method, an effective bound for the denominators of M_Z in the case of general fiber isomorphic to $\mathbb{P}^1$ that is considerably smaller than the one in [41]. For the reader's convenience we present here an argument, due to Todorov [41, Theorem 3.2], valid in the general case.

Theorem 1.6.1. *Let b be a non-negative integer. There exists an integer $m = m(b)$ such that for any klt-trivial fibration $f\colon (X, B) \to Z$ with $\mathrm{Betti}_{\dim E'}(E') = b$ where E' is a non-singular model of the cover of a general fiber of f, $E \to F$ associated to the unique element of $|r(K_F + B|_F)|$ the divisor mM_Z has integral coefficients.*

We begin by reducing the problem to the case where the base Z is a curve.

Proposition 1.6.2. *If Theorem 1.6.1 holds for all fibrations whose bases have dimension one then Theorem 1.6.1 holds for fibrations whose bases have dimension $k \geq 1$.*

Proof. We prove the statement by induction on $k = \dim Z$. If $k = 1$, it follows from the hypothesis. Assume the statement holds for fibrations over bases of dimension $k - 1$ and we consider $f\colon X \to Z$

with $\dim Z = k$. Let H be a hyperplane section of Z as in Lemma 1.5.1. We have thus $M_Z|_H = M_H$. Since H is ample, it meets each component of M_Z and we can choose it such that it meets transversally the components of M_Z. It follows that NM_Z has integral coefficients if and only if NM_H does, and we are done by inductive hypothesis. □

Proof of Theorem 1.6.1. By Proposition 1.6.2 we can assume that $\dim Z = 1$.
Consider a finite base change as in Theorem 1.3.12

$$\tau\colon Z' \to Z.$$

If $h'\colon V' \to Z'$ is the induced morphism then $R^d h'_* \mathbb{C}_{V'_0}$ has unipotent monodromies. The covering τ is Galois and let G be its Galois group. Then we have an action of G on Z'

$$G \quad \to \quad \mathrm{Bir}(Z') = \mathrm{Aut}(Z'). \tag{1.6.1}$$

By abuse of notation we will denote by g both an element of G and its image in $\mathrm{Aut}(Z')$.

Let $p' \in Z'$ be a point and let e be the ramification order of τ at p'. Let $G_{p'}$ be the stabilizer of p' with respect to the action (1.6.1). Set $\mu_e = \{x \in \mathbb{C} \,|\, x^e = 1\}$.
There exists an analytic open set $p' \in U \subseteq Z'$ and a local coordinate z on U centered in p' such that for any $g \in G_{p'}$ there exists $x \in \mu_e$ such that

$$\begin{array}{rccc} g|_U\colon & U & \longrightarrow & U \\ & z & \longmapsto & xz. \end{array}$$

This induces a natural homomorphism

$$G_{p'} \to \mu_e.$$

Then the actions of $G_{p'}$ given by Proposition 1.3.11 factorize through actions of μ_e:

$$\Phi\colon \mu_e \to GL((R^d h'_* \mathbb{C}_{V'})_{p'}),$$

$$\Psi\colon \mu_e \to GL((h'_* \omega_{V'/Z'})_{p'})$$

which commute with the inclusion $(h'_* \omega_{V'/Z'})_{p'} \subseteq (R^d h'_* \mathbb{C}_{V'})_{p'}$, that is, such that for any $\zeta \in \mu_e$ the restriction of $\Phi(\zeta)$ to $(h'_* \omega_{V'/Z'})_{p'}$ equals $\Psi(\zeta)$.
Thus on

$$(h'_* \omega_{V'/Z'})_{p'} = \bigoplus_{i=0}^{r-1} f_* \mathcal{O}_X(\lceil (1-i)K_{X/Z} - iB + if^* B_Z + if^* M_Z \rceil)$$

we have two actions:

- one by the group μ_e that acts on φ by a multiplication by an e-th root of unity,
- one by the group μ_r that acts on $\sqrt[r]{\varphi}$ by a multiplication by an r-th root of unity.

Then there is a $\mu_r \rtimes \mu_e$-action on $(h'_* \omega_{V'/Z'})_{p'}$ and we can define a μ_l-action on $(h'_* \omega_{V'/Z'})_{p'}$ where $l = er/(e, r)$. Since $\mu_r \subseteq \mu_l$ and this second group is commutative, the action of μ_l preserves the eigensheaves with respect to the action of μ_r. By Proposition 1.3.16, the divisor $M_{Z'}$ is an eigensheaf with respect to the action of μ_r. Then μ_l acts on the stalk $\mathcal{O}_{Z'}(M_{Z'}) \otimes \mathbb{C}_{p'}$ by a character $\chi_{p'}$.

If for every p' and for every character $\chi_{p'}$ the order of $\chi_{p'}$ divides an integer N then

$$NM_Z = (\tau_*\mathcal{O}(NM_{Z'}))^G$$

because by Proposition 1.2.16 we have $M_{Z'} = \tau^* M_Z$. Thus NM_Z is a Cartier divisor.

Let $\mathcal{H}'$ be the canonical extension of the sheaf $(R^d(h'_0)_*\mathbb{C})_{\text{prim}} \otimes \mathcal{O}_{Z'_0}$ to Z', where d is the dimension of the fiber of h', the subscript prim stands for the primitive part of the cohomology and $h'_0 \colon V'_0 \to Z'_0$ is the restriction to the smooth locus. The Hodge filtration also extends and its bottom piece is $h'_*\mathcal{O}_{V'}(K_{V'/Z'})$. Then all the characters that are conjugate to $\chi_{p'}$ must appear as subrepresentations of $\mathcal{H}'_{p'}$ (see also [10, Corollary 4.2.8(ii)(b)] or Corollary 1.3.14).

If $\chi_{p'}$ acts by a primitive k-th root of unity, then its conjugate subrepresentations are $\phi(k)$ where ϕ is the Euler function. This bounds k because then $\phi(l) \leq B_d$, where $B_d = h^d(E', \mathbb{C})$ is the d-th Betti number.
Set $m(x) = \operatorname{lcm}\{k \mid \phi(k) \leq x\}$. Then $m(B_d)M_Z$ has integral coefficients. □

1.7 The case $M_Z \equiv 0$

In this section Z will always be a smooth projective variety.

1.7.1 KLT-trivial fibrations with numerically trivial moduli part

The goal of this subsection is the proof of Theorem 1.1.5. As in Theorem 1.1.4 the problem can be reduced to the case where the base is a curve.

Proposition 1.7.1. *Let b be a non-negative integer. Assume that there exists an integer $m = m(b)$ such that for any klt-trivial fibration $f \colon (X, B) \to Z$ with*

- $\dim Z = 1$*;*
- $M_Z \equiv 0$*;*
- $\text{Betti}_{\dim E'}(E') = b$ *where E' is a non-singular model of the cover $E \to F$ associated to the unique element of $|r(K_F + B|_F)|$;*

we have $mM_Z \sim 0$.

Then the same holds for bases Z of arbitrary dimension.

Proof. We proceed by induction on $\dim Z = k$. The base of induction is the hypothesis of the theorem.
Let us assume the statement holds for bases of dimension $k-1$ and prove it for a klt-fibration $f \colon (X, B) \to Z$ with $\dim Z = k$. Let H be a hyperplane section, given by Lemma 1.5.1, such that $M_Z|_H = M_H$. Let m be the integer given by the inductive hypothesis. Since $M_Z \equiv 0$ the divisor $H - mM_Z$ is ample. By taking the long exact sequence associated to

$$0 \to \mathcal{O}_Z(mM_Z - H) \to \mathcal{O}_Z(mM_Z) \to \mathcal{O}_H(mM_Z|_H) \to 0$$

we obtain $H^0(Z, mM_Z) \cong H^0(H, mM_H)$ because $H^i(Z, mM_Z - H) = 0$ for any $i < \dim Z$. Then $H^0(Z, mM_Z) \cong \mathbb{C}$, which implies $mM_Z \sim 0$. □

Proof of Theorem 1.1.5. By Proposition 1.7.1, it is sufficient to prove the statement when the base Z is a curve.

Let us write the canonical bundle formula for f:

$$K_X + B + \frac{1}{r}(\varphi) = f^*(K_Z + B_Z + M_Z).$$

Let V be a smooth model of the normalization of X in $\mathbb{C}(X)(\sqrt[r]{\varphi})$. Let $h\colon V \to Z$ be the induced morphism and let Z_0 be the open set where h is smooth. Let $V_0 = h^{-1}Z_0$ and $h_0 = h|_{V_0}$.

(i) Let us suppose that $R^d h_{0*}\mathbb{C}_{V_0}$ has unipotent monodromies. We argue as in [2], Theorems 4.5 and 0.1.
The line bundle $\mathcal{O}_{Z_0}(M_Z)$ is a direct summand in $h_{0*}\omega_{V_0/Z_0}$ (see [2, Lemma 5.2]) and since $\deg M_Z = 0$ by [19] it defines a local subsystem of the variation of Hodge structure $R^d h_{0*}\mathbb{C}_{V_0}$. By Corollary 1.3.14, applied to $\mathcal{H}_{\mathbb{C}} = R^d h_{0*}\mathbb{C}_{V_0}$ and $W = \mathcal{O}_{Z_0}(M_Z)$, there exists m such that $\mathcal{O}_{Z_0}(mM_Z)$ is a trivial local system where

$$m = m(b) = \mathrm{lcm}\{k | \phi(k) \leq b\}$$

with ϕ the Euler function and $b = h^d(E', \mathbb{C})$. Since $R^d h_{0*}\mathbb{C}_{V_0}$ has unipotent monodromies, the canonical extension commutes with the tensor product, thus the isomorphism $\mathcal{O}_{Z_0}(mM_Z) \cong \mathcal{O}_{Z_0}$ extends to

$$\mathcal{O}_Z(mM_Z) \cong \mathcal{O}_Z.$$

(ii) Unipotent reduction: Consider a finite base change as in Proposition 1.3.12

$$\tau\colon Z' \to Z$$

such that $R^d h'_*\mathbb{C}_{V'_0}$ has unipotent monodromies, where $h'\colon V' \to Z'$ is the induced morphism. We have $\tau = \tau_k \circ \cdots \circ \tau_1$ where

$$\tau\colon Z' = Z_{k+1} \xrightarrow{\tau_k} Z_k \to \cdots \to Z_2 \xrightarrow{\tau_1} Z_1 = Z.$$

The morphism τ_j is a cyclic covering defined by building data

$$\delta_j A_j \sim H_j,$$

where A_j is a very ample divisor on Z_j. We know by case **(i)** that $m(b)M_{Z'} \sim 0$. By Theorem 1.6.1 $m(b)M_{Z_k}$ is a Cartier divisor. We have thus the following isomorphisms:

$$\mathbb{C} \cong H^0(Z_{k+1}, m(b)M_{Z_{k+1}}) \cong H^0(Z_{k+1}, \tau^* m(b)M_{Z_k}) \cong H^0(Z_k, m(b)M_{Z_k} \otimes \tau_*\mathcal{O}_{Z_{k+1}}).$$

The second isomorphism is by Proposition 1.2.16 and the third follows from the projection formula. From the general theory of cyclic covers we have an isomorphism

$$\tau_*\mathcal{O}_{Z_{k+1}} \cong \bigoplus_{l=0}^{\delta_k - 1} \mathcal{O}_{Z_{k+1}}(-lA_k).$$

Then we obtain

$$H^0(Z_{k+1}, m(b)M_{Z_{k+1}}) \cong \bigoplus_{l=0}^{\delta_k - 1} H^0(Z_k, m(b)M_{Z_k} - lA_k).$$

Since $M_Z \equiv 0$, the divisor $m(b)M_{Z_k} - lA_k$ has negative degree on Z_k for any $l > 0$, thus

$$\mathbb{C} \cong H^0(Z_{k+1}, m(b)M_{Z_{k+1}}) \cong H^0(Z_k, m(b)M_{Z_k})$$

and $m(b)M_{Z_k} \sim 0$ We can conclude by induction on k.

□

Remark 1.7.2. Note that the same proof as in point **(ii)** of the proof of Theorem 1.1.5 implies a statement on **Effective non-vanishing** (see also Remark 1.5.4). Indeed let $\tau \colon Z' \to Z$ be as in Proposition 1.3.12. Assume that $H^0(Z', m(b)M_{Z'}) \neq 0$. Then the reasoning above implies that $H^0(Z, m(b)M_Z) \neq 0$.

1.7.2 LC-trivial fibrations with numerically trivial moduli part

In this section we prove Theorem 1.1.6. Let $f \colon (X, B) \to Z$ be an lc-trivial fibration. Let V be a smooth model of the normalization of X in $\mathbb{C}(X)(\sqrt[r]{\varphi})$, let $g \colon V \to X$ be induced morphism and let $B_V = g^*(K_X + B) - K_V$. Assume moreover that (V, B_V) is log smooth and let P_V be the divisor given by the sum of the components E of B_V of coefficient one and such that $h(E) = Z$. Let $h = f \circ g$ and let Z_0 be the open set where h is smooth. Let $V_0 = h^{-1}Z_0$ and $h_0 = h|_{V_0}$. Let d be the dimension of the generic fiber.

By [40, §5], $R^d h_{0*}\mathbb{C}_{V_0 \setminus P_V}$ is the support of a variation of mixed Hodge structure on Z_0,

$$(R^d h_{0*}\mathbb{C}_{V_0 \setminus P_V}, \{\mathcal{F}^p\}, \{\mathcal{W}_k\})$$

Such that the bottom piece of the Hodge filtration $\{\mathcal{F}^p\}$ is

$$\mathcal{F}^d = h_{0*}\omega_{V_0/Z_0}(P_V).$$

We recall that, by the definition of variation of mixed Hodge structure, the filtration induced by $\{\mathcal{F}^p\}$ on $\mathcal{W}_k/\mathcal{W}_{k-1}$ determines a variation of Hodge structure of weight k on Z_0. Moreover the weight filtration on $\mathcal{F}^b$ is the weight filtration defined in (1.3.1)

$$\mathcal{W}_k((h_0)_*\omega_{V_0/Z_0}(P_V)) = (h_0)_*\Omega^k_{V_0/Z_0}(\log D) \wedge \Omega^{d-k}_{V_0/Z_0}.$$

In order to prove Theorem 1.1.6, we prove that $\mathcal{O}_{Z_0}(M_Z)$ is a subsystem of a variation of Hodge structure related to the variation of mixed Hodge structure on $R^d h_{0*}\mathbb{C}_{V_0 \setminus P_V}$. We start with the following two results.

Proposition 1.7.3. *Assume that for any lc-trivial fibration $f \colon (X, B) \to Z$ with*

- $\dim Z = 1$;

- *$M_Z \equiv 0$;*

there exists an integer m such that $mM_Z \sim 0$.

Then the same conclusion holds for bases Z of arbitrary dimension.

Proof. The proof follows the same lines as the proof of Proposition 1.7.1. □

Lemma 1.7.4 (Lemma 21, [25])**.** *Let $\mathcal{L}$ be an invertible sheaf over a non-singular projective curve C, let C_0 be an open subset of C and let h be a metric on $\mathcal{L}|_{C_0}$. Let p be a point of $C\backslash C_0$ and let t be a local parameter of C centered at p. We assume that for a uniformizing section v of $\mathcal{L}$ we have $h(v,v) = O(t^{-2\alpha_p}|\log t|^{\beta_p})$ for some real numbers α_p, β_p. Then*

$$\deg_C \mathcal{L} = \frac{i}{2\pi}\int_{C_0}\Theta + \sum_{p\in C\backslash C_0}\alpha_p$$

where Θ is the curvature associated to h.

The following is a generalization of [3, Prop 3.4].

Proposition 1.7.5. *Let V and Z be smooth projective varieties. Let $h\colon V \to Z$ be a fibration and let Σ_Z be a simple normal crossing divisor such that*

- *h is smooth over $Z_0 = Z\backslash\Sigma_Z$,*
- *$\mathcal{W}_l/\mathcal{W}_{l-1}$ has unipotent monodromies, where $\{\mathcal{W}_k\}$ is the weight filtration.*

Let $\mathcal{L}$ be an invertible sheaf such that $\mathcal{L}|_{Z_0}$ is a direct summand of $\mathcal{W}_l/\mathcal{W}_{l-1}$ for some l. Assume that $\mathcal{L} \equiv 0$. Then $\mathcal{L}|_{Z_0}$ is a local subsystem of $\mathcal{W}_l/\mathcal{W}_{l-1}$.

Proof. Since $\mathcal{W}_l/\mathcal{W}_{l-1}$ is a geometric variation of Hodge structure, there is on it a flat bilinear form Q and thus a metric and a metric connection. Then $\mathcal{L}|_{Z_0}$ has an induced hermitian metric h, a metric connection and a curvature Θ. To prove that $\mathcal{L}|_{Z_0}$ defines a flat subsystem of $\mathcal{W}_l/\mathcal{W}_{l-1}$ it is sufficient to prove that the induced metric connection is flat, i.e. that $\Theta = 0$.
The relation between the matrix Γ of the connection and the matrix H that represents the metric is $\Gamma = \bar{H}^{-1}\partial\bar{H}$. By Remark 1.3.9, the order near p of the elements of Γ is $O(|t|^{-1}|\log t|^{\beta_p})$. Let v be a uniformizing section of $\mathcal{L}$. Then the order of $h(v,v)$ near p is $O(|\log t|^{\beta_p})$. Let $C \subseteq Z$ be a curve such that $C \cap Z_0 \neq \emptyset$. Let $\nu\colon \hat{C} \to C$ be its normalization and $C_0 = C \cap Z_0$. We apply Lemma 1.7.4 and we obtain

$$\deg_C \mathcal{L} = \frac{i}{2\pi}\int_{\nu^{-1}C_0}\nu^*\Theta.$$

Since $\mathcal{L}$ is numerically zero, we obtain

$$\int_{\nu^{-1}C_0}\nu^*\Theta = 0$$

for every C and therefore $\Theta = 0$. □

Let us recall that we are working with a cyclic covering $V \to X$ of degree r and that the Galois group of the field extension $\mathbb{C}(X) \subseteq \mathbb{C}(V)$ is the group of r-th roots of unity that we denote by μ_r. Moreover we have an induced action of μ_r on the sheaves of relative differentials. Let P_V be the divisor given by the sum of the components E of B_V of coefficient one and such that $h(E) = Z$.

Lemma 1.7.6. *The action of μ_r on $(h_0)_*\omega_{V_0/Z_0}(P_V)$ preserves the weight filtration*

$$\{\mathcal{W}_k((h_0)_*\omega_{V_0/Z_0}(P_V))\}.$$

Proof. A generator of μ_r determines a birational map $\sigma\colon V \dashrightarrow V$. Consider a resolution of σ

$$\begin{array}{ccc} V_1 & & \\ {\scriptstyle \sigma_1}\downarrow & \searrow^{\sigma_2} & \\ V & \overset{}{\underset{\sigma}{\dashrightarrow}} & V. \end{array}$$

Let $V_{1,0} \subseteq V$ be the locus where $h \circ \sigma_1$ is smooth. Let us consider the weight filtrations on $\Omega^\bullet_{V_0/Z_0}(\log P_V)$ and on $\Omega^\bullet_{V_{1,0}/Z_0}(\log P_{V_1})$ where

$$P_{V_1} = \operatorname{Supp}(\sigma_2^{-1} P_V).$$

The morphism σ_2 induces, for any m, the following:

$$\sigma_2^*\colon \Omega^m_{V_0/Z_0}(\log P_V) \to \Omega^m_{V_{1,0}/Z_0}(\log P_{V_1}).$$

We want to prove that σ_2 preserves the weight filtration. Since σ_2 is a composition of blow-ups of smooth centers it is sufficient to prove the property for one blow-up.
Let $z_1, \dots, z_n$ be a system of coordinates on $U \subseteq V_0$ such that

$$P_V \cap U = \{z_1 \cdot \ldots \cdot z_k = 0\}.$$

Then

$$\sigma_2^*\left(\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_h}{z_h} \wedge dz_{h+1} \wedge \cdots \wedge dz_n\right) = \sigma_2^*\left(\frac{dz_1}{z_1}\right) \wedge \cdots \wedge \sigma_2^*\left(\frac{dz_h}{z_h}\right) \wedge \sigma_2^*(dz_{h+1}) \wedge \cdots \wedge \sigma_2^*(dz_n).$$

Let C be the center of the blow-up, let z_i be one of the coordinates. There are two cases

(i) locally C is contained in the zero locus of z_i;

(ii) C is not contained in the zero locus of z_i.

In case **(i)**, let t be an equation of the exceptional divisor and let z_i' be an equation of the strict transform of z_i. Then $\sigma_2^*(dz_i) = d(z_i' \cdot t) = t \cdot dz_i' + z_i' \cdot dt$ and

$$\sigma_2^*\left(\frac{dz_i}{z_i}\right) = \frac{t \cdot dz_i' + z_i' \cdot dt}{z_i' t} = \frac{dz_i'}{z_i'} + \frac{dt}{t}.$$

In case **(ii)**, we simply have

$$\sigma_2^*(dz_i) = dz_i' \text{ and } \sigma_2^* \frac{dz_i}{z_i} = \frac{dz_i'}{z_i'}.$$

Finally, the morphism σ_1 acts by pushforward and that does not increase the number of poles. □

Proof of Theorem 1.1.6. By Proposition 1.7.3 we can assume that the base Z is a curve.
Let us suppose that $\mathcal{W}_k/\mathcal{W}_{k-1}$ has unipotent monodromies for every k. Since by Lemma 1.7.6 the action of μ_r preserves the weight filtration

$$\{\mathcal{W}_k((h_0)_*\omega_{V_0/Z_0}(P_V))\}$$

on V, then for any k the sheaf

$$\mathcal{W}_k((h_0)_*\omega_{V_0/Z_0}(\log P_V)) = h_*(\Omega^k_{V_0/Z_0}(\log P_V) \wedge \Omega^{n-k}_{V_0/Z_0})$$

decomposes as sum of eigensheaves. In particular, since $\mathcal{O}_Z(M_Z)$ is an eigensheaf of rank one of $h_*\omega_{V/Z}(P_V)$, there exist l such that $\mathcal{O}_Z(M_Z)|_{Z_0} \subseteq W_l$ and $\mathcal{O}_Z(M_Z)|_{Z_0} \not\subseteq \mathcal{W}_{l-1}$. Thus there exists $\mathcal{V}$ containing $\mathcal{W}_{l-1}$ such that $\mathcal{W}_l = \mathcal{O}_Z(M_Z)|_{Z_0} \oplus \mathcal{V}$. By Proposition 1.7.5 thus $\mathcal{O}_Z(M_Z)|_{Z_0}$ defines a local subsystem of $\mathcal{W}_l/\mathcal{W}_{l-1}$. By Corollary 1.3.14 we have $\mathcal{O}_Z(M_Z)|_{Z_0}^{m(h)} \sim \mathcal{O}_Z$ with $h = \mathrm{rk}\mathcal{W}_l/\mathcal{W}_{l-1}$. Since if the monodromies are unipotent the canonical extension commutes with tensor product, we have $\mathcal{O}_Z(M_Z)^{m(h)} \sim \mathcal{O}_Z$.

The general situation, when $\mathcal{W}_k/\mathcal{W}_{k-1}$ has non unipotent monodromies for some k, can be reduced to the unipotent situation. We take a covering $\tau\colon Z' \to Z$ such that on Z' we have unipotent monodromies. Then $m(h)M_{Z'} \sim 0$ and since $M_{Z'} = \tau^* M_Z$ we have $\deg \tau \cdot m(h) M_Z \sim 0$. □

Remark 1.7.7. It follows from the proof that it is possible to bound the torsion index of $\mathcal{O}_Z(M_Z)$ in terms of the rank of $\mathcal{W}_l/\mathcal{W}_{l-1}$. This rank is for instance less or equal than $h^d(E \backslash P_E, \mathbb{C})$ where $\rho\colon E \to F$ is a smooth model of the covering of F induced by $|r(K_F + B|_F)|$ and P_E are the places of the pair (E, B^E) obtained by $\rho^*(K_F + B|_F) = K_E + B^E$. It would be useful to determine exactly the l such that $\mathcal{O}_Z(M_Z)$ is a subline bundle of $\mathcal{W}_l/\mathcal{W}_{l-1}$ in order to have a bound that is easier to compute.

1.8 Bounds on the denominators when the fiber is a rational curve

1.8.1 A useful result on blow-ups on surfaces

Let X be a smooth surface. Let $\delta\colon \hat{X} \to X$ be a sequence of blow-ups, $\delta = \varepsilon_h \circ \cdots \circ \varepsilon_1$ and denote by p_i the point blown-up by ε_i. In what follows by abuse of notation we will denote by E_i the exceptional curve of ε_i as well as its birational transform in further blow-ups. **In what follows we will suppose that in** $\mathrm{Exc}(\delta)$ **there is just one** (-1)**-curve.** Since the exceptional curve E_h of ε_h is a (-1)-curve it is the only (-1)-curve of $\mathrm{Exc}(\delta)$. Suppose that the first point p_1 that is blown-up belongs to a smooth curve F. We will denote by $\tilde{F}$ the strict transform of F by $\varepsilon_i \circ \cdots \circ \varepsilon_1$ for each i.

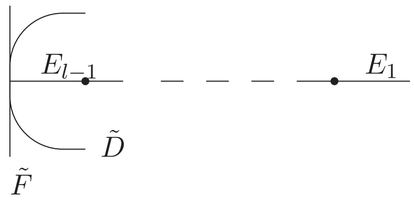
Lemma 1.8.1. *Let $f\colon (X, B) \to Z$ be an lc-trivial fibration such that $f\colon X \to Z$ is a $\mathbb{P}^1$-bundle on a smooth curve Z and suppose that $B = (2/d)D$ where D is a reduced divisor such that $D \cdot F = d$. Suppose moreover that there is a point $p \in Z$ such that D is tangent to $F = f^*p$ at a smooth point q of D with contact of order $l \in [d/2, d)$. Then the log canonical threshold*

$$\gamma := \gamma_p = \sup\{t \in \mathbb{R} |\, ((X, B), tf^*p) \text{ is lc over } p\}$$

has the following expression

$$\gamma = 1 + \frac{1}{l} - \frac{2}{d}.$$

Proof. A log resolution for the pair $(X, 2/dD + \gamma_p F)$ over p is a sequence of blow ups $\delta = \varepsilon_l \circ \cdots \circ \varepsilon_1$ such that ε_1 blows up q and for any $i \in \{2, \ldots, l\}$ the morphism ε_i is the blow up of the point $\tilde{F} \cap E_{i-1}$. A picture of the $(l-1)$-th step is



Then

$$\delta^* D = \tilde{D} + \sum_{j=1}^{l} j E_j,$$

$$\delta^*(\frac{2}{d} D) = \frac{2}{d}\tilde{D} + \frac{2}{d}\sum_{j=1}^{l} j E_j,$$

and

$$\delta^* K_X = K_{\tilde{X}} + \sum_{j=1}^{l} j E_j.$$

By Remark 1.2.13 γ is computed by

$$\begin{aligned} \gamma &= \min\{1, \min_{i \in \{1 \ldots l\}} \{1 + \frac{1}{i} - \frac{2}{d}\}\} \\ &= \min\{1, 1 + \frac{1}{l} - \frac{2}{d}\} \end{aligned}$$

we obtain

$$\gamma = 1 + \frac{1}{l} - \frac{2}{d}.$$

□

1.8.2 Local results

In this section we will always be in the situation where the fibers have dimension 1. In this case, if $B = 0$ the condition that K_F is torsion implies the generic fiber is an elliptic curve. If $B \neq 0$ then F has to be a rational curve and the second condition in the definition of the lc-trivial fibrations implies that the horizontal part of B is effective.

Thanks to the following lemma, studying the denominators of M_Z is the same thing as studying the denominators of B_Z.

Lemma 1.8.2. *Let Z be a smooth projective variety. Let $f\colon (X,B) \to Z$ be an lc-trivial fibration whose general fiber is $\mathbb{P}^1$. Then for all $I \in \mathbb{N}$ the divisor IrB_Z has integral coefficients if and only if IrM_Z has integral coefficients.*

Proof. By cutting with sufficiently general hyperplane sections we can assume that $\dim Z = 1$. We write the canonical bundle formula for $f\colon (X,B) \to Z$:

$$K_X + B + \frac{1}{r}(\varphi) = f^*(K_Z + B_Z + M_Z).$$

Let $\nu\colon \hat{X} \to X$ be a desingularization of X, let $\hat{B}$ be the divisor defined by

$$K_{\hat{X}} + \hat{B} = \nu^*(K_X + B)$$

and $\hat{f} = f \circ \nu$. Then $\hat{f}\colon (\hat{X}, \hat{B}) \to Z$ is lc-trivial by Remark 1.2.10 and has the same discriminant as f. Moreover it has the same moduli part, since

$$K_{\hat{X}} + \hat{B} + \frac{1}{r}(\varphi) = \nu^*(K_X + B) + \frac{1}{r}(\varphi) = \hat{f}^*(K_Z + B_Z + M_Z).$$

The surface $\hat{X}$ is smooth and $\hat{X} \to Z$ has generic fiber $\mathbb{P}^1$ hence there exists a birational morphism defined over Z

$$\begin{array}{ccc} \hat{X} & \longrightarrow & X' \\ {\scriptstyle \hat{f}}\downarrow & \swarrow {\scriptstyle f'} & \\ Z & & \end{array}$$

where $f'\colon X' \to Z$ is a $\mathbb{P}^1$-fibration. It follows that each fiber of $\hat{f}$ has an irreducible component with coefficient one. Then the statement follows from the equality

$$r(K_{\hat{X}} + \hat{B}) + (\varphi) = r\hat{f}^*(K_Z + B_Z + M_Z).$$

Indeed the divisor

$$r(K_{\hat{X}} + \hat{B}) + (\varphi) - r\hat{f}^*K_Z$$

is integral, then so is $r\hat{f}^*(B_Z + M_Z)$. Let $p \in Z$ be a point, let G be a component of the fiber $\hat{f}^*p$ with coefficient 1. Then

$$\operatorname{coeff}_G(r\hat{f}^*(B_Z + M_Z)) = r\operatorname{coeff}_p(B_Z + M_Z) \in \mathbb{Z}.$$

It follows that $rI\operatorname{coeff}_p(B_Z)$ is an integer if and only if $rI\operatorname{coeff}_p(M_Z)$ is. □

Theorem 1.8.3. *Let $f\colon (X,B) \to Z$ be an lc-trivial fibration such that X is a smooth projective surface, Z is a smooth projective curve and $f\colon X \to Z$ is a $\mathbb{P}^1$-bundle. Let $p \in Z$ be a point and γ be the log canonical threshold of f^*p with respect to (X,B). Then there is a constant $m \le 2r^2$ such that $m\gamma$ is an integer. Such an m is of the form lr where $l \le 2r$.*

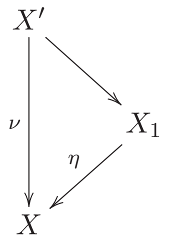
Proof. The pair $(X, B + \gamma F)$ is lc and not klt, that is, it has an lc center. There are now two cases.
The center has dimension one.
If the center has dimension one, then it is the whole fiber because all the fibers are irreducible. In this case we have

$$1 = \mathrm{mult}_F(B + \gamma F) = \mathrm{mult}_F(B) + \gamma$$

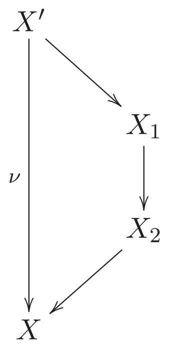
and since $r\mathrm{mult}_F(B) \in \mathbb{Z}$ also $r\gamma \in \mathbb{Z}$.
The center has dimension zero.
Step 1 Take $\nu\colon X' \to X$ a log resolution of $(X, B + \gamma F)$. Notice that the fiber over p is a tree of $\mathbb{P}^1$. Since $(X, B+\gamma F)$ is lc and not klt, one of the components of $\nu^* f^* p$ is a place for the pair $(X, B+\gamma F)$, that is, the coefficient of the divisor

$$\nu^*(K_X + B + \gamma F) - K_{\hat{X}}$$

along it is one. Write ν as a composition of blow-ups, set $\nu = \varepsilon_N \circ \cdots \circ \varepsilon_1$ and let k be the minimum of the indices such that the exceptional curve E_k of ε_k is a place for $(X, B + \gamma F)$. Let η be the composition $\varepsilon_k \circ \cdots \circ \varepsilon_1 \colon X_1 \to X$. We have:



If the only η-exceptional (-1)-curve in X_1 is E_k then we set $\hat{X} = X_1$ and $\delta := \eta$. Otherwise, if there is another (-1)-curve, by the Castelnuovo's theorem we can contract it in a smooth way:



This process ends because in X' there were finitely many ν-exceptional curves. Then we obtain a smooth surface $\hat{X}$ such that the only (-1)-curve in X is P. Set $\delta\colon \hat{X} \to X$. Modulo renumbering the indices we can assume that $\delta = \varepsilon_h \circ \cdots \circ \varepsilon_1$.

Step 2 We have obtained $\hat{X}$ smooth with a diagram

$$\begin{array}{ccc} X' & & \\ & \searrow & \\ \downarrow \nu & & \hat{X} \\ & \swarrow \delta & \\ X & & \end{array}$$

where there is only one δ-exceptional (-1)-curve that is a place for the pair $(X, B+\gamma F)$. Let p_i be the point blown up by ε_i. Let $\tilde{B}_i^j$ be the strict transform of the component B_i of B at the step j and $\tilde{B}^j$ be the strict transform of B. By abuse of notation we will denote by $\tilde{F}$ the strict transform of F by every ε_i and by E_i the exceptional curve of ε_i as well as its strict transform in the further blow-ups. Notice that E_h is the unique place. In what follows we will adopt the following notation:

$$B = \sum b_i B_i;$$

$$\delta^* K_X = K_{\hat{X}} - \sum e_i E_i; \quad \delta^* B = \tilde{B} + \sum \alpha_i E_i; \quad \delta^* F = \tilde{F} + \sum a_i E_i.$$

Here $\tilde{B}$ and $\tilde{F}$ denote the strict transform of B and F. Since $b_i \in 1/r\mathbb{Z}$ for any i, we have

$$\alpha_i \in \frac{1}{r}\mathbb{Z} \text{ for any i.} \tag{1.8.1}$$

Since E_h is a place, we have

$$1 = \mathrm{mult}_{E_h}(\delta^*(K_X + B + \gamma F) - K_{\hat{X}}) = -e_h + \alpha_h + \gamma a_h.$$

Since e_h is an integer and $\alpha_h \in 1/r\mathbb{Z}$, if we prove that $a_h \leq 2r$ we are done. By the minimality of δ there exists a component B_1 of B such that the strict transform $\tilde{B}_1^h$ of B_1 meets E_h, that is, $\tilde{B}_1^h \cdot E_h > 0$. Then

$$\begin{aligned} 2r &\geq B_1 \cdot F = \delta^* B_1 \cdot \delta^* F = \tilde{B}_1^h \cdot \delta^* F = \tilde{B}_1^h \cdot (\tilde{F} + \sum a_i E_i) \\ &\geq a_h \tilde{B}_1^h \cdot E_h \geq a_h. \end{aligned}$$

□

We can finally prove the main local result.

Theorem 1.8.4. *Let X and Z be smooth projective varieties. Let $f\colon (X, B) \to Z$ be an lc-trivial fibration whose generic fiber is $\mathbb{P}^1$. Let $B_Z = \sum \beta_i P_i$ be the discriminant. Then for every i there exists $l_i \leq 2r$ such that $rl_i\beta_i \in \mathbb{Z}$.*

Proof. The statement in dimension 2 follows from Theorem 1.8.3 and [2, Lemma 2.6]. Indeed if $X \to Z$ is a fibration whose general fiber is a $\mathbb{P}^1$ and X is smooth, then by the general theory of smooth surfaces there exists a birational morphism $\sigma\colon X \to X'$ where X' is a $\mathbb{P}^1$-bundle. More

precisely X' is a relatively minimal model of X that is unique if the genus of Z is positive.
The general result follows from the one in dimension 2 by induction on the dimension of the base. Suppose now that the statement is true in dimension $n-1$ and let $X \to Z$ be a fibration of dimension n. The set

$$\mathcal{S} = \left\{ \begin{array}{l} P \subseteq Z \text{ prime divisor such that the log canonical} \\ \text{threshold of } f^*P \text{ with respect to } (X,B) \text{ is different from } 1 \end{array} \right\}$$

is a finite set.
We fix then $P \in \mathcal{S}$. By the Bertini theorem, since Z is smooth, we can find a hyperplane section $H \subseteq Z$ such that

1. H is smooth;
2. H intersects P transversally;
3. H does not contain any intersection $P \cap P'$ where $P' \in \mathcal{S} \backslash \{P\}$.

Set

$$X_H = f^{-1}(H); \quad f_H = f|_{X_H}; \quad B_H = B|_{X_H}; \quad P_H = P \cap H.$$

The restriction $f_H \colon (X_H, B_H) \to H$ is again an lc-trivial fibration. By [30, Theorem 7.5] the pair $(X_H, B_H + t f_H^* P_H)$ is lc if and only if the pair $(X, B + X_H + t f^* P)$ is lc and the pair $(X_H, B_H + t f_H^* P_H)$ is klt if and only if the pair $(X, B + X_H + t f^* P)$ is plt. Hence the log canonical threshold of $f_H^* P_H$ with respect to (X_H, B_H) is equal to the log canonical threshold of $f^* P$ with respect to (X, B) and the theorem follows from the inductive hypothesis. □

Notice that even if in many cases $m = r$ is sufficient for mM_Z to have integral coefficients there exist cases in which a greater coefficient is needed.

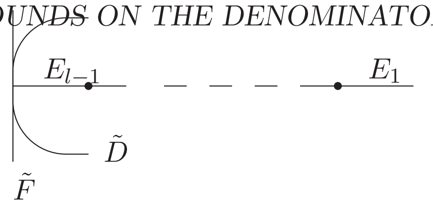
Example 1.8.5. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $\pi \colon X \to \mathbb{P}^1$ be the first projection. Let t be a coordinate on the first copy of $\mathbb{P}^1$ and let $[x : y]$ be homogeneous coordinates on the second copy of $\mathbb{P}^1$. Set

$$D = \{t y^d - x^l y^{d-l} - x^d = 0\}$$

and let $\bar{D}$ be the Zariski closure of D in X. Let q be the point $(0, [0:1]) \in D$.
Consider the pair $(X, 2/d\bar{D})$. Then we have $\deg(K_X + 2/d\bar{D})|_F = 0$ and there exists a rational function φ such that we can write

$$K_X + 2/d\bar{D} + \frac{1}{r}(\varphi) = f^*(K_{\mathbb{P}^1} + B_{\mathbb{P}^1} + M_{\mathbb{P}^1})$$

where $r = d$ if d is odd and $r = d/2$ if d is even. We want to compute now the coefficient of the divisor $B_{\mathbb{P}^1}$ at the point $t = 0$. Its coefficient is $1 - \gamma$ where γ is the log canonical threshold of $((X, 2/d\bar{D}), F)$. A log resolution for the pair $(X, 2/d\bar{D})$ over the point $t = 0$ is given by $\delta \colon \hat{X} \to X$ such that $\delta = \varepsilon_l \circ \ldots \circ \varepsilon_1$ is a composition of l blow-ups. Let E_i be the exceptional curve of ε_i. Then ε_1 blows up q and for any $i \geq 2$ the morphism ε_i blows up the intersection of the strict transform of F and E_{i-1}. At the $(l-1)$-th step the picture is as follows



We call $\delta\colon \hat{X} \to X$ this composition of blow-ups. We have

$$\delta^* K_X = K_{\hat{X}} - \sum_{i=1}^{l} iE_i \quad \delta^*\bar{D} = \tilde{D} + \sum_{i=1}^{l} iE_i \quad \delta^* F = \tilde{F} + \sum_{i=1}^{l} iE_i,$$

where by abuse of notation we denote by E_i the exceptional divisor of the i-th blow-up as well as its strict transforms after the following blow-ups. Thus

$$\delta^*(K_X + 2/d\bar{D} + \gamma F) = K_{\hat{X}} + 2/d\tilde{D} + \gamma\tilde{F} + \sum_{i=1}^{l} i(-1+\gamma+2/d)E_i.$$

By Lemma 1.8.1 we have

$$\gamma = 1 + \frac{1}{l} - \frac{2}{d}.$$

To prove that the bound stated in Theorem 1.8.3 is not far from being sharp, we take d even such that $d/2$ is odd and $l = d-1$. Then $r = d/2$ and

$$\gamma = 1 - \frac{2l-d}{ld} = 1 - \frac{2(2r-1)-2r}{2r^2-r} = 1 - \frac{2(2r-1)-2r}{2r^2-r} = 1 - \frac{2(r-1)}{(2r-1)r}.$$

Since $2(r-1)$ and $(2r-1)r$ are coprime, the smallest integer m such that $m\gamma$ is integral is $m = 2r^2 - r$. Notice that for any $r \geq 7$ we have $2r^2 - r > 12r$, thus the example gives a counterexample to the Prokhorov and Shokurov expectation.

For any algebraic curve C we can obtain, by performing a base change, an example of lc-trivial fibration whose base is C and such that $12rM_C$ is not an integral divisor.
Let C be an algebraic curve. There exists a finite morphism $\tau\colon C \to \mathbb{P}^1$ and modulo composing τ with an automorphism of $\mathbb{P}^1$ we can assume that the support of the ramification divisor on $\mathbb{P}^1$ is disjoint from the support of $B_{\mathbb{P}^1}$. We have the following commutative diagram

$$\begin{array}{ccc} C \times \mathbb{P}^1 & \xrightarrow{T} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow{\tau} & \mathbb{P}^1 \end{array}$$

where f is the first projection. Let $B = T^*(2/d\bar{D})$.
Then we are done if we consider $f\colon (C \times \mathbb{P}^1, B) \to C$.

1.8.3 Global results

Lemma 1.8.6. *Let $f\colon X \to Z$ be a $\mathbb{P}^1$-bundle on a smooth curve Z. Let $D \subseteq X$ be a reduced divisor such that $f|_D\colon D \to Z$ is a ramified covering of degree d with at least N ramification points $p_1, \ldots, p_N$ that are smooth points for D. Suppose that d is even. Suppose moreover that the ramification indices $l_1, \ldots, l_N$ at $p_1, \ldots, p_N$ satisfy the following properties:*

1. $2l_i \geq d$ *for any* i*;*
2. l_i *and* l_j *are coprime for any* $i \neq j$*;*
3. l_i *and* d *are coprime for any* i*.*

Then

(i) *the fibration*

$$f\colon (X, 2/dD) \to Z$$

is an lc-trivial fibration, in particular there exists a rational function φ *such that*

$$K_X + \frac{2}{d}D + \frac{1}{r}(\varphi) = f^*(K_Z + M_Z + B_Z).$$

(ii) *The Cartier index of the fiber is* $r = d/2$*.*

(iii) *Let* V *be the smallest integer such that* VM_Z *has integral coefficients. Then* $V \geq r^{N+1}$*.*

Proof. The first part of the statement follows easily from the fact the degree of $(K_X + 2/dD)|_F$ is 0. The Cartier index of the fiber is

$$r = \min\{m|\, m(K_X + 2/dD)|_F \text{ is a Cartier divisor}\}.$$

But since F is a smooth rational curve this is

$$r = \min\{m|\, m(K_X + 2/dD)|_F \text{ has integral coefficients}\} = \frac{d}{2}$$

and the second part of the statement is proved. In order to prove the third part of the statement we remark that since D is smooth at p_i and $f|_D$ ramifies at p_i the only possibility is that D is tangent to F at p_i with order of tangency exactly l_i.
Then we can apply Lemma 1.8.1 and by Equation (1.8.1) an expression for γ is

$$\gamma = 1 + \frac{1}{l_i} - \frac{2}{d}.$$

Since l_i and d are coprime, $l_i d$ divides V for any i. Again since l_i and l_j are coprime for any $i \neq j$

$$l_1 \cdot \ldots \cdot l_N \cdot d \mid V.$$

Since $l_i \geq d/2 = r$ for any i we have

$$V \geq l_1 \cdot \ldots \cdot l_N \cdot d \geq 2r^{N+1}.$$

□

Proof of Theorem 1.1.3 (1). Let N be a positive integer and $f\colon X \to Z$ be a $\mathbb{P}^1$-bundle on a smooth rational curve. Let $U \subseteq Z$ be an open set that trivializes the $\mathbb{P}^1$-bundle and such that we have a local holomorphic coordinate t on it. Take $d, l_1, \dots, l_N \in \mathbb{N}$ be such that

$$l_0 := 0 < l_1 < \cdots < l_N < l_{N+1} := d$$

and such that they verify conditions (1)(2)(3) of Lemma 1.8.6. Let $o_1, \dots, o_N$ be distinct points in U. Let $[u : v]$ be the coordinates on the fiber and $x = u/v$ the local coordinate on the open set $\{v \neq 0\}$. Let D be the Zariski closure in X of

$$D_0 = \left\{ \sum_{k=1}^{N+1} \left((x^{l_{k-1}} + \cdots + x^{l_k - 1}) \prod_{i=k}^{N} (t - o_i) \right) \right\}.$$

The restriction of D to the fiber over o_i is the zero locus of a polynomial of the form

$$h_i(x) = x^{l_i} q_i(x)$$

such that x does not divide q_i. Notice that D is smooth at the points $p_i = (0, o_i)$ because the derivative with respect to t of the polynomial that defines D_0 is nonzero at those points. This insures that D is tangent to the fiber $f^* o_i$ with multiplicity exactly l_i and then that

$$f|_D \colon D \to Z$$

has ramification index exactly l_i at p_i. The fibration $f\colon (X, 2/dD) \to Z$ satisfies all the hypotheses of Lemma 1.8.6. Therefore if V is the minimum positive integer such that VM_Z has integral coefficients we have $V \geq r^{N+1}$. □

Proof of Theorem 1.1.3 (2). Let $B_Z = \sum \beta_i P_i$ be the discriminant divisor. Let V be the minimum integer such that VB_Z has integral coefficients. If we write $\beta_i = u_i/v_i$ with $u_i, v_i \in \mathbb{N}$ and coprime, it is clear that $V = lcm\{v_i\}$. By Theorem 1.8.4 v_i divides $l_i r$ for some $l_i \leq 2r$. Then

$$V = \operatorname{lcm}\{v_i\} \mid \operatorname{lcm}\{l_i r\}.$$

Moreover

$$\operatorname{lcm}\{l_i r\} \mid r\operatorname{lcm}\{l | l \leq 2r\}$$

Thus V divides $N(r) = r\operatorname{lcm}\{l | l \leq 2r\}$ and we are done. □

The bound of Theorem 1.1.3 is not far from being sharp thanks to the following example.

Example 1.8.7. Let r be an odd integer. For a prime integer q set

$$s(q) = \max\{s \mid q^s \leq 2r\}.$$

Notice that

$$\frac{N(r)}{r} = \operatorname{lcm}\{l | l \leq 2r\} = \prod q^{s(q)}.$$

Set

$$h(q) = \max\{h \mid r \leq 2^h q^{s(q)} \leq 2r\}$$

and set

$$\{l_1 < \ldots < l_N\} = \{2^{h(q)} q^{s(q)} |\ q < 2r,\ q \text{ prime}\},$$

$$l_0 = 0, l_{N+1} = d = 2r.$$

Let $r = \prod q^{k(q)}$ be the decomposition of r into prime factors and remark that $s(q) \geq k(q)$ for any prime integer q. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $\pi \colon X \to \mathbb{P}^1$ be the first projection. Consider the divisor $\bar{D}$ defined as the Zariski closure of

$$D_0 = \left\{ \sum_{k=1}^{N+1} \left((x^{l_{k-1}} + \ldots + x^{l_k - 1}) \prod_{i=k}^{N} (t - o_i) \right) \right\}.$$

Consider now $B = 1/r\bar{D}$. The fibration $f \colon (X, B) \to \mathbb{P}^1$ is lc-trivial. Let V be the minimum integer such that VM_Z has integral coefficients.

Then for each $i \in \{1, \ldots, N\}$ by Lemma 1.8.1 we have the following expression for γ_i:

$$\gamma_i = 1 - \frac{2l_i - d}{l_i d} = 1 + \frac{r - l_i}{l_i r}.$$

For every q there exists $i \in \{1, \ldots, N\}$ such that $l_i = 2^{h(q)} q^{s(q)}$. Assume $q \neq 2$. Since r is odd

$$\gcd\{2^{h(q)} q^{s(q)}, r\} = q^{k(q)},$$

then

$$\gamma_i = 1 - \frac{l_i - r}{l_i r} = 1 + \frac{r/q^{k(q)} - 2^{h(q)} q^{s(q)-k(q)}}{2^{h(q)} q^{s(q)-k(q)} r}.$$

We show now that $r/q^{k(q)} - 2^{h(q)} q^{s(q)-k(q)}$ and $2^{h(q)} q^{s(q)-k(q)} r$ are coprime. Notice that

$$2^{h(q)} q^{s(q)-k(q)} r = 2^{h(q)} \cdot q^{s(q)} \prod_{q' \neq q} q'^{k(q')}$$

is the decomposition into prime factors. Since $r/q^{k(q)}$ is not divisible by q,

$$r/q^{k(q)} - 2^{h(q)} q^{s(q)-k(q)} \neq 0 (\text{mod } q).$$

Let q' be a prime integer such that $q' \neq q$ and $q' \mid r$, then

$$r/q^{k(q)} - 2^{h(q)} q^{s(q)-k(q)} = -2^{h(q)} q^{s(q)-k(q)} \neq 0 (\text{mod } q').$$

Then for any q such that $q \leq 2r$ we have

$$q^{s(q)-s'(q)} r | V$$

which implies that

$$\text{lcm}\{q^{s(q)-s'(q)} r\} | V \text{for all } q \neq 2.$$

If $q = 2$, let l_i be such that $l_i = 2^{s(2)}$. Then

$$\gamma_i = 1 - \frac{l_i - r}{l_i r} = 1 + \frac{r - 2^{s(2)}}{2^{s(2)} r}.$$

Since r is odd, $r - 2^{s(2)}$ and $2^{s(2)}r$ are coprime. Thus

$$2^{s(2)} \mid V.$$

Since

$$2^{s(2)} \cdot \operatorname{lcm}\{q^{s(q)-s'(q)}r\} = \frac{N(r)}{r}$$

we are done.

Let C be an algebraic curve. By performing a base change as in Example 1.8.5 we can obtain an example of lc-trivial fibration with base C such that the minimum integer that clears the denominators of M_C is $N(r)/r$.

Chapter 2

On the Fujita-Zariski decomposition on threefolds

2.1 Introduction

Let S be a smooth projective surface defined over $\mathbb{C}$. Let D be an effective divisor on S. In 1962 O. Zariski proved in [46] the existence of two divisors P, N such that

1. $N = \sum a_i N_i$ is effective, P is nef and $D = P + N$;
2. either $N = 0$ or the matrix $(N_i \cdot N_j)$ is negative definite;
3. $(P \cdot N_i) = 0$ for any i.

Such a decomposition is unique and is called the *Zariski decomposition of D*.
Fujita in [20] generalized the statement to pseudoeffective divisors. Moreover he noticed in [21] that the divisor P is the unique divisor that satisfies the following property:

(α) for any birational model $f\colon X' \to X$ and any nef divisor L on X' such that $f_*L \leq D$ we have $f_*L \leq P$.

Due to the importance of the Zariski decomposition on surfaces, several generalizations to higher dimensional varieties were studied. A very nice survey that collects the different definitions and their main properties is [39]. The property (α) gives rise to the following generalization.

Definition 2.1.1 (Definition 6.1, [39])**.** *Let X be a smooth complex projective variety and D a pseudoeffective divisor. A decomposition $D = P_f + N_f$ is called a Zariski decomposition in the Fujita sense (or simply Fujita-Zariski decomposition) if*

1. $N_f \geq 0$;

2. P_f is nef;

3. for any birational model $\mu\colon X' \to X$ and any nef divisor L on X' such that $\mu_*L \leq D$ we have $\mu_*L \leq P_f$.

It follows from the definition that, if a Fujita-Zariski decomposition exists, then it is unique (see Remark 2.2.1).

The importance of the Fujita-Zariski decomposition is very well illustrated by the results by Birkar [4] and Birkar-Hu [5] who proved the equivalence between the existence of log minimal model for pairs and the existence of the Fujita-Zariski decomposition for log canonical divisors. We refer to [4, Theorem 1.5] and [5, Theorem 1.2] for the precise statements.

In [35] several partial results were proved on the Zariski decomposition in dimension 3 that correlated the existence of the Fujita-Zariski decomposition *over a curve* Σ (see [35, III.4] for a complete definition) to the stability of the conormal bundle of Σ. More precisely, let D be a pseudoeffective divisor on X and Σ a smooth curve such that $D \cdot \Sigma < 0$. Let I_Σ be the ideal defining Σ in X. If the conormal bundle I_Σ/I_Σ^2 is semistable, then, Nakayama proved (cf. [35, Lemma III.4.5]) the existence of a decomposition $\varphi^*D = P + N$ such that $N \geq 0$ and the divisor P has positive intersection with every curve of the exceptional divisor of

$$\varphi\colon \mathrm{Bl}_\Sigma X \to X.$$

If I_Σ/I_Σ^2 is unstable then, again by Nakayama (cf. Lemma 2.2.10), there exists a short exact sequence

$$0 \to \mathcal{L} \to I_\Sigma/I_\Sigma^2 \to \mathcal{M} \to 0$$

such that $\deg \mathcal{L} > \deg \mathcal{M}$. By [35, Lemma III.4.6] if the conormal bundle is not "too unstable", namely if $2 \deg \mathcal{M} \geq \deg \mathcal{L}$, then there exists a birational model $\varphi\colon X' \to X$ such that φ^*D has a Fujita-Zariski decomposition over Σ. Therefore the study of the semistability properties of the conormal bundle of a curve in a threefold plays an important role in the theory of Zariski decomposition. With this respect a key intermediate technical step is the following.

Theorem 2.1.2. *Let X be a smooth complex projective variety of dimension 3. Let $\Sigma \subseteq X$ be a smooth curve and assume that the conormal bundle*

$$I_\Sigma/I_\Sigma^2$$

is not semistable as a vector bundle of rank two on Σ. Then there exists a sequence of blow-ups $\varphi\colon \hat{X} \to X$ along smooth curves not contained in Σ such that, if $\hat{\Sigma}$ is the strict transform of Σ in $\hat{X}$, then $I_{\hat{\Sigma}}/I_{\hat{\Sigma}}^2$ is semistable.

Actually, we will prove a statement that is much more precise than Theorem 2.1.2, namely Theorem 2.3.3, which also gives a control over the degree of the conormal bundle. Such a control over the degree of the conormal bundle could be useful for instance in order to apply results as [35, Lemma III.4.5] where the coefficients of a certain Zariski decomposition over Σ is computed in terms of $\deg I_\Sigma/I_\Sigma^2$.
We will then show that Theorem 2.3.3 leads to the existence of a birational model $\varphi\colon \tilde{X} \to X$ and a decomposition $\varphi^*D = P + N$ such that $N \geq 0$ and such that P has some positivity properties on the exceptional locus of φ.

This chapter is organized as follows: Section 2.2 collects some preliminary definitions and results about the Fujita-Zariski decomposition, the σ-decomposition and the semistability of vector bundles on curves. Section 2.3 is devoted to the proof of Theorem 2.3.3, with which we make the conormal bundle of a curve semistable and of degree arbitrarly big.

2.2 Preliminaries

In this section we collect some definitions and basic facts about the Fujita-Zariski decomposition and the σ-decomposition. Moreover we state various results on curves that will be used later.

2.2.1 Fujita-Zariski decomposition and σ-decomposition

Remark 2.2.1. It follows from the definition that, if a Fujita-Zariski decomposition exists, then it is unique. Indeed, if $D = P'_f + N'_f$ is another Fujita-Zariski decomposition, then, from the property (3) of the definition applied to the two decompositions $D = P'_f + N'_f$ and $D = P_f + N_f$, we obtain $P_f \le P'_f$ and $P'_f \le P_f$.

In [35] we have the following definitions. Let us denote by $|B|_{\text{num}}$ the set of effective $\mathbb{R}$-divisors Δ numerically equivalent to B.

Definition 2.2.2 (Definition III.1.1, [35])**.** *Let D be a pseudoeffective divisor of a smooth projective variety. Let Γ be a prime divisor and A an ample divisor. We define*

$$\sigma_\Gamma(D) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \inf\{\text{mult}_\Gamma \Delta \mid \Delta \in |D + \varepsilon A|_{\text{num}}\}.$$

The limit does not depend on the choice of the ample divisor A by [35, Lemma III.1.5] and thus it depends only on the numerical equivalence class of D. Moreover, by [35, Corollary III.1.11] there is only a finite number of prime divisors Γ satisfying $\sigma_\Gamma(D) > 0$. Thus the expression

$$\sum_\Gamma \sigma_\Gamma(D)\Gamma$$

defines a divisor.

Definition 2.2.3 (Definition III.1.12, [35])**.** *Let D be a pseudoeffective divisor of a smooth projective variety. We define*

$$N_\sigma(D) = \sum \sigma_\Gamma(D)\Gamma \quad \text{and} \quad P_\sigma(D) = D - N_\sigma(D).$$

The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is called the σ-decomposition of D.

Definition 2.2.4 (Definition III.1.16, [35])**.** *The σ-decomposition $D = P_\sigma(D) + N_\sigma(D)$ for a pseudo-effective $\mathbb{R}$-divisor is called the Zariski decomposition in Nakayama's sense (or simply the Nakayama-Zariski decomposition) if $P_\sigma(D)$ is nef.*

Remark 2.2.5. By [35, Proposition III.1.14, Remark III.1.17(2)], if the Nakayama-Zariski decomposition exists then it is the Fujita-Zariski decomposition. The converse is not known.

Definition 2.2.6. *Let D be a pseudoeffective divisor. The diminished base locus of D is defined as follows*

$$\mathbb{B}_-(D) = \bigcup_{A \text{ ample}} \mathbb{B}(D + A) \quad \text{where} \quad \mathbb{B}(D + A) = \bigcap\{\text{Supp}(D + A) | \Delta \ge 0, \Delta \sim_{\mathbb{R}} D + A\}.$$

Remark 2.2.7. If D is a pseudoeffective divisor that has birationally a Nakayama-Zariski decomposition then its diminished base locus is closed. Indeed let $f: Y \to X$ be a birational model such that $f^*D = P_\sigma(f^*D) + N_\sigma(f^*D)$ is a Nakayama-Zariski decomposition. Then

$$\mathbb{B}_-(f^*D) = \bigcup_A \mathbb{B}(N_\sigma(f^*D) + P_\sigma(f^*D) + A)$$

and, since $P_\sigma(f^*D) + A$ is ample,

$$\mathbb{B}(N_\sigma(f^*D) + P_\sigma(f^*D) + A) \subseteq \mathrm{Supp}N_\sigma(f^*D)$$

for any A, showing that $\mathbb{B}_-(f^*D) \subseteq \mathrm{Supp}N_\sigma(f^*D)$. The other containment follows from the definitions of σ-decomposition and diminished base locus. Then $\mathbb{B}_-(f^*D) = \mathrm{Supp}N_\sigma(f^*D)$ is closed and so is $\mathbb{B}_-(D)$ because by [33, Proposition 2.5]

$$f(\mathbb{B}_-(f^*D)) = \mathbb{B}_-(D).$$

Remark 2.2.8. If D admits birationally a Nakayama-Zariski decomposition, then the diminished base locus of $P_\sigma(D)$ is the union of a finite number of subvarieties of codimension at least two. Indeed it follows easily from the definitions and from [13, Proposition 1.19] that the diminished base locus of the positive part $\mathbb{B}_-(P_\sigma(D))$ does not have any component of codimension one. Moreover, if there exists a birational model $\mu\colon \tilde{X} \to X$ such that $\mu^*P_\sigma(D)$ has a Fujita-Zariski decomposition on $\tilde{X}$

$$\mu^*P_\sigma(D) = \bar{P} + \bar{N},$$

then the decomposition

$$\mu^*D = \bar{P} + \bar{N} + \mu^*N_\sigma(D)$$

gives a Fujita-Zariski decomposition for μ^*D on $\tilde{X}$.

2.2.2 Useful results on curves

Definition 2.2.9. *A vector bundle $\mathcal{E}$ on a smooth projective curve is said to be semistable if for any vector bundle $0 \neq \mathcal{F} \subseteq \mathcal{E}$ the following inequality is true*

$$\frac{\deg \det\mathcal{F}}{\mathrm{rank}\mathcal{F}} \leq \frac{\deg \det\mathcal{E}}{\mathrm{rank}\mathcal{E}}.$$

Lemma 2.2.10 (Lemma 1.1, [36])**.** *Let $\mathcal{E}$ be a vector bundle of rank two on a smooth compact curve C.*

1. *If $\mathcal{E}$ is a semistable vector bundle then there exist no curves Γ on the ruled surface $\mathbb{P}_C(\mathcal{E})$ with $\Gamma^2 < 0$.*

2. *If $\mathcal{E}$ is unstable, then there exists a unique (up to isomorphisms) exact sequence*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0 \tag{2.2.1}$$

which satisfies the following two conditions:

- *$\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on C,*
- $\deg \mathcal{L} > \deg \mathcal{M}$.

Remark 2.2.11. The sequence 2.2.1 is the Harder-Narashiman filtration and $\mathcal{L}$ is the maximal destabilizing subsheaf.

Definition 2.2.12. *The sequence (2.2.1) is called the characteristic exact sequence of* $\mathcal{E}$*. We set* $\delta(\mathcal{E}) = \deg \mathcal{L} - \deg \mathcal{M}$*. If* $\mathcal{E} = I_C/I_C^2$ *is the conormal bundle of a smooth curve in a smooth threefold, then we adopt the notation* $\delta(C) = \delta(I_C/I_C^2)$*.*

The following lemma is probably well known to experts. Since we could not find a reference in the literature, we put a proof here for the reader's convenience.

Lemma 2.2.13. *Let* X *be a smooth variety of dimension 3.*

1. *Let* $C \subseteq X$ *be a curve. Then there exists a birational morphism* $\eta\colon W \to X$*, composition of blow-ups along smooth curves, such that*

$$\eta^{-1}C = \tilde{C} \cup \bigcup_i G_i$$

where $\tilde{C}$ *is the strict transform of* C *and it is smooth and* G_i *is a smooth curve for any* i*.*

2. *Let* C_j*, for* $j = 1, \ldots, l$*, be smooth curves in* X*. Then there exists a birational morphism* $\eta\colon W \to X$*, composition of blow-ups along smooth curves, such that*

$$\eta^{-1}(C_1 \cup \ldots \cup C_l) = \bigcup_i G_i$$

where G_i *is a smooth curve for any* i *and for any* $j_1 \neq j_2$ *the curves* $G_{j_1} \cap G_{j_2}$ *intersect transversally in at most one point.*

Proof. (1) If C is smooth there is nothing to prove. Then assume that C is singular. Let $p \in C$ be a singular point. In a local analytic neighborhood U of p we can write C as a union of irreducible components

$$C = C_1 \cup \ldots \cup C_k.$$

We first reduce to the case where C_i is smooth at p for every i. Let C' be one of the C_i. Modulo shrinking U, we can assume that it is isomorphic to an open neighborhood of the origin in $\mathbb{C}^3$ and by [23, Theorem 2.26] we can find a map

$$\begin{array}{rcl} \gamma\colon \mathbb{C} & \to & C' \\ 0 & \mapsto & p \end{array}$$

that is injective and such that the derivative of γ is nonzero for any $t \neq 0$. If we write the expansion of each component of γ as a Laurent series we have

$$\gamma(t) = \left(t^l, \sum a_i t^{m_i}, \sum b_i t^{n_i}\right).$$

We can assume that the first component is monomial by composing with a suitable biholomorphism of the source $\mathbb{C}$. We can also assume that

$$l \leq m_1 \leq n_1.$$

The injectivity of γ implies that l, the m_i and the n_i are coprime. The order of γ at zero is the minimum of the orders of the three components. We prove by induction on the order that we can

desingularize C' with blow-ups of smooth curves.
If $l = 1$ then C' is smooth.
Assume that $l > 1$. Since l, the m_i and the n_i are coprime, there exists an exponent m_i or n_i that is not divisible by l. Without loss of generality we can assume that the smaller of such exponents is one of the m_i. Then there exists a biholomorphism of the target $\mathbb{C}^3$ of the form

$$\Psi(x, y, z) = (x, y - p_1(x), z)$$

with the following properties: p_1 is a polynomial and

$$\Psi \circ \gamma(t) = \left(t^l, \sum a_i t^{m'_i}, \sum b_i t^{n_i}\right) = (t^l, t^{m'_1}u(t), t^{n_1}v(t))$$

where l does not divide m'_1 and u and v are invertible functions. Let

$$\tilde{X} \to X$$

be the blowing up of a smooth curve Γ such that Γ has the same tangent direction at p as $\{x = y = 0\} \subseteq U$. Then a parametrization for the strict transform of C' is

$$\tilde{\gamma}(t) = (t^l, t^{m'_1 - l}u(t), t^{n_1}v(t)).$$

Let

$$m'_1 = l \cdot q + r$$

be the result of the euclidean division of m'_1 by l. If we blow-up q times a curve of local equation

$$\{x = y = 0\},$$

a parametrization for the strict transform of C' is

$$\tilde{\tilde{\gamma}}(t) = (t^l, t^r u(t), t^{n_1}v(t)).$$

The order of $\tilde{\tilde{\gamma}}$ at the singular point is thus $r < l$. Then we apply the inductive hypothesis and we conclude.
We separate the irreducible components. Now we can assume that C_i is smooth in p for every i. Let C_1 and C_2 be two irreducible components and let τ_i be the tangent of C_i at p.
If τ_1 and τ_2 are not colinear then we blow-up along a curve Γ whose tangent does not lie in the plane generated by τ_1 and τ_2. If $\tilde{C}_i$ is the strict transform of C_i then

$$\tilde{C}_1 \cap \tilde{C}_2 = \emptyset.$$

If C_1 and C_2 have the same tangent direction then we can find two parametrizations of the following form:

$$\begin{array}{rcl} \gamma_1 \colon \mathbb{C} & \to & C_1 \\ t & \mapsto & (t, 0, 0) \end{array}$$

of C_1 and

$$\begin{array}{rcl} \gamma_2 \colon \mathbb{C} & \to & C_1 \\ t & \mapsto & (tw_1(t), t^m w_2(t), t^n w_3(t)) \end{array}$$

of C_2 where w_i is an invertible function and $1 \leq m \leq n$. Since C_2 has the same tangent direction as C_1, we have $m > 1$. We prove by induction on m that we can separate C_1 and C_2. We blow-up along a curve whose tangent direction at p is the same as the tangent direction of is

$$\{x = y = 0\} \subseteq U$$

and we obtain

$$\tilde{X} \to X.$$

Parametrizations for the strict transforms $\tilde{C}_1$ and $\tilde{C}_2$ are

$$\begin{array}{rcl} \tilde{\gamma}_1 \colon \mathbb{C} & \to & C_1 \\ t & \mapsto & (t, 0, 0) \end{array}$$

for $\tilde{C}_1$ and

$$\begin{array}{rcl} \gamma_2 \colon \mathbb{C} & \to & C_1 \\ t & \mapsto & (tw_1(t), t^{m-1}w_2'(t), t^n w_3(t)) \end{array}$$

for $\tilde{C}_2$ where $w_2' = w_2/w_1$. Then we conclude by inductive hypothesis. We notice that the preimage of the singular point p in $\tilde{X}$ is a curve of local equation $\{x = z = 0\}$. Thus it meets $\tilde{C}_1$ and $\tilde{C}_2$ transversally.
(2) The proof of this second item follows the same line as the proof of the first. The statement is proved by blowing-up generic smooth curves through $C_{j_1} \cap C_{j_2}$. □

2.3 Making the conormal bundle semistable

Lemma 2.3.1. *Let X be a smooth complex projective variety of dimension 3. Let $\Sigma \subseteq X$ be an irreducible smooth curve. Let*

$$0 \to \mathcal{A} \to I_\Sigma/I_\Sigma^2 \to \mathcal{B} \to 0$$

be a short exact sequence where $\mathcal{A}$ and $\mathcal{B}$ are line bundles. Let Γ be a smooth curve that meets Σ transversally in one point p and such that the composition

$$\mathcal{A}_p \to (I_\Sigma/I_\Sigma^2)_p \to \Omega_{X,p} \to \Omega_{\Gamma,p}$$

is nonzero. Let $\varphi \colon X_1 \to X$ be the blow-up of Γ and let Σ_1 be the strict transform of Σ. Then the conormal bundle of Σ_1 has a presentation

$$0 \to \tilde{\mathcal{A}} \to I_{\Sigma_1}/I_{\Sigma_1}^2 \to \tilde{\mathcal{B}} \to 0 \qquad (2.3.1)$$

where $\tilde{\mathcal{A}} = \varphi^*\mathcal{A}$, $\deg I_{\Sigma_1}/I_\Sigma^2 = \deg I_\Sigma/I_{\Sigma_1}^2 + 1$ and $\deg \tilde{\mathcal{B}} = \deg \mathcal{B} + 1$.

Proof. We have the following short exact sequence of sheaves on X_1

$$0 \longrightarrow \varphi^*\Omega_X \xrightarrow{\Phi} \Omega_{X_1} \longrightarrow \Omega_{X_1/X} \longrightarrow 0.$$

The morphism of sheaves Φ is an isomorphism over $X \backslash \Gamma$.

Since $\Sigma \subseteq X$ is a smooth curve, we have the following exact sequence

$$0 \to I_\Sigma/I_\Sigma^2 \to \Omega_X \otimes \mathcal{O}_\Sigma \to \Omega_\Sigma \to 0. \tag{2.3.2}$$

Analogously for $\Sigma_1 \subseteq X_1$, the strict transform of Σ, we have

$$0 \to I_{\Sigma_1}/I_{\Sigma_1}^2 \to \Omega_{X_1} \otimes \mathcal{O}_{\Sigma_1} \to \Omega_{\Sigma_1} \to 0. \tag{2.3.3}$$

The restriction of the blow-up $\varphi\colon \Sigma_1 \to \Sigma$ is an isomorphism. Then, sequence (2.3.2) pulls back to an exact sequence of vector bundles on Σ_1

$$0 \to \varphi^* I_\Sigma/I_\Sigma^2 \to \varphi^*\Omega_X \otimes \mathcal{O}_{\Sigma_1} \to \varphi^*\Omega_\Sigma \to 0. \tag{2.3.4}$$

We claim that

$$\Phi(\varphi^*(I_\Sigma/I_\Sigma^2)) \subseteq I_{\Sigma_1}/I_{\Sigma_1}^2.$$

Indeed we have the following commutative diagram with exact columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \varphi^*(I_\Sigma/I_\Sigma^2) & & I_{\Sigma_1}/I_{\Sigma_1}^2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \varphi^*\Omega_X \otimes \mathcal{O}_{\Sigma_1} & \xrightarrow{\Phi} & \Omega_{X_1} \otimes \mathcal{O}_{\Sigma_1} & \longrightarrow & \Omega_{X_1/X} \otimes \mathcal{O}_{\Sigma_1} & \longrightarrow & 0 \\
 & & \alpha\downarrow & & \beta\downarrow & & & & \\
 & & \varphi^*\Omega_\Sigma & \xrightarrow{\cong} & \Omega_{\Sigma_1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Since the diagram commutes, we have $\Phi(\ker\alpha) \subseteq \ker\beta$, and the claim is proved.

Moreover the sheaf $\Omega_{X_1/X} \otimes \mathcal{O}_{\Sigma_1}$ is the skyscreaper sheaf supported on p,

$$\Omega_{X_1/X} \otimes \mathcal{O}_{\Sigma_1} \cong \mathbb{C}_p.$$

Thus we have

$$0 \to \varphi^*(I_\Sigma/I_\Sigma^2) \xrightarrow{\Phi} I_{\Sigma_1}/I_{\Sigma_1}^2 \to \mathbb{C}_p \to 0. \tag{2.3.5}$$

By sequence (2.3.5) we have

$$\deg I_{\Sigma_1}/I_{\Sigma_1}^2 = \deg I_\Sigma/I_\Sigma^2 + 1. \tag{2.3.6}$$

The morphism Φ has the property that

$$\Phi|_{\varphi^*\mathcal{A}} \text{ is injective.} \tag{2.3.7}$$

Indeed Φ is an isomorphism over $\Sigma\backslash\{p\}$ and, if we consider the stalk over p, on $\varphi^*\mathcal{A}_p$ it is nonzero by hypothesis. The sheaf defined by $\tilde{\mathcal{A}} := \Phi(\varphi^*\mathcal{A})$ is a sub vector bundle of rank one of $I_{\Sigma_1}/I^2_{\Sigma_1}$. Set $\tilde{\mathcal{B}}$ for the quotient, so that we have

$$0 \to \tilde{\mathcal{A}} \to I_{\Sigma_1}/I^2_{\Sigma_1} \to \tilde{\mathcal{B}} \to 0.$$

The condition on the degree of $\tilde{\mathcal{B}}$ follows from the choice of $\tilde{\mathcal{A}}$ and from (2.3.6). □

Lemma 2.3.2. *Let X be a smooth complex projective variety of dimension 3. Let $\Sigma \subseteq X$ be an irreducible smooth curve. Assume that the conormal bundle of Σ is unstable and that $\delta(\Sigma) = 1$, where $\delta(\Sigma)$ is defined in Definition 2.2.12. Let*

$$0 \to \mathcal{L} \to I_\Sigma/I^2_\Sigma \to \mathcal{M} \to 0 \tag{2.3.8}$$

be the characteristic exact sequence. Let $\varphi\colon X_1 \to X$ be the blow-up of a smooth curve Γ as in Lemma 2.3.1 for the sequence (2.3.8). Let Σ_1 be the strict transform of Σ in X_1. Then the conormal bundle of Σ_1 is semistable.

Proof. Let

$$0 \to \tilde{\mathcal{L}} \to I_{\Sigma_1}/I^2_{\Sigma_1} \to \tilde{\mathcal{M}} \to 0$$

be the sequence given by Lemma 2.3.1. Then $\deg\tilde{\mathcal{L}} = \deg\tilde{\mathcal{M}}$. Assume that that $I_{\Sigma_1}/I^2_{\Sigma_1}$ is unstable, so by Lemma 2.2.10(2) we have the characteristic sequence

$$0 \to \mathcal{L}' \to I_{\Sigma_1}/I^2_{\Sigma_1} \to \mathcal{M}' \to 0 \tag{2.3.9}$$

and, by definition of characteristic sequence, $\deg\mathcal{L}' > \deg\mathcal{M}'$. Consider now the morphism of sheaves $\chi\colon \tilde{\mathcal{L}} \to \mathcal{M}'$ given by the composition of the injective arrow of (2.3.1) and the surjective arrow of (2.3.9). If χ is identically zero, then $\tilde{\mathcal{L}} \cong \mathcal{L}'$, which is a contradiction because then also $\tilde{\mathcal{M}} \cong \mathcal{M}'$, but $\deg\mathcal{L} = \deg\mathcal{M}$ and $\deg\mathcal{L}' > \deg\mathcal{M}'$. Then χ is nonzero, which implies the inequalities

$$\deg\mathcal{L}' > \deg\mathcal{M}' \geq \deg\tilde{\mathcal{L}} = \deg\tilde{\mathcal{M}}.$$

But this leads again to a contradiction because

$$\deg\mathcal{L}' + \deg\mathcal{M}' = \deg(\det I_{\Sigma_1}/I^2_{\Sigma_1}) = \deg\tilde{\mathcal{L}} + \deg\tilde{\mathcal{M}}.$$

Therefore, if $\delta(\Sigma) = 1$, the conormal bundle $I_{\Sigma_1}/I^2_{\Sigma_1}$ is semistable.

□

Theorem 2.3.3. *Let X be a smooth complex projective variety of dimension 3. Let $\Sigma \subseteq X$ be an irreducible smooth curve and N an integer. Then there exists a birational model $\varphi\colon \hat{X} \to X$ given by a sequence of blow-ups along smooth curves not contained in Σ with the following properties. Let $\hat{\Sigma}$ be the strict transform of Σ in $\hat{X}$.*

1. *The degree of $I_{\hat{\Sigma}}/I^2_{\hat{\Sigma}}$ is at least N.*

2. *The vector bundle $I_{\hat{\Sigma}}/I^2_{\hat{\Sigma}}$ is semistable.*

3. $\varphi^{-1}\Sigma = \hat{\Sigma} \cup \bigcup_{i=1}^{n} F_i^s \cup \bigcup_{i=1}^{m} F_i^d$ *is a chain of curves and* $\bigcup_{i=1}^{n} F_i^s$ *and* $\bigcup_{i=1}^{m} F_i^d$ *are chains of smooth rational curves. These two chains both intersect* $\hat{\Sigma}$ *in one point.*

Proof. **(1)** Write the conormal bundle as extension of two vector bundles of rank one.

$$0 \to \mathcal{A} \to I_\Sigma/I_\Sigma^2 \to \mathcal{B} \to 0.$$

Let Γ_1 be a smooth curve as in Lemma 2.3.1. Let $\psi_1 \colon X_1 \to X$ be the blow up of Γ_1. If Σ_1 is the strict transform of Σ, then $\deg I_{\Sigma_1}/I_{\Sigma_1}^2 = I_\Sigma/I_\Sigma^2 + 1$. Let E_1 be the exceptional divisor of ψ_1. It is easy to verify that a section Γ_2 of $E_1 \to \Gamma_1$ meeting Σ_1 verifies the hypothesis of Lemma 2.3.1. Then we blow-up Γ_2. We continue this process until we reach degree N.
(2) Assume that we already have the condition on the degree of the conormal bundle of Σ. If I_Σ/I_Σ^2 is unstable, consider its characteristic sequence

$$0 \to \mathcal{L} \to I_\Sigma/I_\Sigma^2 \to \mathcal{M} \to 0. \qquad (2.3.10)$$

We apply Lemma 2.3.1 to sequence (2.3.10): we blow-up a curve Γ_1 and we obtain $\varphi_1 \colon X_1 \to X$. Let E_1 be the exceptional divisor of φ_1. Then, as in item (1), we blow-up a section Γ_2 of $E_1 \to \Gamma_1$ meeting Σ and we repeat this process $n = \delta(\Sigma)$ times. We obtain $\hat{X} \to X$. Let $\hat{\Sigma}$ be the strict transform of Σ in $\hat{X}$. We prove that the conormal bundle of $\hat{\Sigma}$ is semistable by induction on n. Let us first suppose that $\delta(\Sigma) = 1$. Then it follows from Lemma 2.3.2. Assume that $\delta(\Sigma) > 1$. It is sufficient to prove that $\delta(\Sigma_1) = \delta(\Sigma) - 1$. By Lemma 2.2.10(2), the sequence

$$0 \to \varphi_1^*\mathcal{L} \to I_{\Sigma_1}/I_{\Sigma_1}^2 \to \tilde{\mathcal{M}} \to 0$$

given by Lemma 2.3.1 is the characteristic exact sequence of Σ_1. Since, again by Lemma 2.3.1, $\deg \tilde{\mathcal{M}} = \deg \mathcal{M} + 1$, we have that $\delta(\Sigma_1) = \delta(\Sigma) - 1$. We remark that at each step the degree of the conormal sheaf grows by one:

$$\deg I_{\Sigma_1}/I_{\Sigma_1}^2 = \deg I_\Sigma/I_\Sigma^2 + 1$$

so item (1) is preserved.
(3) Let F_i^d be the intersection of the preimage of Σ with the i-th exceptional divisor of the blow-ups made in order to reach degree N. Let F_i^s be the intersection of the preimage of Σ with the i-th exceptional divisor of the blow-ups made in order to reach semistability (see Figure 2.1). Then both F_i^d and F_i^s are rational curves because they are contained in fibers of the respective blow-ups. It follows from the construction that the F_i^s and the F_i^d form two chains of rational curves.

□

Remark 2.3.4. Theorem 2.1.2 is exactly Theorem 2.3.3(2).

Remark 2.3.5. Notice that we cannot perform the blow-ups needed to achieve Theorem 2.3.3(2) before those needed to achieve item (1). Indeed, after item (1) the conormal bundle could not be semistable anymore, even if it had this property before starting the process. On the other hand, the "semistabilization" naturally increases the degree of the conormal bundle.

If we do not assume that Σ is smooth we have the following.

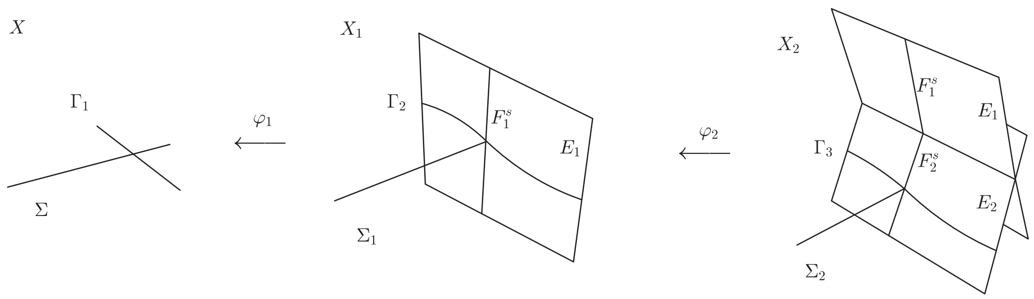


Figure 2.1:

Corollary 2.3.6. *Let X be a smooth complex projective variety of dimension 3. Let $\Sigma \subseteq X$ be an irreducible curve and N an integer. Then there exists a birational model $\varphi\colon \hat{X} \to X$ given by a sequence of blow-ups along smooth curves not contained in Σ with the following properties. Let $\hat{\Sigma}$ be the strict transform of Σ in $\hat{X}$.*

1. *$\varphi^{-1}\Sigma = \hat{\Sigma} \cup \bigcup_{i=1}^{n} F_i$ where $\hat{\Sigma}$ is smooth and F_i is a smooth rational curve for any i.*
2. *The degree of $I_{\hat{\Sigma}}/I_{\hat{\Sigma}}^2$ is at least N.*
3. *The vector bundle $I_{\hat{\Sigma}}/I_{\hat{\Sigma}}^2$ is semistable.*

Proof. By Lemma 2.2.13 there exists a birational morphism $\eta\colon W \to X$ such that

$$\eta^{-1}\Sigma = \tilde{\Sigma} \cup \bigcup G_i$$

where $\tilde{\Sigma}$ is the strict transform of Σ and is smooth and G_i is a smooth rational curve for any i. By Theorem 2.3.3 there exists $\mu\colon \hat{X} \to W$ that satisfies properties (2) and (3). Moreover

$$\mu^{-1}\eta^{-1}\Sigma = \hat{\Sigma} \cup \bigcup G_i \cup \bigcup_{i=1}^{n} F_i^s \cup \bigcup_{i=1}^{m} F_i^d$$

is the union of the strict transform of Σ and some smooth rational curves. □

Since in our process of making the conormal bundle semistable we are creating new curves, it should be useful to control also their conormal bundles. This is done by the following two results.

Lemma 2.3.7. *Let X be a smooth variety of dimension three, let $F \subseteq X$ be a smooth rational curve. Let*

$$I_F/I_F^2 = \mathcal{O}(a) \oplus \mathcal{O}(b)$$

be the conormal bundle of F in X and suppose that $a > b$. Let $\varphi\colon X_1 \to X$ be a blow-up given by Lemma 2.3.1 for the sequence

$$0 \to \mathcal{O}(a) \to I_F/I_F^2 \to \mathcal{O}(b) \to 0.$$

Let F_1 be the strict transform of F in X_1. Then

$$I_{F_1}/I_{F_1}^2 = \mathcal{O}(a) \oplus \mathcal{O}(b+1).$$

Proof. Since F_1 is a rational curve, we have

$$I_{F_1}/I_{F_1}^2 = \mathcal{O}(a') \oplus \mathcal{O}(b')$$

for some integers a', b'. By Lemma 2.3.1 $a' + b' = a + b + 1$. By the proof of Theorem 2.3.3, we know that $a' - b' = a - b - 1$, leaving as the only possibility $a' = a$ and $b' = b + 1$. □

Proposition 2.3.8. *Notation as in Theorem 2.3.3. The conormal bundle of F_i^s, and respectively of the F_i^d, is isomorphic to*

$$I_{F_i^s}/I_{F_i^s}^2 = \begin{cases} \mathcal{O} \oplus \mathcal{O}(1) & i = n \\ \mathcal{O} \oplus \mathcal{O}(2) & i < n, \end{cases}$$

and

$$I_{F_i^d}/I_{F_i^d}^2 = \begin{cases} \mathcal{O} \oplus \mathcal{O}(1) & i = m \\ \mathcal{O} \oplus \mathcal{O}(2) & i < m. \end{cases}$$

Proof. We prove the statement for the curves F_i^s, the proof for the F_i^d being completely analogous. Let

$$X_n \xrightarrow{\varphi_n} X_{n-1} \xrightarrow{\varphi_{n-1}} \dots \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X$$

be the sequence of blow-ups performed in order to achieve semistability. By abuse of notation we denote by F_i^s the curve in X_i as well as its strict transform in X_j for any $j > i$ and in $\hat{X}$. If $i = n$, then F_n^s is the fiber of a blow-up and the statement a well-known fact. If $i < n$, then $F_i^s \subseteq X_i$ has conormal bundle $\mathcal{O} \oplus \mathcal{O}(1)$. By Lemma 2.3.7, its strict transform in X_{i+1} has conormal bundle $\mathcal{O} \oplus \mathcal{O}(2)$. Then the statement follows because φ_j is an isomorphism on F_i^s for any $j > i + 1$. □

The diminished base locus depends only on the numerical equivalence class of D by [13, Proposition 1.19]. Arguing as in Nakayama [35, Lemma III.4.5] we prove the following.

Corollary 2.3.9. *Let X be a smooth projective threefold. Let D be a pseudoeffective divisor such that $\mathbb{B}_-(D)$ does not have any component of dimension two. Let Σ be a smooth curve such that $D \cdot \Sigma < 0$. Then for any point $p \in \Sigma$ there exists a birational morphism $\mu\colon \tilde{X} \to X$ such that*

1. $\mu(\mathrm{Exc}(\mu)) = \Sigma$
2. *for any curve $C \subseteq \mathrm{Exc}(\mu)$ we have $P_\sigma(\mu^* D) \cdot C \geq$ unless $\mu(C) = p$.*

Proof. Let $\varphi\colon \hat{X} \to X$ be a birational morphism, that exists by Theorem 3.3, such that $I_{\hat{\Sigma}}/I_{\hat{\Sigma}}^2$ is semistable, where $\hat{\Sigma}$ is the strict transform of Σ in X. We have

$$\mathbb{B}_-(\varphi^* D) \cap \mathrm{Exc}(\varphi) = \hat{\Sigma} \cup \bigcup F_i^s.$$

We can chose φ in Theorem 3.3 such that $\varphi(\mathrm{Exc}(\varphi)) \cap \hat{\Sigma} = \{p\}$. Let $\varepsilon\colon \tilde{X} \to \hat{X}$ be the blow-up of $\hat{\Sigma}$ and let E be the exceptional divisor of ε. Set $\mu = \varepsilon \circ \varphi$. The intersection of the diminished base locus with $\mathrm{Exc}(\mu)$ is

$$\mathbb{B}_-(\mu^* D) \cap \mathrm{Exc}(\mu) = E \cup \bigcup \tilde{F_i^s},$$

where $\tilde{F_i^s}$ is the strict transform of F_i^s in $\tilde{X}$. By [35, Proposition III.1.14] the restriction $P_\sigma(\mu^* D)|_E$ is pseudoeffective. Since by [34, Theorem 3.1] any pseudoeffective divisor on E is nef, for any curve $C \subseteq E$ we have $P_\sigma(\mu^* D) \cdot C \geq 0$. Thus for any curve $C \subseteq \mathrm{Exc}(\mu)$, we have $P_\sigma(\mu^* D) \cdot C \geq 0$ unless C is one of the F_i^s and, if $C = F_i^s$ for some i, then $\mu(C) = p$. □

Corollary 2.3.10. *Let X be a smooth projective threefold. Let D be a pseudoeffective divisor such that $\mathbb{B}_-(D)$ does not have any component of dimension two. Let Σ be a curve such that $D \cdot \Sigma < 0$. Then for any point $p \in \Sigma \backslash \mathrm{Sing}(\Sigma)$ there exists a birational morphism $\mu\colon \tilde{X} \to X$ such that*

1. $\mu(\mathrm{Exc}(\mu)) = \Sigma$

2. *for any curve* $C \subseteq \mathrm{Exc}(\mu)$ *we have* $P_\sigma(\mu^* D) \cdot C \geq$ *unless* $\mu(C) \subseteq \{p\} \cup \mathrm{Sing}(\Sigma)$.

Proof. By Lemma 2.2.13 there exists a birational morphism $\eta \colon W \to X$ such that

$$\eta^{-1}\Sigma = \tilde{\Sigma} \cup \bigcup_i G_i$$

such that $\tilde{\Sigma}$ is smooth and G_i is a rational curve for any i. By Corollary 2.3.9 there exists a birational morphism $\mu \colon \tilde{X} \to W$ such that $\mu(\mathrm{Exc}(\mu)) = \tilde{\Sigma}$ and for any curve $C \subseteq \mathrm{Exc}(\mu)$ we have $P_\sigma(\mu^* \eta^* D) \cdot C \geq 0$ unless if $\mu(C) = p$. Thus, if $C \subseteq \mu^{-1}\eta^{-1}\Sigma$ is a curve, we have $P_\sigma(\mu^* \eta^* D) \cdot C \geq 0$ unless if $\eta(\mu(C)) = p$ or C is the strict transform in $\tilde{X}$ of one of the G_i. If C is the strict transform in $\tilde{X}$ of one of the G_i, then $\eta(\mu(C)) \subseteq \mathrm{Sing}(\Sigma)$. □

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Deux aspects de la géométrie birationnelle des variétés algébriques : la formule du fibré canonique et la décomposition de Zariski

Résumé

Insérer votre résumé en français suivi des mots-clés

La formule du fibré canonique et la décomposition de Fujita-Zariski sont deux outils très importants en géométrie birationnelle. La formule du fibré canonique pour une fibration f:(X,B)→ Z consiste à écrire K_X+B comme le tiré en arrière de K_Z+B_Z+M_Z où K_Z est le diviseur canonique, B_Z contient des informations sur les fibres singulières et M_Z est appelé partie modulaire. Il a été conjecturé qu'il existe une modification birationnelle Z' de Z telle que M_Z' est semiample sur Z' , où M_Z' est la partie modulaire induite par le changement de base. Un diviseur pseudoeffectif D admet une décomposition de Fujita-Zariski s'il existent un diviseur nef P et un diviseur effectif N tels que D=P+N et P est "le plus grand diviseur nef" avec la propriété que D−P est effectif.

formule du fibré canonique, partie modulaire, b-semiamplitude effective, décomposition de Fujita-Zariski, sigma-décomposition

Résumé en anglais

Insérer votre résumé en anglais suivi des mots-clés

The canonical bundle formula and the Fujita-Zariski decomposition are two very important tools in birational geometry. The canonical bundle formula for a fibration f:(X, B)→Z consists in writing K_X+B as the pullback of K_Z+B_Z+M_Z where K_Z is the canonical divisor, B_Z contains informations on the singular fibres and M_Z is called moduli part. It was conjectured that there exists a birational modification Z' of Z such that M_Z' is semiample on Z', where M_Z' is the moduli part induced by the base change. A pseudoeffective divisor D admits a Fujita-Zariski decomposition if there exist a nef divisor P and an effective divisor N such that D=P+N and P is "the biggest nef divisor" such that D−P is effectve.

canonical bundle formula, moduli part, effective b-semiampleness, Fujita-Zariski decomposition, sigma-decomposition