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On the minimal number of periodic Reeb orbits on a contact manifold

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Contents

Prolegomenon

As the title indicates, periodic Reeb orbits are at the centre of this work. The general framework is contact geometry. A *contact structure* on a manifold M of dimension $2n-1$ is a hyperplane field ξ which is maximally non integrable; i.e if we write locally $\xi = \ker \alpha$ where α is a differential 1-form, then $\alpha \wedge (d\alpha)^{n-1} \neq 0$ everywhere. If α is globally defined, we say that ξ is *coorientable* and we call α a *contact form*. In this thesis we consider only coorientable contact structures. The 1-form α is not unique; for any function $f : M \to \mathbb{R}$, the 1-form $e^f\alpha$ defines the same contact structure. The *Reeb vector field* R_α associated to a contact form α is the unique vector field on M characterized by: $\iota(R_\alpha)d\alpha = 0$ and $\alpha(R_\alpha) = 1$. Since this vector field does not vanish anywhere, there are no fixed points of its flow. Periodic orbits are thus the most noticeable objects in the flow. Poincaré pointed out their interest in his "traité de la mécanique céleste":
"Ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable."

Does there exist a periodic Reeb orbit for any contact form on any manifold?

The answer to this question is negative as shown by the following example. Let $\mathbb{R}^{2n-1}$ be endowed with the "standard" contact form

$$\alpha = dz - \sum_{i=1}^{n-1} x^i dy^i.$$

The Reeb vector field is given by $R_\alpha = \partial_z$, and there are no periodic Reeb orbits.

The same question for a compact oriented manifold endowed with a contact 1-form is still open. It is called the Weinstein conjecture. Some partial results are known. Taubes answered positively this question for manifolds in dimension 3.

Theorem 0.0.1 ([Tau07]) *If Y is a closed oriented three-manifold with a contact form, then the associated Reeb vector field has a closed orbit.*

In higher dimensions, only partial results have been obtained. Let us mention a first result due to Viterbo:

Theorem 0.0.2 ([Vit87]) *If $\Sigma \subset (\mathbb{R}^{2n}, \omega_0)$ is a compact hypersurface of "contact type", then Σ has at least one periodic Reeb orbit.*

More recently Albers and Hofer proved:

Theorem 0.0.3 ([AH09]) *Let (M, ξ) be a closed "PS-overtwisted[1]" contact manifold. Then the Reeb vector field associated to any contact form α inducing ξ has a contractible periodic orbit.*

One could also quote many particular results related for example to Hamiltonian dynamics.

Another natural question is to ask how many periodic Reeb orbits can exist. Since any periodic orbit can be iterated any number of times, it is more reasonable to ask how many geometrically distinct periodic orbits can exist. Consider the sphere S^{2n-1}, naturally embedded in $\mathbb{R}^{2n}$, endowed with the standard contact structure,

$$\alpha = \alpha^{std}\Big|_{S^{2n-1}} = \frac{1}{2}\sum_{i=1}^{n}(x^i dy^i - y^i dx^i))\Big|_{S^{2n-1}}.$$

The Reeb vector field is $R_\alpha = \sum_{i=1}^n 2(x^i\partial_{y^i} - y^i\partial_{x^i})$ and all Reeb orbits are periodic. On the other hand, if we look on the same sphere at the "deformed" contact form

$$\alpha' := \frac{1}{2}\sum_{i=1}^{n} a_i(x^i dy^i - y^i dx^i)\Big|_{S^{2n-1}},$$

where all the a_i are rationally independent, the Reeb vector field is $R_{\alpha'} = \sum_{i=1}^n \frac{2}{a_i}(x^i\partial_{y^i} - y^i\partial_{x^i})$. There are only n distinct periodic Reeb orbits, one in each "coordinate plane". Now the contact forms α and α' on the sphere are isotopic (i.e. there exists a smooth path of contact forms on the sphere joining them). Gray's stability theorem asserts that there exists a diffeomorphism of the sphere, which maps the contact structure $\xi' = \ker\alpha'$ to the contact structure $\xi = \ker\alpha$. Hence, depending on the contact form defining the standard contact structure on the sphere, one can get different answers concerning the number of distinct periodic orbits. In view of this fact, a natural question is:

If (M, ξ) is a compact contact manifold, can one say something about the minimal number of geometrically distinct periodic Reeb orbits for any contact form α (eventually in a subclass) defining the contact structure ξ?

In particular what is the answer for the sphere? Some results are known in this case; in particular, it was solved in dimension 3 by Hofer, Wysocki and Zehnder for the class of "dynamically convex" contact forms.

[1] We refer to the cited reference for a precise definition. It will not be used in the sequel

Theorem 0.0.4 ([HWZ98]) *Assume the contact form $\alpha = f \cdot \alpha_{std}$ on S^3 is dynamically convex*[2]*, where $f : S^3 \to (0, \infty)$ is a smooth, positive function. Let R_α be the associated Reeb vector field. Then there are either precisely 2 or infinitely many periodic orbits of R_α.*

In higher dimensions, less is known :

Theorem 0.0.5 ([LZ02]) *Any strictly convex, compact hypersurface $\Sigma \subset \mathbb{R}^{2n}$ carries at least $\lfloor \frac{n}{2} \rfloor + 1$ geometrically distinct periodic Reeb orbits.*

Theorem 0.0.6 ([WHL07]) *Any strictly convex, compact hypersurface $\Sigma \subset \mathbb{R}^6$ carries at least 3 geometrically distinct periodic Reeb orbits.*

Theorem 0.0.7 ([EL80, BLMR85]) *Let Σ be a contact type hypersurface in $\mathbb{R}^{2n}$. Let $\xi = \ker \alpha$ be the contact structure induced by the standard contact form on $\mathbb{R}^{2n}$. Assume there exists a point $x_0 \in \mathbb{R}^{2n}$ and numbers $0 < r \leq R$ such that:*

$$\forall x \in \Sigma, \quad r \leq \|x - x_0\| \leq R \quad \textit{with} \quad \frac{R}{r} < \sqrt{2}.$$

Assume also that $\forall x \in \Sigma, \quad \langle \nu_\Sigma(x), x \rangle > r$ where $\nu_\Sigma(x)$ is the exterior unit normal vector of Σ at x. Then Σ carries at least n geometrically distinct periodic Reeb orbits.

A first result of this thesis is an alternative (geometric) proof of this result when all periodic Reeb orbits on Σ are non degenerate[3]. A reasonable conjecture is that any starshaped hypersurface in $\mathbb{R}^{2n}$ carries at least n distinct periodic Reeb orbits.
For other manifolds than the sphere, very little is known; in dimension 3, Cristofaro-Gardiner and Hutchings proved :

Theorem 0.0.8 ([CGH12]) *Every (possibly degenerate*[4]*) contact form on a closed three-manifold has at least two embedded periodic Reeb orbits.*

In higher dimensions, a recent result is the following:

Theorem 0.0.9 ([Kan13]) *Suppose that a closed contact manifold (M, ξ) of dimension $2n-1$ admits a displaceable exact contact embedding into a symplectic manifold $(W, \omega = d\lambda)$ which is convex at infinity*[5] *and satisfies $\langle c_1(W), \pi_2(W) \rangle = 0$.*

Assume that at least one of the following conditions is satisfied

1. *$H_*(W_0, M; \mathbb{Q}) \neq 0$ for some $* \in 2\mathbb{N} - 1$*

[2] We refer to the cited reference for precise definitions. It will not be used in the sequel.
[3] The definition of a non degenerate orbit is given in Definition 1.1.5.
[4] A contact form is non degenerate if all its periodic Reeb orbits are non degenerate.
[5] We refer to the cited reference for precise definitions. It will not be used in the sequel.

2. $H_*(W_0, M; \mathbb{Q}) = 0$ *for all even degree* $* \leq 2n-4$

where W_0 *is the relatively compact domain of* W *bounded by* M*. Then there are at least two periodic Reeb orbits contractible in* W *for any nondegenerate contact form* α *on* (M,ξ) *such that* $\alpha - \lambda_{|_M}$ *is exact.*

In this thesis I present new results on the minimal number of periodic Reeb orbits on some contact type hypersurfaces in negative line bundles (Proposition 0.0.15 and Theorem 0.0.16).

Strategy followed in this thesis

The problem of finding periodic Reeb orbits on a contact manifold which is embedded in a symplectic manifold can often be translated into the problem of finding periodic orbits of a Hamiltonian vector field on a prescribed energy level. For instance, if C is a starshaped domain in $\mathbb{R}^{2n}$ such that $0 \in \operatorname{Int} C$, finding periodic orbits of the Reeb vector field on the boundary of C (for the standard contact form α_C^{std}) amounts to finding periodic orbits of the Hamiltonian vector field defined by a power of the gauge function, on the boundary of C which is a level set of this Hamiltonian. Indeed, the *gauge function* of C, $j_C : \mathbb{R}^{2n} \to [0,\infty)$ is defined by

$$j_C(x) := \min\left\{\lambda \middle| \tfrac{x}{\lambda} \in C\right\}$$

and the Hamiltonian vector field associated to $H_\beta = j_C(x)^\beta$ is $X_{H_\beta} = \frac{\beta}{2} R_{\alpha_C^{std}}$.

The first idea to tackle the question of the minimal number of periodic Reeb orbits was to use a homological invariant of the contact structure, constructed from periodic Reeb orbits. To build such an invariant is the aim of contact homology. At the time of this writing, contact homology is still in development and encounters “transversality” problems. Instead we consider positive S^1-equivariant symplectic homology which is built from periodic orbits of Hamiltonian vector fields in a symplectic manifold whose boundary is the given contact manifold. In this spirit, Bourgeois and Oancea, in [BO12], relate, in the case where it could be defined, the linearised contact homology of the boundary to the positive S^1-equivariant symplectic homology of the symplectic manifold. The positive S^1-equivariant symplectic homology is one of the main objects considered in this thesis.

Our first aim is to analyse the relation between the symplectic homologies of an exact compact symplectic manifold with contact type boundary (also called Liouville domain) and the periodic Reeb orbits on the boundary. The next point is to prove some properties of these homologies. For a Liouville domain embedded into another one, we construct a morphism between their homologies. We study the invariance of these homologies with respect to the choice of the contact form on the boundary. Finally, we use the positive S^1-equivariant symplectic homology to give a new proof of Theorem 0.0.7 and see how it can extend to the framework of hypersurfaces in negative line bundles.

Another approach to solve the question of the minimal number of periodic Reeb orbits on hypersurfaces in $\mathbb{R}^{2n}$, developed by Long, uses variational methods and a thorough study of their Conley-Zehnder index. With this in mind, we study the generalisation of the Conley-Zehnder index defined for any path of symplectic matrices. This led us to analyse in details normal forms of symplectic matrices. Those results could be useful to study degenerate orbits.

Content of the thesis and statements of the results

The first four chapters develop the approach using positive S^1-equivariant symplectic homology.

In Chapter 1, we recall the definition of positive S^1-equivariant symplectic homology, first describing Floer homology (Section 1.1), symplectic homology (Section 1.2.2), positive symplectic homology (Section 1.2.4), then recalling two equivalent definitions of S^1-equivariant symplectic homology in Sections 1.3.2 and 1.4.

The link between the generators of positive S^1-equivariant symplectic homology and periodic Reeb orbits is explained in Chapter 2. The explicit computation gives the following result:

Theorem 0.0.10 *Let (W, λ) be a Liouville domain. Assume there exists a contact form α on the boundary ∂W such that the Conley-Zehnder index of all periodic Reeb orbits have the same parity. Then*

$$SH^{S^1,+}(W, \mathbb{Q}) = \bigoplus_{\gamma \in \mathcal{P}(R_\alpha)} \mathbb{Q}\langle\gamma\rangle$$

where $\mathcal{P}(R_\alpha)$ denotes the set of periodic Reeb orbits on ∂W.

In Chapter 3, we show that positive S^1-equivariant symplectic homology has good functorial properties. In the first section, we construct a "transfer morphism" between all the above mentioned variants of symplectic homology in the case of two Liouville domains embedded one into the other. This construction generalises a construction given by Viterbo ([Vit99]). We prove that this morphism has nice composition properties:

Theorem 0.0.11 *Let $(V_1, \lambda_{V_1}) \subseteq (V_2, \lambda_{V_2}) \subseteq (V_3, \lambda_{V_3})$ be Liouville domains with Liouville embeddings. Then the following diagram commutes:*

$$\begin{tikzcd} SH^\dagger(V_3, \lambda_{V_3}) \arrow[r, "\phi^\dagger_{V_3,V_2}"] \arrow[rr, bend right, "\phi^\dagger_{V_3,V_1}"'] & SH^\dagger(V_2, \lambda_{V_2}) \arrow[r, "\phi^\dagger_{V_2,V_1}"] & SH^\dagger(V_1, \lambda_{V_1}) \end{tikzcd}$$

where $\dagger$ can be any of the following symbol: $\emptyset, +, S^1, (S^1, +)$.

where SH denotes the symplectic homology, SH^+, the positive symplectic homology, SH^{S^1} the S^1-equivariant symplectic homology and $SH^{S^1,+}$ the positive S^1-equivariant symplectic homology.

The second section of Chapter 3 is dedicated to the invariance of the different variants of symplectic homology. In particular, we prove

Theorem 0.0.12 *Let (W_0, λ_0) and (W_1, λ_1) be two Liouville manifolds*[6] *of finite type such that there exists a symplectomorphism $f : (W_0, \lambda_0) \to (W_1, \lambda_1)$. Then*

$$SH^\dagger(W_0, \lambda_0) \cong SH^\dagger(W_1, \lambda_1).$$

Theorem 0.0.13 *Let (M_0, ξ_0) and (M_1, ξ_1) be two contact manifolds that are exactly fillable; i.e. there exist Liouville domains (W_0, λ_0) and (W_1, λ_1) such that $\partial W_0 = M_0$, $\xi_0 = \ker(\lambda_{0|_{M_0}})$, $\partial W_1 = M_1$ and $\xi_1 = \ker(\lambda_{1|_{M_1}})$. Assume there exists a contactomorphism $\varphi : (M_0, \xi_0) \to (M_1, \xi_1)$. Assume moreover that ξ_0 admits a contact form α_0 such that all periodic Reeb orbits are nondegenerate and their Conley-Zehnder indices have all the same parity. Then*

$$SH^{S^1,+}(W_0, \lambda_0) \cong SH^{S^1,+}(W_1, \lambda_1).$$

This Theorem, together with Theorem 0.0.10 reproves Ustilovsky's result on the existence of non diffeomorphic contact structures on the spheres S^{4m+1}. The original proof depends on a theory of cylindrical contact homology, which is not yet rigorously established due to transversality problems.

Theorem 0.0.14 ([Ust99]) *For each natural number m, there exist infinitely many pairwise non isomorphic contact structures on S^{4m+1}.*

In Chapter 4 we use positive S^1-equivariant symplectic homology to give a new proof of Theorem 0.0.7, about the minimal number of periodic Reeb orbits on some hypersurfaces in $\mathbb{R}^{2n}$, when all periodic Reeb orbits on Σ are non degenerate. It appears as Theorem 4.1.1 in the following.

We extend the definitions of positive symplectic homology and positive S^1-equivariant symplectic homology in Sections 4.2.1 and 4.2.2 to the non exact case. It allows us to extend the techniques developed for the proof of Theorem 0.0.7 to start the study of hypersurfaces in negative line bundles. This framework is a natural generalisation of hypersurfaces in $\mathbb{C}^n$. Indeed the sphere is the boundary of the ball in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ but also the boundary of the ball blown up[7] at the origin. The blown up ball, at the origin, in $\mathbb{C}^n$ is

$$\hat{B}^{2n} := \Big\{\big(z, [t]\big) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \,|\, z \in [t]\Big\}.$$

[6] We refer to Definition 3.2.8 for a precise definition of Liouville manifold.

[7] We refer to [MS98] for a detailed definition of blow ups

It is canonically isomorphic to the canonical disk bundle over $\mathbb{C}P^{n-1}$ which is a subbundle of the tautological complex (negative) line bundle over $\mathbb{C}P^{n-1}$

$$\mathcal{O}(-1) \longrightarrow \mathbb{C}P^{n-1}.$$

This generalisation gives:

Proposition 0.0.15 *Let Σ be a contact type hypersurface in a negative line bundle over a closed symplectic manifold $\mathcal{L} \to B$ such that the intersection of Σ with each fiber is a circle. The contact form is the restriction of $r^2\theta^\nabla$ where θ^∇ is the transgression form on $\mathcal{L}$ and r is the radial coordinate on the fiber. Then Σ carries at least $\sum_{i=0}^{2n}\beta_i$ geometrically distinct periodic Reeb orbits (the β_i are the Betti numbers of B).*

Theorem 0.0.16 *Let Σ be a contact type hypersurface in a negative line bundle $\mathcal{L}$, over a symplectic manifold B. Suppose that there exists a Liouville domain W' (such that its first Chern class vanishes on all tori) whose boundary coincides with the circle bundle of radius R_1 in $\mathcal{L}$, denoted $S_{R_1^2}$. Suppose there exists a Morse function $f : B \to \mathbb{R}$ such that all critical points of f have a Morse index of the same parity. Let α be the contact form on Σ induced by $r^2\theta^\nabla$ on $\mathcal{L}$ (θ^∇ is the transgression form on $\mathcal{L}$ and r is the radial coordinate on the fiber). Assume that Σ is "pinched" between two circle bundles $S_{R_1^2}$ and $S_{R_2^2}$ of radii R_1 and R_2 such that $0 < R_1 < R_2$ and $\frac{R_2}{R_1} < \sqrt{2}$. Assume that the minimal period of any periodic Reeb orbit on Σ is bounded below by R_1^2. Then Σ carries at least $\sum_{i=0}^{2n}\beta_i$ geometrically distinct periodic Reeb orbits, where the β_i denote the Betti numbers of B.*

In this Theorem, the assumption on the existence of a Morse function all of whose critical points have Morse indices of the same parity is of a technical nature. Its purpose is to bring the situation within the scope of Theorem 0.0.10, which is our tool for computing the positive S^1-equivariant symplectic homology. The lower bound on the period of any periodic Reeb orbit is semi-technical; it is now the only way we have to distinguish the images of the orbits. The "pinching" assumption is more conceptual, its main implication is that the "n first generators" of the positive S^1-equivariant symplectic homology are simple orbits.

The techniques developed in the thesis should prove, extending the homologies to the setup of monotone compact symplectic manifolds using coefficients in the Novikov ring, the following:

Conjecture 0.0.17 *Let Σ be a contact type hypersurface in a negative line bundle, $\mathcal{L}$, over a closed monotone symplectic manifold B. The bundle is endowed with a hermitian structure and a connection. Suppose there exists a Morse function $f : B \to \mathbb{R}$ such that all critical points of f have a Morse index of the same parity. Let α be the contact form on Σ induced by $r^2\theta^\nabla$ on $\mathcal{L}$. Assume that Σ is "pinched" between two circle bundles S_{R_1} and S_{R_2} of radii R_1 and R_2 respectively such that $0 < R_1 < R_2$ and $\frac{R_2}{R_1} < \sqrt{2}$. Assume that the* *minimal action of any periodic Reeb orbit on Σ is bounded below by R_1^2. Then Σ carries at least $\sum_{i=0}^{2n}\beta_i$ geometrically distinct periodic Reeb orbits.*

The last two chapters are a contribution to a detailed study of indices. I hope it can be used in the future to tackle the study of degenerate periodic orbits. The Conley-Zehnder index is an integer associated to a path of symplectic matrices. It relies on a precise description of symplectic matrices. Chapter 5, which will appear as a paper in Portugaliae Mathematica, gives new normal forms for symplectic matrices. Let us present here the normal form on the generalised eigenspace of eigenvalue ± 1.

Theorem 0.0.18 *Any symplectic endomorphism A of a finite dimensional symplectic vector space (V,Ω) is the direct sum of its restrictions $A_{|V_{[\lambda]}}$ to the real A-invariant symplectic subspace $V_{[\lambda]}$ whose complexification is the direct sum of the generalized eigenspaces of eigenvalues $\lambda, \frac{1}{\lambda}, \overline{\lambda}$ and $\frac{1}{\overline{\lambda}}$:*

$$V_{[\lambda]}^{\mathbb{C}} := E_\lambda \oplus E_{\frac{1}{\lambda}} \oplus E_{\overline{\lambda}} \oplus E_{\frac{1}{\overline{\lambda}}}.$$

If $\lambda \in \{\pm 1\}$, there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of matrices of the form

$$\begin{pmatrix} J(\lambda, r_j)^{-1} & C(r_j, s_j, \lambda) \\ 0 & J(\lambda, r_j)^\tau \end{pmatrix}$$

where $C(r_j, s_j, \lambda) := J(\lambda, r_j)^{-1} \operatorname{diag}\big(0,\ldots,0,s_j\big)$ with $s_j \in \{0,1,-1\}$ and where $J(\lambda, r_j)$ is the elementary Jordan matrix of dimension r_j associated to λ. If $s_j = 0$, then r_j is odd. The dimension of the eigenspace of the eigenvalue λ is given by $2\operatorname{Card}\{j \mid s_j = 0\} + \operatorname{Card}\{j \mid s_j \neq 0\}$.
The number of s_j equal to $+1$ (resp. -1) arising in blocks of dimension $2k$ (i.e. with corresponding $r_j = k$) is equal to the number of positive (resp. negative) eigenvalues of the symmetric 2-form

$$\hat{Q}_{2k}^{\lambda} : \operatorname{Ker}\big((A-\lambda\operatorname{Id})^{2k}\big) \times \operatorname{Ker}\Big((A-\lambda\operatorname{Id})^{2k}\Big) \to \mathbb{R}$$

$$(v,w) \mapsto \lambda\,\Omega\big((A-\lambda\operatorname{Id})^k v, (A-\lambda\operatorname{Id})^{k-1} w\big).$$

The decomposition is unique up to a permutation of the blocks and is determined by λ, by the dimension $\dim\big(\operatorname{Ker}(A-\lambda\operatorname{Id})^r\big)$ for each $r \geq 1$, and by the rank and the signature of the symmetric bilinear 2-form $\hat{Q}_{2k}^{\lambda}$ for each $k \geq 1$.

The last Chapter, which will appear as a paper in Annales de la faculté des Sciences de Toulouse, is devoted to the study of a generalised version of the Conley-Zehnder index defined by Robbin and Salamon in [RS93]. We start by giving a new formula for the "classical" Conley-Zehnder index.

Theorem 0.0.19 *Let $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be a continuous path of matrices linking the matrix* Id *to a matrix which does not admit* 1 *as an eigenvalue. Let $\widetilde{\psi} : [0,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be an extension such that $\widetilde{\psi}$ coincides with ψ on the interval $[0,1]$, such that $\widetilde{\psi}(s)$ does not admit* 1 *as an eigenvalue for all $s \geqslant 1$ and such that the path ends either at $\widetilde{\psi}(2) = W^+ := -\mathrm{Id}$ either at $\widetilde{\psi}(2) = W^- := \mathrm{diag}(2, -1, \ldots, -1, \frac{1}{2}, -1, \ldots, -1)$. The Conley-Zehnder index of ψ is equal to the integer given by the degree of the map $\tilde{\rho}^2 \circ \widetilde{\psi} : [0,2] \to S^1$:*

$$\mu_{CZ}(\psi) = \deg(\tilde{\rho}^2 \circ \widetilde{\psi}) \tag{1}$$

for ANY continuous map $\tilde{\rho} : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to S^1$ which coincides with the (complex) determinant $\det_{\mathbb{C}}$ on $U(n) = O\left(\mathbb{R}^{2n}\right) \cap \mathrm{Sp}\left(\mathbb{R}^{2n}, \Omega_0\right)$; so that $\tilde{\rho}(W^-) \in \{\pm 1\}$ and so that $\deg\left(\tilde{\rho}^2 \circ \psi_{2-}\right) = n - 1$ for $\psi_{2-} : t \in [0,1] \mapsto \exp t\pi J_0 \begin{pmatrix} 0 & 0 & -\frac{\log 2}{\pi} & 0 \\ 0 & \mathrm{Id}_{n-1} & 0 & 0 \\ -\frac{\log 2}{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{n-1} \end{pmatrix}$. In particular, two alternative ways to compute the Conley-Zehnder index are :

- *Using the polar decomposition of a matrix, $\mu_{CZ}(\psi) = \deg({\det_{\mathbb{C}}}^2 \circ U \circ \widetilde{\psi})$ where $U : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to U(n) : A \mapsto AP^{-1}$ with P the unique symmetric positive definite matrix such that $P^2 = A^\tau A$.*

- *Using the normalized determinant of the $\mathbb{C}$-linear part of a symplectic matrix, $\mu_{CZ}(\psi) = \deg(\hat{\rho}^2 \circ \widetilde{\psi})$ where $\hat{\rho} : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to S^1 : A \mapsto \hat{\rho}(A) = \frac{\det_{\mathbb{C}}\left(\frac{1}{2}(A - J_0 A J_0)\right)}{\left|\det_{\mathbb{C}}\left(\frac{1}{2}(A - J_0 A J_0)\right)\right|}$ with $J_0 = \left(\begin{smallmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{smallmatrix}\right)$ the standard complex structure on $\mathbb{R}^{2n}$.*

We give a characterisation of the generalised the Conley-Zehnder index defined by Robbin and Salamon.

Theorem 0.0.20 *The Robbin-Salamon index for a continuous path of symplectic matrices is characterized by the following properties:*

- (Homotopy) *it is invariant under homotopies with fixed end points;*

- (Catenation) *it is additive under catenation of paths;*

- (Zero) *it vanishes on any path $\psi : [a,b] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega)$ of matrices such that $\dim \mathrm{Ker}\big(\psi(t) - \mathrm{Id}\big) = k$ is constant on $[a,b]$;*

- (Normalization) *if $S = S^\tau \in \mathbb{R}^{2n \times 2n}$ is a symmetric matrix with all eigenvalues of absolute value $< 2\pi$ and if $\psi(t) = \exp(J_0 S t)$ for $t \in [0,1]$, then $\mu_{RS}(\psi) = \frac{1}{2} \mathrm{Sign}\, S$ where $\mathrm{Sign}\, S$ is the signature of S.*

We give a new way to compute this index.

Theorem 0.0.21 *Let $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be a path of symplectic matrices. Decompose $\psi(0) = \psi^\star(0) \oplus \psi^{(1)}(0)$ and $\psi(1) = \psi^\star(1) \oplus \psi^{(1)}(1)$ where $\psi^\star(\cdot)$ does not admit 1 as eigenvalue and $\psi^{(1)}(\cdot)$ is the restriction of $\psi(\cdot)$ to its generalized eigenspace of eigenvalue 1. Consider a continuous extension $\Psi : [-1,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ of ψ such that*

- $\Psi(t) = \psi(t)$ *for* $t \in [0,1]$*;*
- $\Psi\big(-\frac{1}{2}\big) = \psi^\star(0) \oplus \big(\begin{smallmatrix} e^{-1}\,\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id}\end{smallmatrix}\big)$ *and* $\Psi(t) = \psi^\star(0) \oplus \phi_0(t)$ *where* $\phi_0(t)$ *has only real positive eigenvalues for* $t \in \big[-\frac{1}{2}, 0\big]$*;*
- $\Psi\big(\frac{3}{2}\big) = \psi^\star(1) \oplus \big(\begin{smallmatrix} e^{-1}\,\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id}\end{smallmatrix}\big)$ *and* $\Psi(t) = \psi^\star(1) \oplus \phi_1(t)$ *where* $\phi_1(t)$ *has only real positive eigenvalues for* $t \in \big[1, \frac{3}{2}\big]$*;*
- $\Psi(-1) = W^\pm$, $\Psi(2) = W^\pm$ *and* $\Psi(t)$ *does not admit 1 as an eigenvalue for* $t \in \big[-1, -\frac{1}{2}\big]$ *and for* $t \in \big[\frac{3}{2}, 2\big]$.

Then the Robbin Salamon index is given by

$$\mu_{RS}(\psi) = \deg(\tilde{\rho}^2 \circ \Psi) + \frac{1}{2}\sum_{k\geq 1} \mathrm{Sign}\Big(\hat{Q}_k^{(\psi(0))}\Big) - \frac{1}{2}\sum_{k\geq 1} \mathrm{Sign}\Big(\hat{Q}_k^{(\psi(1))}\Big)$$

with $\tilde{\rho}$ as in theorem 0.0.19, and with

$$\begin{aligned} \hat{Q}_k^A \; : \; & \mathrm{Ker}\Big((A - \mathrm{Id})^{2k}\Big) \times \mathrm{Ker}\Big((A - \mathrm{Id})^{2k}\Big) \to \mathbb{R} \\ & (v,w) \mapsto \Omega\big((A-\mathrm{Id})^k v, (A-\mathrm{Id})^{k-1} w\big). \end{aligned}$$

The advantage of this new formula is that we can compute the index of any path without perturbing the path. The drawback is that we have to extend the initial path.

Prolégomènes

Les orbites périodiques de Reeb sont au centre de ce travail. Le cadre général est la géométrie de contact. Une *structure de contact* sur une variété M de dimension $2n-1$ est un champ d'hyperplans ξ maximalement non intégrable; i.e. si on écrit, localement, $\xi = \ker \alpha$ ou α est une 1-forme différentielle, alors $\alpha \wedge (d\alpha)^{n-1}$ est partout non nulle. Si α est globalement définie, on dit que ξ est *coorientée* et on appelle α une *forme de contact.* Dans cette thèse, nous ne considérons que des structures de contact coorientées. La 1-forme α n'est pas unique; pour toute fonction $f : M \to \mathbb{R}$, la 1-forme $e^f\alpha$ définit la même structure de contact. Le *champ de vecteurs de Reeb* R_α associé à une forme de contact α est l'unique champ de vecteurs sur M caractérisé par: $\iota(R_\alpha)d\alpha = 0$ et $\alpha(R_\alpha) = 1$. Ce champ de vecteurs ne s'annulant nulle part, son flot n'a pas de point fixe. Les orbites périodiques sont donc les objets les plus remarquables de ce flot. Poincaré en a montré l'intérêt dans son "Traité de la mécanique céleste":
"Ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable."

Existe t-il une orbite de Reeb périodique pour toute forme de contact sur n'importe quelle variété?

La réponse à cette question est négative comme illustré par l'exemple suivant. Soit $\mathbb{R}^{2n-1}$ muni de la forme de contact "standard"

$$\alpha = dz - \sum_{i=1}^{n-1} x^i dy^i.$$

Le champ de vecteurs de Reeb est donné par $R_\alpha = \partial_z$, et il n'y a pas d'orbites de Reeb périodiques.

La même question pour des variétés compactes orientées munies d'une 1-forme de contact est toujours ouverte; c'est la conjecture de Weinstein. Quelques résultats partiels sont connus. Taubes a répondu de manière affirmative à cette question pour les variétés de dimension 3.

Theorem 0.0.22 ([Tau07]) *Si Y est une variété fermée, orientée, de dimension 3, munie d'une forme de contact, alors le champ de vecteurs de Reeb associé possède une orbite périodique.*

En plus grande dimension, seuls des résultats partiels sont connus. Nous commençons par mentionner un résultat dû à Viterbo.

Theorem 0.0.23 ([Vit87]) *Si $\Sigma \subset (\mathbb{R}^{2n}, \omega_0)$ est une hypersurface de "type contact", alors Σ possède au moins une orbite de Reeb périodique.*

Plus récemment, Albers et Hofer ont prouvé:

Theorem 0.0.24 ([AH09]) *Soit (M, ξ) une variété de contact fermée et "PS-vrillée[8]". Alors le champ de vecteurs de Reeb associé à n'importe quelle forme de contact α déterminant ξ, possède au moins une orbite de Reeb périodique et contractible.*

Nous pourrions citer beaucoup d'autres résultats, par exemple liés à la dynamique Hamiltonienne.

Une autre question naturelle est de demander combien d'orbites de Reeb périodiques peuvent exister. Comme toute orbite de Reeb périodique peut être itérée un nombre arbitraire de fois, il est plus raisonnable de demander combien d'orbites de Reeb périodiques géométriquement distinctes peuvent exister. Considérons la sphère S^{2n-1} naturellement plongée dans $\mathbb{R}^{2n}$, munie de la structure de contact standard,

$$\alpha = \alpha^{std}\Big|_{S^{2n-1}} = \frac{1}{2}\sum_{i=1}^{n}(x^i dy^i - y^i dx^i))\Big|_{S^{2n-1}}.$$

Le champ de Reeb est $R_\alpha = \sum_{i=1}^{n} 2(x^i\partial_{y^i} - y^i\partial_{x^i})$ et toutes les orbites de Reeb sont périodiques. D'un autre côté, si nous regardons la même sphère munie de la forme de contact "déformée"

$$\alpha' := \frac{1}{2}\sum_{i=1}^{n} a_i(x^i dy^i - y^i dx^i)\Big|_{S^{2n-1}},$$

où tous les a_i sont rationnellement indépendants, le champ de Reeb est $R_{\alpha'} = \sum_{i=1}^{n} \frac{2}{a_i}(x^i\partial_{y^i} - y^i\partial_{x^i})$. Il y a seulement n orbites de Reeb géométriquement distinctes, une dans chaque "plan de coordonnées". Les formes de contact α et α' sur la sphère sont isotopes (i.e. il existe un chemin lisse de formes de contact sur la sphère les reliant). Le théorème de stabilité de Gray assure l'existence d'un difféomorphisme de la sphère envoyant la structure de contact $\xi' = \ker\alpha'$ sur la structure de contact $\xi = \ker\alpha$. Donc, en fonction de la forme de contact définissant la structure de contact standard sur la sphère, nous pouvons avoir différentes réponses concernant le nombre d'orbites périodiques distinctes. En vue de quoi, une question naturelle est

[8]Nous référons à la référence citée pour une définition précise qui ne sera pas utilisée dans la suite de ce travail.

Si (M, ξ) est une variété de contact compacte, que pouvons-nous dire sur le nombre minimal d'orbites de Reeb périodiques et géométriquement distinctes pour toute forme de contact α (éventuellement dans une sous-classe) définissant la structure de contact ξ?

En particulier, quelle est la réponse pour la sphère? Quelques résultats sont connus dans ce cas. Le problème a été résolu en dimension 3 par Hofer, Wysocki et Zehnder pour la classe des formes de contact "dynamiquement convexes".

Theorem 0.0.25 ([HWZ98]) *Supposons que la forme de contact $\alpha = f \cdot \alpha_{std}$ sur S^3 est dynamiquement convexe*[9]*, où $f : S^3 \to (0, \infty)$ est une fonction lisse positive. Soit R_α le champ de vecteurs de Reeb associé. Alors il y a soit exactement 2 soit une infinité d'orbites périodiques de R_α.*

En grande dimension, moins de choses sont connues:

Theorem 0.0.26 ([LZ02]) *Toute hypersurface $\Sigma \subset \mathbb{R}^{2n}$ compacte et strictement convexe possède au moins $\lfloor \frac{n}{2} \rfloor + 1$ orbites de Reeb périodiques géométriquement distinctes.*

Theorem 0.0.27 ([WHL07]) *Toute hypersurface $\Sigma \subset \mathbb{R}^6$ compacte et strictement convexe possède au moins 3 orbites de Reeb périodiques géométriquement distinctes.*

Theorem 0.0.28 ([EL80, BLMR85]) *Soit Σ une hypersurface de type contact dans $\mathbb{R}^{2n}$. Soit $\xi = \ker \alpha$ la structure de contact induite par la forme de contact standard sur $\mathbb{R}^{2n}$. Supposons qu'il existe un point $x_0 \in \mathbb{R}^{2n}$ et des nombres $0 < r \leq R$ tels que:*

$$\forall x \in \Sigma, \quad r \leq \|x - x_0\| \leq R \quad \textit{avec} \quad \frac{R}{r} < \sqrt{2}.$$

Supposons également que $\forall x \in \Sigma, \quad \langle \nu_\Sigma(x), x \rangle > r$ où $\nu_\Sigma(x)$ est le vecteur unitaire normal extérieur à Σ en x. Alors Σ possède au moins n orbites de Reeb périodiques géométriquement distinctes.

Un premier résultat de cette thèse est une preuve alternative (géométrique) de ce résultat quand toutes les orbites de Reeb périodiques sur Σ sont non dégénérées[10]. Une conjecture raisonnable est que toute hypersurface étoilée dans $\mathbb{R}^{2n}$ possède au moins n orbites de Reeb périodiques géométriquement distinctes.
Pour d'autres variétés que la sphère, fort peu est connu; en dimension 3, Cristofaro-Gardiner et Hutchings ont prouvé:

[9]Nous référons à la référence citée pour une définition précise qui ne sera pas utilisée dans la suite de ce travail.
[10]La définition d'orbites non dégénérée est donnée Définition 1.1.5.

Theorem 0.0.29 ([CGH12]) *Toute forme de contact (possiblement dégénérée*[11]*) sur une variété de dimension* 3 *possède au moins* 2 *orbites de Reeb périodiques plongées.*

En plus grande dimension, un résultat récent est le suivant:

Theorem 0.0.30 ([Kan13]) *Supposons que la variété de contact fermée* (M, ξ) *de dimension* $2n-1$ *admet un plongement de contact exact et déplaçable dans une variété symplectique* $(W, \omega = d\lambda)$ *convexe à l'infini*[12] *et satisfaisant* $\langle c_1(W), \pi_2(W)\rangle = 0$*. Supposons qu'au moins une des conditions suivantes est satisfaite*

1. $H_*(W_0, M; \mathbb{Q}) \neq 0$ *pour un* $* \in 2\mathbb{N} - 1$
2. $H_*(W_0, M; \mathbb{Q}) = 0$ *pour tout degré pair* $* \leq 2n - 4$

où W_0 *est le domaine relativement compact de* W *bordé par* M*. Alors il y a au moins* 2 *orbites de Reeb périodiques contractibles dans* W *pour toute forme de contact non dégénérée* α *sur* (M, ξ) *telle que* $\alpha - \lambda_{|M}$ *est exacte.*

Dans cette thèse, je présente de nouveaux résultats sur le nombre minimal d'orbites de Reeb périodiques géométriquement distinctes sur certaines hypersurface dans des fibrés en droites négatifs (Proposition 0.0.36 and Théorème 0.0.37).

Stratégie suivie dans cette thèse

Le problème de trouver des orbites de Reeb périodiques sur une variété de contact plongée dans une variété symplectique peut souvent être traduit en le problème de trouver des orbites périodiques d'un champ de vecteurs Hamiltonien sur un niveau d'énergie fixé. Par exemple, si C est un domaine étoilé dans $\mathbb{R}^{2n}$ tel que $0 \in \operatorname{Int} C$, trouver les orbites de Reeb périodiques sur le bord de C (pour la forme de contact standard α_C^{std}) revient à trouver les orbites périodiques du champ de vecteurs Hamiltonien défini comme une puissance de la fonction de jauge, sur le bord de C qui est un niveau de ce Hamiltonien. En effet, la *fonction de jauge* de C, $j_C : \mathbb{R}^{2n} \to [0, \infty)$, est définie par

$$j_C(x) := \min\{\lambda | \tfrac{x}{\lambda} \in C\}$$

et le champ de vecteurs Hamiltonien associé à $H_\beta = j_C(x)^\beta$ est $X_{H_\beta} = \frac{\beta}{2} R_{\alpha_C^{std}}$.

La première idée pour aborder la question du nombre minimal d'orbites de Reeb périodiques était d'utiliser un invariant homologique de la structure de contact construit à partir des orbites de Reeb périodiques. La construction d'un tel invariant est le but de l'homologie de contact. Actuellement, l'homologie de contact est toujours en développement

[11] Une forme de contact est non dégénérée si toutes ses orbites de Reeb périodiques sont non dégénérées.

[12] Nous référons à la référence citée pour les définitions précises qui ne seront pas utilisées dans la suite de ce travail.

et rencontre des problèmes de "transversalité". Nous considérons, à la place, l'homologie symplectique S^1-équivariante positive construite à partir d'orbites périodiques de champs de vecteurs Hamiltoniens sur une variété symplectique dont le bord est la variété de contact considérée. Dans cet esprit, Bourgeois et Oancea [BO12] ont lié, l'homologie de contact linéarisée du bord (dans le cas où elle peut être définie) avec l'homologie S^1-équivariante positive de la variété symplectique. L'homologie symplectique S^1-équivariante positive est un des objets principaux considérés dans les quatre premiers chapitres de cette thèse.

Notre premier but est d'analyser les relations entre les homologies symplectiques d'une variété symplectique exacte avec un bord de type contact (également appelé domaine de Liouville) et les orbites de Reeb périodiques sur le bord. Le point suivant est de prouver quelques propriétés de ces homologies. Pour un domaine de Liouville plongé dans un autre, nous construisons un morphisme entre leurs homologies. Nous étudions l'invariance de ces homologies par rapport au choix d'une forme de contact sur le bord. Finalement, nous utilisons l'homologie S^1-équivariante positive pour donner une nouvelle preuve du Théorème 0.0.28 et regardons comment elle peut s'étendre au cadre d'hypersurfaces dans des fibrés en droites négatifs.

Une autre approche à la question du nombre minimal d'orbites de Reeb périodiques sur des hypersurfaces dans $\mathbb{R}^{2n}$, développée par Long, utilise des méthodes variationnelles et une étude détaillée de l'indice de Conley-Zehnder. Dans cette optique, nous étudions une généralisation de l'indice de Conley-Zehnder définie pour tout chemin de matrices symplectiques. Ceci nous a mené à une analyse détaillée de formes normales de matrices symplectiques. Ces résultats peuvent être utiles pour une étude d'orbites dégénérées.

Contenu de la thèse et énoncés de résultats

Les quatre premiers chapitres développent l'approche utilisant l'homologie symplectique S^1-équivariante positive.

Dans le Chapitre 1, nous rappelons la définition de l'homologie symplectique S^1-équivariante positive. Nous présentons l'homologie de Floer (Section 1.1), l'homologie symplectique (Section 1.2.2), l'homologie symplectique positive (Section 1.2.4). Ensuite nous exposons deux définitions équivalentes de l'homologie symplectique S^1-équivariante positive dans les Sections 1.3.2 et 1.4.

Le lien entre les générateurs de l'homologie symplectique S^1-équivariante positive et les orbites de Reeb périodiques est expliqué dans le Chapitre 2. Un calcul explicite donne:

Theorem 0.0.31 *Soit (W, λ) un domaine de Liouville. Supposons qu'il existe une forme de contact α sur le bord ∂W telle que les indices de Conley-Zehnder de toutes les orbites de Reeb périodiques ont la même parité. Alors*

$$SH^{S^1,+}(W, \mathbb{Q}) = \bigoplus_{\gamma \in \mathcal{P}(R_\alpha)} \mathbb{Q}\langle\gamma\rangle$$

où $\mathcal{P}(R_\alpha)$ est l'ensemble des orbites de Reeb périodiques sur ∂W.

Dans le Chapitre 3, nous montrons que l'homologie symplectique S^1-équivariante positive a de bonnes propriétés de fonctorialité. Dans la première Section, nous construisons un "morphisme de transfert" pour toutes les variantes précitées d'homologie symplectique dans le cas de deux domaines de Liouville emboîtés. Cette construction généralise une construction de Viterbo ([Vit99]). Nous prouvons que ce morphisme possède de bonnes propriétés de composition:

Theorem 0.0.32 *Soit* $(V_1, \lambda_{V_1}) \subseteq (V_2, \lambda_{V_2}) \subseteq (V_3, \lambda_{V_3})$ *des domaines de Liouville avec des plongements de Liouville. Alors le diagramme suivant commute:*

$$\begin{tikzcd} SH^\dagger(V_3, \lambda_{V_3}) \arrow[r, "\phi^\dagger_{V_3,V_2}"] \arrow[rr, bend right, "\phi^\dagger_{V_3,V_1}"'] & SH^\dagger(V_2, \lambda_{V_2}) \arrow[r, "\phi^\dagger_{V_2,V_1}"] & SH^\dagger(V_1, \lambda_{V_1}) \end{tikzcd}$$

où $\dagger$ *est l'un des symboles suivants:* $\emptyset, +, S^1, (S^1, +)$.

où SH dénote l'homologie symplectique, SH^+, l'homologie symplectique positive, SH^{S^1} l'homologie symplectique S^1-équivariante et $SH^{S^1,+}$ l'homologie symplectique S^1-équivariante positive.

La seconde Section du Chapitre 3 est consacrée à l'invariance des différentes variantes d'homologie symplectique. En particulier, nous prouvons

Theorem 0.0.33 *Soit* (W_0, λ_0) *et* (W_1, λ_1) *deux variétés de Liouville*[13] *de type fini tels que il existe un symplectomorphisme* $f : (W_0, \lambda_0) \to (W_1, \lambda_1)$*. Alors*

$$SH^\dagger(W_0, \lambda_0) \cong SH^\dagger(W_1, \lambda_1).$$

Theorem 0.0.34 *Soit* (M_0, ξ_0) *and* (M_1, ξ_1) *deux variétés de contact exactement remplissables; i.e. il existe des domaines de Liouville* (W_0, λ_0) *et* (W_1, λ_1) *tels que* $\partial W_0 = M_0$, $\xi_0 = \ker(\lambda_{0|_{M_0}})$, $\partial W_1 = M_1$ *and* $\xi_1 = \ker(\lambda_{1|_{M_1}})$*. Supposons qu'il existe un contactomorphisme* $\varphi : (M_0, \xi_0) \to (M_1, \xi_1)$*. Supposons de plus que* ξ_0 *admet une forme de contact* α_0 *telle que toute les orbites de Reeb périodiques sont non dégénérées et leurs indices de Conley-Zehnder ont tous la même parité. Alors*

$$SH^{S^1,+}(W_0, \lambda_0) \cong SH^{S^1,+}(W_1, \lambda_1).$$

Ce Théorème, couplé au Théorème 0.0.31 donne une preuve du résultat d'Ustilovsky sur l'existence de structures de contact non difféomorphes sur les sphères S^{4m+1}. La preuve originelle repose sur une théorie d'homologie de contact cylindrique qui n'est pas encore établie rigoureusement.

[13] Nous référons à la Définition 3.2.8 pour une définition précise de variété de Liouville.

Theorem 0.0.35 ([Ust99]) *Pour tout nombre naturel m, il existe une infinité de structures de contact non isomorphes sur S^{4m+1}.*

Dans le Chapitre 4, nous utilisons l'homologie symplectique S^1-équivariante positive pour donner une nouvelle preuve du Théorème 0.0.28 sur le nombre minimal d'orbites de Reeb périodiques sur certaines hypersurfaces dans $\mathbb{R}^{2n}$ quand toutes les orbites de Reeb sont non dégénérées. Cela apparaît dans la suite comme le Théorème 4.1.1.

Nous étendons les définitions d'homologie symplectique positive et d'homologie symplectique S^1-équivariante positive au cas non exact dans les Sections 4.2.1 et 4.2.2. Ceci nous permet d'étendre les techniques développées pour la preuve du Théorème 0.0.28 pour commencer l'étude d'hypersurfaces dans des fibrés en droites négatifs. Ce cadre est la généralisation naturelle d'hypersurfaces dans $\mathbb{C}^n$. En effet la sphère est le bord de la boule dans $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ mais également de la boule éclatée[14] à l'origine. La boule éclatée en l'origine dans $\mathbb{C}^n$ est

$$\hat{B}^{2n} := \Big\{ \big(z, [t]\big) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \,|\, z \in [t] \Big\}.$$

Elle est canoniquement isomorphe au fibré en disques canonique au-dessus de $\mathbb{C}P^{n-1}$ qui est un sous-fibré du fibré en droites complexes négatif tautologique au-dessus de $\mathbb{C}P^{n-1}$

$$\mathcal{O}(-1) \longrightarrow \mathbb{C}P^{n-1}.$$

Cette généralisation donne:

Proposition 0.0.36 *Soit Σ une hypersurface de type contact dans un fibré en droites négatif au-dessus d'une variété symplectique fermée $\mathcal{L} \to B$ tel que l'intersection de Σ avec chaque fibre est un cercle. Alors Σ possède au moins $\sum_{i=0}^{2n} \beta_i$ orbites de Reeb périodiques et géométriquement distinctes (les β_i étant les nombres de Betti de B).*

Theorem 0.0.37 *Soit Σ une hypersurface de type contact dans un fibré en droites négatif $\mathcal{L}$ au-dessus d'une variété symplectique B. Supposons qu'il existe un domaine de Liouville W' (tel que sa première classe de Chern s'annule sur tous les tores) dont le bord coincide avec le fibré en cercles de rayon R_1 dans $\mathcal{L}$, denoté $S_{R_1^2}$. Supposons qu'il existe une fonction de Morse $f : B \to \mathbb{R}$ telle que tous les points critiques de f ont un indice de même parité. Soit $\xi = \ker \alpha$ la structure de contact sur Σ induite par $r^2\theta^\nabla$ sur $\mathcal{L}$ (θ^∇ étant la forme de transgression sur $\mathcal{L}$ et r est la coordonnée radiale dans la fibre). Supposons que Σ est "pincée" entre deux fibrés en cercles $S_{R_1^2}$ et $S_{R_2^2}$ de rayon R_1 et R_2 tels que $\frac{R_2}{R_1} < \sqrt{2}$. Supposons que la période minimale de toute orbite de Reeb périodique sur Σ est bornée inférieurement par R_1^2. Alors Σ possède au moins $\sum_{i=0}^{2n} \beta_i$ orbites de Reeb périodiques et géométriquement distinctes, où les β_i sont les nombres de Betti de B.*

[14] Nous référons à [MS98] pour une définition détaillée d'éclatement.

Dans ce Théorème, l'hypothèse de l'existence d'une fonction de Morse dont tous les points critiques ont un indice de même parité est de nature technique. Son but est de nous amener dans les hypothèse du Théorème 0.0.31, qui est notre outil pour calculer l'homologie symplectique S^1-équivariante positive. La borne inférieure sur la période de toute orbite de Reeb périodique est semi-technique; c'est le seul moyen actuellement de distinguer les différentes images des orbites. L'hypothèse de "pincement" est plus conceptuelle, son implication majeure est que les "n premiers générateurs" de l'homologie symplectique S^1-équivariante positive sont des orbites simples.

Les techniques développées dans cette thèse devraient prouver, en étendant les homologies au cadre des variétés symplectiques compactes et monotones, et en introduisant l'anneau de Novikov comme anneau de coefficients, le résultat suivant:

Conjecture 0.0.38 *Soit Σ une hypersurface de type contact dans un fibré en droites négatif $\mathcal{L}$ au-dessus d'une variété symplectique fermée et monotone B. Le fibré est muni d'une structure hermitienne et d'une connexion. Supposons qu'il existe une fonction de Morse $f : B \to \mathbb{R}$ telle que tous les points critiques de f ont un indice de même parité. Soit $\xi = \ker \alpha$ la structure de contact sur Σ induite par $r^2\theta^\nabla$ sur $\mathcal{L}$ (θ^∇ étant la forme de transgression sur $\mathcal{L}$ et r est la coordonnée radiale dans la fibre). Supposons que Σ est "pincée" entre deux fibrés en cercles $S_{R_1^2}$ et $S_{R_2^2}$ de rayon R_1 et R_2 tels que $\frac{R_2}{R_1} < \sqrt{2}$. Supposons que la période minimale de toute orbite de Reeb périodique sur Σ est bornée inférieurement par R_1^2. Alors Σ possède au moins $\sum_{i=0}^{2n} \beta_i$ orbites de Reeb périodiques et géométriquement distinctes, où les β_i sont les nombres de Betti de B.*

Les deux derniers chapitres sont une contribution à une étude d'indices de type Conley-Zehnder. J'espère que cela pourra être utilisé pour l'étude des orbites périodiques dégénérées. L'indice de Conley-Zehnder généralisé est un entier associé à un chemin de matrices symplectiques. Notre étude s'appuie sur une description détaillée de matrices symplectiques. Le Chapitre 5, qui apparaîtra comme article dans Portugaliae Mathematica, donne de nouvelles formes normales des matrices symplectiques. Nous présentons ici la forme normale sur l'espace propre généralisé de valeur propre ± 1.

Theorem 0.0.39 *Tout endomorphisme symplectique A d'un espace vectoriel symplectique de dimension finie (V, Ω) est la somme directe de ses restrictions $A_{|V_{[\lambda]}}$ au sous-espace symplectique réel A-invariant $V_{[\lambda]}$ dont la complexification est la somme directe de ses espaces propres généralisés de valeur propre $\lambda, \frac{1}{\lambda}, \overline{\lambda}$ et $\frac{1}{\overline{\lambda}}$:*

$$V_{[\lambda]}^{\mathbb{C}} := E_\lambda \oplus E_{\frac{1}{\lambda}} \oplus E_{\overline{\lambda}} \oplus E_{\frac{1}{\overline{\lambda}}}.$$

Si $\lambda \in \{\pm 1\}$, il existe une base symplectique de $V_{[\lambda]}$ dans laquelle la matrice représentant la restriction de A à $V_{[\lambda]}$ est une somme directe symplectique de matrices de la forme:

$$\begin{pmatrix} J(\lambda, r_j)^{-1} & C(r_j, s_j, \lambda) \\ 0 & J(\lambda, r_j)^\tau \end{pmatrix}$$

où $C(r_j, s_j, \lambda) := J(\lambda, r_j)^{-1}\operatorname{diag}\big(0, \ldots, 0, s_j\big)$ *avec* $s_j \in \{0, 1, -1\}$ *et où* $J(\lambda, r_j)$ *est la matrice de Jordan élémentaire de dimension* r_j *associée à* λ. *Si* $s_j = 0$, *alors* r_j *est impair. La dimension de l'espace propre de valeur propre* λ *est donnée par* $2\operatorname{Card}\{j \mid s_j = 0\} + \operatorname{Card}\{j \mid s_j \neq 0\}$.
Le nombre de s_j *égaux à* $+1$ *(reps.* -1*) apparaissant dans des blocs de dimension* $2k$ *(i.e. avec* $r_j = k$*) est égal au nombre de valeurs propres positives (resp. négatives) de la* 2*-forme symétrique*

$$\hat{Q}^{\lambda}_{2k} : \operatorname{Ker}\big((A - \lambda\operatorname{Id})^{2k}\big) \times \operatorname{Ker}\Big((A - \lambda\operatorname{Id})^{2k}\Big) \to \mathbb{R}$$

$$(v, w) \mapsto \lambda\,\Omega\big((A - \lambda\operatorname{Id})^{k} v, (A - \lambda\operatorname{Id})^{k-1} w\big).$$

La décomposition est unique à permutation des blocs près et est déterminée par λ, *par la dimension* $\dim\big(\operatorname{Ker}(A - \lambda\operatorname{Id})^r\big)$ *pour tout* $r \geq 1$ *et par le rang et la signature de la* 2*-forme bilinéaire symétrique* $\hat{Q}^{\lambda}_{2k}$ *pour tout* $k \geq 1$.

Le dernier Chapitre, qui apparaîtra comme article dans les Annales de la faculté des Sciences de Toulouse, est dévolu à l'étude d'une version généralisée de l'indice de Conley-Zehnder définie par Robbin et Salamon dans [RS93]. Nous commençons par donner une nouvelle formule pour l'indice de Conley-Zehnder "classique".

Theorem 0.0.40 *Soit* $\psi : [0, 1] \to \operatorname{Sp}(\mathbb{R}^{2n}, \Omega_0)$ *un chemin continu de matrices symplectiques liant la matrice* Id *à une matrice n'admettant pas* 1 *comme valeur propre. Soit* $\widetilde{\psi} : [0, 2] \to \operatorname{Sp}(\mathbb{R}^{2n}, \Omega_0)$ *une extension telle que* $\widetilde{\psi}$ *coincide avec* ψ *sur l'intervalle* $[0, 1]$, *telle que* $\widetilde{\psi}(s)$ *n'admette pas* 1 *comme valeur propre pour tout* $s \geqslant 1$ *et telle que le chemin se termine soit en* $\widetilde{\psi}(2) = W^+ := -\operatorname{Id}$, *soit en* $\widetilde{\psi}(2) = W^- := \operatorname{diag}(2, -1, \ldots, -1, \frac{1}{2}, -1, \ldots, -1)$.
L'indice de Conley-Zehnder de ψ *est égal à l'entier donné par le degré de l'application* $\tilde{\rho}^2 \circ \widetilde{\psi} : [0, 2] \to S^1$:

$$\mu_{CZ}(\psi) = \deg(\tilde{\rho}^2 \circ \widetilde{\psi}) \tag{2}$$

pour TOUTE application continue $\tilde{\rho} : \operatorname{Sp}(\mathbb{R}^{2n}, \Omega_0) \to S^1$ *coïncidant avec le déterminant complexe* $\det_{\mathbb{C}}$ *sur* $U(n) = O\big(\mathbb{R}^{2n}\big) \cap \operatorname{Sp}\big(\mathbb{R}^{2n}, \Omega_0\big)$; *telle que* $\tilde{\rho}(W^-) \in \{\pm 1\}$ *et telle que* $\deg\left(\tilde{\rho}^2 \circ \psi_{2-}\right) = n - 1$ *pour* $\psi_{2-} : t \in [0, 1] \mapsto \exp t\pi J_0 \begin{pmatrix} 0 & 0 & -\frac{\log 2}{\pi} & 0 \\ 0 & \operatorname{Id}_{n-1} & 0 & 0 \\ -\frac{\log 2}{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{Id}_{n-1} \end{pmatrix}$. *En particulier, deux manières alternatives de calculer l'indice de Conley-Zehnder sont:*

- *En utilisant la décomposition polaire des matrices,* $\mu_{CZ}(\psi) = \deg({\det_{\mathbb{C}}}^2 \circ U \circ \widetilde{\psi})$ *où* $U : \operatorname{Sp}(\mathbb{R}^{2n}, \Omega_0) \to U(n) : A \mapsto AP^{-1}$ *avec* P *l'unique matrice symétrique déifie positive telle que* $P^2 = A^{\tau}A$.

- *En utilisant le déterminant normalisé de la partie* $\mathbb{C}$*-linéaire d'une matrice sympplectique,*

$\mu_{CZ}(\psi) = \deg(\hat{\rho}^2 \circ \widetilde{\psi})$ *où* $\hat{\rho} : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to S^1 : A \mapsto \hat{\rho}(A) = \frac{\det_{\mathbb{C}}\left(\frac{1}{2}(A - J_0AJ_0)\right)}{\left|\det_{\mathbb{C}}\left(\frac{1}{2}(A - J_0AJ_0)\right)\right|}$ *avec* $J_0 = \left(\begin{smallmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{smallmatrix}\right)$ *la structure complexe standard sur* $\mathbb{R}^{2n}$.

Nous donnons une caractérisation de l'indice de Conley-Zehnder généralisé défini par Robbin et Salamon.

Theorem 0.0.41 *L'indice de Robbin-Salamon pour un chemin continu de matrices symplectiques est caractérisé par les propriétés suivantes:*

- (Homotopie) *il est invariant par homologies à extrémités fixées;*
- (Caténation) *il est additif sous caténation de chemins;*
- (Zéro) *il s'annule sur tout chemin de matrices* $\psi : [a, b] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega)$ *tel que* $\dim \mathrm{Ker}\big(\psi(t) - \mathrm{Id}\big) = k$ *est constant sur* $[a, b]$*;*
- (Normalisation) *si* $S = S^\tau \in \mathbb{R}^{2n \times 2n}$ *est une matrice symétrique dont toutes les valeurs propres sont en valeur absolue* $< 2\pi$ *et si* $\psi(t) = \exp(J_0St)$ *pour* $t \in [0, 1]$*, alors* $\mu_{RS}(\psi) = \frac{1}{2} \mathrm{Sign}\, S$ *où* $\mathrm{Sign}\, S$ *est la signature de* S.

Nous donnons une nouvelle façon de calculer cet indice.

Theorem 0.0.42 *Soit* $\psi : [0, 1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ *un chemin de matrices symplectiques. Décomposons* $\psi(0) = \psi^\star(0) \oplus \psi^{(1)}(0)$ *et* $\psi(1) = \psi^\star(1) \oplus \psi^{(1)}(1)$ *où* $\psi^\star(\cdot)$ *n'admet pas* 1 *comme valeur propre et* $\psi^{(1)}(\cdot)$ *est la restriction de* $\psi(\cdot)$ *à son espace propre généralisé de valeur propre* 1*. Considérons une extension continue* $\Psi : [-1, 2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ *de* ψ *telle que*

- $\Psi(t) = \psi(t)$ *pour* $t \in [0, 1]$*;*
- $\Psi\big(-\frac{1}{2}\big) = \psi^\star(0) \oplus \left(\begin{smallmatrix} e^{-1}\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id} \end{smallmatrix}\right)$ *et* $\Psi(t) = \psi^\star(0) \oplus \phi_0(t)$ *où* $\phi_0(t)$ *n'a que des valeurs propres réelles pour* $t \in \big[-\frac{1}{2}, 0\big]$*;*
- $\Psi\big(\frac{3}{2}\big) = \psi^\star(1) \oplus \left(\begin{smallmatrix} e^{-1}\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id} \end{smallmatrix}\right)$ *et* $\Psi(t) = \psi^\star(1) \oplus \phi_1(t)$ *où* $\phi_1(t)$ *n'a que des valeurs propres réelles pour* $t \in \big[1, \frac{3}{2}\big]$*;*
- $\Psi(-1) = W^\pm$, $\Psi(2) = W^\pm$ *et* $\Psi(t)$ *n'admet pas* 1 *comme valeur propre pour* $t \in \big[-1, -\frac{1}{2}\big]$ *and for* $t \in \big[\frac{3}{2}, 2\big]$.

Alors, l'indice de Robbin-Salamon est donné par

$$\mu_{RS}(\psi) = \deg(\tilde{\rho}^2 \circ \Psi) + \frac{1}{2} \sum_{k \geq 1} \mathrm{Sign}\Big(\hat{Q}_k^{(\psi(0))}\Big) - \frac{1}{2} \sum_{k \geq 1} \mathrm{Sign}\Big(\hat{Q}_k^{(\psi(1))}\Big)$$

avec $\tilde{\rho}$ comme dans le Théorème 0.0.40, et avec

$$\begin{aligned}\hat{Q}_k^A \quad : \quad & \operatorname{Ker}\Big((A-\mathrm{Id})^{2k}\Big) \times \operatorname{Ker}\Big((A-\mathrm{Id})^{2k}\Big) \to \mathbb{R} \\ & (v,w) \mapsto \Omega\big((A-\mathrm{Id})^k v, (A-\mathrm{Id})^{k-1} w\big).\end{aligned}$$

L'avantage de cette nouvelle formule est de pouvoir calculer l'indice de tout chemin sans avoir à la perturber. L'inconvénient est que nous devons étendre le chemin initial.

1 Background on symplectic homology

1.1 Floer Homology

Let (W, ω) be a compact symplectic manifold. For simplicity of the presentation, we assume

Assumption 1.1.1 that W is symplectically aspherical, i.e the symplectic form vanishes on the second fundamental group

$$\langle \omega, \pi_2(W) \rangle = 0,$$

Assumption 1.1.2 and that the first Chern class of the manifold (i.e. the first Chern class of its tangent bundle, endowed with a compatible almost complex structure) vanishes on the second fundamental group

$$\langle c_1(W), \pi_2(W) \rangle = 0.$$

Assumption 1.1.1 will ensure that the action of a contractible loop is well-defined. To deal with other free homotopy classes of loops, one has to assume a stronger version of atoroidality. Assumption 1.1.2 implies that the Conley-Zehnder index of a 1-periodic orbit of a Hamiltonian is well-defined on $\mathbb{Z}$. One can get rid of this assumption by looking at the homology with coefficients in the Novikov ring. Both assumptions will ensure that there are no holomorphic spheres, which is a necessary requirement for the moduli spaces of Floer trajectories to be nice manifolds with boundaries.

Floer homology for W is a kind of Morse homology on the loop space of W, $\mathcal{L}(W)$. It has been developed in the late eighties [Flo89, FHS95]; a detailed account with proofs can be found in the book [AD10]. This homology is based on functionals defined on the space of contractible loops $\mathcal{L}_{contr}(W)$; this is the connected component of the loop space containing the constant loops. A functional $\mathcal{A}_H$ on $\mathcal{L}_{contr}(W)$ is associated to a time dependent Hamiltonian on W, $H : \mathbb{R} \times W \to \mathbb{R}$ such that $H(t+1, x) = H(t, x)$ for all $t \in \mathbb{R}$. Since the Hamiltonian is periodic in the $\mathbb{R}$ variable, we will see H as a function $S^1 \times W \to \mathbb{R}$ and denote by θ the variable in $S^1 \cong \mathbb{R}/_{\mathbb{Z}}$.

Definition 1.1.3 The *Hamiltonian action functional* $\mathcal{A}_H : \mathcal{L}_{contr}(W) \to \mathbb{R}$ is defined by

$$\mathcal{A}_H(\gamma) := -\int_D u^\star\omega - \int_0^1 H\big(\theta, \gamma(\theta)\big)d\theta$$

where D denotes the disk, $D = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$, and where u is an extension of γ to the disk : $u : D \to W$ with $u(e^{2\pi i\theta}) = \gamma(\theta)$.

This functional is well-defined (independent of the choice of u) thanks to assumption 1.1.1.

Lemma 1.1.4 *A loop γ is a critical point of $\mathcal{A}_H$ if and only if it is induced by a 1-periodic solution $\underline{\gamma} : \mathbb{R} \to W$ of the Hamiltonian system*

$$\dot{\underline{\gamma}}(t) = X_H^t\big(\underline{\gamma}(t)\big)$$

where the vector field X_H^t is the Hamiltonian vector field corresponding to the function $H(t, \cdot)$, i.e. $\iota(X_H^t)\omega = dH(t, \cdot)$.

Such a 1-periodic solution will be called *a 1-periodic orbit of* X_H and we shall denote by $\mathcal{P}(H)$ the set of contractible 1-periodic orbits of X_H. If γ is a 1-periodic orbit of X_H with $\gamma(0) = x$ and if φ^{X_H} denotes the flow of the time dependent vector field X_H, then x is a fixed point of the flow after time 1, $\varphi_1^{X_H}(x) = x$, and the differential of $\varphi_1^{X_H}$ at x yields an endomorphism of T_xW which preserves ω_x; it is called the Poincaré return map.

Definition 1.1.5 A 1-periodic orbit γ of X_H is *non degenerate* if 1 is not an eigenvalue of the Poincaré return map.

We look only at Hamiltonians whose 1-periodic orbits are all non degenerate; this implies that the 1-periodic orbits are isolated. Such a Hamiltonian is called *non degenerate*. We associate to each 1-periodic orbit its Conley-Zehnder index which is an integer, defined as follows. We choose as above an extension u of the orbit γ to the disk, $u : D \to W$; we choose a symplectic trivialization of the bundle on the disk defined by the pullback by u of the tangent bundle to W; since the differential of the flow of the Hamiltonian vector field $\varphi_t^{X_H}$ at $x = \gamma(0)$ yields a symplectic endomorphism from $T_{\gamma(0)}W$ to $T_{\gamma(t)}W$, it is represented in the trivialization by a symplectic matrix. We associate in this way a path $\psi : [0, 1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ of symplectic matrices, starting from the identity and such that 1 is not an eigenvalue of $\psi(1)$. The Conley-Zehnder index of the orbit γ is defined to be the Conley-Zehnder index of the path ψ as defined in section 6.2.

This index does not depend on the chosen trivialisation of u^*TW, and is invariant under a continuous deformation of the extension u. If two extensions u and u' differ, up to homotopy, by an element S in $\pi_2(W)$, the difference between the Conley-Zehnder indices of γ is equal to twice the evaluation of the first Chern class of W on this element S (see, for instance, [MS98]). The assumption 1.1.2 ensures that the Conley-Zehnder index does not depend on the choice of the extension u.

The Floer complex is the $\mathbb{Z}$-vector space generated by the contractible 1-periodic orbits of X_H, graded by minus their Conley-Zehnder index.

In Morse homology, to define the boundary operator, one has to count negative gradient trajectories between critical points. In our setting, to define negative gradient trajectories of $\mathcal{A}_H$, one needs a metric on the loop space. One chooses a smooth loop $J : S^1 \to \mathrm{End}(TW) : \theta \mapsto J^\theta$ of almost complex structures on W which are compatible with ω, i.e $\omega(J^\theta X, J^\theta Y) = \omega(X,Y)$ for all $X, Y \in TW$ and $\omega(X, J^\theta X) > 0$ for all $0 \neq X \in TW$. The resulting inner product on the tangent space to the loop space at the loop γ, $T_\gamma\big(\mathcal{L}(W)\big) = \Gamma^\infty(S^1, \gamma^* TW)$, is defined by

$$\langle \xi, \eta\rangle := \int_{S^1} \omega_{\gamma(\theta)}\big(\xi(\theta), J^\theta(\gamma(\theta))\eta(\theta)\big) d\theta \qquad .$$

Lemma 1.1.6 *Negative gradient trajectories of $\mathcal{A}_H$ correspond to maps $u : \mathbb{R} \times S^1 \to W$ satisfying perturbed nonlinear Cauchy-Riemann equations called* Floer equations*:*

$$\frac{\partial u}{\partial s}(s,\theta) + J^\theta\big(u(s,\theta)\big)\left(\frac{\partial u}{\partial \theta}(s,\theta) - X_H^\theta\big(u(s,\theta)\big)\right) = 0 \tag{1.1}$$

with $\iota(X_H^\theta)\omega = dH(\theta, \cdot)$.

Such maps are called *Floer trajectories*. As in Morse theory, we want to "count" the number of negative gradient trajectories between some pairs of critical points.
A first important issue is to know whether a Floer trajectory converges to 1-periodic orbits. We define the energy of a Floer trajectory as

$$E(u) := \tfrac{1}{2}\int_{S^1}\int_{\mathbb{R}} \Big(\big|\tfrac{\partial u}{\partial s}\big|^2 + \Big|\tfrac{\partial u}{\partial \theta} - X_H^\theta \circ u\Big|^2 \Big) ds d\theta = \int_{S^1}\int_{\mathbb{R}} \big|\tfrac{\partial u}{\partial s}\big|^2 \, ds d\theta.$$

Proposition 1.1.7 ([Flo89], see also [AD10] theorem 6.5.6) *A Floer trajectory with finite energy converges at $\pm\infty$ to 1-periodic orbits of X_H, assuming that all the 1-periodic orbits of X_H are non degenerate.*

Let us consider the space $\mathcal{M}$ of contractible smooth Floer trajectories with finite energy. By contractible, we mean that $u(s,\cdot) : S^1 \to W$ is contractible for one, and hence all, $s \in \mathbb{R}$. This contractibility assumption is considered only when studying contractible 1-periodic orbits.

$$\mathcal{M} := \{u : \mathbb{R} \times S^1 \to W \,|\, u \text{ is a contractible solution of (1.1)}, C^\infty \text{ and } E(u) < \infty\}.$$

Theorem 1.1.8 ([Flo89], see also [AD10], theorem 6.5.4) *$\mathcal{M}$ is compact in $C^\infty_{loc}(\mathbb{R}\times S^1, W)$, where C^∞_{loc} is the space of smooth maps endowed with the topology of uniform convergence on compact subsets.*

Let γ^-, γ^+ be two 1-periodic orbits of X_H and let $\mathcal{M}(\gamma^-, \gamma^+, H, J)$ denote the space of $u \in \mathcal{M}$ such that

$$\lim_{s\to-\infty} u(s,\cdot) = \gamma^- \quad \text{and} \quad \lim_{s\to+\infty} u(s,\cdot) = \gamma^+.$$

A naive but crucial remark is that one can "count" the points of a 0-dimensional compact set. One shows that the space of Floer trajectories between two 1-periodic orbits of the Hamiltonian vector field X_H is a manifold, provided one has perturbed a little -if needed- the Hamiltonian. The way to prove that is to describe $\mathcal{M}(\gamma^-, \gamma^+, H, J)$ as the zero set of a smooth Fredholm map between two Banach manifolds. The perturbation of H is introduced so that the differential of the map is surjective at all solutions. Indeed recall :

Proposition 1.1.9 *Let $\mathcal{E}$ (resp. $\mathcal{F}$) be a connected Banach manifold, locally modelled on the Banach space E (resp. F). Let $\mathcal{D}: \mathcal{E} \to \mathcal{F}$ be a smooth map which is Fredholm, i.e. such that the differential at each point $x \in \mathcal{E}$, $T_x\mathcal{D} : E \to F$ is a Fredholm operator. Let y be an element in $\mathcal{F}$. If, for any $u \in \mathcal{D}^{-1}(y)$, the differential -also called linearization- $T_u\mathcal{D} : E \to F$ is surjective, then $\mathcal{D}^{-1}(y)$ is a submanifold; its dimension is the index of $T_u\mathcal{D}$ (which is independent of u) and its tangent space at u is the kernel of $T_u\mathcal{D}$.*

For $\mathcal{D}^{-1}(y)$ to be a manifold, it is enough, (since the set of Fredholm operators is open in the set of bounded linear maps), to show that $T_u\mathcal{D}$ is Fredholm and surjective at any point $u \in \mathcal{D}^{-1}(y)$, and to prove that the index of $T_u\mathcal{D}$ is constant on $\mathcal{D}^{-1}(y)$.

To do this in the Floer homology context, one considers the Floer operator $\bar\partial$ defined, given a Hamiltonian H and a loop of almost complex structures J, by

$$\begin{aligned} \bar\partial : C^\infty_{loc}(\mathbb{R}\times S^1, W) &\to C^\infty_{loc}(\mathbb{R}\times S^1, W) \\ u &\mapsto \frac{\partial u}{\partial s} + J^\theta \circ u\Big(\frac{\partial u}{\partial \theta} - X^\theta_H \circ u\Big) \end{aligned}$$

and one extends it to suitable Banach spaces.

Theorem 1.1.10 ([Flo89]; see also [AD10] theorem 8.1.5) *The linearization D_u of the Floer operator at the point u is a Fredholm operator whose index is equal to the difference of the Conley-Zehnder indices of the limiting 1-periodic orbits*

$$\mathrm{index}(D_u) = \mu_{CZ}(\gamma^+) - \mu_{CZ}(\gamma^-).$$

To apply Proposition 1.1.9, one shows the existence of a pair (H, J) so that the corresponding linearisation D_u is surjective for all $u \in \mathcal{M}(\gamma^-, \gamma^+, H, J)$ and for all γ^-, γ^+ 1-periodic orbits of X_H. One first shows that the set of all Floer trajectories $\mathcal{X}(\gamma^-, \gamma^+, J)$ defined for a fixed J and for a Hamiltonian varying in a class $H_0 + h$ for H_0 fixed, non degenerate, and h in a Banach space $\mathcal{B}$ (so that the 1-periodic orbits remain unchanged -and non degenerate) is a Banach manifold. For this, one considers $\mathcal{X}(\gamma^-, \gamma^+, J)$ as the set of zeroes of the section σ defined by Floer equations

$$\sigma(u,h) = \frac{\partial u}{\partial s} + J^\theta \circ u\Big(\frac{\partial u}{\partial \theta} - X^\theta_{H_0+h} \circ u\Big)$$

and one uses the following proposition.

Proposition 1.1.11 *Let $\mathcal{Z} \xrightarrow{\pi} \mathcal{P}$ be a Banach vector bundle over a Banach manifold and let $\sigma : \mathcal{P} \to \mathcal{Z}$ be a smooth section. Then the intersection of σ and the 0-section is a Banach manifold, whenever the section σ is transversal to the 0-section at every point of the intersection.*

One then considers the natural projection $\pi(\sigma(u,h)) = h$ from this Banach manifold $\mathcal{X}(\gamma^-, \gamma^+, J)$ on the Banach manifold $\mathcal{B}$ and one uses Sard-Smale theorem.

Proposition 1.1.12 (Sard-Smale) *Let $\pi : \mathcal{X} \to \mathcal{B}$ be a smooth map between Banach manifolds, whose differential is a Fredholm operator. Then the set of regular values of π is of second Baire category, i.e. is the intersection of a numerable set of open dense subsets.*

Let us observe that the regular values of π are exactly the elements $h \in \mathcal{B}$ such that, for any $u \in \mathcal{M}(\gamma^-, \gamma^+, H, J)$, the linearized map D_u is surjective. Let us denote by $(\mathcal{H}\mathcal{J})_{reg}$ the space of all pairs (H, J) such that the 1-periodic orbits of X_H are all nondegenerate and the operator D_u is surjective for all $u \in \mathcal{M}(\gamma^-, \gamma^+, H, J)$ and all 1-periodic orbits γ^-, γ^+. One gets

Theorem 1.1.13 ([FHS95]) *Let (W, ω) be a compact manifold. Let $J \in \mathcal{J}(W, \omega)$ and $H_0 : S^1 \times W \to \mathbb{R}$ be a Hamiltonian such that the 1-periodic orbits of X_{H_0} are all non-degenerate. Denote by $C^\infty(H_0)$ the set of Hamiltonians that coincide with H_0 up to the second order on the 1-periodic orbits of X_{H_0}. Then the set*

$$\mathcal{H}_{reg} = \{H \in C^\infty(H_0) \,|\, (H, J) \in (\mathcal{H}\mathcal{J})_{reg}\}$$

is dense (of second Baire category) in $C^\infty(H_0)$.

This implies :

Theorem 1.1.14 ([Flo89, FHS95]; see also [AD10] Theorem 8.1.2) *For a generic choice of the Hamiltonian H (by this we mean a H in $\mathcal{H}_{reg}$) and every γ^-, γ^+, 1-periodic orbits of X_H, each $\mathcal{M}(\gamma^-, \gamma^+, H, J)$ is a smooth compact manifold of dimension $\mu_{CZ}(\gamma^+) - \mu_{CZ}(\gamma^-)$.*

We assume in what follows that the Hamiltonian H is generic in the above sense. There is an $\mathbb{R}$-action on $\mathcal{M}(\gamma^-,\gamma^+,H,J)$ defined by reparametrization of u in the s coordinate. Thus, if $\mu_{CZ}(\gamma^+)=\mu_{CZ}(\gamma^-)+1$, the quotient $\mathcal{M}(\gamma^-,\gamma^+,H,J)/_{\mathbb{R}}$ of $\mathcal{M}(\gamma^-,\gamma^+,H,J)$ under this $\mathbb{R}$-action is 0-dimensional. In order to get compact spaces, one studies the limit of a sequence of Floer trajectories.

Theorem 1.1.15 ([Flo89] or for instance [AD10] Theorem 9.1.6) *Let (u_n) be a sequence of elements in $\mathcal{M}(\gamma_1,\gamma_2,H,J)$. There exists*

1. *a subsequence of (u_n);*
2. *a finite number of critical points $x_0=\gamma_1,\ x_1,\dots,x_{l+1}=\gamma_2$ of $\mathcal{A}_H$;*
3. *sequences of real numbers (s_n^i) for $0\leq i\leq l$;*
4. *elements $u^i\in\mathcal{M}(x_i,x_{i+1},H,J)$ for $0\leq i\leq l$;*

such that, for $0\leq i\leq l$,

$$\lim_{n\to\infty} u_n\cdot s_n^i=u^i \qquad \text{in } C^\infty_{loc}$$

with $(u\cdot s)(s',t):=u(s+s',t)$.

This means that, up to the $\mathbb{R}$-action, the limit of a sequence of Floer trajectories can be a broken Floer trajectory. We define the closure of $\mathcal{M}(\gamma^-,\gamma^+,H,J)/_{\mathbb{R}}$ as the union of $\mathcal{M}(\gamma^-,\gamma^+,H,J)/_{\mathbb{R}}$ and all those broken trajectories (up to reparametrization) so that those closures are compact. In particular, when $\mu_{CZ}(\gamma^-)=\mu_{CZ}(\gamma^+)-1$ the space $\mathcal{M}(\gamma^-,\gamma^+,H,J)/_{\mathbb{R}}$ is equal to its closure and is thus compact. This means it consists of a finite number of points.

Each of these points comes with a sign induced by the choice of a system of coherent orientations; for simplicity, we postpone their description to section 1.1.1. Without those orientations, everything is well defined for a $\mathbb{Z}/_{2\mathbb{Z}}$ valued Floer homology.

Definition 1.1.16 The *Floer complex* is the $\mathbb{Z}$-vector space generated by the 1-periodic orbits of X_H, graded by minus their Conley-Zehnder index

$$FC(H,J):=\bigoplus_{\gamma\in\mathcal{P}(H)}\mathbb{Z}\langle\gamma\rangle.$$

The *Floer differential* $\partial: FC_*(H,J)\to FC_{*-1}(H,J)$ is defined by

$$\partial(\gamma^-):=\sum_{\substack{\gamma^+\in\mathcal{P}(H)\\ -\mu_{CZ}(\gamma^-)=-\mu_{CZ}(\gamma^+)+1}}\#\mathcal{M}(\gamma^-,\gamma^+,H,J)/_{\mathbb{R}}\ \gamma^+$$

where # is a count of points with signs. (Those signs are defined in section 1.1.1).

The fact that $\partial^2 = 0$ follows from the study of the boundary of $\mathcal{M}(\gamma^-, \gamma^+, H, J)/_{\mathbb{R}}$ when the difference of the Conley-Zehnder indices is 2. When $-\mu_{CZ}(\gamma^-) = -\mu_{CZ}(\gamma^+) + 2$, the space $\mathcal{M}(\gamma^-, \gamma^+, H, J)/_{\mathbb{R}}$ is of dimension 1. Its boundary $\partial\mathcal{M}/_{\mathbb{R}}$ consists of an even number of points and is given by

$$\bigcup_{\{x \mid -\mu_{CZ}(x) = -\mu_{CZ}(\gamma^+)+1\}} \mathcal{M}(\gamma^-, x, H, J)/_{\mathbb{R}} \times \mathcal{M}(x, \gamma^+, H, J)/_{\mathbb{R}}.$$

The coefficient of γ^+ in $\partial^2(\gamma^-)$ is given by

$$\sum_{\{x \mid -\mu_{CZ}(x) = -\mu_{CZ}(\gamma^+)+1\}} \#\mathcal{M}(\gamma^-, x, H, J)/_{\mathbb{R}} \ \#\mathcal{M}(x, \gamma^+, H, J)/_{\mathbb{R}} = \#\partial\mathcal{M}/_{\mathbb{R}} = 0.$$

(see the end of section 1.1.1).

Definition 1.1.17 *Floer homology* is defined as the homology of the Floer complex with the Floer differential, for a pair $(H, J) \in (\mathcal{HJ})_{reg}$; it is denoted $FH_*(H, J)$:

$$FH_*(H, J) := H_*\big(FC_*(H, J), \partial\big).$$

Theorem 1.1.18 ([Flo89], see also [AD10] chapter 11) *Floer homology does not depend on the choice of the regular pair (H, J). It is isomorphic to the singular homology of W.*

Corollary 1.1.19 ([Flo89]) *The number of 1-periodic orbits of a non degenerate time dependent periodic Hamiltonian on a compact symplectic manifold W is bounded below by the sum of the Betti numbers of W.*

Remark 1.1.20 One can replace assumptions 1.1.1 and 1.1.2 by a monotonicity condition namely there exist $k \geq 0$ such that $[\omega] = k[c_1]$. Then the action functional is defined on a cover of the loop space and the Conley-Zehnder index depends on the choice of the trivialisation disk. The way to deal with this is to look at Floer homology with coefficients in the Novikov ring. We shall not use this generalisation here.

1.1.1 Parenthesis : A glimpse on signs

We indicate in this section how to define the signs attached to Floer trajectories, and mention the steps which prove that $\#\partial\mathcal{M}/_{\mathbb{R}} = 0$ (and hence, by the above, that $\partial^2 = 0$).

Operator gluing lemma

We look at operators

$$D : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n}),$$

with $p > 2$, which are of the form

$$D = \partial_s + J\partial_t + S(s,t)$$

with $S(s,\cdot) \to S_\pm(\cdot)$ for $s \to \pm\infty$, where $S_\pm$ belongs to the following space $\mathcal{S}$ of loops of symmetric matrices. Given a loop S of symmetric matrices, one considers the corresponding path ψ of symplectic matrices defined by the differential equations $\dot{\psi} = JS\psi$. The loop S belongs to the space $\mathcal{S}$ if and only if 1 is not an eigenvalue of $\psi(1)$, i.e. $\det\big(\psi(1) - \mathrm{Id}\big) \neq 0$. We denote by $\mathcal{O}(\mathbb{R} \times S^1; S_-, S_+)$ the space of such operators.

For $R >> 0$ big enough and $S_\pm, S \in \mathcal{S}$, one defines a *gluing operation*

$$\mathcal{O}(\mathbb{R} \times S^1; S_-, S) \times \mathcal{O}(\mathbb{R} \times S^1; S, S_+) \to \mathcal{O}(\mathbb{R} \times S^1; S_-, S_+) : (D_1, D_2) \mapsto D_1\#_R D_2$$

in the following way. Fix a smooth function $\beta : \mathbb{R} \to [0,1]$ such that $\beta(s) = 0$ for $s \leq 0$, $\beta(s) = 1$ for $s \geq 1$.Define,

$$D_i^R := \partial_s + J\partial_t + S(t) + \beta(-s+R)\big(S_i(s,t) - S(t)\big) \quad \text{for } i = 1,2$$

The glued operator, $D_1\#_R D_2$ is defined by

$$D_1\#_R D_2 := \begin{cases} D_1^R(s+R) & \text{if } s \leq 0 \\ D_2^R(s-R) & \text{if } s \geq 0 \end{cases}.$$

Theorem 1.1.21 (Operator gluing lemma) *Assume D_1 and D_2 are surjective. Then $D_1\#_R D_2$ is surjective for $R >> 0$ and has uniformly bounded right inverse.*

Proof: Choose Q_1, Q_2, right inverse for D_1, D_2. We first construct an approximate right inverse T_R for $D_R = D_1\#_R D_2$ i.e an operator

$$T_R : L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

such that

$$\|D_R T_R - \mathrm{Id}\| < \tfrac{1}{2} \quad \text{and} \quad T_R \text{ is uniformly bounded.}$$

Then $Q_R := T_R(D_R T_R)^{-1}$ is a genuine right inverse for D_R and is uniformly bounded.

We construct T_R according to the following diagram:

$$\begin{tikzcd}
L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})\times L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n}) \arrow[r, "Q_1\times Q_2"] & W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})\times W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n}) \arrow[d, "G_R"] \\
L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n}) \arrow[u, "S_R"] \arrow[r, dotted, "T_R"] & W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})
\end{tikzcd}$$

We define $T_R := G_R \circ Q_1 \times Q_2 \circ S_R$ where G_R is a "gluing map" and S_R is a "splitting map". The splitting map is defined as

$$S_R(\zeta) = (\zeta_1, \zeta_2) \quad \text{with} \begin{cases} \zeta_1(s,\cdot) = \big(1-\beta(s-R)\big)\zeta(s-R,\cdot) \\ \zeta_2(s,\cdot) = \beta(s+R)\zeta(s+R,\cdot) \end{cases}.$$

Given $L > 0$, we define $\beta_L(s) := \beta\big(\frac{s}{L}\big)$ and we assume that β is such that $\beta_L(s) = 0$ for $s \in [0,1]$ if L is big. We define the gluing map to be

$$G_R(\xi_1,\xi_2) = \Big(1-\beta_{\frac{R}{2}}(s)\Big)\xi_1(s+R) + \Big(1-\beta_{\frac{R}{2}}(-s)\Big)\xi_2(s-R).$$

Note that since G_R and S_R are uniformly bounded, so is T_R.
To conclude the theorem, one can show that

$$\|D_R T_R - \mathrm{Id}\| \to 0 \text{ as } R \to \infty.$$

Coherent orientations

Let $D : X \to Y$ be a Fredholm operator. Its determinant is the 1-dimensional vector space

$$\det(D) := \Lambda^{\max}\ker(D) \otimes \Lambda^{\max}\big(\operatorname{coker}(D)\big)^{\vee}.$$

An orientation of D is an orientation of this vector space.
One considers the real line bundle $\det \to \mathcal{F}(X,Y)$ over the space of Fredholm operators, whose fiber above D is $\det(D)$.

Lemma 1.1.22 *Assume $D_1 \in \mathcal{O}(\mathbb{R}\times S^1; S_-, S)$ and $D_2 \in \mathcal{O}(\mathbb{R}\times S^1; S, S_+)$ are surjective. For R big enough, if $D_R := D_1 \#_R D_2$ and Q_R is its right inverse, there is a canonical isomorphism*

$$\phi_R : \ker(D_1) \oplus \ker(D_2) \to \ker(D_R)$$

defined by

$$\phi_R := (\mathrm{Id} - Q_R D_R) \circ G_R$$

where G_R is the gluing map defined above.

A consequence of this Lemma is that there is a canonical isomorphism

$$\det(D_1) \otimes \det(D_2) \to \det(D_R) \quad \text{for } R \gg 0. \tag{1.2}$$

Definition 1.1.23 A *system of coherent orientations on the space of operators*

$$\{\mathcal{O}(\mathbb{R} \times S^1; S_-, S_+) | S_-, S_+ \text{ as above } \}$$

is an orientation of the determinant line bundle over each $\mathcal{O}(\mathbb{R} \times S^1; S_-, S_+)$, which is compatible with the gluing operation via the canonical isomorphism (1.2):

$$\det(D_1 \#_R D_2) \simeq \det(D_1) \otimes \det(D_2).$$

This can be done because $\mathcal{O}(\mathbb{R} \times S^1; S_-, S_+)$ is contractible.

Theorem 1.1.24 ([FH94, BM04, BO09b]) *There exists a system of coherent orientations.*

Definition 1.1.25 Assume a system of coherent orientations is given. We shall define the sign attached in Floer coboundary operator to a Floer trajectory between two 1- periodic orbits with a difference of Conley-Zehnder index equal to 1. The space of trajectories is of dimension 1, its quotient by the action of $\mathbb{R}$, $\mathcal{M}(\gamma^-, \gamma^+)$, is of dimension 0. Given $[u] \in \mathcal{M}(\gamma^-, \gamma^+)$, the dimension of $\ker(D_u)$ is equal to 1; this $\ker(D_u)$ is spanned by $\langle \partial_s u \rangle$. The *sign associated to* $[u]$, $\epsilon\big([u]\big)$, is given by

$$\begin{cases} +1 & \text{if orientation of } \ker(D_u) \text{ given by } \langle \partial_s u \rangle \text{ coincides with the coherent orientation} \\ -1 & \text{if it is not the case.} \end{cases}$$

Proposition 1.1.26 *Consider two broken Floer trajectories* $([u], [v])$ *and* $([u'], [v'])$ *which are the two ends of a* 1*-dimensional moduli space* $\mathcal{M}(x, y)$*. Then*

$$\epsilon\big([u]\big) \cdot \epsilon\big([v]\big) + \epsilon\big([u']\big) \cdot \epsilon\big([v']\big) = 0.$$

This shows that $\#\partial\mathcal{M}(x, y) = 0$.

1.2 Symplectic Homology

Symplectic homology is defined for a compact symplectic manifold W with boundary of contact type. It is defined as a direct limit of Floer homologies of the symplectic completion of W, using some special Hamiltonians. This homology was developed by Viterbo in [Vit99], using works of Cieliebak, Floer, Hofer [FH94, CFH95]. The class of admissible Hamiltonians was extended by Oancea in his PhD thesis [Oan08]. The case of autonomous Hamiltonians is treated by Cieliebak-Floer-Hofer-Wysocki, Hermann, Bourgeois-Oancea in [CFHW96, Her98, BO09b].

1.2.1 Setup

Let (W, ω) be a compact *symplectic manifold with contact type boundary* $M := \partial W$. This means that there exists a Liouville vector field X (i.e. a vector field X such that $\mathcal{L}_X \omega = \omega$) defined on a neighbourhood of the boundary M, and transverse to M. In the sequel, we shall assume that the Liouville vector field has been chosen and we shall denote by (W, ω, X) such a manifold. We denote by λ the 1-form defined in a neighbourhood of M by

$$\lambda := \iota(X)\omega$$

and by α the *contact* 1-*form* on M which is the restriction of λ to M:

$$\alpha := \big(\iota(X)\omega\big)_{|_M}.$$

We denote by ξ the *contact structure* defined by α, i.e $\xi := \ker \alpha$. The *Reeb vector field* R_α is the vector field on M defined by :

$$\begin{cases} \iota(R_\alpha) d\alpha = 0 \\ \alpha(R_\alpha) = 1. \end{cases}$$

The *action spectrum* of (M, α) is the set of all periods of the Reeb vector field

$$\mathrm{Spec}(M, \alpha) := \{T \in \mathbb{R}^+ \mid \exists \gamma \text{ periodic orbit of } R_\alpha \text{ of period } T\}.$$

The *symplectic completion* of (W, ω, λ) is the symplectic manifold defined by

$$\widehat{W} := W \bigcup_G (M \times \mathbb{R}^+) := \big(W \sqcup (M \times [-\delta, +\infty])\big)/\sim_G$$

with the symplectic form

$$\widehat{\omega} := \begin{cases} \omega & \text{on } W \\ d(e^\rho \alpha) & \text{on } M \times [-\delta, +\infty] \end{cases}.$$

The equivalence $\sim_G$, between a neighbourhood U of M in W and $M \times [-\delta, 0]$, is defined by the diffeomorphism

$$G : M \times [-\delta, 0] \to U : \quad (p, \rho) \mapsto \varphi^X_\rho(p)$$

where φ^X is the flow of the Liouville vector field X. This is always possible since

$$G^\star \omega = e^\rho(d\alpha + d\rho \wedge \alpha) = d(e^\rho \alpha).$$

Observe indeed that $G_{\star(y,\rho)}\left(\frac{\partial}{\partial\rho}\right) = X_{G(y,\rho)}$, $\quad G_{\star(y,\rho)}(Y_y) = \left(\varphi_\rho^X\right)_{\star y} Y_y$. Since $\lambda(X) = (\iota(X)\omega)(X) = 0$, we also have $\mathcal{L}_X\lambda = \lambda$ and $\left(\varphi_\rho^X\right)^\star \lambda = e^\rho\lambda$. Hence, $\forall Y_y, Z_y \in T_yM$:

$$
\begin{aligned}
(G^\star\omega)_{(y,\rho)}\left(\frac{\partial}{\partial\rho}, Y_y\right) &= \omega_{\varphi_\rho^X y}\left(X_{G(y,\rho)}, \left(\varphi_\rho^X\right)_{\star y} Y_y\right)\\
&= \left(\left(\varphi_\rho^X\right)^\star\omega\right)_y (X_y, Y_y) = e^\rho d\lambda_y(X_y, Y_y) = e^\rho\left(\mathcal{L}_X\lambda\right)_y (Y_y)\\
&= e^\rho(d\alpha + d\rho\wedge\alpha)\left(\frac{\partial}{\partial\rho}, Y_y\right)\\
(G^\star\omega)_{(y,\rho)}(Y_y, Z_y) &= \omega_{\varphi_\rho^X y}\left(\left(\varphi_\rho^X\right)_{\star y} Y_y, \left(\varphi_\rho^X\right)_{\star y} Y_z\right)\\
&= \left(\left(\varphi_\rho^X\right)^\star\omega\right)_y (Y_y, Z_y) = e^\rho d\lambda_y(X_y, Y_y) = e^\rho d\alpha_y(X_y, Y_y)\\
&= e^\rho(d\alpha + d\rho\wedge\alpha)(Y_y, Z_y)\,.
\end{aligned}
$$

We still assume throughout that ω is symplectically aspherical and that the first Chern class vanishes on the second fundamental group.

1.2.2 Symplectic homology

Given a time-dependent Hamiltonian $H : S^1 \times \widehat{W} \to \mathbb{R}$, we define for each $\theta \in S^1$ the Hamiltonian vector field X_H^θ by

$$\widehat{\omega}(X_H^\theta, .) = dH(\theta, \cdot), \qquad \theta \in S^1.$$

Definition 1.2.1 The class $\mathcal{H}_{\mathrm{std}}$ of *admissible Hamiltonians* consists of smooth functions $H : S^1 \times \widehat{W} \to \mathbb{R}$ satisfying the following conditions:

1. H is negative and C^2-small on $S^1 \times W$;
2. there exists $\rho_0 \geq 0$ such that $H(\theta, p, \rho) = \beta e^\rho + \beta'$ for $\rho \geq \rho_0$, with $0 < \beta \notin \mathrm{Spec}(M, \alpha)$ and $\beta' \in \mathbb{R}$;
3. $H(\theta, p, \rho)$ is C^2-close to $h(e^\rho)$ on $S^1 \times M \times [0, \rho_0]$, for h a convex increasing function.

We say furthermore that it is non degenerate if all 1-periodic orbits of X_H are nondegenerate, i.e the Poincaré return map has no eigenvalue equal to 1.

The complex $SC(H, J)$ considered is the Floer complex, i.e. the complex generated by 1-periodic orbits of the Hamiltonian vector field X_H with boundary δ defined as before through Floer trajectories.

Remark 1.2.2 Condition 1 implies that the only 1-periodic orbits of X_H in W are constants; they correspond to critical points of H. On $S^1 \times M \times [0, +\infty[$, if a Hamiltonian is the pullback of a function on $[0, +\infty[$, $H_1(\theta, p, \rho) = h_1(e^\rho)$, then the corresponding Hamiltonian vector field is proportional to the Reeb vector field, $X^\theta_{H_1}(p, \rho) = -h_1'(e^\rho)R_\alpha(p)$. Hence, for such a Hamitonian H_1, with h_1 increasing, the image of a 1-periodic orbit of X_{H_1} is the image of a periodic orbit of the Reeb vector field $-R_\alpha$ of period $T := h_1'(e^\rho)$ located at level $M \times \{\rho\}$. In particular, condition 2 implies that there is no 1-periodic orbit of X_H in $M \times [\rho_0, +\infty[$ for a Hamiltonian H in $\mathcal{H}_{\rm std}$; indeed; $h(e^\rho) = \beta e^\rho + \beta'$ so $h'(e^\rho) = \beta$ which is assumed to be different from the period of any closed orbit of the Reeb vector field. Condition 3 ensures that for any non constant 1-periodic orbit γ_H of X_H for a Hamiltonian H in $\mathcal{H}_{\rm std}$, there exists a closed orbit of the Reeb vector field R_α of period $T < \beta$ (with β the slope of H "at ∞"), such that γ_H is close to this closed orbit of (minus) the Reeb vector field located in $M \times \{\rho\}$ with $T = h'(e^\rho)$.

Remark 1.2.3 We can consider a larger class of admissible Hamiltonians, removing conditions 1 and 3. It will not change the definition of the symplectic homology. However condition 1 allows to identify 1-periodic orbits of small action with critical points of H in W. It will be important in the definition of positive symplectic homology.

We denote again by $\mathcal{P}(H)$ the set of 1-periodic orbits of X_H .

Definition 1.2.4 The *class $\mathcal{J}$ of admissible* $J : S^1 \to \mathrm{End}(T\widehat{W}) : \theta \mapsto J^\theta$ consists of smooth loops of compatible almost complex structures J^θ on $\widehat{W}$, such that, at infinity (i.e. for ρ large enough) J is autonomous (i.e. independent of θ), invariant under translations in the ρ variable, and satisfies

$$J^\theta \xi = \xi \qquad J^\theta(\partial_\rho) = R_\alpha.$$

The space $\widehat{W}$ is not compact; it is proven in [FH94, CFH95, FHS95, Vit99, Oan08] that the Floer homology $SH(H, J) := HSC((H, J), \partial)$ is well-defined for a pair (H, J) in a set of so-called regular pairs $(\mathcal{HJ})_{reg}$ which is of second Baire category in the set of pairs of admissible non degenerate Hamitonians and admissible loops of almost complex structures. One can even fix a non degenerate admissible H and the set of H-regular J's, i.e. admissible J so that $(H, J) \in (\mathcal{HJ})_{reg}$, is of second Baire category in the set of admissible loops of almost complex structures. Furthermore, $SH(H, J)$ is independent of the choice of the H- regular J [Flo89]. It is however dependent of the choice of H. An important point in the proofs is that the Floer trajectories with finite energy are confined due to a maximum principle.

The symplectic homology is defined as a direct limit of $SH(H, J)$ over H non degenerate in $\mathcal{H}_{\rm std}$. To define the direct limit one needs a partial order $\leq$ on $\mathcal{H}_{\rm std}$ and morphisms $SH(H_1, J_1) \to SH(H_2, J_2)$ whenever $H_1 \leq H_2$ are non degenerate, and these morphisms should have nice composition rules.

- The partial order on $\mathcal{H}_{\text{std}}$ is given by $H_1 \leq H_2$ if $H_1(\theta, x) \leq H_2(\theta, x)$ for all $(\theta, x) \in S^1 \times \widehat{W}$ (for more general Hamiltonians, it is enough to have $H_1(\theta, x) \leq H_2(\theta, x)$ for all (θ, x) outside a compact domain).
- Let (H_1, J_1) and (H_2, J_2) be two regular pairs for $(\widehat{W}, \widehat{\omega})$ with $H_1 \leq H_2$, and consider a smooth increasing homotopy of regular pairs between them, (H_s, J_s) where $s \in \mathbb{R}$ and (H_s, J_s) is constant for $|s|$ large. By increasing, we mean $\partial_s H_s \geq 0$ (again for more general H it is enough to consider $\partial_s H_s \geq 0$ outside a compact subset to be able to define as below a continuation map). The morphism $SH(H_1, J_1) \to SH(H_2, J_2)$ is the continuation map induced by this increasing homotopy, when it is regular, as described below.

Consider the Floer equation

$$\frac{\partial u}{\partial s} + J_s^\theta \circ u \Big(\frac{\partial u}{\partial \theta} - X_{H_s}^\theta \circ u \Big) = 0 \tag{1.3}$$

defined on the set of maps $u : \mathbb{R} \times S^1 \to \widehat{W}$ of class C^∞_{loc}. If $u : \mathbb{R} \times S^1 \to \widehat{W}$ is a solution of (1.3), we define its energy to be

$$E(u) := \int_{-\infty}^{+\infty} \int_{S^1} \left\| \frac{\partial u}{\partial s} \right\|^2 d\theta ds. \tag{1.4}$$

The fact that the homotopy is increasing insures that Floer trajectories with finite energy are confined. Let $\gamma_1 \in \mathcal{P}(H_1)$ and $\gamma_2 \in \mathcal{P}(H_2)$, we denote by $\mathcal{M}(\gamma_1, \gamma_2, H_s, J_s)$ the space of solutions $u : \mathbb{R} \times S^1 \to \widehat{W}$ of (1.3) with finite energy, $E(u) < \infty$ such that $\lim_{s \to -\infty} u(s, \cdot) = \gamma_1$ and $\lim_{s \to +\infty} u(s, \cdot) = \gamma_2$. Remark that in this case there is no $\mathbb{R}$-action on the space $\mathcal{M}(\gamma_1, \gamma_2, H_s, J_s)$.

Theorem 1.2.5 ([Oan08, FHS95]) *For a generic choice of the homotopy (H_s, J_s), the spaces $\mathcal{M}(\gamma_1, \gamma_2, H_s, J_s)$, are manifolds of dimension $(-\mu_{CZ}(\gamma_1)) - (-\mu_{CZ}(\gamma_2))$ for all $\gamma_1 \in \mathcal{P}(H_1)$ and $\gamma_2 \in \mathcal{P}(H_2)$. Moreover, if $\mu_{CZ}(\gamma_1) = \mu_{CZ}(\gamma_2)$, then $\mathcal{M}(\gamma_1, \gamma_2, H_s, J_s)$ is compact.*

We will call such a homotopy, regular.

Definition 1.2.6 Let (H_1, J_1) and (H_2, J_2) be two regular pairs for $(\widehat{W}, \widehat{\omega})$ with $H_1 \leq H_2$, and consider a smooth regular increasing homotopy (H_s, J_s) between them. Such a regular homotopy induces a morphism, called a *continuation morphism*

$$\begin{aligned} \phi^{(H_s, J_s)} : SC_*(H_1, J_1) &\to SC_*(H_2, J_2) \\ \gamma &\mapsto \sum_{\substack{\gamma' \in \mathcal{P}(H_2) \\ -\mu_{CZ}(\gamma') = -\mu_{CZ}(\gamma)}} \#\mathcal{M}(\gamma, \gamma', H_s, J_s) \gamma' \end{aligned}$$

where, again, $\#$ is a count of points in a compact 0-dimensional space, with signs.

The fact that it is a chain map relies on the study of the boundary of the 1-dimensional manifold $\mathcal{M}(\gamma_1, \gamma_2, H_s, J_s)$ with $-\mu_{CZ}(\gamma_1) = -\mu_{CZ}(\gamma_2) + 1$.

Theorem 1.2.7 (for instance [AD10] theorem 11.3.10) *Let (u_n) be a sequence of elements in $\mathcal{M}(\gamma_1, \gamma_2, H_s, J_s)$. There exists*

1. *a subsequence of (u_n),*
2. *$x_0 = \gamma_1$, $x_1, \ldots, x_k$ critical points of $\mathcal{A}_{H_1}$,*
3. *$y_0, \ldots, y_{l-1}, y_l = \gamma_2$ critical points of $\mathcal{A}_{H_2}$,*
4. *sequences of real numbers (s_n^i) for $0 \leq i \leq k-1$ and $(s_n'^j)$ for $0 \leq j \leq l-1$ such that $s_n^i \to_{n\to\infty} -\infty$ and $s_n'^j \to_{n\to\infty} +\infty$,*
5. *elements $u^i \in \mathcal{M}(x_i, x_{i+1}, H_1, J_1)$ for $0 \leq i \leq k-1$ and, for $0 \leq j \leq l-1$, elements $v^j \in \mathcal{M}(y_j, y_{j+1}, H_2, J_2)$,*
6. *an element $w \in \mathcal{M}(x_k, y_0, H_s, J_s)$*

such that, for $0 \leq i \leq k-1$ and for $0 \leq j \leq l-1$,

$$\lim_{n\to+\infty} u_n \cdot s_n^i = u^i, \qquad \lim_{n\to+\infty} u_n \cdot s_n'^j = v^j \qquad \text{in } C^\infty_{loc}$$

with $(u \cdot s)(s', t) = u(s + s', t)$, and such that

$$\lim_{n\to+\infty} u_n = w \qquad \text{in } C^\infty_{loc}.$$

Theorem 1.2.8 (for instance [AD10] proposition 11.2.8) *At the homological level, the induced morphism, $\phi^{(H_s,J_s)}$ is independent of the choice of the regular homotopy between (H_1, J_1) and (H_2, J_2).*

The theorem above is one of the versions of the so-called "homotopy of homotopies" theorem; we give a proof in section 1.2.3.
The fact that the continuation morphisms $SH(H_1, J_1) \to SH(H_2, J_2)$ for $H_1 \leq H_2$ compose nicely results from the following theorem.

Theorem 1.2.9 (for instance [AD10] proposition 11.2.9) *Consider three regular pairs (H_1, J_1), (H_2, J_2) and (H_3, J_3) for $(\widehat{W}, \widehat{\omega})$. Let (H_s, J_s) and (H_s', J_s') be two regular homotopies between (H_1, J_1) and (H_2, J_2) and (H_2, J_2) and (H_3, J_3) respectively. Then there exists a regular homotopy (H_s'', J_s'') between (H_1, J_1) and (H_3, J_3) such that $\phi^{(H_s,J_s)} \circ \phi^{(H_s',J_s')}$ and $\phi^{(H_s'',J_s'')}$ induces the same homomorphism in homology.*

This theorem is based on the gluing lemma as explained in subsection 1.1.1 (see also Theorem 3.1.13).

Definition 1.2.10 The *symplectic homology* of (W,ω) is defined as the direct limit

$$SH_*(W,\omega,X) := \varinjlim_{H\in\mathcal{H}_{\mathrm{std}}} SH_*(H,J)$$

where, for each H, $J : S^1 \to End\big((\Gamma(T\widehat{W})\big)$ is chosen so that (H,J) is a regular pair.

Example: the ball B^{2n}

We consider the ball B^{2n} with the symplectic form which is the restriction of the standard symplectic 2-form $\omega_{std} = \frac{i}{2}dz\wedge d\bar{z}$ on $\mathbb{C}^n$ and with the Liouville radial vector field defined by $X_{rad} = \frac{1}{2}(z\partial_z + \bar{z}\partial_{\bar{z}})$. The completion is given by $\widehat{B^{2n}} = \mathbb{C}^n$ with the standard symplectic form $\omega_{std} = \frac{i}{2}dz \wedge d\bar{z}$. We look at Hamiltonians

$$H_C : \mathbb{C}^n \to \mathbb{R} : z \mapsto C\,\|z\|^2$$

such that $\frac{C}{\pi} \notin \mathbb{Z}$. These Hamiltonians are not in $\mathcal{H}_{\mathrm{std}}$ but form an admissible cofinite family by Remark 1.2.3. For each C, the Hamiltonian vector field is

$$X_{H_C} = -2iC(z\partial_z - \bar{z}\partial_{\bar{z}}).$$

The integral trajectories are of the form $z(t) = e^{-2iCt}z_0$; therefore, the only 1-periodic orbit of X_H is the critical point $z = 0$. The Floer chain groups are thus

$$SC_*(H_C) = \begin{cases} \mathbb{Z} & \text{if } * = -\mu_{CZ}(0) \\ 0 & \text{otherwise} \end{cases}$$

and, since the differential is 0, the homology groups are the same as the chain groups.

The Conley-Zehnder index of the constant orbit at $z = 0$ depends on C and is given by

$$-\mu_{CZ}(0) = 2n\left\lfloor \frac{C}{\pi} \right\rfloor + n.$$

Let $C_k := k\pi + \epsilon$ where $k \in \mathbb{N}_{\geq 0}$ and $\epsilon > 0$. The continuation maps $\varphi_k : SC_*(H_{C_k}) \to SC_*(H_{C_{k+1}})$ are all identically zero, thus the symplectic homology is also 0

$$SH_*(B^{2n},\omega_{std},X_{rad}) = \varinjlim_{k} SH_*(H_{C_k}) = 0.$$

1.2.3 Parenthesis on the homotpy of homotopies Theorem

This classical material can be found, for instance, in [AD10, Sal99, Rit].

Definition 1.2.11 Let H_1 and H_2 be two Hamiltonians. We say that an increasing homotopy H_s between H_1 and H_2 is *regular* if for all 1-periodic orbits $\gamma_1 \in \mathcal{P}(H_1)$ and $\gamma_2 \in \mathcal{P}(H_2)$, $\mathcal{M}(\gamma_1,\gamma_2,H_s,J_s)$ is a manifold of dimension $-\mu_{CZ}(\gamma_1) + \mu_{CZ}(\gamma_2)$.

Theorem 1.2.12 *The morphism*

$$\phi : SH(H_1, J_1) \to SH(H_2, J_2)$$

is independent of the choice of the regular homotopy between H_1 and H_2.

Proof: Consider two regular homotopies K_0 and K_1 joining H_1 and H_2. We are going to construct an homotopy between ϕ^{K_0} and ϕ^{K_1} in other word a

$$S : SC_*(H_1, J_1) \to SC_{*+1}(H_2, J_2)$$

satisfying the relation

$$\phi^{K_1} - \phi^{K_0} = S \circ \partial_{H_1} + \partial_{H_2} \circ S.$$

Consider a homotopy of homotopies K_η, $\eta \in [0, 1]$ such that in a neighbourhood of 0, $K_\eta \equiv K_0$ and in a neighbourhood of 1, $K_\eta \equiv K_1$. For $\gamma_1 \in \mathcal{P}(H_1)$, $\gamma_2 \in \mathcal{P}(H_2)$ and η fixed, we denote by $\mathcal{M}(\gamma_1, \gamma_2, K_\eta)$ the space of Floer trajectories $u : \mathbb{R} \times S^1 \to \mathbb{R}$

$$\partial_s u + J_\eta(\partial_\theta u - X_{K_\eta}) = 0$$

and define the parametrized moduli space

$$\mathcal{M}^K(\gamma_1, \gamma_2) := \bigcup_{\eta \in [0,1]} \mathcal{M}(\gamma_1, \gamma_2, K_\eta).$$

We now use the following two theorems:

Theorem 1.2.13 *If $\mu_{CZ}(\gamma_1) - \mu_{CZ}(\gamma_2) + 1 = 0$, then $\mathcal{M}^K(\gamma_1, \gamma_2)$ is a compact manifold of dimension 0.*

Theorem 1.2.14 ([AD10]) *Let us define*

$$\Pi^K(\gamma_1, \gamma_2) := \left(\bigcup_{\substack{\gamma_1' \in \mathcal{P}(H_1) \\ \mu_{CZ}(\gamma_1) - \mu_{CZ}(\gamma_1') = 1}} \mathcal{M}(\gamma_1, \gamma_1', H_1, J_1)/_{\mathbb{R}} \times \mathcal{M}^K(\gamma_1, \gamma_1') \right) \bigcup \left(\bigcup_{\substack{\gamma_2' \in \mathcal{P}(H_2) \\ \mu_{CZ}(\gamma_2') - \mu_{CZ}(\gamma_2) = 1}} \mathcal{M}^K(\gamma_2', \gamma_2') \times \mathcal{M}(\gamma_2', \gamma_2, H_2, J_2)/_{\mathbb{R}} \right).$$

If $\mu_{CZ}(\gamma_1) = \mu_{CZ}(\gamma_2)$, then $\mathcal{M}^K(\gamma_1, \gamma_2) \cup \Pi^K(\gamma_1, \gamma_2)$ is a compact manifold of dimension 1 with boundary equal to

$$\Pi^K(\gamma_1, \gamma_2) \cup \underbrace{\big(\{0\} \times \mathcal{M}(\gamma_1, \gamma_2, K_0)\big)}_{\textit{with opposite orientation}} \cup \big(\{1\} \times \mathcal{M}(\gamma_1, \gamma_2, K_1)\big).$$

We are now ready to proceed with the proof of Theorem 1.2.12. We define the homotopy $S : SC_*(H_1, J_1) \to SC_{*+1}(H_2, J_2)$ as follows: if $\gamma_1 \in \mathcal{P}(H_1)$ such that $\mu_{CZ}(\gamma_1) = k$ then

$$S_k(\gamma_1) = \sum_{\substack{\gamma_2 \in \mathcal{P}(H_2) \\ \mu_{CZ}(\gamma_2)=k+1}} \#\mathcal{M}^K(\gamma_1, \gamma_2)\gamma_2.$$

We have, for $\gamma_1 \in \mathcal{P}(H_1)$ such that $\mu_{CZ}(\gamma_1) = k$,

$$\begin{aligned}
&S \circ \partial_{H_1}(\gamma_1) + \partial_{H_2} \circ S(\gamma_1) \\
&\quad = S_{k-1} \sum_{\substack{\gamma_1' \in \mathcal{P}(H_1) \\ \mu_{CZ}(\gamma_1')=k-1}} \left(\#\mathcal{M}(\gamma_1, \gamma_1', H_1, J_1)/_{\mathbb{R}}\right)\gamma_1' + \partial_{H_2} \sum_{\substack{\gamma_2' \in \mathcal{P}(H_2) \\ \mu_{CZ}(\gamma_2')=k+1}} \#\mathcal{M}^K(\gamma_1, \gamma_2')\gamma_2' \\
&\quad = \sum_{\substack{\gamma_2 \in \mathcal{P}(H_2) \\ \mu_{CZ}(\gamma_2)=k}} \sum_{\substack{\gamma_1' \in \mathcal{P}(H_1) \\ \mu_{CZ}(\gamma_1')=k-1}} \left(\#\mathcal{M}(\gamma_1, \gamma_1', H_1, J_1)/_{\mathbb{R}}\right)\#\mathcal{M}^K(\gamma_1, \gamma_2)\gamma_2 \\
&\qquad + \sum_{\substack{\gamma_2 \in \mathcal{P}(H_2) \\ \mu_{CZ}(\gamma_2)=k}} \sum_{\substack{\gamma_2' \in \mathcal{P}(H_2) \\ \mu_{CZ}(\gamma_2')=k+1}} \#\mathcal{M}^K(\gamma_1, \gamma_2')\left(\#\mathcal{M}(\gamma_2', \gamma_2, H_2, J_2)/_{\mathbb{R}}\right)\gamma_2 \\
&\quad = \sum_{\substack{\gamma_2 \in \mathcal{P}(H_2) \\ \mu_{CZ}(\gamma_2)=k}} \#\Pi^K(\gamma_1, \gamma_2)\gamma_2.
\end{aligned}$$

On the other side

$$\phi^{K_1} - \phi^{K_0} = -\sum_{\substack{\gamma_2 \in \mathcal{P}(H_2) \\ \mu_{CZ}(\gamma_2)=k}} \left(\#\mathcal{M}(\gamma_1, \gamma_2, K_0)\right)\gamma_2 + \sum_{\substack{\gamma_2 \in \mathcal{P}(H_2) \\ \mu_{CZ}(\gamma_2)=k}} \left(\#\mathcal{M}(\gamma_1, \gamma_2, K_1)\right)\gamma_2.$$

Therefore we reach the conclusion using theorem 1.2.14.

1.2.4 Positive symplectic homology

Let (W, ω, X) be a compact symplectic manifold with contact type boundary, satisfying assumptions 1.1.1 and 1.1.2 and let $(\widehat{W}, \widehat{\omega})$ be its symplectic completion. The idea of positive symplectic homology is to "remove" the data of constant 1-periodic orbits from symplectic homology.

We assume in this section that (W, ω, X) *is an exact symplectic manifold*, i.e. there exists a globally defined 1-form λ such that $d\lambda = \omega$. We need this assumption in order to identify the set of critical points of a Hamiltonian with its 1-periodic orbits of small action.

Let $H : S^1 \times \widehat{W} \to \mathbb{R}$ be a Hamiltonian in $\mathcal{H}_{\text{std}}$ (cf. Definition 1.2.1). Recall that the Hamiltonian action functional $\mathcal{A}_H : C^\infty_{\text{contr}}(S^1, \widehat{W}) \to \mathbb{R}$ is defined as

$$\mathcal{A}_H(\gamma) := -\int_{D^2} \sigma^\star \widehat{\omega} - \int_{S^1} H\big(\theta, \gamma(\theta)\big) d\theta$$

where $\sigma : D^2 \to \widehat{W}$ is an extension of γ to the disc D^2. When the symplectic form is exact, $\omega = d\lambda$, the action becomes

$$\mathcal{A}_H(\gamma) := -\int_{S^1} \gamma^\star \widehat{\lambda} - \int_{S^1} H\big(\theta, \gamma(\theta)\big) d\theta.$$

The 1-periodic orbits of $H \in \mathcal{H}_{\text{std}}$ fall into two classes (see Remark 1.2.2)

1. critical points in W; whose action is strictly less than some small positive constant ϵ; indeed, if (θ, x) is a critical point of H, the action of the constant orbit is equal to $-H(\theta, x)$;

2. non-constant periodic orbits lying in $\widehat{W} \setminus W$ whose action is strictly greater than ϵ; indeed, the action of such an orbit is close, for a given ρ in $[0, \rho_0]$ with $T = h'(e^\rho)$ in $\text{Spec}(M, \alpha)$, to the action of the orbit of the vector field $-h'(e^\rho)R_\alpha$ located in $M \times \{\rho\}$; this is given by $-\int_{S^1} e^\rho \alpha(-h'(e^\rho)R_\alpha)d\theta - \int_{S^1} h(e^\rho)d\theta = e^\rho h'(e^\rho) - h(e^\rho) = e^\rho T - h(e^\rho)$; it is positive since h is convex.

The ϵ above is chosen (for instance) as half the minimal value of the periods of closed orbits of the Reeb vector field on $M = \partial W$. Functions H are chosen so that the value of $|H|$ in $S^1 \times W$ is less than ϵ, so that $h(e^\rho)$ is less than $\frac{1}{2}\epsilon$ (hence $e^\rho T - h(e^\rho)$ is greater than $\frac{3}{2}\epsilon$) and the C^2-closeness to an autonomous function is such that the actions differ at most by $\frac{1}{2}\epsilon$.

Lemma 1.2.15 *The action decreases along Floer trajectories, i.e. if $u : \mathbb{R} \times S^1 \to \widehat{W}$ is a solution of equation* (1.1) *such that*

$$\lim_{s \to -\infty} u(s, \cdot) = \gamma^-(\cdot) \quad \text{and} \quad \lim_{s \to +\infty} u(s, \cdot) = \gamma^+(\cdot)$$

then

$$\mathcal{A}(\gamma^-) \geq \mathcal{A}(\gamma^+).$$

For a proof we refer to the more general case of lemma 1.3.18.

Let $SC^{\leq \epsilon}(H, J)$ be the complex generated by the 1-periodic orbits of action no greater than ϵ. It is built out of critical points of H and it is a subcomplex of $SC(H, J)$. It has been proven by Viterbo in [Vit99, Proposition 1.3] that

$$H_*(SC^{\leq \epsilon}(H, J), \partial) \cong H_{*+n}(W, \partial W).$$

Definition 1.2.16 The *positive Floer complex* is defined as the quotient of the total complex by the subcomplex of critical points;

$$SC^+(H,J) := SC(H,J)/_{SC^{\leq\epsilon}(H,J)}$$

Remark that the differential induces a differential on the quotient which we still denote ∂.

Positive symplectic homology is defined as a direct limit over non degenerate $H \in \mathcal{H}_{\text{std}}$ of the homology of $SC^+(H,J)$. The continuation morphisms defined in Definition 1.2.6 descend to the quotient since the action decreases along a solution of (1.3) cfr lemma 1.3.18 (when the homotopy is increasing everywhere).

Definition 1.2.17 The *positive symplectic homology* of (W,ω) is defined as

$$SH^+(W,\omega,X) := \varinjlim_{H\in\mathcal{H}_{\text{std}}} H_*\big(SC^+_*(H,J),\partial\big)$$

Remark 1.2.18 The short exact sequence

$$0 \to SC^{\leq\epsilon}(H,J) \to SC(H,J) \to SC^+(H,J) \to 0$$

induces a long exact sequence in homology

$$\begin{tikzcd}[column sep=small] H_{*+n}(W,\partial W) \arrow[rr] & & SH_*(W,\omega,X) \arrow[dl] \\ & SH^+_*(W,\omega,X) \arrow[ul, "{[-1]}"'] & \end{tikzcd} \qquad (1.5)$$

Positive symplectic homology can be defined in a wider context. We refer to section 4.2.1 for an explicit construction on compact symplectic manifold with contact type boundary satisfying assumptions 1.1.1 and 1.1.2.

Example: the ball B^{2n}

We use the long exact sequence :

$$\begin{tikzcd}[column sep=small] H_{*+n}(B^{2n},\partial B^{2n}=S^{2n-1}) \arrow[rr] & & SH_*(B^{2n},\omega_{std},X_{rad}) \arrow[dl] \\ & SH^+_*(B^{2n},\omega_{std},X_{rad}) \arrow[ul, "{[-1]}"'] & \end{tikzcd}$$

and the fact that $SH_*(B^{2n}) = 0$ to deduce that

$$SH^+_*(B^{2n},\omega_{std},X_{rad}) \cong H_{*+n-1}(B^{2n},S^{2n-1}).$$

To compute the relative homology $H_*(B^{2n}, S^{2n-1})$, we use the exact sequence

$$\begin{tikzcd} H_*(S^{2n-1}) \arrow[rr] & & H_*(B^{2n}) \arrow[dl] \\ & H_*(B^{2n}, S^{2n-1}) \arrow[ul, "{[-1]}"] & \end{tikzcd}.$$

Thus we have

$$H_*(B^{2n}, S^{2n-1}) = \begin{cases} \mathbb{Z} & \text{if } * = 2n \\ 0 & \text{otherwise} \end{cases}$$

and this implies

$$SH_*^+(B^{2n}, \omega_{std}, X_{rad}) = \begin{cases} \mathbb{Z} & \text{if } * = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular the positive symplectic homology of the ball has only one generator. This shows that this homology invariant cannot detect all distinct periodic Reeb orbits on the sphere (with a contact structure which is non degenerate).

1.3 S^1-equivariant symplectic homology

1.3.1 S^1-equivariant homology

Let X be a topological space endowed with an S^1-action. If the S^1-action is free, $X/_{S^1}$ is a topological space. The aim of S^1-equivariant homology is to build on the space X a homology which coincides, when the action is free, with the singular homology of the quotient. One considers the universal principal S^1-bundle $ES^1 \to BS^1$.The diagonal action on $X \times ES^1$ is free and one denotes by $X \times_{S^1} ES^1$ the quotient $(X \times ES^1)/_{S^1}$.

Definition 1.3.1 (Borel) Let X be a topological space endowed with an S^1-action. The *S^1-equivariant homology of X* with $\mathbb{Z}$-coefficients is

$$H_*^{S^1}(X) := H_*(X \times_{S^1} ES^1, \mathbb{Z}).$$

An axiomatic definition of equivariant homology was stated later by Basu, [Bas], based on the following Proposition:

Proposition 1.3.2 *The S^1-equivariant homology with $\mathbb{Z}$-coefficients is a functor $H_*^{S^1}$ from the category of S^1-spaces and S^1-maps to the category of abelian groups and homomorphisms. Let X be a topological space endowed with a S^1-action, $H_*^{S^1}$ associates to X a sequence of abelian groups: $H_i^{S^1}(X, \mathbb{Z}), i \geq 0$. Let $f : X \to Y$ be an S^1-equivariant map between topological spaces endowed with an S^1-action. It induces homomorphisms $f_i^{S^1} : H_i^{S^1}(X, \mathbb{Z}) \to H_i^{S^1}(Y, \mathbb{Z})$. The functor $H_*^{S^1}$ satisfy the two following conditions:*

1. *If the S^1-action on X is free, then $H_*^{S^1}(X,\mathbb{Z}) = H_*(X/_{S^1},\mathbb{Z})$ (the singular homology of $X/_{S^1}$).*

2. *If $f : X \to Y$ induces an isomorphism $f_* : H_*(X,\mathbb{Z}) \to H_*(Y,\mathbb{Z})$, then it also induces an isomorphism $f_*^{S^1} : H_*^{S^1}(X,\mathbb{Z}) \to H_*^{S^1}(Y,\mathbb{Z})$.*

Any functor satisfying the two conditions of Proposition 1.3.2 is given by Definition 1.3.1. Indeed, the projection $pr_1 : X \times ES^1 \to X : (x,e) \mapsto x$ is an S^1-equivariant map which induces an isomorphism

$$pr_{1*} : H_*(X \times ES^1,\mathbb{Z}) \to H_*(X,\mathbb{Z})$$

since ES^1 is contractible. By 2, pr_{1*} induces an isomorphism

$$pr_{1*}^{S^1} : H_*^{S^1}(X \times ES^1,\mathbb{Z}) \to H_*^{S^1}(X,\mathbb{Z}).$$

Condition 1 then implies

$$H_*^{S^1}(X,\mathbb{Z}) \cong H_*(X \times_{S^1} ES^1,\mathbb{Z}).$$

1.3.2 S^1-equivariant symplectic homology

The setup is the same as for symplectic homology (cf. section 1.2.1). The S^1-equivariant symplectic homology is defined for any compact symplectic manifold with contact type boundary (W,ω,X). The S^1-action we are referring to in this section is the reparametrization action on the loop space,

$$\varphi \cdot \gamma(\theta) = \gamma(\theta - \varphi)$$

not an action on W. This homology was first introduced by Viterbo in [Vit99]; a different approach, which will be presented in section 1.4 was sketched by Seidel in [Sei08] and a detailed study by Bourgeois and Oancea appears in [BO12, BO10, BO13b].

The functional

Viterbo's idea is to adapt Borel's construction for Morse theory to the space of contractible loops in $\widehat{W}$ with the S^1-action. We consider the model of ES^1 given as a limit of spheres S^{2N+1} for N going to ∞ with the Hopf S^1-action. To provide S^1-invariant functionals, we use S^1-invariant Hamiltonians : $H : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$. The S^1-invariance condition reads,

$$H(\theta+\varphi, x, \varphi z) = H(\theta, x, z), \qquad \forall \theta, \varphi \in S^1, \forall z \in S^{2N+1}.$$

The action functional $\mathcal{A} : C^\infty(S^1,\widehat{W}) \times S^{2N+1} \to \mathbb{R}$, called the parametrised action functional, is defined as

$$\mathcal{A}(\gamma,z) := -\int_{D^2} \sigma^\star\widehat{\omega} - \int_{S^1} H\big(\theta,\gamma(\theta),z\big)d\theta. \tag{1.6}$$

where $\sigma : D^2 \to \widehat{W}$ is an extension of γ to the disc D^2. It is invariant under the diagonal S^1-action on $C^\infty(S^1, \widehat{W}) \times S^{2N+1}$.

The critical points of the parametrised action functional are pairs (γ, z) such that

$$\gamma \in \mathcal{P}(H_z) \quad \text{and} \quad \int_{S^1} \frac{\partial H_z}{\partial z}\big(\theta, \gamma(\theta)\big) d\theta = 0, \tag{1.7}$$

where H_z is the function on $S^1 \times \widehat{W}$ defined by $H_z(\theta, x) := H(\theta, x, z)$ and where $\mathcal{P}(H_z)$ denote, as before, the set of 1-periodic orbits of X_{H_z}. The set of critical points of $\mathcal{A}$, denoted by $\mathcal{P}^{S^1}(H)$, is S^1-invariant. If $q = (\gamma, z) \in \mathcal{P}^{S^1}(H)$, we denote by S_q the S^1-orbit of q

$$S_q := \{\varphi \cdot q := \varphi \cdot (\gamma, z) = (\varphi \cdot \gamma, \varphi z) \,|\, \varphi \in S^1\}.$$

Such an S_q is called nondegenerate if the Hessian $d^2\mathcal{A}(\gamma, z)$ has a 1-dimensional kernel for some (and hence any) $(\gamma, z) \in S_q$.

Definition 1.3.3 An S^1*-invariant Hamiltonian* H *is called admissible* if H_z is in $\mathcal{H}_{\text{std}}$ (as in Definition 1.2.1) with constant slope independent of z for all $z \in S^{2N+1}$ and if for any critical point $q \in \mathcal{P}^{S^1}(H)$, the S^1-orbit S_q is non degenerate. Let $\mathcal{H}^{S^1,N}$ be the family of such hamiltonians.

Again, one can consider more general Hamiltonian in $\mathcal{H}_{\text{std}}$; the main point is that it coincides with a linear function with constant slope outside a compact set.

Proposition 1.3.4 ([BO10], Proposition 5.1) *The set* $\mathcal{H}^{S^1,N}$ *is of second Baire category in the space of* S^1*-invariants Hamiltonians* H *such that* H_z *is in* $\mathcal{H}_{std}$ *with constant slope independent of* z *for all* $z \in S^{2N+1}$.

The chain complex

The chain complex of our homology will be generated by the set of S^1-orbits of critical points of $\mathcal{A}$.

Definition 1.3.5 ([BO13b]) The *parametrized index of a non degenerate circle of critical points* S_q, with $q = (\gamma, z)$, is defined as follows. The Hamiltonian $H : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$ is extended to a function $\tilde{H} : S^1 \times \widehat{W} \times T^*S^{2N+1} \to \mathbb{R}$ by pullback

$$\tilde{H}\big(\theta, x, (z, \zeta)\big) = H(\theta, x, z) = H_z(\theta, x).$$

The cotangent bundle T^*S^{2N+1} is endowed with its canonical symplectic structure $dz \wedge d\zeta$ and one considers the Hamiltonian vector field $X_{\tilde{H}} = X_{H_z} - \frac{\partial H}{\partial z}\partial_\zeta$. The 1-periodic orbits of $X_{\tilde{H}}$ are of the form $\widetilde{\gamma} := \big(\gamma(\cdot), z, \zeta(\cdot)\big)$ with γ a 1-periodic orbit of H_z and $\zeta(\theta) = \zeta(0) - \int_0^\theta \frac{\partial H}{\partial z}\big(\theta', \gamma(\theta'), z\big) d\theta'$ with (γ, z) a critical point of the parametrized action functional and where $\zeta(0)$ can be chosen arbitrarily in $T_z^*S^{2N+1}$.

The parametrized index of the circle of critical points $S_{(\gamma,z)}$, $\mu_{param}\big(S_{(\gamma,z)}\big)$, is defined as the Robbin-Salamon index (see section 6.4) of the path of symplectic matrices defined by the differential of the flow of $X_{\tilde{H}}$ along $\widetilde{\gamma}$ using a trivialisation of $T\big(\widehat{W}\times T^*S^{2N+1}\big)$ over a disk bounded by $\widetilde{\gamma}$.

The *grading in the chain complex* of the element S_q is equal to $-\mu_{param}(S_q)+N$.

Floer trajectories

To define negative gradient trajectories of the parametrized action functional $\mathcal{A}$, one needs a metric on the loop space and on the sphere. One chooses again a parametrized smooth loop $J : S^1\times S^{2N+1}\to \mathrm{End}(TW)$ of almost complex structures on W which are compatible with ω, and a metric g on S^{2n+1}; since the functional is S^1 invariant, we want the following S^1 invariance.

Definition 1.3.6 A *parametrized loop of almost complex structures*

$$J : S^1\times S^{2N+1}\to \mathrm{End}(T\widehat{W}), (\theta,z)\mapsto J_z^\theta$$

is called S^1-invariant if

$$J_{\varphi z}^{\theta+\varphi}=J_z^\theta, \qquad \forall\theta,\varphi\in S^1, \forall z\in S^{2N+1}$$

and *is called admissible* if for all z in S^{2N+1}, the loop of almost complex structures J_z is in $\mathcal{J}$ as defined in Definition 1.2.4.

We denote by $\mathcal{J}^{S^1,N}$ the set of pairs (J,g) consisting of an admissible S^1-invariant parametrised loop of almost complex structures and an S^1-invariant Riemannian metric g on S^{2N+1}.

Definition 1.3.7 Given $H\in\mathcal{H}^{S^1,N}$, $(J,g)\in\mathcal{J}^{S^1,N}$ and $q^-,q^+\in\mathcal{P}^{S^1}(H)$, we denote by $\widehat{\mathcal{M}}(S_{q^-},S_{q^+};H,J,g)$ the set of S^1-equivariant Floer trajectories, consisting of pairs (u,z) with $u:\mathbb{R}\times S^1\to\widehat{W}$ and $z:\mathbb{R}\to S^{2N+1}$ such that

$$\begin{cases} \partial_s u + J^\theta_{z(s)}\circ u\big(\partial_\theta u - X^\theta_{H_{z(s)}}\circ u\big)=0\\ \dot{z}(s)-\displaystyle\int_{S^1}\vec{\nabla}_z H\big(\theta,u(s,\theta),z(s)\big)d\theta = 0\end{cases} \tag{1.8}$$

with the asymptotic conditions

$$\lim_{s\to-\infty}\big(u(s,\cdot),z(s)\big)\in S_{q^-}, \qquad \lim_{s\to+\infty}\big(u(s,\cdot),z(s)\big)\in S_{q^+}.$$

By $\vec{\nabla}_z H\big(\theta,x,z\big)$, we mean the gradient at the point z with respect to the metric g, of the function on S^{2n+1} defined by $H(\theta,x,\cdot)$.

Proposition 1.3.8 ([BO10], Proposition 5.2) *Assume $S_{q^-}, S_{q^+} \subset \mathcal{P}^{S^1}(H)$ are non-degenerate. Then for any $(u,z) \in \widehat{\mathcal{M}}(S_{q^-}, S_{q^+}; H, J, g)$, the linearisation $D_{u,z}$ of the equation* (1.8)*, extended to suitable Banach spaces is Fredholm of index $-\mu_{param}\big(S_{q^-}\big) + \mu_{param}\big(S_{q^+}\big) + 1$.*

Transversality results

Bourgeois and Oancea have proven a transversality result involving the two following classes of Hamiltonians (section 7 of [BO10]):

1. *Generic Hamiltonians.* The set, denoted $\mathcal{H}_{gen}$, of Hamiltonians $H \in \mathcal{H}^{S^1,N}$ such that:

   a) For all $(\gamma, z) \in \mathcal{P}^{S^1}(H)$, γ is a simple embedded curve ;

   b) For all distinct elements $(\gamma_1, z_1), (\gamma_2, z_2) \in \mathcal{P}^{S^1}(H)$, we have $\gamma_1 \neq \gamma_2$.

2. *Split Hamiltonians.* The set, denoted $\mathcal{H}_{split}$, of Hamiltonians $H \in \mathcal{H}^{S^1,N}$ of the form $K(x) + f(z)$, with K C^2-small on W such that K has either constant and non degenerate 1-periodic orbits, or non constant and transversally non degenerate ones and f is an S^1-invariant function.

We denote by $\mathcal{H}_*$ the union of those two sets: $\mathcal{H}_* = \mathcal{H}_{gen} \cup \mathcal{H}_{split}$.

Definition 1.3.9 ([BO12]) An admissible Hamiltonian $H \in \mathcal{H}^{S^1,N}$ is called strongly admissible if

1. For every $(\gamma, z) \in \mathcal{P}^{S^1}(H)$ such that γ is not constant, we have

$$X^{\theta}_{H_z}\big(\gamma(\theta)\big) \neq 0, \quad \forall \theta \in S^1.$$

2. For every $(\gamma, z) \in \mathcal{P}^{S^1}(H)$ such that γ is constant (equal to $x \in \widehat{W}$), there exists a neighbourhood U of $\{x\} \times (S^1 \cdot z)$ in $\widehat{W} \times S^{2N+1}$ such that $H(\theta, x', z') = K(x') + f(z')$ for all $\theta \in S^1$ and for all $(x', z') \in U$. Moreover, x is an isolated critical point of K.

Remark that any Hamiltonian in $\mathcal{H}_*$ is strongly admissible.

Definition 1.3.10 ([BO12]) Given a strongly admissible Hamiltonian H, a pair $(J, g) \in \mathcal{J}^{S^1,N}$ is called adapted to H if the following hold :

1. For every $(\gamma, z_0) \in \mathcal{P}^{S^1}(H)$, we have

$$\big[J^{\theta}_z X^{\theta}_{H_z}, X^{\theta}_{H_z}\big]\big(\gamma(\theta)\big) \notin Span\big(J^{\theta}_z X^{\theta}_{H_z}, X^{\theta}_{H_z}\big), \quad \forall \theta \in S^1, z \in S^1 \cdot z_0.$$

2. for every $(\gamma, z_0) \in \mathcal{P}^{S^1}(H)$ such that γ is constant (equal to $x \in \widehat{W}$), there exists a neighbourhood U of $\{x\} \times (S^1 \cdot z)$ in $\widehat{W} \times S^{2N+1}$ such that J_z^θ is independent of θ and z on U, i.e. $J_z^\theta(x') = J(x')$ for all $(x', z) \in U$ and $\theta \in S^1$.

We denote by $\mathcal{HJ}'$ the set of triples (H, J, g) such that H is a strongly admissible Hamiltonian and (J, g) is adapted to H and we denote by $\mathcal{H}_*\mathcal{J}'$ the subset of $\mathcal{HJ}'$ corresponding to elements $H \in \mathcal{H}_*$, asking, furthermore, if $H \in \mathcal{H}_{split}$, that J is independent of θ, of ρ and of z for $\rho \geq 1$.

Given $H \in \mathcal{H}^{S^1,N}$, we say that a pair $(J, g) \in \mathcal{J}^{S^1,N}$ is regular for H if the linearisation $D_{u,z}$ of the equation (1.8), extended to suitable Banach spaces, is surjective for any $q^-, q^+ \in \mathcal{P}^{S^1}(H)$ and any $(u, z) \in \widehat{\mathcal{M}}(S_{q^-}, S_{q^+}; H, J, g)$. We denote the set of those regular pairs by $\mathcal{J}_{reg}^{S^1,N}(H)$.

Theorem 1.3.11 ([BO10], Theorem 7.4) *There exists an open subset $\mathcal{HJ}'_{reg} \subset \mathcal{HJ}'$ which is dense in a neighbourhood of $\mathcal{H}_*\mathcal{J}' \subset \mathcal{HJ}'$ and consisting of triples (H, J, g) such that $H \in H^{S^1,N}$ and $(J, g) \in J_{reg}^{S^1,N}(H)$.*

For $(H, J, g) \in \mathcal{HJ}'_{reg}$, the moduli space $\widehat{\mathcal{M}}(S_{q^-}, S_{q^+}; H, J, g)$ is a manifold whose dimension is $-\mu_{param}(S_{q^-}) + \mu_{param}(S_{q^+}) + 1$; it carries an action of $\mathbb{R}$ (by reparametrization in the s-variable) and an action by S^1 coming from the S^1-invariance of the action $\mathcal{A}$ and of the almost complex structure (J, g). We denote by $\mathcal{M}^{S^1}(S_{q^-}, S_{q^+}; H, J, g)$ the moduli space quotiented by those two actions. It is a smooth manifold of dimension $-\mu_{param}(S_{q^-}) + \mu_{param}(S_{q^+}) - 1$.

Definition 1.3.12 The S^1*-equivariant Floer complex* $SC_*^{S^1,N}(H, J, g)$ is the following chain complex:

$$SC_*^{S^1,N}(H) := SC_*^{S^1,N}(H, J, g) := \bigoplus_{S_q \in \mathcal{P}^{S^1}(H)} \mathbb{Z}\langle S_q \rangle$$

where the grading is defined by $|S_q| = -\mu_{param}(S_q) + N$. The S^1*-equivariant differential* is defined as

$$\partial^{S^1} : SC_*^{S^1,N}(H, J, g) \longrightarrow SC_{*-1}^{S^1,N}(H, J, g)$$

$$\partial^{S^1}(S_{q^-}) := \sum_{\substack{S_{q^+} \subset \mathcal{P}^{S^1}(H) \\ |S_{q^-}| - |S_{q^+}| = 1}} \#\mathcal{M}^{S^1}(S_{q^-}, S_{q^+}; H, J, g) S_{q^+}$$

where $\#$ is a count with signs defined in [BO12], obtained by comparing the coherent orientations on $\mathcal{M}^{S^1}(S_{q^-}, S_{q^+}; H, J, g)$ with the orientation induced by the infinitesimal generator of the action.

The fact that it is indeed a differential follows from:

Proposition 1.3.13 ([BO13a], Proposition 2.2) *The map ∂^{S^1} satisfies $\partial^{S^1} \circ \partial^{S^1} = 0$.*

Definition 1.3.14 The S^1*-equivariant Floer homology groups* are defined as

$$SH_*^{S^1,N}(H,J,g) := H_*\big(SC_*^{S^1,N}(H), \partial^{S^1}\big)$$

The S^1-equivariant Floer homology groups are independent of the choice of the regular pair (J,g).

Proposition 1.3.15 ([BO13a]) *Given a Hamiltonian in $\mathcal{H}^{S^1,N}$, and two regular pairs for H, (J_1,g_1) and (J_2,g_2) in $\mathcal{J}_{reg}^{S^1,N}$, there exists a canonical isomorphism*

$$SH_*^{S^1,N}(H,J_1,g_1) \simeq SH_*^{S^1,N}(H,J_2,g_2).$$

In the sequel we shall denote $SH_*^{S^1,N}(H,J,g)$ by $SH_*^{S^1,N}(H)$.

Let $H_1, H_2 \in \mathcal{H}_{reg}^{S^1,N}$ be two S^1-invariant Hamiltonians and let H_s be an increasing homotopy between them (i.e $\partial_s H_s \geq 0$ and there exists s_0 such that for $|s| \geq s_0$, we have $H_{-|s|} = H_1$ and $H_{|s|} = H_2$). We consider the solutions of

$$\begin{cases} \partial_s u + J^\theta_{s,z(s)}\big(\partial_\theta u - X_{H_{s,z(s)}}(u)\big) = 0 \\ \dot{z}(s) - \displaystyle\int_{S^1} \vec{\nabla}_z H\big(s,\theta,u(s,\theta),z(s)\big) d\theta = 0 \end{cases} \tag{1.9}$$

with the asymptotic conditions

$$\lim_{s\to-\infty}\big(u(s,\cdot),z(s)\big) \in S_{q_1}, \qquad \lim_{s\to+\infty}\big(u(s,\cdot),z(s)\big) \in S_{q_2}$$

where $q_1 \in \mathcal{P}^{S^1}(H_1)$ and $q_2 \in \mathcal{P}^{S^1}(H_2)$. Let $\mathcal{M}(S_{q_1}, S_{q_2}, H_s, J_s, g_s)$ be the space of solutions of (1.9) ; it carries an S^1-action (but no $\mathbb{R}$-action) and we denote the quotient by $\mathcal{M}^{S^1}(S_{q_1}, S_{q_2}, H_s, J_s, g_s)$.

Proposition 1.3.16 ([BO12], Proposition 2.1) *For all $q_1 \in \mathcal{P}^{S^1}(H_1)$ and $q_2 \in \mathcal{P}^{S^1}(H_2)$, the $\mathcal{M}^{S^1}(S_{q_1}, S_{q_2}, H_s, J_s, g_s)$ are smooth manifolds of dimension $-\mu_{param}(S_{q_1})+\mu_{param}(S_{q_2})$.*

We can define the *continuation morphism* :

$$\phi : SC_*^{S^1,N}(H_1,J_1,g_1) \to SC_*^{S^1,N}(H_2,J_2,g_2)$$

by

$$\phi(S_{q_1}) := \sum_{\substack{S_{q_2} \subset \mathcal{P}^{S^1}(H_2) \\ |S_{q_1}|-|S_{q_2}|=1}} \#\mathcal{M}^{S^1}(S_{q_1}, S_{q_2}; H_s, J_s, g_s)\, S_{q_2} \tag{1.10}$$

where $q_1 \in \mathcal{P}^{S^1}(H_1)$ and $q_2 \in \mathcal{P}^{S^1}(H_2)$. These morphisms are chain maps and thus pass to the quotient where we still denote them by ϕ. As previously, we define the S^1-equivariant homology groups of W to be the direct limit over continuation maps of the S^1-equivariant Floer homology groups

$$SH_*^{S^1,N}(W,\omega,X) := \varinjlim_{H \in \mathcal{H}^{S^1,N}} SH_*^{S^1,N}(H). \tag{1.11}$$

We then take the direct limit over N with respect to the S^1-equivariant embeddings $S^{2N+1} \hookrightarrow S^{2N+3}$ which induce maps $SH_*^{S^1,N}(W,\omega,X) \to SH_*^{S^1,N+1}(W,\omega,X)$ for each N.

Definition 1.3.17 The S^1*-equivariant symplectic homology* of W is

$$SH_*^{S^1}(W,\omega,X) := \varinjlim_N SH_*^{S^1,N}(W,\omega,X).$$

Example: the ball B^{2n}

The idea is the same as in section 1.2.2. We take a Hamiltonian

$$H_C : \mathbb{C}^n \times S^{2N+1} \to \mathbb{R} : (z,p) \mapsto C\left\|z\right\|^2 + \tilde{f}(p)$$

where $\tilde{f}$ is the S^1-invariant lift of a perfect Morse function on $\mathbb{C}P^{n-1}$. The critical points of $\mathcal{A}_H$ are $(0,p_0),\dots,(0,p_{N-1})$ where $\{p_i \,|\, i = 0\dots N-1\} = Crit(\tilde{f})$. The index of $(0,p_i)$ is

$$-\mu_{param}(0,p_i) + N = 2n\left\lfloor \frac{C}{\pi} \right\rfloor + n + 2i.$$

Therefore

$$SC_*^{S^1,N}(H_C) = \begin{cases} \mathbb{Z} & \text{if } * = 2n\left\lfloor \frac{C}{\pi}\right\rfloor + n + 2i, i \in \{0,\dots N-1\} \\ 0 & \text{otherwise} \end{cases}$$

and, the differential is 0 since the complex is lacunary, thus $SH_*^{S^1,N}(H_C) \simeq SC_*^{S^1,N}(H_C)$. If we let $C_k = k\pi(1+2N) + \epsilon$ for $\epsilon > 0$, the continuation maps $\varphi_k : SC_*^{S^1,N}(H_{C_k}) \to SC_*^{S^1,N}(H_{C_{k+1}})$ are identically 0 and thus

$$SH_*^{S^1,N}(B^{2n},\omega_{std},X_{rad}) = \varinjlim_N \varinjlim_k SH_*^{S^1,N}(H_{C_k}) = 0.$$

1.3.3 Positive S^1-equivariant symplectic homology

As in section 1.2.4, we assume that (W, ω) is an exact compact symplectic manifold with contact type boundary in order to identify the complex generated by 1-periodic orbits of X_H of action $\leq \epsilon$ with the complex generated by the critical points of H. To see that it is a subcomplex, we study the action along the Floer trajectories.

Proposition 1.3.18 *The action decreases along a parametrized Floer trajectory; i.e a solution of* (1.9).

PROOF: Since the action for the Hamiltonian H_s along the pair $\big(u(s,\cdot), z(s)\big)$ is given by

$$\mathcal{A}_{H_{s,z(s)}}\big(u(s,\cdot), z(s)\big) = -\int_{D^2} \sigma_s^\star \widehat{\omega} - \int_{S^1} H\big(s, \theta, u(s,\theta), z(s)\big)$$

where $\sigma_s : D^2 \to \widehat{W}$ is an extension of $\gamma_s = u(s,\cdot)$ to the disc D^2. By the asphericity condition, $\int_{D^2} \sigma_s^\star \widehat{\omega} = \int_{D^2} \sigma_{s_0}^\star \widehat{\omega} + \int_{S^1 \times [s_0, s]} u^\star \widehat{\omega}$ so that

$$\begin{aligned}
\frac{\partial}{\partial s}\mathcal{A}_{H_{s,z(s)}}\big(u(s,\cdot), z(s)\big) &= -\int_{S^1} \omega(\partial_s u, \partial_\theta u) d\theta - \int_{S^1} \tfrac{\partial}{\partial u} H\big(s, \theta, u(s,\theta), z(s)\big) \tfrac{\partial}{\partial s} u(s,\theta) d\theta \\
&\quad - \int_{S^1} \vec{\nabla}_z H\big(s, \theta, u(s,\theta), z(s)\big) \cdot \dot{z}(s) d\theta \\
&\quad - \int_{S^1} \tfrac{\partial}{\partial s} H\big(s, \theta, u(s,\theta), z(s)\big) d\theta \\
&= -\int_{S^1} \omega(\partial_s u, \partial_\theta u) d\theta - \int_{S^1} dH(\partial_s u) d\theta \\
&\quad - \int_{S^1} \vec{\nabla}_z H\big(s, \theta, u(s,\theta), z(s)\big) d\theta \cdot \int_{S^1} \vec{\nabla}_z H\big(s, \theta, u(s,\theta), z(s)\big) d\theta \\
&\quad - \int_{S^1} \tfrac{\partial}{\partial s} H\big(s, \theta, u(s,\theta), z(s)\big) d\theta
\end{aligned}$$

The last two terms $-\left\| \int_{S^1} \vec{\nabla}_z H\big(s, \theta, u(s,\theta), z(s)\big) d\theta \right\|^2$ and $-\int_{S^1} \frac{\partial}{\partial s} H\big(s, \theta, u(s,\theta), z(s)\big) d\theta$ are clearly non positive. The first line can be rewritten as

$$\begin{aligned}
-\int_{S^1} \omega(\partial_s u, \partial_\theta u) d\theta - \int_{S^1} \omega(X_H, \partial_s u) d\theta &= -\int_{S^1} \omega(\partial_s u, \partial_\theta u - X_{H_{z(s)}}) d\theta \\
&= -\int_{S^1} \omega(\partial_s u, J^\theta_{z(s)} \partial_s u) d\theta = -\left\| \partial_s u \right\|^2_{g J^\theta_{z(s)}} \leq 0.
\end{aligned}$$

Corollary 1.3.19 *The action decreases along a solution of* (1.8).

Let $H \in \mathcal{H}^{S^1,N}$. By the above Corollary, the complex generated by 1-periodic orbits of X_H of action $\leq \epsilon$ (denoted $SC^{S^1,N,\leq\epsilon}(H,J,g)$) is a subcomplex of $SC^{S^1,N}(H,J,g)$. It is built out of critical points of H. As in Viterbo [Vit99, Proposition 1.3] we have

$$H_*(SC^{S^1,N,\leq\epsilon}(H,J,g),\partial^{S^1}) \cong H^{S^1}_{*+n}(W,\partial W).$$

where the S^1-action on the pair $(W,\partial W)$ is the trivial one.

Definition 1.3.20 Let $H \in \mathcal{H}^{S^1,N}_{reg}$ be a Hamiltonian. The *positive S^1-equivariant complex* is defined as

$$SC^{S^1,N,+}(H,J,g) := SC^{S^1,N}(H,J,g) \big/ _{SC^{S^1,N,\leq\epsilon}(H,J,g)}.$$

Remark that the differential induces a differential on the quotient which we still denote ∂^{S^1}. The continuation morphisms defined in equation (1.10) descend to the quotient since the action decreases along a solution of (1.9) cfr Proposition 1.3.18. The *positive S^1-equivariant Floer groups* are defined as

$$SH^{S^1,N,+}(H) := H\big(SC^{S^1,N,+}(H),\partial^{S^1}\big).$$

Definition 1.3.21 The *positive S^1-equivariant symplectic homology* is defined as

$$SH^{S^1,+}_*(W,\omega,X) := \varinjlim_N \varinjlim_{H\in\mathcal{H}^{S^1,N}} SH^{S^1,N,+}_*(H).$$

Example: the ball B^{2n}

The short exact sequence

$$0 \to SC^{S^1,N,\leq\epsilon}(H,J,g) \to SC^{S^1,N}(H,J,g) \to SC^{S^1,N,+}(H,J,g) \to 0$$

induces a long exact sequence in homology

$$\begin{tikzcd}
H^{S^1}_{*+n}(B^{2n},\partial B^{2n}=S^{2n-1}) \arrow[rr] & & SH^{S^1}_*(B^{2n},\omega_{std},X_{rad}) . \arrow[dl] \\
& SH^{S^1,+}_*(B^{2n},\omega_{std},X_{rad}) \arrow[ul, "{[-1]}"] &
\end{tikzcd} \tag{1.12}$$

The fact that $SH^{S^1,N}_*(B^{2n},\omega_{std},X_{rad}) = 0$ implies

$$SH^{S^1,+}_*(B^{2n},\omega_{std},X_{rad}) \cong H^{S^1}_{*+n-1}(B^{2n},S^{2n-1}).$$

The S^1-action on the pair (B^{2n}, S^{2n-1}) is trivial ; therefore

$$H_*^{S^1}(B^{2n}, S^{2n-1}) = H_*(B^{2n}, S^{2n-1}) \otimes H_*(BS^1).$$

We have as in 1.2.4 that

$$H_*(B^{2n}, S^{2n-1}) = \begin{cases} \mathbb{Z} & \text{if } * = 2n \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$SH_*^{S^1,+}(B^{2n}, \omega_{std}, X_{rad}) = \begin{cases} \mathbb{Z} & \text{if } * = n+1+2i, \quad i \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

1.4 An alternative presentation of the S^1-equivariant symplectic homology

The definition of S^1equivariant symplectic homology that we have presented fits nicely in the general picture of symplectic homologies but is hard to tackle. An alternative homology has been suggested by Seidel in [Sei08] and developed by Bourgeois and Oancea in [BO12]. It has the advantage to use a special class of Hamiltonians and simplified equations for Floer trajectories, so that properties and computations are often feasible. The important point is that this homology coincides with the S^1equivariant symplectic homology defined above.

We shall look at a subfamily of the S^1 equivariant Hamiltonians $\mathcal{H}^{S^1,N}$ as defined in Definition 1.3.3. We shall define the generators of our complex and the equations for the Floer trajectories considered to define a differential.

The data

We consider a compact symplectic manifold with compact type boundary (W, ω, X). We choose a perfect Morse function $f : \mathbb{C}P^N \to \mathbb{R}$ and a Riemannian metric $\bar{g}$ on $\mathbb{C}P^N$ for which the gradient flow of f has the Morse-Smale property (and this will be for each N). Let $\tilde{f} : S^{2N+1} \to \mathbb{R}$ be the S^1-invariant lift of f. Let g be the lifted S^1 invariant metric on the sphere S^{2N+1}. We denote by $Crit(\tilde{f})$ the critical set of $\tilde{f}$; it is a union of circles. We choose a point z_0 on each critical circle and we fix a local slice transverse in S^{2N+1} to the circle in $Crit(\tilde{f})$ at z_0, considering the hypersurface T_{z_0} spanned by the stable and the unstable manifold at z_0 (with respect to the gradient $\vec{\nabla}\tilde{f}$ of $\tilde{f}$ with respect to g). Let U be a neighbourhood of $Crit(\tilde{f})$ and let $\check{\rho} : S^{2N+1} \to \mathbb{R}$ be a cut-off function on U which is equal to 1 in a neighbourhood $U' \subset U$ of $Crit(\tilde{f})$ and 0 outside U. We define

$$\epsilon := \min_{z \in S^{2N+1} \setminus U'} \left\| \vec{\nabla}\tilde{f}(z) \right\| > 0.$$

Class of admissible Hamiltonians

We look at the subfamily $\mathcal{H}^{S^1,N}(f) \subset \mathcal{H}^{S^1,N}$ consisting of Hamiltonians of the form $H + \tilde{f}$ with $H : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$ in $\mathcal{H}^{S^1,N}$ (cf. Definition 1.3.3) such that

1. Each critical point (γ, z) of the parametrized action functional $\mathcal{A}_{H+\tilde{f}}$ defined by $H + \tilde{f}$ lies over a z which is a critical point of $\tilde{f}$;
2. For every $z \in Crit(\tilde{f})$, $H(\cdot, \cdot, z)$ has non degenerate periodic orbits;
3. $H + \tilde{f}$ has nondegenerate S^1-orbits;
4. $\left\| \vec{\nabla}_z H\big(\theta, x, z\big) \right\| < \epsilon$, for all $z \in S^{2N+1} \setminus U'$;
5. For all $z \in U'$, $\vec{\nabla}_z H \cdot \vec{\nabla} \tilde{f}(z) = 0$.

Remark 1.4.1 Condition 3 can be replaced by the following : near every critical orbit of $\tilde{f}$, we have $H(\theta, x, z) = H'(\theta - \phi_z, x)$, where $\phi_z \in S^1$ is the unique element such that the action of its inverse brings z into T_{z_0}, i.e. $\phi_z^{-1} \cdot z \in T_{z_0}$ and $H' \in \mathcal{H}_{\text{std}}$. In fact, we shall consider elements H which are built from an $H' : S^1 \times \widehat{W} \to \mathbb{R}$ in $\mathcal{H}_{\text{std}}$ as in Definition 1.2.1, close to an autonomous Hamiltonian; we shall develop this in the next chapter.

The chain complex

Given an admissible $H + \tilde{f}$, the set $\mathcal{P}^{S^1}(H + \tilde{f})$ of critical points (γ, z) of the parametrized action functional $\mathcal{A}_{H+\tilde{f}}$ arise in circles and each one of those circle gives a generator of the chain complex. So the complex is generated by the set

$$\Big\{ S_{(\gamma,z)} := S^1 \cdot \big(\gamma, z\big) \Big\}.$$

The index of the generator $S_{(\gamma,z)}$ is defined to be

$$\mu(S_{\gamma,z}) := -\mu_{CZ}(\gamma) + \mu_{Morse}(z; -\tilde{f}).$$

The differential operator

Let (J_z^θ) be an S^1-invariant family of almost complex structures independent of z along each local slice. Let $p^- = (\gamma^-, z^-)$ and $p^+ = (\gamma^+, z^+)$ be two critical points of $\mathcal{A}_{H+\tilde{f}}$. We denote by $\widehat{\mathcal{M}}(S_{p^-}, S_{p^+}; H, f, J_z^\theta, g)$ the space of solutions (u, z), $u : \mathbb{R} \times S^1 \to \widehat{W}$, $z : \mathbb{R} \to S^{2N+1}$ to the system of equations

$$\begin{cases} \partial_s u + J^\theta_{z(s)} \circ u\big(\partial_\theta u - X_{H_{z(s)}} \circ u\big) = 0 \\ \dot{z} - \vec{\nabla} \tilde{f}(z) = 0 \end{cases}$$

with the conditions

$$\lim_{s\to-\infty}\big(u(s,\cdot),z(s)\big)\in S_{p^-}\qquad \lim_{s\to\infty}\big(u(s,\cdot),z(s)\big)\in S_{p^+}. \tag{1.13}$$

This system is to be compared to the system (1.9).
If $S_{p^-}\neq S_{p^+}$, we denote by $\mathcal{M}(S_{p^-},S_{p^+};H,f,J_z^\theta,g)$ the quotient of $\widehat{\mathcal{M}}(S_{p^-},S_{p^+};H,f,J_z^\theta,g)$ by the reparametrization $\mathbb{R}$-action. $\mathcal{M}(S_{p^-},S_{p^+};H,f,J_z^\theta,g)$ carries a free S^1-action and we denote by $\mathcal{M}^{S^1}(S_{p^-},S_{p^+};H,f,J_z^\theta,g)$ the quotient of $\mathcal{M}(S_{p^-},S_{p^+};H,f,J_z^\theta,g)$ by this S^1-action.

Proposition 1.4.2 ([BO12]) *For generically chosen J_z^θ and g, the spaces $\mathcal{M}^{S^1}(S_{p^-},S_{p^+};H,f,J_z^\theta,g)$ are smooth manifolds of dimension $-\mu(S_{p^-})+\mu(S_{p^+})-1$.*

The *chain complex* is defined as:

$$S\widetilde{C}_*^{S^1,N}(H,f):=\bigoplus_{S_p\subset\mathcal{P}^{S^1}(H+\tilde{f})}\mathbb{Z}\langle S_p\rangle.$$

The *differential* $\widetilde{\partial}^{S^1}:S\widetilde{C}_*^{S^1,N}(H,f)\to S\widetilde{C}_{*-1}^{S^1,N}(H,f)$ is defined by

$$\widetilde{\partial}^{S^1}(S_{p^-}):=\sum_{\substack{S_p^+\subset\mathcal{P}^{S^1}(H+\tilde{f})\\ \mu(S_{p^-})-\mu(S_{p^+})=1}}\#\mathcal{M}^{S^1}(S_{p^-},S_{p^+};H,f,J_z^\theta,g)S_{p^+}$$

with signs defined in [BO12]. The fact that the homology defined above is the same as the one defined in section 1.3.2 is proven in:

Theorem 1.4.3 ([BO12], Proposition 2.7) *For any non degenerate $K\in\mathcal{H}^{S^1,N}$ that coincides with $H+\tilde{f}$ outside a compact set and for any pair $(J',g')\in\mathcal{J}^{S^1,N}$ which is regular for K and coincides with (J,g) outside a compact set, there is a canonical isomorphism*

$$SH_*^{S^1,N}(K,J',g')\simeq H_*\big(S\widetilde{C}_*^{S^1,N}(H,f),\widetilde{\partial}^{S^1}\big).$$

Continuation maps are defined as usual, using the space of solutions (u,s) of

$$\begin{cases}\partial_s u+J^\theta_{s,z(s)}\circ u\big(\partial_\theta u-X_{H_{s,z(s)}}\circ u\big)=0\\ \dot{z}-\vec{\nabla}\tilde{f}(z)=0\end{cases}\tag{1.14}$$

with $H_s+\tilde{f}$ an increasing homotopy between $H_0+\tilde{f}$ and $H_1+\tilde{f}$. The isomorphisms of Theorem 1.4.3 commute with continuation maps when $H_0\leq H_1$ and when $N\to\infty$.
Hence we have an alternative definition:

Definition 1.4.4 The S^1*-equivariant Floer homology groups* are defined as

$$SH_*^{S^1,N}(H,f,J,g) := H_*\big(\widetilde{SC}_*^{S^1,N}(H,f), \widetilde{\partial}^{S^1}\big)$$

The S^1*-equivariant symplectic homology groups of* W are defined as

$$SH_*^{S^1}(W,\omega,X) := \varinjlim_{N} \varinjlim_{H\in\mathcal{H}^{S^1,N}(f)} SH_*^{S^1,N}(H,f,J,g)$$

We show here below that the action decreases along these new trajectories. This allows to define $SH^{S^1,+}$ in the context of exact compact symplectic manifolds with contact type boundary.

Proposition 1.4.5 *Let $H_0+\tilde f$ and $H_1+\tilde f$ be Hamiltonians in $\mathcal{H}^{S^1,N}(f)$ and let $\tilde H_s := H_s+\tilde f$ be an increasing homotopy between $H_0+\tilde f$ and $H_1+\tilde f$. If (u,z), $u:\mathbb{R}\times S^1\to\widehat{W}$ and $z:\mathbb{R}\to S^{2N+1}$ is a solution of*

$$\begin{cases} \partial_s u + J^\theta_{s,z(s)}\circ u\big(\partial_\theta u - X_{H_{s,z(s)}}\circ u\big) = 0 \\ \dot z - \vec\nabla \tilde f(z) = 0 \end{cases}$$

with

$$\lim_{s\to-\infty}\big(u(s,\cdot),z(s)\big) = \big(\gamma^-(\cdot),z^-\big) \quad \textit{and} \quad \lim_{s\to+\infty}\big(u(s,\cdot),z(s)\big) = \big(\gamma^+(\cdot),z^+\big)$$

then

$$\mathcal{A}(\gamma^-,z^-) \geq \mathcal{A}(\gamma^+,z^+).$$

Proof: The proof proceeds as in proposition 1.3.18. The parametrized action for the Hamiltonian $H_s+\tilde f$ on the pair $(u(s,\cdot),z(s))$ is given by

$$-\int_{D^2}\sigma_s^\star\widehat{\omega} - \int_{S^1}(H_s+\tilde f)\,(\theta,u(s,\theta),z(s))\,d\theta$$

where $\sigma_s : D^2 \to \widehat{W}$ is an extension of $\gamma_s = u(s,\cdot)$ to the disc D^2. By the asphericity condition, $\int_{D^2} \sigma_s^\star \widehat{\omega} = \int_{D^2} \sigma_{s_0}^\star \widehat{\omega} + \int_{S^1\times[s_0,s]} u^\star \widehat{\omega}$ so that

$$\begin{aligned}\frac{\partial}{\partial s}\mathcal{A}_{H_s+\tilde{f}}\big(u(s,\cdot),z(s)\big) &= -\int_{S^1}\omega(\partial_s u,\partial_\theta u)d\theta - \int_{S^1}\tfrac{\partial}{\partial u}H_s\big(\theta,u(s,\theta),z(s)\big)\tfrac{\partial}{\partial s}u(s,\theta)d\theta\\ &\quad -\int_{S^1}\vec{\nabla}_z(H_s+\tilde{f})\big(\theta,u(s,\theta),z(s)\big)\cdot\dot{z}(s)d\theta\\ &\quad -\int_{S^1}\big(\tfrac{\partial}{\partial s}(H_s+\tilde{f})\big)\big(\theta,u(s,\theta),z(s)\big)d\theta\\ &= -\int_{S^1}\omega(\partial_s u,\partial_\theta u)d\theta - \int_{S^1} dH_{s,z(s)}(\partial_s u)d\theta\\ &\quad -\int_{S^1}\vec{\nabla}_z(H_s+\tilde{f})\big(s,\theta,u(s,\theta),z(s)\big)\cdot\vec{\nabla}f(z)d\theta\\ &\quad -\int_{S^1}\tfrac{\partial}{\partial s}(H_s+\tilde{f})\big(\theta,u(s,\theta),z(s)\big)d\theta.\end{aligned}$$

The first line can be rewritten as

$$\begin{aligned}-\int_{S^1}\omega(\partial_s u,\partial_\theta u)d\theta - \int_{S^1}\omega(X_{H_{s,z(s)}},\partial_s u)d\theta &= -\int_{S^1}\omega(\partial_s u,\partial_\theta u - X_{H_{s,z(s)}})d\theta\\ &= -\int_{S^1}\omega(\partial_s u, J^\theta_{z(s)}\partial_s u)d\theta = -\left\|\partial_s u\right\|^2_{gJ^\theta_{z(s)}} \leq 0.\end{aligned}$$

The last two terms rewrite as

$$\begin{aligned}&-\int_{S^1}\big(\tfrac{\partial}{\partial s}(H_s+\tilde{f})\big)\big(\theta,u(s,\theta),z(s)\big)d\theta\\ &\qquad -\int_{S^1}\vec{\nabla}_z H\big(s,\theta,u(s,\theta),z(s)\big)\cdot\vec{\nabla}f(z)d\theta\\ &\qquad -\int_{S^1}\left\|\vec{\nabla}f(z)\right\|^2 d\theta\\ &\leq 0 \qquad .\end{aligned}$$

Condition 4 and 5 conclude.

Remark 1.4.6 With the assumptions of Proposition, 1.4.5, it appears in the proof above that

$$\int \left\|\partial_s u\right\|^2_{gJ^\theta_{z(s)}} dsd\theta \leq \mathcal{A}(\gamma^-,z^-) - \mathcal{A}(\gamma^+,z^+).$$

Definition 1.4.7 Let $H \in \mathcal{H}^{S^1,N}(f)$ be a Hamiltonian. The *positive* S^1*-equivariant complex* is defined as

$$S\widetilde{C}^{S^1,N,+}(H,f) := S\widetilde{C}^{S^1,N}(H,f) / {S\widetilde{C}^{S^1,N,\leq\epsilon}(H,f)}$$

where $S\widetilde{C}^{S^1,N,\leq\epsilon}(H,f)$ is the set of critical points of $\mathcal{A}_{H+\tilde{f}}$ of action less than ϵ. The differential passes to the quotient where we still denote it $\widetilde{\partial}^{S^1}$ and the positive S^1-equivariant Floer groups are defined as

$$SH^{S^1,N,+}(H,f) := H\big(S\widetilde{C}^{S^1,N,+}(H,f), \widetilde{\partial}^{S^1}\big).$$

Observe that the f should more precisely read f_N in all the above construction.

Definition 1.4.8 The *positive* S^1*-equivariant symplectic homology* is defined by

$$SH_*^{S^1,+}(W,\omega,X) := \varinjlim_N \varinjlim_{H\in\mathcal{H}^{S^1,N}(f_N)} SH_*^{S^1,N,+}(H,f_N).$$

We assume (W,ω,X) to be exact and we assume the function f to be small in order to a identify 1-periodic orbits of small action with a pair (p,z), p a critical points of H.

2 $SH^{S^1,+}$ and periodic Reeb orbits

The goal of this section is to relate the positive S^1-equivariant homology of an exact compact symplectic manifold $(W, d\lambda)$, which is a so-called Liouville domain, to the Reeb orbits on $(M = \partial W, \alpha = \lambda_{|_M})$; this relation is expressed in Theorem 2.2.2. We use the alternative description of S^1-equivariant symplectic homology ([Sei08] , [BO12]) presented in section 1.4, for a nice subclass of Hamiltonians.

Liouville domains

Definition 2.0.9 A *Liouville domain* (also called compact symplectic manifold with restricted contact type boundary) is a compact manifold W with boundary $\partial W = M$, together with a 1-form λ such that $\omega := d\lambda$ is symplectic and the Liouville vector field X defined by $\iota(X)\omega = \lambda$ points strictly outwards along ∂W. The Liouville domain will be denoted (W, λ).

Let us observe that the asphericity condition is automatically satisfied. We still assume that $\langle c_1(TW), \pi_2(W)\rangle = 0$. We have defined the (symplectic) completion of a compact symplectic manifold with contact type boundary in section 1.2.1. We consider the completion

$$\widehat{W} = W \cup (\partial W \times \mathbb{R}^+)$$

of a Liouville domain (W, λ), built from the flow of the Liouville vector field X. We denote by $\widehat{\lambda}$ the 1-form on $\widehat{W}$ defined by λ on W and by $e^\rho \alpha$ on $\partial W \times \mathbb{R}^+$ with $\alpha := \lambda_{|_{\partial V}}$. The completion will be denoted $(\widehat{W}, \widehat{\lambda})$.

2.1 The multicomplex defining positive S^1-equivariant homology

The "nice subclass" of Hamiltonians that we use was introduced in [BO12]. The Hamiltonians are constructed using elements in $\mathcal{H}_{\text{std}}$ which are small perturbations of autonomous Hamiltonians as we shall now indicate.

2.1.1 Construction of admissible Hamiltonians from elements in $\mathcal{H}_{\mathrm{std}}$.

As in section 1.4, we fix a sequence of perfect Morse functions $f_N : \mathbb{C}P^N \to \mathbb{R}$, which are C^2-small, together with a Riemannian metric $\bar{g}_N$ for which the gradient flow of f_N has the Morse-Smale property. For instance, we can take

$$f_N\big([w^0 : \ldots : w^N]\big) = C\frac{\sum_{j=0}^N a_j \left|w^j\right|^2}{\sum_{j=0}^N \left|w^j\right|^2} \quad \text{with } a_{i+1} > a_i \in \mathbb{R} \text{ and } C < 0 \in \mathbb{R}$$

and the standard metric.

We denote by $\tilde{f}_N : S^{2N+1} \to \mathbb{R}$ their S^1-invariant lift, and by $Crit(\tilde{f}_N)$ the set of critical points of $\tilde{f}_N$ (which is a union of circles).

We choose a point z_j on the critical circle which projects on the critical point of $-f_N$ of index $2j$. In our example, the point $z_j \in S^{2N+1}$ can be taken as the point $(w^0, \ldots, w^N)$ with $w^i = \delta^i_j$.

We fix a local slice transverse in S^{2N+1} to the circle in $Crit(\tilde{f}_N)$ at z_j. This local slice is the hypersurface T_{z_j} spanned by the stable and the unstable manifold at z_j (with respect to the gradient $\vec{\nabla}\tilde{f}_N$ of $\tilde{f}_N$ with respect to g_N). In our example, $T_{z_j} = \{(w^0 \ldots, w^N) \in S^{2N+1} \,|\, w^j \in \mathbb{R}^+\}$.

We consider U_N a neighbourhood of $Crit(\tilde{f}_N)$ and $\check{\rho}_N : S^{2N+1} \to \mathbb{R}$ a S^1 invariant cut-off function on U_N which is equal to 1 in a neighbourhood $U'_N \subset U_N$ of $Crit(\tilde{f}_N)$ and 0 outside U_N. We set $\epsilon_N := \min_{z \in S^{2N+1} \setminus U'_N} \left\|\vec{\nabla}\tilde{f}_N(z)\right\| > 0$.

If $H' \in \mathcal{H}_{\mathrm{std}}$, we can create an S^1-invariant Hamiltonian $H : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$ from H' and $\tilde{f}$. Define $\widetilde{H} : S^1 \times \widehat{W} \times U_N \to \mathbb{R}$ by $\widetilde{H}(\theta, x, z) := H'(\theta - \phi_z, x)$ where $\phi_z \in S^1$ is the unique element such that $\phi_z^{-1} \cdot z \in T_{z_j}$ when z is close to the critical circle including z_j. We extend $\widetilde{H}$ to $H : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$, by

$$H(\theta, x, z) := \check{\rho}_N(z)\widetilde{H}(\theta, x, z) + \big(1 - \check{\rho}_N(z)\big)\beta(x)H'(\theta, x) \tag{2.1}$$

using the cutoff function $\check{\rho}_N$ on S^{2N+1} and a function $\beta : \widehat{W} \to \mathbb{R}$ which is 0 where H' is time-dependent and equal to 1 outside a compact set. The element H is admissible, i.e. is in $\mathcal{H}^{S^1,N}(f)$ as defined in section 1.4, when assumption 4 is satisfied; this will be the case when we restrict ourselves to a subclass of $H' \in \mathcal{H}_{\mathrm{std}}$, consisting of small perturbations of some autonomous functions. This will be developed in section 2.1.3.

2.1.2 The complex for a subclass of special Hamiltonians

Let $H' : S^1 \times \widehat{W} \to \mathbb{R}$ in $\mathcal{H}_{\mathrm{std}}$ be fixed, with non degenerate 1-periodic orbits. We consider a sequence $H_N \in \mathcal{H}^{S^1,N}, N \geq 1$ such that $H_N(\theta, x, z) = H'(\theta - \phi_z, x)$ for every $z \in Crit(\tilde{f}_N)$ (see for instance construction 2.1) and a sequence $J_N \in \mathcal{J}^{S^1,N}$ such that J_N is regular for H_N.

Let $i_0 : \mathbb{C}P^N \hookrightarrow \mathbb{C}P^{N+1} : [w^0 : \ldots : w^{N-1}] \mapsto [w^0 : \ldots : w^{N-1} : 0]$ and $i_1 : \mathbb{C}P^N \hookrightarrow \mathbb{C}P^{N+1} : [w^0 : \ldots : w^{N-1}] \mapsto [0 : w^0 : \ldots : w^{N-1}]$ and denote by $\tilde{i}_0 : S^{2N+1} \to S^{2N+3} : z \mapsto (z, 0)$ and $\tilde{i}_1 : S^{2N+1} \to S^{2N+3} : z \mapsto (0, z)$ their lifts. We assume:

1. $\mathrm{Im}(i_0)$ and $\mathrm{Im}(i_1)$ are invariant under the gradient flow of f_{N+1};
2. $f_N = f_{N+1} \circ i_0 = f_{N+1} \circ i_1 + cst$ and $i_1^\star \bar{g}_{N+1} = i_0^\star \bar{g}_{N+1} = \bar{g}_N$ (this will be true for our example when $a_{i+1} = a_i + 1$ for all i);
3. $H_{N+1}\big(\cdot, \cdot, \tilde{i}_1(z)\big) = H_{N+1}\big(\cdot, \cdot, \tilde{i}_0(z)\big) = H_N(\cdot, \cdot, z)$;
4. $J_{N+1, \tilde{i}_1(z)} = J_{N+1, \tilde{i}_0(z)} = J_{N,z}$.

The critical points of $\mathcal{A}_{H_N + \tilde{f}_N}$ are pairs (γ_z, z) where z is a critical point of $\tilde{f}_N$ and where γ_z is a ϕ_z-translation of a 1-periodic orbit γ of H' in $\widehat{W}$ (i.e $\gamma_z(\theta) = \gamma(\theta - \phi_z)$ which writes $\gamma_z = \phi_z \cdot \gamma$). We have thus a natural identification (with gradings)

$$\begin{aligned} S\widetilde{C}_*^{S^1,N}(H_N, f_N) &\simeq \mathbb{Z}[u]/_{u^{N+1}} \otimes_{\mathbb{Z}} SC_*(H', J) \\ S^1 \cdot (\gamma_{z_j}, z_j) &\mapsto u^j \otimes \gamma =: u^j \gamma \end{aligned}$$

where z_j is the chosen critical point of $-\tilde{f}_N$ of index $2j$ and u is a formal variable of degree 2.

The differential, under this identification of complexes, writes

$$\widetilde{\partial}^{S^1}(u^l \otimes \gamma) = \sum_{j=0}^{l} u^{l-j} \otimes \varphi_j(\gamma). \tag{2.2}$$

for maps

$$\varphi_j : SC_*(H') \to SC_{*+2j-1}(H'), \quad j = 0, \ldots N$$

defined by counting elements of $\mathcal{M}^{S^1}(S_{(\gamma_{z_j}^-, z_j)}, S_{(\gamma_{z_0}^+, z_0)}; H_N, f_N, J_N, g_N)$ which is the quotient by the $\mathbb{R}$ and the S^1-action of the space of solutions of

$$\begin{cases} \partial_s u + J^\theta_{z(s)} \circ u\big(\partial_\theta u - X_{H_{N,z(s)}} \circ u\big) = 0 \\ \dot{z} - \vec{\nabla} \tilde{f}(z) = 0 \end{cases}$$

going from $S^1 \cdot (\gamma_{z_j}^-, z_j)$ to $S^1 \cdot (\gamma_{z_0}^+, z_0)$.

It follows from the assumptions (1), (2), (3) and (4) that for a fixed j, the maps φ_j obtained for varying values of $N \geq j$ coincide. Therefore we can encode the limit as $N \to \infty$ of all the $S\widetilde{C}_*^{S^1,N}(H_N, f_N)$ into a complex denoted

$$S\widehat{C}_*^{S^1}(H') := \mathbb{Z}[u] \otimes_{\mathbb{Z}} SC_*(H')$$

with differential induced by (2.2) that we can formally write as

$$\widehat{\partial}^{S^1} = \varphi_0 + u^{-1}\varphi_1 + u^{-2}\varphi_2 + \dots$$

As before, there are well-defined continuation maps induced by increasing homotopies of Hamiltonians and we have

Proposition 2.1.1 *[BO12] The S^1 equivariant homology of W is given by:*

$$SH_*^{S^1}(W) := \varinjlim_{H\in\mathcal{H}_{std}} H\Big(S\widehat{C}_*^{S^1}(H'), \widehat{\partial}^{S^1}\Big).$$

2.1.3 Perturbation of Morse-Bott Hamiltonians

We show that we have a good control on the generators of the complex $SC_*(H')$ defining symplectic homology when we choose the admissible Hamiltonian H' in $\mathcal{H}_{\rm std}$ to be close to an autonomous Hamiltonian. We shall use techniques taken from [BO09b] where symplectic homology is computed directly from autonomous Hamiltonians.

This we do in the context of a compact symplectic manifold with contact type boundary (W, ω, X). We denote by M the boundary ∂W. As before, we denote by $(\widehat{W}, \widehat{\omega})$ the completion, by ρ the second coordinate on $M \times \mathbb{R}^+$, and by α the contact form on M defined by $\alpha = \iota(X)\omega_{|M}$. We denote by R_α the corresponding Reeb vector field.

Definition 2.1.2 Let $\mathcal{H}_{\rm MB}$ be the set of Hamiltonians $H : \widehat{W} \to \mathbb{R}$ such that

1. $H_{|W}$ is a negative C^2-small Morse function;
2. $H(p, \rho) = h(\rho)$ outside W, where h is a strictly increasing function, which coincides with $h(\rho) = ae^\rho + b$ for $\rho > \rho_0$, $a, b \in \mathbb{R}$ and $a \notin \mathrm{Spec}(M, \alpha)$, and we assume that $h'' - h' > 0$ on $[0, \rho_0)$.

Note that the 1-periodic orbits of X_H in W, for $H \in \mathcal{H}_{\rm MB}$, are constant and non degenerate by assumption 1. For $(p, \rho) \in M \times \mathbb{R}^+$ and $H \in \mathcal{H}_{\rm MB}$, we have

$$X_H(p, \rho) = -e^\rho h'(\rho)R_\alpha. \tag{2.3}$$

Thus the 1-periodic orbit of X_H are either critical points of H in W or non constant 1-periodic orbits, located on levels $M \times \{\rho\}, \rho \in (0, \rho_0)$, which are in correspondence with periodic $-R_\alpha$-orbits of period $e^\rho h'(\rho)$.

Since H is autonomous, every 1-periodic orbit, γ_H of X_H, corresponding to the periodic Reeb orbit γ, gives birth to a S^1 family of 1-periodic orbits of X_H which is denoted by S_γ.

We shall modify an element $H \in \mathcal{H}_{\rm MB}$, as in [CFHW96], to deform this autonomous Hamiltonian into a time-dependent Hamiltonian H_δ with only non degenerate 1-periodic

orbits. The Hamiltonian $H_\delta(\theta, p)$ will coincide with $H(p)$ outside a neighbourhood of the image of the non-constant 1-periodic orbits of X_H. We proceed as follows:
We choose a perfect Morse function on the circle, $\check{f} : S^1 \to \mathbb{R}$.
For each 1-periodic orbit γ_H of X_H, we consider the integer l_{γ_H} so that γ_H is a l_{γ_H}-fold cover of a simple periodic orbit:

$$l_{\gamma_H} := \max\bigl\{k \in \mathbb{N} \,|\, \gamma_H(\theta + \tfrac{1}{k}) = \gamma_H(\theta) \ \ \forall \theta \in S^1\bigr\}.$$

This number l_{γ_H} is constant on the S^1-family of 1-periodic orbits of X_H corresponding to the periodic Reeb orbit γ. We set $l_\gamma = l_{\gamma_H} = \frac{1}{T}$ where T is the period of γ.
We choose a symplectic trivialization $\psi := (\psi_1, \psi_2) : U_\gamma \to V \subset S^1 \times \mathbb{R}^{2n-1}$ between open neighborhoods $U_\gamma \subset \partial W \times \mathbb{R}^+ \subset \widehat{W}$ of the image of γ_H and V of $S^1 \times \{0\}$ such that $\psi_1\bigl(\gamma_H(\theta)\bigr) = l_\gamma \theta$. Here $S^1 \times \mathbb{R}^{2n-1}$ is endowed with the standard symplectic form. Let $\check{g} : S^1 \times \mathbb{R}^{2n-1} \to [0,1]$ be a smooth cutoff function supported in a small neighborhood of $S^1 \times \{0\}$ such that $\check{g}_{|_{S^1 \times \{0\}}} \equiv 1$. We denote by $\check{f}_\gamma$ the function defined on S_γ by $\check{f} \circ \psi_{1|_{S_\gamma}}$.

For $\delta > 0$ and $(\theta, p, \rho) \in S^1 \times U_\gamma$, we define

$$H_\delta(\theta, p, \rho) := h(\rho) + \delta \check{g}\bigl(\psi(p,\rho)\bigr)\check{f}\bigl(\psi_1(p,\rho) - l_\gamma \theta\bigr). \tag{2.4}$$

The Hamiltonian H_δ coincides with H outside the open sets $S^1 \times U_\gamma$.

Lemma 2.1.3 ([CFHW96, BO09b]) *The 1-periodic obits of H_δ, for δ small enough, are either constant orbits (the same as those of H) or nonconstant orbits which are non degenerate and form pairs $(\gamma_{\min}, \gamma_{\mathrm{Max}})$ which coincide with the orbits in S_γ starting at the minimum and the maximum of $\check{f}_\gamma$ respectively, for each Reeb orbit γ such that S_γ appears in the 1-periodic orbits of H. Their Conley-Zehnder index is given by $\mu_{CZ}(\gamma_{\min}) = \mu_{CZ}(\gamma) - 1$ and $\mu_{CZ}(\gamma_{\mathrm{Max}}) = \mu_{CZ}(\gamma)$.*

2.2 Computing $SH^{S^1,+}$

We consider now the symplectic homologies with coefficients in $\mathbb{Q}$, denoted $SH^\dagger(W, \mathbb{Q})$ on a Liouville domain (W, λ). The goal of this section is to show that if the Conley-Zehnder indices of all periodic Reeb orbits on $M = \partial W$ have the same parity, then the positive S^1 equivariant symplectic homology is generated by those periodic Reeb orbits. Let $f_N : \mathbb{C}P^N \to \mathbb{R}$ be as before a sequence of perfect Morse functions, which we assume here to be C^2-small.

The class of Hamiltonians

We consider a Hamiltonian denoted $H_{\delta,N}$ which is a S^1-equivariant lift, as in construction 2.1, of a Hamiltonian H_δ which is a perturbation, as in section 2.1.3, of a Hamiltonian H in $\mathcal{H}_{\mathrm{MB}}$ (cf definition 2.1.2) such that the slope a is big and ρ_0 is small.

As mentioned before, the non constant critical points of $\mathcal{A}_{H_{\delta N}+\tilde{f}_N}$ are pairs (γ_z, z) where z is a critical point of $\tilde{f}_N$ and where γ_z is a ϕ_z-translation of a non constant 1-periodic orbit γ' of H_δ in $\widehat{W}$. Such a γ' is of the form $\gamma_{\min}$ or γ_{Max}, located on a level $M \times \{\rho\}, \rho \in (0, \rho_0)$ corresponding to a periodic orbit of $-R_\alpha$ of period $T = e^\rho h'(\rho)$.

Remark 2.2.1 The action of this critical point is given by

$$-\int_{S^1} \gamma_z^\star \widehat{\lambda} - \int_{S^1} (H_N + \tilde{f}_N)(\theta, \gamma_z(\theta), z)\, d\theta.$$

With our assumptions (f small, ρ_0 small), the second term is close to zero. The first term is equal to $-\int_{S^1} (\gamma')^\star \widehat{\lambda}$ and this is given by $e^\rho T$. Recall that T is called the contact action of the Reeb orbit γ of period T. Hence the action of this critical point is close to T.

We shall take f and H so that the difference between the action of a non constant critical orbit and the period T of the corresponding Reeb orbit is, for any critical orbit, smaller than a quarter of the smallest period, the smallest spectral gap and the smallest distance between two geometrically distinct periodic Reeb orbits.

Theorem 2.2.2 *Let (W, λ) be a Liouville domain. Assume there exists a contact form α on ∂W such that the Conley-Zehnder index of all periodic Reeb orbits have the same parity. Then*

$$SH^{S^1,+}(W, \mathbb{Q}) = \bigoplus_{\gamma \in \mathcal{P}(R_\alpha)} \mathbb{Q}\langle \gamma \rangle$$

where $\mathcal{P}(R_\alpha)$ denotes the set of periodic Reeb orbits on ∂W.

PROOF: Let H be a Hamiltonian in $\mathcal{H}_{\mathrm{MB}}$ such that the action is distinct for S^1-families of orbits corresponding to Reeb orbits of different period. This is possible by Remark 2.2.1. We consider, as mentionned above, the S^1-equivariant functions $H_{\delta,N}$ which are lifts of a perturbation H_δ of H. We use the natural identification, described in section 2.1.2:

$$S\widetilde{C}^{S^1,N,+}(H_{\delta,N}, f_N) \simeq \mathbb{Z}[u]/_{u^{N+1}} \otimes SC^+(H_\delta)$$

and the description of $SC^+(H_\delta)$ given by Lemma 2.1.3.

The complex $S\widetilde{C}^{S^1,N,+}(H_{\delta,N}, f_N)$ is filtered by the action thanks to Proposition 1.4.5. We take the filtration $F_p S\widetilde{C}^{S^1,N,+}(H_{\delta,N}, f_N), p \in \mathbb{Z}$ such that for every $p \in \mathbb{Z}$, the quotient $F_{p+1} S\widetilde{C}^{S^1,N,+}(H_{\delta,N}, f_N) / F_p S\widetilde{C}^{S^1,N,+}(H_{\delta,N}, f_N)$ is a union of sets

$$\{1 \otimes \gamma_{\mathrm{Max}}, \ldots, u^N \otimes \gamma_{\mathrm{Max}}, 1 \otimes \gamma_{\min}, \ldots, u^N \otimes \gamma_{\min}\}$$

corresponding to underlying Reeb orbits γ of the same period T.

We consider the zero page of the associated spectral sequence.

$$E^{0,N}_{p,q} := F_{p+1} S\widetilde{C}^{S^1,N,+}_{p+q}(H_{\delta,N}, f_N) / F_p S\widetilde{C}^{S^1,N,+}_{p+q}(H_{\delta,N}, f_N)$$

We have "twin towers of generators", one tower corresponding to each periodic Reeb orbit of period T on ∂W,

$$\begin{tikzcd}
u^N \otimes \gamma_{\mathrm{Max}} \arrow[dr, "u^{-1}\varphi_1"] & u^N \otimes \gamma_{\min} \arrow[l, "\varphi_0"'] \\
\vdots & \vdots \\
u^2 \otimes \gamma_{\mathrm{Max}} \arrow[dr, "u^{-1}\varphi_1"] & u^2 \otimes \gamma_{\min} \arrow[l, "\varphi_0"'] \\
u \otimes \gamma_{\mathrm{Max}} \arrow[dr, "u^{-1}\varphi_1"] & u \otimes \gamma_{\min} \arrow[l, "\varphi_0"'] \\
1 \otimes \gamma_{\mathrm{Max}} & 1 \otimes \gamma_{\min} \arrow[l, "\varphi_0"']
\end{tikzcd}$$

with induced differential as in the above diagram with the notation of section 2.1.2. The differential between two elements in distinct towers vanishes. Indeed the corresponding Reeb orbits are geometrically distinct in ∂W so any Floer trajectory linking elements in distinct towers should satisfy

$$\int \|\partial_s u\|^2_{gJ^\theta_{z(s)}} \, dsd\theta \geq \mathrm{dist}(\gamma^-, \gamma^+).$$

On the other hand, by Remark 1.4.6,

$$\begin{aligned}
\int \|\partial_s u\|^2_{gJ^\theta_{z(s)}} \, dsd\theta &\leq \mathcal{A}(\gamma^-, z^-) - \mathcal{A}(\gamma^+, z^+) \\
&\leq T_{\gamma^-} - T_{\gamma^+} + \tfrac{1}{2}\min\{\text{distance between two distinct Reeb orbits}\}.
\end{aligned}$$

So there cannot be a Floer trajectory between elements in different towers.

To study any given tower, we use the explicit description of φ_0 and φ_1.

1. [BO09b, Lemma 4.28] Let $\gamma_{\min}$, γ_{Max} and H_δ be as above. For δ small enough, the moduli space $\mathcal{M}(\gamma_{\min}, \gamma_{\mathrm{Max}}; H_\delta, J)/\mathbb{R}$ consists of two elements; they have opposite signs, due to the choice of a system of coherent orientations, if and only of the underlying Reeb orbit γ is good. This implies that in our case,

$$\varphi_0 = 0.$$

2. [BO12, Lemma 3.3] The map $\varphi_1 : SC^+_*(H_\delta) \to SC^+_{*+1}(H_\delta)$ acts by

$$\varphi_1(\gamma_{\mathrm{Max}}) = \begin{cases} k_\gamma \gamma_{\min} & \text{if } \gamma \text{ is good,} \\ 0 & \text{if } \gamma \text{ is bad.} \end{cases}$$

where k_γ is the multiplicity of the underlying Reeb orbit γ i.e. γ is a k_γ-fold cover of a simple periodic Reeb orbit. A Reeb orbit is called bad if its Conley-Zehnder index is not of the same parity as the Conley-Zehnder index of the simple Reeb orbit with same image.
The factor k_γ comes from the fact that $\gamma_{\min}(\theta) = \gamma_{\mathrm{Max}}(\theta + \tau_0)$ for a real number τ_0 and that if the underlying Reeb orbit is of multiplicity k_γ, $\gamma_{\min}(\theta) = \gamma_{\min}\big(\theta + \frac{1}{k_\gamma}\big)$. So that $\gamma_{\min}(\theta) = \gamma_{\mathrm{Max}}(\theta + \tau_0 + \frac{m}{k_\gamma})$ for any integer $0 \le m \le k_\gamma$. We have here k_γ trajectories and they all appear with the same sign.

To compute the first page $E^{1;N}_{p,q}$ of the spectral sequence we have the complex $E^{0;N}_{p,q}$:

$$\mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{(\times k_\gamma)} \dots \xrightarrow{(\times k_\gamma)} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{(\times k_\gamma)} \mathbb{Q} \xrightarrow{0} \mathbb{Q}$$

and thus, in the homology $E^{1;N}_{p,q}$, we are left with one copy of $\mathbb{Q}$ in degree $-\mu_{CZ}(\gamma)$ and one copy of $\mathbb{Q}$ in degree $-\mu_{CZ}(\gamma) + 2N$. The first page is given by

$$E^{1;N} = \bigoplus_{\gamma \in \mathcal{P}(H_\delta)} \mathbb{Q}\langle \gamma_{\mathrm{Max}} \rangle \oplus \mathbb{Q}\langle u^N \otimes \gamma_{\min} \rangle.$$

Due to the assumption, the differential on the first page of the spectral sequence vanishes (because of the same parity of the Conley-Zehnder indices) therefore, for N large enough, it gives the homology

$$SH^{S^1,N,+}(H_{\delta,N}) = \bigoplus_{\gamma \in \mathcal{P}(H_\delta)} \mathbb{Q}\langle \gamma_{\mathrm{Max}} \rangle \oplus \mathbb{Q}\langle u^N \otimes \gamma_{\min} \rangle.$$

The morphism induced by a regular homotopy between two such Hamiltonians (built from standard Hamiltonians close to Morse Bott Hamiltonians) respects the filtration, thanks to proposition 1.4.5. We can therefore take the direct limit on the pages over those Hamiltonians which form a cofinal family. The inclusion $S^{2N+1} \hookrightarrow S^{2N+3}$ induces a map

$$E^{1;N} = \bigoplus_{\gamma \in \mathcal{P}(R_\alpha)} \mathbb{Q}\langle \gamma_{\mathrm{Max}} \rangle \oplus \mathbb{Q}\langle u^N \otimes \gamma_{\min} \rangle \to E^{1;N+1}.$$

which is the identity on the first factor and zero on the second factor. Taking the direct limit over the inclusion $S^{2N+1} \hookrightarrow S^{2N+3}$ we have

$$SH^{S^1,+}(W;\mathbb{Q}) = \varinjlim_N E^{1;N} = \bigoplus_{\gamma \in \mathcal{P}(R_\alpha)} \mathbb{Q}\langle \gamma \rangle.$$

Remark 2.2.3 Stricto sensu, in the above Theorem, we have only proven that

$$SH^{S^1,+}(W,\mathbb{Q}) = \bigoplus_{\gamma\in\widetilde{\mathcal{P}}(R_\alpha)} \mathbb{Q}\langle\gamma\rangle$$

where $\widetilde{\mathcal{P}}(R_\alpha)$ are the periodic Reeb orbits contractible in the Liouville domain W. Nonetheless Theorem 2.2.2 is true after extending the definition of $SH^{S^1,+}(H)$ to all 1-periodic orbits of H. This is done in section 2.2.2.

An orbit γ_H is bad if the underlying Reeb orbit is bad.

Proposition 2.2.4 *There are no bad orbits in the generators of the S^1-equivariant symplectic homology.*

Proof: In the spectral sequence, as above, the twin tower over a bad orbit is as follows:

$$\begin{tikzcd}
u^N \otimes \gamma_{\mathrm{Max}} \arrow[dr, "0"] & u^N \otimes \gamma_{\mathrm{min}} \arrow[l, "\times(\pm 2)"'] \\
\vdots & \vdots \\
u^2 \otimes \gamma_{\mathrm{Max}} \arrow[dr, "0"] & u^2 \otimes \gamma_{\mathrm{min}} \arrow[l, "\times(\pm 2)"'] \\
u \otimes \gamma_{\mathrm{Max}} \arrow[dr, "0"] & u \otimes \gamma_{\mathrm{min}} \arrow[l, "\times(\pm 2)"'] \\
1 \otimes \gamma_{\mathrm{Max}} & 1 \otimes \gamma_{\mathrm{min}} \arrow[l, "\times(\pm 2)"']
\end{tikzcd}$$

So the complex is:

$$\mathbb{Q} \xrightarrow{\times(\pm 2)} \mathbb{Q} \xrightarrow{0} \ldots \xrightarrow{0} \mathbb{Q} \xrightarrow{\times(\pm 2)} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{\times(\pm 2)} \mathbb{Q}\,.$$

Therefore, on the first page of the spectral sequence, $E^{1;N}_{p,q} = 0$.

Corollary 2.2.5 *The only generators that may appear in the positive S^1-equivariant homology are of the form $u^0 \otimes \gamma_{\mathrm{Max}}$ with γ_{Max} a good orbit.*

Corollary 2.2.6 *The number of good periodic Reeb orbits of periods $\leq T$ is bounded below by the rank of the positive S^1-equivariant symplectic homology of action $\leq T$.*

Theorem 2.2.2 establishes a link between periodic orbits of the Reeb vector field of a contact form on M and the positive S^1-equivariant symplectic homology of an exact symplectic filling W of M (i.e. a compact symplectic manifold $(M, \omega = d\lambda)$ with contact type boundary such that $\partial W = M$, $\alpha = \lambda_{|M}$)). We shall study, in chapter 3, the invariance of positive S^1-equivariant symplectic homology. The idea is to use this homology to get information on Reeb orbits on some contact manifolds.

2.2.1 The example of Brieskorn spheres

The Brieskorn manifold $\Sigma(a_0, \dots, a_n)$, with all $a_i \geq 2$ positive integers is defined as the intersection of the singular hypersurface $\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \dots + z_n^{a_n} = 0\}$ in $\mathbb{C}^{n+1}$ with the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$. It is a smooth $2n-1$-dimensional manifold which admits a contact form

$$\alpha = \frac{i}{8} \sum_{j=0}^{n} a_j (z_j d\overline{z_j} - \overline{z_j} dz_j)$$

with corresponding Reeb vector field

$$R_\alpha = \left(\frac{4i}{a_0} z_0, \dots, \frac{4i}{a_n} z_n \right).$$

For any odd number $n = 2m + 1$ and any $p \equiv \pm 1 \mod 8$, the Brieskorn manifold $\Sigma(p, 2, \dots, 2)$ is diffeomorphic to the standard sphere S^{4m+1} [Bri66]. One defines the contact structures ξ_p on S^{4m+1} defined as the kernel of the contact form α_p with

$$\alpha_p := \frac{ip}{8}(z_0 d\overline{z_0} - \overline{z_0} dz_0) + \frac{i}{4} \sum_{j=1}^{n} (z_j d\overline{z_j} - \overline{z_j} dz_j).$$

The fact that the Brieskorn sheres are exactly fillable can be found, for instance, in the book of Geiges [Gei06].

Proposition 2.2.7 *For $p_1 \neq p_2$, the positive S^1 equivariant homologies of symplectic fillings of the Brieskorn spheres are different.*

PROOF: We consider the description of the chain complex for those homologies in terms of good periodic orbits of the Reeb vector field, graded by minus their Conley indices. We shall show that all Conley-Zehnder indices are even. To compute them, the first thing to do is to build an explicit perturbation of the contact form so that all periodic Reeb orbits are non degenerate. We proceed as in [Ust99]. For that one makes the change of coordinate

$$w_0 = z_0,\ w_1 = z_1 \begin{pmatrix} w_{2j} \\ w_{2j+1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} z_{2j} \\ z_{2j+1} \end{pmatrix},\ \text{for} j \geq 1.$$

In these coordinates

$$\Sigma(p,2,\dots 2) = \left\{ w \in \mathbb{C}^{n+1} \,\middle|\, w_0^p + w_1^2 + 2\sum_{j=1}^{m} w_{2j}w_{2j+1} = 0,\ |w|^2 = 1 \right\}.$$

Consider the real positive function $f : \Sigma(p,2,\dots 2) \to \mathbb{R}$ given by

$$f(w) = |w|^2 + \sum_{j=1}^{m} \epsilon_j \big(|w_{2j}|^2 - |w_{2j+1}|^2\big), \quad \text{where } 0 < \epsilon_j < 1.$$

The contact form $f\alpha$ defines the same contact structure on $\Sigma(p,2,\dots 2)$ as α and its associated Reeb vector field is given by

$$R_{f\alpha}(w) = \Big(\tfrac{4i}{p}w_0, 2iw_1, 2i(1+\epsilon_1)w_2, 2i(1-\epsilon_1)w_3, \dots, 2i(1+\epsilon_m)w_{n-1}, 2i(1-\epsilon_m)w_n\Big).$$

If all the ϵ_j are irrational and linearly independent over $\mathbb{Q}$, the only periodic orbits are

$$\begin{aligned}
\gamma_0(t) &= \Big(re^{\frac{4it}{p}}, ir^{\frac{p}{2}}e^{2it}, 0, \dots, 0\Big), \quad r > 0, r^p + r^2 = 1,\ 0 \le t \le p\pi;\\
\gamma_j^+(t) &= \Big(0, \dots, 0, \underbrace{e^{2it(1+\epsilon_j)}}_{2j}, 0, \dots, 0\Big), \quad 0 \le t \le \frac{\pi}{1+\epsilon_j},\ j = 1, \dots, m;\\
\gamma_j^-(t) &= \Big(0, \dots, 0, \underbrace{e^{2it(1-\epsilon_j)}}_{2j+1}, 0, \dots, 0\Big), \quad 0 \le t \le \frac{\pi}{1-\epsilon_j},\ j = 1, \dots, m
\end{aligned}$$

and all their iterates, γ_0^N, γ_j^{+N}, γ_j^{-N}, for all $N \ge 1$. Their Conley-Zehnder index is given by

$$\begin{aligned}
&\mu_{CZ}\big(\gamma_0^N\big) = 2Np(n-2) + 4N;\\
&\mu_{CZ}\Big(\gamma_j^{\pm N}\Big) = 2\left\lfloor \frac{2N}{p(1\pm\epsilon_j)} \right\rfloor + 2\left\lfloor \frac{N}{1\pm\epsilon_j} \right\rfloor + 2\sum_{\substack{k=1\\k\neq j}}^{m} \left(\left\lfloor \frac{N(1+\epsilon_k)}{1\pm\epsilon_j} \right\rfloor + \left\lfloor \frac{N(1-\epsilon_k)}{1\pm\epsilon_j} \right\rfloor \right) + n - 1.
\end{aligned}$$

All indices have the same parity, thus applying Theorem 2.2.2, theS^1-equivariant positive symplectic homologies are generated by the periodic orbits of the Reeb vector field graded by their Conley indices. If $p_1 \neq p_2$, those positive S^1-equivariant symplectic homologies are different as proven in [Ust99].

2.2.2 Homology with non contractible orbits

To deal with non contractible orbits, one chooses for any free homotopy class of loops a, a representative l_a and one chooses a trivialisation of the tangent space along that curve.

For the free homotopy class of contractible loop, l_0 is chosen to be constant loop with constant trivialisation. One ask moreover that $l_{a^{-1}}$ is l_a in the reverse order and with the corresponding trivialisation. One replaces assumption 1.1.1 of asphericity by an assumption of atoroidality namely for any $v : T^2 \to W$

$$\int_{T^2} v^\star \omega = 0.$$

We also replace assumption 1.1.2, asking that the first Chern class of the tangent bundle vanishes on all toruses inside W. The action functional induced by a Hamiltonian H becomes

$$\mathcal{A}(\gamma) := -\int_{[0,1]\times S^1} u^\star \omega - \int_{S^1} H(\theta, \gamma(\theta))d\theta$$

where $u : [0,1] \times S^1 \to W$ is a homotopy from l_a to γ.

For any loop γ belonging to the free homotopy class a, one chooses a homotopy $u : [0,1] \times S^1 \to W$ from l_a to γ and one considers the trivialisation of TW on γ induced by u and by the choice of the trivialisation along l_a. Let us observe that any Floer trajectory can only link two orbits in the same free homotopy class and as before, the action decreases along Floer trajectories.

As before, the Floer complex is generated by the 1-periodic orbits of H graded by minus their Conley-Zehnder index. The differential "counts" Floer trajectories between two orbits whose difference of grading is 1.

The positive version of symplectic homology is defined as before since the set of critical points of H is still a subcomplex : Floer trajectories can only link a critical point to a contractible orbit.

All the results stated above extend to this framework.

3 Structural properties of symplectic homology

3.1 Transfer morphism

In this section, we prove that symplectic homology, positive symplectic homology, S^1-equivariant symplectic homology and positive S^1-equivariant symplectic homology are functors (reversing the arrows) defined on the category where objects are Liouville domains, and morphisms are embeddings. Precisely, we construct a morphism between the (S^1- equivariant positive) symplectic homologies when one Liouville domain is embedded in another one, and we show that those morphisms compose nicely. Such a morphism, called a transfer morphism, has been studied by Viterbo [Vit99] in the case of the symplectic homology. We adapt his construction to extend it to all the variants of the symplectic homology considered above.

Recall definition 2.0.9: A *Liouville domain* (W, λ) is a compact manifold W with boundary $\partial W = M$, together with a 1-form λ such that $\omega := d\lambda$ is symplectic and such that the Liouville vector field X defined by $\iota(X)\omega = \lambda$ points strictly outwards along ∂W. We still assume that $\langle c_1(TW), \pi_2(W)\rangle = 0$. We have defined the (symplectic) completion of a compact symplectic manifold with contact type boundary in section 1.2.1. We consider the completion

$$\widehat{W} = W \cup (\partial W \times \mathbb{R}^+)$$

of a Liouville domain (W, λ), built from the flow of the Liouville vector field X. We denote by $\widehat{\lambda}$ the 1-form on $\widehat{W}$ defined by λ on W and by $e^\rho\alpha$ on $\partial W \times \mathbb{R}^+$ with $\alpha := \lambda_{|\partial V}$. We shall denote by $SH^\dagger(W, \lambda)$ the symplectic homology $SH^\dagger(W, d\lambda, X)$.

Definition 3.1.1 Let (V, λ_V) and (W, λ_W) be two Liouville domains. A *Liouville embedding* $j : (V, \lambda_V) \to (W, \lambda_W)$ is a symplectic embedding $j : V \to W$ with V and W of codimension 0 such that $j^\star\lambda_W = \lambda_V$. (One can consider, more generally, a symplectic embedding j of codimension 0 such that λ_W coincides in a neighbourhood of $j(\partial V)$ in W with $\widehat{\lambda_V} + df$.)

The goal of this section is to adapt Viterbo's definition of transfer morphisms between symplectic homology [Vit99] so that it extends to

$$SH^{\dagger}(W,\lambda_W) \to SH^{\dagger}(V,\lambda_V)$$

with $\dagger = +, S^1$ or $(S^1,+)$.

We first present the construction for symplectic and positive symplectic homology. The idea, as in [Vit99], is to use increasing homotopies between $H_1 : S^1 \times \widehat{W} \to \mathbb{R} \in \mathcal{H}_{\text{std}}(W)$ and an $H_2 : S^1 \times \widehat{W} \to \mathbb{R}$ in a special class $\mathcal{H}_{stair}(V,W)$.

Let U be a neighbourhood of ∂V in $W \setminus \mathring{V}$ so that (U,ω_W) is symplectomorphic to $\big(\partial V \times [0,\delta], d(e^{\rho}\alpha_V)\big)$.

Definition 3.1.2 A Hamiltonian $H_2 : S^1 \times \widehat{W} \to \mathbb{R}$ is in $\mathcal{H}_{stair}(V,W)$ if and only if

- on $S^1 \times V$, H_2 is negative and C^2-small ;
- on $S^1 \times U \cong S^1 \times \partial V \times [0,\delta]$, with ρ the last coordinate, H_2 is of the following form

  -there exists $0 < \rho_0 \ll \delta$ such that $H_2(\theta,p,\rho) = \beta e^{\rho} + \beta'$ for $\rho_0 \leq \rho \leq \delta - \rho_0$, with $0 < \beta \notin \operatorname{Spec}(\partial V,\alpha) \cup \operatorname{Spec}(\partial W,\alpha)$ and $\beta' \in \mathbb{R}$;

  -$H_2(\theta,p,\rho)$ is C^2-close on $S^1 \times \partial V \times [0,\rho_0]$ to a convex increasing function of e^{ρ} which is independent of θ and p;

  -$H_2(\theta,p,\rho)$ is C^2-close on $S^1 \times \partial V \times [\delta-\rho_0,\delta]$ to a concave increasing function of e^{ρ} which is independent of θ and p;
- on $S^1 \times W \setminus (V \cup U)$, H_2 is C^2-close to a constant ;
- on $S^1 \times \partial W \times [0,+\infty[$, with ρ' the $\mathbb{R}^+$ coordinate on $\partial W \times \mathbb{R}^+$, H_2 is of the following form

  -there exists $\rho'_1 > 0$ such that $H_2(\theta,p,\rho') = \mu e^{\rho'} + \mu'$ for $\rho' \geq \rho'_1$, with $0 < \mu \notin \operatorname{Spec}(\partial V,\alpha) \cup \operatorname{Spec}(\partial W,\alpha)$, $\mu \leq \beta$, $\mu' \in \mathbb{R}$;

  -$H_2(\theta,p,\rho')$ is C^2-close on $S^1 \times \partial W \times]0,\rho'_1]$ to a concave increasing function of $e^{\rho'}$ which is independent of θ and p;
- all 1-periodic orbits of $X^{\theta}_{H_2}$ are non-degenerate, i.e the Poincaré return map has no eigenvalue equal to 1.

A representation of H_2 is given in Figure 3.1.

The 1-periodic orbits of H_2 lie either in the interior $\mathring{V}$ (which we call region I), either in $\partial V \times [0,\rho_0]$ (region II), either in $\partial V \times [\delta-\rho_0,\delta]$ (region III), either in $W \setminus (V \cup U)$ (region IV) or in $\partial W \times [0,\rho_1]$ (region V). We consider their action using the following obvious lemma:

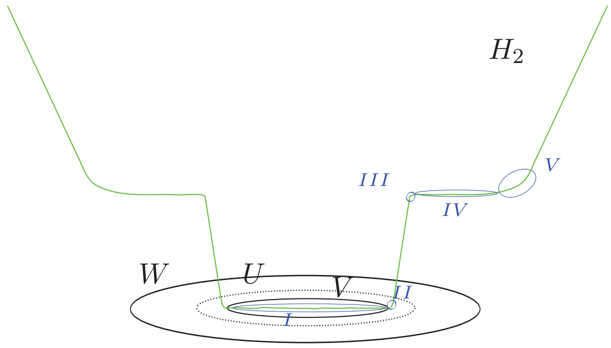

Figure 3.1: Example of H_2 on $\widehat{W}$

Lemma 3.1.3 *Let H and $\widetilde{H}$ be two C^2-close Hamiltonians and let $\gamma \in \mathcal{P}(H)$ and $\widetilde{\gamma} \in \mathcal{P}(\widetilde{H})$ be C^2-close. Then*

$$\mathcal{A}(\gamma) \text{ is close to } \mathcal{A}(\widetilde{\gamma})$$

PROOF:

$$\begin{aligned}
|\mathcal{A}_H(\gamma) - \mathcal{A}_{\widetilde{H}}(\widetilde{\gamma})| &= \Big| \int_{S^1} \widetilde{\gamma}^\star \lambda + \int_{S^1} \widetilde{H}\big(\theta, \widetilde{\gamma}(\theta)\big) - \int_{S^1} \gamma^\star \lambda - \int_{S^1} H\big(\theta, \gamma(\theta)\big) \Big| \\
&\leq \int_{S^1} \Big| \lambda_{\widetilde{\gamma}(\theta)}\big(\dot{\widetilde{\gamma}}(\theta)\big) - \lambda_{\gamma(\theta)}\big(\dot{\gamma}(\theta)\big) \Big| d\theta \\
&\quad + \int_{S^1} \Big| \widetilde{H}\big(\theta, \widetilde{\gamma}(\theta)\big) - H\big(\theta, \gamma(\theta)\big) \Big| d\theta \\
&\leq \epsilon
\end{aligned}$$

I In region I, there are only critical points so the action of the critical point q is non negative and small ($< \epsilon$).

II In region II, H_2 is C^2-close to a convex function $H = h(r)$ (with $r = e^\rho$); since

$$dH = h'(r)dr = \iota(X_H)\omega_W = \iota(X_H)d(r\alpha_V) = \iota(X_H)(dr \wedge \alpha_V + rd\alpha_V),$$

we have $X_H = -h'(r)R_{\alpha_V}$ where R_{α_V} is the Reeb vector field on ∂V associated to the contact form $\alpha_V = \lambda_{V|_{\partial V}}$. So an orbit of X_H lies on a constant level for r and its action is given by:

$$\begin{aligned}\mathcal{A}(\gamma) &= -\int_{S^1} \gamma^\star(r\alpha_V) - \int_{S^1} H\big(\gamma(\theta)\big)d\theta \\ &= -\int_{S^1} (r\alpha_V)_{\gamma(\theta)}\big(\dot{\gamma}(\theta)\big) - \int_{S^1} h(r)d\theta \\ &= -\int_{S^1} r\alpha_V\big(-h'(r)R_{\alpha_V}\big) - h(r) \\ &= h'(r)r - h(r).\end{aligned}$$

Since ρ_0 is small we have $e^{\rho_0} \sim 1$ and $h(e^{\rho_0}) \sim 0$, so the actions of 1-periodic orbits of H_2 in this region are close to the periods of closed orbits of the Reeb vector field on the boundary of V of periods $T < \beta$ and they are greater than ϵ.

III In region III, the computation is similar to the case of region II:

$$\mathcal{A}(\gamma_{H_2}) \text{ is close to } h'(r)r - h(r) \text{ which is close to } e^\delta(T-\beta) < 0$$

hence the actions of 1-periodic orbits of H_2 in this region are negative.

IV In region IV, there are only critical points so the action of the critical point q is given by $-H_2(q)$ which is close to $-e^\delta\beta$.

V In region V, the computation of the action is similar to the case of region II:

$$\mathcal{A}(\gamma) \text{ is close to } h'(r)r - h(r) \qquad \text{with } r = e^{\rho'}.$$

Observe that here the 1-periodic orbits are close to 1-periodic orbits of $-h'(r)R_{\alpha_W}$ where now R_{α_W} is the Reeb vector field on ∂W. The action of any 1-periodic orbit of H_2 in this region is close to $e^{\rho'}T' - h(e^{\rho'})$ where T' is the period of a closed orbit of the Reeb vector field on the boundary of W with $T' < \mu < \beta$ and where $h(e^{\rho'}) > e^\delta\beta$.

So, for nice parameters (for instance $\rho_1' < \delta$), we have

$$\mathcal{A}(IV) < \mathcal{A}(III), \mathcal{A}(V) < 0 < \mathcal{A}(I) < \epsilon < \mathcal{A}(II).$$

We denote by $C^{IV,III,V,I}(H_2,J)$ the subcomplex of the Floer complex for H_2 generated by critical orbits lying in regions IV, III, V, and I and by $C^{IV,III,V}(H_2,J)$ the subcomplex of the Floer complex for H_2 generated by critical orbits lying in regions IV, III and V. Observe that $C^{IV,III,V,I}(H_2,J)$ coincides with the subcomplex generated by 1-periodic orbits of action $\leq \epsilon$ and $C^{IV,III,V}(H_2,J)$ coincides with subcomplex generated by 1-periodic orbits of action $\leq -\eta$ (for a well chosen small positive η). With similar notations, we have the identifications:

$$C^{I,II}(H_2,J) = C^{IV,III,V,I,II}(H_2,J)/_{C^{IV,III,V}(H_2,J)} = SC(H_2,J),\partial)/_{SC^{\leq -\eta}(H_2,J),\partial)},$$

$$C^{II}(H_2,J) = C^{IV,III,V,I,II}(H_2,J)/_{C^{IV,III,V,I}(H_2,J)} = SC(H_2,J),\partial)/_{SC^{\leq\epsilon}(H_2,J),\partial)}.$$

Since the action decreases along Floer trajectories, the Floer differential passes to the quotient where we still denote it ∂. Remark that the function H_2 is not in $\mathcal{H}_{\mathrm{std}}(V)$. We want to relate the homology of $\big(C^{I,II}(H_2,J),\partial\big)$ to the homology of a function in $\mathcal{H}_{\mathrm{std}}(V)$.

Definition 3.1.4 Let $H_2 \in \mathcal{H}_{stair}(V,W)$; we denote by β the slope of the linear part close to ∂V, as in Definition 3.1.2. The associated function $H = \iota_V(H_2) \in \mathcal{H}_{\mathrm{std}}(V)$, defined on $S^1 \times \widehat{V}$, is the function which coincides with H_2 on $V \cup (\partial V \times [0, \delta - \rho_0])$ and which is linear with slope β "further" in the completion: $H(\theta, e^\rho) = \beta\rho + \beta'$ for all $\rho \geq \delta - \rho_0$.

Proposition 3.1.5 *Let H_2 be an function in $\mathcal{H}_{stair}$ and let $H = \iota_V H_2$ be the associated function in $\mathcal{H}_{std}(V)$ as defined above. We assume furthermore that the Hamiltonians are generic in the sense that the homologies are well-defined for a good choice of J's. Then*

$$H\big(C^{I,II}(H_2,J),\partial\big) = H\big(SC(H,J)\big) \qquad \textit{and} \qquad H\big(C^{II}(H_2,J),\partial\big) = H\big(SC^+(H,J)\big).$$

Proof: We need to check that there is no Floer trajectory $u : \mathbb{R} \times S^1 \to \widehat{W}$ going from an orbit in $C^{I,II}$ (resp. C^{II}) to an orbit in $C^{I,II}$ (resp. C^{II}) with points in $\widehat{W} \setminus (U \cup V)$. We prove it by contradiction, as a direct application of Abouzaid maximum principle which we prove below as theorem 3.1.6. Assume that $u : \mathbb{R} \times S^1 \to \widehat{W}$ is a Floer trajectory whose image intersects $\widehat{W} \setminus (U \cup V)$. We consider the intersection of the image with a slice $\partial V \times \{\rho\}$ for any $\rho_0 < \rho < \delta - \rho_0$ and we choose a regular value $\rho_0 + \epsilon$ of $\rho \circ u$. The manifold $W' := \widehat{W} \setminus (V \cup (\partial V \times [0, \rho_0 + \epsilon[))$ is symplectic with contact type with boundary $\partial V \times \{\rho_0 + \epsilon\}$ and Liouville vector field pointing inwards. Let S be the inverse image of W' under the map u; it is a compact Riemann surface with boundary ; the complex structure j is the restriction to S of the complex structure j on the cylinder defined by $j(\partial_s) = \partial_\theta$. We define β to be the restriction of $d\theta$ to S. The fact that u is a Floer trajectory is equivalent to $(du - X_H \otimes \beta)^{0,1} := \frac{1}{2}\left((du - X_H \otimes \beta) + J(du - X_H \otimes \beta)j\right) = 0$, where du is the differential of the map u viewed as a section of $T^*S \otimes u^*TW'$. Then theorem 3.1.6, which is slight generalisation of a theorem of Abouzaid, concludes.

Theorem 3.1.6 (Abouzaid, [Rit13]) *Let $(W', \omega' = d\lambda')$ be an exact symplectic manifold with contact type boundary $\partial W'$, such that the Liouville vector field points inwards. Let ρ be the coordinate near $\partial W'$ defined by the flow of the Liouville vector field starting from the boundary and let $r := e^\rho$; near the boundary the symplectic form writes $\omega' = d(r\alpha)$ with α the contact form on $\partial W'$ given by the restriction of λ'. Let J be a compatible almost complex structure such that $J^*\lambda' = dr$ on the boundary. Let $H : W' \to \mathbb{R}$ be non negative, and such that $H = h(r)$ where h is a convex increasing function near the boundary. Let S be a compact Riemann surface with boundary and let β be a 1-form such that $d\beta \geq 0$.*

Then any solution $u : S \to W'$ of $(du - X_H \otimes \beta)^{0,1} = 0$ with $u(\partial S) \subset \partial W'$ is entirely contained in $\partial W'$.

Proof: The energy of a map $u : S \to W'$ is defined as

$$E(u) := \frac{1}{2}\int_S \|du - X_H \otimes \beta\|^2 vol_S$$

where du is viewed as a section of $T^*S \otimes u^*TW'$. If $s + it$ is a local holomorphic coordinate on S, so that $j(\partial_s) = \partial_t$ and $vol_S = ds \wedge dt$ we have

$$\begin{aligned}\frac{1}{2}\|du - X_H \otimes \beta\|^2 vol_S &= \omega'\left(\partial_s u - X_H\beta(\partial_s), \partial_t u - X_H\beta(\partial_t)\right) ds \wedge dt \\ &= \big(\omega'(\partial_s u, \partial_t u) - dH(\partial_t u)\beta(\partial_s) + dH(\partial_s u)\beta(\partial_t)\big)\, ds \wedge dt \\ &= u^*\omega' + u^*(dH) \wedge \beta.\end{aligned}$$

It is obviously non negative for any path. Since $d(u^\star H\beta) = u^\star(dH) \wedge \beta + \underbrace{u^\star H d\beta}_{\geq 0}$, we have

$$\begin{aligned}E(u) = \int_S u^\star d\lambda' + u^\star(dH)\wedge\beta \leq \int_S d(u^\star\lambda') + d(u^\star H\beta) \leq \int_{\partial S} u^\star\lambda' - \lambda'(X_H)\beta \\ \text{using Stokes's theorem and } H = h(r) \leq rh'(r) = r\alpha\big(h'(r)R_\alpha\big) = -\lambda'(X_H) \text{ on } u(\partial S) \subset \partial V \\ = \int_{\partial S} \lambda'(du - X_H \otimes \beta) = \int_{\partial S} -\lambda' J(du - X_H \otimes \beta) j \quad \text{since } (du - X_H \otimes \beta)^{0,1} = 0 \\ = \int_{\partial S} -dr(du - X_H \otimes \beta)j \quad \text{since } J^*\lambda' = dr \text{ along } u(\partial S) \subset \partial W' \\ = \int_{\partial S} -dr\, du\, j \quad \text{since } dr \text{ vanishes on } X_H \text{ on } u(\partial S) \subset \partial W'\end{aligned}$$

Let ν be the outward normal direction along ∂S. Then $(\nu, j\nu)$ is an oriented frame, so ∂S is oriented by $j\nu$. Now $dr(du)j(j\nu) = d(r \circ u)(-\nu) \geq 0$ since in the inward direction, $-\nu$, $r \circ u$ can only increase because r is minimum on $\partial W'$. So $E(u) \leq 0$ hence $E(u) = 0$. This implies that $du - X_H \otimes \beta = 0$ which shows that the image of du is in the span of X_H which is the span of $R_\alpha \in T\partial W'$ on $\partial W'$. Hence the image of u is entirely in contained in $\partial W'$.

For any element $H_1 \in \mathcal{H}_{\mathrm{std}}(W)$, one can consider an element in $H_2 \in \mathcal{H}_{stair}(V, W)$ such that H_1 and H_2 coincide "far in the completion", i.e. on $\partial W \times [\rho_2', +\infty[\subset \widehat{W}$. Let $H = \iota_V(H_2) \in \mathcal{H}_{\mathrm{std}}(V)$. We want to build a morphism from the homology defined by H_1 to the homology defined by H. We shall first construct a morphism in the homology defined by H_2. With $H_1 \in \mathcal{H}_{\mathrm{std}}(W)$ and $H_2 \in \mathcal{H}_{stair}(V, W)$ as above, we can consider an increasing homotopy $H_s, s \in \mathbb{R}$, between H_1 and H_2, i.e $\frac{d}{ds}H_s \geq 0$, with the property that there exists s_0 such that $H_s \equiv H_1$ for $s \leq -s_0$ and $H_s \equiv H_2$ for $s \geq s_0$. We define a morphism $SC(H_1, J_1) \to SC(H_2, J_2)$ by counting Floer trajectories for the homotopy.

Denote by $\mathcal{M}(\gamma_1, \gamma_2, H_s, J_s)$ the space of Floer trajectories from γ_1 to γ_2 i.e maps $u : \mathbb{R} \times S^1 \to \widehat{W}$ such that:

$$\bar{\partial}_{J_s,H_s}(u) := \partial_s u + J_s^\theta \circ u(\partial_\theta u - X_{H_s}^\theta \circ u) = 0 \tag{3.1}$$

$$\lim_{s\to-\infty} u(s,\cdot) = \gamma_1(\cdot) \qquad \text{and} \qquad \lim_{s\to\infty} u(s,\cdot) = \gamma_2(\cdot).$$

Again it is proven in [Oan08, FHS95] that for a generic choice of the pair (H_s, J_s), the spaces $\mathcal{M}(\gamma_1, \gamma_2, H_s, J_s)$ are manifolds of dimension $\mu_{CZ}(\gamma_2) - \mu_{CZ}(\gamma_1)$ for any γ_1 in $\mathcal{P}(H_1)$ and γ_2 in $\mathcal{P}(H_2)$. Let us observe that there is no general $\mathbb{R}$-action on this space.

The homotopy H_s gives rise to a morphism

$$\begin{aligned} \phi_{H_s} : SC(H_1, J_1) &\to SC(H_2, J_2) \\ \gamma_1 &\mapsto \sum_{\substack{\gamma_2 \in \mathcal{P}(H_2) \\ \mu_{CZ}(\gamma_2) = \mu_{CZ}(\gamma_1)}} \#\mathcal{M}(\gamma_1, \gamma_2, H_s, J_s)\gamma_2 \end{aligned}$$

where the count involves, as always, signs.

Proposition 3.1.7 *The morphism ϕ_{H_s} is a chain map.*

Proof: As before this follows from the study of the boundary of a space of Floer trajectories. Let $\gamma_1 \in \mathcal{P}(H_1)$ and $\gamma_2 \in \mathcal{P}(H_2)$ be such that $\mu_{CZ}(\gamma_1) = \mu_{CZ}(\gamma_2) + 1$. The 1-dimensional manifold $\mathcal{M}(\gamma_1, \gamma_2, H_s, J_s)$ has the following boundary

$$\cup_{\gamma \in \mathcal{P}(H_1)} \mathcal{M}(\gamma_1, \gamma, H_1, J_1) \times \mathcal{M}(\gamma, \gamma_2, H_s, J_s) \bigcup \cup_{\widetilde{\gamma} \in \mathcal{P}(H_2)} \mathcal{M}(\gamma_1, \widetilde{\gamma}, H_s, J_s) \times \mathcal{M}(\widetilde{\gamma}, \gamma_2, H_2, J_2).$$

The first part yields the coefficient of γ_2 in $\phi_{H_s} \circ \partial_{H_1}(\gamma_1)$ and the second part corresponds to $\partial_{H_2} \circ \phi_{H_s}(\gamma_1)$.

So ϕ_{H_s} induces a morphism in homology, still denoted by ϕ_{H_s}

$$\phi_{H_s} : SH(H_1, J) \to SH(H_2, J).$$

The fact that ϕ_{H_s} is independent of the choice of the homotopy is a consequence of the homotopy of homotopies theorem (section 1.2.3). We denote it by ϕ_{H_1,H_2}.

Definition 3.1.8 Given an element H_1 in $\mathcal{H}_{\mathrm{std}}(W)$, consider an element $H_2 \in \mathcal{H}_{stair}(V, W)$ such that H_1 and H_2 coincide "far in the completion", and let $H = \iota_V(H_2) \in \mathcal{H}_{\mathrm{std}}(V)$. We define the *transfer morphism*

$$SH(H_1, J) \to SH(H, J') = SH(H_2, J), \partial)/_{SH^{\leq -\eta}(H_2, J), \partial)} = H\big(C^{I,II}(H_2, J), \partial\big)$$

which is the composition of ϕ_{H_1,H_2} followed by the natural projection.The action decreases along Floer trajectories, so this maps $SH^{\leq\epsilon}(H_1,J)$ to $SH^{\leq\epsilon}(H_2,J),\partial)/_{SH^{\leq-\eta}(H_2,J),\partial)}$ and induces a *transfer morphism* for the positive homology

$$SH^+(H_1,J) \to SH^+(H,J') = SH^{\leq\epsilon}(H_2,J),\partial)/_{SH^{\leq-\eta}(H_2,J),\partial)} = H\big(C^{II}(H_2,J),\partial\big).$$

With our identification, the map is obtained by counting solutions of

$$\frac{\partial u}{\partial s} + J_s\circ u\Big(\frac{\partial u}{\partial\theta} - X_{H_s}\circ u\Big) = 0$$

going from a 1-periodic orbit of X_{H_1} to a 1-periodic orbit of X_{H_2} lying in region I or II.

The homotopy of homotopies theorem shows that the map does not depend on the choice of stair function H_2 such that $\iota_V H_2 = H$ and such that H_1 and H_2 coincide far in the completion; we shall denote it $\phi^H_{H_1}$. It also shows that the map ϕ_{H_1,H_2} commutes with continuation, i.e if $\rho_1 : SH(H_1) \to SH(H_1')$ is a continuation for H_1 and $\rho_2 : SH(H_2) \to SH(H_2')$ is a continuation for H_2 then

$$\phi_{H_1',H_2'}\circ\rho_1 = \rho_2\circ\phi_{H_1,H_2}.$$

Proposition 3.1.9 *The transfer map $\phi^H_{H_1} : SH(H_1,J) \to SH(H,J')$ commutes with continuations.*

Proof: To show this, we still have to show that a continuation map built in W from $SH(H_2,J)$ to $SH(H_2',J')$, defined by an increasing homotopy $H_s : S^1\times\widehat{W}\to\mathbb{R}$, induces a continuation map in V from $SH(H=\iota_V(H_2),J)$ to $SH(H'=\iota_V(H_2'),J')$. For this, it is enough to check that there is no Floer trajectory corresponding to the homotopy, i.e. $u:\mathbb{R}\times S^1\to\widehat{W}$ solution of

$$\frac{\partial u}{\partial s} + J_s\circ u\Big(\frac{\partial u}{\partial\theta} - X_{H_s}\circ u\Big) = 0,$$

going from an orbit in $C^{I,II}(H_2,J)$ (resp. $C^{II}(H_2,J)$) to an orbit in $C^{I,II}(H_2',J')$ (resp. $C^{II}(H_2',J')$) with points in $\widehat{W}\setminus(U\cup V)$. We prove it by contradiction, proceeding as in the proof of Proposition 3.1.5, using a generalized Abouzaid maximum principle which we prove below as proposition 3.1.10. Assume that $u:\mathbb{R}\times S^1\to\widehat{W}$ is a Floer trajectory whose image intersects $\widehat{W}\setminus(U\cup V)$. We consider the intersection of the image with a slice $\partial V\times\{\rho\}$ for any $\rho_0<\rho<\delta-\rho_0$ and we choose a regular value $\rho_0+\epsilon$ of $\rho\circ u$. The manifold $W' := \widehat{W}\setminus(V\cup(\partial V\times[0,\rho_0+\epsilon[))$ is symplectic with contact type with boundary $\partial V\times\{\rho_0+\epsilon\}$ and the Liouville vector field pointing inwards. Let S be the inverse image of W' under the map u; it is a compact Riemann surface embedded in the cylinder with boundary ; the complex structure j is the restriction to S of the complex structure j on the cylinder defined by $j(\partial_s) = \partial_\theta$. The fact that u is a Floer trajectory is equivalent to $(du - X_{H_s} \otimes d\theta)^{0,1} := \frac{1}{2}\left((du - X_{H_s} \otimes d\theta) + J(du - X_{H_s} \otimes d\theta)j\right) = 0$, where du is the differential of the map u viewed as a section of $T^*S \otimes u^*TW'$. Then Proposition 3.1.10 concludes.

Proposition 3.1.10 *Let $(W', \omega' = d\lambda')$ be an exact symplectic manifold with contact type boundary $\partial W'$, such that the Liouville vector field points inwards. Let ρ be the coordinate near $\partial W'$ defined by the flow of the Liouville vector field starting from the boundary and let $r := e^\rho$. Let J be a compatible almost complex structure such that $J^*\lambda' = dr$ on the boundary. Let $H : \mathbb{R} \times S^1 \times W' \to \mathbb{R}$ be an increasing homotopy, such that $H(s, \theta, p, \rho) = H_s^\theta(p, \rho) = h_s(r)$ where h_s are convex increasing functions near the boundary. Let S be a compact Riemann surface with boundary embedded in the cylinder. Then any solution $u : S \to W'$ of $(du - X_{H_s} \otimes d\theta)^{0,1} = 0$ with $u(\partial S) \subset \partial W'$ is entirely contained in $\partial W'$.*

Proof: The proof starts as in Theorem 3.1.6. The energy of u is non negative and given by

$$E(u) := \frac{1}{2}\int_S \|du - X_{H_s} \otimes d\theta\|^2 vol_S = \int_S u^*\omega' + u^*(dH_s^\theta) \wedge d\theta.$$

We have $u^\star(dH_s^\theta) \wedge d\theta = d(u'^\star H) \wedge d\theta - \underbrace{u^\star\partial_s H_s^\theta ds \wedge d\theta}_{\geq 0}$, for $u' : S \to \mathbb{R} \times S^1 \times W'$ which maps an element $(\theta, s) \in S$ to the element $(s, \theta, u'(\theta, s))$. Hence

$$\begin{aligned}
E(u) &= \int_S u^\star d\lambda' + u^\star(dH) \wedge d\theta \\
&\leq \int_S d(u^\star\lambda') + d(u'^\star H d\theta) \leq \int_{\partial S} u^\star\lambda' - \lambda'(X_{H_s})d\theta \\
&\quad \text{using Stokes's theorem and } H = h_s(r) \leq r\alpha\big(h_s'(r)R_\alpha\big) = -\lambda'(X_{H_s}) \text{ on } u(\partial S) \subset \partial V \\
&= \int_{\partial S} \lambda'(du - X_{H_s} \otimes d\theta)
\end{aligned}$$

and the proof proceeds as in Theorem 3.1.6.

Corollary 3.1.11 *The transfer maps $\{\phi_{H_1}^H\}$induce a transfer map:*

$$\phi_{W,V} : SH(W, \lambda_W) \to SH(V, \lambda_V).$$

and, on the quotient, the morphism

$$\phi^+ = \phi_{W,V}^+ : SH^+(W, \lambda_W) \to SH^+(V, \lambda_V).$$

Theorem 3.1.12 (Composition) *Let $(V_1, \lambda_{V_1}) \subseteq (V_2, \lambda_{V_2}) \subseteq (V_3, \lambda_{V_3})$ be Liouville domains with Liouville embeddings. Then the following diagram commutes:*

$$\begin{tikzcd} SH^+(V_3, \lambda_{V_3}) \arrow[r, "\phi^+_{V_3,V_2}"] \arrow[rr, bend right, "\phi^+_{V_3,V_1}"'] & SH^+(V_2, \lambda_{V_2}) \arrow[r, "\phi^+_{V_2,V_1}"] & SH^+(V_1, \lambda_{V_1}) \end{tikzcd} \tag{3.2}$$

Proof: The proof results from the comparison of a count of Floer trajectories. On one hand, one counts Floer trajectories corresponding to an increasing homotopy H_{13}, going from a 1-periodic orbit of X_{H_1} for an admissible Hamiltonian H_1 on $S^1 \times \widehat{V_3}$ to the $C^{II,I}$ part of a stair Hamiltonian H_3 with two "steps". On the other hand, one counts trajectories relative to the composition of two increasing homotopies, H_{12} going from H_1 to H_2 (a stait hamiltonian with one step) and H_{23} going from H_2 to H_3. The property is a consequence of the composition of homotopies that we now present.

Let H_1, H_2, H_3 be three Hamiltonians on the completion of a symplectic manifold (W, ω) with contact type boundary such that there exist two increasing homotopies H_{12} from H_1 to H_2 and H_{23} from H_2 to H_3. We assume, as always here, that there exists s_0 such that $H_{12} \equiv H_1$ for $s \leq -s_0$ and $H_{12} \equiv H_2$ for $s \geq s_0$ and similarly for H_{23} . Denote, for $R \in \mathbb{R},\ R \geq s_0 >> 0$, by $H_{12}\#_R H_{23}$ the gluing of the two homotopies;

$$H_{12}\#_R H_{23} = \begin{cases} H_{12}(s+R,\cdot,\cdot) & s \leq 0 \\ H_{23}(s-R,\cdot,\cdot) & s \geq 0 \end{cases}$$

The almost complex structures J_{12} and J_{23} are glued similarly. We choose J_{12} and J_{23} such that the operators D^{12} and D^{23}, which linearize the equations (3.1) on the Banach tangent space to a Floer trajectory, are surjective at any solution. By the operator gluing lemma (cfr section 1.1.1), the operator $D^{12}\#_R D^{23}$, corresponding to the linearisation of the Floer equation for $H_{12}\#_R H_{23}$, is surjective and the index of $D^{12}\#_R D^{23}$, is the sum of the indices of D^{12} and D^{23}. Hence the space of Floer trajectories $\mathcal{M}(\gamma_1, \gamma_3, H_{12}\#_R H_{23}, J_{12}\#_R J_{23})$, for $\gamma_1 \in \mathcal{P}(H_1)$ and $\gamma_3 \in \mathcal{P}(H_3)$, is a smooth manifold of dimension $\mu_{CZ}(\gamma_1) - \mu_{CZ}(\gamma_3)$.

Theorem 3.1.13 *Let $\gamma_1 \in \mathcal{P}(H_1)$ and $\gamma_3 \in \mathcal{P}(H_3)$ such that $\mu_{CZ}(\gamma_1) = \mu_{CZ}(\gamma_3)$. Then for R' large enough,*

$$\mathcal{M}(\gamma_1, \gamma_3, H_{12}\#_{R'} H_{23}, J_{12}\#_{R'} J_{23}) = \bigcup_{\gamma_2 \in \mathcal{P}(H_2)} \mathcal{M}(\gamma_1, \gamma_2, H_{12}, J_{12}) \times \mathcal{M}(\gamma_2, \gamma_3, H_{23}, J_{23})$$

Proof: Let us consider the 1-dimensional manifold defined as the union of 0-dimensional manifolds:

$$\bigsqcup_{R \geq R' \geq s_0} \mathcal{M}(\gamma_1, \gamma_3, H_{12}\#_R H_{23}, J_{12}\#_R J_{23}).$$

Its boundary consists of $\mathcal{M}(\gamma_1, \gamma_3, H_{12}\#_{R'}H_{23}, J_{12}\#_{R'}J_{23})$ and of broken trajectories. It can not include

$$\bigsqcup_{\gamma\in\mathcal{P}(H_1)} \mathcal{M}(\gamma_1,\gamma,H_1,J_1)\times\mathcal{M}(\gamma,\gamma_3,H_{12}\#_R H_{23},J_{12}\#_R J_{23}).$$

since the elements of $\mathcal{M}(\gamma_1,\gamma,H_1,J_1)$ have index at least equal to 1 and the elements in $\mathcal{M}(\gamma,\gamma_3,H_{12}\#_R H_{23},J_{12}\#_R J_{23})$ are of index at least 0. Thus this boundary is just

$$\mathcal{M}(\gamma_1,\gamma_3,H_{12}\#_{R'}H_{23},J_{12}\#_{R'}J_{23})\bigcup\mathcal{M}(\gamma_1,\gamma_2,H_{12},J_{12})\times\mathcal{M}(\gamma_2,\gamma_3,H_{23},J_{23}).$$

The conclusion follows.

3.1.1 Transfer morphism for S^1-equivariant symplectic homology

We extend the definition of the transfer morphisms of the previous section to S^1-equivariant and positive S^1-equivariant symplectic homology.
We consider two embedded Liouville domains $(V,\lambda_V)\subset(W,\lambda_W)$ and we want to define a morphism $SH^{S^1}(W,\lambda_W)\to SH^{S^1}(V,\lambda_V)$. We use the alternative definition of the S^1 equivariant symplectic homology, considering the cofinal family of Hamiltonians which are built as in section 2.2 : starting with autonomous Hamiltonians H in $\mathcal{H}_{\mathrm{std}}$, we do small Morse Bott type deformations H_δ as presented in section 2.1.3 and then lift those to S^1-equivariant functions H_δ^N as in section 2.1.1. In this setting, the S^1-equivariant symplectic homology can be computed by a simplified complex as described in proposition 2.1.1:

$$\widehat{SC}_*^{S^1}(H_\delta):=\mathbb{Z}[u]\otimes_{\mathbb{Z}}SC_*(H_\delta)$$

with differential

$$\widehat{\partial}^{S^1}=\varphi_0+u^{-1}\varphi_1+u^{-2}\varphi_2+\dots$$

where the maps φ_j counts Floer trajectories for parametrized Hamiltonians

$$\begin{cases}\partial_s u+J^\theta_{z(s)}\circ u\big(\partial_\theta u-X_{H^N_{\delta,z(s)}}\circ u\big)=0\\ \dot z-\vec{\nabla}\tilde f(z)=0\end{cases}$$

going from $S^1\cdot(\gamma^-,z_j)$ to $S^1\cdot(\gamma^+,z_0)$ with z_j the critical point of f of index $-2j$.

We have seen in section 2.2 that the action of the element represented by $u^k\otimes\gamma$ is very close to the action of γ. To define transfer morphisms, we start with an autonomous Hamiltonian H_1 in $\mathcal{H}_{\mathrm{std}}(W)$ and an autonomous H_2 in $\mathcal{H}_{stair}(W)$, and we do small Morse Bott type deformations $H_{1\delta}$ et $H_{2\delta}$. We define as in the previous section the subcomplex $\mathbb{Z}[u]\otimes_{\mathbb{Z}}(C^{III,IV,V}(H_{2\delta}))$ corresponding to points with negative action and we identify the quotient $\mathbb{Z}[u]\otimes_{\mathbb{Z}}SC_*(H_{2\delta})/\mathbb{Z}[u]\otimes_{\mathbb{Z}}(C^{III,IV,V}(H_{2\delta}))$ to $\mathbb{Z}[u]\otimes_{\mathbb{Z}}C^{I,II}(H_{2\delta})$. We consider the Hamiltonian $\iota_V H_{2\delta}$ in $\mathcal{H}_{\mathrm{std}}(W)$.

Proposition 3.1.14 *For δ small enough, the S^1 equivariant homology of the quotients coincide with the S^1 equivariant homology of the small domain:*

$$H\big(\mathbb{Z}[u]\otimes_{\mathbb{Z}} C^{I,II}(H_{2\delta}),\partial\big) = H\big(SC(\mathbb{Z}[u]\otimes_{\mathbb{Z}} SC(\iota_V H_{2\delta}))\big)$$
$$H\big(\mathbb{Z}[u]\otimes_{\mathbb{Z}} C^{II}(H_{2\delta}),\partial\big) = H\big(SC^{+}(\mathbb{Z}[u]\otimes_{\mathbb{Z}} SC(\iota_V H_{2\delta}))\big)$$

Proof: What remains to be checked is again there is no parametrized Floer trajectory $u:\mathbb{R}\times S^1\to\widehat{W}$ going from an orbit in $C^{I,II}(H_{2\delta})$ to an orbit in $C^{I,II}(H_{2\delta})$ with points in $\widehat{W}\setminus(U\cup V)$. This is proven by contradiction. If there was a parametrized trajectory going from an orbit in $C^{I,II}(H_{2\delta})$ to an orbit in $C^{I,II}(H_{2\delta})$ with points in $\widehat{W}\setminus(U\cup V)$ for all δ's, then, by a theorem of Bourgeois and Oancea [BO09b, Proposition 4.7], there would be such a broken trajectory for the autonomous Hamiltonian and we have proven in Proposition 3.1.5 that this can not exist.

To get a transfer map, we use an autonomous increasing homotopy between H_1 and H_2 and we deform it into an increasing homotopy between $H_{1\delta}$ and $H_{2\delta}$; this induces a map

$$\mathbb{Z}[u]\otimes_{\mathbb{Z}} SC_*(H_{1\delta})\to\mathbb{Z}[u]\otimes_{\mathbb{Z}} SC_*(H_{2\delta}).$$

This map decreases the action (which is defined on the second factor) and commutes with the differential so it induces a map going to the quotient

$$H\big((\mathbb{Z}[u]\otimes_{\mathbb{Z}} SC_*(H_{1\delta},\partial)\big)\to H\big(\mathbb{Z}[u]\otimes_{\mathbb{Z}} C^{II}(H_{2\delta}),\partial\big).$$

This maps commutes with continuation maps.

Proposition 3.1.15 *For δ small enough, a continuation map in the homology defined from an $H_{2\delta}$ induces a continuation continuation map in the homology defined from $\iota_V H_{2\delta}$.*

Proof: One checks again that there is no parametrized Floer trajectory, corresponding to a homotopy, going from an orbit in $C^{I,II}(H_{2\delta})$ to an orbit in $C^{I,II}(H'_{2\delta})$ with points in $\widehat{W}\setminus(U\cup V)$. This is done as in the former proposition, using the fact that the existence of such a trajectory for all δ's would imply the existence of such a broken trajectory for the autonomous Hamiltonian and we have proven in Proposition 3.1.9 that this can not exist.

We thus get a transfer morphism

$$\phi^{S^1}_{W,V}: SH^{S^1}(W,\lambda_W)\to SH^{S^1}(V,\lambda_V).$$

and, on the quotient, the morphism

$$\phi^{S^1,+}=\phi^{S^1,+}_{W,V}: SH^{S^1,+}(W,\lambda_W)\to SH^{S^1,+}(V,\lambda_V).$$

By the same arguments as before, those morphisms compose nicely.

Theorem 3.1.16 (Composition) *Let $(V_1, \lambda_{V_1}) \subseteq (V_2, \lambda_{V_2}) \subseteq (V_3, \lambda_{V_3})$ be Liouville domains with Liouville embeddings. Then the following diagram commutes:*

$$\begin{tikzcd} SH^{S^1,+}(V_3,\lambda_{V_3}) \arrow[r, "\phi^{S^1,+}_{V_3,V_2}"] \arrow[rr, bend right, "\phi^{S^1,+}_{V_3,V_1}"] & SH^{S^1,+}(V_2,\lambda_{V_2}) \arrow[r, "\phi^{S^1,+}_{V_2,V_1}"] & SH^{S^1,+}(V_1,\lambda_{V_1}) \end{tikzcd} \tag{3.3}$$

3.2 Invariance of symplectic homology

In this section, we study the invariance of the (S^1-equivariant) positive symplectic homology with respect to the choice of the Liouville vector field in a neighbourhood of the boundary. This has been studied by Viterbo [Vit99], Cieliebak [Cie02] and Seidel [Sei08] in the case of the symplectic homology, in the framework of Liouville domains.

Lemma 3.2.1 *Let (W, ω, X) be a compact symplectic manifold with contact type boundary and let k be a positive real number. Then*

$$SH^{\dagger}(W, \omega, X) = SH^{\dagger}(W, k\omega, X).$$

Where $\dagger$ denotes any of the variants that we have considered $\emptyset, +, S^1$ or $(S^1, +)$.

Proof: The symplectic completions are $(\widehat{W}, \widehat{\omega})$ and $(\widehat{W}, k\widehat{\omega})$; the chain complexes for a pair (H, J) on $(\widehat{W}, \widehat{\omega})$ and the pair (kH, J) on $(\widehat{W}, k\widehat{\omega})$ are the same, since the 1 periodic orbits are the same, and the Floer trajectories satisfy the same equations; indeed $X^{\omega}_{H} = X^{k\omega}_{kH}$. Similarly, continuation maps are equivalent taking as homotopies H_s and kH_s. The result follows, observing that kH form a cofinal family.

For positive or S^1-equivariant positive homology, we clearly assume that we are in a framework where it is well-defined.

Lemma 3.2.2 *Let (W, ω, X) and (W', ω', X') be two compact symplectic manifolds with contact type boundary. If there exists a symplectomorphism $\varphi : W \to W'$ such that $\varphi(\partial W) = \partial W'$, and such that $\varphi_\star(X) = X'$ on a neighbourhood of ∂W then*

$$SH^{\dagger}(W, \omega, X) \cong SH^{\dagger}(W', \omega', X').$$

Proof: We can extend φ to a symplectomorphism $\widehat{\varphi} : \widehat{W} \to \widehat{W'}$ of the completions. For J' an almost complex structure on $\widehat{W'}$, we take the corresponding almost complex structure J on $\widehat{W}$ defined by

$$J_x := \widehat{\varphi}^{-1}_{\star_x} \circ J'_{\widehat{\varphi}(x)} \circ \widehat{\varphi}_{\star_x}$$

and if H' is a Hamiltonian on $\widehat{W'}$, we take the Hamiltonian H on $\widehat{W}$ defined by $H := \widehat{\varphi}^{\star} H'$. Then the 1 periodic orbirs are in bijection and so are the Floer trajectories. The subfamily

$\{\widehat{\varphi}^{\star}H'\}$ of Hamiltonians is cofinal and thus we reach the conclusion. ⬛

We now restrict our attention to Liouville domains, also called symplectic manifolds with restricted contact type boundary. Recall that a *Liouville domain* is a compact manifold W with boundary, together with a one form λ such that $\omega := d\lambda$ is symplectic and the Liouville vector field X defined by $\iota(X)\omega = \lambda$ points strictly outwards along ∂W.

The asphericity condition being satisfied, we assume that $\langle c_1(TW), \pi_2(W)\rangle = 0$. Recall that the completion of a Liouville domain (W, λ) is $\widehat{W} = W \cup (\partial W \times \mathbb{R}^+)$ with symplectic form given by $\widehat{\omega} = \omega$ on W and $\widehat{\omega} = de^{\rho}\lambda_{|\partial W}$ on $\partial W \times \mathbb{R}^+$. We refer to section 1.2.1 for more details. We denote, as in the previous chapter, by $SH^{\dagger}(W, \lambda)$ the symplectic homology $SH^{\dagger}(W, d\lambda, X)$.

Lemma 3.2.3 *Let (W, λ) be a Liouville domain. Then for all $R \in \mathbb{R}^+$, we have*

$$SH^{\dagger}(W, \lambda) \cong SH^{\dagger}\big(W \cup (\partial W \times [0, R]), \lambda'\big)$$

where the 1-form λ' on $\partial W \times [0, R]$ is the restriction of the 1-form $\widehat{\lambda}$, thus $(e^{\rho}\alpha)$ with $\alpha := \lambda_{|\partial W}$.

Proof: Denote by φ^X_t the flow of X; since $\mathcal{L}_X\lambda = \lambda$ we have $\varphi^{X\star}_t\lambda = e^t\lambda$. This gives a symplectomorphism

$$\varphi^X_R : (W, e^R\omega) \to \big(W \cup (\partial W \times [0, R]), \omega'\big)$$

mapping the boundary ∂W to the boundary $\{R\} \times \partial W$ and such that $\varphi^{X*}_R\lambda = e^R\lambda$. One concludes by the two lemmas above. Explicitely, the diffeomorphism $\varphi^X_R : \widehat{W} \to \widehat{W}$ maps Hamiltonian vector fields as follows : $(\varphi^X_R)_*(X_{H'}) = X_H$ when $H' = e^{-R}(\varphi^X_R)^*H$; hence φ^X_R gives a bijection between 1-periodic orbits of $X_{H'}$ and 1-periodic orbits of X_H, and, with suitable choices of J's, a bijection between Floer trajectories between 1-periodic orbits of $X_{H'}$ and Floer trajectories between 1-periodic orbits of X_H. Hence it yields an isomorphism

$$SH^{\dagger}(W, e^{-R}(\varphi^X_R)^*H) \cong SH^{\dagger}\big(W \cup (\partial W \times [0, R]), H\big).$$

Furthermore, the diffeomorphism φ^X_R intertwines a continuation morphism defined by a homotopy H'_s to the corresponding continuation morphism defined by H_s when again $H'_s = e^{-R}(\varphi^X_R)^*H_s$. This yields the isomorphism mentionned above. ⬛

Lemma 3.2.4 *The transfer morphism*

$$SH^{\dagger}\big(W \cup (\partial W \times [0, R]), \lambda'\big) \mapsto SH^{\dagger}(W, \lambda)$$

is an isomorphism which coincides with the natural identification of Lemma 3.2.3.

PROOF: Let H be an admissible Hamiltonian for $W \cup (\partial W \times [0, R])$. Consider the homotopy $H^1_s := e^{-f(s)} {\varphi^X_{f(s)}}^\star H$ with $f : \mathbb{R} \to [0, R]$ a smooth function so that $H^1_s = H$ for large negative s and $H^1_s = \widetilde{H} := e^{-R} (\varphi^X_R)^* H$ for large positive s. The set of 1 periodic orbits for H^1_s is constant (since, as in the Lemma above, the diffeomorphism $\varphi^X_{f(s)}$ of the completion is a bijection between 1-periodic orbits of $X_{H^1_s}$ and 1-periodic orbits of X_H). This homotopy defines the "transfer morphism"

$$\phi : SH\big(W \cup (\partial W \times [0, R]), H\big) \to SH(W, \widetilde{H}).$$

Let $\{H^\eta_s\}_{\eta \in [0,1]}$ be a family of homotopies (with non fixed endpoint) such that H^0_s is the constant homotopy $H^0_s = H$ for all s, and such that all H^η_s are of the form $e^{-f'(s,\eta)} {\varphi^X_{f'(s,\eta)}}^\star H$ with $f'(., \eta) : \mathbb{R} \to [0, \eta R]$ and $f'(., 1) = f$. We have $H^\eta_{+\infty} = e^{-\eta R} {\varphi^X_{\eta R}}^\star H = H^1_{f^{-1}(\eta R)}$. The set of 1-periodic orbits of H^η_s is in bijection with the set of orbits of H. We consider, for a given η, the space of Floer trajectories

$$\mathcal{M}(H^\eta_s, J^\eta_s) := \bigcup_{\substack{(\gamma^\eta_-, \gamma^\eta_+) \in \mathcal{P}(H^\eta_{-\infty}) \times \mathcal{P}(H^\eta_{+\infty}) \\ \mu_{CZ}(\gamma^\eta_-) = \mu_{CZ}(\gamma^\eta_+)}} \mathcal{M}(\gamma^\eta_-, \gamma^\eta_+, H^\eta_s, J^\eta_s)$$

and the parametrized moduli space

$$\mathcal{M}(\{H^\eta_s, J^\eta_s\}) := \bigcup_{\eta \in [0,1]} \mathcal{M}(H^\eta_s, J^\eta_s)$$

which could have boundaries for some $\eta \neq 0, 1$. It defines a cobordism between $\mathcal{M}(H^0_s, J^0_s)$ and $\mathcal{M}(H^1_s, J^1_s)$. Now $\mathcal{M}(H^0_s, J^0_s) = \mathcal{M}(H, J)$ is the space of constant trajectories $\{\, u(s, \cdot) = \gamma_0(\cdot) \,|\, \gamma_0 \in \mathcal{P}(H)\}$. Thus for small η's, say $\eta \leq \eta_0$, the cobordism is a bijection, $\mathcal{M}(H^\eta_s, J^\eta_s)$ consists of exactly one Floer trajectory starting from each orbit in $\mathcal{P}(H)$ and arriving at the corresponding orbit in $\mathcal{P}(H^\eta_{+\infty})$. The morphism induced by $H^{\eta_0}_s$ is thus the natural identification of periodic orbits. Hence the transfer

$$\phi : SH\big(W \cup (\partial W \times [0, R]), H\big) \to SH\big(W \cup (\partial W \times [0, R - \epsilon]), e^\epsilon {\varphi^X_\epsilon}^\star H\big)$$

is the natural identification for $\epsilon = \eta_0 R$. Now we use the flow of the Liouville vector field, φ^X_ϵ, to carry all this construction further and we get the natural identification as the transfer morphism

$$\phi : SH\big(W \cup (\partial W \times [0, R - \epsilon]), e^\epsilon {\varphi^\epsilon_X}^\star H\big) \to SH\big(W \cup (\partial W \times [0, R - 2\epsilon]), e^{2\epsilon} {\varphi^{2\epsilon}_X}^\star H\big).$$

By induction and functoriality, we get the result.

Lemma 3.2.5 *Let W be a compact symplectic manifold with contact type boundary. Let λ_t, $t \in [0,1]$ be an isotopy of Liouville forms on W such that in a neighbourhood U of the boundary, $\lambda_t = \lambda_0$. Then*

$$SH^\dagger(W, \lambda_0) \cong SH^\dagger(W, \lambda_1).$$

Proof: Remark that we do not require the $d\lambda_t$ to be equal.
Let X_t be the time dependent vector field defined by

$$\iota(X_s)(d\lambda_s) = -\left(\tfrac{d}{dt}\lambda(t)_{|s}\right)$$

and let φ_t be its flow. In the neighbourhood U, the vector field vanishes, $X_s = 0$, and so $\varphi_1^\star\lambda_1 = \lambda_1 = \lambda_0$ on U . Furthermore $\varphi_1^\star d\lambda_1 = d\lambda_0$ because

$$\begin{aligned}
\frac{d}{dt}\varphi_t^\star\lambda_t\Big|_s &= \varphi_s^\star\Big(\frac{d\lambda_t}{dt}\Big|_s\Big) + \varphi_s^\star\mathcal{L}_{X_s}\lambda_s \\
&= \varphi_s^\star\Big(\frac{d\lambda_t}{dt}\Big|_s\Big) + \varphi_s^\star\big(\iota(X_s)d\lambda_s + d\iota(X_s)\lambda_s\big) \\
&= d\Big(\varphi_s^\star\big(\lambda_s(X_s)\big)\Big).
\end{aligned}$$

This implies that the completions for λ_0 and $\varphi_1^\star\lambda_1$ are the same, therefore, by lemma 3.2.2,

$$SH^\dagger(W, \lambda_1) = SH^\dagger(W, \varphi_1^\star\lambda_1) = SH^\dagger(W, \lambda_0).$$

Theorem 3.2.6 *Let W be a compact symplectic manifold with contact type boundary. Let λ_t, $t \in [0,1]$ be a homotopy of Liouville forms on W. Then*

$$SH^\dagger(W, \lambda_0) \cong SH^\dagger(W, \lambda_1).$$

To prove this Proposition, we use the following Proposition from Cieliebak and Eliashberg:

Proposition 3.2.7 ([CE12], Proposition 11.8) *Let W be a compact symplectic manifold with contact type boundary. Let λ_t, $t \in [0,1]$ be a homotopy of Liouville forms on W. Then there exists a diffeomorphism of the completions $f : \widehat{W_0} \to \widehat{W_1}$ such that $f^\star\widehat{\lambda_1} - \widehat{\lambda_0} = dg$ where g is a compactly supported function.*

Proof of Theorem 3.2.6: There exists a positive real ρ_0 such that $supp(g) \subset W \cup (\partial W \times [0, \rho_0])$. We choose positive real numbers ρ_1, ρ_0' and ρ_1' such that $f^{-1}(W \cup (\partial W \times [0, \rho_1])$ contains $W \cup (\partial W \times [0, \rho_0])$, $f^{-1}(W \cup (\partial W \times [0, \rho_1])) \subset W \cup (\partial W \times [0, \rho_0'])$ and $W \cup (\partial W \times [0, \rho_0']) \subset f^{-1}(W \cup (\partial W \times [0, \rho_1']))$. The situation is represented in Figure 3.2

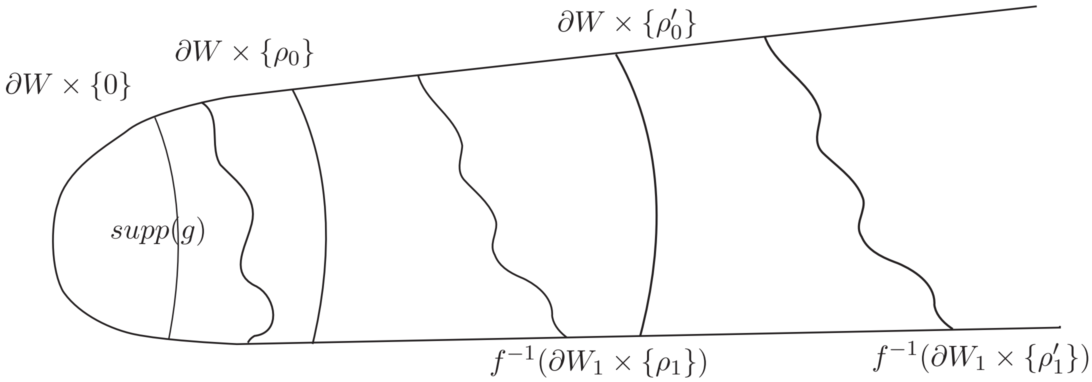

Figure 3.2: The choice of ρ_0, ρ_1, ρ'_0 and ρ'_1

The diffeomorphism f and the flow of X_1 on $\widehat{W_1}$ give

$$\big(f^{-1}(W \cup (\partial W \times [0,\rho_1])), f^\star\widehat{\lambda_1}\big) \cong \big(W \cup (\partial W \times [0,\rho_1], \widehat{\lambda_1}\big) \cong (W, e^{\rho_1}\lambda_1).$$

The completion of $\big(f^{-1}(W \cup (\partial W \times [0,\rho_1])), f^\star\widehat{\lambda_1}\big)$ coincides with $(\widehat{W_0}, \widehat{\lambda_0})$ since close to the boundary $f_\star X_0 = X_1$.

$$\begin{aligned} SH(W,\lambda_1) &\cong SH\big(W \cup (\partial W \times [0,\rho_1]), \widehat{\lambda_1}\big) \quad \text{by Lemma 3.2.3} \\ &\cong SH\big(f^{-1}(W \cup (\partial W \times [0,\rho_1])), f^\star\widehat{\lambda_1}\big) \quad \text{by Lemma 3.2.2} \\ &\cong SH\big(f^{-1}(W \cup (\partial W \times [0,\rho_1])), \widehat{\lambda_0} + dg\big) \quad \text{by Proposition 3.2.7} \\ &\cong SH\big(\underbrace{f^{-1}(W \cup (\partial W \times [0,\rho_1]))}_{=:W_1}, \widehat{\lambda_0}\big) \quad \text{by Lemma 3.2.5.} \end{aligned}$$

Denoting by $\varphi_t^{X_0}$ the flow of X_0 and by W_0 the manifold $W \cup (\partial W \times [0,\rho_0])$, we have

$$\varphi^{X_0}_{\rho'_1-\rho_1}(W_1) = f^{-1}(W \cup (\partial W \times [0,\rho'_1]))$$

and

$$\varphi^{X_0}_{\rho'_0-\rho_0}(W_0) = W \cup (\partial W \times [0,\rho'_0])$$

Using the functoriality of the transfer morphism,

$$\begin{tikzcd} SH(\varphi^{X_0}_{\rho'_1-\rho_1}(W_1), \widehat{\lambda_0}) \arrow[r] \arrow[rr, bend right, "\cong"'] & SH(\varphi^{X_0}_{\rho'_0-\rho_0}(W_0), \widehat{\lambda_0}) \arrow[r] \arrow[rr, bend left, "\cong"] & SH(W_1, \widehat{\lambda_0}) \arrow[r] & SH(W_0, \widehat{\lambda_0}); \end{tikzcd}$$

therefore

$$SH(W,\lambda_1) \cong SH(W_1,\widehat{\lambda_0}) \cong SH(W_0,\widehat{\lambda_0}) \cong SH(W,\lambda_0).$$

Seidel in [Sei08] has extended the definition of symplectic homology (and all its variants) to Liouville manifolds.

Definition 3.2.8 (see for instance [CE12]) A *Liouville manifold* is an exact symplectic manifold (W,ω,X), where the vector field X is an expanding Liouville vector field, i.e $\mathcal{L}_X\omega=\omega$ and $\varphi_t^X\omega=e^t\omega$ such that

- the vector field X is complete and
- the manifold is convex in the sense that there exists an exhaustion $W=\cup_{k=1}^{\infty}W^k$ by compact domains $W_k\subset W$ with smooth boundaries along which X is outward pointing.

In the following we will denote a Liouville manifold either by (W,ω,X) or by $(W,\lambda := \iota(X)\omega)$.

The set

$$\mathrm{Skel}(V,\omega,X) := \bigcup_{k=1}^{\infty}\bigcap_{t>0}\varphi_{-t}^X(W^k)$$

is called the *skeleton* of the Liouville manifold (W,ω,X). It is independent of the choice of the exhausting sequence of compact sets W^k. A Liouville manifold (W,ω,X) is said to be of *finite type* if its skeleton is compact. Every finite type Liouville manifold is the completion of a Liouville domain[1].

Definition 3.2.9 ([Sei08]) Let (W,ω,X) be a Liouville manifold non necessarily of finite type and let W^k be an exhaustion by compact domains $W_k\subset W$ with smooth boundaries along which X is outward pointing such that $W^k\subset W^{k+1}$. The symplectic homology (and its variants) of (W,λ) is defined as the inverse limit of the symplectic homologies of $(W^k,\lambda_{|_{W^k}})$

$$SH^\dagger(W,\lambda) := \varprojlim SH^\dagger(W^k,\lambda_{|_{W^k}}).$$

The morphisms appearing in this inverse limit are the transfer morphisms.

This definition is independent of the chosen exhaustion. Remark that in the case of finite type Liouville manifolds, this definition coincides with the previous one.

[1] We refer to the book by Cieliebak and Eliashberg for more details, [CE12, Chapter 11]

Proposition 3.2.10 *Let (W_0, λ_0) and (W_1, λ_1) be two Liouville manifolds not necessarily of finite type. Assume there exists an exact symplectomorphism $f : W_0 \to W_1$ i.e. such that $f^\star \lambda_1 - \lambda_0 = dg$ with g a function on W_0. Then*

$$SH^\dagger(W_0, \lambda_0) \cong SH^\dagger(W_1, \lambda_1).$$

Proof: Let W_0^k be an exhaustion for W_0 and W_1^k be an exhaustion for W_1 such that for all k,

$$W_0^k \subset f^{-1}(W_1^k) \subset W_0^{k+1}$$

where the inclusion at each level means the inclusion in the interior of the next compact space. Let η be a smooth function $\eta : W_0 \to [0, 1]$ such that $\eta = 1$ in a neighbourhood of $\cup_{k=1}^\infty f^{-1}(\partial W_1^k)$ and $\eta = 0$ in a neighbourhood of $\cup_{k=1}^\infty \partial W_0^k$. We define the 1-form λ on W_0 to be

$$\lambda := \lambda_0 + d(\eta g).$$

We have

$$SH(W_0^k, \lambda_0) \cong SH(W_0^k, \lambda) \quad \text{and} \quad SH(W_1^k, \lambda_1) \cong SH\big(f^{-1}(W_1^k), \lambda\big).$$

The functoriality of the transfer morphism implies that the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[rr, bend left] & SH(f^{-1}(W_1^{k+1}), \lambda) \arrow[r] \arrow[rr, bend right] & SH(W_0^{k+1}, \lambda) \arrow[r] \arrow[rr, bend left] & SH(f^{-1}(W_1^k), \lambda) \arrow[r] \arrow[rr, bend right] & SH(W_0^k, \lambda) \arrow[r] & \cdots .
\end{tikzcd}$$

Therefore,

$$\begin{aligned}
SH(W_0, \lambda_0) &\cong \varprojlim SH(W_0^k, \lambda_0) \cong \varprojlim SH(W_0^k, \lambda) \\
&\cong \varprojlim SH\big(f^{-1}(W_1^k), \lambda\big) \cong \varprojlim SH(W_1^k, f_\star \lambda) \\
&\cong \varprojlim SH(W_1^k, \lambda_1) \cong SH(W_1, \lambda_1).
\end{aligned}$$

The above result may be extended thanks to the following Lemma:

Lemma 3.2.11 ([BEE12], see also [CE12], Lemma 11.2) *Any symplectomorphism between finite type Liouville manifolds $f : (W_0, \lambda_0) \to (W_1, \lambda_1)$ is diffeotopic to an exact symplectomorphism.*

We have thus

Theorem 3.2.12 *Let (W_0, λ_0) and (W_1, λ_1) be two Liouville manifolds of finite type such that there exists a symplectomorphism $f : (W_0, \lambda_0) \to (W_1, \lambda_1)$. Then*

$$SH^\dagger(W_0, \lambda_0) \cong SH^\dagger(W_1, \lambda_1).$$

3.2.1 Invariance of the homology of contact fillings

In this section we shall prove:

Theorem 3.2.13 *Let (M_0, ξ_0) and (M_1, ξ_1) be two contact manifolds that are exactly fillable; i.e. there exist Liouville domains (W_0, λ_0) and (W_1, λ_1) such that $\partial W_0 = M_0$, $\xi_0 = \ker(\lambda_{0|_{M_0}})$, $\partial W_1 = M_1$ and $\xi_1 = \ker(\lambda_{1|_{M_1}})$. Assume there exists a contactomorphism $\varphi : (M_0, \xi_0) \to (M_1, \xi_1)$ which is "oriented" in the sense that $\varphi^\star \lambda_1\big|_{M_1} = e^f \lambda_0\big|_{M_0}$. Assume moreover that there exists a contact form $\tilde{\alpha}_0$ on M_0 such that all periodic Reeb orbits are nondegenerate and their Conley-Zehnder index have all the same parity. Then*

$$SH^{S^1,+}(W_0, \lambda_0) \cong SH^{S^1,+}(W_1, \lambda_1).$$

Lemma 3.2.14 ([Cie02]) *Let $(\alpha_t)_{t\in[0,1]}$ be a smooth family of contact forms on a closed manifold M of dimension $2n-1$. Then there exists a $R > 0$ and a non-decreasing function $f : [0, R] \to [0, 1]$ such that $f \equiv 0$ close to $\rho = 0$ and $f \equiv 1$ close to $\rho = R$ and*

$$d\big(e^\rho \alpha_{f(\rho)}\big) \text{ is symplectic on } M \times [0, R]$$

Proof: The proof is a computation:

$$d\big(e^\rho \alpha_{f(\rho)}\big) = e^\rho d\rho \wedge \alpha_{f(\rho)} + e^\rho d\alpha_{f(\rho)} + e^\rho f'(\rho) d\rho \wedge \dot{\alpha}_{f(\rho)}.$$

$$\Big(d\big(e^\rho \alpha_{f(\rho)}\big)\Big)^n = ne^{n\rho}\Big(d\rho \wedge \big(\alpha_{f(\rho)} + f'(\rho)\dot{\alpha}_{f(\rho)}\big) \wedge \big(d\alpha_{f(\rho)}\big)^{n-1}\Big)$$

and thus $d\big(e^\rho \alpha_{f(\rho)}\big)$ is symplectic if and only if $\big(\alpha_{f(\rho)} + f'(\rho)\dot{\alpha}_{f(\rho)}\big)\big(R_{\alpha_{f(\rho)}}\big) > 0$. This is true if f' is small.

Lemma 3.2.15 *If (M, ξ) is a compact contact manifold which is exactly fillable by a Liouville domain (W, λ_0) (i.e. $\partial W = M$ and $\xi = \ker \alpha_0$ where $\alpha_0 = \lambda_0\big|_M$) then, for any contact form α_1 such that $\xi = \ker \alpha_1$ (and α_1 defines the same orientation on M), there exists a homotopy of Liouville form $\lambda_s, s \in [0, 1]$ on W such that $\lambda_1\big|_M = \alpha_1$.*

Proof: Since $\alpha_1 = e^g \alpha_0$, for a smooth function g on M, we consider the smooth family of contact forms $\alpha_t = e^{tg}\alpha_0, t \in [0, 1]$. We define on $W \cup M \times [0, R] \subset \widehat{W}$ the 1-form $\tilde{\lambda}$:

$$\tilde{\lambda} = \begin{cases} \lambda_0 & \text{on } M \\ e^\rho \alpha_{f(\rho)} & \text{on } M \times [0, R] \end{cases}$$

with f as in Lemma 3.2.14, so that $d\tilde{\lambda}$ is symplectic. The flow of the vector field X_0, where $\iota(X_0)d\lambda_0 = \lambda_0$, $\varphi^{X_0}_{-r}$ induces a diffeomorphism from $W \cup M \times [0, r]$ to W. The pull-back by this flow of $e^{-r}\tilde{\lambda}$ gives the desired $\lambda_{f(r)}$.

Combining with Theorem 3.2.6 and Theorem 2.2.2, this yields

Lemma 3.2.16 *Let (M_0, ξ_0) be a contact manifold that is exactly fillable by the Liouville domains (W_0, λ_0). Assume that there exists a (oriented) contact form $\tilde{\alpha}_0$ on M_0 such that all periodic Reeb orbits are nondegenerate and their Conley-Zehnder index have all the same parity. Then*

$$SH^{S^1,+}(W_0, \lambda_0) = \bigoplus_{\gamma \in \mathcal{P}(R_{\tilde{\alpha}_0})} \mathbb{Q}\langle \gamma \rangle$$

where $\mathcal{P}(R_{\tilde{\alpha}_0})$ denotes the set of periodic Reeb orbits on $(M_0, \tilde{\alpha}_0)$.

Proof of Theorem 3.2.13: Given the contactomorphism $\varphi : (M_0, \xi_0) \to (M_1, \xi_1)$ and the contact form $\tilde{\alpha}_0$, we define the form $\tilde{\alpha}_1 := (\varphi^{-1})^\star \tilde{\alpha}_0$; it is a contact form on M_1 and its periodic orbits are non degenerate, in bijection with those of $\tilde{\alpha}_0$ with the same Conley-Zehnder index. We apply twice Lemma 3.2.16; once for $(W_0, \lambda_0, \tilde{\alpha}_0)$ and for $(W_1, \lambda_1, \tilde{\alpha}_1)$.

This gives a tool to prove Ustilovsky's Theorem.

Corollary 3.2.17 (Ustilovsky, [Ust99]) *For each natural number m, there exist infinitely many pairwise non isomorphic contact structures on S^{4m+1}.*

Proof: We see that one can build contact structures on S^{4m+1}, which are exactly fillable, but which do not yield isomorphic $SH^{S^1,+}$ homologies of the filling. The result then follows from Theorem 3.2.13. The contact structures in question are those defined by the Brieskorn sheres; see section 2.2.1. The fact that the homologies are different follows from proposition 2.2.7.

4 On the minimal number of periodic Reeb orbits

We shall now use the properties of positive S^1-equivariant symplectic homology to get results on the minimal number of geometrically distinct periodic Reeb orbits on some contact manifolds. We first give an alternative proof of a result of Ekeland and Lasry on the minimal number of distinct periodic Reeb orbits on a hypersurface in $\mathbb{R}^{2n}$. We also obtain information on the minimal number of simple periodic Reeb orbits on some hypersurfaces in some negative complex bundles over a compact symplectic manifold. We extend our machinery to some non exact symplectic manifolds with contact type boundary.

4.1 Minimal number of periodic Reeb orbits on a hypersurface in $\mathbb{R}^{2n}$

We show how to use the transfer morphism to give an alternative proof of a result by Ekeland and Lasry concerning the number of simple periodic Reeb orbits on a hypersurface in $\mathbb{R}^{2n}$, pinched between two spheres, endowed with the restriction of the standard contact form on $\mathbb{R}^{2n}$.

Theorem 4.1.1 (Ekeland, Lasry, [EL80, Eke90]) *Let Σ be a contact type hypersurface in $\mathbb{R}^{2n}$. Let $\xi = \ker\alpha$ be the contact structure induced by the standard contact form on $\mathbb{R}^{2n}$. Assume there exists a point $x_0 \in \mathbb{R}^{2n}$ and numbers $0 < R_1 \leq R_2$ such that:*

$$\forall x \in \Sigma, \quad R_1 \leq \|x - x_0\| \leq R_2 \qquad \textit{with } \frac{R_2}{R_1} < \sqrt{2}$$

Assume also that $\forall x \in \Sigma, \quad T_x\Sigma \cap B_{R_1}(x_0) = \emptyset$. Assume moreover that all periodic Reeb orbits are non degenerate. Then Σ carries at least n geometrically distinct periodic Reeb orbits.

Remark 4.1.2 The assumption $\forall x \in \Sigma, \quad T_x\Sigma \cap B_{R_1}(x_0) = \emptyset$ (which is weaker than convexity) can be stated as

$$\langle \nu_\Sigma(z), z\rangle > R_1, \quad \forall z \in \Sigma \tag{4.1}$$

where $\nu_\Sigma(z)$ is the exterior normal vector of Σ at point z and $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product on $\mathbb{R}^{2n}$.

Proof: We consider ellipsoids, very close to the spheres, defined by

$$S'_{R_1} = \left\{ \sum_{i=1}^{n} a_i^{-1}\big((x^i)^2 + (y^i)^2\big) = R_1^2 \right\}$$

with $a_1 < \cdots < a_n$ real numbers arbitrarily close to 1 and rationally independent, and $S'_{R_2} = \{\sum_{i=1}^n a_i^{-1}\big((x^i)^2+(y^i)^2\big) = R_2^2\}$, and we denote by $\widetilde{S'_{R_1}}$, $\widetilde{\Sigma}$ and $\widetilde{S'_{R_2}}$ the compact regions in $\mathbb{R}^{2n}$ bounded respectively by S'_{R_1}, Σ and S'_{R_2}, endowed with the restriction of the standard symplectic form ω on $\mathbb{R}^{2n}$. We take the parameters a_i sufficiently close to 1 so that we have the inclusion

$$\widetilde{S'_{R_1}} \subset \widetilde{\Sigma} \subset \widetilde{S'_{R_2}}$$

of Liouville domains. The contact form on the boundaries is the one induced by $\iota(X_{rad})\omega$, where X_{rad} is the radial vector field $X_{rad} = \frac{1}{2}\sum x^i\partial_{x^i} + y^i\partial_{y^i}$. The completion of those Liouville domain is $(\mathbb{R}^{2n},\omega)$. By Theorem 3.1.16, the transfer morphisms yields the following commutative diagram:

$$\begin{tikzcd} SH^{S^1,+}(\widetilde{S'_{R_2}},\omega) \arrow[r, "\phi"] \arrow[rr, bend right, "\cong"'] & SH^{S^1,+}(\widetilde{\Sigma},\omega) \arrow[r] & SH^{S^1,+}(\widetilde{S'_{R_1}},\omega). \end{tikzcd} \tag{4.2}$$

We can consider the positive S^1-equivariant symplectic homology truncated by the action at level $\leq T$, $SH^{S^1,+,T}$. Since all Floer trajectories inducing the morphisms lower the action, we still have the commutative diagram for the truncated positive invariant symplectic homology:

$$\begin{tikzcd} SH^{S^1,+,T}(\widetilde{S'_{R_2}},\omega) \arrow[r, "\phi"] \arrow[rr, bend right, "\cong"'] & SH^{S^1,+,T}(\widetilde{\Sigma},\omega) \arrow[r] & SH^{S^1,+,T}(\widetilde{S'_{R_1}},\omega). \end{tikzcd} \tag{4.3}$$

where we have chosen a number T such that

$$\pi a_n R_2^2 < T < 2\pi a_1 R_1^2. \tag{4.4}$$

This is possible thanks to the "pinching" hypothesis $\frac{R_2}{R_1} < \sqrt{2}$.

By Theorem 2.2.2, $SH^{S^1,+,T}(\widetilde{S'_{R_2}},\omega)$ is generated by n elements $u^0\otimes\gamma^1_{\text{Max}},\dots u^0\otimes\gamma^n_{\text{Max}}$ corresponding to n simple periodic Reeb orbits on S'_{R_2}, $\gamma^1,\dots,\gamma^n$ of action $\pi a_1 R_2^2,\dots,\pi a_n R_2^2$. The analogous is true for $SH^{S^1,+,T}(\widetilde{S'_{R_1}},\omega)$ with actions $\pi a_1 R_1^2,\dots,\pi a_n R_1^2$.

By (4.3), $SH^{S^1,+,T}(\widetilde{\Sigma},\omega)$ is thus of rank at least n. All applications in the above diagrams decrease the action thus the action of each of those n generators in $SH^{S^1,+,T}(\widetilde{\Sigma},\omega)\cap \mathrm{Im}(\phi)$ is pinched between $\pi a_1 R_1^2$ and $\pi a_n R_2^2 < 2\pi a_1 R_1^2$.

By Corollary 2.2.5, the only generators that may appear in $SH^{S^1,+,T}(\widetilde{\Sigma},\omega)$ are elements of the form $u^0 \otimes \gamma_{\mathrm{Max}}$ with γ a good Reeb orbit on Σ.

It remains to prove that the n elements in the image of ϕ are geometrically distinct. By the pinching condition on their action, we know that they are not iterate one from another but we still need to prove that two of them can not be the iterates of a same orbit of smaller action. This we do by proving that the smallest possible action for any periodic Reeb orbit on Σ is greater than $\pi a_1 R_1^2$.
Let $\gamma : [0,T] \to \Sigma$ be a simple periodic Reeb orbit. We have :

$$\begin{aligned}
2T &= \int_0^T \alpha_{\gamma(t)}\big(\dot\gamma(t)\big)dt \\
&= \int_0^T \langle \dot\gamma(t), J\gamma(t)\rangle dt \qquad \text{since } \alpha_x(X_x) = \tfrac{1}{2}\langle X_x, Jx\rangle \\
&= \int_0^T \langle \dot\gamma(t), J\bar\gamma(t)\rangle dt \qquad \text{with } \bar\gamma(t) := \gamma(t) - \int_0^T \gamma(t)dt \\
&\leq \|\dot\gamma\|_{L^2}\|\bar\gamma\|_{L^2} \\
&\leq \|\dot\gamma\|_{L^2}^2 \tfrac{T}{2\pi} \qquad \text{via the Wirtinger's inequality} \\
&= \frac{T}{2\pi}\int_0^T \|\dot\gamma(t)\|^2 dt \\
&= \frac{T}{2\pi}\int_0^T \|(R_\alpha)_{\gamma(t)}\|^2 dt
\end{aligned} \tag{4.5}$$

For any point x in Σ, the norm of the Reeb vector field is bounded by $\|(R_\alpha)_x\| \leq \frac{2}{R_1}$. Indeed, R_α is proportional to $J\nu_\Sigma$ since $\iota(J\nu_\Sigma)d\alpha = 0$ because $\iota(J\nu_\Sigma)d\alpha(Y) = \omega(J\nu_\Sigma, Y) = -\langle \nu_\Sigma, Y\rangle = 0$ for all $Y \in T\Sigma$. Thus $R_\alpha = cJ\nu_\Sigma$ with $|c| = \|R_\alpha\|$. But $\alpha_x(R_{\alpha x}) = 1 = \frac{1}{2}\langle c_x J\nu_\Sigma(x), Jx\rangle = \frac{c_x}{2}\langle \nu_\Sigma(x), x\rangle$. Therefore, by assumption (4.1), $c_x = \frac{2}{\langle \nu_\Sigma(x), x\rangle} \leq \frac{2}{R_1}$. And thus (4.5) $\leq \frac{4}{R_1^2}T\frac{T}{2\pi}$. Then $2T \leq 2T\frac{T}{\pi R_1^2}$ and we reach the conclusion

$$T \geq \pi R_1^2.$$

Hence the conclusion of the Theorem.

The original proof of Theorem 4.1.1 uses variational methods that work only in $\mathbb{R}^{2n}$. This new proof may extend to other cases such as hypersurfaces in negative line bundles. For this, we have to extend our machinery to some non exact compact symplectic manifolds with contact type boundaries.

4.2 Extension of the definitions

Let (W, ω, X) be a compact symplectic manifold with contact type boundary, satisfying assumptions 1.1.1 and 1.1.2 and let $(\widehat{W}, \widehat{\omega})$ be its symplectic completion. We want to define the positive symplectic homology and the positive S^1-equivariant symplectic homology of (W, ω). In this general situation, the action does not distinguish between constant and non-constant 1-periodic orbits since we have no control on the disk bounded by a 1-periodic orbit.

4.2.1 Positive symplectic homology

This section is joint results with Strom Borman.

Definition 4.2.1 Let $\tilde{H}$ be a Hamiltonian in $\mathcal{H}_{\mathrm{std}}$. The submodule of the Floer complex generated by the critical points of $\tilde{H}$ is denoted by $SC_*^-(\tilde{H}, J)$.

We restrict ourselves to the cofinal family of perturbation of Morse-Bott Hamiltonians as defined in section 2.1.3. We use the same notations as in section 2.1.3.

Theorem 4.2.2 *Let H be a Hamiltonian in $\mathcal{H}_{MB}$. There exists a real number $\delta_0 > 0$ such that for all $\delta < \delta_0$, the submodule $SC_*^-(H_\delta, J)$ is a subcomplex of the Floer complex $SC_*(H_\delta, J)$.*

Proof: We prove this by contradiction, assuming that no such δ_0 exists. Then there is a decreasing sequence δ_n converging to 0 such that $SC_*^-(H_{\delta_n}, J)$ is not a subcomplex of $SC_*(H_{\delta_n}, J)$. This implies that for all n, there exists a Floer trajectory for H_{δ_n} going from a fixed critical point of H_{δ_n} to a nonconstant 1-periodic orbit of H_{δ_n}. We can assume (considering, if needed, a subsequence) that the 1-periodic orbit is the same for all n. This sequence of Floer trajectories converges to a Floer trajectory for the autonomous Hamiltonian by the following theorem due to Bourgeois and Oancea:

Theorem 4.2.3 ([BO09b], Proposition 4.7) *Let $v_n \in \mathcal{M}(\gamma^-, \gamma^+, H_{\delta_n}, J)$ be a sequence of Floer trajectories with $\delta_n \to 0$ as $n \to \infty$. Then there exists a Broken Floer trajectory with gradient fragments u and a subsequence (still denoted by v_n) such that $v_n \to u$.*

To conclude the proof of theorem 4.2.2, we shall now prove that it is impossible for an autonomous Hamiltonian H to have such a Floer trajectory. Precisely, we shall show that, for any critical point p of H and any nonconstant 1-periodic orbit γ of H, the moduli space $\mathcal{M}(p, \gamma; H, J)$ of solutions $u : \mathbb{R} \times S^1 \to \widehat{W}$ of the Floer equation

$$\left\{ \begin{array}{rcl} \partial_s u + J \circ u\big(\partial_\theta u - X_H \circ u\big) & = & 0 \\ u(-\infty, \cdot) & = & p \text{ (critical point)} \\ u(+\infty, \cdot) & = & \gamma(\cdot) \subset \partial W \times \{\rho_\gamma\} \end{array} \right. \tag{4.6}$$

is empty. (We have considered the image of γ sitting in $\partial W\times\{\rho_\gamma\}$.) We choose a compatible almost complex structure J i.e a $J \in \mathrm{End}\big(T(\partial W \times \mathbb{R}^+)\big)$ such that $d\alpha(\cdot, J\cdot) > 0$ on $\xi \subset \partial W \times \mathbb{R}^+$, such that $J\xi = \xi$ and such that $J\partial_\rho = R_\alpha$ near the Hamiltonian orbits. We assume u is a Floer trajectory, solution of (4.6), and we consider a real number $s_0 \in \mathbb{R}$ such that $u(s,\cdot)$ stays in a neighbourhood of γ for all $s \geq s_0$. We can decompose u:

$$u(s,\theta) := \big(f(s,\theta), a(s,\theta)\big) \in \partial W \times \mathbb{R}^+$$

with $a(s,\theta)$ close to ρ_γ for $s \geq s_0$. We distinguish two cases:

Case 1: there exist s and θ such that $a(s,\theta) > \rho_\gamma$;

Case 2: $a(s,\theta) \leq \rho_\gamma \quad$ for all s and θ.

We shall see that neither case can arise, and this will conclude our proof.

Case 1

We prove by contradiction that it can not happen, as a direct application of Abouzaid maximum principle (proven above as theorem 3.1.6). Assume that $u : \mathbb{R} \times S^1 \to \widehat{W}$ is a Floer trajectory whose image intersects $\widehat{W} \setminus \big(W \cup (\partial W \times [0,\rho_\gamma])\big)$. There exists $\delta > 0$ such that the intersection of the image of u with a slice $\partial W \times \{\rho\}$ is non-empty for any $\rho_\gamma < \rho < \rho_\gamma + \delta$ and we choose between them a regular value $\rho_\gamma + \epsilon$ of $\rho \circ u$.

The manifold $W' := \widehat{W} \setminus (W \cup (\partial W \times [0, \rho_\gamma + \epsilon[))$ is symplectic with contact type boundary $\partial W \times \{\rho_0 + \epsilon\}$ and Liouville vector field pointing inwards. Let S be the inverse image of W' under the map u; it is a compact Riemann surface with boundary embedded in $\mathbb{R} \times S^1$; the complex structure j is the restriction to S of the complex structure j on the cylinder defined by $j(\partial_s) = \partial_\theta$. We define β to be the restriction of $d\theta$ to S. The fact that u is a Floer trajectory is equivalent to $(du - X_H \otimes \beta)^{0,1} := \frac{1}{2}((du - X_H \otimes \beta) + J(du - X_H \otimes \beta)j) = 0$, where du is the differential of the map u viewed as a section of $T^*S \otimes u^*TW'$. Then theorem 3.1.6 concludes.

Case 2

To prove that this situation can not happen, we use an argument taken from Bourgeois and Oancea [BO09a]. We restrict to $s \in [s_0, \infty)$ and we define $\bar{a}(s) := \int_{S^1} a(s,\theta)d\theta$. Equation (4.6) decomposes as:

$$\left\{\begin{array}{ccc} \partial_s a - \alpha(\partial_\theta f) - e^{-a}h'(a) & = & 0 \\ \alpha(\partial_s f) + \partial_\theta a & = & 0 \\ \pi_\xi(\partial_s f) + \pi_\xi(J\partial_\theta f) & = & 0 \end{array}\right. \tag{4.7}$$

where $\pi_\xi : T(\partial W \times \mathbb{R}) \to \xi$ is the projection on ξ along $\mathbb{R}\langle\partial_\rho\rangle \oplus \mathbb{R}\langle R_\alpha\rangle$. The first equation of (4.7) implies

$$\partial_s \bar{a}(s) = \int_{S^1} \alpha(\partial_\theta f)d\theta + \int_{S^1} h'(a)e^{-a}d\theta \leq \int_{S^1} \alpha(\partial_\theta f)d\theta + T \tag{4.8}$$

where $T = h'(\rho_\gamma)e^{\rho_\gamma}$. The inequality is true since $h'(b)e^{-b}$ is increasing in b.

For $s_0 \le s \le s'$, we have, by Stokes,

$$\int_{S^1} \alpha\big(\partial_\theta f(s',\theta)\big)d\theta - \int_{S^1} \alpha\big(\partial_\theta f(s,\theta)\big)d\theta = \int_{[s,s']\times S^1} f^\star d\alpha \tag{4.9}$$

and

$$\int_{[s,s']\times S^1} f^\star d\alpha > 0 \tag{4.10}$$

because $d\alpha(\cdot, J\cdot) > 0$ on the contact distribution and $d\alpha$ "kills" ∂_ρ and R_α; we assume of course that f is non constant. Hence the map $s \mapsto \int_{S^1} \alpha\big(\partial_\theta f(s,\theta)\big)d\theta$ is an increasing function. Since we know that $\lim_{s\to\infty} f(s,\theta) = \gamma'(-T\theta)$, where γ' is the corresponding Reeb orbit, we have

$$\int_{S^1} \alpha\big(\partial_\theta f(+\infty,\theta)\big)d\theta = -T. \tag{4.11}$$

Hence

$$\partial_s \bar{a}(s) \le \int_{S^1} \alpha\big(\partial_\theta f(s,\theta)\big)d\theta + T \le 0. \tag{4.12}$$

Now $\bar{a}(+\infty) = \rho_\gamma$ since $\big(a(+\infty,\theta), f(+\infty,\theta)\big) = \big(\rho_\gamma, \gamma'(-T\theta)\big)$. Since $a \le \rho_\gamma$ by assumption and since $\bar{a}(s)$ is non increasing by the above, we have

$$a(s,\theta) \equiv \rho_\gamma. \tag{4.13}$$

So (4.7) becomes:

$$\begin{cases} \alpha(\partial_\theta f) - h'(a)e^{-a} & = & 0 \\ \alpha(\partial_s f) & = & 0 \\ \pi_\xi(df\circ j) - J\pi_\xi(df) & = & 0 \end{cases} \tag{4.14}$$

Thus $\alpha(\partial_\theta f) = h'(a)e^{-a} = h'(\rho_\gamma)e^{\rho_\gamma} =$ constant. Using (4.9), we get

$$\int_{[s_0,\infty)\times S^1} f^\star d\alpha = 0 \tag{4.15}$$

which implies that f is constant, which yields a contradiction.

We denote by $\mathcal{H}'_{\mathrm{std}}$ the subfamily of Hamiltonians in $\mathcal{H}_{\mathrm{std}}$ such that the conclusion of Theorem 4.2.2 holds.

Definition 4.2.4 Let H_δ be a Hamiltonian in $\mathcal{H}'_{\mathrm{std}}$, we define the positive Floer complex to be the quotient

$$SC_*^+(H_\delta, J) := SC_*(H_\delta, J)\big/_{SC_*^-(H_\delta, J)}.$$

We would like to define as in section 1.2.4 the direct limit over Hamiltonians of the positive homologies defined by those quotients.

Theorem 4.2.5 *Let $H_{0\delta}$ and $H_{1\delta}$ be two Hamiltonian in $\mathcal{H}'_{std}$ such that $H_{0\delta} \leq H_{1\delta}$. Let H_s be a increasing homotopy between them. The continuation map maps $SC^-_*(H_0, J_0)$ to $SC^-_*(H_1, J_1)$.*

PROOF: Let H_1 be the Hamiltonian in $\mathcal{H}_{\mathrm{MB}}$ from which $H_{1\delta}$ is the perturbation. It suffices to show, as before, that for any nonconstant orbit γ of H_1 the moduli space $\mathcal{M}(p, \gamma; H_s, J_s)$ of solutions $u : \mathbb{R} \times S^1 \to \widehat{W}$ to the Floer continuation equation

$$\begin{cases} \partial_s u + J_s \circ u\big(\partial_\theta u - X_{H_s} \circ u\big) &= 0 \\ u(-\infty, \theta) &= p \text{ (critical point)} \\ u(+\infty, \theta) &= \big(t_\gamma, \gamma(-T\theta)\big) \end{cases} \tag{4.16}$$

is empty.

We consider again a real number $s_0 \in \mathbb{R}$ such that $u\big(s, \theta\big)$ stays in a neighbourhood of the image of γ for all $s \geq s_0$ and such that H_s is constant (in s) for $s \geq s_0$. We can again decompose u in

$$u(s, \theta) = \big(f(s, \theta), a(s, \theta)\big) \in M \times \mathbb{R}^+.$$

Again,we distinguish two cases:

1. there exist s and θ such that $a(s, \theta) > \rho_\gamma$;
2. $a(s, \theta) \leq \rho_\gamma \quad \forall s, \theta$.

The proof to rule out the second case is the same as in Theorem 4.2.2.
For the first case, the argument is the same and the conclusion comes from the generalised version of "Abouzaid maximum principle" stated in proposition 3.1.10.

We are now ready to define the positive symplectic homology in the framework of a compact symplectic manifold with contact type boundary (W, ω) which is aspherical and such that its first Chern class vanishes on the second homotopy group :

Definition 4.2.6 The positive symplectic homology of (W, ω) is defined as

$$SH^+(W, \omega) := \varinjlim_{H \in \mathcal{H}'_{\mathrm{std}}} H_*(SC^+_*(H, J), d).$$

4.2.2 Positive S^1-equivariant symplectic homology

We shall use the special complex in the alternative definition of S^1-equivariant symplectic homology presented in section 2.1.2. One considers an autonomous Hamiltonian H in $\mathcal{H}_{\mathrm{MB}}$ (cf definition 2.1.2) such that the slope a is big and ρ_0 is small; one denotes by H_δ a small perturbation, as in section 2.1.3. One defines the positive S^1-equivariant Floer complex of H_δ by

$$\widetilde{SC}_*^{S^1,+}(H_\delta) := \mathbb{Z}[u] \otimes_{\mathbb{Z}} SC_*^+(H_\delta)$$

with differential induced by the differential given by (2.2). The fact that $SC_*^{S^1,-}(H_\delta)$ is a subcomplex of $SC_*^{S^1}(H_\delta)$ for δ small enough is a consequence of Theorem 4.2.3, extended to parametrized Floer trajectories in chapter 5.2 of [BO13a]; one reduces the problem to the situation of Theorems 4.2.5 and 4.2.2. Similarly, there are well-defined continuation maps induced by increasing homotopies of Hamiltonians and we have

$$SH_*^{S^1,+}(W) := \lim_{\substack{\longrightarrow \\ H_\delta \in \mathcal{H}'_{\mathrm{std}}}} SH_*^{S^1,+}(H_\delta).$$

4.3 Reeb orbits on hypersurfaces in negative line bundles

We sketch a procedure to detect a minimal number of geometrically distinct periodic Reeb orbits for hypersurfaces in some negative line bundles.

Symplectic structure on the complement of the zero section

Our framework here is a complex line bundle $\mathcal{L} \xrightarrow{\pi} B^{2n}$ over a closed symplectic manifold (B^{2n}, ω_B), endowed with a Hermitian structure h and a connection ∇. We assume $\mathcal{L}$ to be *negative* i.e.

$$c_1(\mathcal{L}) = -k[\omega_B]$$

for a real number $k > 0$. The *transgression 1-form*, $\theta^\nabla \in \Omega^1\big(\mathcal{L} \setminus O_{\mathcal{L}}, \mathbb{R}\big)$ is defined by

$$\begin{cases} \theta_u^\nabla(u) &= 0, \quad \theta_u^\nabla(iu) = \frac{1}{2\pi} \qquad u \in \mathcal{L} \setminus O_{\mathcal{L}} \\ \theta^\nabla|_{H^\nabla} &\equiv 0 \quad \text{where } H^\nabla \text{ is the horizontal distribution.} \end{cases} \tag{4.17}$$

We have

$$d\theta^\nabla = k\pi^\star\omega_B.$$

We denote by r the radial function on the fiber, i.e. $r : \mathcal{L} \to \mathbb{R} : u \mapsto h_{\pi(u)}(u,u)^{\frac{1}{2}} =: |u|$.

Observe that $d(r^2\theta^\nabla)$ is symplectic except on the zero section $O_{\mathcal{L}}$. We want to have information about the minimal number of periodic orbits of the Reeb vector field on a hypersurface in $\mathcal{L} \setminus O_{\mathcal{L}}$ endowed with the contact form defined by the restriction of $(r^2\theta^\nabla)$ to Σ.

Reeb orbits on the circle bundle with varying radius

Let $f : B \to \mathbb{R}$ be a smooth function. Define the contact hypersurface

$$\left(S_{e^f} = \{u \in \mathcal{L} \,|\, |u| = e^{\frac{1}{2}f(\pi(u))}\}, \alpha := (r^2\theta^\nabla)_{|S_{e^f}}\right)$$

The Reeb vector field on S_{e^f} is given by:

$$R_\alpha = e^{-f(\pi(u))}\left(2\pi\partial_\theta + \bar{X}_f\right) \tag{4.18}$$

where ∂_θ is the infinitesimal rotation in the fiber (∂_θ at the point u identifies with iu), where X_f is the Hamiltonian vector field on B corresponding to the function f (i.e. $\iota(X_f)\omega_B = df$) and where $\bar{X}$ denotes the horizontal lift of a vector $X \in TB$. The periodic Reeb orbits correspond to the critical points of f. The contact action (i.e. the period) of a simple orbit γ which lies above a critical point p is

$$\mathcal{A}(\gamma) = e^{f(p)}. \tag{4.19}$$

The Conley-Zehnder index of an orbit which is a k iterate of a simple orbit over the critical point p is given by

$$\mu_{CZ}(\gamma) = 2k - \frac{1}{2}\operatorname{Sign}(\operatorname{Hess}_p f) \tag{4.20}$$

It is given by the Conley-Zehnder index of the path of symplectic matrices,

$$\phi : \left[0, e^{f(p)}\right] \to \operatorname{Sp}(\mathbb{R}^{2n}, \Omega_0) \quad \phi(t) := \left(\varphi_t^{R_\alpha}\right)_{\star_q}\Big|_{\xi_q}$$

given by the expression of the differential of the flow in a symplectic trivialisation of the contact structure ξ along γ.

We have thus proven the following, using Morse's inequalities:

Proposition 4.3.1 *Let Σ be a contact type hypersurface in $\mathcal{L}$ such that the intersection of Σ with each fiber is a circle. The contact form is the restriction of $r^2\theta^\nabla$. Then Σ carries at least $\sum_{i=0}^{2n}\beta_i$ geometrically distinct periodic Reeb orbits, where β_i denote the Betti numbers of B.*

We do not make here any assumptions on B except that B is closed.

Symplectic structure on a negative line bundle

Definition 4.3.2 Let $\rho : [0, \infty[\to [0, 1]$ be a smooth decreasing function such that on a neighbourhood of 0, ρ is equal to 1 and ρ vanishes outside a compact set. Let $\epsilon > 0$ be such that $2r + \epsilon\rho'(r) > 0$. The two form defined by

$$\omega_{\rho,\epsilon} := d(r^2\theta^\nabla) + \epsilon d(\rho(r)\theta^\nabla) \tag{4.21}$$

is smooth and well defined on $\mathcal{L}$; indeed, it is obviously well defined (and exact) on $\mathcal{L} \setminus O_{\mathcal{L}}$. On the zero section, it coincides with the sum of $\epsilon\pi^*(\omega_B)$ and the standard form on the fiber. It is symplectic, non exact. For a fixed ϵ those forms are cohomologous. For two different choices of ϵ and ρ, one can interpolate between $\omega_{\rho_0,\epsilon_0}$ and $\omega_{\rho_1,\epsilon_1}$ staying in the class of elements of the form $\omega_{\rho,\epsilon}$.

When the circle bundle is the boundary of a Liouville domain

Theorem 4.3.3 *Let Σ be a contact type hypersurface in $\mathcal{L}$, negative line bundle over a symplectic manifold. Suppose that there exists a Liouville domain W' (such that its first Chern class vanishes on all tori) whose boundary coincides with the circle bundle $S_{R_1^2}$. Suppose there exists a Morse function $f : B \to \mathbb{R}$ such that all critical points of f have a Morse index of the same parity. Let α be the contact form on Σ induced by $r^2\theta^\nabla$ on $\mathcal{L}$. Assume that Σ is "pinched" between two circle bundles $S_{R_1^2}$ and $S_{R_2^2}$ of radii R_1 and R_2 such that $0 < R_1 < R_2$ and $\frac{R_2}{R_1} < \sqrt{2}$. Assume that the minimal action of any periodic Reeb orbit on Σ is bounded below by R_1^2. Then Σ carries at least $\sum_{i=0}^{2n} \beta_i$ geometrically distinct periodic Reeb orbits, where the β_i denote the Betti numbers of B.*

In this Theorem, the assumption on the existence of a Morse function all of whose critical points have Morse indices of the same parity is of a technical nature. Its purpose is to bring the situation within the scope of Theorem 2.2.2, which is our tool for computing the positive S^1-equivariant symplectic homology. The lower bound on the period of any periodic Reeb orbit is semi-technical; it is now the only way we have to distinguish the images of the orbits. The "pinching" assumption is more conceptual, its main implication is that the "n first generators" of the positive S^1-equivariant symplectic homology are simple orbits.

Proof: The proof is the same as for Theorem 4.1.1 using transfer morphisms for Liouville domains. We see the hypersurfaces as lying in the completion of the Liouville domain W' which we assumed to exist. We find a small ϵ so that the convex domain $\widetilde{\Sigma}$ bounded by the hypersurface Σ is such that

$$\widetilde{S}_{R_1^2 e^{\epsilon f}} \subset \widetilde{\Sigma} \subset \widetilde{S}_{R_2^2 e^{\epsilon f}} \subset \widehat{W'}$$

where $\widetilde{S_f}$ is the domain bounded by S_f. We can compute the positive S^1-equivariant symplectic homology, which is spanned by periodic orbits of the Reeb vector field by Theorem 2.2.2.

This is possible by the pinching condition. One uses then the transfer morphisms with truncated action. We have seen that there are $\sum_{i=0}^{2n} \beta_i$ simple periodic orbits on $S_{R_1 e^{\epsilon f}}$ whose actions are very close to R_1^2 and the same number of simple periodic orbits on $S_{R_2 e^{\epsilon f}}$ whose actions are very close to R_2^2. The transfer morphism imply the existence of at least $\sum_{i=0}^{2n} \beta_i$ periodic orbits on Σ with action between R_1^2 et R_2^2. Since we have assumed here that the minimal action of any periodic Reeb orbit on Σ is bounded below by R_1^2, those orbits are geometrically distinct.

Symplectic homology for a hypersurface in a negative line bundle

Let $W \subset \mathcal{L}$ be a compact symplectic manifold of codimension 0 in $\mathcal{L}$ with contact type boundary such that $O_{\mathcal{L}} \subset W$; the symplectic form on W is $\omega_{\rho,\epsilon_{|W}}$. We choose the function ρ such that ρ vanishes in a neighbourhood of the boundary of W. For the symplectic homology to be well-defined, we assume that the closed symplectic manifold B is atoroidal and that the first Chern class of its tangent bundle vanishes on the tori. We first observe that symplectic homology does not depend on the choice of such a function ρ. The completions $(\mathcal{L})$ are all symplectomorphic outside a neighbourhood of $O_{\mathcal{L}}$.

Lemma 4.3.4 *The (positive, S^1-equivariant, positive S^1-equivariant) symplectic homology of W is independent of ρ and ϵ.*

The proof is the same as that of Lemma 3.2.5, using the fact that for a fixed ϵ the difference of two of those symplectic forms is exact.

We even have a stronger result:

Lemma 4.3.5 *Let β be a positive real number and consider Hamiltonians $H_\beta : S^1 \times \widehat{W} \to \mathbb{R}$ and $\widetilde{H}_\beta : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$ which are of the form $\beta r + \beta'$ for r big. Then $SH(H_\beta)$ and $SH^{S^1}(\widetilde{H}_\beta)$ are independent of ρ and ϵ.*

Proof: We consider a curve ω_t, $t \in [0,1]$ of symplectic structures in our class and two 1-periodic orbits γ_-, γ_+ of H_β. Remark the those orbits are the same for all t since the symplectic forms coincide on $\partial W \times \mathbb{R}_+$ in the completion. We consider the moduli spaces $\mathcal{M}_t(\gamma_-, \gamma_+, H_\beta^{\omega_t}, J_t,)$ built from Floer trajectories which are solutions of

$$\frac{\partial u}{\partial s} + J_t^\theta \circ u\Big(\frac{\partial u}{\partial \theta} - X^t_{H_\beta} \circ u\Big) = 0 \tag{4.22}$$

going from one of these orbits to the other, where X^t_H is the Hamiltnian symplectic vector field corresponding to H for the symplectic structure ω_t. Observe that this equation coincides with the classical Floer equation (with no t dependance) outside a compact set. So it behaves in the same way as the continuation equation 1.3 for a homotopy with compact support. Hence it defines a map which intertwines the differential. This gives an isomorphism in homology, since an inverse is defined by following the path of symplectic matrices in the reverse way.

One does a similar reasoning to see that this isomorphism commutes with continuation maps. Hence we get

Lemma 4.3.6 *The symplectic homology $SH(W)$ of W is independent of ρ and ϵ.*

Remark that in this case of negative line bundle, the first term in the action i.e. $\int_{D^2}\sigma^\star\omega_{\rho,\epsilon}$ where $\sigma : D^2 \to \widehat{W}$ with $\sigma_{|_{\partial D^2}} = \gamma$ is given by

$$\begin{aligned}\int_{D^2}\sigma^\star\omega_{\rho,\epsilon} &= \int_{D^2}\sigma^\star\big(d(r^2\theta^\nabla) + \epsilon d(\rho(r)\theta^\nabla)\big)\\ &= \int_{D^2}\sigma^\star d(r^2\theta^\nabla) + \int_{D^2}\sigma^\star\epsilon d(\rho(r)\theta^\nabla)\\ &= \int_{S^1}\gamma^\star d(r^2\theta^\nabla) + \epsilon\int_{D^2}\sigma^\star d(\rho(r)\theta^\nabla).\end{aligned}$$

For a chosen Hamiltonian H_β (or an S^1 equivariant lift $\widetilde{H}_\beta$), one chooses ϵ sufficiently small so that the actions are close to

$$\mathcal{A}_{H_\beta}(\gamma) = -\int_{S^1}\gamma^\star d(r^2\theta^\nabla) - \int_{S^1} H_\beta\big(\theta,\gamma(\theta)\big)$$

$$\mathcal{A}_{\widetilde{H}_\beta}(\gamma,z) = -\int_{S^1}\gamma^\star d(r^2\theta^\nabla) - \int_{S^1}\widetilde{H}_\beta\big(\theta,\gamma(\theta),z\big).$$

and by close we mean that the difference with the action is smaller than the smallest value of the action and smaller than the difference of two actions (this is possible since, for a given β, we only have a finite number of 1-periodic orbits). We proceed equivalently for non contractible orbits for pullbacks on tori linking the orbit to a given loop in the same homotopy class defined on the manifold B. This allows, when looking at a "stair Hamiltonian" to define a transfer morphism, to use the action to distinguish the different subcomplexes. The transfer is then defined as in section 3.1. All reasonings are the same, since the exactness of the symplectic form was only used to distinguish different classes of orbits by the value of their action. To see that the transfer does not depend on ϵ, one proceeds as in Lemma 4.3.5.

At this point, extending the above results to the positive S^1 equivariant homology, we would have a theorem for hypersurfaces in negative line bundles over atoroidal symplectic closed manifolds endowed with a Morse function such that all critical points of f have a Morse index of the same parity. This is asking too much. So we should use the wider context of symplectic manifolds which are monotone and work in the homology with coefficients in the Novikov ring. Everything should extend to this new setup and should lead to

Conjecture 4.3.7 *Let Σ be a contact type hypersurface in $\mathcal{L}$, negative line bundle over a closed monotone symplectic manifold. The bundle is endowed with a hermitian structure and a connection. Suppose there exists a Morse function $f : B \to \mathbb{R}$ such that all critical points of f have a Morse index of the same parity. Let α be the contact form on Σ induced by $r^2\theta^\nabla$ on $\mathcal{L}$. Assume that Σ is "pinched" between two circle bundles S_{R_1} and S_{R_2} of radii*

R_1 and R_2 respectively such that $0 < R_1 < R_2$ and $\frac{R_2}{R_1} < \sqrt{2}$. Assume that the minimal action of any periodic Reeb orbit on Σ is s bounded below by R_1^2. Then Σ carries at least $\sum_{i=0}^{2n} \beta_i$ geometrically distinct periodic Reeb orbits; where the β_i are the Betti numbers of B.

5 Normal forms for symplectic matrices

This chapter is to appear as an homonymous paper in Portugaliae Mathematica [Gutb].

We give here a self contained and elementary description of normal forms for symplectic matrices, based on geometrical considerations. The normal forms in question are expressed in terms of elementary Jordan matrices and integers with values in $\{-1, 0, 1\}$ related to signatures of quadratic forms naturally associated to the symplectic matrix.

Let V be a real vector space of dimension $2n$ with a non degenerate skewsymmetric bilinear form Ω. The symplectic group $\mathrm{Sp}(V, \Omega)$ is the set of linear transformations of V which preserve Ω:

$$\mathrm{Sp}(V, \Omega) = \{\, A : V \to V \mid A \text{ linear and } \Omega(Au, Av) = \Omega(u, v) \text{ for all } u, v \in V \,\}\,.$$

A *symplectic basis* of the symplectic vector space (V, Ω) is a basis $\{e_1, \ldots, e_{2n}\}$ in which the matrix representing the symplectic form is $\Omega_0 = \left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right)$. In a symplectic basis, the matrix A' representing an element $A \in \mathrm{Sp}(V, \Omega)$ belongs to

$$\mathrm{Sp}(2n, \mathbb{R}) = \left\{\, A' \in Mat(2n \times 2n, \mathbb{R}) \mid A'^{\tau} \Omega_0 A' = \Omega_0 \,\right\}$$

where $(\cdot)^{\tau}$ denotes the transpose of a matrix.
Given an element A in the symplectic group $\mathrm{Sp}(V, \Omega)$, we want to find a symplectic basis of V in which the matrix A' representing A has a distinguished form; to give a *normal form* for matrices in $\mathrm{Sp}(2n, \mathbb{R})$ means to describe a distinguished representative in each conjugacy class. In general, one cannot find a symplectic basis of the complexified vector space for which the matrix representing A has Jordan normal form.

The normal forms considered here are expressed in terms of elementary Jordan matrices and matrices depending on an integer $s \in \{-1, 0, 1\}$. They are closely related to the forms given by Long in [LD00, Lon02] ; the main difference is that, in those references, some indeterminacy was left in the choice of matrices in each conjugacy class, in particular when the matrix admits 1 as an eigenvalue. We speak in this case of *quasi-normal forms.* Other constructions can be found in [Wim91, LM74, LMX99, Spe72, MT99] but they are either quasi-normal or far from Jordan normal forms. Closely related are the constructions of normal forms for real matrices that are selfadjoint, skewadjoint or unitary with respect to an indefinite inner product where sign characteristics are introduced; they have been studied in many sources; for instance -mainly for selfadjoint and skewadjoint matrices- in the monograph of I. Gohberg, P. Lancaster and L. Rodman [GLR05], and for unitary matrices in the papers [AYLR04, GR91, Meh06b, Rod06]. Normal forms for symplectic matrices have been given by C. Mehl in [Meh06a] and by V. Sergeichuk in [Ser87] ; in those descriptions, the basis producing the normal form is not required to be symplectic.

We construct here normal forms using elementary geometrical methods.

The choice of representatives for normal (or quasi normal) forms of matrices depends on the application one has in view. Quasi normal forms were used by Long to get precise formulas for indices of iterates of Hamiltonian orbits in [Lon00]. The forms obtained here were useful for us to give new characterisations of Conley-Zehnder indices of general paths of symplectic matrices [Guta]. We have chosen to give a normal form in a symplectic basis. The main interest of our description is the natural interpretation of the signs appearing in the decomposition, and the description of the decomposition for matrices with 1 as an eigenvalue. It also yields an easy natural characterization of the conjugacy class of an element in $\mathrm{Sp}(2n,\mathbb{R})$. We hope it can be useful in other situations.

Assume that V decomposes as a direct sum $V = V_1 \oplus V_2$ where V_1 and V_2 are Ω-orthogonal A-invariant subspaces. Suppose that $\{e_1,\ldots,e_{2k}\}$ is a symplectic basis of V_1 in which the matrix representing $A|_{V_1}$ is $A' = \begin{pmatrix} A'_1 & A'_2 \\ A'_3 & A'_4 \end{pmatrix}$. Suppose also that $\{f_1,\ldots,f_{2l}\}$ is a symplectic basis of V_2 in which the matrix representing $A|_{V_2}$ is $A'' = \begin{pmatrix} A''_1 & A''_2 \\ A''_3 & A''_4 \end{pmatrix}$. Then $\{e_1,\ldots,e_k,f_1,\ldots,f_l,e_{k+1},\ldots,e_{2k},f_{l+1},\ldots,f_{2l}\}$ is a symplectic basis of V and the matrix representing A in this basis is

$$\begin{pmatrix} A'_1 & 0 & A'_2 & 0 \\ 0 & A''_1 & 0 & A''_2 \\ A'_3 & 0 & A'_4 & 0 \\ 0 & A''_3 & 0 & A''_4 \end{pmatrix}.$$

The notation $A' \diamond A''$ is used in Long [Lon00] for this matrix. It is "a direct sum of matrices with obvious identifications". We call it the *symplectic direct sum* of the matrices A' and A''.

We $\mathbb{C}$-linearly extend Ω to the complexified vector space $V^{\mathbb{C}}$ and we $\mathbb{C}$-linearly extend any $A \in \mathrm{Sp}(V,\Omega)$ to $V^{\mathbb{C}}$. If v_λ denotes an eigenvector of A in $V^{\mathbb{C}}$ of the eigenvalue λ, then $\Omega(Av_\lambda, Av_\mu) = \Omega(\lambda v_\lambda, \mu v_\mu) = \lambda\mu\Omega(v_\lambda, v_\mu)$, thus $\Omega(v_\lambda, v_\mu) = 0$ unless $\mu = \frac{1}{\lambda}$. Hence the eigenvalues of A arise in "quadruples"

$$[\lambda] := \left\{\lambda, \frac{1}{\lambda}, \overline{\lambda}, \frac{1}{\overline{\lambda}}\right\}. \tag{5.1}$$

We find a symplectic basis of $V^{\mathbb{C}}$ so that A is a symplectic direct sum of block-upper-triangular matrices of the form

$$\begin{pmatrix} J(\lambda,k)^{-1} & 0 \\ 0 & J(\lambda,k)^{\tau} \end{pmatrix}\begin{pmatrix} \mathrm{Id} & D(k,s) \\ 0 & \mathrm{Id} \end{pmatrix},$$

or

$$\begin{pmatrix} J(\overline{\lambda},k)^{-1} & & & 0 \\ & J(\lambda,k)^{-1} & & \\ & & J(\overline{\lambda},k)^{\tau} & \\ 0 & & & J(\lambda,k)^{\tau} \end{pmatrix}\begin{pmatrix} \mathrm{Id} & 0 & 0 & D(k,s) \\ & \mathrm{Id} & D(k,s) & 0 \\ & & \mathrm{Id} & 0 \\ 0 & & & \mathrm{Id} \end{pmatrix},$$

or

$$\begin{pmatrix} J(\overline{\lambda},k)^{-1} & & & 0 \\ & J(\lambda,k+1)^{-1} & & \\ & & J(\overline{\lambda},k)^{\tau} & \\ 0 & & & J(\lambda,k+1)^{\tau} \end{pmatrix}\begin{pmatrix} \mathrm{Id} & 0 & 0 & S(k,s,\lambda) \\ & \mathrm{Id} & S(k,s,\lambda)^{\tau} & 0 \\ & & \mathrm{Id} & 0 \\ 0 & & & \mathrm{Id} \end{pmatrix}.$$

Here, $J(\lambda,k)$ is the elementary $k\times k$ Jordan matrix corresponding to an eigenvalue λ, $D(k,s)$ is the diagonal $k\times k$ matrix

$$D(k,s) = \mathrm{diag}(0,\dots 0,s),$$

and $S(k,s,\lambda)$ is the $k\times(k+1)$ matrix defined by

$$S(k,s,\lambda) := \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & \frac{1}{2}is & \lambda is \end{pmatrix},$$

with s an integer in $\{-1,0,1\}$. Each $s\in\{\pm 1\}$ is called a sign and the collection of such signs appearing in the decomposition of a matrix A is called the sign characteristic of A.

More precisely, on the real vector space V, we shall prove:

Theorem 5.0.8 (Normal forms for symplectic matrices) *Any symplectic endomorphism A of a finite dimensional symplectic vector space (V,Ω) is the direct sum of its restrictions $A_{|V_{[\lambda]}}$ to the real A-invariant symplectic subspace $V_{[\lambda]}$ whose complexification is the direct sum of the generalized eigenspaces of eigenvalues $\lambda, \frac{1}{\lambda}, \overline{\lambda}$ and $\frac{1}{\overline{\lambda}}$:*

$$V^{\mathbb{C}}_{[\lambda]} := E_{\lambda}\oplus E_{\frac{1}{\lambda}}\oplus E_{\overline{\lambda}}\oplus E_{\frac{1}{\overline{\lambda}}}.$$

We distinguish three cases : $\lambda\notin S^1$, $\lambda=\pm 1$ and $\lambda\in S^1\setminus\{\pm 1\}$.

Normal form for $A_{|V_{[\lambda]}}$ ***for*** $\lambda\notin S^1$:

Let $\lambda\notin S^1$ be an eigenvalue of A. Let $k := \dim_{\mathbb{C}}\mathrm{Ker}(A-\lambda\,\mathrm{Id})$ (on $V^{\mathbb{C}}$) and q be the smallest integer so that $(A-\lambda\,\mathrm{Id})^q$ is identically zero on the generalized eigenspace E_{λ}.

- *If λ is a real eigenvalue of A ($\lambda \notin S^1$ so $\lambda \neq \pm 1$), there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of k matrices of the form*
$$\begin{pmatrix} J(\lambda, q_j)^{-1} & 0 \\ 0 & J(\lambda, q_j)^{\tau} \end{pmatrix}$$
with $q = q_1 \geq q_2 \geq \cdots \geq q_k$ and $J(\lambda, m)$ is the elementary $m \times m$ Jordan matrix associated to λ
$$J(\lambda, m) = \begin{pmatrix} \lambda & 1 & & & & & \\ & \lambda & 1 & & & \mathit{0} & \\ & & \lambda & 1 & & & \\ & & & \ddots & \ddots & & \\ & \mathit{0} & & & \lambda & 1 & \\ & & & & & \lambda & 1 \\ & & & & & & \lambda \end{pmatrix}.$$
This decomposition is unique, when λ has been chosen in $\{\lambda, \lambda^{-1}\}$. It is determined by the chosen λ and by the dimension $\dim\bigl(\mathrm{Ker}(A - \lambda\,\mathrm{Id})^r\bigr)$ for each $r > 0$.

- *If $\lambda = re^{i\phi} \notin (S^1 \cup \mathbb{R})$ is a complex eigenvalue of A, there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of k matrices of the form*
$$\begin{pmatrix} J_{\mathbb{R}}\bigl(re^{-i\phi}, 2q_j\bigr)^{-1} & 0 \\ 0 & J_{\mathbb{R}}\bigl(re^{-i\phi}, 2q_j\bigr)^{\tau} \end{pmatrix}$$
with $q = q_1 \geq q_2 \geq \cdots \geq q_k$ and $J_{\mathbb{R}}(re^{i\phi}, 2m)$ is the $2m \times 2m$ block upper triangular matrix defined by
$$J_{\mathbb{R}}(re^{i\phi}, 2m) := \begin{pmatrix} R(re^{i\phi}) & \mathrm{Id} & & & & & \\ & R(re^{i\phi}) & \mathrm{Id} & & & \mathit{0} & \\ & & R(re^{i\phi}) & \mathrm{Id} & & & \\ & & & \ddots & \ddots & & \\ & \mathit{0} & & & R(re^{i\phi}) & \mathrm{Id} & \\ & & & & & R(re^{i\phi}) & \mathrm{Id} \\ & & & & & & R(re^{i\phi}) \end{pmatrix}$$
with $R(re^{i\phi}) = \begin{pmatrix} r\cos\phi & -r\sin\phi \\ r\sin\phi & r\cos\phi \end{pmatrix}$.
This decomposition is unique, when λ has been chosen in $\{\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}\}$. It is is determined by the chosen λ and by the dimension $\dim\bigl(\mathrm{Ker}(A - \lambda\,\mathrm{Id})^r\bigr)$ for each $r > 0$.

Normal form for $A_{|V_{[\lambda]}}$ for $\lambda = \pm 1$:
Let $\lambda = \pm 1$ be an eigenvalue of A. There exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of matrices of the form
$$\begin{pmatrix} J(\lambda, r_j)^{-1} & C(r_j, s_j, \lambda) \\ 0 & J(\lambda, r_j)^{\tau} \end{pmatrix}$$

where $C(r_j,s_j,\lambda) := J(\lambda,r_j)^{-1}\operatorname{diag}\big(0,\ldots,0,s_j\big)$ *with* $s_j \in \{0,1,-1\}$*. If* $s_j = 0$*, then* r_j *is odd. The dimension of the eigenspace of the eigenvalue* λ *is given by* $2\operatorname{Card}\{j \mid s_j = 0\} + \operatorname{Card}\{j \mid s_j \neq 0\}$*.*
The number of s_j *equal to* $+1$ *(resp.* -1*) arising in blocks of dimension* $2k$ *(i.e. with corresponding* $r_j = k$*) is equal to the number of positive (resp. negative) eigenvalues of the symmetric* 2*-form*

$$\hat{Q}^{\lambda}_{2k} : \operatorname{Ker}\big((A-\lambda\operatorname{Id})^{2k}\big) \times \operatorname{Ker}\Big((A-\lambda\operatorname{Id})^{2k}\Big) \to \mathbb{R}$$

$$(v,w) \mapsto \lambda\,\Omega\big((A-\lambda\operatorname{Id})^{k}v,(A-\lambda\operatorname{Id})^{k-1}w\big).$$

The decomposition is unique up to a permutation of the blocks and is determined by λ*, by the dimension* $\dim\big(\operatorname{Ker}(A-\lambda\operatorname{Id})^{r}\big)$ *for each* $r \geq 1$*, and by the rank and the signature of the symmetric bilinear* 2*-form* $\hat{Q}^{\lambda}_{2k}$ *for each* $k \geq 1$*.*

Normal form for $A_{|V_{[\lambda]}}$ ***for*** $\lambda \in S^1 \setminus \{\pm 1\}$ **:**
Let $\lambda \in S^1, \lambda \neq \pm 1$ *be an eigenvalue of* A*. There exists a symplectic basis of* $V_{[\lambda]}$ *in which the matrix representing the restriction of* A *to* $V_{[\lambda]}$ *is a symplectic direct sum of* $4k_j \times 4k_j$ *matrices* ($k_j \geq 1$) *of the form*

$$\left(\begin{array}{c|c}\Big(J_{\mathbb{R}}(\overline{\lambda},2k_j)\Big)^{-1} & \begin{matrix}0 & \cdots & 0 & & \\ \vdots & & \vdots & s_j\,V^1_{k_j}(\phi) & s_j\,V^2_{k_j}(\phi)\\ 0 & \cdots & 0 & & \end{matrix} \\ \hline 0 & \Big(J_{\mathbb{R}}(\overline{\lambda},2k_j)\Big)^{\tau}\end{array}\right) \tag{5.2}$$

and $(4k_j+2)\times(4k_j+2)$ *matrices* ($k_j \geq 0$) *of the form*

$$\left(\begin{array}{c|c|ccccc|c}\Big(J_{\mathbb{R}}(\overline{\lambda},2k_j)\Big)^{-1} & s_j\,U^2_{k_j}(\phi) & \begin{matrix}0 & \ldots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{matrix} & & \frac{s_j}{2}V^2_{k_j}(\phi) & & \frac{-s_j}{2}V^1_{k_j}(\phi) & U^1_{k_j}(\phi)\\ \hline 0 & \cos\phi & 0 \;\ldots\; 0 & & 1 & & 0 & s_j\sin\phi\\ \hline 0 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & & \Big(J_{\mathbb{R}}(\overline{\lambda},2k_j)\Big)^{\tau} & & & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ \hline 0 & -s_j\sin\phi & 0 \;\ldots\; 0 & & 0 & & -s_j & \cos\phi\end{array}\right) \tag{5.3}$$

where $J_{\mathbb{R}}(e^{i\phi},2k)$ *is defined as above, where* $\Big(V^1_{k_j}(\phi)\,V^2_{k_j}(\phi)\Big)$ *is the* $2k_j \times 2$ *matrix defined by*

$$\Big(V^1_{k_j}(\phi)\,V^2_{k_j}(\phi)\Big) = \begin{pmatrix}(-1)^{k_j-1}R(e^{ik_j\phi})\\ \vdots\\ R(e^{i\phi})\end{pmatrix} \tag{5.4}$$

with $R(e^{i\phi}) = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}$*, where*

$$\Big(U^1_{k_j}(\phi)\,U^2_{k_j}(\phi)\Big) = \Big(V^1_{k_j}(\phi)\,V^2_{k_j}(\phi)\Big)\Big(R(e^{i\phi})\Big) \tag{5.5}$$

and where $s_j = \pm 1$. The complex dimension of the eigenspace of the eigenvalue λ in $V^{\mathbb{C}}$ is given by the number of such matrices.
The number of s_j equal to $+1$ (resp. -1) arising in blocks of dimension $2m$ in the normal decomposition given above is equal to the number of positive (resp. negative) eigenvalues of the Hermitian 2-form $\hat{Q}^\lambda_m$ defined on $\operatorname{Ker}\big((A-\lambda\operatorname{Id})^m\big)$ by:

$$\begin{aligned}\hat{Q}^\lambda_m :\quad & \operatorname{Ker}\big((A-\lambda\operatorname{Id})^m\big)\times\operatorname{Ker}\big((A-\lambda\operatorname{Id})^m\big)\to\mathbb{C}\\ & (v,w)\mapsto \tfrac{1}{\lambda}\Omega\big((A-\lambda\operatorname{Id})^k v,(A-\overline{\lambda}\operatorname{Id})^{k-1}\overline{w}\big) \qquad \text{if } m=2k\\ & (v,w)\mapsto i\,\Omega\big((A-\lambda\operatorname{Id})^k v,(A-\overline{\lambda}\operatorname{Id})^{k}\overline{w}\big) \qquad \text{if } m=2k+1.\end{aligned}$$

This decomposition is unique up to a permutation of the blocks, when λ has been chosen in $\{\lambda,\overline{\lambda}\}$. It is determined by the chosen λ, by the dimension $\dim\big(\operatorname{Ker}(A-\lambda\operatorname{Id})^r\big)$ for each $r\geq 1$ and by the rank and the signature of the Hermitian bilinear 2-form $\hat{Q}^\lambda_m$ for each $m\geq 1$.

The normal form for $A_{|V_{[\lambda]}}$ is given in Theorem 5.2.1 for $\lambda\notin S^1$, in Theorem 6.5.1 for $\lambda=\pm 1$, and in Theorem 5.4.2 for $\lambda\in S^1\setminus\{\pm 1\}$. The characterisation of the signs is given in Proposition 5.3.3 for $\lambda=\pm 1$ and in Proposition 5.4.4 for $\lambda\in S^1\setminus\{\pm 1\}$.

A direct consequence of Theorem 5.0.8 is the following characterization of the conjugacy class of a matrix in the symplectic group.

Theorem 5.0.9 *The conjugacy class of a matrix $A\in\operatorname{Sp}(2n,\mathbb{R})$ is determined by the following data:*

- *the eigenvalues of A which arise in quadruples $[\lambda]=\{\lambda,\lambda^{-1},\overline{\lambda},\overline{\lambda}^{-1}\}$;*
- *the dimension $\dim\big(\operatorname{Ker}(A-\lambda\operatorname{Id})^r\big)$ for each $r\geq 1$ for one eigenvalue in each class $[\lambda]$;*
- *for $\lambda=\pm 1$, the rank and the signature of the symmetric form $\hat{Q}^\lambda_{2k}$ for each $k\geq 1$ and for an eigenvalue λ in $S^1\setminus\{\pm 1\}$ chosen in each $[\lambda]$, the rank and the signature of the Hermitian form $\hat{Q}^\lambda_m$ for each $m\geq 1$, with*

$$\begin{aligned}\hat{Q}^\lambda_m :\quad & \operatorname{Ker}\big((A-\lambda\operatorname{Id})^m\big)\times\operatorname{Ker}\big((A-\lambda\operatorname{Id})^m\big)\to\mathbb{C}\\ & (v,w)\mapsto \tfrac{1}{\lambda}\Omega\big((A-\lambda\operatorname{Id})^k v,(A-\overline{\lambda}\operatorname{Id})^{k-1}\overline{w}\big) \qquad \text{if } m=2k\\ & (v,w)\mapsto i\,\Omega\big((A-\lambda\operatorname{Id})^k v,(A-\overline{\lambda}\operatorname{Id})^{k}\overline{w}\big) \qquad \text{if } m=2k+1.\end{aligned}$$

□

5.1 Preliminaries

Lemma 5.1.1 *Consider $A\in\operatorname{Sp}(V,\Omega)$ and let $0\neq\lambda\in\mathbb{C}$. Then $\operatorname{Ker}(A-\lambda\operatorname{Id})^j$ in $V^{\mathbb{C}}$ is the symplectic orthogonal complement of $\operatorname{Im}(A-\frac{1}{\lambda}\operatorname{Id})^j$.*

PROOF:

$$\begin{aligned}\Omega\big((A-\lambda\,\mathrm{Id})u, Av\big) &= \Omega(Au,Av)-\lambda\Omega(u,Av)=\Omega(u,v)-\lambda\Omega(u,Av)\\ &= -\lambda\Omega\Big(u,\big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)v\Big)\end{aligned}$$

and by induction

$$\Omega\big((A-\lambda\,\mathrm{Id})^j u, A^j v\big)=(-\lambda)^j\Omega\Big(u,\big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)^j v\Big). \tag{5.6}$$

The result follows from the fact that A is invertible.

Corollary 5.1.2 *If E_λ denotes the generalized eigenspace of eigenvalue λ, i.e $E_\lambda := \big\{v \in V^{\mathbb{C}} \mid (A-\lambda\,\mathrm{Id})^j v = 0$ for an integer $j>0\big\}$, we have*

$$\Omega(E_\lambda, E_\mu)=0 \quad \text{when} \quad \lambda\mu\neq 1.$$

Indeed the symplectic orthogonal complement of $E_\lambda=\cup_j \mathrm{Ker}(A-\lambda\,\mathrm{Id})^j$ is the intersection of the $\mathrm{Im}(A-\frac{1}{\lambda}\,\mathrm{Id})^j$. By Jordan normal form, this intersection is the sum of the generalized eigenspaces corresponding to the eigenvalues which are not $\frac{1}{\lambda}$.

If $v=u+iu'$ is in $\mathrm{Ker}(A-\lambda\,\mathrm{Id})^j$ with u and u' in V then $\overline{v}=u-iu'$ is in $\mathrm{Ker}(A-\overline{\lambda}\,\mathrm{Id})^j$ so that $E_\lambda\oplus E_{\overline{\lambda}}$ is the complexification of a real subspace of V. From this remark and corollary 5.1.2 the space

$$W_{[\lambda]} := E_\lambda \oplus E_{\frac{1}{\lambda}} \oplus E_{\overline{\lambda}} \oplus E_{\frac{1}{\overline{\lambda}}} \tag{5.7}$$

is the complexification of a real and symplectic A-invariant subspace $V_{[\lambda]}$ and

$$V = V_{[\lambda_1]} \oplus V_{[\lambda_2]} \oplus \ldots \oplus V_{[\lambda_K]} \tag{5.8}$$

where we denote by $[\lambda]$ the set $\{\lambda, \overline{\lambda}, \frac{1}{\lambda}, \frac{1}{\overline{\lambda}}\}$ and by $[\lambda_1], \ldots, [\lambda_K]$ the distinct such sets exhausting the eigenvalues of A.
We denote by $A_{[\lambda_i]}$ the restriction of A to $V_{[\lambda_i]}$. It is clearly enough to obtain normal forms for each $A_{[\lambda_i]}$ since A will be a symplectic direct sum of those.

We shall construct a symplectic basis of $W_{[\lambda]}$ (and of $V_{[\lambda]}$) adapted to A for a given eigenvalue λ of A. We assume that $(A-\lambda\,\mathrm{Id})^{p+1}=0$ and $(A-\lambda\,\mathrm{Id})^p\neq 0$ on the generalized eigenspace E_λ. Since A is real, this integer p is the same for $\overline{\lambda}$. By lemma 5.1.1, $\mathrm{Ker}(A-\lambda\,\mathrm{Id})^j$ is the symplectic orthogonal complement of $\mathrm{Im}\big(A-\frac{1}{\lambda}\,\mathrm{Id}\big)^j$ for all j, thus $\dim\mathrm{Ker}(A-\lambda\,\mathrm{Id})^j=\dim\mathrm{Ker}\big(A-\frac{1}{\lambda}\,\mathrm{Id}\big)^j$; hence the integer p is the same for λ and $\frac{1}{\lambda}$.

We decompose $W_{[\lambda]}$ (and $V_{[\lambda]}$) into a direct sum of A-invariant symplectic subspaces. Given a symplectic subspace Z of $V_{[\lambda]}$ which is A-invariant , its orthogonal complement (with respect to the symplectic 2-form) $V' := Z^{\perp_\Omega}$ is again symplectic and A-invariant.

The generalized eigenspace for A on $V'^{\mathbb{C}}$ are $E'_\mu = V'^{\mathbb{C}} \cap E_\mu$, and the smallest integer p' for which $(A - \lambda\,\mathrm{Id})^{p'+1} = 0$ on E'_λ is such that $p' \leq p$.

Hence, to get the decomposition of $W_{[\lambda]}$ (and $V_{[\lambda]}$) it is enough to build a symplectic subspace of $W_{[\lambda]}$ which is A-invariant and closed under complex conjugation and to proceed inductively. We shall construct such a subspace, containing a well chosen vector $v \in E_\lambda$ so that $(A - \lambda\,\mathrm{Id})^p v \neq 0$.

We shall distinguish three cases; first $\lambda \notin S^1$ then $\lambda = \pm 1$ and finally $\lambda \in S^1 \setminus \{\pm 1\}$.

We first present a few technical lemmas which will be used for this construction.

5.1.1 A few technical lemmas

Let (V, Ω) be a real symplectic vector space. Consider $A \in \mathrm{Sp}(V, \Omega)$ and let λ be an eigenvalue of A in $V^{\mathbb{C}}$.

Lemma 5.1.3 *For any positive integer j, the bilinear map*

$$\widetilde{Q}_j : E_\lambda / \mathrm{Ker}(A - \lambda\,\mathrm{Id})^j \times E_{\frac{1}{\lambda}} / \mathrm{Ker}\big(A - \tfrac{1}{\lambda}\,\mathrm{Id}\big)^j \to \mathbb{C}$$

$$\big([v],[w]\big) \mapsto \widetilde{Q}_j\big([v],[w]\big) := \Omega\big((A - \lambda\,\mathrm{Id})^j v, w\big) \qquad v \in E_\lambda, w \in E_{\frac{1}{\lambda}} \tag{5.9}$$

is well defined and non degenerate. In the formula, $[v]$ denotes the class containing v in the appropriate quotient.

Proof: The fact that $\widetilde{Q}_j$ is well defined follows from equation (5.6); indeed, for any integer j, we have

$$\Omega\big((A - \lambda\,\mathrm{Id})^j u, v\big) = (-\lambda)^j \Omega\Big(A^j u, \big(A - \tfrac{1}{\lambda}\,\mathrm{Id}\big)^j v\Big). \tag{5.10}$$

The map is non degenerate because $\widetilde{Q}_j\big([v],[w]\big) = 0$ for all w if and only if $(A - \lambda\,\mathrm{Id})^j v = 0$ since Ω is a non degenerate pairing between E_λ and $E_{\frac{1}{\lambda}}$, thus if and only if $[v] = 0$. Similarly, $\widetilde{Q}_j\big([v],[w]\big) = 0$ for all v if and only if w is Ω-orthogonal to $\mathrm{Im}(A - \lambda\,\mathrm{Id})^j$, thus if and only if $w \in \mathrm{Ker}\big(A - \frac{1}{\lambda}\,\mathrm{Id}\big)^j$ hence $[w] = 0$.

Lemma 5.1.4 *For any $v, w \in V$, any $\lambda \in \mathbb{C} \setminus \{0\}$ and any integers $i \geq 0$, $j > 0$ we have:*

$$\begin{aligned}\Omega\Big((A - \lambda\,\mathrm{Id})^i v, \big(A - \tfrac{1}{\lambda}\,\mathrm{Id}\big)^j w\Big) &= -\frac{1}{\lambda}\Omega\Big((A - \lambda\,\mathrm{Id})^{i+1} v, \big(A - \tfrac{1}{\lambda}\,\mathrm{Id}\big)^j w\Big) \\ &\quad -\frac{1}{\lambda^2}\Omega\Big((A - \lambda\,\mathrm{Id})^{i+1} v, \big(A - \tfrac{1}{\lambda}\,\mathrm{Id}\big)^{j-1} w\Big).\end{aligned} \tag{5.11}$$

In particular, if λ is an eigenvalue of A, if $v \in E_\lambda$ is such that $p \geq 0$ is the largest integer for which $(A - \lambda\,\mathrm{Id})^p v \neq 0$, we have for any integers $k, j \geq 0$:

$$\Omega\big((A - \lambda\,\mathrm{Id})^{p+k} v, w\big) = (-\lambda^2)^j \Omega\Big((A - \lambda\,\mathrm{Id})^{p+k-j} v, \big(A - \tfrac{1}{\lambda}\,\mathrm{Id}\big)^j w\Big) \tag{5.12}$$

so that

$$\Omega\big((A-\lambda\,\mathrm{Id})^p v, w\big) = (-\lambda^2)^p \Omega\Big(v, \big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)^p w\Big) \tag{5.13}$$

and

$$\Omega\Big((A-\lambda\,\mathrm{Id})^k v, \big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)^j w\Big) = 0 \textit{ if } \ k+j>p. \tag{5.14}$$

Proof: We have:

$$\begin{aligned}
&\Omega\Big((A-\lambda\,\mathrm{Id})^i v, \big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)^j w\Big)\\
&\quad = -\frac{1}{\lambda}\Omega\Big(\big(A-\lambda\,\mathrm{Id}-A\big)(A-\lambda\,\mathrm{Id})^i v, \big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)^j w\Big)\\
&\quad = -\frac{1}{\lambda}\Omega\Big((A-\lambda\,\mathrm{Id})^{i+1} v, \big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)^j w\Big)\\
&\qquad +\frac{1}{\lambda}\Omega\Big(A(A-\lambda\,\mathrm{Id})^i v, \big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)\big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)^{j-1} w\Big)\\
&\quad = -\frac{1}{\lambda}\Omega\Big((A-\lambda\,\mathrm{Id})^{i+1} v, \big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)^j w\Big)\\
&\qquad +\frac{1}{\lambda}\Omega\Big((A-\lambda\,\mathrm{Id})^i v, \big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)^{j-1} w\Big)\\
&\qquad -\frac{1}{\lambda^2}\Omega\Big(A(A-\lambda\,\mathrm{Id})^i v, \big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)^{j-1} w\Big)
\end{aligned}$$

and formula (5.11) follows.
For any integers $k, j \geq 0$ and any v such that $(A-\lambda\,\mathrm{Id})^p v = 0$, we have, by (5.6),

$$(-\lambda)^j \Omega\Big((A-\lambda\,\mathrm{Id})^{p+k+1-j} v, \big(A-\tfrac{1}{\lambda}\,\mathrm{Id}\big)^j w\Big) = \Omega\big((A-\lambda\,\mathrm{Id})^{p+k+1} v, A^j w\big) = 0.$$

Hence, applying formula (5.11) with a decreasing induction on j, we get formula (5.12). The other formulas follow readily.

Definition 5.1.5 For $\lambda \in S^1$ an eigenvalue of A and $v \in E_\lambda$ a generalized eigenvector, we define

$$T_{i,j}(v) := \frac{1}{\lambda^i \overline{\lambda}^j}\Omega\big((A-\lambda\,\mathrm{Id})^i v, (A-\overline{\lambda})^j \overline{v}\big). \tag{5.15}$$

We have, by equation (5.11) :

$$T_{i,j}(v) = -T_{i+1,j}(v) - T_{i+1,j-1}(v), \tag{5.16}$$

and also,

$$T_{i,j}(v) = -\overline{T_{j,i}(v)}. \tag{5.17}$$

Lemma 5.1.6 *Let $\lambda \in S^1$ be an eigenvalue of A and $v \in E_\lambda$ be a generalised eigenvector such that the largest integer p so that $(A - \lambda\,\mathrm{Id})^p v \neq 0$ is odd, say, $p = 2k-1$. Then, in the A-invariant subspace E_λ^v of E_λ generated by v, there exists a vector v' generating the same A-invariant subspace $E_\lambda^{v'} = E_\lambda^v$, so that $(A - \lambda\,\mathrm{Id})^p v' \neq 0$ and so that*

$$T_{i,j}(v') = 0 \quad \text{for all } i, j \leq k-1.$$

If λ is real (i.e. ± 1), and if v is a real vector (i.e. in V), the vector v' can be chosen to be real as well.

Proof: Observe that

$$\begin{aligned} T_{k,k-1}(v) &= -T_{k,k}(v) - T_{k-1,k}(v) \quad \text{by (5.11)} \\ &= -T_{k-1,k}(v) \quad \text{by (5.14)} \\ &= \overline{T_{k,k-1}(v)} \quad \text{by (5.17)} \end{aligned}$$

is real and can be put to $d = \pm 1$ by rescaling the vector. We use formulas (5.11) and (5.17) and we proceed by decreasing induction on $i + j$ as follows:

- if $T_{k-1,k-1}(v) = \alpha_1$, this α_1 is purely imaginary, we replace v by

  $$v' := v - \frac{\alpha_1}{2\lambda d}(A - \lambda\,\mathrm{Id})v;$$

  clearly $E_\lambda^{v'} = E_\lambda^v$ and $T_{i,j}(v') = T_{i,j}(v)$ for $i + j \geq 2k-1$ but now

  $$T_{k-1,k-1}(v') = \alpha_1 - \frac{\alpha_1}{2d}T_{k,k-1}(v) - \frac{\overline{\alpha_1}}{2d}T_{k-1,k}(v) = 0;$$

  so we can now assume $T_{k-1,k-1}(v) = 0$; observe that if λ is real and v is in V, then $\alpha_1 = 0$ and $v' = v$;

- if $T_{k-2,k-1}(v) = \alpha_2 = -T_{k-1,k-2}(v)$, this α_2 is real and we replace v by

  $$v - \frac{\alpha_2}{2\lambda^2 d}(A - \lambda\,\mathrm{Id})^2 v;$$

  the space E_λ^v does not change and the quantities $T_{i,j}(v)$ do not vary for $i+j \geq 2k-2$; now

  $$T_{k-2,k-1}(v') = \alpha_2 - \frac{\alpha_2}{2d}T_{k,k-1}(v) - \frac{\overline{\alpha_2}}{2d}T_{k-2,k+1}(v) = 0,$$

  hence also $T_{k-1,k-2}(v') = 0$; observe that if λ is real and v is in V, then v' is in V.

- we now assume by induction to have a $J > 0$ so that $T_{i,j}(v) = 0$ for all $0 \leq i, j \leq k-1$ so that $i + j > 2k - 1 - J$;

- if $T_{k-J,k-1}(v) = \alpha_J$, then $T_{k-J,k-1}(v) = (-1)^{J-1}T_{k-1,k-J}(v)$ so that α_J is real when J is even and is imaginary when J is odd; we replace v by
$$v - \frac{\alpha_J}{2\lambda^J d}(A - \lambda\,\mathrm{Id})^J v;$$
the space E^v_λ does not change and the quantities $T_{i,j}(v)$ do not vary for $i+j \geq 2k-J$; but now
$$\begin{aligned} T_{k-J,k-1}(v') &= \alpha_J - \frac{\alpha_J}{2d}T_{k,k-1}(v) - \frac{\overline{\alpha_J}}{2d}T_{k-J,k+J-1}(v) \\ &= \alpha_J - \frac{\alpha_J}{2} - (-1)^J\frac{\overline{\alpha_J}}{2} = 0. \end{aligned}$$
Hence also $T_{k-J+1,k-2}(v') = 0, \ldots\ T_{k-1,k-J+1}(v') = 0$; so the induction proceeds. Observe that if λ is real and v is in V then v' is in V.

We shall use repeatedly that a $n \times n$ block triangular symplectic matrix is of the form
$$A' = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R}) \Leftrightarrow \begin{cases} B = (D^\tau)^{-1} \\ C = (D^\tau)^{-1}S \ \text{ with}\, S \ \text{ symmetric.} \end{cases} \tag{5.18}$$

5.2 Normal forms for $A_{|V_{[\lambda]}}$ when $\lambda \notin S^1$.

As before, p denotes the largest integer such that $(A - \lambda\,\mathrm{Id})^p$ does not vanish identically on the generalized eigenspace E_λ. Let us choose an element $v \in E_\lambda$ and an element $w \in E_{\frac{1}{\lambda}}$ such that
$$\widetilde{Q}_p\big([v],[w]\big) = \Omega\big((A - \lambda\,\mathrm{Id})^p v, w\big) \neq 0.$$
Let us consider the smallest A-invariant subspace E^v_λ of E_λ containing v; it is of dimension $p+1$ and a basis is given by
$$\big\{a_0 := v, \ldots, a_i := (A - \lambda\,\mathrm{Id})^i v, \ldots, a_p := (A - \lambda\,\mathrm{Id})^p v\big\}.$$
Observe that $Aa_i = (A - \lambda\,\mathrm{Id})a_i + \lambda a_i$ so that $Aa_i = \lambda a_i + a_{i+1}$ for $i < p$ and $Aa_p = a_p$.

Similarly, we consider the smallest A-invariant subspace $E^w_{\frac{1}{\lambda}}$ of $E_{\frac{1}{\lambda}}$ containing w; it is also of dimension $p+1$ and a basis is given by
$$\left\{b_0 := w, \ldots, b_j := \left(A - \tfrac{1}{\lambda}\,\mathrm{Id}\right)^j w, \ldots b_p := \left(A - \tfrac{1}{\lambda}\,\mathrm{Id}\right)^p w\right\}.$$
One has

$\Omega(a_i, a_j) = 0$ and $\Omega(b_i, b_j) = 0$ because $\Omega(E_\lambda, E_\mu) = 0$ if $\lambda\mu \neq 1$;

$\Omega(a_i, b_j) = 0$ if $i + j > p$ by equation (5.14) ;

$\Omega(a_i, b_{p-i}) = \big(\frac{-1}{\lambda^2}\big)^{p-i}\Omega\Big(\big(A - \lambda \operatorname{Id}\big)^p v, w\Big)$ by equation (5.12) and is non zero by the choice of v, w.

The matrix representing Ω in the basis $\{b_p, \ldots, b_0, a_0, \ldots, a_p\}$ is thus of the form

$$\left(\begin{array}{cc|cc} 0 & 0 & \overline{*} & 0 \\ & \ddots & \ddots & \\ 0 & 0 & * & \overline{*} \\ \hline \overline{*} & * & 0 & 0 \\ & \ddots & \ddots & \\ 0 & \overline{*} & 0 & 0 \end{array}\right)$$

with non vanishing $\overline{*}$. Hence Ω is non degenerate on $E^v_\lambda \oplus E^w_{\frac{1}{\lambda}}$ which is thus a symplectic A-invariant subspace.

We now construct a symplectic basis $\big\{b'_p, \ldots, b'_0, a_0, \ldots, a_p\big\}$ of $E^v_\lambda \oplus E^w_{\frac{1}{\lambda}}$, extending $\{a_0, \ldots, a_p\}$, using a Gram-Schmidt procedure on the b_i's. This gives a normal form for A on $E^v_\lambda \oplus E^w_{\frac{1}{\lambda}}$.

If λ is real, we take v, w in the real generalized eigenspaces $E^{\mathbb{R}}_\lambda$ and $E^{\mathbb{R}}_{\frac{1}{\lambda}}$ and we obtain a symplectic basis of the real A-invariant symplectic vector space , $E^{\mathbb{R}v}_\lambda \oplus E^{\mathbb{R}w}_{\frac{1}{\lambda}}$. If λ is not real, one considers the basis of $E^{\overline{v}}_{\overline{\lambda}} \oplus E^{\overline{w}}_{\frac{1}{\overline{\lambda}}}$ defined by the conjugate vectors $\{\overline{b'_p}, \ldots, \overline{b'_0}, \overline{a_0}, \ldots, \overline{a_p}\}$ and this yields a conjugate normal form on $E_{\overline{\lambda}} \oplus E_{\frac{1}{\overline{\lambda}}}$, hence a normal form on $W_{[\lambda]}$ and this will induce a real normal form on $V_{[\lambda]}$.

We choose v and w such that $\Omega\Big(\big(A - \frac{1}{\lambda} \operatorname{Id}\big)^p w, v\Big) = 1$. We define inductively on j

$b'_p := \frac{1}{\Omega(b_p, a_0)} b_p = b_p$;

$b'_{p-j} = \frac{1}{\Omega(b_{p-j}, a_j)}\big(b_{p-j} - \sum_{k<j} \Omega(b_{p-j}, a_k) b'_{p-k}\big)$,
so that any b'_j is a linear combination of the b_r with $r \geq j$.

In the symplectic basis $\big\{b'_p, \ldots, b'_0, a_0, \ldots, a_p\big\}$ the matrix representing A is

$$\begin{pmatrix} B & 0 \\ 0 & J(\lambda, p+1)^\tau \end{pmatrix}$$

where

$$J(\lambda, m) = \begin{pmatrix} \lambda & 1 & & & & & \\ & \lambda & 1 & & & 0 & \\ & & \lambda & 1 & & & \\ & & & \ddots & \ddots & & \\ & 0 & & & \lambda & 1 & \\ & & & & & \lambda & 1 \\ & & & & & & \lambda \end{pmatrix} \tag{5.19}$$

is the elementary $m \times m$ Jordan matrix associated to λ. Since the matrix is symplectic, B is the transpose of the inverse of $J(\lambda, p+1)^\tau$ by (5.18), so $B = J(\lambda, p+1)^{-1}$.
This is the normal form for A restricted to $E^v_\lambda \oplus E^w_{\frac{1}{\lambda}}$.
If $\lambda = re^{i\phi} \notin \mathbb{R}$ we consider the symplectic basis $\{b'_p, \ldots, b'_0, a_0, \ldots, a_p\}$ of $E^v_\lambda \oplus E^w_{\frac{1}{\lambda}}$ as above and the conjugate symplectic basis $\{\overline{b'_p}, \ldots, \overline{b'_0}, \overline{a_0}, \ldots, \overline{a_p}\}$ of $E^{\overline{v}}_{\overline{\lambda}} \oplus E^{\overline{w}}_{\frac{1}{\overline{\lambda}}}$. Writing $b'_j = \frac{1}{\sqrt{2}}(u_j + iv_j)$ and $a_j = \frac{1}{\sqrt{2}}(w_j - ix_j)$ for all $0 \leq j \leq p$ with the vectors u_j, v_j, w_j, x_j in the real vector space V, we get a symplectic basis $\{u_p, v_p \ldots, u_0, v_0, w_0, x_0 \ldots, w_p, x_p\}$ of the real subspace of V whose complexification is $E^v_\lambda \oplus E^w_{\frac{1}{\lambda}} \oplus E^{\overline{v}}_{\overline{\lambda}} \oplus E^{\overline{w}}_{\frac{1}{\overline{\lambda}}}$. In this basis, the matrix representing A is

$$\begin{pmatrix} J_{\mathbb{R}}\big(\overline{\lambda}, 2(p+1)\big)^{-1} & 0 \\ 0 & J_{\mathbb{R}}\big(\overline{\lambda}, 2(p+1)\big)^{\tau} \end{pmatrix}$$

where $J_{\mathbb{R}}(re^{i\phi}, 2m)$ is the $2m \times 2m$ matrix written in terms of 2×2 matrices as

$$J_{\mathbb{R}}(re^{i\phi}, 2m) := \begin{pmatrix} R(re^{i\phi}) & \mathrm{Id} & & & & & \\ & R(re^{i\phi}) & \mathrm{Id} & & & 0 & \\ & & R(re^{i\phi}) & \mathrm{Id} & & & \\ & & & \ddots & \ddots & & \\ & 0 & & & R(re^{i\phi}) & \mathrm{Id} & \\ & & & & & R(re^{i\phi}) & \mathrm{Id} \\ & & & & & & R(re^{i\phi}) \end{pmatrix} \tag{5.20}$$

with $R(re^{i\phi}) = \begin{pmatrix} r\cos\phi & -r\sin\phi \\ r\sin\phi & r\cos\phi \end{pmatrix}$. By induction, we get

Theorem 5.2.1 (Normal form for $A_{|V_{[\lambda]}}$ for $\lambda \notin S^1$.) *Let $\lambda \notin S^1$ be an eigenvalue of A. Denote $k := \dim_{\mathbb{C}} \mathrm{Ker}(A - \lambda\,\mathrm{Id})$ (on $V^{\mathbb{C}}$) and p the smallest integer so that $(A - \lambda\,\mathrm{Id})^{p+1}$ is identically zero on the generalized eigenspace E_λ.*

- *If $\lambda \neq \pm 1$ is a real eigenvalue of A, there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of k matrices of the form*

$$\begin{pmatrix} J(\lambda, p_j+1)^{-1} & 0 \\ 0 & J(\lambda, p_j+1)^{\tau} \end{pmatrix}$$

  with $p = p_1 \geq p_2 \geq \cdots \geq p_k$ and $J(\lambda, k)$ defined by (5.19). To eliminate the ambiguity in the choice of λ in $[\lambda] = \{\lambda, \lambda^{-1}\}$ we can consider the real eigenvalue such that $\lambda > 1$. The size of the blocks is determined knowing the dimension $\dim\left(\mathrm{Ker}(A - \lambda\,\mathrm{Id})^r\right)$ for each $r \geq 1$.

- *If $\lambda = re^{i\phi} \notin (S^1 \cup \mathbb{R})$ is a complex eigenvalue of A, there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct* *sum of k matrices of the form*

$$\begin{pmatrix} J_{\mathbb{R}}\big(re^{-i\phi},2(p_j+1)\big)^{-1} & 0 \\ 0 & J_{\mathbb{R}}\big(re^{-i\phi},2(p_j+1)\big)^{\tau} \end{pmatrix}$$

with $p=p_1\geq p_2\geq\cdots\geq p_k$ and $J_{\mathbb{R}}(re^{i\phi},k)$ defined by (5.20). To eliminate the ambiguity in the choice of λ in $[\lambda]=\{\lambda,\lambda^{-1},\overline{\lambda},\overline{\lambda}^{-1}\}$ we can choose the eigenvalue λ with a positive imaginary part and a modulus greater than 1. The size of the blocks is determined, knowing the dimension $\dim_{\mathbb{C}}(\mathrm{Ker}(A-\lambda\,\mathrm{Id})^r)$ for each $r\geq 1$.

This normal form is unique, when a choice of λ in the set $[\lambda]$ is fixed.

5.3 Normal forms for $A_{|V_{[\lambda]}}$ when $\lambda=\pm 1$.

In this situation $[\lambda]=\{\lambda\}$ and $V_{[\lambda]}$ is the generalized real eigenspace of eigenvalue λ, still denoted –with a slight abuse of notation– E_λ. Again, p denotes the largest integer such that $(A-\lambda\,\mathrm{Id})^p$ does not vanish identically on E_λ. We consider $\widetilde{Q}_p : E_\lambda/\mathrm{Ker}(A-\lambda\,\mathrm{Id})^p\times E_\lambda/\mathrm{Ker}(A-\lambda\,\mathrm{Id})^p\to\mathbb{R}$ the non degenerate form defined by $\widetilde{Q}_p\big([v],[w]\big)=\Omega\big((A-\lambda\,\mathrm{Id})^p v,w\big)$. We see directly from equation (5.13) that $\widetilde{Q}_p$ is symmetric if p is odd and antisymmetric if p is even.

5.3.1 If $p=2k-1$ is odd

we choose $v\in E_\lambda$ such that

$$\widetilde{Q}\big([v],[v]\big)=\Omega\big((A-\lambda\,\mathrm{Id})^p v,v\big)\neq 0$$

and consider the smallest A-invariant subspace E^v_λ of E_λ containing v; it is spanned by

$$\big\{a_p:=(A-\lambda\,\mathrm{Id})^p v,\ldots,a_i:=(A-\lambda\,\mathrm{Id})^i v,\ldots,a_0:=v\big\}.$$

We have

$\Omega(a_i,a_j)=0$ if $i+j\geq p+1(=2k)$ by equation (5.14);

$\Omega(a_i,a_{p-i})\neq 0$; by equation(5.12) and by the choice of v.

Hence E^v_λ is a symplectic subspace because, in the basis defined by the e_i's, Ω has the triangular form $\begin{pmatrix} 0 & & \overline{*} \\ & \cdot^{\cdot^{\cdot}} & \\ \overline{*} & & * \end{pmatrix}$ and has a non-zero determinant.

We can choose v in $E_\lambda \subset V$ so that $\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\lambda\,\mathrm{Id})^{k-1}v\big) = \lambda s$ with $s = \pm 1$ by rescaling the vector and one may further assume, by lemma 5.1.6, that

$$T_{i,j}(v) = \frac{1}{\lambda^i}\frac{1}{\lambda^j}\Omega\big((A-\lambda\,\mathrm{Id})^i v, (A-\lambda\,\mathrm{Id})^j v\big) = 0 \quad \text{for all } \ 0 \le i,j \le k-1.$$

We now construct a symplectic basis $\{a'_p, \ldots, a'_k, a_0, \ldots, a_{k-1}\}$ of E^v_λ, extending $\{a_0, \ldots, a_{k-1}\}$, by a Gram-Schmidt procedure, having chosen v as above. We define inductively on $0 \le j \le k-1$

$a'_p := \frac{1}{\Omega(a_p, a_0)} a_p$;

$a'_{p-j} = \frac{1}{\Omega(a_{p-j}, a_j)}\big(a_{p-j} - \sum_{k<j}\Omega(a_{p-j}, a_k)a'_{p-k}\big)$,
so that any a'_j is a linear combination of the a_r's with $r \ge j$ and in particular $a'_k = \frac{1}{s\lambda}a_k + \sum_{j=1}^{k-1} c_j a_{k+j}$.

In the symplectic basis $\{a'_p, \ldots, a'_k, a_0, \ldots, a_{k-1}\}$ the matrix representing A is

$$A' = \begin{pmatrix} B & C \\ 0 & J(\lambda,k)^\tau \end{pmatrix}$$

with $J(\lambda, m)$ defined by (5.19) and with C identically zero except for the last column, and the coefficient $C^k_k = s\lambda$. Since the matrix is symplectic, B is the transpose of the inverse of $J(\lambda, p+1)^\tau$ by (5.18), so $B = J(\lambda,k)^{-1}$ and $J(\lambda,k)C$ is symmetric with zeroes except in the last column, hence diagonal of the form $\mathrm{diag}\big(0, \ldots, 0, s\big)$. Thus

$$\begin{pmatrix} J(\lambda,k)^{-1} & J(\lambda,k)^{-1}\mathrm{diag}\big(0,\ldots,0,s\big) \\ 0 & J(\lambda,k)^\tau \end{pmatrix},$$

with $s = \pm 1$, is the normal form of A restricted to E^v_λ. Recall that

$$s = \lambda^{-1}\,\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\lambda\,\mathrm{Id})^{k-1}v\big).$$

5.3.2 If $p = 2k$ is even

we choose v and w in E_λ such that

$$\widetilde{Q}\big([v],[w]\big) = \Omega\big((A-\lambda\,\mathrm{Id})^p v, w\big) = \lambda^p = 1$$

and we consider the smallest A-invariant subspace $E^v_\lambda \oplus E^w_\lambda$ of E_λ containing v and w. It is of dimension $4k+2$. Remark that $\Omega\big((A-\lambda\,\mathrm{Id})^p v, v\big) = 0$. We can choose v so that

$$T_{r,s}(v) = \frac{1}{\lambda^{r+s}}\Omega\big((A-\lambda\,\mathrm{Id})^r v, (A-\lambda\,\mathrm{Id})^s v\big) = 0 \quad \text{for all } \ r,s.$$

Indeed, by formula (5.11) we have $T_{i,j}(v) = -T_{i+1,j}(v) - T_{i+1,j-1}(v)$. Observe that $T_{i,j}(v) = -T_{j,i}(v)$ so that $T_{i,i}(v) = 0$ and $T_{j,i}(v) = -T_{j,i+1}(v) - T_{j-1,i+1}(v)$. We proceed by induction, as in lemma 5.1.6 :

- $T_{p,0}(v) = 0$ implies $T_{p-r,r}(v) = 0$ for all $0 \leq r \leq p$ by equation (5.12).
- We assume by decreasing induction on J, starting from $J = p$, that we have $T_{i,j}(v) = 0$ for all $i + j \geq J$. Then we have $T_{J-1-s,s}(v) = -T_{J-1-s,s+1}(v) - T_{J-2-s,s+1}(v)$; the first term on the righthand side vanishes by the induction hypothesis, so $T_{J-1,0}(v) = (-1)^s T_{J-1-s,s}(v) = (-1)^{J-1} T_{0,J-1}(v) = (-1)^J T_{J-1,0}$.

  If $T_{J-1,0}(v) = \alpha \neq 0$, J must be even and we replace v by
$$v' = v + \tfrac{\alpha}{2\lambda^{p-J+1}}(A - \lambda\,\mathrm{Id})^{p-J+1}w.$$
  Then $v' \in E^v_\lambda \oplus E^w_\lambda$, $E^v_\lambda \oplus E^w_\lambda = E^{v'}_\lambda \oplus E^w_\lambda$, $\Omega\big((A - \lambda\,\mathrm{Id})^p v', w\big) = \lambda^p$ and $T_{i,j}(v') = T_{i,j}(v) = 0$ for all $i + j \geq J$ but now
$$\begin{array}{rcl} T_{J-1,0}(v') & = & T_{J-1,0}(v) + \tfrac{\alpha}{2\lambda^p}\Omega\big((A - \lambda\,\mathrm{Id})^p w, v\big) \\ & & +\tfrac{\alpha}{2\lambda^p}\Omega\big((A - \lambda\,\mathrm{Id})^{J-1}v, (A - \lambda\,\mathrm{Id})^{p-J+1}w\big) \\ & & +\tfrac{\alpha^2}{4\lambda^p}\Omega\big((A - \lambda\,\mathrm{Id})^p w, (A - \lambda\,\mathrm{Id})^{p-J+1}w\big) \\ & = & \alpha - \dfrac{\alpha}{2} - \dfrac{\alpha}{2} = 0 \end{array}$$
  so that $T_{i,j}(v') = 0$ for all $i + j \geq J - 1$ and the induction proceeds.

We assume from now on that we have chosen v and w in E_λ so that
$\Omega\big((A - \lambda\,\mathrm{Id})^p v, w\big) = 1$ and $\Omega\big((A - \lambda\,\mathrm{Id})^r v, (A - \frac{1}{\lambda}\,\mathrm{Id})^s v\big) = 0$ for all r, s.
We can proceed similarly with w so we can thus furthermore assume that
$\Omega\Big((A - \lambda\,\mathrm{Id})^j w, \big(A - \lambda\,\mathrm{Id}\big)^k w\Big) = 0$ for all j, k.

A basis of $E^v_\lambda \oplus E^w_\lambda$ is given by
$$\big\{a_p = (A - \lambda\,\mathrm{Id})^p v, \ldots, a_0 = v, b_0 = w, \ldots, b_p = (A - \lambda\,\mathrm{Id})^p w\big\}.$$
We have

> $\Omega(a_i, a_j) = 0$ and $\Omega(b_i, b_j) = 0$ by the choice of v and w;
>
> $\Omega(a_i, b_j) = 0$ if $i + j > p$ by equation (5.14) ;
>
> $\Omega(a_i, b_{p-i}) \neq 0$ by equation (5.12) and the choice of of v, w.

The matrix representing Ω has the form $\left(\begin{array}{ccc|ccc} & & & \bar{*} & & 0 \\ & 0 & & & \ddots & \\ & & & * & & \bar{*} \\ \hline \bar{*} & & * & & & \\ & \ddots & & & 0 & \\ 0 & & \bar{*} & & & \end{array}\right)$ hence is non singular and the subspace $E^v_\lambda \oplus E^w_\lambda$ is symplectic. We now construct a symplectic basis $\big\{a'_p, \ldots, a'_0, b_0, \ldots, b_p\big\}$ of $E^v_\lambda \oplus E^w_{\frac{1}{\lambda}}$, extending $\{b_0, \ldots, b_p\}$, using a Gram-Schmidt procedure on the a_i's. We define inductively on j

$a'_p := \frac{1}{\Omega(a_p,b_0)} a_p$;

$a'_{p-j} = \frac{1}{\Omega(a_{p-j},b_j)} \big(a_{p-j} - \sum_{k<j} \Omega(a_{p-j}, b_k) a'_{p-k}\big)$,
so that any a'_j is a linear combination of the a'_k with $k \geq j$.

In the symplectic basis $\{a'_p, \ldots, a'_0, b_0, \ldots, b_p\}$ the matrix representing A is

$$\begin{pmatrix} B & 0 \\ 0 & J(\lambda, p+1)^\tau \end{pmatrix}.$$

Hence, the matrix

$$\begin{pmatrix} J(\lambda, p+1)^{-1} & 0 \\ 0 & J(\lambda, p+1)^\tau \end{pmatrix}$$

is a normal form for A restricted to $E^v_\lambda \oplus E^w_\lambda$. Thus we have:

Theorem 5.3.1 (Normal form for $A_{|V_{[\lambda]}}$ for $\lambda = \pm 1$.) *Let $\lambda = \pm 1$ be an eigenvalue of A. There exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of matrices of the form*

$$\begin{pmatrix} J(\lambda, r_j)^{-1} & C(r_j, s_j, \lambda) \\ 0 & J(\lambda, r_j)^\tau \end{pmatrix}$$

where $C(r_j, s_j, \lambda) := J(\lambda, r_j)^{-1} \operatorname{diag}\big(0, \ldots, 0, s_j\big)$ with $s_j \in \{0, 1, -1\}$. If $s_j = 0$, then r_j is odd. The dimension of the eigenspace of eigenvalue 1 is given by $2\operatorname{Card}\{j \mid s_j = 0\} + \operatorname{Card}\{j \mid s_j \neq 0\}$.

Definition 5.3.2 Given $\lambda \in \{\pm 1\}$, we define, for any integer $k \geq 1$, a bilinear form $\hat{Q}^\lambda_{2k}$ on $\operatorname{Ker}\big((A - \lambda \operatorname{Id})^{2k}\big)$:

$$\begin{aligned} \hat{Q}^\lambda_{2k} \ : \ & \operatorname{Ker}\Big((A - \lambda \operatorname{Id})^{2k}\Big) \times \operatorname{Ker}\Big((A - \lambda \operatorname{Id})^{2k}\Big) \to \mathbb{R} \\ & (v, w) \mapsto \lambda\, \Omega\big((A - \lambda \operatorname{Id})^k v, (A - \lambda \operatorname{Id})^{k-1} w\big). \end{aligned} \tag{5.21}$$

It is symmetric.

Proposition 5.3.3 *Given $\lambda \in \{\pm 1\}$, the number of positive (resp. negative) eigenvalues of the symmetric 2-form $\hat{Q}^\lambda_{2k}$ is equal to the number of s_j equal to $+1$ (resp. -1) arising in blocks of dimension $2k$ (i.e. with corresponding $r_j = k$) in the normal decomposition of A on $V_{[\lambda]}$ given in theorem 6.5.1.*
On $V_{[\lambda]}$, we have:

$$\sum_j s_j = \sum_{k=1}^{dimV} \operatorname{Signature}(\hat{Q}^\lambda_{2k}) \tag{5.22}$$

Proof: On the intersection of Ker $\big((A-\lambda\,\mathrm{Id})^{2k}\big)$ with one of the symplectically orthogonal subspaces E^v_λ constructed above for an odd $p \neq 2k-1$, the form $\hat{Q}^\lambda_{2k}$ vanishes identically. On the intersection of Ker $\big((A-\lambda\,\mathrm{Id})^{2k}\big)$ with a subspace E^v_λ for a v so that $p = 2k-1$ and $\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\lambda\,\mathrm{Id})^{k-1}v\big) = \lambda s$ the only non vanishing component is $\hat{Q}^\lambda_{2k}(v,v) = s$.
Indeed, Ker $\big((A-\lambda\,\mathrm{Id})^{2k}\big) \cap E^v_\lambda$ is spanned by

$$\{(A-\lambda\,\mathrm{Id})^r v\,;\, r \geq 0 \text{ and } r+2k > p\},$$

and $\Omega\big((A-\lambda\,\mathrm{Id})^{k+r}v, (A-\lambda\,\mathrm{Id})^{k-1+r'}v\big) = 0$ when $2k+r+r'-1 > p$ so the only non vanishing cases arise when $r = r' = 0$ and $p = 2k-1$.
Similarly, the 2 form $\hat{Q}^\lambda_{2k}$ vanishes on the intersection of Ker $\big((A-\lambda\,\mathrm{Id})^{2k}\big)$ with a subspace $E^v_\lambda \oplus E^w_\lambda$ constructed above for an even p.

The numbers s_j appearing in the decomposition of A are thus invariant of the matrix.

Corollary 5.3.4 *The normal decomposition described in theorem 6.5.1 is determined by the eigenvalue λ, by the dimension $\dim\big(\mathrm{Ker}(A-\lambda\,\mathrm{Id})^r\big)$ for each $r \geq 1$, and by the rank and the signature of the symmetric bilinear 2-forms $\hat{Q}^\lambda_{2k}$ for each $k \geq 1$. It is unique up to a permutation of the blocks.* $\square$

5.4 Normal forms for $A_{|V_{[\lambda]}}$ when $\lambda = e^{i\phi} \in S^1 \setminus \{\pm 1\}$.

We denote again by p the largest integer such that $(A-\lambda\,\mathrm{Id})^p$ does not vanish identically on E_λ and we consider the non degenerate sesquilinear form

$$\widehat{Q} : E_\lambda/\mathrm{Ker}(A-\lambda\,\mathrm{Id})^p \times E_\lambda/\mathrm{Ker}(A-\lambda\,\mathrm{Id})^p \to \mathbb{C}$$
$$\widehat{Q}\big([v],[w]\big) = \overline{\lambda^p}\,\Omega\big((A-\lambda\,\mathrm{Id})^p v, \overline{w}\big).$$

Since $\widehat{Q}$ is non degenerate, we can choose $v \in E_\lambda$ such that $\widehat{Q}([v],[v]) \neq 0$ thus $(A-\lambda\,\mathrm{Id})^p v \neq 0$ and we consider the smallest A-invariant subspace, stable by complex conjugaison, and containing v : $E^v_\lambda \oplus E^{\overline{v}}_{\overline{\lambda}} \subset E_\lambda \oplus E_{\overline{\lambda}}$. A basis is given by

$$\big\{a_i := (A-\lambda\,\mathrm{Id})^i v, b_j := (A-\overline{\lambda}\,\mathrm{Id})^j \overline{v} \quad 0 \leq i,j \leq p\big\}.$$

We have $a_i = \overline{b_i}$ and

- $\Omega(a_i,a_j) = 0,\ \Omega(b_i,b_j) = 0$ because $\Omega(E_\lambda,E_\lambda) = 0$;
- $\Omega(a_i,b_k) = 0$ if $i+k \geq p+1$ by equation (5.14);
- $\Omega(a_i,b_k) \neq 0$ if $p = i+k$ by equation (5.12) and by the choice of v.

We conclude that $E^v_\lambda \oplus E^{\overline{v}}_{\overline{\lambda}}$ is a symplectic subspace.

5.4.1 If $p = 2k-1$ is odd

observe that $T_{k,k-1}(v) := \frac{1}{\lambda}\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\overline{\lambda}\,\mathrm{Id})^{k-1}\overline{v}\big) = s$ is real and can be put to ± 1 by rescaling the vector (we could even put it to 1 exchanging if needed λ and its conjugate). One may further assume, by lemma 5.1.6 that

$$T_{i,j}(v) = \frac{1}{\lambda^i}\frac{1}{\overline{\lambda}^j}\Omega\big((A-\lambda\,\mathrm{Id})^i v, (A-\overline{\lambda}\,\mathrm{Id})^j\overline{v}\big) = 0 \quad \text{for all } \ 0 \le i,j \le k-1.$$

We consider the basis $\{a_{2k-1}, \ldots, a_k, b_p, \ldots, b_k, b_0, \ldots b_{k-1}, a_0, \ldots a_{k-1}\}$ for such a vector v with $T_{k,k-1}(v) = s = \pm 1$ and $T_{i,j}(v) = 0$ for all $0 \le i,j \le k-1$; the matrix representing Ω has the form

$$\left(\begin{array}{c|c}
0 & \begin{array}{c|c} \begin{matrix} \overline{*} & & 0 \\ & \ddots & \\ * & & \overline{*} \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \overline{*} & & 0 \\ & \ddots & \\ * & & \overline{*} \end{matrix} \end{array} \\ \hline
\begin{array}{c|c} \begin{matrix} \overline{*} & & * \\ & \ddots & \\ 0 & & \overline{*} \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \overline{*} & & * \\ & \ddots & \\ 0 & & \overline{*} \end{matrix} \end{array} & 0
\end{array}\right)$$

and we transform it by a Gram-Schmidt method into a symplectic basis composed of pairs of conjugate vectors, extending $\{b_0, \ldots, b_{k-1}, a_0, \ldots, a_{k-1}\}$ on which Ω identically vanishes. We define

$$\begin{aligned}
a'_{2k-1} &= \frac{1}{\Omega(a_{2k-1}, b_0)} a_{2k-1}, \\
b'_{2k-1} &= \frac{1}{\Omega(b_{2k-1}, a_0)} b_{2k-1} = \overline{a'_{2k-1}}
\end{aligned}$$

and, inductively on increasing j with $1 < j \le k$

$$\begin{aligned}
a'_{2k-j} &= \frac{1}{\Omega(a_{2k-j}, b_{j-1})}\left(a_{2k-j} - \sum_{r=1}^{j-1}\Omega(a_{2k-j}, b_{r-1})\, a'_{2k-r}\right), \\
b'_{2k-j} &= \overline{a'_{2k-j}}.
\end{aligned}$$

Any a'_{2k-j} is a linear combination of the a_{2k-i} for $1 \le i \le j$; reciprocally any a_{2k-j} can be written as a linear combination of the a'_{2k-i} for $1 \le i \le j$, and the coefficient of a'_{2k-j} is equal to $\Omega(a_{2k-j}, b_{j-1})$.
The basis $\{a'_{2k-1}, \ldots, a'_k, b'_{2k-1}, \ldots, b'_k, b_0, \ldots, b_{k-1}, a_0, \ldots, a_{k-1}\}$ is symplectic, and in that

basis, since $A(a_r) = \lambda a_r + a_{r+1}$ and $A(b_r) = \overline{\lambda} b_r + b_{r+1}$ for all $r < 2k-2$, the matrix representing A is of the block upper triangular form

$$\begin{pmatrix} * & 0 & 0 & C \\ & * & \overline{C} & 0 \\ & & J(\overline{\lambda},k)^\tau & 0 \\ 0 & & & J(\lambda,k)^\tau \end{pmatrix}$$

where C is a $k \times k$ matrix such that the only non vanishing terms are on the last column ($C^i_j = 0$ when $j < k$) and $C^k_k = \Omega(a_k, b_{k-1}) = s\lambda$. The fact that the matrix is symplectic implies that $S := J(\overline{\lambda},k)C$ is hermitean; since $S^i_j = 0$ when $j \neq k$, we have,

$$C = J(\overline{\lambda},k)^{-1} \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & s \end{pmatrix} = C(k,s,\overline{\lambda})$$

and the matrix of the restriction of A to the subspace $E^v_\lambda \oplus E^{\overline{v}}_{\overline{\lambda}}$ has the block triangular normal form

$$\begin{pmatrix} J(\overline{\lambda},k)^{-1} & 0 & 0 & C(k,s,\overline{\lambda}) \\ & J(\lambda,k)^{-1} & C(k,s,\lambda) & 0 \\ & & J(\overline{\lambda},k)^\tau & 0 \\ 0 & & & J(\lambda,k)^\tau \end{pmatrix}. \tag{5.23}$$

Writing $a'_{2k-j} = \frac{1}{\sqrt{2}}(e_{2j-1} - ie_{2j})$, $b'_{2k-j} = \overline{a'_{2k-j}} = \frac{1}{\sqrt{2}}(e_{2j-1} + ie_{2j})$, as well as $a_{j-1} = \frac{1}{\sqrt{2}}(f_{2j-1} - if_{2j})$ and $b_{j-1} = \overline{a_{j-1}} = \frac{1}{\sqrt{2}}(f_{2j-1} + if_{2j})$ for $1 \leq j \leq k$, the vectors e_i, f_j all belong to the real subspace denoted $V^v_{[\lambda]}$ of V whose complexification is $E^v_\lambda \oplus E^{\overline{v}}_{\overline{\lambda}}$ and we get a symplectic basis

$$\{e_1, \dots, e_{2k}, f_1, \dots, f_{2k}\}$$

of this real subspace $V^v_{[\lambda]}$. The matrix representing A in this basis is :

$$\begin{pmatrix} \big(J_{\mathbb{R}}(\overline{\lambda},2k)\big)^{-1} & C_{\mathbb{R}}(k,s,\overline{\lambda}) \\ 0 & \big(J_{\mathbb{R}}(\overline{\lambda},2k)\big)^\tau \end{pmatrix} \tag{5.24}$$

where $J_{\mathbb{R}}(e^{i\phi},2k)$ is defined as in (5.20) and where $C_{\mathbb{R}}(k,s,e^{i\phi})$ is the $(p+1)\times(p+1)$ matrix written in terms of two by two matrices as

$$C_{\mathbb{R}}(k,s,e^{i\phi})^\tau = s \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ (-1)^{k-1}R(e^{ik\phi}) & \dots & -R(e^{i2\phi}) & R(e^{i\phi}) \end{pmatrix} \tag{5.25}$$

with $R(e^{i\phi}) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ as before and $s = \pm 1$. This is the normal form of A restricted to $V^v_{[\lambda]}$; recall that

$$s = \lambda^{-1}\,\Omega\big((A - \lambda\,\mathrm{Id})^k v, (A - \overline{\lambda}\,\mathrm{Id})^{k-1}\overline{v}\big).$$

5.4.2 If $p = 2k$ is even

we observe that $\Omega\big((A - \overline{\lambda}\,\mathrm{Id})^k\overline{v}, (A - \lambda\,\mathrm{Id})^k v\big)$ is purely imaginary and we choose v so that it is $\Omega\big((A - \overline{\lambda}\,\mathrm{Id})^k\overline{v}, (A - \lambda\,\mathrm{Id})^k v\big) = si$ where $s = \pm 1$ (remark that the sign changes if one permutes λ and $\overline{\lambda}$). We can further choose the vector v so that :

$$\begin{aligned} \Omega\left((A - \lambda\,\mathrm{Id})^k v, (A - \overline{\lambda}\,\mathrm{Id})^{k-1}\overline{v}\right) &= \frac{1}{2}\lambda si \qquad (5.26)\\ T_{i,j}(v) := \frac{1}{\lambda^i\overline{\lambda}^j}\Omega\left((A - \lambda\,\mathrm{Id})^i v, (A - \overline{\lambda}\,\mathrm{Id})^j\overline{v}\right) &= 0 \qquad \text{for all } 0 \leq i,j \leq k-1; \end{aligned}$$

Indeed, as before, by (5.11), we have $T_{i,j}(v) = -T_{i+1,j}(v) - T_{i+1,j-1}(v)$ and $T_{i,j}(v) = -\overline{T_{j,i}(v)}$ and we proceed as in lemma 5.1.6 by decreasing induction on $i + j$:

- if $T_{k,k-1}(v) = \alpha_1$, since $T_{k-1,k}(v) = si - T_{k,k-1}(v)$ the imaginary part of α_1 is equal to $\frac{1}{2}si$ and we replace v by $v - \frac{\alpha_1}{2\lambda si}(A - \lambda\,\mathrm{Id})v$; it generates the same A-invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i+j \geq 2k$ but now $T_{k,k-1}(v) = \alpha_1 - \frac{\alpha_1}{2si}T_{k+1,k-1}(v) + \frac{\overline{\alpha_1}}{2si}T_{k,k}(v) = \alpha_1 - \frac{1}{2}\alpha_1 - \frac{1}{2}\overline{\alpha_1} = \frac{1}{2}si$ since $T_{k,k}(v) = -T_{k+1,k-1}(v) = -si$; so we can now assume $T_{k,k-1}(v) = \frac{1}{2}si$;

- if $T_{k-1,k-1}(v) = \alpha_2$, this α_2 is purely imaginary and we replace v by $v - \frac{\alpha_2}{2\lambda^2 si}(A - \lambda\,\mathrm{Id})^2 v$; it generates the same A-invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i + j \geq 2k - 1$; now $T_{k-1,k-1}(v) = \alpha_2 - \frac{\alpha_2}{2si}T_{k+1,k-1}(v) + \frac{\overline{\alpha_2}}{2si}T_{k-1,k+1}(v) = \alpha_2 - \frac{1}{2}\alpha_2 + \frac{1}{2}\overline{\alpha_2} = 0$. We may thus assume this property to hold for v.

- if $T_{k-2,k-1}(v) = \alpha_3 = -T_{k-1,k-1}(v) - T_{k-1,k-2}(v) = \overline{T_{k-2,k-1}(v)}$, this α_3 is real and we replace v by $v - \frac{\alpha_3}{2\lambda^3 si}(A - \lambda\,\mathrm{Id})^3 v$; it generates and the the same A-invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i + j \geq 2k - 2$; now $T_{k-2,k-1}(v) = \alpha_3 - \frac{\alpha_3}{2si}T_{k+1,k-1}(v) + \frac{\overline{\alpha_3}}{2si}T_{k-2,k+2}(v) = 0$, since $T_{k+1,k-1}(v) = -T_{k,k}(v) = -T_{k-2,k+2}(v) = si$; hence also $T_{k-1,k-2}(v) = 0$;

- we now assume by induction to have a $J > 1$ so that $T_{i,j}(v) = 0$ for all $0 \leq i,j \leq k-1$ so that $i + j > 2k - 1 - J$;

- if $T_{k-J,k-1}(v) = \alpha_{J+1}$, then $T_{k-J,k-1}(v) = (-1)^{J-1}\overline{T_{k-1,k-J}(v)}$ so that α_{J+1} is real when J is even and is imaginary when J is odd; we replace v by $v - \frac{\alpha_{J+1}}{2\lambda^{J+1}si}(A - \lambda\,\mathrm{Id})^{J+1}v$; it sgenerates the same A-invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i + j \geq 2k - J$, but now $T_{k-J,k-1}(v) = \alpha_{J+1} - \frac{\alpha_{J+1}}{2si}T_{k+1,k-1}(v) + \frac{\overline{\alpha_{J+1}}}{2si}T_{k-J,k+J}(v) = \alpha_{J+1} - \frac{\alpha_{J+1}}{2} + (-1)^{J+1}\frac{\overline{\alpha_{J+1}}}{2} = 0$.
  Hence also $T_{k-J+1,k-2}(v) = 0, \ldots\ T_{k-1,k-J+1}(v) = 0$; so the induction step is proven.

Remark 5.4.1 For such a v, all $T_{i,j}(v)$ are determined inductively and we have

$$
\begin{array}{rcl}
T_{i,j}(v) & = & 0 \quad \text{if } i+j \geq 2k+1 \quad \text{and} \quad \text{for all } 0 \leq i,j \leq k-1 \\
T_{k-r,k+r}(v) & = & (-1)^{r+1} si \qquad \text{for all } \ 0 \leq r \leq k \\
T_{k-r,k+m}(v) & = & (-1)^{r+1} \dfrac{si}{2} \dfrac{(r+m)(r-1)!}{m!(r-m)!} \qquad \text{for all } 0 \leq m \leq r \leq k,\ r > 1 \\
T_{i,j}(v) & = & T_{j,i}(v) \qquad \text{for all } i,j.
\end{array}
$$

With the notation $a_i = (A - \lambda\,\mathrm{Id})^i v, b_i = (A - \overline{\lambda}\,\mathrm{Id})^i \overline{v}$, we consider the basis

$$\{a_{2k}, \ldots, a_{k+1}, b_{2k}, \ldots, b_{k+1}, b_k; b_0, \ldots, b_{k-1}, a_0, \ldots, a_{k-1}, a_k\}$$

for such a vector v; the matrix representing Ω in this basis has the form

$$
\left(\begin{array}{c|c|c|c|c|c}
0 & 0 & 0 & \begin{matrix} \overline{*} & & 0 \\ & \ddots & \\ * & & \overline{*} \end{matrix} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \begin{matrix} \overline{*} & & 0 \\ & \ddots & \\ * & & \overline{*} \end{matrix} & 0 \\
\hline
0 & 0 & 0 & 0 & * \ldots * & si \\
\hline
\begin{matrix} \overline{*} & & * \\ & \ddots & \\ 0 & & \overline{*} \end{matrix} & 0 & 0 & 0 & 0 & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\
\hline
0 & \begin{matrix} \overline{*} & & * \\ & \ddots & \\ 0 & & \overline{*} \end{matrix} & \begin{matrix} * \\ \vdots \\ * \end{matrix} & 0 & 0 & 0 \\
\hline
0 & 0 & -si & * \cdots * & 0 & 0
\end{array}\right).
$$

We transform (by a Gram-Schmidt method) the basis above into a symplectic basis, composed of pairs of conjugate vectors (up to a factor) and extending

$$b_0, \ldots, b_{k-1}, a_0, \ldots, a_{k-1}$$

on which Ω identically vanishes. We define inductively, for increasing j with $1 \leq j \leq k-1$

$$\begin{aligned}
a'_{2k} :&= \frac{1}{\Omega\big((A-\lambda\,\mathrm{Id})^{2k}v,\overline{v}\big)}(A-\lambda\,\mathrm{Id})^{2k}v = \frac{1}{\Omega(a_{2k},b_0)}a_{2k}\\
b'_{2k} :&= \frac{1}{\Omega\big((A-\overline{\lambda}\,\mathrm{Id})^{2k},\overline{v},v\big)}(A-\overline{\lambda}\,\mathrm{Id})^{2k}\overline{v} = \frac{1}{\Omega(b_{2k},a_0)}b_{2k} = \overline{a'_{2k}}\\
a'_{2k-j} &= \frac{1}{\Omega(a_{2k-j},b_j)}\left(a_{2k-j} - \sum_{r=0}^{j-1}\Omega(a_{2k-j},b_r)a'_{2k-r}\right)\\
b'_{2k-j} &= \frac{1}{\Omega(b_{2k-j},a_j)}\left(b_{2k-j} - \sum_{r=0}^{j-1}\Omega(b_{2k-j},a_r)b'_{2k-r}\right) = \overline{a'_{2k-j}}\\
a'_k &= a_k - \sum_{r=0}^{k-1}\Omega(a_k,b_r)a'_{2k-r}\\
b'_k &= \frac{1}{\Omega(b_k,a_k)}\left(b_k - \sum_{r=0}^{k-1}\Omega(b_k,a_r)b'_{2k-r}\right) = \frac{1}{is}\overline{a'_k}.
\end{aligned}$$

Each a'_{2k-j} is a linear combination of the $(A-\lambda\,\mathrm{Id})^{2k-r}v$ for $0 \leq r \leq j$. The basis

$$\{a'_{2k},\ldots,a'_{k+1},b'_{2k},\ldots,b'_{k+1},b'_k;b_0,\ldots,b_{k-1},a_0,\ldots,a_{k-1},a'_k\}$$

is now symplectic. Since $A(a_r) = \lambda a_r + a_{r+1}$ for all $r < 2k$, and $A(a_{2k}) = \lambda a_{2k}$, the matrix representing A in that basis is of the form

$$\begin{pmatrix}
A_1 & 0 & 0 & \begin{pmatrix} 0 & \begin{matrix} c^{2k} & d^{2k}\\ \vdots & \vdots \\ c^{k+1} & d^{k+1}\end{matrix}\end{pmatrix}\\
0 & A_2 & \begin{pmatrix} 0 & \begin{matrix} e^{2k}\\ \vdots\\ e^{k+1}\\ e^k\end{matrix}\end{pmatrix} & 0\\
0 & 0 & J(\overline{\lambda},k)^\tau & 0\\
0 & 0 & 0 & J(\lambda,k+1)^\tau
\end{pmatrix}$$

with $A(b_{k-1}) = \overline{\lambda}b_{k-1} + \sum_{j=0}^k e^{k+j}b'_{k+j}$, $A(a_{k-1}) = \lambda a_{k-1} + a'_k + \sum_{j=1}^k c^{k+j}a'_{k+j}$ and $A(a'_k) = \lambda a'_k + \sum_{j=1}^k d^{k+j}a'_{k+j}$.

Since a matrix $\begin{pmatrix} A' & E\\ 0 & D\end{pmatrix}$ is symplectic if and only if $A' = (D^\tau)^{-1}$ and $D^\tau E$ is symmetric, we have

$$A_1 = J(\overline{\lambda},k)^{-1} \qquad\qquad A_2 = J(\lambda,k+1)^{-1}$$

and

$$J(\overline{\lambda},k)\begin{pmatrix} & c^{2k} & d^{2k} \\ 0 & \vdots & \vdots \\ & c^{k+1} & d^{k+1}\end{pmatrix} = \left(J(\lambda,k+1)\begin{pmatrix} & e^{2k} \\ 0 & \vdots \\ & e^{k+1} \\ & e^{k}\end{pmatrix}\right)^{\tau}.$$

This implies

$$J(\overline{\lambda},k)\begin{pmatrix} c^{2k} & d^{2k} \\ \vdots & \vdots \\ c^{k+2} & d^{k+2} \\ c^{k+1} & d^{k+1}\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ s_1 & s_2\end{pmatrix} \qquad J(\lambda,k+1)\begin{pmatrix} e^{2k} \\ \vdots \\ e^{k+2} \\ e^{k+1} \\ e^{k}\end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ s_1 \\ s_2\end{pmatrix}$$

so that $s_1 = \overline{\lambda}c^{k+1}$ and $s_2 = \overline{\lambda}d^{k+1}$. Now

$$\begin{aligned} A(a'_k) &= A\Big(a_k + \sum_{j\geq 1} F_k^j a_{k+j}\Big) = \lambda a'_k + a_{k+1} + \sum_{j\geq 1} F_k^j a_{k+j+1} \\ &= \lambda a'_k + a'_{k+1}\Omega(a_{k+1}, b_{k-1}) + \sum_{j\geq 1} F_k^{\prime j} a'_{k+j+1}\end{aligned}$$

so that $d^{k+1} = \Omega(a_{k+1}, b_{k-1}) = \lambda^2 is$ and $s_2 = \lambda is$. We also have

$$A(a_{k-1}) = \lambda a_{k-1} + a_k = \lambda a_{k-1} + a'_k + \Omega(a_k, b_{k-1})a'_{k+1} + \sum_{j\geq 2} G^j a'_{k+j}$$

so that $c^{k+1} = \Omega(a_k, b_{k-1}) = \lambda\frac{1}{2}is$ and $s_1 = \frac{1}{2}is$.

We have thus shown that the matrix representing A in the chosen basis has the block upper-triangular normal form

$$\begin{pmatrix} J(\overline{\lambda},k)^{-1} & 0 & 0 & J(\overline{\lambda},k)^{-1}S \\ & J(\lambda,k+1)^{-1} & J(\lambda,k+1)^{-1}S^{\tau} & 0 \\ & & J(\overline{\lambda},k)^{\tau} & 0 \\ & 0 & & J(\lambda,k+1)^{\tau}\end{pmatrix} \tag{5.27}$$

where S is the $k\times(k+1)$ matrix defined by

$$S = S(k,d,\lambda) := \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & \frac{1}{2}is & \lambda is\end{pmatrix}. \tag{5.28}$$

We write $a'_{2k+1-j} = \frac{1}{\sqrt{2}}(e_{2j-1} - ie_{2j})$, $b'_{2k+1-j} = \overline{a'_{2k+1-j}} = \frac{1}{\sqrt{2}}(e_{2j-1} + ie_{2j})$, as well as $a_{j-1} = \frac{1}{\sqrt{2}}(f_{2j-1} - if_{2j})$ and $b_{j-1} = \overline{a_{j-1}} = \frac{1}{\sqrt{2}}(f_{2j-1} + if_{2j})$ for $1 \leq j \leq k$, and $a'_k = \frac{1}{\sqrt{2}}(e_{2k+1} + id\, f_{2k+1}), b'_k = -id\overline{a'_k} = \frac{1}{\sqrt{2}}(-f_{2k+1} - id\, e_{2k+1})$. The vectors e_i, f_j all belong to the real subspace $V^v_{[\lambda]}$ of V whose complexification is $E^v_\lambda \oplus E^{\overline{v}}_{\overline{\lambda}}$ and we get a symplectic basis

$$\{e_1, \ldots, e_{2k+1}, f_1, \ldots, f_{2k+1}\}$$

of $V^v_{[\lambda]}$. In this basis, the matrix representing A is :

$$\left(\begin{array}{c|c|ccc|c}
\left(J_{\mathbb{R}}(\overline{\lambda},2k)\right)^{-1} & s\,U^2(\phi) & \begin{matrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{matrix} & \frac{s}{2}V^2(\phi) & \frac{-s}{2}V^1(\phi) & U^1(\phi) \\
\hline
0 & \cos\phi & 0 \;\ldots\; 0 & 1 & 0 & s\sin\phi \\
\hline
0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & \left(J_{\mathbb{R}}(\overline{\lambda},2k)\right)^{\tau} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\
\hline
0 & -s\sin\phi & 0 \;\ldots\; 0 & 0 & -s & \cos\phi
\end{array}\right)$$

where $s = \pm 1$, $U^1(\phi), U^2(\phi), V^1(\phi)$ and $V^2(\phi)$ are real $2k \times 1$ column matrices such that

$$\left(V^1(\phi)\, V^2(\phi)\right) = \begin{pmatrix} (-1)^{k-1} R(e^{ik\phi}) \\ \vdots \\ R(e^{i\phi}) \end{pmatrix}$$

$$\left(U^1(\phi)\, U^2(\phi)\right) = \begin{pmatrix} (-1)^{k-1} R(e^{i(k+1)\phi}) \\ \vdots \\ R(e^{i2\phi}) \end{pmatrix} = \left(V^1(\phi)\, V^2(\phi)\right)\left(R(e^{i\phi})\right).$$

This is the normal form of A restricted to $V^v_{[\lambda]}$. Recall that

$$s = i\Omega\big((A - \lambda\,\mathrm{Id})^k v, (A - \overline{\lambda}\,\mathrm{Id})^k \overline{v}\big).$$

Theorem 5.4.2 (Normal form for $A_{|V_{[\lambda]}}$ for $\lambda \in S^1 \setminus \{\pm 1\}$.) *Let $\lambda \in S^1 \setminus \{\pm 1\}$ be an eigenvalue of A. There exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of $4k_j \times 4k_j$ matrices ($k_j \geq 1$) of the form*

$$\left(\begin{array}{c|ccc}
\left(J_{\mathbb{R}}(\overline{\lambda},2k_j)\right)^{-1} & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} & s_j\, V^1_{k_j}(\phi) & s_j\, V^2_{k_j}(\phi) \\
\hline
0 & & \left(J_{\mathbb{R}}(\overline{\lambda},2k_j)\right)^{\tau} &
\end{array}\right) \tag{5.29}$$

and $(4k_j+2)\times(4k_j+2)$ *matrices* $(k_j \geq 0)$ *of the form*

$$\left(\begin{array}{c|c|ccc|c}
\Big(J_{\mathbb{R}}(\overline{\lambda},2k_j)\Big)^{-1} & s_j\,U^2_{k_j}(\phi) & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} & \frac{s_j}{2}V^2_{k_j}(\phi) & \frac{-s_j}{2}V^1_{k_j}(\phi) & U^1_{k_j}(\phi) \\
\hline
0 & \cos\phi & 0 \;\ldots\; 0 & 1 & 0 & s_j\sin\phi \\
\hline
0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \multicolumn{3}{c|}{\Big(J_{\mathbb{R}}(\overline{\lambda},2k_j)\Big)^{\tau}} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\
\hline
0 & -s_j\sin\phi & 0 \;\ldots\; 0 & 0 & -s_j & \cos\phi
\end{array}\right) \tag{5.30}$$

where $J_{\mathbb{R}}(e^{i\phi},2k)$ *is defined as in* (5.20), *where* $\Big(V^1_{k_j}(\phi)\,V^2_{k_j}(\phi)\Big)$ *is the* $2k_j\times 2$ *matrix defined by*

$$\Big(V^1_{k_j}(\phi)\,V^2_{k_j}(\phi)\Big) = \begin{pmatrix} (-1)^{k_j-1}R(e^{ik_j\phi}) \\ \vdots \\ R(e^{i\phi}) \end{pmatrix} \tag{5.31}$$

with $R(e^{i\phi}) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, *where*

$$\Big(U^1_{k_j}(\phi)\,U^2_{k_j}(\phi)\Big) = \Big(V^1_{k_j}(\phi)\,V^2_{k_j}(\phi)\Big)\Big(R(e^{i\phi})\Big) \tag{5.32}$$

and where $s_j = \pm 1$. *The complex dimension of the eigenspace of eigenvalue* λ *in* $V^{\mathbb{C}}$ *is given by the number of such matrices.*

Definition 5.4.3 Given $\lambda \in S^1 \setminus \{\pm 1\}$, we define, for any integer $m \geq 1$, a Hermitian form $\hat{Q}^{\lambda}_m$ on $\mathrm{Ker}\,((A-\lambda\,\mathrm{Id})^m)$ by:

$$\begin{array}{ll}
\hat{Q}^{\lambda}_m : \; \mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big)\times\mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big) \to \mathbb{C} & \\
\qquad (v,w) \mapsto \frac{1}{\lambda}\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\overline{\lambda}\,\mathrm{Id})^{k-1}\overline{w}\big) & \text{if } m=2k \\
\qquad (v,w) \mapsto i\,\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\overline{\lambda}\,\mathrm{Id})^{k}\overline{w}\big) & \text{if } m=2k+1.
\end{array}$$

Proposition 5.4.4 *For* $\lambda \in S^1\setminus\{\pm 1\}$, *the number of positive (resp. negative) eigenvalues of the Hermitian* 2*-form* $\hat{Q}^{\lambda}_m$ *is equal to the number of* s_j *equal to* $+1$ *(resp.* -1*) arising in blocks of dimension* $2m$ *in the normal decomposition of* A *on* $V_{[\lambda]}$ *given in theorem 5.4.2.*

Proof: On the intersection of $\mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big)$ with one of the symplectically orthogonal subspaces $E^v_\lambda \oplus E^{\overline{v}}_{\overline{\lambda}}$ constructed above from a v such that $(A-\lambda\,\mathrm{Id})^p v \neq 0$ and $(A-\lambda\,\mathrm{Id})^{p+1}v = 0$, the form $\hat{Q}^{\lambda}_m$ vanishes identically, except if $p = m-1$ and the only non vanishing component is $\hat{Q}^{\lambda}_m(v,v) = s$.
Indeed, $\mathrm{Ker}\big((A-\lambda\,\mathrm{Id})^m\big)\cap E^v_\lambda$ is spanned by

$$\{(A-\lambda\,\mathrm{Id})^r v\,;\, r\geq 0 \text{ and } r+m>p\},$$

and $\hat{Q}^\lambda_m\big((A-\lambda\,\mathrm{Id})^r v, (A-\lambda\,\mathrm{Id})^{r'} v\big) = 0$ when $m+r+r'-1 > p$ so the only non vanishing cases arise when $r = r' = 0$ and $m = p+1$ so for $\hat{Q}^\lambda_m(v,v)$. This is equal to $\frac{1}{\lambda}\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\overline{\lambda}\,\mathrm{Id})^{k-1}\overline{v}\big) = \frac{1}{\lambda}\lambda s = s$ if $m = 2k$, and to $i\,\Omega\big((A-\lambda\,\mathrm{Id})^k v, (A-\overline{\lambda}\,\mathrm{Id})^k \overline{v}\big) = i(-is) = s$ if $m = 2k+1$.

The numbers s_j appearing in the decomposition are thus invariant of the matrix.

Corollary 5.4.5 *The normal decomposition described in theorem 5.4.2 is unique up to a permutation of the blocks when the eigenvalue λ has been chosen in $\{\lambda, \overline{\lambda}\}$, for instance by specifyng that its imaginary part is positive. It is completely determined by this chosen λ, by the dimension $\dim_{\mathbb{C}}\big(\mathrm{Ker}(A-\lambda\,\mathrm{Id})^r\big)$ for each $r \geq 1$ and by the rank and the signature of the Hermitian bilinear 2-forms $\hat{Q}^\lambda_m$ for each $m \geq 1$.* □

6 Generalised Conley-Zehnder index

This chapter will appear as an homonymous paper in Annales de la Faculté des Sciences de Toulouse, [Guta].

The Conley-Zehnder index associates an integer to any continuous path of symplectic matrices starting from the identity and ending at a matrix which does not admit 1 as an eigenvalue. We give new ways to compute this index. Robbin and Salamon define a generalization of the Conley-Zehnder index for any continuous path of symplectic matrices; this generalization is half integer valued. It is based on a Maslov-type index that they define for a continuous path of Lagrangians in a symplectic vector space $(W, \overline{\Omega})$, having chosen a given reference Lagrangian V. Paths of symplectic endomorphisms of $(\mathbb{R}^{2n}, \Omega_0)$ are viewed as paths of Lagrangians defined by their graphs in $(W = \mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \overline{\Omega} = \Omega_0 \oplus -\Omega_0)$ and the reference Lagrangian is the diagonal. Robbin and Salamon give properties of this generalized Conley-Zehnder index and an explicit formula when the path has only regular crossings. We give here an axiomatic characterization of this generalized Conley-Zehnder index. We also give an explicit way to compute it for any continuous path of symplectic matrices.

6.1 Introduction

The Conley-Zehnder index associates an integer to any continuous path ψ defined on the interval $[0,1]$ with values in the group $\mathrm{Sp}\Big(\mathbb{R}^{2n}, \Omega_0 = \left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right)\Big)$ of $2n \times 2n$ symplectic matrices, starting from the identity and ending at a matrix which does not admit 1 as an eigenvalue. This index is used in the definition of the grading of Floer homology theories. If the path ψ were a loop with values in the unitary group, one could define an integer by looking at the degree of the loop in the circle defined by the (complex) determinant -or an integer power of it. The construction [SZ92, Sal99, AD10] of the Conley-Zehnder index is based on this idea. One uses a continuous map ρ from the sympletic group $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ into S^1 and an "admissible" extension of ψ to a path $\widetilde{\psi} : [0,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ in such a way that $\rho^2 \circ \widetilde{\psi} : [0,2] \to S^1$ is a loop. The Conley-Zehnder index of ψ is defined as the degree of this loop

$$\mu_{\mathrm{CZ}}(\psi) := \deg(\rho^2 \circ \widetilde{\psi}).$$

We recall this construction in section 6.2 with the precise definition of the map ρ. The value of $\rho(A)$ involves the algebraic multiplicities of the real negative eigenvalues of A and the signature of natural symmetric 2-forms defined on the generalised eigenspaces of A for the non real eigenvalues lying on S^1. We give alternative ways to compute this index :

Theorem 6.1.1 *Let $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be a continuous path of matrices linking the matrix Id to a matrix which does not admit 1 as an eigenvalue. Let $\widetilde{\psi} : [0,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be an extension such that $\widetilde{\psi}$ coincides with ψ on the interval $[0,1]$, such that $\widetilde{\psi}(s)$ does not admit 1 as an eigenvalue for all $s \geqslant 1$ and such that the path ends either at $\widetilde{\psi}(2) = W^+ := -\mathrm{Id}$ either at $\widetilde{\psi}(2) = W^- := \mathrm{diag}(2, -1, \ldots, -1, \frac{1}{2}, -1, \ldots, -1)$. The Conley-Zehnder index of ψ is equal to the integer given by the degree of the map $\tilde{\rho}^2 \circ \tilde{\psi} : [0,2] \to S^1$:*

$$\mu_{CZ}(\psi) = \deg(\tilde{\rho}^2 \circ \widetilde{\psi}) \tag{6.1}$$

for ANY continuous map $\tilde{\rho} : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to S^1$ with the following properties:

1. *$\tilde{\rho}$ coincides with the (complex) determinant $\det_{\mathbb{C}}$ on $U(n) = O\left(\mathbb{R}^{2n}\right) \cap \mathrm{Sp}\left(\mathbb{R}^{2n}, \Omega_0\right)$;*
2. *$\tilde{\rho}(W^-) \in \{\pm 1\}$;*
3. *$\deg\left(\tilde{\rho}^2 \circ \psi_{2-}\right) = n - 1$*

   for $\psi_{2-} : t \in [0,1] \mapsto \exp t\pi J_0 \begin{pmatrix} 0 & 0 & -\frac{\log 2}{\pi} & 0 \\ 0 & \mathrm{Id}_{n-1} & 0 & 0 \\ -\frac{\log 2}{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{n-1} \end{pmatrix}.$

In particular, two alternative ways to compute the Conley-Zehnder index are :

- *Using the polar decomposition of a matrix,*

$$\mu_{CZ}(\psi) = \deg({\det}_{\mathbb{C}}{}^2 \circ U \circ \widetilde{\psi}) \tag{6.2}$$

  where $U : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to U(n) : A \mapsto AP^{-1}$ with P the unique symmetric positive definite matrix such that $P^2 = A^\tau A$.

- *Using the normalized determinant of the $\mathbb{C}$-linear part of a symplectic matrix,*

$$\mu_{CZ}(\psi) = \deg(\hat{\rho}^2 \circ \widetilde{\psi}) \tag{6.3}$$

  where $\hat{\rho} : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to S^1 : A \mapsto \hat{\rho}(A) = \frac{\det_{\mathbb{C}}\left(\frac{1}{2}(A - J_0 A J_0)\right)}{\left|\det_{\mathbb{C}}\left(\frac{1}{2}(A - J_0 A J_0)\right)\right|}$ with $J_0 = \left(\begin{smallmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{smallmatrix}\right)$ the standard complex structure on $\mathbb{R}^{2n}$.

In [RS93], Robbin and Salamon define a Maslov-type index for a continuous path Λ from the interval $[a,b]$ to the space $\mathcal{L}_{(W,\overline{\Omega})}$ of Lagrangian subspaces of a symplectic vector space $(W, \overline{\Omega})$, having chosen a reference Lagrangian L. They give a formula of this index for a path having only regular crossings. A crossing for Λ is a number $t \in [a,b]$ for which $\dim \Lambda_t \cap L \neq 0$, and a crossing t is regular if the crossing form $\Gamma(\Lambda, L, t)$ is nondegenerate. We recall the precise definitions in section 6.3.

Robbin and Salamon define the index of a continuous path of symplectic matrices $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \psi_t$ as the index of the corresponding path of Lagrangians in $(W := \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \overline{\Omega} = -\Omega_0 \times \Omega_0)$ defined by their graphs,

$$\Lambda = \mathrm{Gr}\, \psi : [0,1] \to \mathcal{L}_{(W,\overline{\Omega})} : t \mapsto \mathrm{Gr}\, \psi_t = \{(x, \psi_t x) | x \in \mathbb{R}^{2n}\}.$$

The reference Lagrangian is the diagonal $\Delta = \{(x,x) | x \in \mathbb{R}^{2n}\}$. They prove that this index coincide with the Conley Zehnder index on continuous paths of symplectic matrices which start from the identity and end at a matrix which does not admit 1 as an eigenvalue. To be complete, we include this in section 6.4. They also prove that this index vanishes on a path of symplectic matrices with constant dimensional 1-eigenspace. Robbin and Salamon present also another way to associate an index to a continuous path ψ of symplectic matrices. One chooses a Lagrangian L in $\mathcal{L}_{(\mathbb{R}^{2n},\Omega_0)}$ and one considers the index of the path of Lagrangians $t \mapsto \psi_t L$, with L as the reference Lagrangian. We show in section 6.4.2 that those two indices do not coincide in general.

We use the normal form of the restriction of a symplectic endomorphism to the generalized eigenspace of eigenvalue 1 obtained in [Gutb] to construct special paths of symplectic endomorphisms with a constant dimension of the eigenspace of eigenvalue 1. This leads in section 6.5 to a characterization of the generalized half-integer valued Conley Zehnder index defined by Robbin and Salamon :

Theorem 6.1.2 *The Robbin-Salamon index for a continuous path of symplectic matrices is characterized by the following properties:*

- (Homotopy) *it is invariant under homotopies with fixed end points;*
- (Catenation) *it is additive under catenation of paths;*
- (Zero) *it vanishes on any path* $\psi : [a,b] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega)$ *of matrices such that* $\dim \mathrm{Ker}\big(\psi(t) - \mathrm{Id}\big) = k$ *is constant on* $[a,b]$*;*
- (Normalization) *if* $S = S^\tau \in \mathbb{R}^{2n \times 2n}$ *is a symmetric matrix with all eigenvalues of absolute value* $< 2\pi$ *and if* $\psi(t) = \exp(J_0 S t)$ *for* $t \in [0,1]$, *then* $\mu_{RS}(\psi) = \frac{1}{2} \mathrm{Sign}\, S$ *where* $\mathrm{Sign}\, S$ *is the signature of* S.

The same techniques lead in section 6.6 to a new formula for this index :

Theorem 6.1.3 *Let* $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ *be a path of symplectic matrices. Decompose* $\psi(0) = \psi^\star(0) \oplus \psi^{(1)}(0)$ *and* $\psi(1) = \psi^\star(1) \oplus \psi^{(1)}(1)$ *where* $\psi^\star(\cdot)$ *does not admit* 1 *as eigenvalue and* $\psi^{(1)}(\cdot)$ *is the restriction of* $\psi(\cdot)$ *to its generalized eigenspace of eigenvalue* 1. *Consider a continuous extension* $\Psi : [-1,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ *of* ψ *such that*

- $\Psi(t) = \psi(t)$ *for* $t \in [0,1]$;
- $\Psi\big(-\frac{1}{2}\big) = \psi^\star(0) \oplus \left(\begin{smallmatrix} e^{-1}\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id}\end{smallmatrix}\right)$ *and* $\Psi(t) = \psi^\star(0) \oplus \phi_0(t)$ *where* $\phi_0(t)$ *has only real positive eigenvalues for* $t \in \big[-\frac{1}{2}, 0\big]$;
- $\Psi\big(\frac{3}{2}\big) = \psi^\star(1) \oplus \left(\begin{smallmatrix} e^{-1}\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id}\end{smallmatrix}\right)$ *and* $\Psi(t) = \psi^\star(1) \oplus \phi_1(t)$ *where* $\phi_1(t)$ *has only real positive eigenvalues for* $t \in \big[1, \frac{3}{2}\big]$;
- $\Psi(-1) = W^\pm$, $\Psi(2) = W^\pm$ *and* $\Psi(t)$ *does not admit* 1 *as an eigenvalue for* $t \in \big[-1, -\frac{1}{2}\big]$ *and for* $t \in \big[\frac{3}{2}, 2\big]$.

Then the Robbin Salamon index is given by

$$\mu_{RS}(\psi) = \deg(\tilde{\rho}^2 \circ \Psi) + \frac{1}{2}\sum_{k\geq 1} \mathrm{Sign}\Big(\hat{Q}_k^{(\psi(0))}\Big) - \frac{1}{2}\sum_{k\geq 1} \mathrm{Sign}\Big(\hat{Q}_k^{(\psi(1))}\Big)$$

with $\tilde{\rho}$ *as in theorem 6.1.1, and with*

$$\begin{aligned} \hat{Q}_k^A \quad : \quad & \mathrm{Ker}\left((A-\mathrm{Id})^{2k}\right) \times \mathrm{Ker}\left((A-\mathrm{Id})^{2k}\right) \to \mathbb{R} \\ & (v,w) \mapsto \Omega\big((A-\mathrm{Id})^k v, (A-\mathrm{Id})^{k-1} w\big). \end{aligned}$$

In the theorem above, we have used the notation $A \oplus B$ for the symplectic direct sum of two symplectic endomorphisms with the natural identification of $\mathrm{Sp}(V', \Omega') \times \mathrm{Sp}(V'', \Omega'')$ as a subgroup of $\mathrm{Sp}(V' \oplus V'', \Omega' \oplus \Omega'')$. This writes in symplectic basis as

$$A \oplus B := \begin{pmatrix} A_1 & 0 & A_2 & 0 \\ 0 & B_1 & 0 & B_2 \\ A_3 & 0 & A_4 & 0 \\ 0 & B_3 & 0 & B_4 \end{pmatrix} \qquad \text{for } A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

We recall the definition of the Conley-Zehnder index in section 6.2 and obtain a new way of computing this index in Proposition 6.2.7 and its corrolaries (stated above as Theorem 6.1.1). In sections 6.3 and 6.4, we present known results about the Robbin Salamon index of a path of Lagrangians and the Robbin Salamon index of a path of symplectic matrices, including the fact that it is a generalization of the Conley-Zehnder index; in section 6.4.2, we stress the fact that another index introduced by Robbin and Salamon does not coincide with this generalization of the Conley-Zehnder index. In section 6.5, we give a characterization of the generalization of the Conley-Zehnder index (stated above as Theorem 6.1.2). Section 6.6 gives a new formula to compute this index (stated above as Theorem 6.1.3).

6.2 The Conley-Zehnder index

The Conley-Zehnder index is an application which associates a integer to a continuous path of symplectic matrices starting from the identity and ending at a matrix in the set $\mathrm{Sp}^\star(\mathbb{R}^{2n}, \Omega_0)$ of symplectic matrices which do not admit 1 as an eigenvalue.

Definition 6.2.1 ([SZ92, Sal99]) We consider the set $\mathrm{SP}(n)$ of continuous paths of matrices in $\mathrm{Sp}(\mathbb{R}^{2n},\Omega_0)$ linking the matrix Id to a matrix in $\mathrm{Sp}^\star(\mathbb{R}^{2n},\Omega_0)$:

$$\mathrm{SP}(n) := \left\{ \psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n},\Omega_0) \;\middle|\; \begin{array}{l} \psi(0) = \mathrm{Id} \text{ and} \\ 1 \text{ is not an eigenvalue of } \psi(1) \end{array} \right\}.$$

Definition 6.2.2 ([SZ92, AD10]) Let $\rho : \mathrm{Sp}(\mathbb{R}^{2n},\Omega_0) \to S^1$ be the continuous map defined as follows. Given $A \in \mathrm{Sp}(\mathbb{R}^{2n},\Omega)$, we consider its eigenvalues $\{\lambda_i\}$. For an eigenvalue $\lambda = e^{i\varphi} \in S^1 \setminus \{\pm 1\}$, let $m^+(\lambda)$ be the number of positive eigenvalues of the symmetric non degenerate 2-form Q defined on the generalized eigenspace E_λ by

$$Q : E_\lambda \times E_\lambda \to \mathbb{R} : \ (z,z') \mapsto Q(z,z') := \mathcal{I}m\ \Omega_0(z,\overline{z'}).$$

Then

$$\rho(A) := (-1)^{\frac{1}{2}m^-} \prod_{\lambda \in S^1\setminus\{\pm 1\}} \lambda^{\frac{1}{2}m^+(\lambda)} \tag{6.4}$$

where m^- is the sum of the algebraic multiplicities $m_\lambda = \dim_{\mathbb{C}} E_\lambda$ of the real negative eigenvalues.

Proposition 6.2.3 ([SZ92, AD10]) *The map $\rho : \mathrm{Sp}(\mathbb{R}^{2n},\Omega_0) \to S^1$ has the following properties:*

1. *[determinant] ρ coincides with $\det_{\mathbb{C}}$ on the unitary subgroup*

$$\rho(A) = \det{}_{\mathbb{C}} A \text{ if } A \in \mathrm{Sp}(\mathbb{R}^{2n},\Omega_0) \cap O(2n) = U(n);$$

2. *[invariance] ρ is invariant under conjugation :*

$$\rho(kAk^{-1}) = \rho(A)\ \forall k \in \mathrm{Sp}(\mathbb{R}^{2n},\Omega_0);$$

3. *[normalisation] $\rho(A) = \pm 1$ for matrices which have no eigenvalue on the unit circle;*

4. *[multiplicativity] ρ behaves multiplicatively with respect to direct sums : if $A = A' \oplus A''$ with $A' \in \mathrm{Sp}(\mathbb{R}^{2m},\Omega_0)$, $A'' \in \mathrm{Sp}(\mathbb{R}^{2(n-m)},\Omega_0)$ and $\oplus$ expressing as before the obvious identification of $\mathrm{Sp}(\mathbb{R}^{2m},\Omega_0) \times \mathrm{Sp}(\mathbb{R}^{2(n-m)},\Omega_0)$ with a subgroup of $\mathrm{Sp}(\mathbb{R}^{2n},\Omega_0)$ then*

$$\rho(A) = \rho(A')\rho(A'').$$

The construction [Sal99, AD10] of the Conley-Zehnder index is based on the two following facts

- $\mathrm{Sp}^\star(\mathbb{R}^{2n}, \Omega_0)$ has two connected components, one containing the matrix $W^+ := -\mathrm{Id}$ and the other containing
$$W^- := \mathrm{diag}(2, -1, \dots, -1, \frac{1}{2}, -1, \dots, -1);$$
- any loop in $\mathrm{Sp}^\star(\mathbb{R}^{2n}, \Omega_0)$ is contractible in $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$.

Thus any path $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ in $\mathrm{SP}(n)$ can be extended to a path $\widetilde{\psi}[0,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ so that

- $\widetilde{\psi}(t) = \psi(t)$ for $t \leq 1$;
- $\widetilde{\psi}(t)$ is in $\mathrm{Sp}^\star(\mathbb{R}^{2n}, \Omega_0)$ for any $t \geq 1$;
- $\widetilde{\psi}(2) = W^\pm$.

Observe that $\big(\rho(\mathrm{Id})\big)^2 = 1$ and $\big(\rho(W^\pm)\big)^2 = 1$ so that $\rho^2 \circ \widetilde{\psi} : [0,2] \to S^1$ is a loop in S^1 and the contractibility property shows that its degree does not depend on the extension chosen.

Definition 6.2.4 The Conley-Zehnder index of ψ is defined by:

$$\mu_{\mathrm{CZ}} : \mathrm{SP}(n) \to \mathbb{Z} : \ \psi \mapsto \mu_{\mathrm{CZ}}(\psi) := \deg(\rho^2 \circ \widetilde{\psi}) \tag{6.5}$$

for an extension $\widetilde{\psi}$ of ψ as above.

Proposition 6.2.5 ([Sal99, AD10]) *The Conley-Zehnder index has the following properties:*

1. (Naturality) *For all path $\phi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ we have*
$$\mu_{CZ}(\phi\psi\phi^{-1}) = \mu_{CZ}(\psi);$$
2. (Homotopy) *The Conley-Zehnder index is constant on the components of* $\mathrm{SP}(n)$;
3. (Zero) *If $\psi(s)$ has no eigenvalue on the unit circle for $s > 0$ then*
$$\mu_{CZ}(\psi) = 0;$$
4. (Product) *If $n' + n'' = n$, , if ψ' is in $\mathrm{SP}(n')$ and ψ''in $\mathrm{SP}(n'')$, then*
$$\mu_{CZ}(\psi' \oplus \psi'') = \mu_{CZ}(\psi') + \mu_{CZ}(\psi'');$$
with the identification of $\mathrm{Sp}(\mathbb{R}^{2n'}, \Omega_0) \times \mathrm{Sp}(\mathbb{R}^{2n''}, \Omega_0)$ *with a subgroup of* $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$*;*

5. (Loop) *If* $\phi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ *is a loop with* $\phi(0) = \phi(1) = \mathrm{Id}$, *then*

$$\mu_{CZ}(\phi\psi) = \mu_{CZ}(\psi) + 2\mu(\phi)$$

where $\mu(\phi)$ *is the Maslov index of the loop* ϕ, *i.e.* $\mu(\phi) = \deg(\rho \circ \phi)$;

6. (Signature) *If* $S = S^\tau \in \mathbb{R}^{2n\times 2n}$ *is a symmetric non degenerate matrix with all eigenvalues of absolute value* $< 2\pi$ ($\|S\| < 2\pi$) *and if* $\psi(t) = \exp(J_0 S t)$ *for* $t \in [0,1]$, *then* $\mu_{CZ}(\psi) = \frac{1}{2}\,\mathrm{Sign}(S)$ (*where* $\mathrm{Sign}(S)$ *is the signature of* S).

7. (Determinant) $(-1)^{n-\mu_{CZ}(\psi)} = \mathrm{sign}\det\big(\mathrm{Id} - \psi(1)\big)$

8. (Inverse) $\mu_{CZ}(\psi^{-1}) = \mu_{CZ}(\psi^\tau) = -\mu_{CZ}(\psi)$

Proposition 6.2.6 ([Sal99, AD10]) *The properties 2, 5 and 6 of homotopy, loop and signature characterize the Conley-Zehnder index.*

Proof: Assume $\mu' : \mathrm{SP}(n) \to \mathbb{Z}$ is a map satisfying those properties. Let $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be an element of $\mathrm{SP}(n)$; Since ψ is in the same component of $\mathrm{SP}(n)$ as its prolongation $\tilde{\psi} : [0,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ we have $\mu'(\psi) = \mu'(\tilde{\psi})$.

Observe that $W^+ = \exp \pi(J_0 S^+)$ with $S^+ = \mathrm{Id}$ and $W^- = \exp \pi(J_0 S^-)$ with

$$S^- = \begin{pmatrix} 0 & 0 & -\frac{\log 2}{\pi} & 0 \\ 0 & \mathrm{Id}_{n-1} & 0 & 0 \\ -\frac{\log 2}{\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{n-1} \end{pmatrix}.$$

The catenation of $\tilde{\psi}$ and ψ_2^- (the path ψ_2 in the reverse order, i.e followed from end to beginning) when $\psi_2 : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ $t \mapsto \exp t\pi J_0 S^\pm$ is a loop ϕ. Hence $\tilde{\psi}$ is homotopic to the catenation of ϕ and ψ_2, which is homotopic to the product $\phi\psi_2$ (see, for instance, [Gutb]).

Thus we have $\mu'(\psi) = \mu'(\phi\psi_2)$. By the loop condition $\mu'(\phi\psi_2) = \mu'(\psi_2) + 2\mu(\phi)$ and by the signature condition $\mu'(\psi_2) = \frac{1}{2}\,\mathrm{Sign}(S^\pm)$. Thus

$$\mu'(\psi) = 2\mu(\phi) + \frac{1}{2}\,\mathrm{Sign}(S^\pm).$$

Since the same is true for $\mu_{CZ}(\psi)$, this proves uniqueness.

Remark that we have only used the signature property to know the value of the Conley-Zehnder index on the paths $\psi_{2\pm} :\ t \in [0,1] \mapsto\ \exp t\pi J_0 S^\pm$. Hence we have :

Proposition 6.2.7 *Let* $\psi \in \mathrm{SP}(n)$ *be a continuous path of matrices in* $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ *linking the matrix* Id *to a matrix in* $\mathrm{Sp}^\star(\mathbb{R}^{2n}, \Omega_0)$ *and let* $\widetilde{\psi} : [0,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ *be an extension such that* $\widetilde{\psi}$ *coincides with* ψ *on the interval* $[0,1]$, *such that* $\widetilde{\psi}(s) \in \mathrm{Sp}^\star(\mathbb{R}^{2n}, \Omega_0)$ *for all*

$s \geqslant 1$ and such that the path ends either in $\widetilde{\psi}(2) = -\mathrm{Id} = W^+$ either in $\widetilde{\psi}(2) = W^- :=$ $\mathrm{diag}(2, -1, \ldots, -1, \frac{1}{2}, -1, \ldots, -1)$. The Conley-Zehnder index of ψ is equal to the integer given by the degree of the map $\tilde{\rho}^2 \circ \widetilde{\psi} : [0, 2] \to S^1$:

$$\mu_{CZ}(\psi) := \deg(\tilde{\rho}^2 \circ \widetilde{\psi}) \tag{6.6}$$

for any continuous map $\tilde{\rho} : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to S^1$ which coincide with the (complex) determinant $\det_{\mathbb{C}}$ on $U(n) = O\left(\mathbb{R}^{2n}\right) \cap \mathrm{Sp}\left(\mathbb{R}^{2n}, \Omega_0\right)$, such that $\tilde{\rho}(W^-) = \pm 1$, and such that

$$\deg\left(\tilde{\rho}^2 \circ \psi_{2-}\right) = n - 1 \quad \textit{for } \psi_{2-} : t \in [0, 1] \mapsto \exp t\pi J_0 S^-.$$

Proof: This is a direct consequence of the fact that the map defined by $\deg(\tilde{\rho}^2 \circ \widetilde{\psi})$ has the homotopy property, the loop property (since any loop is homotopic to a loop of unitary matrices where ρ and $\det_{\mathbb{C}}$ coincide) and we have added what we need of the signature property to characterize the Conley-Zehnder index. Indeed $\frac{1}{2}\,\mathrm{Sign}\, S^- = n - 1, S^+ =$ $\mathrm{Id}_{2n}, \frac{1}{2}\,\mathrm{Sign}\, S^+ = n$ and
$\exp t\pi J_0 S^+ = \exp t\pi \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix} = \begin{pmatrix} \cos \pi t\, \mathrm{Id}_n & -\sin \pi t\, \mathrm{Id}_n \\ \sin \pi t\, \mathrm{Id}_n & \cos \pi t\, \mathrm{Id}_n \end{pmatrix}$ is in $\mathrm{U}(n)$
so that $\tilde{\rho}^2 \left(\exp t\pi \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}\right) = e^{2\pi i n t}$ and $\deg(\tilde{\rho}^2 \circ \psi_{2+}) = n$.

Corollary 6.2.8 *The Conley-Zehnder index of a path $\psi \in \mathrm{SP}(n)$ is given by*

$$\mu_{CZ}(\psi) := \deg({\det}_{\mathbb{C}}{}^2 \circ U \circ \widetilde{\psi}) \tag{6.7}$$

where $U : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to U(n)$ is the projection defined by the polar decomposition $U(A) =$ AP^{-1} with P the unique symmetric positive definite matrix such that $P^2 = A^\tau A$.

Proof: The map $\tilde{\rho} := \det_{\mathbb{C}} \circ U$ satisfies all the properties stated in proposition 6.2.7; it is indeed continuous, coincides obviously with $\det_{\mathbb{C}}$ on $\mathrm{U}(n)$ and we have that
$\exp t\pi J_0 \begin{pmatrix} 0 & -\frac{\log 2}{\pi} \\ -\frac{\log 2}{\pi} & 0 \end{pmatrix} = \begin{pmatrix} 2^t & 0 \\ 0 & 2^{-t} \end{pmatrix}$ is a positive symmetic matrix so that $U(\exp t\pi J_0 S^-) =$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \pi t\, \mathrm{Id}_{n-1} & 0 & -\sin \pi t\, \mathrm{Id}_{n-1} \\ 0 & 0 & 1 & 0 \\ 0 & \sin \pi t\, \mathrm{Id}_{n-1} & 0 & \cos \pi t\, \mathrm{Id}_{n-1} \end{pmatrix}$;
hence $\det^2_{\mathbb{C}} \circ U(\exp t\pi J_0 S^-) = e^{2\pi i (n-1) t}$ and $\deg({\det}_{\mathbb{C}}{}^2 \circ U \circ \psi_{2-}) = n - 1$.

Formula (6.7) is the definition of the Conley-Zehnder index used in [dG09, HWZ95]. Another formula is obtained using the parametrization of the symplectic group introduced in [RR89]:

Corollary 6.2.9 *The Conley-Zehnder index of a path $\psi \in \mathrm{SP}(n)$ is given by*

$$\mu_{CZ}(\psi) := \deg(\hat{\rho}^2 \circ \widetilde{\psi}) \tag{6.8}$$

where $\hat\rho : \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \to S^1$ is the normalized complex determinant of the $\mathbb{C}$-linear part of the matrix:

$$\hat\rho(A) = \frac{\det_{\mathbb{C}}\left(\frac{1}{2}(A - J_0AJ_0)\right)}{\left|\det_{\mathbb{C}}\left(\frac{1}{2}(A - J_0AJ_0)\right)\right|}. \tag{6.9}$$

PROOF: Remark that for any $A \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ the element $C_A := \frac{1}{2}(A - J_0AJ_0)$, which clearly defines a complex linear endomorphism of $\mathbb{C}^n$ since it commutes with J_0, is always invertible. Indeed for any non-zero $v \in V$

$$4\Omega_0(C_Av, J_0C_Av) \;=\; 2\Omega_0(v, J_0v) + \Omega_0(Av, J_0Av) + \Omega_0(AJ_0v, J_0AJ_0v) > 0.$$

If $A \in \mathrm{U}(n)$, then $C_A = A$ so that $\hat\rho(A) = \det_{\mathbb{C}}(A)$ hence $\hat\rho$ is a continuous map which coincide with $\det_{\mathbb{C}}$ on $\mathrm{U}(n)$. Furthermore $\frac{1}{2}\left(\begin{pmatrix} 2^t & 0 \\ 0 & 2^{-t}\end{pmatrix} - J_0\begin{pmatrix} 2^t & 0 \\ 0 & 2^{-t}\end{pmatrix}J_0\right) = \frac{1}{2}\begin{pmatrix} 2^t+2^{-t} & 0 \\ 0 & 2^t+2^{-t}\end{pmatrix}$ hence its complex determinant is equal to $\frac{1}{2}(2^t+2^{-t})$ and its normalized complex determinant is equal to 1 so that $\hat\rho(\exp t\pi J_0S^-) = e^{\pi i(n-1)t}$ and $\deg(\hat\rho^2 \circ \psi_{2-}) = n-1$.

6.3 The Robbin-Salamon index for a path of Lagrangians

A *Lagrangian* in a symplectic vector space (V, Ω) of dimension $2n$ is a subspace L of V of dimension n such that $\Omega|_{L\times L} = 0$. Given any Lagrangian L in V, there exists a Lagrangian M (not unique!) such that $L \oplus M = V$. With the choice of such a supplementary M any Lagrangian L' in a neighborhood of L (any Lagrangian supplementary to M) can be identified to a linear map $\alpha : L \to M$ through $L' = \{v + \alpha(v) \mid v \in L\}$, with α such that $\Omega\big(\alpha(v), w\big) + \Omega\big(v, \alpha(w)\big) = 0\ \forall v, w \in L$. Hence it can be identified to a symmetric bilinear form $\underline{\alpha} : L \times L \to \mathbb{R} : (v, v') \mapsto \Omega\big(v, \alpha(v')\big)$. In particular the tangent space at a point L to the space $\mathcal{L}_{(V,\Omega)}$ of Lagrangians in (V, Ω) can be identified to the space of symmetric bilinear forms on L.

If $\Lambda : [a, b] \to \mathcal{L}_n := \mathcal{L}_{(\mathbb{R}^{2n},\Omega_0)} : t \mapsto \Lambda_t$ is a smooth curve of Lagrangian subspaces in $(\mathbb{R}^{2n}, \Omega_0)$, we define $Q(\Lambda_{t_0}, \dot\Lambda_{t_0})$ to be the symmetric bilinear form on Λ_{t_0} defined by

$$Q(\Lambda_{t_0}, \dot\Lambda_{t_0})(v, v') = \frac{d}{dt}\underline{\alpha}_t(v, v')\Big|_{t_0} = \frac{d}{dt}\Omega\big(v, \alpha_t(v')\big)\Big|_{t_0} \tag{6.10}$$

where $\alpha_t : \Lambda_{t_0} \to M$ is the map corresponding to Λ_t for a decomposition $\mathbb{R}^{2n} = \Lambda_{t_0} \oplus M$ with M Lagrangian. Then [RS93] :

- the symmetric bilinear form $Q(\Lambda_{t_0}, \dot\Lambda_{t_0}) : \Lambda_{t_0} \times \Lambda_{t_0} \to \mathbb{R}$ is independent of the choice of the supplementary Lagrangian M to Λ_{t_0};

- if $\psi \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ then
$$Q(\psi\Lambda_{t_0}, \psi\dot{\Lambda}_{t_0})(\psi v, \psi v') = Q(\Lambda_{t_0}, \dot{\Lambda}_{t_0})(v, v') \quad \forall v, v' \in \Lambda_{t_0}. \tag{6.11}$$

Let us choose and fix a Lagrangian L in $(\mathbb{R}^{2n}, \Omega_0)$. Consider a smooth path of Lagrangians $\Lambda : [a, b] \to \mathcal{L}_n$. A *crossing* for Λ is a number $t \in [a, b]$ for which $\dim \Lambda_t \cap L \neq 0$. At each crossing time $t \in [a, b]$ one defines the *crossing form*

$$\Gamma(\Lambda, L, t) = Q\big(\Lambda_t, \dot{\Lambda}_t\big)\Big|_{\Lambda_t \cap L}. \tag{6.12}$$

A crossing t is called *regular* if the crossing form $\Gamma(\Lambda, L, t)$ is nondegenerate. In that case $\Lambda_s \cap L = \{0\}$ for $s \neq t$ in a neighborhood of t.

Definition 6.3.1 ([RS93]) For a curve $\Lambda : [a, b] \to \mathcal{L}_n$ with only regular crossings the *Robbin-Salamon index* is defined as

$$\mu_{RS}(\Lambda, L) = \frac{1}{2} \operatorname{Sign} \Gamma(\Lambda, L, a) + \sum_{\substack{a<t<b \\ t \text{ crossing}}} \operatorname{Sign} \Gamma(\Lambda, L, t) + \frac{1}{2} \operatorname{Sign} \Gamma(\Lambda, L, b). \tag{6.13}$$

Robbin and Salamon show (Lemmas 2.1 and 2.2 in [RS93]) that two paths with only regular crossings which are homotopic with fixed endpoints have the same Robbin-Salamon index and that every continuous path of Lagrangians is homotopic with fixed endpoints to one having only regular crossings. These two properties allow to define the Robbin-Salamon index for every continuous path of Lagrangians and this index is clearly invariant under homotopies with fixed endpoints. It depends on the choice of the reference Lagrangian L. Robbin and Salamon show ([RS93], Theorem 2.3):

Theorem 6.3.2 ([RS93]) *The index μ_{RS} has the following properties:*

1. *(Naturality) For $\psi \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega)$ $\quad \mu_{RS}(\psi\Lambda, \psi L) = \mu_{RS}(\Lambda, L)$.*
2. *(Catenation) For $a < c < b$, $\mu_{RS}(\Lambda, L) = \mu_{RS}(\Lambda_{|_{[a,c]}}, L) + \mu_{RS}(\Lambda_{|_{[c,b]}}, L)$.*
3. *(Product) If $n' + n'' = n$, identify $\mathcal{L}_{n'} \times \mathcal{L}_{n''}$ as a submanifold of $\mathcal{L}_n$ in the obvious way. Then $\mu_{RS}(\Lambda' \oplus \Lambda'', L' \oplus L'') = \mu_{RS}(\Lambda', L') + \mu_{RS}(\Lambda'', L'')$.*
4. *(Localization) If $L = R^n \times \{0\}$ and $\Lambda(t) = \mathrm{Gr}(A(t))$ where $A(t)$ is a path of symmetric matrices, then the index of Λ is given by*
   $\mu_{RS}(\Lambda, L) = \frac{1}{2} \operatorname{Sign} A(b) - \frac{1}{2} \operatorname{Sign} A(a)$.
5. *(Homotopy) Two paths $\Lambda_0, \Lambda_1 : [a, b] \to \mathcal{L}_n$ with $\Lambda_0(a) = \Lambda_1(a)$ and $\Lambda_0(b) = \Lambda_1(b)$ are homotopic with fixed endpoints if and only if they have the same index.*
6. *(Zero) Every path $\Lambda : [a, b] \to \Sigma_k(V)$, with $\Sigma_k(V) = \{\, M \in \mathcal{L}_n \mid \dim M \cap L = k \,\}$, has index $\mu_{RS}(\Lambda, L) = 0$.*

6.4 The Robbin-Salamon index for a path of symplectic matrices

6.4.1 Generalized Conley-Zehnder index

Consider the symplectic vector space $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \overline{\Omega} = -\Omega_0 \times \Omega_0)$. Given any linear map $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, its graph

$$\operatorname{Gr}\psi = \{(x, \psi x) | x \in \mathbb{R}^{2n}\}$$

is a $2n$-dimensional subspace of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ which is Lagrangian if and only if ψ is symplectic $\big(\psi \in \operatorname{Sp}(\mathbb{R}^{2n}, \Omega_0)\big)$.

A particular Lagrangian is given by the diagonal

$$\Delta = \operatorname{Gr}\operatorname{Id} = \{(x, x) | x \in \mathbb{R}^{2n}\}. \tag{6.14}$$

Remark that $\operatorname{Gr}(-\psi)$ is a Lagrangian subspace which is always supplementary to $\operatorname{Gr}\psi$ for $\psi \in \operatorname{Sp}(\mathbb{R}^{2n}, \Omega_0)$. In fact $\operatorname{Gr}\phi$ and $\operatorname{Gr}\psi$ are supplementary if and only if $\phi - \psi$ is invertible.

Definition 6.4.1 ([RS93]) The *Robbin-Salamon index* of a continuous path of symplectic matrices $\psi : [0, 1] \to \operatorname{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \psi_t$ is defined as the Robbin-Salamon index of the path of Lagrangians in $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \overline{\Omega})$,

$$\Lambda = \operatorname{Gr}\psi : [0, 1] \to \mathcal{L}_{2n} : t \mapsto \operatorname{Gr}\psi_t$$

when the fixed Lagrangian is the diagonal Δ:

$$\mu_{RS}(\psi) := \mu_{RS}(\operatorname{Gr}\psi, \Delta). \tag{6.15}$$

Note that this index is defined for any continuous path of symplectic matrices but can have half integer values.

A crossing for a smooth path $\operatorname{Gr}\psi$ is a number $t \in [0, 1]$ for which 1 is an eigenvalue of ψ_t and

$$\operatorname{Gr}\psi_t \cap \Delta = \{\, (x, x) \,|\, \psi_t x = x \,\}$$

is in bijection with $\operatorname{Ker}(\psi_t - \operatorname{Id})$.

The properties of homotopy, catenation and product of theorem 6.3.2 imply that [RS93]

- μ_{RS} is invariant under homotopies with fixed endpoints,
- μ_{RS} is additive under catenation of paths and
- μ_{RS} has the product property $\mu_{RS}(\psi' \oplus \psi'') = \mu_{RS}(\psi') + \mu_{RS}(\psi'')$ as in proposition 6.2.5.

The zero property of the Robbin-Salamon index of a path of Lagrangians becomes:

Proposition 6.4.2 *If* $\psi : [a,b] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega)$ *is a path of matrices such that* $\dim \mathrm{Ker}(\psi(t) - \mathrm{Id}) = k$ *for all* $t \in [a,b]$ *then* $\mu_{RS}(\psi) = 0$.

Indeed, $\mathrm{Gr}\,\psi_t \cap \Delta = \{v \in \mathbb{R}^{2n} | \psi_t v = v\}$ so $\dim(\mathrm{Gr}\,\psi_t \cap \Delta) = k$ if and only if $\dim \mathrm{Ker}(\psi(t) - \mathrm{Id}) = k$.

Proposition 6.4.3 (Naturality) *Consider two continuous paths of symplectic matrices* $\psi, \phi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ *and define* $\psi' = \phi\psi\phi^{-1}$. *Then*

$$\mu_{RS}(\psi') = \mu_{RS}(\psi)$$

PROOF: One has

$$\begin{aligned} \Lambda'_t := \mathrm{Gr}\,\psi'_t &= \{(x, \phi_t\psi_t\phi_t^{-1}x) \,|\, x \in \mathbb{R}^{2n}\} \\ &= \{(\phi_t y, \phi_t\psi_t y) \,|\, y \in \mathbb{R}^{2n}\} \\ &= (\phi_t \times \phi_t)\,\mathrm{Gr}\,\psi_t \\ &= (\phi_t \times \phi_t)\Lambda_t \end{aligned}$$

and $(\phi_t \times \phi_t)\Delta = \Delta$. Furthermore $(\phi_t \times \phi_t) \in \mathrm{Sp}(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \overline{\Omega})$.
Hence $t \in [0,1]$ is a crossing for the path of Lagrangians $\Lambda' = \mathrm{Gr}\,\psi'$ if and only if $\dim \mathrm{Gr}\,\psi'_t \cap \Delta \neq 0$ if and only if $\dim(\phi_t \times \phi_t)(\mathrm{Gr}\,\psi_t \cap \Delta) \neq 0$ if and only if t is a crossing for the path of Lagrangian $\Lambda = \mathrm{Gr}\,\psi$.

By homotopy with fixed endpoints, we can assume that Λ has only regular crossings and ϕ is locally constant around each crossing t so that

$$\frac{d}{dt}(\phi\psi\phi^{-1})(t) = \phi_t\dot{\psi}_t\phi_t^{-1}.$$

Then at each crossing

$$\begin{aligned} \Gamma(\mathrm{Gr}\,\psi', \Delta, t) &= Q(\Lambda'_t, \dot{\Lambda}'_t)|_{\mathrm{Gr}\,\psi'_t \cap \Delta} \\ &= Q((\phi_t \times \phi_t)\Lambda_t, (\phi_t \times \phi_t)\dot{\Lambda}_t)|_{(\phi_t \times \phi_t)\,\mathrm{Gr}\,\psi_t \cap \Delta} \\ &= Q(\Lambda_t, \dot{\Lambda}_t)|_{\mathrm{Gr}\,\psi_t \cap \Delta} \circ (\phi_t^{-1} \times \phi_t^{-1}) \otimes (\phi_t^{-1} \times \phi_t^{-1}) \end{aligned}$$

in view of (6.11), so that

$$\mathrm{Sign}\,\Gamma(\mathrm{Gr}\,\psi', \Delta, t) = \mathrm{Sign}\,\Gamma(\mathrm{Gr}\,\psi, \Delta, t).$$

Definition 6.4.4 For any smooth path ψ of symplectic matrices, define a path of symmetric matrices S through

$$\dot{\psi}_t = J_0 S_t \psi_t.$$

This is indeed possible since $\psi_t \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)\,\forall t$, thus $\psi_t^{-1}\dot{\psi}_t$ is in the Lie algebra $sp(\mathbb{R}^{2n}, \Omega_0)$ and every element of this Lie algebra may be written in the form $J_0 S$ with S symmetric.

The symmetric bilinear form $Q\big(\mathrm{Gr}\,\psi, \frac{d}{dt}\mathrm{Gr}\,\psi\big)$ is given as follows. For any $t_0 \in [0,1]$, write $\mathbb{R}^{2n} \times \mathbb{R}^{2n} = \mathrm{Gr}\,\psi_{t_0} \oplus \mathrm{Gr}(-\psi_{t_0})$. The linear map $\alpha_t : \mathrm{Gr}\,\psi_{t_0} \to \mathrm{Gr}(-\psi_{t_0})$ corresponding to $\mathrm{Gr}\,\psi_t$ is obtained from:

$$(x, \psi_t x) = (y, \psi_{t_0} y) + \alpha_t(y, \psi_{t_0} y) = (y, \psi_{t_0} y) + (\widetilde{\alpha}_t y, -\psi_{t_0}\widetilde{\alpha}_t y)$$

if and only if $(\mathrm{Id} + \widetilde{\alpha}_t) y = x$ and $\psi_{t_0}(\mathrm{Id} - \widetilde{\alpha}_t) y = \psi_t x$, hence $\psi_{t_0}^{-1}\psi_t(\mathrm{Id} + \widetilde{\alpha}_t) = \mathrm{Id} - \widetilde{\alpha}_t$ and

$$\widetilde{\alpha}_t = (\mathrm{Id} + \psi_{t_0}^{-1}\psi_t)^{-1}(\mathrm{Id} - \psi_{t_0}^{-1}\psi_t) \quad ; \qquad \frac{d}{dt}\widetilde{\alpha}_t\Big|_{t_0} = -\frac{1}{2}\psi_{t_0}^{-1}\dot{\psi}_{t_0}.$$

Thus

$$\begin{aligned}
&Q\Big(\mathrm{Gr}\,\psi_{t_0}, \frac{d}{dt}\,\mathrm{Gr}\,\psi_{t_0}\Big)\big((v, \psi_{t_0} v), (v', \psi_{t_0} v')\big) \\
&= \frac{d}{dt}\overline{\Omega}\big((v, \psi_{t_0} v), \alpha_t(v', \psi_{t_0} v')\big)\Big|_{t_0} \\
&= \frac{d}{dt}\overline{\Omega}\big((v, \psi_{t_0} v), (\widetilde{\alpha}_t v', -\psi_{t_0}\widetilde{\alpha}_t v')\big)\Big|_{t_0} \\
&= -2\Omega_0\Big(v, \frac{d}{dt}\widetilde{\alpha}_t\Big|_{t_0} v'\Big) \\
&= \Omega_0(v, \psi_{t_0}^{-1}\dot{\psi}_{t_0} v') \\
&= \Omega_0(\psi_{t_0} v, J_0 S_{t_0} \psi_{t_0} v').
\end{aligned}$$

Hence the restriction of Q to $\mathrm{Ker}(\psi_{t_0} - \mathrm{Id})$ is given by

$$Q\Big(\mathrm{Gr}\,\psi_{t_0}, \frac{d}{dt}\,\mathrm{Gr}\,\psi_{t_0}\Big)\big((v, \psi_{t_0} v), (v', \psi_{t_0} v')\big) = v^\tau S_{t_0} v' \quad \forall v, v' \in \mathrm{Ker}(\psi_{t_0} - \mathrm{Id}).$$

A crossing $t_0 \in [0,1]$ is thus regular for the smooth path $\mathrm{Gr}\,\psi$ if and only if the restriction of S_{t_0} to $\mathrm{Ker}(\psi_{t_0} - \mathrm{Id})$ is nondegenerate.

Definition 6.4.5 ([RS93]) Let $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \psi_t$ be a smooth path of symplectic matrices. Write $\dot{\psi}_t = J_0 S_t \psi_t$ with $t \mapsto S_t$ a path of symmetric matrices. A number $t \in [0,1]$ is called a *crossing* if $\det(\psi_t - \mathrm{Id}) = 0$. For $t \in [0,1]$, the *crossing form* $\Gamma(\psi, t)$ is defined as the quadratic form which is the restriction of S_t to $\mathrm{Ker}(\psi_t - \mathrm{Id})$. A crossing t_0 is called *regular* if the crossing form $\Gamma(\psi, t_0)$ is nondegenerate.

Proposition 6.4.6 ([RS93]) *For a smooth path $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \psi_t$ having only regular crossings, the Robbin-Salamon index introduced in definition 6.4.1 is given by*

$$\mu_{RS}(\psi) = \frac{1}{2}\,\mathrm{Sign}\,\Gamma(\psi, 0) + \sum_{\substack{t \text{ crossing},\\ t\, \in]0,1[}} \mathrm{Sign}\,\Gamma(\psi, t) + \frac{1}{2}\,\mathrm{Sign}\,\Gamma(\psi, 1). \tag{6.16}$$

Proposition 6.4.7 ([RS93]) *Let $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be a continuous path of symplectic matrices such that $\psi(0) = \mathrm{Id}$ and such that 1 is not an eigenvalue of $\psi(1)$ (i.e. $\psi \in \mathrm{SP}(n)$). The Robbin-Salamon index of ψ defined by (6.15) coincides with the Conley-Zehnder index of ψ In particular, for a smooth path $\psi \in \mathrm{SP}(n)$ having only regular crossings, the Conley-Zehnder index is given by*

$$\begin{aligned}\mu_{CZ}(\psi) &= \frac{1}{2}\,\mathrm{Sign}\,\Gamma(\psi,0) + \sum_{\substack{t \text{ crossing},\\ t\in]0,1[}} \mathrm{Sign}\,\Gamma(\psi,t)\\ &= \frac{1}{2}\,\mathrm{Sign}(S_0) + \sum_{\substack{t \text{ crossing},\\ t\in]0,1[}} \mathrm{Sign}\,\Gamma(\psi,t) \qquad (6.17)\end{aligned}$$

with $S_0 = -J_0\dot{\psi}_0$.

Proof: Since the Robbin-Salamon index for paths of Lagrangians is invariant under homotopies with fixed end points, the Robbin-Salamon index for paths of symplectic matrices is also invariant under homotopies with fixed endpoints.

Its restriction to $\mathrm{SP}(n)$ is actually invariant under homotopies of paths in $\mathrm{SP}(n)$ since for any path in $\mathrm{SP}(n)$, the starting point $\psi_0 = \mathrm{Id}$ is fixed and the endpoint ψ_1 can only move in a connected component of $\mathrm{Sp}^*(\mathbb{R}^{2n}, \Omega_0)$ where no matrix has 1 as an eigenvalue.

To show that this index coincides with the Conley-Zehnder index, it is enough, in view of proposition 6.2.6, to show that it satisfies the loop and signature properties.

Let us prove the signature property. Let $\psi_t = \exp(tJ_0S)$ with S a symmetric nondegenerate matrix with all eigenvalues of absolute value $< 2\pi$, so that $\mathrm{Ker}(\exp(tJ_0S) - \mathrm{Id}) = \{0\}$ for all $t \in]0,1]$. Hence the only crossing is at $t = 0$, where $\psi_0 = \mathrm{Id}$ and $\dot{\psi}_t = J_0S\psi_t$ so that $S_t = S$ for all t and

$$\mu_{CZ}(\psi) = \frac{1}{2}\,\mathrm{Sign}\,S_0 = \frac{1}{2}\,\mathrm{Sign}\,S.$$

To prove the loop property, note that μ_{RS} is additive for catenation and invariant under homotopies with fixed endpoints. The path $(\phi\psi)$ is homotopic to the catenation of ϕ and ψ; it is thus enough to show that the Robbin-Salamon index of a loop is equal to $2\deg(\rho\circ\phi)$. Since two loops ϕ and ϕ' are homotopic if and only if $\deg(\rho\circ\phi) = \deg(\rho\circ\phi')$, it is enough to consider the loops ϕ_n defined by

$$\phi_n(t) := \begin{pmatrix}\cos 2\pi nt & -\sin 2\pi nt\\ \sin 2\pi nt & \cos 2\pi nt\end{pmatrix} \oplus \begin{pmatrix} a(t)\,\mathrm{Id} & 0\\ 0 & a(t)^{-1}\,\mathrm{Id}\end{pmatrix}$$

with $a : [0,1] \to \mathbb{R}^+$ a smooth curve with $a(0) = a(1) = 1$ and $a(t) \neq 1$ for $t \in]0,1[$. Since $\rho\big(\phi_n(t)\big) = e^{2\pi int}$, we have $\deg(\phi_n) = n$.
The crossings of ϕ_n arise at $t = \frac{m}{n}$ with m an integer between 0 and n. At such a crossing, $\mathrm{Ker}\big(\phi_n(t)\big)$ is $\mathbb{R}^2$ for $0 < t < 1$ and is $\mathbb{R}^{2n}$ for $t = 0$ and $t = 1$. We have

$$\dot{\phi}_n(t) = \left(\begin{pmatrix}0 & -2\pi n\\ 2\pi n & 0\end{pmatrix} \oplus \begin{pmatrix}\frac{\dot{a}(t)}{a(t)}\,\mathrm{Id} & 0\\ 0 & -\frac{\dot{a}(t)}{a(t)}\,\mathrm{Id}\end{pmatrix}\right)\phi_n(t)$$

so that, extending $\oplus$ to symmetric matrices in the obvious way,

$$S(t) = \begin{pmatrix} 2\pi n & 0 \\ 0 & 2\pi n \end{pmatrix} \oplus \begin{pmatrix} 0 & -\frac{\dot a(t)}{a(t)}\,\mathrm{Id} \\ -\frac{\dot a(t)}{a(t)}\,\mathrm{Id} & 0 \end{pmatrix}.$$

Thus $\operatorname{Sign}\Gamma(\phi_n, t) = 2$ for all crossings $t = \frac{m}{n},\ 0 \leq m \leq n$. From equation (6.16) we get

$$\begin{aligned} \mu_{RS}(\phi_n) &= \frac{1}{2}\operatorname{Sign}\Gamma(\phi_n, 0) + \sum_{0<m<n} \operatorname{Sign}\Gamma\big(\phi_n, \tfrac{m}{n}\big) + \frac{1}{2}\operatorname{Sign}\Gamma(\phi_n, 1) \\ &= 1 + 2(n-1) + 1 = 2n = 2\deg(\rho \circ \phi_n) \end{aligned}$$

and the loop property is proved. Thus the Robbin-Salamon index for paths in $\mathrm{SP}(n)$ coincides with the Conley-Zehnder index.

The formula for the Conley-Zehnder index of a path $\psi \in \mathrm{SP}(n)$ having only regular crossings, follows then from (6.16). Indeed, we have $\operatorname{Ker}(\psi_1 - \mathrm{Id}) = \{0\}$, while $\operatorname{Ker}(\psi_0 - \mathrm{Id}) = \mathbb{R}^{2n}$ and $\Gamma(\psi, 0) = S_0$.

6.4.2 Another index defined by Robbin and Salamon

Definition 6.4.8 A *symplectic shear* is a path of symplectic matrices of the form $\psi_t = \begin{pmatrix} \mathrm{Id} & B(t) \\ 0 & \mathrm{Id} \end{pmatrix}$ with $B(t)$ symmetric.

Proposition 6.4.9 *The Robbin-Salamon index of a symplectic shear $\psi_t = \begin{pmatrix} \mathrm{Id} & B(t) \\ 0 & \mathrm{Id} \end{pmatrix}$, with $B(t)$ symmetric, is equal to*

$$\mu_{RS}(\psi) = \frac{1}{2}\operatorname{Sign} B(0) - \frac{1}{2}\operatorname{Sign} B(1).$$

Proof: We write $B(t) = A(t)^\tau D(t) A(t)$ with $A(t) \in O(\mathbb{R}^n)$ and $D(t)$ a diagonal matrix. The matrix $\phi_t = \begin{pmatrix} A(t)^\tau & 0 \\ 0 & A(t) \end{pmatrix}$ is in $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ and

$$\psi'_t := \phi_t \psi_t \phi_t^{-1} = \begin{pmatrix} \mathrm{Id} & D(t) \\ 0 & \mathrm{Id} \end{pmatrix}.$$

By proposition 6.4.3 $\mu_{\mathrm{RS}}(\psi) = \mu_{\mathrm{RS}}(\psi')$; by the product property it is enough to show that $\mu_{\mathrm{RS}}(\psi) = \frac{1}{2}\operatorname{Sign} d(0) - \frac{1}{2}\operatorname{Sign} d(1)$ for the path

$$\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^2, \Omega_0) : t \mapsto \psi_t = \begin{pmatrix} 1 & d(t) \\ 0 & 1 \end{pmatrix}.$$

Since μ_{RS} is invariant under homotopies with fixed end points, we may assume $\psi_t = \begin{pmatrix} a(t) & d(t) \\ c(t) & a(t)^{-1}\big(1 + d(t)c(t)\big) \end{pmatrix}$ with a and c smooth functions such that $a(0) = 1$, $a(1) = 1$, $\dot a(0) \neq$

0, $\dot{a}(1) \neq 0$ and $a(t) > 1$ for $0 < t < 1$; $c(0) = c(1) = 0$, $c(t)d(t) \geq 0$ $\forall t$ and $\dot{c}(t) \neq 0$ (resp.$= 0$) when $d(t) \neq 0$ (resp.$= 0$) for $t = 0$ or 1.
The only crossings are $t = 0$ and $t = 1$ since the trace of $\psi(t)$ is > 2 for $0 < t < 1$. Now, at those points ($t = 0$ and $t = 1$) $\dot{\psi}_t = \begin{pmatrix} \dot{a}(t) & \dot{d}(t) \\ \dot{c}(t) & -\dot{a}(t)+d(t)\dot{c}(t) \end{pmatrix}$ so that $S_t = -J_0\dot{\psi}_t\psi_t^{-1} = \begin{pmatrix} \dot{c}(t) & -\dot{a}(t) \\ -\dot{a}(t) & \dot{a}(t)d(t)-\dot{d}(t) \end{pmatrix}$.

Clearly, at the crossings, we have $\operatorname{Ker}\psi_t = \mathbb{R}^2$ iff $d(t) = 0$ and $\operatorname{Ker}\psi_t$ is spanned by the first basis element iff $d(t) \neq 0$, so that from definition 6.4.5 $\Gamma(\psi, t) = \begin{pmatrix}\dot{c}(t)\end{pmatrix}$ when $d(t) \neq 0$ and $\Gamma(\psi, t) = \begin{pmatrix} 0 & -\dot{a}(t) \\ -\dot{a}(t) & 0 \end{pmatrix}$ when $d(t) = 0$. Hence both crossings are regular and $\operatorname{Sign}\Gamma(\psi, t) = \operatorname{Sign}\dot{c}(t)$ when $d(t) \neq 0$ and $\operatorname{Sign}\Gamma(\psi, t) = 0$ when $d(t) = 0$. Since $d(t)c(t) \geq 0$ for all t, we clearly have $\operatorname{Sign}\dot{c}(0) = \operatorname{Sign} d(0)$ and $\operatorname{Sign}\dot{c}(1) = -\operatorname{Sign} d(1)$. Proposition 6.4.6 then gives $\mu_{\mathrm{RS}}(\psi) = \frac{1}{2}\Gamma(\psi, 0) + \frac{1}{2}\operatorname{Sign}\Gamma(\psi, 1) = \frac{1}{2}\operatorname{Sign} d(0) - \frac{1}{2}\operatorname{Sign} d(1)$.

Remark 6.4.10 Robbin and Salamon introduce another index μ'_{RS} for paths of symplectic matrices built from their index for paths of Lagrangians. Consider the fixed Lagrangian $L = \{0\} \times \mathbb{R}^n$ in $(\mathbb{R}^{2n}, \Omega_0)$, observe that AL is Lagrangian for any $A \in \operatorname{Sp}(\mathbb{R}^{2n}, \Omega_0)$, and define, for $\psi : [0, 1] \to \operatorname{Sp}(\mathbb{R}^{2n}, \Omega_0)$

$$\mu'_{\mathrm{RS}}(\psi) := \mu_{\mathrm{RS}}(\psi L, L). \tag{6.18}$$

This index has the following properties [RS93] :

- it is invariant under homotopies with fixed endpoints and two paths with the same endpoints are homotopic with fixed endpoints if and only if they have the same μ'_{RS} index;
- it is additive under catenation of paths;
- it has the product property $\mu_{RS}(\psi' \oplus \psi'') = \mu_{RS}(\psi') + \mu_{RS}(\psi'')$;
- it vanishes on a path whose image lies in

  $$\{A \in \operatorname{Sp}(\mathbb{R}^{2n}, \Omega_0) \mid \dim AL \cap L = k\}$$

  for a given $k \in \{0, \ldots, n\}$;
- $\mu'_{RS}(\psi) = \frac{1}{2}\operatorname{Sign} B(0) - \frac{1}{2}\operatorname{Sign} B(1)$ when $\psi_t = \begin{pmatrix} \mathrm{Id} & B(t) \\ 0 & \mathrm{Id} \end{pmatrix}$.

Robbin and Salamon [RS93] prove that those properties characterize this index.

The two indices μ_{RS} and μ'_{RS} defined on paths of symplectic matrices DO NOT coincide in general. Indeed, consider the path $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \psi_t = \begin{pmatrix} \mathrm{Id} & 0 \\ C(t) & \mathrm{Id} \end{pmatrix}$. Since $\psi_t L \cap L = L \quad \forall t$, $\mu'_{\mathrm{RS}}(\psi) = 0$. On the other hand, if $\phi = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$ and $\psi' = \phi\psi\phi^{-1}$, then $\psi'_t = \begin{pmatrix} \mathrm{Id} & -C(t) \\ 0 & \mathrm{Id} \end{pmatrix}$. Then

$$\mu'_{\mathrm{RS}}(\psi') = \frac{1}{2}\operatorname{Sign} C(1) - \frac{1}{2}\operatorname{Sign} C(0)$$

which is in general different from $\mu'_{\mathrm{RS}}(\psi)$. Whereas, by (6.4.3), $\mu_{\mathrm{RS}}(\psi) = \mu_{\mathrm{RS}}(\psi')$.

The index μ'_{RS} vanishes on a path whose image lies into one of the $(n+1)$ strata defined by $\{A \in \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) \mid \dim AL \cap L = k\}$ for $0 \leq k \leq n$, whereas the index μ_{RS} vanishes on a path whose image lies into one of the $(2n+1)$ strata defined by the set of symplectic matrices whose eigenspace of eigenvalue 1 has dimension k (for $0 \leq k \leq 2n$).

However, the two indices μ_{RS} and μ'_{RS} coincide on symplectic shears.

6.5 Characterization of the Robbin-Salamon index

In this section, we prove theorem 6.1.2 stated in the introduction. Before proving this theorem, we show that the Robbin-Salamon index is characterized by the fact that it extends Conley-Zehnder index and has all the properties stated in the previous section. This is made explicit in Lemma 6.5.2. We then use the characterization of the Conley-Zehnder index given in Proposition 6.2.6 to give in Lemma 6.5.3 a characterization of the Robbin-Salamon index in terms of six properties. We use explicitly the normal form of the restriction of a symplectic endomorphism to its generalised eigenspace of eigenvalue 1 that we have proven in [Gutb] and that we summarize in the following proposition

Proposition 6.5.1 (Normal form for $A_{|V_{[\lambda]}}$ for $\lambda = \pm 1$.) *Let $\lambda = \pm 1$ be an eigenvalue of $A \in Sp(\mathbb{R}^{2n}, \Omega_0)$ and let $V_{[\lambda]}$ be the generalized eigenspace of eigenvalue λ. There exists a symplectic basis of $V_{[\lambda]}$ in which the matrix associated to the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of matrices of the form*

$$\begin{pmatrix} J(\lambda, r_j)^{-1} & C(r_j, d_j, \lambda) \\ 0 & J(\lambda, r_j)^{\tau} \end{pmatrix}$$

where $C(r_j, d_j, \lambda) := J(\lambda, r_j)^{-1}\operatorname{diag}\big(0, \ldots, 0, d_j\big)$ with $d_j \in \{0, 1, -1\}$. If $d_j = 0$, then r_j is odd. The dimension of the eigenspace of eigenvalue 1 is given by $2\operatorname{Card}\{j \mid d_j = 0\} + \operatorname{Card}\{j \mid d_j \neq 0\}$.
For any integer $k \geq 1$, the bilinear form on $\operatorname{Ker}\big((A - \lambda\,\mathrm{Id})^{2k}\big)$ defined by

$$\begin{aligned} \hat{Q}_k \quad : \quad & \operatorname{Ker}\Big((A - \lambda\,\mathrm{Id})^{2k}\Big) \times \operatorname{Ker}\Big((A - \lambda\,\mathrm{Id})^{2k}\Big) \to \mathbb{R} \\ & (v, w) \mapsto \Omega((A - \lambda\,\mathrm{Id})^{k} v, (A - \lambda\,\mathrm{Id})^{k-1} w) \end{aligned} \tag{6.19}$$

is symmetric and we have

$$\sum_j d_j = \lambda \sum_{k\geq 1} \text{Signature}(\hat{Q}_k) \tag{6.20}$$

Lemma 6.5.2 *The Robbin-Salamon index is characterized by the following properties:*

1. *(Generalization) it is a correspondence μ_{RS} which associates a half integer to any continuous path $\psi : [a,b] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ of symplectic matrices and it coincides with μ_{CZ} on paths starting from the identity matrix and ending at a matrix for which 1 is not an eigenvalue;*

2. (Naturality) *if $\phi, \psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$, we have $\mu_{RS}(\phi\psi\phi^{-1}) = \mu_{RS}(\psi)$;*

3. (Homotopy) *it is invariant under homotopies with fixed end points;*

4. (Catenation) *it is additive under catenation of paths;*

5. (Product) *it has the product property $\mu_{RS}(\psi' \oplus \psi'') = \mu_{RS}(\psi') + \mu_{RS}(\psi'')$;*

6. (Zero) *it vanishes on any path $\psi : [a,b] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega)$ of matrices such that $\dim \mathrm{Ker}(\psi(t) - \mathrm{Id}) = k$ is constant on $[a,b]$;*

7. (Shear)*on a symplectic shear , $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ of the form*

$$\psi_t = \begin{pmatrix} \mathrm{Id} & -tB \\ 0 & \mathrm{Id} \end{pmatrix} = \exp t \begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix} = \exp tJ_0 \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

with B symmetric, it is equal to $\mu_{RS}(\psi) = \frac{1}{2} \,\mathrm{Sign}\, B$.

Proof: We have seen in the previous section that the index μ_{RS} defined by Robbin and Salamon satisfies all the above properties. To see that those properties characterize this index, it is enough to show (since the group $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ is connected and since we have the catenation property) that those properties determine the index of any path starting from the identity. Since it must be a generalization of the Conley-Zehnder index and must be additive for catenations of paths, it is enough to show that any symplectic matrix A which admits 1 as an eigenvalue can be linked to a matrix B which does not admit 1 as an eigenvalue by a continuous path whose index is determined by the properties stated. From proposition 6.5.1, there is a basis of $\mathbb{R}^{2n}$ such that A is the symplectic direct sum of a matrix which does not admit 1 as eigenvalue and matrices of the form

$$A^{(1)}_{r_j,d_j} := \begin{pmatrix} J(1,r_j)^{-1} & J(1,r_j)^{-1}\,\mathrm{diag}(0,\ldots,0,d_j) \\ 0 & J(1,r_j)^{\tau} \end{pmatrix};$$

with d_j equal to $0, 1$ or -1. The dimension of the eigenspace of eigenvalue 1 for $A^{(1)}_{r_j,d_j}$ is equal to 1 if $d_j \neq 0$ and is equal to 2 if $d_j = 0$. In view of the naturality and the product property of the index, we can consider a symplectic direct sum of paths with the constant path on the symplectic subspace where 1 is not an eigenvalue and we just have to build a path in $\mathrm{Sp}(\mathbb{R}^{2r_j},\Omega_0)$ from $A^{(1)}_{r_j,d_j}$ to a matrix which does not admit 1 as eigenvalue and whose index is determined by the properties given in the statement. This we do by the catenation of three paths : we first build the path $\psi_1 : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2r_j},\Omega_0)$ defined by

$$\psi_1(t) := \begin{pmatrix} D(t,r_j)^{-1} & D(t,r_j)^{-1}\operatorname{diag}\big(c(t),0,\ldots,0,d(t)\big) \\ 0 & D(t,r_j)^{\tau} \end{pmatrix}$$

with $D(t,r_j) = \begin{pmatrix} 1 & 1-t & 0 & \ldots & \ldots & 0 \\ 0 & e^t & 1-t & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & \vdots \\ 0 & \ldots & 0 & e^t & 1-t & 0 \\ 0 & \ldots & \ldots & 0 & e^t & 1-t \\ 0 & \ldots & \ldots & \ldots & 0 & e^t \end{pmatrix}$,
and with $c(t) = td_j$, $d(t) = (1-t)d_j$. Observe that $\psi_1(0) = A^{(1)}_{r_j,d_j}$ and $\psi_1(1)$ is the symplectic direct sum of $\begin{pmatrix} 1 & c(1)=d_j \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} e^{-1}\,\mathrm{Id}_{r_j-1} & 0 \\ 0 & e\,\mathrm{Id}_{r_j-1} \end{pmatrix}$ and this last matrix does not admit 1 as eigenvalue.

Clearly $\dim\ker\big(\psi_1(t)-\mathrm{Id}\big) = 2$ for all $t\in[0,1]$ when $d_j = 0$; we now prove that $\dim\ker(\psi_1(t)-\mathrm{Id}) = 1$ for all $t\in[0,1]$ when $d_j \neq 0$. Hence the index of ψ_1 must always be zero by the zero property.
To prove that $\dim\ker(\psi_1(t)-\mathrm{Id}) = 1$ we have to show the non vanishing of the determinant of the $2r_j-1\times 2r_j-1$ matrix

$$\begin{pmatrix}
E^t_{12} & \ldots & \ldots & E^t_{1r_j} & c(t) & 0 & \ldots & 0 & E^t_{1r_j}d(t) \\
e^{-t}-1 & E^t_{23} & \ddots & E^t_{2r_j} & 0 & 0 & \ldots & 0 & E^t_{2r_j}d(t) \\
0 & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\vdots & \ddots & e^{-t}-1 & E^t_{r_j-1\,r_j} & 0 & 0 & \ldots & 0 & E^t_{r_j-1\,r_j}d(t) \\
0 & \ldots & 0 & e^{-t}-1 & 0 & 0 & \ldots & 0 & e^{-t}d(t) \\
0 & \ldots & 0 & 0 & 1-t & e^t-1 & 0 & \ddots & 0 \\
\vdots & \ldots & \vdots & 0 & 0 & 1-t & e^t-1 & \ddots & 0 \\
\vdots & \ldots & \vdots & 0 & \ldots & 0 & \ddots & \ddots & 0 \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1-t & e^t-1
\end{pmatrix}$$

where $E^t := D(t,r_j)^{-1}$ is upper triangular. This determinant is equal to

$$(-1)^{r_j+1}c(t)(e^{-t}-1)^{r_j-1}(e^t-1)^{r_j-1} + (-1)^{r_j-1}d(t)(1-t)^{r_j-1}\det E'(t)$$

where $E'(t)$ is obtained by deleting the first column and the last line in $E(t)-\mathrm{Id}$ so given by the $(r_j-1)\times(r_j-1)$ matrix

$$\begin{pmatrix} (t-1)e^{-t} & (t-1)^2e^{-2t} & \dots & \dots & (t-1)^{r_j-1}e^{-(r_j-1)t} \\ e^{-t}-1 & (t-1)e^{-2t} & (t-1)^2e^{-3t} & \dots & (t-1)^{r_j-2}e^{-(r_j-1)t} \\ 0 & e^{-t}-1 & (t-1)e^{-2t} & \ddots & (t-1)^{r_j-3}e^{-(r_j-2)t} \\ \vdots & \ddots & \ddots & \ddots & \ddots \quad \vdots \\ \vdots & & \ddots \quad e^{-t}-1 & (t-1)e^{-2t} & (t-1^2)e^{-3t} \\ 0 & \dots & 0 & e^{-t}-1 & (t-1)e^{-2t} \end{pmatrix}.$$

Thus $\det E'(t)=(t-1)(e^{-t}-(e^{-t}-1))\det F_{r_j-2}(t)$ where

$$F_m(t):=\begin{pmatrix} (t-1)e^{-2t} & (t-1)^2e^{-3t} & \dots & \dots & (t-1)^{r_j-2}e^{-(r_j-1)t} \\ e^{-t}-1 & (t-1)e^{-2t} & \ddots & \dots & (t-1)^{r_j-3}e^{-(r_j-2)t} \\ 0 & e^{-t}-1 & (t-1)e^{-2t} & \ddots & (t-1)^{r_j-3}e^{-(r_j-2)t} \\ \vdots & \ddots & \ddots & \ddots & \ddots \quad \vdots \\ \vdots & & \ddots \quad e^{-t}-1 & (t-1)e^{-2t} & (t-1^2)e^{-3t} \\ 0 & \dots & 0 & e^{-t}-1 & (t-1)e^{-2t} \end{pmatrix}.$$

and we have $\det F_m(t)=((t-1)e^{-2t}-(e^{-t}-1)(t-1)e^{-t})\det F_{m-1}(t)=(t-1)e^{-t}\det F_{m-1}(t)$ so that, by induction on m, $\det F_m(t)=(t-1)^m e^{-(m+1)t}$ hence the determinant we have to study is
$(-1)^{r_j-1}c(t)(2-e^t-e^{-t})^{r_j-1}+d(t)(t-1)^{r_j}\det F_{r_j-2}(t)$ which is equal to
$(-1)^{r_j-1}c(t)(2-e^t-e^{-t})^{r_j-1}+d(t)(t-1)^{r_j}(t-1)^{r_j-2}e^{-(r_j-1)t}$ hence to

$$c(t)(e^t+e^{-t}-2)^{r_j-1}+d(t)(1-t)^{2r_j-2}e^{-(r_j-1)t}$$

which never vanishes if $c(t)=td_j$ and $d(t)=(1-t)d_j$ since $e^t+e^{-t}-2$ and $(1-t)$ are ≥ 0.

We then construct a path $\psi_2:[0,1]\to \mathrm{Sp}(\mathbb{R}^{2r_j},\Omega_0)$ which is constant on the symplectic subspace where 1 is not an eigenvalue and which is a symplectic shear on the first two dimensional symplectic vector space, i.e.

$$\psi_2(t):=\begin{pmatrix} 1 & (1-t)d_j \\ 0 & 1\end{pmatrix}\oplus\begin{pmatrix} e^{-1}\,\mathrm{Id}_{r_j-1} & 0 \\ 0 & e\,\mathrm{Id}_{r_j-1}\end{pmatrix};$$

then the index of ψ_2 is equal to $\frac{1}{2}\operatorname{Sign} d_j$. Observe that ψ_2 is constant if $d_j=0$; then the index of ψ_2 is zero. In all cases $\psi_2(1)=\mathrm{Id}_2\oplus\begin{pmatrix} e^{-1}\,\mathrm{Id}_{r_j-1} & 0 \\ 0 & e\,\mathrm{Id}_{r_j-1}\end{pmatrix}$.
We then build $\psi_3:[0,1]\to \mathrm{Sp}(\mathbb{R}^{2r_j},\Omega_0)$ given by

$$\psi_3(t):=\begin{pmatrix} e^{-t} & 0 \\ 0 & e^t\end{pmatrix}\oplus\begin{pmatrix} e^{-1}\,\mathrm{Id}_{r_j-1} & 0 \\ 0 & e\,\mathrm{Id}_{r_j-1}\end{pmatrix}$$

which is the direct sum of a path whose Conley-Zehnder index is known and a constant path whose index is zero. Clearly 1 is not an eigenvalue of $\psi_3(1)$.

Combining the above with the characterization of the Conley-Zehnder index, we now prove:

Lemma 6.5.3 *The Robbin-Salamon index for a path of symplectic matrices is characterized by the following properties:*

- (Homotopy) *it is invariant under homotopies with fixed end points;*
- (Catenation) *it is additive under catenation of paths;*
- (Zero) *it vanishes on any path* $\psi : [a,b] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega)$ *of matrices such that* $\dim \mathrm{Ker}(\psi(t) - \mathrm{Id}) = k$ *is constant on* $[a,b]$*;*
- (Product) *it has the product property* $\mu_{RS}(\psi' \oplus \psi'') = \mu_{RS}(\psi') + \mu_{RS}(\psi'')$*;*
- (Signature) *if* $S = S^\tau \in \mathbb{R}^{2n\times 2n}$ *is a symmetric non degenerate matrix with all eigenvalues of absolute value* $< 2\pi$ *and if* $\psi(t) = \exp(J_0 S t)$ *for* $t \in [0,1]$, *then* $\mu_{RS}(\psi) = \frac{1}{2} \mathrm{Sign}\, S$ *where* $\mathrm{Sign}\, S$ *is the signature of* S*;*
- (Shear) *if* $\psi_t = \exp t J_0 \left(\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}\right)$ *for* $t \in [0,1]$, *with* B *symmetric, then* $\mu_{RS}(\psi) = \frac{1}{2} \mathrm{Sign}\, B$.

PROOF: Remark first that the invariance by homotopies with fixed end points, the additivity under catenation and the zero property imply the naturality; they also imply the constancy on the components of $\mathrm{SP}(n)$. The signature property stated above is the signature property which arose in the characterization of the Conley-Zehnder index given in proposition 6.2.6. To be sure that our index is a generalization of the Conley-Zehnder index, there remains just to prove the loop property. Since the product of a loop ϕ and a path ψ starting at the identity is homotopic to the catenation of ϕ and ψ, it is enough to prove that the index of a loop ϕ with $\phi(0) = \phi(1) = \mathrm{Id}$ is given by $2\deg(\rho \circ \phi)$. Since two loops ϕ and ϕ' are homotopic if and only if $\deg(\rho\circ\phi) = \deg(\rho\circ\phi')$, it is enough to consider the loops ϕ_n defined by $\phi_n(t) := \left(\begin{smallmatrix} \cos 2\pi nt & -\sin 2\pi nt \\ \sin 2\pi nt & \cos 2\pi nt \end{smallmatrix}\right) \oplus \mathrm{Id}$; since $\phi_n(t) = \big(\phi_1(t)\big)^n$, it is enough to show, using the homotopy, catenation, product and zero properties that the index of the loop given by $\phi(t) = \left(\begin{smallmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{smallmatrix}\right)$ for $t \in [0,1]$ is equal to 2. This is true, using the signature property, writing ϕ as the catenation of the path $\psi_1(t) := \phi(\frac{t}{2}) = \exp t J_0 \left(\begin{smallmatrix} \pi & 0 \\ 0 & \pi \end{smallmatrix}\right)$ for $t \in [0,1]$ whose index is 1 and the path $\psi_2(t) := \phi(\frac{t}{2}) = \exp t J_0 \left(\begin{smallmatrix} \pi & 0 \\ 0 & \pi \end{smallmatrix}\right)$ for $t \in [1,2]$. We introduce the path in the reverse direction $\psi_2^-(t) := \exp -t J_0 \left(\begin{smallmatrix} \pi & 0 \\ 0 & \pi \end{smallmatrix}\right)$ for $t \in [0,1]$ whose index is -1; since the catenation of ψ_2^- and ψ_2 is homotopic to the constant path whose index is zero, the index of ϕ_1 is given by the index of ψ_1 minus the index of ψ_2^- hence is equal to 2.

We are now ready to prove the characterization of the Robbin-Salamon index stated in the introduction.

Proof of theorem 6.1.2: Observe that any symmetric matrix can be written as the symplectic direct sum of a non degenerate symmetric matrix S and a matrix S' of the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}\right)$ where B is symmetric and may be degenerate. The index of the path $\psi_t = \exp tJ_0S'$ is equal to the index of the path $\psi'_t = \exp t\lambda J_0 S'$ for any $\lambda > 0$. Hence the signature and shear conditions, in view of the product condition, can be simultaneously written as: if $S = S^\tau \in \mathbb{R}^{2n\times 2n}$ is a symmetric matrix with all eigenvalues of absolute value $< 2\pi$ and if $\psi(t) = \exp(J_0St)$ for $t \in [0,1]$, then $\mu_{\mathrm{RS}}(\psi) = \frac{1}{2}\operatorname{Sign} S$. This is the normalization condition stated in the theorem.

From Lemma 6.5.3, we just have to prove that the product property is a consequence of the other properties. We prove it for paths with values in $\mathrm{Sp}(\mathbb{R}^{2n},\Omega_0)$ by induction on n, the case $n=1$ being obvious. Since $\psi' \oplus \psi''$ is homotopic with fixed endpoints to the catenation of $\psi' \oplus \big(\psi''(0)\big)$ and $\big(\psi'(1)\big) \oplus \psi''$, it is enough to show that the index of $A \oplus \psi$ is equal to the index of ψ for any fixed $A \in \mathrm{Sp}(\mathbb{R}^{2n'},\Omega_0)$ with $n' < n$ and any continuous path $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n''},\Omega_0)$ with $n'' < n$.

Using the proof of lemma 6.5.2, any symplectic matrix A can be linked by a path $\phi(s)$ with constant dimension of the 1-eigenspace to a matrix of the form $\exp(J_0S')$ with S' a symmetric $n' \times n'$ matrix with all eigenvalues of absolute value $< 2\pi$. The index of $A \oplus \psi$ is equal to the index of $\exp(J_0S') \oplus \psi$; indeed $A \oplus \psi$ is homotopic with fixed endpoints to the catenation of the three paths $\phi_s \oplus \psi(0)$, $\exp(J_0S') \oplus \psi$ and the path $\phi_s \oplus \psi(1)$ in the reverse order, and the index of the first and third paths are zero since the dimension of the 1-eigenspace does not vary along those paths.

Hence it is enough to show that the index of $\exp(J_0S') \oplus \psi$ is the same as the index of ψ. This is true because the map μ sending a path ψ in $\mathrm{Sp}(\mathbb{R}^{2n''},\Omega_0)$ (with $n'' < n$) to the index of $\exp(J_0S') \oplus \psi$ has the four properties stated in the theorem, and these characterize the Robbin-Salamon index for those paths by induction hypothesis. It is clear that μ is invariant under homotopies, additive for catenation and equal to zero on paths ψ for which the dimension of the 1-eigenspace is constant. Furthermore $\mu(\exp t(J_0S))$ which is the index of $\exp(J_0S') \oplus \exp t(J_0S)$ is equal to $\frac{1}{2}\operatorname{Sign} S$, because the path $\exp tJ_0(S' \oplus S)$ whose index is $\frac{1}{2}\operatorname{Sign}(S' \oplus S) = \frac{1}{2}\operatorname{Sign} S' + \frac{1}{2}\operatorname{Sign} S$ is homotopic with fixed endpoints with the catenation of $\exp t(J_0S') \oplus \mathrm{Id} = \exp tJ_0(S' \oplus 0)$, whose index is $\frac{1}{2}\operatorname{Sign} S'$, and the path $\exp(J_0S') \oplus \exp t(J_0S)$.

6.6 A formula for the Robbin-Salamon index

Let $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be a path of symplectic matrices. The symplectic transformation $\psi(1)$ of $V = \mathbb{R}^{2n}$ decomposes as

$$\psi(1) = \psi^\star(1) \oplus \psi^{(1)}(1)$$

where $\psi^\star(1)$ does not admit 1 as eigenvalue and $\psi^{(1)}(1)$ is the restriction of $\psi(1)$ to the generalized eigenspace of eigenvalue 1

$$\psi(1)|_{V_{[1]}} .$$

By proposition 6.5.1, there exists a symplectic matrix A such that $A\psi^{(1)}(1)A^{-1}$ is equal to

$$\begin{aligned}\psi^\star(1) \oplus \begin{pmatrix} J(1,r_1)^{-1} & C\big(r_1,d_1^{(1)},1\big) \\ 0 & J(1,r_1)^\tau \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} J(1,r_k)^{-1} & C\big(r_k,d_k^{(1)},1\big) \\ 0 & J(1,r_k)^\tau \end{pmatrix} \\ \oplus \begin{pmatrix} J(1,s_1)^{-1} & 0 \\ 0 & J(1,s_1)^\tau \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} J(1,s_l)^{-1} & 0 \\ 0 & J(1,s_l)^\tau \end{pmatrix}\end{aligned} \tag{6.21}$$

with each $d_j^{(1)} = \pm 1$. Since $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ is connected, there is a path $\varphi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ such that $\varphi(0) = \mathrm{Id}$ and $\varphi(1) = A$. We define

$$\psi_I : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0) : t \mapsto \varphi(t)\psi(t)\big(\varphi(t)\big)^{-1}.$$

It is a path from $\psi(1)$ to the matrix defined in 6.21. Clearly, $\mu_{RS}(\psi_I) = 0$ and ρ is constant on ψ_I.

Let $\psi_{II} : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be the path from $\psi_I(1)$ to

$$\psi^\star(1) \oplus \begin{pmatrix} 1 & d_1^{(1)} \\ 0 & 1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & d_k^{(1)} \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} e^{-1}\,\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id} \end{pmatrix}$$

defined as in the proof of lemma 6.5.2 in each block by

$$\begin{pmatrix} D(t,r_j)^{-1} & D(t,r_j)^{-1}\,\mathrm{diag}\big(td_j^{(1)},0,\ldots,0,(1-t)d_j^{(1)}\big) \\ 0 & D(t,r_j)^\tau \end{pmatrix}$$

with $D(t,r_j) = \begin{pmatrix} 1 & 1-t & 0 & \ldots & \ldots & 0 \\ 0 & e^t & 1-t & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & \vdots \\ 0 & \ldots & 0 & e^t & 1-t & 0 \\ 0 & \ldots & \ldots & 0 & e^t & 1-t \\ 0 & \ldots & \ldots & \ldots & 0 & e^t \end{pmatrix}$. Note that $\mu_{RS}(\psi_{II}) = 0$ since the eigenspace of eigenvalue 1 has constant dimension and ρ is constant on ψ_{II}.

We define $\psi_{III} : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ from $\psi_{II}(1)$ to

$$\psi^\star(1) \oplus \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \oplus \begin{pmatrix} e^{-1}\,\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id} \end{pmatrix}$$

which is given on each block $\begin{pmatrix} 1 & d_j^{(1)} \\ 0 & 1 \end{pmatrix}$ by $\begin{pmatrix} 1 & (1-t)d_j^{(1)} \\ 0 & 1 \end{pmatrix}$. Note that $\mu_{RS}(\psi_{III}) = \frac{1}{2}\sum_j d_j^{(1)}$ by proposition 6.4.9 and ρ is constant on ψ_{III}.
Finally, consider $\psi_{IV} : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ from $\psi_{III}(1)$ to

$$\psi^\star(1) \oplus \left(\begin{smallmatrix} e^{-1}\,\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id} \end{smallmatrix}\right)$$

which is given by $\psi^\star(1) \oplus \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \oplus \left(\begin{smallmatrix} e^{-1}\,\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id} \end{smallmatrix}\right)$. Note that $\mu_{RS}(\psi_{IV}) = 0$, ρ is constant on ψ_{IV} and $\psi_{IV}(1)$ is in $\mathrm{Sp}^\star(\mathbb{R}^{2n}, \Omega_0)$. Since two paths of matrices with fixed ends are homotopic if and only if their image under ρ are homotopic, the catenation of the paths ψ_{III} and ψ_{IV} is homotopic to any path from $\psi_I(1)$ to $\psi^\star(1) \oplus \left(\begin{smallmatrix} e\,\mathrm{Id} & 0 \\ 0 & e^{-1}\,\mathrm{Id} \end{smallmatrix}\right)$ of the form $\psi^\star(1)\oplus) \oplus \phi_1(t)$ where $\phi_1(t)$ has only real positive eigenvalues. We proceed similarly for $\psi(0)$ and we get

Theorem 6.6.1 *Let $\psi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ be a path of symplectic matrices. Decompose $\psi(0) = \psi^\star(0) \oplus \psi^{(1)}(0)$ and $\psi(1) = \psi^\star(1) \oplus \psi^{(1)}(1)$ where $\psi^\star(0)$ (resp. $\psi^\star(1)$) does not admit 1 as eigenvalue and $\psi^{(1)}(0)$ (resp. $\psi^{(1)}(1)$) is the restriction of $\psi(0)$ (resp. $\psi(1)$) to the generalized eigenspace of eigenvalue 1 of $\psi(0)$ (resp. $\psi(1)$). Consider a prolongation $\Psi : [-1,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ of ψ such that*

- *$\Psi(t) = \psi(t)\, \forall t \in [0,1]$;*
- *$\Psi\left(-\frac{1}{2}\right) = \psi^\star(0) \oplus \left(\begin{smallmatrix} e^{-1}\,\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id} \end{smallmatrix}\right)$ and $\Psi(t) = \psi^\star(0) \oplus \phi_0(t)$ where $\phi_0(t)$ has only real positive eigenvalues for $t \in \left[-\frac{1}{2}, 0\right]$;*
- *$\Psi\left(\frac{3}{2}\right) = \psi^\star(1) \oplus \left(\begin{smallmatrix} e^{-1}\,\mathrm{Id} & 0 \\ 0 & e\,\mathrm{Id} \end{smallmatrix}\right)$ and $\Psi(t) = \psi^\star(1) \oplus \phi_1(t)$ where $\phi_1(t)$ has only real positive eigenvalues for $t \in \left[1, \frac{3}{2}\right]$;*
- *$\Psi(-1) = W^{\pm}$, $\Psi(2) = W^{\pm}$ and $\Psi(t) \in \mathrm{Sp}^\star(\mathbb{R}^{2n}, \Omega_0)$ for $t \in \left[-1, -\frac{1}{2}\right] \cup \left[\frac{3}{2}, 2\right]$.*

Then

$$\mu_{RS}(\psi) = \deg(\rho^2 \circ \Psi) + \frac{1}{2}\sum d_i^{(0)} - \frac{1}{2}\sum d_j^{(1)}.$$

Remark that we can replace in the formula above ρ by $\tilde{\rho}$ as in proposition 6.2.7.
By proposition 6.5.1, we have theorem 6.1.3 :

$$\mu_{RS}(\psi) = \deg(\rho^2 \circ \Psi) + \frac{1}{2}\sum_{k=1}^{dimV} \mathrm{Sign}\Big(\hat{Q}_k^{(\psi(0))}\Big) - \frac{1}{2}\sum_{k=1}^{dimV} \mathrm{Sign}\Big(\hat{Q}_k^{(\psi(1))}\Big).$$

Remark 6.6.2 The advantage of this new formula is that to compute the index of a path whose crossing with the Maslov cycle is non transverse we do not need to perturb the path. The drawback is that we have to extend the initial path.

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Summary

This thesis deals with the question of the minimal number of distinct periodic Reeb orbits on a contact manifold which is the boundary of a compact symplectic manifold.

The positive S^1-equivariant symplectic homology is one of the main tools considered in this thesis. It is built from periodic orbits of Hamiltonian vector fields in a symplectic manifold whose boundary is the given contact manifold.

Our first result describes the relation between the symplectic homologies of an exact compact symplectic manifold with contact type boundary (also called Liouville domain), and the periodic Reeb orbits on the boundary. We then prove some properties of these homologies. For a Liouville domain embedded into another one, we construct a morphism between their homologies. We study the invariance of the homologies with respect to the choice of the contact form on the boundary. We use the positive S^1-equivariant symplectic homology to give a new proof of a Theorem by Ekeland and Lasry about the minimal number of distinct periodic Reeb orbits on some hypersurfaces in $\mathbb{R}^{2n}$. We indicate how it extends to some hypersurfaces in some negative line bundles.

We also give a characterisation and a new way to compute the generalized Conley-Zehnder index defined by Robbin and Salamon for any path of symplectic matrices. A tool for this is a new analysis of normal forms for symplectic matrices.

Résumé

Le sujet de cette thèse est la question du nombre minimal d'orbites de Reeb distinctes sur une variété de contact qui est le bord d'une variété symplectique compacte.

L'homologie symplectique S^1-équivariante positive est un des outils principaux de cette thèse; elle est construite à partir d'orbites périodiques de champs de vecteurs hamiltoniens sur une variété symplectique dont le bord est la variété de contact considérée.

Nous analysons la relation entre les différentes variantes d'homologie symplectique d'une variété symplectique exacte compacte (domaine de Liouville) et les orbites de Reeb de son bord. Nous démontrons certaines propriétés de ces homologies. Pour un domaine de Liouville plongé dans un autre, nous construisons un morphisme entre leurs homologies. Nous étudions ensuite l'invariance de ces homologies par rapport au choix de la forme de contact sur le bord. Nous utilisons l'homologie symplectique S^1-équivariante positive pour donner une nouvelle preuve d'un théorème de Ekeland et Lasry sur le nombre minimal d'orbites de Reeb distinctes sur certaines hypersurfaces dans $\mathbb{R}^{2n}$. Nous indiquons comment étendre au cas de certaines hypersurfaces dans certains fibrés en droites complexes négatifs.

Nous donnons une caractérisation et une nouvelle façon de calculer l'indice de Conley-Zehnder généralisé, défini par Robbin et Salamon pour tout chemin de matrices symplectiques. Ceci nous a mené à développer de nouvelles formes normales de matrices symplectiques.