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Géométrie de la longueur extrémale sur les espaces de Teichmüller

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Géométrie de la longueur extrémale sur les espaces de Teichmüller

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À ma Famille....

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La violence est le dernier refuge de l'incompétence.
—Isaac Asimov, *Foundation*

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[1] souvent stupides, il est vrai...
[2] Sumimasen encore !

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Je veux aussi remercier Sofiane, même si ce n'est pas un doctorant, pour son soutien et ses discussions franches et joyeuses.

[3] J'ai traduit depuis l'anglais, donc l'exactitude de ces propos peut être sujette à interprétation.

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[4] Je voulais remercier Gustave, mais ce matin il a été insupportable, du coup niet.

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Introduction

Un chemin de mille lieues commence toujours par un pas.

—Lao Tseu

Cette introduction a pour but d'introduire des objets qui seront utilisés tout au long de ce manuscrit. On donnera donc, de manière brève et orientée, un large aperçu de la théorie dite classique de Teichmüller.

1 Généralités

1 .1 Définitions

On va d'abord donner le point de vue conforme de cet espace. Fixons donc une surface de Riemann compacte sans bord X_0 de genre $g \geq 2$[5]. Un *marquage* sera la donnée d'une surface de Riemann X et d'une application quasiconforme $f : X_0 \to X$. Un marquage sera noté (X, f). On dira alors que deux marquages (X, f) et (Y, g) sont équivalents, s'il existe une application holomorphe $h : X \to Y$ qui est isotope à $g \circ f^{-1}$. *L'espace de Teichmüller* de X_0, noté $\mathcal{T}(X_0)$, sera l'ensemble des marquages, modulo cette relation d'équivalence. On notera la classe d'équivalence de (X, f) par $[X, f]$. On peut munir $\mathcal{T}(X_0)$ d'une structure d'espace métrique uniquement géodésique. La métrique utilisée, appelée *métrique de Teichmüller*, est définie comme suit. Pour $x = [X, f] \in \mathcal{T}(X_0)$ et $y = [Y, g] \in \mathcal{T}(X_0)$ on pose

$$d_T(x, y) = \log \inf_h K_h, \tag{1}$$

où $h : X \to Y$ est une application quasiconforme isotope à $g \circ f^{-1}$ et K_h représente sa constante quasiconforme. Pour les définitions de “quasiconforme” et “constante quasiconforme”, on renvoie à [3, 19] ou à la sous-section 2.1.3 du chapitre 2.

Le théorème d'uniformisation nous permet de donner un autre point de vue sur l'espace de Teichmüller, le point de vue hyperbolique. En effet, si S est une surface orientée et compacte de genre $g \geq 2$, alors l'espace de Teichmüller peut être vu comme l'ensemble des classes d'isotopie de structures hyperboliques complètes d'aire finie sur S. Par simplicité, on continuera de le noter $\mathcal{T}(X_0)$. Ce point de

[5]Pour éviter quelques détails techniques, on ne considèrera pas les points marqués.

vue permet de définir une métrique asymétrique appelée *métrique de Thurston*. Rappelons brièvement la définition. Pour x et y, deux points de $\mathcal{T}(X_0)$, on pose

$$d_{Th}(x, y) = \log \inf_{\phi} L_{\phi}, \tag{2}$$

où ϕ est un difféomorphisme de S isotope à l'identité et L_{ϕ} sa constante de Lipschitz.

On peut déjà noter la ressemblance entre les relations (1) et (2). On renvoie le lecteur à [58] pour des détails sur ce sujet.

Dans tout ce qui va suivre, on jonglera entre ces deux points de vues et on essaiera de mettre en évidence les points communs qui en résultent.

1 .2 Paramétrisation à la Teichmüller

L'idée lumineuse de Teichmüller fut de voir le lien entre les applications réalisant l'infimum dans (1) et les différentielles quadratiques.

On rappelle qu'une différentielle quadratique holomorphe (ou plus simplement différentielle quadratique) q sur une surface de Riemann X est une section holomorphe du fibré canonique (ou cotangent) tensorisé deux fois. L'ensemble de telles sections est noté $\Gamma_{\text{hol}}\left(X, K_X^{\otimes 2}\right)$ ou $\mathcal{Q}(X)$. La première notation étant conventionnelle en géométrie complexe, on se bornera à utiliser la deuxième. L'ensemble $\mathcal{Q}(X)$ est muni de manière canonique d'une norme $\|\cdot\|$ (voir la formule (2.4) ci-dessous). De plus, une différentielle quadratique q possède énormément de propriétés géométriques. En effet, elle définit une *métrique plate singulière* sur la surface de Riemann dont l'aire par rapport à cette métrique est égale à $\|q\|$. Elle définit aussi une paire de feuilletages mesurés transverses appelés feuilletage horizontal et feuilletage vertical et notés respectivement $F_{h,q}$ et $F_{v,q}$ (voir les relations (2.3) et (2.2) ci-dessous). Cette paire transverse induit un nouveau système de coordonnées pour X où q s'écrit au voisinage d'un point régulier (i.e. un point où la différentielle quadratique ne s'annule pas) comme

$$q = d\xi^2 = (d\xi_1 + i\, d\xi_2)^2 .$$

Un tel système de coordonnées pour q sera appelé *coordonnées normales* de q. Pour plus de détails, on renvoie le lecteur à la section 11 du chapitre 11 de [14]. Un bref aperçu est aussi donné à la sous-section 2.1.2 du chapitre 2 de cette thèse.

On note l'ensemble des feuilletages mesurés par $\mathcal{MF}$. Précisons qu'il existe une définition indépendante des différentielles quadratiques (voir par exemple le paragraphe 3 de [67]). Il existe une action naturelle de $\mathbb{R}_+^*$ sur $\mathcal{MF} \setminus \{0\}$ qui induit l'*espace projectif des feuilletages mesurés* noté $\mathcal{PMF}$. Pour $F \in \mathcal{MF}$, on notera sa classe projective par $[F]$. On peut aussi montrer que l'espace $\mathcal{PMF}$ admet l'ensemble des classes d'homotopie libre de courbes fermées simples, noté $\mathcal{S}$, comme sous ensemble dense.

On peut montrer qu'il existe un lien profond entre les feuilletages mesurés et les différentielles quadratiques. Ce lien est donné par le résultat suivant :

Théorème 1 (Hubbard et Masur)**.** *Soient* $x = [X, f] \in \mathcal{T}(X_0)$ *et* $F \in \mathcal{MF}$*. Alors, il existe une unique différentielle quadratique notée* $q_{x,F}$ *sur* X *telle que* $F_{h,q_{x,F}} = f(F)$*.*

Il est important de préciser ici qu'il y a un abus de notation. En effet, l'égalité dans ce théorème signifie qu'à un "mouvement de Whitehead" près, F provient d'une différentielle quadratique (voir l'exemple du §2 du chapitre 2 de [18]). De plus, la version donnée par Hubbard et Masur est plus forte. Elle dit que l'espace $\mathcal{Q}(X)$ est homéomorphe à $\mathcal{MF}$.

On a donc vu que pour une différentielle quadratique q sur X, on peut toujours associer des coordonnées $\xi = \xi_1 + i\xi_2$ dites normales. Pour $K \geq 1$, on définit au voisinage d'un point régulier de q, l'application

$$f_q^K : \xi \mapsto K^{-\frac{1}{2}}\xi_1 + iK^{\frac{1}{2}}\xi_2.$$

On peut montrer qu'il existe une expression analogue au voisinage des zéros de q. Ceci entraine que l'on peut définir une application quasiconforme sur X, toujours notée f_q^K, qui definit une nouvelle surface de Riemann $X_{q,K}$. De plus, cette application atteint le minimum dans (1) quand on considère le problème extrémal pour X et $X_{q,K}$. Ces applications sont appelées applications de Teichmüller et permettent en outre de donner une paramétrisation de l'espace de Teichmüller. Cette paramétrisation est donnée par l'homéomorphisme suivant :

$$\begin{aligned} \mathcal{R}_x : \mathcal{Q}(X) &\to \mathcal{T}(X_0) \\ q &\mapsto \begin{cases} \left[X_{q,\|q\|}, f_q^{\|q\|} \circ g\right] & \text{si } q \neq 0, \\ x & \text{sinon,} \end{cases} \end{aligned} \tag{3}$$

où $x = [X, g] \in \mathcal{T}(X_0)$. Ceci montre en particulier que l'espace $\mathcal{T}(X_0)$ est une variété topologique de dimension réelle $6g - 6$. On peut aussi vérifier qu'à re-normalisation près, l'application (3) définit des rayons géodésiques pour la métrique de Teichmüller, appelés *rayons de Teichmüller*. Par abus de langage on dira aussi *déformation de Teichmüller*.

On utilisera dans cette thèse une autre formulation. Pour $[F] \in \mathcal{PMF}$ et $t \geq 0$, on notrera $\mathcal{R}_{[F]}^t(x)$ au lieu de $\mathcal{R}_x(q)$ où $F_{h,q} = F$ et $t = \|q\|$. On dira alors que $\mathcal{R}_{[F]}^t(x)$ est le rayon de Teichmüller centré en x, *dirigé* par $[F]$ et de paramètre t.

1 .3 Paramétrisation à la Thurston

Donnons maintenant une paramétrisation qui utilise le point de vue hyperbolique. On fixe donc une structure hyperbolique $x \in \mathcal{T}(X_0)$. On considèrera ici l'espace des laminations géodésiques mesurées $\mathcal{ML}$. L'espace $\mathcal{ML}$ dépend de la structure hyperbolique que l'on s'est fixée au préalable, mais on peut montrer qu'il existe une correspondance bijective avec $\mathcal{MF}$. Une lamination géodésique mesurée λ, est la donnée d'un sous-ensemble fermé de X_0 qui est une réunion disjointe de géodésiques complètes et simples $|\lambda|$, appelé support de λ, et d'une mesure transverse, invariante le long de $|\lambda|$ et dont le support de cette mesure est exactement $|\lambda|$. Comme $\mathbb{R}_+^*$ agit canoniquement sur $\mathcal{ML}$, on définit $\mathcal{PML}$, l'espace projectif des laminations mesurées projectives comme $\mathcal{ML} \setminus \{0\} \big/ \mathbb{R}_+^*$.

À tout $\mu \in \mathcal{ML}$, on peut associer un *complété* noté $\overline{\mu}$. Ce procédé de complétion (qui n'est pas unique !) consiste à rajouter des géodésiques complètes et simples à $|\mu|$, de telle sorte que le complémentaire de ce nouvel ensemble soit une réunion de triangles idéaux. En feuilletant les triangles idéaux ainsi obtenus par des bouts d'horocycles, on peut construire un feuilletage mesuré $F_{\overline{\mu}}(x)$, totalement transverse à μ, que l'on appelle *feuilletage horocyclique* associé à $\overline{\mu}$. Pour plus de détails et de meilleures explications, on renvoie le lecteur à l'introduction de [63] ou encore à la section 3 de l'exposé [59]. Cette correspondance entre lamination complète et feuilletage horocyclique permet de donner une autre paramétrisation de l'espace de Teichmüller. En effet, Thurston a démontré au paragraphe 9 de [68] que pour une lamination géodésique mesurée μ on a un homéomorphisme

$$\begin{aligned} F_{\overline{\mu}} : \mathcal{T}(X_0) &\to \mathcal{MF}(\mu) \\ x &\mapsto F_{\overline{\mu}}(x)\,. \end{aligned} \tag{4}$$

L'ensemble $\mathcal{MF}(\mu)$ consiste en les feuilletages mesurés qui sont (totalement) transverses à μ. Cet homéomorphisme permet de définir ce que Thurston appelle les *coordonnées cataclysmiques* de $\mathcal{T}(X_0)$. Une conséquence est que cette application permet de définir des lignes géodésiques pour d_{Th} appelées lignes d'étirement. En effet, soient $x \in \mathcal{T}(X_0)$, $\mu \in \mathcal{ML}$ et $t \in \mathbb{R}$. On appelle *étirement* de x de direction $\overline{\mu}$ et de paramètre t le point de $\mathcal{T}(X_0)$,

$$\mathcal{S}_{\overline{\mu}}^t(x) = F_{\overline{\mu}}^{-1}\left(e^t \cdot F_{\overline{\mu}}(x)\right). \tag{5}$$

On peut montrer que $\left(\mathcal{S}_{\overline{\mu}}^t(x)\right)_t$, appelée *ligne d'étirement* dirigée par $\overline{\mu}$ est une ligne géodésique paramétrée par la longueur d'arc au sens suivant :

$$\forall s < t,\ d_{Th}\left(\mathcal{S}_{\overline{\mu}}^s(x), \mathcal{S}_{\overline{\mu}}^t(x)\right) = t - s. \tag{6}$$

On peut dire ici que la notion d'étirement joue le rôle de déformation de Teichmüller dans le contexte hyperbolique. Cette analogie est motivée par le fait que ces lignes sont géodésiques pour d_{Th} qui réalisent l'infimum dans (2). On rappelle que les lignes de Teichmüller sont géodésiques pour d_T et qu'elles réalisent l'infimum dans (1).

2 Outils géométriques

Suivant le point de vue adopté sur l'espace de Teichmüller, on sera amené à manipuler des objets géométriques, tels que la longueur hyperbolique ou la longueur extrémale.

2 .1 Longueur hyperbolique

Utilisons ici encore, le point de vue hyperbolique de l'espace de Teichmüller. Toujours par le théorème d'uniformisation, un point de $\mathcal{T}(X_0)$ peut être vu comme une métrique hyperbolique sur la surface topologique sous-jacente. On rappelle que métrique hyperbolique signifie métrique riemannienne de courbure constante égale à -1. Ainsi, si on fixe $x \in \mathcal{T}(X_0)$ et $\alpha \in \mathcal{S}$, la longueur hyperbolique de α sur x a bien un sens et sera notée $l_x(\alpha)$. Soyons un peu plus précis ici car on utilise la définition par les marquages de l'espace de Teichmüller. Si x est la classe d'équivalence de la paire (X, f), alors

$$l_x(\alpha) = l_X(f(\alpha)),$$

où X est vue comme une surface hyperbolique.

Thurston a montré dans [68] (Theorem 8.5) que la métrique de Thurston pouvait s'exprimer en fonction d'un quotient de longueurs hyperboliques. Cette relation s'exprime comme suit :

$$\forall x, y \in \mathcal{T}(X_0), \ d_{Th}(x, y) = \log \sup_{\alpha \in \mathcal{S}} \frac{l_y(\alpha)}{l_x(\alpha)}. \tag{7}$$

Une des applications de cette formule est ladite *formule de Wolpert* qui dit que

$$\forall x, y \in \mathcal{T}(X_0), \ d_{Th}(x, y) \leq d_T(x, y). \tag{8}$$

On attribue cette formule à Wolpert (voir [71], Lemma 3.1), mais il est intéressant de préciser que Teichmüller, dans le paragraphe 35 de [61], avait déjà observé cette inégalité.

2 .2 Longueur extrémale

En utilisant maintenant le point de vue conforme de $\mathcal{T}(X_0)$, on va définir un invariant conforme, la longueur extrémale.

Définition 2. Soient $x = [X, f] \in \mathcal{T}(X_0)$ et $\alpha \in \mathcal{S}$. La *longueur extrémale* de α sur x notée $\mathrm{Ext}_x(\alpha)$ est définie par la relation suivante :

$$\mathrm{Ext}_x(\alpha) = \inf_A \frac{1}{\mathrm{Mod}(A)},$$

où A est un cylindre euclidien conformément plongé dans X et dont l'image du coeur cylindrique par cette application holomorphe est isotope à $f(\alpha)$. On note $\mathrm{Mod}(A)$ le rapport entre la hauteur et la circonférence du cylindre A et on l'appelle le *module* de A (voir aussi la figure 1 ci-dessous).

Cette définition est appelée "définition géométrique" de la longueur extrémale. Il existe une autre définition, appelée "définition analytique" et dont l'énoncé se trouve

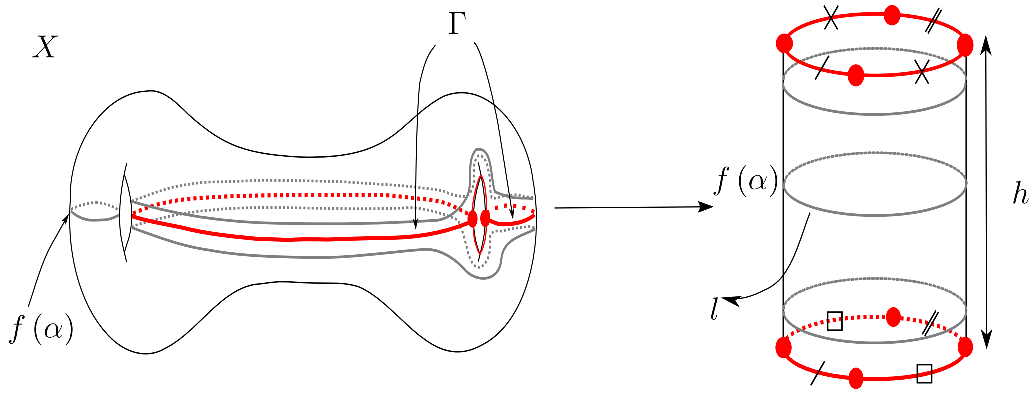


FIGURE 1 – Une manière grossière de voir le cylindre qui réalise l'infimum dans la définition 2.12 de la longueur extrémale de $\alpha \in \mathcal{S}$ sur $x = [X, f] \in \mathcal{T}(X_0)$. Le graphe rouge Γ représente le graphe critique du feuilletage associé à $f(\alpha)$ et donc $X \setminus \Gamma$ est biholomorphe à un cylindre euclidien de module $\frac{h}{l}$.

au début de la sous-section 2.1.4 du chapitre 2. Il est intéressant de préciser qu'historiquement c'est la définition analytique qui est apparue en premier sous l'impulsion de Ahlfors et Beurling[6]. C'est Jenkins qui dans [22] a montré l'équivalence entre les deux définitions.

Le principal intérêt de la longueur extrémale est son application aux espaces de Teichmüller. En effet, Kerckhoff a montré dans [25] que si pour $x \in \mathcal{T}(X_0)$, on pose pour tout $t \in \mathbb{R}_+$ et tout $\alpha \in \mathcal{S}$

$$\mathrm{Ext}_x(t \cdot \alpha) = t^2 \cdot \mathrm{Ext}_x(\alpha), \tag{9}$$

alors $\mathrm{Ext}_x(\cdot)$ s'étend continûment à l'espace $\mathcal{MF}$ des feuilletages mesurés. Il a de plus montré que si $F \in \mathcal{MF}$, alors

$$\mathrm{Ext}_x(X) = \|q_{x,F}\|. \tag{10}$$

Tout ceci a permis à Kerckhoff de déduire ce que l'on appelle aujourd'hui la *formule de Kerckhoff* et qui est le point de départ de la *Géométrie de la Longueur Extrémale.* Cette formule est la suivante :

$$\forall x, y \in \mathcal{T}(X_0), \; d_T(x,y) = \log \sup_{\alpha \in \mathcal{S}} \frac{\mathrm{Ext}_y(\alpha)}{\mathrm{Ext}_x(\alpha)}. \tag{11}$$

Le lecteur prendra soin de remarquer l'analogie avec la formule (7).

[6]Il est intéressant de connaître l'évolution historique de cette notion tant elle est intrinsèquement liée à la théorie classique de Teichmüller et à des problèmes de représentation conforme du plan. On ne fera que citer la page 394 du volume premier des oeuvres complètes (collected papers) d'Ahlfors ainsi que les articles associés.

Même si, contrairement à la longueur hyperbolique, la longueur extrémale n'est pas homogène, il existe une inégalité les comparant, appelée *inégalité de Maskit* (voir le corollaire 3 dans [39]) qui est

$$\mathrm{Ext}_x(\alpha) \leq \frac{l_x(\alpha)}{2} e^{\frac{l_x(\alpha)}{2}}, \tag{12}$$

où $x \in \mathcal{T}(X_0)$ et $\alpha \in \mathcal{S}$. La preuve de cette inégalité utilise le *lemme du collier* (voir [24] pour l'énoncé originel) et donc des calculs de géométrie hyperbolique élémentaires. L'inconvénient d'une telle formule est qu'elle ne s'étend pas à $\mathcal{MF}$. Cependant, on verra qu'elle permet de montrer que les lignes d'étirement ou les lignes de tremblement de terre convergent dans certains cas dans la compactification de Gardiner-Masur (voir la sous-section 4 .2 ci-dessous).

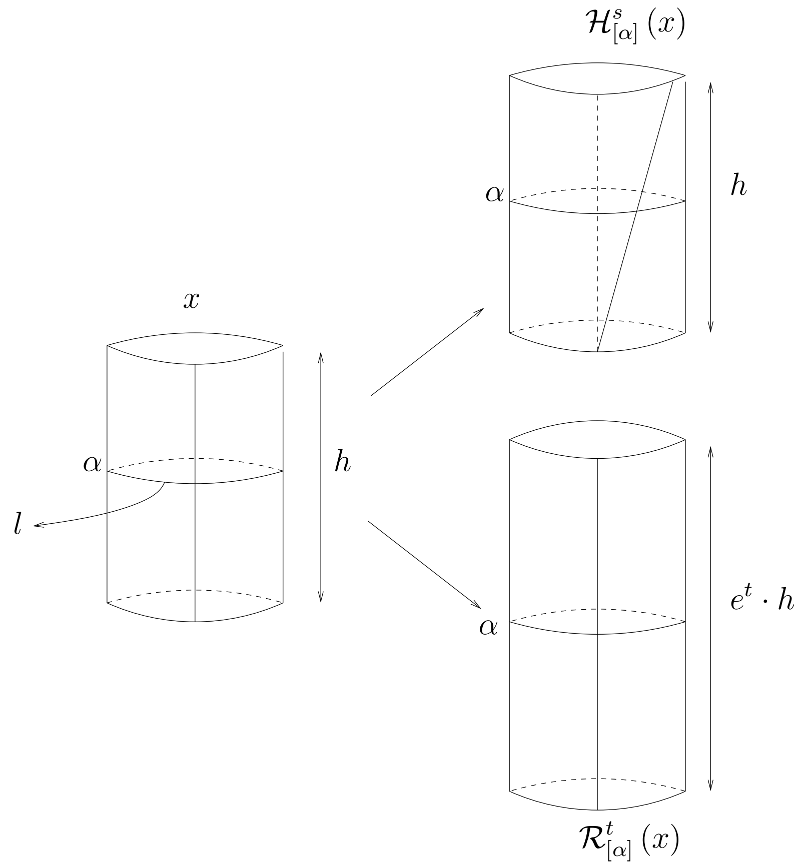


FIGURE 2 – Description géométrique de la déformation de Teichmüller et de la déformation horocyclique. Dans les deux cas la direction est donnée par une courbe fermée simple $\alpha \in \mathcal{S}$. On considère le cylindre euclidien maximale associé à α qui détermine $\mathrm{Ext}_x(\alpha)$. La façon dont on identifie des parties du bord du cylindre dépend de la topologie de la surface.

3 Compactifications de l'espace de Teichmüller

Comme on l'a déjà énoncé, l'espace de Teichmüller est une boule ouverte de dimension réelle $6g-6$. Cet espace peut donc être muni de différentes métriques, mais aussi de différentes compactifications. Toutes ces compactifications étant naturelles suivant le point de vue adopté.

3 .1 Compactification de Thurston

On considère ici le point de vue hyperbolique.

Thurston a introduit un plongement défini par

$$\Phi_{Th} : x \in \mathcal{T}(X_0) \mapsto [l_x(\cdot)] \in \mathrm{P}\mathbb{R}^{\mathcal{S}}_{\geq 0}, \tag{13}$$

et a montré que son image était relativement compacte. On rappelle que $\mathbb{R}^{\mathcal{S}}_{\geq 0}$ représente l'ensemble des fonctionnelles définies sur $\mathcal{S}$ à valeurs dans $\mathbb{R}_+$, et $\mathrm{P}\mathbb{R}^{\mathcal{S}}_{\geq 0}$ le quotient de $\mathbb{R}^{\mathcal{S}}_{\geq 0} \setminus \{0\}$ par l'action naturelle de $\mathbb{R}_+$. La compactification de Thurston, notée ici $\overline{\mathcal{T}(X_0)}^{Th}$, est donc $\overline{\Phi_{Th}(\mathcal{T}(X_0))}$. De plus, Thurston a montré que la frontière de ce compact s'identifiait avec l'espace $\mathcal{PMF}$ des classes projectives des feuilletages mesurés. On peut aussi réécrire cette compactification de la manière suivante :

$$\overline{\mathcal{T}(X_0)}^{Th} = \mathcal{T}(X_0) \cup \mathcal{PMF}. \tag{14}$$

Pour simplifier les notations dans cette introduction, on notera $y_n \xrightarrow[n\to+\infty]{Th} \mathfrak{y}$, pour une suite $(y_n)_n$ qui converge vers $\mathfrak{y} \in \overline{\mathcal{T}(X_0)}^{Th}$.

De plus, Thurston a défini l'intersection géométrique entre deux feuilletages mesurés F et G notée $i(F,G)$. Cette intersection est une généralisation du nombre d'intersection(s) minimale(s) entre deux courbes fermées simples sur la surface. On obtient donc une application continue

$$i(\cdot,\cdot) : \mathcal{MF} \times \mathcal{MF} \to \mathbb{R}_+. \tag{15}$$

Pour plus de détails sur cette compactification, on renvoie le lecteur à [15]. Cependant il est pertinent de rappeler qu'un feuilletage mesuré F définit une application continue

$$\begin{aligned} \mathcal{MF} &\to \mathbb{R}_+ \\ G &\mapsto i(F,G). \end{aligned}$$

De plus, on peut associer à tout feuilletage F, son ensemble de zéros, que l'on appellera "null-set", comme

$$\mathcal{N}(F) = \{G \in \mathcal{MF} \mid i(F,G) = 0\}. \tag{16}$$

On dira que deux feuilletages F et G sont null-set équivalents, et on notera $F \sim_{\mathcal{N}} G$, si $\mathcal{N}(F) = \mathcal{N}(G)$. Ceci définit une relation d'équivalence sur $\mathcal{MF}$ (ou $\mathcal{PMF}$), et permet d'introduire l'espace des *feuilletages réduits*[7] noté $\mathcal{NMF}$, qui muni de la topologie quotient, n'est pas séparé (voir la proposition 1.10 ci-dessous). Une étude approfondie de cet espace sera faite au chapitre 1 où l'on montrera principalement un résultat de rigidité sur l'action du groupe modulaire (voir le théorème 1.12 ci-dessous). On peut aussi remarquer que cela permet de définir $\overline{\mathcal{T}(X_0)}^{Th,red}$, appelée *compactification réduite de Thurston* de $\mathcal{T}(X_0)$.

3 .2 Compactification de Gardiner-Masur

On reprend désormais le point de vue conforme de $\mathcal{T}(X_0)$. En suivant l'idée de Thurston et du plongement donné par la relation (13), Gardiner et Masur ont défini dans [16] un nouveau plongement de $\mathcal{T}(X_0)$ en utilisant la longueur extrémale. Ils ont introduit l'application

$$\Phi_{GM} : x \in \mathcal{T}(X_0) \mapsto \left[\mathrm{Ext}_x^{\frac{1}{2}}(\cdot)\right] \in \mathrm{P}\mathbb{R}_{\geq 0}^{\mathcal{S}}, \tag{17}$$

et ont montré qu'elle était injective et que son image était relativement compacte. On obtient donc ce qu'on appelle désormais la compactification de Gardiner-Masur que l'on note $\overline{\mathcal{T}(X_0)}^{GM}$.

On nonnera dans la partie III de cette thèse une construction analogue dans le cas des surfaces à bord. Les idées étant essentiellement les mêmes que pour le cas sans bords, les personnes désirant connaitre les détails techniques peuvent consulter le papier originel de Gardiner et Masur ou la partie III.

La racine carré qui apparait dans (17) a été introduite pour comparer le bord $\partial_{GM}\mathcal{T}(X_0)$ avec $\mathcal{PMF}$. Gardiner et Masur ont montré que si la complexité de la surface était assez grande alors $\mathcal{PMF} \subsetneq \partial_{GM}\mathcal{T}(X_0)$. Miyachi, à travers les papiers [46, 47, 51] et [52], a considérablement amélioré la connaissance de cette compactification. Permettons-nous de citer quelques exemples.

Fixons $x \in \mathcal{T}(X_0)$. Pour $y \in \mathcal{T}(X_0)$, on définit la fonction continue suivante :

$$\mathcal{E}_y^x : F \in \mathcal{MF} \mapsto \left(\frac{\mathrm{Ext}_y(F)}{e^{d_T(x,y)}}\right)^{\frac{1}{2}}. \tag{18}$$

Miyachi a commencé par montrer que $\left\{\mathcal{E}_y^x\right\}_{y\in\mathcal{T}(X_0)}$ formait une famille normale afin d'obtenir un résultat d'existence qui s'écrit comme suit :

Théorème 3 (Théorème 1.1 de [46])**.** *Soit* $p \in \partial_{GM}\mathcal{T}(X_0)$*. Alors il existe une unique fonction continue* $\mathcal{E}_p^x : \mathcal{MF} \to \mathbb{R}_+$ *telle que*

1. $\mathcal{E}_p^x$ *est un représentant de* p *dans* $\mathbb{R}_{\geq 0}^{\mathcal{S}}$,

[7] Le terme anglais est "null-set foliation space".

2. $\max\left\{\mathcal{E}_p^x(F) \mid \operatorname{Ext}_x(F) = 1\right\} = 1$,
3. *si* y_n *converge vers* p*, alors* $\mathcal{E}_{y_n}^x$ *converge uniformément sur tout compact de* $\mathcal{MF}$ *vers* $\mathcal{E}_p^x$*.*

Pour simplifier les notations dans cette introduction, on notera $y_n \underset{n\to+\infty}{\overset{GM}{\longrightarrow}} p$, pour une suite $(y_n)_n$ qui converge vers $p \in \overline{\mathcal{T}(X_0)}^{GM}$.

Une des premières applications du théorème 3 est le résultat suivant.

Lemme 4. *Soient* $(y_n) \subset \mathcal{T}(X_0)$ *et* $(z_n) \subset \mathcal{T}(X_0)$ *deux suites telles que*

- $y_n \underset{n\to+\infty}{\overset{GM}{\longrightarrow}} p \in \overline{\mathcal{T}(X_0)}^{GM}$,
- $z_n \underset{n\to+\infty}{\overset{GM}{\longrightarrow}} q \in \overline{\mathcal{T}(X_0)}^{GM}$,
- $d_T(y_n, z_n) \underset{n\to+\infty}{\longrightarrow} 0$.

Alors $p = q$*.*

Preuve. Si p ou q appartient à $\mathcal{T}(X_0)$, alors la preuve est immédiate.

Supposons donc que $p, q \in \partial_{GM}\mathcal{T}(X_0)$. D'après le théorème précédent (i.e. le théorème 3), il suffit de montrer que $\mathcal{E}_p^{x_0} = \mathcal{E}_q^{x_0}$.

D'après la formule (11), pour tout $n \in \mathbb{N}$ et tout $\alpha \in \mathcal{S}$, on a

$$0 \leq \mathcal{E}_{x_n}^2(\alpha) \leq \mathcal{E}_{y_n}^2(\alpha)\, e^{d_T(x_n,y_n)} e^{d_T(x_0,y_n)-d_T(x_0,x_n)}.$$

Et donc quand n tend vers $+\infty$, on obtient

$$\mathcal{E}_p(\alpha) \leq \mathcal{E}_q(\alpha).$$

On utilise la symétrie de la métrique de Teichmüller et donc pour tout $\alpha \in \mathcal{S}$, on obtient

$$\mathcal{E}_p(\alpha) = \mathcal{E}_q(\alpha).$$

Le lemme est maintenant démontré. □

Rajoutons quelques commentaires sur le lemme 4. On ne peut pas seulement supposer que la distance entre ces deux suites soit bornée. En effet, Masur a démontré dans [40] que deux rayons de Teichmüller partant du même point sont à distance bornée s'ils sont dirigés par des feuilletages rationnels de même support. Or on sait depuis Miyachi (voir le théorème 1 de [47]) que deux rayons de direction différente ont des points limites distincts dans $\partial_{GM}\mathcal{T}(X_0)$. Rappelons qu'un feuilletage est dit rationnel s'il décompose la surface en union finie de cylindre. De plus, sous cette forme la réciproque n'est pas vraie. En effet, il suffit de considérer un rayon de Teichmüller paramétré de deux manières différentes. Par exemple, pour $[F] \in \mathcal{PMF}$ il suffit de considérer $y_n = \mathcal{R}_{nq}(x)$ et $x_n = \mathcal{R}_{n^2q}(x)$. On peut tout de même se demander si modulo une hypothèse supplémentaire la réciproque est vraie. Il est donc naturel de se poser la question suivante.

Question 1. Peut-on trouver deux suites (y_n) et (z_n) dans $\mathcal{T}(X_0)$ telles que

- $y_n \underset{n\to+\infty}{\overset{GM}{\longrightarrow}} p$ et $z_n \underset{n\to+\infty}{\overset{GM}{\longrightarrow}} p$, où $p \in \partial_{GM}\mathcal{T}(X_0)$,
- $d_T(y_n, x_0)/d_T(z_n, x_0) \underset{n\to+\infty}{\longrightarrow} 1$,
- $d_T(y_n, z_n) \underset{n\to+\infty}{\not\longrightarrow} 0$?

On donnera une réponse affirmative dans le chapitre 4, en considérant les déformations horocycliques (voir en particulier le théorème 4.3).

Dans [51], Miyachi a développé un analogue des courants géodésiques (notion introduite par Bonahon dans [7]) afin d'étendre la notion d'intersection géométrique au bord de Gardiner-Masur. Ce résultat, très utile dans cette thèse, est le suivant.

Théorème 5 (Proposition 7 de [51])**.** *Il existe une unique fonction continue et symétrique*

$$i_x(\cdot,\cdot) : \overline{\mathcal{T}(X_0)}^{GM} \times \overline{\mathcal{T}(X_0)}^{GM} \to \mathbb{R}_+$$

telle que pour tout $p \in \overline{\mathcal{T}(X_0)}^{GM}$ *et tout* $[G] \in \mathcal{PMF}$ *on a*

$$i_x(p,[G]) = \mathcal{E}_p^x(G).$$

On renvoie aussi à la sous-section 2.2.4 pour une explication un peu plus détaillée.

Ceci permet de définir pour un point $p \in \overline{\mathcal{T}(X_0)}^{GM}$, son *null-set* comme

$$\mathcal{N}_{GM}(p) = \left\{ q \in \overline{\mathcal{T}(X_0)}^{GM} \mid i_x(p,q) = 0 \right\}.$$

On peut montrer que cela ne dépend pas du point base x choisi.

Rajoutons enfin un dernier résultat de Miyachi qui fut à l'origine de [5] (article écrit en collaboration avec Miyachi et Ohshika) et donc de la partie I de cette thèse.

Théorème 6 (Théorème 7.1 de [52])**.** *Soit* $p \in \mathcal{T}(X_0)$*. Alors il existe* $G \in \mathcal{MF}$ *tel que*

$$\mathcal{N}_{GM}(p) = \mathcal{N}_{GM}(G).$$

Comme pour l'espace des feuilletages mesurés, on peut définir une relation d'équivalence sur $\overline{\mathcal{T}(X_0)}^{GM}$ et ainsi obtenir la *compactification réduite de Gardiner-Masur*, notée $\overline{X_0}^{GM,red}$. De plus, le théorème 6 implique que $\overline{X_0}^{Th,red}$ et $\overline{X_0}^{GM,red}$ sont ensemblistement identiques, mais qu'en est-il des topologies quotients induites ?

4 Convergence aux bords

On a déjà défini “grossièrement” deux types de déformations, à savoir, la déformation de structures conformes déterminée par des rayons de Teichmüller et la déformation de structures hyperboliques déterminée par des étirements. Il existe cependant un autre type de déformation de structures hyperboliques introduite par Thurston. C’est ce qu’on appelle les *tremblements de terre*. Pour un élément x de $\mathcal{T}(X_0)$, le tremblement de terre de x de paramètre $t \in \mathbb{R}$ et de direction $[\mu] \in \mathcal{PML}$ est noté $E^t_{[\mu]}(x)$. Par convention, un tremblement de terre de paramètre positif (resp. négatif) sera un tremblement de terre vers la gauche (resp. vers la droite). De plus, on on considère $\mathcal{PML}$ pour la direction seulement pour faire l’analogie avec les déformations horocycliques. Ceci ne change rien car on peut identifier $\mathcal{PML}$ avec l’ensemble des laminations mesurées de longueur 1 par rapport à une structure hyperbolique fixée. C’est juste une normalisation. Il existe de nombreuses propriétés sur ce type de déformation que l’on ne donnera pas. On rappellera seulement les propriétés qui ont été à la base de la partie II de cette thèse. Le lecteur intéressé pourra consulter [72, 8, 55, 56, 27] ; une liste bien évidemment non exhaustive. La première propriété est un résultat de Thurston, démontré par Kerckhoff dans [26] afin de résoudre le “problème de réalisation de Nielsen”.

Théorème 7 (Théorème 2 de [26])**.** *Soient x et y deux éléments distincts de $\mathcal{T}(X_0)$. Alors il existe un unique couple $([\mu], t) \in \mathcal{PML} \times \mathbb{R}_+$ tel que $y = E^t_{[\mu]}(x)$.*

Une autre propriété est un résultat de Théret se trouvant dans sa thèse (voir [63]). Il stipule que les actions d’étirement et de tremblement de terre commutent si les directions sont bien choisies. De manière plus précise, l’énoncé est le suivant :

Théorème 8 (Théorème 10 du chapitre 3 de [63])**.** *Soient $x \in \mathcal{T}(X_0)$ et $\mu \in \mathcal{ML}$. Alors*

$$\forall s, t \in \mathbb{R},\ \mathcal{S}^t_{\overline{\mu}}\left(E^s_{[\mu]}(x)\right) = E^s_{[\mu]}\left(\mathcal{S}^t_{\overline{\mu}}(x)\right),$$

où $\overline{\mu}$ est une completion de μ.

Enfin, la dernière propriété qui est bien connue des géomètres, nous dit que la longueur hyperbolique d’une lamination mesurée[8] reste invariante le long d’un tremblement de terre dirigé par cette même lamination. Cela se réécrit de la manière suivante.

Propriété 9. *Soient $x \in \mathcal{T}(X_0)$ et $\mu \in \mathcal{ML}$. Alors pour tout $t \in \mathbb{R}$,*

$$l_{E^t_{[\mu]}(x)}(\mu) = l_x(\mu).$$

[8] On renvoie le lecteur à [58] pour la définition de longueur hyperbolique d’une lamination mesurée.

En ayant en tête ces résultats, l'auteur de cette thèse a essayé de trouver un analogue conforme au tremblement de terre. Ce type de déformation existait déjà et avait été (en partie) déjà étudié par Marden et Masur dans [37]. S'inspirant de cet article, nous avons donc défini les déformations *horocycliques*. Une telle déformation est paramétrée par la classe projective d'un feuilletage mesuré $[F]$ et par un réel t. On la notera $\mathcal{H}^t_{[F]}(x)$ et on dira que c'est la déformation horocyclique de x de direction $[F]$ et de paramètre t. Une description explicite existe si la direction, c'est-à-dire la classe projective du feuilletage, est une courbe fermée simple. Celle-ci a été donnée par Marden et Masur. On pourra aussi consulter la figure 2. On renvoie à la définition 3.4 ci-dessous pour une définition plus explicite. On verra aux chapitres 3 et 4, que ces déformations conformes vérifient des propriétés analogues aux théorèmes 7 et 8 et à la propriété 9. On devra alors considérer la longueur extrémale au lieu de la longueur hyperbolique (voir respectivement les propriétés 3.5, 3.6 et 3.9).

4 .1 Convergence dans le bord de Thurston

D'après la définition de Φ_{Th}, une manière de déterminer les limites éventuelles d'une suite de $\mathcal{T}(X_0)$ est d'étudier le comportement asymptotique des longueurs hyperboliques. Récemment, Walsh dans voir [70], a montré un théorème que l'on peut considérer comme un analogue hyperbolique du théorème 3 ci-dessus.

Fixons $x \in \mathcal{T}(X_0)$. On définit pour tout $y \in \mathcal{T}(X_0)$, l'application

$$\begin{aligned} \mathcal{L}^x_y : \mathcal{MF} &\to \mathbb{R}_+ \\ F &\mapsto \frac{l_x(F)}{e^{d_{Th}(x,y)}}. \end{aligned} \tag{19}$$

Walsh a démontré que $(x_n)_n \subset \mathcal{T}(X_0)$ converge vers $[G]$ dans $\overline{\mathcal{T}(X_0)}^{Th}$ si, et seulement si, $\mathcal{L}^x_{x_n}$ converge uniformément sur tout compact de $\mathcal{MF}$ vers $F \in \mathcal{MF} \mapsto C \cdot i(G,F)$; la constante C, ne dépendant que de x et G.

Dans cette thèse, une des applications de cette caractérisation sera pour montrer le corollaire 4.4 ci-dessous.

Cependant, de façon immédiate ce résultat implique que le tremblement de terre dirigé par une courbe fermée simple converge dans le bord de Thurston vers la classe projective de cette même courbe. Il suffit de constater que si $\alpha \in \mathcal{S}$ représente la direction de notre tremblement de terre, alors la longueur hyperbolique de $F \in \mathcal{N}(\alpha)$ reste invariante le long de cette déformation. Ainsi, en utilisant l'application (19), on en déduit que si $[G]$ est un point d'accumulation de $\left(\mathcal{E}^t_\alpha(x)\right)_t$, alors

$$i(G,F) = 0,$$

et donc G est topologiquement le même feuilletage que α, d'où la convergence annoncée. Le même raisonnement marche si la direction est uniquement ergodique.

On peut raisonner de même avec les lignes d'étirements. En effet, ceci résulte des résultats de Théret disant la chose suivante. Soient $x \in \mathcal{T}(X_0)$ et $\mu \in \mathcal{ML}$. Alors pour $\alpha \in \mathcal{ML}$,

1. $\left(l_{\mathcal{S}^t_{\overline{\mu}}(x)}\left(\alpha\right)\right)_{t\geq 0}$ est bornée si $i\left(\alpha, F_{\overline{\mu}}\left(x\right)\right) = 0$,
2. $\left(l_{\mathcal{S}^t_{\overline{\mu}}(x)}\left(\alpha\right)\right)_{t\leq 0}$ est bornée si $i\left(\alpha, \mu\right) = 0$.

On déduit de cela des résultats déjà observés par Papadopoulos dans [56] (voir la proposition 5.2 pour le cas uniquement ergodique) et Théret dans [64] (voir le théorème 3.2 pour le cas des courbes fermée simples), à savoir que si μ est une géodésique fermée simple ou une lamination uniquement ergodique, alors

$$\mathcal{S}^t_{\overline{\mu}}\left(x\right) \xrightarrow[t\to-\infty]{Th} \left[\mu\right].$$

4 .2 Convergence dans le bord de Gardiner-Masur

S'intéresser au null-set d'un point du bord de Gardiner-Masur n'est pas seulement intéressant pour étudier la compactification réduite ; cela permet aussi de déterminer les points du bord qui correspondent à des feuilletages mesurés. En effet, Miyachi a démontré dans [46] un résultat qui permet de déterminer les points du bord connaissant le null-set associé.

Théorème 10 (Théorème 3 de [46])**.** *Soient $x \in \mathcal{T}(X_0)$ et $F \in \mathcal{MF}$ un feuilletage uniquement ergodique ou une courbe fermée simple (éventuellement pondérée). Soit $p \in \partial_{GM}\mathcal{T}\left(X_0\right)$ tel que*

$$\forall G \in \mathcal{N}\left(F\right),\ \mathcal{E}^x_p\left(G\right) = 0.$$

Alors

$$p = \left[F\right].$$

Donnons cependant un exemple d'application de ce théorème sur des exemples provenant du point de vue hyperbolique de l'espace de Teichmüller, à savoir les tremblements de terre et les lignes d'étirement. Le résultat est le suivant.

Propriété 11. *Soient $x \in \mathcal{T}(X_0)$ et $\alpha \in \mathcal{S}$. Alors*

$$E^t_{[\alpha]} \xrightarrow[t\to\pm\infty]{GM} \left[\alpha\right] \ \text{ et } \mathcal{S}^t_{\overline{\alpha}}\left(x\right) \xrightarrow[t\to-\infty]{GM} \left[\alpha\right].$$

De plus, si pour $\mu \in \mathcal{ML}$, $F_{\overline{\mu}}\left(x\right)$ est un élément de $\mathcal{S}$, alors

$$\mathcal{S}^t_{\overline{\mu}}\left(x\right) \xrightarrow[t\to+\infty]{GM} \left[F_{\overline{\mu}}\left(x\right)\right].$$

Preuve. Nous ne ferons la démonstration que pour la ligne d'étirement $\left(\mathcal{S}^t_{\overline{\alpha}}(x)\right)_{t\leq 0}$. Le même raisonnement s'appliquant au tremblement de terre dirigé par α. Dans le cas où $F_{\overline{\mu}}(x)$ est un élément de $\mathcal{S}$, on utilise aussi le même raisonnement avec la remarque donnée par l'item 1.

Soit $p \in \partial_{GM}\mathcal{T}(X_0)$ un point d'accumulation de $\left(\mathcal{S}^t_{\overline{\alpha}}(x)\right)_{t\leq 0}$. Soit $\beta \in \mathcal{N}(\alpha)\cap\mathcal{S}$. Alors par l'observation de Théret (voir item 2 ci-dessus) et l'inégalité de Maskit (inégalité (12)) on en déduit par passage à la limite que

$$\mathcal{E}^x_p(\beta) = 0.$$

Or par le théorème 6, il existe $G \in \mathcal{MF}$ tel que $\mathcal{N}_{GM}(p) = \mathcal{N}_{GM}(G)$ et donc, comme la restriction de $i_x(\cdot,\cdot)$ coincide avec $i(\cdot,\cdot)$ (à multiplication par un réel non nul près), on a

$$\mathcal{N}(G)\cap\mathcal{S} = \mathcal{N}(\alpha)\cap\mathcal{S}. \tag{20}$$

A priori, même si $\mathcal{S}$ est dense dans $\mathcal{PMF}$, il n'y a aucune raison que $\mathcal{N}(G) = \mathcal{N}(\alpha)$. Cependant, on peut montrer que c'est le cas car $\alpha \in \mathcal{S}$. On en déduit finalement

$$\mathcal{N}_{GM}(p)\cap\mathcal{MF} = \mathcal{N}_{GM}(G)\cap\mathcal{MF} = \mathcal{N}(G) = \mathcal{N}(\alpha),$$

et donc par le théorème 10 on déduit le résultat. □

Rajoutons que la convergence des tremblements de terre dans le bord de Gardiner-Masur a été démontrée d'une autre manière par Jiang et Su dans [23]. Au passage, ils démontrent aussi que les déformations horocycliques convergent si la direction est uniquement ergodique, un résultat qui est démontré d'une autre manière dans la section 4.2 du chapitre 4. Jiang et Su utilisent pour cela le plongement des métriques plates dans l'espace des courants géodésiques introduit par Duchin, Leininger et Rafi dans [13].

Une autre application du théorème 10 sera faite pour étudier la convergence des déformations horocycliques. De façon plus précise, on montrera que les déformations horocycliques convergent dans le bord de Gardiner-Masur vers la direction associée, si cette direction est ou bien un feuilletage uniquement ergodique ou bien une courbe fermée simple. On renvoie le lecteur aux théorèmes 4.3 et 4.6 du chapitre 4. On montrera aussi que ceci entraîne la convergence vers la même limite dans le bord de Thurston.

Ces résultats sur la convergence des déformations horocycliques est, à mon sens, à mettre en corrélation avec la convergence (dans le bord de Gardiner-Masur) des rayons de Teichmüller vers la classe projective d'un feuilletage mesuré. En effet, un rayon de Teichmüller dirigé par une courbe fermée simple ou un feuilletage mesuré uniquement ergodique, converge dans la compactification de Gardiner-Masur (et aussi dans la compactiffcation de Thurston) vers la classe projective de sa direction (voir [25, 16, 47] et [40]).

Outline of the Thesis

Aco es un modesto trabail, cal estre indulgent.

- In Chapter 1, we define $\mathcal{NML}$, the null-set lamination space, and we prove that the extended mapping class group is rigid on this space.
- In Chapter 2, we introduce three reduced compactifications of Teichmüller space and we produce a bijective continuous mapping between two of them.
- In Chapter 3, we consider horocyclic deformations and we give some elementary properties.
- In Chapter 4, we are interested in convergence of the horocyclic deformations towards the Thurston compactification and the Gardiner-Masur compactification of Teichmüller space.
- In Chapter 5, we consider the reduced Teichmüller space of a surface with non-empty boundary and we give a compactification *à la* Gardiner-Masur of this space.

Part I

Reduced compactifications of Teichmüller space

Chapter 1

On the Reduced Thurston compactification

In this chapter we shall investigate what we call the reduced Thurston compactification. Let us recall that the Thurston compactification is the closure $\overline{\Phi_{Th}(\mathcal{T}(X_0))}$, where Φ_{Th} is defined by Relation (13). The Thurston boundary is the set $\mathcal{PMF}$ of projective measured foliations. Using the geometric intersection number, we shall define an equivalence relation on $\mathcal{PMF}$, and taking the quotient of that space by this relation we shall obtain the reduced Thurston boundary, denoted here by $\mathcal{NMF}$. The main result of this chapter is that the reduced Thurston boundary has the *rigidity* property, that is, the group of self-homeomorphism of this boundary is canonically isomorphic (except in some exceptional cases) to the extended mapping class group $\mathrm{MCG}^*(X_0)$.

The study on such a reduced boundary has been inspired by works from Papadopoulos and Ohshika on the set $\mathcal{UML}$ of unmeasured laminations (see respectively [57] and [53]). The inspiration also comes from Ohshika's results on the "reduced Bers boundary" in [54]. We can also cite work by Charitos, Papadoperakis and Papadopoulos in [12], where they prove that the set $\mathcal{GL}$ of the geodesic laminations endowed with the so-called *Thurston topology* has also the rigidity property.

As we already wrote in the introduction, $\mathcal{MF}$ (resp. $\mathcal{PMF}$) is canonically homeomorphic to $\mathcal{ML}$ (resp. $\mathcal{PML}$). Due to the fact that this chapter and the following one are based on the paper [5], a joint work with Professor Miyachi and Professor Ohshika, we shall consider the set of measured laminations $\mathcal{ML}$ instead of $\mathcal{MF}$ but, all will work if we consider $\mathcal{MF}$. Moreover, in this chapter, X_0 shall be considered as a closed hyperbolic surface of genus g.

1.1 Notation

In this section, we only give definitions we need in the study of the reduced Thurston boundary.

We recall that a measured lamination λ on X_0 is a pair of a closed subset consisting of disjoint simple complete geodesics denoted by $|\lambda|$ and a transverse measure supported on $|\lambda|$ which is invariant under translations along $|\lambda|$. The set of measured laminations is denoted by $\mathcal{ML}$[1] and the set of simple closed geodesics on X_0 is denoted by $\mathcal{S}$. For $\lambda \in \mathcal{ML}$, the set $|\lambda|$ is called the *support* of λ and can be viewed either as an element of $\mathcal{UML}$, the set of *unmeasured laminations*, or an element of $\mathcal{GL}$, the set of *geodesic laminations*. Let us recall that the space of geodesic laminations can be endowed with (at least) two different topologies, the *Hausdorff* one and the Thurston one. The Thurston topology is non-Hausdorff. For more details, we refer to [66], [11] and [12]. We recall that the unmeasured lamination space $\mathcal{UML}$ is the quotient space obtained from $\mathcal{ML}$ (or $\mathcal{PML}$, see the definition below) by forgetting transverse measures, and this space endowed with the quotient topology is non-Hausdorff. We denote by $\Pi_{\mathcal{U}}$ the canonical projection from $\mathcal{ML}$ to $\mathcal{UML}$. The rigidity problem on that space has already been studied by Papadopoulos in [57] and Ohshika in [53].

For two simple closed geodesics α and β on X_0, we define the *geometric intersection number* of these two geodesics as the number of intersection points between them, and we denote this number by $i(\alpha, \beta)$. Moreover, it is well known that the intersection number $i(\cdot, \cdot)$ has a natural continuous extension to $\mathcal{ML} \times \mathcal{ML}$ (see for example the introduction of [63] or Subsection 2.1.1). Furthermore, it is well-known that $\mathcal{ML}$ is homeomorphic to $\mathbb{R}^{6g-6}$.

Definition 1.1. A measured lamination is said to be *minimal*, if it does not contain a proper non-empty sublamination.

The simplest examples of such measured laminations are simple closed geodesics on X_0. However, there exist minimal measured laminations which are not simple closed geodesics on X_0, as well. For such a measured lamination λ, there is a unique minimal connected compact subsurface $\Sigma(\lambda)$ with totally geodesic boundary such that λ has a non-zero geometric intersection number with any simple closed geodesic living in the interior of $\Sigma(\lambda)$. The subsurface $\Sigma(\lambda)$ is called the *supporting surface* of λ.

Remark 1.2. When we mention the supporting surface of a given measured lamination, it only concerns minimal measured laminations which are not simple closed geodesics. Moreover, such a lamination fills the interior of its supporting surface.

The interesting fact that we shall use deeply in this chapter, is the decomposition of any measured lamination. Indeed, for any $\lambda \in \mathcal{ML}$, there exist $(n_1, n_2, n_3) \in \mathbb{N}^3$ and measured laminations $(\alpha_i)_{1 \leq i \leq n_1}$, $(\beta_i)_{1 \leq i \leq n_2}$ and $(\gamma_i)_{1 \leq i \leq n_3}$ such that

$$\lambda = \sum_{i=1}^{n_1} \alpha_i + \sum_{i=1}^{n_2} \beta_i + \sum_{i=1}^{n_3} \gamma_i, \tag{1.1}$$

where

[1] This notion (and also the following ones) depends on the hyperbolic structure X_0, but it seems reasonable for the author to remove this dependance in his notation.

- $\forall 1 \leq i \leq n_1$, α_i is a minimal measured lamination contained in λ and whose support is not an element of $\mathcal{S}$,
- $\forall 1 \leq j \leq n_2$, β_j is a weighted-simple closed geodesic contained in λ which is not a boundary component of the supporting surface of any α_i,
- $\forall 1 \leq k \leq n_3$, γ_k is a weighted-simple closed geodesic contained in λ which is a boundary component of the supporting surface of some α_i.

We shall say that γ_i is a *peripheral curve* of λ and that λ is *saturated* if its support contains all peripheral curves. We then observe that for any measured lamination, there is a saturated measured lamination which is obtained by adding all peripheral curves of λ. Such a measured lamination is unique up to transverse measures given on these added geodesics. For more details we refer for example to [31].

Note that Relation (1.1) means that for any $\mu \in \mathcal{S}$,

$$i(\lambda, \mu) = \sum_{i=1}^{n_1} i(\alpha_i, \mu) + \sum_{i=1}^{n_2} i(\beta_i, \mu) + \sum_{i=1}^{n_3} i(\gamma_i, \mu).$$

Let us recall that $\mathbb{R}_{>0}$ acts on $\mathcal{ML}$ by multiplying the transverse measure. The induced quotient space $\mathcal{ML} - \{0\} \big/ \mathbb{R}_{>0}$ is called the *projective measured lamination* space on X_0 and is denoted by $\mathcal{PML}$.

1.2 Null-sets

The aim of this section is to define the reduced Thurston boundary $\mathcal{NML}$. To do so, let us first define an equivalence relation on $\mathcal{ML}$ (or $\mathcal{PML}$). This relation is given by the *null-set* of a measured lamination which is as follows.

Definition 1.3. Let λ be a measured lamination on X_0. The null-set of λ is the set

$$\mathcal{N}(\lambda) = \{\mu \in \mathcal{ML} \mid i(\lambda, \mu) = 0\}.$$

By Definition 1.3, we obtain the following easy observation.

Lemma 1.4 ([5], Lemma 2.1)**.** *Let λ and μ be two measured laminations such that $i(\lambda, \mu) = 0$. Then the following assertions are equivalent:*

1. $\mathcal{N}(\lambda) = \mathcal{N}(\mu)$,
2. $\mathcal{N}(\lambda) \cap \mathcal{S} = \mathcal{N}(\mu) \cap \mathcal{S}$.

Even if it is well known, we give the proof below.

Proof. $\boxed{\Rightarrow}$ It is obvious.

$\boxed{\Leftarrow}$ Assume that $\mathcal{N}(\lambda)$ differs from $\mathcal{N}(\mu)$. By interchanging the role of λ and μ if necessary, we can assume that $\mathcal{N}(\mu) \subsetneq \mathcal{N}(\lambda)$ and then there exists a measured lamination ν such that

$$i(\lambda, \nu) > 0 \tag{1.2}$$

whereas

$$i(\mu, \nu) = 0.$$

We can assume ν to be minimal by picking up a component intersecting λ. From the decomposition of measured lamination and the fact that $i(\mu, \nu) = 0$, we deduce that either ν is a minimal component of μ or ν is disjoint from μ. If $|\nu| \subset |\mu|$, then by the assumption on λ and μ, we have $i(\lambda, \nu) = 0$, which is according to Equation (1.2) a contradiction. If ν is disjoint from μ, then by approximating ν by simple closed geodesics in the Hausdorff topology of $\mathcal{GL}$, we have the existence of $\alpha \in \mathcal{S}$ such that $i(\mu, \alpha) = 0$, whereas $i(\lambda, \alpha) > 0$ and hence a contradiction. □

In fact, this also proves the lemma

Lemma 1.5. *Let λ and μ be two measured laminations such that $i(\lambda, \mu) = 0$. Then the following assertions are equivalent:*

1. $\mathcal{N}(\mu) \subset \mathcal{N}(\lambda)$,
2. $\mathcal{N}(\mu) \cap \mathcal{S} \subset \mathcal{N}(\lambda) \cap \mathcal{S}$.

The next proposition follows from Lemma 1.5. It gives an interesting geometric characterisation of inclusion of a null-set in another one.

Proposition 1.6 ([5], Lemma 2.2)**.** *Let λ and μ be two measured laminations. Then the following assertions are equivalent:*

1. $\mathcal{N}(\mu) \subset \mathcal{N}(\lambda)$.
2. *Any minimal component of $|\lambda|$ which is not associated with a peripheral curve is also a component of $|\mu|$.*

Proof. $\boxed{\Rightarrow}$ Let λ_0 be a minimal component of λ.

Case 1: If λ_0 is not a (weighted) simple closed geodesic, then we have a well-defined supporting surface of λ_0 denoted by $\Sigma(\lambda_0)$. Let μ_0 be any component of μ, then $i(\mu_0, \lambda) = 0$ (because $i(\mu_0, \mu) = 0$) and then $i(\lambda_0, \mu_0) = 0$. We deduce that μ_0 is either disjoint from λ_0 or $|\mu_0| = |\lambda_0|$. Thus, the intersection between $|\mu|$ and the interior of $\Sigma(\lambda_0)$ is λ_0 or empty. In the latter case, any simple closed geodesic γ in $\Sigma(\lambda_0)$ that is not a boundary component, has a non-zero geometric intersection number with λ_0 and hence $i(\gamma, \mu) > 0$ which is a contradiction because the intersection is assumed to be empty.

Case 2: If λ_0 is a (weighted) simple closed geodesic which is not a peripheral curve for λ, then there exists a simple closed geodesic η which intersects λ_0 and which is disjoint from the other components of λ. Necessarily, $|\lambda_0|$ is included to $|\mu|$ because otherwise $i(\eta, \mu) = 0$ and $i(\eta, \lambda_0) > 0$.

$\boxed{\Leftarrow}$ Let α be a simple closed geodesic such that $i(\mu, \alpha) = 0$.

Case 1: If α is a boundary component of a supporting surface of some minimal component of μ, then α is either a peripheral curve of μ or is disjoint from μ. In all these cases, $i(\lambda, \alpha) = 0$ by assumptions.

Case 2: If α is not a boundary component of some supporting surface, then it is either a minimal component (among simple close geodesics) of μ and hence by assumption on λ, we deduce that $i(\alpha, \lambda) = 0$; or it is disjoint from μ and again by assumption, α is disjoint from λ and then $i(\alpha, \lambda) = 0$.

We now just have to apply Lemma 1.5 to complete the proof. □

The following characterization is a direct corollary.

Corollary 1.7 ([5], Corollary 2.3)**.** *Let λ and μ be two measured laminations. Then we have that $\mathcal{N}(\lambda) = \mathcal{N}(\mu)$ if, and only if, $|\mu|$ is obtained from $|\lambda|$ by adding or removing the support of peripheral curve of λ.*

Remark 1.8. Using the correspondence between measured laminations and measured foliations, this corollary is expressed as follows for measured foliations. We say that two measured or projective foliations are in the same *generalised Whitehead equivalence class* if the support of one of them is obtained from that of the other by repeating the following three operations:

1. Isotoping the foliation.
2. Shrinking an arc on a singular leaf connecting two singularities or inserting such an arc to split a singularity in two.
3. Removing an open annulus foliated by compact leaves homotopic to some peripheral simple closed curve or inserting such an annulus.

This notion defines a new equivalence relation on $\mathcal{MF}$ and by the correspondence with measured laminationspace this implies that two measured foliations are in the same Whitehead equivalence class if and only if the corresponding measured laminations have the same null-sets. Thus, given two measured foliations F and G, we have that $\mathcal{N}(F) = \mathcal{N}(G)$ if and only if F and G are in the same Whitehead equivalence class.

We have all ingredients to define the *reduced Thurston boundary* of Teichmüller space. Indeed, we shall say that two measured laminations are *null-set equivalent*, if they have the same null-sets. This defines an equivalence relation on $\mathcal{ML}$ denoted by $\sim_{\mathcal{N}}$. Moreover, since the condition "$i(\cdot, \cdot) = 0$" does not depend on choices of transverse measures, the relation $\sim_{\mathcal{N}}$ also defines an equivalence relation on $\mathcal{PML}$ or $\mathcal{UML}$, still denoted by $\sim_{\mathcal{N}}$.

Definition 1.9. The *null-set lamination* space is

$$\mathcal{NML} = \mathcal{PML}\big/_{\sim_{\mathcal{N}}}.$$

We can also define the null-set lamination space as $\mathcal{ML}\big/_{\sim_{\mathcal{N}}}$. For a measured lamination λ, we denote by $[\lambda]_{\mathcal{N}}$ its corresponding element in $\mathcal{NML}$. In other words, $[\lambda]_{\mathcal{N}} = \Pi_{\mathcal{N}}(\lambda)$.

In order to justify the title of this chapter, we also call $\mathcal{NML}$, the *reduced Thurston boundary*. The justification of this term will be given in the next chapter.

We endow $\mathcal{NML}$ with the topology induced by the canonical projection map $\Pi_{\mathcal{N}} : \mathcal{ML} \to \mathcal{NML}$. This space with respect to this topology is non-Hausdorff. Indeed, we have the following statement.

Proposition 1.10. *The null-set lamination space is a non Hausforff compact topological space.*

Proof. The compactness directly follows from the quotient topology and the compacity of $\mathcal{PML}$. Let us show that it is a non-Hausdorff space. Let α_0 and α_1 be two disjoint simple closed geodesics. For any $t \in [0,1]$, we set

$$\alpha_t = (1-t)\cdot\alpha_0 + t\cdot\alpha_1.$$

We then have a new measured lamination such that

$$\forall 0 < t < 1,\ [\alpha_t]_{\mathcal{N}} = \left[\alpha_{\frac{1}{2}}\right]_{\mathcal{N}}.$$

Hence, we have a pair of distinct points $[\alpha_0]_{\mathcal{N}}$ and $[\alpha_1]_{\mathcal{N}}$ in the closure of a single point $\left[\alpha_{\frac{1}{2}}\right]_{\mathcal{N}}$. It follows that any neighbourhood of $[\alpha_1]_{\mathcal{N}}$ contains $\left[\alpha_{\frac{1}{2}}\right]_{\mathcal{N}}$. Thus, we have proved that $\mathcal{NML}$ is not a Fréchet (or T_1) space and hence, it is non-Hausdorff. ☐

We have also the following property.

Property 1.11 ([5], Lemma 3.1)**.** *There exists a continuous map $\Pi_{\mathcal{U}}^{\mathcal{N}} : \mathcal{UML} \to \mathcal{NML}$ such that $\Pi_{\mathcal{N}} = \Pi_{\mathcal{U}}^{\mathcal{N}} \circ \Pi_{\mathcal{U}}$.*

Proof. This is quite obvious. Indeed, let λ and μ be two measured laminations such that $|\lambda| = |\mu|$. Then, from Lemma 1.4 we have that $\lambda \sim_{\mathcal{N}} \mu$ and hence, the (natural) map $\Pi_{\mathcal{U}}^{\mathcal{N}} : \mathcal{UML} \to \mathcal{NML}$ such that $\Pi_{\mathcal{N}} = \Pi_{\mathcal{U}}^{\mathcal{N}} \circ \Pi_{\mathcal{U}}$ is well-defined. Moreover, since $\Pi_{\mathcal{U}}$ and $\Pi_{\mathcal{N}}$ are continuous and surjective, we obtain the continuity of $\Pi_{\mathcal{U}}^{\mathcal{N}}$. ☐

1.3 Rigidity

Let us start this section by recalling some well-known definitions. We denote by $\mathrm{Diff}(X_0)$ the group of diffeomorphisms on X_0. The set $\mathrm{Diff}^+(X_0)$ (resp. $\mathrm{Diff}_0(X_0)$) is defined to be the group of diffeomorphisms on X_0 which preserve the orientation (resp. which are isotopic to the identity map). The *mapping class group* of X_0, denoted by $\mathrm{MCG}(X_0)$, is the group

$$\mathrm{Diff}^+(X_0) \big/ \mathrm{Diff}_0(X_0)$$

and the *extended mapping class group*, denoted by $\mathrm{MCG}^*(X_0)$, is the group

$$\mathrm{Diff}(X_0) \big/ \mathrm{Diff}_0(X_0).$$

The aim of this chapter is to prove the following theorem.

Theorem 1.12 ([5], Theorem 3.2)**.** *Any homeomorphism of $\mathcal{NML}$ is induced by an element of $MCG^*(X_0)$. Such an element is unique if the genus g of X_0 is strictly greater than* 2*.*

To obtain this result we adopt the same strategy as in [57], [53], [12] and hence in [54]. Indeed, we first consider some topological notions on a non-Hausdorff space, such as the adherence height. Then we show that any homeomorphism of $\mathcal{NML}$ induces an automorphism of the complex of curves on X_0. We conclude by using the well-known Ivanov result (see below) and a density argument which is the same as Ohshika in [53].

Before getting further, let us recall that the *complex of curves* $\mathcal{C}(X_0)$ of X_0 is a simplicial complex of dimension $3g-4$ where k-simplexes are the data of $(k+1)$ disjoint simple closed geodesics. The main result that we need, which deals with the group $\mathrm{Aut}(\mathcal{C}(X_0))$ of automorphisms of $\mathcal{C}(X_0)$, is the following.

Theorem 1.13 ([20], Theorem 1 and [36])**.** *Any element of $Aut(\mathcal{C}(X_0))$ is induced by an element of $MCG^*(X_0)$. Such an element is unique if the genus g of X_0 is strictly greater than* 2*.*

1.3.1 On the topology of $\mathcal{NML}$

In order to follow the strategy in [57] and [53], we need to introduce the notion of unilateral adherence. This notion uses deeply the fact that $\mathcal{NML}$ is not a T_1-space.

Definition 1.14. Let x and y be two distinct points in $\mathcal{NML}$. We say that x is *unilaterally adherent* to y, if any neighbourhood of y contains x.

We first observe the following:

Lemma 1.15 ([5], Lemma 3.3)**.** *Let λ and μ be measured laminations on X_0. Then $[\mu]_{\mathcal{N}}$ is unilaterally adherent to $[\lambda]_{\mathcal{N}}$ if and only if $\mathcal{N}(\mu)$ is properly contained in $\mathcal{N}(\lambda)$.*

Proof. $\boxed{\Rightarrow}$ Suppose that $\mathcal{N}(\mu)$ is not contained in $\mathcal{N}(\lambda)$. Then by definition, there is a measured lamination η such that $i(\lambda, \eta) > 0$ whereas $i(\mu, \eta) = 0$.

We set

$$U_\eta = \{\nu \in \mathcal{ML} \mid i(\nu, \eta) > 0\},$$

which is an open set of $\mathcal{ML}$. We shall show that ${\Pi_{\mathcal{N}}}^{-1}\Pi_{\mathcal{N}}(U_\eta) = U_\eta$.

The inclusion $U_\eta \subset {\Pi_{\mathcal{N}}}^{-1}\Pi_{\mathcal{N}}(U_\eta)$ is obvious. Let ν be an element of ${\Pi_{\mathcal{N}}}^{-1}\Pi_{\mathcal{N}}(U_\eta)$. Then there is a measured lamination $\nu' \in U_\eta$ such that $[\nu]_{\mathcal{N}} = [\nu']_{\mathcal{N}}$. Since $i(\nu', \eta) > 0$, there is a component ν'_0 of ν' with $i(\nu'_0, \eta) > 0$. If $|\nu'_0|$ is a boundary component of $\Sigma(\nu'_1)$ for some component ν'_1 of ν', then we have that $i(\nu'_1, \eta) > 0$. Therefore, we can assume that ν'_0 is not such a simple closed geodesic. By Corollary 1.7, we see that $|\nu'_0|$ is contained in ν, and hence $i(\nu, \eta) > 0$. This shows that ν is contained in U_η. Thus we have shown that ${\Pi_{\mathcal{N}}}^{-1}\Pi_{\mathcal{N}}(U_\eta) = U_\eta$, and therefore $\Pi_{\mathcal{N}}(U_\eta)$ is an open set in $\mathcal{NML}$. Since $[\lambda]_{\mathcal{N}} \in \Pi_{\mathcal{N}}(U_\eta)$ and $[\mu]_{\mathcal{N}} \notin \Pi_{\mathcal{N}}(U_\eta)$, we see that $[\mu]_{\mathcal{N}}$ is not unilaterally adherent to $[\lambda]_{\mathcal{N}}$.

$\boxed{\Leftarrow}$ Suppose that $\mathcal{N}(\mu)$ is properly contained in $\mathcal{N}(\lambda)$. Still by Corollary 1.7, we can assume that $|\mu|$ contains every boundary component of the minimal supporting surfaces of its minimal components. For short, we assume that μ is saturated. Then by Proposition 1.6, $|\mu|$ contains every minimal component of $|\lambda|$. Let μ_0 be the union of the components of μ whose supports are also contained in $|\lambda|$, i.e. $|\mu_0| = |\lambda|$. Now we consider for $t \in [0, 1]$, the measured lamination

$$\mu_t = (1-t)(\mu \setminus \mu_0) \cup \mu_0.$$

As in the proof of Proposition 1.10, we get $[\mu_t]_{\mathcal{N}} = [\mu]_{\mathcal{N}}$ for every $t \in [0, 1)$, and $[\mu_1]_{\mathcal{N}} = [\lambda]_{\mathcal{N}}$. This shows that every neighbourhood of $[\lambda]_{\mathcal{N}}$ contains $[\mu]_{\mathcal{N}}$, and hence $[\mu]_{\mathcal{N}}$ is unilaterally adherent to $[\lambda]_{\mathcal{N}}$. □

Let us give another definition.

Definitions 1.16. Let $[\lambda]_{\mathcal{N}} \in \mathcal{NML}$. An *adherence tower* of length n for $[\lambda]_{\mathcal{N}}$ is a finite sequence $([\lambda_0]_{\mathcal{N}}, \cdots, [\lambda_n]_{\mathcal{N}})$ where $[\lambda_0]_{\mathcal{N}} = [\lambda]_{\mathcal{N}}$ and for all $i = 0, \cdots, n-1$, $[\lambda_i]_{\mathcal{N}}$ is unilaterally adherent to $[\lambda_{i+1}]_{\mathcal{N}}$.

The *adherence height* of $[\lambda]_{\mathcal{N}}$ is the supremum of the lengths of adherence towers for $[\lambda]_{\mathcal{N}}$ and we denote it by $a.h.r([\lambda]_{\mathcal{N}})$.

The first easy obvervation is

Corollary 1.17. *Let x and y be two points in $\mathcal{NML}$ such that x is unilaterally adherent to y. Then*

$$a.h.r(y) < a.h.r(x).$$

Proof. This follows from the definition. □

There is an upper bound for the adherence height which only depends on the topology of X_0. Indeed,

Lemma 1.18 ([5], Lemma 3.4)**.** *Let $[\lambda]_{\mathcal{N}} \in \mathcal{NML}$. Then*

$$a.h.r([\lambda]_{\mathcal{N}}) \leq 3g - 3$$

and equality holds if and only if $|\lambda|$ is a pants decomposition of X_0.

Proof. By Proposition 1.6, Corollary 1.7 and Lemma 1.15, for a sequence $([\lambda]_{\mathcal{N}} = [\lambda_0]_{\mathcal{N}}, \cdots, [\lambda_n]_{\mathcal{N}})$ defining the adherence height, representatives can be taken so that $|\lambda_{i+1}| \subset |\lambda_i|$. This immediately implies the inequality. The number of components of a measured lamination is equal to $3g-3$ only if its support is a pants decomposition. This proves the second part of the statement. □

Thus, we have characterised points in $\mathcal{NML}$ represented by pants decompositions in terms of aherence height. Also, we can characterise points in $\mathcal{NML}$ corresponding to multicurves as points appearing in the adherence tower of length exactly equal to $3g-3$. Indeed, if $\lambda \in \mathcal{ML}$ is a (weighted) multi simple closed geodesic, there exists a pants decomposition $\mathcal{P}(X_0)$ such that $\lambda \in \mathcal{P}(X_0)$.

We now have everything we need to prove the rigidity theorem.

1.3.2 Proof of Theorem 1.12

As we have already pointed out, we shall start by using the result of Ivanov and Luo.

Since both adherence towers and adherence lengths are preserved by the autohomeomorphisms of $\mathcal{NML}$, we have the following.

Lemma 1.19 ([5], Lemma 3.5)**.** *Let $f : \mathcal{NML} \to \mathcal{NML}$ be a homeomorphism. Then f induces a simplicial automorphism of the complex of curves $\mathcal{C}(X_0)$.*

Thus, by applying Theorem 1.13, we have that for any self-homeomorphism $f : \mathcal{NML} \to \mathcal{NML}$, there is a diffeomorphism $\phi : X_0 \to X_0$ which induces the same action on $\mathcal{C}(X_0)$ as f, and such a ϕ is unique, up to isotopy, if $g \geq 3$. Now the aim is to show that in fact ϕ induces f on $\mathcal{NML}$. We use the same symbol ϕ to denote the self-homeomorphism induced by ϕ on $\mathcal{NML}$. To prove that $\phi = f$ on $\mathcal{NML}$, we use the same technique as in [53]. We shall first show the following lemma, which corresponds to Lemma 3.1 in [57].

Lemma 1.20 ([5], Lemma 3.6)**.** *For two measured laminations λ, μ with $i(\lambda, \mu) > 0$, there are neighbourhoods $\mathcal{U}_\lambda$ of $[\lambda]_{\mathcal{N}}$ and $\mathcal{U}_\mu$ of $[\mu]_{\mathcal{N}}$ in $\mathcal{NML}$ such that $\mathcal{U}_\lambda \cap \mathcal{U}_\mu = \emptyset$.*

Proof. By the continuity of the intersection number, there are disjoint neighbourhoods V_λ, V_μ of λ, μ in $\mathcal{ML}$ such that

$$\forall\,(\lambda', \mu') \in V_\lambda \times V_\mu,\ i(\lambda', \mu') > 0.$$

We set

$$\tilde{U}_\lambda = \{\lambda' \in \mathcal{ML} \mid \forall \mu' \in V_\mu,\ i(\lambda', \mu') > 0\}\,,$$

which is an open set of $\mathcal{ML}$. By the same argument as in the proof of Lemma 1.15, we see that ${\Pi_{\mathcal{N}}}^{-1}\,\Pi_{\mathcal{N}}(\tilde{U}_\lambda) = \tilde{U}_\lambda$. Since $\tilde{U}_\lambda$ contains V_λ, it is an open neighbourhood of λ, and hence $\Pi_{\mathcal{N}}(U_\lambda)$ is an open neighbourhood of $[\lambda]_{\mathcal{N}}$.

Next we set

$$\tilde{U}_\mu = \left\{\mu' \in \mathcal{ML} \mid \forall \lambda' \in \tilde{U}_\lambda,\ i(\mu', \lambda') > 0\right\},$$

which is for the same reason, an open set of $\mathcal{ML}$. Again, we have ${\Pi_{\mathcal{N}}}^{-1}\,\Pi_{\mathcal{N}}(\tilde{U}_\mu) = \tilde{U}_\mu$, and also $V_\mu \subset \tilde{U}_\mu$ by the definition of $\tilde{U}_\lambda$. Therefore $\Pi_{\mathcal{N}}\left(\tilde{U}_\mu\right)$ is an open neighbourhood of μ. We remark that $\tilde{U}_\lambda \cap \tilde{U}_\mu = \emptyset$ by the definition of $\tilde{U}_\mu$.

By setting $\mathcal{U}_\lambda = \Pi_{\mathcal{N}}(\tilde{U}_\lambda)$ and $\mathcal{U}_\mu = \Pi_{\mathcal{N}}(\tilde{U}_\mu)$, we finish the proof. ◻

Now, we state a lemma which is similar to Lemma 2 in [53].

Lemma 1.21 ([5], Lemma 3.7)**.** *Let $\{K_i\}$ be a sequence of weighted multi simple closed geodesics converging to a measured lamination λ such that $\{|K_i|\}$ converges to $|\lambda|$ in the Hausdorff topology of the geodesic lamination space $\mathcal{GL}$. If $\{[K_i]_{\mathcal{N}}\}$ converges to a point $[\mu]_{\mathcal{N}}$ in $\mathcal{NML}$, then either $[\lambda]_{\mathcal{N}} = [\mu]_{\mathcal{N}}$ or $[\lambda]_{\mathcal{N}}$ is unilaterally adherent to $[\mu]_{\mathcal{N}}$.*

Proof. Suppose that $[\mu]_{\mathcal{N}}$ and $[\lambda]_{\mathcal{N}}$ are distinct and that $[\lambda]_{\mathcal{N}}$ is not unilaterally adherent to $[\mu]_{\mathcal{N}}$. Then by Lemma 1.15, $\mathcal{N}(\lambda)$ is not contained in $\mathcal{N}(\mu)$. This means that there is a connected measured lamination η such that $i(\lambda, \eta) = 0$ whereas $i(\mu, \eta) > 0$. If $|\eta|$ is a minimal component of $|\lambda|$, we have $i(\lambda, \mu) > 0$. Then by the previous lemma, there are neighbourhoods $\mathcal{U}_\lambda$ of $[\lambda]_{\mathcal{N}}$ and $\mathcal{U}_\mu$ of $[\mu]_{\mathcal{N}}$ in $\mathcal{NML}$ with $\mathcal{U}_\lambda \cap \mathcal{U}_\mu = \emptyset$. This implies that $\{[K_i]_{\mathcal{N}}\}$ which converges to $[\lambda]_{\mathcal{N}}$ cannot converge to $[\mu]_{\mathcal{N}}$ at the same time.

Now, we assume that $|\eta|$ is not a minimal component of $|\lambda|$. This means that η is disjoint from λ. We set $U_\eta = \{\nu \in \mathcal{ML} \mid i(\nu, \eta) > 0\}$. As shown in the proof of Lemma 1.15, $U_\eta = {\Pi_{\mathcal{N}}}^{-1}\,\Pi_{\mathcal{N}}(U_\eta)$, which implies that $\Pi_{\mathcal{N}}(U_\eta)$ is an open neighbourhood of $[\mu]_{\mathcal{N}}$. Since $|K_i|$ converges to $|\lambda|$ in the Hausdorff topology, and η is disjoint from λ, we see that $i(K_i, \eta) = 0$ for i large enough. Therefore, $[K_i]_{\mathcal{N}}$ is not contained in $\Pi_{\mathcal{N}}(U_\eta)$ for i large enough. This shows that $\{[K_i]_{\mathcal{N}}\}$ cannot converge to $[\mu]_{\mathcal{N}}$. ◻

Having proved Lemma 1.21, the rest of the proof of Theorem 1.12 is the same as that of the main theorem of [53]. For the convenience of the reader we provide a proof.

Proof of Theorem 1.12. As we pointed out, a self-homeomorphism f of $\mathcal{NML}$ induces an element of Aut $(\mathcal{C}(X_0))$, and hence there exists a diffeomorphism $\Phi : X_0 \to X_0$ which induces the same action on $\mathcal{C}(X_0)$ as f. Now, let λ be a measured lamination. There exists a sequence $\{K_i\}$ of (weighted) multi simple closed geodesics which satisfies hypothesis of Lemma 1.21. Since $[K_i]_{\mathcal{N}}$ can be viewed as a simplex of $\mathcal{C}(X_0)$, we have that

$$f([K_i]_{\mathcal{N}}) = \phi([K_i]_{\mathcal{N}}).$$

□

Recall that for a self-homeomorphism $f : \mathcal{NML} \to \mathcal{NML}$, we have a homeomorphism $\phi : X_0 \to X_0$ which induces the same map on $\mathcal{C}(X_0)$. For $\lambda \in \mathcal{ML}$, we take a sequence of multi-curves K_i as in Lemma 1.21. Then $f([K_i]_{\mathcal{N}}) = \phi([K_i]_{\mathcal{N}})$ since $[K_i]_{\mathcal{N}}$ is regarded as a simplex in $\mathcal{C}(S)$. Note that $\phi(\lambda)$ is the Hausdorff limit of $\phi(K_i)$. By the continuity of f, we see that $f[\lambda]_{\mathcal{N}}$ must be one of the limits of $f([K_i]_{\mathcal{N}}) = \phi([K_i]_{\mathcal{N}})$. By Lemma 1.21, this implies that either $f([\lambda]_{\mathcal{N}}) = \phi([\lambda]_{\mathcal{N}})$ or $\phi([\lambda]_{\mathcal{N}})$ is unilaterally adherent to $f([\lambda]_{\mathcal{N}})$. Since

$$a.h.r(f([\lambda]_{\mathcal{N}})) = a.h.r([\lambda]_{\mathcal{N}}) = a.h.r(\phi([\lambda]_{\mathcal{N}})),$$

the latter cannot happen. Thus we have shown that $f([\lambda]_{\mathcal{N}}) = \phi([\lambda]_{\mathcal{N}})$ for every $[\lambda]_{\mathcal{N}} \in \mathcal{NML}$, which completes the proof of Theorem 1.12.

As noted in Remark 1.8, all of the above remains valid if we consider the set of measured foliations $\mathcal{MF}$ instead of $\mathcal{ML}$, and hence we just have to replace $\mathcal{NML}$ by $\mathcal{NMF}$. Before moving to Chapter 2, let us specify that in the rest of this thesis, we shall only deal with $\mathcal{MF}$ and so with $\mathcal{NMF}$. Moreover, in order to avoid useless details, we only consider measured laminations on a closed hyperbolic surface, but all results remain true if X_0 is a complete hyperbolic surface of genus $g \geq 0$ with $n \geq 0$ cusps. Indeed, the same strategy works, including the result of Ivanov and Luo. In addition, we need to cite Kormaz with his paper [28], which deals with the rigidity in exceptional cases.

Furthermore, the study of such a space $\mathcal{NML}$, or $\mathcal{NMF}$, was motivated by the understanding of differences between compactifications of the Teichmüller space of X_0. This will be explained in the next chapter.

Chapter 2

Other Reduced compactifications of Teichmüller space

In this chapter, we shall use the conformal point of view on Teichmüller space. It means that we shall only consider Riemann surfaces of genus $g \geq 1$ with $n \geq 0$ marked points. From now, X_0 denotes such a Riemann surface. This chapter is organized as follows. In the first section, we shall recall some well-known results on Teichmüller space. In the second section, we shall introduce compactifications of this space and we shall especially give a large overview on the Gardiner-Masur one. This will be useful in Chapter 4 when we focus on the convergence of deformation of some conformal structures toward the Gardiner-Masur boundary. We shall recall that all considered compactifications (except the visual one) carry a notion of intersection number and then a "null-set" equivalence relation as in the previous chapter. We shall consider the quotient spaces, which will be non-Hausdorff spaces, and we shall obtain some relations between these new spaces. We recall that results in that chapter are from the paper [5].

2.1 Background

As we already mentioned, we shall use the measured foliation space $\mathcal{MF}$ instead of $\mathcal{ML}$. Let us start this section by recalling the definition of this space.

2.1.1 Measured foliations

We say that a simple closed curve on X_0 is *essential* if it is not homotopic to a point. We denote the set of homotopy classes of essential simple closed curves by $\mathcal{S}(X_0)$ or by $\mathcal{S}$. The notation is similar to the one in Section 1.1 of the previous chapter because in each homotopy class of essential simple closed curve, there is a unique element which is a simple closed geodesic (when X_0 is considered as a hyperbolic surface). Given two elements α and β of $\mathcal{S}$, we define their *geometric intersection number*, denoted (also) by $i(\alpha, \beta)$, as the minimal intersection number of two essential simple closed curves in the homotopy classes α and β. It is well known that this number corresponds to the intersection number of the corresponding simple closed geodesics.

We set $\mathbb{R}_+ \times \mathcal{S} = \{t \cdot \alpha \mid t \geq 0 \text{ and } \alpha \in \mathcal{S}\}$ and we call it the set of *weighted simple closed curves*. It is known that

$$\begin{aligned} i_\star : \mathbb{R}_+ \times \mathcal{S} &\to \mathbb{R}_+^{\mathcal{S}} \\ t \cdot \alpha &\mapsto t \cdot i(\alpha, \cdot) \end{aligned} \tag{2.1}$$

is an embedding. We denote the closure of the image of this map by $\mathcal{MF}$ and we call it the set of *measured foliations*. We define the space $\mathcal{PMF}$ of *projective measured foliations* as the quotient of $\mathcal{MF} \setminus \{0\}$ by the natural action of $\mathbb{R}_+$. We denote by $[F]$ the projective class of $F \in \mathcal{MF}$. It is well known that $\mathcal{MF}$ and $\mathcal{PMF}$ are respectively homeomorphic to $\mathbb{R}^{6g-6+2n}$ and $\mathbb{S}^{6g-7+2n}$. Furthermore, it is known that the geometric intersection number can be extended to a continuous function from $\mathcal{MF} \times \mathcal{MF}$ to $\mathbb{R}_+$. Thus, any measured foliation F can be seen as a continuous function from $\mathcal{MF}$ to $\mathbb{R}_+$. We refer to [15] and [67] for a more geometric interpretation. The main advantage of the geometric interpretation is to allow the introduction of the notion of critical points and critical graph associated with a measured foliation.

In comparison with the measured lamination space, the set $\mathcal{MF}$ only depends on the topology of X_0, but we can show by considering X_0 as a hyperbolic structure, that these two spaces are homeomorphic and this homeomorphism preserves the intersection number. Moreover, for a measured foliation F, we denote by $|F|$ the foliation F without the transverse measure and hence it corresponds to an element of $\mathcal{UMF}$, the unmeasured foliation space.

A measured foliation F is *rational* if it is determined by a system of positive real numbers $\{w_k\}_{1 \leq i \leq k}$ and a system of distinct simple closed curves $\{\alpha_i\}_{1 \leq i \leq k}$ such that

$$\forall G \in \mathcal{MF},\ i(F, G) = \sum_{1 \leq i \leq k} w_i i(\alpha_i, G).$$

A measured foliation F is *uniquely ergodic* if

$$\mathcal{N}(F) = \{t \cdot F \mid t \in \mathbb{R}_+\}.$$

Let us add one more definition. A pair (F, G) of measured foliations is said to be *transverse* if for any $H \in \mathcal{MF} \setminus \{0\}$,

$$i(F, H) + i(G, H) > 0.$$

We now give another point of view on measured foliations.

2.1.2 Quadratic differentials

An *admissible quadratic differential* q on X_0 is an object which is locally of the form $q = q(z)\, dz^2$ such that $q(z)$ is meromorphic with poles of order at most 1 at the marked points. Such a quadratic differential determines a pair of transverse measured foliations, $F_{v,q}$ and $F_{h,q}$ which are respectively called the *vertical foliation* and the *horizontal foliation* of q on X_0. To be more precise, these foliations are defined as follows. Let $\alpha \in \mathcal{S}$, then

$$i\left(F_{v,q}, \alpha\right) = \inf_{\alpha' \in \alpha} \int_{\alpha'} |\operatorname{Re} \sqrt{q}| \tag{2.2}$$

and

$$i\left(F_{h,q}, \alpha\right) = \inf_{\alpha' \in \alpha} \int_{\alpha'} |\operatorname{Im} \sqrt{q}|. \tag{2.3}$$

Notice that these foliations have critical points which correspond to the zeros or poles of q.

We denote by $\mathcal{Q}\left(X_0\right)$ the space of such quadratic differentials on X_0. This space is endowed with an L^1-norm $\|\cdot\|$ which is defined as follows. For any $q \in \mathcal{Q}\left(X_0\right)$,

$$\|q\| = \iint_{X_0} |q|. \tag{2.4}$$

We set $\mathcal{Q}_1\left(X_0\right)$ (resp. $\mathcal{Q}_{\leq 1}\left(X_0\right)$), the space of admissible quadratic differentials which are of norm 1 (resp. ≤ 1). By abuse of langage we still call elments of $\mathcal{Q}\left(X_0\right)$, quadratic differentials. Furthermore, for any $q \in \mathcal{Q}\left(X_0\right)$, there exists a system of local coordinates $z = x + iy$ on X_0 where away from the critical points of q, we have $q = dz^2$. These local coordinates are called the *natural coordinates* of q. This notion will be used in the definition of the Teichmüller disc in Section 3.1 of the next chapter.

We consider the space of quadratic differentials on X_0, but the definition works for any Riemann surface X. Moreover, it is well known that a transverse pair of measured foliations (F, G) defines a Riemann surface X and a quadratic differential q on X where F (resp. G) corresponds to the horizontal (resp. vertical) foliation of q. Such a pair also determines a point in Teichmüller space (see the definition below).

Another link between quadratic differentials and measured foliations is given by the following result of Hubbard and Masur.

Theorem 2.1 ([18], Theorem 2.3)**.** *Let X be a Riemann surface and $F \in \mathcal{MF}$. Then there exists a unique quadratic differential q_F on X such that $F_{h,q_F} = F$.*

Let us add that the equality above means that F, up to Whitehead moves, comes from the quadratic differential q_F. This theorem is what we shall call the *Hubbard-Masur theorem.*

Actually, Hubbard and Masur (and Kerckhoff in [25]) proved a stronger result which says that $\mathcal{MF}$ is homeomorphic to $\mathcal{Q}(X)$ when we consider these two spaces with the topology respectively induced by the geometric intersection and the norm $\|\cdot\|$. An equivalent statement is that $\mathcal{PMF}$ is homeomorphic to $\mathcal{Q}_1\left(X_0\right)$. For more details about quadratic differentials, we refer for example to [14] and [60].

One of the main application of the space of measured foliations (or the space of quadratic differentials) is that it gives a global parametrization of the space called Teichmüller space.

2.1.3 Teichmüller space

The *Teichmüller space* of X_0 is a space which "classifies" Riemann surfaces of the same topological type as X_0. Teichmüller introduced this space in [61] to solve the *Riemann moduli problem* and called it the set of "topologically determined principal regions."

We say that (X_1, f_1) and (X_2, f_2), where $f_i : X_0 \to X_i$ ($i = 1, 2$) is a quasiconformal homeomorphism,[1] are equivalent if there exists a conformal map $h : X_1 \to X_2$ which is homotopic to $f_2 \circ f_1^{-1}$. The *Teichmüller space* of X_0, denoted by $\mathcal{T}(X_0)$, is the set of equivalence classes of pairs (X, f). For a pair (X, f), we denote the corresponding point in $\mathcal{T}(X_0)$ by $[X, f]$ and we call $x_0 = [X_0, \mathrm{id}]$ the base point of $\mathcal{T}(X_0)$. This definition of $\mathcal{T}(X_0)$ is what we call the *conformal point of view* on Teichmüller space.

Using the Uniformization Theorem, the Teichmüller space can also be viewed as the set of all isotopy classes of complete hyperbolic structures with finite area on the underlying topological surface of X_0. We shall say that it is the *hyperbolic point of view* of $\mathcal{T}(X_0)$.

There is a natural distance function on $\mathcal{T}(X_0)$, called the *Teichmüller distance* defined as follows. Let $x = [X, f]$ and $y = [Y, g]$ be two points in $\mathcal{T}(X_0)$. The Teichmüller distance between x and y is

$$d_T(x, y) = \inf \log K_h, \tag{2.5}$$

where h is taken over all quasiconformal homeomorphisms homotopic to $g \circ f^{-1}$ and K_h denotes the *quasiconformal dilatation* of h. We recall that the quasiconformal dilatation of h is defined as the essential supremum of

$$p \in X \mapsto \frac{|\partial_{\bar{z}} h(p)| + |\partial_z h(p)|}{|\partial_z h(p)| - |\partial_{\bar{z}} h(p)|}, \tag{2.6}$$

where whith the local coordinates $z = x + iy$,

$$\begin{cases} \partial_z h &= \dfrac{1}{2}\left(\dfrac{\partial h}{\partial x} - i\dfrac{\partial h}{\partial y}\right), \\ \partial_{\bar{z}} h &= \dfrac{1}{2}\left(\dfrac{\partial h}{\partial x} + i\dfrac{\partial h}{\partial y}\right). \end{cases}$$

The starting point of the *classical Teichmüller theory* is the Teichmüller theorem. This theorem, originally stated in [61] and proved for closed surfaces in [62] (see

[1] We refer to [3] for an introduction to this notion.

also [9] and [10]), was given as a solution of the *moduli problem* of Riemann. Before presenting this theorem here, we shall introduce some more notations.

Let us start by explaining the way we use Theorem 2.1 in the context of Teichmüller space. Let $x = [X, g] \in \mathcal{T}(X_0)$ and $F \in \mathcal{MF}$. Then there exists a unique quadratic differential on X, denoted by $q_{x,F}$ or q_F for short, such that $F_{h,q_F} = f(F)$. Moreover, we recall that for any $[F] \in \mathcal{PMF}$ and for any $0 \leq k < 1$, there exists a unique quasiconformal map $f_k^{[F]} : X_0 \to f_k^{[F]}(X_0)$ which is the solution of the *Beltrami equation*

$$\partial_{\bar{z}} f = -k \frac{\overline{q_F}}{|q_F|} \partial_z f. \tag{2.7}$$

Let us recall that $f_k^{[F]}$ is now called the *Teichmüller map* associated with q_F and whose quasiconformal dilatation is equal to $\dfrac{1+k}{1-k}$. With such notations, for any $x = [X, g] \in \mathcal{T}(X_0)$, $t \geq 0$ and $f \in \mathcal{MF}$, we set

$$\mathcal{R}^t_{[F]}(x) = \left[f^{[F]}_{\tanh\left(\frac{t}{2}\right)}(X), f^{[F]}_{\tanh\left(\frac{t}{2}\right)} \circ g \right]. \tag{2.8}$$

Hence, we can define the following map:

$$\begin{aligned} \mathbb{R}_+ \times \mathcal{PMF} &\to \mathcal{T}(X_0) \\ (t, [F]) &\mapsto \mathcal{R}^t_{[F]}(x). \end{aligned} \tag{2.9}$$

The *Teichmüller theorem* states that this map induces a homeomorphism from $\mathbb{R}_+ \times \mathcal{PMF} \big/_{(0,[F]) \sim (0,[G])}$ to $\mathcal{T}(X_0)$. Furthermore, Teichmüller (already in [61]) proved that $t \geq 0 \mapsto \mathcal{R}^t_{[F]}(x)$ is a geodesic ray (with respect to the Teichmüller distance) parametrized by arc length. Such a ray is called the *Teichmüller ray* emanating from x and directed by $[F]$. We shall also use the term *Teichmüller deformation*, especially in Part II of this thesis. If G represents the vertical foliation of q_F, then the set $\left(\mathcal{R}^t_{[G]}(x)\right)_{t \geq 0} \cup \left(\mathcal{R}^t_{[F]}(x)\right)_{t \geq 0}$ forms a geodesic line called the *Teichmüller geodesic line* through x and directed by (F, G). By abuse of notation we will denote it by $\left(\mathcal{R}^t_{[F]}(x)\right)_{t \in \mathbb{R}}$.

2.1.4 Extremal length

For this subsection, we fix $x = [X, g] \in \mathcal{T}(X)$.

For a *measurable conformal metric* $\rho = \rho(z)|dz|$ on X, we set

$$A_\rho = \iint_X \rho^2.$$

Furthermore, for $\alpha \in \mathcal{S}$ we set $L_\rho(\alpha) = \inf_{\alpha'} \int_{\alpha'} \rho(z)|dz|$ where α' belongs to α. The *extremal length* of α on x is defined as

$$\mathrm{Ext}_x(\alpha) = \sup_\rho \frac{L_\rho(\alpha)^2}{A_\rho}, \tag{2.10}$$

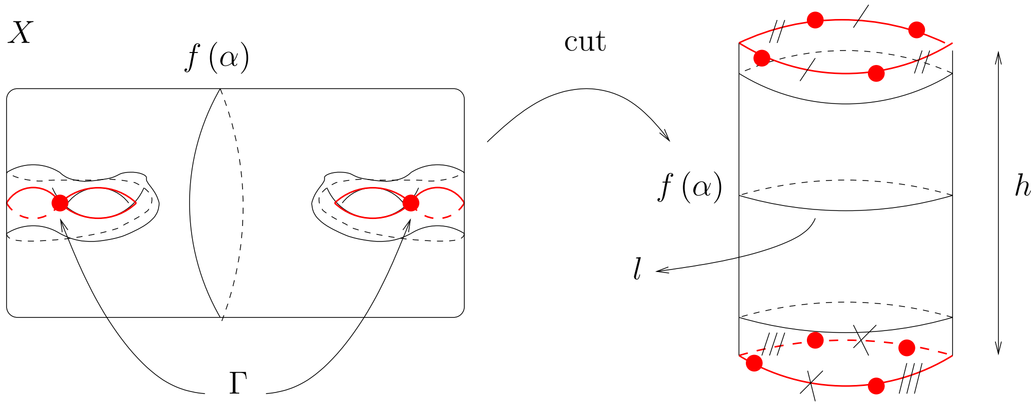


Figure 2.1: A visual example to obtain the Euclidean cylinder which realises the infimum in (2.12). The term "cut" means that $X\setminus\Gamma$ is biholomorphic to the Euclidean cylinder. With this notation, the extremal length of $\alpha\in\mathcal{S}$ on $x=[X,f]\in\mathcal{T}(X_0)$ is equal to $\dfrac{l}{h}$.

where ρ is taken over all measurable conformal metrics such that $A_\rho\neq 0,+\infty$. We refer to [3] for an introduction to this notion. This definition is called the *analytic definition* of extremal length.

There exists an equivalent definition of extremal length which is called the *geometric definition*. The definition is as follows. Let $\alpha\in\mathcal{S}$. Then the extremal length of α on x is

$$\operatorname{Ext}_x(\alpha)=\inf_A\frac{1}{\operatorname{Mod}(A)},\tag{2.11}$$

where A is taken over all Euclidean cylinders that can be conformally embedded into X such that the image of the core curve by this embedding is in the class $g(\alpha)$. We recall that the modulus of a Euclidean cylinder is the ratio between its height and circumference (see also Figure 2.1).

Moreover, by setting for any $t\geq 0$, $\operatorname{Ext}_x(t\cdot)=t^2\operatorname{Ext}_x(\cdot)$, Kerckhoff has proved in [25] that $\operatorname{Ext}_x(\cdot)$ extends continuously to $\mathcal{MF}$ and

$$\operatorname{Ext}_x(F)=\|q_{x,F}\|,\tag{2.12}$$

where $q_{x,F}$ is the unique quadratic differential on X whose horizontal foliation is $g(F)$. Kerckhoff proved also the following result now called the *Kerckhoff formula*:

Theorem 2.2 ([25], Theorem 4). *Let $x,\,y\in\mathcal{T}(X_0)$. Then*

$$d_T(x,y)=\log\sup_{\alpha\in\mathcal{S}}\frac{\operatorname{Ext}_y(\alpha)}{\operatorname{Ext}_x(\alpha)}.\tag{2.13}$$

To prove this theorem, Kerckhoff observes the following. Let $F \in \mathcal{MF}$ and $t \geq 0$. Then, as the Teichmüller map $f^{[F]}_{\tanh\left(\frac{t}{2}\right)}$ determines a quadratic differential q_t on $f^{[F]}_{\tanh\left(\frac{t}{2}\right)}(X)$ such that $F_{h,q_t} = e^{\frac{t}{2}} \cdot F$ and $F_{v,q_t} = e^{-\frac{t}{2}} \cdot F_{h,q_F}$, we have

$$\begin{cases} \mathrm{Ext}_{\mathcal{R}^t_{[F]}(x)}\left(F\right) & = e^{-t}\,\mathrm{Ext}_x\left(F\right), \\ \mathrm{Ext}_{\mathcal{R}^t_{[F]}(x)}\left(F_{v,q_F}\right) & = e^{t}\,\mathrm{Ext}_x\left(F_{v,q_F}\right). \end{cases} \tag{2.14}$$

Let us recall a property which is called the *Minsky inequality.*

Property 2.3 ([42], Lemma 5.1)**.** *Let x be a point in $\mathcal{T}(X_0)$ and (F, G) be a pair of measured foliations. Then*

$$i\left(F, G\right)^2 \leq \mathrm{Ext}_x\left(F\right) \cdot \mathrm{Ext}_x\left(G\right)$$

with equality if and only if F and G are realized as horizontal and vertical foliation of some quadratic differential on x.

The main consequence of this property is that for any $x \in \mathcal{T}(X_0)$ and $F \in \mathcal{MF}$ we have

$$\mathrm{Ext}_x\left(F\right) = \sup_{G \in \mathcal{MF}\setminus\{0\}} \frac{i\left(F, G\right)^2}{\mathrm{Ext}_x\left(G\right)}. \tag{2.15}$$

As we shall see, Equality (2.15) plays a central role in defining the intersection number between two boundary points of the Gardiner-Masur compactification (see Theorem 2.8 below).

Let us close this subsection with a notation. We set

$$\mathcal{MF}_1^{x_0} = \left\{F \in \mathcal{MF} \mid \mathrm{Ext}_{x_0}\left(F\right) = 1\right\},$$

where x_0 is the basepoint. It can be shown that $\mathcal{MF}_1^{x_0}$ is homeomorphic to $\mathcal{PMF}$.

2.2 Different compactifications of Teichmüller space

There exist several distinct compactifications of Teichmüller space and we shall recall a few of them. We start with the Thurston compactiffcation. The corresponding boundary has already been defined in the previous chapter.

2.2.1 The Thurston compactification

The definition of this compactification uses the hyperbolic point of view on $\mathcal{T}(X_0)$. Therefore, for each point $x = [X, g] \in \mathcal{T}(X_0)$ and $\alpha \in \mathcal{S}$ (α being a simple closed geodesic), we denote by $l_x\left(\alpha\right)$ the hyperbolic length on X of $g\left(\alpha\right)$. Therefore, we can define

$$\Phi_{Th} : x \in \mathcal{T}(X_0) \mapsto \left[l_x\left(\cdot\right)\right] \in \mathrm{P}\mathbb{R}^{\mathcal{S}}_{\geq 0}, \tag{2.16}$$

where $\mathrm{P}\mathbb{R}^{\mathcal{S}}_{\geq 0} = \mathbb{R}^{\mathcal{S}}_{\geq 0} \setminus \{0\} \big/ \mathbb{R}_{>0}$. Thurston showed that Φ_{Th} is an embedding whose image is relatively compact. We denote the closure of this image by $\overline{\mathcal{T}(X_0)}^{Th}$ and we call it the *Thurston compactification* of $\mathcal{T}(X_0)$. An important fact is that the boundary $\partial_{Th}\mathcal{T}(X_0)$ of that closure is exactly $\mathcal{PMF}$ or $\mathcal{PML}$ (see [15] for more details), and so we can write

$$\overline{\mathcal{T}(X_0)}^{Th} = \mathcal{T}(X_0) \cup \mathcal{PMF}.$$

We know that this compactification is homeomorphic to the unit closed ball in $\mathbb{R}^{6g-6+2n}$. In the previous chapter, we studied the null-set foliation space $\mathcal{NML}$ which is homeomorphic to $\mathcal{NMF}$. We also define the *reduced Thurston compactification* of $\mathcal{T}(X_0)$ as

$$\overline{\mathcal{T}(X_0)}^{Th,red} = \mathcal{T}(X_0) \cup \mathcal{NMF} \tag{2.17}$$

and we endow it with the quotient topology. We call $\mathcal{NMF}$, the *reduced Thurston boundary* of $\mathcal{T}(X_0)$.

2.2.2 The Teichmüller compactification

Let $x = [X, g]$ be a point in $\mathcal{T}(X_0)$. Using the Teichmüller theorem and so the map defined by Relation 2.9, we identify $\mathcal{PMF}$ with points at infinity of $\mathcal{T}(X_0)$ in order to get a compactification of $\mathcal{T}(X_0)$. This compactification denoted by $\overline{\mathcal{T}(X_0)}^{T,x}$, is called the *Teichmüller compactification* based at x. We denote the boundary by $\partial_{T,x}\mathcal{T}(X_0)$. Actually, in the literature (see for example [19] or [14]) the boundary is identified with $\mathcal{Q}_1(X)$, but we know from Theorem 2.1 that it is homeomorphic to $\mathcal{PMF}$. Thus, there is a canonical homeomorphism between $\partial_{T,x}\mathcal{T}(X_0)$ and $\partial_{Th}\mathcal{T}(X_0)$.

However, the Teichmüller compactification depends on the basepoint. It is one reason for which the Teichmüller compactification is not the same as Thurston's one. Indeed, the identity map on $\mathcal{T}(X_0)$ does not extend continuously to a map from $\partial_{T,x}\mathcal{T}(X_0)$ to $\overline{\mathcal{T}(X_0)}^{Th}$. By using the action of MCG (X_0) on $\mathcal{T}(X_0)$, Kerckhoff proved in [25] that this action does not extend continuously to $\partial_{T,x}\mathcal{T}(X_0)$, whereas it does to $\partial_{Th}\mathcal{T}(X_0)$. Another proof is given by Lenzhen. In [29], she provides an example of a Teichmüller ray that does not converge to a point in the Thurston boundary.

Let us introduce the most important notion of this subsection. The notion of geometric intersection number on $\partial_{T,x}\mathcal{T}(X_0)$ defined as follows:

$$\forall q_1, q_2 \in \mathcal{Q}_1(X)\,,\ i_T(q_1\,,\,q_2) = i\left(F_{h,q_1}, F_{h,q_2}\right). \tag{2.18}$$

Hence, we can associate to any boundary point its null-set. Indeed, if $q \in \partial_{T,x}\mathcal{T}(X_0) = \mathcal{Q}_1(X)$, then

$$\mathcal{N}_T(q) = \{h \in \partial_{T,x}\mathcal{T}(X_0) \mid i_T(q\,,\,h) = 0\}\,.$$

We say that two Teichmüller boundary points are equivalent if they have the same null-set. This defines an equivalence relation on $\partial_{T,x}\mathcal{T}(X_0)$ denoted by $\sim_{\mathcal{N}_T}$ and we set

$$\partial^{red}_{T,x}\mathcal{T}(X_0) = \partial_{T,x}\mathcal{T}(X_0) \Big/ \sim_{\mathcal{N}_T}. \tag{2.19}$$

This quotient space is called the *reduced Teichmüller boundary* of $\mathcal{T}(X_0)$. We extend this equivalence relation to Teichmüller space by setting for any $x, y \in \mathcal{T}(X_0)$, $x \sim_{\mathcal{N}_T} y$ if and only if $x = y$. Finally we define the *reduced Teichmüller compactification* based at x as

$$\overline{\mathcal{T}(X_0)}^{T,x,red} = \overline{\mathcal{T}(X_0)}^{T,x} \Big/ \sim_{\mathcal{N}_T} = \mathcal{T}(X_0) \cup \partial^{red}_{T,x}\mathcal{T}(X_0) \tag{2.20}$$

and the topology is endowed with the quotient topology. The canonical quotient map is denoted by $\Pi_{\mathcal{N}_T}$. For a boundary point q, we denote by $[q]_{\mathcal{N}_T}$ its corresponding element in $\partial^{red}_{T,x}\mathcal{T}(X_0)$. In other terms, $[q]_{\mathcal{N}_T} = \Pi_{\mathcal{N}_T}(q)$.

From the identification between $\partial_{T,x}\mathcal{T}(X_0)$ and $\mathcal{PMF}$, we have that $\overline{\mathcal{T}(X_0)}^{T,x,red}$ (resp. $\partial^{red}_{T,x}\mathcal{T}(X_0)$) is equal (as a set) to $\overline{X_0}^{Th,red}$ (resp. $\mathcal{NMF}$). The main question concerns topologies:

Question 2.4. Are these spaces homeomorphic to each other?

If the answer is affirmative, then the dependance on x in the reduced compactification is inessential. Let us note that Ohshika proved in [54] that although the Bers compactification depends on the basepoint, the corresponding reduced compactification does not.

2.2.3 The asymptotic visual compactification

Let x be a fixed point in $\mathcal{T}(X_0)$. We say that two Teichmüller rays emanating from x and directed respectively by $[F_1]$ and $[F_2]$ are *asymptotic* if the function

$$t \in \mathbb{R}_+ \mapsto d_T\left(\mathcal{R}^t_{[F_1]}(x), \mathcal{R}^t_{[F_2]}(x)\right)$$

is bounded. This notion defines an equivalence relation on the set of Teichmüller rays emanating from x. We denote by $\partial_{vis,x}\mathcal{T}(X_0)$ the quotient space of the space of Teichmüller rays emanating from x by that equivalence relation. We endow $\mathcal{T}(X_0) \cup \partial_{vis,x}\mathcal{T}(X_0)$ with the quotient topology induced from that of $\overline{\mathcal{T}(X_0)}^{T,x}$. We denote this new space by $\overline{\mathcal{T}(X_0)}^{vis,x}$ and we call it the *asymptotic visual compactification* of $\mathcal{T}(X_0)$. The quotient map shall be denoted by Π_{vis}. The reason we mention in Proposition 1.10 explains also that it is a compact set whose *visual boundary* is non-Hausdorff.

Let us note that McCarthy and Papadopoulos used in [44] the fact that $\partial_{vis,x}\mathcal{T}(X_0)$ is non-Hausdorff in order to prove that $(\mathcal{T}(X_0), d_T)$ is not a Gromov hyperbolic space. This result was already proved (in a different way) by Masur and Wolf in [41]. Miyachi gave in [52] another proof (see §6.4 in that paper).

We shall use this compactification to prove Theorem 2.15 below. Indeed, even without a notion of intersection number on it, we shall show (see Subsection 2.3.2) that there exists a relation with the reduced Teichmüller compactification.

2.2.4 The Gardiner-Masur compactification

In contrast with the Teichmüller compactification and the Thurston compactification, the Gardiner-Masur compactification is less known. Gardiner and Masur defined this compactification in [16] and Miyachi, through [46, 47, 50, 51, 48] and [52] proved many of the known results. Let us review some of them.

In the same way as the map given by Relation (2.16) we define

$$\Phi_{GM} : x \in \mathcal{T}(X_0) \mapsto \left[\mathrm{Ext}_x^{\frac{1}{2}}(\cdot)\right] \in \mathrm{P}\mathbb{R}_{\geq 0}^{\mathcal{S}}. \tag{2.21}$$

Gardiner and Masur showed in [16] that Φ_{GM} is also an embedding with relatively compact image. The closure of the image is denoted by $\overline{\mathcal{T}(X_0)}^{GM}$ and called the *Gardiner-Masur compactification* of $\mathcal{T}(X_0)$. Gardiner and Masur also showed that if $\dim_{\mathbb{C}} \mathcal{T}(X_0) = 3g - 3 + n \geq 2$, then $\mathcal{PMF} \subsetneq \partial_{GM}\mathcal{T}(X_0)$. Furthermore, Miyachi proved in [45] that in the case of the once-punctured torus, the Gardiner-Masur and the Thurston boundaries coincide.

As we pointed out in Subsection 2.1.1, Thurston boundary points can be represented by continuous functions from $\mathcal{MF}$ to $\mathbb{R}_+$. Actually this is still the case for Gardiner-Masur boundary points. Indeed, if we set for any $y \in \mathcal{T}(X_0)$,

$$\mathcal{E}_y^{x_0} : F \in \mathcal{MF} \mapsto \left(\frac{\mathrm{Ext}_y(F)}{e^{d_T(x_0,y)}}\right)^{\frac{1}{2}}, \tag{2.22}$$

where x_0 denotes the basepoint, then we can show that $\left\{\mathcal{E}_y^{x_0}(\cdot)\right\}_{y\in\mathcal{T}(X_0)}$ forms a normal family of continuous functions which "determines" boundary points. More precisely:

Theorem 2.5 ([46], Theorem 1.1)**.** *Let $p \in \partial_{GM}\mathcal{T}(X_0)$. There exists a unique continuous map $\mathcal{E}_p^{x_0} : \mathcal{MF} \to \mathbb{R}_+$ such that*

1. *$\mathcal{E}_p^{x_0}$ represents (as element of $\mathbb{R}_{\geq 0}^{\mathcal{S}}$) the point p,*
2. *$\max_{F\in\mathcal{MF}_1^{x_0}} \mathcal{E}_p^{x_0}(F) = 1$,*
3. *if y_n converges to p, then $\mathcal{E}_{y_n}^{x_0}$ converges uniformly to $\mathcal{E}_p^{x_0}$ on any compact set of $\mathcal{MF}$.*

This is an existence theorem, but we have to keep in mind that if a boundary point is an element of $\mathcal{PMF}$, denoted by $[F]$, then

$$\mathcal{E}_{[F]}^{x_0}(\cdot) = \frac{1}{\operatorname{Ext}_{x_0}^{\frac{1}{2}}(F)} \cdot i(F, \cdot). \tag{2.23}$$

Remark 2.6. The fact that x_0 represents the basepoint of $\mathcal{T}(X_0)$ does not matter; we can rewrite Theorem 2.5 by considering any point in $\mathcal{T}(X_0)$ instead of x_0.

One of the major advances on *extremal length geometry*, namely, the study of the Gardiner-Masur compactification, was achieved by Miyachi in his pleasant paper [51]. In that paper he introduces, following the idea of Bonahon on *geodesic currents* and the extension of the geometric intersection number (see [7]), an extension of the geometric intersection number on the Gardiner-Masur compactification. Let us sketch this construction.

We set

$$\mathcal{C}_{GM} = \left\{ \mathfrak{p} \in \mathbb{R}_{\geq 0}^{\mathcal{S}} \mid [\mathfrak{p}] \in \overline{\mathcal{T}(X_0)}^{GM} \right\} \cup \{0\}.$$

This set is the analogue of the space of *geodesic currents* and since $\mathcal{PMF} \subset \partial_{GM}\mathcal{T}(X_0)$, we have $\mathcal{MF} \subset \mathcal{C}_{GM}$. Considering Relation (2.15), Miyachi extends the notion of extremal length on $\mathcal{C}_{GM}$ as follows.

Let $x \in \mathcal{T}(X_0)$ and $p \in \partial_{GM}\mathcal{T}(X_0)$. We set

$$\mathcal{E}xt_x^{x_0}(p) = \sup_{G \in \mathcal{MF}\setminus\{0\}} \frac{\mathcal{E}_p^{x_0}(G)^2}{\operatorname{Ext}_x(G)}. \tag{2.24}$$

Now, we define a continuous extension of extremal length. For any $x \in \mathcal{T}(X_0)$ and any $\mathfrak{a} \in \mathcal{C}_{GM}$, we set

$$\operatorname{Ext}_x(\mathfrak{a}) = t^2 \cdot \mathcal{E}xt_x^{x_0}(p), \tag{2.25}$$

where $\mathfrak{a} = t \cdot \mathcal{E}_p^{x_0}$ for some $t > 0$ and $p \in \overline{\mathcal{T}(X_0)}^{GM}$. We can show that this notion does not depend on x_0 (see Theorem 3 in [51]). Moreover, the restriction of this function on $\mathcal{MF}$ coincides with the original definition of extremal length. Indeed, if $\mathfrak{a} = F \in \mathcal{MF}$, then

$$\begin{aligned}
\operatorname{Ext}_x(\mathfrak{a}) &= \operatorname{Ext}_x(F) \\
&= \operatorname{Ext}_x\left(\operatorname{Ext}_{x_0}^{\frac{1}{2}}(F) \cdot \frac{1}{\operatorname{Ext}_{x_0}^{\frac{1}{2}}(F)} F\right) \\
&= \operatorname{Ext}_{x_0}(F) \cdot \mathcal{E}xt_x^{x_0}\left(\frac{1}{\operatorname{Ext}_{x_0}^{\frac{1}{2}}(F)} F\right) \\
&= \operatorname{Ext}_{x_0}(F) \cdot \sup_{G \in \mathcal{MF}\setminus\{0\}} \frac{\mathcal{E}_{[F]}^{x_0}(G)^2}{\operatorname{Ext}_x(G)} \\
&= \operatorname{Ext}_x(F).
\end{aligned}$$

Keeping in mind Relation (2.22), if we set for any $y \in \mathcal{T}(X_0)$

$$\xi_y^{x_0} : p \in \overline{\mathcal{T}(X_0)}^{GM} \mapsto \left(\frac{\mathcal{E}xt_y^{x_0}(p)}{e^{d_T(x_0,y)}} \right)^{\frac{1}{2}}, \tag{2.26}$$

we deduce that $\left\{ \xi_y^{x_0}(\cdot) \right\}_{y \in \mathcal{T}(X_0)}$ forms a normal family (Section 7 in [51]). Hence, we have the following proposition.

Proposition 2.7 ([51], Proposition 7)**.** *There exists a unique symmetric continuous function*

$$i_{x_0}(\cdot, \cdot) : \overline{\mathcal{T}(X_0)}^{GM} \times \overline{\mathcal{T}(X_0)}^{GM} \to \mathbb{R}_+$$

such that for any pair $(y, p) \in \mathcal{T}(X_0) \times \overline{\mathcal{T}(X_0)}^{GM}$, we have

$$i_{x_0}(y, p) = \xi_y^{x_0}(p).$$

In particular, we can show that if $p \in \partial_{GM}\mathcal{T}(X_0)$ and $[F] \in \mathcal{PMF}$, then

$$i_{x_0}(p, [F]) = \frac{\mathcal{E}_p^{x_0}(F)}{\operatorname{Ext}_{x_0}^{\frac{1}{2}}(F)}. \tag{2.27}$$

Using a density argument, we can see that for any $y \in \mathcal{T}(X_0)$ and any $\mathfrak{a} \in \mathcal{C}_{GM}$, we have

$$\operatorname{Ext}_y(\mathfrak{a}) = e^{d_T(x_0,y)} \cdot \operatorname{Ext}_{x_0}(\mathfrak{a}) \cdot i_{x_0}(y, [\mathfrak{a}])^2. \tag{2.28}$$

This intersection number function leads to the definition of an intrinsic intersection number on $\mathcal{C}_{GM}$. This is given by the following theorem:

Theorem 2.8 ([52], Theorem 4)**.** *There exists a unique symmetric continuous function*

$$i_{GM}(\cdot\,,\,\cdot) : \mathcal{C}_{GM} \times \mathcal{C}_{GM} \to \mathbb{R}_+$$

such that for any $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{C}_{GM} \times \mathcal{C}_{GM}$,

$$i_{GM}(\mathfrak{a}\,,\,\mathfrak{b}) = \operatorname{Ext}_{x_0}^{\frac{1}{2}}(\mathfrak{a}) \cdot \operatorname{Ext}_{x_0}^{\frac{1}{2}}(\mathfrak{b}) \cdot i_{x_0}([\mathfrak{a}], [\mathfrak{b}]). \tag{2.29}$$

From the definitions, it is not difficult to prove that the restriction of $i_{GM}(\cdot\,,\,\cdot)$ to $\mathcal{MF} \times \mathcal{MF}$ coincides with the geometric intersection number. Furthermore, this intersection number satisfies the so-called *generalised Minsky inequality* (see Corollary 3 of [51]). This inequality states that for any $y \in \mathcal{T}(X_0)$, any $\mathfrak{a} \in \mathcal{C}_{GM}$ and any $\mathfrak{b} \in \mathcal{C}_{GM}$, we have

$$i_{GM}(\mathfrak{a}\,,\,\mathfrak{b})^2 \leq \operatorname{Ext}_y(\mathfrak{a}) \cdot \operatorname{Ext}_y(\mathfrak{b}). \tag{2.30}$$

We refer to Corollary 3 of [51]. For any $p \in \partial_{GM}\mathcal{T}(X_0)$, we define the associated null-set as

$$\mathcal{N}_{GM}(p) = \{q \in \partial_{GM}\mathcal{T}(X_0) \mid i_{x_0}(p, q) = 0\}.$$

Note that from (2.29), the null-set of p does not depend on the choice of x_0. As before, we have an equivalence relation on $\partial_{GM}\mathcal{T}(X_0)$ denoted by $\sim_{\mathcal{N}_{GM}}$ which leads to define the *reduced Gardiner-Masur boundary*

$$\partial_{GM}^{red}\mathcal{T}(X_0) = \partial_{GM}\mathcal{T}(X_0) \Big/ \sim_{\mathcal{N}_{GM}}. \tag{2.31}$$

We extend this equivalence relation to Teichmüller space by setting for any $x, y \in \mathcal{T}(X_0)$, $x \sim_{\mathcal{N}_{GM}} y$ if and only if $x = y$. Therefore, we define the *reduced Gardiner-Masur compactification*

$$\overline{\mathcal{T}(X_0)}^{GM,red} = \overline{\mathcal{T}(X_0)}^{GM} \Big/ \sim_{\mathcal{N}_{GM}} = \mathcal{T}(X_0) \cup \partial_{GM}^{red}\mathcal{T}(X_0). \tag{2.32}$$

The topology on these sets is given by the quotient topology. We denote the canonical quotient map by $\Pi_{\mathcal{N}_{GM}}$. We denote the equivalence class of $p \in \partial_{GM}\mathcal{T}(X_0)$ by $[p]_{\mathcal{N}_{GM}}$.

2.3 Relations between compactifications

In the previous section, we recalled some compactifications of Teichmüller space. These compactifications are well known to be different from each other. Indeed, we have the following:

Proposition 2.9 ([5], Lemma 4.1)**.** *If X_0 is neither of genus one with one marked point nor of genus zero with four marked points, then the identity mapping on $\mathcal{T}(X_0)$ does not extend continuously between any two of these above four compactifications.*

In order to distinguish the convergence to the Thurston compactification and the convergence to the Gardiner-Masur compactification, we shall write $\xrightarrow{Th}$ and $\xrightarrow{GM}$.

Proof. Since the asymptotic visual boundary $\partial_{vis,x_0}\mathcal{T}(X_0)$ is a non-Hausdorff space and the other three compactifications are Hausdorff spaces, we have for free that the asymptotic visual compactification is different from the others.

As already pointed out, the Teichmüller compactification $\overline{\mathcal{T}(X_0)}^{T,x_0}$ is different from the Thurston one $\overline{\mathcal{T}(X_0)}^{Th}$, since $\mathrm{MCG}(X_0)$ does not extend continuously to $\overline{\mathcal{T}(X_0)}^{T,x_0}$. For the same reason, $\overline{\mathcal{T}(X_0)}^{T,x_0}$ is different from the Gardiner-Masur one $\overline{\mathcal{T}(X_0)}^{GM}$. Indeed, Miyachi proved in [46] (Theorem 5.1) that $\mathrm{MCG}(X_0)$ extends continuously to $\overline{\mathcal{T}(X_0)}^{GM}$.

Now, let us assume that the identity mapping on $\mathcal{T}(X_0)$ extends continuously to a map from $\overline{\mathcal{T}(X_0)}^{Th}$ to $\overline{\mathcal{T}(X_0)}^{GM}$. Let α_1 and α_2 be two disjoint simple closed curves. We set $F = \alpha_1 + \alpha_2$ and $G = \alpha_1 + 2 \cdot \alpha_2$. Hence, we obtain two rational measured foliations which are not projectively equivalent. Following a result of Masur in [40], we have

$$\mathcal{R}^t_{[F_1]}(x_0) \xrightarrow[t\to+\infty]{Th} [F_1]$$

and

$$\mathcal{R}^t_{[F_2]}(x_0) \xrightarrow[t\to+\infty]{Th} [F_1] .$$

Then we have two different Teichmüller rays which converge to the same limit in the Thurston compactification. Now we use a result of Miyachi. In [47],[2] he proves that Teichmüller rays always converge towards the gardiner-Masur boundary and that if the corresponding directions are different, then the limit points are also different. Thus, the identity map cannot extend continuously. □

The main goal of this chapter is to understand the difference between the above compactifications. According to the previous section, all these compactifications (except the asymptotic visual one) have a notion of intersection number. The reduced Teichmüller compactification is the same (as a set) as the reduced Thurston one, but the two spaces might be endowed with different topologies. We shall see below that the reduced Gardiner-Masur compactification can be also identified with these two reduced compactifications. Let us state an easy observation.

Property 2.10. *The three reduced compactifications are non-Hausdorff compact spaces.*

Proof. This uses the same proof as Proposition 1.10. □

2.3.1 Null-sets in $\partial_{GM}\mathcal{T}(X_0)$

In order to see the relationship between $\partial^{red}_{GM}\mathcal{T}(X_0)$ and $\mathcal{NMF}$ we have to introduce one more notation.

Definition 2.11. Let p be an element of $\partial_{GM}\mathcal{T}(X_0)$. We say that $G \in \mathcal{MF}$ is an *associated foliation* for p if there exist $x \in \mathcal{T}(X_0)$, a sequence $([G_n]) \subset \mathcal{PMF}$ and a sequence $(t_n) \subset \mathbb{R}_+$ which tends to $+\infty$ such that

$$[G_n] \xrightarrow[n\to+\infty]{} [G] \text{ and } \mathcal{R}^{t_n}_{[G_n]}(x) \xrightarrow[n\to+\infty]{GM} p.$$

This notion of associated foliation leads to the following technical result.

Theorem 2.12 ([52], Theorem 7.1)**.** *Let $p \in \partial_{GM}\mathcal{T}(X_0)$. Then any associated foliation $G \in \mathcal{MF}$ for p statisfies*

$$\mathcal{N}_{GM}(p) = \mathcal{N}_{GM}(G) .$$

[2] Actually in this case, we should use another result from the paper [46] by Miyachi, where he gives the explicit limit of such rays in the Gardiner-Masur compactification.

This theorem and Definition 2.11 imply that if we denote by p the limit point in $\partial_{GM}\mathcal{T}(X_0)$ of the Teichmüller ray directed by $[F] \in \mathcal{PMF}$, then

$$\mathcal{N}_{GM}(p) = \mathcal{N}_{GM}(F)\,. \tag{2.33}$$

This theorem implies that the three reduced compactifications are equal as sets. This is why it seems natural to wonder if the identity mapping on $\mathcal{T}(X_0)$ extends to a homeomorphism between any two of them.

Let us close this section by considering boundary points that are projective measured foliations.

Property 2.13 ([5], Proposition 2.9)**.** *Let $[G]$ be a projective measured foliation regarded as an element of $\partial_{GM}\mathcal{T}(X_0)$. Then any associated foliation for $[G]$ is projectively equivalent to G.*

Proof. Let $H \in \mathcal{MF} \subset \mathcal{C}_{GM}$ be an associated foliation for $[G]$. Let x, $(t_n) \subset \mathbb{R}_+$ and $([H_n]) \subset \mathcal{PMF}$ be as in Definition 2.11. We can assume that $(H_n) \subset \mathcal{C}_{GM}$ is such that $H_n \to H$. From (2.14), (2.28), (2.29), (2.30) and the continuity of $i_{GM}(\cdot\,,\,\cdot)$ we have for any $\mathfrak{a} \in \mathcal{C}_{GM}$

$$\begin{aligned}
i_{GM}(H\,,\,\mathfrak{a}) &= \lim_{n\to+\infty} i_{GM}(H_n\,,\,\mathfrak{a})\\
&\leq \lim_{n\to+\infty} \operatorname{Ext}^{\frac{1}{2}}_{\mathcal{R}^{t_n}_{[H_n]}(x)}(H_n)\cdot \operatorname{Ext}^{\frac{1}{2}}_{\mathcal{R}^{t_n}_{[H_n]}(x)}(\mathfrak{a})\\
&= \lim_{n\to+\infty} e^{-\frac{1}{2}t_n}\operatorname{Ext}^{\frac{1}{2}}_{x}(H_n)\cdot \operatorname{Ext}^{\frac{1}{2}}_{\mathcal{R}^{t_n}_{[H_n]}(x)}(\mathfrak{a})\\
&= \operatorname{Ext}^{\frac{1}{2}}_{x}(H)\cdot \lim_{n\to+\infty} e^{-\frac{1}{2}t_n}\operatorname{Ext}^{\frac{1}{2}}_{\mathcal{R}^{t_n}_{[H_n]}(x)}(\mathfrak{a})\\
&= \operatorname{Ext}^{\frac{1}{2}}_{x}(H)\cdot \lim_{n\to+\infty} \operatorname{Ext}^{\frac{1}{2}}_{x}(\mathfrak{a})\, i_x\left(\mathcal{R}^{t_n}_{[H_n]}(x)\,,[\mathfrak{a}]\right)\\
&= \operatorname{Ext}^{\frac{1}{2}}_{x}(H)\cdot \frac{1}{\operatorname{Ext}^{\frac{1}{2}}_{x}(G)}\cdot i_{GM}(G\,,\,\mathfrak{a})\,.
\end{aligned}$$

Since $i_{GM}(\cdot\,,\,\cdot)$ coincides with the intersection number, we deduce that for any $\alpha \in \mathcal{S}$

$$\frac{1}{\operatorname{Ext}^{\frac{1}{2}}_{x}(H)}\cdot i(H,\alpha) \leq \frac{1}{\operatorname{Ext}^{\frac{1}{2}}_{x}(G)}\cdot i(G,\alpha)\,. \tag{2.34}$$

Moreover, since the extremal length of a given measured foliation is related to a norm of some quadratic differential (see Relation (2.12)), we can use the so-called *height theorem* ([38], Theorem 3.2) for quadratic differentials in order to conclude that $H = G$. ☐

Another easy observation is the following one, which is a direct consequence of Theorem 3 in [47] (see Theorem 4.2 below for the statement).

Lemma 2.14. *Let p be a Gardiner-Masur boundary point. If there exists $F \in \mathcal{MF}$ a uniquely ergodic measured foliation or a simple closed curve such that*

$$\mathcal{N}_{GM}(p) = \mathcal{N}_{GM}(F),$$

then

$$p = [F].$$

2.3.2 A continuous bijective mapping

The main aim of this subsection is to prove the following result.

Theorem 2.15 ([5], Proposition 4.2)**.** *Let x be a point in $\mathcal{T}(X_0)$. Then, the identity mapping on $\mathcal{T}(X_0)$ extends to a continuous bijective mapping from $\overline{\mathcal{T}(X_0)}^{T,x,red}$ to $\overline{\mathcal{T}(X_0)}^{GM,red}$.*

We fix $x = [x, G] \in \mathcal{T}(X_0)$. In order to prove this theorem we shall consider the asymptotic visual compactification.

Property 2.16 ([5], proof of Proposition 4.2)**.** *There exists a continuous map*

$$\Pi^{\mathcal{N}_T}_{vis} : \overline{\mathcal{T}(X_0)}^{vis,x} \to \overline{\mathcal{T}(X_0)}^{T,x,red}$$

such that

$$\Pi_{\mathcal{N}_T} = \Pi^{\mathcal{N}_T}_{vis} \circ \Pi_{vis}.$$

Proof. We define $\Pi^{\mathcal{N}_T}_{vis}$ as follows. The restriction of $\Pi^{\mathcal{N}_T}_{vis}$ on $\mathcal{T}(X_0)$ is the identity mapping. Now, let r_1 and r_2 be two Teichmüller rays such that $\Pi_{vis}(r_1) = \Pi_{vis}(r_2)$. We assume that r_1 (resp. r_2) is given by $q_1 \in \mathcal{Q}_1(X)$ (resp. q_2). Following Ivanov's work in [21], Lenzhen and Masur proved in [30], that two rays are asymptotic if and only if their given directions are absolutely continuous with respect to each other,[3] and so they have the same null-set. We deduce that

$$[q_1]_{\mathcal{N}_T} = [q_2]_{\mathcal{N}_T}.$$

Hence, we define the stated map $\Pi^{\mathcal{N}_T}_{vis}$ of our proposition. This map is well-defined and since $\Pi_{\mathcal{N}_T}$ and Π_{vis} are continuous and surjective, it is also continuous. ☐

Now, we define four new maps which shall be the identity mapping on $\mathcal{T}(X_0)$. We define $\phi_1 : \overline{\mathcal{T}(X_0)}^{T,x} \to \overline{\mathcal{T}(X_0)}^{GM}$ as the map which coincides with the identity map on $\mathcal{T}(X_0)$ and which maps any boundary element to the limit in the Gardiner-Masur boundary of Teichmüller ray directed by that element. Miyachi proved ([47], Theorem 1) that such a map is injective but not continuous. We set $\phi_2 = \Pi_{\mathcal{N}_{GM}} \circ \phi_1$.

[3] The if part was proved by Ivanov in [21].

At this step, there is no reason for this map to be continuous. We define $\phi_3 : \overline{\mathcal{T}(X_0)}^{vis,x} \to \overline{\mathcal{T}(X_0)}^{GM,red}$ such that

$$\phi_2 = \phi_3 \circ \Pi_{vis}\,.$$

Let us give more details. Let r_1 and r_2 be two asymptotic Teichmüller rays whose associated directions are given respectively by $[F_1]$ and $[F_2]$. As we already noticed in (2.33), we have

$$\mathcal{N}_{GM}\left(\phi_1\left(r_1\right)\right) = \mathcal{N}_{GM}\left(F_1\right) \text{ and } \mathcal{N}_{GM}\left(\phi_1\left(r_2\right)\right) = \mathcal{N}_{GM}\left(F_2\right).$$

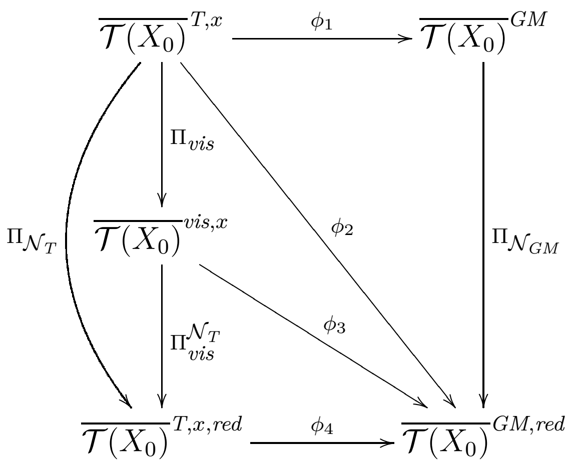
Using once more the argument following the Lenzhen-Masur result, we have $\mathcal{N}_{GM}\left(F_1\right) = \mathcal{N}_{GM}\left(F_2\right)$ and so, ϕ_3 is well-defined. Finally we define

$$\phi_4 : \overline{\mathcal{T}(X_0)}^{T,x,red} \to \overline{\mathcal{T}(X_0)}^{GM,red}$$

as the map such that

$$\phi_2 = \phi_4 \circ \Pi_{\mathcal{N}_{GM}}\,.$$

By Theorem 2.12, this map is well-defined. All these maps lead to the following commutative diagram:



Proof of Theorem 2.15. Since the injectivity is almost for free, we just have to prove that the map $\phi_4 : \overline{\mathcal{T}(X_0)}^{T,x,red} \to \overline{\mathcal{T}(X_0)}^{GM,red}$ is surjective and continuous. Indeed, let p_1 and $p_2 \in \partial_{GM}\mathcal{T}\left(X_0\right)$ such as $[p_1]_{\mathcal{N}_{GM}} = [p_2]_{\mathcal{N}_{GM}}$ and two measured foliations F_1 and F_2 such that

$$\mathcal{R}^t_{[F_1]}\left(x\right) \xrightarrow[t\to+\infty]{GM} p_1 \text{ and } \mathcal{R}^t_{[F_2]}\left(x\right) \xrightarrow[t\to+\infty]{GM} p_2.$$

Thus,

$$\begin{aligned}\mathcal{N}_T\left(q_{F_1}\right) &= \mathcal{N}\left(F_1\right)\\ &= \mathcal{N}_{GM}\left(F_1\right)\cap\mathcal{MF}\\ &= \mathcal{N}_{GM}\left(p_1\right)\cap\mathcal{MF}\\ &= \mathcal{N}_{GM}\left(p_2\right)\cap\mathcal{MF}\\ &= \mathcal{N}_{GM}\left(F_2\right)\cap\mathcal{MF}\\ &= \mathcal{N}_T\left(q_{F_2}\right).\end{aligned}$$

We start by proving that $\phi_2 : \overline{\mathcal{T}(X_0)}^{T,x} \to \overline{\mathcal{T}(X_0)}^{GM,red}$ is continuous and surjective.

First, we are interested in the surjectivity of ϕ_2. Let $p \in \overline{\mathcal{T}(X_0)}^{GM}$. There exists $F \in \mathcal{MF}$ such that $\mathcal{N}_{GM}(p) = \mathcal{N}_{GM}(F)$ and so, by considering the Teichmüller ray directed by $[F]$ emanating from x, it implies the surjectivity.

Now, we argue by contradiction to deduce the continuity. Assume that ϕ_2 is not continuous. Then there is an open set $\mathcal{U}$ in $\overline{\mathcal{T}(X_0)}^{GM}$ such that $\phi_2^{-1}(\mathcal{U})$ is not open. Hence, there exist $q_0 \in \partial_{T,x}\mathcal{T}(X_0)$ and a sequence $(q_n) \subset \overline{\mathcal{T}(X_0)}^{T,x}$ such that $q_n \to q_0$, $\phi_2(q_0) \in \mathcal{U}$ and $(\phi_2(q_n)) \not\subset \mathcal{U}$. We have that $F_{h,q_n} \to F_{h,q_0}$. Moreover, up to a subsequence, we can assume that $(\phi_1(q_n))$ converges to $p \in \partial_{GM}\mathcal{T}(X_0)$. It means that F_{h,q_0} is an associated foliation for p and then

$$\mathcal{N}_{GM}(p) = \mathcal{N}_{GM}(\phi_1(q_0)).$$

This should be rewritten as

$$\Pi_{\mathcal{N}_{GM}}(p) = \phi_2(q_0).$$

Since $\Pi_{\mathcal{N}_{GM}}$ is continuous, we have ${\Pi_{\mathcal{N}_{GM}}}^{-1}(\mathcal{U})$ is an open neighbourhood of p. Hence for n large enough, we have $\phi_1(q_n) \in {\Pi_{\mathcal{N}_{GM}}}^{-1}(\mathcal{U})$, i.e.

$$\phi_2(q_n) = \Pi_{\mathcal{N}_{GM}}(\phi_1(q_n)) \in \mathcal{U},$$

which is a contradiction. We complete this proof by checking that ϕ_4 is surjective and continuous. This follows from

$$\phi_2 = \phi_4 \circ \Pi_{\mathcal{N}_T}$$

and the continuity and the surjectivity of ϕ_2. □

Part II

Horocyclic deformation on Teichmüller space

Chapter 3

Horocyclic deformation

As before, in this chapter and the following one, all Riemann surfaces considered shall be oriented closed surfaces of type (g, n), where g represents the genus and n the number of marked points. To avoid unimportant considerations, we assume that such Riemann surfaces have a strictly negative Euler characteristic, which means that we only consider hyperbolic surfaces.

Let us recall that Teichmüller space $\mathcal{T}(X_0)$ can be viewed from the conformal point of view or from the hyperbolic point of view. Depending on the point of view we use, there are two natural tools, the extremal length and the hyperbolic length, which lead to endow $\mathcal{T}(X_0)$ with respectively the Teichmüller metric (see Relation (2.13) above) and the Thurston metric (see Relation (4.1) below). Moreover, these two geometric quantities also lead to the definition of respectively the Gardiner-Masur compactification and the Thurston compactification.

Using the hyperbolic point of view, we consider two natural one-parameter deformations in Teichmüller space, stretches and earthquakes. A stretch line is directed by a complete geodesic lamination[1] on a hyperbolic surface and defines a geodesic line with respect to the so-called Thurston metric on $\mathcal{T}(X_0)$. For any complete geodesic lamination, and so for any stretch line, we can associate a measured lamination which is called the stump of the given direction. Théret showed in [64] that if the stump is either a simple closed geodesic or a uniquely ergodic measured lamination, then the associated stretch line converges in the reverse direction in the Thurston boundary to the projective class of the stump. Moreover, Théret also solved in [65] the negative convergence of stretch line whose associated stump is a weighted multi-geodesic (or a rational foliation). In such cases the limit is the barycenter of the stump. Note that the question of the convergence in the positive direction was solved by Papadopoulos in [56]. The earthquake deformation, introduced by Thurston, generalizes the Fenchel-Nielsen deformation. It is directed by a measured lamination. It is well known that the hyperbolic length of the direction remains constant along this deformation. Moreover, the earthquake converges to the projective class of the corresponding direction in the Thurston boundary. Earthquake deformations are not geodesics as the stretch lines are, but they have a natural behaviour relative to each other. Indeed, under suitable assumptions on directions, Théret

[1] A complete geodesic lamination is a geodesic lamination such that its complementary regions are all ideal triangles.

showed in [63] that performing first a stretch and then an earthquake is the same as performing first an earthquake and then a stretch. In the following two chapters, we shall keep in mind all these properties and consider other deformations and their convergence to the Thurston boundary and to the Gardiner-Masur boundary.

From the conformal point of view of $\mathcal{T}(X_0)$, we already considered the Teichmüller deformation and such a deformation plays the role of a stretch line in that point of view. We recall that this deformation is given by a direction which is the projective class of measured foliations, and is geodesic with respect to the Teichmüller metric (see Relation (2.8) above). Some investigations about the convergence in the Thurston compactification or the Gardiner-Masur compactification have already been done. Indeed, in the case of Thurston's compactification, it is well known that a Teichmüller deformation directed by a simple closed curve converges to its direction (see [25, 16]). Masur showed in [40] that this is also the case if the direction is uniquely ergodic. Later, Lenzhen in [29] constructed an example of such a deformation which does not converge in that compactification. However, in the Gardiner-Masur compactification, the convergence is most natural. Liu and Su in [35] and Miyachi in [47], proved that any Teichmüller deformation converges. Using Kerckhoff's computations in [25], Miyachi also gave in [46] the explicit limit when the direction is given by a rational measured foliation. Walsh in [69] generalized this result by giving the limit for any direction.

In this chapter, we shall define another deformation, called the horocyclic deformation. This is the natural analogue of the earthquake deformation from the conformal point of view. Like for the Teichmüller deformation, the horocyclic deformation is directed by the projective class of measured foliations and stays in some Teichmüller disc. Moreover, this deformation also has a nice property with respect to the Teichmüller deformation. Indeed, seen as maps, these two deformations commute if they have the same direction. We shall also see that the extremal length of the direction stays invariant along the associated horocyclic deformation.

In the next chapter, we shall prove that this deformation converges in the Gardiner-Masur compactification if the given direction is either a simple closed curve or a projective class of a uniquely ergodic measured foliation. In these two cases, we shall show that they correspond to limits of associated Teichmüller deformations. We shall also see that in these two particular cases the horocyclic deformations also converge to the Thurston compactification. In contrast with the convergence of earthquakes to the Thurston boundary, we shall see through an example that there exists a horocyclic deformation which does not converge to the same limit as the corresponding Teichmüller deformation.

Finally, we add that this part of this thesis is based on the paper [4]. In [43], Mirzakhani deals with the dynamics of the "Teichmüller horocyclic flow" on a bundle of Teichmüller space, the bundle of holomorphic quadratic differentials. Her main result is different but related to our work.

3.1 Backgrounds on Teichmüller discs

We start this section by recalling the notion of Teichmüller discs and their known properties.

Let $x = [X, f] \in \mathcal{T}(X_0)$ and $F \in \mathcal{MF}$. By Theorem 2.1, we can associate to F a unique quadratic differential q on X whose horizontal foliation is $f(F)$. It is well known that

$$\imath_{(x,[F])} : \mathbb{D} \to \mathcal{T}(X_0) \tag{3.1}$$

$$r \cdot e^{i\theta} \mapsto \mathcal{R}^{2\tanh^{-1}(r)}_{\left[F_{h,e^{-i\theta}q}\right]}(x) \tag{3.2}$$

is an isometric embeding, considering the Poincaré metric on $\mathbb{D}$. We denote by $\mathbb{D}(x, [F])$ the image of $\mathbb{D}$ by $\imath_{(x,[F])}$ and we call it the *Teichmüller disc* associated with $(x, [F])$. Note that the notion of Teichmüller disc already appears in the famous Teichmüller paper [61] under the name "complex geodesic" (see §121). Since the upper half-plane is biholomorphic to the unit disc, we shall consider $\mathbb{H}$ instead of $\mathbb{D}$.

There exists another point of view on the Teichmüller disc which is more geometric. The point $x \in \mathcal{T}(X_0)$ is determined by the transverse pair $(f(F), F_{v,q})$. Such a pair gives a system of coordinates which are natural coordinates for q. An element of $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ acts on such coordinates and defines a new transverse pair of measured foliations and therefore a new point in Teichmüller space. Furthermore, $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ is isomorphic to the upper half plane and the orbit of x under the action of this group is the Teichmüller disc $\mathbb{D}(x, [F])$. For more details, we refer to [17].

We deduce from this second point of view the following elementary result.

Lemma 3.1. *Let x, $y \in \mathcal{T}(X_0)$ and $F \in \mathcal{MF}$. If $y \in \mathbb{D}(x, [F])$, then $\mathbb{D}(x, [F])$ and $\mathbb{D}(y, [F])$ are identical up to an automorphism of the disc.*

Even if this result is well known, we sketch a proof.

Proof. As $y \in \mathbb{D}(x, [F])$, there exists a pair $(s, t) \in \mathbb{R}^2$ such that y is determined by

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{-\frac{t}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix}. \tag{3.3}$$

These two matrices which act on natural coordinates, preserve (projectively) the measured foliation F, and so they determine a quadratic differential on y whose horizontal foliation is projectively the same as F. Using the inverse matrix of (3.3) on these new natural coordinates, we obtain x and so $x \in \mathbb{D}(y, [F])$. Using Lemma 3.1 of [37], which states that if two Teichmüller discs have at least two common points, then these two discs are the "same," we complete the proof. ☐

From the proof of Lemma 3.1, we observe that for any $t \in \mathbb{R}$, $\mathcal{R}^t_{[F]}(\cdot)$ is identified with the diagonal matrix of (3.3), and so it preserves $\mathbb{D}(x,[F])$. Thus, by pulling back the Teichmüller disc to $\mathbb{H}$, we can consider this Teichmüller ray as a map from $\mathbb{H}$ to $\mathbb{H}$ such that for any $t \in \mathbb{R}$ and any $z = x + iy \in \mathbb{H}$,

$$\mathcal{R}^t_{[F]}(z) = x + i e^t y. \tag{3.4}$$

The parabolic element in (3.3) corresponds, up to normalization, to what we shall call the horocyclic deformation directed by F. A study of such deformations in Teichmüller space is done in the next section and the next chapter.

Before investigating such a deformation, we state the following question which should be understood as an analogue of Lemma 3.1 of [37].

Question 3.2. If two Teichmüller discs have a common point and a common boundary point (in the Gardiner-Masur boundary), can we say that these two discs are equal?

At this moment, we cannot fully solve this question, but if the boundary point is an element of $\mathcal{PMF}$, then we can give an answer. Indeed, we have the following property:

Property 3.3. *Let D_1 and D_2 be two Teichmüller discs such that*

- *the intersection of these discs is not empty,*
- *there exists an element of $\mathcal{PMF}$ which belongs to $\partial_{GM} D_1 \cap \partial_{GM} D_2$.*

Then, D_1 is equal to D_2.

Proof. Let us recall the meaning of $\partial_{GM} D_1$ (resp. $\partial_{GM} D_2$). We consider the closure of $\phi_{GM}(D_1)$, which is a compact set, and since Φ_{GM} is an embedding we have

$$\overline{\Phi_{GM}(D_1)} = D_1 \cup \partial_{GM} D_1.$$

Let us denote by x a common point of D_1 and D_2 and by $[F]$ a common boundary point. Up to a biholomorphism of $\mathbb{D}$, we can assume that x is the center of D_1 and D_2 and therefore there exists $(F_1, F_2) \in \mathcal{MF} \times \mathcal{MF}$ such that

$$D_1 = \mathbb{D}(x, [F_1])$$

and

$$D_2 = \mathbb{D}(x, [F_2]).$$

Hence, there exist $\left(x^1_n\right) \subset D_1$ and $\left(x^2_n\right) \subset D_2$ by assumption such that

$$x^1_n \xrightarrow[n\to+\infty]{GM} [F] \text{ and } x^2_n \xrightarrow[n\to+\infty]{GM} [F].$$

By the definition of associated foliation and Property 2.13 we deduce that there exists $(\theta_1, \theta_2) \in [0, 2\pi[$ such that

$$q_{x,F} = e^{i\cdot\theta_1} \cdot q_{x,F_1} \text{ and } q_{x,F} = e^{i\cdot\theta_2} \cdot q_{x,F_2}.$$

Therefore that the two Teichmüller discs are equal. □

3.2 First approach to horocyclic deformations

3.2.1 Definition

We define the horocyclic deformation as follows.

Definition 3.4. Let $t \in \mathbb{R}$ and $F \in \mathcal{MF}$. The *horocyclic deformation* directed by F of parameter t is the map

$$\begin{aligned}\mathcal{H}_{[F]}^t : \mathcal{T}(X_0) &\to \mathcal{T}(X_0)\\ x &\mapsto \imath_{(x,[F])}\left(k_t e^{i\theta_t}\right),\end{aligned}$$

where $k_t = \dfrac{1}{\sqrt{1+\frac{4\,\mathrm{Ext}_{x_0}(F)^2}{t^2\,\mathrm{Ext}_x(F)^2}}}$ and $\theta_t = \arctan\left(\dfrac{2\,\mathrm{Ext}_{x_0}(F)}{t\,\mathrm{Ext}_x(F)}\right)$.

We observe that for a fixed real number t, the horocyclic deformation depends only on the projective class of the given measured foliation. Thus, we can suppose that the foliation F belongs to $\mathcal{MF}_1^{x_0}$.

As for Teichmüller rays, by pulling back $\mathbb{D}(x,[F])$ to $\mathbb{H}$, one can check applying our normalization that for any $t \in \mathbb{R}$,

$$\mathcal{H}_{[F]}^t(i) = i - s \cdot \mathrm{Ext}_x(F). \tag{3.5}$$

Thus, the image of $t \in \mathbb{R} \mapsto \mathcal{H}_{[F]}^t(x)$ coincides with the image under $\imath_{(x,[F])}$ of a certain horocycle (see Figure 4.1 below). Such a ray is called the *horocyclic deformation* emanating from x and directed by $[F]$.

Moreover, since for any point in $\mathbb{D}(x,[F])$, the Teichmüller ray at this point directed by F stays in this disc, we deduce that for any $s \in \mathbb{R}$, $\mathcal{H}_{[F]}^t(\cdot)$ preserves the associated Teichmüller disc.

Like in the case of the Teichmüller line, we can give an explicit expression of the action of the horocyclic deformation on the upper half-plane. We conjugate it by an appropriate automorphism in order to bring back the problem in i. We can also deduce from Relation (3.5) (and even from the definition) that for any point $x \in \mathcal{T}(X_0)$ and for any $F \in \mathcal{MF}_1^{x_0}$, the map $s \in \mathbb{R} \mapsto \mathcal{H}_{[F]}^s(x)$ is continuous.

In the case where F is a simple closed curve α, it is important to note that for any point $x \in \mathcal{T}(X_0)$, $\left(\mathcal{H}_{[\alpha]}^n(x)\right)_{n\in\mathbb{Z}}$ corresponds to the orbit of x under the action of the group generated by the Dehn twist along α. Such a Dehn twist is denoted by τ_α. This fact was observed by Marden and Masur in [37] and it will be used below. Marden and Masur also gave a description of $\mathcal{H}_{[\alpha]}^t(x)$ when t is real. In what follows such a point is called *conformal twist* along α of parameter t (see the proof of Property 3.9 below for the definition and also Figure 3.1 for a geometric description.).

As we shall see below, the horocyclic deformation has some similarities with the earthquake map.

3.2.2 Elementary properties

In this subsection we present several elementary properties of horocyclic deformations. The first one states the existence of a horocyclic deformation between any two given points of Teichmüller space. It is comparable to a theorem of Thurston (see Theorem 2 in [26] for the statement and a proof). The statement is the following.

Property 3.5. *Let x and y be two distinct points in $\mathcal{T}(X_0)$. Then there exists a unique $F \in \mathcal{MF}_1^{x_0}$ and a unique $s > 0$ such that*

$$y = \mathcal{H}^s_{[F]}(x) .$$

Proof. By (2.9), there exists a unique $(G, s) \in \mathcal{MF}_1^{x_0} \times \mathbb{R}_+^*$ such that $y = \mathcal{R}^s_{[G]}(x)$. Thus, it suffices to consider $e^{-i\tau} \cdot q_G$ for a some τ and set $F = F_{v,e^{-i\tau}q_G}$ and in this case the proof is clear. □

As the horocyclic deformation and the Teichmüller deformation preserve the Teichmüller disc for a given foliation, we obtain the following elementary result (see Figure 3.1 for a geometric description).

Property 3.6. *Let $F \in \mathcal{MF}$. Then for any $s \in \mathbb{R}$ and any $t \in \mathbb{R}$ we have*

$$\mathcal{H}^s_{[F]} \circ \mathcal{R}^t_{[F]} = \mathcal{R}^t_{[F]} \circ \mathcal{H}^s_{[F]}.$$

Proof. Let $x \in \mathcal{T}(X_0)$. We fix $(s,t) \in \mathbb{R}^2$. As the transformations we consider preserve the Teichmüller disc $\mathbb{D}(x, [F])$, the computations will be done in the upper half-plane. Thus, x corresponds to i. From Relations (3.4) and (3.5), we get

$$\mathcal{R}^t_{[F]}\left(\mathcal{H}^s_{[F]}(i)\right) = -s \cdot \mathrm{Ext}_x(F) + i \cdot e^t.$$

Moreover, using Relation (2.14) and conjugating the horocyclic deformation of $\mathcal{R}^t_{[F]}(i)$ by $z \mapsto e^{-t} \cdot z$, we get

$$\begin{aligned}\mathcal{H}^s_{[F]}\left(\mathcal{R}^t_{[F]}(i)\right) &= e^t \cdot \left(i - s \cdot \mathrm{Ext}_{\mathcal{R}^t_{[F]}(x)}(F)\right) \\ &= -s \cdot \mathrm{Ext}_x(F) + i\, e^t.\end{aligned}$$

The proof is complete. □

Note that this result is analogous to a result of Théret. In [63], he proves that a earthquake and stretch commute if their directions are the same.

Remark 3.7. If the directions of the Teichmüller deformation and of the horocyclic deformation are not the same, we do not have necessarily Property 3.6. Indeed, let α and β be two distinct simple closed curves such that $i(\alpha, \beta) \neq 0$. Assume that for any $s, t \in \mathbb{R}$,

$$\mathcal{R}^t_{[\alpha]} \circ \mathcal{H}^s_{[\beta]} = \mathcal{H}^s_{[\beta]} \circ \mathcal{R}^t_{[\alpha]}.$$

In particular this is true when $s = 1$. As we said before, the horocyclic deformation of parameter 1 corresponds to the Dehn twist along β. If we fix a point $x \in \mathcal{T}(X_0)$, we get for any $t \geq 0$

$$\mathcal{R}^t_{[\alpha]}\left(\tau_\beta \cdot x\right) = \tau_\beta \cdot \mathcal{R}^t_{[\alpha]}\left(x\right). \tag{3.6}$$

We recall that for any $y = [Y, g] \in \mathcal{T}(X_0)$, $\tau_\beta \cdot y = \left[Y, g \circ \tau_\beta^{-1}\right]$. However, Gardiner and Masur showed in [16] that for any $y \in \mathcal{T}(X_0)$, $\mathcal{R}^t_{[\alpha]}\left(y\right) \xrightarrow[t \to +\infty]{GM} [\alpha]$. Miyachi proved in [46] that the mapping class group extends continuously to the Gardiner-Masur boundary. Thus, when t tends to $+\infty$ in Equality (3.6), we obtain

$$[\alpha] = \tau_\beta \cdot [\alpha],$$

which is obviously not true.

Since the Teichmüller rays directed by simple closed curves converge in the Thurston boundary, we obtain a similar result by using the convergence in $\mathcal{PMF}$.

Another interesting fact is that the horocyclic deformation is continuous with respect to the direction. The statement is the following.

Lemma 3.8. *Let $x = [X, f] \in \mathcal{T}(X_0)$ and $s \in \mathbb{R}$. Then*

$$\mathcal{H}^s_{[\cdot]}\left(x\right) : \mathcal{PMF} \to \mathcal{T}(X_0)$$

is continuous.

Proof. This is a simple consequence of the Teichmüller theorem given by (2.9). Indeed, let $[F_n]$ be a sequence of elements in $\mathcal{PMF}$ which converges to $[F]$ for the topology induced by the geometric intersection. Let $(q_n)_n$ and q_F be the corresponding elements in $\mathcal{Q}_1\left(X\right)$. For any n, the point $\mathcal{H}^s_{[F_n]}\left(x\right)$ is determined by the Teichmüller deformation of parameter $2\tanh^{-1}\left(k_s\right)$ directed by the horizontal foliation of $e^{-i\theta_s} \cdot q_n$. We have the same description for $\mathcal{H}^s_{[F]}\left(x\right)$ using q_F instead of q_n. As the map given in (2.9) is a homeomorphism, a fortiori it is continuous with respect to $\mathcal{PMF}$ and so the lemma is proved. □

Actually, using the same idea as the proof of Lemma 3.8, we can prove a stronger result which is the following. Let $x \in \mathcal{T}(X_0)$, then

$$\begin{aligned} \mathbb{R}_+ \times \mathcal{PMF} &\to \mathcal{T}(X_0) \\ (s, [F]) &\mapsto \begin{cases} \mathcal{H}^s_{[F]}\left(x\right) & \text{if } s > 0 \\ x & \text{if } s = 0 \end{cases} \end{aligned} \tag{3.7}$$

induces a homeomorphism from $\mathbb{R}_+ \times \mathcal{PMF} \big/_{(0,[F]) \sim (0,[G])}$ to $\mathcal{T}(X_0)$.

Moreover, Lemma 3.8 will be useful for proving that the extremal length of a particular foliation does not change along a horocyclic deformation. Indeed, we have the following property.

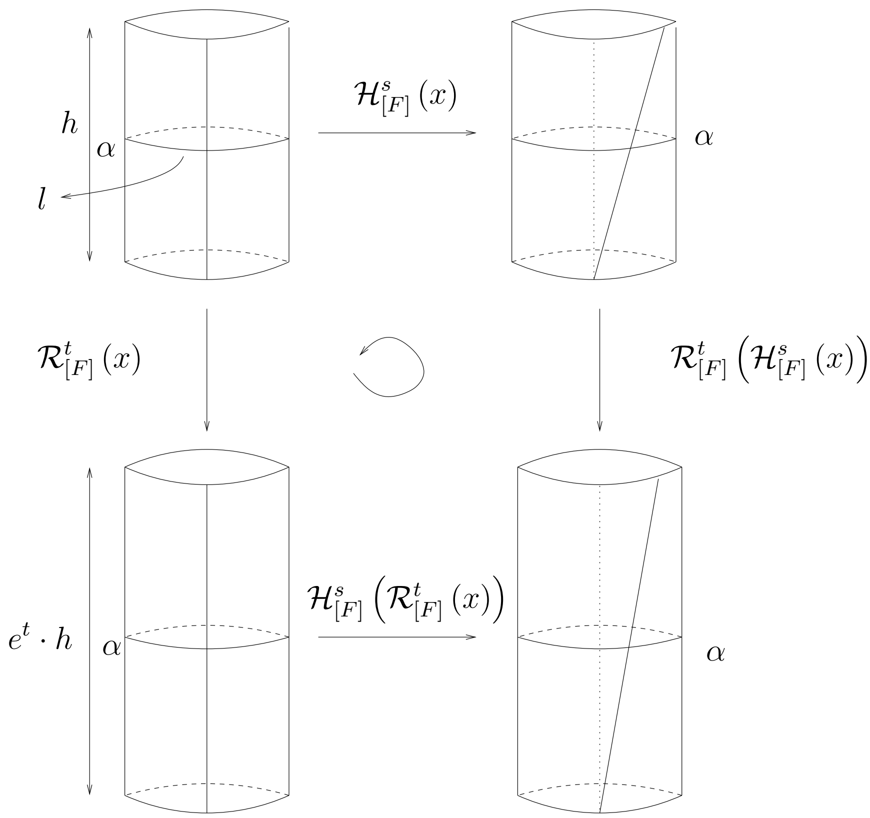


Figure 3.1: Geometric description of a Teichmüller deformation and a horocyclic deformation directed by $\alpha \in \mathcal{S}$. We have to consider the maximal cylinder of core curve α which gives $\operatorname{Ext}_x(\alpha)$. The way we identify boundary components depends on the topology of the underlying surface.

Property 3.9. *Let $F \in \mathcal{MF}$ and $x = [X, f] \in \mathcal{T}(X_0)$. Then*

$$\forall s \in \mathbb{R},\ \operatorname{Ext}_{\mathcal{H}^s_{[F]}(x)}(F) = \operatorname{Ext}_x(F).$$

Proof. First we prove the property in the particular case of simple closed curves. Then we deduce the general case using the continuity of extremal length and the density of $\mathcal{S}$.

Let $\alpha \in \mathcal{S}$ and q_α the corresponding quadratic differential on X. We recall that the horizontal foliation of q_α is exactly $f(\alpha)$. Thus, the complement in X of the corresponding critical graph is biholomorphic to a cylinder A of modulus $M = \dfrac{1}{\operatorname{Ext}_x(\alpha)}$. We can consider that A is the planar annulus of inner radius 1 and outer radius $\exp(2\pi M)$. Now we fix $s \in \mathbb{R}$ and we denote by x_s the image of x by $\mathcal{H}^s_{[\alpha]}$. Following the description given by Marden and Masur in [37], $x_s = [X_s, f_s]$, where f_s is the quasiconformal map which lifts to $\tilde{f}_s : A \to A$; $z \mapsto z|z|^{i\frac{s}{M}}$. We say that the map f_s is the conformal twist along α of parameter t. The surface X_s is obtained by identifying parts of boundary components of A. Therefore, if s is not an integer, then X_s and X are different. As the map $\tilde{f}_s$ does not change the modulus of A and preserves the core curve which is in the class of α, we deduce from the geometric definition of extremal length that

$$\mathrm{Ext}_{x_s}(\alpha) = \mathrm{Ext}_x(\alpha).$$

□

We deduce the following result.

Corollary 3.10. *Let $F \in \mathcal{MF}$ and $x \in \mathcal{T}(X_0)$. Then for any $s,\, t \in \mathbb{R}$*

$$\mathcal{H}^{s+t}_{[F]}(x) = \mathcal{H}^{s}_{[F]}\left(\mathcal{H}^{t}_{[F]}(x)\right).$$

In particular, $\mathcal{H}^{t}_{[F]} : \mathcal{T}(X_0) \to \mathcal{T}(X_0)$ is a bijection.

Proof. On the one hand, Relation (3.5) implies that

$$\mathcal{H}^{s+t}_{[F]}(x) = i - (s+t)\cdot \mathrm{Ext}_x(F).$$

On the other hand, using Property 3.9 and an appropriate conjugation, $\varphi : z \in \mathbb{H} \mapsto z + t \cdot \mathrm{Ext}_x(F)$, we get

$$\begin{aligned}\mathcal{H}^{s}_{[F]}\left(\mathcal{H}^{t}_{[F]}(x)\right) &= \varphi^{-1}\left(\mathcal{H}^{s}_{[F]}\left(\varphi\left(i - t\cdot \mathrm{Ext}_x(F)\right)\right)\right)\\ &= i - (s+t)\,\mathrm{Ext}_x(F).\end{aligned}$$

□

Chapter 4

Convergence towards compactifications

In this chapter, we will focus on the asymptotic behaviour of the horocyclic deformation to the Gardiner-Masur boundary. This is formulated by the following question.

Question 4.1. Does $\left(\mathcal{H}^t_{[F]}(x)\right)_t$ converge towards the Gardiner-Masur boundary as $t \to \pm\infty$?

Looking at Figure 4.1, a naive guess would be that $\left(\mathcal{H}^t_{[F]}(x)\right)_t$ converges and the limit would be exactly the limit of the Teichmüller ray determined by F. This is the case if $\dim_{\mathbb{C}} \mathcal{T}(X_0) = 1$. Indeed, the embedding (3.1) is a homeomorphism and from [45], this homeomorphism can be continuously extended to the boundary. However, Miyachi proved in [50] (Subsection 8.1) that if $\dim_{\mathbb{C}} \mathcal{T}(X_0) \geq 2$, then the embedding $\imath_{(x,[F])}$ does not extend continuously to the Gardiner-Masur boundary. However, as we shall see below that the result holds in at least two particular cases. Moreover, in these two cases corresponding Teichmüller rays (i.e. those with the same direction) also converge to the same point. In Section 4.3, we provide an example of such a deformation which cannot converge to the same limit as the corresponding Teichmüller ray. At this moment we do not know if it converges or not, but we conjecture that it does not converge because the limit point of the corresponding Teichmüller ray is not an element of $\mathcal{PMF}$. We close this chapter by proving that horocyclic deformations converge to the reduced Gardiner-Masur boundary, and the limit is the same as the direction and therefore, it is the same limit as the Teichmüller ray given by the same direction.

4.1 The simple closed curves case

Let $x = [X, f] \in \mathcal{T}(X_0)$ and $\alpha \in \mathcal{S}$. We are interested in the convergence of $\left(\mathcal{H}^t_{[\alpha]}(x)\right)_t$ as $t \to \pm\infty$. We recall that τ_α denotes the Dehn twist along α. To simplify notation we set for any $t \in \mathbb{R}$, $x_t = \mathcal{H}^t_{[\alpha]}(x)$. As we remarked in the proof of Property 3.9, $x_t = [X_t, f_t]$ where f_t is the conformal twist along α of parameter t. We also recall that if $t \in \mathbb{Z}$, then $f_t = f \circ \tau_\alpha^{-t}$.

In order to study the convergence of $(x_t)_t$, we will have to use another result of Miyachi.

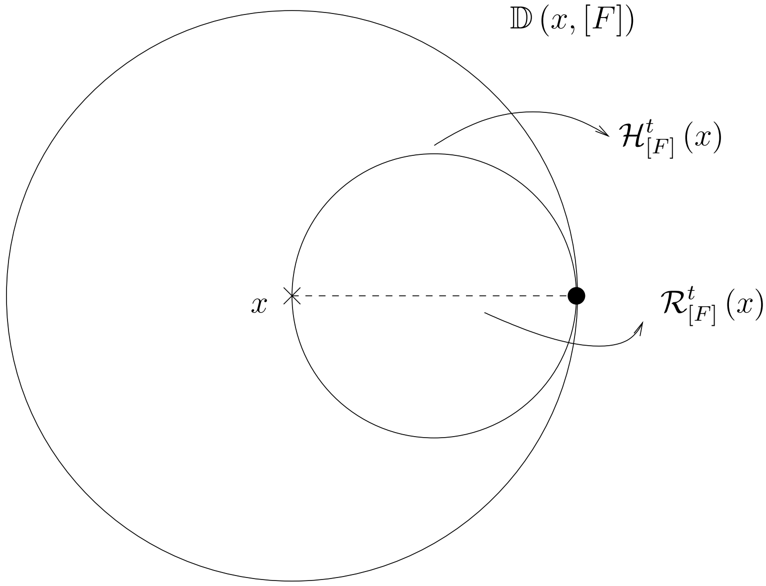


Figure 4.1: The Teichmüller disc $\mathbb{D}(x,[F])$. Points of the circle passing through x are horocyclic deformations of x directed by $[F]$ and the dotted segment represents the Teichmüller deformation of x directed by $[F]$.

Theorem 4.2 ([47], Theorem 3)**.** *Let $F \in \mathcal{MF}$ be either a uniquely ergodic measured foliation or a simple closed curve. Let $p \in \partial_{GM}\mathcal{T}(X_0)$. If for any $G \in \mathcal{N}(F)$, we have $\mathcal{E}_p^{x_0}(G) = 0$, then*

$$\mathcal{E}_p^{x_0}(\cdot) = \frac{1}{\operatorname{Ext}_{x_0}^{\frac{1}{2}}(F)} \cdot i(F,\cdot).$$

We have all the elements to establish the following result.

Theorem 4.3. *With the above notation,*

$$x_t \xrightarrow[t\to\pm\infty]{GM} [\alpha].$$

Proof. Let $p \in \partial_{GM}\mathcal{T}(X_0)$ be any accumulation point of $(x_t)_t$. Up to a subsequence, we can assume that $x_t \xrightarrow[t\to\infty]{GM} p$. By Property 3.9, we already have

$$\begin{aligned}
\mathcal{E}_p^x(\alpha) &= \lim_{t\to\infty} \mathcal{E}_{x_t}^x(\alpha) \\
&= \lim_{t\to\infty}\left(\frac{\operatorname{Ext}_{x_t}(\alpha)}{e^{2\cdot\tanh^{-1}(k_t)}}\right)^{\frac{1}{2}} \\
&= \lim_{t\to\infty}\left(\frac{\operatorname{Ext}_{x}(\alpha)}{e^{2\cdot\tanh^{-1}(k_t)}}\right)^{\frac{1}{2}} \\
&= 0.
\end{aligned}$$

Now, let $G \in \mathcal{MF}$ such that $i(\alpha, G) = 0$. For any $t \in \mathbb{R}$, we have, by the quasiconformal distorsion (or the Kerckhoff formula),

$$\begin{aligned} \mathrm{Ext}_{x_t}(G) &\leq e^{d_T\left(x_{\lfloor t \rfloor}, x_t\right)} \cdot \mathrm{Ext}_{x_{\lfloor t \rfloor}}(G) \\ &\leq e^{d_T\left(x_{\lfloor t \rfloor}, x_{\lceil t \rceil}\right)} \cdot \mathrm{Ext}_X\left(\tau_\alpha^{-\lfloor t \rfloor}(G)\right) \\ &\leq e^{d_T\left(x_{\lfloor t \rfloor}, x_{\lceil t \rceil}\right)} \cdot \mathrm{Ext}_x(G). \end{aligned}$$

Furthermore, the mapping class group acts by isometries with respect to the Teichmüller distance, then $d_T\left(x_{\lfloor t \rfloor}, x_{\lceil t \rceil}\right) = d_T(x, x_1)$. Thus, $\left(\mathrm{Ext}_{x_t}(G)\right)_t$ is bounded from above and we deduce that

$$\mathcal{E}_p^x(G) = \lim_{t \to \infty} \left(\frac{\mathrm{Ext}_{x_t}(G)}{e^{2 \cdot \tanh^{-1}(k_t)}} \right)^{\frac{1}{2}} = 0.$$

The conclusion follows from Theorem 4.2. ☐

This result is analogous to the convergence of Fenchel-Nielsen deformations in the Thurston boundary. Indeed, it is well known that a Fenchel-Nielsen deformation determined by a simple closed curve, converges to this simple closed curve. By the way, from the proof of Theorem 4.3, we can deduce the following corollary.

Corollary 4.4. *Let α be a simple closed curve. Let $x \in \mathcal{T}(X_0)$. Then*

$$\mathcal{H}^t_{[\alpha]}(x) \xrightarrow[t \to \pm\infty]{Th} [\alpha].$$

To prove this corollary, first we have to recall some facts. We recall that the *Thurston asymmetrimetric metric* $d_{\mathrm{Th}}(\cdot, \cdot)$ can be defined as follows:

$$\forall x, y \in \mathcal{T}(X_0), \; d_{Th}(x, y) = \log \sup_{\alpha \in \mathcal{S}} \frac{l_y(\alpha)}{l_x(\alpha)}. \tag{4.1}$$

This metric was introduced by Thurston in [68] and some investigations about it can be found in [59], [63] and [34]. We refer also to [58]. There are some similarities with the Kerckhoff formula. Furthermore, by setting for any $x \in \mathcal{T}(X_0)$,

$$\begin{aligned} \mathcal{L}_x : \mathcal{MF} &\to \mathbb{R}_+ \\ F &\mapsto \frac{l_x(F)}{e^{d_{Th}(x_0, x)}}, \end{aligned}$$

Walsh proved in [70] that a sequence x_n in the Teichmüller space converges to the projective class of G in the Thurston boundary, if and only if, $\mathcal{L}_{x_n}$ converges to $F \in \mathcal{MF} \mapsto C \cdot i(G, F)$ uniformly on compact sets of $\mathcal{MF}$. The constant C depends on x_0 and G (see [70] for more details).

Proof of Corollary 4.4. Let us denote by $(x_t)_t$ the sequence $\left(\mathcal{H}^t_{[\alpha]}(x)\right)_t$. Let $[G] \in \mathcal{PMF}$ be any accumulation point of $(x_t)_t$. By the analytic definition of extremal length and the Gauss-Bonnet formula, we have that for any $\beta \in \mathcal{S}$ and any $t \in \mathbb{R}$,

$$l^2_{x_t}(\beta) \leq 2\pi|\chi(X_0)| \operatorname{Ext}_{x_t}(\beta). \tag{4.2}$$

Thus, for any $\beta \in \mathcal{S}$ such that $i(\alpha, \beta) = 0$, we know from the proof of Theorem 4.3 that $(\operatorname{Ext}_{x_t}(\beta))_t$ is bounded from above. Hence, from (4.2) we have that $(l_{x_t}(\beta))_t$ is also bounded from above. Then, we deduce that

$$\frac{l_{x_t}(\beta)}{e^{d_{Th}(x,x_t)}} \underset{t\to\pm\infty}{\longrightarrow} 0,$$

and by the result of Walsh, we can say that

$$i(G, \beta) = 0.$$

As this equality is true for any simple closed curve whose geometric intersection with α is zero, we deduce that G is topologically the same foliation as α. Therefore G is projectively equivalent to α. This fact is true for any accumulation point of x_t and this completes the proof. □

Remark 4.5. In contrast to the uniquely ergodic case, the author does not know if for a sequence x_n in the Teichmüller space and a simple closed curve α we have

$$x_n \underset{n\to+\infty}{\overset{GM}{\longrightarrow}} [\alpha] \Leftrightarrow x_n \underset{n\to+\infty}{\overset{Th}{\longrightarrow}} [\alpha].$$

We know this property only in a few cases, namely when the sequence is given by the Teichmüller deformation or the horocyclic deformation directed by a simple closed curve. These examples come from the conformal point of view of Teichmüller space, but we can show that it is also true for Fenchel-Nielsen deformations and for stretch lines if the direction or the associated horocyclic foliation is a simple closed curve (see Property 11 in the introduction).

From Theorem 4.3, we also deduce that we can find two sequences y_n and z_n which are at the same distance from x, converge to the same point in the Gardiner-Masur boundary and such that $d_T(y_n, z_n) \to +\infty$. For example consider $y_n = \mathcal{H}^n_{[\alpha]}(x)$ and $x_n = \mathcal{R}^{2\cdot\tanh^{-1}(k_n)}_{[\alpha]}(x)$. As a result we provide an affirmative answer to Question 1 which was about the existence or not of two sequences converging to the same point in the Gardiner-Masur boundary such that the (Teichmüller) distance between them tends to infinity while distances from the base point are asymptotically the same.

4.2 The uniquely ergodic case

Let $x \in \mathcal{T}(X_0)$ and $F \in \mathcal{MF}$ be a uniquely measured foliation. Before studying the asymptotic behaviour of $\left(\mathcal{H}^t_{[F]}(x)\right)_t$ we recall some facts about the *Gromov product*. The Gromov product of y and z with basepoint x for d_T is defined by

$$\langle x \,|\, y\rangle_z = \frac{1}{2}\left(d_T(x,y) + d_T(x,z) - d_T(y,z)\right).$$

Miyachi proved in [51], that the Gromov product at x has a continuous extension to $\overline{\mathcal{T}(X_0)}^{GM} \times \overline{\mathcal{T}(X_0)}^{GM}$ with value in $[0,+\infty]$. He also gave an explicit expression in terms of intersection number (see Proposition 2.7). For any p and q in $\partial_{GM}\mathcal{T}(X_0)$, the Gromov product of p and q with basepoint x is

$$\langle p \,|\, q\rangle_x = -\frac{1}{2}\log\left(i_x(p,q)\right).$$

We recall that if $q = [F] \in \mathcal{PMF}$, then from Relation (2.27) we have

$$\langle p \,|\, [F]\rangle_x = -\frac{1}{2}\log\left(\frac{\mathcal{E}^x_p(F)}{\mathrm{Ext}_x^{\frac{1}{2}}(F)}\right). \tag{4.3}$$

For any $t \in \mathbb{R}$ we set $y_t = \mathcal{H}^t_{[F]}(x)$ and $z_t = \mathcal{R}^{|t|}_{[F]}(x)$. Using the isometric embedding of the disc into Teichmüller space, we have:

$$\langle y_t \,|\, z_t\rangle_x \underset{t\to\infty}{\longrightarrow} +\infty.$$

Then, for any accumulation point p of y_t we have, using Relation (4.3), $\mathcal{E}^x_p(F) = 0$. By Theorem 4.2 and the fact that F is uniquely ergodic, we conclude that $p = [F]$. Thus, we have proved:

Theorem 4.6. *The horocyclic deformation directed by a uniquely ergodic measured foliation F converges in the Gardiner-Masur boundary to the associated projective foliation.*

We deduce from Miyachi's results (Corollary 5.1 in [46] and Theorem 4.2 above) the following equivalence relation: a sequence y_n in Teichmüller space converges to a class of uniquely ergodic measured foliation with respect to the Thurston embedding if and only if it converges to the same class of foliation with respect to the Gardiner-Masur embedding. Thus we have

Corollary 4.7. *Let F be a uniquely ergodic measured foliation and x be a point in $\mathcal{T}(X_0)$. Then*

$$\mathcal{H}^t_{[F]}(x) \underset{t\to\pm\infty}{\overset{Th}{\longrightarrow}} [F].$$

We have seen that the horocyclic deformation directed by a simple closed curve or a uniquely ergodic measured foliation converges in the Gardiner-Masur boundary. In this case, its limit is the same as the limit of the Teichmüller ray directed by the same foliation. Thus, in these two particular cases we have given a positive answer to Question 4.1. In the most general case, one could expect that the horocyclic deformation for a given direction converges to the same limit as the Teichmüller ray with the same direction. However, a negative result is given below. In some sense, it was already observed by Gardiner and Masur in [16] in their proof that $\mathcal{PMF}$ is properly contained in the Gardiner-Masur boundary.

4.3 An example of rational foliation

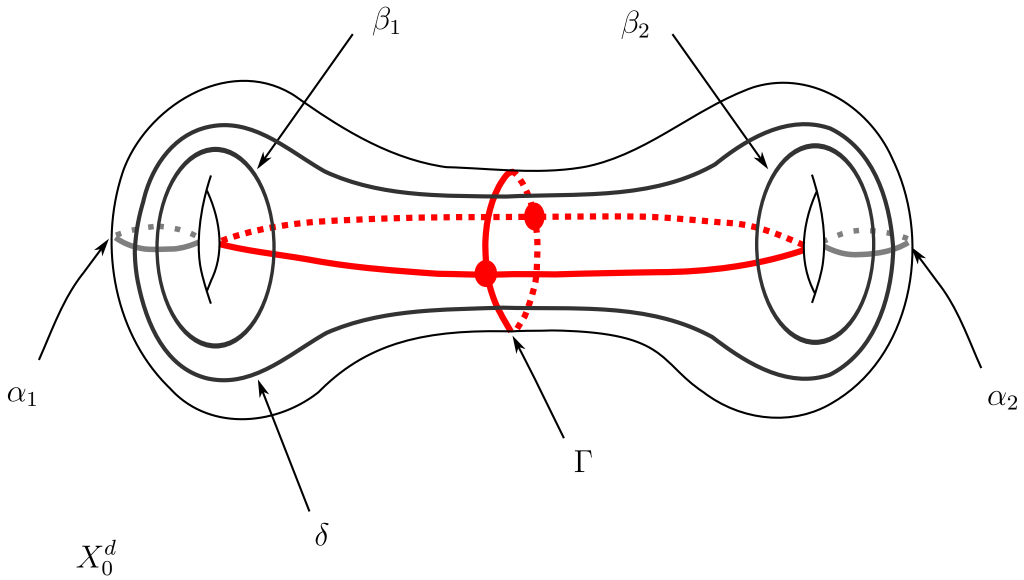


Figure 4.2: On the symmetric Riemann surface X_0^d, we draw the critical graph Γ of the measured foliation $F = \alpha_1 + \alpha_2$. Up to a Whitehead move, we can assume that this measured foliation has two critical points of order 4.

Let X_0^d be a Riemann surface of genus 2 obtained by gluing two tori with one boundary component along their boundary, one of them being the mirror conformal structure of the other. We denote them by T and $\overline{T}$. Thus, we get a natural anti-holomorphic involution on X_0^d which can be seen as a complex conjugation. We denote it by $i_{X_0^d}$. We fix α_1 and α_2, two disjoints simple closed curves as in Figure 4.2, such that α_2 is obtained by conjugating α_1. To be more precise, $\alpha_2 = i^d_{X_0}(\alpha_1)$. Up to changing T, we can assume without loss of generality that

$$\mathrm{Ext}_T(\alpha_1) = \mathrm{Ext}_{\overline{T}}(\alpha_2) = 1. \tag{4.4}$$

We set $F = \alpha_1 + \alpha_2$ and $x_0^d = \left[X_0^d, \mathrm{id}\right]$. Thus, by symmetry, we deduce that the quadratic differential q_F is invariant by $i_{X_0^d}$ and that

$$\mathrm{Ext}_{x_0^d}(F) = \|q_F\| = 2. \tag{4.5}$$

By works of Marden and Masur in Section 2 of [37], we deduce that

$$\forall n \in \mathbb{N},\ \mathcal{H}^{2n}_{[F]}\left(x_0^d\right) = \left(\tau_{\alpha_1}^n \circ \tau_{\alpha_2}^n\right) \cdot x_0^d. \tag{4.6}$$

For the sake of simplicity, we set for any $n \in \mathbb{Z}$, $x_n = \mathcal{H}^{2n}_{[F]}\left(x_0^d\right)$.

Following Kerckhoff's computations in [25] (see precisely §4), Miyachi gave in [46] (see Theorem 6.1) the expression of the limit of Teichmüller rays directed by a rational foliation (see also the appendix of [46], for more details). Thus, in our case, using Equality (4.4) above, we conclude that

$$\mathcal{R}^t_{[F]}\left(x_0^d\right) \xrightarrow[t\to+\infty]{GM} \left[\left(i\left(\alpha_1,\cdot\right)^2 + i\left(\alpha_2,\cdot\right)^2\right)^{\frac{1}{2}}\right] \in \mathrm{P}\mathbb{R}^{\mathcal{S}}_{\geq 0}. \tag{4.7}$$

As Miyachi proved in [46], this limit is not an element of $\mathcal{PMF}$.

Following [16], let us show that any accumulation point of (x_n) in $\partial_{GM}\mathcal{T}(X_0)$ is different from the limit in (4.7). Let $q \in \partial_{GM}\mathcal{T}(X_0)$ be any accumulation point of (x_n). Assume that q is equal to the limit of the Teichmüller ray directed by F. Then there exists $\lambda > 0$ such that

$$\forall \gamma \in \mathcal{S},\ \mathcal{E}_q^{x_0^d}(\gamma)^2 = \lambda \cdot \left(i\left(\alpha_1,\gamma\right)^2 + i\left(\alpha_2,\gamma\right)^2\right). \tag{4.8}$$

However, if β_1, β_2 and δ are as in Figure 4.2, then for any integer n

$$\begin{cases} \mathcal{E}_{x_n}^{x_0^d}(\beta_1)^2 &= \dfrac{1}{4n^2} \cdot \mathrm{Ext}_{X_0^d}\left(\tau_{\alpha_1}^{-n}(\beta_1)\right), \\ \mathcal{E}_{x_n}^{x_0^d}(\beta_2)^2 &= \dfrac{1}{4n^2} \cdot \mathrm{Ext}_{X_0^d}\left(\tau_{\alpha_2}^{-n}(\beta_2)\right), \\ \mathcal{E}_{x_n}^{x_0^d}(\delta)^2 &= \dfrac{1}{4n^2} \cdot \mathrm{Ext}_{X_0^d}\left(\left(\tau_{\alpha_1}^{-n} \circ \tau_{\alpha_2}^{-n}\right)(\delta)\right). \end{cases}$$

Moreover, since $\frac{1}{n} \cdot \tau_{\alpha_i}^{-n}(\beta_i)$ tends to α_i (for $i = 1, 2$) and $\frac{1}{n} \cdot \left(\tau_{\alpha_1}^{-n} \circ \tau_{\alpha_2}^{-n}\right)(\delta)$ tends to F as $n \to \infty$, we deduce by continuity of the extremal length that

$$\begin{cases} \mathcal{E}_q^{x_0^d}(\beta_1)^2 &= \dfrac{1}{4} \cdot \mathrm{Ext}_{X_0^d}(\alpha_1), \\ \mathcal{E}_q^{x_0^d}(\beta_2)^2 &= \dfrac{1}{4} \cdot \mathrm{Ext}_{X_0^d}(\alpha_2), \\ \mathcal{E}_q^{x_0^d}(\delta)^2 &= \dfrac{1}{4} \cdot \mathrm{Ext}_{X_0^d}(F) = \dfrac{1}{2}. \end{cases}$$

By symmetry, we have $\operatorname{Ext}_{X_0^d}(\alpha_1) = \operatorname{Ext}_{X_0^d}(\alpha_1) = c$ and by the geometric definition of extremal length, we necessarily have $c < 1$. Thus, by comparing with Relation (4.8), we get a contradiction.

We just saw that the horocyclic deformation directed by F cannot converge to the same limit as the Teichmüller ray directed by the same foliation. However, the author does not know if in this particular case, the horocyclic deformation converges in the Gardiner-Masur boundary.

4.4 Relation with the reduced Gardiner-Masur compactification

From the previous section, we found an example which conjecturally does not converge to the Gardiner-Masur boundary. An interesting question is the characterization of the limit set of such a deformation. This question, in some sense, is inspired by a result of Lenzhen. In [29], Lenzhen proves that the limit set of a Teichmüller ray directed by a particular foliation is a simplex of dimension one (see Corollary 2 of that paper).

However, there exists a compactification where the previous horocyclic deformation, and even all horocyclic deformations converge to $\partial^{red}_{GM}\mathcal{T}(X_0)$. Indeed,

Theorem 4.8. *Let $F \in \mathcal{MF}$ and $x \in \mathcal{T}(X_0)$. Then*

$$\mathcal{H}^t_{[F]}(x) \xrightarrow[t\to\pm\infty]{GM,red} [F]_{\mathcal{N}_{GM}}.$$

Proof. By definition of horocyclic deformations, this is obvious that the corresponding direction is an associated foliation for any accumulation point of such a sequence. ☐

Another natural question is about the convergence of horocyclic deformations (or Teichmüller deformations) towards $\partial^{red}_{Th}X_0$.

Part III

On the Teichmüller space of a surface with boundary

Chapter 5

A compactification following Gardiner and Masur

In this last chapter, we shall study the (reduced) Teichmüller space of a surface with non-empty boundary. Thus, we shall only consider Riemann surfaces of finite type (g, n, b), which means that all Riemann surfaces shall be surfaces of genus $g \geq 0$ with $n \geq 0$ marked points in the interior and $b > 0$ boundary components. Furthermore, we shall assume that the corresponding Euler characteristic is negative. In other terms, we shall assume that $2 - 2g - n - b < 0$. For each such Riemann surface X, we can define its *mirror* Riemann surface denoted by $\overline{X}$. An atlas for such a surface, which determines a Riemann surface of type (g, n, b), is obtained by composing each local coordinate for X with the conjugation mapping $z \mapsto \overline{z}$. We let $X^d = X \cup \overline{X}$ denote the *double* of X, obtained by gluing X and $\overline{X}$ along their corresponding boundary components. We then obtain a Riemann surface of genus $2g+b-1$ with $2n$ marked points. Moreover, there exists a natural anti-holomorphic mapping on X^d denoted by i_{X^d} which holds $\partial X = \partial\overline{X}$ pointwise fixed, when the boundary is considered as embedded in X^d.

In the rest of this chapter, X_0 shall denote a Riemann surface of type (g, n, b). After introducing the Teichmüller space of X_0, we shall define a compactification using extremal length. This is the analogue of the Gardiner-Masur compactification for surfaces without boundary. The case of surfaces with boundary requires some further work, as we shall see throughout this chapter. We shall close this chapter, and then this thesis, with some questions about this new compactification. Let us add that this chapter can be viewed as a continuation of papers [32, 33] and [6].

5.1 Notation

5.1.1 Measured foliations

We start by recalling a few facts about measured foliations for surface with non-empty boundary. A *measured foliation* on X_0 is a foliation with isolated singularities, endowed with a measure on transverse arcs which is invariant if we slide the arcs along the leaves. Since X_0 has boundary components, we allow the leaves of a given measured foliation to be either transverse or tangent to a boundary component.

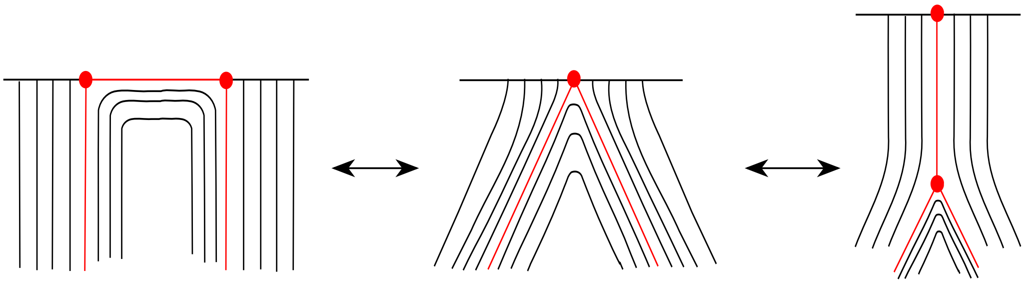


Figure 5.1: Whitehead move in a boundary component.

Two measured foliations are said to be equivalent if they are isotopic (relative to the boundary), up to Whitehead moves (i.e. a move which collapses or creates an arc joining two singular points). Since we deal with surfaces with boundary, we have two kinds of Whitehead moves depending on the place where arcs that join two singular points are. If such an arc is in the interior then the Whitehead move is the same as for surfaces without boundary. If the arc is on the boundary then we collapse such an arc to a point of the boundary and we create an arc joining that point to a point of the interior. An example of a Whitehead move is given in Figure 5.1. We still denote the set of equivalence classes of measured foliations by $\mathcal{MF} = \mathcal{MF}(X_0)$. Actually, using Whitehead moves, we can give the geometric description of measured foliations in the neighborhood of a boundary component. This is given by the following lemma.

Lemma 5.1. *Up to Whitehead moves, in the neighborhood of a boundary component, leaves are either parallel, transverse or parallel with only one transverse singular leave.*

These three cases are respectively called type I, type II and type III and illustrated in Figure 5.2.

Since X_0 has a negative Euler characteristic, we can equip X_0 with a hyperbolic structure, and therefore $\mathcal{MF}$ can be identified with the measured lamination space on X_0 denoted by $\mathcal{ML} = \mathcal{ML}(X_0)$. Such a space has been studied by Alessandrini et al. in [6] where they proved (see Proposition 3.9 of that paper) that $\mathcal{ML}$ is homeomorphic to $\mathbb{R}^{6g-6+2n+3b}$. To prove this, they showed that $\mathcal{ML}$ is homeomorphic to $\mathcal{ML}^{sym}\left(X_0^d\right)$, the space of symmetric measured laminations. We set $\mathcal{MF}^{sym}\left(X_0^d\right)$, called the symmetric measured foliation space, the set corresponding to $\mathcal{ML}^{sym}\left(X_0^d\right)$. We then have a homeomorphism between $\mathcal{MF}$ and $\mathcal{MF}^{sym}\left(X_0^d\right)$.

We define the space $\mathcal{PMF} = \mathcal{PMF}(X_0)$ of projective measured foliations as the quotient of $\mathcal{MF} \setminus \{0\}$ by the action of $\mathbb{R}_+$ on transverse measures. Like in the case without boundary, elements of $\mathcal{PMF}$ shall be denoted by $[F]$, where $F \in \mathcal{MF}$.

Let us add other definitions we shall use in the next section.

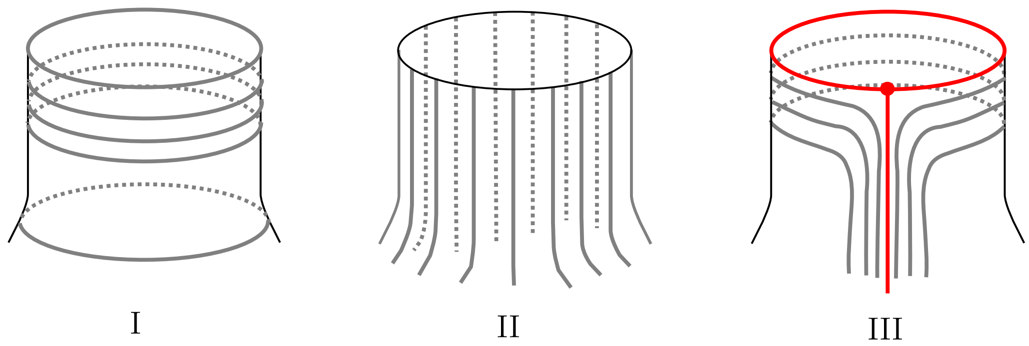


Figure 5.2: The three types of measured foliations in the neighborhood of a boundary component. For the type III, the red graph represents the critical graph.

We let $\mathcal{A} = \mathcal{A}(X_0)$ denote the set of homotopy (relative to ∂X_0) classes of essential arcs on X_0. By "essential," we mean that the arc have endpoints on the boundary of X_0 and cannot be deformed (by a homotopy relative to ∂X_0) to a point of ∂X_0 (see α in Figure 5.3 for an example of an essential arc). For any element $\alpha \in \mathcal{A}$, we can associate an element α^d of $\mathcal{S}\left(X_0^d\right) \cap \mathcal{MF}^{sym}\left(X_0^d\right)$ by setting

$$\alpha^d = \alpha + i_{X_0^d}(\alpha). \tag{5.1}$$

We let $\mathcal{B} = \mathcal{B}(X_0)$ denote the set of homotopy classes of simple closed curves which are homotopic to a boundary component (see β in Figure 5.3 for an example of such a curve). For any element $\beta \in \mathcal{B}$, we can associate β^d, a symmetric weighted simple closed curve, by setting

$$\beta^d = 2 \cdot \beta. \tag{5.2}$$

We let $\mathcal{C} = \mathcal{C}(X_0)$ denote the set of homotopy classes of essential simple closed curves which are not homotopic to a boundary component. We recall that an essential simple closed curve is a simple closed curve which is not homotopic to either a point in the interior or a marked point (see for example γ in Figure 5.3). For any element $\gamma \in \mathcal{C}$, we can associate a symmetric rational measured foliation on X_0^d denoted by γ^d. This measured foliation is defined as follows:

$$\gamma^d = \gamma + i_{X_0^d}(\gamma). \tag{5.3}$$

Any element δ of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ defines a measured foliation. Indeed, we take a foliated cylinder (if $\delta \in \mathcal{B} \cup \mathcal{C}$) or a foliated quadrilateral (if $\delta \in \mathcal{A}$) denoted by C, which is embedded in X_0 and whose leaves are in the free homotopy class of δ. We then collapse the closure of each connected component of $X_0 \setminus C$ in order to obtain a graph that we call the *critical graph* (see Figure 5.4 for an example of such a process). Moreover, the transverse measure of δ is defined as follows. We choose an arc c that joins the two boundary components of C and which is transverse to

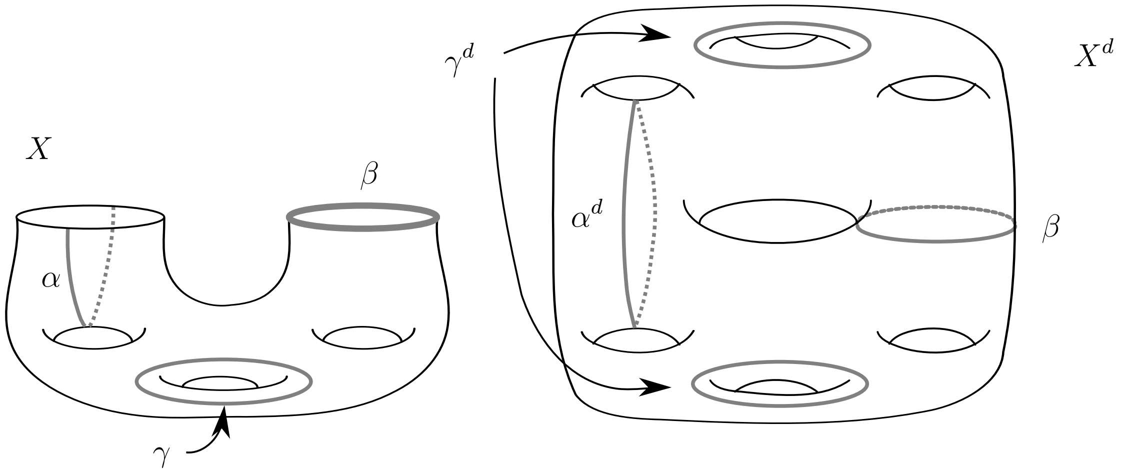


Figure 5.3: On the left hand side, we have three examples of elements in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. On the right hand side, we have their corresponding elements in X^d.

the foliations. We also choose a homeomorphism between this arc and the interval $[0, 1]$ and we take the only invariant transverse measure for the foliation on C that induces the given Lebesgue measure on the arc c.

Furthermore, the *geometric intersection function* is defined as follows:

$$\begin{aligned} i(\cdot, \cdot) : \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \times \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} &\to \mathbb{R}_+ \\ (\delta_1, \delta_2) \qquad\qquad &\mapsto \min \# \left\{ \tilde{\delta_1} \cap \tilde{\delta_1} \right\}, \end{aligned} \tag{5.4}$$

where the minimum is taken over all curves $\tilde{\delta_1}$ and $\tilde{\delta_2}$ in the free homotopy class of respectively δ_1 and δ_2. Furthermore, using the geometric intersection on X_0^d, we easily observe that

$$i(\delta_1, \delta_2) = \frac{1}{2} \cdot i\left(\delta_1^d, \delta_2^d\right). \tag{5.5}$$

A measured foliation F is said to be *rational* if it is determined by a system of positive real numbers $\{w_i\}_{1 \leq i \leq k}$ and a system $\{\delta_i\}_{1 \leq i \leq k}$ of disjoint elements of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ such that

$$\forall \gamma \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}, i(F, \gamma) = \sum_{i=1}^{k} w_i \cdot i(\delta_i, \gamma).$$

We shall write for such a foliation

$$F = \sum_{i=1}^{k} w_i \cdot \delta_i.$$

The set of rational measured foliations shall be denoted by $\mathcal{RMF}$. One of the particularities of this set is given by the following proposition.

Proposition 5.2 ([6], Lemma 3.2)**.** *The set of rational measured foliations on X_0 is dense in $\mathcal{MF}$.*

Let us sketch a proof.

Proof. Let $F \in \mathcal{MF}$. By using the natural anti-holomorphic mapping, we can define F^d, a symmetric measured foliation on X_0^d. Such a symmetric foliation is associated with a symmetric weighted train track. Using the density of $\mathbb{Q}$, we can find a sequence of symmetric weighted train tracks which converges to that train track and whose weights are rational. Such a sequence corresponds to a sequence of symmetric rational measured foliations on X_0, and therefore by taking the corresponding part in X_0 we complete the proof. □

Moreover, Lemma 2.14 of [33] says that any element of $\mathcal{C}$ is a limit of elements of $\mathcal{A}$ and therefore, by Proposition 5.2 we can consider only rational measured foliation without elements of $\mathcal{C}$. We denote the space of such rational foliations by $\mathcal{A}^{\mathfrak{m}} \cup \mathcal{B}^{\mathfrak{m}}$.

Lastly, like for the case without boundary, the geometric intersection extends continuously to a symmetric function from $\mathcal{MF} \times \mathcal{MF} \to \mathbb{R}_+$ and satisfies for any $(F, G) \in \mathcal{MF} \times \mathcal{MF}$,

$$i\left(F,G\right) = \frac{1}{2} \cdot i\left(F^d, G^d\right), \tag{5.6}$$

where F^d (resp. G^d) denotes the corresponding symmetric measured foliation on X_0^d.

We shall see below that any measured foliation arises from a quadratic differential.

5.1.2 Quadratic differentials

An *admissible quadratic differential* (or, simply, quadratic differential since there will be no confusion) q on X_0 is locally the data of $q = q\left(z\right) dz^2$ such that $q\left(z\right)$ is meromorphic in the interior X_0 with at most poles of order 1 at marked points. Moreover, we assume that q is real along the boundary components.

We keep the same notation as previously. The space of such quadratic differentials is denoted by $\mathcal{Q}\left(X_0\right)$ and is equipped with the L^1-norm (see Formula (2.4) for the definition). It is known that $\mathcal{Q}\left(X_0\right)$ is homeomorphic (and isomorphic as vector space) to the space of quadratic differentials on X_0^d which are symmetric (i.e. invariant by the natural action of $i_{X_0^d}$). We denote this space by $\mathcal{Q}^{sym}\left(X_0^d\right)$. Thus, $\mathcal{Q}\left(X_0\right)$ is a (real) vector space of dimension $6g-6+2n+3b$. Furthermore, fof any quadratic differential q, one can also associate a singular flat metric, and a pair of transverse measured foliations called vertical and horizontal foliation. These measured foliations are still denoted by $F_{v,q}$ and $F_{h,q}$. Let us note that from the assumption on the boundary, the vertical (or horizontal) foliations are either perpendicular or tangent to the boundary. Using the doubling process we deduce the Hubbard-Masur theorem (see Theorem 2.1 for the statement) for the case with boundary.

Theorem 5.3. *Let $F \in \mathcal{MF}$. Then there exists a unique quadratic differential whose corresponding horizontal foliation is, up to Whitehead moves, equal to F. We denote this quadratic differential by q_F.*

Actually, since $\mathcal{Q}(X_0)$ is homeomorphic to $\mathcal{Q}^{sym}\left(X_0^d\right)$, which itself is homeomorphic to $\mathcal{MF}^{sym}\left(X_0^d\right)$, we deduce that

$$\mathcal{Q}(X_0) \ni q \mapsto F_{h,q} \in \mathcal{MF} \tag{5.7}$$

is a homeomorphism.

Moreover, we have the following useful observations. Let $q \in \mathcal{Q}(X_0)$. We denote by q^d the corresponding element in $\mathcal{Q}^{sym}\left(X_0^d\right)$. Then,

$$\begin{cases} \|q\| &= \frac{1}{2} \cdot \|q^d\|, \\ F_{v,q^d} &= (F_{v,q})^d, \\ F_{h,q^d} &= (F_{h,q})^d. \end{cases} \tag{5.8}$$

We shall recall that the space of quadratic differential gives a parametrization of Teichmüller space.

5.1.3 Teichmüller space

We shall use the same notation as for surfaces without boundary.

We say that (X_1, f_1) and (X_2, f_2), where $f_i : X_0 \to X_i$ ($i = 1, 2$) is a quasiconformal homeomorphism, are equivalent if there exists a conformal map $h : X_1 \to X_2$ which is homotopic (relative to the boundary) to $f_2 \circ f_1^{-1}$. The *reduced Teichmüller space* of X_0, denoted by $\mathcal{T}(X_0)$, is the set of equivalence classes of pairs (X, f). For a pair (X, f), we denote the corresponding point in $\mathcal{T}(X_0)$ by $[X, f]$ and we call $x_0 = [X_0, \mathrm{id}]$ the base point of $\mathcal{T}(X_0)$. We shall omit the term "reduced."

This space can be equipped with the so-called Teichmüller metric. We still denote this metric by d_T. The definition is essentially the same as in (2.5). The term "essentially" means that homotopy must be relative to the boundary. Moreover, we can define in the same way, Teichmüller maps (see (2.7)) and Teichmüller rays (see (2.8)) and obtain the Teichmüller theorem (see (2.9)). We shall use for these tools the same notations as Parts I and II. The space $\mathcal{T}(X_0)$ is then homeomorphic to $\mathcal{Q}(X_0)$.

The reference for the (reduced) Teichmüller space is the book [1], written by Abikoff. However, it seems interesting to remark that Teichmüller had already observed classical results on that space (see in particular Chapter 20 of [61]) and promised in [62] to write the correct proof of the Teichmüller theorem for surfaces with boundary. Since he died in 1943, this promise was broken. For more details, we refer also to [9, 10] and [2].[1]

Moreover, there is a canonical isometric embedding from $\mathcal{T}(X_0)$ to $\mathcal{T}\left(X_0^d\right)$ which is given by what is called the *process of doubling*. Let us give more details. Let $x = [X, f]$ be a point in $\mathcal{T}(X_0)$. We set $f^d : X_0^d \to X^d$ such that

$$f^d_{|X_0} = f \text{ and } f^d_{|\overline{X_0}} = i_{X^d} \circ f \circ i_{X_0^d}.$$

We set $x^d = \left[X^d, f^d\right]$ and we verify that it is well-defined. Thus, we have the map

$$\begin{aligned} \imath : \mathcal{T}(X_0) &\to \mathcal{T}\left(X_0^d\right) \\ x &\mapsto x^d \end{aligned} \tag{5.9}$$

which is known to be an isometric embedding. In particular, geodesics in $\mathcal{T}(X_0)$ are still geodesics after the process of doubling.

Let us add that Teichmüller space of a surface with boundary has already been studied

5.2 Extremal length geometry

As we already pointed out, the extremal length geometry on Teichmüller space of surface without boundary is based on the Kerckhoff formula. Hence, after recalling the notion of extremal length we shall prove the "Kerckhoff formula" for surface with non-empty boundary.

5.2.1 The Kerckhoff formula

In order to state the Kerckhoff formula we shall extend the notion of extremal length to a suitable dense subset of measured foliations. This set shall be $\mathcal{A}^{\mathfrak{m}} \cup \mathcal{B}^{\mathfrak{m}}$. Althought we have a well-defined notion of extremal length for elements in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ or even in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, definitions given above (see Relations (2.10) and (2.11)) do not work for rational measured foliations and therefore for $\mathcal{A}^{\mathfrak{m}} \cup \mathcal{B}^{\mathfrak{m}}$. Indeed, we must pay attention to weights. We then define:

Definition 5.4. Let $x = [X, f]$ be a point in $\mathcal{T}(X_0)$ and $\delta \in \mathcal{A}^{\mathfrak{m}} \cup \mathcal{B}^{\mathfrak{m}}$. Then there exist a finite set of real positive numbers $\{w_i\}_{1 \le i \le k}$ and a system $\{\delta_i\}_{1 \le i \le k}$ of disjoint elements of $\mathcal{A} \cup \mathcal{B}$ such that

$$\delta = \sum_{i=1}^{k} w_i \cdot \delta_i.$$

[1] We have to pointed out that Ahlfors used a "variational proof" for the so-called *Teichmüller existence theorem* and according to him, this proof has a "flaw, and [...] does not convince me today."

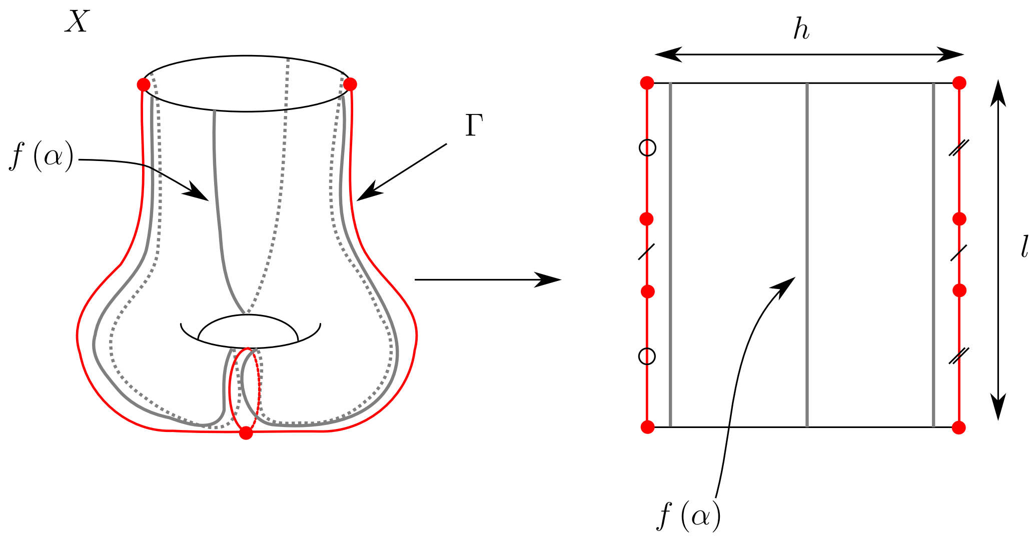


Figure 5.4: An example of quadrilateral which realizes the infimum in (5.10). If $x = [X, f] \in \mathcal{T}(X_0)$ and $\alpha \in \mathcal{A}$ then $\mathrm{Ext}_x(\alpha) = \frac{l}{h}$. To see this, we cut X along Γ and therefore $X - \Gamma$ is biholomorphic to that rectangle. Moreover, this quadrilateral shows the way to obtain from an element of $\mathcal{A}$ a foliation on X.

The extremal length of δ on x is

$$\mathrm{Ext}_x(\alpha) = \inf \sum_{i=1}^{k} \frac{w_i^2}{\mathrm{Mod}(A_i)}, \tag{5.10}$$

where the infimum is taken over all disjoint Euclidean cylinders or rectangles A_i that can be conformally embedded into X and whose image of the core curve by this embedding is in the class $f(\delta_i)$. We recall that the modulus of a Euclidean rectangle is the ratio between the height and the length of the core curve (see also Figure 5.4).

Remark 5.5. It is easy to see that Definition 5.10 is a "generalization" of the geometric definition of extremal length given in (2.11). However, it would be interesting to generalize also the analytic definition given by (2.10).

Moreover, it is well known that if $f : X_0 \to X$ is a quasiconformal mapping whose quasiconformal dilatation is equal to K, then for any modulus or quadrilateral A of X_0 we have

$$\frac{1}{K} \cdot \mathrm{Mod}(A) \le \mathrm{Mod}(f(A)) \le K \cdot \mathrm{Mod}(A),$$

and therefore

$$\forall x, y \in \mathcal{T}(X_0), \forall \delta \in \mathcal{A}^{\mathrm{m}} \cup \mathcal{B}^{\mathrm{m}}, \ \mathrm{Ext}_y(\delta) \le e^{d_{\mathcal{T}}(x,y)} \cdot \mathrm{Ext}_x(\delta). \tag{5.11}$$

Using once again the process of doubling and works of Jenkins, Strebel and others (see for example [22] and §21 of Chapter VI in [60]) we can show that the infimum is realized by a singular flat metric. We also refer to Subsection 2.3.3 of [49]. More precisely.

Proposition 5.6. *Let $x = [X, f] \in \mathcal{T}(X_0)$ and $\delta \in \mathcal{RMF}$. Then*

$$\mathrm{Ext}_x(\delta) = \|q_{x,\delta}\|,$$

where $q_{x,\delta} \in \mathcal{Q}(X)$ denotes the unique quadratic differential which satisfies

$$F_{h,q_{x,\delta}} = f(\delta).$$

We then obtain the following result which is analogous to one of Kerckhoff (see Relation (2.12)).

Corollary 5.7. *Let $x \in \mathcal{T}(X_0)$. Then the extremal length extends continuously to $\mathcal{MF}$ and we have for any $F \in \mathcal{MF}$*

$$\mathrm{Ext}_x(F) = \|q_{x,F}\|.$$

Proof. This follows from the homeomorphism (5.7). □

The first consequence of that corollary is the following.

$$\forall x \in \mathcal{T}(X_0), \forall F \in \mathcal{MF},\ \mathrm{Ext}_x(F) = \frac{1}{2} \cdot \mathrm{Ext}_{x^d}\left(F^d\right). \tag{5.12}$$

We now have all the ingredients to state the Kerckhoff formula.

Theorem 5.8. *Let $x, y \in \mathcal{T}(X_0)$. Then*

$$d_T(x, y) = \log \sup_{\delta \in \mathcal{A}^{\mathfrak{m}} \cup \mathcal{B}^{\mathfrak{m}}} \frac{\mathrm{Ext}_y(\delta)}{\mathrm{Ext}_x(\delta)}.$$

Proof. We just have to follow the strategy used by Kerckhoff in [25]. Let x and y be two points of $\mathcal{T}(X_0)$. From (5.11), we obtain a first inequality which is

$$\log \sup_{\delta \in \mathcal{A}^{\mathfrak{m}} \cup \mathcal{B}^{\mathfrak{m}}} \frac{\mathrm{Ext}_y(\delta)}{\mathrm{Ext}_x(\delta)} \leq d_T(x, y). \tag{5.13}$$

Moreover, there exists a unique $q \in \mathcal{Q}(X_0)$ such that y is obtained from x by a Teichmüller deformation of x with direction $F_{h,q}$. Since the map $\imath$ defined in (5.9) is an isometry, we have that y^d is obtained from x^d by a Teichmüller deformation of direction F_{h,q^d}. From (2.14), (5.8), (5.12) and Proposition 5.2, we obtain the existence of a sequence $(\delta_n) \subset \mathcal{A}^{\mathsf{m}} \cup \mathcal{B}^{\mathsf{m}}$ such that

$$\begin{aligned} d_T(x,y) &= d_T\left(x^d, y^d\right) \\ &= \log \frac{\operatorname{Ext}_{y^d}\left(F_{v,q^d}\right)}{\operatorname{Ext}_{x^d}\left(F_{v,q^d}\right)} \\ &= \log \frac{\operatorname{Ext}_{y^d}\left((F_{v,q})^d\right)}{\operatorname{Ext}_{x^d}\left((F_{v,q})^d\right)} \\ &= \log \frac{\operatorname{Ext}_{y}\left((F_{v,q})\right)}{\operatorname{Ext}_{x}\left((F_{v,q})\right)} \\ &= \lim_{n\to+\infty} \log \frac{\operatorname{Ext}_y(\delta_n)}{\operatorname{Ext}_x(\delta_n)}, \end{aligned}$$

and therefore

$$d_T(x,y) \leq \log \sup_{\delta \in \mathcal{A}^{\mathsf{m}} \cup \mathcal{B}^{\mathsf{m}}} \frac{\operatorname{Ext}_y(\delta)}{\operatorname{Ext}_x(\delta)}. \tag{5.14}$$

By using the two above relations, we conclude the proof. □

Another simple generalization of the case without boundary is the Minsky inequality. Indeed, using (5.6) we have for any $x \in \mathcal{T}(X_0)$ and any pair $(F,G) \in \mathcal{MF} \times \mathcal{MF}$ the following:

$$\begin{aligned} i(F,G) &= \frac{1}{2} \cdot i\left(F^d, G^d\right) \\ &\leq \frac{1}{2} \cdot \operatorname{Ext}_{x^d}^{\frac{1}{2}}\left(F^d\right) \operatorname{Ext}_{x^d}^{\frac{1}{2}}\left(G^d\right) \end{aligned}$$

and therefore, once again by using (5.12),

$$i(F,G) \leq \operatorname{Ext}_x^{\frac{1}{2}}(F) \cdot \operatorname{Ext}_x^{\frac{1}{2}}(G). \tag{5.15}$$

5.2.2 A compactification of $\mathcal{T}(X_0)$

We shall give here a compactification of $\mathcal{T}(X_0)$ using extremal length. We shall follow exactly the same strategy as Gardiner and Masur used in [16].

Consider the mapping

$$\tilde{\Psi}_{GM} : x \in \mathcal{T}(X_0) \mapsto \operatorname{Ext}_x^{\frac{1}{2}}(\cdot) \in \mathbb{R}_{\geq 0}^{\mathcal{A}^{\mathsf{m}} \cup \mathcal{B}^{\mathsf{m}}}. \tag{5.16}$$

By Proposition 5.6, for any $x \in \mathcal{T}(X_0)$, we have $\tilde{\Psi}_{GM}(x) \in \mathbb{R}_{>0}^{\mathcal{A}^{\mathsf{m}} \cup \mathcal{B}^{\mathsf{m}}}$ and therefore we can define the map

$$\Psi_{GM} : x \in \mathcal{T}(X_0) \mapsto \left[\operatorname{Ext}_x^{\frac{1}{2}}(\cdot)\right] \in P\mathbb{R}_{\geq 0}^{\mathcal{A}^{\mathsf{m}} \cup \mathcal{B}^{\mathsf{m}}}, \tag{5.17}$$

where $\mathrm{P}\mathbb{R}_{\geq 0}^{\mathcal{A}^{\mathfrak{m}}\cup\mathcal{B}^{\mathfrak{m}}} = \mathbb{R}_{\geq 0}^{\mathcal{A}^{\mathfrak{m}}\cup\mathcal{B}^{\mathfrak{m}}} \setminus \{0\} \big/ \mathbb{R}_+$. We denote the projection map from $\mathbb{R}_{\geq 0}^{\mathcal{A}^{\mathfrak{m}}\cup\mathcal{B}^{\mathfrak{m}}}$ to $\mathrm{P}\mathbb{R}_{\geq 0}^{\mathcal{A}^{\mathfrak{m}}\cup\mathcal{B}^{\mathfrak{m}}}$ by $\mathfrak{pr}$.

The main goal of this subsection is to show the following theorem.

Theorem 5.9. *The image of Teichmüller space by Ψ_{GM} is relatively compact.*

To prove it we shall need three lemmas.

Lemma 5.10. *The map Ψ_{GM} is injective.*

Proof. Let x_1 and x_2 be two points of $\mathcal{T}(X_0)$ such that $\Psi_{GM}(x_1) = \Psi_{GM}(x_2)$. Then there exists a strictly positive number t such that

$$\forall \delta \in \mathcal{A}^{\mathfrak{m}} \cup \mathcal{B}^{\mathfrak{m}},\ \mathrm{Ext}_{x_1}(\delta) = t \cdot \mathrm{Ext}_{x_2}(\delta)\,.$$

Thus, from Theorem 5.8 we deduce that $d_T(x_1, x_2) = \log t$. However, the Teichmüller distance is symmetric and therefore $t = 1$. The proof is complete. □

The second lemma is the same as Lemma 6.1 of [16]. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_N\}$ be a system of arcs which fill X_0 in the following sense

$$\forall F \in \mathcal{MF} \setminus \{0\}\,,\ \sum_{j=1}^{N} i(F, \alpha_j) > 0. \tag{5.18}$$

Let us note that the doubling of such a system fill up X_0^d.

Lemma 5.11. *There exists $c > 0$ such that for any $q \in \mathcal{Q}(X_0)$ of norm 1 there exists $j \in \{1, \cdots, N\}$ such that*

$$i(F_{h,q}, \alpha_j) \geq c.$$

The constant only depends on the set $\{\alpha_i\}_{1\leq j\leq N}$.

Proof. We argue by contradiction. Let us assume that there exists a sequence (q_n) of quadratic differentials of norm 1 such that

$$\forall 1 \leq j \leq N,\ i(F_{h,q_n}, \alpha_j) \underset{n\to+\infty}{\longrightarrow} 0.$$

Since the set of quadratic differentials of norm 1 is compact, we can assume, up to a subsequence, that (q_n) converges to q. Using the continuity of intersection number we deduce that for any $1 \leq j \leq N$,

$$i(F_{h,q}, \alpha_j) = 0.$$

Since q has a norm equal to 1, this equality contradicts Relation (5.18). □

The third lemma also appears in [16] but the proof we give is slightly different. Indeed, Gardiner and Masur use the analytic definition of the extremal length.

Lemma 5.12. *Let $\delta \in \mathcal{A}^{\mathrm{m}} \cup \mathcal{B}^{\mathrm{m}}$. Then there exists $D = D(c,\delta) > 0$ such that*

$$\forall x \in \mathcal{T}(X_0)\,,\ \mathrm{Ext}_x(\delta) \leq D \cdot \max_{1\leq j\leq N} \mathrm{Ext}_x(\alpha_j)\,.$$

Proof. Let $x = [X,f] \in \mathcal{T}(X_0)$. We recall that $x_0 = [X_0, \mathrm{id}]$ denotes the basepoint of Teichmüller space. Since $\mathcal{T}(X_0)$ is (uniquely) geodesic, there exist $t_0 \geq 0$ and $[F] \in \mathcal{PMF}$ such that

$$x = \mathcal{R}^{t_0}_{[F]}(x_0)\,.$$

Since we only consider the projective class of F, we can assume that F satisfies

$$\mathrm{Ext}_{x_0}(F) = 1$$

and therefore,

$$\mathrm{Ext}_x(F) = e^{-t_0}.$$

Using Inequality (5.15) and the relations above we have for $1 \leq j \leq N$:

$$\begin{aligned}\mathrm{Ext}_x(\alpha_j) &\geq \frac{i(F,\alpha_j)^2}{\mathrm{Ext}_x(F)}\\ &= \frac{i(F,\alpha_j)^2}{e^{-t_0}}\\ &= e^{t_0} i(F,\alpha_j)^2\,.\end{aligned}$$

Using Lemma 5.11 we deduce that

$$\max_{1\leq j\leq N} \mathrm{Ext}_x(\alpha_j) \geq e^{t_0}c^2 > 0,$$

and therefore,

$$\begin{aligned}\mathrm{Ext}_x(\delta) &\leq e^{t_0} \cdot \mathrm{Ext}_{x_0}(\delta)\\ &\leq e^{t_0}\,\mathrm{Ext}_{x_0}(\delta)\,\frac{\max_{1\leq j\leq N}\mathrm{Ext}_x(\alpha_j)}{e^{t_0}c^2}\\ &= \frac{\mathrm{Ext}_{x_0}(\delta)}{c^2}\max_{1\leq j\leq N}\mathrm{Ext}_x(\alpha_j)\,.\end{aligned}$$

In order to complete the proof, we just set $D = \dfrac{\mathrm{Ext}_{x_0}(\delta)}{c^2}$. □

We have now all the ingredients to conclude.

Proof of Theorem 5.9. We consider the mapping

$$\begin{aligned}\Theta_{GM} : \mathcal{T}(X_0) &\to \mathbb{R}^{\mathcal{A}^{\mathrm{m}}\cup\mathcal{B}^{\mathrm{m}}}_{\geq 0}\\ x &\mapsto \frac{1}{\max_{1\leq j\leq N}\mathrm{Ext}_x^{\frac{1}{2}}(\alpha_j)}\,\mathrm{Ext}_x^{\frac{1}{2}}(\cdot)\,.\end{aligned}$$

This map satisfies the following relation:

$$\Phi_{GM} = \mathfrak{pr} \circ \Theta_{GM}. \tag{5.19}$$

Let us fix $\delta \in \mathcal{A}^{\mathfrak{m}} \cup \mathcal{B}^{\mathfrak{m}}$. Lemma 5.12 implies that $\Theta_{GM}(\mathcal{T}(X_0))(\delta) \subset [0, D]$, which is a compact set, and then by Tychonoff's theorem we deduce that $\Theta_{GM}(\mathcal{T}(X_0))$ is relatively compact. Since we consider the quotient topology on $\mathrm{P}\mathbb{R}_{\geq 0}^{\mathcal{A}^{\mathfrak{m}} \cup \mathcal{B}^{\mathfrak{m}}}$, we conclude that $\Phi_{GM}(\mathcal{T}(X_0))$ is relatively compact. $\square$

We then define the *Gardiner-Masur compactification* as the closure $\overline{\Psi_{GM}(\mathcal{T}(X_0))}$, and we denote it by $\overline{\mathcal{T}(X_0)}^{GM}$.

5.3 Problems

This compactification leads to generalizations of known results in the case with boundary. Unfortunately, since each thesis has to be written in a finite time, we collected these generalizations as a list of problems.

Problem I The first generalization concerns characterisation of boundary points. Indeed, we would like to say that any point $p \in \partial_{GM}\mathcal{T}(X_0)$ corresponds to a unique continuous function $\mathcal{E}_p^{x_0} : \mathcal{MF} \to \mathbb{R}_+$. In other terms, we want to state a theorem analogous to Theorem 2.5. Since Miyachi uses the Minsky inequality, an inequality which is still valid by Relation (5.12), this problem seems to be reasonably solvable.

If this characterisation is true, then a natural question arises. Indeed, it concerns the continuous extension of $\imath : \mathcal{T}(X_0) \to \mathcal{T}\left(X_0^d\right)$ to boundaries. The main problem is that there is no reason why a sequence which converges toward a point of $\partial_{GM}\mathcal{T}(X_0)$ also converges to a point of $\partial_{GM}\mathcal{T}\left(X_0^d\right)$. Up to characterisation, we are able to only prove that if $y_n \xrightarrow{GM} p$, then any accumulation point of y_n^d coincides on $\mathcal{MF}^{sym}\left(X_0^d\right)$.

Problem II The other generalisation concerns the notion of the intersection number. As we already pointed out, in the case without boundary, Miyachi extends the geometric intersection number by using the Minsky inequality. We can therefore use the same strategy for the case with boundary.

Problem III Another problem, which seems to be for the moment the most difficult, is about the comparison between $\mathcal{PMF}$ and $\partial_{GM}\mathcal{T}(X_0)$. Indeed, it seems reasonable that a Teichmüller ray converges towards the Gardiner-Masur boundary, but if the direction is an element of $\mathcal{A}^{\mathfrak{m}} \cup \mathcal{B}^{\mathfrak{m}}$ and if the map $\imath$ extends continuously then the potential limit point is not an element of $\mathcal{PMF}$. Thus, the strategy of Gardiner and Masur cannot be used.

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Il faut être toujours ivre. Tout est là: c'est l'unique question. Pour ne pas sentir l'horrible fardeau du Temps qui brise vos épaules et vous penche vers la terre, il faut vous enivrer sans trêve.
Mais de quoi? De vin, de poésie, ou de vertu, à votre guise, mais enivrez-vous.
Et si quelquefois, sur les marches d'un palais, sur l'herbe verte d'un fossé, dans la solitude morne de votre chambre, vous vous réveillez, l'ivresse déjà diminuée ou disparue, demandez au vent, à la vague, à l'étoile, à l'oiseau, à l'horloge; à tout ce qui fuit, à tout ce qui gémit, à tout ce qui roule, à tout ce qui chante, à tout ce qui parle, demandez quelle heure il est. Et le vent, la vague, l'étoile, l'oiseau, l'horloge, vous répondront: "il est l'heure de s'enivrer; pour ne pas être les esclaves martyrisés du Temps, enivrez-vous; enivrez-vous sans cesse! De vin, de poésie ou de vertu, à votre guise."

—Charles Baudelaire, *Les petits poèmes en proses*

Vincent ALBERGE

Géométrie de la longueur extrémale sur les espaces de Teichmüller

Résumé

Dans ce travail nous nous intéressons à la géométrie de l'espace de Teichmüller via la longueur extrémale et à sa relation avec d'autres géométries. En effet, via le théorème d'uniformisation de Poincaré, l'espace de Teichmüller d'une surface orientable de type finie est un espace qui "classifie" aussi bien les structures hyperboliques de cette surface que les structures conformes. Suivant la classification utilisée, on obtient deux compactifications différentes de cet espace, qui sont respectivement la compactification de Thurston et la compactification de Gardiner-Masur. La première étant induite par la longueur hyperbolique et la deuxième par la longueur extrémale. Dans une première partie, on considère les compactifications dites "réduites" de Thurston et Gardiner-Masur. On montre qu'il existe une bijection naturelle entre les deux et que le groupe des auto-homéomorphismes du bord réduit de Thurston est canoniquement isomorphe au groupe modulaire étendu de la surface sous-jacente. Dans une deuxième partie, on étudie la convergence de certaines déformations de structures conformes aussi bien sur le bord de Thurston que sur celui de Gardiner-Masur. Ces déformations, appelées déformations horocycliques, sont un analogue des tremblements de terre de structures hyperboliques. Enfin, dans une troisième et dernière partie, on introduit une compactification à la Gardiner-Masur de l'espace de Teichmüller d'une surface à bord. On généralise des résultats obtenus dans le cas sans bord, et on établit quelques différences.

Mots clés : espace de Teichmüller – longueur extrémale – compactification de Thurston – compactification de Gardiner-Masur – déformation horocyclique – compactification réduite.

Résumé en anglais

In this thesis we are interested in the extremal length geometry of Teichmüller space and the links with other geometries. In particular, we work on two different compactifications of Teichmüller space, namely, the Thurston compactification and the Gardiner-Masur compactification. In the first part, we consider the so-called reduced compactifications of Thurston and Gardiner-Masur. We show that there exists a canonical bijection between them and that the group of self-homeomorphisms of the reduced Thurston boundary is canonicaly isomorphic (except for a few cases) to the extended mapping class group of the corresponding surface. In the second part, we study the asymptotic behaviour of some conformal structure deformations to the Thuston boundary and to the Gardiner-Masur boundary. These deformations are called horocyclic deformations and they are analogous to earthquakes of hyperbolic structures. Finally, in the last part, using extremal length we extend the notion of Gardiner-Masur compactification to surfaces with non-empty boundary, and we investigate differences with the case without boundary.

Keywords : Teichmüller space – extremal length – Thurston compactification – Gardiner-Masur compactification – horocyclic deformations – reduced compactification.