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Sur l'aire et le volume en géométrie sphérique et hyperbolique

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1 Introduction (en français)

L'essence de cette thèse se situe dans l'étude de la transition entre la géométrie sphérique et la géométrie hyperbolique, au niveau de certains théorèmes. Notre but était de prouver quelques théorèmes en géometrie hyperbolique, en s'appuyant sur les méthodes d'Euler, Schubert et de Steiner en géométrie sphérique. Ces théorèmes sont intéressants pour eux-mêmes mais aussi pour comprendre comment les méthodes des preuves se transmettent d'une géométrie à l'autre.
Il faut se rappeler à ce propos que même si la géométrie sphérique et hyperbolique font toutes les deux partie de ce qu'on appelle "géométrie non-Euclidienne", ces deux géométries sont différentes et on ne peut pas toujours espérer que les théorèmes de l'une ont un analogue dans l'autre. Par exemple, sur la sphère, deux géodésiques se rencontrent toujours en deux points, et elles sont de longueur finie. C'est une propriété que l'on utilise souvent dans les preuves des théorèmes sur la sphère, et elle n'a evidemment pas d'analogue dans le plan hyperbolique.
Nous commençons par des considérations basées sur deux mémoires d'Euler. Dans le premier mémoire [6], Euler donne les preuves d'un ensemble complet de formules trigonométriques sphériques pour les triangles rectangles en utilisant une méthode variationnelle. L'avantage de cette méthode est qu'elle n'utilise pas l'espace euclideen ambiant. Par conséquent, la méthode est en quelque sorte intrinsèque et peut être utilisée avec quelques modifications dans le cadre de la géométrie hyperbolique. Afin d'obtenir les formules trigonométriques, Euler travaille dans les coordonnées dites équidistantes sur la sphère et il dérive l'élément de longueur ds dans ces coordonnées. Il utilise à de nombreux endroits le fait que la géométrie sphérique est infinitésimalement euclidienne, c'est-à-dire que les relations euclidiennes sont satisfaites au niveau des différentielles.
Nous travaillons (théorème 1) dans le cadre de la géométrie de Lobachevsky introduite dans [14]. Il y a quelques intersections entre les résultats de Lobachevsky et ceux d'Euler. Tous les deux, ils travaillent dans des systèmes de coordonnées analogues – le premier adapté au plan hyperbolique, et l'autre à la sphère. Une différence entre les deux approches est que Lobachevsky utilise la trigonométrie, qu'il développe à partir des principes premiers, afin de trouver l'élément de longueur ds tandis qu'Euler utilise l'élément de longueur ds et la propriété de la géométrie sphérique d'être infinitésiment Euclidienne, afin de reconstruire la trigonométrie.
Nous donnons l'analogue hyperbolique des formules de la trigonometrie des triangles rectangles hyperboliques utilisant les méthodes du calcul des variations. C'est un travail en collaboration avec Weixu Su.

Théorème 1 (Frenkel, Su; [9]). *Dans le plan hyperbolique, soit ABC un triangle rectangle avec longueurs des côtés a, b, c et angles opposés α, β,* $\frac{\pi}{2}$. *Alors, ces quantités satisfont aux relations trigonométriques suivantes:*

$$\sinh b = \frac{\tanh a}{\tan\alpha}; \quad \cosh b = \frac{u}{\sin\alpha\cosh a}; \quad \tanh b = \frac{\cos\alpha\sinh a}{u}, \tag{1}$$
$$\sinh c = \frac{\sinh a}{\sin\alpha}; \quad \cosh c = \frac{u}{\sin\alpha}; \quad \tanh c = \frac{\sinh a}{u}, \tag{2}$$
$$\sin\beta = \frac{\cos\alpha}{\cosh a}; \quad \cos\beta = \frac{u}{\cosh a}; \quad \tan\beta = \frac{\cos\alpha}{u}, \tag{3}$$

où $u = \sqrt{\cosh 2a - \cos^2 \alpha}$.
Le second mémoire d'Euler, sur lequel j'ai travaillé, [7], concerne une formule de l'aire en géométrie sphérique en termes des longueurs des côtés d'un triangle, ainsi qu'un problème lié, connu sous le nom *le problème de Lexell*. Ce problème fut résolu par Lexell dans [13] dans le cas sphérique et par A'Campo et Papadopoulos dans [1] dans le cas hyperbolique.
Nous utilisons l'analogue hyperbolique d'une formule d'Euler pour l'aire dans la formulation et la résolution du problème de Lexell dans le cas hyperbolique. Le théorème suivant est l'analogue hyperbolique de cette formula d'Euler. C'est aussi un travail qui j'ai fait en collaboration avec Weixu Su.
Théorème 2 (Frenkel, Su; [9]). *Dans le plan hyperbolique, l'aire $\mathcal{A}$ d'un triangle de côtés de longueurs a, b, c est donnée par*

$$\cos \frac{\mathcal{A}}{2} = \frac{1 + \cosh a + \cosh b + \cosh c}{4 \cosh \frac{1}{2} a \cosh \frac{1}{2} b \cosh \frac{1}{2} c}.$$

Après avoir prouvé le théorème 2, nous le revisitons pour donner une forme courte et une interpretation géometrique de la formule pour l'aire. Ces considérations sont liées au problème de Lexell suivant:
Problème de Lexell: En géométrie hyperbolique, étant donnés deux points distincts A, B, déterminer le lieu des points P de telle sorte que l'aire du triangle dont les sommets A, B et P est égal à une constante donnée $\mathcal{S}$.
La construction géométrique qu'on utilise dans la preuve du problème de Lexell est très utile pour donner une forme courte de la formule d'Euler. Nous donnons au passage une preuve du problème de Lexell, un peu differente de la preuve du Théorème 5.11 dans [1].
Théorème 3 (Frenkel, Su; [9]). *Dans le plan hyperbolique, l'aire $\mathcal{A}$ d'un triangle dont la base est de longueur a et dont le segment qui relie les milieux de deux côtés restants du triangle est de longueur m_a, est donnée par*

$$\cos \frac{\mathcal{A}}{2} = \frac{\cosh m_a}{\cosh \frac{a}{2}}.$$

Ensuite, nous considerons les *analogies de Néper*[1]. Nous donnons l'analogue hyperbolique de ces relations. Nous les utilisons pour déduire des formules alternatives pour l'aire d'un triangle hyperbolique (théorèmes 5 et 6 ci-dessous) en analogie avec le cas sphérique. Ensuite, nous appliquons ces formules pour donner l'equation de la courbe de Lexell (théorème 7) qui est la solution du problème de Lexell[2].

Théorème 4. *Soit ABC un triangle hyperbolique avec longueurs de côtés a, b, c et angles opposés α, β et γ, respectivement. Alors les identités suivantes sont vraies:*

$$\tan \frac{1}{2}(\alpha + \beta) = \cot \frac{1}{2}\gamma \frac{\cosh \frac{1}{2}(a-b)}{\cosh \frac{1}{2}(a+b)}. \tag{4}$$

$$\tan \frac{1}{2}(\alpha - \beta) = \cot \frac{1}{2}\gamma \frac{\sinh \frac{1}{2}(a-b)}{\sinh \frac{1}{2}(a+b)}. \tag{5}$$

[1] "l'analogie", en grec, signifie "le rapport".
[2] Voir Note X dans [12] pour le cas sphérique.

Théorème 5. *Un triangle hyperbolique étant donné, avec longueurs de côtés a, b et c et angles opposés α, β et γ, respectivement, soit $\mathcal{A}$ l'aire de ce triangle. Alors*

$$\cot \frac{1}{2}\mathcal{A} = \frac{\coth \frac{1}{2}a \, \coth \frac{1}{2}b - \cos\gamma}{\sin\gamma}. \tag{6}$$

Théorème 6. *Soit ABC un triangle hyperbolique avec longueurs de côtés a, b et c et angles opposés α, β et γ, respectivement. Soit $\mathcal{A}$ l'aire de ce triangle. Alors*

$$\cot \frac{1}{2}\mathcal{A} = \frac{1 + \cosh a + \cosh b + \cosh c}{\sin\gamma \sinh a \sinh b}. \tag{7}$$

Pour le résultat suivant, nous devons introduire les *coordonnées équidistantes.* Nous introduisons les coordonnées dans le plan hyperbolique comme suit. Soit O un point et Ox et Oy deux lignes orthogonales, qui se croisent en O. Nous choisissons un côté *positif* sur chaque ligne par rapport à O. Ces lignes sont les axes x y du repère. Soit M un point quelconque dans le plan hyperbolique. Nous désignons par P_x le pied de la perpendiculaire de M à Ox. En analogie avec les coordonnées cartésiennes habituelles dans le plan Euclidien, nous associons à M le couple de nombres réels (x, y), données par

$$x = \pm\text{Long}\,(OP_x), \tag{8}$$
$$y = \pm\text{Long}\,(P_xM). \tag{9}$$

Nous rappelons cependant que contrairement au plan euclidien, le plan hy-

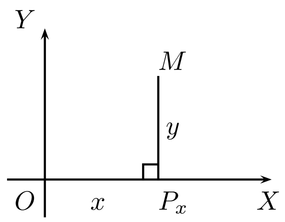

Figure 1: Les coordonnée equidistantes (x, y)

perbolique et la sphère ne sont pas munis de deux feuilletages orthogonaux géodésiques.

Nous posons x (ou y) positfif, si M se trouve dans le même demi-plan limité par la ligne Oy (ou OX) que le côté positif de Ox (ou le côté positif de Oy). De même, nous posons x (ou y) négatif, si M et le côté positif de Ox (ou le côté positif de Oy) se trouvent dans les demi-plans différents avec la ligne du bord Oy (ou OX).

Théorème 7. *Dans les coordonnées equidistantes (x, y), l'equation de la courbe de Lexell a la forme*

$$\cosh x \, \cosh y = \coth \frac{1}{2}\mathcal{A} \sinh \frac{c}{2} \sinh y - \cosh \frac{c}{2}. \tag{10}$$

Une application du théorème de Lexell est la *construction d'une famille continue des figures de même aire.* Si cette famille est paramétrée par $t \in [0, 1]$, on a pour $t = 0$ un quadrilatère hyperbolique et pour $t = 1$ un triangle. On donne aussi une condition angulaire pour l'existence d'une telle famille.

Les formules trigonométriques suivantes donnent une nouvelle formule pour l'aire et trouvent leur application dans les théorèmes 9 et 10 ci-dessous, voir [3] pour la version spherique.

Théorème 8 (Théorème de Cagnoli). *Dans le plan hyperbolique, étant donné un triangle avec longueurs de côtés a, b, c et angles opposés α, β, γ, son aire $\mathcal{A}$ est donné par*

$$\sin\frac{\mathcal{A}}{2} = \frac{\sinh\frac{b}{2}\sinh\frac{c}{2}\sin\alpha}{\cosh\frac{a}{2}}. \tag{11}$$

En particulier,

$$\begin{aligned}
\sinh\frac{a}{2} &= \sqrt{\frac{\sin\frac{\mathcal{A}}{2}\,\sin\left(\frac{\mathcal{A}}{2}+\alpha\right)}{\sin\beta\sin\gamma}}; \quad \cosh\frac{a}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\gamma\right)\,\sin\left(\frac{\mathcal{A}}{2}+\beta\right)}{\sin\beta\sin\gamma}}\\
\sinh\frac{b}{2} &= \sqrt{\frac{\sin\frac{\mathcal{A}}{2}\,\sin\left(\frac{\mathcal{A}}{2}+\beta\right)}{\sin\alpha\sin\gamma}}; \quad \cosh\frac{b}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\alpha\right)\,\sin\left(\frac{\mathcal{A}}{2}+\gamma\right)}{\sin\alpha\sin\gamma}}\\
\sinh\frac{c}{2} &= \sqrt{\frac{\sin\frac{\mathcal{A}}{2}\,\sin\left(\frac{\mathcal{A}}{2}+\gamma\right)}{\sin\alpha\sin\beta}}; \quad \cosh\frac{c}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\alpha\right)\,\sin\left(\frac{\mathcal{A}}{2}+\beta\right)}{\sin\alpha\sin\beta}}.
\end{aligned}$$

Théorème 9 (Théorème de Steiner). *Pour tous les triangles hyperboliques, les droites qui passent par un sommet et qui coupent le triangle en deux parties de même aire, sont concourantes.*

Théorème 10 (Théorème de Neuberg). *Parmi tous les triangles avec deux côtés données, b et c, celui qui a l'aire maximale $\mathcal{A}^*$ satisfait les relations*

$$\sin\frac{\mathcal{A}^*}{2} = \tanh\frac{b}{2}\tanh\frac{c}{2} \tag{12}$$

et

$$\alpha = \beta + \gamma. \tag{13}$$

Nous passons maintenant au problème de Schubert.

Problème de Schubert: Trouver les maxima et les minima de l'aire d'un triangle quand la base et la hauteur sont fixées.

La résolution de ce problème dans les cas sphérique et hyperbolique est faite en collaboration avec Vincent Alberge. Nous nous sommes inspirés de l'article de Schubert [22]. Nous résolvons ce problème de deux manières différentes; la première preuve est essentiellement la même que celle donnée par Schubert dans [22]. La deuxième preuve est différente.

Pour résoudre ce problème, Schubert étudie les variations de la fonction d'aire, qui est donnée en fonction des angles. Il décompose le triangle en deux triangles rectangles et utilise des relations trigonométriques (comme dans le théorème 1 ci-dessus), afin de calculer les variations des angles et donc de l'aire. Schubert

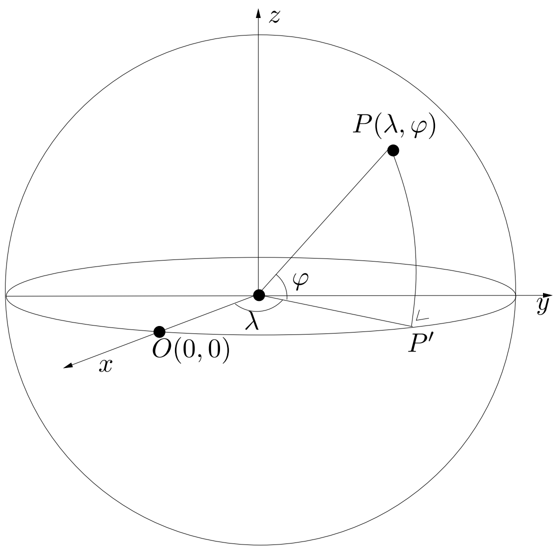

Figure 2: Les coordonnées sphériques (λ,φ) sur la sphère unité

trouve deux points critiques pour sa fonction d'aire et il étudie le signe de la dérivée seconde à ces points. Les deux points réalisent le minimum et le maximum, respectivement (voir la figure 2). Bien que les calculs de Schubert soient naturels et élémentaires, nous utilisons dans notre preuve une autre relation pour l'aire d'un triangle rectangulaire. Cela nous permet d'éviter l'étude de la dérivée seconde.
Cette preuve est faite en coordonnées sphériques (λ, φ), où $\lambda \in [0, 2\pi[$ et $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, voir Figure 2. Nous utilisons également la formule de conversion bien connue des coordonnées sphériques à cartésiennes sur la sphère unité :

$$(\lambda, \varphi) \mapsto (\cos\varphi \, \cos\lambda, \cos\varphi \, \sin\lambda, \sin\varphi).$$

Notre deuxième preuve est purement géométrique et utilise les courbes de Lexell, qui sont les solutions du problème de Lexell. Une telle courbe a d'abord été étudiée par Lexell dans [13]. Plus tard, Euler dans [7] et Steiner dans [23] ont donné d'autres constructions de ces courbes.
Nous avons obtenu les résultats suivants:
Théorème 11 (Alberge, Frenkel, version sphérique). *Soit C_0 le point de $\mathcal{E}(h)$ tel que le pied de la hauteur de ce point est le milieu de AB. Soit C_π le point tel que le pied de la hauteur de ce point est le milieu de $A'B'$, où A' et B' sont les points antipodaux de A et de B, respectivement. Alors*

$$\forall C \in \mathcal{E}(h), \ \text{Aire}(ABC_0) \leq \text{Aire}(ABC) \leq \text{Aire}(ABC_\pi).$$

Théorème 12 (Alberge, Frenkel, version hyperbolique). *Soit AB un segment sur une droite donnée $\mathcal{G}$ et soit $\mathcal{E}(h)$ la courbe equidistante à $\mathcal{G}$ à distance h. Alors, il y a une unique position C^* sur $\mathcal{E}(h)$ telle que l'aire du triangle ABC^* est extrémale. De plus, le triangle ABC^* est isoscèle et l'aire du triangle ABC^* est maximale.*

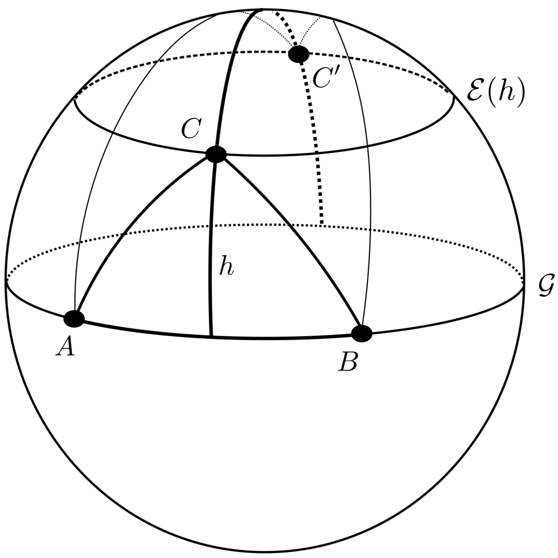

Figure 3: Nous traçons une géodesique $\mathcal{G}$ passant par deux points A et B. Nous traçons une courbe equidistante $\mathcal{E}\,(h)$ qui est à distance sphérique h de cette géodésique. Les points C et C' sont les lieux où l'aire du triangle ayant pour base AB et un sommet sur $\mathcal{E}(h)$ est minimale et maximale, respectivement. De plus, les sommets de deux triangles appartiennent à la géodesique qui passe par le pôle de $\mathcal{G}$ (et ainsi de $\mathcal{E}(h)$) et le milieu de AB.

Dans le cas hyperbolique, nous donnons aussi deux preuves, deux analogues du cas sphérique. La première preuve est faite dans le modèle de Poincaré. La seconde est faite en coordonnées équidistantes. L'idée, comme dans le cas sphérique, est d'étudier les points d'intersection des courbes de Lexell avec un horocycle fixe. Le choix des coordonnées correspond à notre but, car l'équation de l'horocycle en coordonnées équidistantes a la forme simple: $y = const$. Malgré le fait que dans la géométrie hyperbolique il n'y ait pas de points antipodaux et que la construction de la courbe de Lexell par Steiner n'y fonctionne pas, les courbes de Lexell peuvent être définies d'une manière différente, de manière à ce que la méthode puisse toujours être utilisée.
De plus, nous revisitons la formule de l'aire d'un triangle rectangle hyperbolique en donnant deux preuves. La première utilise la méthode d'Euler et la deuxième utilise les relations du Théorème de Cagnoli.
Théorème 13 (Formule de l'aire pour le triangle rectangle). *Soit ABC un triangle rectangle hyperbolique avec longueurs de côtés a et h. Alors l'aire de ce triangle satisfait à l'équation*

$$\tan\frac{\mathcal{A}}{2} = \tanh\frac{a}{2}\,\tanh\frac{h}{2}.$$

Ensuite, nous considérons le théorème de Steiner suivant ([23], p.109):
Entre tous les triangles isopérimètres et de même base le triangle isoscèle est un maximum.
Soit AB un segment de longueur (hyperbolique) a sur une droîte $\mathcal{G}$, et soit $\mathcal{H}(d)$

un ensemble de points C tel que la somme de longueurs (hyperboliques) AC et BC soit égale à une constante donnée $d > 0$. C'est l'analogue hyperbolique de l'ellipse.

Théorème 14. *Dans les coordonnées equidistantes (x, y), l'equation de l'ellipse a la forme*

$$\operatorname{acosh}\Big(\cosh(x + \frac{a}{2}) \cosh y\Big) + \operatorname{acosh}\Big(\cosh y \cosh(x - \frac{a}{2})\Big) = d. \tag{14}$$

L'analogue hyperbolique du problème de Steiner est:

Théorème 15. *Soit AB un segment de longueur (hyperbolique) a sur une droîte donnée $\mathcal{G}$ et soit $d \geq a$. Il y a alors quatre positions sur $\mathcal{H}(d)$ pour lesquelles la surface de ABC est extrémale. Ce sont les deux triangles isocèles ABC et ABC' et les deux triangles dégénérés ABL et ABR. Dans le cas de ABC et ABC', l'aire est maximale. Dans les deux cas dégénérés, nous obtenons des minima.*

Tous les problèmes sur l'aire que j'ai addressés dans ma thèse peuvent être adressés dans le cas du volume, et tous sont des problèmes ouverts en dimension ≥ 3.

La variation du volume d'une famille lisse (P_t) dépendante d'un paramètre réel de polyèdres sur la sphère S^n, $n \geq 3$, a été donnée par Schläfli et étendue dans l'espace hyperbolique par Sforza (voir [16] pour plus de références). Le résultat de Schläfli est en fait une généralisation au premier ordre du calcul de l'aire par excès ou défaut des polygones sur la sphere S^2 ou plan hyperbolique d'après la formule classique d'Albert Girard. Dans un exposé à Cagliari, 2015, A'Campo a esquissé le fait qu'en utilisant la géométrie intégrale le résultat classique en dimension 2 implique le théorème de Schläfli. Dans notre travail nous donnons les grandes lignes de ce résultat de A'Campo.

2 Introduction

The essence of this thesis is to study the transition from spherical to hyperbolic geometry at the level of certain theorems. Our goal is to prove some theorems in the hyperbolic geometry based on the methods of Euler, Schubert and Steiner in the spherical geometry. These theorems, interesting in their own right, exhibit a way to adapt methods of proofs from spherical to hyperbolic geometry.

We need to recall that even if the spherical and hyperbolic geometries belong both to the common field of "Non-Euclidean geometry", these two geometries are different and we can not always expect that the theorems of one have analogues in another one. For example, on the sphere, two lines intersect always in two points and each line has a finite length. This property is often used in the proofs of the theorems on the sphere, and it has clearly no analogues in the hyperbolic plane.

We start by considerations based on two memoirs of Euler. In the first memoir [6] Euler gives the proofs of a complete set of spherical trigonometric formulae for right triangles using a variational method. The advantage of his method is that it does not use the ambient Euclidean 3-space. Therefore, the method is in some sense intrinsic and may be used with some modifications in the setting of hyperbolic geometry. In order to obtain the trigonometric formulae, Euler works in the so-called equidistant coordinates on the sphere and derives the length element ds in these coordinates. He uses at many places the fact that spherical geometry is infinitesimally Euclidean, i.e. the Euclidean relations are satisfied at the level of differentials.

We work (Theorem 1) in the setting of Lobachevsky's geometry as introduced in [14]. There are some intersections between Lobachevsky's and Euler's results. Both use "equidistant" coordinates which are appropriately defined in each case; Lobachevsky adapts them to the hyperbolic case and Euler on the sphere. One difference between the two approaches is that Lobachevsky uses trigonometry, which he develops from the first principles, in order to find the length element ds, whereas Euler uses ds and the property of spherical geometry of being infinitesimally Euclidean, in order to reconstruct the trigonometry.

We give the hyperbolic analogues of trigonometric formulae for right hyperbolic triangles. This is a joint work with Weixu Su.

Theorem 1 (Frenkel, Su; [9]). *In the hyperbolic plane, let ABC be a right triangle with the sides a, b, c and angles α, β, $\frac{\pi}{2}$. Then the following trigonometric identities hold:*

$$\sinh b = \frac{\tanh a}{\tan \alpha}; \quad \cosh b = \frac{u}{\sin \alpha \cosh a}; \quad \tanh b = \frac{\cos \alpha \sinh a}{u}, \tag{1}$$

$$\sinh c = \frac{\sinh a}{\sin \alpha}; \quad \cosh c = \frac{u}{\sin \alpha}; \quad \tanh c = \frac{\sinh a}{u}, \tag{2}$$

$$\sin \beta = \frac{\cos \alpha}{\cosh a}; \quad \cos \beta = \frac{u}{\cosh a}; \quad \tan \beta = \frac{\cos \alpha}{u}, \tag{3}$$

where $u = \sqrt{\cosh^2 a - \cos^2 \alpha}$.

We prove these hyperbolic formulae using the methods of Euler.

The second memoir of Euler that we use, [7], deals with an area formula in spherical geometry in terms of side lengths and a related problem, known as

Lexell's problem. This problem was solved by Lexell (a young collaborator of Euler) in [13] for the spherical case and by A'Campo and Papadopoulos in [1] for the hyperbolic case.
We give here Euler's area formula and the formulation of Lexell's problem for the hyperbolic case.
The following theorem is the hyperbolic analogon of the Euler's formula mentionned above. This is also a work that I did with Weixu Su.

Theorem 2 (Frenkel, Su; [9]). *In the hyperbolic plane, the area $\mathcal{A}$ of a triangle with side lengths a, b, c is given by*

$$\cos\frac{\mathcal{A}}{2} = \frac{1+\cosh a+\cosh b+\cosh c}{4\cosh\frac{1}{2}a\cosh\frac{1}{2}b\cosh\frac{1}{2}c}.$$

After proving Theorem 2, we revisit it in order to give a short form and geometric interpretation of the area formula. Our considerations are related to Lexell's problem, which is stated as follows.
Lexell's Problem. In hyperbolic geometry, given two distinct points A, B, determine the locus of points P such that the area of the triangle with vertices A, B and P is equal to a given constant $\mathcal{S}$.
The geometric construction that we use in the proof of Lexell's problem is useful for giving a short form of Euler's area formula. We give casually a proof of Lexell's problem, which differs slightly from the proof of Theorem 5.11 in [1].
Among the many mathematicians working on Lexell's problem, a special place have Euler (see [7]), Steiner (see [23]), Legendre (see [12]), and Lebesgue (see [11]). A recent work on spherical Lexell's problem was done by Maehara and Martini in [15].

Theorem 3 (Frenkel, Su; [9]). *In the hyperbolic plane, the area $\mathcal{A}$ of a triangle with the base of length a and the length of a segment joining the midpoints of two other sides of the triangle m_a, is given by*

$$\cos\frac{\mathcal{A}}{2} = \frac{\cosh m_a}{\cosh\frac{a}{2}}.$$

Further, we consider *Néper analogies.*[3] We give the hyperbolic analogue of these relations. We use them in order to deduce alternative formulae for the area of a hyperbolic triangle (Theorems 5 and 6 below) in analogy with the spherical case. Next, we apply these formulae in order to give the equation of a Lexell's curve (Theorem 7) which is the solution of Lexell's problem, see Note X in [12] for a spherical case.

Theorem 4. *Given a hyperbolic triangle with the side lengths a, b, c and opposite angles α, β and γ, respectively. Then* Néper analogies *hold:*

$$\tan\frac{1}{2}(\alpha+\beta) = \cot\frac{1}{2}\gamma\,\frac{\cosh\frac{1}{2}(a-b)}{\cosh\frac{1}{2}(a+b)}. \tag{4}$$

$$\tan\frac{1}{2}(\alpha-\beta) = \cot\frac{1}{2}\gamma\,\frac{\sinh\frac{1}{2}(a-b)}{\sinh\frac{1}{2}(a+b)}. \tag{5}$$

[3]"l'analogie" signifies "the rapport"

Theorem 5. *Given a hyperbolic triangle with side lengths a, b and c and with opposite angles α, β and γ, respectively. Let $\mathcal{A}$ be the area of this triangle. Then*

$$\cot\frac{1}{2}\mathcal{A} = \frac{\coth\frac{1}{2}a\,\coth\frac{1}{2}b - \cos\gamma}{\sin\gamma} \tag{6}$$

Theorem 6. *Given a hyperbolic triangle with side lengths a, b and c with opposite angles α, β and γ, respectively, let $\mathcal{A}$ be the area of this triangle. Then*

$$\cot\frac{1}{2}\mathcal{A} = \frac{1+\cosh a+\cosh b+\cosh c}{\sin\gamma\sinh a\sinh b} \tag{7}$$

For the following result, we have to introduce the equidistant coordinates:
We introduce coordinates in the hyperbolic plane as follows. Let O be a point and OX and OY two orthogonal lines which intersect at O. We choose a side on each line with respect to O, which we call *positive*. These lines will become $x-$ and $y-$ axes of the coordinate system (see Figure 1). Let M be a point in the hyperbolic plane. We denote by P_x the foot of the perpendicular from M to OX. In analogy with the usual Cartesian coordinates in the Euclidean plane, we associate to M the pair of real numbers (x, y), given by

$$x = \pm\text{Length}\,(OP_x), \tag{8}$$
$$y = \pm\text{Length}\,(P_xM). \tag{9}$$

The signs of x and y are chosen as follows. We take x (or y) positive, if M lies in the same halfplane bounded by the line OY (or OX) as the positive side of OX (or the positive side of OY). Likewise, we take x (or y) negative, if M and the positive side of OX(or the positive side of OY) lie in the different halfplanes with the boundary line OY (or OX). Note that the hyperbolic plane and the

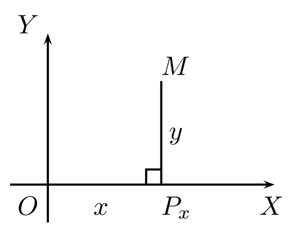

Figure 1: Equidistant coordinates (x, y).

sphere do not admit orthogonal foliations by geodesics, unlike the Euclidean plane.

Theorem 7. *In the "equidistant coordinates" (x, y) the equation of Lexell curve has a form*

$$\cosh x\,\cosh y = \cot\frac{1}{2}\mathcal{A}\,\sinh\frac{a}{2}\,\sinh y - \cosh\frac{a}{2}. \tag{10}$$

One application of Lexell's theorem is the *construction of a continuous family of figures with the same area.* If this family is parametrized by $t \in [0, 1]$, we have at $t = 0$ a hyperbolic quadrilateral and at $t = 1$ a triangle. We give also an angular condition for the existence of such a family.

The following trigonometric formulae give a new area fomula and find their application in Theorem 9 and 10 below, see [3] for the spherical version.

Theorem 8. *Cagnoli's theorem. In hyperbolic plane, given a triangle with side lengths a, b, c and the opposite angles α, β, γ, the area $\mathcal{A}$ of this triangle is given by*

$$\sin\frac{\mathcal{A}}{2} = \frac{\sinh\frac{b}{2}\sinh\frac{c}{2}\sin\alpha}{\cosh\frac{a}{2}}. \tag{11}$$

In particular, the following relations hold

$$\begin{aligned}
\sinh\frac{a}{2} &= \sqrt{\frac{\sin\frac{\mathcal{A}}{2}\,\sin\left(\frac{\mathcal{A}}{2}+\alpha\right)}{\sin\beta\sin\gamma}}; \quad \cosh\frac{a}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\gamma\right)\,\sin\left(\frac{\mathcal{A}}{2}+\beta\right)}{\sin\beta\sin\gamma}}\\
\sinh\frac{b}{2} &= \sqrt{\frac{\sin\frac{\mathcal{A}}{2}\,\sin\left(\frac{\mathcal{A}}{2}+\beta\right)}{\sin\alpha\sin\gamma}}; \quad \cosh\frac{b}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\alpha\right)\,\sin\left(\frac{\mathcal{A}}{2}+\gamma\right)}{\sin\alpha\sin\gamma}}\\
\sinh\frac{c}{2} &= \sqrt{\frac{\sin\frac{\mathcal{A}}{2}\,\sin\left(\frac{\mathcal{A}}{2}+\gamma\right)}{\sin\alpha\sin\beta}}; \quad \cosh\frac{c}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\alpha\right)\,\sin\left(\frac{\mathcal{A}}{2}+\beta\right)}{\sin\alpha\sin\beta}}.
\end{aligned}$$

Theorem 9 (Steiner's Theorem). *In a hyperbolic triangle, the lines that pass through vertices and bisect the area, are concurrent.*

Theorem 10. *The maximal area $\mathcal{A}^*$ among the triangles with two given sides b and c is given by*

$$\sin\frac{\mathcal{A}^*}{2} = \tanh\frac{b}{2}\tanh\frac{c}{2}. \tag{12}$$

Moreover, the triangle with maximal area satisfies an equation

$$\alpha = \beta + \gamma. \tag{13}$$

Now we pass to Schubert's problem.
Schubert's Problem: To find the maxima and the minima for the area of spherical triangles with a given base length and altitude's length.
The solution of this problem in the spherical and the hyperbolic cases is done in collaboration with Vincent Alberge. We are inspired by the paper of Schubert [22]. We solve this problem in two different ways; the first proof is essentially the same as the one given by Schubert dans [22]. The second proof is different. In order to solve this problem Schubert studies the variations of the area function, which is given in terms of the angles. He decomposes the triangle into two right-angled triangles and uses trigonometric relations (as in Theorem 1 above), in order to compute the variations of angles and therefore of the area. Schubert finds two critical points for his area function and he studies the sign of the second derivative at these points. The two points realize the minimum and the maximum, respectively (see figure 3). Although the computations of Schubert are natural and elementary, we use in our proof another relation for the area of a right-angled triangle. This permits us to avoid the study of the second derivative.
This proof is done in the *spherical coordinates* (λ, φ), where $\lambda \in [0, 2\pi[$ and

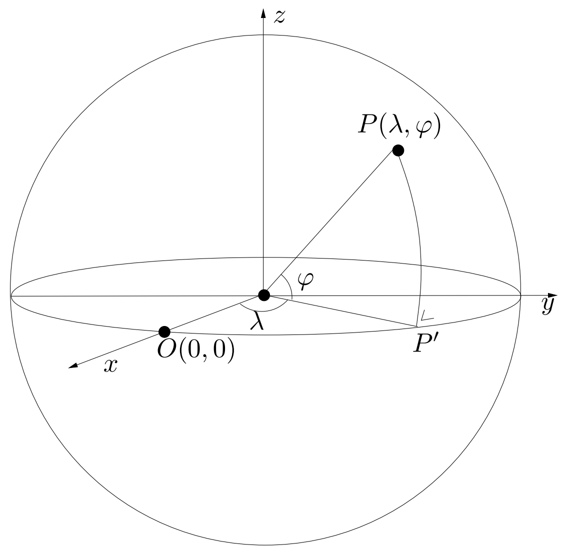

Figure 2: Spherical coordinates (λ,φ) on the unit sphere.

$\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, see Figure 2. We use also the well-known conversion formula from spherical to Carthesian coordinates of the ambient space $\mathbb{R}^3$ on the unit sphere:

$$(\lambda, \varphi) \mapsto (\cos\varphi\,\cos\lambda, \cos\varphi\,\sin\lambda, \sin\varphi)$$

Our second proof is purely geometric and uses the Lexell curves that are solutions of Lexell's problem. Such a curve was studied firstly by Lexell in [13]. Later, Euler in [7] and Steiner in [23] gave another constructions of Lexell's curves.

We obtained the following results:

Theorem 11 (Alberge, Frenkel, spherical version). *Let C_0 be the point of $\mathcal{E}(h)$ such that the foot of the altitude from it onto the line AB is the midpoint of AB. Let C_π be the point of $\mathcal{E}(h)$ such that the foot of the altitude from it onto the line is the midpoint of $A'B'$, where A' and B' are the antipodal points of A and B, respectively. Then*

$$\forall C \in \mathcal{E}(h),\ \text{Area}(ABC_0) \le \text{Area}(ABC) \le \text{Area}(ABC_\pi).$$

Theorem 12 (Alberge, Frenkel, hyperbolic version). *Among the hyperbolic triangles with a base AB and with the third vertex belonging to the equidistant set $\mathcal{E}(h)$, the isosceles triangles ABC and ABC' are the only triangles with the maximal area and the only triangles with the extremal area.*

In the hyperbolic case we give also two proofs, two analogues of the spherical case. The first proof is made in the Poincaré model. The second one is made in equidistant coordinates. The idea, as in the spherical case, is to study the intersection points of the Lexell curves with a fixed horocycle. The choice of coordinates corresponds to our goal, because the equation of horocycle in equidistant coordinates has a simple form: $y = const$. Although in the hyperbolic geometry there are no antipodal points and the construction of the Lexell

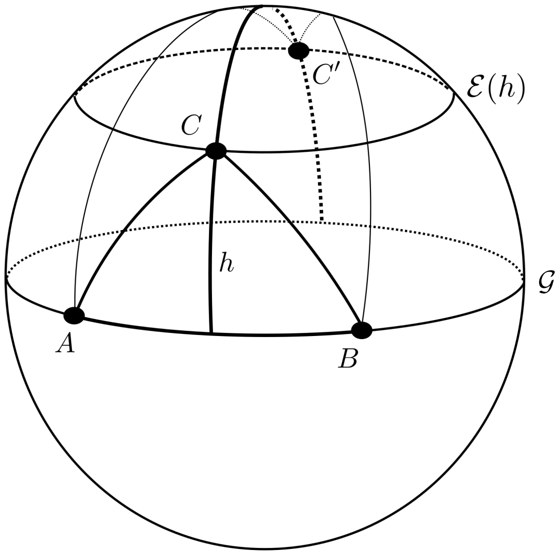

Figure 3: We draw a line $\mathcal{G}$ with two points A and B on it. We draw an equidistant curve $\mathcal{E}(h)$ which is at the (spherical) distance h. The points C and C' are the loci where the area of the corresponding triangle is minimal and maximal, respectively. Furthermore, they belong to the line which passes through the pole of $\mathcal{G}$ (and thus of $\mathcal{E}(h)$) and the midpoint of AB.

curve due to Steiner does not work, the Lexell curves can be defined in the different manner, such that the method is still applicable.
Moreover, we revisit the area formula of a right hyperbolic triangle and give two proofs. The first one uses Euler's method and the second one uses the relations of Cagnoli's theorem.

Theorem 13 (Area Formula for Right Triangle). *Let ABC be a right hyperbolic triangle with the lengths of catheta a and h. Then*

$$\tan\frac{\mathcal{A}}{2} = \tanh\frac{a}{2}\tanh\frac{h}{2}.$$

Further, we consider the following Steiner's theorem ([23], p.109):
Among all the isoperimetric triangles with a fixed base the isoscelle triangle has the maximal area.
Let AB be a segment of (hyperbolic) length a on a line $\mathcal{G}$, and let $\mathcal{H}(d)$ be a set of points C such that the sum of (hyperbolic) lengths AC and BC is equal to a given constant $d > 0$. This is a hyperbolic analogon of an ellipse.

Theorem 14. *In the equidistant coordinates (x, y), the ellipse equation has a form*

$$\operatorname{acosh}\Big(\cosh(x+\frac{a}{2})\cosh y\Big) + \operatorname{acosh}\Big(\cosh y\cosh(x-\frac{a}{2})\Big) = d. \tag{14}$$

The hyperbolic analogon of Steiner's problem is:

Theorem 15. *In the hyperbolic plane, let AB be a segment of length a on a given line $\mathcal{G}$ and let $d \geq a$. Then there are four positions on the ellipse $\mathcal{H}$ such* *that the area of ABC is extremal at these points. These are the two isosceles triangles ABC and ABC' and two degenerate triangles ABL and ABR. In the case of ABC and ABC' the area is maximal. At the two degenerate cases we get the minima.*

All the preceeding problems and theorems on the area that we treated in this thesis can be generalized to the case of higher dimensions ($n \geq 3$), where they are all open problems.

The variation of the volume for a smooth one-parametric family P_t of compact polyhedra depending on a real parameter t on the sphere S^n was given by Schläfli, and extended by Sforza to the hyperbolic space $H^n, n \geq 2$ (see [16] for more references). The result of Schläfli is a generalisation of first order area calculation by angle excess or defect of polygons on the sphere S^2 or on the hyperbolic plane after the classical formula of Albert Girard. In a talk in Cagliari, 2015, A'Campo gave the sketch of the proof using integral geometry which says that the classical result in dimension 2 for area implies the Schläfli theorem in higher dimensions. In our work, we give this result of A'Campo.

3 An area formula for hyperbolic triangles

3.1 The length element in Lobachevsky's work Euler introduced in [6] the coordinates on the sphere, such that $y = const$-curves are a chosen great circle G and the small circles equidistant to it, $x = const$-curves are the great circles, which are orthogonal to G. Thereafter, he deduced the formula for the length element ds in these coordinates

$$ds^2 = \cos^2 y\, dx^2 + dy^2. \tag{1}$$

Lobachevsky proved in [14] the analogue of this formula

$$ds^2 = \cosh^2 y\, dx^2 + dy^2. \tag{2}$$

In this Section we will explain this result and introduce the related concepts and notions, which will be also used in the proof of Theorem 1.

3.1.1 The theory of parallels and the angle of parallelism. In the *Pangeometry* Lobachevsky presents a summary of his lifelong work on hyperbolic geometry. He starts with first principles and develops from there the theory of parallel lines. He uses first the characteristic property of hyperbolic geometry: the sum of the angles in a right triangle is strictly less than two right angles. Based on this fact he defines a function Π, called *the angle of parallelism* as follows.
First, we recall the notion of parallel lines in the sense of Lobachevsky. Consider a line l and a point P not belonging to l. There is the unique perpendicular from P to the line l. We denote it by PP' and its length by p. Let $\mathcal{L}$ be the set of lines passing through P. Each line from $\mathcal{L}$ can be parametrized by the angle $\theta \in [0, \pi[$, which this line makes with the perpendicular PP' on a given side of a line containing PP'. Then *the parallel line to l through P (on a given side of PP')* is the line $l' = l_{\theta^*}$ characterized by the property that for $\theta < \theta^*$ the lines l_θ intersect l and for $\theta \geq \theta^*$, the lines l_θ and l do not intersect. Therefore, a parallel line l' to l is the limiting position of lines through P, which intersect l on a given side of P'. The angle θ^* is called *the angle of parallelism* (see Figure 1).

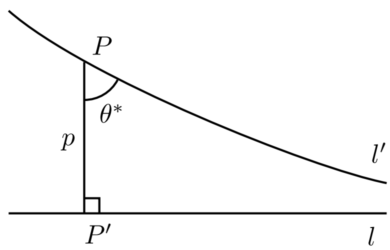

Figure 1: Angle of parallelism $\theta^* = \Pi(p)$

Definition 1. Let l be a line and P a point not on l. Let P' be the foot of the perpendicular dropped from P to l and p the length of PP'. A parallel line to l

through P makes with the perpendicular PP' two angles, one of which is acute. This angle is called *the angle of parallelism*, and it is denoted by $\Pi(p)$.

This notation, used by Lobachevsky, emphasizes the dependance of the angle of parallelism on the length p of the perpendicular PP', which can be seen as *the angle of parallelism function*.
The first analytic formula in [14] precises the relation $\Pi(p)$ ([14], p. 25):

$$\cos \Pi(p) = \tanh(p). \tag{3}$$

3.1.2 Equidistant coordinates.

We introduce coordinates in the hyperbolic plane as follows. Let O be a point and OX and OY two orthogonal lines, which intersect at O. We choose a side on each line with respect to O, which we call *positive*. These lines will become $x-$ and $y-$ axes of the coordinate system (see Figure 2). Let M be a point in the hyperbolic plane. We denote by P_x the foot of the perpendicular from M to OX. In analogy with the usual Cartesian coordinates in the Euclidean plane, we associate to M the pair of real numbers (x, y), given by

$$x = \pm \text{Length}\,(OP_x), \tag{4}$$
$$y = \pm \text{Length}\,(P_x M). \tag{5}$$

The signs of x and y are chosen as follows. We take x (or y) positive, if M lies in the same halfplane bounded by the line OY (or OX) as the positive side of OX (or the positive side of OY). Likewise, we take x (or y) negative, if M and the positive side of OX(or the positive side of OY) lie in the different halfplanes with the boundary line OY (or OX).

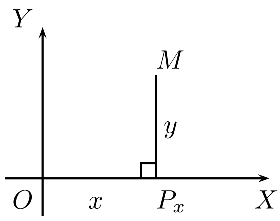

Figure 2: Equidistant coordinates (x, y)

Remark 1. In these coordinates, the y = const–lines are the line OX and the equidistant curves to OX, and the x = const–lines are the lines which are orthogonal to OX. Interesting facts on these coordinates systems in Euclidean, spherical and hyperbolic geometries can be found in [25].

3.1.3 The length element ds in coordinates.

In this Section we introduce ds in equidistant coordinates (x, y), due to Lobachevsky ([14], p.43)

$$ds = \sqrt{\frac{dx^2}{\sin^2 \Pi(y)} + dy^2}. \tag{6}$$

From (3),

$$\frac{1}{\sin \Pi(y)} = \cosh y.$$

Therefore, we can rewrite (6) as (2).

3.2 Trigonometry in Right Triangles In this Section we will see, how to reconstruct trigonometry from ds as in (2) and from the assumption that hyperbolic geometry is at the infinitesimal level Euclidean. Practically, we assume in the proof of the following result that the hyperbolic relations limit to the Euclidean ones.

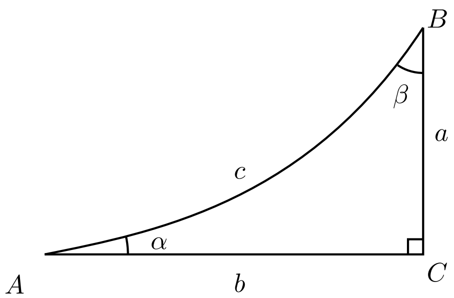

Figure 3: Right hyperbolic triangle

Theorem 1. *In the hyperbolic plane, let ABC be a right triangle with the sides a, b, c and angles α, β, $\frac{\pi}{2}$ as indicated in Figure 3). Then the following trigonometric identities hold:*

$$\sinh b = \frac{\tanh a}{\tan \alpha}; \quad \cosh b = \frac{u}{\sin \alpha \cosh a}; \quad \tanh b = \frac{\cos \alpha \sinh a}{u}, \tag{7}$$

$$\sinh c = \frac{\sinh a}{\sin \alpha}; \quad \cosh c = \frac{u}{\sin \alpha}; \quad \tanh c = \frac{\sinh a}{u}, \tag{8}$$

$$\sin \beta = \frac{\cos \alpha}{\cosh a}; \quad \cos \beta = \frac{u}{\cosh a}; \quad \tan \beta = \frac{\cos \alpha}{u}, \tag{9}$$

where $u = \sqrt{\cosh^2 a - \cos^2 \alpha}$.

Proof. We work in the equidistant coordinates (x, y). We take A as the origin O of the coordinate system and the line containing AC as OX. The positive side of OX (with respect to O) is a side containing the point C. We choose the positive side of OY such that the ordinate of the point B is positive.
The line AB in coordinates (x, y) can be seen as the graph of a function $g : [0, b] \to \mathbb{R}$

$$AB = \mathrm{Graph}(g) = \{(x, g(x)) | x \in [0, b]\}.$$

This provides a parametrization of AB by $\gamma : [0, b] \to \mathbb{R}^2$, given by $x \mapsto (x, g(x))$, and the length of AB can be computed by means of ds (compare with (2)):

$$\text{Length (AB)} = \int_0^b \sqrt{\cosh^2 g(x) + (g'(x))^2}\, dx. \tag{10}$$

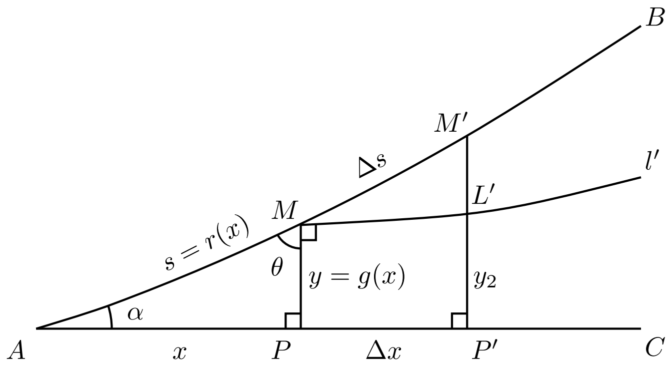

Figure 4: Proof of Theorem

Since $g(b) = a$, the function g is useful in the search for the relations between the sides of the triangle ABC. We used in (10) implicitly another function of this type, $r : [0, b] \to [0, c]$, given by $r(x) = \text{Length}(AM)$, where M has the coordinates $(x, g(x))$. We have $r(b) = c$. The function under the integral sign in (10) is the derivative of r,

$$r'(x) = \sqrt{\cosh^2 g(x) + (g'(x))^2}. \tag{11}$$

The idea of the proof is to deduce a differential equation in terms of $g(x)$ and $g'(x)$ using method of variations. This equation combined with (11) will allow us to find $r(b)$ and the other relations of this Theorem.
Among all the curves connecting A and B, which can be represented in coordinates as graphs of differentiable functions, the line segment AB has the minimal length. Therefore, the function $g(x)$ minimizes the length functional L, given by

$$L(f) = \text{Length}\,(c_f) = \int_0^b \sqrt{\cosh^2 f(x) + (f'(x))^2}\, dx \tag{12}$$

and defined on the functional space

$$C^1_{0,a}([0, b], \mathbb{R}) = \{f : [0, b] \to \mathbb{R} |\, f \text{ is differentiable, } f(0) = 0 \text{ and } f(b) = c\}.$$

In (12), we denote by c_f a curve connecting A and B, such that it can be represented in coordinates as Graph (f).
The differential of L vanishes necessarily at g and in particular we have

$$\frac{d}{d\varepsilon} L(g + \epsilon h)|_{\epsilon=0} = 0 \tag{13}$$

for every differentiable function $h : [0, b] \to \mathbb{R}$ with $g + \varepsilon h \in C^1_{0,a}([0, b], \mathbb{R})$.
For the sake of brevity, we introduce a function $F : \mathbb{R}^2 \to \mathbb{R}$, given by

$$F(x_1, x_2) = \sqrt{\cosh^2 x_1 + x_2^2},$$

such that the functional L becomes

$$L(f) = \int_0^b F(f(x), f'(x))\, dx.$$

The directional derivative in the left-hand side of (13) after integration by parts becomes

$$\frac{d}{d\varepsilon}L(g+\varepsilon h)|_{\varepsilon=0} = \int_0^b \left(\frac{\partial F}{\partial x_1}(g(x),g'(x)) - \frac{d}{dx}\frac{\partial F}{\partial x_2}(g(x),g'(x))\right) h\,dx.$$

This relation vanishes if and only if the term in brackets is identically zero. We obtain this result by taking a positive function φ for h. The next relation was named later on *Euler-Lagrange Formula*

$$\frac{\partial F}{\partial x_1}(g(x),g'(x)) - \frac{d}{dx}\frac{\partial F}{\partial x_2}(g(x),g'(x)) = 0 \text{ for all } x \in [0,b] \tag{14}$$

We simplify further the notation by means of the functions

$$G_i(x) = \frac{\partial F}{\partial x_i}(g(x),g'(x)),\ i = 1,2. \tag{15}$$

The Euler-Lagrange Formula (14) gives the simple relation

$$G_1 = G_2'. \tag{16}$$

On the other hand, we manipulate the function $r'(x)$, which stands under the integral sign in $L(g)$ (compare with (11))

$$r''(x) = \frac{\partial F}{\partial x_1}(g(x),g'(x))\,g'(x) + \frac{\partial F}{\partial x_2}(g(x),g'(x))\,g''(x),$$

or in terms of the functions G_i and using further the relation for G_i, given by the Euler-Lagrange Formula (16)

$$r'' = G_1\,g' + G_2\,g'' = G_2'\,g' + G_2\,g'' = (G_2\,g')',$$

we finally get

$$r' = G_2\,g' + C,$$

where C is a constant with respect to the variable x.
We rewrite this relation using the definition of G_2 (see (15)) and the relation $r'(x)$ (see (11))

$$\sqrt{(g')^2 + \cosh^2 g} = \frac{(g')^2}{(g')^2 + \cosh^2 g} + C.$$

Resolving this equation in g' gives

$$g'(x) = \frac{\cosh g(x)\sqrt{\cosh^2 g(x) - C^2}}{C} \tag{17}$$

and (11) takes the form

$$r'(x) = \frac{\cosh^2 g(x)}{C}. \tag{18}$$

Let M and M' be two points on AB with coordinates (x,y) and (x',y'), respectively. We denote by P and P' the feet of the perpendiculars dropped from M and M'. Let l' be a halfline, orthogonal to MP. We denote the intersection of

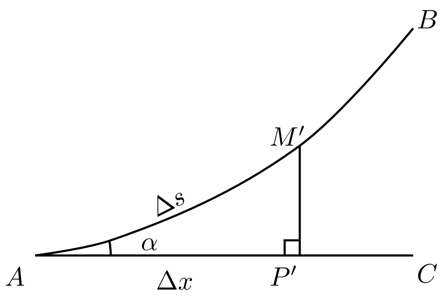

Figure 5: Case $M = A$.

l' with $M'P'$ by L'. We obtain the trirectangular quadrilateral $PML'P'$ with the acute angle L'. We denote the length of $L'P'$ by y_2, the length of AM by s, which coincides with $r(x)$, the length of MM' by Δs, $x' - x$ by Δx and $y' - y$ by Δy (see Figure 4).
Taking $M = A$, we determine the value of a constant C from a limit Euclidean relation for $\cos\alpha$ in the triangle $AM'P$ (see Figure 5).
In a right hyperbolic triangle holds a limit Euclidian relation:

$$\cos\alpha = \lim_{\Delta s \to 0} \frac{\Delta x}{\Delta s} = \frac{1}{r'(0)} = C.$$

We obtain

$$C = \cos\alpha. \tag{19}$$

We return to the general situation again, i.e. M is an arbitrary point on AB. We denote the angle $\angle AMP$ by θ. In the triangle $MM'L'$, the angle $\angle M'ML' = \frac{\pi}{2} - \theta$. If Length$(MM') \to 0$, the angle $\angle M'L'M$ tends to $\frac{\pi}{2}$ and for $MM'L'$ holds a limit Euclidean relation for right triangles

$$\sin(\angle M'ML') = \lim_{\Delta s \to 0} \frac{y' - y_2}{\Delta s}.$$

In $MM'L'$, we have $\chi = \frac{\pi}{2} - \theta$. The value of $\theta = \angle AMP$ depends on the position of M. We define a function $w : [0, b] \to [0, \beta]$, such that $w(x) = \angle AMP$.

$$\cos w(x) = \lim_{\Delta s \to 0} \frac{y' - y_2}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta y}{\Delta s} = \frac{g'(x)}{r'(x)} = \frac{\sqrt{\cosh^2 g(x) - C^2}}{\cosh g(x)}.$$

Taking $x = b$ in this relation and replacing C by (19), we get

$$\cos\beta = \frac{\sqrt{\cosh^2 a - \cos^2\alpha}}{\cosh a}. \tag{20}$$

We get the relations for $r(b)$ and $g^{-1}(a)$ by means of (17) and (18)

$$c = r(b) = \int_0^b r'(x)\,dx = \int_0^a \frac{\cosh g}{\sqrt{\cosh^2 g - C^2}}\,dg = \operatorname{acosh}\frac{\cosh^2 a - C^2}{1 - C^2}$$

$$b = \int_0^b dx = \int_0^a \frac{C}{\cosh g\sqrt{\cosh^2 g - C^2}}\,dg = \operatorname{asinh}\frac{C\tanh a}{\sqrt{1 - C^2}}$$

Replacing C by (19), we finally obtain:

$$\cosh c = \frac{\sqrt{\cosh^2 a - \cos^2 \alpha}}{\sin \alpha}, \tag{21}$$

$$\sinh b = \frac{\cos \alpha \tanh a}{\sin \alpha} = \frac{\tanh a}{\tan \alpha}. \tag{22}$$

The remaining relations of Theorem follow easily from (20), (21), (22) and from the properties of the functions cosh, sinh and cos. □

3.3 An area formula by a method of Euler

Theorem 2 (Hyperbolic Euler Formula). *In the hyperbolic plane, the area $\mathcal{A}$ of a triangle with side lengths a, b, c is given by*

$$\cos \frac{\mathcal{A}}{2} = \frac{1 + \cosh a + \cosh b + \cosh c}{4 \cosh \frac{1}{2}a \cosh \frac{1}{2}b \cosh \frac{1}{2}c}.$$

Beforehand, we deduce a formula for the area of the hyperbolic circular region $K_{r,\varphi}$ with radius r and angle φ (see Figure 6), which we will use in the proof.

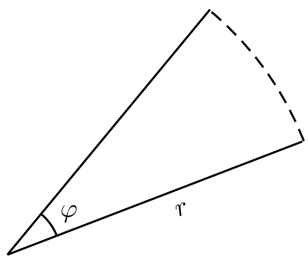

Figure 6: Region $K_{r,\varphi}$

For this, we cite the result of J.-M. de Tilly from his work on hyperbolic geometry [4], see also [5] for the description of a cinematical approach of de Tilly. De Tilly introduced a function $circ(r)$, which is defined as the circumference of the hyperbolic circle of radius r and showed without use of trigonometry that

$$circ(r) = 2\pi \sinh r. \tag{23}$$

Lemma 1. *The area of the hyperbolic circular region $K_{r,\varphi}$ with radius r and angle φ is*

$$Area(K_{r,\varphi}) = |\varphi|(\cosh r - 1).$$

Proof. By (23), the circumference of the hyperbolic circle of radius x is $2\pi \sinh x$ and hence, the length $l(x)$ of the hyperbolic circular arc of the same radius and angle φ is $\sinh x|\varphi|$. Then, for the area of circular segment, we have

$$\text{Area}(K_{r,\varphi}) = |\varphi| \int_0^r \sinh x \, dx = |\varphi| \, (\cosh r - 1).$$

□

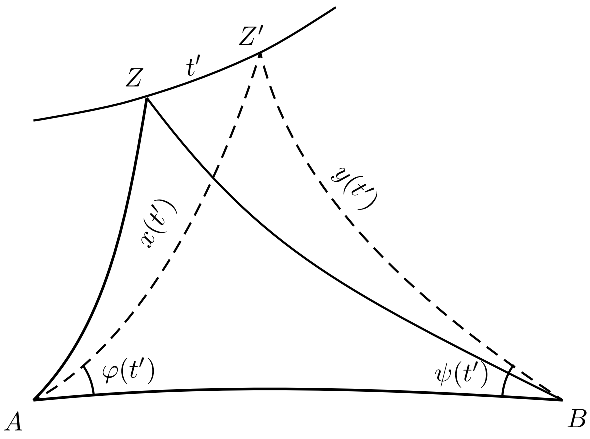

Figure 7: Proof of Theorem 1

Proof of the Hyperbolic Euler Formula. Let AB be a segment of fixed length a. Let Z be a point in a chosen connected component of $\mathbb{H}^2\backslash\{l\}$, where l is a line containing A and B. We fix a ray r, starting at Z and consider the area function $\mathcal{A} : [0, \epsilon) \to \mathbb{R}_{>0}$, given by $\mathcal{A}(t') =$ Area (ABZ'), where Z' is a point on r such that the length of ZZ' is t'.
Besides the function $\mathcal{A}$, we define in the same manner the functions x, y, φ and ψ, such that $x(t')$, $y(t')$, $\varphi(t')$ and $\psi(t')$ are the values of the side lengths AZ', BZ' and of the angles $\angle Z'AB$ and $\angle ABZ'$, respectively (see Figure 5) .
We are looking for the expression for the differential $d\mathcal{A}$. We consider the area change

$$\Delta\mathcal{A}(t') = \text{Area } (ABZ') - \text{Area } (ABZ).$$

The area change is given by

$$\Delta\mathcal{A} = \operatorname{sgn}(\Delta\varphi) \text{ Area}(AZZ') + \operatorname{sgn}(\Delta\psi) \text{ Area}(BZZ') \,. \tag{24}$$

In order to see this, we distinguish the four cases

1. $\operatorname{sgn}(\Delta\varphi) = \operatorname{sgn}(\Delta\psi) = 1$,
2. $\operatorname{sgn}(\Delta\varphi) = \operatorname{sgn}(\Delta\psi) = -1$,
3. $\operatorname{sgn}(\Delta\varphi) = -1$, $\operatorname{sgn}(\Delta\psi) = 1$,
4. $\operatorname{sgn}(\Delta\varphi) = 1$, $\operatorname{sgn}(\Delta\psi) = -1$.

Since $\Delta\mathcal{A}$ corresponding to the cases 1 and 3 is equal to $-\Delta\mathcal{A}$ in the cases 2 and 4, respectively, we reduce our consideration to the cases 1 and 3.
In the case 1, one triangle is entirely contained in the other one and Formula (24) can be easily seen from Figure 8.
In the case 3 a side of one triangle intersects a side of another triangle. We denote the intersection point by O (see Figure 5 for the case 3).
First, the area difference does not change, if one removes from the triangles ABZ' and ABZ the common triangle AOB

$$\text{Area}(ABZ') - \text{ Area}(ABZ) = \text{ Area}(BOZ') - \text{ Area}(AOZ).$$

Further, adding ZOZ' to both triangles BOZ' and AOZ does not affect the area difference again, therefore we finally get

$$\text{Area}(ABZ') - \text{Area}(ABZ) = \text{Area}(BZZ') - \text{Area}(AZZ'),$$

which is Formula (24) for the case 3.
We consider

$$\lim_{t'\to+0} \frac{\Delta\mathcal{A}(t')}{t'} =$$
$$= \text{sgn}(\Delta\varphi) \lim_{t'\to+0} \frac{\text{Area}(AZZ')}{t'} + \text{sgn}(\Delta\psi) \lim_{t'\to+0} \frac{\text{Area}(BZZ')}{t'}. \tag{25}$$

Since the area of AZZ' is enclosed by the areas of the hyperbolic circular segments $K_{x(0),\varphi}$ and $K_{x(t'),\varphi}$, by Lemma 1 we have an estimation

$$|\Delta\varphi|(\cosh x(0) - 1) \leq \text{Area}(AZZ') \leq |\Delta\varphi|(\cosh x(t') - 1),$$

from which it follows

$$\lim_{t'\to+0} \frac{\text{Area}(AZZ')}{t'} = |\Delta\varphi|\,(\cosh x(0) - 1). \tag{26}$$

For the sake of brevity, we denote $x(0)$ by x and $y(0)$ by y.
Combining (26) with (25), we get the following relations in terms of the differentials $d\mathcal{A}$, $d\varphi$ and $d\psi$

$$d\mathcal{A} = (\cosh x - 1)\, d\varphi + (\cosh y - 1)\, d\psi. \tag{27}$$

In order to rewrite this relation in terms of the differentials of the sides dx and dy, we use the First Cosine Law. Applying it to ABZ', we obtain

$$\cos\varphi(t') = \frac{\cosh a \cosh x(t') - \cosh y(t')}{\sinh a \sinh x(t')}.$$

By differentiation, we get

$$-\sin\varphi\, d\varphi = \frac{(-\cosh a + \cosh y \cosh x)\, dx - \sinh x \sinh y\, dy}{\sinh a \sinh^2 x},$$

where x, y and φ denote $x(0)$, $y(0)$ and $\varphi(0)$.
From now on and until the relation (31), we follow the computations of Euler, in order to find a well-suited substitution $v = f(a, x, y)$ for the integration of $d\mathcal{A}$.
Since

$$\sin\varphi = \sqrt{1 - \cos^2\varphi} = \frac{w}{\sinh a \sinh x},$$

where

$$w := \sqrt{1 - \cosh^2 x - \cosh^2 a - \cosh^2 y + 2\cosh a \cosh x \cosh y},$$

we have

$$d\varphi = \frac{(\cosh a - \cosh y \cosh x)\, dx + \sinh x \sinh y\, dy}{\sinh x\, w}.$$

Analogously, we obtain

$$d\psi = \frac{(\cosh a - \cosh x \cosh y)\, dy + \sinh x \sinh y\, dx}{\sinh y\, w}.$$

Therefore (27) becomes

$$d\mathcal{A} = \frac{(\cosh a - \cosh y \cosh x)\, dx + \sinh x \sinh y dy}{\sinh x\, w}(\cosh x - 1)$$
$$+ \frac{(\cosh a - \cosh x \cosh y)\, dy + \sinh x \sinh y\, dx}{\sinh y\, w}(\cosh y - 1).$$

Now, using the trigonometric formulae

$$\tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1},$$

we simplify our equation

$$w\, d\mathcal{A} = \tanh \frac{x}{2}(\cosh a + \cosh y - \cosh x - 1)\, dx$$

$$+ \tanh \frac{y}{2}(\cosh a + \cosh x - \cosh y - 1)\, dy.$$

Taking the symmetric expression

$$s := \cosh a + \cosh x + \cosh y,$$

we get

$$w\, d\mathcal{A} = (-1+s)(\tanh \frac{x}{2}\, dx + \tanh \frac{y}{2}\, dy) - 2 \tanh \frac{x}{2} \cosh x\, dx - 2 \tanh \frac{y}{2} \cosh y\, dy.$$

Applying again the (former) expression of $\tanh \frac{x}{2}$, we have

$$\tanh \frac{x}{2} \cosh x = - \tanh \frac{x}{2} + \sinh x,$$

so that

$$\frac{w\, d\mathcal{A}}{1+s} = \tanh \frac{x}{2}\, dx + \tanh \frac{y}{2}\, dy - 2 \frac{\sinh x\, dx + \sinh y\, dy}{1+s}. \tag{28}$$

Using $\int \tanh \frac{1}{2} x\, dx = 2 \ln(\cosh \frac{1}{2} x)$ and

$$\int \frac{\sinh x\, dx + \sinh y\, dy}{1+s} = \int \frac{d(1+s)}{1+s} = \ln(1+s),$$

we can find the primitive of the right-hand side of the equation (28) and we get

$$\int \frac{w\, d\mathcal{A}}{1+s} = 2 \ln \frac{q}{(1+s) \cosh \frac{a}{2}}, \tag{29}$$

where q is given by the symmetric expression

$$q := \cosh \frac{a}{2} \cosh \frac{x}{2} \cosh \frac{y}{2}.$$

Taking $p := \frac{q}{1+s}$ and derivating the equation, we obtain

$$\frac{w\, d\mathcal{A}}{1+s} = 2\frac{dp}{p},$$

or equivalently,

$$d\mathcal{A} = \frac{2dp}{p}\frac{1+s}{w}. \tag{30}$$

The term $\frac{1+s}{w}$ can be expressed as a function of p. Indeed, we note that

$$\begin{aligned} w^2 + (1+s)^2 &= 2(1+\cosh a)(1+\cosh x)(1+\cosh y) \\ &= 16\cosh^2\frac{a}{2}\cosh^2\frac{x}{2}\cosh^2\frac{y}{2} \\ &= 16q^2. \end{aligned}$$

Hence,

$$w = \sqrt{16q^2 - (1+s)^2}$$

and therefore

$$\frac{w}{1+s} = \sqrt{16p^2 - 1}.$$

Then (30) takes the form

$$d\mathcal{A} = \frac{2dp}{p\sqrt{16p^2-1}}.$$

Letting now $v = \frac{1}{4p}$, we obtain

$$d\mathcal{A} = -\frac{2dv}{\sqrt{1-v^2}}. \tag{31}$$

In terms of v, the primitive of $d\mathcal{A}$ can be computed, so that we get

$$\begin{aligned} A &= C + 2\arccos v \\ &= C + 2\arccos\frac{1+\cosh a+\cosh x+\cosh y}{4\cosh\frac{1}{2}a\cosh\frac{1}{2}x\cosh\frac{1}{2}y}. \end{aligned}$$

We determine the constant C by considering the degenerate Euclidean case $\mathcal{A} = 0$. If $x = a$, we get $y = 0$, such that the form of our equation becomes

$$0 = C + 2\arccos\frac{2+2\cosh a}{4\cosh^2\frac{1}{2}a}.$$

From $4\cosh^2\frac{1}{2}a = 2+2\cosh a$ follows $C = 0$.
Taking now $x = b$ and $y = c$, we finally get the result

$$\cos\frac{1}{2}\mathcal{A} = \frac{1+\cosh a+\cosh b+\cosh c}{4\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}}.$$

□

Lemma 2. *The solution of the Lexell problem is a smooth curve.*

Proof. Let $\mathcal{P}_1$ be a chosen connected component of $\mathbb{H}\backslash\{l\}$, where l contains the points A and B. The solution of the Lexell problem is the set

$$L = \{Z \in \mathcal{P}_1 | \text{ Area}(ABZ) = \mathcal{A}\}.$$

L is a smooth curve, if the area function $\mathcal{B} : \mathcal{P}_1 \to \mathbb{R}_{>0}$, given by

$$\mathcal{B}(Z) = \text{ Area}(ABZ)$$

is regular or, equivalently, for every point $Z \in \mathcal{P}_1$ there is a ray r starting at Z, such that the value of $\mathcal{B}$ changes in the direction of this ray.

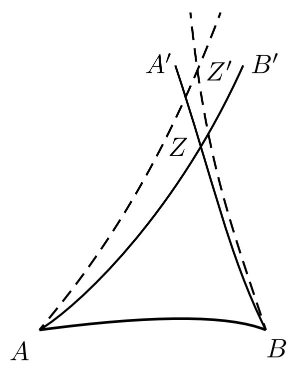

Figure 8: Area function $\mathcal{A}$ is regular.

For a fixed $Z \in \mathcal{P}_1$ consider $Z' \in \mathcal{P}_1$ in the interior of $\angle A'ZB'$, where the points A' and B' belong to the extensions of the sides AZ and BZ respectively (see Figure 8). Using Pasch's axiom of neutral geometry, one can deduce that ABZ is contained in ABZ', which gives $\mathcal{B}(Z') > \mathcal{B}(Z)$. This gives that the area value changes along ZZ'. Therefore, the function $\mathcal{B}$ is regular. □

Lemma 2 also follows from the fact that the area of triangles is a function that depends real analytically on the three vertices and that it is non-degenerate.

3.4 The area formula revisited and Lexell's problem We start this Section by recalling the Girard theorem, which states that the area of a spherical triangle on a sphere of radius 1 is the angular excess, i.e. the (positive) difference between sum of the angles of this triangle and π. In the case of the hyperbolic plane, the following is the hyperbolic analogon of this theorem.

Theorem 3 (Hyperbolic Girard Theorem). *In the hyperbolic plane, the area of a triangle with angles α, β and γ is given by the angular deficit*

$$\mathcal{A} = \pi - (\alpha + \beta + \gamma).$$

A proof of this Theorem without use of trigonometry can be found in [1], Sec. 5.3.

In his proof of Area formula, Euler fixes the length of one side of triangle, being inspired by the Lexell problem, where one is looking for the configuration set of all triangles with the same base and the same area. The quantity v, which appears at the end of the proof of Theorem 2 has the property that it varies or remains constant at the same time with the area $\mathcal{A}$, according to the form of (31). Recall that

$$v = \frac{1 + \cosh a + \cosh b + \cosh c}{4 \cosh \frac{a}{2} \cosh \frac{b}{2} \cosh \frac{c}{2}}. \tag{32}$$

In order to give a geometrical interpretation of the Euler formula, one is led then to look for the quantities, which will not be affected by the change of the area.

The crucial construction here is the dissection of the hyperbolic triangle explained in [1], Ch. 5.5.

Before giving this construction, let us note that we use here the term *congruence* of triangles or more generally, of plane figures F_1 and F_2, as a primary notion of hyperbolic geometry, developped from the first principles (see [1] Ch. 2 for more details). We will write $F_1 \equiv F_2$ for this relation. Moreover, we say that two figures F_1 and F_2 are *scissors equivalent*, if one of these figures can be cut into a finite number of pieces and reassembled into another one. It is easy to show that the relation of scissors congruence defines an equivalence relation on plane figures and that the scissors equivalent figures have the same area, in whatever reasonable sense area is defined. (In fact, in Euclid's *Elements* the notion of "having the same area" is defined as scissors equivalence of figures.)

Let ABZ be a triangle in the hyperbolic plane, let A' and B' the midpoints of AZ and BZ respectively, and let F, H, G be the feet of the perpendiculars dropped from A, B, Z respectively on the line $A'B'$ (see Figure 9). By construction, the following congruences hold: $AFA' \equiv A'GZ$ and $ZGB' \equiv B'BH$. From this one deduces easily that the triangle ABZ is scissors equivalent to the *Khayyam-Saccheri quadrilateral* $BHFA$ (i.e. a quadrilateral, which has two opposite edges of the same length and makes right angles with the third common edge ([1] Def. 3.28)).

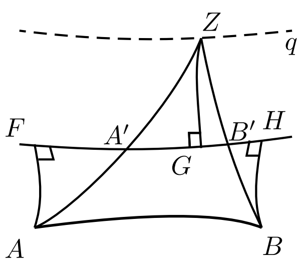

Figure 9: Dissection of Hyperbolic Triangle and the Lexell Problem.

This construction is closely related to the solution of the Lexell problem. Indeed, consider a triangle with notation as in Figure 9. Let $\mathcal{A}$ denote the area of this triangle. Let q be an equidistant curve to the line $A'B'$, which contains Z. Then every triangle ABZ' such that Z' lies on q and close to Z has the same area $\mathcal{A}$. This follows immediately from the consideration that all such triangles are scissors equivalent to the same quadrilateral $BHFA$. By Lemma 2, the solution of the Lexell problem is a smooth curve. Thus, the solution is exactly the curve q.

Remark 2. Lexell's problem in $\mathbb{H}^3$. Given a tetrahedron $ABCD$ with the base

ABC in hyperbolic space $\mathbb{H}^3$. To find the locus of points P such that

$$\mathrm{Vol}_3(ABCP) = \mathrm{Vol}_3(ABCD).$$

The natural conjecture, seen the result in the dimension 2, is that the solution of Lexell's problem in the dimension 3 is an equidistant surface passing through the point D to the plane determined by the three middle points M_{AD}, M_{BD} and M_{CD}.

Let m denote the hyperbolic length of $A'B'$. By construction, in the quadrilateral $BHFA$ the length of HF is $2m$, and the length of AF is the same as the length of BH and is equal to h and the length of AB is a. Therefore, these quantities can be related to v, since they remain constant if the area stays constant. A computation shows that

$$\cosh m = \frac{1 + \cosh a + \cosh b + \cosh c}{4 \cosh \frac{b}{2} \cosh \frac{c}{2}}. \tag{33}$$

Thus the quantity v turns out to be

$$v = \frac{\cosh m}{\cosh \frac{a}{2}}$$

and we get immediately an alternative elegant expression for the hyperbolic Euler formula

$$\cos \frac{\mathcal{A}}{2} = \frac{\cosh m}{\cosh \frac{a}{2}}. \tag{34}$$

We will justify the formula (33) in a Theorem (see [24] of Terquem for a spherical version).

Theorem 4. *In a hyperbolic triangle with notations as in Figure 10, the length of the mediane is given by*

$$\cosh AN = \frac{\cosh a + \cosh b}{2 \cosh \frac{c}{2}} \tag{35}$$

and the length of the middle line is given by

$$\cosh MN = \frac{1 + \cosh a + \cosh b + \cosh c}{4 \cosh \frac{b}{2} \cosh \frac{c}{2}}. \tag{36}$$

Proof. 1.To find the length of the mediane AN. Let $|AN| = x$. Applying First Cosine Law to the triangles ABN, BCA gives:

$$\cos \angle B = -\frac{\cosh x - \cosh a \cosh \frac{c}{2}}{\sinh a \sinh \frac{c}{2}} = -\frac{\cosh b - \cosh a \cosh c}{\sinh a \sinh c}$$

$$(\cosh x - \cosh a \cosh \frac{c}{2}) \sinh a \sinh c = (\cosh b - \cosh a \cosh c) \sinh a \sinh \frac{c}{2}.$$

$$(\cosh x - \cosh a \cosh \frac{c}{2}) \sinh c = (\cosh b - \cosh a \cosh c) \sinh \frac{c}{2}.$$

$$\cosh x \sinh c = \cosh a (\cosh \frac{c}{2} \sinh c - \cosh c \sinh \frac{c}{2}) + \cosh b \sinh \frac{c}{2}$$

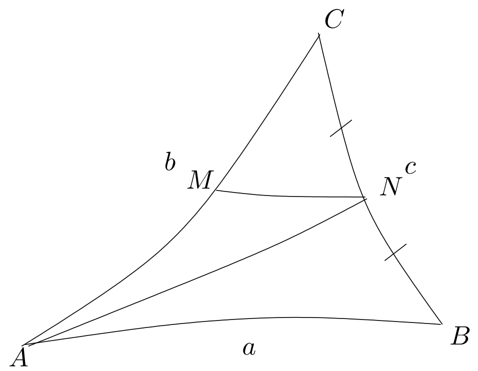

Figure 10: middle line MN

$$\cosh x = \frac{\cosh a \sinh \frac{c}{2} + \cosh b \sinh \frac{c}{2}}{\sinh c}$$
$$\cosh x = \frac{\cosh a + \cosh b}{2\cosh \frac{c}{2}}.$$

2. To find the length of the middle segment we use the first part of this proof. From CAN:

$$\cosh MN = \frac{\cosh AN + \cosh \frac{c}{2}}{2cosh\frac{b}{2}}.$$

From ABC:

$$\cosh AN = \frac{\cosh a + \cosh b}{2\cosh \frac{c}{2}}$$

$$\cosh MN = \frac{\frac{\cosh a + \cosh b}{2\cosh \frac{c}{2}} + \cosh \frac{c}{2}}{2\cosh \frac{b}{2}} =$$
$$= \frac{\cosh a + \cosh b + 2\cosh^2 \frac{c}{2}}{2\cosh \frac{c}{2}\cosh \frac{b}{2}} =$$
$$= \frac{1 + \cosh a + \cosh b + \cosh c}{4\cosh \frac{b}{2}\cosh \frac{c}{2}}.$$

□

Further, we formulate the hyperbolic analogue of Gudermann's theorem in spherical geometry, which is a corollary of short form of area (34). The spherical version of Gudermann's Theorem can be found in [19].

Corollary 1 (Gudermann's Theorem in [19]). *In hyperbolic geometry, let $\mathcal{T}$ be the set of triangles with a fixed side length a and a fixed area $\mathcal{A}$. Then the length m_a of a segment that joins the midpoints of two other sides is also fixed.*

As a conclusion, we will give a geometric interpretation of (34) on the quadrilateral $BHFA$. The perpendicular bisector of AB divides the Khayyam-Saccheri quadrilateral $BHFA$ into two congruent trirectangular quadrilaterals ([1], Chap.

3.6). On the one hand, by the dissection argument, the area of such a quadrilateral is equal to $\frac{\mathcal{A}}{2}$ and on the other hand, by the Hyperbolic Girard Theorem, it is given by the angle defect $2\pi - (\alpha + 3\frac{\pi}{2}) = \frac{\pi}{2} - \alpha$. So we have

$$\cos\frac{\mathcal{A}}{2} = \cos(\frac{\pi}{2} - \alpha) = \sin\alpha.$$

Thus the formula (34) reflects the trigonometrical relation in the trirectangular quadrilaterals

$$\sin\alpha = \frac{\cosh m}{\cosh\frac{a}{2}},$$

which can be found in [1], Chap. 6.6. The proof is a trigonometric computation which uses the trigonometric formulas on triangles introduced in this Chapter.

Remark 3. Volume formula in $\mathbb{H}^3$*.* Several significant mathematicians were working on the volume formula. Among them, Lobachevsky in [14], Thurston and Milnor in [17], Vinberg in [25]. Finding of a simple volume formula in the case of a tetrahedron is still an open question.

4 Area formulae and their Application to Lexell's problem

4.1 Background Recall (can be skipped by the reader familiar with trigonometry)

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y. \tag{37}$$

We take $x = \frac{1}{2}(\alpha + \beta)$ and $y = \frac{1}{2}(\alpha - \beta)$ then $x + y = \alpha$ and we have

$$\sinh(\alpha) = \sinh \frac{1}{2}(\alpha + \beta) \cosh \frac{1}{2}(\alpha - \beta) + \cosh \frac{1}{2}(\alpha + \beta) \sinh \frac{1}{2}(\alpha - \beta).$$

$$\sinh(\beta) = \sinh \frac{1}{2}(\alpha + \beta) \cosh \frac{1}{2}(\alpha - \beta) - \cosh \frac{1}{2}(\alpha + \beta) \sinh \frac{1}{2}(\alpha - \beta).$$

Adding and subtracting these two formulae, we get

$$\sinh \alpha + \sinh \beta = 2 \sinh \frac{1}{2}(\alpha + \beta) \cosh \frac{1}{2}(\alpha - \beta) \tag{38}$$

$$\sinh \alpha - \sinh \beta = 2 \cosh \frac{1}{2}(\alpha + \beta) \sinh \frac{1}{2}(\alpha - \beta) \tag{39}$$

Analogously, with addition theorem for Cosines. Finally we get

$$\cosh \alpha + \cosh \beta = 2 \cosh \frac{1}{2}(\alpha + \beta) \cosh \frac{1}{2}(\alpha - \beta) \tag{40}$$

$$\cosh \alpha - \cosh \beta = 2 \sinh \frac{1}{2}(\alpha + \beta) \sinh \frac{1}{2}(\alpha - \beta) \tag{41}$$

We recall here the corresponding spherical formulae

$$\sin \alpha + \sin \beta = 2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta) \tag{42}$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta) \tag{43}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta) \tag{44}$$

$$\cos \alpha - \cos \beta = -2 \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta) \tag{45}$$

Another formula that we use is

$$\tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x} \tag{46}$$

It can be proven by passing to half angle.

4.2 Néper analogies

Lemma 3. *Let ABC be a hyperbolic triangle with the side lengths a, b and c and with the opposite angles α, β and γ. Then the following holds:*

$$\cos \alpha \sinh c = \cosh a \sinh b - \cos \gamma \sinh a \cosh b. \tag{47}$$

Proof. Replacing $\cos\alpha$ and $\cos\gamma$ in the formula (47) by First Cosine Law:

$$\cos\alpha = -\frac{\cosh a - \cosh b\cosh c}{\sinh b\sinh c},$$
$$\cos\gamma = -\frac{\cosh c - \cosh a\cosh b}{\sinh a\sinh b},$$

we get

$$-\frac{\cosh a - \cosh b\cosh c}{\sinh b} = \cosh a\sinh b + \frac{\cosh c - \cosh a\cosh b}{\sinh b}\cosh b,$$
$$-\frac{\cosh a - \cosh b\cosh c}{\sinh b} = \frac{\cosh a\,(\sinh^2 b - \cosh^2 b) + \cosh b\cosh c}{\sinh b},$$
$$-\frac{\cosh a - \cosh b\cosh c}{\sinh b} = -\frac{\cosh a\ - \cosh b\cosh c}{\sinh b}.$$

□

Another equation is obtained by permutation of a and b

$$\cos\beta\sinh c = \cosh b\sinh a - \cos\gamma\sinh b\cosh a \tag{48}$$

Taking a sum of relations (47) and (48) and reducing, we have

$$\sinh c\,(\cos\alpha + \cos\beta) = (1-\cos\gamma)\sinh(a+b) \tag{49}$$

Another formula is obtained using Sine Law

$$\frac{\sinh c}{\sin\gamma} = \frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta}.$$

We have

$$\sinh c\sin\alpha = \sinh a\sin\gamma$$
$$\sinh c\sin\beta = \sinh b\sin\gamma$$

Adding/Substracting these two relations, we get

$$\sinh c\,(\sin\alpha + \sin\beta) = \sin\gamma(\sinh a + \sinh b), \tag{50}$$
$$\sinh c\,(\sin\alpha - \sin\beta) = \sin\gamma(\sinh a - \sinh b). \tag{51}$$

Dividing succesively these two relations by relation (49), we obtain

$$\frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \frac{\sin\gamma}{1-\cos\gamma}\cdot\frac{\sinh a + \sinh b}{\sinh(a+b)} \tag{52}$$
$$\frac{\sin\alpha - \sin\beta}{\cos\alpha + \cos\beta} = \frac{\sin\gamma}{1-\cos\gamma}\frac{\sinh a - \sinh b}{\sinh(a+b)}. \tag{53}$$

Using the formulae (38), (39), (42), (43), (44) and (46), we pass from (52)-(53) to (54)-(55). The former identities are called *Néper analogies.*

Theorem 5. *Given a hyperbolic triangle with the side lengths a, b, c and the opposite angles α, β and γ, respectively. Then* Néper analogies *hold:*

$$\tan\frac{1}{2}(\alpha+\beta) = \cot\frac{1}{2}\gamma\,\frac{\cosh\frac{1}{2}(a-b)}{\cosh\frac{1}{2}(a+b)}. \tag{54}$$

$$\tan\frac{1}{2}(\alpha-\beta) = \cot\frac{1}{2}\gamma\,\frac{\sinh\frac{1}{2}(a-b)}{\sinh\frac{1}{2}(a+b)}. \tag{55}$$

4.3 Area Formula

Theorem 6. *Given a hyperbolic triangle with side lengths a, b and c and with the opposite angles α, β and γ, respectively. Let $\mathcal{A}$ be the area of this triangle. Then*

$$\cot\frac{1}{2}\mathcal{A} = \frac{\coth\frac{1}{2}a\,\coth\frac{1}{2}b - \cos\gamma}{\sin\gamma} \tag{56}$$

Proof. From Girard Theorem 3,

$$\mathcal{A} = \pi - (\alpha + \beta + \gamma).$$

Moreover,

$$\cot\frac{1}{2}\,\mathcal{A} = \cot\frac{1}{2}(\pi - (\alpha + \beta + \gamma)) = \\ = \tan\frac{1}{2}(\alpha + \beta + \gamma).$$

By Addition Theorem for Tangens, we have

$$\tan\frac{1}{2}((\alpha + \beta) + \gamma) = \frac{\tan\frac{1}{2}(\alpha + \beta) + \tan\frac{\gamma}{2}}{1 - \tan\frac{\alpha+\beta}{2}\tan\frac{\gamma}{2}}. \tag{57}$$

Applying Néper analogy (54) to (57), we become

$$\frac{\tan\frac{1}{2}(\alpha + \beta) + \tan\frac{\gamma}{2}}{1 - \tan\frac{\alpha+\beta}{2}\tan\frac{\gamma}{2}} = \\ = \frac{\cot\frac{\gamma}{2}\cosh\frac{1}{2}(a - b) + \tan\frac{\gamma}{2}\cosh\frac{1}{2}(a + b)}{\cosh\frac{1}{2}(a + b) - \cosh\frac{1}{2}(a - b)}.$$

With $\omega = \cot\frac{\gamma}{2}$, the former relation becomes

$$\frac{\omega\cosh\frac{1}{2}(a - b) + \frac{1}{\omega}\cosh\frac{1}{2}(a + b)}{\cosh\frac{1}{2}(a + b) - \cosh\frac{1}{2}(a - b)}.$$

By aid of Addition Theorems, we continue to simplify this relation

$$\frac{\cosh\frac{1}{2}a\cosh\frac{1}{2}b\,(\omega + \frac{1}{\omega}) - \sinh\frac{1}{2}a\sinh\frac{1}{2}b\,(\omega - \frac{1}{\omega})}{2\sinh\frac{1}{2}a\sinh\frac{1}{2}b}$$

Since

$$\frac{\omega + \frac{1}{\omega}}{2} = \frac{1}{\sin\gamma}$$

and

$$\frac{\omega - \frac{1}{\omega}}{2} = \frac{\cos\gamma}{\sin\gamma},$$

we finally get

$$\frac{\coth\frac{1}{2}a\coth\frac{1}{2}b - \cos\gamma}{\sin\gamma}.$$

Thus,

$$\tan\frac{1}{2}(\alpha + \beta + \gamma) = \frac{\cot\frac{1}{2}a\cot\frac{1}{2}b - \cos\gamma}{\sin\gamma}$$

and therefore,

$$\cot\frac{1}{2}\mathcal{A} = \frac{\cot\frac{1}{2}a\cot\frac{1}{2}b - \cos\gamma}{\sin\gamma}.$$

□

4.4 Modified Area Formula

Theorem 7. *Given a hyperbolic triangle with side lengths a, b and c with the opposite angles α, β and γ, respectively. Let $\mathcal{A}$ be the area of this triangle. Then*

$$\cot\frac{1}{2}\mathcal{A} = \frac{1+\cosh a+\cosh b+\cosh c}{\sin\gamma\sinh a\sinh b} \tag{58}$$

Proof. We start with (56):

$$\cot\frac{1}{2}\mathcal{A} = \frac{\cot\frac{1}{2}a\ \cot\frac{1}{2}b - \cos\gamma}{\sin\gamma}.$$

Applying First Cosine Law to $\cos\gamma$ and using formulae

$$\coth\frac{a}{2} = \frac{1+\cosh a}{\sinh a}$$

$$\coth\frac{b}{2} = \frac{1+\cosh b}{\sinh b},$$

we get for

$$\sin\gamma\cot\frac{1}{2}\mathcal{A} = \frac{(1+\cosh a)(1+\cosh b)}{\sinh a\sinh b} - \cos\gamma =$$

$$= \frac{(1+\cosh a)(1+\cosh b)}{\sinh a\sinh b} + \frac{\cosh c-\cosh a\cosh b}{\sinh a\sinh b} = \frac{1+\cosh a+\cosh b+\cosh c}{\sinh a\sinh b}.$$

Therefore,

$$\cot\frac{1}{2}\mathcal{A} = \frac{1+\cosh a+\cosh b+\cosh c}{\sinh a\sinh b\sin\gamma}.$$

□

4.5 Equation of Lexell curve

Theorem 8. *In the "equidistant coordinates" (x, y) the equation of Lexell curve has a form*

$$\cosh x\ \cosh y = \cot\frac{1}{2}\mathcal{A}\ \sinh\frac{a}{2}\ \sinh y - \cosh\frac{a}{2}. \tag{59}$$

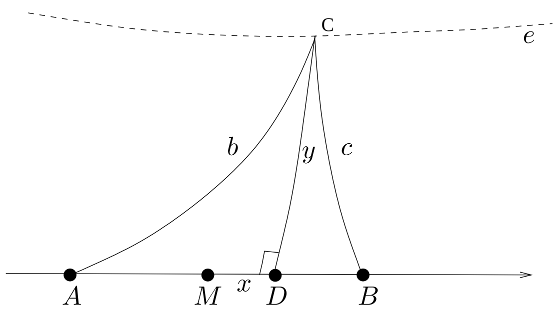

Figure 11: equation of Lexell curve

Proof. Let ABC be a triangle with a given side a, let M be a midpoint of AB. We put this triangle in the coordiante system of the equidistant coordinates such that the origin O coincides with a point M and the OX-axis coincides with the line connecting A and B (see Figure 11). Let the area of ABC be given, we denote it by $\mathcal{A}$. Moreover, let CD be an altitude of ABC to the base AB. Let $|MD| = x$ and $|DC| = y$. In other words, C has coordinates (x, y).
Then $|AD| = x + \frac{a}{2}$ and $|BD| = \frac{a}{2} - x$.
These two relations hold independent of whether D lies inside or outside of the segment AB.
We consider the two right triangles, ADC and DBC and apply Pythagoras:

$$\cosh b = \cosh y \cosh(x - \frac{a}{2}),$$

$$\cosh c = \cosh y \cosh(x + \frac{a}{2}).$$

Summing the last two relations, we get

$$\cosh b + \cosh c = \cosh y\, (\cosh(x - \frac{a}{2}) + \cosh(x + \frac{a}{2})) =$$

$$= \cosh y\, (2 \cosh x \cosh \frac{a}{2}),$$

We obtain

$$\cosh b + \cosh c = \cosh y\, (2 \cosh x \cosh \frac{a}{2}). \tag{60}$$

Together with $1 + \cosh a = 2\cosh^2 \frac{a}{2}$ we transform Modified Area Formula (58):

$$\cot \frac{1}{2}\mathcal{A} = \frac{1 + \cosh a + \cosh b + \cosh c}{\sin \alpha \sinh b \sinh c}$$

$$\cot \frac{1}{2}\mathcal{A} = \frac{2\cosh^2 \frac{a}{2} + 2\cosh \frac{a}{2} \cosh x \cosh y}{\sin \alpha \sinh b \sinh c} =$$

$$= \frac{2\cosh \frac{a}{2}(\cosh \frac{a}{2} + \cosh x \cosh y)}{\sin \alpha \sinh b \sinh c} =$$

$$= \frac{2\cosh \frac{a}{2}(\cosh \frac{a}{2} + \cosh x \cosh y)}{\sinh b \sin \gamma \sinh a} =$$

$$= \frac{2\cosh \frac{1}{2}a(\cosh \frac{a}{2} + \cosh x \cosh y)}{\sinh b \sin \gamma\, 2 \sinh \frac{a}{2} \cosh \frac{a}{2}}$$

$$= \frac{\cosh \frac{a}{2} + \cosh x \cosh y}{\sinh b \sin \gamma \sinh \frac{a}{2}}.$$

By relation for right hyperbolic triangle

$$\sinh b \sin \gamma = \sinh y$$

we finally get:

$$\cot \frac{1}{2}\mathcal{A} = \frac{\cosh \frac{a}{2} + \cosh x \cosh y}{\sinh \frac{a}{2} \sinh y}$$

or

$$\cosh x \cosh y = \cot \frac{1}{2}\mathcal{A} \sinh \frac{a}{2} \sinh y - \cosh \frac{a}{2}.$$

□

5 Construction of a family of figures using Lexell's problem

In the hyperbolic plane, given an arbitrary quadrilateral Q with angles w_1, w_2, w_3, w_4 and area $\mathcal{A}$, the question is whether it is possible to find a continuous family of figures of the same area that turns this quadrilateral into a triangle.
A brief observation shows that this deformation is not always possible, due to the fact that the area of a hyperbolic triangle is bounded by π, where the maximum area is realized by an ideal triangle. Thus, such a quadrilateral of area $\mathcal{A} > \pi$ does not support this continuous family.
We will propose in this section a construction of such a family of figures. We will give afterwards an exact condition in terms of angles for the existence of this construction. This condition is summarized in Theorem 10 below.
Construction. We divide the quadrilateral Q into two triangles T_1 and T_2 by drawing a diagonal AC (see Figure 12). Then, we consider the Lexell curve q for the triangle ACD with base AC, that is the locus of vertices of triangles of fixed area (of triangle ACD) and fixed base (AC). The line l_{BC}, which is an extension of a side BC, may intersect the curve q or may not. In the case of intersection, let D' denote the intersection point. We obtain then the triangle ABD', which has the same area as Q. In order to get the associated family of figures with the desired properties, we parametrize the piece $(DD')_q$ of the Lexell curve q by $\gamma : [0,1] \to (DD')_q$, $\gamma(0) = D$. We obtain $\{Q_t = ABCD_t\}$, $t \in [0,1]$, which is the continuous family of congruent figures that transforms a quadrilateral Q into the triangle ABD'.
Criterion of existence. Let us note that given a quadrilateral, we have eight possibilities to apply the above construction. These possibilities can be parametrized by a pair, consisting of a vertex of the quadrilateral and a side that does not contain this vertex (compare with the pair (D, BC) on Figure 12).

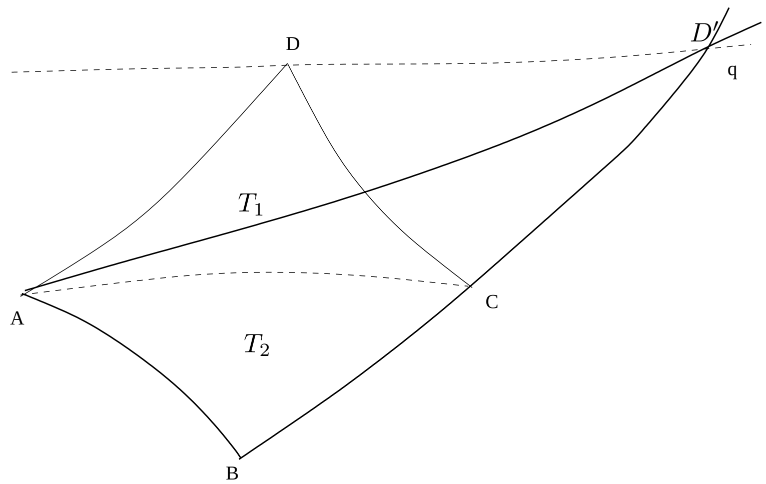

Figure 12: ABD' is a triangle of the same area as $ABCD$

In order to give an angular condition for the existence of this construction, we

need to single out the elements of our construction, which are responsible for the existence of the intersection point D'.
First, we recall the construction of the Lexell curve. We draw the midline l_m, i.e. the line joining the midpoints of sides AD and CD of the triangle ACD. The Lexell curve is then the equidistant curve to l_m passing through D. If the line l_{BC} intersects the curve q, then it necessarily intersects the midline l_m, and vice versa. We justify this fact with the following lemma.

Lemma 4. *Let q and q' be the two connected components of an equidistant set to the line l. Let H_q and $H_{q'}$ be two open half-planes, which are the sides of l containing q and q', respectively. A line l' having a point S in $H_{q'}$ intersects the line l if and only if it intersects the curve q.*

Proof. Assume that l' intersects l with some angle θ, but does not intersect q. Let M be the intersection point of l' and l, N a point on l' in H_q and P the foot of the perpendicular from N to l. The family of right triangles MN_tP_t, depending on a parameter $t > 0$, which is the hyperbolic length of MP_t (see Figure 13).
Let δ_t be the hyperbolic length of P_tN_t. Then for every $t > 0$ we have the following relation in the triangle MN_tP_t (compare with first relation in Theorem 1):

$$\tan\theta = \frac{\tanh\delta_t}{\sinh t}. \tag{61}$$

By assumption, l' does not intersect q, so we have $\delta_t < d$ for all $t > 0$, where d denotes the distance between l and q. Then the right-hand side of (61) tends to 0 for $t \to +\infty$, whereas the left-hand side of (61) remains a strictly positive constant. This gives a contradiction.
Conversely, if l' intersects q in a point in H_q and moreover, it has a point in $H_{q'}$, then it necessarily intersects the line l which separates H_q from $H_{q'}$. $\square$

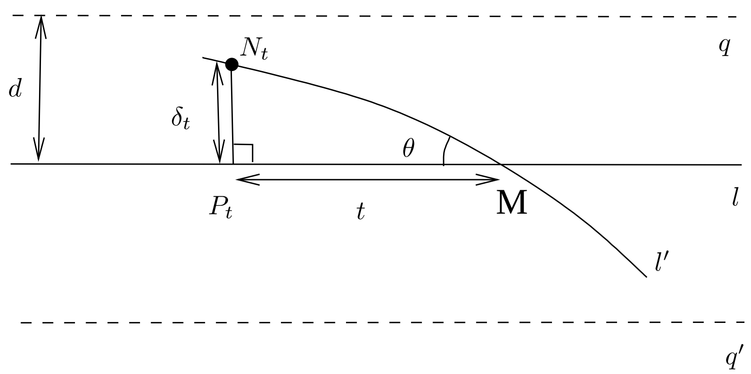

Figure 13: Proof of Lemma

We refer again to Figure 12. By Lemma 4, the existence of D' is equivalent to the existence of the intersection point of l_{BC} and l_m, denoted by M.
Secondly, we consider a Khayyam-Saccheri quadrilateral $CHFA$, obtained by drawing the perpendiculars AF and CH from A and C to the midline l_m (see Figure 14). Recall that in neutral geometry, the *Khayyam-Saccheri quadrilateral*

is a quadrilateral with two equal opposite sides, which are both perpendicular to the third side.
This quadrilateral has the same area as the triangle ACD, since it is the result

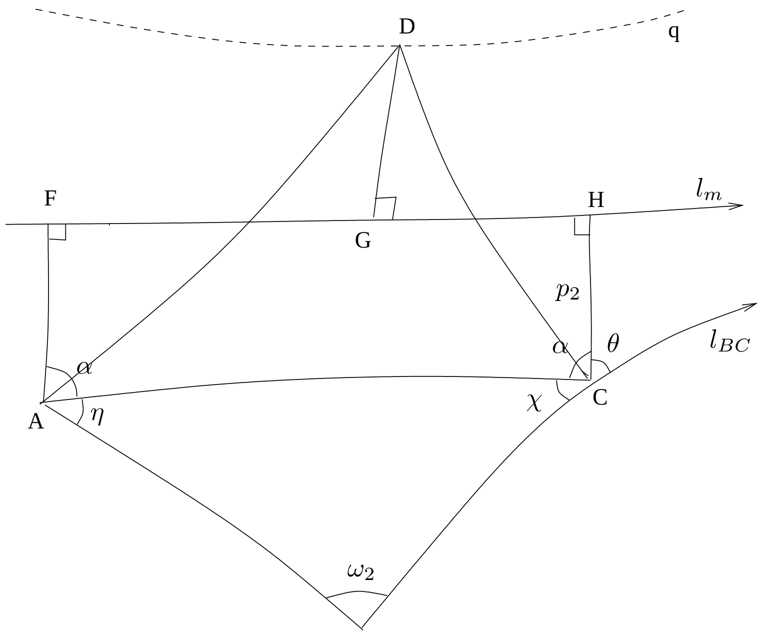

Figure 14: Non-euclidean dissection

of a non-Euclidean dissection of this triangle. For completeness, we justify here this fact. Let G be a perpendicular from D to a line l_m and let A' and C' be the intersection points of l_m with AD and CD, respectively, hence the midpoints of AD and CD. We have then the following congruences: $AFA' \equiv A'GD$ and $DGC' \equiv C'CH$, from which follows the result.
This way of cutting a triangle ACD into the finite number of pieces and rearranging them to the Khayyam-Saccheri quadrilateral $CHFA$ is the non-Euclidean analogue of a classical construction of Euclid (see Chapter 5.5 of [1] for more details).
Further, if the equal acute angles in $CHFA$ have measure α, then the area of $CHFA$ is given by

$$\text{Area (CFHA)} = 2\pi - (\pi + 2\alpha) = \pi - 2\alpha. \tag{62}$$

The computation of the area of the quadrialateral $CHFA$ uses a straightforward generalization of Hyperbolic Girard theorem for triangles to the case of polygons:

Theorem 9. *In the hyperbolic plane, the area of a polygon with* n *vertices and angles* $\{\alpha_i\}_{i=1}^n$ *is given by the angular deficit*

$$\text{Area}(P) = (n-2)\pi - \sum_i \alpha_i.$$

Thirdly, we observe that the intersection of the lines l_m and l_{BC} is also controlled by the angle θ, which the line l_{BC} makes with the perpendicular CH. Since the

angle sum in the triangle CHM is less than π, the intersection point M does not exist for $\theta \geq \frac{\pi}{2}$. We can obtain a more precise estimation using Lobachevsky's angle of parallelism function $\Pi(p)$. Before giving this estimation, let us recall some notions of the theory of parallel lines, introduced by Lobachevsky (compare Ch. 7 in [1]).
In hyperbolic geometry, the line l' through a point P not on l is *parallel* to l', if it is the limiting position of lines passing through P and points on a given side of l with respect to the point P', which is the foot of perpendicular from P on l. More precisely, we can consider a family of lines $\{l_\theta\}_\theta$ passing through P and making an angle $\theta \in [0, \pi[$ with the perpendicular PP' on a given side of PP' (see Figure 15). Then a parallel line $l' = l^*_\theta$ to l is a line characterized by the property that for all $\theta < \theta^*$ the lines l_θ intersect l on a given side of P', and for $\theta > \theta^*$, they do not intersect l. The lines l_θ with $\theta > \theta^*$ are called *ultra-parallel*. By this definition, through a given point P not belonging to l pass exactly two parallel lines to l (corresponding to two sides of P'), and infinitely many ultra-parallel lines.

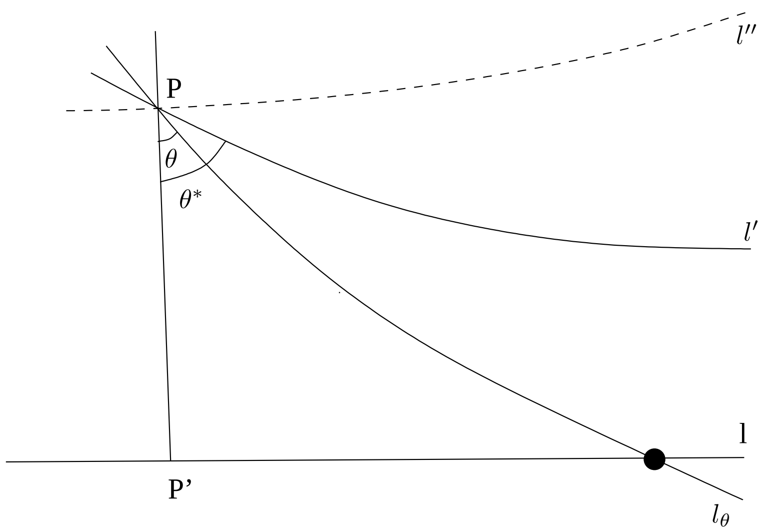

Figure 15: l' and l are parallel, l'' and l are ultra-parallel

The angle θ^*, called *Lobachevsky's angle of parallelism*, depends only on the hyperbolic length p of perpendicular PP'. This dependence called *Lobachevsky's angle of parallelism function* and denoted by $\Pi(p)$ (see Ch. 7.6 of [1]) is given by Lobachevsky's relation

$$\cos \Pi(p) = \tanh p.$$

Applying Lobachevsky's angle of parallelism function to our situation, we have that the lines l_m and l_{BC} intersect, if $\theta < \Pi(p_2) < \frac{\pi}{2}$, where p_2 is the hyperbolic length of the perpendicular CH, and do not intersect otherwise.
With notation as in Figure 15, the angle $\theta = \pi - (\chi + \alpha)$. Then the lines l_m and l_{BC} do not intersect for

$$\pi - (\chi + \alpha) \geq \Pi(p_2). \tag{63}$$

Applying the same argument to the lines l_m and l_{BA}, we get for the nonintersection of these lines the analogous condition

$$\pi - (\eta + \alpha) \geq \Pi(p_2). \tag{64}$$

Note that because of the symmetry of the Khayyam-Saccheri quadrilateral $CHFA$ with respect to the perpendicular bisector of AC, we have $|AF| = |CH| = p_2$ and $\angle A = \angle C = \alpha$.
Let $\mathcal{A}_1$ denote the area of the triangle ACD and $\mathcal{A}_2$ the area of the triangle ABC. For the area $\mathcal{A}_1$ we have

$$\mathcal{A}_1 = \text{Area}\,(\text{CFHA}) = \pi - 2\alpha \tag{65}$$

and

$$\mathcal{A}_2 = \pi - (\chi + \eta + w_2). \tag{66}$$

Combining (65) and (66) with conditions (63) and (64), we obtain

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 = \pi - (\chi + \alpha) + \pi - (\eta + \alpha) - w_2 \geq 2\Pi(p_2) - w_2. \tag{67}$$

Let us recall that the non-existence of the continuous family $\{Q_t\}_{t\in[0,1]}$, obtained by the construction explained above is garanteed by the non-existence of intersection points in all 8 possible situations of applying this construction. Condition (67) controls the failing only of 2 of them, namely those corresponding to the choice of D and of the extension lines BC and BA. This answer can be parametrized by the angle w_2 and the length p_2 of the perpendicular $|AF|$. Repeating our argument for the remaining cases, we get

$$\mathcal{A} \geq \max_{i=1,..,4} (2\Pi(p_i) - w_i). \tag{68}$$

We are able now to formulate the main result of this section.

Theorem 10. *In the hyperbolic plane, let Q be a quadrilateral with angles w_1, w_2, w_3, w_4 and area $\mathcal{A}$. Let w_i be the angle at the vertex A_i and p_i the length of* semialtitude *(i.e. the perpendicular to a midline of a triangle) drawn from the vertex A_{i+2} (opposite vertex to A_i). The condition for the existence of continuous family $\{Q_t\}_{t\in[0,1]}$ of figures of the same area $\mathcal{A}$ such that $Q_0 = Q$ and Q_1 is a triangle, is:*

$$\mathcal{A} < \max_{i=1,..,4} (2\Pi(p_i) - w_i), \tag{69}$$

where $\Pi(p_i)$ denotes the Lobachevsky's angle of parallelism associated to p_i.

Spherical analogue. Let us note that in spherical geometry, the Construction described above is always possible. The reason is a remarkable property of spherical geometry that two lines within it always intersect. Indeed, by deriving the angular estimation in the hyperbolic case we observed that the possibility of the construction depends on the existence of an intersection point between the midline l_m and the line l_{BC}. The same observation is also valid in spherical geometry. Since the lines l_m and l_{BC} always intersect, the construction is always possible.
Euclidean analogue. In the case of Euclidean geometry the base line l_{AB} and the midline l_m are parallel. Since l_{BC} intersects l_{AB}, it intersects necessarily l_m. Therefore, also in the Euclidean case, the construction is always applicable, though the reason is different from that of spherical geometry.

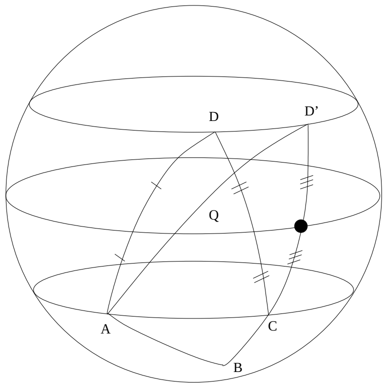

Figure 16: spherical case

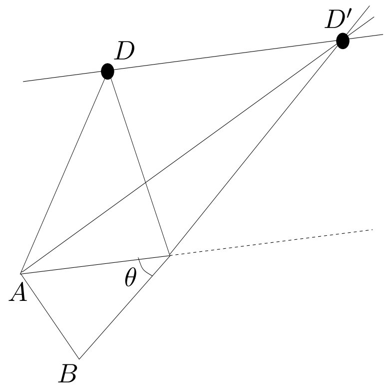

Figure 17: Euclidean case

Theorem 11. *In the spherical or Euclidean plane, let Q be a quadrilateral. There exists a continuous family $\{Q_t\}_{t\in[0,1]}$ of figures of the same area $\mathcal{A}$ such that $Q_0 = Q$ and Q_1 is a triangle.*

6 Cagnoli's identities and their Applications

6.1 Cagnoli's identities and Area formula

Theorem 12. *Cagnoli's theorem. In hyperbolic plane, given a triangle with side lengths a, b, c and the opposite angles α, β, γ, the area $\mathcal{A}$ of this triangle is given by*

$$\sin\frac{\mathcal{A}}{2} = \frac{\sinh\frac{b}{2}\sinh\frac{c}{2}\sin\alpha}{\cosh\frac{a}{2}}. \tag{70}$$

In particular, the following relations hold

$$\begin{aligned}
\sinh\frac{a}{2} &= \sqrt{\frac{\sin\frac{\mathcal{A}}{2}\,\sin\left(\frac{\mathcal{A}}{2}+\alpha\right)}{\sin\beta\sin\gamma}}; \quad \cosh\frac{a}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\gamma\right)\,\sin\left(\frac{\mathcal{A}}{2}+\beta\right)}{\sin\beta\sin\gamma}}\\
\sinh\frac{b}{2} &= \sqrt{\frac{\sin\frac{\mathcal{A}}{2}\,\sin\left(\frac{\mathcal{A}}{2}+\beta\right)}{\sin\alpha\sin\gamma}}; \quad \cosh\frac{b}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\alpha\right)\,\sin\left(\frac{\mathcal{A}}{2}+\gamma\right)}{\sin\alpha\sin\gamma}}\\
\sinh\frac{c}{2} &= \sqrt{\frac{\sin\frac{\mathcal{A}}{2}\,\sin\left(\frac{\mathcal{A}}{2}+\gamma\right)}{\sin\alpha\sin\beta}}; \quad \cosh\frac{c}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\alpha\right)\,\sin\left(\frac{\mathcal{A}}{2}+\beta\right)}{\sin\alpha\sin\beta}}.
\end{aligned}$$

Proof. 1. We consider the Second Law of Cosines

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a$$

and the Addition Theorem for Cosines

$$\cos(\beta+\gamma) = \cos\beta\cos\gamma - \sin\beta\sin\gamma.$$

We take their sum:

$$\cos\alpha + \cos(\beta+\gamma) = (\cosh\, a - 1)\sin\beta\sin\gamma$$

Using the trigonometric identity $\cosh(a) - 1 = 2\sinh^2 \frac{a}{2}$, we get

$$\sinh \frac{a}{2} = \sqrt{\frac{\cos \alpha + \cos(\beta + \gamma)}{2 \sin \beta \sin \gamma}}$$

By relation (44), we get

$$\sinh \frac{a}{2} = \sqrt{\frac{\cos \frac{\alpha+\beta+\gamma}{2} \cos \frac{\alpha-(\beta+\gamma)}{2}}{\sin \beta \sin \gamma}}$$

By Girard's theorem, the area $\mathcal{A}$ of a triangle ABC is given by the angular defect formula $\mathcal{A} = \pi - \alpha + \beta + \gamma$. We have

$$\cos \frac{\alpha + \beta + \gamma}{2} = \cos \left(\frac{(\pi - \alpha - \beta - \gamma) - \pi}{2} \right) = -\sin \frac{\mathcal{A}}{2}$$

and

$$\cos \frac{\alpha - (\beta + \gamma)}{2} = \cos \left(\frac{(\pi - \alpha - \beta - \gamma) + 2\alpha - \pi}{2} \right) = -\sin (\frac{\mathcal{A}}{2} + \alpha)$$

Therefore,

$$\sinh \frac{a}{2} = \sqrt{\frac{\sin \frac{\mathcal{A}}{2} \sin (\frac{\mathcal{A}}{2} + \alpha)}{\sin \beta \sin \gamma}}. \tag{71}$$

Analogously,

$$\sinh \frac{b}{2} = \sqrt{\frac{\sin \frac{\mathcal{A}}{2} \sin (\frac{\mathcal{A}}{2} + \beta)}{\sin \alpha \sin \gamma}}$$

and

$$\sinh \frac{c}{2} = \sqrt{\frac{\sin \frac{\mathcal{A}}{2} \sin (\frac{\mathcal{A}}{2} + \gamma)}{\sin \alpha \sin \beta}}.$$

2. Again, summing up the Second Law of Cosine and another Addition Theorem

$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cosh a$$

$$\cos(\beta - \gamma) = \cos \beta \cos \gamma + \sin \beta \sin \gamma$$

we get:

$$\cos \alpha + \cos(\beta - \gamma) = (1 + \cosh a) \sin \beta \sin \gamma$$

$$1 + \cosh a = 2\cosh^2 \frac{a}{2}.$$

Resolving in $\cosh \frac{a}{2}$ gives

$$\cosh \frac{a}{2} = \sqrt{\frac{\cos \frac{\alpha+\beta-\gamma}{2} \cos \frac{\alpha-\beta+\gamma}{2}}{\sin \beta \sin \gamma}}.$$

Taking in consideration

$$\cos \frac{\alpha + \beta - \gamma}{2} = \cos \left(\frac{(\pi - \alpha - \beta - \gamma) + 2\gamma - \pi}{2} \right) = -\sin (\frac{\mathcal{A}}{2} + \gamma)$$

and

$$\cos\frac{\alpha-\beta+\gamma}{2} = \cos\left(\frac{(\pi-\alpha-\beta-\gamma)+2\beta-\pi}{2}\right) = -\sin\left(\frac{\mathcal{A}}{2}+\beta\right)$$

we finally get

$$\cosh\frac{a}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\gamma\right)\,\sin\left(\frac{\mathcal{A}}{2}+\beta\right)}{\sin\beta\sin\gamma}}.$$

Analogously,

$$\cosh\frac{b}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\alpha\right)\,\sin\left(\frac{\mathcal{A}}{2}+\gamma\right)}{\sin\alpha\sin\gamma}} \tag{72}$$

and

$$\cosh\frac{c}{2} = \sqrt{\frac{\sin\left(\frac{\mathcal{A}}{2}+\alpha\right)\,\sin\left(\frac{\mathcal{A}}{2}+\beta\right)}{\sin\alpha\sin\beta}}. \tag{73}$$

Using formulae (70), (102) and (103) for $\sinh\frac{b}{2}$, $\sinh\frac{c}{2}$ and $\cosh\frac{a}{2}$, we get the Area formula

$$\sin\frac{\mathcal{A}}{2} = \frac{\sinh\frac{b}{2}\sinh\frac{c}{2}\sin\alpha}{\cosh\frac{a}{2}}.$$

□

6.2 Steiner's theorem The following result is an analogue of a Theorem of Steiner in spherical geometry.

Theorem 13 (Steiner's Theorem). *In a hyperbolic triangle, the lines that pass through vertices and bisect the area, are concurrent.*

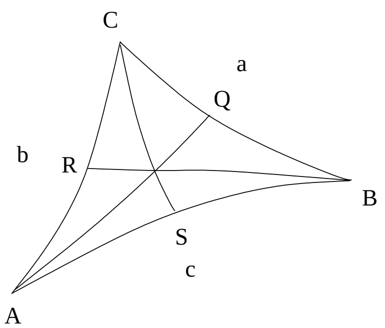

Figure 18: Lines that bisect the area are concurrent.

Proof. Let ABC be a triangle with side lenghts a, b, c and opposite angles α, β, γ, respectively. Let AQ, BR and CS be the segments that bisect the area of ABC (see Figure 18). The segments AQ, BR, CS bisect the area of ABC. First, the triangles ABQ and ACQ have the same area. We use formula (70).

$$\frac{\sinh\frac{c}{2}\sinh\frac{|BQ|}{2}\sin\beta}{\cosh\frac{|AQ|}{2}} = \frac{\sinh\frac{b}{2}\sinh\frac{|CQ|}{2}\sin\gamma}{\cosh\frac{|AQ|}{2}}$$

Hence,

$$\frac{\sinh\frac{|BQ|}{2}}{\sinh\frac{|CQ|}{2}} = \frac{\cosh\frac{c}{2}}{\cosh\frac{b}{2}}.$$

The last relation holds because

$$\frac{\cosh\frac{c}{2}}{\cosh\frac{b}{2}} = \frac{\sinh\frac{b}{2}\sin\gamma}{\sinh\frac{c}{2}\sin\beta},$$

which is equivalent to the Law of Sines

$$\frac{\sinh c}{\sinh b} = \frac{\sin\gamma}{\sin\beta}.$$

We proceed in the same way with the two other pairs of triangles: ABR, BRC and ACS, BCS and we get the formulae

$$\frac{\sinh\frac{|AR|}{2}}{\sinh\frac{|CR|}{2}} = \frac{\cosh\frac{c}{2}}{\cosh\frac{a}{2}}$$

and

$$\frac{\sinh\frac{|AS|}{2}}{\sinh\frac{|BS|}{2}} = \frac{\cosh\frac{b}{2}}{\cosh\frac{a}{2}}.$$

It follows that

$$\sinh\frac{|BQ|}{2}\sinh\frac{|CR|}{2}\sinh\frac{|AS|}{2} = \sinh\frac{|QC|}{2}\sinh\frac{|RA|}{2}\sinh\frac{|SB|}{2}. \tag{74}$$

Secondly, the triangles ABQ, SBC have equal areas. We use another area hyperbolic formula that was proven here in Theorem Sec6:

$$\cot\frac{\mathcal{A}}{2} = \frac{\coth\frac{a}{2}\coth\frac{b}{2} - \cos\gamma}{\sin\gamma}.$$

Since the triangles BCR and CQA have a common angle γ, we get

$$\coth\frac{a}{2}\coth\frac{|CR|}{2} = \coth\frac{|QC|}{2}\coth\frac{b}{2}. \tag{75}$$

Analogously, we get the relations for both other pairs of triangles. We get for ASC and ARB (with common angle α):

$$\coth\frac{c}{2}\coth\frac{|AR|}{2} = \coth\frac{|AS|}{2}\coth\frac{b}{2}. \tag{76}$$

And we get for BCS and ABQ (with common angle β):

$$\coth\frac{c}{2}\coth\frac{|BQ|}{2} = \coth\frac{|SB|}{2}\coth\frac{a}{2}. \tag{77}$$

From these three relations we get

$$\coth\frac{|BQ|}{2}\coth\frac{|CR|}{2}\coth\frac{|AS|}{2} = \coth\frac{|QC|}{2}\coth\frac{|RA|}{2}\coth\frac{|SB|}{2}. \tag{78}$$

From (74) and (78) we get

$$\cosh\frac{|BQ|}{2}\cosh\frac{|CR|}{2}\cosh\frac{|AS|}{2} = \cosh\frac{|QC|}{2}\cosh\frac{|RA|}{2}\cosh\frac{|SB|}{2}. \quad (79)$$

From (74) and (79) we get

$$\sinh|BQ|\sinh|CR|\sinh|AS| = \sinh|QC|\sinh|RA|\sinh|SB|.$$

By hyperbolic analogue of Ceva's theorem (see **??**), the segments AQ, BR and CS are concurrent. □

6.3 Neuberg's theorem

Theorem 14. *The maximal area $\mathcal{A}^*$ among the triangles with two given sides b and c is given by*

$$\sin\frac{\mathcal{A}^*}{2} = \tanh\frac{b}{2}\tanh\frac{c}{2}. \quad (80)$$

Moreover, the triangle with maximal area satisfies an equation

$$\alpha = \beta + \gamma, \quad (81)$$

The following proof is based on Neuberg's trigonometric proof in spherical geometry (see [3], Chap. V).

Proof. We consider a relation of Cagnoli's theorem 12 with b, c being given

$$\sin(\alpha + \frac{\mathcal{A}}{2}) = \sin\frac{\mathcal{A}}{2}\coth\frac{b}{2}\coth\frac{c}{2}. \quad (82)$$

Firstly, we observe that an inequality

$$\coth\frac{b}{2}\coth\frac{c}{2} > 1 \quad (83)$$

holds for every pair $(b, c) \in \mathbb{R}^2$.
Indeed, (83) is equivalent to

$$\cosh\frac{b-c}{2} > 0,$$

which is true for all pairs (b, c).
Secondly, in order that the triangle is possible, due to Formula (82), we must restrict the value of $\sin\frac{\mathcal{A}}{2}$. If

$$\sin\frac{\mathcal{A}}{2} \leq \tanh\frac{b}{2}\tanh\frac{c}{2},$$

the triangle with two given sides b, c is possible.
If

$$\sin\frac{\mathcal{A}}{2} = \tanh\frac{b}{2}\tanh\frac{c}{2},$$

the area is maximal, equation (82) becomes $\sin(\alpha + \frac{\mathcal{A}}{2}) = 1$, or $\alpha + \frac{\mathcal{A}}{2} = \frac{\pi}{2}$. By Girard's theorem, this is equivalent to

$$\alpha = \beta + \gamma. \quad (84)$$

□

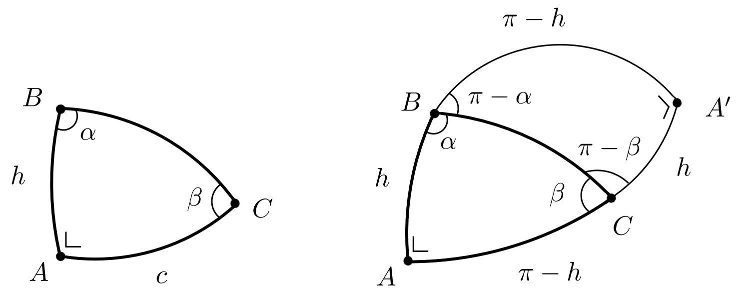

Figure 19: On the left hand side we have a right-angled triangle whose vertices are A, B and C. The lengths of AB and AC are denoted by c and h respectively. On the right hand side, we draw the particular right-angled triangle whose length AC is equal to $\pi - h$. The point A' is the antipodal point of A.

7 Schubert's spherical problem

7.1 Background in spherical geometry In this Section, we deal with the unit sphere $\mathbb{S}^2$ in the Euclidean space $\mathbb{R}^3$. We recall that $\mathbb{S}^2$ is equipped with the so-called *angular metric* and that the lines are the *great circles*. The parallels (or the equidistant curves) to a given line are the *small circles* (or latitudes) whose the corresponding pole is the same as the one of the line. In the rest of this section, we shall recall a trigonometric formula along with an application for the area of a (spherical) triangle.
We recall without proof that the area of a spherical triangle ABC is

$$\text{Area}(ABC) = \widehat{ABC} + \widehat{ACB} + \widehat{BAC} - \pi, \tag{85}$$

and then if the triangle is right at A we have

$$\text{Area}(ABC) = \widehat{ABC} + \widehat{ACB} - \frac{\pi}{2}. \tag{86}$$

Formula (85) is known as the Girard formula. There are well-known proofs of this formula, and one of them is contained in Euler's paper [8].
We now give a useful trigonometric formula. Let ABC be a right-angled triangle which is right at A and whose lengths AB, and AC are c and h, respectively, If we set $\alpha = \angle B$ and $\beta = \angle C$, then

$$\tan(\alpha) = \frac{\tan(c)}{\sin(h)} \text{ and } \tan(\beta) = \frac{\tan(h)}{\sin(c)}. \tag{87}$$

For more details about these relations we can refer to [1].

Lemma 5 (Area formula). *Let T be a right-angled spherical triangle whose length of the base is c and whose altitude is h. Then*

$$\text{Area}(T) = 2\arctan\left(\tan\left(\frac{h}{2}\right)\tan\left(\frac{c}{2}\right)\right).$$

Although the proof is elementary, we provide it for the convenience of the reader.

Proof. Let ABC be a right-angled triangle with a right angle γ. See Figure 19 for a picture. We shall deal with the four different cases, depending on the values of α and β.

Case I: Assume that $\alpha = \beta = \frac{\pi}{2}$. Then we have a triangle with three right angles. It implies that the triangle ABC is equilateral with the side length $\frac{\pi}{2}$. Then the area is $\frac{\pi}{2}$. On the other hand, $c = h = \frac{\pi}{2}$ and therefore $\arctan\left(\tan\left(\frac{h}{2}\right)\tan\left(\frac{c}{2}\right)\right) = \frac{\pi}{4}$, so the lemma is proved in this case.

Case II: Assume that either α or β is right. The triangle is then isosceles. Without loss of generality we assume that $\alpha = \frac{\pi}{2}$. Then, the area is equal to β. On the other hand, $h = \frac{\pi}{2}$ and $c = \alpha$ so the lemma is also proved here.

The last two cases depend on whether or not $\alpha + \beta = \pi$. But because of the law of sines applied to the triangle ABC, we have $\beta + \gamma = \pi$ if and only if $c + h = \pi$ or $c = h$. However, because of (87), we cannot have $c = h$, so $\alpha + \beta = \pi$ if and only if $c + h = \pi$.

Case III: Assume now that $\alpha + \beta = \pi$, or equivalently $c = \pi - h$. On the one hand we then have $\text{Area}(ABC) = \frac{\pi}{2}$, and on the other hand we have $\tan\left(\frac{h}{2}\right)\tan\left(\frac{\pi-h}{2}\right) = 1$. Therefore, the lemma is also proved in this case.

Case IV: Assume finally that $\alpha + \beta \neq \pi$, which by what we saw above is equivalent to $c \neq \pi - h$. From classical relations for trigonometric functions and from Relations (86) and (87) we have

$$\begin{aligned}
\tan\left(\text{Area}(ABC)\right) &= \frac{-1}{\tan\left(\alpha+\beta\right)} \\
&= \frac{-1 + \tan\left(\alpha\right)\tan\left(\beta\right)}{\tan\left(\alpha\right) + \tan\left(\beta\right)} \\
&= \frac{-1 + \frac{1}{\cos(h)\cos(c)}}{\frac{\tan(h)}{\sin(c)} + \frac{\tan(c)}{\sin(h)}} \\
&= \frac{\sin\left(h\right)\sin\left(c\right)\left(1 - \cos\left(h\right)\cos\left(c\right)\right)}{\sin^2\left(h\right)\cos\left(c\right) + \sin^2\left(c\right)\cos\left(h\right)} \\
&= \frac{\sin\left(h\right)\sin\left(c\right)\left(1 - \cos\left(h\right)\cos\left(c\right)\right)}{\cos\left(c\right) - \cos^2\left(h\right)\cos\left(c\right) + \cos\left(h\right) - \cos^2\left(c\right)\cos\left(h\right)} \\
&= \frac{\sin\left(h\right)\sin\left(c\right)}{\cos\left(h\right) + \cos\left(c\right)}. \qquad (88)
\end{aligned}$$

In the same vein we have

$$\begin{aligned}
\tan\left(2\arctan\left(\tan\left(\tfrac{h}{2}\right)\tan\left(\tfrac{c}{2}\right)\right)\right) &= \frac{2\tan\left(\frac{h}{2}\right)\tan\left(\frac{c}{2}\right)}{1 - \tan^2\left(\frac{h}{2}\right)\tan^2\left(\frac{c}{2}\right)} \\
&= \frac{\sin\left(h\right)\sin\left(c\right)}{2\left(\cos^2\left(\frac{h}{2}\right)\cos^2\left(\frac{c}{2}\right) - \sin^2\left(\frac{h}{2}\right)\sin^2\left(\frac{c}{2}\right)\right)} \\
&= \frac{\sin\left(h\right)\sin\left(c\right)}{2\left(1 + \frac{\cos(h)-1}{2} + \frac{\cos(c)-1}{2}\right)} \\
&= \frac{\sin\left(h\right)\sin\left(c\right)}{\cos\left(h\right) + \cos\left(c\right)}. \qquad (89)
\end{aligned}$$

Comparing (88) and (89), the lemma is proved in this case and so is in all cases. □

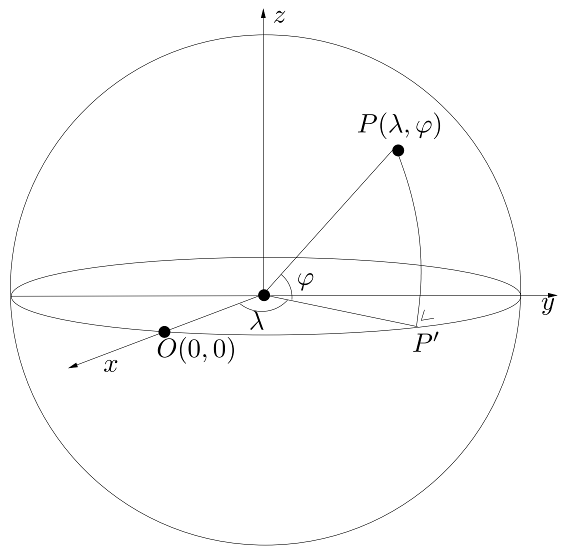

Figure 20: Spherical coordinates (λ,φ) on the unit sphere.

Our proof is made in spherical coordinates (λ, φ), where $\lambda \in [0, 2\pi[$ and $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, see Figure 20. We use also the well-known conversion formula from spherical to carthesian coordinates on the unit sphere:

$$(\lambda, \varphi) \mapsto (\cos\varphi \, \cos\lambda, \cos\varphi \, \sin\lambda, \sin\varphi)$$

7.2 An analytic solution In order to solve Schubert's extremal problem, we shall give a useful expression for the area. We fix two points A and B on the sphere which are at distance $c > 0$. We denote by $\mathcal{G}$ the line through these two points. Let $h \in \left(0, \frac{\pi}{2}\right)$. We denote by $\mathcal{E}(h)$, one of the equidistant lines (a small circle) from $\mathcal{G}$ at distance h. We can assume that $\mathcal{G}$ is the equator and $\mathcal{E}(h)$ is the small circle at distance h in the northern hemisphere. By using the well-known spherical coordinates on the unit sphere, we have without loss of generality

$$A = \left(\cos\left(\tfrac{c}{2}\right), -\sin\left(\tfrac{c}{2}\right), 0\right) \text{ and } B = \left(\cos\left(\tfrac{c}{2}\right), \sin\left(\tfrac{c}{2}\right), 0\right).$$

Furthermore, the equidistant curve $\mathcal{E}(h)$ is seen as the image of the mapping

$$t \in \mathbb{R} \mapsto (\cos(t)\cos(h), \sin(t)\cos(h), \sin(h)). \tag{90}$$

Hence, a triangle with base AB and a third vertex on $\mathcal{E}(h)$, has an area which depends on t. We denote by $A(t)$ the area of the triangle ABC_t where

$$C_t = (\cos(t)\cos(h), \sin(t)\cos(h), \sin(h)).$$

We then have a 2π-periodic function $A(t)$ and since the symmetry with respect to the plane $\{y = 0\}$ is an isometry, this function is an even function. Thus, we shall study $A(t)$ on the interval $[0, \pi]$. Let us first find the general expression for this function. The idea is quite elementary and consists of dividing the triangle

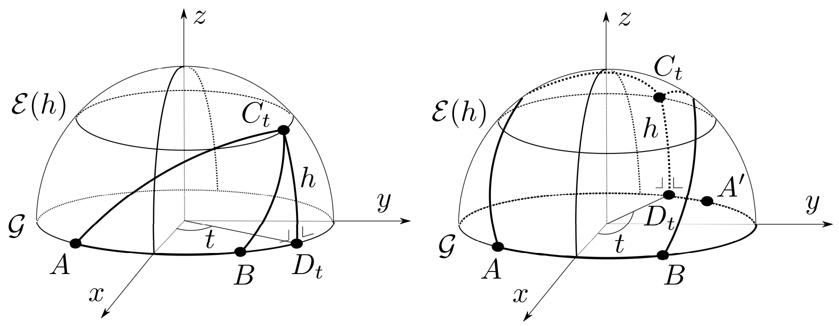

Figure 21: We draw using the spherical coordinates, two different sorts of triangle ABC_t. The point D_t is the foot of the altitude from C_t onto AB and the point A' is the antipodal point of A. On the left hand side, we have $\frac{c}{2} < t < \pi - \frac{c}{2}$. In this case the area of ABC_t is the same as the difference between the area of AD_tC_t and BD_tC_t. On the right hand side, since $\pi - \frac{c}{2} < t < \pi$ the segment AD_t has length equal to $2\pi - t - \frac{c}{2}$, and the sum of the areas of ABC_t, AD_tC_t and BD_tC_t is equal to the area of the northern hemisphere which is 2π.

ABC_t into two right-angled triangles and using the formula given by Lemma 17.

Let $t \in [0, \pi]$ and C_t be the corresponding point on $\mathcal{E}(h)$. By drawing the altitude from C_t, we obtain a point D_t on the line $\mathcal{G}$, and then two right-angled triangles AD_tC_t and D_tBC_t. Note that $D_t = (\cos(t), \sin(t), 0)$. Depending on the position of D_t (and then on the values of t) we have by using twice Lemma 17 (see also Figure 21) the following.
If $0 \leq t \leq \frac{c}{2}$, this means: if D_t is situated between the midpoint of AB and B, then

$$A(t) = \text{Area}(AD_tC_t) + \text{Area}(D_tBC_t),$$

where the length of AD_t is equal to $t + \frac{c}{2}$ and of BD_t is $\frac{c}{2} - t$. This gives us by Lemma 17:

$$A(t) = 2\arctan\left(\tan\left(\tfrac{h}{2}\right)\tan\left(\tfrac{t}{2} + \tfrac{c}{4}\right)\right) + 2\arctan\left(\tan\left(\tfrac{h}{2}\right)\tan\left(\tfrac{c}{4} - \tfrac{t}{2}\right)\right). \quad (91)$$

If $\frac{c}{2} \leq t < \pi - \frac{c}{2}$, this means: if D_t is between B and the antipodal point A' to A. In this case

$$A(t) = \text{Area}(AD_tC_t) - \text{Area}(D_tBC_t),$$

where length of AD_t is equal to $t + \frac{c}{2}$ and of BD_t is $t - \frac{c}{2}$. This gives us the same relation (91). This is the situation of Figure 21 on the left side.
And if $\pi - \frac{c}{2} < t \leq \pi$, this means that D_t lies between A' and the antipodal point to the midpoint of AB. In this case the sum of the areas of ABC_t, AD_tC_t and BD_tC_t is equal to the area of the northern hemisphere which is 2π. The length of AD_t is equal to $2\pi - t - \frac{c}{2}$ and of BD_t is $t - \frac{c}{2}$. This is situation of Figure 21 on the right side.

$$A(t) = 2\pi + 2\arctan\left(\tan\left(\tfrac{h}{2}\right)\tan\left(\tfrac{t}{2} + \tfrac{c}{4}\right)\right) + 2\arctan\left(\tan\left(\tfrac{h}{2}\right)\tan\left(\tfrac{c}{4} - \tfrac{t}{2}\right)\right). \quad (92)$$

We can now solve the problem by study of the function $A(t)$. Note that from (91), (92) and the parity, $A(t)$ is at least of class $\mathcal{C}^1$ on $\mathbb{R}$. Straight-forward

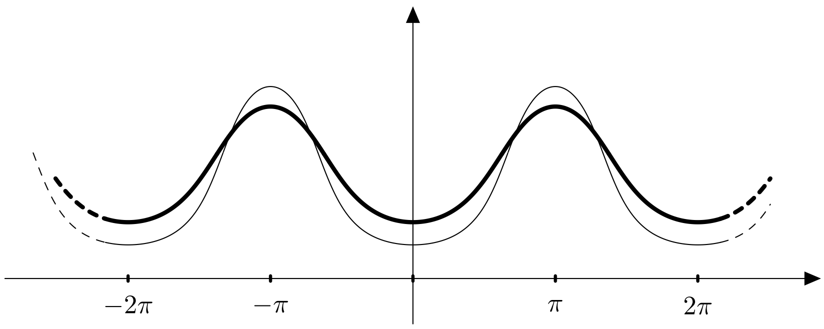

Figure 22: The graphs of the area function for two different pairs of values for c and h. The thick curve is for $h = \frac{2\pi}{7}$ and $c = \frac{3\pi}{4}$. The thin curve is for $h = \frac{\pi}{5}$ and $c = \frac{2\pi}{3}$.

computations give us that on $[0, \pi] \setminus \left\{\pi - \frac{c}{2}\right\}$

$$A'(t) = \frac{\tan\left(\frac{h}{2}\right)\left(1+\tan^2\left(\frac{t}{2}+\frac{c}{4}\right)\right)}{1+\tan^2\left(\frac{h}{2}\right)\tan^2\left(\frac{t}{2}+\frac{c}{4}\right)} - \frac{\tan\left(\frac{h}{2}\right)\left(1+\tan^2\left(\frac{t}{2}-\frac{c}{4}\right)\right)}{1+\tan^2\left(\frac{h}{2}\right)\tan^2\left(\frac{t}{2}-\frac{c}{4}\right)}$$

and then by setting

$$g(t) = \frac{\tan\left(\frac{h}{2}\right)}{\left(1+\tan^2\left(\frac{h}{2}\right)\tan^2\left(\frac{t}{2}+\frac{c}{4}\right)\right)\left(1+\tan^2\left(\frac{h}{2}\right)\tan^2\left(\frac{t}{2}-\frac{c}{4}\right)\right)}, \tag{93}$$

we have

$$\begin{aligned}\forall t \in [0, \pi] \setminus \left\{\pi - \frac{c}{2}\right\}, \ \frac{A'(t)}{g(t)} &= \left(1+\tan^2\left(\tfrac{t}{2}+\tfrac{c}{4}\right)\right)\left(1+\tan^2\left(\tfrac{h}{2}\right)\tan^2\left(\tfrac{t}{2}-\tfrac{c}{4}\right)\right) - \\ &\quad \left(1+\tan^2\left(\tfrac{t}{2}-\tfrac{c}{4}\right)\right)\left(1+\tan^2\left(\tfrac{h}{2}\right)\tan^2\left(\tfrac{t}{2}+\tfrac{c}{4}\right)\right) \\ &= \left(1+\tan^2\left(\tfrac{h}{2}\right)\right)\left(\tan^2\left(\tfrac{t}{2}+\tfrac{c}{4}\right) - \tan^2\left(\tfrac{t}{2}-\tfrac{c}{4}\right)\right).\end{aligned} \tag{94}$$

Finally from (93) and (94), we deduce that

$$\forall t \in (0, \pi), \ A'(t) > 0 \text{ and } A'(0) = A'(\pi) = 0.$$

As an illustration we give in Figure 22 two examples of the graph of $A(t)$.
In conclusion, we proved the following result which solves the Schubert's problem.

Theorem 15. *Let C_0 be the point of $\mathcal{E}(h)$ such that the foot of the altitude from it onto the line AB is the midpoint of AB. Let C_π be the point of $\mathcal{E}(h)$ such that the foot of the altitude from it to the line AB is the midpoint of $A'B'$, where A' and B' are the antipodal points of A and B, respectively. Then*

$$\forall C \in \mathcal{E}(h), \ \text{Area}(ABC_0) \leq \text{Area}(ABC) \leq \text{Area}(ABC_\pi).$$

Although the preceding proof is elementary, it uses some computations, so the natural question is about the existence of a proof which is entirely geometric. Such a proof exists and uses the notion of Lexell curve.

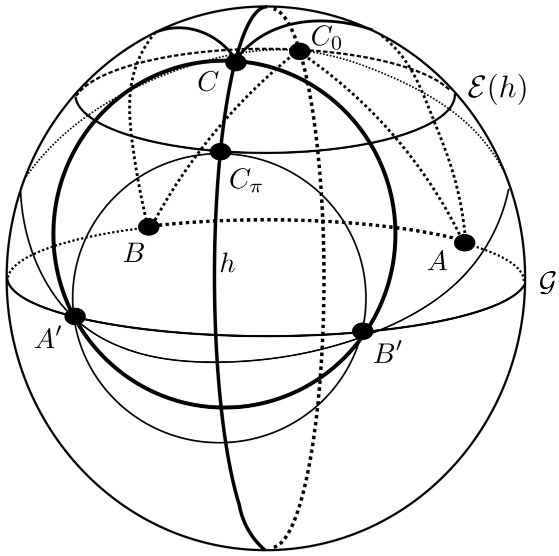

Figure 23: Three Lexell curves for a given base AB. The thickest Lexell curve intersects the equidistant curve $\mathcal{E}(h)$ at two points. The two other Lexell curves are extremal cases, where the intersection with $\mathcal{E}(h)$ consists of only one points.

7.3 The use of the Lexell curves The Lexell curve is the locus of third vertices of triangles with a given base and area. Let us recall Steiner's construction of Lexell curve. Let AB be a fixed segment on the unit sphere and C a third point on it which does not belong to the great circle containing the points A and B. We recall that A' and B' represent the antipodal points of A and B, respectively. The intersection between the sphere and the plane containing A', B' and C is a small circle which is the Lexell curve. More precisely, any point D on this circle defines a triangle ABD which has the same area as ABC. Furthermore, the corresponding pole of the Lexell curve belongs to the perpendicular bisector of AB.

Let us see now how Lexell curves are used in the resolution of the Schubert's problem. As before, we fix two points A, B and an equidistant curve $\mathcal{E}(h)$ which is at distance h from the line $\mathcal{G}$ which passes through A and B. Again, we denote by A' and B' the corresponding antipodal points. Let C_0 and C_π be the intersection points between $\mathcal{E}(h)$ and the perpendicular bisector of AB. The point C_0 being the closest point to the base AB. Let us take now a point C which lies inside the segment C_0C_π. We draw the corresponding Lexell curve; it is clear that such a curve intersects $\mathcal{E}(h)$ at two points. This process gives us a natural one to one correspondence between the segment C_0C_π and the equidistant curve $\mathcal{E}(h)$. Since the area strictly increases as C moves from C_0 to C_π, the minimal area is attained at C_0 and the maximal area is attained at C_π. This gives another proof of the Schubert's problem.

8 Schubert's hyperbolic problem

Theorem 16. *Among the hyperbolic triangles with a base AB and with the third vertex belonging to the equidistant set $\mathcal{E}(h)$, the isosceles triangles ABC and ABC' are the only triangles with the maximal area and the only triangles with the extremal area.*

8.1 Background in hyperbolic geometry The area of a hyperbolic triangle ABC is given, up to a scalar factor, by the angular dificit

$$\text{Area}(ABC) = \pi - \angle A - \angle B - \angle C. \tag{95}$$

For a simple proof we refer to [2]. In particular, the area of a right hyperbolic triangle with $\angle B = \frac{\pi}{2}$ becomes

$$\text{Area}(ABC) = \frac{\pi}{2} - \angle A - \angle C. \tag{96}$$

We continue with hyperbolic right-angled triangles by recalling some trigonometric relations. They were proven in Section 3.2. Let ABC be a right-angled hyperbolic triangle with the right angle B and with the (hyperbolic) lengths of catheta $|AB| = a$ and $|BC| = h$ and the (hyperbolic) measures of angles $\angle C = \alpha$ and $\angle A = \beta$ (see Figure 24), then

$$\tan\,\alpha = \frac{\tanh\,a}{\sinh\,h} \text{ and } \tan\,\beta = \frac{\tanh\,h}{\sinh\,a}. \tag{97}$$

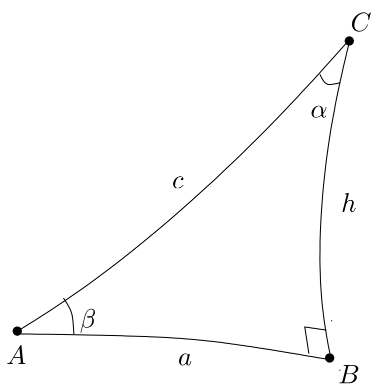

Figure 24: Right-angled hyperbolic triangle.

These trigonometric formulae can be proven in many different ways. In particular, a proof, which uses cinematic approach, can be found in [4] by J.-M. de Tilly. See also the essay [5] by D. Slutskiy.

Lemma 6 (Area Formula for Right Triangle). *Let T be a right hyperbolic triangle with the lengths of catheta a and h (see Figure 24). Then*

$$\tan\frac{\text{Area}(T)}{2} = \tanh\frac{a}{2}\,\tanh\frac{h}{2}.$$

Proof. Let ABC be a right-angled triangle as in Figure 24. From classical relations for hyperbolic[4] and trigonometric functions and from Relations (96) and (97) we get

$$\begin{aligned}\tan\left(\operatorname{Area}(ABC)\right) = \frac{1}{\tan\left(\alpha+\beta\right)} &= \frac{1-\tan\alpha\tan\beta}{\tan\,\alpha+\tan\,\beta}\\ &= \frac{1-\frac{1}{\cosh\,h\,\cosh\,a}}{\frac{\tanh\,h}{\sinh\,a}+\frac{\tanh\,a}{\sinh\,h}}\\ &= \frac{\sinh\,h\,\sinh\,a\left(\cosh\,h\,\cosh\,a-1\right)}{\sinh^2\,h\,\cosh\,a+\sinh^2\,a\,\cosh\,h}\\ &= \frac{\sinh\,h\,\sinh\,a\left(\cosh\,h\,\cosh\,a-1\right)}{\cosh^2\,h\,\cosh\,a-\cosh\,a+\cosh^2\,a\,\cosh\,h-\cosh\,h}\\ &= \frac{\sinh\,h\,\sinh\,c}{\cosh\,h+\cosh\,c}. \end{aligned} \tag{98}$$

Analogously,

$$\begin{aligned}\tan\left(2\arctan\left(\tanh\,\frac{h}{2}\tanh\,\frac{a}{2}\right)\right) &= \frac{2\,\tanh\,\frac{h}{2}\,\tanh\,\frac{a}{2}}{1-\tanh^2\frac{h}{2}\,\tanh^2\frac{a}{2}}\\ &= \frac{2\left(\cosh\,h-1\right)\left(\cosh\,a-1\right)}{\sinh\,h\,\sinh\,a\left(1-\left(\frac{\cosh\,h-1}{\sinh\,h}\right)^2\left(\frac{\cosh\,a-1}{\sinh\,a}\right)^2\right)}\\ &= \frac{2\,\sinh\,h\,\sinh\,a\left(\cosh\,h-1\right)\left(\cosh\,a-1\right)}{\sinh^2 h\,\sinh^2 a-\left(\cosh\,h-1\right)^2\left(\cosh\,a-1\right)^2}\\ &= \frac{\sinh\,h\,\sinh\,a}{\cosh\,h+\cosh\,a}. \end{aligned} \tag{99}$$

Comparing the two computations, we deduce the lemma. ☐

The two other proofs of this area formula are given later in Section 8.4. Lemma 17 is used to obtain a remarkable area formula for hyperbolic triangles in terms of altitude length $|CH| = h$ and of oriented segment lengths $|AH| = p_1$ and $|BH| = p_2$.

Corollary 2 (Area Formula for Arbitrary Triangle). *Let T be a hyperbolic triangle with an altitude of length h that divides a side of triangle into the two segments of lengths p_1 and p_2. Then*

$$\operatorname{Area}(T) = 2\arctan\left(\tanh\frac{p_1}{2}\,\tanh\frac{h}{2}\right) + 2\arctan\left(\tanh\frac{p_2}{2}\,\tanh\frac{h}{2}\right)$$

or equivalently,

$$\tan\frac{\operatorname{Area}(T)}{2} = \frac{\left(\tanh\frac{p_1}{2}+\tanh\frac{p_2}{2}\right)\,\tanh\frac{h}{2}}{1-\tanh\frac{p_1}{2}\,\tanh\frac{p_2}{2}\,\tanh^2\frac{h}{2}}.$$

Finally, let us recall a few facts about Poincaré half-plane model of hyperbolic plane. Poincaré half-model is a half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$, which is equipped with a hyperbolic metric $d_{\mathbb{H}}$:

$$d_{\mathbb{H}}\left(z_1, z_2\right) = 2\operatorname{arctanh}\left(\frac{|z_1 - z_2|}{|z_1 - \overline{z_2}|}\right) \quad \text{for all } z_1, z_2 \in \mathbb{H}.$$

[4] We essentially meant $\cosh^2\,x - \sinh^2\,x = 1$ and $\tanh\left(2x\right) = \frac{\cosh\,x-1}{\sinh\,x}$.

With respect to this metric, the isometries are either of the form

$$z \in \mathbb{H} \mapsto \frac{az+b}{cz+d} \text{ such that } (a,b,c,d) \in \mathbb{R}^4 \text{ and } ad-bc=1, \tag{100}$$

or of the form

$$z \in \mathbb{H} \mapsto \frac{a\overline{z}+b}{c\overline{z}+d} \text{ such that } (a,b,c,d) \in \mathbb{R}^4 \text{ and } ad-bc=-1. \tag{101}$$

In this model, the geodesic lines are represented by the half-circles with the centers situated on the real axis or by the vertical lines. The hypercycles to a given geodesic line are the circular arcs or the straight lines that share the end points with those of the geodesic line. These two kinds of hypercycles are elements of Figure 25 below.

8.2 An analytic solution of Schubert's problem We work in the Poincaré half-plane model. Let A and B be the two points at the hyperbolic distance a on a line l. Up to conjugacy by an isometry (a mapping of the form (100) or (101)) we can assume that B has a coordinate i and A has a coordinate $e^a i$. We reduce our consideration to the connected component m_h of $\mathcal{E}(h)$, since by aid of reflection with respect to the axis l the results can be extended to another connected component m_h'. In the Poincaré model, we assume that the hypercycle m_h is the half-straight line $\left\{\alpha e^{i\theta_h} \mid \alpha > 0\right\}$ where $\theta_h = 2\arctan\left(e^h\right)$. For our problem, we reparametrize m_h by setting C_t as a point on m_h with a coordinate $e^{\frac{a}{2}+2t}e^{i\theta_h}$ for $t \in \mathbb{R}$. According to this parametrization the area function $A(t) = \text{Area}(ABC_t)$ is an even function, since the area $A(t)$ remains invariant under reflection with respect to the perpendicular bisector of AB (corresponds to the isometry $z \mapsto \frac{e^l}{\overline{z}}$). Let D_t be the foot of the perpendicular from C_t to l. The coordinate of D_t is then $e^{\frac{a}{2}+2t}i$. By Corollary 2, the area of ABC_t is

$$A(t) = 2\arctan\left(\tanh\left(t+\frac{a}{4}\right)\tanh\frac{h}{2}\right) - 2\arctan\left(\tanh\left(t-\frac{a}{4}\right)\tanh\frac{h}{2}\right). \tag{102}$$

An immediate consequence of this formula is

$$\lim_{t\to-\infty} A(t) = 0 = \lim_{t\to+\infty} A(t). \tag{103}$$

A is a continuous even function that vanishes at infinity. We shall prove that the area function attains its unique maximum at $t=0$. It suffices to show that $t=0$ is the unique critical point of A.
We have for all $t \in \mathbb{R}$,

$$A'(t) = \frac{2\tanh\frac{h}{2}\left(1-\tanh^2\left(t+\frac{l}{4}\right)\right)}{1+\tanh^2\frac{h}{2}\tanh^2\left(t+\frac{l}{4}\right)} - \frac{2\tanh\frac{h}{2}\left(1-\tanh^2\left(t-\frac{l}{4}\right)\right)}{1+\tanh^2\frac{h}{2}\tanh^2\left(t-\frac{l}{4}\right)}.$$

By setting

$$g(t) = \left(1+\tanh^2\frac{h}{2}\tanh^2\left(t+\frac{l}{4}\right)\right)\left(1+\tanh^2\frac{h}{2}\tanh^2\left(t-\frac{l}{4}\right)\right), \tag{104}$$

we get

$$A'(t) = \frac{2\tanh\frac{h}{2}\left(1+\tanh^2\frac{h}{2}\right)\left(\tanh^2\left(t-\frac{l}{4}\right)-\tanh^2\left(t+\frac{l}{4}\right)\right)}{g(t)}. \tag{105}$$

The straightforward computation gives that $A'(t) = 0$ is equivalent to $t = 0$ if we exclude the degenerate cases (i.e. $a = 0$ or $h = 0$).
The unique maximum of A corresponds to the triangle ABC_0 with the property that the altitude C_0D_0 bisects AB. Thus, the triangle ABC_0 is isosceles, and Theorem 16 follows.

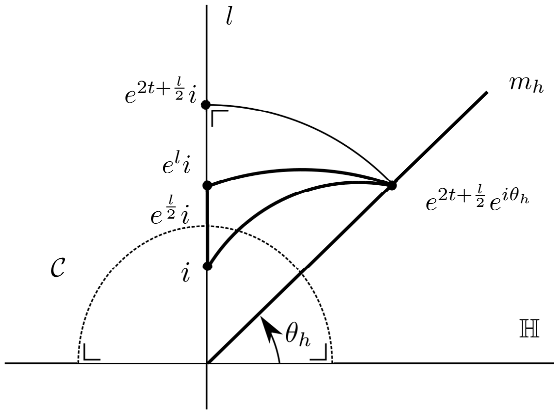

Figure 25: l is a geodesic line passing through the points A and B with the coordinates $e^a i$ and i, respectively. The (hyperbolic) distance between A and B is then a. m_h is an equidistant curve to l at (hyperbolic) distance h. Each point C_t of coordinate $e^{2t+\frac{a}{2}}e^{i\theta_h}$ on m_h gives two right-angled triangles. Our extremal problem is invariant under reflection with respect to the perpendicular bisector of AB, represented as the half-circle $\mathcal{C}$ with the center at the origin and the radius $e^{\frac{a}{2}}$.

8.3 A solution using Lexell curves Let us recall the definition of a *hyperbolic Lexell curve*. We fix a hyperbolic segment AB and we take a third point C, which is not on the geodesic containing AB. The question is about the locus of points that define a triangle of base AB and with the same area as ABC. Such a problem in spherical geometry is called the *Lexell problem*. We will call *Lexell curve* the solution of Lexell problem. Our proof is made in equidistant coordinates. We will use the equation of Lexell curve, given in Section 4.5.
Let A and B be two fixed points in the hyperbolic plane which lie on the geodesic line $\mathcal{G}$, and let h be a positive real number. We denote by C one of the two points at distance h from the segment AB such that ABC is isosceles. We still use the notation m_h for the equidistant curve of axis $\mathcal{G}$ which passes through C. We will give a solution using equidistant coordinates.
We introduce the coordinate system such that OX coincides with $\mathcal{G}$ and OY is on a perpendicular bisector of AB, (the origin is in O).
For every point D on an altitude OC, except of O and C, the corresponding Lexell curve intersects m_h in two points, D_1 and D_2. Indeed, the intersection points of corresponding Lexell curve

$$\cosh x \cosh y = \cot \frac{1}{2}\mathcal{A}(ABD) \sinh \frac{a}{2} \sinh y - \cosh \frac{a}{2}$$

and of equidistant curve m_h:

$$y = h$$

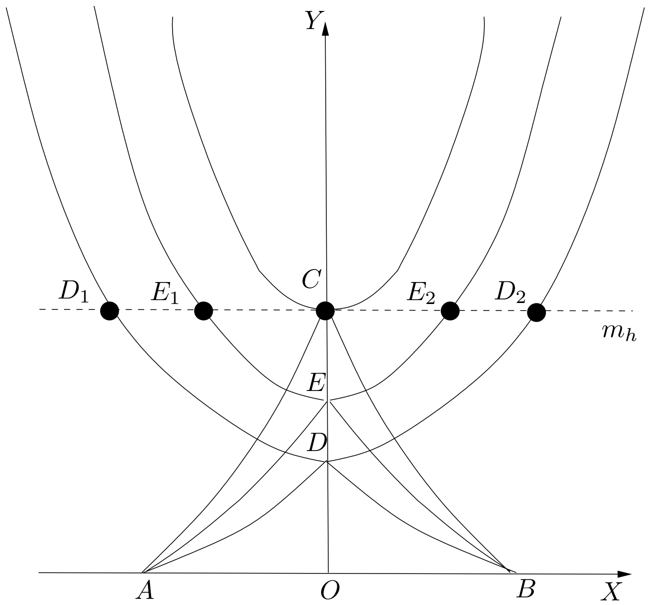

Figure 26: Solution of Schubert's problem: isoscele triangle ABC.

are $D_1 = (-x_1, h)$ and $D_2 = (x_1, h)$, where $\pm x_1$ satisfy

$$\cosh(\pm x_1) = \frac{\cot \frac{1}{2}\mathcal{A}(ABD) \sinh \frac{a}{2} \sinh h - \cosh \frac{a}{2}}{\cosh h}. \tag{106}$$

Then for every D on OC, the triangles ABD_1, ABD and ABD_2 have the same area.

Moreover, the bigger is the distance from AB to D, the bigger is the area of ABD and the closer the points of intersection D_1 and D_2 to C. Indeed, consider two Lexell curves $\mathcal{L}_D$ and $\mathcal{L}_E$ that correspond to the isosceles triangles ABD and ABE, where $|OD| < |OE|$ (see Figure 26). From (106) we have

$$\cosh(\pm x_1^D) = \frac{\cot \frac{1}{2}\mathcal{A}(ABD) \sinh \frac{a}{2} \sinh h - \cosh \frac{a}{2}}{\cosh h}. \tag{107}$$

$$\cosh(\pm x_1^E) = \frac{\cot \frac{1}{2}\mathcal{A}(ABE) \sinh \frac{a}{2} \sinh h - \cosh \frac{a}{2}}{\cosh h}. \tag{108}$$

Since $|OD| < |OE|$, the areas $\mathcal{A}(ABD) < \mathcal{A}(ABE)$ and $\cot \frac{1}{2}\mathcal{A}(ABD) > \cot \frac{1}{2}\mathcal{A}(ABE)$[5]. From (107) and (108) follows that $\cosh(\pm x_1^D) > \cosh(\pm x_1^E)$, i.e. $[-x_1^E, x_1^E] \subset [-x_1^D, x_1^D]$ in $\mathbb{R}$. Equivalently, the segment $[E_1, E_2]$ lies inside of $[D_1, D_2]$ on m_h.

We deduce that the maximum of area will be attained in a point $x_C = 0$, where the Lexell curve $\mathcal{L}_C$ is tangent to m_h. The corresponding triangle ABC is isoscele.

[5] $\cot(\cdot)$ is a monotonically decreasing function

Remark 4. Schubert's problem in $\mathbb{H}^3$*.* Given a tetrahedron $ABCD$ with a fixed base triangle ABC and a vertex D on an equidistant surface $\mathcal{E}(h)$ to the plane determined by ABC at distance $h > 0$. To find a position of D on $\mathcal{E}(h)$ such that $\mathrm{Vol}_3(ABCD)$ is maximal.
The natural conjecture for Schubert's problem is that the solution is *isoscele* tetrahedron, i.e. the tetrahedron with

$$\mathrm{Area}(ABD) = \mathrm{Area}(ACD) = \mathrm{Area}(BCD).$$

8.4 An Area formula for Right Triangles revisited In this Section we will give another two proofs for The Area formula for Right Triangles (Lemma 17). We will use

$$\tan\alpha = \frac{\tanh a}{\sinh h} \text{ and } \tan\beta = \frac{\tanh h}{\sinh a}. \tag{109}$$

Theorem 17 (Area Formula for Right Triangle). *Let ABC be a right hyperbolic triangle with the lengths of catheta a and h (see Figure 3). Then*

$$\tan\frac{\mathcal{A}}{2} = \tanh\frac{a}{2}\tanh\frac{h}{2}.$$

This proof is inspired by the proof of Euler's formula in [9]. This proof uses Relation (109) above.

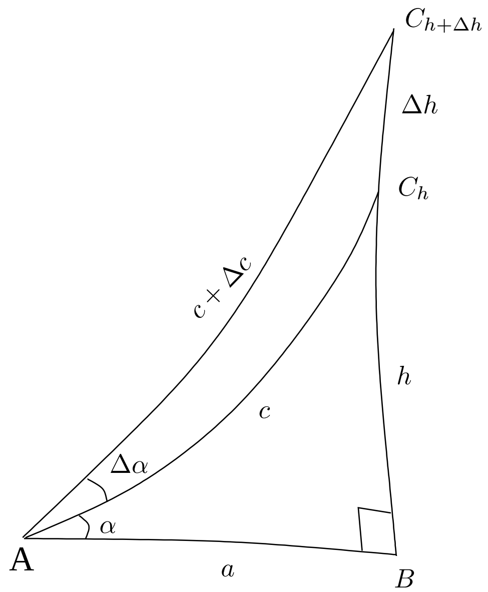

Figure 27: Right-angled hyperbolic triangle

Proof. Let ABC be a right hyperbolic triangle with a fixed cathetus a. We consider the area function $\mathcal{A}(h)$, which to every length of the second cathetus h assigns the area of a triangle $\mathrm{ABC_h}$ and in the same manner the functions $c(h)$

and $\alpha(h)$ (see Figure 27).
We have

$$\text{Area}(ABC_{h+\Delta h}) - \text{Area}(ABC_h) = \text{sgn}(\Delta\alpha)\,\text{Area}(AC_hC_{h+\Delta h}).$$

Moreover, $AC_hC_{h+\Delta h}$ is enclosed between two circular regions $K_{c,\Delta\alpha}$ and $K_{c+\Delta c,\Delta\alpha}$. Then by Lemma 1,

$$|\Delta\alpha|(\cosh c - 1) \leq \text{Area}(AC_hC_{h+\Delta h}) \leq |\Delta\alpha|(\cosh(c+\Delta c) - 1)$$

and

$$\Delta\alpha\,(\cosh c - 1) \leq \text{sgn}(|\Delta\alpha|)\text{Area}(AC_hC_{h+\Delta h}) \leq \Delta\alpha\,(\cosh(c+\Delta c) - 1)$$

Dividing by Δh and passing to the limit $\Delta h \to 0$ gives

$$A'(h) = \alpha'(h)(\cosh c(h) - 1)$$

or

$$d\mathcal{A} = (\cosh\ c - 1)\,d\alpha.$$

As next, we derive the trigonometric relation

$$\tan\ \alpha = \frac{\tanh\ h}{\sinh\ a} \tag{110}$$

to obtain

$$(1+\tan^2\alpha)\,d\alpha = \frac{1-\tanh^2\ h}{\sinh\ a}\,dh$$

and using (110)

$$\left(1+\frac{\tanh^2 h}{\sinh^2 a}\right)\,d\alpha = \frac{1-\tanh^2 h}{\sinh a}\,dh.$$

Then we obtain the differential relation

$$d\alpha = \frac{1-\tanh^2 h}{\sinh a\,(1+\frac{\tanh^2 h}{\sinh^2 a})}\,dh = \frac{\sinh a}{\cosh a\ \cosh h + 1}\,dh,$$

which can be integrated so that we finally get

$$T = \int_0^b \frac{\sinh a}{\cosh a\ \cosh h + 1}\,dh = 2\arctan\left(\tanh\frac{a}{2}\ \tanh\frac{h}{2}\right).$$

The last integral is computed using the "universal" trigonometric substitution $s = \tanh\frac{h}{2}$. ☐

Another proof uses Cagnoli's identities

Proof.

$$\tanh\frac{a}{2}\tanh\frac{h}{2} = \frac{\sinh\frac{a}{2}}{\cosh\frac{a}{2}}\frac{\sinh\frac{b}{2}}{\cosh\frac{b}{2}}$$

By Cagnoli's formulae,

$$\sinh\frac{a}{2}\sinh\frac{b}{2} = \frac{\sin\frac{\mathcal{A}}{2}}{\sin\gamma}\sqrt{\frac{\sin(\frac{\mathcal{A}}{2}+\alpha)\sin(\frac{\mathcal{A}}{2}+\beta)}{\sin\alpha\sin\beta}} \tag{111}$$

and

$$\cosh\frac{a}{2}\cosh\frac{b}{2} = \frac{\sin(\frac{\mathcal{A}}{2}+\gamma)}{\sin\gamma}\sqrt{\frac{\sin(\frac{\mathcal{A}}{2}+\alpha)\sin(\frac{\mathcal{A}}{2}+\beta)}{\sin\alpha\sin\beta}} \tag{112}$$

Dividing (111) by (112) we get

$$\tanh\frac{a}{2}\tanh\frac{b}{2} = \frac{\sin\frac{\mathcal{A}}{2}}{\sin(\frac{\mathcal{A}}{2}+\gamma)}.$$

Since $\gamma = \frac{\pi}{2}$, we obtain

$$\tanh\frac{a}{2}\tanh\frac{b}{2} = \frac{\sin\frac{\mathcal{A}}{2}}{\cos\frac{\mathcal{A}}{2}} = \tan\frac{\mathcal{A}}{2}.$$

□

9 Steiner's problem

9.1 Equation of Ellipse in equidistant coordinates

Definition 2. Let ABC be a triangle with lengths of base a and of two other sides, b and c. Let $d > a$. We call *ellipse* and denote by $\mathcal{H}(d)$ a geometric locus of points P such that $|AP| + |BP| = d$, or equivalently, $b + c = d$.

We position the coordinate system such that the origin is the midpoint of AB, the x-axis on a line passing through A and B, we call it $\mathcal{G}$ and the y-axis coincides with the perpendicular bisector (see Figure 28).
Let P be a point with coordinates (x, y), i.e. $|OP'| = x$ and $|MP'| = y$, where P' is a foot of a perpendicular from M to the x-axis.
From Pythagoras' Theorem, we have

$$\cosh(x+\frac{a}{2})\cosh y = \cosh b$$

$$\cosh(\frac{a}{2}-x)\cosh y = \cosh c$$

Then $b + c = d$ is equivalent to

$$\operatorname{acosh}\Big(\cosh(x+\frac{a}{2})\cosh y\Big) + \operatorname{acosh}\Big(\cosh y\cosh(x-\frac{a}{2})\Big) = d. \tag{113}$$

9.2 An analytic solution of Steiner's problem

Theorem 18. *In the hyperbolic plane, let AB be a segment of length a on a given line $\mathcal{G}$ and let $d \geq a$. Then there are four positions on the ellipse $\mathcal{H}$ such that the area of ABC is extremal at these points. These are the two isosceles triangles ABC and ABC' and two degenerate triangles ABL and ABR. In the case of ABC and ABC' the area is maximal. At the two degenerate cases we get the minima.*

The following proof does not use models.

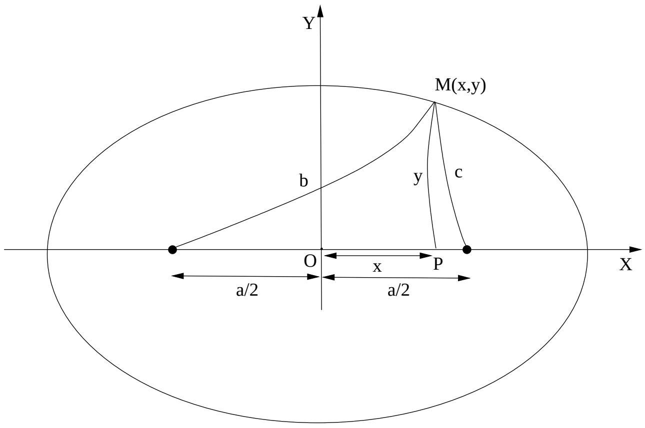

Figure 28: Ellipse $\mathcal{H}(d)$, $b + c = d$.

Proof. Given a triangle ABC. We assume that the length of AB is equal to some $a > 0$. The main idea of this proof is to vary the third point C of the triangle along the ellipse $\mathcal{H}(d)$. We recall that $\mathcal{H}(d)$ is given by the relation $b + c = d$, where b and c are the remaining sides of ABC. (see Figure 29).
We consider the area function $\mathcal{A} : \mathbb{H}^2 \to \mathbb{R}_{>0}$, which to every point $C \in \mathbb{H}^2$ assigns the area of ABC. We do not exclude the degenerate case, where C lies on a line $\mathcal{G}$ passing through A and B.
Analogous to (27) we have

$$d\mathcal{A} = (\cosh\, b - 1)\, d\alpha + (\cosh\, c - 1)\, d\beta. \tag{114}$$

Taking the derivative of the First Cosine Law, we want to pass from the differentials of the angles $d\alpha$ and $d\beta$ to the differentials of the sides db and dc

$$-\sin\,\alpha\, d\alpha = \frac{(-\cosh\, a + \cosh\, b\, \cosh\, c)\, db - \sinh\, b\, \sinh\, c\, dc}{\sinh\, a\, \sinh^2\, b}$$

and

$$-\sin\,\beta\, d\beta = \frac{(-\cosh\, a + \cosh\, b\, \cosh\, c)\, db - \sinh\, b\, \sinh\, c\, dc}{\sinh\, a\, \sinh^2\, c}.$$

Applying again the Cosine Law, we get an expression for $\sin\,\alpha$ and $\sin\,\beta$

$$\sin\alpha = \frac{w}{\sinh\, a\, \sinh\, b}$$

and

$$\sin\beta = \frac{w}{\sinh\, a\, \sinh\, c},$$

where w denotes

$$w = \sqrt{1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2\, \cosh\, a\, \cosh\, b\, \cosh\, c}.$$

This gives us the expressions for $d\alpha$ and $d\beta$, which we put into the formula (114) for $d\mathcal{A}$

$$d\mathcal{A} = \frac{(\cosh\, a - \cosh\, b\, \cosh\, c)\, db + \sinh\, b\, \sinh\, c\, dc}{\sinh\, b\, w}\, (\cosh b - 1)$$

$$+\frac{(\cosh\, a-\cosh\, b\, \cosh\, c)\, dc+\sinh\, b\, \sinh\, c\, db}{\sinh\, c\, w}\, (\cosh\, c-1).$$

Taking into consideration

$$db=-dc,$$

which comes from the relation $b+c=d$, the problem simplifies to one-dimensional. In the following, $c=c(b)=d-b$

$$d\mathcal{A}=(\frac{\cosh\, a-\cosh\, b\, \cosh\, c-\sinh\, b\, \sinh\, c}{\sinh\, b\, w}(\cosh\, b-1)+$$

$$\frac{\cosh\, b\, \cosh\, c+\sinh\, b\, \sinh\, c-\cosh a}{\sinh\, c\, w}(\cosh c-1))\, db,$$

which leads to

$$d\mathcal{A}=\frac{[\cosh(a)-\cosh(d)]\left(\frac{\cosh\, b-1}{\sinh\, b}-\frac{\cosh\, c-1}{\sinh\, c}\right)}{w}\, db.$$

Now, the critical points of the area function are the solutions of $d\mathcal{A}=0$. This relation is equivalent to

$$\cosh\, a=\cosh\, d \tag{115}$$

or

$$\sinh\,(b-c)=\sinh\, b-\sinh\, c. \tag{116}$$

Equation (115) gives the solution $a=\pm d$ and Equation (116) holds whenever $b=c$ or $b=0$ for all $c\in\mathbb{R}_{\geq 0}$ or $c=0$ for all $b\in\mathbb{R}_{\geq 0}$. These solutions correspond to the degenerate cases ($C=A$ or $C=B$) and the isosceles triangles (case where C lies on a perpendicular bisector of AB), see Figure 29.
It remains to verify the types of extrema at these points. We investigate the second derivative

$$\mathcal{A}''(b)=(\cosh\, a-\cosh\, d)\,\frac{\left(\frac{1}{2\cosh^2\frac{b}{2}}+\frac{1}{2\cosh^2\frac{c}{2}}\right)\, w-(\tanh\,\frac{b}{2}-\tanh\,\frac{c}{2})\, w'}{w^2},$$

therefore

$$\mathcal{A}''(b)|_{b=c}=(\cosh\, a-\cosh\, d)\,\frac{1}{\cosh^2\frac{b}{2}}<0,$$

since $d>a$. Thus, the isosceles triangle is a maximum for the area function among all the triangles with a fixed base and the fixed sum of two other sides d.
Moreover, we obtain that in the degenerate cases the second derivative does not exist. □

Remark 5. Steiner's problem in $\mathbb{H}^3$. Given a tetrahedron $ABCD$ with a fixed base triangle ABC. Let $\mathcal{H}(d)$ be a set of points D such that $\text{Area}(ABD)+\text{Area}(BCD)+\text{Area}(ACD)$ is constant and equal to $d>0$. To find position(s) of D on $\mathcal{H}(d)$ such that $\text{Vol}_3(ABCD)$ is maximal.
The natural conjecture for Steiner problem is that the solution is *isoscele* tetrahedron, i.e. the tetrahedron with

$$\text{Area}(ABD)=\text{Area}(ACD)=\text{Area}(BCD).$$

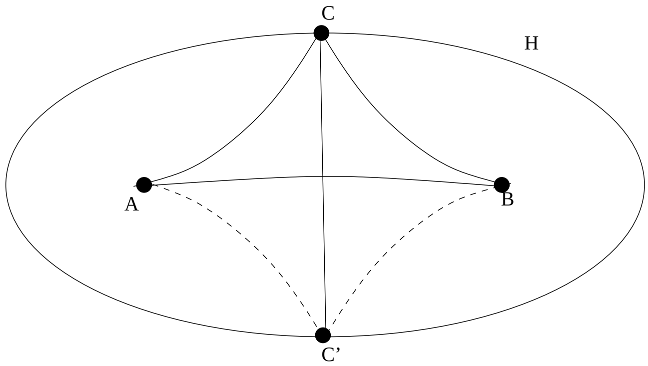

Figure 29: Two isosceles triangles, ABC and ABC', give the maximal area among the triangles with third vertex on $\mathcal{H}(d)$.

10 On a result of A'Campo concerning the variation of volume of polyhedra in non-Euclidean geometry (Schläfli formula)

The variation of the volume for a smooth one-parametric family P_t of compact polyhedra depending on a real parameter t on the sphere S^n was given by Schläfli, and extended by Sforza to the hyperbolic space $H^n, n \geq 2$ (see [16] for more references). The result of Schläfli is a generalisation of first order area calculation by angle excess or defect of polygons on the sphere S^2 or on the hyperbolic plane after the classical formula of Albert Girard. In a talk in Cagliari, 2015, A'Campo gave the sketch of the proof using integral geometry which says that the classical result in dimension 2 for area implies the Schläfli theorem in higher dimensions. In our work, we give this result of A'Campo.
The classical Crofton formula is a formula in Euclidean integral geometry. It gives the length of a segment in Euclidean n-space in terms of the measure of the set of hyperplanes that intersect this segment. For this, one needs a measure on the set of hyperplanes in $\mathbb{R}^n$.
Let I be a segment in $\mathbb{E}^n$, we denote by $\#(I \cap E)$ the number of intersections of the segment I with an Euclidean hyperplane E. The formula in the Euclidean setting is the following:

$$\text{Length}(I) = c_{1,n-1} \int_{\text{Gr}_{n-1}(\mathbb{E}^n)} \#(I \cap E)\, d\mu_{Gr_{n-1}(\mathbb{E}^n)}(E).$$

The case of dimension 2 is treated in [20], p.12-13. In this case, the constant $c_{1,2} = 2$. There is a generalization of the Crofton formula for the sphere and the hyperbolic space. The hyperbolic case of dimension 2 is treated in [21], p. 691. In this case, the constant is $c_{1,2} = 2$ as above.
Let us consider the principle for construction of Crofton formula. Let P be a compact polyhedron or submanifold of dimension $l \leq n$ in $\mathbb{M}^n$, where $\mathbb{M}^n = \mathbb{S}^n$ or $\mathbb{H}^n$, $n \geq 2$. Let $Gr_k(\mathbb{M}^n)$ be the space of k-dimensional totally geodesic connected maximal subspaces E in $\mathbb{M}^n$. Let $\mu_{Gr_k(M^n)}$ be a measure on $Gr_k(\mathbb{M}^n)$, which is invariant by isometries of $\mathbb{M}^n$. This measure is defined up to multiplication by a constant factor $c \in \mathbb{R} \setminus \{0\}$. Let f_P be the function on $Gr_k(\mathbb{M}^n)$ that

assigns to every $E \in Gr_k(\mathbb{M}^n)$ the volume $\mathrm{Vol}_u(P \cap E)$, where $u = k + l - n$ and this volume is induced from the Riemannian metric g of Riemannian manifold $\mathbb{M}^n$. This function is integrable with respect to $\mu_{Gr_k(\mathbb{M}^n)}$. *The Crofton formula* says that there exists a constant $c_{k,l}(\mathbb{M}^n)$ such that

$$\int_{Gr_k(\mathbb{M}^n)} f_P(E)\, d\mu_{Gr_k(\mathbb{M}^n)}(E) = c_{k,l}(\mathbb{M}^n)\,\mathrm{Vol}_l(P) \tag{117}$$

The Crofton formula is valid for $u \geq 0$.
For $n \geq 2$, let $(P_t)_{t\in\mathbb{R}}$ be a smooth family of n-dimensional compact convex polyhedra in $\mathbb{H}^n$. We assume that the family is generic in the sense that for every t, the interior of P_t is dense in P_t. For each $t \in \mathbb{R}$, and for each $0 \leq l \leq n$, let $\mathcal{F}_l(P_t)$ be the set of l-dimensional faces of P_t. An $(n-2)$-face $K_t \in \mathcal{F}_{n-2}(P_t)$ is the intersection of two $(n-1)$-faces K'_t, $K''{}_t \in \mathcal{F}_{n-1}(P_t)$. Let $\theta(K_t) \in [0, \pi[$ be the angle of P_t along K_t, this is the dihedral angle between K'_t, K''_t and also between the hyperplanes that carry the faces K'_t, K''_t.
Schläfli's formula is a formula for the variation of the volume of (P_t), that is, the derivative $\frac{d}{dt}\mathrm{Vol}_n(P_t)$:

$$\frac{d}{dt}\mathrm{Vol}_n(P_t) = -\frac{1}{n-1} \sum_{K_t\in\mathcal{F}_{n-2}(P_t)} \mathrm{Vol}_{n-2}(K_t)\frac{d}{dt}\theta(K_t).$$

Our aim is to give a proof of this formula using integral geometry. In a first step it is shown that there exists a constant s_n that does not depend on the family (P_t) such that the following formula holds:

$$\frac{d}{dt}\mathrm{Vol}_n(P_t) = -s_n \sum_{K_t\in\mathcal{F}_{n-2}(P_t)} \mathrm{Vol}_{n-2}(K_t)\frac{d}{dt}\theta(K_t).$$

The value $s_n = \frac{1}{n-1}$ will be determined later.
The main tools for the proof are two Crofton type formulae and *Girard's theorem.* Girard's formula is a formula for the area of a non- Euclidean triangle given by angle defect $\pi - (\alpha + \beta + \gamma)$ in the hyperbolic case, where α, β and γ are the angles of the triangle.
In integral geometry, one first needs a measure. Let $Gr_2(\mathbb{H}^n)$ be the manifold consisting of the two dimensional totally geodesic connected maximal subspaces E in $\mathbb{H}^n$. In $\mathrm{Gr}_2(\mathrm{H}^n)$ the notation stands for *Grassmanian.* It reminds us of the name Grassmanian of a finite-dimensional vector space, which is a space parametrizing the set of vector spaces of a certain dimension. Here, $Gr_2(\mathbb{H}^n)$ is the set of 2-planes of $\mathbb{H}^n$. It is more complicated than the usual Grassmannian because the setting of hyperbolic geometry is , unlike $\mathbb{R}^n$, a non-linear setting. The group of isometries G_n of $\mathbb{H}^n$ acts transitively on $Gr_2(\mathbb{H}^n)$. We shall use the fact that up to scaling there exists a unique positive measure $\mu_{Gr_2(\mathbb{H}^n)}$ on $Gr_2(\mathbb{H}^n)$ that is G_n-invariant.
Let U be an open bounded subset in $\mathbb{H}^n$. Define

$$\nu(U) = \int_{Gr_2(\mathbb{H}^n)} \mathrm{Vol}_2(E \cap U)\, d\mu_{Gr_2(\mathbb{H}^n)}(E)$$

The assignment $U \mapsto \nu(U)$ is strictly increasing with respect to inclusion of open subsets in $\mathbb{H}^n$, additive with respect to unions of disjoint open sets and hence generates a measure on $\mathbb{H}^n$. This measure is invariant by G_n and hence equal up to scaling to the Riemannian measure Vol_n of hyperbolic space $\mathbb{H}^n$. By rescaling the measure $\mu_{Gr_2(\mathbb{H}^n)}$ one obtains a first Crofton type formula

$$Vol_n(P_t) = \nu(P_t) = \int_{Gr_2(\mathbb{H}^n)} \mathrm{Vol}_2(E \cap P_t)\, d\mu_{Gr_2(\mathbb{H}^n)}(E).$$

A half-space in $\mathbb{H}^n$ is one of the two components bounded by a codimension-1 totally geodesic maximal connected subspace. Define a *sector* S as the union of two half spaces A, B in $\mathbb{H}^n$. Define a *roof* R as a triple (A, B, K) where K is a polyhedron of dimension $(n-2)$ that is contained in the intersection $\partial A \cap \partial B$ of the boundaries of the half spaces A, B. We call $\partial A \cap \partial B$ the *back* of the sector S.

Let R be a roof. Define $\alpha(R)$ to be the integral

$$\alpha(R) = \int_{Gr_2(\mathbb{H}^n)} \alpha(E; A, B, K)\, d\mu_{Gr_2(\mathbb{H}^n)},$$

where $\alpha(E; A, B, K) \in [0, \pi]$ equals 0 if $E \cap K = \emptyset$ and the angle of the two-dimensional sector $E \cap (A \cup B)$ (see Figure 30).

The quantity $\alpha(E; A, B, K) \in [0, \pi]$ is equal to the dihedral angle $\theta(A, B)$ of

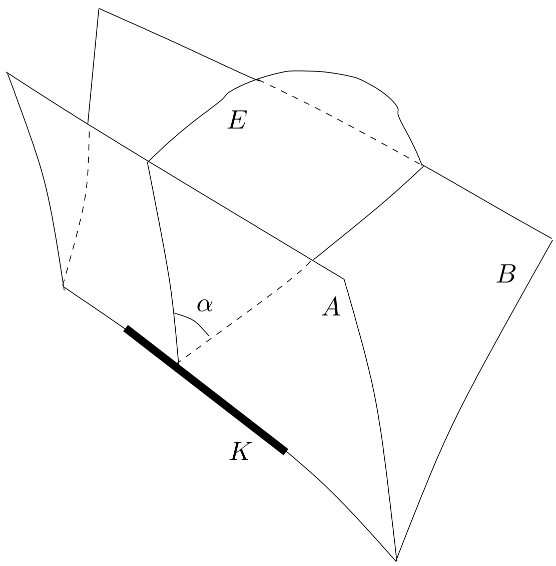

Figure 30: Roof $R = (A, B, K)$ and surface angle $\alpha(E; A, B, K)$.

the sector $A \cup B$ if E intersects back of the sector $A \cup B$ under a right angle and else $\alpha(E; A, B, K)$ may exceed the dihedral angle $\theta(A, B)$.

The quantity $\alpha(E; A, B, K)$ is monotone and additive with respect to K and moreover additive if one subdivides the sector $A \cup B$ by the hyperplanes that contain the back of the sector. It follows that there exists a constant s_n such that a second type of Crofton formula

$$\alpha(E; A, B, K) = s_n \mathrm{Vol}_{n-2}(K)\theta(A, B)$$

holds.
In the above integrals one can neglect all E's that do not intersect the faces of various dimensions of P_t transversally. We denote the remaining domain of integration by $Gr_{2,T}(\mathbb{H}^n)$. Indeed, the set of E's that intersect a face F_t of P_t non-transversally is of $\mu_{Gr_2(\mathbb{H}^n)}$ - measure zero. More clearly, $E \in Gr_{2,T}(\mathbb{H}^n)$ if and only if $E \cap F_t = \emptyset$ for all $F_t \in \mathcal{F}_{n-3}(P_t)$.
By using the Dominated Convergence Theorem of Lebesgue one gets the possibility of inverting differentiation and integration. Hence

$$\frac{d}{dt}\mathrm{Vol_n(P_t)} = \frac{d}{dt}\int_{Gr_2(\mathbb{H}^n)} \mathrm{Vol}_2(E \cap P_t)\, d\mu_{Gr_2(\mathbb{H}^n)} =$$

$$\int_{Gr_2(\mathbb{H}^n)} \frac{d}{dt}\mathrm{Vol_2(E \cap P_t)}\, d\mu_{Gr_2(\mathbb{H}^n)} =$$

$$-\sum_{K_t \in \mathcal{F}_{n-2}(P_t)} \int_{Gr_{2,T}(\mathbb{H}^n)} \frac{d}{dt}\alpha(E; K_t', K_t'', K_t)\, d\mu_{Gr_2(\mathbb{H}^n)} =$$

$$-s_n \sum_{K_t \in \mathcal{F}_{n-2}(P_t)} \mathrm{Vol}_{n-2}(K_t)\, \frac{d}{dt}\theta(K_t', K_t'').$$

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Elena FRENKEL

Sur l’aire et le volume en géométrie sphérique et hyperbolique

Résumé

L’objet de ce travail est de prouver des théorèmes de géométrie hyperbolique en utilisant des méthodes développées par Euler, Schubert et Steiner en géométrie sphérique. On donne des analogues hyperboliques de certaines formules trigonométriques en utilisant la méthode des variations et une formule pour l’aire d’un triangle. Euler utilisa cette idée en géométrie sphérique. On résout ensuite le problème de Lexell en géométrie hyperbolique. Cette partie est basée sur un travail en collaboration avec Weixu Su. En utilisant l’analogue hyperbolique des identités de Cagnoli, on prouve deux résultats classiques en géométrie hyperbolique. Ensuite, on donne les solutions aux problèmes de Schubert (en collaboration avec Vincent Alberge) et de Steiner. En suivant les idées de Norbert A’Campo, on donne l’ébauche de la preuve de la formule de Schläfli en utilisant la géométrie intégrale. Cette recherche peut être généralisée partiellement au cas de la dimension 3.

Mots-clés : aire – volume – géométrie hyperbolique – géométrie sphérique – problème de Lexell – formule de l’aire d’Euler – problème de Schubert – formule de Schläfli

Résumé en anglais

Our aim is to prove some theorems in hyperbolic geometry based on the methods of Euler, Schubert and Steiner in spherical geometry. We give the hyperbolic analogues of some trigonometric formulae by method of variations and an area formula in terms of sides of triangles, both due to Euler in spherical case. We solve Lexell’s problem. This is a joint work with Weixu Su. We give a shorter formula than Euler’s area formula. Using hyperbolic analogues of Cagnoli’s identities, we prove two classical results in hyperbolic geometry. Further, we give solutions of Schubert’s and Steiner’s problems. The study of Schubert’s problem is a joint work with Vincent Alberge. Finally, following ideas of Norbert A’Campo, we give the sketch of the proof of Schläfli formula using integral geometry. The mentioned theorems can be generalized to the case of dimension 3 partially by means of the techniques used developed in this thesis.

Keywords: area – volume – hyperbolic geometry – spherical geometry – Lexell’s problem – Euler’s area formula – Schubert’s problem – Schläfli formula