Thèse

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE

UMR 7501

Strasbourg

présentée pour obtenir le grade de docteur de
l'Université de Strasbourg
Spécialité MATHÉMATIQUES

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Stabilisation et asymptotique spectrale de l'équation des ondes amorties vectorielle

Soutenue le 12 décembre 2018
devant la commission d'examen

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Remerciements

Le moment fatidique d'écrire mes remerciements est enfin arrivé, c'est l'occasion de regarder le chemin parcouru et de repenser à tous ces moments qui ont constitué mes trois années (et quelques) de thèse. Cette période de ma vie aura vraiment été merveilleuse pour moi, à tout point de vue. je souhaite donc remercier ici tous ceux qui ont rendu cette thèse à la fois possible et si agréable.

Je voudrais évidemment commencer par exprimer toute ma gratitude à ma directrice de thèse Nalini Anantharaman. Merci de m'avoir encadré tout au long de cette thèse dont le sujet m'intéresse tant, tes explications ont toujours été éclairantes et j'ai beaucoup appris grâce à toi. Je voudrais aussi te remercier pour tes qualités humaines, tu as toujours su faire preuve de gentillesse et de bienveillance à mon égard, même lorsque je mettais ta patience à rude épreuve. Sans toi cette thèse n'aurait pas vu le jour et sans ces qualités elle n'aurait pu être agréable, merci encore une fois.

Un merci très chaleureux à Thomas Duyckaerts et Michael Hitrik qui ont accepté de rapporter ma thèse, mais aussi de m'apporter sans hésiter leur aide et leur soutien pour des candidatures de post-doc bien tardives. Je vous dois une fière chandelle ! Merci aussi à Karine Beauchard, Raphaël Côte et Johannes Sjöstrand qui me font l'honneur d'avoir accepté de faire partie de mon jury de thèse.

Puisque cette thèse marque la fin de mes études à l'Université de Strasbourg je voudrais en profiter pour remercier les divers professeurs que j'ai pu avoir et qui ont su m'encourager, me conseiller et m'aider lorsque j'en avais besoin. Votre pédagogie et vos enseignements on grandement contribué à mon amour pour les mathématiques. Je remercie donc en particulier (par ordre d'apparition) : Claudine Mitschi, Vincent Blanloeil, Pierre Guillot, Gaël Collinet, Adriano Marmora, Michèle Audin, Vilmos Komornik et Reinhard Schäfke. Un merci tout spécial à Patrick Gérard (de l'Université d'Orsay) pour avoir encadré mon mémoire de M2 et m'avoir mis sur la piste de mon sujet de thèse. Pour finir, et même s'il n'a pas été mon professeur, je remercie Fabricio Macià pour son invitation à Madrid, le séjour fut des plus agréables et enrichissants.

C'est assez inhabituel et, bien qu'il y ait peu de chance qu'ils me lisent un jour, je souhaite remercier les étudiants que j'ai pu avoir pendant mon monitorat. Même s'ils ont parfois été pénibles j'ai, dans l'ensemble, beaucoup aimé faire cours à l'Université de Strasbourg et, après tout, c'est bien à eux que je le dois. Je remercie aussi les canards du patio d'avoir égayé mes nombreuses pauses et le SUAPS pour m'avoir permis de pratiquer l'escalade à deux pas de mon bureau. Enfin, je profite de ce paragraphe un peu hors catégorie pour remercier Myriam Pepino pour son aide avec le papier, mes origamis lui doivent beaucoup.

Cette thèse n'aurait pas eu la même saveur (houblon, kimchi, mexikaner...) sans les nombreux doctorants, post-doc et assimilés que j'ai pu côtoyer à l'UFR. La liste de personnes est longue, autant ne pas perdre de temps ! Merci d'avoir fait partie de ma vie pendant ces 3 ans à : Guillaume, Audrey, Nicolas et Agathe, Vincent, Simon, Amaury, Florian et Claire, Stéphane et Élodie, Mohamad, Arthur, Philippe et Claire, Anderson, Pierre, P.A., Yann, Emmanuelle, Alix, Marie, Martin, Laura, Francisco, Cuong, Philippe et Aurélia, Gatien et Pauline, Alexander, Frédéric, Georgios, Thomas... et ceux que j'oublie sans doute. Mention spéciale à mes co-bureaux et aux fidèles du groupe de travail du jeudi soir ! Je n'oublie pas non plus tous mes amis «hors fac» ou «hors thèse» qui m'accompagnent pour la plupart depuis bien avant la thèse et, je l'espère, continueront de le faire bien après.

Je voudrais maintenant remercier ma famille : mon père, ma mère et mes deux sœurs. Je ne sais pas vraiment par quoi commencer, il y aurait tant à dire ! Que dire sinon que c'est grâce à vous que je suis devenu la personne que je suis aujourd'hui, je vous dis donc merci pour tout.

Pour finir, mes derniers remerciements sont évidemment pour toi, Élise. Tu m'apportes plus que je ne saurais l'écrire en ces quelques lignes qu'il me reste. Merci à toi de partager ma vie jour après jour, de m'apporter ton amour et, tout simplement, de me rendre heureux.

Table des matières

Introduction **1**

1 Quelques résultats préliminaires **9**
1.1 Stabilisation faible 9
1.2 Opérateurs pseudodifférentiels semi-classiques 11
1.3 Mesures de défaut microlocales 15

2 Stabilisation de l'équation des ondes vectorielle **19**
2.1 Introduction 20
2.2 Propagation of the microlocal defect measure 25
2.3 Estimation of the best decay rate 30
2.3.1 Upper bound for α 31
2.3.2 Lower bound for α 35
2.3.3 End of the proof of Theorem 2.3 and proof of Theorem 2.2 37
2.4 Behavior of C_∞ 39
2.5 Addenda to the second chapter 45

3 Asymptotique spectrale pour l'opérateur A_a **49**
3.1 Introduction 50
3.1.1 Setting 50
3.1.2 Results 51
3.1.3 An open question 54
3.1.4 Plan of the article 54
3.2 Proof of Theorem 3.3 55
3.3 Proof of Proposition 3.1 59
3.3.1 Idea of the proof 59
3.3.2 Construction of the operator $R(z)$ 60
3.3.3 Proof of Proposition 3.2 62
3.3.4 End of the proof of Proposition 3.1 69
3.4 Proof of Proposition 3.4 70

A Semi-classical anti-Wick quantization **73**

B Multiplicative ergodic theorem of Oseledets **75**

C A formula for the Lyapunov exponents of G **77**

Bibliographie **81**

Introduction

Équation des ondes amorties

Considérons une membrane vibrante Ω, comme par exemple une peau de tambour, fixée à son bord et notons $u(t,x)$ le déplacement transversal de la membrane à la position x et au temps t. On sait alors que la fonction u est solution de l'équation suivante

$$(\partial_t^2 - \Delta)u = 0$$

où $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ représente le laplacien à deux dimensions. Cette équation est naturellement nommée équation des ondes. On peut alors rajouter un terme à cette équation, étant donné une fonction lisse $a : \Omega \to \mathbf{R}$ on considère l'équation

$$(\partial_t^2 - \Delta + 2a\partial_t)u = 0.$$

On parle alors d'équation des ondes amorties et la fonction a est appelée amortisseur. Nous verrons un peu plus tard que lorsque a est une fonction positive le mot «amortisseur» est justifié. Cette équation admet diverses généralisations, voyons maintenant celle qui va nous intéresser.

Soit (M,g) une variété riemannienne sans bord, compacte, connexe, de classe $\mathscr{C}^\infty$ et de dimension d. Soit $a : M \to \mathscr{H}_n^+(\mathbf{C})$ une fonction de classe $\mathscr{C}^\infty$ à valeurs dans les matrices hermitiennes positives. L'équation aux dérivées partielles que nous étudierons est la suivante :

$$\begin{cases} (\partial_t^2 - \Delta + 2a(x)\partial_t)u = 0 \text{ dans } \mathcal{D}'(\mathbf{R}\times M)^n \\ u_{|t=0} = u_0 \in H^1(M)^n \text{ et } \partial_t u_{|t=0} = u_1 \in L^2(M)^n \end{cases} \tag{1}$$

où Δ est l'opérateur de Laplace-Beltrami sur M. Notons $H = H^1(M)^n \oplus L^2(M)^n$ l'espace des conditions initiales, on définit sur cet espace l'opérateur non borné

$$A_a = \begin{pmatrix} 0 & \mathrm{Id}_n \\ \Delta & -2a \end{pmatrix} \text{ de domaine } D(A_a) = H^2(M)^n \oplus H^1(M)^n.$$

Le théorème de Hille-Yosida garantit alors l'existence d'une unique solution de (1) dans l'espace $C^0(\mathbf{R}, H^1(M)^n) \cap C^1(\mathbf{R}, L^2(M)^n)$. À partir de maintenant nous identifierons l'espace H des conditions initiales avec l'espace des solutions de (1). Soit u une solution de (1), on définit $E(u,t)$ son énergie au temps t comme

$$E(u,t) = \frac{1}{2} \int_M |\partial_t u(t,x)|^2 + |\nabla u(t,x)|_g^2 \mathrm{d}x$$

où $|\partial_t u(t,x)|$ est la norme euclidienne de $\partial_t u(t,x)$ et $|\nabla u(t,x)|_g^2 = g_x(\nabla u(t,x), \nabla u(t,x))$. Lorsque la fonction a est hermitienne positive on montre par une intégration par parties que l'énergie est décroissante. L'équation (1) permet donc de modéliser les vibrations d'une membrane ou d'un milieu en présence d'un amortisseur.

L'équation des ondes amorties est étudiée activement depuis maintenant plusieurs décennies. Les questions qui lui sont liées sont variées et les techniques utilisées pour répondre à ces questions trouvent des applications dans de nombreux problèmes d'analyse et plus spécifiquement de contrôle des équations aux dérivées partielles. Citons par exemple les problèmes de stabilisation, d'observabilité et de contrôle de l'équation des ondes qui font appel à des techniques et des outils communs.

Bien que l'équation des ondes amorties ait été étudiée activement et dans divers cadres, la majeure partie des résultats concernent l'équation *scalaire*, c'est à dire le cas $n = 1$. Dans cette thèse nous nous intéresserons à ce qu'il se passe pour une équation *vectorielle*, c'est à dire lorsque $n \geq 1$. L'étude de (1) pour $n \geq 1$ est intéressante pour plusieurs raisons. Tout d'abord, il s'agit d'une généralisation tout à fait naturelle du cas $n = 1$. C'est aussi une équation des ondes vectorielles simple et son étude peut permettre d'améliorer la compréhension d'autres équations d'ondes vectorielles telles que l'équation de Lamé ou les équations de Maxwell. Pour finir il s'agit d'un problème mathématique possédant un intérêt intrinsèque. Ceci est d'ailleurs vérifié *a posteriori* par l'apparition de nouveaux phénomènes lorsque $n > 1$.

Le premier chapitre de cette thèse contient quelques résultats classiques sur l'équation des ondes amorties. C'est aussi l'occasion d'introduire deux outils qui nous seront utiles par la suite, à savoir les opérateur pseudodifférentiels et les mesures de défaut microlocales.

Le deuxième chapitre constitue la première moitié des travaux de cette thèse et est une reproduction de l'article [Kle17], accepté pour publication chez SIAM Journal on Control and Optimization. Il porte sur le problème de la stabilisation de l'équation des ondes, c'est à dire l'étude du comportement de l'énergie des solutions de (1) lorsque le temps t tend vers l'infini.

Le troisième chapitre constitue la seconde moitié des travaux de cette thèse, il s'agit d'une reproduction de [Kle18] qui n'a pas encore été soumise pour publication. Cette partie porte sur le spectre de l'opérateur A_a, on y présente quelques résultats sur la répartition asymptotique des valeurs propres de A_a.

Nous allons maintenant présenter les principaux résultats des Chapitres 2 et 3.

Stabilisation de l'équation des ondes

Si l'on dérive $E(u,t)$ par rapport au temps et que l'on fait une intégration par parties on trouve la formule de l'énergie :

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u,t) = -\int_M \langle 2a(x)\partial_t u(t,x), \partial_t u(t,x)\rangle_{\mathbf{C}^n}\,\mathrm{d}x. \tag{2}$$

L'énergie est donc décroissante au cours du temps si a est hermitienne positive, ce qui justifie le choix du mot «amortisseur». Commençons par énoncer un résultat classique.

Théorème 1 *Les conditions suivantes sont équivalentes :*

(i) $\forall u \in H, \ \lim_{t\to\infty} E(u,t) = 0.$

(ii) *La seule valeur propre de A_a sur l'axe imaginaire est 0.*

(iii) *Soit $u \in H^2(M)^n$ une fonction propre de Δ, si $au = 0$ alors $u = 0$.*

Si l'une de ces conditions est vérifiée on parle alors de stabilisation faible. Il est à noter que lorsque a est définie positive sur un ouvert la condition (iii) est automatiquement vérifiée. En particulier lorsque $n = 1$ il y a stabilisation faible si et seulement si a n'est pas identiquement nulle, ce n'est plus vrai lorsque $n > 1$. Une démonstration du Théorème 1 est donnée au Chapitre 1. On peut obtenir une bien meilleure information sur la décroissance de l'énergie en rajoutant une hypothèse géométrique.

Théorème 2 ([Kle17]) *Les conditions suivantes sont équivalentes :*

(i) *Il y a stabilisation faible et pour toute géodésique maximale $(x_s)_{s\in\mathbf{R}}$ de M on a*

$$\bigcap_{s\in\mathbf{R}} \ker(a(x_s)) = \{0\}. \tag{GCC}$$

(ii) *Il existe deux constantes $C, \beta > 0$ telles que pour tout $u \in H$ et pour tout temps t*

$$E(u,t) \leq Ce^{-\beta t}E(u,0). \tag{3}$$

Si l'une de ces conditions est vérifiée on dit qu'il y a stabilisation forte de l'équation des ondes, la condition sur l'intersection des noyaux de a est quant à elle appelée condition de contrôle géométrique. Pour $n = 1$ ce théorème fut d'abord démontré par Rauch et Taylor (voir [RaTa74]) puis, dans le cadre d'une variété *à bord*, par Bardos, Lebeau et Rauch (voir [BLR92]). La forme de la condition de contrôle géométrique pour $n \geq 1$ était déjà connue des spécialistes du domaine mais il n'existait pas de preuve de ce théorème dans la littérature. Ce théorème sera démontré au Chapitre 2 comme un corollaire du Théorème 3. Lorsque $n > 1$ la condition de contrôle géométrique n'implique pas la stabilisation faible et nous montrerons que cette hypothèse est bien nécessaire dans l'énoncé du Théorème 2. En effet nous verrons que lorsque $n > 1$ il est possible de trouver une variété M et un amortisseur a tels que (GCC) soit vérifiée mais qu'il n'y ait même pas stabilisation faible. Lorsqu'il y a effectivement stabilisation forte on peut se demander quel est le plus grand β admissible dans (3).

Définition On note le meilleur taux de décroissance exponentielle de l'énergie α, il est définit par la formule suivante :

$$\alpha = \sup\{\beta \in \mathbf{R} : \exists C > 0, \forall u \in H, \forall T > 0, E(u,T) \le Ce^{-\beta T} E(u,0)\}.$$

On souhaite décrire α en fonction de l'amortisseur a, nous allons voir qu'il s'exprime comme le minimum d'une quantité spectrale et d'une quantité dynamique. En utilisant la théorie des opérateurs de Fredholm on peut démontrer que $\mathrm{sp}(A_a)$, le spectre de A_a, est uniquement constitué de valeurs propres isolées et de multiplicité finie.

Définition On définit D_0, l'abscisse spectrale de A_a, comme la quantité suivante :

$$D_0 = \sup\{\mathfrak{Re}(\lambda) : \lambda \in \mathrm{sp}(A_a)\backslash\{0\}\}.$$

Soit (x_0, ξ_0) un point de $S^*M = \{(x,\xi) \in T^*M : \sqrt{g^x(\xi,\xi)} = 1/2\}$, on note ϕ le flot géodésique sur S^*M parcouru à vitesse 1 et on note $\phi_t(x_0,\xi_0) = (x_t, \xi_t)$. On définit alors la fonction $G_t : S^*M \to \mathscr{M}_n(\mathbf{C})$ comme la solution de l'équation différentielle

$$\left\{ \begin{array}{l} G_0(x_0,\xi_0) = \mathrm{Id}_n \\ \partial_s G_s(x_0,\xi_0) = -a(x_s) G_s(x_0,\xi_0). \end{array} \right. \tag{4}$$

Définition Pour tout temps $t > 0$ on définit

$$C(t) = \frac{-1}{t} \sup_{(x_0,\xi_0) \in S^*M} \ln\left(\|G_t(x_0;\xi_0)\|_2\right) \quad \text{et} \quad C_\infty = \lim_{t \to \infty} C(t)$$

où $\|\cdot\|_2$ est la norme d'opérateur associée à la norme euclidienne.

On peut montrer par un argument de sous additivité que C_∞ existe bien. Dans le cas où $n = 1$ on trouve simplement

$$G_t(x_0,\xi_0) = \exp\left(-\int_0^t a(x_s)\mathrm{d}s\right) \quad \text{et} \quad C(t) = \inf_{(x_0,\xi_0) \in S^*M} \frac{1}{t} \int_0^t a(x_s)\mathrm{d}s.$$

Le fait que cette formule ne soit plus vraie pour $n > 1$ est, comme la majeure partie des autres différences avec le cas scalaire, une conséquence de la non commutativité des matrices. Nous pouvons maintenant énoncer le résultat principal du Chapitre 2.

Théorème 3 ([Kle17]) *Le meilleur taux de décroissance exponentielle est donné par la formule*

$$\alpha = 2\min\{-D_0, C_\infty\}.$$

De plus on peut trouver une variété M et un amortisseur a tels que $D_0 < 0$ mais $C_\infty = 0$. On peut aussi trouver une variété M et un amortisseur a tels que $C_\infty > 0$ et $D_0 = 0$, mais uniquement si $n > 1$.

Pour $n = 1$ ce théorème a été démontré par Lebeau (voir [Leb93]) pour une variété riemannienne *à bord*. La démonstration de ce théorème reste proche de celle de Lebeau, elle repose sur l'utilisation de faisceaux gaussiens et de mesures de défaut microlocales.

Le Chapitre se poursuit par une étude du comportement de C_∞ en fonction de a. On y démontre notamment l'apparition d'un phénomène de sur-amortissement haute fréquence lorsque $n > 1$. Pour finir on trouvera une compilation de quelques généralisations rapides des Théorèmes 2 et 3 qui n'étaient pas incluses dans l'article dont est tiré le Chapitre 2. Il s'agit principalement de voir ce qu'il se passe lorsque a n'est plus hermitienne positive, on constatera une fois de plus l'apparition de nouveaux phénomènes lorsque $n > 1$.

Asymptotique spectrale

On se place ici dans le même cadre que précédemment à l'exception d'une chose : la matrice a n'est plus nécessairement positive. Les variations de l'énergie sont toujours données par (2) mais l'énergie n'est plus forcément décroissante. On s'intéresse maintenant à la répartition des valeurs propres de A_a. Rappelons que le spectre de A_a est constitué uniquement de valeurs propres discrètes et de multiplicités finies. En utilisant la formule de l'énergie on voit aisément que toutes les valeurs propres non nulles sont contenues dans une bande parallèle à l'axe des ordonnées, cette bande dépend des valeurs maximales et minimales du spectre de a sur M (voir section 2.5). Quitte à exclure un nombre fini de valeurs propres on peut affiner cette bande mais avant de pouvoir énoncer ce résultat nous devons généraliser un peu la définition de C_∞.

Définition Pour tout temps $t > 0$ on définit les quantités suivantes

$$C^-(t) = \frac{-1}{t} \sup_{(x_0,\xi_0)\in S^*M} \ln\left(\|G_t(x_0,\xi_0)\|_2\right)$$

$$\text{et } \; C^+(t) = \frac{-1}{t} \inf_{(x_0,\xi_0)\in S^*M} \ln\left(\|G_t(x_0,\xi_0)^{-1}\|_2^{-1}\right).$$

On définit de plus $C_\infty^\pm = \lim_{t\to+\infty} C_t^\pm$.

Notons que C_∞^- est égal à la quantité C_∞ définie à la section précédente et que lorsque $n = 1$ on a simplement

$$C^+(t) = \sup_{(x_0,\xi_0)\in S^*M} \frac{1}{t}\int_0^t a(x_s)\mathrm{d}s \text{ et } C^-(t) = \inf_{(x_0,\xi_0)\in S^*M} \frac{1}{t}\int_0^t a(x_s)\mathrm{d}s.$$

Théorème 4 ([Kle18]) *Soit $\varepsilon > 0$, il n'existe qu'un nombre fini de valeurs propres de A_a en dehors de la bande $\{z \in \mathbf{C} : -C^+_\infty - \varepsilon < \mathfrak{Re}(z) < -C^-_\infty + \varepsilon[\}$.*

Ce théorème a été démontré dans le cas scalaire par Lebeau (voir [Leb93]) puis d'une autre façon par Sjöstrand (voir [Sjö00]). La démonstration de Sjöstrand est reproduite au Chapitre 1, il semblerait cependant qu'elle ne se généralise pas au cas $n > 1$. La démonstration de ce théorème pour n quelconque peut être trouvée dans les addenda du Chapitre 2 et utilise les mesures de défaut microlocales. Énonçons maintenant un analogue de la loi de Weyl pour l'opérateur A_a.

Théorème 5 *Le nombre de valeurs propres λ de A_a satisfaisant $\mathfrak{Im}(\lambda) \in [0; R]$ est équivalent à*

$$n\left(\frac{R}{2\pi}\right)^d C_M$$

lorsque R tend vers l'infini, où C_M désigne le volume de $\{(x,\xi) \in T^*M : g^x(\xi,\xi) \leq 1\}$ pour la mesure de Liouville. De plus le reste est un $\mathcal{O}(R^{d-1})$.

Pour $n = 1$ ce théorème est dû à Markus et Matsaev (voir [MaMa82]) mais fut redémontré indépendamment par Sjöstrand (voir [Sjö00]). La démonstration s'adapte aisément à l'équation des ondes amorties vectorielle. Notons que comme le spectre de A_a est symétrique par rapport à l'axe réel ce théorème décrit aussi la répartitions des valeurs propres de partie imaginaire négative.

On voudrait encore affiner la bande donnée par le Théorème 4, malheureusement on ne peut en général pas le faire sans exclure une infinité de valeurs propres. Nous allons cependant voir qu'il est possible de trouver une bande plus fine contenant «la majorité» des valeurs propres de A_a. Le Théorème 5 nous permettra justement de justifier ce terme de «majorité» de valeurs propres. La première chose à dire est que la fonction G est un cocycle, cela veut dire que pour tout $(x,\xi) \in S^*M$ et tout temps $s, t \in \mathbf{R}$ on a

$$G_{t+s}(x,\xi) = G_t(\phi_s(x,\xi))G_s(x,\xi).$$

On peut donc appliquer le théorème ergodique sous-additif de Kingman pour montrer que les limites

$$\lambda_n(x,\xi) = \lim_{t\to\infty} \frac{1}{t}\log \|G_t(x,\xi)\|_2 \text{ et } \lambda_1(x,\xi) = \lim_{t\to\infty}\frac{1}{t}\log\left(\left\|G_t(x,\xi)^{-1}\right\|_2^{-1}\right)$$

existent pour presque tout $(x,\xi) \in S^*M$. De plus les fonctions λ_1 et λ_n sont bornées, mesurables et invariantes par le flot géodésique. Notons que λ_1 et λ_n correspondent respectivement au plus petit et plus grand exposant de Lyapunov définis par le théorème d'Osseledets (voir Annexe B). On pose finalement

$$\Lambda^- = \operatorname*{ess\,inf}_{(x,\xi)\in S^*M} \lambda_1(x,\xi) \text{ et } \Lambda^+ = \operatorname*{ess\,sup}_{(x,\xi)\in S^*M} \lambda_n(x,\xi).$$

Lorsque $n = 1$ cela revient à

$$\Lambda^- = \operatorname*{ess\,inf}_{(x_0,\xi_0)\in S^*M} \lim_{t\to\infty} \frac{-1}{t}\int_0^t a(x_s)\mathrm{d}s \text{ et } \Lambda^+ = \operatorname*{ess\,sup}_{(x_0,\xi_0)\in S^*M} \lim_{t\to\infty} \frac{-1}{t}\int_0^t a(x_s)\mathrm{d}s.$$

De façon générale on a alors les majorations suivantes

$$C_\infty^- \leq -\Lambda^+ \leq -\Lambda^- \leq C_\infty^+$$

et chaque inégalité est habituellement stricte. Nous pouvons maintenant énoncer le résultat principal du Chapitre 3.

Théorème 6 ([Kle18]) *Pour tout $\varepsilon > 0$ fixé le nombre de valeurs propres λ de A_a vérifiant $\mathfrak{Im}(\lambda) \in [R; R+1]$ et $\mathfrak{Re}(\lambda) \notin]\Lambda^- - \varepsilon; \Lambda^+ + \varepsilon[$ est un $o(R^{d-1})$ lorsque R tend vers l'infini.*

Ce théorème est dû à Sjöstrand lorsque $n = 1$ (voir [Sjö00]). La démonstration du Théorème 6 occupe la quasi totalité du Chapitre 3 et repose sur la construction d'une résolvante approchée. En combinant les théorèmes 5 et 6 on voit que la majorité des valeurs propres de A_a se situent près de la bande $\{z \in \mathbf{C} : \mathfrak{Re}(z) \in [\Lambda^-; \Lambda^+]\}$.

Chapitre 1

Quelques résultats préliminaires

Dans ce chapitre nous présentons quelques premiers résultats sur la stabilisation de l'équation des ondes et sur la répartition des valeurs propres de A_a. C'est aussi l'occasion d'introduire deux outils que nous utiliserons par la suite, à savoir les opérateurs pseudo-différentiels et les mesures de défaut microlocales.

1.1 Stabilisation faible

Comme expliqué dans l'introduction, la matrice a et les variations de l'énergie d'une solution u de (1) sont reliées par la formule de l'énergie :

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u,t) = -2\int_M \langle a(x)\partial_t u(t,x), \partial_t u(t,x)\rangle_{\mathbf{C}^n}\,\mathrm{d}x.$$

L'énergie est donc décroissante lorsque a est hermitienne positive, il est donc naturel de qualifier a d'amortisseur. Puisque l'énergie est décroissante et positive elle admet une limite, on peut alors se demander à quelles conditions sur a cette limite est-elle nulle. Le théorème suivant répond à cette question.

Théorème 1.1 *Les conditions suivantes sont équivalentes :*

(i) $\forall u \in H,\ \lim_{t\to\infty} E(u,t) = 0.$

(ii) *La seule valeur propre de A_a sur l'axe imaginaire est 0.*

(iii) *Soit $u \in H^2(M)^n$ une fonction propre de Δ, si $au = 0$ alors $u = 0$.*

Démonstration. Si $\lambda \neq 0$ est une valeur propre de A_a alors il existe $u \in H$ tel que $E(u,0) = 1$ et $e^{tA_a}u = e^{t\lambda}u$. On en déduit $E(u,t) = e^{2t\Re(\lambda)}$ et donc (i) $\Longrightarrow$ (ii).

Supposons que (iii) soit fausse, il existe alors une fonction u non nulle telle que $\Delta u = -\lambda^2 u$ avec $\lambda \in \mathbf{R}^*$ et $au = 0$. On voit alors que $(u, i\lambda u)$ est un vecteur propre de A_a associé à la valeur propre $i\lambda$ d'où (ii) $\Longrightarrow$ (iii).

Remarquons que pour avoir (i) il faut et il suffit en fait que (i) soit vérifiée pour tout $u = (u_0, u_1) \in D(A_a)$, le résultat pour $u \in H$ s'en déduit par densité. Supposons donc que (iii) soit vraie mais que (i) soit fausse. Il existe alors une fonction $u \in D(A_a)$ telle que $\lim_{t\to\infty} E(u,t) = 1$, on définit alors la suite $(u_k)_{k\in\mathbf{N}} = (e^{kA_a}u)_{k\in\mathbf{N}}$. Cette suite est bornée dans $D(A_a)$ et comme $D(A_a)$ s'injecte de façon compacte dans H il existe une sous suite $(u_{n_k})_k$ convergente dans H, on note $v = (v_0, v_1)$ sa limite et $e^{tA_a}v = (v(t), \partial_t v(t))$. Pour tout temps $s \geq 0$ on a

$$e^{(s+n_k)A_a}u = e^{sA_a}e^{n_kA_a}u \underset{k\to\infty}{\longrightarrow} e^{sA_a}v$$

et puisque $\lim_{t\to\infty} E(u,t) = 1$ on en déduit $E(v,s) = 1$. En utilisant la formule de l'énergie on voit alors que $a\partial_t v(t) = 0$ et donc que v est aussi solution de l'équation des ondes non amorties :

$$e^{tA_a}v = e^{tA_0}v.$$

On peut alors décomposer $v(t)$ selon une base hilbertienne de $L^2(M)$ formée de vecteurs propres du laplacien. En effet il existe une famille orthonormée $(\psi_k^\pm)_k$ telle que

$$v(t) = v_0 + \sum_{k=1}^{\infty} e^{it\lambda_k}\psi_k^+ + \sum_{k=1}^{\infty} e^{-it\lambda_k}\psi_k^-$$

où $\Delta\psi_k^\pm = -\lambda_k^2\psi_k^\pm \neq 0$ et $j \neq k \implies \lambda_j \neq \lambda_k$. Fixons un entier $N \geq 1$ et posons

$$w(t,x) = \frac{1}{t}\int_0^t \partial_s v(s,x)e^{-is\lambda_N}\mathrm{d}s,$$

la fonction $w(t,\cdot) - i\lambda_N\psi_N^+(\cdot)$ tend alors vers 0 dans $L^2(M)^n$ lorsque t tend vers l'infini. Par conséquent nous avons aussi

$$\lim_{t\to\infty} \left\| a(\cdot)\left(w(t,\cdot) - i\lambda_N\psi_N^+(\cdot)\right)\right\|_{L^2(M)^n} = 0$$

mais puisque

$$a(x)w(t,x) = \frac{1}{t}\int_0^t a(x)\partial_s v(s,x)e^{-is\lambda_N}\mathrm{d}s = 0$$

on en déduit $a\psi_N^+ = 0$. D'après la propriété (iii) cela implique que $\psi_N^+ = 0$. Le même raisonnement appliqué à tous les $\psi_i^\pm$ montre que v est constante, ce qui est absurde puisque l'on avait $E(v,0) = 1$. C'est donc bien que (iii) $\implies$ (i) et la preuve est terminée.

□

Lorsque l'une de ces trois conditions est satisfaite on dit qu'il y a stabilisation faible de l'équation des ondes. Dans le cas scalaire on dispose du résultat suivant.

Corollaire 1.2 *Si $n = 1$ alors il y a stabilisation faible si et seulement si a n'est pas la fonction nulle.*

Démonstration. Si $a = 0$ alors l'énergie de toute solution de l'équation des ondes est constante, il n'y a donc pas stabilisation faible. La réciproque est une conséquence du théorème précédent et du principe de prolongement unique pour les fonctions propres du laplacien. □

Pour les mêmes raisons on sait que si a est définie positive en un point et donc, par continuité, sur un ouvert alors il y a stabilisation faible. La réciproque est cependant fausse pour $n \geq 2$, pour le voir il suffit de considérer un amortisseur du type

$$a(x) = \begin{pmatrix} a_1(x) & 0 \\ 0 & a_2(x) \end{pmatrix}$$

où a_1 et a_2 sont deux fonctions positives, non nulles et à supports disjoints.

Supposons maintenant que l'on ait un peu plus que la stabilisation faible et que la décroissance soit uniforme sur H. Autrement dit supposons qu'il existe une fonction positive $f : \mathbf{R} \to \mathbf{R}$ telle que $\lim_{t\to\infty} f(t) = 0$ et

$$\forall u \in H,\ \forall t \geq 0,\ E(u,t) \leq f(t)E(u,0).$$

Il existe alors un temps $T > 0$ tel que $f(T) \leq 1/2$, en notant $u_k = e^{kA_a}u$ on voit que $E(u_k,T) \leq E(u_k,0)/2$ et donc $E(u,kT) \leq E(u,0)/2^k$. Puisque l'énergie est décroissante il existe alors deux constantes $C, \beta > 0$ telles que

$$\forall u \in H,\ \forall t \geq 0,\ E(u,t) \leq Ce^{-\beta t}E(u,0). \tag{1.1}$$

Les propriétés de semi groupe de l'équation des ondes entrainent donc une dichotomie forte : soit l'énergie des solutions décroit uniformément exponentiellement soit il existe des solutions dont l'énergie décroit arbitrairement lentement. Si la condition (1.1) est satisfaite on dit alors qu'il y a stabilisation forte de l'équation des ondes. Le Théorème 2 donne une condition nécessaire et suffisante à la stabilisation forte de l'équation des ondes, cela permet d'ailleurs de démontrer que stabilisation faible et forte ne sont pas équivalentes.

1.2 Opérateurs pseudodifférentiels semi-classiques

Dans cette section nous allons présenter quelques résultats standards concernant les opérateurs pseudodifférentiels semi-classiques. Nous terminerons par une courte application de ce concept à l'équation des ondes amorties. La plupart des résultats seront énoncés sans preuves, le lecteur intéressé par les opérateurs pseudodifférentiels pourra par exemple consulter [Hör85] ou encore [Zwo12] pour le cadre semi-classique.

Définition Soit $m \in \mathbf{R}$, on définit l'ensemble des symboles d'ordre m comme l'espace des fonctions $b \in C^\infty(\mathbf{R}^{2d})$ telles que pour tout $\alpha, \beta \in \mathbf{N}^d$ il existe une constante $C_{\alpha,\beta}$

satisfaisant
$$\forall(x,\xi)\in\mathbf{R}^{2d},\ \ |\partial_x^\alpha\partial_\xi^\beta b(x,\xi)|\le C_{\alpha,\beta}(1+\|\xi\|_2^2)^{(m-|\beta|)/2}. \tag{1.2}$$
Nous noterons S^m cet espace.

Nous aurons fréquemment besoin d'utiliser des fonctions $b\in S^m$ dépendant d'un paramètre $h\in]0;1]$, on imposera alors que la majoration (1.2) soit uniforme en $h\in]0;1]$. On définit S^m_{cl} comme l'ensemble des symboles $b\in S^m$ vérifiant
$$\exists b_m\in S^m,\ \ b-b_m\in hS^{m-1}.$$
Le symbole b_m est unique et est appelé symbole principal de b.

Définition Soit $b=b_h\in S^m$ un symbole et $u\in\mathscr{S}(\mathbf{R}^d)$ une fonction de la classe de Schwartz. On définit alors l'opérateur $\mathrm{Op}_h(b):\mathscr{S}(\mathbf{R}^d)\to\mathscr{S}(\mathbf{R}^d)$ par
$$\mathrm{Op}_h(b)u(x)=\frac{1}{(2\pi h)^d}\iint_{\mathbf{R}^{2d}}e^{\frac{i}{h}\langle x-y,\xi\rangle}b_h(x,\xi)u(y)\mathrm{d}y\mathrm{d}\xi. \tag{1.3}$$
$\mathrm{Op}_h(b)$ est un opérateur pseudodifférentiel semi-classique (d'ordre m), on dit qu'il s'agit de la quantification gauche (ou standard) semi-classique de b. Il est aisé de montrer que si b est un polynôme en ξ l'opérateur $\mathrm{Op}_h(b)$ est un opérateur différentiel et que l'on peut obtenir tout opérateur différentiel de cette façon.

On peut montrer que $\mathrm{Op}_h(b)$ s'étend de façon unique en un opérateur continu de $H^s(\mathbf{R}^d)$ dans $H^{s-m}(\mathbf{R}^d)$.

Proposition 1.3 *Soient $b\in S^{m_1}$ et $c\in S^{m_2}$ deux symboles, alors $\mathrm{Op}_h(b)\mathrm{Op}_h(c)$ est un opérateur pseudodifférentiel semi-classique de symbole $b\#c$, d'ordre m_1m_2 et on a*
$$b\#c-bc-\frac{h}{2i}\{b,c\}\in h^2S^{m_1m_2-2} \tag{1.4}$$
où $\{b,c\}=\partial_\xi b\partial_x c-\partial_x b\partial_\xi c$ est le crochet de Poisson de b et c.

En particulier si b_{m_1} et c_{m_2} sont respectivement les symboles principaux de b et c alors $b\#c$ admet $b_{m_1}c_{m_2}$ pour symbole principal. De plus $[\mathrm{Op}_h(b),\mathrm{Op}_h(c)]$ admet $\frac{h}{i}\{b_{m_1},c_{m_2}\}\in hS^{m_1m_2-1}$ comme symbole principal. Il ne s'agit ici que des deux premiers termes de l'expansion de $b\#c$ en puissances de h mais on aurait pu donner un développement à tout ordre.

On s'intéresse maintenant au lien entre la positivité d'un opérateur et celle de son symbole. En général la positivité de b n'implique pas celle de $\mathrm{Op}_h(b)$, c'est à dire que l'on peut trouver un symbole $b\in S^0$ positif et une fonction $u\in L^2(\mathbf{R}^d)$ tels que $\mathrm{Op}_h(b)$ soit auto-adjoint mais
$$\langle\mathrm{Op}_h(b)u,u\rangle_{L^2(\mathbf{R}^d)}<0.$$
Il existe cependant un lien un peu plus faible qui s'avèrera très utile.

Proposition 1.4 (Inégalité de Gårding) *Soit $b_h = b \in S^0$ un symbole réel et positif tel que $\mathrm{Op}_h(b)$ soit auto-adjoint, il existe alors deux constantes $C, h_0 > 0$ telles que*

$$\forall u \in L^2(\mathbf{R}^d),\ \forall h \in]0, h_0],\ \langle \mathrm{Op}_h(b)u, u\rangle_{L^2(\mathbf{R}^d)} \geq -Ch\|u\|^2_{L^2(\mathbf{R}^d)}. \tag{1.5}$$

De la même façon on peut s'intéresser au lien entre l'existence de $1/b$ et de $\mathrm{Op}_h(b)^{-1}$.

Définition Soit $b \in S^m$ un symbole, on dit que b est elliptique s'il existe une constante $C > 0$ indépendante de h telle que

$$\forall (x,\xi) \in \mathbf{R}^{2d},\ |b(x,\xi)| \geq (1 + \|\xi\|_2^2)^{m/2}.$$

Proposition 1.5 *Soit $b \in S^m$ un symbole elliptique, il existe alors une constante $h_0 > 0$ telle que l'opérateur $\mathrm{Op}_h(b) : H^s(\mathbf{R}^d) \to H^{s-m}(\mathbf{R}^d)$ admette un inverse continu pour tout $h \in]0; h_0]$. De plus la norme de cet inverse peut être uniformément bornée pour $h \in]0; h_0]$.*

Les définitions que nous avons vues jusqu'à présent s'étendent de façon naturelle à des symboles définis sur des variétés riemanniennes lisses et à valeurs dans $\mathscr{M}_n(\mathbf{C})$. Les résultats ne sont alors pas très différents, à une exception notable près. Si b et c sont deux symboles matriciels d'ordre respectif m_1 et m_2 et de symbole principaux respectifs b_{m_1} et c_{m_2} alors $[\mathrm{Op}_h(b), \mathrm{Op}_h(c)]$ est en général d'ordre $m_1 m_2$ et son symbole principal est $b_{m_1}c_{m_2} - c_{m_2}b_{m_1}$ qui est non nul si b_{m_1} et c_{m_2} ne commutent pas.

Exemple Soit (M, g) une variété riemannienne C^∞, compacte et sans bords. Notons Δ l'opérateur de Laplace-Beltrami sur M, $-h^2\Delta$ est alors un opérateur pseudo-différentiel (et même différentiel) d'ordre 2 et de symbole principal $p : (x,\xi) \mapsto g^x(\xi,\xi)$.

Nous terminons cette section par un présentation de la démonstration du Théorème 4 pour $n = 1$ donnée par Sjöstrand dans [Sjö00]. Commençons par faire une réduction semi-classique de notre problème. Soit τ un nombre complexe, $i\tau$ est une valeur propre de A_a si et seulement s'il existe une fonction $u \in H^2(M)$ telle que

$$(-\Delta - \tau^2 + 2ia\tau)v = 0,$$

on s'intéresse à la limite $\mathfrak{Re}(\tau) \to +\infty$. Notons $0 < h \ll 1$ notre paramètre semi-classique tendant vers 0 et prenons $i\tau$ une valeur propre de A_a telle que $h^{-1} - 1 \leq |\tau| \leq h^{-1} + 1$. On écrit alors $\tau = \lambda/h$ avec $||\lambda| - 1| \ll 1$ et l'équation précédente devient

$$(-h^2\Delta - \lambda^2 + 2ia\lambda h)v = 0.$$

Si l'on pose maintenant $z = \lambda^2$, et $\lambda = \sqrt{z}$ avec $\mathfrak{Re}(z) > 0$ l'équation devient

$$(-h^2\Delta + 2iha\sqrt{z} - z)v = 0.$$

On peut finalement la réécrire

$$(\mathcal{P} - z)v = 0 \tag{1.6}$$

où $\mathcal{P} = \mathcal{P}(z) = P + ihQ(z)$, $P = -h^2\Delta$ est le laplacien semi-classique sur M et $Q(z) = 2a\sqrt{z}$ est auto-adjoint lorsque z est réel positif. On note de plus p et q les symboles respectifs de P et Q, on a alors le lemme suivant.

Lemme 1.6 *Supposons qu'il existe une solution non identiquement nulle de* (1.6) *pour un certain* $z = 1 + \mathcal{O}(h)$*, alors*

$$h \inf_{(x,\xi)\in p^{-1}(\mathfrak{Re}(z))} q(\mathfrak{Re}(z))(x,\xi) - \mathcal{O}(h^2) \leq \mathfrak{Im}(z) \leq h \sup_{(x,\xi)\in p^{-1}(\mathfrak{Re}(z))} q(\mathfrak{Re}(z))(x,\xi) + \mathcal{O}(h^2). \tag{1.7}$$

Nous renvoyons à [Sjö00] pour la démonstration de ce lemme. On voudrait maintenant améliorer le lemme précédent en conjuguant $\mathcal{P} = P + ihQ(z)$ par un opérateur pseudodifférentiel. Soit B un opérateur pseudodifférentiel elliptique, auto-adjoint, d'ordre 0 et de symbole principal $b = e^g$. On a

$$B^{-1}PB = P + B^{-1}[P,B] = P + ihC$$

où C est un opérateur pseudodifférentiel semi-classique de symbole principal

$$c = b^{-1}\{p,b\} = \{p,g\} \in S^1.$$

Ainsi, puisque $z = \mathfrak{Re}(z) + \mathcal{O}(h)$,

$$B^{-1}(P + ihQ(z))B = P + ih\mathrm{Op}_h\left(q(\mathfrak{Re}(z)) - \{p,g\}\right) + h^2R(z)$$

où $R(z)$ est un opérateur pseudodifférentiel d'ordre 1. On cherche donc à choisir $g = g_{\mathfrak{Re}(z)}$ de sorte à ce que $\inf_{(x,\xi)\in p^{-1}(\mathfrak{Re}(z))} q(\mathfrak{Re}(z))(x,\xi) - \{p,g\}(x,\xi)$ devienne plus grand et que $\sup_{(x,\xi)\in p^{-1}(\mathfrak{Re}(z))} q(\mathfrak{Re}(z))(x,\xi) - \{p,g\}(x,\xi)$ devienne plus petit. On peut en fait choisir g pour que $\{p,g\} = q - \langle q\rangle_T$ sur $p^{-1}(\mathfrak{Re}(z))$ où

$$\langle q\rangle_T = \frac{1}{2T}\int_{-T}^{T} q\circ\phi_t \mathrm{d}t$$

et ϕ désigne le flot géodésique parcouru à vitesse 1. Pour cela on pose

$$f : t \mapsto \mathbf{1}_{[-1;0]}(t)\frac{t+1}{2} + \mathbf{1}_{[0;1]}(t)\frac{t-1}{2} \quad \text{et} \quad f_T : t \mapsto (t/T),$$

on a alors $f_T' = \frac{1}{2T}\mathbf{1}_{[-T;T]} - \delta_0$. En définissant sur $p^{-1}(\mathfrak{Re}(z))$ la fonction g_T comme une intégrale indéfinie

$$g_T = -\int f_T(s) q\circ\phi_{t-s}\mathrm{d}s$$

on obtient bien $\{p,g\} = q - \langle q\rangle_T$ sur $p^{-1}(\mathfrak{Re}(z))$. On prolonge alors g_T sur T^*M en un symbole appartenant à S^0 et on choisit un opérateur B_T elliptique, autoadjoint et de symbole principal e^{g_T}. Si v est un vecteur propre de $P + ihQ$ associé à la valeur propre z alors $B_T^{-1}v$ est un vecteur propre de $B_T^{-1}(P + ihQ)B_T = P + ih\mathrm{Op}_h(\langle q\rangle_T) + h^2R(z)$ associé à la valeur propre z. On remarque alors que le Lemme 1.6 est encore valable avec le reste $h^2R(z)$ et donc que

$$h \inf_{(x,\xi)\in p^{-1}(\mathfrak{Re}(z))} \langle q(\mathfrak{Re}(z))\rangle_T(x,\xi) - \mathcal{O}(h^2) \leq \mathfrak{Im}(z)$$

$$\mathfrak{Im}(z) \leq h \sup_{(x,\xi)\in p^{-1}(\mathfrak{Re}(z))} \langle q(\mathfrak{Re}(z))\rangle_T(x,\xi) + \mathcal{O}(h^2).$$

Puisque ceci est vrai pour tout temps T et que, par sous additivité,

$$\sup_{T>0} \inf_{(x,\xi)\in p^{-1}(\mathfrak{Re}(z))} \langle q\rangle_T(x,\xi) = \lim_{T\to\infty} \inf_{(x,\xi)\in p^{-1}(\mathfrak{Re}(z))} \langle q\rangle_T(x,\xi)$$

$$\text{et} \inf_{T>0} \sup_{(x,\xi)\in p^{-1}(\mathfrak{Re}(z))} \langle q\rangle_T(x,\xi) = \lim_{T\to\infty} \sup_{(x,\xi)\in p^{-1}(\mathfrak{Re}(z))} \langle q\rangle_T(x,\xi),$$

on en déduit le Théorème 4 pour $n = 1$.

Il semblerait que cette preuve ne s'adapte pas bien au cas vectoriel. En effet pour des fonctions à valeurs matricielles on n'aura plus forcément l'existence d'un g tel que $a^{-1}\{p, a\} = \{p, g\}$. Nous verrons à la fin du Chapitre 2 comment démontrer ce théorème pour tout $n \geq 1$ à l'aide des mesures de défaut microlocales.

1.3 Mesures de défaut microlocales

Soit Ω un ouvert de $\mathbf{R}^d$ et soit (u_k) une suite bornée dans $L^2(\Omega)$. Quitte à extraire une sous suite on peut supposer que (u_k) est faiblement convergente vers une limite u. On cherche alors à mesurer le défaut de convergence forte de la suite (u_k). Une façon de faire est par exemple de considérer la suite de mesures $(|u_k - u|^2\mathrm{d}\lambda)_k$, cette suite est bornée et, quitte à extraire une nouvelle fois, on peut supposer la suite convergente vers une mesure de Radon ν au sens suivant :

$$\forall f \in C_c^0(\Omega),\ \lim_{k\to\infty} \int_\Omega f|u_k - u|^2\mathrm{d}\lambda = \int_\Omega f\mathrm{d}\nu.$$

On peut alors montrer que (u_k) converge fortement vers u dans $L^2_{\mathrm{loc}}(\Omega)$ si et seulement si ν est la mesure nulle. Par exemple pour $\Omega = \mathbf{R}^d$ en prenant $u_k(x) = k^{d/2}v(k(x - x_0))$ où $v \in L^2(\mathbf{R}^d)$ on trouve $u = 0$ et $\nu = \|v\|^2_{L^2}\delta_{x_0}$ avec δ_{x_0} la mesure de Dirac au point x_0. Si l'on prend maintenant $u_k(x) = e^{ikx\cdot\xi_0}v(x)$ avec $\xi \neq 0$ on trouve $u = 0$ et $\mathrm{d}\nu = |v(x)|^2\mathrm{d}\lambda$. Ce deuxième exemple illustre une des limites de la mesure ν : elle ne permet pas de différentier les différentes directions d'oscillation ξ_0.

Nous allons maintenant définir une nouvelle mesure qui nous permettra cette fois ci de distinguer les différentes directions d'oscillations. Soit Ω un ouvert de $\mathbf{R}^d$, on définit l'ensemble $S^*\Omega = \{(x,\xi) \in T^*\Omega : \|\xi\|_2 = 1\}$ que l'on appellera fibré cotangent unitaire de Ω.

Théorème 1.7 *Soit (u_k) une suite bornée d'éléments de $L^2_{\text{loc}}(\Omega)$ convergeant faiblement vers 0. Il existe alors une suite extraite $(u_{n_k})_k$ et une mesure de Radon μ sur $S^*\Omega$ telles que, pour tout opérateur pseudodifférentiel A d'ordre 0, à support compact et de symbole principal a on ait*

$$\lim_{k\to\infty} \langle Au_{n_k}, u_{n_k}\rangle_{L^2} = \int_{S^*\Omega} a \mathrm{d}\mu.$$

De la même façon que pour les opérateurs pseudodifférentiels il est possible d'étendre ce résultat au cas où Ω est une variété riemannienne lisse. Donnons une démonstration succincte de ce théorème.

Démonstration. Cette démonstration repose sur l'inégalité de Gårding, en effet on sait grâce à elle que si a est réel et positif alors

$$\liminf_{k\to\infty} \mathfrak{Re}\langle Au_n, u_n\rangle_{L^2} \geq 0 \quad \text{et} \quad \lim_{k\to\infty} \mathfrak{Im}\langle Au_n, u_n\rangle_{L^2} = 0. \tag{1.8}$$

De la même façon on peut montrer que

$$\limsup_{k\to\infty} \mathfrak{Re}\langle Au_n, u_n\rangle_{L^2} \leq \sup_{k\in\mathbf{N}} \|u_k\|^2_{L^2} \|a\|_{L^\infty(S^*\Omega)}. \tag{1.9}$$

On sait donc qu'il existe une suite extraite $(u_{\sigma(k)})_k$ telle que $\langle Au_{\sigma(k)}, u_{\sigma(k)}\rangle_{L^2}$ converge. Comme $C^\infty_c(S^*\Omega)$ est séparable on peut, par un argument diagonal, trouver une sous-suite $(u_{n_k})_k$ telle que $\langle Au_{n_k}, u_{n_k}\rangle_{L^2}$ soit convergente pour tout a dans un sous espace dense de $C^\infty_c(S^*\Omega)$. Les inégalités (1.8) et (1.9) nous permettent alors de voir que cette limite est une fonctionnelle positive et continue sur ce sous espace dense de $C^\infty_c(S^*\Omega)$. Cette fonctionnelle s'étend donc de façon unique en une forme linéaire positive et continue sur tout $C^0_c(S^*\Omega)$ qui, par le théorème de représentation de Riesz, s'identifie à une mesure de Radon sur $S^*\Omega$: il s'agit de la mesure μ recherchée.

□

La mesure μ est appelée mesure de défaut microlocale (MDM) de la suite $(u_{n_k})_k$. Encore une fois on peut montrer que la suite $(u_{n_k})_k$ converge fortement vers 0 dans L^2_{loc} si et seulement si $\mu = 0$. Si la mesure de défaut microlocale existe sans extraction préalable on dira que la suite (u_k) est pure. Reprenons maintenant notre exemple précédent : $u_k(x) = e^{ikx\cdot\xi_0}v(x)$, la suite (u_k) est pure et sa mesure de défaut microlocale est $|v(x)|^2\mathrm{d}x \otimes \delta_{\xi_0/|\xi_0|}$. Notons que la mesure ν définie au début de cette section est simplement la projection de μ sur Ω par l'application $\pi : (x,\xi) \mapsto x$ et qu'on la retrouve lorsque A est un opérateur de multiplication par une fonction a ne dépendant pas de

ξ :

$$\lim_{k\to\infty}\langle au_{n_k}, u_{n_k}\rangle_{L^2} = \int_{S^*\Omega} a\mathrm{d}\mu = \int_{\Omega} a\mathrm{d}\nu.$$

Pour plus d'informations sur les mesures de défaut microlocales on pourra consulter [Bur97] ou [Gér91].

Pour les applications qui nous intéressent nous aurons besoin d'une version un peu différente du Théorème 1.7. Soit (M, g) une variété riemannienne compacte, connexe, sans bord et de classe C^∞.

Théorème 1.8 *Soit* $(u_n)_n$ *une suite bornée de fonctions de* $H^1_{\mathrm{loc}}(\mathbf{R}\times M)^n$ *faiblement convergente vers* 0. *Il existe alors*

- *une sous-suite* $(u_{n_k})_k$*,*
- *une mesure de Radon positive* ν *sur* $S^*(\mathbf{R}\times M)$*,*
- *une matrice* M *de fonctions* ν*-intégrables sur* $S^*(\mathbf{R}\times M)$ *telle que* M *soit hermitienne positive semi-définie* ν*-p.p. et* $\mathrm{Tr}(M) = 1$ ν*-p.p.,*

telles que, pour tout opérateur pseudodifférentiel à support compact B*, de symbole principal matriciel* b *de dimension* n *et d'ordre* 2 *on ait*

$$\lim_{k\to+\infty}\langle Bu_{n_k}|u_{n_k}\rangle_{H^{-1},H^1} = \int_{S^*(\mathbf{R}\times M)}\mathrm{Tr}(bM)\mathrm{d}\nu. \tag{1.10}$$

Pour illustrer l'intérêt des mesures de défaut microlocales dans le problème de stabilisation de l'équation des ondes nous allons maintenant montrer le lien entre mesure de défaut microlocale et énergie d'une suite de solution de l'équation des ondes. Soit $(u_k)_k$ une suite d'éléments de $C^0(\mathbf{R}, H^1(M)^n)\cap C^1(\mathbf{R}, L^2(M)^n)$ solutions de l'équation des ondes amorties $(\partial_t^2 - \Delta + 2a\partial_t)u_k = 0$ et soit $\psi : \mathbf{R}\to\mathbf{R}$ une fonction positive, C^∞ et à support dans $]0;\eta[$. Par définition de l'énergie on a

$$\int_0^\eta \psi(t)E(u_k,t)\mathrm{d}t = \frac{1}{2}\int_0^\eta \psi(t)\int_M |\partial_t u_k(t,x)|^2 + g_x\left(\nabla u_k(t,x), \nabla u_k(t,x)\right)\mathrm{d}x\mathrm{d}t,$$

où $|\cdot|^2$ désigne la norme euclidienne sur $\mathbf{C}^n$. En intégrant par parties on trouve

$$\int_0^\eta \psi(t)E(u_k,t)\mathrm{d}t = \frac{-1}{2}\left\langle\left[\psi'\partial_t + \psi\left(\partial_t^2+\Delta\right)\right]u_k, u_k\right\rangle_{H^{-1}(\mathbf{R}\times M), H^1(\mathbf{R}\times M)}.$$

Supposons maintenant que (u_k) tende faiblement vers 0 dans H^1_{loc} et qu'elle soit pure, on note $\mu = M\nu$ sa mesure de défaut microlocale. On sait alors que $\psi'\partial_t u_k$ converge fortement vers 0 dans H^{-1} et donc

$$\begin{aligned}\lim_{k\to\infty}\int_0^\eta \psi(t)E(u_k,t)\mathrm{d}t &= \frac{1}{2}\lim_{k\to\infty}\left\langle-\psi\left(\partial_t^2+\Delta\right)u_k, u_k\right\rangle_{H^{-1}(\mathbf{R}\times M), H^1(\mathbf{R}\times M)}\\ &= \frac{1}{2}\int_{S^*(\mathbf{R}\times M)}\mathrm{Tr}[\sigma_2(-\psi(\partial_t^2+\Delta))M]\mathrm{d}\nu\end{aligned}$$

où $\sigma_2(-\psi(\partial_t^2+\Delta))$ désigne le symbole principal de $-\psi(\partial_t^2+\Delta)$. Si l'on note (t,τ,x,ξ) les points de $T^*(\mathbf{R}\times M)$ on trouve alors

$$\sigma_2(-\psi(\partial_t^2+\Delta))(t,\tau,x,\xi)=\psi(t)(\tau^2+g^x(\xi,\xi))\mathrm{Id}_n$$

et donc, par définition de $S^*(\mathbf{R}\times M)$,

$$\lim_{k\to\infty}\int_0^\eta \psi(t)E(u_k,t)\mathrm{d}t=\frac{1}{2}\int_{S^*(\mathbf{R}\times M)}\psi\mathrm{d}\nu.$$

Puisque ceci est vrai pour toute fonction positive $\psi\in C_c^\infty(]0;\eta[)$ on en déduit finalement

$$\lim_{k\to\infty}\int_0^\eta E(u_k,t)\mathrm{d}t=\frac{1}{2}\nu(\{(t,\tau,x,\xi)\in S^*(\mathbf{R}\times M):t\in]0;\eta[\}).$$

Remarquons que nous n'avons jusqu'à présent pas réellement utilisé le fait que u_k soit solution de l'équation des ondes. Nous utiliserons cette hypothèse dans le prochain chapitre pour démontrer un résultat de propagation de la mesure μ par le flot hamiltonien engendré par le symbole principal de $\partial_t^2-\Delta$. Cela nous permettra de comparer $\nu(\{t\in]0;\eta[\})$ et $\nu(\{t\in]T,T+\eta[\})$ et donc d'obtenir des estimées sur la décroissance de l'énergie.

Chapitre 2

Stabilisation de l'équation des ondes vectorielle

Ce chapitre est une reproduction de l'article [Kle17], accepté pour publication au journal *SIAM Journal on Control and Optimization*.

Résumé : L'énergie des solutions de l'équation des ondes amorties scalaire décroit uniformément exponentiellement lorsque la condition de contrôle géométrique est vérifiée. Un théorème de Lebeau (voir [Leb93]) donne une expression de ce taux de décroissance exponentiel en fonction des moyennes du terme d'amortissement le long des géodésiques et du spectre du générateur infinitésimal de l'équation. Le but de cet article est de généraliser ce résultat au cas d'une équation des ondes amorties vectorielle sur une variété Riemannienne compacte sans bords. Nous obtenons une expression similaire à celle de Lebeau mais de nouveaux phénomènes tels qu'un sur-amortissement haute fréquence apparaissent dans le cas vectoriel. Nous démontrons aussi une condition nécessaire et suffisante pour la stabilisation forte de l'équation des ondes amorties vectorielle.

Abstract : The energy of solutions of the scalar damped wave equation decays uniformly exponentially fast when the geometric control condition is satisfied. A theorem of Lebeau (see [Leb93]) gives an expression of this exponential decay rate in terms of the average value of the damping terms along geodesics and of the spectrum of the infinitesimal generator of the equation. The aim of this text is to generalize this result in the setting of a vectorial damped wave equation on a Riemannian manifold with no boundary. We obtain an expression analogous to Lebeau's one but new phenomena like high frequency overdamping arise in comparison to the scalar setting. We also prove a necessary and sufficient condition for the strong stabilization of the vectorial wave equation.

2.1 Introduction

Let (M,g) be a $\mathscr{C}^\infty$, connected, compact Riemannian manifold of dimension d without boundary . Let Δ be the Laplace-Beltrami operator on M for the metric g and let a be a $\mathscr{C}^\infty$ function from M to $\mathscr{H}_n^+(\mathbf{C})$, the space of positive semidefinite Hermitian matrices of dimension n. We are interested in the following system of equations

$$\begin{cases} (\partial_t^2 - \Delta + 2a(x)\partial_t)u = 0 \ \text{ in } \ \mathcal{D}'(\mathbf{R}\times M)^n \\ u_{|t=0} = u_0 \in H^1(M)^n \ \text{ and } \ \partial_t u_{|t=0} = u_1 \in L^2(M)^n. \end{cases} \tag{2.1}$$

Let $H = H^1(M)^n \oplus L^2(M)^n$ and define on H the unbounded operator

$$A_a = \begin{pmatrix} 0 & \mathrm{Id}_n \\ \Delta & -2a \end{pmatrix} \text{ of domain } D(A_a) = H^2(M)^n \oplus H^1(M)^n.$$

By application of the Hille-Yosida theorem to A_a the system (2.1) has a unique solution in the space $C^0(\mathbf{R}, H^1(M)^n) \cap C^1(\mathbf{R}, L^2(M)^n)$; from now on we will identify H with the space of solutions of (2.1). The Euclidean norm on $\mathbf{R}^n$ or $\mathbf{C}^n$ will be written $|\cdot|$ and if $\mathcal{H}$ is a Hilbert space we will write $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ for its inner product or simply $\langle\cdot,\cdot\rangle$ when there is no possible confusion. Let us define $E(u,t)$, the energy of a solution u of (2.1) at time t, by the formula

$$E(u,t) = \frac{1}{2}\int_M |\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 \mathrm{d}x$$

where $|\nabla u(t,x)|^2 = g_x(\nabla u(t,x), \nabla u(t,x))$. We then have the relation

$$E(u,T) = E(u,0) - \int_0^T \int_M \big\langle 2a(x)\partial_t u(t,x), \partial_t u(t,x)\big\rangle_{\mathbf{C}^n} \mathrm{d}x\mathrm{d}t. \tag{2.2}$$

The energy is thus a nonincreasing function of time. We are interested in the problem of stabilization of the wave equation ; that is, determining the long time behavior of the energy of solutions of (2.1). This has been well studied in the scalar case ($n = 1$) but not so much in the vectorial case ($n > 1$). However the equation (2.1) is a very natural generalization of the scalar problem and as such represents an interesting problem. A posteriori this problem is also interesting since new phenomena arise in the vectorial case. We also hope that the good understanding of the stabilization of (2.1) can help to better understand stabilization for similar but more complex equations like the electromagnetic wave equation or the Lamé equation. The aim of this article is to adapt and prove some classical results of scalar stabilization to the vectorial case, and we will also highlight the main differences between the two settings. The most basic result about stabilization of the wave equation is probably the following.

Theorem 2.1 *The following conditions are equivalent :*

(i) $\forall u \in H \quad \lim_{t\to\infty} E(u,t) = 0$

(ii) *The only eigenvalue of A_a on the imaginary axis is 0.*

Moreover, if a is positive definite at one point (and thus on an open set) then the two conditions above are satisfied.

Condition **(i)** is called weak stabilization of the damped wave equation. For a succinct proof of this result see the introduction of [Leb93], for a more detailed proof in a simpler setting see Theorem 4.2 of [BuGé01]. Note that when $n = 1$, condition **(i)** is equivalent to $a \neq 0$; it would be interesting to find such an explicit condition when $n \geq 2$.

Theorem 2.2 *The following conditions are equivalent :*

(i) *There is weak stabilization and for every maximal geodesic $(x_s)_{s\in\mathbf{R}}$ of M we have*

$$\bigcap_{s\in\mathbf{R}} \ker(a(x_s)) = \{0\}. \tag{GCC}$$

(ii) *There exist two constants $C, \beta > 0$ such that for all $u \in H$ and for every time t*

$$E(u,t) \leq Ce^{-\beta t}E(u,0).$$

The condition on the intersections of the kernels of $a(x_s)$ is called the geometric control condition (GCC) and condition **(ii)** is called strong stabilization of the damped wave equation. For $n = 1$ this theorem has been proved in the setting of a Riemannian manifold with or without boundary by Bardos, Lebeau, Rauch, and Taylor (see [RaTa74] and [BLR92]). Note that when $n = 1$, the weak stabilization hypothesis is not needed because it is a consequence of the geometric control condition. However, when $n > 1$, the geometric condition alone does not imply strong or even weak stabilization, as we shall see later, so this hypothesis is necessary. It is still an open problem to find a purely geometric condition equivalent to strong stabilization of the vectorial wave equation. To the knowledge of the author, Theorem 2.2 has not been proved in the existing literature, but it seems that it was already known by people well acquainted with the field. The question of observability and controllability for a system of coupled wave equations a bit more general than (2.1) was recently studied by Y. Cui, C. Laurent and Z. Wang in [CLW18]. We will get a proof of Theorem 2.2 as a corollary of Theorem 2.3.

Definition We denote the best exponential decay rate of the energy by α, defined as follows :

$$\alpha = \sup\{\beta \in \mathbf{R} : \exists C > 0, \forall u \in H, \forall T > 0, E(u,T) \leq Ce^{-\beta T}E(u,0)\}.$$

The main result of this article is Theorem 2.3, its aim being to express α as the minimum of two quantities. The first quantity depends on the spectrum of A_a, and the second depends on a differential equation described by the values of a along geodesics. However, we still need to define a few things before stating Theorem 2.3.

It is well known that $\mathrm{sp}(A_a)$, the spectrum of A_a, is discrete and solely contains eigenvalues λ_j satisfying $\mathfrak{Re}(\lambda_j) \in [-2\sup_{x\in M} \|a(x)\|_2; 0]$ and $|\lambda_j| \to \infty$. This comes from the fact that $D(A_a)$ is compactly embedded in H and that, for $\mathfrak{Re}(\lambda) \notin [-2\sup_{x\in M} \|a(x)\|_2; 0]$, the operator $(A_a - \lambda \mathrm{Id})$ is bijective from $D(A_a)$ to H and has a continuous inverse. Moreover the spectrum of A_a is invariant by complex conjugation. We will denote by E_{λ_j} the generalized eigenvector subspace of A_a associated with λ_j ; this subspace is defined as

$$E_{\lambda_j} = \left\{ u \in D(A_a) : \exists k \in \mathbf{N}, (A_a - \lambda_j)^k u = 0 \right\}$$

and is of finite dimension. We next define the following quantities.

$$D(R) = \sup\{\mathfrak{Re}(\lambda_j) : \lambda_j \in \mathrm{sp}(A_a), |\lambda_j| > R\},\ \ D_0 = \lim_{R\to 0^+} D(R)$$

$$\text{and}\ \ D_\infty = \lim_{R\to +\infty} D(R)$$

These quantities are all nonnegative and for every $R > 0$ we have $D_0 \geq D(R) \geq D_\infty$. The quantity D_0 is sometimes called the spectral abscissa of A_a.

Since M is a Riemannian manifold there is a natural isometry between T_xM and T_x^*M *via* the scalar product g_x. The scalar product defined on T_x^*M by this isometry is called g^x, and if $\xi \in T_x^*M$, we will write $|\xi|_g$ for $\sqrt{g^x(\xi,\xi)}$. Let us call S^*M the cotangent sphere bundle of M, that is, the subset $\{(x,\xi) \in T^*M : |\xi|_g = 1/2\}$ of T^*M. We call ϕ the geodesic flow on S^*M and recall that it corresponds to the Hamiltonian flow generated by $|\xi|_g^2$. In everything that follows $(x_0;\xi_0)$ will denote a point of S^*M, and we will write (x_t,ξ_t) for $\phi_t(x_0,\xi_0)$. We now introduce the function $G_t^+ : S^*M \to \mathscr{M}_n(\mathbf{C})$, where t is a real number. It is defined as the solution of the differential equation

$$\begin{cases} G_0^+(x_0,\xi_0) = \mathrm{Id}_n \\ \partial_t G_t^+(x_0,\xi_0) = -a(x_t) G_t^+(x_0,\xi_0). \end{cases} \tag{2.3}$$

We shall see later that G_t^+ is a cocycle map ; this means that it satisfies the relation $G_{s+t}^+(x,\xi) = G_t^+(\phi_s(x,\xi))G_s^+(x,\xi)$. In the scalar-like case where $a(x)$ is a diagonal matrix for every x the matrix G_t^+ is simply described by the formula

$$G_t^+(x_0,\xi_0) = \exp\left(-\int_0^t a(x_s)\mathrm{d}s\right). \tag{2.4}$$

As we will see, the fact that this formula is no longer true in the general setting is the main reason why new phenomena arise in comparison to the scalar case (see, for example, Proposition 2.4). Let us define for every $t > 0$, the quantities

$$C(t) \overset{\text{def}}{=} \frac{-1}{t} \sup_{(x_0,\xi_0)\in S^*M} \ln\left(\|G_t^+(x_0;\xi_0)\|_2\right) \quad \text{and} \quad C_\infty = \lim_{t\to\infty} C(t). \tag{2.5}$$

We will see later that this limit does exist. In the scalar case one also has the simpler formula

$$C(t) = \frac{1}{t} \inf_{(x_0,\xi_0)\in S^*M} \int_0^t a(x_s)\mathrm{d}s. \tag{2.6}$$

There is a similar but more complex formula when $n \geq 1$. Denote by y_t a vector of $\mathbf{C}^n$ of Euclidean norm 1 such that

$$G_t^+(x_0,\xi_0)G_t^+(x_0,\xi_0)^* y_t = \|G_t^+(x_0,\xi_0)\|_2^2 y_t. \tag{2.7}$$

The vector y_t obviously depends on (x_0,ξ_0) even though it is not explicitly written. We then have for every $t > 0$

$$C(t) = \frac{1}{t} \inf_{(x_0,\xi_0)\in S^*M} \int_0^t \langle a(x_s)y_s, y_s\rangle \mathrm{d}s. \tag{2.8}$$

This formula is a direct consequence of Proposition 2.8 and does not depend on the choice of y_s. Since a is Hermitian positive semidefinite it follows from (2.8) that $C(t) \geq 0$ and $C_\infty \geq 0$. Recall also that $D(0) \leq 0$ and we can finally state the main result of this article.

Theorem 2.3 *The best exponential decay rate is given by the formula*

$$\alpha = 2\min\{-D_0; C_\infty\}. \tag{2.9}$$

Moreover we have the following properties :

(i) $C_\infty \leq -D_\infty$

(ii) *One can have* $-D_0 > 0$ *and* $C_\infty = 0$.

(iii) *One can have* $C_\infty > 0$ *and* $D_0 = 0$*, but only if* $n > 1$.

This result has already been proved by G. Lebeau [Leb93] for $n = 1$ on a Riemannian manifold *with boundary*. The novelty of this article thus comes from the fact that we are dealing with vectorial waves with a matrix damping term, and this leads to the appearance of interesting new phenomena in comparison to the scalar setting (see, for example, section 4). The proof of Theorem 2.3 stays close to the one of Lebeau, and so it is very likely that it would extend to the case where $\partial M \neq \emptyset$ if one would be willing to adapt Corollary 2.5. Let us also point out a similar result about the asymptotic behavior of the observability constant of the scalar wave equation in Theorem 2 and Corollary 4 of [HPT16].

Remark – We will show in the proof of Theorem 2.2 that the geometric control condition is in fact equivalent to $C_\infty > 0$. Combining this with point **(iii)** of Theorem 2.3 we already see that (GCC) is not equivalent to strong stabilization when $n > 1$. Moreover, using point **(i)** of Theorem 2.3, we see that when $C_\infty > 0$ and $D_0 = 0$ we have (GCC), but weak stabilization still fails. In fact when (GCC) is satisfied the strong stabilization fails if and only if the weak stabilization also fails.

Remark – Proposition 2.9 and the results on Gaussian beams show that C_∞ is taking account of the energy decay of the high frequency solutions of (2.1). On the other hand we have $D_0 \geq D_\infty$ and $-C_\infty \geq D_\infty$, so if $-D_0 < C_\infty$ there exists an eigenfunction u of A_a such that $E(u,t) = e^{-2D_0 t}E(u,0) = e^{-\alpha t}E(u,0)$. This means that D_0 is taking into account the energy decay of low frequency solutions of (2.1).

High frequency overdamping A natural question to ask is how does α behave as a function of the damping term a ? Let us write, respectively, $\alpha(a)$, $D_0(a)$, and $C_\infty(a)$ for the quantities α, D_0, and C_∞ associated with a damping term a. An interesting fact is that the function $a \mapsto \alpha(a)$ is not monotonous, even in the simplest case. Indeed in [CoZu93] S. Cox and E. Zuazua showed that [1] in the case of a scalar damped wave equation on a string of length one, the decay rate is given by $\alpha(a) = -2D_0(a)$. They also calculated the spectral abscissa $D_0(a)$ in the case of a constant damping term and found $D_0(a) = -a + \mathfrak{Re}(\sqrt{a^2 - \pi^2})$. This shows that increasing the constant damping term above π actually reduces $\alpha(a)$, a phenomenon called "overdamping".

Theorem 2 of [Leb93] shows that for a scalar damped wave equation on a general manifold the decay rate $\alpha(a)$ is governed by $D_0(a)$ *and* $C_\infty(a)$. However in that case the overdamping can only come from D_0 since $a \mapsto C_\infty(a)$ is obviously monotonous, subadditive and positively homogeneous from (2.6). In view of the previous remark it makes sense to call this phenomenon "low frequency overdamping".

On the other hand with the *vectorial* damped wave equation the situation is different. We will show that $a \mapsto C_\infty(a)$ is neither monotonous nor subadditive nor homogeneous and thus an overdamping phenomenon can also come from the C_∞ term. Once again in view of the previous remark we call this phenomenon "high frequency overdamping". To be more precise we will prove the following result.

Proposition 2.4 *The function $a \mapsto C_\infty(a)$ is neither homogeneous nor monotonous; indeed it is possible to have $C_\infty(2a) < C_\infty(a)$ or $2C_\infty(a) < C_\infty(2a)$. It is also not subadditive, $C_\infty(a+b)$ can be strictly greater or smaller than $C_\infty(a) + C_\infty(b)$.*

This result will be proved by computing explicitly the quantity C_∞ in simple settings. Figure 1 shows the results of some of those computations and illustrates the nonlinear behavior of the map $a \mapsto C_\infty(a)$.

However it seems that C_∞ still has some kind of linear behavior. Namely, on $M = S^1$ and with a particular kind of damping term (see Section 2.4), we are able to show that

$$\lim_{\lambda\to\infty} \frac{C_\infty(\lambda a)}{\lambda} \quad \text{and} \quad \lim_{\lambda\to 0^+} \frac{C_\infty(\lambda a)}{\lambda}\,;$$

1. Provided a is of bounded variation.

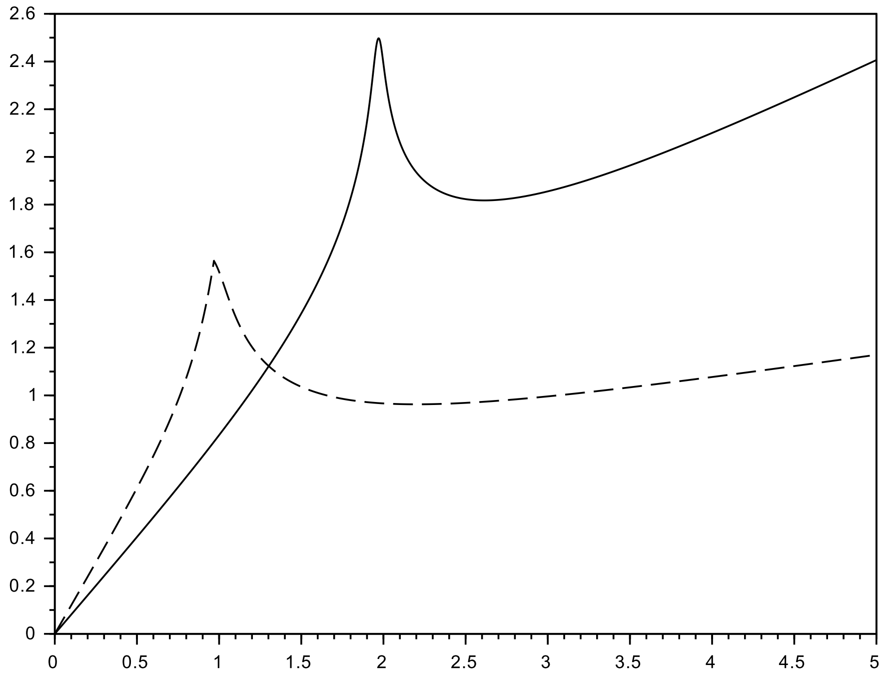

FIGURE 2.1 – Plot of the function $\lambda \mapsto C_\infty(\lambda a)$ for two different damping term a on S^1. See section 2.4 for the definition of these damping terms.

both exist and are finite. This result is proven in section 2.4, but it remains open whether this is still true for any damping term on a general manifold M.

The remainder of this article is organized as follows. Section 2.2 contains definitions and results about the propagation of the microlocal defect measures associated with a sequence of solutions of (2.1). These results will play an important role while bounding α from below. The next section is devoted to the proof of Theorems 2.2 and 2.3. Establishing the formula for α is the most difficult part, the proof for the lower bound makes use of Gaussian beams, while for the upper bound we will use the result of section 2.2 conjointly with a decomposition in high and low frequencies. Finally, in the last section we study the behavior of C_∞ and prove Proposition 2.4.

2.2 Propagation of the microlocal defect measure

Let us work with the manifold $\mathbf{R} \times M$ endowed with the product metric induced by the ones of $\mathbf{R}$ and M. We will denote by (t, τ, x, ξ) the points of $T^*(\mathbf{R} \times M)$, where

$(t,\tau) \in T^*\mathbf{R}$ and $(x,\xi) \in T^*M$. Given a point $(x,\xi) \in T^*M$ we will write $|\xi|_g^2 = g^x(\xi,\xi)$ to denote the square of the norm of ξ. We moreover define $S^*(\mathbf{R} \times M)$ as the subset of points of $T^*(\mathbf{R} \times M)$ such that $\tau^2 + |\xi|_g^2 = 1/2$ and recall that $S^*M = \{(x,\xi) \in T^*M : |\xi|_g^2 = 1/4\}$. We call ϕ the geodesic flow on T^*M, that is, the Hamiltonian flow generated by $|\xi|_g^2$ and φ the Hamiltonian flow on $T^*(\mathbf{R} \times M)$ generated by $|\xi|_g^2 - \tau^2$. In other words

$$\varphi_s(t,\tau,x,\xi) = (t - 2s\tau, \tau, \phi_s(x,\xi)).$$

In everything that follows (x_0,ξ_0) will denote a point of S^*M, and we will write $(x_t,\xi_t) = \phi_t(x_0,\xi_0)$.

Throughout this section we call P the differential operator $\partial_t^2 - \Delta$. We know that P is self-adjoint on $L^2(\mathbf{R} \times M)^n$ and has $(|\xi|_g^2 - \tau^2)\mathrm{Id}_n = p \cdot \mathrm{Id}_n$ for principal symbol, note that p is a scalar valued function. If b is a smooth function from $T^*(\mathbf{R} \times M)$ to $\mathscr{M}_n(\mathbf{C})$, we denote by $\{p,b\}$ the Poisson bracket of p and b; it is defined as the matrix whose coefficients are the usual Poisson bracket $\{p,b_{ij}\}$. With this definition the basic properties of the Poisson bracket are still true. Namely, we have a Leibniz rule $\{p,bc\} = \{p,b\}c + b\{p,c\}$, and it is linked to the Hamiltonian flow of p in the usual way, that is $\partial_s(b \circ \varphi_s)(\rho) = \{p,b\}(\varphi_s(\rho))$. Moreover, if B is a pseudodifferential operator of order m and of principal symbol $\sigma_m(B) = b$ then $[P,B]$ is a pseudo-differential operator of order $\leq m+1$ and of principal symbol $-i\{p,b\}$. Note that this is only possible because $p \cdot \mathrm{Id}$ commutes with every matrix of $\mathscr{M}_n(\mathbf{C})$. For more details about pseudodifferential operators see [Hör85].

We now recall some results about microlocal defect measures. For proofs and more details see the original article of P. Gérard [Gér91].

Proposition 2.1 *Let $I \subset \mathbf{R}$ be an interval, and let $(u_n)_n$ be a sequence of functions of $H^m_{\mathrm{loc}}(I \times M)$ weakly converging to 0. Then there exists*

- *a sub-sequence $(u_{n_k})_k$;*
- *a positive Radon measure ν on $S^*(\mathbf{R} \times M)$;*
- *a matrix M of ν-integrable functions on $S^*(\mathbf{R} \times M)$ such that M is Hermitian positive semidefinite ν-a.e. and $\mathrm{Tr}(M) = 1$ ν-a.e.;*

such that, for every compactly supported pseudodifferential operator B with principal symbol b of order $2m$ we have

$$\lim_{k\to+\infty} \langle Bu_{n_k}|u_{n_k}\rangle_{H^{-m},H^m} = \int_{S^*(\mathbf{R}\times M)} \mathrm{Tr}(bM)\mathrm{d}\nu. \tag{2.10}$$

Note that here b is a matrix of dimension n depending on (t,τ,x,ξ). One crucial property is that $(u_{n_k})_k$ strongly converges to 0 in $H^m_{\mathrm{loc}}(I \times M)$ if and only if $\mu = 0$.

Definition In the setting of the previous theorem we will call $\mu = M\nu$ the microlocal defect measure of the subsequence $(u_{n_k})_k$, and we will say that $(u_n)_n$ is "pure" if it has a microlocal defect measure without preliminary extraction of a subsequence.

Proposition 2.2 *Let $I \subset \mathbf{R}$ be a compact interval, and let (u_n) be a pure sequence of $H^1(I \times M)$ weakly converging to 0 with $M\mathrm{d}\nu$ as microlocal defect measure. Recalling that $P = \partial_t^2 - \Delta$ and that its principal symbol is $p \cdot \mathrm{Id}$, the following properties are equivalent :*

(i) $Pu_n \underset{n\to\infty}{\to} 0$ *strongly in $H^{-1}(I \times M)$.*

(ii) *ν is supported on the set $\{p = 0\}$.*

Proposition 2.3 *Let $(u_k)_k$ be a bounded sequence of $H^1(I \times M)$ weakly converging to 0. Assume that u_k is solution of the damped wave equation for every k and let b be a smooth function on $S^*(I \times M)$ to $\mathscr{M}_n(\mathbf{C})$, 1-homogeneous in the (τ, ξ) variable. If $(u_k)_k$ is pure with microlocal defect measure $\mu = M\nu$ then*

$$\int_{S^*(I\times M)} \mathrm{Tr}\left[(\{b,p\} - 2\tau(ab+ba))M\right]\mathrm{d}\nu = 0.$$

Proof. Let B be a pseudodifferential operator of order 1 and with principal symbol b. We then have

$$\lim_{k\to\infty} \langle [B,P]u_k, u_k\rangle_{H^{-1},H^1} = \int \mathrm{Tr}[\sigma_2([B,P])M]\mathrm{d}\nu = \frac{1}{i}\int \mathrm{Tr}\left[\{b,p\}M\right]\mathrm{d}\nu,$$

but we moreover know that $\langle [B,P]u_k, u_k\rangle = -2\langle (Ba\partial_t + a\partial_t B)u_k, u_k\rangle$, which tends to

$$-2i\int \mathrm{Tr}[\tau(ab+ba)M]\mathrm{d}\nu,$$

thus finishing the proof. □

In what follows $\mu = M\mathrm{d}\nu$ will denote the microlocal defect measure of a pure sequence $(u_k)_k$ of solutions of the damped wave equation on $\mathbf{R} \times M$. Here our aim is to give a relation between $\varphi_s^*\mu$ and μ. The measure $\varphi_s^*\mu$ is the push forward of μ by φ_s, and is defined by the following property :

$$\text{for every } \mu\text{-integrable function } b \text{ we have } \int \mathrm{Tr}[(b\circ\varphi_s)\mathrm{d}\mu] = \int \mathrm{Tr}[b\mathrm{d}\varphi_s^*\mu].$$

Definition For every $s \in \mathbf{R}$ we define the function $G_s : T^*(\mathbf{R} \times M) \to \mathscr{M}_n(\mathbf{C})$ as the solution of the following differential equation :

$$\begin{cases} G_0(t,\tau,x,\xi) = \mathrm{Id}_n \\ \partial_s G_s(t,\tau,x,\xi) = \{p, G_s\}(t,\tau,x,\xi) + 2\tau G_s(t,\tau,x,\xi)a(x). \end{cases}$$

The matrix G_t is a cocycle map ; that is, it satisfies the relation $G_{s+t}(\rho) = G_t(\varphi_s(\rho))G_s(\rho)$. The proof of this fact is given for G_t^+ at the end of the section.

Proposition 2.4 *The propagation of the measure is given by the formula $\varphi_s^*\mu = G_{-s}\mu G_{-s}^*$. More precisely, this means that for every continuous function b compactly supported in the (t,x) variable we have*

$$\int_{S^*(\mathbf{R}\times M)} \mathrm{Tr}[(b\circ\varphi_s)G_sMG_s^*]\mathrm{d}\nu = \int_{S^*(\mathbf{R}\times M)} \mathrm{Tr}[bM]\mathrm{d}\nu$$

or, equivalently, for every continuous function c compactly supported in the (t,x) variable

$$\int_{S^*(\mathbf{R}\times M)} \mathrm{Tr}[c\,G_{-\sigma}MG_{-\sigma}^*]\mathrm{d}\nu = \int_{S^*(\mathbf{R}\times M)} \mathrm{Tr}[c\circ\varphi_\sigma M]\mathrm{d}\nu.$$

Proof. In order to show the first equality, it suffices to verify that

$$\partial_s \int \mathrm{Tr}[(b\circ\varphi_s)G_sMG_s^*]\mathrm{d}\nu = 0. \tag{2.11}$$

We know that we can differentiate under the integral sign,

$$\partial_s \int \mathrm{Tr}\left[(b\circ\varphi_s)G_sMG_s^*\right]\mathrm{d}\nu = \int \mathrm{Tr}\left[\partial_s\big((b\circ\varphi_s)G_sMG_s^*\big)\right]\mathrm{d}\nu = \int \mathrm{Tr}\left[\partial_s\big(G_s^*(b\circ\varphi_s)G_s\big)M\right]\mathrm{d}\nu.$$

Denoting by $'$ the differentiation with respect to s, we then get

$$\begin{aligned}\partial_s\big(G_s^*(b\circ\varphi_s)G_s\big) &= G_s^{*\prime}(b\circ\varphi_s)G_s + G_s^*\{p, b\circ\varphi_s\}G_s + G_s^*(b\circ\varphi_s)G_s' \\ &= \{p, G_s^*(b\circ\varphi_s)G_s\} - \{p, G_s^*\}(b\circ\varphi_s)G_s - G_s^*(b\circ\varphi_s)\{p, G_s\} \\ &\quad + G_s^{*\prime}(b\circ\varphi_s)G_s + G_s^*(b\circ\varphi_s)G_s',\end{aligned}$$

and by application of the previous proposition

$$\int \mathrm{Tr}[\{p, G_s^*(b\circ\varphi_s)G_s\}M]\mathrm{d}\nu = -\int \mathrm{Tr}[(2\tau aG_s^*(b\circ\varphi_s)G_s + 2\tau G_s^*(b\circ\varphi_s)G_sa)M]\mathrm{d}\nu.$$

Collecting all the terms together we see that in order to have (2.11) it suffices that

$$\partial_sG_s = \{p, G_s\} + 2\tau G_sa \text{ and } \partial_sG_s^* = \{p, G_s^*\} + 2\tau aG_s^*,$$

which coincides with the definition of G and proves the first formula. The last formula is obtained from the first one by simply writing $c = b\circ\varphi_s$ and $\sigma = -s$. □

Proposition 2.5 *The measure ν is supported on the set $\{\tau = \pm 1/2\}$.*

Proof. It is a consequence of Proposition 2.2 : ν is a measure on $S^*(\mathbf{R}\times M)$ so $\tau^2+|\xi|_g^2 = 1/2$ and it is supported on the set $\{p=0\}$ because $(\partial_t^2-\Delta)u_k = -2a\partial_tu_k$ strongly converges to 0 in H^{-1}. □

Definition This encourages us to consider the two connected components

$$SZ^+ = S^*(\mathbf{R}\times M)\cap\{\tau=-1/2\} \text{ and } SZ^- = S^*(\mathbf{R}\times M)\cap\{\tau=1/2\},$$

as well as $\mu^+ = M^+\nu^+$ and $\mu^- = M^-\nu^-$, the restrictions of μ to SZ^+ and SZ^- respectively. Moreover we will denote by G_s^+ and G_s^- the restrictions of G_s to SZ^+ and SZ^-, respectively.

With this notation we get

$$\partial_s G_s^+ = \{p, G_s^+\} - G_s^+ a.$$

Remark – Since the functions a and p do not depend on t we see that $G_s^+(t_1,\tau,x,\xi)$ and $G_s^+(t_2,\tau,x,\xi)$ satisfy the same differential equation (with respect to s). Consequently $G_s^+(t_1,\tau,x,\xi) = G_s^+(t_2,\tau,x,\xi)$ and since the τ variable is constant on SZ^+ the values of G_s^+ depend only on (x,ξ) so we can consider it as a functions on S^*M. The same obviously goes for G_s^-.

Corollary 2.5 *Letting B be a Borel set of SZ^+, we have* $\nu^+(\varphi_s(B)) = \int_B \mathrm{Tr}[G_s^+ M G_s^{+*}]\mathrm{d}\nu^+$.

Proof.

$$\nu^+(\varphi_s(B)) = \int_{SZ^+}\mathbf{1}_{\varphi_s(B)}\mathrm{d}\nu^+ = \int_{SZ^+}\mathbf{1}_B\circ\varphi_{-s}\mathrm{d}\nu^+ = \int_{SZ^+}\mathbf{1}_B\,\mathrm{Tr}[G_s^+ M G_s^{+*}]\mathrm{d}\nu^+$$

□

The cocycle G^+ thus plays an important role here since it completely describes the evolution of the microlocal defect measure. We finish this section with a few useful facts about G^+.

A direct calculation shows that the matrix G^+ satisfies the following cocycle formula :

$$\forall\rho\in S^*M,\ \forall s,t\in\mathbf{R},\ \ G_{s+t}^+(\rho) = G_t^+(\phi_s(\rho))G_s^+(\rho). \tag{2.12}$$

Indeed if we differentiate the right side with respect to s, we get

$$\begin{aligned}\partial_s G_t^+(\phi_s(\rho))G_s^+(\rho) &= G_t^+(\phi_s(\rho))\big[\{p,G_s^+\}(\rho) - G_s^+(\rho)a(\rho)\big] + \{p, G_t^+\circ\phi_s\}(\rho)G_s^+(\rho)\\ &= \{p,(G_t^+\circ\phi_s)G_s^+\}(\rho) - G_t^+(\phi_s(\rho))G_s^+(\rho)a(\rho).\end{aligned}$$

The matrices $(G_t^+\circ\phi_s)G_s^+$ and G_{s+t}^+ thus satisfy the same differential equation with the same initial condition and are consequently equal. This cocycle formula gives us a second differential equation satisfied by G^+. For every $(x_0,\xi_0)\in S^*M$

$$\partial_t G_t^+(x_0,\xi_0) = \lim_{h\to 0}\frac{G_{t+h}^+(x_0,\xi_0) - G_t^+(x_0,\xi_0)}{h} \quad\text{and}\quad G_{t+h}^+(x_0,\xi_0) = G_h^+(\phi_t(x_0,\xi_0))G_t^+(x_0,\xi_0)$$

and hence $\partial_t G_t^+(x_0,\xi_0) = \partial_s G_s^+(\phi_t(x_0,\xi_0))\big|_{s=0} \cdot G_t^+(x_0,\xi_0) = -a(x_t)G_t^+(x_0,\xi_0)$,

where $(x_t,\xi_t) = \phi_t(x_0,\xi_0)$. In accordance with the definition of G^+ given in the introduction we see that it is the solution of the differential equation

$$\begin{cases} G_0^+(x_0,\xi_0) = \mathrm{Id}_n \\ \partial_t G_t^+(x_0,\xi_0) = -a(x_t)G_t^+(x_0,\xi_0). \end{cases} \tag{2.13}$$

Let us add one final formula which will be useful later. If we define $j:(x,\xi)\mapsto(x,-\xi)$ we have $\phi_s(j(\rho)) = j(\phi_{-s}(\rho))$ and deduce that $\partial_s(G_s^-\circ j) = -\{p, G_s^-\circ j\} + (G_s^-\circ j)a$.

2.3 Estimation of the best decay rate

We recall some definitions from the introduction of this article. The following quantities are nonpositive :

$$D(R) = \sup\{\mathfrak{Re}(\lambda_j) : \lambda_j \in \mathrm{sp}(A_a), |\lambda_j| > R\},\ D_0 = \lim_{R\to 0^+} D(R)$$

$$\text{and } D_\infty = \lim_{R\to+\infty} D(R).$$

For every $t \geq 0$ we choose y_t a vector of $\mathbf{C}^n$ of Euclidean norm 1 such that

$$G_t^+(x_0,\xi_0)G_t^+(x_0,\xi_0)^* y_t = \|G_t^+(x_0,\xi_0)\|_2^2 y_t. \tag{2.14}$$

The vector y_t depends on (x_0,ξ_0), even though it is not written. We then define for every $t > 0$ the quantities

$$C(t) = \frac{1}{t}\inf_{(x_0,\xi_0)\in S^*M}\int_0^t \langle a(x_s)y_s, y_s\rangle \mathrm{d}s = \frac{-1}{t}\sup_{(x_0,\xi_0)\in S^*M}\ln\left(\|G_t^+(x_0;\xi_0)\|_2\right) \tag{2.15}$$

$$\text{and } C_\infty = \lim_{t\to\infty} C(t).$$

We will see later that these definitions make sense and that they do not depend on the choice of y_s. Remember that $C(t)$ is nonnegative.

The remainder of this section is mainly dedicated to the proof of the formula

$$\alpha = 2\min\{C_\infty, -D_0\}.$$

Before starting let us just indicate the main steps of the proof. We first give an upper bound for α using Gaussian beams. These are particular approximate solutions of the damped wave equation that are concentrated near a geodesic. In order to prove the lower bound for α we will use a high frequency inequality (Proposition 2.9) together with a decomposition of solutions of (2.1) in high and low frequencies.

2.3.1 Upper bound for α

Let $\lambda_j \in \mathrm{sp}(A_a)\backslash\{0\}$ and $u = (u_0, u_1) \in E_{\lambda_j}\backslash\{0\}$ be such that $A_a u = \lambda_j u$. The solution of (2.1) then is $u(t,x) = e^{t\lambda_j} u_0(x)$, and we have $E(u,t) = e^{2t\mathfrak{Re}(\lambda_j)} E(u,0)$. Since $E(u,0) \neq 0$ we know that $\alpha \leq -2D(0)$.

Showing that $\alpha \leq 2C_\infty$ is a bit more difficult as it requires us to construct Gaussian beams. We will start by constructing them on $\mathbf{R}^d$ endowed with a Riemannian metric g. Gaussian beams are approximate solutions of the wave equation (in a sense made precise by (2.16)) whose energy may be arbitrarily concentrated along a geodesic up to a fixed time $T > 0$ (see (2.18)). They will allow us to construct exact solutions to the damped wave equation whose energy is also arbitrarily concentrated along a geodesic up to some time T. As always we will call $(x_t; \xi_t) = \phi_t(x_0, \xi_0)$ the points of the geodesic. We will follow and adapt the construction given in [Ral82] or [MaZu02].

We consider for every integer k a function $u_k : \mathbf{R}^d \to \mathbf{R}^n$ given by the formula

$$u_k(t,x) = k^{-1+d/4} b(t,x) \exp(ik\psi(t,x))\omega$$

where $\psi(t,x) = \langle \xi(t), (x - x(t)) \rangle + \frac{1}{2}\langle M_t(x - x(t)), x - x(t) \rangle$, with M_t a $d \times d$ symmetric matrix with positive definite imaginary part, b is a bounded continuous function and ω is a vector of $\mathbf{C}^n$. In what follows C represents a positive constant that can vary from one line to another but does not depend on k ; however, C can depend on T.

Theorem 2.6 ([Ral82]) *It is possible to chose M_t and b such that*

$$\sup_{t \in [0;T]} \|\partial_t^2 u_k(t,\cdot) - \Delta_g u_k(t,\cdot)\|_{L^2(\mathbf{R}^d)} \leq C k^{-1/2}, \tag{2.16}$$

$$\forall t \in [0;T] \ \lim_{k \to \infty} E(u_k, t) \ \text{ is positive, finite, and does not depend on } t, \tag{2.17}$$

$$\sup_{t \in [0;T]} \int_{\mathbf{R}^d \backslash B(x_t, k^{-1/4})} |\partial_t u_k(t,\cdot)|^2 + |\nabla u_k(t,\cdot)|_g^2 \mathrm{d}x \leq C \exp(-\beta\sqrt{k}). \tag{2.18}$$

Under these conditions we say that u_k is a Gaussian beam. We also need the following lemma from [Ral82].

Lemma 2.7 ([Ral82]) *Let $c \in L^\infty(\mathbf{R}^d)$ be a function satisfying $|x - x_0|^{-\alpha} c(x) \in L^\infty(\mathbf{R}^d)$ for some $\alpha \geq 0$ and some $x_0 \in \mathbf{R}^d$, and let A be a symmetric, positive definite, real $d \times d$ matrix. Then*

$$\int_{\mathbf{R}^d} \left| c(x) \exp\big(-k\langle A(x - x_0), x - x_0 \rangle \big) \right|^2 \mathrm{d}x \leq C k^{-d/2-\alpha} \tag{2.19}$$

for some $C > 0$ that does not depend on k.

Using Lemma 2.7 with $c = |b(t,\cdot)|$ and $\alpha = 0$ we see that $\|u_k(t,\cdot)\|_{L^2(\mathbf{R}^d)} \leq Ck^{-1/2}$. Let us now define the function $v_k(t,x) = G_t^+(x_0,\xi_0)u_k(t,x)$. As we shall see, it is an approximate solution of the damped wave equation. Indeed we have

$$(\partial_t^2 - \Delta_g + 2a\partial_t)v_k(t,x) = G_t^+(x_0,\xi_0)\left(\partial_t^2 - \Delta_g\right)u_k(t,x) + 2(a(x) - a(x_t))G_t^+(x_0,\xi_0)\partial_t u_k(t,x)$$

$$+ \left(a(x_t)^2 - \partial_t a(x_t) - 2a(x)a(x_t)\right)G_t^+(x_0,\xi_0)u_k(t,x) \stackrel{\text{def}}{=} f_k(t,x)$$

and we need to show that $\|f_k(t,\cdot)\|_{L^2} \leq Ck^{-1/2}$. In order to do that we only need to prove $\|2(a(\cdot) - a(x_t))G_t^+(x_0,\xi_0)\partial_t u_k(\cdot,t)\|_{L^2} \leq Ck^{-1/2}$ because the other terms obviously satisfy the bound. Now since the function $x \mapsto |x - x_t|^{-1}\|a(x) - a(x_t)\|_2$ is in L^∞ we can use Lemma 2.7 on $2(a(\cdot) - a(x_t))G_t^+(x_0,\xi_0)\partial_t u_k(\cdot,t)$ and finally get

$$\sup_{t\in[0;T]} \|(\partial_t^2 - \Delta_g + 2a\partial_t)v_k(t,\cdot)\|_{L^2(\mathbf{R}^d)} \leq Ck^{-1/2}.$$

Moreover we see that v_k still satisfies properties (2.17) and (2.18), although now the limit of the energy of v_k may vary with t because $G_t^+(x_0,\xi_0)$ does. We finally define w_k as the solution of (2.1) with initial conditions $w_k(0,\cdot) = v_k(0,\cdot)$ and $\partial_t w_k(0,\cdot) = \partial_t v_k(0,\cdot)$. By definition of w_k we have $(\partial_t^2 - \Delta_g + 2a\partial_t)v_k = (\partial_t^2 - \Delta_g + 2a\partial_t)(v_k - w_k) = f_k$ and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}E(v_k - w_k, t) = -2\int_{\mathbf{R}^d}\langle a\partial_t(v_k - w_k), \partial_t(v_k - w_k)\rangle\mathrm{d}x + \int_{\mathbf{R}^d}\mathfrak{Re}\langle f_k, \partial_t(v_k - w_k)\rangle\mathrm{d}x.$$

The first term on the right hand side is negative and, using Cauchy–Schwarz, we can bound the second term by $Ck^{-1/2}$. Indeed we already know that $\|f_k\|_{L^2} \leq Ck^{-1/2}$ and $\|\partial_t(v_k - w_k)\|_{L^2}$ is uniformly bounded in $k \in \mathbf{N}$ and $t \in [0;T]$. Since $E(w_k - v_k, 0) = 0$ by integrating we get

$$\sup_{t\in[0;T]} E(v_k - w_k, t) \leq CTk^{-1/2}.$$

In combination with the estimate (2.18) of u_k we see that $w_k(t,\cdot)$ is concentrated around x_t. More precisely we have

$$\sup_{t\in[0;T]}\int_{\mathbf{R}^d\setminus B(x_t,k^{-1/4})} |\partial_t w_k(t,\cdot)|^2 + |\nabla w_k(t,\cdot)|_g^2\mathrm{d}x \leq CTk^{-1/2}. \tag{2.20}$$

Then we set ω such that $\lim_{k\to\infty} E(v_k,0) = 1$ and $G_T^{+*}(x_0,\xi_0)G_T^+(x_0,\xi_0)\omega = \|G_T^+(x_0,\xi_0)\|_2^2\omega$. According to the definition of v_k we have

$$E(v_k,T) = \frac{1}{2}\int_M |G_T^+(x_0,\xi_0)\partial_t u_k(T,\cdot) - a(x_T)G_T^+(x_0,\xi_0)u_k(T,\cdot)|^2 + |G_T^+(x_0,\xi_0)\nabla u_k(T,\cdot)|^2\mathrm{d}x$$

but $\|u_k(T,\cdot)\|_{L^2} \leq Ck^{-1/2}$ so the term $a(x_T)G_T^+(x_0,\xi_0)u_k(T,\cdot)$ vanishes and we get

$$\lim_{k\to\infty} E(v_k,T) = \|G_T^+(x_0,\xi_0)\|_2^2.$$

This in turn implies that $(w_k)_k$ is a sequence of solutions of (2.1) which satisfies $\lim_{k\to\infty} E(w_k, 0) = 1$ and $\lim_{k\to\infty} E(w_k, T) = \|G_T^+(x_0, \xi_0)\|_2^2$. Summing up the discussion so far, we have the following.

Proposition 2.6 *Let (M, g) be a compact Riemannian manifold without boundary. For any time $T > 0$, any $\varepsilon > 0$ and any $(x_0, \xi_0) \in S^*\mathbf{R}^d$ there exists a solution u of the damped wave equation such that $E(u, 0) = 1$ and $\left|E(u, T) - \|G_T^+(x_0, \xi_0)\|_2^2\right| < \varepsilon$.*

Using charts, this result extends to any compact Riemannian manifold (M, g) without boundary and we finally get the following.

Proposition 2.7 *For any time $T > 0$, any $\varepsilon > 0$, and any $(x_0, \xi_0) \in S^*M$ there exists a solution u of the damped wave equation such that $E(u, 0) = 1$ and $\left|E(u, T) - \|G_T^+(x_0, \xi_0)\|_2^2\right| < \varepsilon$.*

Define $\Gamma_t = G_t^+(x_0, \xi_0) G_t^+(x_0, \xi_0)^*$ and, for every time t, choose y_t a vector of Euclidean norm 1 such that $\Gamma_t y_t = \|\Gamma_t\|_2 y_t$. Let us emphasize again that y_t and Γ_t both implicitly depend on (x_0, ξ_0).

Proposition 2.8

$$\|G_t^+(x_0, \xi_0)\|_2^2 = \|\Gamma_t\|_2 = \exp\left(-2\int_0^t \langle a(x_s)y_s, y_s\rangle \mathrm{d}t\right)$$

Proof. The only thing to prove is the second equality. The map $t \mapsto \Gamma_t$ is the solution of the differential equation

$$\begin{cases} \Gamma_0 = \mathrm{Id}_n \\ \partial_t \Gamma_t = -a(x_t)\Gamma_t - \Gamma_t a(x_t). \end{cases} \tag{2.21}$$

It is hence C^∞ and *a fortiori* locally Lipschitz. Consequently the map $t \mapsto \|\Gamma_t\|_2$ is also locally Lipschitz [2], which implies that it is differentiable for almost every t. Since Γ_t is Hermitian positive definite, $\|\Gamma_t\|_2 = \langle \Gamma_t y_t, y_t\rangle$, and if z is any other vector of norm 1, then $\|\Gamma_t\|_2 \geq \langle \Gamma_t z, z\rangle$. Fixing a time t_0, we then have

$$\begin{aligned} \partial_t \langle \Gamma_t y_{t_0}, y_{t_0}\rangle|_{t=t_0} &= -\left\langle [a(x_{t_0})\Gamma_{t_0} + \Gamma_{t_0} a(x_{t_0})] y_{t_0}, y_{t_0}\right\rangle \\ &= -2\|\Gamma_{t_0}\|_2 \langle a(x_{t_0}) y_{t_0}, y_{t_0}\rangle. \end{aligned}$$

We know that $\langle \Gamma_t y_t, y_t\rangle \geq \langle \Gamma_t y_{t_0}, y_{t_0}\rangle$ for every t and there is equality when $t = t_0$. If $\|\Gamma_t\|_2$ is differentiable at t_0 we deduce that at this point the derivatives of the two functions $t \mapsto \langle \Gamma_t y_t, y_t\rangle$ and $t \mapsto \langle \Gamma_t y_{t_0}, y_{t_0}\rangle$ must be the same. Hence for almost every time t

$$\partial_t \|\Gamma_t\|_2 = \partial_t \langle \Gamma_t y_t, y_t\rangle = -2\|\Gamma_t\|_2 \langle a(x_t) y_t, y_t\rangle.$$

2. We cannot really do better than that in terms of regularity.

To finish the proof we just need to see that the function

$$\Phi : t \mapsto \frac{\|\Gamma_t\|_2}{\exp\left(-2\int_0^t \langle a(x_s)y_s, y_s\rangle \mathrm{d}s\right)}$$

is Lipschitz on every bounded interval $[0;T]$ and *a fortiori* absolutely continuous. From $\Phi' = 0$ almost everywhere we deduce that Φ is constant and since $\Phi(0) = 1$ this finishes the proof.

□

Notice that the choice of y_t is not unique and that $t \mapsto y_t$ is not continuous in general. On the other hand, the derivative of $\|\Gamma_t\|_2$ is uniquely defined almost everywhere, so that the choice of y_t has no importance. Therefore we have

$$C(t) \stackrel{\text{def}}{=} \frac{-1}{t} \sup_{(x_0,\xi_0)\in S^*M} \ln\left(\|G_t^+(x_0;\xi_0)\|_2\right) = \frac{1}{t} \inf_{(x_0,\xi_0)\in S^*M} \int_0^t \langle a(x_s)y_s, y_s\rangle \mathrm{d}s.$$

This function is obviously nonnegative, but in order to prove other properties it is easier to work with $\exp(-tC(t)) = \sup_{\rho\in S^*M} \|G_t^+(\rho)\|_2$. The function a is continuous on M and the geodesic flow ϕ is continuous on $\mathbf{R}\times S^*M$; since G^+ is defined as the solution of (2.13) the function $\|G^+\|$ is in turn continuous on $\mathbf{R}\times S^*M$. As S^*M is compact, $t \mapsto \exp(-tC(t))$ is continuous and so is $t \mapsto C(t)$. We now show that $t \mapsto tC(t)$ is subadditive : letting t and s be two nonnegative reals, we have the following equivalences :

$$(t+s)C(t+s) \geq tC(t)+sC(s) \iff \exp(-2(t+s)C(t+s)) \leq \exp(-2tC(t))\exp(-2sC(s))$$

$$\iff \sup_{(x,\xi)\in S^*M} \|G_{t+s}^+ {G_{t+s}^+}^*\|_2 \leq \big(\sup_{(x,\xi)\in S^*M} \|G_t^+ {G_t^+}^*\|_2\big)\cdot\big(\sup_{(x,\xi)\in S^*M} \|G_s^+ {G_s^+}^*\|_2\big). \quad (2.22)$$

Recalling the cocycle formula $G_{s+t}^+(\rho) = G_t^+(\phi_s(\rho))G_s^+(\rho)$, it follows that

$$G_{t+s}^+(\rho)G_{t+s}^+(\rho)^* = G_t^+(\phi_s(\rho))G_s^+(\rho)G_s^+(\rho)^*G_t^+(\phi_s(\rho))^*$$

and since for any two matrices R and S we have $\|S^*R^*RS\|_2 \leq \|S^*S\|_2\|R^*R\|_2$, inequality (2.22) is satisfied and $t \mapsto tC(t)$ is indeed subadditive. By application of Fekete's subadditive lemma we deduce that $C(t)$ admits a limit when $t \to \infty$ and that $C(t) \leq C_\infty$ for every positive t.

By combining the results of this section it is now easy to prove that $\alpha \leq 2C_\infty$. Assuming that $\alpha = 2C_\infty + 4\eta$ for some $\eta > 0$, this means that there exists some constant $C > 0$ such that

$$\forall t \geq 0,\ \forall u \in H,\ E(u,t) \leq CE(u,0)\exp(-2t(C_\infty+\eta)). \quad (2.23)$$

Now pick some T such that $C\exp(-2T(C_\infty+\eta)) < \exp(-T(2C_\infty+\eta))$. Since $C_\infty \geq$

$C(T)$ we have $\exp(-T(2C_\infty+\eta)) \leq \exp(-T(2C(T)+\eta))$, but using Proposition 2.7 there exists some $u \in H$ such that

$$E(u,T) > E(u,0)\exp(-T(2C(T)+\eta)) > CE(u,0)\exp(-2T(C_\infty+\eta))$$

which contradicts (2.23) and concludes the proof that $\alpha \leq 2C_\infty$.

2.3.2 Lower bound for α

We are now going to use the results of Section 2 in order to prove the following energy inequality for high frequencies.

Proposition 2.9 *For every time $T > 0$ and every $\varepsilon > 0$ there exists a constant $C(\varepsilon,T)$ such that for every $u = (u_0; u_1)$ in H we have*

$$E(u,T) \leq (1+\varepsilon)e^{-2TC(T)}E(u,0) + C(\varepsilon,T)\|u_0,u_1\|^2_{L^2\oplus H^{-1}}. \tag{2.24}$$

Proof. Assume that (2.24) is false. In this case for some T, some ε, and every integer $k \geq 1$ there is a solution $u^k = (u_0^k, u_1^k)$ of (2.1) satisfying

$$E(u^k,T) \geq (1+\varepsilon)e^{-2TC(T)}E(u,0) + k\|u_0^k,u_1^k\|^2_{L^2\oplus H^{-1}} \text{ and } E(u^k,0)=1. \tag{2.25}$$

First we show that the sequence (u^k) is bounded in $H^1(I\times M)$, where $I = [-2T; 2T]$. Indeed $E(u^k,0) = 1$ and (2.2) implies that $E(u^k,-2T)$ is bounded uniformly in k. Since the energy is nonincreasing the sequence (u^k) must be bounded in $H^1(I\times M)$. Moreover $\|u_0^k,u_1^k\|^2_{L^2\oplus H^{-1}} \leq E(u^k,T)/k \leq 1/k$, so (u^k) converges to 0 in $L^2(I\times M)$ and so it weakly converges to 0 in $H^1(I\times M)$. If we are to extract a subsequence we might as well assume that (u^k) admits $\mu = M\nu$ (with $\mathrm{Tr}(M) = 1$) as microlocal defect measure. As the energy is nonincreasing it follows from (2.25) that for every $\eta \in]0;T[$ and every nonnegative function $\psi \in C_0^\infty(]0;\eta[)$

$$\int_{T-\eta}^{T}\psi(T-t)E(u^k,t)\mathrm{d}t \geq (1+\varepsilon)e^{-2TC(T)}\int_0^\eta \psi(t)E(u^k,t)\mathrm{d}t.$$

Since this is true for every function ψ, taking the limit $k\to\infty$ in the previous inequality gives

$$\nu(S^*(\mathbf{R}\times M)\cap t\in]T-\eta,T[) \geq (1+\varepsilon)e^{-2TC(T)}\nu(S^*(\mathbf{R}\times M)\cap t\in]0;\eta[). \tag{2.26}$$

On the other hand Corollary 2.5 gives

$$\begin{aligned}
\nu^+(SZ^+ \cap t \in]T-\eta; T[) &= \nu^+(\varphi_{T-\eta}(SZ^+ \cap t \in]0;\eta[)) \\
&= \int_{SZ^+\cap t\in]0;\eta[} \mathrm{Tr}[G^+_{T-\eta} M {G^+_{T-\eta}}^*]\mathrm{d}\nu \\
&\leq \sup_{(x;\xi)\in S^*(M)} \|G^+_{T-\eta}(x,\xi)\|_2^2 \nu^+(SZ^+ \cap t \in]0;\eta[) \\
&= e^{-2(T-\eta)C(T-\eta)}\nu^+(SZ^+ \cap t \in]0;\eta[).
\end{aligned}$$

To get this upper bound we used the following properties.

$$\mathrm{Tr}[G^+_{T-\eta} M {G^+_{T-\eta}}^*] = \mathrm{Tr}[{G^+_{T-\eta}}^* G^+_{T-\eta} M] \leq \|G^+_{T-\eta}\|_2^2 \,\mathrm{Tr}(M) = \|G^+_{T-\eta}\|_2^2$$

We then use the same argument on SZ^-. With the relation $\partial_s(G^-_s \circ j) = -\{p, G^-_s \circ j\} + 2(G^-_s \circ j)a$ given at the end of Section 2.2, we find

$$\nu^-(SZ^- \cap t \in]T-\eta; T[) \leq e^{-2(T-\eta)C(T-\eta)}\nu^-(SZ^- \cap t \in]0;\eta[).$$

By combining ν^+ and ν^- together, we get

$$\nu(S^*(\mathbf{R}\times M) \cap t \in]T-\eta, T[) \leq e^{-2(T-\eta)C(T-\eta)}\nu(S^*(\mathbf{R}\times M) \cap t \in]0;\eta[). \tag{2.27}$$

Recall that $s \mapsto e^{-2sC(s)}$ is continuous, so for η sufficiently small the inequalities (2.26) and (2.27) imply that $\nu(S^*(\mathbf{R}\times M) \cap t \in]0;\eta[) = 0$. Consequently the sequence (u^k) strongly converges to 0 in $H^1(]0;\eta[\times M)$, and thus it also strongly converges to 0 in $H^1(I\times M)$. This contradicts the hypothesis $E(u^k, 0) = 1$ and finishes the proof.

□

The remainder of the proof for the formula of α is completely borrowed from the article of Lebeau [Leb93]. Indeed, it works verbatim [3]. Letting A_a^* be the adjoint of A_a, we have $-A_a^* = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta & +2a \end{pmatrix}$, and the spectrum of A_a^* is the complex conjugate of the spectrum of A_a. Let us call $E^*_{\lambda_j}$ the generalized eigenvector space of A_a^* associated with the spectral value $\overline{\lambda_j}$. For $N \geq 1$ we set

$$H_N = \left\{ x \in H : \langle x|y\rangle_H = 0,\ \forall y \in \bigoplus_{|\lambda_j|\leq N} E^*_{\lambda_j} \right\}.$$

The space H_N is invariant by the evolution operator e^{tA_a}. To see this take $x \in H_N$ and

3. The article of Lebeau is in French so any translation error that may occur is due to the author of the present article.

$\{y_l\}$ a basis of the finite dimensional vector space $\bigoplus_{|\lambda_j|\le N} E^*_{\lambda_j} \subset D(A^*_a)$, giving us

$$\partial_t \langle e^{tA}x, y_l\rangle = \langle e^{tA}x, A^*_a y_l\rangle = \sum c_{l,k}\langle e^{tA}x, y_k\rangle \text{ and so } \langle e^{tA}x, y_l\rangle = 0.$$

Set $H' = L^2 \oplus H^{-1}$, and let θ_n be the norm of the embedding of H_N in H'. The operator $-A^*_a$ is a compact perturbation of the skew-adjoint operator A_0, implying that the family $\{E^*_{\lambda_j}\}_j$ is total in H (see [GoKr69], Chapter 5 Theorem 10.1) and thus that $\lim \theta_N = 0$. Let us assume that $2\min\{-D_0, C_\infty\} > 0$, or otherwise there is nothing to prove. Fix $\eta > 0$ small enough so that $\beta = 2\min\{-D_0, C_\infty\} - \eta$ is positive. Now take T such that $4|C_\infty - C(T)| < \eta$ and $2\log(3) < \eta T$ and finally take N such that $C(1,T)\theta_N^2 \le e^{-2TC(T)}$. It follows from the previous proposition that

$$\forall u \in H_N,\ E(u,T) \le 3e^{-2TC(T)}E(u,0),$$

and since H_N is stable by the evolution

$$\forall k \in \mathbf{N},\ \forall u \in H_N,\ E(u,kT) \le 3^k e^{-2kTC(T)}E(u,0).$$

The energy is nonincreasing, so there exists a real $B > 0$ such that

$$\forall t \ge 0,\ \forall u \in H_N,\ E(u,t) \le Be^{-\beta t}E(u,0). \tag{2.28}$$

Let γ be a path circling around $\{\lambda_j : |\lambda_j| \le N\}$ clockwise, and let $\Pi = \frac{1}{2i\pi}\int_\gamma \frac{\mathrm{d}\lambda}{\lambda - A_a}$ be the spectral projector on $W_N = \bigoplus_{|\lambda_j|\le N} E_{\lambda_j}$. In this case Π^* is the spectral projector of A^*_a on $\bigoplus_{|\lambda_j|\le N} E^*_{\lambda_j}$ and so for every $u \in H$, one has

$$v = \Pi u \in W_N,\ w = (1-\Pi)u \in H_N \ \text{ and } \ u = v + w. \tag{2.29}$$

Now W_N is of finite dimension and since $\beta \le -2D(0)$ there exists some C such that

$$\forall u \in W_N,\ \forall t \ge 0,\ E(u,t) \le Ce^{-\beta t}E(u,0). \tag{2.30}$$

Finally, since the decomposition (2.29) is continuous, there exists some C_0 such that $E(v,0) + E(w,0) \le C_0 E(u,0)$. Combining (2.30) and (2.28) we get $\alpha \ge \beta$, thus finishing the proof of the formula for α.

2.3.3 End of the proof of Theorem 2.3 and proof of Theorem 2.2

We still need to prove properties **(i)**, **(ii)**, and **(iii)** of Theorem 2.3. For **(ii)** there is nothing to do since it has already been done in [Leb93] in the case $n = 1$, which is sufficient. For **(i)** we can assume $C_\infty > 0$, or otherwise there is nothing to prove, and we can fix some $\beta < 2C_\infty$. Notice that $E_{\lambda_j} \subset H_N$ as soon as $|\lambda_j| > N$ and so, by reasoning as for (2.28) we see that for N large enough we get

$$|\lambda_j| > N \Rightarrow 2\mathfrak{Re}(\lambda_j) \leq -\beta.$$

Since $\beta < 2C_\infty$ was arbitrary this implies $D_\infty \leq -C_\infty$ and proves **(i)**. Before we get to the last point of Theorem 2.3 we are going to prove Theorem 2.2.

Proof of Theorem 2.2. We start by proving **(ii)⇒(i)** by contraposition. Assume that **(i)** is not satisfied. If there is no weak stabilization, then obviously **(ii)** is false. We can thus assume that there exist a point $(x_0, \xi_0) \in S^*M$ and a vector $y \in \mathbf{C}^n$ of Euclidean norm 1 such that $a(x_s)y = 0$ for every time s. This means we have

$$\partial_t G_t^{+*}(x_0, \xi_0)y = -G_t^{+*}(x_0, \xi_0)a(x_t)y = 0$$

hence $\|G_t^+(x_0, \xi_0)G_t^{+*}(x_0, \xi_0)\|_2 = \|G_t^{+*}(x_0, \xi_0)\|_2^2 = \sup_{|v|=1} (G_t^{+*}(x_0, \xi_0)v, G_t^{+*}(x_0, \xi_0)v) = 1.$

This implies that for every positive t one has $C(t) = C_\infty = 0$, and thus, by Theorem 2.3, it implies $\alpha = 0$. This in turn shows that there is not strong stabilization and proves **(ii)⇒(i)**.

Reciprocally, assume that condition **(i)** is satisfied. Then by a compactness argument there exists $T > 0$ such that for all $(x_0, \xi_0) \in S^*M$ and all $y \in \mathbf{C}^n$ of euclidean norm 1

$$\int_0^T \langle a(x_t)y, y\rangle \mathrm{d}t > 0.$$

We begin by proving $C_\infty > 0$. Since $C_\infty \geq C(t)$ for every t it suffices to show that $C(T)$ is positive. Let us assume that $C(T) = 0$. Then there exist $(x_0, \xi_0) \in S^*M$ and $y \in \mathbf{C}^n$ of norm 1 such that $\Gamma_0(x_0; \xi_0)y = \Gamma_T(x_0, \xi_0)y = y$. We recall that $\Gamma_t = G_t^+ G_t^{+*}$ and that, according to Proposition 2.8, $\|\Gamma_t\|_2$ is nonincreasing. As $\|\Gamma_0(x_0, \xi_0)\|_2 = \|\Gamma_T(x_0, \xi_0)\|_2 = 1$ we know that $\|\Gamma_t(x_0, \xi_0)\|_2 = 1$ for every $t \in [0; T]$. Using Gaussian beams in section 3.1, we have proved that, for every $\varepsilon > 0$, there exists a solution u of the damped wave equation such that $|E(u, t) - \|\Gamma_t(x_0; \xi_0)y\|_2| < \varepsilon$ for every $t \in [0; T]$. Since the energy is nonincreasing it means that, for every $t \in [0; T]$ we have $\|\Gamma_t(x_0; \xi_0)y\|_2 = 1$ and thus that $\Gamma_t(x_0; \xi_0)y = \|\Gamma_t(x_0; \xi_0)\|_2 y = y$. In view of Proposition 2.8 this means that

$$0 = C(T) = \frac{-1}{T}\log\left(\|\Gamma_T(x_0, \xi_0)\|_2\right) = \frac{1}{T}\int_0^T \langle a(x_t)y, y\rangle \mathrm{d}t,$$

which is absurd, so we must have $C_\infty > 0$. The weak stabilization assumption implies that A_a has no eigenvalue (except for 0) on the line $\{\mathfrak{Re}(z) = 0\}$. It follows that the only possibility for D_0 to be zero is that D_∞ is also zero. However we showed that $C_\infty > 0$ and $C_\infty \leq -D_\infty$ so we have $-D_0 > 0$ and, by Theorem 2.3, we have strong stabilization. □

With this proof we see why $C_\infty > 0$ is equivalent to (GCC), the geometric control condition. In dimension $n = 1$ the geometric control condition is equivalent to strong stabilization [BLR92], which is in turn equivalent to $\alpha > 0$. This means that the situation **(iii)** of Theorem 2.3 cannot occur when $n = 1$. To show that the situation $C_\infty > 0$ and $D(0) = 0$ does occur we will work on the circle $M = \mathbf{R}/2\pi\mathbf{Z}$. Let $k > 0$ be a fixed integer and set $u_1(t,x) = e^{ikt}\sin(kx)$ and $u_2(t,x) = e^{ikt}\sin(kx+1)$. The function u defined by

$$u(t,x) = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix} \quad \text{is a solution of } \partial_t^2 u - \Delta u = 0.$$

We now define $a(x)$ as the orthogonal projector on $u(0,x)^\perp$. In this way we get

$$\forall (t,x) \in \mathbf{R} \times M, \quad a(x)\partial_t u(t,x) = ike^{ikt}a(x)u(0,x) = 0.$$

The function u is thus a solution of the damped wave equation and we see that ik is an eigenvalue of A_a. By construction $D_0 = 0$; however, $\ker(a(x))$ is of dimension 1 and not constant so the geometric control condition is satisfied. This forces C_∞ to be positive and finishes the proof of Theorem 2.3.

Remark – Let us emphasize once again that, in the scalar case ($n = 1$), the geometric control condition implies $a > 0$ on an open set, and thus it also implies weak stabilization. On the other hand, when $n > 1$ we can have the geometric control condition and no weak stabilization. This means that when $n = 1$ Theorem 2.2 can be stated without the weak stabilization condition, but it is necessary whenever $n > 1$.

2.4 Behavior of C_∞

In this section we are interested in the behavior of C_∞ as a function of the damping term a. For this reason we will denote by $C_\infty(a)$ the constant C_∞ associated with the damping term a when needed. In the scalar case, things are pretty simple. If a and b are two damping terms and $\lambda \geq 0$ a real number we have $C_\infty(\lambda a) = \lambda C_\infty(a)$ and $C_\infty(a+b) \geq C_\infty(a) + C_\infty(b)$, which is a direct consequence of (2.6). Moreover, if a and b are such that $a \geq b$ pointwise, then $C_\infty(a) \geq C_\infty(b)$. The vector case is more complicated since there is no simple expression for the matrix G_t^+. We will thus limit ourselves to the study of a one-dimensional example.

We will work on the circle $M = \mathbf{R}/2\pi\mathbf{Z}$. Using the cocycle formula of G^+ it's easy to see that $\lim \frac{-1}{t}\ln(\|G_t^+(x,\pm 1)\|_2^2)$ does not depend on x, which will be taken equal to 0 from now on. Still using this cocycle formula, we see that if p and q are integers, then

$$C_\infty(a) = \lim_{t\to\infty} \frac{-1}{t}\ln(\|G_t^+(0,\pm 1)\|_2^2) = \lim_{p\to\infty} \frac{-1}{2p\pi}\ln(\|G_{2p\pi}^+(0,\pm 1)\|_2^2)$$

and also

$$G^+_{2(p+q)\pi}(0,\pm 1) = G^+_{2p\pi}((0,\pm 1))G^+_{2q\pi}((0,\pm 1)).$$

Combining the above, we find

$$\begin{aligned}\lim_{t\to\infty}\frac{-1}{t}\ln\left(\left\|G^+_t(x,\pm 1)\right\|_2^2\right) &= \lim_{p\to\infty}\frac{-1}{2p\pi}\ln\left(\left\|\left[G^+_{2\pi}(0,\pm 1)\right]^p\right\|_2^2\right)\\ &= \frac{-1}{\pi}\ln\left(\rho\left(G^+_{2\pi}(0,\pm 1)\right)\right)\end{aligned}$$

where $\rho(M)$ denotes the spectral radius of the matrix M. This equality also shows that the limit does exist and that

$$C_\infty(a) = \frac{-1}{\pi}\max\left\{\ln\left(\rho\left(G^+_{2\pi}(0,1)\right)\right);\ln\left(\rho\left(G^+_{2\pi}(0,-1)\right)\right)\right\}.$$

In other words the problem of finding C_∞ is simply reduced to the analysis of two spectral radii. In fact it can be proved that $G^+_{2\pi}(0,1) = G^+_{2\pi}(0,-1)^*$ so there is really only one spectral radius here. To prove this equality it suffices to remark that $G^+_s(x_0,\xi_0)$ and $G^+_s(x_s,-\xi_s)^*$ satisfy the same differential equation. Equivalently, it is easy to prove this equality when a is piecewise constant and by an argument of density the result is also true for every smooth function a. Notice that when $n = 1$ the two matrices $G^+_{2\pi}(0,1)$ and $G^+_{2\pi}(0,-1)$ are equal but this is not true in the general case since G^+ need not be Hermitian. In conclusion we proved that

$$C_\infty(a) = \frac{-1}{\pi}\ln\left(\rho\left(G^+_{2\pi}(0,1)\right)\right). \tag{2.31}$$

We are only going to deal with a particular case of damping terms, but it will be general enough to exhibit all the behaviors we want. Take A_1, A_2, and A_3 three positive definite Hermitian matrices with their eigenvalues in $(0;1]$; we know there exist three positive semidefinite matrices a_1, a_2 and a_3 such that $\exp(-a_j) = A_j$. Now take ψ a smooth, nonnegative cut-off function such that $\int_{S^1}\psi \mathrm{d}\lambda = 1$ and $\operatorname{supp}\psi \subset (0;2\pi/3)$. The damping terms we are interested in are of the form

$$a(x) = a_1\psi(x) + a_2\psi(x+2\pi/3) + a_3\psi(x+4\pi/3) \tag{2.32}$$

and with this condition we simply have $G^+_{2\pi}(0,1) = A_1A_2A_3$ and $G^+_{2\pi}(0,-1) = A_3A_2A_1 = G^+_{2\pi}(0,1)^*$. Let us now compare $C_\infty(a)$ and $C_\infty(2a)$; according to (2.31) we have

$$C_\infty(a) = \frac{-1}{\pi}\ln\left(\rho(A_1A_2A_3)\right) \quad\text{and}\quad C_\infty(2a) = \frac{-1}{\pi}\ln\left(\rho(A_1^2A_2^2A_3^2)\right).$$

If we use a program to randomly generate the A_j, it is not hard to find some function a such that $C_\infty(2a) > 2C_\infty(a)$, for example, taking

$$A_1 = \begin{pmatrix} 0.87 & 0.21+0.09i \\ 0.21-0.09i & 0.51 \end{pmatrix}, A_2 = \begin{pmatrix} 0.35 & -0.23+0.08i \\ -0.23-0.08i & 0.61 \end{pmatrix}$$

$$\text{and } A_3 = \begin{pmatrix} 0.23 & 0.11-0.21i \\ 0.11+0.21i & 0.25 \end{pmatrix}.$$

It is even possible to have $C_\infty(2a) < C_\infty(a)$, for example, taking

$$A_1 = \begin{pmatrix} 0.49 & 0.46-0.11i \\ 0.46+0.11i & 0.52 \end{pmatrix}, A_2 = \begin{pmatrix} 0.49 & -0.02+0.3i \\ -0.02-0.3i & 0.58 \end{pmatrix}$$

$$\text{and } A_3 = \begin{pmatrix} 0.52 & -0.3-0.33i \\ -0.3+0.33i & 0.37 \end{pmatrix}.$$

This proves that C_∞ is neither monotonous nor positively homogeneous. Note that

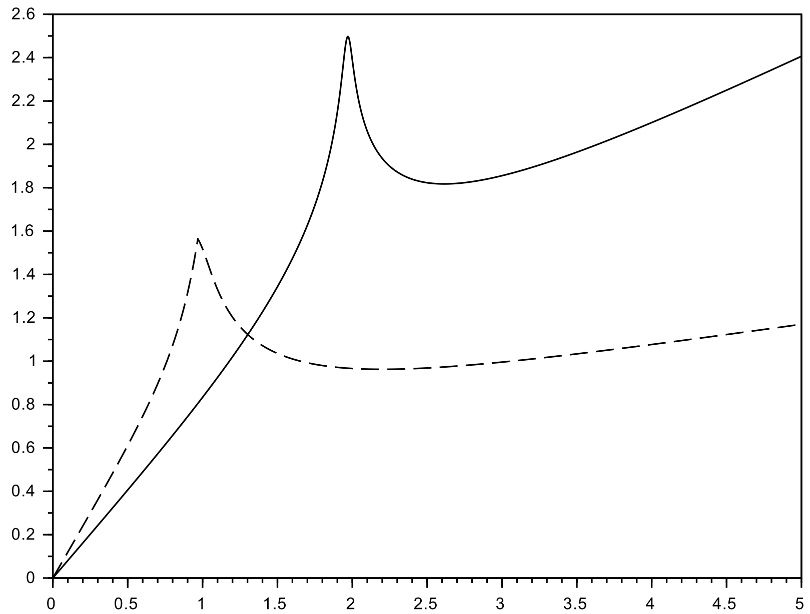

FIGURE 2.2 – Graphs of $\lambda \mapsto C_\infty(\lambda a)$ for the first example (solid line) and second example (dotted line).

even with $A_i \in \mathscr{M}_n(\mathbf{R})$ there are examples of damping terms a such that $C_\infty(2a) < C_\infty(a)$ or $C_\infty(2a) > 2C_\infty(a)$. Figure 2.2 shows the behavior of $\lambda \mapsto C_\infty(\lambda a)$ for the two previous examples.

We are going to use the same method to study the additivity of C_∞. Assuming now that $\operatorname{supp}\psi \subset (0;\pi/2)$, we look at two damping terms defined by

$$a(x) = a_1\psi(x) + a_2\psi(x+\pi) \text{ and } b(x) = b_1\psi(x+\pi/2) + b_2\psi(x+3\pi/2).$$

By equality (2.31) we get

$$C_\infty(a+b) = \frac{-1}{\pi}\ln\left(\rho(A_1B_1A_2B_2)\right) \text{ and } C_\infty(a)+C_\infty(b) = \frac{-1}{\pi}\left(\ln\left(\rho(A_1A_2)\right) + \ln\left(\rho(B_1B_2)\right)\right).$$

Then again, using a program to randomly generate the A_j and the B_j, it's not hard to find a and b such that $C_\infty(a+b) < C_\infty(a) + C_\infty(b)$; for example, taking

$$A_1 = \begin{pmatrix} 0.27 & -0.15-0.15i \\ -0.15+0.15i & 0.18 \end{pmatrix},\ A_2 = \begin{pmatrix} 0.31 & 0.25+0.3i \\ 0.25-0.3i & 0.54 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.65 & 0.35-0.28i \\ 0.35+0.28i & 0.38 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0.05 & -0.04+0.05i \\ -0.04-0.05i & 0.08 \end{pmatrix}$$

we find $C_\infty(a+b) \approx 1.45$ and $C_\infty(a) + C_\infty(b) \approx 2.99$. Conversely, it is possible to have $C_\infty(a+b) > C_\infty(a) + C_\infty(b)$; for example, taking

$$A_1 = \begin{pmatrix} 0.17 & 0.07-0.11i \\ 0.07+0.11i & 0.12 \end{pmatrix},\ A_2 = \begin{pmatrix} 0.32 & -0.09-0.35i \\ -0.09+0.35i & 0.61 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.13 & -0.19+0.04i \\ -0.19-0.04i & 0.4 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0.18 & 0.01+0.13i \\ 0.01-0.13i & 0.23 \end{pmatrix}$$

we find $C_\infty(a+b) \approx 1.87$ and $C_\infty(a) + C_\infty(b) \approx 1.20$.

However, C_∞ still has some kind of homogeneous behavior as λ tends to infinity. Assume, for example, that a is of the form (2.32) but with any finite number of a_i instead of only 3. In this case there exist some positive definite Hermitian matrices A_i with eigenvalues in $(0;1]$ such that

$$C_\infty(a) = \frac{-1}{\pi}\ln\left(\rho(A_1\dots A_j)\right)$$

and such that for every real $\lambda \geq 0$ we have

$$C_\infty(\lambda a) = \frac{-1}{\pi}\ln\left(\rho(A_1^\lambda\dots A_j^\lambda)\right).$$

We are going to prove that in this case $\lim_{\lambda\to\infty} C_\infty(\lambda a)/\lambda$ exists, is nonnegative, and is finite. The first thing to note is that every A_i^λ converges to some orthogonal projector

P_i, so $A_1^\lambda \dots A_j^\lambda$ converges to $P_1 \dots P_j$, which has a spectral radius which is either zero or nonzero. If $\rho(P_1 \dots P_j) = r \neq 0$ then $\rho(A_1^\lambda \dots A_j^\lambda)$ also converges to r and thus $C_\infty(\lambda a)/\lambda$ converges to 0. We may thus assume from now on that the spectral radius of $P_1 \dots P_j$ is 0. Let us call $\chi_\lambda = X^n + \sum_{i=0}^{n-1} b_i(\lambda) X^i$ the characteristic polynomial of $A_1^\lambda \dots A_j^\lambda$ and recall that if ξ is an eigenvalue of A_i, then ξ^λ is an eigenvalue of A_i^λ. Now since the determinant is a polynomial function, by diagonalizing each A_i^λ, we get that each coefficient of χ_λ can be written as

$$b_i(\lambda) = \sum_{l=0}^{k_i} c_{i,l} \beta_{i,l}^\lambda \text{ with } c_{i,l} \in \mathbf{C}^* \text{ and } \beta_{i,0} > \beta_{i,1} > \cdots \beta_{i,k_i} > 0.$$

Since $\rho(A_1^\lambda \dots A_j^\lambda)$ converges to 0 we know that χ_λ converges to X^n, and so every $\beta_{i,l}$ must be in $(0;1)$. Now, looking at the polynomial $\widehat{\chi}_\lambda(X) = \gamma^{\lambda n} \chi(X/\gamma^\lambda)$, we have

$$\widehat{\chi}_\lambda(X) = X^n + \sum_{i=0}^{n-1} \gamma^{\lambda(n-i)} b_i(\lambda) X^i \text{ and}$$

$$\gamma^{\lambda(n-i)} b_i(\lambda) = \left(\gamma^{n-i}\beta_{i,0}\right)^\lambda \left(c_{i,0} + \sum_{l=1}^{k_i} c_{i,l} \left(\frac{\beta_{i,l}}{\beta_{0,l}}\right)^\lambda\right).$$

For this reason there exists a unique [4] real number $\gamma > 1$ such that $\widehat{\chi}_\lambda(X) = \gamma^{\lambda n}\chi(X/\gamma^\lambda)$ converges to some unitary polynomial $Q \neq X^n$. This means that the roots of $\widehat{\chi}_\lambda$ converge to the roots of Q. Letting ξ be a root of Q with maximal modulus, recall that $\xi \neq 0$ because $Q \neq X^n$. A complex number z is a root of χ_λ if and only if $\gamma^\lambda z$ is a root of $\widehat{\chi}_\lambda$ and these roots are converging to those of Q. We deduce from this that $\gamma^\lambda \rho(A_1^\lambda \dots A_j^\lambda)$ converges to $|\xi|$, and we finally have

$$\lim_{\lambda\to\infty} \frac{C_\infty(\lambda a)}{\lambda} = \frac{-1}{\pi} \ln(\gamma^{-1}), \tag{2.33}$$

which is exactly what we wanted. The very same kind of argument also shows that

$$\lim_{\lambda\to 0^+} \frac{C_\infty(\lambda a)}{\lambda}$$

exists and is finite. Numerical simulations seem to indicate that we always have

$$\lim_{\lambda\to\infty} \frac{C_\infty(\lambda a)}{\lambda} \leq \lim_{\lambda\to 0^+} \frac{C_\infty(\lambda a)}{\lambda}$$

but the function $\lambda \mapsto C_\infty(\lambda a)/\lambda$ need not be monotonous. Figure 3 and Figure 4 illustrate some possible behavior of the functions $\lambda \mapsto C_\infty(\lambda a)$ and $\lambda \mapsto C_\infty(\lambda a)/\lambda$.

4. $\gamma = \min_i \beta_{i,0}^{1/(i-n)}$

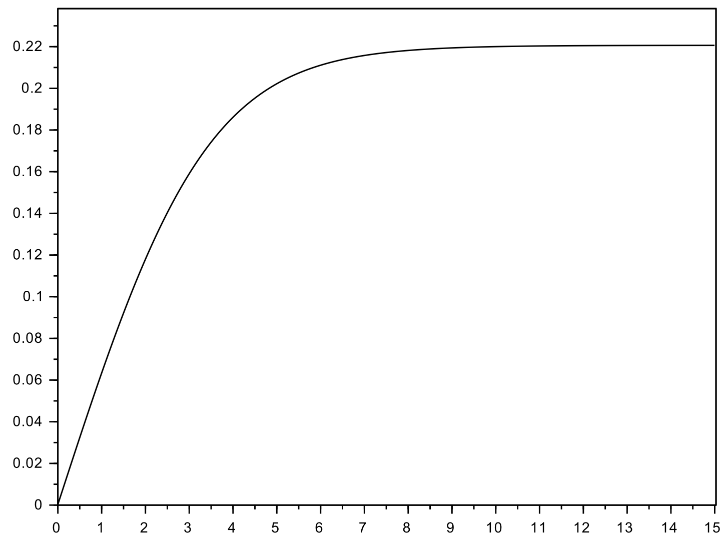

FIGURE 2.3 – Graph of $\lambda \mapsto C_\infty(\lambda a)$ for some damping term a with $\lim_\lambda C_\infty(\lambda a)/\lambda = 0$.

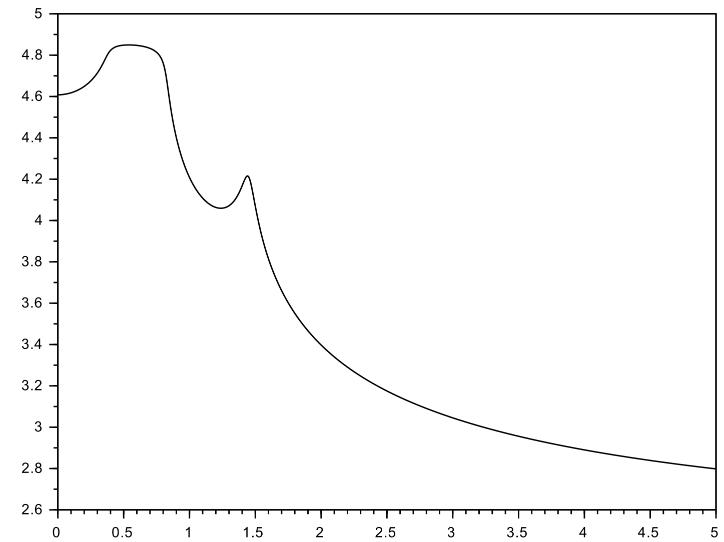

FIGURE 2.4 – Graph of $\lambda \mapsto C_\infty(\lambda a)/\lambda$ for some damping term a of the form (2.32). Here $\lim_\lambda C_\infty(\lambda a)/\lambda$ is positive and $\lambda \mapsto C_\infty(\lambda a)/\lambda$ is not monotonous.

A very natural question to ask is whether property (2.33) is still true in a more general setting, that is, is it still true with any smooth a on a general manifold? Unfortunately several difficulties prevent us from answering this question. For example, notice that on a general manifold there is no equivalent of formula (2.31) and that it is not even clear that $\|a_k - a\|_\infty \to 0$ implies $C_\infty(a_k) \to C_\infty(a)$ on a general manifold. Even on the circle where this is true it does not mean that

$$\lim_{k\to\infty} \lim_{\lambda\to\infty} \frac{C_\infty(\lambda a_k)}{\lambda} = \lim_{\lambda\to\infty} \frac{C_\infty(\lambda a)}{\lambda}$$

and so it is not clear that $\lim_{\lambda\to\infty} C_\infty(\lambda a)/\lambda$ exists for a smooth a even in the simple case of the circle.

2.5 Addenda to the second chapter

Geometrical control condition and strong stabilization

As we have already seen, the geometrical condition is not equivalent to the strong stabilization when $n \geq 1$. More precisely, if (GCC) is true and if we don't have strong stabilization then $C_\infty > 0$ and $D_0 = 0$. Since $D_\infty \leq -C_\infty$ we know there is only a finite number of eigenvalues of A_a on the imaginary axis. Consequently if (GCC) is true we can still expect to have strong stabilization for solutions with initial conditions in a subspace $H' \subset H$ of finite co-dimension.

Assume that (GCC) is true, we call $\lambda_1, \dots \lambda_N$ the non-zero eigenvalues of A_a on the imaginary axis. In the same manner as in the proof of $\alpha \geq 2\min\{-D_0, C_\infty\}$ we define

$$H' = \left\{ x \in H : \langle x|y\rangle_H = 0,\ \forall y \in \bigoplus_{j=1}^{N} E^*_{\lambda_j} \right\},$$

this space is stable by A_a. Let γ be a loop circling clockwise around $\lambda_1, \dots, \lambda_n$ but no other eigenvalue of A_a, we define the spectral projector of A_a on $\bigoplus_{j=1}^{N} E_{\lambda_j}$ by

$$\Pi = \frac{1}{2i\pi} \int_\gamma (\lambda - A_a)^{-1} \mathrm{d}\lambda.$$

Then Π^* is the spectral projector of A_a^* on $\bigoplus_{j=1}^{N} E^*_{\lambda_j}$ and so H' is of finite co-dimension because

$$H = (1-\Pi)(H) \oplus \Pi(H) = H' \oplus \left(\bigoplus_{j=1}^{N} E_{\lambda_j} \right).$$

Assume that there exists some non-constant $u \in H'$ such that $A_a u = \lambda u$. Then $\lambda = \lambda_j$ for some $j = 1, \dots, N$, this implies $u \in \bigoplus_{j=1}^{N} E_{\lambda_j}$ and so $u = 0$, a contradiction. Consequently $A_{a|H'}$ has no non-zero eigenvalue on the imaginary line. We assumed (GCC) to be true so $C_\infty > 0$ and so $D_\infty < 0$, this implies that the spectral abscissa of $A_{a|H'}$ is strictly positive. We can thus use the same proof we used for $\alpha \geq 2\min\{-D_0, C_\infty\}$ to show that there exists some constants $C, \beta > 0$ such that

$$\forall t \geq 0,\ \forall u \in H',\ E(u,t) \leq Ce^{-\beta t}E(u,0).$$

Proof of Theorem 3.1

In the next chapter we will be interested in the repartitions of the eigenvalues of A_a. We have already encountered two results on this subject. The first result is that the eigenvalues of A_a are all contained in the strip $\{z \in \mathbf{C} : \mathfrak{Re}(z) \in [-2\sup_{x\in M} \|a(x)\|_2; 0]\}$. The second result is the point **(i)** of Theorem 2.3 : for every $\varepsilon > 0$ there is only a finite number of eigenvalues λ of A_a satisfying $\mathfrak{Re}(\lambda) \leq -C_\infty - \varepsilon$. We would like to generalize a bit these results.

Take a $\mathscr{C}^\infty$ smooth function $a : M \to \mathscr{M}_n(\mathbf{C})$ then $a = c + id$ where both c and d are Hermitian matrices valued functions. The energy formula then becomes

$$\begin{aligned} \frac{d}{dt}E(u,t) &= -2\mathfrak{Re}\int_M \langle a(x)\partial_t u(t,x); \partial_t u(t,x)\rangle_{\mathbf{C}^n}\, dx \\ &= -2\int_M \langle c(x)\partial_t u(t,x); \partial_t u(t,x)\rangle_{\mathbf{C}^n}\, dx. \end{aligned}$$

If $\lambda \in \mathbf{C}^*$ is an eigenvalue of A_a then there exists $u \in H$ with $E(u,0) = 1$ and $u(t,x) = e^{t\lambda}u(0,x)$. Using this and the energy formula we get

$$\mathfrak{Re}(\lambda) \in [-2c^+; -2c^-]$$

where

$$c^- = \inf_{x\in M} \min \mathrm{sp}(c(x)) \ \text{ and } \ c^+ = \sup_{x\in M} \max \mathrm{sp}(c(x))$$

and $\mathrm{sp}(c(x))$ is the spectrum of $c(x)$.

We now want to extend the point **(i)** of Theorem 2.3 in the form of Theorem 3.1, we will thus assume that a is Hermitian but not necessarily positive semi-definite. In this setting the propagation of the microlocal defect measure is still given by the matrix G but it is posible to have $\|G_t^+(x,\xi)\|_2 > 1$ since the energy is not necessarily decreasing. Let $(\lambda_k)_{k\in\mathbf{N}}$ be a sequence of eigenvalues of A_a such that $|\lambda_k| \to \infty$ and $\mathfrak{Re}(\lambda_k) \to D_\infty$ and let η be a positive real. Let $u^k = (u_0^k, u_1^k)$ be an eigenvector of A_a associated with the eigenvalue λ_k such that $E(u_k, 0) = 1$, we know that $\|(u_0^k, u_1^k)\|_{L^2\oplus H^{-1}}$ converges to 0 (see [GoKr69], Chapter 5) and so u^k weakly converges to 0 in $H^1(]0;\eta[\times M)$. We can thus extract a pure subsequence that we will still call $(u_k)_{k\in\mathbf{N}}$ and the associated microlocal defect measure will be called $d\mu = Md\nu$. For any $T > 0$ we have

$$\lim_{k\to\infty} \int_0^\eta E(u_k,t)dt = \nu(S^*(\mathbf{R}\times M)\cap t \in]0;\eta[)$$

$$\text{and} \quad \lim_{k\to\infty} \int_T^{T+\eta} E(u_k,t)dt = \nu(S^*(\mathbf{R}\times M)\cap t \in]T;T+\eta[).$$

Proceeding as in the proof of Proposition 2.9 we see that

$$\nu(S^*(\mathbf{R}\times M)\cap t \in]T;T+\eta[) \leq e^{-2TC(T)}\nu(S^*(\mathbf{R}\times M)\cap t \in]0;\eta[).$$

If we fix T and let η go to zero we get

$$\forall T>0, \quad \lim_{k\to\infty} E(u_k,T) \leq e^{-2TC(T)}$$

and so

$$\lim_{k\to\infty} \mathfrak{Re}(\lambda_k) = D_\infty \geq -\sup_{t>0} C(t) = -C_\infty.$$

The proof is exactly the same if we take a sequence of eigenvalues λ_k such that

$$\lim_{k\to\infty} \mathfrak{Re}(\lambda_k) = \lim_{R\to+\infty} \inf\{\mathfrak{Re}(\lambda) : \lambda \in \mathrm{sp}(A_a) \text{ and } |\lambda| > R\}$$

thus proving Theorem 3.1.

Addition of a skew-Hermitian matrix to the damping term

In this subsection we assume that $a = c + id$ with c Hermitian positive semi-definite and d Hermitian. Since the matrix c is positive semi-definite we know that the energy is non increasing. We would like to expand Theorem 2.2 and Theorem 2.3 to this setting. The first thing we need to do is to look at the propagation of the microlocal defect measure. For this purpose it suffices to look at Proposition 2.3, with $a = c + id$ it becomes :

Proposition 2.10 *Let $(u_k)_k$ be a bounded sequence of $H^1(I\times M)$ weakly converging to 0. Assume that u_k is solution of the damped wave equation for every k and let b be a smooth function on $S^*(I\times M)$ to $\mathscr{M}_n(\mathbf{C})$, 1-homogeneous in the (τ,ξ) variable. If $(u_k)_k$ is pure with microlocal defect measure $\mu = M\nu$ then*

$$\int_{S^*(I\times M)} \mathrm{Tr}\left[(\{b,p\} - 2\tau(a^*b + ba))M\right]\mathrm{d}\nu = 0.$$

Proof. Recall that $P = \partial_t^2 - \Delta$ and that p is the principal symbol of P. Let B be a pseudo-differential operator of order 1 with principal symbol b, we then have

$$\lim_{k\to\infty} \langle [B,P]u_k, u_k\rangle_{H^{-1},H^1} = \int \mathrm{Tr}[\sigma_2([B,P])M]\mathrm{d}\nu = \frac{1}{i}\int \mathrm{Tr}\left[\{b,p\}M\right]\mathrm{d}\nu.$$

On the other hand

$$\langle [B,P]u_k, u_k\rangle_{H^{-1},H^1} = -2\langle (Ba\partial_t + a^*\partial_t B)u_k, u_k\rangle$$

which tends to

$$-2i\int \mathrm{Tr}[\tau(a^*b+ba)M]\mathrm{d}\nu,$$

thus finishing the proof. □

We can then proceed as with $d=0$ and we recover $\varphi_s^*\mu = G_{-s}\mu G_{-s}^*$ with G^+ satisfying the relation $\partial_t G_t^+(x_0,\xi_0) = -(c(x_t)+id(x_t))G_t^+(x_0,\xi_0)$. Notice that, when $n=1$, the skew-Hermitian part does not influence the propagation of the measure. Indeed, if we look at the previous proposition for $n=1$ we have

$$a^*b+ba = (c+id)b + b(c-id) = cb+bc$$

since d and b are scalar valued and thus commute. Obviously this is not true in general when $n>1$. It is easy to see that the rest of the proof of Theorem 2.3 still works exactly the same with $d\neq 0$. Finally the statement of Theorem 2.3 given for $d=0$ is still valid with $d\neq 0$. Notice that for $n=1$ the quantity C_∞ does not depend on d, on the contrary for $n>1$ it is possible to find some c and d such that, for example, $C_\infty(c)=0$ and $C_\infty(c+id)>0$.

Now that we have proved Theorem 2.3 for $d\neq 0$ it is easy to adapt the proof of Theorem 2.2 to the case $d\neq 0$. By doing so we get the following modified Geometric control condition.

Theorem 2.8 *The following conditions are equivalent.*

(i) *There is weak stabilization and*

$$\forall (x_0,\xi_0)\in S^*M,\ \forall v\in \mathbf{C}^n\backslash\{0\},\ \exists s\in\mathbf{R},\ G_s(x,\xi)v\notin \ker(c(x_s)) \tag{GCC}$$

(ii) *There exists two constants $C,\beta>0$ such that for all $u\in H$ and for every time t*

$$E(u,t)\le Ce^{-\beta t}E(u,0).$$

Chapitre 3

Asymptotique spectrale pour l'opérateur A_a

Ce chapitre est une reproduction de l'article [Kle18], qui n'a pas encore été soumis.

Résumé : Les fréquences propres associées à une équation des ondes amorties scalaire sont contenues dans une bande parallèle à l'axe réel. Dans [Sjö00] Sjöstrand à montré que, à un sous ensemble de densité nulle près, les fréquences propres sont contenues dans une plus fine bande définie pas les limites de Birkhoff du terme d'amortissement.Dans cet article nous montrons que ce résultat est encore vrai pour une équation des ondes amorties vectorielle. Dans ce cas là ce sont les exposants de Lyapunov d'un cocycle associé au terme d'amortissement qui remplacent les limites de Birkhoff.

Abstract : The eigenfrequencies associated to a scalar damped wave equation are known to belong to a band parallel to the real axis. In [Sjö00] Sjöstrand showed that up to a set of density 0, the eigenfrequencies are confined in a thinner band determined by the Birkhoff limits of the damping term. In this article we show that this result is still true for a vectorial damped wave equation. In this setting the Lyapunov exponents of the cocycle given by the damping term play the role of the Birkhoff limits of the scalar setting.

3.1 Introduction

3.1.1 Setting

Let (M, g) be a compact Riemannian manifold of dimension d without boundary, we will moreover assume that (M, g) is connected and C^∞. Let Δ be the Laplace-Beltrami operator on M for the metric g and let a be a smooth function from M to $\mathscr{H}_n(\mathbf{C})$, the space of hermitian matrices of dimension n. We are interested in the following system of equations

$$\begin{cases} (\partial_t^2 - \Delta + 2a(x)\partial_t)u = 0 \text{ in } \mathcal{D}'(\mathbf{R} \times M)^n \\ u_{|t=0} = u_0 \in H^1(M)^n \text{ and } \partial_t u_{|t=0} = u_1 \in L^2(M)^n. \end{cases} \tag{3.1}$$

Let $H = H^1(M)^n \oplus L^2(M)^n$ and define on H the unbounded operator

$$A_a = \begin{pmatrix} 0 & \mathrm{Id}_n \\ \Delta & -2a \end{pmatrix} \text{ of domain } D(A_a) = H^2(M)^n \oplus H^1(M)^n.$$

By application of Hille-Yosida theorem to A_a the system (3.1) has a unique solution in the space $C^0(\mathbf{R}, H^1(M)^n) \cap C^1(\mathbf{R}, L^2(M)^n)$. If u is a solution of (3.1) then we can define the energy of u at time t by the formula

$$E(u, t) = \frac{1}{2} \int_M |\partial_t u(t, x)|^2 + |\nabla u(t, x)|_g^2 dx \tag{3.2}$$

where $|v|_g^2 = g_x(v, v)$. An integration by parts show that

$$\frac{d}{dt} E(u, t) = -\int_M \langle 2a(x)\partial_t u(t, x), \partial_t u(t, x)\rangle_{\mathbf{C}^n} \, dx, \tag{3.3}$$

so if a is not null then the energy is not constant. In particular if a is positive semi-definite the energy is non-increasing and a can be seen as a dampener. In the problem of stabilisation of the wave equation one is interested with the long time behavior of the energy of solutions to (3.1). The spectrum of A_a is obviously related to this long time behavior although it is not sufficient to completely describe it (see for example [Leb93]). In this article we will thus be interested in the asymptotic repartition of the eigenvalues of A_a. Besides the possible applications to stabilisation the problem of determining an asymptotic repartitions of eigenvalues for a non-self-adjoint operator is an interesting problem by itself. This problem has already been studied in the scalar case ($n = 1$) for example in [Sjö00]. The aim of this article is to adapts some results of [Sjö00] to the case where $n \geq 1$.

3.1.2 Results

It is a classical fact that the spectrum of A_a is symmetric with respect to the real axis and that it contains only discrete eigenvalues with finite multiplicities, this is proved using Fredholm theory. If u is a stationary solution of (3.1) we can write $u(t,x) = e^{it\tau}v(x)$ and we are lead to the equation

$$(-\Delta - \tau^2 + 2ia(x)\tau)v(x) = 0. \tag{3.4}$$

We say that $\tau \in \mathbf{C}$ is an eigenvalue for (3.4) if there is a non-zero solution v of (3.4). Note that τ is an eigenvalue of (3.4) if and only if $i\tau$ is an eigenvalue of A_a.

Definition We note $\mathrm{sp}(a(x))$ the set of all the eigenvalues of $a(x)$ and define

$$a^- = \inf_{x\in M} \min \mathrm{sp}(a(x)) \text{ and } a^+ = \sup_{x\in M} \max \mathrm{sp}(a(x)).$$

By reasoning on the energy with (3.3) it is easy to see that $\mathfrak{Im}(\tau)$ can only be in the interval $[2a^-; 2a^+]$. We are now interested in the asymptotic repartition of $\mathfrak{Im}(\tau)$ when $|\tau|$ goes to infinity. Since the spectrum of A_a is invariant under complex conjugation we will only be interested in the limit $\mathfrak{Re}(\tau) \to +\infty$.

We define the function $p : T^*M \to \mathbf{R}$ by $p(x,\xi) = g^x(\xi,\xi)$, notice that p is the principal symbol of $-\Delta$. We also define ϕ as the Hamiltonian flow generated by p, it can also be interpreted as the geodesic flow on T^*M travelled at twice the speed. In what follows (x_0,ξ_0) will be a point of T^*M and we will write (x_t,ξ_t) for $\phi_t(x_0,\xi_0)$.

Definition Let t be a real number, we define the function $G_t : p^{-1}(1/2) \to \mathscr{M}_n(\mathbf{C})$ as the solution of the differential equation

$$\begin{cases} G_0(x_0,\xi_0) = \mathrm{Id}_n \\ \partial_t G_t(x_0,\xi_0) = -a(x_t)G_t(x_0,\xi_0). \end{cases} \tag{3.5}$$

This definition naturally extends to $(x_0,\xi_0) \in T^*M$.

Notice that G is a cocycle map, this means that for every $s,t \in \mathbf{R}$ and every $(x,\xi) \in T^*M$ we have the equality $G_{t+s}(x,\xi) = G_t(\phi_s(x,\xi))G_s(x,\xi)$. When $n = 1$ the function a is real valued and we simply have

$$G_t(x_0,\xi_0) = \exp\left(-\int_0^t a(x_s)ds\right). \tag{3.6}$$

Throughout the entire article if E is a vector space and $\|\cdot\|_*$ is a norm on E we will also write $\|\cdot\|_*$ for the associated operator norm on $\mathcal{L}(E)$.

Definition For every positive time t we define the following quantities

$$C_t^- = \frac{-1}{t} \sup_{(x_0,\xi_0)\in p^{-1}(1/2)} \ln\left(\|G_t^+(x_0;\xi_0)\|_2\right)$$

$$\text{and } C_t^+ = \frac{-1}{t} \inf_{(x_0,\xi_0)\in p^{-1}(1/2)} \ln\left(\|G_t^+(x_0;\xi_0)^{-1}\|_2^{-1}\right).$$

We will also note $C_\infty^\pm = \lim_{t\to+\infty} C_t^\pm$.

It is easy to show that this limit always exists using a sub-additivity argument. Note that the existence and the value of the limit does not depend on the choice of the norm since they are all equivalent, however it is sometimes easier to work with the operator norm associated with the euclidian norm on $\mathbf{C}^n$. Again, if $n = 1$ we get the simpler expression

$$C_t^- = \inf_{(x,\xi)\in p^{-1}(1/2)} \frac{1}{t}\int_0^t a(x_s)ds \ \text{ and } \ C_t^+ = \sup_{(x,\xi)\in p^{-1}(1/2)} \frac{1}{t}\int_0^t a(x_s)ds.$$

For an analogue to this expressions when $n \geq 1$ see Annex C.

Theorem 3.1 *For every $\varepsilon > 0$ there is only a finite number of eigenvalues of* (3.4) *outside the strip $\mathbf{R} + i]C_\infty^- - \varepsilon; C_\infty^+ + \varepsilon[$.*

In the scalar case this was first proved by Lebeau [Leb93] using microlocal defect measures. In fact, because of the setting of his article, he only proved it for the upper bound and $a \leq 0$ but his proof easily extends to Theorem 3.1 with $n = 1$. Theorem 3.1 was also proved by Sjöstrand in [Sjö00] for $n = 1$ using different techniques. The argument of Sjöstrand relies on a conjugation by pseudo-differential operators to replace the damping term a by its average on geodesics of length T. Because of the non commutativity of matrices it seems that this argument cannot be modified in a straight forward manner to prove the vectorial case ($n \geq 1$). In [Kle17], using the same technique as Lebeau, we proved the upper bound of Theorem 3.1 in the general case $n \geq 1$ for a function a valued in $\mathscr{H}_n^+(\mathbf{C})$, the space of Hermitian positive semi-definite matrices. Once again the argument used there can easily be adapted to prove Theorem 3.1 in its full generality [1].

Theorem 3.2 *The number of eigenvalues τ with $\mathfrak{Re}(\tau) \in [0;\lambda]$ is equivalent to*

$$n\left(\frac{\lambda}{2\pi}\right)^d \iint_{p^{-1}([0;1])} 1 dx d\xi$$

when λ goes to $+\infty$. Moreover the remainder is a $\mathcal{O}(\lambda^{d-1})$.

For $n = 1$ this result was first proved by [MaMa82] and then independently in [Sjö00]. Once again the proof can easily be adapted to our case $n \geq 1$.

1. see section 2.5 of this thesis

We now want to make further estimations of the asymptotics of the imaginary part of the eigenvalues of (3.4). Recall that the matrix G is a cocycle and that the geodesic flow on $p^{-1}(1)$ preserves Liouville's measure. Thus we can use Kingman's subadditive ergodic theorem to show that the limits

$$\lambda_n(x,\xi) = \lim_{t\to\infty} \frac{1}{t} \log \|G_t(x,\xi)\|_2 \quad \text{and} \quad \lambda_1(x,\xi) = \lim_{t\to\infty} \frac{1}{t} \log \left(\left\|G_t(x,\xi)^{-1}\right\|_2^{-1} \right)$$

exist for almost every $(x,\xi) \in p^{-1}(1)$. Moreover the functions $(x,\xi) \mapsto \lambda_1(x,\xi)$ and $(x,\xi) \mapsto \lambda_n(x,\xi)$ are both measurable and bounded. Note that the existence and the value of the limit does not depend on the choice of the norm because they are all equivalent. Note also that λ_1 and λ_n are respectively the smallest and largest Lyapunov exponents defined by the multiplicative ergodic theorem of Oseledets. The statement of Oseledets theorem can be found in Annex B. We now define

$$\Lambda^- = \operatorname{ess\,inf} \lambda_1(x,\xi) \text{ and } \Lambda^+ = \operatorname{ess\,sup} \lambda_n(x,\xi).$$

In the scalar case $\lambda_1 = \lambda_n$ and we have

$$\Lambda^+ = - \operatorname*{ess\,inf}_{(x,\xi)\in p^{-1}(1/2)} \lim_{t\to\infty} \frac{1}{t} \int_0^t a(x_s)ds \quad \text{and} \quad \Lambda^- = - \operatorname*{ess\,sup}_{(x,\xi)\in p^{-1}(1/2)} \lim_{t\to\infty} \frac{1}{t} \int_0^t a(x_s)ds.$$

Notice the minus sign in comparison to the definition of $C_\infty^\pm$. In general we have

$$C_\infty^- \leq -\Lambda^+ \leq -\Lambda^- \leq C_\infty^+$$

and every inequality is usually strict. We can now state the main theorem of this article

Theorem 3.3 *For every $\varepsilon > 0$ the number of eigenvalues τ satisfying $\mathfrak{Re}(\tau) \in [\lambda; \lambda + 1]$ and $\mathfrak{Im}(\tau) \notin] - \Lambda^+ - \varepsilon; -\Lambda^- + \varepsilon[$ is $o(\lambda^{d-1})$ when λ tends to infinity.*

In view of Theorem 3.2 and Theorem 3.3 we see that "most" of the eigenvalues have their imaginary part in the interval $] - \Lambda^+ - \varepsilon; -\Lambda^- + \varepsilon[$. Theorem 3.3 was proved by Sjöstrand [Sjö00] when $n = 1$ and the asymptotics was then refined by Anantharaman for a negatively curved manifold in [Ana10]. The main goal of this article is to prove Theorem 3.3 in the general case $n \geq 1$. As for Theorem 3.1 the arguments used by Sjöstrand in [Sjö00] seem to not work anymore when $n \geq 1$. This mainly comes from the fact that matrices do not commute and thus formula (3.6) is no longer true when $n \geq 1$.

3.1.3 An open question

If we take $n = 1$ then there is only one Lyapunov exponent for G which is simply the opposite of the Birkhoff average of a :

$$\lambda_1(x_0, \xi_0) = \lim_{t\to\infty} \frac{-1}{t} \int_0^t a(x_s)ds.$$

If we make the assumption that the geodesic flow is ergodic on M we get

$$\Lambda^- = \Lambda^+ = \frac{-1}{\mathrm{vol}(M)} \int_M a(x)dx = \lambda_1(x, \xi) \text{ a.e.}$$

and Theorem 3.3 tells us that most of the eigenvalues of A_a are concentrated around the vertical line of imaginary part $\frac{-1}{\mathrm{vol}(M)} \int_M a(x)dx$. Now if we drop the assumption $n = 1$ and keep the ergodic assumption we do not necessarily have $\Lambda^+ = \Lambda^-$ but the Lyapunov exponents λ_i defined by Theorem B.1 will be constant almost everywhere. We write λ_i for the almost sure value of the function $(x, \xi) \mapsto \lambda_i(x, \xi)$ and we thus have $\lambda_1 = \Lambda^-$ and $\lambda_n = \Lambda^+$. Theorem 3.3 tells us that most of the eigenvalues of A_a will be concentrated around the strip $\{z \in \mathbf{C} : \mathfrak{Im}(z) \in [-\lambda_n; -\lambda_1]\}$ but it seems natural to ask if the following stronger property holds.

Question *Is it true that for every $\varepsilon > 0$ the number of eigenvalues τ satisfying $\mathfrak{Re}(\tau) \in [\lambda; \lambda + 1]$ and*

$$\mathfrak{Im}(\tau) \notin \bigcup_{i=1}^{n}]\lambda_i - \varepsilon; \lambda_i + \varepsilon[$$

is a $o(\lambda^{d-1})$ when λ tends to infinity ?

It seems that at the present time, the techniques developed in this article do not allow us to answer this question but we plan to investigate it in a near future. Nevertheless here is a partial result in this direction.

Proposition 3.4 *Let $M = \mathbf{R}/2\pi\mathbf{Z}$ and $\lambda_1 \leq \dots \lambda_n$ be the Lyapunov exponents of G, which do not depend on (x, ξ). For every $\varepsilon > 0$ there is only a finite number of eigenvalues outside the strips*

$$\mathbf{R} + i]\lambda_i - \varepsilon; \lambda_i + \varepsilon[, \ i = 1, \dots, n.$$

The proof can be found in Section 3.4 and relies on microlocal defect measures. Notice that, as in Theorem 3.1, we get that only a finite number of eigenvalues are outside of certain set. This is very specific to the circle for which the Lyapunov exponents do not depend on (x, ξ).

3.1.4 Plan of the article

Section 3.2 is dedicated to the proof of Theorem 3.3 which starts by a semi-classical reduction. The general idea of the proof is to express the eigenvalues of (3.4) as zeros of some Fredholm determinant depending holomorphically in z and then to use Jensen's formula to bound the number of these zeros.

In order to construct the aforementioned Fredholm determinant and to get the appropriate bound we need to construct some approximate resolvent for $-\Delta - \tau^2 + 2ia(x)\tau$, this is the object of Proposition 3.1. The proof of Proposition 3.1 is postponed to Section 3.3 and represents the core of this article. Section 3.3 starts by a sketch of the proof.

The article then ends with the proof of Proposition 3.4 and three annexes. The first one presents the semi-classical anti-Wick quantization and its basic properties. The second annex presents the multiplicative ergodic theorem of Oseledets. The last annex show how to express the Lyapunov exponents of the cocycle G in terms of the values of a along geodesics.

3.2 Proof of Theorem 3.3

The first step of the proof is to perform a semi-classical reduction borrowed from [Sjö00]. Recall from the Introduction that $i\tau$ is an eigenvalue of A_a if and only if there exists some non zero $v : M \to \mathbf{C}^n$ such that

$$(-\Delta - \tau^2 + 2ia\tau)v = 0.$$

We are interested in the asymptotic behaviour of the eigenvalues of A_a and since its spectrum is invariant by complex conjugation we can restrict ourself to the case $\mathfrak{Re}(\tau) \to +\infty$. Let us call h our semiclassical parameter tending to zero and let $i\tau$ be an eigenvalue of A_a, depending on h, such that $h\tau = 1 + o(1)$ when h goes to zero. If we write $\tau = \kappa/h$ the previous equation becomes

$$(-h^2\Delta - \kappa^2 + 2ia\kappa h)v = 0.$$

Now if we write $z = \kappa^2$, and $\kappa = \sqrt{z}$ with $\mathfrak{Re}(z) > 0$ the equation becomes

$$(-h^2\Delta + 2iha\sqrt{z} - z)v = 0.$$

We might finally rewrite it as

$$(\mathcal{P} - z)v = 0 \tag{3.7}$$

with $\mathcal{P} = \mathcal{P}(z) = P + ihQ(z)$, $P = -h^2\Delta$ is the semiclassical Laplacian and $Q(z) = 2a\sqrt{z}$. Note that P is self adjoint, Q depends holomorphically on z in a neighbourhood of 1 and it is self adjoint whenever z is a positive real number. Throughout the rest of the article we will use differential operators depending on the semi-classical parameter h, an exposition of the theory of h-pseudo-differential operators is given in [Zwo12].

Remark – Notice that $z = \kappa^2$ and that $(1+x)^2 = 1 + 2x + o(x)$ so we have

$$h^{-1}\mathfrak{Im}(z) = h^{-1}2\mathfrak{Im}(\kappa) + o(1) = 2\mathfrak{Im}(\tau) + o(1).$$

This explains the appearance of some multiplications by two in the rest of the article.

According to this semi-classical reduction, finding an upper bound on the number of eigenvalues of $\mathcal{P}$ in an open set $1 + h\widetilde{\Omega}$ yields an upper bound on the number of eigenvalues of (3.4) in an open set $h^{-1} + \widetilde{\Omega}/2 + o(1)$.

Definition Let $\varepsilon > 0$ be fixed, we then put

$$\widetilde{\Omega} = \{z \in \mathbf{C} : \mathfrak{Re}(z) \in]-2; 2[,\ \mathfrak{Im}(z) \in]2a^- - 3; -\Lambda^+ - \varepsilon/2[\}$$

$$\widetilde{\omega} = \{z \in \mathbf{C} : \mathfrak{Re}(z) \in]-1; 1[,\ \mathfrak{Im}(z) \in]2a^- - 2; -\Lambda^+ - \varepsilon[\}$$

$$\widetilde{z_0} = i(2a^- - 1).$$

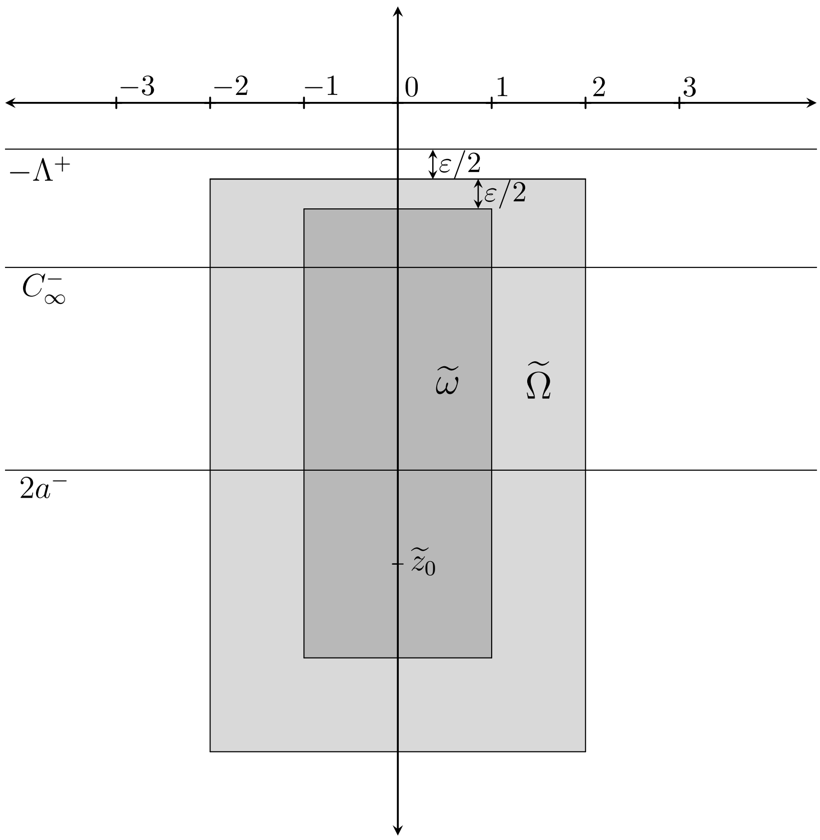

FIGURE 3.1 – Drawing of $\widetilde{\Omega}$, $\widetilde{\omega}$ and $\widetilde{z_0}$.

We are going to prove that the number of eigenvalues of $\mathcal{P}$ in $\omega_h = 1 + 2h\widetilde{\omega}$ is a $o(h^{1-d})$. Since there are no eigenvalues of (3.4) with imaginary part smaller than $2a^-$

this will prove that for every $\varepsilon > 0$ the number of eigenvalues τ satisfying $\mathfrak{Re}(\tau) \in [h^{-1}-1; h^{-1}+1]$ and $\mathfrak{Im}(\tau) \leq -\Lambda^+ - \varepsilon$ is a $o(h^{1-d})$ when h tends to zero. The proof is exactly the same for eigenvalues satisfying $\mathfrak{Im}(\tau) \geq -\Lambda^- + \varepsilon$ and this will thus prove Theorem 3.3.

The key ingredient here is the next proposition but, in order to improve clarity, its proof is postponed until the next section.

Proposition 3.1 *For every z in $\Omega_h = 1 + 2h\widetilde{\Omega}$ of $\mathbf{C}$ there exists an operator $R(z) \in \mathcal{L}(L^2)$ depending holomorphically on $z \in \Omega_h$ such that $R(z)(\mathcal{P} - z) = \mathrm{Id} + R_1(z) + R_2(z)$ where $R_1, R_2 \in \mathcal{L}(L^2)$, $\|R_1(z)\|_{L^2} < 1/2$ and $\|R_2(z)\|_{\mathrm{tr}} = o(h^{1-d})$. Moreover for $z_0 = 1+2h\widetilde{z_0} \in \omega_h$ the operator $R(z_0)(\mathcal{P}(z_0) - z_0)$ is invertible in L^2 and $\left\|(R(z_0)(\mathcal{P}(z_0) - z_0))^{-1}\right\|_{L^2}$ is uniformly bounded in h.*

Using this proposition we want to bound the number of eigenvalues of $\mathcal{P}$ in Ω_h. First of all notice that if $\mathcal{P} - z$ has a non zero kernel then so does $R(z)(\mathcal{P} - z)$, thus we only need to bound the dimension of the kernel of $R(z)(\mathcal{P} - z)$. Now since $\|R_1(z)\|_{L^2} \leq 1/2$ the operator $\mathrm{Id} + R_1(z)$ is invertible and there exists an invertible operator $Q(z)$ such that $Q(z)R(z)(\mathcal{P} - z) = \mathrm{Id} + K(z)$ where $K(z) = Q(z)R_2(z)$. Since Q is invertible we have $\dim\ker(R(z)(\mathcal{P} - z)) = \dim\ker(\mathrm{Id} + K(z))$.

The operator R_2 is of trace class and thus K is also of trace class with

$$\|K\|_{\mathrm{tr}} \leq \|Q\|_{L^2}\|R_2\|_{\mathrm{tr}} \leq 2\|R_2\|_{\mathrm{tr}} = o(h^{1-d}).$$

It follows that z is an eigenvalue of $\mathcal{P}$ only if $D(z) \stackrel{\mathrm{def}}{=} \det(1 + K(z))$ is equal to 0. Moreover the multiplicity of an eigenvalue z of $\mathcal{P}$ is less than the multiplicity of the zero of D. Using a general estimate on Fredholm determinants we get

$$|D(z)| \leq \exp(\|K(z)\|_{\mathrm{tr}})$$

On the other hand for z_0 we have

$$|D(z_0)|^{-1} = \det((1 + K(z_0))^{-1}) \text{ and } (1 + K(z_0))^{-1} = 1 - K(z_0)(1 + K(z_0))^{-1}.$$

Using this and the estimate on Fredholm determinant we get

$$|D(z_0)|^{-1} = |\det(1 - K(z_0)(1 + K(z_0))^{-1})| \leq \exp\left(\|K(z_0)(1 + K(z_0))^{-1}\|_{\mathrm{tr}}\right)$$
$$\leq \exp\left(\|(1 + K(z_0))^{-1}\|\|K(z_0)\|_{tr}\right)$$

which, according to Proposition 3.1, is smaller than $\exp(C\|K(z_0)\|_{\mathrm{tr}})$ for some constant $C > 0$ independent of h. So far we have proved that

$$\log|D(z_0)| \geq -C\|K(z_0)\|_{\mathrm{tr}} \text{ and } \forall z \in \Omega_h,\ \log|D(z)| \leq \|K(z)\|_{\mathrm{tr}}. \tag{3.8}$$

As we will see in section 3.3 the operators $R(z)$ and $R_2(z)$ both depend holomorphically in z on Ω_h. Since $\mathcal{P} - z$ is holomorphic we get that $\mathrm{Id} + R_1(z)$ and $Q(z)$ are also holomorphic and finally that $K(z)$ is holomorphic in z on Ω_h. Consequently the function $z \mapsto D(z)$ is holomorphic on Ω_h, we want to apply Jensen's inequality in order to bound the number of zeros of D on a subset of Ω_h containing z_0.

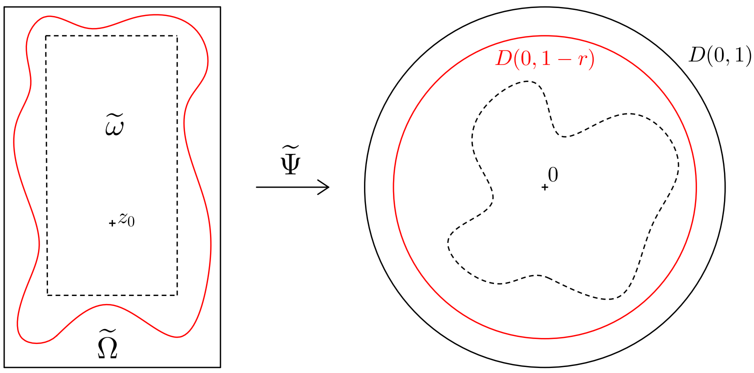

FIGURE 3.2 – If r is sufficiently close to 0 then $\widetilde{\Psi}(\widetilde{\omega}) \subset D(0, 1 - r)$.

Since $\widetilde{\Omega}$ is a simply connected open set there exists a Riemannian mapping $\widetilde{\Psi} : \widetilde{\Omega} \to D(0; 1)$ between $\widetilde{\Omega}$ and the open unit disk which also satisfy $\widetilde{\Psi}(\widetilde{z}_0) = 0$. If we put $\Psi_h : z \mapsto \widetilde{\Psi}(\frac{z/2-1}{h})$ then $\Psi_h : \Omega_h \to D(0, 1)$ is a Riemannian mapping which maps z_0 to 0. For every $0 < t < 1$ let us call $n(t)$ the number of zeros (with multiplicity) of $D \circ \Psi_h^{-1}$ in $D(0, t)$; Jensen's formula states that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |D \circ \Psi_h^{-1}(te^{i\theta})| d\theta - \log |D \circ \Psi_h(0)| = \int_0^t \frac{n(s)}{s} ds.$$

For r close enough to 0 we have $\Psi_h(\omega_h) \subset D(0; 1 - r)$, we then use (3.8) to obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \log |D \circ \Psi_h^{-1}((1 - r/2)e^{i\theta})| d\theta \leq \sup_{z \in \Omega_h} \|K(z)\|_{\mathrm{tr}} = o(h^{1-d})$$

$$\text{and } \log |D \circ \psi_h^{-1}(0)| \geq -C\|K(z_0)\|_{\mathrm{tr}}.$$

Combining this estimates with Jensen's formula for $D \circ \Psi_h^{-1}$ gives us

$$\int_0^{1-r/2} \frac{n(s)}{s} ds \leq (1+C) \sup_{z \in \Omega_h} \|K(z)\|_{\mathrm{tr}} = o(h^{1-d})$$

and since the map $s \mapsto n(s)$ is increasing we have

$$n(1-r) \leq \frac{2}{r} \int_0^{1-r/2} \frac{n(s)}{s} ds \leq \frac{2}{r}(1+C) \sup_{z \in \Omega_h} \|K(z)\|_{\mathrm{tr}}$$

The number of zeros of D in ω_h is equal to the number of zeros of $D \circ \Psi_h^{-1}$ in $\Psi_h(\omega_h)$ which is obviously less than $n(1-r)$.

Notice that r does not depend on h because Ω_h, ω_h, z_0 and Ψ_h are just rescaled versions on $\widetilde{\Omega}$, $\widetilde{\omega}$, $\widetilde{z}_0$ and $\widetilde{\Psi}$. We therefore obtain the desired bound : the number of zeros of D in ω_h is a $o(h^{1-d})$ and the proof of Theorem 3.3 is complete.

□

3.3 Proof of Proposition 3.1

In order to ease the notations we will use the Landau notation (or "big O" notation) directly for operators throughout all this section. It must always be interpreted as a Landau notation for the L^2 norm of the operator when the semi-classical parameter h goes to zero. For example if we write $\mathcal{P} - z = P - 1 + \mathcal{O}(h)$ we mean that

$$\|\mathcal{P} - z - P + 1\|_{L^2} = \mathcal{O}(h).$$

3.3.1 Idea of the proof

Notice that $\mathcal{P} - z = P - 1 + \mathcal{O}(h)$ and that the principal symbol of $P-1$ is $|\xi|_g^2 - 1$. Using functional calculus (see (3.15)) it is easy to find some pseudo-differential operator A_3 such that the principal symbol of $A_3(\mathcal{P} - z)$ is 1 where $||\xi|_g^2 - 1| \geq Ch$ for some fixed but large enough C. On the set where $||\xi|_g^2 - 1| < Ch$ we use a different approach. We start with the formula

$$\frac{-i}{h} \int_0^T e^{it(\mathcal{P}-z)/h} dt (\mathcal{P} - z) = 1 - e^{iT(\mathcal{P}-z)/h}$$

and we would like to have $\|e^{iT(\mathcal{P}-z)/h}\|_{L^2} < 1$ for some time T so $1 - e^{iT(\mathcal{P}-z)/h}$ would be invertible. As we will show in Lemma 3.6 the operator $e^{iT(\mathcal{P}-z)/h}$ behaves like

$$e^{-iTz/h} e^{iTh\Delta} \mathrm{Op}_h^{\mathrm{AW}}(G_{2T}(x, \xi/2))$$

for functions that are concentrated around the energy layer $p^{-1}(1)$. Here $\mathrm{Op}_h^{\mathrm{AW}}$ denotes the anti-Wick quantization [2] and ϕ_t is the Hamiltonian flow associated with p. We thus see that we cannot hope to have $\|e^{iT(\mathcal{P}-z)/h}\|_{L^2} < 1$ for every $z \in \Omega_h$. More precisely we cannot hope to have $\|e^{iT(\mathcal{P}-z)/h}\|_{L^2} < 1$ when $\mathfrak{Im}(z/h) \geq 2C_\infty^-$. However, for almost every (x,ξ) in $p^{-1}(1)$ we have $\lim_{t\to\infty} \frac{1}{t}\log\|G_{2t}(x,\xi/2)\| \leq 2\Lambda^+$ and so for T large enough we have

$$\frac{1}{T}\log\|G_{2T}(x,\xi/2)\| \leq 2\Lambda^+ + \varepsilon/2 \tag{3.9}$$

for every (x,ξ) of a compact set $K \subset p^{-1}(1)$ with large measure. If χ is a C^∞ function with compact support in ξ such that $\chi(x,\xi) = 0$ for every (x,ξ) that does not satisfy (3.9) then

$$\|e^{iT(\mathcal{P}-z)/h}\mathrm{Op}_h^{\mathrm{AW}}(\chi)\|_{L^2} < 1.$$

In order to localize this to the energy layer $||\xi|_g - 1| < Ch$ we will use operators of the form $f\left(\frac{P-1}{h}\right)$ where $f \in \mathcal{S}(\mathbf{R})$ is properly chosen. For example we will show that the operator

$$f\left(\frac{P-1}{h}\right)\mathrm{Op}_h^{\mathrm{AW}}(1-\chi)f\left(\frac{P-1}{h}\right)$$

has a small trace norm compared to h^{1-d}. Then, by choosing carefully f, χ and T depending on h we can define an operator $R(z)$ which has the properties presented in Proposition 3.1.

3.3.2 Construction of the operator $R(z)$

According to Oseledec's theorem for almost every $(x,\xi) \in T^*M$ the Lyapunov exponents of $(G_t(x,\xi))_{t\geq 0}$ are well defined. Recall that

$$\Lambda^- = \operatorname*{ess\,inf}_{(x,\xi)\in T^*M} \lambda_1(x,\xi) \text{ and } \Lambda^+ = \operatorname*{ess\,sup}_{(x,\xi)\in T^*M} \lambda_n(x,\xi),$$

where $\lambda_1(x,\xi) \leq \ldots \leq \lambda_n(x,\xi)$ are (if they exist) the Lyapunov exponents of $(G_t(x,\xi))_{t\geq 0}$. So for almost every (x,ξ) in $\{(x,\xi) \in T^*M : |\xi|_g = 1]\}$ we have

$$\lim_{t\to+\infty}\frac{1}{t}\log(\|G_{2t}(x,\xi/2)\|) = 2\lambda_n(x,\xi) \leq 2\Lambda^+.$$

According to Egorov's theorem for every $\varepsilon, \eta > 0$ there exist a compact set $K \subset p^{-1}(1)$ and a time T_0 such that for every $T \geq T_0$ and every (x,ξ) in K we have

$$\frac{1}{T}\log(\|G_{2T}(x,\xi/2)\|) < 2\Lambda^+ + \varepsilon/2$$

and the Liouville's measure of $p^{-1}(1)\backslash K$ is smaller than η. Now remark that the geodesic flow is continuous and so, for a fixed T, the matrix G_T depends continuously on

2. See Annex A for a definition.

(x,ξ). This means that if a point (x,ξ) is close enough to K then we still have

$$\frac{1}{T}\log(\|G_{2T}(x,\xi/2)\|) < 2\Lambda^+ + \varepsilon/2.$$

Consequently there exists some $\mathscr{C}^\infty$ smooth function $\widetilde{\chi}_K^{(T)} : T^*M \to \mathbf{R}_+$ such that $\widetilde{\chi}_K^{(T)}$ equals 1 on K, $\mathrm{supp}(\widetilde{\chi}_K^{(T)}) \subset \{(x,\xi) \in T^*M : |\xi|_g \in [1/2;3/2]\}$ and

$$(x,\xi) \in \mathrm{supp}(\widetilde{\chi}_K^{(T)}) \implies \frac{1}{T}\log(\|G_{2T}(x,\xi/2)\|) < 2\Lambda^+ + \varepsilon/2.$$

We then define the function $\chi_K^{(T)} = \widetilde{\chi}_K^{(T)} \circ j \circ \phi_T$ where $j(x,\xi) = (x,-\xi)$ and ϕ is the hamiltonian flow generated by p. Notice that the functions j and ϕ_T both preserve the Liouville measure on $p^{-1}(1)$.

We now choose some non negative function $f \in \mathcal{S}(\mathbf{R})$ such that $f(0) = 1$ and $\mathrm{supp}(\hat{f})$ is compact. Let C be some positive constant which will be fixed later and let us define the three operators A_1, A_2 and A_3 as

$$\begin{aligned} A_1(z) &= -f\left(\frac{P-1}{Ch}\right)\mathrm{Op}_h^{\mathrm{AW}}(\chi_K^{(T)})f\left(\frac{P-1}{Ch}\right)\frac{i}{h}\int_0^T e^{it(\mathcal{P}-z)/h}dt, \\ A_2 &= -f\left(\frac{P-1}{Ch}\right)\mathrm{Op}_h^{\mathrm{AW}}(1-\chi_K^{(T)})f\left(\frac{P-1}{Ch}\right)\frac{i}{h}\int_0^T e^{it(\mathcal{P}-z_0)/h}dt, \\ \text{and } A_3 &= \left(1-f\left(\frac{P-1}{Ch}\right)\right)w(P-1)\left(1-f\left(\frac{P-1}{Ch}\right)\right). \end{aligned}$$

Here $\mathrm{Op}_h^{\mathrm{AW}}(b)$ is the h-anti-Wick quantization of a symbol b and w is defined by

$$w : x \mapsto \frac{1}{x}\left(1-\chi\left(\frac{x}{Ch}\right)\right)$$

where χ is some smooth cut-off function with $\chi(x) = 1$ around zero and some small support that will be chosen later. For a definition of the anti-Wick quantization see Annex A. We finally define $R(z)$ as the sum of A_1, A_2 and A_3 and we are now ready to state a more precise version of Proposition 3.1.

Proposition 3.2 *We have the following equalities.*

$$A_1(z)(\mathcal{P}-z) = f\left(\frac{P-1}{Ch}\right)\mathrm{Op}_h^{\mathrm{AW}}(\chi_K^{(T)})f\left(\frac{P-1}{Ch}\right)+\mathcal{O}\left(\frac{1}{C}\right)+\mathcal{O}\left(e^{-T\varepsilon/2}\right)+\mathcal{O}_T(h^{1/2}) \tag{3.10}$$

$$A_2(\mathcal{P}-z_0) = f\left(\frac{P-1}{Ch}\right)\mathrm{Op}_h^{\mathrm{AW}}(1-\chi_K^{(T)})f\left(\frac{P-1}{Ch}\right)+\mathcal{O}\left(\frac{1}{C}\right)+\mathcal{O}\left(e^{-T\varepsilon/2}\right)+\mathcal{O}_T(h^{1/2}) \tag{3.11}$$

$$\left\|A_3(\mathcal{P}-z)-\left(1-f\left(\frac{P-1}{Ch}\right)\right)^2\right\|_{L^2} \leq \mathcal{O}\left(\frac{1}{C}\right)+\delta \tag{3.12}$$

Where $\delta > 0$ depends only on χ and f and can be made arbitrarily small. We moreover have the following bounds

$$\|A_2(\mathcal{P} - z)\|_{\mathrm{Tr}} \leq \left(\int_{p^{-1}(1)} 1 - \chi_K^{(T)} dL_0\right) \cdot \mathcal{O}_C(h^{d-1}) \tag{3.13}$$

$$\left\| f\left(\frac{P-1}{Ch}\right) \mathrm{Op}_h^{\mathrm{AW}}(1 - \chi_K^{(T)}) f\left(\frac{P-1}{Ch}\right)\right\|_{\mathrm{Tr}} \leq \left(\int_{p^{-1}(1)} 1 - \chi_K^{(T)} dL_0\right) \cdot \mathcal{O}_C(h^{d-1}) \tag{3.14}$$

where L_0 is the Liouville measure on $p^{-1}(1)$.

3.3.3 Proof of Proposition 3.2

We start by proving a lemma on commutators with $f\left(\frac{P-1}{Ch}\right)$.

Lemma 3.5 *Let U be a pseudo-differential operator on M with symbol u compactly supported in the ξ variable. We have the following bound.*

$$\left\|\left[f\left(\frac{P-1}{Ch}\right), U\right]\right\|_{L^2} = \mathcal{O}\left(\frac{1}{C}\right).$$

Proof. We first recall the formula

$$f\left(\frac{P-1}{Ch}\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{f}(t) e^{it\frac{P-1}{Ch}} dt. \tag{3.15}$$

According to this formula

$$\left[f\left(\frac{P-1}{Ch}\right), U\right] = \int_{\mathbf{R}} \hat{f}(t)\left[e^{it\frac{P-1}{Ch}}, U\right] dt = \int_{\mathbf{R}} \hat{f}(t) e^{\frac{-it}{Ch}}\left[e^{it\frac{P}{Ch}}, U\right] dt.$$

Since $e^{it\frac{P}{Ch}}$ is an isometry we have

$$\left\|\left[e^{it\frac{P}{Ch}}, U\right]\right\|_{L^2} = \left\|e^{it\frac{P}{Ch}} U e^{-it\frac{P}{Ch}} - U\right\|_{L^2}$$

and by Egorov's theorem we know that $e^{it\frac{P}{Ch}} U e^{-it\frac{P}{Ch}} = \mathrm{Op}_h(u \circ \phi_{t/C}) + \mathcal{O}(h)$ where ϕ is the Hamiltonian flow on T^*M associated with p. Notice that if we write $(y, \eta) = \phi_{t/C}(x, \xi)$ then $|\xi|_x^2 = |\eta|_y^2$ so u and $u \circ \phi_{t/C}$ are both compactly supported in ξ. Consequently we have

$$\left\|u - u \circ \phi_{t/C}\right\|_\infty = \mathcal{O}\left(\frac{1}{C}\right)$$

uniformly in $t \in \mathrm{supp}(\hat{f})$ and the same estimates goes with the derivatives of $u - u \circ \phi_{t/C}$.

According to Calderon-Vaillancourt's theorem we then have

$$\left\| \left[e^{it\frac{P}{Ch}}, U \right] \right\|_{L^2} = \mathcal{O}\left(\frac{1}{C}\right)$$

uniformly in $t \in \mathrm{supp}(\hat{f})$ and we finally get

$$\left\| \left[f\left(\frac{P-1}{Ch}\right), U \right] \right\|_{L^2} = \mathcal{O}\left(\frac{1}{C}\right).$$

□

We also need a lemma to approximate $e^{it(\mathcal{P}-z)/h}$.

Lemma 3.6 *Let $t \in [0;T]$ be a fixed positive real number and u be a symbol compactly supported, we have*

$$\left\| \mathrm{Op}_h^{\mathrm{AW}}(u) \left[e^{it\mathcal{P}/h} - \mathrm{Op}_h^{\mathrm{AW}}(G_{2t} \circ j \circ \phi_t(x, \xi/2)) e^{-ith\Delta} \right] \right\|_{L^2(M)} = \mathcal{O}_T(h^{1/2})$$

where $j(x,\xi) = (x, -\xi)$ for every $(x,\xi) \in T^*M$ and ϕ_t is the Hamiltonian flow generated by p.

Proof. We are only going to prove this result with $M = \mathbf{R}^d$ with a metric g, the extension to any compact Riemannian manifold is straightforward. The first step of the proof is to precisely describe the action of $e^{it\mathcal{P}/h}$ on coherent state, this is a classical result and we will follow the presentation and notations of [Rob06]. Let $g : \mathbf{R}^d \to \mathbf{C}$ be the function defined by

$$g : x \mapsto \frac{1}{\pi^{d/4}} \exp\left(-\|x\|_2^2/2\right).$$

and we note $\varphi_0 = \Lambda_h g$ where Λ_h is the dilatation operator defined by $\Lambda_h f(x) = h^{-d/4} f(h^{-1/2}x)$. In other words we have

$$\varphi_0(x) = \frac{1}{(\pi h)^{d/4}} \exp\left(\frac{-\|x\|^2}{2h}\right).$$

If $\rho = (x_0, \xi_0)$ is a point of $T^*\mathbf{R}^d = \mathbf{R}^{2d}$ we define φ_ρ by

$$\varphi_\rho = \mathcal{T}(\rho)\varphi_0$$

where $\mathcal{T}(\rho) = \exp\left(\frac{i}{h}(\xi_0 \cdot x + ihx_0 \cdot \partial_x)\right)$ is the Weyl operator, in other words

$$\varphi_\rho(x) = e^{ix\cdot\xi_0/h}\varphi_0(x - x_0).$$

The function φ_ρ is called the coherent state associated with (x_0, ξ_0). Finally, if $v \in \mathbf{C}^n$ we define $\varphi_{\rho,v} = \varphi_\rho \cdot v$. According to [Rob06], for every integer N, every ρ in a compact set K and every $t \in [0;T]$ we have

$$\left\| e^{it\mathcal{P}/h}\varphi_{\rho,v} - \psi^{(N)}_{\rho,v}(t) \right\|_{L^2} = \mathcal{O}_{K,T}(h^{\frac{N+1}{2}})$$

where

$$\psi^{(N)}_{\rho,v}(t) = e^{i\delta_t/h}\mathcal{T}(\rho_t)\Lambda_h \widetilde{G}_{2t}(j(\widetilde{\rho}))\mathcal{M}[F_t]\left(\sum_{0\le j\le N} h^{j/2}b_j(t)g\right).$$

We will only give a partial description of the terms of $\psi^{(N)}_{\rho,v}(t)$ here, for a complete definition see [Rob06]. The point $\rho_t \in \mathbf{T}^*\mathbf{R}^d$ is simply given by the inverse Hamiltonian flow : $\rho_t = (x_{-t}, \xi_{-t}) = \phi_{-t}(x_0,\xi_0) = \phi_{-t}\rho$ and $\widetilde{\rho} = (x_0, \xi_0/2)$. The quantity δ_t is real and only depends on t . The function $\widetilde{G}_t : T^*\mathbf{R}^d \to \mathscr{M}_n(\mathbf{C})$ is defined as the solution of the following differential equation :

$$\begin{cases} \widetilde{G}_0(x_0,\xi_0) = \mathrm{Id}_n \\ \partial_t \widetilde{G}_t(x_0,\xi_0) = -a(x_t)\sqrt{z}\widetilde{G}_t(x_0,\xi_0) \end{cases} \tag{3.16}$$

we thus have $\partial_t \widetilde{G}_{2t}(j(\widetilde{\rho})) = -2a(x_{-t})\sqrt{z}\widetilde{G}_{2t}(j(\widetilde{\rho}))$. The function $j : T^*\mathbf{R}^d \to T^*(\mathbf{R}^d)$ is defined by $j(x,\xi) = (x,-\xi)$. The functions $b_j(t)$ are polynomial functions in the x variable with coefficients in $\mathbf{C}^n$, the first term $b_0(t)$ is constant and equal to v. The term $\mathcal{M}[F_t]$ describe the sprawl of the Gaussian g under the action of the propagator, F_t is a flow of linear symplectic transformation and $\mathcal{M}$ is a realization of the metaplectic representation. We can apply the same result when $a = 0$ and we find

$$\left\| e^{-iht\Delta}\varphi_{\rho,v} - e^{i\delta_t/h}\mathcal{T}(\rho_t)\Lambda_h\mathcal{M}[F_t]\left(\sum_{0\le j\le N} h^{j/2}c_j(t)g\right)\right\|_{L^2} = \mathcal{O}_{K,T}(h^{\frac{N+1}{2}}),$$

the only difference with $e^{it\mathcal{P}/h}\varphi_{\rho,v}$ is in the polynomials c_j and in the absence of $\widetilde{G}_t$, note that we also have $c_0(t) = v$. Since the function $v \mapsto \psi^{(N)}_{\rho,v}(t)$ is linear and since $b_0(t) = c_0(t) = v$ we can define by induction some matrices $q_1(t,\rho),\dots,q_N(t,\rho)$ depending polynomially on x such that

$$\left(1 + h^{1/2}q_1(t,\rho) + \ldots + h^{N/2}q_N(t,\rho)\right)\left(\sum_{0\le j\le N} h^{j/2}c_j(t)g\right) =$$

$$\left(\sum_{0\le j\le N} h^{j/2}b_j(t)g\right) + \mathcal{O}(h^{\frac{N+1}{2}}) \tag{3.17}$$

for every $v \in \mathbf{C}^n$, $t \in [0;T]$ and $\rho \in K$. Indeed the matrix $q_i(t,\rho)$ must satisfy the relation

$$c_i(t) + \sum_{j=1}^{i} q_j(t,\rho)c_{i-j}(t) = b_i(t) \Longleftrightarrow q_i(t,\rho)v = b_i(t) - c_i(t) - \sum_{j=1}^{i-1} q_j(t,\rho)c_{i-j}(t)$$

for every $v \in \mathbf{C}^n$ and since all the polynomials $b_j(t)$ and $c_j(t)$ depend linearly on v this defines uniquely the matrix $q_i(t,\rho)$ if the $q_j(t,\rho),\ j = 1,\dots,i-1$ are fixed. Now let $f_1, f_2 : \mathbf{R}^{2d} \to \mathscr{M}_n(\mathbf{C})$ be two symbols of order $\leq m \in \mathbf{R}$; then we have

$$\mathcal{M}[F_t]\mathrm{Op}_1^{\mathrm{W}}(f_1) = \mathrm{Op}_1^{\mathrm{W}}\left(f_1 \circ F_t^{-1}\right)\mathcal{M}[F_t]$$

where $\mathrm{Op}_1^{\mathrm{W}}$ is the Weyl quantization for $h = 1$. A proof of this result can be found in [Rob06]. We also have

$$\Lambda_h(f_1 \cdot f_2) = h^{d/4}\Lambda_h(f_1) \cdot \Lambda_h(f_2),$$

$$\text{and } \ \mathcal{T}(z_t)(f_1 \cdot f_2) = \mathcal{T}(z_t)(f_1) \cdot \mathcal{T}(z_t)(f_2).$$

Using the previous relations and (3.17) we see that there exist some matrices $Q_1(t,\rho),\dots,Q_N(t,\rho)$ depending polynomially on x such that

$$\left\| \psi_{\rho,v}^{(N)} - \left(\widetilde{G}_{2t}(j(\widetilde{\rho})) + \sum_{i=1}^{N} h^{i/2}Q_i(t,\rho)\right) e^{i\delta_t/h}\mathcal{T}(\rho_t)\Lambda_h\mathcal{M}[F_t]\left(\sum_{0\leq j\leq N} h^{j/2}c_j(t)g\right)\right\|_{L^2} = \mathcal{O}(h^{\frac{N+1}{2}}).$$

Notice that the matrix $\widetilde{G}_t(j(\rho))$ does not depend on x, this is due to the fact that $b_0(t) = c_0(t)$. Consequently we have

$$\left\| e^{it\mathcal{P}/h}\varphi_{\rho,v} - \left(\widetilde{G}_{2t}(j(\widetilde{\rho})) + \sum_{i=1}^{N} h^{i/2}Q_i(t,\rho)\right) e^{-iht\Delta}\varphi_{\rho,v}\right\|_{L^2} = \mathcal{O}(h^{\frac{N+1}{2}}) \tag{3.18}$$

and the $\mathcal{O}(h^{\frac{N+1}{2}})$ is uniform in $\rho \in K$ and $t \in [0;T]$. If we take some symbol u with $\mathrm{supp}(u) \Subset K$ then the estimates become uniform in ρ :

$$\left\| \mathrm{Op}_h^{\mathrm{AW}}(u)e^{it\mathcal{P}/h}\varphi_{\rho,v} - \mathrm{Op}_h^{\mathrm{AW}}(u)\left(\widetilde{G}_{2t}(j(\widetilde{\rho})) + \sum_{i=1}^{N} h^{i/2}Q_i(t,\rho)\right) e^{-iht\Delta}\varphi_{\rho,v}\right\|_{L^2} = \mathcal{O}(h^{\frac{N+1}{2}}) \tag{3.19}$$

uniformly in $t \in [0;T]$ and $\rho \in \mathbf{R}^{2d}$. The next step is to remark that $e^{-iht\Delta}\varphi_{\rho,v}$ is a sum of derivatives of anisotropic coherent states associated with ρ_t. So, by using $\rho = \phi_t\rho_t$, there exist some symbols $g_t^{(i)}$ such that

$$\left\| \mathrm{Op}_h^{\mathrm{AW}}(u)\left[\widetilde{G}_{2t}(j(\widetilde{\rho}))e^{-iht\Delta} - \left(\mathrm{Op}_h^{\mathrm{AW}}(\widetilde{G}_{2t} \circ j \circ \phi_{2t}) + \sum_{i=1}^{N} h^{j/2}\mathrm{Op}_h^{\mathrm{AW}}(g_t^{(i)})\right) e^{-iht\Delta}\right]\varphi_{\rho,v}\right\|_{L^2} = \mathcal{O}(h^{\frac{N+1}{2}})$$

uniformly in $\rho \in \mathbf{R}^{2d}$ and $t \in [0;T]$. The same is true for the matrices Q_i and thus there exist some symbols $G_t^{(i)}$ such that

$$\left\| \mathrm{Op}_h^{\mathrm{AW}}(u) \left[e^{it\mathcal{P}/h} - \left(\mathrm{Op}_h^{\mathrm{AW}}(\widetilde{G}_{2t} \circ j \circ \phi_{2t}) + \sum_{i=1}^{N} h^{j/2} \mathrm{Op}_h^{\mathrm{AW}}(G_{2t}^{(i)}) \right) e^{-iht\Delta} \right] \varphi_{\rho,v} \right\|_{L^2} = \mathcal{O}(h^{\frac{N+1}{2}}). \tag{3.20}$$

We now use the fact that for every function $f = (f_1, \dots, f_n) \in L^2(\mathbf{R}^d)^n$ we have

$$f_i = \frac{1}{(2\pi h)^d} \int_{\mathbf{R}^{2d}} \langle f_i, \varphi_{z,e_i} \rangle_{L^2(\mathbf{R}^d)} f_i dz \tag{3.21}$$

where $e_i = (0, \dots, 0, 1, 0 \dots, 0)$ is the i-th vector of the canonical basis of $\mathbf{C}^n$. Combining (3.20) and (3.21) we get that

$$\left\| \mathrm{Op}_h^{\mathrm{AW}}(u) \left[e^{it\mathcal{P}/h} - \left(\mathrm{Op}_h^{\mathrm{AW}}(\widetilde{G}_{2t} \circ j \circ \phi_{2t}) + \sum_{i=1}^{N} h^{j/2} \mathrm{Op}_h^{\mathrm{AW}}(G_{2t}^{(i)}) \right) e^{-iht\Delta} \right] \right\|_{L^2} = \mathcal{O}(h^{\frac{N+1}{2}-d}).$$

Notice that $\|\mathrm{Op}_h^{\mathrm{AW}}(G_{2t}^{(i)})\|_{L^2}$ is bounded uniformly in h and so, if we take $N+1 > 2d$ and only keep the principal terms we get

$$\left\| \mathrm{Op}_h^{\mathrm{AW}}(u) \left(e^{it\mathcal{P}/h} - \mathrm{Op}_h^{\mathrm{AW}}(\widetilde{G}_{2t} \circ j \circ \phi_{2t}(x, \xi/2)) e^{-iht\Delta} \right) \right\|_{L^2} = \mathcal{O}_T(h^{1/2}).$$

The last final step is to remark that $\widetilde{G}_t$ depends smoothly on $z = 1 + \mathcal{O}(h)$: we have $\widetilde{G}_t = G_t$ when $z = 1$ and so $\widetilde{G}_t = G_t + \mathcal{O}_T(h)$. Plugging this in the previous equality, we get what we wanted :

$$\left\| \mathrm{Op}_h^{\mathrm{AW}}(u) \left(e^{it\mathcal{P}/h} - \mathrm{Op}_h^{\mathrm{AW}}(G_t \circ j \circ \phi_t) e^{-iht\Delta} \right) \right\|_{L^2} = \mathcal{O}_T(h^{1/2}).$$

This finishes the proof of the lemma. □

We can now prove (3.10) ; we start by writing

$$\frac{-i}{h} \int_0^T e^{it(\mathcal{P}-z)/h} dt (\mathcal{P} - z) = \mathrm{Id} - e^{iT(\mathcal{P}-z)/h}$$

and so

$$A_1(z)(\mathcal{P} - z) = f\left(\frac{P-1}{Ch}\right) \mathrm{Op}_h^{\mathrm{AW}}(\chi_K^{(T)}) f\left(\frac{P-1}{Ch}\right) \left(\mathrm{Id} - e^{iT(\mathcal{P}-z)/h}\right).$$

Using the fact that $\mathrm{Op}_h^{\mathrm{AW}}(\chi_K^{(T)}) = \mathrm{Op}_h(\chi_K^{(T)}) + \mathcal{O}_T(h)$ twice and applying Lemma 3.5 we get

$$f\left(\frac{P-1}{Ch}\right) \mathrm{Op}_h^{\mathrm{AW}}(\chi_K^{(T)}) f\left(\frac{P-1}{Ch}\right) e^{-iT(\mathcal{P}-z)/h} =$$

$$f\left(\frac{P-1}{Ch}\right)^2 \mathrm{Op}_h^{\mathrm{AW}}(\chi_K^{(T)})e^{-iT(\mathcal{P}-z)/h} + \mathcal{O}_T(h) + \mathcal{O}\left(\frac{1}{C}\right).$$

The operator norm of $f\left(\frac{P-1}{Ch}\right)^2$ is bounded by $\|f^2\|_\infty$ and it only remains to estimate the operator norm of $\mathrm{Op}_h^{\mathrm{AW}}(\chi_K)e^{iT(\mathcal{P}-z)/h}$. According to Lemma 3.6 we have

$$\begin{aligned}\mathrm{Op}_h^{\mathrm{AW}}(\chi_K^{(T)})e^{iT(\mathcal{P}-z)/h} &= e^{-itz/h}\mathrm{Op}_h^{\mathrm{AW}}(\chi_K^{(T)})\mathrm{Op}_h^{\mathrm{AW}}(G_{2T}\circ j\circ\phi_T(x,\xi/2))e^{-ihT\Delta} \\ &\quad +\mathcal{O}_T(h^{1/2}) \\ &= e^{-itz/h}\mathrm{Op}_h^{\mathrm{AW}}(\chi_K^{(T)}G_{2T}\circ j\circ\phi_T(x,\xi/2))e^{-ihT\Delta} + \mathcal{O}_T(h^{1/2})\end{aligned}$$

Since $e^{-ihT\Delta}$ is an isometry we have

$$\left\|\mathrm{Op}_h^{\mathrm{AW}}(\chi_K^{(T)})e^{iT(\mathcal{P}-z)/h}\right\|_{L^2} \le \|\chi_K^{(T)}G_{2T}\circ j\circ\phi_T(x,\xi/2)\|_\infty e^{T\mathfrak{Im}(z/h)} + \mathcal{O}_T(h^{1/2}),$$

recall that $\chi_K^{(T)} = \widetilde{\chi}_K^{(T)}\circ j\circ\phi_T$ so

$$\|\chi_K^{(T)}G_{2T}\circ j\circ\phi_T(x,\xi/2)\|_\infty = \|\widetilde{\chi}_K^{(T)}G_{2T}(x,\xi/2)\|_\infty.$$

If we then use the definition of $\widetilde{\chi}_K^{(T)}$, G_T and Ω_h we see that

$$\|\widetilde{\chi}_K^{(T)}G_{2T}(x,\xi/2)\|_\infty e^{-T\mathfrak{Im}(z/h)} \le e^{-T\varepsilon/2},$$

which finishes the proof of (3.10). The same technique is used to prove (3.11) except that we don't even have to use Lemma 3.6 because $\mathfrak{Im}(z_0/h)$ is small enough :

$$\left\|e^{iT(\mathcal{P}-z_0)}\right\|_{L^2} \le e^{-T}.$$

We continue by proving (3.12). Recall that $\mathcal{P}-z = P-1+\mathcal{O}(h)$ and that $\|w\|_\infty = \mathcal{O}\left(\frac{1}{Ch}\right)$ so

$$\begin{aligned}A_3(z)(\mathcal{P}-z) &= A_3(z)(P-1) + \mathcal{O}\left(\frac{1}{C}\right) \\ &= \left(1-f\left(\frac{P-1}{Ch}\right)\right)w(P-1)(P-1)\left(1-f\left(\frac{P-1}{Ch}\right)\right) + \mathcal{O}\left(\frac{1}{C}\right).\end{aligned}$$

According to the definition of w whe have $w(P-1)(P-1) = 1-\chi\left(\frac{P-1}{Ch}\right)$ and so it only remains to estimate the operator norm of

$$\left(1-f\left(\frac{P-1}{Ch}\right)\right)\chi\left(\frac{P-1}{Ch}\right)\left(1-f\left(\frac{P-1}{Ch}\right)\right).$$

The norm of this operator is bounded by $\|(1-f)\chi(1-f)\|_\infty$, recall that $f(0)=1$ and that f is continuous so by choosing χ with sufficiently small support around 0 we get

$$\|(1-f)\chi(1-f)\|_\infty < \delta$$

for any fixed positive δ and (3.12) is proved.

It only remains now to prove (3.13) and (3.14), we start with (3.14) and we proceed as in [Sjö00]. Since the Anti-Wick quantification is positive and since $1-\chi_K^{(T)} \geq 0$ we know that

$$\begin{aligned} & \left\| f\left(\frac{P-1}{Ch}\right) \mathrm{Op}_h^{\mathrm{AW}}(1-\chi_K^{(T)}) f\left(\frac{P-1}{Ch}\right)\right\|_{\mathrm{Tr}} \\ = \ & \mathrm{Tr}\left[f\left(\frac{P-1}{Ch}\right) \mathrm{Op}_h^{\mathrm{AW}}(1-\chi_K^{(T)}) f\left(\frac{P-1}{Ch}\right)\right] \\ = \ & \mathrm{Tr}\left[f\left(\frac{P-1}{Ch}\right)^2 \mathrm{Op}_h^{\mathrm{AW}}(1-\chi_K^{(T)})\right] \\ = \ & \frac{1}{\sqrt{2\pi}} \mathrm{Tr}\left[\int_{\mathbf{R}} \widehat{f^2}(t) e^{it\frac{P-1}{Ch}} \mathrm{Op}_h^{\mathrm{AW}}(1-\chi_K^{(T)}) dt\right] \\ = \ & \frac{C}{\sqrt{2\pi}} \mathrm{Tr}\left[\int_{\mathbf{R}} \widehat{f^2}(Ct) e^{it\frac{P-1}{h}} \mathrm{Op}_h^{\mathrm{AW}}(1-\chi_K^{(T)}) dt\right]. \end{aligned}$$

For C large enough we have

$$\mathrm{supp}\widehat{f^2}(C\cdot) \subset]-\frac{1}{2}T_{\min}; \frac{1}{2}T_{\min}[\tag{3.22}$$

where $T_{\min}$ is the smallest possible length of a closed trajectory in $p^{-1}(1)$ for the Hamiltonian flow generated by p. Whenever (3.22) is satisfied we know that

$$\lim_{h\to 0} \mathrm{Tr}\left[\int_{\mathbf{R}} \widehat{f^2}(Ct) e^{it\frac{P-1}{h}} \mathrm{Op}_h^{\mathrm{AW}}(1-\chi_K^{(T)}) dt\right] = C_d h^{1-d} \widehat{f^2}(0) \int_{p^{-1}(1)} 1-\chi_K^{(T)} L_0(d\rho) \tag{3.23}$$

where L_0 is the Liouville measure on $p^{-1}(1)$ and C_d only depends on d, the dimension of M. One can find a proof of this classical fact in [DiSj99] for instance. We now use the formula $\|AB\|_{\mathrm{Tr}} \leq \|A\|\|B\|_{\mathrm{Tr}}$ and the fact that $\left\|f\left(\frac{P-1}{Ch}\right)\right\|_{L^2} \leq \|f\|_\infty$ to get (3.14). We use the same technique for (3.13) and so it only remains to show that

$$-\frac{i}{h}\int_0^T e^{it(\mathcal{P}-z_0)/h} dt (\mathcal{P}-z)$$

is uniformly bounded in h. We recall that $\mathcal{P}-z_0$ is invertible and so we can write

$$-\frac{i}{h}\int_0^T e^{it(\mathcal{P}-z_0)/h} dt (\mathcal{P}-z) = \left(\mathrm{Id} - e^{iT(\mathcal{P}-z_0)/h}\right)(\mathcal{P}-z_0)^{-1}(\mathcal{P}-z).$$

By using the energy formula (3.3) we see that $\left\|e^{iT(\mathcal{P}-z_0)}\right\|_{L^2} \leq e^{-T} \leq 1$ and that $\left\|(\mathcal{P}-z_0)^{-1}\right\|_{L^2}$ is uniformly bounded in h. Consequently the operator $(\mathcal{P}-z_0)^{-1}(\mathcal{P}-z)$ is uniformly bounded in L^2 when $h \to 0$. This finishes the proof of Proposition 3.2.

3.3.4 End of the proof of Proposition 3.1

We now use Proposition 3.2 to prove Proposition 3.1. According to Proposition 3.2 we have

$$\begin{aligned} R(z)(\mathcal{P}-z) &= \mathrm{Id} - f\left(\frac{P-1}{h}\right)\mathrm{Op}_h^{\mathrm{AW}}(1-\chi_K^{(T)})f\left(\frac{P-1}{h}\right) + A_2(\mathcal{P}-z) \\ &\quad + \mathcal{O}\left(\frac{1}{C}\right) + \mathcal{O}(\delta) + \mathcal{O}(e^{-\varepsilon T/2}) + \mathcal{O}_T(h^{1/2}). \end{aligned}$$

We define

$$R_2(z) = -f\left(\frac{P-1}{h}\right)\mathrm{Op}_h^{\mathrm{AW}}(1-\chi_K^{(T)})f\left(\frac{P-1}{h}\right) + A_2(\mathcal{P}-z)$$

$$\text{and} \;\; R_1(z) = R(z)(\mathcal{P}-z) - R_2(z).$$

We start by fixing a constant C large enough so the remainder $\mathcal{O}\left(\frac{1}{C}\right)$ is smaller than $1/100$. We then chose χ in the definition of A_3 so that the remainder $\mathcal{O}(\delta)$ is smaller than $1/100$. Fix some arbitrary $\eta > 0$, for T large enough the remainder $\mathcal{O}(e^{-\varepsilon T/2})$ is smaller than $1/100$ and there exists some $\chi_K^{(T)}$ such that

$$\limsup_{h\to 0} h^{n-1}\|R_2(z)\|_{\mathrm{T}r} < \eta.$$

We finally take h small enough so that the remainder $\mathcal{O}_T(h^{1/2})$ is also smaller than $1/100$. By doing so we have constructed an operator $R(z)$ such that $R(z)(\mathcal{P}-z) = \mathrm{Id} + R_1(z) + R_2(z)$ with $\|R_1(z)\|_{L^2} < 1/10$ and $\|R_2(z)\|_{\mathrm{Tr}} \leq \eta h^{1-d}$ for h small enough. When h goes to 0 we can repeat the same process and make η arbitrarily small and get $\|R_2(z)\|_{\mathrm{Tr}} = o(h^{1-d})$. Moreover according to Proposition 3.2 we have

$$\|R_2(z_0)\|_{L^2} = \mathcal{O}\left(\frac{1}{C}\right) + \mathcal{O}(e^{-T\varepsilon/2}) + \mathcal{O}_T(h^{1/2})$$

and as before we can choose C and T in order to also have $\|R_2(z_0)\|_{L^2} < 1/10$ for h small enough. This implies that $R(z_0)(\mathcal{P}-z_0)$ is invertible and that $\left\|(R(z_0)(\mathcal{P}-z_0))^{-1}\right\|$ is uniformly bounded in h. The proof of Proposition 3.1 is thus finished.

3.4 Proof of Proposition 3.4

Let us take $\lambda \in \mathbf{C}^*$ an eigenvalue of A_a and $u \in H\backslash\{0\}$ an associated eigenvector with $E(u,0) = 1$, then

$$\frac{1}{t}\log(E(u,t)) = 2\mathfrak{Re}(\lambda).$$

Let $(\lambda_k)_k$ be a sequence of eigenvalues of A_a with $\mathfrak{Im}(\lambda_k) \to +\infty$ and let $(u_k, \partial_t u_k)_k$ be a sequence of eigenvectors of A_a such that $A_a(u_k, \partial_t u_k) = \lambda_k(u_k, \partial_t u_k)$ and $E(u_k, 0) = 1$. We can thus estimate the variations of energy of u_k when $k \to +\infty$ in order to prove Proposition 3.4. We will use the same techniques as in Chapter 2. We know that the sequence $(u_k)_k$ is weakly converging to 0 in $H^1(I \times M)$ where $I = [-2T; 2T]$ and we can thus extract a pure sub-sequence that we will still call $(u_k)_k$. We call $\mu = M\nu$ the associated microlocal defect measure. As explained in Chapter 1 for every $\eta > 0$ we have

$$\lim_{k\to\infty}\int_0^\eta E(u_k,t)dt = \nu(S^*(M\times I)\cap\{t\in]0;\eta[\})$$

$$\text{and}\quad \lim_{k\to\infty}\int_T^{T+\eta} E(u_k,t)dt = \nu(S^*(M\times I)\cap\{t\in]T;T+\eta[\}).$$

As in the proof of Proposition 2.9 we can connect the two previous quantities using the propagation of the measure. With the notations of Chapter 2 we find

$$\nu^+(t\in]T;T+\eta[) = \int_{SZ^+\cap\{t\in]0;\eta[\}} \mathrm{Tr}[G_T^+ M G_T^{+*}]\mathrm{d}\nu.$$

Recall that the function G_T^+ is defined on $SZ^+ = S^*(\mathbf{R}\times M)\cap\{\tau = -1/2\}$ but we have $G_T^+(t,\tau,x,\xi) = G_T(x,\xi)$. Thus we would like to estimate $\mathrm{Tr}[G_T(x,\xi)MG_T(x,\xi)^*]$.

Consider the point $(0,1) \in S^*M$, since $M = \mathbf{R}/2\pi\mathbf{Z}$ if we put $T = 2l\pi$ with l an integer we have $G_{2l\pi}(x,\xi) = G_{2\pi}(x,\xi)^l$. If we simply write G for the matrix $G_{2\pi}(x,\xi)$ we get

$$\mathrm{Tr}[G_T(x,\xi)MG_T(x,\xi)^*] = \mathrm{Tr}[G^l M G^{l^*}].$$

We are going to show that $\lim_{l\to\infty}\mathrm{Tr}[G^l M G^{l^*}]^{1/2l}$ exists and that it is equal to the modulus of one of the eigenvalue of G. Since M is Hermitian positive semi-definite there exists a matrix N also Hermitian positive semi-definite such that $N^2 = M$, thus

$$\mathrm{Tr}[G^l M G^{l*}] = \mathrm{Tr}[G^l N N G^{l*}] = \mathrm{Tr}[G^l N (G^l N)^*].$$

Let $(e_1, \dots, e_n)$ be an orthonormal basis of eigenvectors of N, we have

$$\mathrm{Tr}[G^l N(G^l N)^*] = \sum_{i=1}^n \langle G^l N e_i, G^l N e_i\rangle_{\mathbf{C}^n} = \sum_{i=1}^n \alpha_i^2 \langle G^l e_i, G^l e_i\rangle_{\mathbf{C}^n} = \sum_{i=1}^n \alpha_i^2 \|G^l e_i\|_2^2$$

where α_i is the eigenvalue of N associated with e_i. We now want to compute the

$\|G^l e_i\|_2^2$, we can decompose the e_i over $(f_1, \dots, f_n)$ a Jordan basis of G :

$$e_i = \sum_{i=j}^{n} \beta_j^i f_j,\ i = 1, \dots, n.$$

Assume for simplicity that all the f_j are eigenvectors associated with the eigenvalues $\mu_1, \dots, \mu_n$ of G, then we have

$$G^l e_i = \sum_{i=j}^{n} \beta_j^i \mu_j^l f_j.$$

If the μ_i are in increasing order of modulus and if we put $J_i = \max\{j : \beta_j^i \neq 0\}$ then

$$G^l e_i = (\mathrm{Id}_n + o(1))\beta_{J_i}^i \mu_{J_i}^l f_{J_i}.$$

Consequently we have

$$\lim_{l\to\infty} \|G^l e_i\|_2^{1/l} = |\mu_{J_i}|.$$

Using the fact that for every n-tuple of complex numbers $(b_1, \dots, b_n)$ we have $\lim_{l\to\infty}(|b_1|^l + \dots + |b_n|^l)^{1/l} = \sup |b_i|$ we see that

$$\left(\sum_{i=1}^{n} \alpha_i^2 \|G^l e_i\|_2^2\right)^{1/2l} = |\mu_J|$$

where $J = \sup\{J_i : \alpha_i \neq 0\}$. The calculations are similar when not every f_j is an eigenvector. So far we have proved that

$$\lim_{l\to\infty,\ l\in\mathbf{N}} \frac{1}{l} \log\left(\mathrm{Tr}[G_{2\pi l}(0,1) M G_{2\pi l}(0,1)^*]\right) = \log|\mu_J|$$

where μ_J is an eigenvalue of $G_{2\pi}(0,1)$. Now we take some time $T = 2\pi l + s$ with $s \in]0; 2\pi[$ and recall that from the cocycle relation

$$G_T(0,1) = G_s(\phi_{2\pi l}(0,1))G_{2\pi l}(0,1) = G_s(0,1)G^l.$$

Therefore

$$\mathrm{Tr}[G_T(0,1) M G_T(0,1)^*] = \mathrm{Tr}[G_s(0,1)^* G_s(0,1) G^l M G^{l*}]$$

and it is a standard exercise to show that

$$d_- \mathrm{Tr}[G^l M G^{l*}] \leq \mathrm{Tr}[G_s(0,1)^* G_s(0,1) G^l M G^{l*}] \leq d_+ \mathrm{Tr}[G^l M G^{l*}]$$

where d_- and d_+ are respectively the smallest and largest eigenvalue of $G_s(0,1)^* G_s(0,1)$.

This means that we also have

$$\lim_{t\to\infty} \frac{2\pi}{t} \log\left(\mathrm{Tr}[G_t(0,1)MG_t(0,1)^*]\right) = \log|\mu_J|$$

for $t \in \mathbf{R}$. The same trick can be used to show that the limit is the same for every $(x,1) \in S^*M$ and that the convergence is uniform in x. The same is equally true for the points of the form $(x,-1) \in S^*M$ and thus

$$\nu^+(t \in]T;T+\eta[) = \exp\left(T\frac{\log|\mu_i| + \varepsilon(T)}{2\pi}\right)\nu^+(t \in]0;\eta[)$$

where μ_i is an eigenvalue of $G_{2\pi}(0,1)$ and $\lim_{t\to\infty}\varepsilon(t) = 0$. We can do the same thing for $\nu^-(t \in]T;T+\eta[)$ and by combining ν^+ and ν^- we find

$$\nu(t \in]T;T+\eta[) = \exp\left(T\frac{\log|\mu_j| + \widetilde{\varepsilon}(T)}{2\pi}\right)\nu(t \in]0;\eta[).$$

where μ_j is an eigenvalue of $G_{2\pi}(0,1)$ and $\lim_{t\to\infty}\widetilde{\varepsilon}(t) = 0$. As this is true for every $\eta > 0$ and every $T \geq 0$ we finally get

$$\lim_{k\to\infty} \mathfrak{Im}(\lambda_k) = \frac{1}{\pi}\log|\mu_i|.$$

Since the $\frac{1}{\pi}\log|\mu_i|$ are the Lyapunov exponents of $(G_t)_{t\geq 0}$ the proof of Proposition 3.4 is complete.

Annexe A

Semi-classical anti-Wick quantization

In this appendix we present, without any proof, a construction of the h-Anti-Wick quantization and some of its basic properties. The proofs for $h = 1$ can be found in [Gos11]. We start by constructing it on $\mathbf{R}^d$ for scalar valued symbols.

Définition Let (x, ξ) be a point of $T^*\mathbf{R}^d = \mathbf{R}^{2d}$, we define the function $e_{(x,\xi)} : \mathbf{R}^d \to \mathbf{R}^d$ by

$$e_{(x,\xi)} : y \mapsto \frac{1}{(h\pi)^{d/4}} e^{-\|x-y\|_2^2/2h} e^{iy\cdot\xi/h}.$$

A standard calculation shows that $\|e_{(x,\xi)}\|_{L^2(\mathbf{R}^d)} = 1$. We now define $\Pi_{(x,\xi)} : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ as the orthogonal projector on the vector subspace generated by $e_{(x,\xi)}$.

Définition Let $a \in S^0_{1,0}(\mathbf{R}^{2d})$, we define its h-anti-Wick quantization $\mathrm{Op}_h^{\mathrm{AW}}(a) : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ by

$$\mathrm{Op}_h^{\mathrm{AW}}(a) = \frac{1}{(2\pi h)^d} \int_{\mathbf{R}^{2d}} a(x,\xi) \Pi_{(x,\xi)} dx d\xi.$$

The anti-Wick quantization has a few convenient properties.

Proposition A.1 *If a is real valued and non negative then $\mathrm{Op}_h^{\mathrm{AW}}(a)$ is self-adjoint and positive :*

$$\forall u \in L^2(\mathbf{R}^d),\ \left\langle \mathrm{Op}_h^{\mathrm{AW}}(a)u, u \right\rangle_{L^2(\mathbf{R}^d)} \geq 0.$$

Moreover we have the following estimates

$$\left\| \mathrm{Op}_h^{\mathrm{AW}}(a) \right\|_{L^2(\mathbf{R}^d)} \leq \|a\|_\infty$$

and

$$\mathrm{Op}_h^{\mathrm{AW}}(1) = \mathrm{Id}_{L^2(\mathbf{R}^d)}.$$

The h-anti-Wick quantization is linked to the h-Weyl quantization in the following way.

Proposition A.2 $\mathrm{Op}_h^{\mathrm{AW}}(a)$ *is a* h*-pseudo differential operator of order* ≤ 0 *and we have*

$$\mathrm{Op}_h^{\mathrm{AW}}(a) = \mathrm{Op}_h^{\mathrm{W}}(a * \varepsilon)$$

where $\varepsilon : (y,\eta) \mapsto (h\pi)^{-n/2} e^{-\|(y,\eta)\|_2^2/h}$.

Consequently we know that $\left\|\mathrm{Op}_h^{\mathrm{AW}}(a) - \mathrm{Op}_h^{\mathrm{W}}(a)\right\|_{L^2(\mathbf{R}^d)} = \mathcal{O}(h)$ when h goes to 0. The same construction can be used for symbols a valued in $\mathscr{M}_n(\mathbf{C})$, the operator $\mathrm{Op}_h^{\mathrm{AW}}(a)$ then acts on $L^2(\mathbf{R}^d)^n$. The previous results still hold in this case but Proposition A.1 needs to be slightly modified :

Proposition A.3 *If* a *is valued in* $\mathscr{H}_n^+(\mathbf{C})$ *the space of Hermitian positive semi-definite matrices then* $\mathrm{Op}_h^{\mathrm{AW}}(a)$ *is self-adjoint and positive :*

$$\forall u \in L^2(\mathbf{R}^d)^n,\ \left\langle \mathrm{Op}_h^{\mathrm{AW}}(a)u, u\right\rangle_{L^2(\mathbf{R}^d)^n} \geq 0.$$

Moreover we have the following estimates

$$\left\|\mathrm{Op}_h^{\mathrm{AW}}(a)\right\|_{L^2(\mathbf{R}^d)^n} \leq \sup_{(x,\xi)\in\mathbf{R}^d} \|a(x,\xi)\|_2$$

and

$$\mathrm{Op}_h^{\mathrm{AW}}(\mathrm{Id}_n) = \mathrm{Id}_{L^2(\mathbf{R}^d)^n}.$$

We can then define a h-anti-Wick quantization on a manifold M using a partition of unity.

Annexe B

Multiplicative ergodic theorem of Oseledets

In this appendix we present, without any proofs, the multiplicative ergodic theorem of Oseledets and some related results. The proofs can be found in [Led84] and [BaPe13]. Let (X, μ) be a probability space and let $(\varphi_t)_{t\in\mathbf{R}}$ be a one parameter group of measure preserving functions from X to X. Let $G : \mathbf{R} \times X \to \mathscr{M}_n(\mathbf{C})$ be a cocycle, this means that G satisfies the following conditions :

- $\forall x \in X$, $G(0, x) = \mathrm{Id}_n$,
- $\forall x \in X$, $\forall s, t \in \mathbf{R}$, $G(s + t, x) = G(s, \varphi_t(x))G(t, x)$.

Theorem B.1 (Oseledets) *Assume that* $\log \|G(t, \cdot)\|$ *and* $\log \|G(t, \cdot)^{-1}\|$ *are both in* $L^1(X, \mu)$ *for every* $t \in [0; 1]$. *For* μ*-almost every* $x \in X$ *there exists real numbers* $\lambda_1(x) < \ldots < \lambda_{k(x)}(x)$ *and a decomposition* $\mathbf{C}^n = V_1^x \oplus \ldots \oplus V_{k(x)}^x$ *such that for every* $v \in V_i^x \setminus \{0\}$

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|G(t, x)v\| = \lambda_i(x).$$

Moreover, the λ_i *are invariant by* φ_t *:* $\lambda_i(x) = \lambda_i(\varphi(x))$ *and*

$$G(t, x)V_i^x = V_i^{\varphi_t(x)}.$$

The numbers λ_i are called the Lyapunov exponents of G and $\dim V_i^x$ is called the multiplicity of the Lyapunov exponent $\lambda_i(x)$. If the dynamical system $(X, \mu, (\varphi_t)_{t\in\mathbf{R}})$ is ergodic then the Lyapunov exponents and their multiplicity are constant on a full measure set of X. Note that the choice of the norm over the space $\mathscr{M}_n(\mathbf{C})$ does not matter since they are all equivalent. Let $x \in X$ be a point for which the Lyapunov exponents are well defined, and let $\mu_1 \le \ldots \le \mu_n(x)$ be the Lyapunov exponents counted with multiplicity, then

$$\sum_{j=0}^{i-1} \mu_{n-j}(x) = \lim_{t\to\pm\infty} \frac{1}{t} \log \|\Lambda^i G_t(t, x)\|$$

where $\Lambda^i G$ acts on $\Lambda^i \mathbf{C}^n$ by

$$\Lambda^i G(t,x)(u_1 \wedge \ldots \wedge u_i) = G(t,x)u_1 \wedge \ldots \wedge G(t,x)u_i.$$

In particular, the greatest Lyapunov exponent is given by the norm of G :

$$\mu_n(x) = \lim_{t\to\pm\infty} \frac{1}{t} \log \|G(t,x)\|.$$

If the matrix $G(t,x)$ is invertible for every t then we also have

$$\mu_1(x) = \lim_{t\to\pm\infty} \frac{1}{t} \log \left(\|G(t,x)^{-1}\| \right).$$

Annexe C

A formula for the Lyapunov exponents of G

The aim of this section is to give a more or less explicit expression for the Lyapunov exponents of g in terms of a. From now on and until the end of the section we fix a point (x_0, ξ_0) for which the Lyapunov exponents of $(G_t(x_0; \xi_0))_{t\geq 0}$ are well defined. In order to simplify notations we will write G_t for $G_t(x_0, \xi_0)$, Γ_t for $G_t G_t^*$ and a_t instead of $a(x_t)$.

Let us call $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ the Lyapunov exponents of G_t with multiplicity. The simplest case is the one of λ_n because we have the following formula

$$\lambda_n = \lim_{t\to\infty} \frac{1}{t} \log \|G_t\|.$$

This equality is true for any norm and thus by choosing the operator norm associated to the Euclidean norm on $\mathbf{C}^n$ we get

$$\lambda_n = \lim_{t\to\infty} \frac{1}{2t} \log \|G_t\|_2^2 = \lim_{t\to\infty} \frac{1}{2t} \log \|G_t G_t^*\|_2. \tag{C.1}$$

We then use the following lemma.

Lemma C.1 *Let E be a $\mathbf{C}$-vector space of finite dimension together with an Hermitian form $\langle\cdot,\cdot\rangle_E$ and let $a : t \mapsto a_t$ be a $\mathscr{C}^\infty$ function with values in the space of hermitian matrices of E. If we define $t \mapsto G_t$ as the solution of the differential equation*

$$\begin{cases} G_0 = \mathrm{Id}_n \\ \partial_t G_t = -a_t G_t. \end{cases}$$

we then have

$$\|G_t G_t^*\|_E = \exp\left(-2\int_0^t \langle a_s y_s, y_s\rangle_E \mathrm{d}t\right)$$

where $y_s \in E$ is any unitary vector 1 *satisfying $G_sG_s^*y_s = \|G_sG_s^*\|_E y_s$.*

Proof. The map $t \mapsto \Gamma_t = G_tG_t^*$ is the solution of the differential equation

$$\begin{cases} \Gamma_0 = \mathrm{Id}_n \\ \partial_t\Gamma_t = -a_t\Gamma_t - \Gamma_t a_t, \end{cases} \tag{C.2}$$

It is thus C^∞ and *a fortiori* locally Lipschitz. Consequently the map $t \mapsto \|\Gamma_t\|_E$ is also locally Lipschitz, and so differentiable for almost every t. Since Γ_t is hermitian positive definite $\|\Gamma_t\|_E = \langle\Gamma_t y_t, y_t\rangle_E$ and if z is any unitary vector then $\|\Gamma_t\|_E \geq \langle\Gamma_t z, z\rangle_E$. Let us fix some time t_0, we then have

$$\begin{aligned} \partial_t\langle\Gamma_t y_{t_0}, y_{t_0}\rangle_E|_{t=t_0} &= -\big\langle[a(x_{t_0})\Gamma_{t_0} + \Gamma_{t_0}a(x_{t_0})]y_{t_0}, y_{t_0}\big\rangle_E \\ &= -2\|\Gamma_{t_0}\|_E\langle a(x_{t_0})y_{t_0}, y_{t_0}\rangle_E. \end{aligned}$$

We know that $\langle\Gamma_t y_t, y_t\rangle \geq \langle\Gamma_t y_{t_0}, y_{t_0}\rangle$ for every t and that there is equality for $t = t_0$. If $t \mapsto \|\Gamma_t\|_E$ is differentiable at t_0 we deduce that at this point the derivative of the two functions $t \mapsto \langle\Gamma_t y_t, y_t\rangle_E$ and $t \mapsto \langle\Gamma_t y_{t_0}, y_{t_0}\rangle_E$ must be equal. Consequently for almost every time t

$$\partial_t\|\Gamma_t\|_E = \partial_t\langle\Gamma_t y_t, y_t\rangle_E = -2\|\Gamma_t\|_E\langle a(x_t)y_t, y_t\rangle_E.$$

In order to finish the proof we just have to see that the function

$$\Phi : t \mapsto \frac{\|\Gamma_t\|_E}{\exp\left(-2\int_0^t \big\langle a(x_s)y_s, y_s\big\rangle_E \mathrm{d}s\right)}$$

is locally Lipschitz and thus locally absolutely continuous. Since $\Phi' = 0$ almost everywhere we deduce that Φ is constant and as $\Phi(0) = 1$ the lemma is proved.

□

By applying this lemma to (C.1) we find

$$\lambda_n = \lim_{t\to\infty}\frac{-1}{t}\int_0^t\langle a_s y_s, y_s\rangle \mathrm{d}t, \tag{C.3}$$

where $y_s \in \mathbf{C}^n$ is any unitary vector satisfying $G_sG_s^*y_s = \|G_sG_s^*\|_2 y_s$. By using the following formula

$$\lambda_1 = \lim_{t\to\infty}\frac{-1}{t}\log\|G_t^{-1}\|$$

and adapting the proof of the previous lemma we can show a formula analogous to (C.3) for λ_1.

We now look at what happens for the other Lyapunov exponents. For every time t we fix an orthonormal basis $(y_t^i)_{1\leq i\leq n}$ made of eigenvectors of $\Gamma_t = G_tG_t^*$. Note γ_t^i the i-th smallest eigenvalue of Γ_t, we add the condition that $\Gamma_t y_t^i = \gamma_t^i y_t^i$. We can still prove (see [Kat82] Theorem 6.8) that the functions $t \mapsto \gamma_t^i$ are all Lipschitz and thus differentiable almost everywhere. However the inequality $\langle \Gamma_t y_{t_0}^i, y_{t_0}^i \rangle \leq \langle \Gamma_t y_t^i, y_t^i \rangle$ has no reason to be true any more and we need to find another proof. We will in fact put ourself in the setting of Lemma C.1 by use of the exterior product.

In order to ease notations we will only write the proof for λ_{n-1}, the other Lyapunov exponents are treated in the exact same way. We start by defining an hermitian form on $\Lambda^2(\mathbf{C}^n)$ as the unique hermitian form on $\Lambda^2(\mathbf{C}^n)$ satisfying

$$\forall u_1, u_2, v_1, v_2 \in \mathbf{C}^n,\ \langle u_1 \wedge u_2, v_1 \wedge v_2 \rangle_{\Lambda^2(\mathbf{C}^n)} = \det\big((\langle u_i, v_j \rangle_{\mathbf{C}^n})_{i,j} \big).$$

Note that if $(e_i)_{1\leq i\leq n}$ is an orthonormal basis of $\mathbf{C}^n$ then $(e_i \wedge e_j)_{1\leq i<j\leq n}$ is an orthonormal basis of $\Lambda^2(\mathbf{C}^n)$ for the hermitian form previously defined. The Hermitian form on $\Lambda^2(\mathbf{C}^n)$ induces a norm that we will denote by $\|\cdot\|_{\Lambda^2(\mathbf{C}^n)}$, we will also denote by $\|\cdot\|_{\Lambda^2(\mathbf{C}^n)}$ the associated operator norm. We now define $\hat{G}_t$ as the endomorphism of $\Lambda^2(\mathbf{C}^n)$ satisfying

$$\forall u, v \in \mathbf{C}^n,\ \hat{G}_t(u \wedge v) = (G_t u) \wedge (G_t v).$$

We can easily show that if $A, B \in \mathscr{M}_n(\mathbf{C})$ then $\hat{A}\hat{B} = \widehat{AB}$ and $\left(\hat{A}\right)^* = \widehat{A^*}$, we thus have $\hat{\Gamma}_t = \hat{G}_t \hat{G}_t^*$. In the same way let us define $\check{a}_t$ as the endomorphism of $\Lambda^2(\mathbf{C}^n)$ satisfying

$$\forall u, v \in \mathbf{C}^n,\ \check{a}_t(u \wedge v) = a_t u \wedge v + u \wedge a_t v.$$

We can now start to search a formula for λ_2. The first thing to observe is that

$$\lambda_n + \lambda_{n-1} = \lim_{t\to\infty} \frac{1}{t} \log \|\hat{G}_t\|_{\Lambda^2(\mathbf{C}^n)} = \lim_{t\to\infty} \frac{1}{2t} \log \|\hat{G}_t \hat{G}_t^*\|_{\Lambda^2(\mathbf{C}^n)}.$$

A proof for this result can be found in [Led84]. Remark that for every $1 \leq i < j \leq n$ we have

$$\partial_t \hat{G}_t(e_i \wedge e_j) = -\check{a}_t \hat{G}_t(e_i \wedge e_j),$$

we deduce that $t \mapsto \hat{G}_t$ is the unique solution of the differential equation

$$\begin{cases} \hat{G}_0 = \mathrm{Id}_n \\ \partial_t \hat{G}_t = -\check{a}_t \hat{G}_t. \end{cases}$$

Take $(e_i)_{1\leq i\leq n}$ an orthonormal basis of $\mathbf{C}^n$ made of eigenvectors of a_t, we see that $(e_i \wedge e_j)_{1\leq i<j\leq n}$ is an orthonormal basis of $\Lambda^2(\mathbf{C}^n)$ made of eigenvectors of $\check{a}_t$ and the eigenvalues of $\check{a}_t$ are the sums of two eigenvalues of a_t. This shows that $\check{a}_t$ is an hermitian matrix acting on $\Lambda^2(\mathbf{C}^n)$ and that we are indeed in the setting of the previous lemma. Thus, if $\hat{y}_s$ is an eigenvector of $\hat{G}_s \hat{G}_s^* = \hat{\Gamma}_s$ associated with its greatest

eigenvalue and that $\hat{y}_s$ is unitary we have

$$\|\hat{G}_t\hat{G}_t^*\|_{\Lambda^2(\mathbf{C}^n)} = \exp\left(-2\int_0^t \langle \check{a}_s\hat{y}_s, \hat{y}_s\rangle_{\Lambda^2(\mathbf{C}^n)}\mathrm{d}s\right). \tag{C.4}$$

Recall that we chose an orthonormal basis $(y_s^i)_{1\leq i\leq n}$ of eigenvectors of Γ_s associated with the eigenvalues of Γ_s in increasing order. As for $\check{a}_s$ the family of vectors $(y_s^i \wedge y_s^j)_{1\leq i<j\leq n}$ form an orthonormal basis of eigenvectors of $\hat{\Gamma}_s$ associated with the eigenvalues $\gamma_s^i\gamma_s^j$. Notice that we can take $\hat{y}_s = y_s^n \wedge y_s^{n-1}$ in (C.4) and by developing the hermitian product we get

$$\|\hat{G}_t\|_{\Lambda^2(\mathbf{C}^n)} = \exp\left(-\int_0^t \langle a_s y_s^n, y_s^n\rangle_{C^n} + \langle a_s y_s^{n-1}, y_s^{n-1}\rangle_{C^n}\mathrm{d}s\right).$$

Since we had

$$\lambda_n + \lambda_{n-1} = \lim_{t\to\infty}\frac{1}{t}\log\|\hat{G}_t\|_{\Lambda^2(\mathbf{C}^n)}$$

and

$$\lambda_n = \lim_{t\to\infty}\frac{-1}{t}\int_0^t \langle a_s y_s^n, y_s^n\rangle_{C^n}\mathrm{d}s$$

we deduce that

$$\lambda_{n-1} = \lim_{t\to\infty}\frac{-1}{t}\int_0^t \langle a_s y_s^{n-1}, y_s^{n-1}\rangle_{C^n}\mathrm{d}s.$$

If we apply the same reasoning for every integer $1 \leq i \leq n$ by letting G_t act on $\Lambda^i(\mathbf{C}^n)$ by

$$\forall u_1, \ldots, u_i \in \mathbf{C}^n,\ \hat{G}_t(u_1 \wedge \ldots \wedge u_i) = (G_t u_1) \wedge \ldots \wedge (G_t u_i)$$

we can prove the following proposition.

Proposition C.1 *For every integer $1 \leq i \leq n$ we have*

$$\lambda_i = \lim_{t\to\infty}\frac{-1}{t}\int_0^t \langle a_s y_s^i, y_s^i\rangle_{C^n}\mathrm{d}s \tag{C.5}$$

where $(y_s^i)_{1\leq i\leq n}$ is an orthonormal basis of eigenvectors of Γ_s associated with the eigenvalues $\gamma_s^1 \leq \ldots \leq \gamma_s^n$ of Γ_s.

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Dans cette thèse nous considérons l'équation des ondes amorties vectorielle sur une variété riemannienne (M, g) compacte, C^∞ et sans bord. Il s'agit de l'équation aux dérivées partielles $(\partial_t^2 - \Delta_g + a\partial_t)u = 0$ où Δ_g est l'opérateur de Laplace Beltrami sur M et $a : M \to H_n$ est une fonction C^∞ à valeurs dans l'espace des matrices hermitiennes de taille n. Les solutions de cette équation sont donc des fonctions de $\mathbf{R} \times M$ à valeurs dans $\mathbf{C}^n$. Nous commençons dans un premier temps par calculer le meilleur taux de décroissance exponentiel de l'énergie en fonction du terme d'amortissement a. Ce résultat nous permet d'en déduire une condition nécessaire et suffisante pour la stabilisation forte de l'équation des ondes amorties vectorielle. Nous mettons aussi en évidence l'apparition d'un phénomène de sur-amortissement haute fréquence qui n'existait pas pour l'équation des ondes amorties scalaires. Dans un second temps nous nous intéressons à la répartitions des fréquences propres de l'équation des ondes amorties vectorielle. Nous démontrons que, à un sous ensemble de densité 0 près, l'ensemble des fréquences propres est contenu dans une bande parallèle à l'axe imaginaire. La largeur de cette bande est déterminée par les exposants de Lyapunov d'un système dynamique définit à partir du coefficient d'amortissement a.

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IRMA 2018/011
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ISSN 0755-3390