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Foncteurs de Long-Moody et homologie stable des groupes de difféotopie

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Introduction

Le groupe de difféotopie d'une variété est un groupe discret associé à cet espace et en définit un invariant fondamental. Dans le cas d'un disque dont n points ont été retirés, il correspond au groupe de tresses sur n brins, noté $\mathbf{B}_n$. Ce groupe, implicitement étudié par Hurwitz dans [Hur91] puis explicitement introduit par Artin dans [Art25], a des liens profonds avec la théorie des nœuds (voir [Bir74a, Jon85, KT08]), la géométrie algébrique et la théorie des groupes finis (voir [Arn70, Bri70]). L'étude des représentations linéaires des groupes de tresses constitue ainsi un thème de recherche très riche, intéragissant avec de nombreux domaines des mathématiques. On renvoie à [BB05] pour une présentation plus détaillée de ce sujet.

Il n'y a dans la littérature que peu d'exemples de représentations linéaires des groupes de tresses ne factorisant pas par les groupes symétriques. Parmi ces rares exemples se trouvent les représentations de Burau [Bur35] et de Lawrence-Krammer [Law90b]. Krammer utilise ces dernières représentations dans [Kra02] pour démontrer la linéarité des groupes de tresses sur $n \geq 5$ brins. Des représentations analogues sont considérées par Bigelow dans [Big01] pour démontrer ce résultat. Enfin, Tong, Yang et Ma ont exhibé dans [TYM96] une famille de représentations simples des groupes de tresses, d'une forme analogue à celles de Burau. De plus, l'état actuel des connaissances ne permet pas d'en établir une classification. Il serait donc intéressant de parvenir à une meilleure compréhension de ces représentations et de s'interroger sur une façon de les relier.

Dans cette optique, le travail découlant d'échanges avec Moody mené par Long en 1994 dans [Lon94] s'avère fécond. En effet, considérant une représentation $\rho : \mathbf{B}_{n+1} \to GL(V)$, Long introduit une construction nouvelle définissant une représentation $\mathcal{LM}(\rho) : \mathbf{B}_n \to GL(V^{\oplus n})$, plus complexe que la représentation initiale ρ. Cette construction sera dite de Long-Moody. Par exemple, en l'appliquant à une représentation de dimension un, on obtient la représentation de Burau (non-réduite). Il est à noter que le principe de cette construction était implicitement déjà présent dans les articles [Lon89a] et [Lon89b] antérieurs de Long. Une interprétation purement matricielle en est également donnée par Bigelow et Tian en 2008 dans [BT08] : ils donnent des démonstrations analogues des résultats de [Lon94] et en étendent certains. Enfin, Birman et Brendle mentionnent la construction de Long-Moody dans leur article de survol sur les tresses [BB05]. Elles insistent en particulier sur l'intérêt qu'en représenterait une étude plus approfondie et posent un problème ouvert (voir [BB05, Open Problem 7]) la concernant, que nous reformulerons de la manière suivante : serait-il possible d'obtenir toutes les représentations unitaires de dimension finie des groupes de tresses par de légères modifications de la construction de Long-Moody ?

De plus, généraliser cette construction pour d'autres familles de groupes s'avérerait fort utile. En effet, l'étude des représentations linéaires des groupes de difféotopie des surfaces ou des variétés de dimension 3 par exemple constitue un sujet de recherche très actif et ces représentations demeurent mal connues. On pourra se référer à [BB05, Section 4.6], [Fun99], [Kor02] ou [Mas08] pour un exposé plus approfondi de ce sujet.

Par ailleurs, Randal-Williams et Wahl démontrent dans [RWW17] la stabilité homologique avec certains types de coefficients tordus pour une grande variété de familles de groupes, parmi lesquels les groupes de difféotopie des surfaces et de 3-variétés. Plus précisément, la stabilité pour ces derniers est étudiée par rapport au nombre de points marqués ou au genre orientable ou non-orientable des surfaces. Ce travail permet en particulier de retrouver les résultats de stabilité à coefficients tordus par rapport au genre orientable d'Ivanov dans [Iva93], généralisés par Cohen et Madsen dans [CM09] puis Boldsen dans [Bol12]. Pour les 3-variétés, la stabilité des groupes de difféotopie est démontrée par rapport à l'itération de la somme connexe sur le bord d'une 3-variété compacte connexe orientée avec une composante de bord.

Les coefficients tordus considérés sont donnés par des foncteurs ayant pour source une catégorie obtenue en appliquant au groupoïde associé à la famille de groupes considérée une construction due à Quillen consistant à ajouter des morphismes au groupoïde (voir [Gra76, p. 219] et ci-dessous pour plus de détails). Ces foncteurs doivent vérifier des conditions de polynomialité et sont alors appelés systèmes de coefficients de degré fini. La notion de polynomialité à laquelle il est fait référence ici a d'abord été introduite par Eilenberg et Mac Lane pour des foncteurs sur des catégories de modules en utilisant la notion d'effets croisés dans [EML54]. Elle a été étendue par Djament et Vespa [DV17] pour des foncteurs ayant pour source une catégorie monoïdale symétrique stricte telle que l'objet unité est objet initial, et ayant pour but une catégorie abélienne. On parle alors de foncteurs fortement polynomiaux. Dans le cas des groupes de difféotopie des surfaces, les catégories associées ne sont pas monoïdales symétriques, mais monoïdales pré-tressées, notion ad hoc introduite par Randal-Williams et Wahl dans [RWW17]. La définition de foncteur fortement polynomial s'adapte à ce type de catégories et sera traitéef dans le premier chapitre de cette thèse, et est reliée à la notion de système de coefficients de [RWW17, Section 4].

La littérature ne présente que peu d'exemples de systèmes de coefficients de degré fini quelconque pour les groupes de tresses et plus généralement les groupes de difféotopie des surfaces. Dans le cas des groupe de tresses, Randal-Williams et Wahl démontrent que les représentations de Burau non-réduites forment un système de coefficients de degré un. Par conséquent, on souhaiterait une manière systématique de construire des systèmes de coefficients de degré fini pour ces familles de groupes. A cette fin, la construction de Long-Moody s'avère particulièrement intéressante. En effet, les représentations de Burau non-réduites se retrouvent via la construction de Long-Moody à partir des représentations de dimension un, ces dernières formant un système de coefficients de degré zéro.

Le premier objectif de cette thèse est de fonctorialiser la construction de Long-Moody dans le sens où elle permet de définir un nouveau système de coefficients de degré fini à partir de n'importe quel système de coefficients de degré fini. Après avoir introduit des variantes de cette construction pour les groupes de tresses, on la généralise pour d'autres familles de groupes. Enfin, l'étude de l'effet des foncteurs ainsi définis sur le degré de polynomialité fournit le résultat principal de cette thèse. Ces travaux font l'objet des chapitres 1 et 2.

Une des motivations principales de ce travail est le calcul de l'homologie stable à coefficients tordus des groupes de difféotopie des surfaces. En effet, très peu de résultats de ce type sont connus. Pour les groupes de tresses, l'homologie à coefficients dans l'anneau des polynômes de Laurent $\mathbb{Z}\left[t^{\pm 1}\right]$ est calculée par Callegaro dans [Cal06] (généralisant les travaux précédents de De Concini, Procesi et Salvetti dans [DCPS01]), celle à coefficients dans la représentation de Tong Yang et Ma complexe est obtenue par Callegaro, Moroni et Salvetti dans [CMS08] et celle à coefficients dans la représentation de Burau réduite complexe est calculée par Chen dans [Che17]. Il est à noter que le calcul de l'homologie stable des groupes de tresses à coefficients dans n'importe quelle puissance tensorielle de la représentation de Burau réduite a des applications en arithmétique. En effet, Chen explique dans [Che17, Section 4] que ce calcul s'avérerait utile pour l'étude de la distribution de $\mathbb{F}_q$-points sur une courbe hyperelliptique. Pour les groupes de difféotopie des surfaces orientables avec une composante de bord, l'homologie stable par rapport au genre à coefficient dans le premier groupe d'homologie de la surface considérée est obtenue par Harer dans [Har91]. Ce résultat est généralisé en cohomologie par Kawazumi pour toute puissance tensorielle du premier groupe de la surface dans [Kaw08]. Enfin, le premier groupe d'homologie stable des groupes de difféotopie des surfaces non-orientables avec une composante de bord et à coefficients dans le premier groupe d'homologie des surfaces considérées est déterminé par Stukow dans [Stu14].

On souhaite ainsi comparer l'homologie stable à coefficients donnés par un foncteur très fortement polynomial F à celle à coefficients donnés par l'image de F par un foncteur de Long-Moody. Cette question en toute généralité s'avère difficile. Néanmoins, dans le chapitre 3, on répond partiellement à ce problème. En associant cette comparaison à des résultats de stabilité homologique à coefficients tordus de Boldsen dans [Bol12], Cohen et Madsen dans [CM09] et Hanbury dans [Han09], on obtient alors des résultats d'homologie stable à coefficients tordus pour des groupes de difféotopie de surfaces orientables et non-orientables ou pour les holomorphes des groupes libres. En particulier, on retrouve ainsi par d'autres méthodes le résultat en cohomologie de [Kaw08] dû à Kawazumi et on généralise les résultats de Jensen de [Jen04].

A la fin des années 1990, il s'est avéré que l'homologie des foncteurs est particulièrement efficace pour calculer l'homologie stable de familles de groupes à coefficients tordus. Ainsi, les travaux de Betley dans [Bet92] et [Bet99], Scorichenko dans [Sco00] et Suslin dans l'appendice de [FFSS99] pour les groupes linéaires montrent que l'homologie stable des groupes linéaires à coefficients tordus est calculée grâce à l'homologie des foncteurs. De plus, des résultats analogues sont obtenus pour les groupes symétriques par Betley dans [Bet02]. Par ailleurs, dans le cadre de familles de groupes associées à des catégories monoïdales symétriques, Djament et Vespa ont développé dans [DV10] un cadre général dans lequel les groupes d'homologie stable à coefficients tordus sont calculés par de l'homologie des foncteurs : l'homologie stable à coefficients tordus s'exprime en fonction de l'homologie stable à coefficients constants et l'homologie de la catégorie source des foncteurs associés aux coefficients tordus. Ce cadre a notamment permis de calculer de l'homologie stable à coefficients tordus des groupes orthogonaux et symplectiques dans [DV10] et [Dja12] et des groupes d'automorphismes des groupes libres dans [DV15] et [Dja15].

L'extension à des familles de groupes associées à des catégories monoïdales pré-tressées du résultat de décomposition de l'homologie stable à coefficients tordus de [DV10] est effectuée au chapitre 3 du présent manuscrit. Cela permet d'établir des résultats d'homologie stable pour des groupes de difféotopie des surfaces ou des groupes d'automorphismes de certains groupes d'Artin à angles droits, à coefficients dans des FI-modules : cette catégorie de foncteurs associée aux groupes symétriques est abondamment étudiée dans la littérature suite aux travaux de Church, Ellenberg et Farb dans [CEF15].

Cette thèse se décompose ainsi en trois parties (rédigées en anglais) dont nous allons donner les résumés détaillés (en français) :

1. la prépublication *The Long-Moody construction and polynomial functors* [Sou17b] ;
2. la prépublication *Generalised Long-Moody functors* [Sou17a] ;
3. le chapitre *On computations of homology with twisted coefficients for mapping class groups [Sou18]* qui donnera lieu à une prépublication.

0.1 Constructions de Long-Moody et foncteurs polynomiaux

Le chapitre 1 de cette thèse porte sur l'étude d'un point de vue fonctoriel et la généralisation de la construction de Long-Moody introduite dans [Lon94] pour les groupes de tresses.

Tout d'abord, on adopte un point de vue issu de la théorie des catégories pour étudier cette construction. On peut considérer les groupes de tresses $\{\mathbf{B}_n\}_{n\in\mathbb{N}}$ comme étant les groupes d'automorphismes d'un groupoïde $\boldsymbol{\beta}$, appelé groupoïde des tresses, ayant pour objets les entiers naturels. L'addition sur les entiers naturels induit une structure monoïdale tressée $\natural$ sur $\boldsymbol{\beta}$ (voir [ML13, Chapter XI] pour plus de détails), ce qui signifie qu'il existe une opération associative $\mathbf{B}_m\natural\mathbf{B}_n \to \mathbf{B}_{m+n}$ compatible avec la composition pour tout $n, m \in \mathbb{N}$. On note $b^{\boldsymbol{\beta}}_{-,-}$ le tressage associé qui permet de commuter les deux termes sur lesquels s'applique l'opération $\natural$. La construction de Quillen $\mathfrak{U}$ (voir [Gra76, p. 219]) permet alors de former une nouvelle catégorie $\mathfrak{U}\boldsymbol{\beta}$ ayant les mêmes objets que $\boldsymbol{\beta}$, où les groupes d'automorphismes sont également les groupes de tresses, mais qui possède des morphismes d'un objet n vers un objet m pour $n \leq m$ contrairement à $\boldsymbol{\beta}$. Plus précisément, l'ensemble des morphismes entre de tels objets n et m est un quotient (ensembliste) $\mathbf{B}_m/\mathbf{B}_{m-n}$ si $m \geq n$ et l'ensemble vide sinon. Un morphisme de n vers m est ainsi noté par une classe d'équivalence $[m-n, \sigma]$ où $\sigma \in \mathbf{B}_m$. La structure monoïdale du groupoïde des tresses s'étend à la catégorie $\mathfrak{U}\boldsymbol{\beta}$. Cette structure n'est pas tressée mais elle satisfait des conditions plus faibles qui en font ainsi une structure monoïdale pré-tressée au sens de Randal-Williams et Wahl dans [RWW17, Section 1]. Les résultats de stabilité homologique à coefficients tordus pour les groupes de tresses de [RWW17, Section 5] sont établis pour un certain type de foncteurs ayant cette catégorie $\mathfrak{U}\boldsymbol{\beta}$ pour source. On note $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ la catégorie des foncteurs de $\mathfrak{U}\boldsymbol{\beta}$ vers la catégorie $\mathbb{K}\text{-}\mathfrak{Mod}$ des $\mathbb{K}$-modules pour $\mathbb{K}$ un anneau commutatif. Un objet de $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ est ainsi la donnée d'une famille de représentations linéaires des groupes de tresses avec, pour tout entier naturel n, une relation de compatibilité de passage de la représentation de $\mathbf{B}_n$ à celle de $\mathbf{B}_{n+1}$ liée aux morphismes de la catégorie $\mathfrak{U}\boldsymbol{\beta}$. Un premier exemple d'objet non-trivial de $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ (pour $\mathbb{K} = \mathbb{C}\left[t^{\pm 1}\right]$) est construit à partir des représentations de Burau non-réduites dans [RWW17, Example 4.3] et est noté $\mathfrak{Bur}_t$. Cette famille de représentations est dite typique des groupes de tresses dans le sens où elle ne provient pas de représentations des groupes symétriques.

On démontre que les représentations de Burau réduites, les représentations de Tong, Yang et Ma et les représentations de Lawrence-Krammer forment également des objets non-triviaux de $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, respectivement notés $\overline{\mathfrak{Bur}}_t$, $\mathfrak{TYM}_t$ et $\mathfrak{LK}$ (voir la section 1.1.2).

La première étape de ce chapitre consiste alors à s'inspirer de la construction de Long-Moody introduite dans [Lon94] afin de construire des endofoncteurs de $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, produisant ainsi de nouveaux objets de cette catégorie.

On note $\mathbf{F}_n$ le groupe libre à $n \in \mathbb{N}$ éléments, $*$ le produit libre de groupes, et $\iota_{\mathbf{F}_n}$ l'unique morphisme $0_{\mathfrak{Gr}} \to \mathbf{F}_n$ (où $0_{\mathfrak{Gr}}$ désigne le groupe trivial). L'idéal d'augmentation de l'anneau du groupe libre à n éléments est noté $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$. On considère des familles de morphismes $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ et $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$. Trois propriétés dites de cohérence sur ces familles de morphismes ont alors été exhibées (on renvoie le lecteur à la section 1.2.1 pour le détail de ces conditions, les familles $\{a_n\}_{n\in\mathbb{N}}$ et $\{\varsigma_n\}_{n\in\mathbb{N}}$ étant alors dites cohérentes) afin de constituer le cadre nécessaire et suffisant au théorème suivant.

Théorème *A* **(Theorem 1.2.20).** *Soit $\mathbb{K}$ un anneau commutatif. Les données suivantes définissent un foncteur exact*

$$\mathbf{LM}_{a,\varsigma} : \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}),$$

appelé foncteur de Long-Moody par rapport aux familles cohérentes de morphismes $\{a_n\}_{n\in\mathbb{N}}$ et $\{\varsigma_n\}_{n\in\mathbb{N}}$, défini pour $F \in Obj(\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}))$ par :

$$\mathbf{LM}_{a,\varsigma}(F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+1),$$

pour tout entier naturel n.

Les morphismes $\{\varsigma_n\}_{n\in\mathbb{N}}$ permettent de définir ce produit tensoriel utilisé pour définir le foncteur de Long-Moody. Celui-ci est défini sur les morphismes grâce aux morphismes $\{a_n\}_{n\in\mathbb{N}}$.

Un premier exemple consiste à considérer les représentations d'Artin des groupes de tresses notées $\{a_{n,1}\}_{n\in\mathbb{N}}$ définies pour $n \geq 1$ pour tout générateur d'Artin σ_i de $\mathbf{B}_n$ par

$$\begin{array}{rcl} a_{n,1}(\sigma_i) : \mathbf{F}_n & \longrightarrow & \mathbf{F}_n \\ g_j & \longmapsto & \begin{cases} g_{i+1} & \text{si } j = i \\ g_{i+1}^{-1} g_i g_{i+1} & \text{si } j = i+1 \\ g_j & \text{si } j \notin \{i, i+1\}. \end{cases} \end{array}$$

et les morphismes $\{\varsigma_{n,1}\}_{n\in\mathbb{N}}$ définis pour $n \geq 1$ par

$$\begin{array}{rcl} \varsigma_{n,1} : \mathbf{F}_n & \longrightarrow & \mathbf{B}_{n+1} \\ g_i & \longmapsto & \begin{cases} \sigma_1^2 & \text{si } i = 1, \\ \sigma_i \circ \varsigma_{n,1}(g_{i-1}) \circ \sigma_i^{-1} & \text{si } i \in \{2, \ldots, n\}. \end{cases} \end{array}$$

Ces familles de morphismes sont cohérentes et le foncteur $\mathbf{LM}_{a_1,\varsigma_1}$ correspond à la construction originale de Long $\mathcal{LM}$ [Lon94]. Plus précisément, en notant $F_{|n+1}$ la restriction d'un objet F de $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ à la sous-catégorie pleine de $\mathfrak{U}\boldsymbol{\beta}$ ayant pour seul objet $n+1$, on a

$$(\mathbf{LM}_{a_1,\varsigma_1}(F))_{|n+1} = \mathcal{LM}\left(F_{|n+1}\right).$$

Le foncteur de Burau non-réduit $\mathfrak{Bur}_t$ est ainsi équivalent au foncteur obtenu en appliquant $\mathbf{LM}_{a_1,\varsigma_1}$ sur un foncteur constant et le foncteur de Lawrence-Krammer $\mathfrak{LK}$ est un sous-foncteur de celui obtenu en appliquant $\mathbf{LM}_{a_1,\varsigma_1}$ sur $\mathfrak{Bur}_t$. Par ailleurs, en utilisant d'autres familles de morphismes $\{a_n\}_{n\in\mathbb{N}}$ (en considérant par exemple les représentations dites de Wada voir [Wad92, Ito13], dont les représentations d'Artin sont des cas particuliers), on peut obtenir d'autres familles de représentations des groupes de tresses : c'est le cas du foncteur de Tong-Yang-Ma $\mathfrak{TYM}_t$ qui s'obtient en appliquant sur un foncteur constant un foncteur de Long-Moody associé à la famille de morphismes $\{a_{n,3}\}_{n\in\mathbb{N}}$ définis pour tout générateur d'Artin σ_i de $\mathbf{B}_n$ par

$$\begin{array}{rcl} a_{n,3}(\sigma_i) : \mathbf{F}_n & \longrightarrow & \mathbf{F}_n \\ g_j & \longmapsto & \begin{cases} g_{i+1} & \text{si } j = i \\ g_i^{-1} & \text{si } j = i+1 \\ g_j & \text{si } j \notin \{i, i+1\}. \end{cases} \end{array}$$

Dans ce premier chapitre, on étend également la notion de forte polynomialité de [DV17] aux foncteurs ayant une catégorie monoïdale pré-tressée pour source et l'unité de la structure monoïdale comme objet initial. Plus précisément, pour $(\mathfrak{M}, \natural, 0)$ une petite catégorie monoïdale stricte pré-tressée et x un objet de $\mathfrak{M}$, on note τ_x l'endofoncteur de $\mathbf{Fct}(\mathfrak{M}, \mathbb{K}\text{-}\mathfrak{Mod})$ (dit de translation) obtenu par précomposition par le foncteur $x\natural-$. Puisque l'objet 0 est initial dans $\mathfrak{M}$, on peut former une transformation naturelle $i_x : Id \to \tau_x$. En considérant le conoyau (respectivement noyau) de i_x noté δ_x (respectivement κ_x), on obtient la suite exacte courte d'endofoncteurs of $\mathbf{Fct}(\mathfrak{M}, \mathbb{K}\text{-}\mathfrak{Mod})$:

$$0 \longrightarrow \kappa_x \longrightarrow Id \longrightarrow \tau_x \longrightarrow \delta_x \longrightarrow 0.$$

Un objet F de $\mathbf{Fct}(\mathfrak{M}, \mathbb{K}\text{-}\mathfrak{Mod})$ est dit fortement polynomial de degré inférieur ou égal à $d \in \mathbb{N}$ si $\delta_x^{d+1}(F) = 0$ pour tout objet x de $\mathfrak{M}$. On note $\mathcal{P}ol_d^{fort}(\mathfrak{M}, \mathbb{K}\text{-}\mathfrak{Mod})$ la sous-catégorie de $\mathbf{Fct}(\mathfrak{M}, \mathbb{K}\text{-}\mathfrak{Mod})$ formée de ces objets. Par exemple, les foncteurs fortement polynomiaux de degré 0 sont les quotients des foncteurs constants et un foncteur strictement monoïdal par rapport à $\natural$ est fortement polynomial de degré 1.

Les foncteurs fortement polynomiaux pour lesquels Randal-Williams et Wahl démontrent la stabilité homologique à coefficients tordus dans [RWW17, Section 5] vérifient des propriétés plus fortes. On baptise de tels objets des foncteurs très fortement polynomiaux. Plus précisément, un objet F de $\mathcal{P}ol_d^{fort}(\mathfrak{M}, \mathbb{K}\text{-}\mathfrak{Mod})$ est dit très fortement polynomial de degré inférieur ou égal à $d \in \mathbb{N}$ si $\kappa_x(F) = 0$ et $\delta_x(F)$ est très fortement polynomial de degré inférieur ou égal à $d-1$, pour tout objet x de $\mathfrak{M}$. Il est ainsi pertinent d'étudier le comportement des foncteurs de Long-Moody sur la (très) forte polynomialité des objets de $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Il est démontré dans [RWW17, Example 4.15] que le foncteur $\mathfrak{Bur}_t$ est très fortement polynomial de degré 1. On démontre :

Proposition. *Le foncteur $\overline{\mathfrak{Bur}_t}$ est fortement polynomial de degré 2, le foncteur $\mathfrak{TYM}_t$ est très fortement polynomial de degré 1 et le foncteur $\mathfrak{LK}$ est très fortement polynomial de degré 2.*

Sous deux légères hypothèses supplémentaires sur la famille de morphismes $\{a_n\}_{n\in\mathbb{N}}$ (on renvoie le lecteur à la section 1.4.1.1 pour le détail de ces conditions), les familles de morphismes $\{a_n\}_{n\in\mathbb{N}}$ et $\{\varsigma_n\}_{n\in\mathbb{N}}$ sont alors dites fiables. Par exemple, les familles $\{a_{n,1}\}_{n\in\mathbb{N}}$ et $\{\varsigma_{n,1}\}_{n\in\mathbb{N}}$ sont fiables. Il s'ensuit alors le résultat suivant :

Théorème *B* (Corollary 1.4.26 et Theorem 1.4.28). *Pour $\{a_n\}_{n\in\mathbb{N}}$ et $\{\varsigma_n\}_{n\in\mathbb{N}}$ des familles fiables de morphismes, le foncteur de Long-Moody associé $\mathbf{LM}_{a,\varsigma}$ induit un foncteur*

$$\mathcal{P}ol_d^{fort}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathcal{P}ol_{d+1}^{fort}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$$

pour tout entier naturel d. De plus, si $F : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ est un foncteur très fortement polynomial de degré d, $\mathbf{LM}(F)$ est très fortement polynomial de degré $d+1$.

La clef de la démonstration de ce théorème est donnée par les résultats techniques suivants. Un foncteur de Long-Moody $\mathbf{LM}_{a,\varsigma}$ associé à des familles fiables de morphismes $\{a_n\}_{n\in\mathbb{N}}$ et $\{\varsigma_n\}_{n\in\mathbb{N}}$ admet la décomposition suivante par rapport au foncteur de translation :

$$\tau_1 \circ \mathbf{LM}_{a,\varsigma} \cong \tau_2 \oplus (\mathbf{LM}_{a,\varsigma} \circ \tau_1).$$

Il en découle alors les équivalences $\delta_1 \circ \mathbf{LM}_{a,\varsigma} \cong \tau_2 \oplus (\mathbf{LM}_{a,\varsigma} \circ \delta_1)$ et $\kappa_1 \circ \mathbf{LM}_{a,\varsigma} \cong \mathbf{LM}_{a,\varsigma} \circ \kappa_1$.

De cette manière, les foncteurs de Long-Moody permettent de produire de nouvelles familles de représentations linéaires des groupes de tresses, formant des foncteurs (très) fortement polynomiaux en n'importe quel degré. Cela fournit ainsi une grande variété d'exemples de coefficients tordus pour lesquelles le résultat de stabilité homologique de [RWW17, Section 5] est vérifié.

Enfin, on montre qu'il n'est pas possible d'obtenir le foncteur de Tong-Yang-Ma $\mathfrak{TYM}_t$ à partir de l'endofoncteur $\mathbf{LM}_{a_1,\varsigma_1}$ associé à la construction originale de Long-Moody. Cela indique qu'il faut au moins considérer les constructions associées à divers morphismes $\{a_n\}_{n\in\mathbb{N}}$ afin de pouvoir répondre positivement au problème soulevé par Birman et Brendle dans [BB05, Open Problem 7].

0.2 Foncteurs de Long-Moody généralisés

Dans le chapitre 2, le travail du chapitre 1 est étendu à d'autres familles de groupes. En effet, les résultats de stabilité homologique à coefficients tordus de [RWW17] sont valables dans un cadre beaucoup plus général que celui des groupes de tresses. Ce cadre inclut notamment les groupes de difféotopie de surfaces de genre (orientable ou non-orientable) non-nul. Ne disposant pas d'une classification des représentations linéaires de ces familles, trouver des foncteurs typiques de ces familles de groupes vérifiant des conditions de polynomialité du type de celles décrites dans la section précédente n'est pas aisé. La construction de diverses familles de représentations linéaires typiques des groupes de tresses dans le chapitre incite donc à généraliser la définition de ce type de foncteurs pour d'autres familles de groupes.

Exposons tout d'abord ce nouveau cadre de travail, s'inspirant de l'exemple qui suit. On note $\Gamma^s_{g,c,1}$ le groupe de difféotopie d'une surface $\Sigma^s_{g,c,1}$ compacte connexe de genre orientable g, de genre non-orientable c, ayant une composante de bord et dont on a retiré s points à l'intérieur, et par $\natural$ la somme connexe sur le bord des surfaces. On définit le groupoïde des surfaces décorées $\mathcal{M}_2$ ayant pour objets les surfaces $\left\{\Sigma^s_{g,c,1}\right\}_{g,c,s\in\mathbb{N}}$ et les groupes de difféotopie $\left\{\Gamma^s_{g,c,1}\right\}_{g,c,s\in\mathbb{N}}$ pour groupes d'automorphismes. Il est muni d'une structure monoïdale tressée $\left(\mathcal{M}_2, \natural, \Sigma^0_{0,0,1}\right)$ grâce à la somme connexe sur le bord et dont le disque $\Sigma^0_{0,0,1}$ est l'unité (on renvoie le lecteur à [RWW17, Section 5.6] ou à la section 2.1 pour plus de détails). Ainsi, $\Sigma^s_{g+1,c,1}$ étant homéomorphe à $\Sigma^1_{1,0,1}\natural\Sigma^s_{g,c,1}$, on a des inclusions canoniques

$$\left[\Sigma^0_{1,0,1}, id_{\Sigma^s_{g,c,1}}\right] : \Gamma^s_{g,c,1} \hookrightarrow \Gamma^s_{g+1,c,1} \text{ et } \left[\Sigma^1_{0,0,1}, id_{\Sigma^s_{g,c,1}}\right] : \Gamma^s_{g,c,1} \hookrightarrow \Gamma^{s+1}_{g,c,1}$$

où les éléments de $\Gamma^s_{g,c,1}$ sont prolongés à $\Sigma^s_{g+1,c,1}$ (respectivement $\Sigma^{s+1}_{g,c,1}$) par l'identité sur $\Sigma^0_{1,0,1}$ (respectivement $\Sigma^1_{0,0,1}$). Il s'agit ainsi de faire varier le genre orientable dans un cas et le nombre de points retirés dans l'autre. Dans le cas général, on considère une famille de groupes $\{G_n\}_{n\in\mathbb{N}}$ telle qu'on dispose d'injections canoniques $G_i \hookrightarrow G_{i+1}$ issues d'une structure monoïdale sous-jacente. Plus précisément, on suppose qu'il existe un groupoïde $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$ monoïdal strict tressé et qu'il existe des objets $\{\underline{n}\}_{n\in\mathbb{N}}$ de $\mathcal{G}'$ tels que $Aut_{\mathcal{G}'}(\underline{n}) = G_n$. La construction de Quillen $\mathfrak{U}\mathcal{G}'$ (voir [Gra76, p. 219]) a les mêmes objets que $\mathcal{G}'$ et l'ensemble des morphismes entre deux objets A et B est donné par $\underset{\mathcal{G}'}{colim}\left[Hom_{\mathcal{G}'}(-\natural A, B)\right]$: cela permet alors d'avoir une catégorie qui possède des morphismes entre objets distincts. Un morphisme entre A et B est ainsi noté $[X, \phi]$ où X est un objet tel que $X\natural A = B$ et $\phi \in Aut_{\mathcal{G}'}(B)$.

Comme le cas des groupes de difféotopie des surfaces le montre, le groupoïde $\mathcal{G}'$ peut avoir trop de groupes d'automorphismes par rapport à la famille de groupes à laquelle on s'intéresse. On considère alors les sous-groupoïdes de $\mathcal{M}_2$ suivants :

- $\mathcal{M}^{s,c}_2$ ayant pour objets $\left\{\Sigma^0_{g,0,1}\natural\Sigma^s_{0,c,1}\right\}_{g\in\mathbb{N}}$ afin d'étudier la stabilité par rapport au genre orientable ;
- $\mathcal{M}^{g,c}_2$ ayant pour objets $\left\{\Sigma^s_{0,0,1}\natural\Sigma^0_{g,c,1}\right\}_{s\in\mathbb{N}}$ afin d'étudier la stabilité par rapport au nombre de points retirés.

Dans le cas général, on souhaite se restreindre au sous-groupoïde dont les groupes d'automorphismes sont exactement les groupes $\{G_n\}_{n\in\mathbb{N}}$. On suppose alors qu'il existe deux objets de $\mathcal{G}'$ (notés 0 et 1) tels que $\underline{n} = 1^{\natural n}\natural 0$ pour tout entier naturel n. On note alors $\mathcal{G}$ (respectivement $\mathfrak{U}\mathcal{G}$) la sous-catégorie pleine de $\mathcal{G}'$ (respectivement de $\mathfrak{U}\mathcal{G}'$) sur les objets $\left\{1^{\natural n}\natural 0\right\}_{n\in\mathbb{N}}$.

Vient alors l'étape clef de la généralisation des foncteurs de Long-Moody : il s'agit de trouver une famille de groupes $\{H_n\}_{n\in\mathbb{N}}$ tels qu'il existe deux familles de morphismes $\{a_n : G_n \to Aut(H_n)\}_{n\in\mathbb{N}}$ et $\{\varsigma_n : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$ vérifiant des conditions de cohérence analogues à celles exhibées dans le chapitre 1. On procède alors de la manière suivante : fixant deux groupes H_0 et H, ce dernier étant supposé non-trivial, on pose $H_m := H^{*m} * H_0$ pour tout entier naturel m et on suppose qu'il existe un foncteur $\mathcal{H} : \mathfrak{U}\mathcal{G} \to \mathfrak{Gr}$ tel que $\mathcal{H}(\underline{n}) = H_n$ pour tout entier naturel n. Ce foncteur $\mathcal{H}$ définit donc une famille de morphismes $\{a_n : G_n \to Aut(H_n)\}_{n\in\mathbb{N}}$. On retrouve le cas étudié dans le chapitre 1 en prenant $G_n = \mathbf{B}_n$ le groupe de tresses sur n brins, $H = \mathbb{Z}$ et H_0 le groupe trivial. Si on s'intéresse aux groupes de difféotopie plus généraux $\left\{\Gamma^s_{g,c,1}\right\}_{g\in\mathbb{N}}$, on prend $H_0 = \pi_1\left(\Sigma^s_{0,c,1}, p\right)$ et $H = \pi_1\left(\Sigma^0_{1,0,1}, p\right)$ si bien que $H_n \cong \pi_1\left(\Sigma^s_{n,c,1}, p\right)$, l'action classique d'un groupe de difféotopie sur le groupe fondamental de la surface permettant de définir le foncteur $\mathcal{H}$. On procède de façon analogue pour les groupes $\left\{\Gamma^s_{g,c,1}\right\}_{s\in\mathbb{N}}$.

Des conditions de cohérence sont alors définies (généralisant celles du chapitre 1) que doivent satisfaire une famille de morphismes $\{\varsigma_n : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$ par rapport au foncteur $\mathcal{H}$ et au tressage du groupoïde $\mathcal{G}'$ nécessaires et suffisantes au théorème suivant (on renvoie le lecteur à la section 2.2.1 pour le détail de ces conditions).

Une des conditions de cohérence technique du chapitre 1 se traduit d'ailleurs dans ce cadre général par le fait que la famille de groupes $\{H_n\}_{n\in\mathbb{N}}$ définisse un foncteur $\mathcal{H}$ sur la catégorie $\mathfrak{U}\mathcal{G}$: cela illustre l'intérêt d'avoir un point de vue fonctoriel sur les objets considérés. Remarquons également que les idéaux d'augmentation des anneaux de groupes $\{R[H_n]\}_{n\in\mathbb{N}}$ définissent alors également un foncteur $\mathcal{I} : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$. On forme alors ce qu'on appelle un système cohérent de Long-Moody noté $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ et on démontre :

Théorème C **(Proposition 2.2.30).** *Pour $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ un système cohérent de Long-Moody, les données suivantes définissent un foncteur exact à droite*

$$\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}} : \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$$

appelé foncteur de Long-Moody associé au système $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$, défini pour $F \in Obj(\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}))$ par :

$$\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(F)(\underline{n}) = \mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} F(\underline{n+1}),$$

pour tout objet $\underline{n}$ de $\mathcal{G}$.

Les morphismes $\{\varsigma_n : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$ permettent de définir ce produit tensoriel utilisé pour définir le foncteur de Long-Moody. Celui-ci est défini sur les morphismes grâce au foncteur $\mathcal{I}$.

Par exemple, considérant un foncteur $\mathcal{H}$, les morphismes triviaux $\{\varsigma_{n,t} : H_n \to 0_{\mathfrak{Gr}} \to G_{n+1}\}_{n\in\mathbb{N}}$ vérifient toujours les conditions de cohérence définissant un système de Long-Moody. De plus, on montre que :

Proposition. *Pour tout objet F de $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, il y a une équivalence naturelle :*

$$\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_t\}}(F) \cong \mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_t\}}(R) \underset{R}{\otimes} F(1\natural -).$$

Par ailleurs, le premier groupe d'homologie formant un foncteur $H_1(-,R) : \mathfrak{Gr} \to R\text{-}\mathfrak{Mod}$ (où $\mathfrak{Gr}$ désigne la catégorie des groupes), la composition $H_1(-,R) \circ \mathcal{H}$ est un objet de $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ noté $H_1(H_-, R)$. On remarque ainsi :

Lemme. *Si H_0 et H sont des groupes libres, alors le foncteur de Long-Moody $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}$ est exact et*

$$\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(R) \cong H_1(H_-, R),$$

où R désigne le foncteur constant égal à R.

Pour les groupes de difféotopie des surfaces $\left\{\Gamma^0_{g,0,1}\right\}_{s\in\mathbb{N}}$, en considérant le foncteur de Long-Moody $\mathbf{LM}_{\{\mathcal{H},\mathcal{M}_2^{s=0,0},\mathcal{M}_2,\varsigma_t\}}$, on obtient alors :

$$\mathbf{LM}_{\{\mathcal{H},\mathcal{M}_2^{s=0,0},\mathcal{M}_2,\varsigma_t\}}(R) \cong H_1\left(\Sigma^0_{-,0,1}, R\right).$$

Ce dernier foncteur $H_1\left(\Sigma^0_{-,0,1}, R\right)$ encode les représentations symplectiques des groupes de difféotopie des surfaces. Il est introduit par Cohen et Madsen dans [CM09] et par Boldsen dans [Bol12], où en particulier la stabilité homologique des groupes $\left\{\Gamma^0_{g,0,1}\right\}_{g\in\mathbb{N}}$ par rapport aux coefficients donnés par ce foncteur est démontrée.

Ainsi, dans le cas où H_0 et H sont des groupes libres, un endofoncteur de Long-Moody $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_t\}}$ pour les morphismes triviaux $\{\varsigma_{n,t}\}_{n\in\mathbb{N}}$ est déterminé par le foncteur $H_1(H_-, R)$. Cette propriété n'est plus vraie lorsqu'on s'intéresse à un endofoncteur de Long-Moody $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}$ pour des morphismes non-triviaux $\{\varsigma_n\}_{n\in\mathbb{N}}$. En général, l'image d'un foncteur donné n'est alors pas déterminée par le foncteur $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(R)$. Par exemple, dans le cas des groupes de tresses, pour les morphismes non-triviaux $\{\varsigma_{n,1}\}_{n\in\mathbb{N}}$, on produit le foncteur $\mathbf{LM}_{a_1,\varsigma_1}$ qui permet d'obtenir le foncteur de Burau à partir d'un foncteur constant.

Obtenir de tels morphismes non-triviaux $\{\varsigma_n\}_{n\in\mathbb{N}}$ vérifiant les conditions de cohérence pour former des foncteurs de Long-Moody constitue une étape difficile. En effet, on ne dispose pas de méthode générale pour en construire en considérant des familles de groupes $\{G_n\}_{n\in\mathbb{N}}$ et $\{H_n\}_{n\in\mathbb{N}}$. Cependant, il peut émerger de tels morphismes dans certaines situations : c'est le cas des groupes de difféotopie des surfaces $\left\{\Gamma^s_{g,0,1}\right\}_{s\in\mathbb{N}}$. On forme alors des morphismes cohérents

$$\left\{\varsigma_n : \pi_1\left(\Sigma^n_{g,0,1}, p\right) \to \Gamma^{n+1}_{g,0,1}\right\}_{n\in\mathbb{N}}$$

en utilisant le scindement d'une suite exacte de Birman (on renvoie le lecteur à la section 2.3.4 pour plus de détails sur cette définition). L'endofoncteur de Long-Moody ainsi construit $\mathbf{LM}_{\left\{\mathcal{H},\mathcal{M}_2^{g,0},\mathcal{M}_2,\varsigma\right\}}$ produit alors de nouvelles familles de représentations des groupes $\left\{\Gamma^s_{g,0,1}\right\}_{s\in\mathbb{N}}$ (voir Remarque 2.3.32).

Les notions de forte et très forte polynomialité présentées dans le chapitre 1 s'étendent au cadre plus général considéré ici. En effet, ces notions sont bien définies pour le groupoïde $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$. On peut étendre les définitions de τ_1, δ_1 et κ_1 pour des objets de $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$ ce qui permet de définir les foncteurs (très) fortement polynomiaux sur la catégorie $\mathfrak{U}\mathcal{G}$. Comme pour les groupes de tresses, la stabilité homologique à coefficients tordus des groupes $\{G_n\}_{n\in\mathbb{N}}$ considérés dans [RWW17, Section 5] est démontrée pour les objets très fortement polynomiaux de $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$.

Par ailleurs, la notion de polynomialité forte n'est pas la plus adaptée pour étudier le comportement stable d'un objet de $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$. En effet, considérons $R : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$ le foncteur constant égal à R et, pour un entier naturel $i \geq 1$, $R_{\geq i} : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$ le foncteur constant égal au groupe trivial sur les objets $\underline{n}$ tels que $n < i$ et le foncteur constant égal à R sur les objets $\underline{n}$ tels que $n \geq i$. Ainsi, R est fortement polynomial de degré 0 et $R_{\geq i}$ est fortement polynomial de degré i alors que ces foncteurs sont égaux pour les objets $\underline{n}$ tels que $n \geq i$. Cela a notamment motivé l'introduction de la notion de faible polynomialité de [DV17], définie dans le cas où $\mathcal{G} = \mathcal{G}'$ et $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$ monoïdal symétrique. On montre que cette notion s'étend au cadre plus général considéré ici. A proprement parler, un objet F de $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$ est stablement nul si $\sum_{n\in\mathbb{N}} \kappa_{1\natural n} F$ est égal à F. Ces objets forment une sous-catégorie localisante (voir [Gab62] pour cette notion) de $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$ dont la catégorie quotient associée est notée $\mathbf{St}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$ et $\pi_{\mathfrak{U}\mathcal{G}}$ la projection associée. L'endofoncteur induit par τ_1 dans $\mathbf{St}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$ a un noyau trivial et son conoyau est de nouveau noté δ_1. Un objet F de $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$ est dit faiblement polynomial de degré inférieur ou égal à $d \in \mathbb{N}$ si $\delta_1^{d+1}\left(\pi_{\mathfrak{U}\mathcal{G}}\left(F\right)\right) = 0$. Par exemple, un foncteur faiblement polynomial de degré 0 est équivalent à un foncteur constant.

Les propriétés sur les foncteurs fortement polynomiaux des foncteurs de Long-Moody du chapitre 1 dans le cas des groupes de tresses a encouragé à étudier les propriétés sur la polynomialité forte et faible d'un foncteur de Long-Moody dans le présent cadre plus général. Pour un système cohérent de Long-Moody $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$, en supposant que le foncteur $\mathcal{H} : \mathcal{G} \to \mathfrak{Gr}$ s'étende à $\mathcal{G}'$ et admette des propriétés de compatibilité par rapport au produit $\natural$ et au tressage $b^{\mathcal{G}'}_{-,-}$ (plus précisément on le suppose monoïdal tressé en considérant le produit $\natural$ à la source et le produit libre $*$ pour la catégorie but, voir la section 2.5.3.1 pour plus de détails), le système $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ est dit fiable. Alors, on généralise le Théorème B et on établit de nouveaux résultats pour les foncteurs faiblement polynomiaux :

Théorème D **(Theorem 2.5.29 et Theorem 2.5.36).** *Pour un système fiable de Long-Moody* $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$*, l'endofoncteur associé* $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_t\}}$ *induit un foncteur*

$$\mathcal{P}ol^{fort}_d\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right) \to \mathcal{P}ol^{fort}_{d+1}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$$

De plus si H_0 et H sont libres, alors le foncteur $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}$ augmente de un le degré de très forte polynomialité. Par ailleurs, si H est libre, alors $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}$ augmente de un le degré de faible polynomialité si H_0 est libre ou si le groupoïde $\mathcal{G}'$ est monoïdal symétrique.

Pour les foncteurs fortement polynomiaux, la démonstration de ce théorème est une généralisation de celle du Théorème B : elle repose en effet sur la décomposition suivante par rapport au foncteur de translation

$$\tau_1 \circ \mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}} \cong \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2\right) \oplus \left(\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}} \circ \tau_1\right)$$

qui permet ensuite de démontrer l'équivalence

$$\delta_1 \circ \mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}} \cong \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2\right) \oplus \left(\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}} \circ \delta_1\right). \tag{0.2.1}$$

En revanche, l'équivalence $\kappa_1 \circ \mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}} \cong \mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}} \circ \kappa_1$ n'est valable qu'en se restreignant au cas où les groupes H_0 et H sont libres. Pour les foncteurs faiblement polynomiaux, on démontre d'abord que les foncteurs

$\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}$ et $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2$ sont bien définis pour la catégorie quotient $\mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ sous les hypothèses additionnelles de l'énoncé, avant d'à nouveau exploiter l'équivalence analogue à (0.2.1) pour $\mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. Le Théorème D permet ainsi par exemple de démontrer que le foncteur $H_1\left(\Sigma^0_{-,0,1}, R\right)$ des représentations symplectiques est très fortement et faiblement polynomial de degré 1. Il indique également que les foncteurs de Long-Moody produisent une grande variété de foncteurs très fortement polynomiaux en n'importe quel degré, pour lesquels les résultats de stabilité homologique de [RWW17] sont vérifiés.

Par ailleurs, dans le cas où le groupoïde $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$ est monoïdal symétrique, on s'intéressse à la compatibilité des foncteurs de Long-Moody avec une construction générale sur $\mathfrak{U}\mathcal{G}'$ introduite dans [DV17]. Plus précisément, en notant $\mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{ini}}$ (respectivement $\mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{nul}}$) la catégorie des catégories monoïdales symétriques avec un objet initial (respectivement nul), on note $\widetilde{-} : \mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{ini}} \to \mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{nul}}$ l'adjoint à gauche du foncteur d'oubli $\mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{nul}} \hookrightarrow \mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{ini}}$. Pour un objet $\mathfrak{M}$ de $\mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{ini}}$ comme par exemple $\mathfrak{U}\mathcal{G}'$, la catégorie $\widetilde{\mathfrak{M}}$ s'avère entre autre utile dans la classification des foncteurs faiblement polynomiaux ayant $\mathfrak{M}$ pour source (voir [DV17, Theorem 3.8]). Dans le cas où on considère le groupoïde $(FB, \sqcup, \emptyset)$ des ensembles finis avec les bijections pour morphismes et la structure monoïdale symétrique induite par l'union disjointe des ensembles, alors $\mathfrak{U}(FB)$ s'identifie à la catégorie FI des ensembles finis avec les injections et $\widetilde{FI}$ est équivalente à la catégorie $FI\sharp$ considérée par Church, Ellenberg et Farb dans [CEF15] afin d'étudier les objets projectifs de la catégorie $\mathbf{Fct}(FI, R\text{-}\mathfrak{Mod})$. On note $\widetilde{\mathfrak{U}\mathcal{G}}$ la sous-catégorie de $\widetilde{\mathfrak{U}\mathcal{G}'}$ sur les objets de $\mathcal{G}$.

En considérant un système fiable de Long-Moody $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$, il suffit alors d'imposer une condition additionnelle sur les morphismes $\{\varsigma_n\}_{n\in\mathbb{N}}$ (voir la section 2.6.2 pour plus de détails, le système $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ est alors dit relevable) afin d'étendre la définition d'un foncteur de Long-Moody $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_t\}}$ à la catégorie $\widetilde{\mathfrak{U}\mathcal{G}}$. A proprement parler, on démontre que :

Théorème E **(Proposition 2.6.24 et Proposition 2.6.25).** *Pour un système relevable de Long-Moody $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ où $\mathcal{G}'$ est monoïdal symétrique, il existe un foncteur $\widetilde{\mathbf{LM}} : \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right) \longrightarrow \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right)$ tel que le diagramme suivant est commutatif :*

$$\begin{array}{ccc} \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right) & \xrightarrow{\widetilde{\mathbf{LM}}} & \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right) \\ {\scriptstyle\left(incl^{\widetilde{\mathfrak{U}\mathcal{G}}}_{\mathfrak{U}\mathcal{G}}\right)^*}\downarrow & & \downarrow{\scriptstyle\left(incl^{\widetilde{\mathfrak{U}\mathcal{G}}}_{\mathfrak{U}\mathcal{G}}\right)^*} \\ \mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right) & \xrightarrow{\mathbf{LM}} & \mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right). \end{array}$$

0.3 Calculs d'homologie à coefficients tordus pour les groupes de difféotopies

Le troisième chapitre porte sur des calculs explicites d'homologie stable à coefficients tordus de groupes de difféotopie de surfaces et de 3-variétés. Nous avons vu dans les sections précédentes que les foncteurs de Long-Moody permettent d'obtenir des coefficients tordus polynomiaux en tout degré pour une famille de groupes $\{G_n\}_{n\in\mathbb{N}}$. Reprenant les notations des sections précédentes, il s'ensuit tout naturellement la question suivante :

Question. *Est-il possible de comparer les homologies stables $H_*(G_\infty, F_\infty)$ et $H_*\left(G_\infty, \mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(F_\infty)\right)$ pour un objet F de $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$?*

Dans un premier temps, on répond partiellement à cette question en s'appuyant sur des structures de produits semi-directs apparaissant naturellement pour les groupes de difféotopie et sur la suite spectrale de Lyndon-Hochschild-Serre. On démontre tout d'abord :

Théorème F **(Corollary 3.2.5).** *Soit $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma_t\}$ un système cohérent de Long-Moody (on rappelle que la famille $\{\varsigma_{n,t}\}_{n\in\mathbb{N}}$ est celle des morphismes triviaux) tel que H_n est un groupe libre pour tout entier naturel n. Alors, pour tout objet F de*

Fct $(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ *et tout entier naturel* $* \geq 1$ *:*

$$H_{*-1}\left(G_{\infty}, \mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_t\}}\left(F_{\infty}\right)\right) \cong H_{*}\left(\underset{n\in(\mathbb{N},\leq)}{Colim}\left(H_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} G_n\right), F_{\infty}\right) / H_{*}\left(G_{\infty}, F_{\infty}\right).$$

Ce résultat général a des applications pour diverses familles de groupes. Tout d'abord, on s'intéresse aux groupes de tresses. Pour un entier naturel n, on note $\mathfrak{Cox}(n)$ la représentation de Coxeter du groupe de tresses $\mathbf{B}_n$ et on rappelle que $\mathfrak{Bur}_t(n)$ désigne la représentation non-réduite de Burau. En utilisant le Théorème F et le résultat de stabilité [Gor78, Theorem C], on démontre que :

Proposition. *Pour tous les entiers naturels n et q tels que $n \geq q+2$:*

$$H_q\left(\mathbf{B}_n, \mathfrak{Cox}(n)\right) \cong \begin{cases} \mathbb{C}^{\oplus 2} & si\ q \geq 2, \\ \mathbb{C} & si\ q = 0, 1. \end{cases}$$

Par ailleurs, en utilisant une suite exacte courte faisant intervenir les représentations de Burau réduites et des résultats de [Che17], on démontre :

Proposition. *Pour tous les entiers naturels $n \geq 3$ et $q \geq 3$:*

$$H_q\left(\mathbf{B}_n, \mathfrak{Bur}_t(n)\right) \cong \begin{cases} \mathbb{C}\left[t^{\pm 1}\right] / (1-t) & si\ 3 \leq q < n{-}2, \\ \mathbb{C}\left[t^{\pm 1}\right] / (1-t) & si\ q = n{-}2\ et\ n\ est\ impair, \\ \mathbb{C}\left[t^{\pm 1}\right] / \left(1-t^2\right) & si\ q = n{-}2\ et\ n\ est\ pair, \\ 0 & sinon. \end{cases}$$

Ensuite, pour les groupes de difféotopie des surfaces, on abrège la notation $\Gamma^{s}_{g,0,1}$ (respectivement $\Gamma^{0}_{g,0,1}$) par $\Gamma^{s}_{g,1}$ (respectivement $\Gamma_{g,1}$) . Alors, en combinant le Théorème F et des résultats de stabilité de [Bol12, CM09], on déduit que :

Proposition. *Pour des entiers naturels m, n et q tels que $2n \geq 3q+m$, on a un isomorphisme :*

$$H_q\left(\Gamma_{n,1}, H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m}\right) \cong \bigoplus_{\left\lfloor \frac{q-1}{2} \right\rfloor \geq k \geq 0} H_{q-(2k+1)}\left(\Gamma_{n,1}, H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m-1}\right).$$

On retrouve ainsi les résultats de [Har91] et [Kaw08].

Enfin, soit $\mathcal{G}^{s}_{n,k}$ l'espace topologique composé d'un bouquet de n cercles, k cercles distingués (ie chacun est relié au point base du bouquet par une arête) et s points distingués (ie chacun est relié au point base du bouquet par une arête). On note $A^{s}_{n,k}$ le groupe des composantes connexes par arcs des équivalences d'homotopies de l'espace $\mathcal{G}^{s}_{n,k}$ (on renvoie à [HW05] pour plus de détails sur ces groupes). Alors, on prouve :

Proposition. *Soient $s \geq 2$ et $q \geq 1$ des entiers naturels et $F : \mathfrak{gr} \to$* **Ab** *(où $\mathfrak{gr}$ est la catégorie des groupes libres de type fini et* **Ab** *est la catégorie des groupes abéliens) réduit (c'est-à-dire nul sur le groupe trivial). Alors, pour tout entier naturel $n \geq 2q+1$, $H_q\left(A^{s}_{n,0}, F(n)\right) = 0$.*

On déduit par ailleurs de cette proposition et des résultats de stabilité de [HW10] que :

Corollaire. *Pour des entiers naturels $n \geq 2q+2$ et $k \geq 0$:*

$$H_q\left(A^{s}_{n,k}, \mathbb{Q}\right) = 0.$$

On recouvre ainsi les résultats de [Jen04] pour les holomorphes des groupes libres.

Dans un second temps, on effectue des calculs d'homologie stable de groupes de difféotopie pour des coefficients tordus donnés par des représentations factorisant par les groupes symétriques ou hyperoctahédraux. Rappelons qu'on note FI la catégorie des ensembles finis avec les injections pour morphismes. Rappelons également que, pour un entier naturel k fixé, les groupes $\left\{Aut\left(\left(\mathbb{Z}^{*k}\right)^{\times n}\right)\right\}_{n\in\mathbb{N}}$ définissent une famille de groupes d'Artin à angles droits. On démontre alors :

Théorème G **(Proposition 3.4.14, Proposition 3.4.26, Corollary 3.4.30).** *Soit* $\mathbb{K}$ *un corps de caractéristique nulle et* d *un entier naturel. Pour* $F : FI \to \mathbb{K}\text{-}\mathfrak{Mod}$ *un foncteur, on a :*

1. $H_d\left(\mathbf{B}_\infty, F_\infty\right) \cong \underset{n\in FI}{Colim}\left(H_d\left(\mathbf{PB}_n, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} F(n)\right)$ *où* $\mathbf{PB}_n$ *désigne le groupe de tresses pures sur* n *brins ;*

2. $H_d\left(\Gamma^\infty_{\infty,1}, F_\infty\right) \cong \underset{n\in FI}{Colim}\left[\bigoplus_{k+l=d}\left(H_k\left(\Gamma_{n,1}, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} H_l\left(\left(\mathbb{C}P^\infty\right)^{\times n}, \mathbb{K}\right)\right) \underset{\mathbb{K}}{\otimes} F(n)\right]$*. En particulier, pour tout entier naturel* k *:*
$$H_{2k+1}\left(\Gamma^\infty_{\infty,1}, F_\infty\right) = 0.$$

3. $H_d\left(Aut\left(\left(\mathbb{Z}^{*k}\right)^{\times\infty}\right), F_\infty\right) = 0$ *pour tout entier naturel* $k \geq 2d+1$.

La démonstration du Théorème G nécessite un résultat général de décomposition de l'homologie stable à coefficients tordus. On reprend les notations et hypothèses de la Section 0.2 : on considère une famille de groupes $\{G_n\}_{n\in\mathbb{N}}$ telle qu'il existe un groupoïde $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$ monoïdal strict tressé et qu'il existe des objets de $\mathcal{G}'$ notés $\underline{n}$ tels que $Aut_{\mathcal{G}'}(\underline{n}) = G_n$ pour tout entier naturel n et on note $\mathcal{G}$ (respectivement $\mathfrak{U}\mathcal{G}$) la sous-catégorie pleine de $\mathcal{G}'$ (respectivement la construction de Quillen $\mathfrak{U}\mathcal{G}'$) ayant pour objets $\{\underline{n}\}_{n\in\mathbb{N}}$. On renvoie le lecteur aux articles [FP03, Section 2] et [DV10, Appendice A] pour une introduction à l'homologie des foncteurs. On démontre alors :

Theorem H **(Theorem 3.3.7).** *Soit* $\mathbb{K}$ *un corps. Pour tout foncteur* $F : \mathfrak{U}\mathcal{G} \to \mathbb{K}\text{-}\mathfrak{Mod}$*, on a un isomorphisme de* $\mathbb{K}$*-modules :*
$$H_*\left(G_\infty, F_\infty\right) \cong \bigoplus_{k+l=*}\left(H_k\left(G_\infty, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} H_l\left(\mathfrak{U}\mathcal{G}, F\right)\right).$$

Ce théorème est une généralisation du résultat analogue dû à Djament et Vespa dans [DV10, Sections 1 et 2] dans le cas où le groupoïde $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$ est monoïdal symétrique.

Chapter 1

The Long-Moody construction and polynomial functors

Abstract: *In 1994, Long and Moody gave a construction on representations of braid groups which associates a representation of $\mathbf{B}_n$ with a representation of $\mathbf{B}_{n+1}$. In this paper, we prove that this construction is functorial and can be extended: it inspires endofunctors, called Long-Moody functors, between the category of functors from Quillen's bracket construction associated with the braid groupoid to a module category. Then we study the effect of Long-Moody functors on strong polynomial functors: we prove that they increase by one the degree of very strong polynomiality.*

Introduction

Linear representations of the Artin braid group on n strands $\mathbf{B}_n$ is a rich subject which appears in diverse contexts in mathematics (see for example [BB05] or [Mar13] for an overview). Even if braid groups are of wild representation type, any new result which allows us to gain a better understanding of them is a useful contribution.

In 1994, as a result of a collaboration with Moody in [Lon94], Long gave a method to construct from a linear representation $\rho : \mathbf{B}_{n+1} \to GL(V)$ a new linear representation $\mathcal{LM}(\rho) : \mathbf{B}_n \to GL(V^{\oplus n})$ of $\mathbf{B}_n$ (see [Lon94, Theorem 2.1]). Moreover, the construction complicates in a sense the initial representation. For example, applying it to a one dimensional representation of $\mathbf{B}_{n+1}$, the construction gives a mild variant of the unreduced Burau representation of $\mathbf{B}_n$. This method was in fact already implicitly present in two previous articles of Long dated 1989 (see [Lon89a, Lon89b]). In the article [BT08] dating from 2008, Bigelow and Tian consider the Long-Moody construction from a matricial point of view. They give alternative and purely algebraic proofs of some results of [Lon94], and they slightly extend some of them. In a survey on braid groups (see the Open Problem 7 in [BB05]), Birman and Brendle underline the fact that the Long-Moody construction should be studied in greater detail.

Our work focuses on the study of the Long-Moody construction $\mathcal{LM}$ from a functorial point of view. More precisely, we consider the category $\mathfrak{U}\boldsymbol{\beta}$ associated with braid groups. This category is an example of a general construction due to Quillen (see [Gra76]) on the braid groupoid $\boldsymbol{\beta}$. In particular, the category $\mathfrak{U}\boldsymbol{\beta}$ has natural numbers $\mathbb{N}$ as objects. For each natural number n, the automorphism group $Aut_{\mathfrak{U}\boldsymbol{\beta}}(n)$ is the braid group $\mathbf{B}_n$. Let $\mathbb{K}$-$\mathfrak{Mod}$ be the category of $\mathbb{K}$-modules, with $\mathbb{K}$ a commutative ring, and $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ be the category of the functors from $\mathfrak{U}\boldsymbol{\beta}$ to $\mathbb{K}$-$\mathfrak{Mod}$. An object M of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ gives by evaluation a family of representations of braid groups $\{M_n : \mathbf{B}_n \to GL(M(n))\}_{n\in\mathbb{N}}$, which satisfies some compatibility properties (see Section 1.1.1). Randal-Williams and Wahl use the category $\mathfrak{U}\boldsymbol{\beta}$ in [RWW17] to construct a general framework to prove homological stability for braid groups with twisted coefficients. Namely, they obtain the stability for twisted coefficients given by objects of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

In Proposition 1.2.20, we prove that a version of the Long-Moody construction is functorial. We fix two families of morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ and $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$, satisfying some coherence properties (see Section 1.2.1). Once this framework set, we show:

Theorem A **(Proposition 1.2.20) .** *There is a functor $\mathbf{LM}_{a,\varsigma} : \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, called the Long-Moody functor with respect to coherent families of morphisms $\{a_n\}_{n\in\mathbb{N}}$ and $\{\varsigma_n\}_{n\in\mathbb{N}}$, which satisfies for $\sigma \in \mathbf{B}_n$ and*

$M \in Obj\left(\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)\right)$

$$\mathbf{LM}_{a,\varsigma}\left(M\right)\left(\sigma\right) = \mathcal{LM}\left(M_n\right)\left(\sigma\right).$$

Among the objects in the category $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$ the strong polynomial functors play a key role. This notion extends the classical one of polynomial functors, which were first defined by Eilenberg and Mac Lane in [EML54] for functors on module categories, using cross effects. This definition can also be applied to monoidal categories where the monoidal unit is a null object. Djament and Vespa introduce in [DV17] the definition of strong polynomial functors for symmetric monoidal categories with the monoidal unit being an initial object. Here, the category $\mathfrak{U}\boldsymbol{\beta}$ is neither symmetric, nor braided, but pre-braided in the sense of [RWW17]. However, we show that the notion of strong polynomial functor extends to the wider context of pre-braided monoidal categories (see Definition 1.3.4). We also introduce the notion of very strong polynomial functor (see Definition 1.3.16). Strong polynomial functors turn out inter alia to be very useful for homological stability problems. For example, in [RWW17], Randal-Williams and Wahl prove their homological stability results for twisted coefficients given by a specific kind of strong polynomial functors, namely coefficient systems of finite degree (see [RWW17, Section 4.4]).

We investigate the effects of Long-Moody functors on very strong polynomial functors. We establish the following theorem, under some mild additional conditions (introduced in Section 1.4.1.1) on the families of morphisms $\{a_n\}_{n\in\mathbb{N}}$ and $\{\varsigma_n\}_{n\in\mathbb{N}}$, which are then said to be reliable.

***Theorem B* (Corollary 1.4.28) .** *Let M be a very strong polynomial functor of $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$ of degree n and let $\{a_n\}_{n\in\mathbb{N}}$ and $\{\varsigma_n\}_{n\in\mathbb{N}}$ be coherent reliable families of morphisms. Then, considering the Long-Moody functor $\mathbf{LM}_{a,\varsigma}$ with respect to the morphisms $\{a_n\}_{n\in\mathbb{N}}$ and $\{\varsigma_n\}_{n\in\mathbb{N}}$, $\mathbf{LM}_{a,\varsigma}\left(M\right)$ is a very strong polynomial functor of degree $n+1$.*

Thus, iterating the Long-Moody functor on a very strong polynomial functor of $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$ of degree d, we generate polynomial functors of $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$, of any degree bigger than d. For instance, Randal-Williams and Wahl define in [RWW17, Example 4.3] a functor $\mathfrak{Bur}_t : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ encoding the unreduced Burau representations. Similarly, we introduce a functor $\mathfrak{TYM}_t : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ corresponding to the representations considered by Tong, Yang and Ma in [TYM96]. These functors $\mathfrak{Bur}_t$ and $\mathfrak{TYM}_t$ are very strong polynomial of degree one (see Proposition 1.3.25), and moreover, we prove that the functor $\mathfrak{Bur}_t$ is equivalent to a functor obtained by applying the Long-Moody construction. Thus, the Long-Moody functors will provide new examples of twisted coefficients corresponding to the framework of [RWW17].

This construction is extended in the forthcoming work [Sou17a] for other families of groups, such as automorphism groups of free groups, braid groups of surfaces, mapping class groups of orientable and non-orientable surfaces or mapping class groups of 3-manifolds. The results proved here for (very) strong polynomial functors will also hold in the adapted categorical framework for these different families of groups.

The paper is organized as follows. Following [RWW17], Section 1.1 introduces the category $\mathfrak{U}\boldsymbol{\beta}$ and gives first examples of objects of $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$. Then, in Section 1.2, we introduce the Long-Moody functors, prove Theorem A and give some of their properties. In Section 1.3, we review the notion of strong polynomial functors and extend the framework of [DV17] to pre-braided monoidal categories. Finally, Section 1.4 is devoted to the proof of Theorem B and to some other properties of these functors. In particular, we tackle the Open Problem 7 of [BB05].

Notation 1.0.1. We will consider a commutative ring $\mathbb{K}$ throughout this work. We denote by $\mathbb{K}\text{-}\mathfrak{Mod}$ the category of $\mathbb{K}$-modules. We denote by $\mathfrak{Gr}$ the category of groups.

Let $\mathfrak{Cat}$ denote the category of small categories. Let $\mathfrak{C}$ be an object of $\mathfrak{Cat}$. We use the abbreviation $Obj\left(\mathfrak{C}\right)$ to denote the objects of $\mathfrak{C}$. For $\mathfrak{D}$ a category, we denote by $\mathbf{Fct}\left(\mathfrak{C}, \mathfrak{D}\right)$ the category of functors from $\mathfrak{C}$ to $\mathfrak{D}$. If 0 is initial object in the category $\mathfrak{C}$, then we denote by $\iota_A : 0 \to A$ the unique morphism from 0 to A. The maximal subgroupoid $\mathscr{G}\mathfrak{r}\left(\mathfrak{C}\right)$ is the subcategory of $\mathfrak{C}$ which has the same objects as $\mathfrak{C}$ and of which the morphisms are the isomorphisms of $\mathfrak{C}$. We denote by $\mathscr{G}\mathfrak{r} : \mathfrak{Cat} \to \mathfrak{Cat}$ the functor which associates to a category its maximal subgroupoid.

1.1 The category $\mathfrak{U}\boldsymbol{\beta}$

The aim of this section is to describe the category $\mathfrak{U}\boldsymbol{\beta}$ associated with braid groups that is central to this paper. On the one hand, we recall some notions and properties about Quillen's construction from a monoidal groupoid and pre-braided monoidal categories introduced by Randal-Williams and Wahl in [RWW17]. On the other hand, we introduce examples of functors over the category $\mathfrak{U}\boldsymbol{\beta}$.

We recall that the braid group on $n \geq 2$ strands denoted by $\mathbf{B}_n$ is the group generated by σ_1, ..., σ_{n-1} satisfying the relations:

- $\forall i \in \{1, \ldots, n-2\}$, $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$;
- $\forall i, j \in \{1, \ldots, n-1\}$ such that $\mid i - j \mid \geq 2$, $\sigma_i \sigma_j = \sigma_j \sigma_i$.

$\mathbf{B}_0$ and $\mathbf{B}_1$ both are the trivial group. The family of braid groups is associated with the following groupoid.

Definition 1.1.1. The braid groupoid $\boldsymbol{\beta}$ is the groupoid with objects the natural numbers $n \in \mathbb{N}$ and morphisms (for $n, m \in \mathbb{N}$):

$$Hom_{\boldsymbol{\beta}}(n, m) = \begin{cases} \mathbf{B}_n & \text{if } n = m \\ \emptyset & \text{if } n \neq m. \end{cases}$$

Remark 1.1.2. The composition of morphisms $\circ$ in the groupoid $\boldsymbol{\beta}$ corresponds to the group operation of the braid groups. So we will abuse the notation throughout this work, identifying $\sigma \circ \sigma' = \sigma\sigma'$ for all elements σ and σ' of $\mathbf{B}_n$ with $n \in \mathbb{N}$ (with the convention that we read from the right to the left for the group operation).

1.1.1 Quillen's bracket construction associated with the groupoid $\boldsymbol{\beta}$

This section focuses on the presentation and the study of Quillen's bracket construction $\mathfrak{U}\boldsymbol{\beta}$ (see [Gra76, p.219]) on the braid groupoid $\boldsymbol{\beta}$. It associates to $\boldsymbol{\beta}$ a monoidal category whose unit is initial. The category $\mathfrak{U}\boldsymbol{\beta}$ has further properties: Quillen's bracket construction on $\boldsymbol{\beta}$ is a pre-braided monoidal category (see Section 1.1.1.2) and $\boldsymbol{\beta}$ is its maximal subgroupoid. For an introduction to (braided) strict monoidal categories, we refer to [ML13, Chapter XI].

Notation 1.1.3. A strict monoidal category will be denoted by $(\mathfrak{C}, \natural, 0)$, where $\mathfrak{C}$ is the category, $\natural$ is the monoidal product and 0 is the monoidal unit.

1.1.1.1 Generalities

In [RWW17], Randal-Williams and Wahl study a construction due to Quillen in [Gra76, p.219], for a monoidal category S acting on a category X in the case $S = X = \mathfrak{G}$ where $\mathfrak{G}$ is a groupoid. It is called Quillen's bracket construction. Our study here is based on [RWW17, Section 1] taking $\mathfrak{G} = \boldsymbol{\beta}$.

Definition 1.1.4. [ML13, Chapter XI, Section 4] A monoidal product $\natural : \boldsymbol{\beta} \times \boldsymbol{\beta} \longrightarrow \boldsymbol{\beta}$ is defined by the usual addition for the objects and laying two braids side by side for the morphisms. The object 0 is the unit of this monoidal product. The strict monoidal groupoid $(\boldsymbol{\beta}, \natural, 0)$ is braided, its braiding is denoted by $b^{\boldsymbol{\beta}}_{-,-}$. Namely, the braiding is defined for all natural numbers n and m such that $n + m \geq 2$ by:

$$b^{\boldsymbol{\beta}}_{n,m} = (\sigma_m \circ \cdots \circ \sigma_2 \circ \sigma_1) \circ \cdots \circ (\sigma_{n+m-2} \circ \cdots \circ \sigma_n \circ \sigma_{n-1}) \circ (\sigma_{n+m-1} \circ \cdots \circ \sigma_{n+1} \circ \sigma_n)$$

where $\{\sigma_i\}_{i \in \{1,\ldots,n+m-1\}}$ denote the Artin generators of the braid group $\mathbf{B}_{n+m}$.

We consider the strict monoidal groupoid $(\boldsymbol{\beta}, \natural, 0)$ throughout this section.

Definition 1.1.5. [RWW17, Section 1.1] Quillen's bracket construction on the groupoid $\boldsymbol{\beta}$, denoted by $\mathfrak{U}\boldsymbol{\beta}$, is the category defined by:

- Objects: $Obj(\mathfrak{U}\boldsymbol{\beta}) = Obj(\boldsymbol{\beta}) = \mathbb{N}$;
- Morphisms: for n and n' two objects of $\boldsymbol{\beta}$, the morphisms from n to n' in the category $\mathfrak{U}\boldsymbol{\beta}$ are given by:

  $$Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n') = \underset{\boldsymbol{\beta}}{colim}\left[Hom_{\boldsymbol{\beta}}(-\natural n, n')\right].$$

  In other words, a morphism from n to n' in the category $\mathfrak{U}\boldsymbol{\beta}$, denoted by $[n' - n, f] : n \to n'$, is an equivalence class of pairs $(n' - n, f)$ where $n' - n$ is an object of $\boldsymbol{\beta}$, $f : (n' - n) \natural n \to n'$ is a morphism of $\boldsymbol{\beta}$, in other words an element of $\mathbf{B}_{n'}$. The equivalence relation $\sim$ is defined by $(n'-n, f) \sim (n'-n, f')$ if and only if there exists an automorphism $g \in Aut_{\boldsymbol{\beta}}(n'-n)$ such that the following diagram commutes.

$$\begin{array}{ccc} (n'-n)\natural n & \xrightarrow{f} & n' \\ {\scriptstyle g\natural id_n}\downarrow & \nearrow_{f'} & \\ (n'-n)\natural n & & \end{array}$$

- For all objects n of $\mathfrak{U}\boldsymbol{\beta}$, the identity morphism in the category $\mathfrak{U}\boldsymbol{\beta}$ is given by $[0, id_n] : n \to n$.
- Let $[n'-n, f] : n \to n'$ and $[n''-n', g] : n' \to n''$ be two morphisms in the category $\mathfrak{U}\boldsymbol{\beta}$. Then, the composition in the category $\mathfrak{U}\boldsymbol{\beta}$ is defined by:

$$\left[n''-n', g\right] \circ \left[n'-n, f\right] = \left[n''-n, g \circ (id_{n'-n}\natural f)\right].$$

The relationship between the automorphisms of the groupoid $\boldsymbol{\beta}$ and those of its associated Quillen's construction $\mathfrak{U}\boldsymbol{\beta}$ is actually clear. First, let us recall the following notion.

Definition 1.1.6. Let $(\mathfrak{G}, \natural, 0)$ be a strict monoidal category. It has no zero divisors if for all objects A and B of $\mathfrak{G}$, $A\natural B \cong 0$ if and only if $A \cong B \cong 0$.

The braid groupoid $(\boldsymbol{\beta}, \natural, 0)$ has no zero divisors. Moreover, by Definition 1.1.1, $Aut_{\boldsymbol{\beta}}(0) = \{id_0\}$. Hence, we deduce the following property from [RWW17, Proposition 1.7].

Proposition 1.1.7. *The groupoid $\boldsymbol{\beta}$ is the maximal subgroupoid of $\mathfrak{U}\boldsymbol{\beta}$.*

In addition, $\mathfrak{U}\boldsymbol{\beta}$ has the additional useful property.

Proposition 1.1.8. *[RWW17, Proposition 1.8 (i)] The unit 0 of the monoidal structure of the groupoid $(\boldsymbol{\beta}, \natural, 0)$ is an initial object in the category $\mathfrak{U}\boldsymbol{\beta}$.*

Remark 1.1.9. Let n be a natural number and $\phi \in Aut_{\boldsymbol{\beta}}(n)$. Then, as an element of $Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n)$, we will abuse the notation $\phi = [0, \phi]$. This comes from the canonical functor:

$$\begin{array}{ccc} \boldsymbol{\beta} & \to & \mathfrak{U}\boldsymbol{\beta} \\ \phi \in Aut_{\boldsymbol{\beta}}(n) & \mapsto & [0, \phi]. \end{array}$$

Finally, we are interested in a way to extend an object of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ to an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. This amounts to studying the image of the restriction $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

Proposition 1.1.10. *Let M be an object of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Assume that for all $n, n', n'' \in \mathbb{N}$ such that $n'' \geq n' \geq n$, there exists an assignment $M([n'-n, id_{n'}]) : M(n) \to M(n')$ such that:*

$$M\left(\left[n''-n', id_{n''}\right]\right) \circ M\left(\left[n'-n, id_{n'}\right]\right) = M\left(\left[n''-n, id_{n''}\right]\right) \tag{1.1.1}$$

Then, we define a functor $M : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ (assigning $M([n'-n, \sigma]) = M(\sigma) \circ M([n'-n, id_{n'}])$ for all $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$) if and only if for all $n, n' \in \mathbb{N}$ such that $n' \geq n$:

$$M\left(\left[n'-n, id_{n'}\right]\right) \circ M(\sigma) = M(\psi\natural\sigma) \circ M\left(\left[n'-n, id_{n'}\right]\right) \tag{1.1.2}$$

for all $\sigma \in \mathbf{B}_n$ and all $\psi \in \mathbf{B}_{n'-n}$.

Remark 1.1.11. Note that for $n' = n$, $M([n'-n, id_{n'}]) = Id_{M(n)}$.

Proof of Proposition 1.1.10. Let us assume that relation (1.1.2) is satisfied. We have to show that the assignment on morphisms is well-defined with respect to $\mathfrak{U}\boldsymbol{\beta}$. First, let us prove that our assignment conforms with the defining equivalence relation of $\mathfrak{U}\boldsymbol{\beta}$ (see Definition 1.1.5). For n and n' natural numbers such that $n' \geq n$, let us consider $[n'-n, \sigma]$ and $[n'-n, \sigma']$ in $Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$ such that there exists $\psi \in \mathbf{B}_{n'-n}$ so that $\sigma' \circ (\psi\natural id_n) = \sigma$. Since M is a functor over $\boldsymbol{\beta}$, $M([n'-n,\sigma]) = M(\sigma')\circ(M(\psi\natural id_n)\circ M([n'-n,id_{n'}]))$. According to the relation (1.1.2) and since M satisfies the identity axiom, we deduce that $M([n'-n,\sigma]) = M(\sigma')\circ M(\psi\natural id_n)\circ M([n'-n,id_{n'}]) = M([n'-n,\sigma'])$.

Now, we have to check the composition axiom. Let n, n' and n'' be natural numbers such that $n''\geq n'\geq n$, let $([n'-n,\sigma])$ and $([n''-n',\sigma'])$ be morphisms respectively in $Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$ and in $Hom_{\mathfrak{U}\boldsymbol{\beta}}(n',n'')$. By our assignment and composition in $\mathfrak{U}\boldsymbol{\beta}$ (see Definition 1.1.5) we have that:

$$M([n''-n',\sigma'])\circ M([n'-n,\sigma]) = M(\sigma')\circ(M([n''-n',id_{n''}])\circ M(\sigma))\circ M([n'-n,id_{n'}]).$$

According to the relation (1.1.2), we deduce that:

$$\begin{aligned}M([n''-n',\sigma'])\circ M([n'-n,\sigma]) &= M(\sigma')\circ(M([n''-n',id_{n''}])\circ M(\sigma))\circ M([n'-n,id_{n'}]).\\ &= M(\sigma')\circ(M(id_{n''-n'}\natural\sigma)\circ M([n''-n',id_{n''}]))\circ M([n'-n,id_{n'}]).\end{aligned}$$

Hence, it follows from relation (1.1.1) that:

$$M([n''-n',\sigma'])\circ M([n'-n,\sigma]) = M(\sigma'\circ(id_{n''-n'}\natural\sigma))\circ M([n''-n,id_n]) = M([n''-n',\sigma']\circ[n'-n,\sigma]).$$

Conversely, assume that the functor $M:\mathfrak{U}\boldsymbol{\beta}\to\mathbb{K}\text{-}\mathfrak{Mod}$ is well-defined. In particular, composition axiom in $\mathfrak{U}\boldsymbol{\beta}$ is satisfied and implies that for all $n,n'\in\mathbb{N}$ such that $n'\geq n$, for all $\sigma\in\mathbf{B}_n$:

$$M([n'-n,id_{n'}])\circ M(\sigma) = M([n'-n,id_{n'-n}\natural\sigma]).$$

It follows from the defining equivalence relation of $\mathfrak{U}\boldsymbol{\beta}$ (see Definition (1.1.5)) that for all $\psi\in\mathbf{B}_{n'-n}$:

$$M([n'-n,id_{n'}])\circ M(\sigma) = M([n'-n,\psi\natural\sigma]).$$

We deduce from the composition axiom that relation (1.1.2) is satisfied. □

Proposition 1.1.12. *Let M and M' be objects of* $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$ *and $\eta: M\to M'$ a natural transformation in the category* $\mathbf{Fct}(\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$. *Then, η is a natural transformation in the category* $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$ *if and only if for all $n,n'\in\mathbb{N}$ such that $n'\geq n$:*

$$\eta_{n'}\circ M([n'-n,id_{n'}]) = M'([n'-n,id_{n'}])\circ\eta_n. \tag{1.1.3}$$

Proof. The natural transformation η extends to the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$ if and only if for all $n,n'\in\mathbb{N}$ such that $n'\geq n$, for all $[n'-n,\sigma]\in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$:

$$M'([n'-n,\sigma])\circ\eta_n = \eta_{n'}\circ M([n'-n,\sigma]).$$

Since η is a natural transformation in the category $\mathbf{Fct}(\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$, we already have $\eta_{n'}\circ M(\sigma) = M'(\sigma)\circ\eta_{n'}$. Hence, this implies that the necessary and sufficient relation to satisfy is relation (1.1.3). □

1.1.1.2 Pre-braided monoidal category

We present the notion of a pre-braided category, introduced by Randal-Williams and Wahl in [RWW17]. This is a generalization of that of a braided monoidal category.

Definition 1.1.13. [RWW17, Definition 1.5] Let $(\mathfrak{C},\natural,0)$ be a strict monoidal category such that the unit 0 is initial. We say that the monoidal category $(\mathfrak{C},\natural,0)$ is pre-braided if:

- The maximal subgroupoid $\mathscr{G}\mathfrak{r}(\mathfrak{C},\natural,0)$ is a braided monoidal category, where the monoidal structure is induced by that of $(\mathfrak{C},\natural,0)$.
- For all objects A and B of $\mathfrak{C}$, the braiding associated with the maximal subgroupoid $b^{\mathfrak{C}}_{A,B}: A\natural B\longrightarrow B\natural A$ satisfies:
  $$b^{\mathfrak{C}}_{A,B}\circ(id_A\natural\iota_B) = \iota_B\natural id_A : A\longrightarrow B\natural A.$$
  Recall that the notation $\iota_B: 0\to B$ was introduced in Notation 1.0.1.

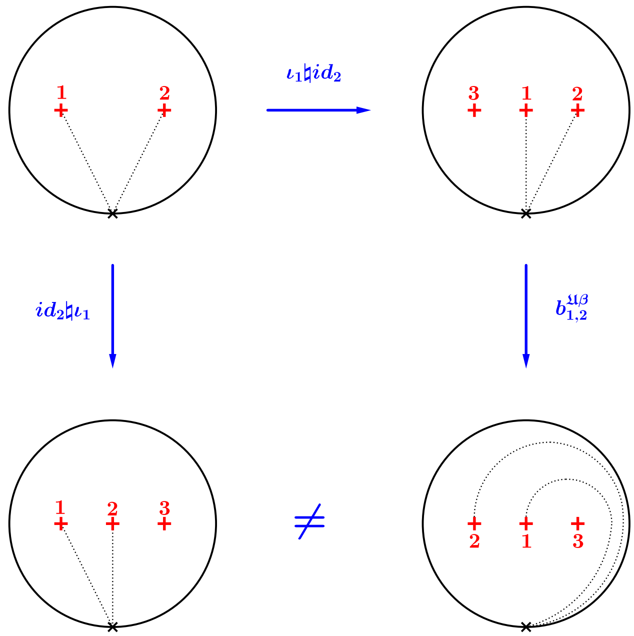


Figure 1.1.1: Failure of the braiding property

Since the groupoid $(\boldsymbol{\beta}, \natural, 0)$ is braided monoidal and it has no zero divisors, we deduce from [RWW17, Proposition 1.8] the following properties.

Proposition 1.1.14. *The category $\mathfrak{U}\boldsymbol{\beta}$ is pre-braided monoidal. The monoidal structure $(\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$ is defined on objects as that of $(\boldsymbol{\beta}, \natural, 0)$ and defined on morphisms letting for $[n'-n, f] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$ and $[m'-m, g] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(m, m')$:*

$$[m'-m, g] \natural [n'-n, f] = \left[(m'-m) \natural (n'-n), (g\natural f) \circ \left(id_{m'-m}\natural \left(b^{\boldsymbol{\beta}}_{m,n'-n}\right)^{-1} \natural id_n\right)\right].$$

In particular, the canonical functor $\boldsymbol{\beta} \to \mathfrak{U}\boldsymbol{\beta}$ is monoidal.

Remark 1.1.15. The category $(\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$ is pre-braided monoidal, but not braided. Indeed, as Figure 1 shows, the pre-braiding defined on $\mathfrak{U}\boldsymbol{\beta}$ is not a braiding: Figure 1 shows that $b^{\boldsymbol{\beta}}_{1,2} \circ (\iota_1 \natural id_2) \neq id_2 \natural \iota_1$ whereas these two morphisms should be equal if $b^{\boldsymbol{\beta}}_{-,-}$ were a braiding.

1.1.2 Examples of functors associated with braid representations

Different families of representations of braid groups can be used to form functors over the pre-braided category $\mathfrak{U}\boldsymbol{\beta}$ to the category $\mathbb{K}\text{-}\mathfrak{Mod}$. Namely, considering $\{M_n : \mathbf{B}_n \to \mathbb{K}\text{-}\mathfrak{Mod}\}_{n\in\mathbb{N}}$ representations of braid groups, or equivalently an object M of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, we are interested in the situations where Proposition 1.1.10 applies so as to define an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

Tong-Yang-Ma results In 1996, in the article [TYM96], Tong, Yang and Ma investigated the representations of $\mathbf{B}_n$ where the i-th generator is sent to a matrix of the form $Id_{i-1} \oplus T \oplus Id_{n-i-1}$, with T a $m \times m$ non-singular matrix and $m \geq 2$. In particular, for $m = 2$, they prove that there exist up to equivalence only two non trivial representations of this type. We give here their result and an interpretation of their work from a functorial point of view, considering the representations over the ring of Laurent polynomials in one variable $\mathbb{C}\left[t^{\pm 1}\right]$.

Notation 1.1.16. Let $\mathfrak{gr}$ denote the full subcategory of $\mathfrak{Gr}$ of finitely generated free groups. The free product $*$: $\mathfrak{gr} \times \mathfrak{gr} \to \mathfrak{gr}$ defines a monoidal structure over $\mathfrak{gr}$, with 0 the unit, denoted by $(\mathfrak{gr}, *, 0)$.

Let $(\mathbb{N}, \leq)$ denote the category of natural numbers (natural means non-negative) considered as a poset. For all natural numbers n, we denote by γ_n the unique element of $Hom_{(\mathbb{N},\leq)}(n, n+1)$. For all natural numbers n

and n' such that $n' \geq n$, we denote by $\gamma_{n,n'} : n \to n'$ the unique element of $Hom_{(\mathbb{N},\leq)}(n,n')$, composition of the morphisms $\gamma_{n'-1} \circ \gamma_{n'-2} \circ \cdots \circ \gamma_{n+1} \circ \gamma_n$. The addition defines a strict monoidal structure on $(\mathbb{N},\leq)$, denoted by $((\mathbb{N},\leq),+,0)$.

Definition 1.1.17. Let $\mathbf{B}_{-} : (\mathbb{N},\leq) \to \mathfrak{Gr}$ and $GL_{-} : (\mathbb{N},\leq) \to \mathfrak{Gr}$ be the functors defined by:

- Objects: for all natural numbers n, $\mathbf{B}_{-}(n) = \mathbf{B}_n$ the braid group on n strands and $GL_{-}(n) = GL_n\left(\mathbb{C}\left[t^{\pm 1}\right]\right)$ the general linear group of degree n.
- Morphisms: let n be a natural number. We define $\mathbf{B}_{-}(\gamma_n) = id_1 \natural - : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$ (where $\natural$ is the monoidal product introduced in Example 1.1.4). We define $GL_{-}(\gamma_n) : GL_n\left(\mathbb{C}\left[t^{\pm 1}\right]\right) \hookrightarrow GL_{n+1}\left(\mathbb{C}\left[t^{\pm 1}\right]\right)$ assigning $GL_{-}(\gamma_n)(\varphi) = id_1 \oplus \varphi$ for all $\varphi \in GL_n\left(\mathbb{C}\left[t^{\pm 1}\right]\right)$.

Notation 1.1.18. For all natural numbers $n \geq 2$, for all $i \in \{1,\ldots,n-1\}$, we denote by $incl_i^n : \mathbf{B}_2 \cong \mathbb{Z} \hookrightarrow \mathbf{B}_n$ the inclusion morphism induced by:

$$incl_i^n(\sigma_1) = \sigma_i.$$

Theorem 1.1.19. *[TYM96, Part II] Let $\eta : \mathbf{B}_{-} \longrightarrow GL_{-}$ be a natural transformation. Assume that for all natural numbers $n \geq 2$, for all $i \in \{1,\ldots,n-1\}$, the following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{B}_n & \xrightarrow{\eta_n} & GL_n\left(\mathbb{C}\left[t^{\pm 1}\right]\right) \\ \uparrow{\scriptstyle incl_i^n} & & \uparrow{\scriptstyle id_{i-1}\oplus - \oplus id_{n-i-1}} \\ \mathbf{B}_2 & \xrightarrow[\eta_2]{} & GL_2\left(\mathbb{C}\left[t^{\pm 1}\right]\right). \end{array}$$

Two such natural transformations η and η' are equivalent if there exists a natural equivalence $\mu : GL_{-} \longrightarrow GL_{-}$ such that $\mu \circ \eta = \eta'$. Then, η is equivalent to one of the following natural transformations.

1. *The trivial natural transformation, denoted by $\mathfrak{id}$: for every generator σ_i of $\mathbf{B}_n$, $\mathfrak{id}_n(\sigma_i) = Id_{GL_n\left(\mathbb{C}\left[t^{\pm 1}\right]\right)}$.*
2. *The unreduced Burau natural transformation, denoted by $\mathfrak{bur}$: for all generators σ_i of $\mathbf{B}_n$,*
$$\mathfrak{bur}_{n,t}(\sigma_i) = Id_{i-1} \oplus B(t) \oplus Id_{n-i-1},$$
with $B(t) = \begin{bmatrix} 1-t & 1 \\ t & 0 \end{bmatrix}$.
3. *The natural transformation denoted by $\mathfrak{tym}$: for every generator σ_i of $\mathbf{B}_n$ if $n \geq 2$,*
$$\mathfrak{tym}_{n,t}(\sigma_i) = Id_{i-1} \oplus TYM(t) \oplus Id_{n-i-1},$$
with $TYM(t) = \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix}$. We call it the Tong-Yang-Ma representation.

The unreduced Burau representation (see [KT08, Section 3.1] or [BB05, Section 4.2] for more details about this family of representations) is reducible but indecomposable, whereas the Tong-Yang-Ma representation is irreducible (see [TYM96, Part II]). We can also consider a natural transformation using the family of reduced Burau representations (see [KT08, Section 3.3] for more details about the associated family of representations): these are irreducible subrepresentations of the unreduced Burau representations.

Definition 1.1.20. Let $GL_{-\text{-}1} : (\mathbb{N},\leq) \to \mathfrak{Gr}$ be the functor defined by:

- Objects: for all natural numbers n, $GL_{-\text{-}1}(n) = GL_{n-1}\left(\mathbb{C}\left[t^{\pm 1}\right]\right)$ the general linear group of degree $n-1$.
- Morphisms: let n be a natural number. We define $GL_{-\text{-}1}(\gamma_n) : GL_{n-1}\left(\mathbb{C}\left[t^{\pm 1}\right]\right) \hookrightarrow GL_n\left(\mathbb{C}\left[t^{\pm 1}\right]\right)$ assigning $GL_{-}(\gamma_n)(\varphi) = id_1 \oplus \varphi$ for all $\varphi \in GL_{n-1}\left(\mathbb{C}\left[t^{\pm 1}\right]\right)$.

Definition 1.1.21. The reduced Burau natural transformation, denoted by $\overline{\mathfrak{bur}}:\mathbf{B}_{-}\to GL_{-\text{-}1}$ is defined by:

- For $n=2$, one assigns $\overline{\mathfrak{bur}}\left(\sigma_{1}\right)=-t$.
- For all natural numbers $n\geq3$, we define for every Artin generator σ_{i} of $\mathbf{B}_{n}$ with $i\in\{2,\ldots,n-2\}$:

$$\overline{\mathfrak{bur}}_{n,t}\left(\sigma_{i}\right)=Id_{i-2}\oplus\overline{B}\left(t\right)\oplus Id_{n-i-2}$$

  with

$$\overline{B}\left(t\right)=\left[\begin{array}{ccc}1 & t & 0\\0 & -t & 0\\0 & 1 & 1\end{array}\right]$$

  and

$$\overline{\mathfrak{bur}}_{n,t}\left(\sigma_{1}\right)=\left[\begin{array}{cc}-t & 0\\1 & 1\end{array}\right]\oplus Id_{n-3}\quad;\quad\overline{\mathfrak{bur}}_{n,t}\left(\sigma_{n-1}\right)=Id_{n-3}\oplus\left[\begin{array}{cc}1 & t\\0 & -t\end{array}\right].$$

Let us use these natural transformations to form functors over the category $\mathfrak{U}\boldsymbol{\beta}$. Indeed, a natural transformation $\eta:\mathbf{B}_{-}\to GL_{-}$ (or $GL_{-\text{-}1}$) provides in particular:

- a functor $\mathfrak{N}:\boldsymbol{\beta}\longrightarrow\mathbb{C}\left[t^{\pm1}\right]\text{-}\mathfrak{Mod}$;
- morphisms $\mathfrak{N}\left([n'-n,id_{n'}]\right):\mathfrak{N}\left(n\right)\to\mathfrak{N}\left(n'\right)$ for all natural numbers $n'\geq n$, such that the relation (1.1.1) of Proposition 1.1.10 is satisfied.

Therefore, according to Proposition 1.1.10, it suffices to show that the relation (1.1.2) is satisfied to prove that $\mathfrak{N}$ is an object of $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},\mathbb{C}\left[t^{\pm1}\right]\text{-}\mathfrak{Mod}\right)$.

Notation 1.1.22. Recall that 0 is a null object in the category of R-modules, and that the notation $\iota_{G}:0\to G$ was introduced in Notation 1.0.1. Let $n\in\mathbb{N}$. For all natural numbers n and n' such that $n'\geq n$, we define $\iota_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'-n}}\oplus id_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}}:\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}\hookrightarrow\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'}$ the embedding of $\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}$ as the submodule of $\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'}$ given by the n last copies of $\mathbb{C}\left[t^{\pm1}\right]$.

Tong-Yang-Ma functor: This example is based on the family introduced by Tong, Yang and Ma (see Theorem 1.1.19). Let $\mathfrak{TYM}_{t}:\boldsymbol{\beta}\to\mathbb{C}\left[t^{\pm1}\right]\text{-}\mathfrak{Mod}$ be the functor defined on objects by $\mathfrak{TYM}_{t}\left(n\right)=\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}$ for all natural numbers n, and for all numbers $n\geq2$, for every Artin generator σ_{i} of $\mathbf{B}_{n}$, by $\mathfrak{TYM}_{t}\left(\sigma_{i}\right)=\mathfrak{tym}_{n,t}\left(\sigma_{i}\right)$ for morphisms. For all natural numbers n and n' such that $n'\geq n$, we assign $\mathfrak{TYM}_{t}\left([n'-n,id_{n'}]\right):\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}\hookrightarrow\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'}$ to be the embedding $\iota_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'-n}}\oplus id_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}}$ (where these morphisms are introduced in Notation 1.1.22). For all natural numbers $n''\geq n'\geq n$, for all Artin generators $\sigma_{i}\in\mathbf{B}_{n}$ and all $\psi_{j}\in\mathbf{B}_{n'-n}$, our assignments give:

$$\begin{aligned}\mathfrak{TYM}_{t}\left(\psi\natural\sigma\right)\circ\mathfrak{TYM}_{t}\left(\left[n'-n,id_{n'}\right]\right) &= \left(Id_{j-1}\oplus TYM\left(t\right)\oplus Id_{(n'-n)-j-1}\oplus Id_{n'-n+i-1}\oplus TYM\left(t\right)\oplus Id_{n'-i-1}\right)\\ &\quad\circ\left(\iota_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'-n}}\oplus id_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}}\right).\end{aligned}$$

Remark that $\left(Id_{j-1}\oplus TYM\left(t\right)\oplus Id_{(n'-n)-j-1}\right)\circ\iota_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus(n'-n)}}=\iota_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus(n'-n)}}$. Hence we deduce that

$$\mathfrak{TYM}_{t}\left(\psi\natural\sigma\right)\circ\mathfrak{TYM}_{t}\left(\left[n'-n,id_{n'}\right]\right)=\mathfrak{TYM}_{t}\left(\left[n'-n,id_{n'}\right]\right)\circ\mathfrak{TYM}_{t}\left(\sigma\right)$$

for all $\sigma\in\mathbf{B}_{n}$ and all $\psi\in\mathbf{B}_{n'-n}$. According to Proposition 1.1.10, our assignment defines a functor $\mathfrak{TYM}_{t}:\mathfrak{U}\boldsymbol{\beta}\to\mathbb{C}\left[t^{\pm1}\right]\text{-}\mathfrak{Mod}$, called the Tong-Yang-Ma functor.

Burau functors: Other examples naturally arise from the Burau representations.

Let $\mathfrak{Bur}_t : \boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ be the functor defined on objects by $\mathfrak{Bur}_t(n) = \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n}$ for all natural numbers n, and for all numbers $n \geq 2$, for every Artin generator σ_i of $\mathbf{B}_n$, by $\mathfrak{Bur}_t(\sigma_i) = \mathfrak{bur}_{n,t}(\sigma_i)$ for morphisms. For all natural numbers n and n' such that $n' \geq n$, we assign $\mathfrak{Bur}_t([n'-n, id_{n'}]) : \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n} \hookrightarrow \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n'}$ to be the embedding $\iota_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n'-n}} \oplus id_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n}}$ (where these morphisms are introduced in Notation 1.1.22).

As for the functor $\mathfrak{TYM}$, the assignment for $\mathfrak{Bur}$ implies that for all natural numbers $n'' \geq n' \geq n$, for all $\sigma \in \mathbf{B}_n$ and all $\psi \in \mathbf{B}_{n'-n}$, $\mathfrak{Bur}_t([n'-n, id_{n'}]) \circ \mathfrak{Bur}_t(\sigma) = \mathfrak{Bur}_t(\psi \natural \sigma) \circ \mathfrak{Bur}_t([n'-n, id_{n'}])$. According to Proposition 1.1.10, our assignment defines a functor $\mathfrak{Bur}_t : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$, called the unreduced Burau functor. This functor $\mathfrak{Bur}_t$ was already considered by Randal-Williams and Wahl in [RWW17, Example 4.3].

Analogously, we can form a functor from the reduced Burau representations. Let $\overline{\mathfrak{Bur}}_t : \boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ be the functor defined on objects by $\overline{\mathfrak{Bur}}_t(0) = 0$ and $\overline{\mathfrak{Bur}}_t(n) = \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n-1}$ for all nonzero natural numbers n, and by $\overline{\mathfrak{Bur}}_t(\sigma_i) = \overline{\mathfrak{bur}}_{n,t}(\sigma_i)$ for morphisms for every Artin generator σ_i of $\mathbf{B}_n$ for all numbers $n \geq 2$.

For all natural numbers n and n' such that $n' \geq n$, we assign $\overline{\mathfrak{Bur}}_t([n'-n, id_{n'}]) : \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n-1} \hookrightarrow \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n'-1}$ to be the embedding $\iota_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n'-n}} \oplus id_{\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n-1}}$ (where these morphisms are introduced in Notation 1.1.22). Repeating mutadis mutandis the work done for the functor $\mathfrak{TYM}$, the assignment for $\overline{\mathfrak{Bur}}_t$ implies that for all natural numbers $n'' \geq n' \geq n$, for all $\sigma \in \mathbf{B}_n$ and all $\psi \in \mathbf{B}_{n'-n}$, $\overline{\mathfrak{Bur}}_t([n'-n, id_{n'}]) \circ \overline{\mathfrak{Bur}}_t(\sigma) = \overline{\mathfrak{Bur}}_t(\psi \natural \sigma) \circ \overline{\mathfrak{Bur}}_t([n'-n, id_{n'}])$. According to Proposition 1.1.10, our assignment defines a functor $\overline{\mathfrak{Bur}}_t : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$, called the reduced Burau functor.

Lawrence-Krammer functor: The family of Lawrence-Krammer representations was notably used to prove that braid groups are linear (see [Big04, Koh12, Kra02]). For this paragraph, we assign $\mathbb{K} = \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$ the ring of Laurent polynomials in two variables and consider the functor GL_- of Definition 1.1.17 with this assignment. Let $\mathfrak{LK} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]\text{-}\mathfrak{Mod}$ be the assignment:

- Objects: for all natural numbers $n \geq 2$, $\mathfrak{LK}(n) = \bigoplus_{1 \leq j < k \leq n} V_{j,k}$, with for all $1 \leq j < k \leq n$, $V_{j,k}$ is a free $\mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$-module of rank one. Hence, $\mathfrak{LK}(n) \cong \left(\mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]\right)^{\oplus n(n-1)/2}$ as $\mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$-modules. Moreover, one assigns $\mathfrak{LK}(1) = 0$ and $\mathfrak{LK}(0) = 0$.
- Morphisms:
  - Automorphisms: for all natural numbers n, for every Artin generator σ_i of $\mathbf{B}_n$ (with $i \in \{1, \ldots, n-1\}$), for all $v_{j,k} \in V_{j,k}$ (with $1 \leq j < k \leq n$),

$$\mathfrak{LK}(\sigma_i) v_{j,k} = \begin{cases} v_{j,k} & \text{if } i \notin \{j-1, j, k-1, k\}, \\ t v_{i,k} + \left(t^2 - t\right) v_{i,i+1} + (1-t) v_{i+1,k} & \text{if } i = j-1, \\ v_{i+1,k} & \text{if } i = j \neq k-1, \\ t v_{j,i} + (1-t) v_{j,i+1} - \left(t^2 - t\right) q v_{i,i+1} & \text{if } i = k-1 \neq j, \\ v_{j,i+1} & \text{if } i = k, \\ -q t^2 v_{i,i+1} & \text{if } i = j = k-1. \end{cases}$$

  - General morphisms: let $n, n' \in \mathbb{N}$, such that $n' \geq n$. For all natural numbers j and k such that $1 \leq j < k \leq n$, we define the embedding $\mathfrak{V}_{j,k}^{n,n'} : V_{j,k} \overset{\sim}{\longrightarrow} V_{j+(n'-n),k+(n'-n)} \hookrightarrow \bigoplus_{1 \leq j < k \leq n'} V_{j,k}$ of free $\mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$-modules. Then we define $\mathfrak{LK}([n'-n, id_{n'}]) : \bigoplus_{1 \leq j < k \leq n} V_{j,k} \to \bigoplus_{1 \leq j < k \leq n'} V_{j,k}$ to be the embedding $\bigoplus_{1 \leq j < k \leq n} \mathfrak{V}_{j,k}^{n,n'}$.

Since we consider a family of representations of $\mathbf{B}_n$ (see [Kra02]), the assignment $\mathfrak{LK}$ defines an object of $\mathbf{Fct}\left(\boldsymbol{\beta}, \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}\right)$.

Let n, n' and n'' be natural numbers such that $n'' \geq n' \geq n$. It follows directly from our definitions of $\mathfrak{LK}([n'-n, id_{n'}])$, $\mathfrak{LK}([n''-n', id_{n''}])$ and $\mathfrak{LK}([n''-n, id_{n''}])$ that relation (1.1.1) of Proposition 1.1.10 is satisfied.

According to the definition of $\mathfrak{LK}(\sigma_l)$ with σ_l an Artin generator of $\mathbf{B}_{n'-n}$, for all $v_{j,k} \in V_{j,k}$ with $1 + (n' - n) \leq j < k \leq n'$, $\mathfrak{LK}(\sigma_l)\, v_{j,k} = v_{j,k}$. Hence for all $\psi \in \mathbf{B}_{n'-n}$:

$$\mathfrak{LK}(\psi \natural id_n) \circ \mathfrak{LK}\left([n'-n, id_{n'}]\right) = \mathfrak{LK}\left([n'-n, id_{n'}]\right).$$

Note also that for all $l \in \{1, \ldots, n-1\}$, for all $v_{j,k} \in V_{j,k}$ with $1 + (n' - n) \leq j < k \leq n'$, it follows from the assignment of $\mathfrak{LK}$ that:

$$\mathfrak{LK}(id_{n'-n} \natural \sigma_l)\left(v_{(n'-n)+j,(n'-n)+k}\right) = \mathfrak{LK}(\sigma_{n'-n+l})\left(v_{(n'-n)+j,(n'-n)+k}\right) = \mathfrak{LK}\left([n'-n, id_{n'}]\right)\left(\mathfrak{LK}(\sigma_l)\left(v_{j,k}\right)\right).$$

Therefore, this implies that for all $\sigma \in \mathbf{B}_n$, $\mathfrak{LK}([n'-n, id_{n'}]) \circ \mathfrak{LK}(\sigma) = \mathfrak{LK}(id_{n'-n} \natural \sigma) \circ \mathfrak{LK}([n'-n, id_{n'}])$. Hence, $\mathfrak{LK}$ satisfies the relation (1.1.2) of Proposition 1.1.10. Hence, the assignment defines a functor $\mathfrak{LK} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]\text{-}\mathfrak{Mod}$, called the Lawrence-Krammer functor.

1.2 Functoriality of the Long-Moody construction

The principle of the Long-Moody construction, corresponding to Theorem 2.1 of [Lon94], is to build a linear representation of the braid group $\mathbf{B}_n$ starting from a representation $\mathbf{B}_{n+1}$. We develop a functorial version of this construction, which leads to the notion of Long-Moody functors (see Section 1.2.2). Beforehand, we need to introduce various tools, which are consequences of the relationships between braid groups and free groups (see Section 1.2.1). Finally, in Section 1.2.3, we investigate examples of functors which are recovered by Long-Moody functors.

1.2.1 Braid groups and free groups

This section recalls some relationships between braid groups and free groups. We also develop tools which will be used throughout our work of Sections 1.2.2 and 1.4.

We consider the free group on n generators, which we denote by $\mathbf{F}_n = \langle g_1, \ldots, g_n \rangle$.

Notation 1.2.1. We denote by $e_{\mathbf{F}_n}$ the unit element of the free group on n generators $\mathbf{F}_n$, for all natural numbers n.

Recall that the category of finitely generated free groups is monoidal using free product of groups (see Notation 1.1.16). The object 0 being null in the category $\mathfrak{gr}$, recall that $\iota_{\mathbf{F}_n} : 0 \to \mathbf{F}_n$ denotes the unique morphism from 0 to $\mathbf{F}_n$ as in Notation 1.0.1.

Definition 1.2.2. Let n be a natural number. We consider $\iota_{\mathbf{F}_1} * id_{\mathbf{F}_n} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n+1}$. This corresponds to the identification of $\mathbf{F}_n$ as the subgroup of $\mathbf{F}_{n+1}$ generated by the n last copies of $\mathbf{F}_1$ in $\mathbf{F}_{n+1}$. Iterating this morphism, we obtain for all natural numbers $n' \geq n$ the morphism $\iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_n} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$.

Let $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n \in \mathbb{N}}$ be a family of group morphisms from the free group $\mathbf{F}_n$ to the braid group $\mathbf{B}_{n+1}$, for all natural numbers n. We require these morphisms to satisfy the following crucial property.

Condition 1.2.3. For all elements $g \in \mathbf{F}_n$, for all natural numbers $n' \geq n$, the following diagram is commutative in the category $\mathfrak{U}\boldsymbol{\beta}$:

$$\begin{array}{ccc}
1 \natural n & \xrightarrow{\varsigma_n(g)} & 1 \natural n \\
{\scriptstyle id_1 \natural [n'-n, id_{n'}]} \downarrow & & \downarrow {\scriptstyle id_1 \natural [n'-n, id_{n'}]} \\
1 \natural n' & \xrightarrow[\varsigma_{n'}\left(e_{\mathbf{F}_{n'-n}} * g\right)]{} & 1 \natural n'.
\end{array}$$

Remark 1.2.4. Condition 1.2.3 will be used to prove that the Long-Moody functor is well defined on morphisms with respect to the tensor product structure in Theorem 1.2.20. Moreover, it will also be used in the proof of Propositions 1.4.14 and 1.4.18.

Lemma 1.2.5. *Condition 2.2.17 is equivalent to assume that for all natural numbers n, for all elements $g \in \mathbf{F}_n$, the morphisms $\{\varsigma_n\}_{n\in\mathbb{N}}$ satisfy the following equality in $\mathbf{B}_{n+2}$:*

$$\left(\left(b^{\beta}_{1,1}\right)^{-1} \natural id_n\right) \circ \left(id_1 \natural \varsigma_n\left(g\right)\right) = \varsigma_{n+1}\left(e_{\mathbf{F}_1} * g\right) \circ \left(\left(b^{\beta}_{1,1}\right)^{-1} \natural id_n\right). \tag{1.2.1}$$

Proof. Let n and n' be natural numbers such that $n' \geq n$. The equality (1.2.1) implies that for all $1 \leq k \leq n' - n$, the following diagram in the category $\boldsymbol{\beta}$ is commutative :

$$\begin{array}{ccc}
1\natural n' & \xrightarrow{id_{n'-(n+k)}\natural\varsigma_{n+k-1}\left(e_{\mathbf{F}_{k-1}}*g\right)} & 1\natural n' \\
\Big\downarrow{\scriptstyle id_{n'-(n+k)}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{(k-1)+n}} & & \Big\downarrow{\scriptstyle id_{n'-(n+k)}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{(k-1)+n}} \\
1\natural n' & \xrightarrow[id_{n'-(n+k)}\natural\varsigma_{n+k}\left(e_{\mathbf{F}_k}*g\right)]{} & 1\natural n'.
\end{array}$$

Hence composing squares, we obtain that the following diagram is commutative in the category $\boldsymbol{\beta}$:

$$\begin{array}{ccccccc}
1\natural\cdots\natural\left(1\natural 1\right)\natural n & \xrightarrow{id_{n'-n-1}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n} & 1\natural\cdots\natural 1\natural\left(1\natural n\right) & \xrightarrow{id_{n'-n-2}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{1+n}} & \cdots & \xrightarrow{\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{n'-1}} & 1\natural n' \\
\Big\downarrow{\scriptstyle id_{n'}\natural\varsigma_n(g)} & & \Big\downarrow{\scriptstyle id_{n'-1}\natural\varsigma_{n+1}\left(e_{\mathbf{F}_1}*g\right)} & & & & \Big\downarrow{\scriptstyle \varsigma_{n'}\left(e_{\mathbf{F}_1}*g\right)} \\
1\natural\cdots\natural 1\natural n & \xrightarrow[id_{n'-n-1}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n]{} & 1\natural\cdots\natural 1\natural\left(1\natural n\right) & \xrightarrow[id_{n'-n-2}\natural\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{1+n}]{} & \cdots & \xrightarrow[\left(b^{\beta}_{1,1}\right)^{-1}\natural id_{n'-1}]{} & 1\natural n'.
\end{array}$$

By definition of the braiding (see Definition 1.1.1), we deduce that the composition of horizontal arrows is the morphism $\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n$ in $\boldsymbol{\beta}$. Recall from Proposition 1.1.14 that $id_1\natural\left[n'-n,\sigma\right] = \left[n'-n, \left(id_1\natural\sigma\right)\circ\left(\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n\right)\right]$. Hence Condition 2.2.17 is satisfied if we assume that the equality (1.2.1) is satisfied for all natural numbers n.

Conversely, assume that Condition 2.2.17 is satisfied. Condition 2.2.17 with $n' = n+1$ ensures that:

$$\left[1, \left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right)\circ\left(id_1\natural\varsigma_n\left(g\right)\right)\right] = \left[1, \varsigma_{n'}\left(e_{\mathbf{F}_1}*g\right)\circ\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right)\right].$$

Since $Aut_{\mathfrak{U}\boldsymbol{\beta}}\left(1\right) = \mathbf{B}_1$ is the trivial group, we deduce from the defining equivalence relation of $\mathfrak{U}\boldsymbol{\beta}$ (see Definition 1.1.5) the equality in $\mathbf{B}_{n+2}$:

$$\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right)\circ\left(id_1\natural\varsigma_n\left(g\right)\right) = \varsigma_{1+n}\left(e_{\mathbf{F}_1}*g\right)\circ\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right).$$

□

Remark 1.2.6. It follows from Lemma 1.2.5 that, for $i \geq 2$, $\varsigma_n(g_i)$ is determined by $\varsigma_k(g_1)$ for $k \leq n$ by the equalities (1.2.1).

Example 1.2.7. The family $\varsigma_{n,1}$, based on what is called the pure braid local system in the literature (see [Lon94, Remark p.223]), is defined by the following inductive assignment for all natural numbers $n \geq 1$.

$$\begin{array}{rcl}
\varsigma_{n,1}:\mathbf{F}_n & \longrightarrow & \mathbf{B}_{n+1} \\
g_i & \longmapsto & \begin{cases} \sigma_1^2 & \text{if } i = 1 \\ \sigma_1^{-1}\circ\sigma_2^{-1}\circ\cdots\circ\sigma_{i-1}^{-1}\circ\sigma_i^2\circ\sigma_{i-1}\circ\cdots\circ\sigma_2\circ\sigma_1 & \text{if } i\in\{2,\ldots,n\}. \end{cases}
\end{array}$$

We assign $\varsigma_{0,1}$ to be the trivial morphism.

Proposition 1.2.8. *The family of morphisms $\{\varsigma_{n,1}\}_{n\in\mathbb{N}}$ satisfies Condition 2.2.17.*

Proof. Relation (2.2.2) is trivially satisfied for $n=0$. Let $n \geq 1$ be a fixed natural number. By definition 1.1.4, we have $\left(b_{1,1}^{\beta}\right)^{-1} = \sigma_1^{-1}$. Moreover, for all $i \in \{2,\ldots,n\}$, we have and $\varsigma_{n+1}\left(e_{\mathbf{F}_1} * g_{i-1}\right) = \varsigma_{n+1}\left(g_i\right)$

$$id_1 \natural \varsigma_{n,1}\left(g_{i-1}\right) = \sigma_2^{-1} \circ \cdots \circ \sigma_{i-1}^{-1} \circ \sigma_i^2 \circ \sigma_{i-1} \circ \cdots \circ \sigma_2.$$

We deduce that:

$$\left(\left(b_{1,1}^{\beta}\right)^{-1} \natural id_n\right) \circ \left(id_1 \natural \varsigma_{n,1}\left(g_{i-1}\right)\right) \circ \left(b_{1,1}^{\beta} \natural id_n\right) = \varsigma_{n,1}\left(g_i\right).$$

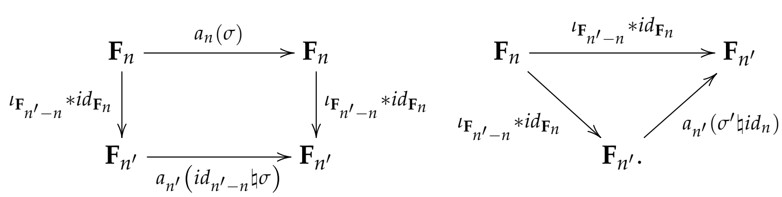
Hence Relation (2.2.2) of Lemma (2.2.18) is satisfied for all natural numbers. □

Example 1.2.9. Let us consider the trivial morphisms $\varsigma_{n,*} : \mathbf{F}_n \to 0_{\mathfrak{Gr}} \to \mathbf{B}_{n+1}$ for all natural numbers n. The relation of Lemma 1.2.5 being easily checked, this family of morphisms $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ satisfies Condition 1.2.3

Action of braid groups on automorphism groups of free groups: There are several ways to consider the group $\mathbf{B}_n$ as a subgroup of $Aut\left(\mathbf{F}_n\right)$. For instance, the geometric point of view of topology gives us an action of $\mathbf{B}_n$ on the free group $\mathbf{F}_n$ (see for example [Bir74a] or [KT08]) identifying $\mathbf{B}_n$ as the mapping class group of a n-punctured disc $\Sigma_{0,1}^n$: fixing a point y on the boundary of the disc $\Sigma_{0,1}^n$, each free generator g_i can be taken as a loop of the disc based y turning around punctures. Each element σ of $\mathbf{B}_n$, as an automorphism up to isotopy of the disc $\Sigma_{0,1}^n$, induces a well-defined action on the fundamental group $\pi_1\left(\Sigma_{0,1}^n\right) \cong \mathbf{F}_n$ called Artin representation (see Example 1.2.15 for more details).

In the sequel, we fix a family of group actions of $\mathbf{B}_n$ on the free group $\mathbf{F}_n$: let $\{a_n : \mathbf{B}_n \to Aut\left(\mathbf{F}_n\right)\}_{n\in\mathbb{N}}$ be a family of group morphisms from the braid group $\mathbf{B}_n$ to the automorphism group $Aut\left(\mathbf{F}_n\right)$. For the work of Sections 1.2.2 and 1.4, we need the morphisms $a_n : \mathbf{B}_n \to Aut\left(\mathbf{F}_n\right)$ to satisfy more properties.

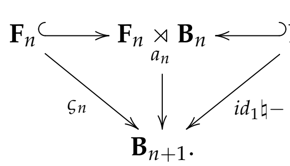
Condition 1.2.10. Let n and n' be natural numbers such that $n' \geq n$. We require $\left(\iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_n}\right) \circ \left(a_n(\sigma)\right) = \left(a_{n'}\left(\sigma' \natural \sigma\right)\right) \circ \left(\iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_n}\right)$ as morphisms $\mathbf{F}_n \to \mathbf{F}_{n'}$ for all elements σ of $\mathbf{B}_n$ and σ' of $\mathbf{B}_{n'-n}$, ie the following diagrams are commutative:



Remark 1.2.11. Condition 1.2.10 will be used to define the Long-Moody functor on morphisms in Theorem 1.2.20. Moreover, it will also be used for the proof of Propositions 1.4.14 and 1.4.18.

We will also require the families of morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut\left(\mathbf{F}_n\right)\}_{n\in\mathbb{N}}$ to satisfy the following compatibility relations.

Condition 1.2.12. Let n be a natural number. We assume that the morphism given by the coproduct $\varsigma_n * (id_1 \natural -) : \mathbf{F}_n * \mathbf{B}_n \to \mathbf{B}_{n+1}$ factors across the canonical surjection to $\mathbf{F}_n \underset{a_n}{\rtimes} \mathbf{B}_n$. In other words, the following diagram is commutative:



where the morphism $\mathbf{F}_n \underset{a_n}{\rtimes} \mathbf{B}_n \to \mathbf{B}_{n+1}$ is induced by the morphism $\mathbf{F}_n * \mathbf{B}_n \to \mathbf{B}_{n+1}$ and the group morphism $id_1 \natural - : \mathbf{B}_n \to \mathbf{B}_{n+1}$ is induced by the monoidal structure. This is equivalent to requiring that, for all elements $\sigma \in \mathbf{B}_n$ and $g \in \mathbf{F}_n$, the following equality holds in $\mathbf{B}_{n+1}$:

$$\left(id_1 \natural \sigma\right) \circ \varsigma_n(g) = \varsigma_n\left(a_n(\sigma)(g)\right) \circ \left(id_1 \natural \sigma\right). \quad (1.2.2)$$

Remark 1.2.13. Condition 1.2.12 is essential in the definition of the Long-Moody functor on objects in Theorem 1.2.20.

We fix a choice for these families of morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$.

Definition 1.2.14. The families $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ are said to be coherent if they satisfy conditions 1.2.3, 1.2.10 and 1.2.12.

Example 1.2.15. A classical family is provided by the Artin representations (see for example [Bir74b, Section 1]). For $n \in \mathbb{N}$, $a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)$ is defined for all elementary braids σ_i where $i \in \{1, \ldots, n-1\}$ by:

$$\begin{array}{rcl} a_{n,1}(\sigma_i) : \mathbf{F}_n & \longrightarrow & \mathbf{F}_n \\ g_j & \longmapsto & \begin{cases} g_{i+1} & \text{if } j = i \\ g_{i+1}^{-1} g_i g_{i+1} & \text{if } j = i+1 \\ g_j & \text{if } j \notin \{i, i+1\}. \end{cases} \end{array}$$

It clearly follows from their definitions that the morphisms $\{a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ satisfy Condition 1.2.10.

The morphisms $\{a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ together with the morphisms $\{\varsigma_{n,1} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ of Example 1.2.7 satisfy Condition 2.2.24.

Proof. Let i be a fixed natural number in $\{1, \ldots, n-1\}$. We prove that the equality (1.2.2) of Condition 2.2.24 is satisfied for all Artin generator σ_i and all generator g_j of the free group (with $j \in \{1, \ldots, n\}$). First, it follows from the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ that:

$$\sigma_{1+i}^{-1} \circ \sigma_i^{-1} \circ \sigma_{1+i}^{-2} \circ \sigma_i^{2} \circ \sigma_{1+i}^{2} \circ \sigma_i \circ \sigma_{1+i} = \sigma_i^{-1} \circ \sigma_{1+i}^{2} \circ \sigma_i,$$

and we deduce that:

$$\sigma_{1+i}^{-1} \circ \varsigma_{n,1}\left(a_{n,1}(\sigma_i)(g_{1+i})\right) \circ \sigma_{1+i} = \varsigma_{n,1}(g_{1+i}).$$

Also, the braid relation $\sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1} = \sigma_i \circ \sigma_{i+1} \circ \sigma_i$ implies that $\sigma_1^{-1} \circ \sigma_{i+1}^{2} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i^{2} \circ \sigma_{i+1}^{-1}$ and a fortiori:

$$\sigma_{1+i}^{-1} \circ \varsigma_{n,1}\left(a_{n,1}(\sigma_i)(g_i)\right) \circ \sigma_{1+i} = \varsigma_{n,1}((g_i)).$$

Finally, for a fixed $j \notin \{i, i+1\}$, the commutation relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ and from the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ give directly:

$$\varsigma_{n,1}(g_j) = \sigma_{1+i}^{-1} \circ \varsigma_{n,1}\left(a_{n,1}(\sigma_i)(g_j)\right) \circ \sigma_{1+i}.$$

□

Corollary 1.2.16. *The families of morphisms* $\{a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ *and* $\{\varsigma_{n,1} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ *are coherent.*

Example 1.2.17. Consider the family of morphisms $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ of Example 1.2.9 and any family of morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$. Then Condition 1.2.12 is always satisfied. As a consequence, these families of morphisms $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ are coherent if and only if the family of morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ satisfies Condition 1.2.10.

1.2.2 The Long-Moody functors

In this section, we prove that the Long-Moody construction of [Lon94, Theorem 2.1] induces a functor

$$\mathbf{LM} : \mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}).$$

We fix families of morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$, which are assumed to be coherent (see Definition 1.2.14).

We first need to make some observations and introduce some tools. Let F be an object of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$ and n be a natural number. A fortiori, the $\mathbb{K}$-module $F(n+1)$ is endowed with a left $\mathbb{K}[\mathbf{B}_{n+1}]$-module structure. Using the morphism $\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}$, $F(n+1)$ is a $\mathbb{K}[\mathbf{F}_n]$-module by restriction.

Let us consider the augmentation ideal of the free group $\mathbf{F}_n$, denoted by $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$. Since it is a (right) $\mathbb{K}[\mathbf{F}_n]$-module, one can form the tensor product $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+1)$. Also, for all natural numbers n and n' such that $n' \geq n$, the morphism $\iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_n} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$ canonically induces a morphism $\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \hookrightarrow \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}$. In addition, the augmentation ideal $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a $\mathbb{K}[\mathbf{B}_n]$-module too:

Lemma 1.2.18. *The action $a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)$ canonically induces an action of $\mathbf{B}_n$ on $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ denoted by $a_n : \mathbf{B}_n \to Aut\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\right)$ (abusing the notation).*

Proof. For any group morphism $H \to Aut(G)$, the group ring $\mathbb{K}[G]$ is canonically an H-module and so is the augmentation ideal $\mathcal{I}_G$, as a submodule of $\mathbb{K}[G]$. □

Remark 1.2.19. If the family of morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ is coherent with respect to the family of morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$, the relation of Condition 1.2.10 remains true mutatis mutandis, for all natural numbers n and n', considering the induced morphisms $a_n : \mathbf{B}_n \to Aut\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\right)$ and $\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \to \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}$.

In the following theorem, we define an endofunctor of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ corresponding to the Long-Moody construction. It will be called the Long-Moody functor with respect to $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$.

Theorem 1.2.20. *Recall that we have fixed coherent families of morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$. The following assignment defines a functor $\mathbf{LM}_{a,\varsigma} : \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.*

- *Objects: for $F \in Obj(\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}))$, $\mathbf{LM}_{a,\varsigma}(F) : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ is defined by:*
  - *Objects: $\forall n \in \mathbb{N}$, $\mathbf{LM}_{a,\varsigma}(F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+1)$.*
  - *Morphisms: for $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$, assign:*

$$\mathbf{LM}_{a,\varsigma}(F)([n'-n,\sigma])\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = a_{n'}(\sigma)\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i) \underset{\mathbb{K}[\mathbf{F}_{n'}]}{\otimes} F(id_1 \natural [n'-n,\sigma])(v),$$

  for all $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n+1)$.

- *Morphisms: let F and G be two objects of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, and $\eta : F \to G$ be a natural transformation. We define $\mathbf{LM}_{a,\varsigma}(\eta) : \mathbf{LM}_{a,\varsigma}(F) \to \mathbf{LM}_{a,\varsigma}(G)$ for all natural numbers n by:*

$$(\mathbf{LM}_{a,\varsigma}(\eta))_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} \eta_{n+1}.$$

In particular, the Long-Moody functor $\mathbf{LM}_{a,\varsigma}$ induces an endofunctor of the category $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

Notation 1.2.21. When there is no ambiguity, once the morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ are fixed, we omit them from the notation $\mathbf{LM}_{a,\varsigma}$ for convenience (especially for proofs).

Proof. For this proof, n, n' and n'' are natural numbers such that $n'' \geq n' \geq n$.

1. First let us show that the assignment of $\mathbf{LM}$ defines an endofunctor of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. The two first points generalize the proof of [Lon94, Theorem 2.1]. Let F, G and H be objects of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.
   (a) We first check the compatibility of the assignment $\mathbf{LM}(F)$ with respect to the tensor product. Consider $\sigma \in \mathbf{B}_n$ $g \in \mathbf{F}_n$, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n+1)$. Since $(id_1 \natural \sigma) \circ \varsigma_n(g) = \varsigma_n(a_n(\sigma)(g)) \circ (id_1 \natural \sigma)$ by

Condition 1.2.12, we deduce that:

$$\begin{aligned}\mathbf{LM}(F)(\sigma)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(\varsigma_n(g))(v)\right) &= a_n(\sigma)(i) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(id_1\natural\sigma)(F(\varsigma_n(g))(v)) \\ &= a_n(\sigma)(i) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} (F(\varsigma_n(a_n(\sigma)(g))) \circ F(id_1\natural\sigma))(v) \\ &= a_n(\sigma)(i\cdot g) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(id_1\natural\sigma)(v) \\ &= \mathbf{LM}(F)(\sigma)\left(i\cdot g \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} (v)\right).\end{aligned}$$

(b) Let us prove that the assignment $\mathbf{LM}(F)$ defines an object of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. According to our assignment and since a_n and $id_1\natural-$ are group morphisms, it follows from the definition that $\mathbf{LM}(F)(id_{\mathbf{B}_n}) = id_{\mathbf{LM}(F)(n)}$. Hence, it remains to prove that the composition axiom is satisfied. Let σ and σ' be two elements of $\mathbf{B}_n$, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n+1)$. From the functoriality of F over $\boldsymbol{\beta}$ and the compatibility of the monoidal structure $\natural$ with composition, we deduce that $F(id_1\natural(\sigma')) \circ F(id_1\natural(\sigma)) = F(id_1\natural(\sigma'\circ\sigma))$. Since a_n is a group morphism, we have:

$$(a_n(\sigma'\circ\sigma))(i) = a_n(\sigma')(a_n(\sigma)(i)).$$

Hence, it follows from the assignment of $\mathbf{LM}$ that:

$$\begin{aligned}\mathbf{LM}(F)(\sigma'\circ\sigma)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) &= (a_n(\sigma'\circ\sigma))(i) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(id_1\natural(\sigma'\circ\sigma))(v) \\ &= a_n(\sigma')(a_n(\sigma)(i)) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} (F(id_1\natural(\sigma')) \circ F(id_1\natural(\sigma)))(v) \\ &= \mathbf{LM}(F)(\sigma') \circ \mathbf{LM}(F)(\sigma)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right).\end{aligned}$$

(c) It remains to check the consistency of our definition of $\mathbf{LM}$ on morphisms of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Let $\eta : F \to G$ be a natural transformation. Hence, we have that:

$$G(id_1\natural\sigma) \circ \eta_{n+1} = \eta_{n'+1} \circ F(id_1\natural\sigma).$$

Hence, it follows from the assignment of $\mathbf{LM}$ that:

$$\mathbf{LM}(G)(\sigma) \circ \mathbf{LM}(\eta)_n = \mathbf{LM}(\eta)_{n'} \circ \mathbf{LM}(F)(\sigma)$$

Therefore $\mathbf{LM}(\eta)$ is a morphism in the category $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Denoting by $id_F : F \to F$ the identity natural transformation, it is clear that $\mathbf{LM}(id_F) = id_{\mathbf{LM}(F)}$. Finally, let us check the composition axiom. Let $\eta : F \to G$ and $\mu : G \to H$ be natural transformations. Let n be a natural number, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n)$. Now, since μ and η are morphisms in the category $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$:

$$\mathbf{LM}(\mu\circ\eta)_n\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} (\mu_{n+1}\circ\eta_{n+1})(v) = \mathbf{LM}(\mu)_n \circ \mathbf{LM}(\eta)_n\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right).$$

2. Let us prove that the assignment $\mathbf{LM}$ lifts to define an endofunctor of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Let F, G and H be objects of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

(a) First, let us check the compatibility of the assignment $\mathbf{LM}(F)$ with respect to the tensor product. In fact, this compatibility being already done for automorphisms (see 1a), the remaining point to prove is the compatibility of $\mathbf{LM}(F)([n'-n, id_{n'}])$. Let $g \in \mathbf{F}_n$, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n+1)$. It follows from Condition 1.2.3 that in $\mathbf{B}_{n+1}$:

$$id_1\natural\left[n'-n, id_{n'-n}\natural\varsigma_n(g)\right] = \varsigma_{n'}\left(e_{\mathbf{F}_{n'-n}} * g\right) \circ \left(id_1\natural\left[n'-n, id_{n'}\right]\right).$$

Since $\left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i\cdot g) = \left(e_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} * i\right)\cdot\left(e_{\mathbf{F}_{n'-n}} * g\right)$, we deduce that:

$$\begin{aligned}
&\mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right)\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F\left(\varsigma_n(g)\right)(v)\right)\\
= & \left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i) \underset{\mathbb{K}\left[\mathbf{F}_{n'}\right]}{\otimes} F\left(id_1 \natural \left[n'-n, id_{n'}\right]\right)\left(F\left(\varsigma_n(g)\right)(v)\right)\\
= & \left(\iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i\cdot g) \underset{\mathbb{K}\left[\mathbf{F}_{n'}\right]}{\otimes} F\left(id_1 \natural \left[n'-n, id_{n'}\right]\right)(v)\\
= & \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right)\left(i\cdot g \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right).
\end{aligned}$$

(b) Let us prove that the assignment $\mathbf{LM}(F)$ defines an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ using Proposition 1.1.10. Recall the compatibility of the monoidal structure $\natural$ with respect to composition and that F is an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Consider $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$. It follows from our assignment, that:

$$\mathbf{LM}(F)\left(\left[n'-n, \sigma\right]\right) = \mathbf{LM}(F)(\sigma) \circ \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right).$$

Moreover, the composition of morphisms introduced in Definition 1.2.2 implies that:

$$\mathbf{LM}(F)\left(\left[n''-n, id_{n''}\right]\right) = \mathbf{LM}(F)\left(\left[n''-n', id_{n''}\right]\right) \circ \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right).$$

Hence, the relation (1.1.1) of Proposition 1.1.10 is satisfied. Let $\sigma \in \mathbf{B}_n$ and $\psi \in \mathbf{B}_{n'-n}$. Since $(\iota_{n'-n} * id_n) \circ (a_n(\sigma)) = (a_{n'}(\psi \natural \sigma)) \circ (\iota_{n'-n} * id_n)$ by Condition 1.2.10, we deduce that:

$$\mathbf{LM}(F)(\psi \natural \sigma) \circ \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right) = \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right) \circ \mathbf{LM}(F)(\sigma).$$

Hence the relation (1.1.2) of Proposition 1.1.10 is also satisfied. Therefore, according to Proposition 1.1.10, since $\mathbf{LM}(F)$ is an object of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, the assignment $\mathbf{LM}(F)$ defines an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

(c) Finally, let us check the consistency of our assignment for $\mathbf{LM}$ on morphisms. Let $\eta : F \to G$ be a natural transformation. We already proved in 1c that $\mathbf{LM}(\eta)$ is a morphism in the category $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Since η is a natural transformation between objects of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, we have that:

$$G\left(id_1 \natural \left[n'-n, id_{n'}\right]\right) \circ \eta_{n+1} = \eta_{n'+1} \circ F\left(id_1 \natural \left[n'-n, id_{n'}\right]\right).$$

Hence, it follows from the assignment of $\mathbf{LM}$ that:

$$\mathbf{LM}(G)\left(\left[n'-n, id_{n'}\right]\right) \circ \mathbf{LM}(\eta)_n = \mathbf{LM}(\eta)_{n'} \circ \mathbf{LM}(F)\left(\left[n'-n, id_{n'}\right]\right).$$

Hence the relation (1.1.3) of Proposition 1.1.12 is satisfied, and we deduce from this last proposition that $\mathbf{LM}(\eta)$ is a morphism in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. The verification of the composition axiom repeats mutatis mutandis the one of 1c.

□

Recall the following fact on the augmentation ideal of the free group $\mathbf{F}_n$ where $n \in \mathbb{N}$.

Proposition 1.2.22. *[Wei94, Chapter 6, Proposition 6.2.6] The augmentation ideal $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a free $\mathbb{K}[\mathbf{F}_n]$-module with basis the set $\{(g_i - 1) \mid i \in \{1, \ldots, n\}\}$.*

This result allows us to prove the following properties.

Proposition 1.2.23. *The functor $\mathbf{LM}_{a,\varsigma} : \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ is reduced and exact. Moreover, it commutes with all colimits and all finite limits.*

Proof. Let $0_{\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ denote the null functor. It follows from the definition of the Long-Moody functor that $\mathbf{LM}\left(0_{\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})}\right) = 0_{\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})}$.
Let n be a natural number. Since the augmentation ideal $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a free $\mathbb{K}[\mathbf{F}_n]$-module (as stated in Proposition 1.2.22), it is therefore a flat $\mathbb{K}[\mathbf{F}_n]$-module. Then, the result follows from the fact that the functor $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} - : \mathbb{K}\text{-}\mathfrak{Mod} \to \mathbb{K}\text{-}\mathfrak{Mod}$ is an exact functor, the naturality for morphisms following from the definition of the Long-Moody functor (see Theorem 1.2.20).

Similarly, the fact that the functor $\mathbf{LM}_{a,\varsigma}$ commutes with all colimits is a formal consequence of the commutation with all colimits of the tensor products $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} -$ for all natural numbers n. The commutation result for finite limits is a property of exact functors (see for example [ML13, Chapter 8, section 3]). $\square$

Remark 1.2.24. Let F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$ and n a natural number. For all $k \in \{1,\ldots,n\}$, we denote $F(n+1)_k = \mathbb{K}[(g_k-1)] \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+1)$ with g_k a generator of $\mathbf{F}_n$. We define an isomorphism

$$\begin{array}{rccc} \Lambda_{n,F} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+1) & \longrightarrow & \bigoplus_{k=1}^{n} F(n+1)_k \cong (F(n+1))^{\oplus n} \\ (g_k-1) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v & \longmapsto & \left(0,\ldots,0,\overbrace{v}^{k\text{-th}},0,\ldots,0\right). \end{array}$$

Thus, for $\eta : F \to G$ a natural transformation, with Λ:

$$\forall n \in \mathbb{N},\ \Lambda_n\left((\mathbf{LM}(\eta))_n\right) = \eta_{n+1}^{\oplus n}.$$

Hence, we can have a matricial point of view on this construction (see [Lon94, Theorem 2.2]). Similarly, the study of Bigelow and Tian in [BT08] is performed from a purely matricial point of view.

Case of trivial ς: Finally, let us consider the family of morphisms $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ of Example 1.2.9.

Remark 1.2.25. As stated in Example 1.2.17, we only need to consider a family of morphisms $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ which satisfies Condition 1.2.10 so that the families $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ are coherent.

Notation 1.2.26. We denote by $\mathfrak{X} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ the constant functor such that $\mathfrak{X}(n) = \mathbb{K}$ for all natural numbers n.

We have the following remarkable property.

Proposition 1.2.27. *Let F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ a family of morphisms which satisfies Condition 1.2.10. Then, as objects of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$, $\mathbf{LM}_{a,\varsigma_*}(F) \cong \mathbf{LM}_{a,\varsigma_*}(\mathfrak{X}) \underset{\mathbb{K}}{\otimes} F(1\natural -)$.*

Proof. Remark 1.2.24 shows that there is an isomorphism of $\mathbb{K}$-modules of the form:

$$\mathbf{LM}_{a,\varsigma_*}(F)(n) \xrightarrow{\Lambda_{n,F}} (F(n+1))^{\oplus n} \xrightarrow{\left(\Lambda_{n,\mathfrak{X}} \underset{\mathbb{K}}{\otimes} id_{F(1\natural n)}\right)^{-1}} \mathbf{LM}_{a,\varsigma_*}(\mathfrak{X})(n) \underset{\mathbb{K}}{\otimes} F(1\natural n).$$

It is straightforward to check that this isomorphism is natural if ς is trivial. $\square$

1.2.3 Evaluation of the Long-Moody functor

A first step to understand the behaviour of a Long-Moody endofunctor is to investigate its effect on the constant functor $\mathfrak{X}$. This is indeed the most basic functor to study. Moreover, as Proposition 2.2.39 shows, the evaluation on this functor is the fundamental information to understand a given Long-Moody endofunctor when we consider the family of morphisms $\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ of Example 2.2.22.

Fixing coherent families of morphisms $\{\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$, we consider the Long-Moody functor

$$\mathbf{LM}_{a,\varsigma} : \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}).$$

For a fixed natural number n, using the isomorphism Λ_n of Remark 1.2.24, we observe that $\mathbf{LM}_{a,\varsigma}(\mathfrak{X})(n) \cong \mathbb{K}^{\oplus n}$.

Notation 1.2.28. Let y be an invertible element of $\mathbb{K}$. Let $y\mathfrak{X} : \boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ be the functor defined for all natural numbers n by $y\mathfrak{X}(n) = \mathbb{K}$ and such that:

- if $n = 0$ or $n = 1$, then $y\mathfrak{X}(id) = id_{\mathbb{K}}$;
- if $n \geq 2$, for every Artin generator σ_i of $\mathbf{B}_n$, $(y\mathfrak{X})(\sigma_i) : \mathbb{K} \to \mathbb{K}$ is the multiplication by y.

For an object F of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, we denote the functor $y\mathfrak{X} \underset{\mathbb{K}}{\otimes} F : \boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ by yF.

1.2.3.1 Computations for $\mathbf{LM}_1$

Let us assume that $\mathbb{K} = \mathbb{C}\left[t^{\pm 1}\right]$. Let us consider the coherent families of morphisms $\{\varsigma_{n,1} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ (introduced in Example 1.2.7) and $\{a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ (introduced in Example 1.2.15). We denote by $\mathbf{LM}_1$ the associated Long-Moody functor. We are interested in the behaviour of the functor $t^{-1}\mathbf{LM}_1(t\mathfrak{X}) : \boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ on automorphisms of the category $\mathfrak{U}\boldsymbol{\beta}$. Indeed, adding a parameter t is necessary to recover functors specifically associated with the category $\mathfrak{U}\boldsymbol{\beta}$, such as $\mathfrak{Bur}_t$ (see Section 1.1.2). Let us fix n a natural number and σ_i an Artin generator of $\mathbf{B}_n$.

Beforehand, let us understand the action $a_{n,1} : \mathbf{B}_n \longrightarrow Aut\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\right)$ induced by $a_{n,1} : \mathbf{B}_n \to Aut(\mathbf{F}_n)$. We compute:

$$\begin{array}{rcl} a_{n,1}(\sigma_i) : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} & \longrightarrow & \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \\ g_j - 1 & \longmapsto & \begin{cases} g_{i+1} - 1 & \text{if } j = i \\ g_{i+1}^{-1} g_i g_{i+1} - 1 = [g_i - 1]\, g_{i+1} + [g_{i+1} - 1]\left(1 - g_{i+1}^{-1} g_i g_{i+1}\right) & \text{if } j = i+1 \\ g_j - 1 & \text{if } j \notin \{i, i+1\}. \end{cases} \end{array}$$

Notation 1.2.29. Let us fix the matrices $r_n = \overbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}}^{n}$ for all natural numbers n.

Hence, we have the following result.

Proposition 1.2.30. *The matrices $\{r_n\}_{n\in\mathbb{N}}$ define a natural equivalence $t^{-1}\mathbf{LM}_1(t\mathfrak{X}) \xrightarrow{r} \mathfrak{Bur}_{t^2}$ as objects of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.*

Proof. Using the isomorphism Λ_n of Remark 1.2.24, we obtain that for σ_i an Artin generator of $\mathbf{B}_n$:

$$t^{-1}\mathbf{LM}_1(t\mathfrak{X})(\sigma_i) = Id_{i-1} \oplus \begin{bmatrix} 0 & t^2 \\ 1 & 1 - t^2 \end{bmatrix} \oplus Id_{n-i-1}.$$

We deduce that $r_n \circ \left(t^{-1}\mathbf{LM}_1(t\mathfrak{X})(\sigma_i)\right) \circ r_n^{-1} = \mathfrak{Bur}_{t^2}(\sigma_i)$. □

Recovering of the Lawrence-Krammer functor: Let us first introduce the following result due to Long in [Lon94]. We assume that $\mathbb{K} = \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$. For this paragraph, we assume that $1 + qt = 0$, q has a square root, $q^2 \neq 1$ and $q^3 \neq 1$.

Notation 1.2.31. We denote by $\mathfrak{X}' : \boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]\text{-}\mathfrak{Mod}$ the constant functor such that $\mathfrak{X}'(n) = \mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$ for all natural numbers n. Generally speaking, for F an object of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ the representation of $\mathbf{B}_n$ induced by F will be denoted by $F_{|\mathbf{B}_n}$.

Proposition 1.2.32. *[Lon94, special case of Corollary 2.10] Let n be a natural number such that $n \geq 4$. Then, the Lawrence-Krammer representation $\mathfrak{LK}_{|\mathbf{B}_n}$ is a subrepresentation of $q^{-1}\left(\mathbf{LM}_1\left(q\left(t^{-1}\mathbf{LM}_1\left(t\mathfrak{X}\right)\right)\right)\right)_{|\mathbf{B}_n}$.*

We first need to introduce new tools. Let n and m be two natural numbers. Let $\underline{w}_n = (w_1, \ldots, w_n) \in \mathbb{C}^n$ such that $w_i \neq w_j$ if $i \neq j$. We consider the configuration space:

$$Y_{\underline{w}_n,m} = \left\{(z_1, \ldots, z_m) \mid z_i \in \mathbb{C},\, z_i \neq w_k \text{ for } 1 \leq k \leq n,\, z_i \neq z_j \text{ if } i \neq j\right\}.$$

The two following results due to Long will be crucial to prove Proposition 1.2.32.

Proposition 1.2.33. *[Lon94, Corollary 2.7] Let n be a natural number and $\rho : \mathbf{B}_{n+1} \to GL(V)$ be a representation of $\mathbf{B}_n$ with V a $\mathbb{C}\left[t^{\pm 1}\right]\left[q^{\pm 1}\right]$-module. Then, the representation defined by Long in [Lon94, Theorem 2.1], which we denote by $\mathcal{LM}$, is a group morphism:*

$$q^{-1}\mathcal{LM}(q\rho) : \mathbf{B}_n \to GL\left(H^1\left(Y_{\underline{w}_n,1}, E_\rho\right)\right)$$

for E_ρ a flat vector bundle associated with ρ (see [Lon94, p. 225-226]).

Lemma 1.2.34. *[Lon94, Lemma 2.9] For all natural numbers m, there is an isomorphism of abelian groups:*

$$H^{m+1}\left(Y_{\underline{w}_n,m+1}, E_{\mathfrak{X}_{|\mathbf{B}_n}}\right) \cong H^1\left(Y_{\underline{w}_n,1}, H^m\left(Y_{\underline{w}_{n+1},m}, E_{\mathfrak{X}_{|\mathbf{B}_n}}\right)\right).$$

In particular, for $m = 1$, $H^2\left(Y_{\underline{w}_n,2}, E_{\mathfrak{X}_{|\mathbf{B}_n}}\right) \cong H^1\left(Y_{\underline{w}_n,1}, H^1\left(Y_{\underline{w}_{n+1},2}, E_{\mathfrak{X}_{|\mathbf{B}_n}}\right)\right)$.

Proof of Proposition 2.33. By Proposition 1.2.33, we can write as a representation:

$$q^{-1}\mathcal{LM}\left(q\left(t^{-1}\mathcal{LM}(t\mathfrak{X})\right)\right) : \mathbf{B}_n \to GL\left(H^1\left(Y_{\underline{w}_n,1}, E_{t^{-1}\mathcal{LM}(t\mathfrak{X})}\right)\right).$$

A fortiori by Lemma 1.2.34, $q^{-1}\mathcal{LM}\left(q\left(t^{-1}\mathcal{LM}\left(t\mathfrak{X}_{|\mathbf{B}_n}\right)\right)\right)$ is an action of $\mathbf{B}_n$ on $H^2\left(Y_{\underline{w}_n,2}, E_{\mathfrak{X}_{|\mathbf{B}_n}}\right)$. In particular, for $m = 2$ and $n \geq 4$, according to [Law90a, Theorem 5.1], the representation of $\mathbf{B}_n$ factoring through the Iwahori–Hecke algebra $H_n(t)$ corresponding to the Young diagram $(n-2, 2)$ is a subrepresentation of $q^{-1}\mathcal{LM}\left(q\left(t^{-1}\mathcal{LM}\left(t\mathfrak{X}_{|\mathbf{B}_n}\right)\right)\right)$. Moreover, this representation is equivalent to the Lawrence-Krammer representation by [Big03, Section 5]. By the definition of the Long-Moody construction (see [Lon94, Theorem 2.1]), $q^{-1}\mathcal{LM}\left(q\left(t^{-1}\mathcal{LM}\left(t\mathfrak{X}_{|\mathbf{B}_n}\right)\right)\right)$ is the representation $q^{-1}\left(\tau_1\mathbf{LM}_1\right)\left(q\left(t^{-1}\mathbf{LM}_1\left(t\mathfrak{X}\right)\right)\right)_{|\mathbf{B}_n}$. ☐

We denote by $\mathfrak{LK}^{\geq 4} : \boldsymbol{\beta} \longrightarrow \left(\mathbb{C}\left[t^{\pm 1}\right]\right)\left[q^{\pm 1}\right]$-$\mathfrak{Mod}$ the subfunctor of the Lawrence-Krammer defined in Example 1.1.2 which is null on the objects such that $n < 4$. The result of Proposition 1.2.32 implies that:

Proposition 1.2.35. *The functor $\mathfrak{LK}^{\geq 4}$ is a subfunctor of $q^{-1}\left(\tau_1\mathbf{LM}_1\right)\left(q\left(t^{-1}\mathbf{LM}_1\left(t\mathfrak{X}\right)\right)\right)^{\geq 4}$.*

1.2.3.2 Computations for other cases

Let us introduce examples of Long-Moody functors which arise using other actions $a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)$.

Wada representations In 1992, Wada introduced in [Wad92] a certain type of family of representations of braid groups. We give here a functorial approach to this work.

Definition 1.2.36. Let $Aut_- : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ be the functor defined by:

- Objects: for all natural numbers n, $Aut_-(n) = Aut(\mathbf{F}_n)$ the automorphism group of the free group on n generators;
- Morphisms: let n be a natural number. We define $Aut_-(\gamma_n) : Aut(\mathbf{F}_n) \hookrightarrow Aut(\mathbf{F}_{n+1})$ assigning $Aut_-(\gamma_n)(\varphi) = id_1 * \varphi$ for all $\varphi \in Aut(\mathbf{F}_n)$, using the monoidal category $(\mathfrak{gr}, *, 0)$ recalled in Notation 2.2.6.

Definition 1.2.37. Let us consider two different non-trivial reduced words $W(g_1, g_2)$ and $V(g_1, g_2)$ on $\mathbf{F}_2$, such that:

- the assignments $g_1 \mapsto W(g_1, g_2)$ and $g_2 \mapsto V(g_1, g_2)$ define a automorphism of $\mathbf{F}_2$;
- the assignment $(W, V) : \mathbf{B}_2 \longrightarrow Aut(\mathbf{F}_2)$:

$$[(W,V)(\sigma_1)](g_j) = \begin{cases} W(g_1,g_2) & \text{if } j = 1 \\ V(g_1,g_2) & \text{if } j = 2 \end{cases}$$

  is a morphism.

Two morphisms $(W, V) : \mathbf{B}_2 \longrightarrow Aut(\mathbf{F}_2)$ and $(W', V') : \mathbf{B}_2 \to Aut(\mathbf{F}_2)$ are said to be swap-dual if $W'(g_1, g_2) = V(g_2, g_1)$ and $V'(g_1, g_2) = W(g_2, g_1)$, backward-dual if $W'(g_1, g_2) = \left(W\left(g_1^{-1}, g_2^{-1}\right)\right)^{-1}$ and $V'(g_1, g_2) = \left(V\left(g_1^{-1}, g_2^{-1}\right)\right)^{-1}$, inverse if $(W', V') = (W, V)^{-1}$.

Definition 1.2.38. [Wad92] Let $W(g_1, g_2)$ and $V(g_1, g_2)$ be two words on $\mathbf{F}_2$. A natural transformation $\mathcal{W} : \mathbf{B}_- \to Aut_-$ is said to be of Wada-type if for all natural numbers n, for all $i \in \{1, \dots, n-1\}$, the following diagram is commutative (we recall that $incl_i^n$ was introduced in Notation 1.1.18 and $Aut_-(\gamma_{2,i})$ in Definition 1.2.2):

$$\begin{array}{ccc} \mathbf{B}_n & \xrightarrow{\mathcal{W}_n} & Aut(\mathbf{F}_n) \\ \uparrow{\scriptstyle incl_i^n} & & \uparrow{\scriptstyle Aut_-(\gamma_{2,i}) * id_{\mathbf{F}_{n-i-1}}} \\ \mathbf{B}_2 & \xrightarrow[(W,V)]{} & Aut(\mathbf{F}_2). \end{array}$$

Remark 1.2.39. Note that therefore a Wada-type natural transformation is entirely determined by the choice of (W, V).

Wada conjectured a classification of these type of representations. This conjecture was proved by Ito in [Ito13].

Theorem 1.2.40. *[Ito13] There are seven classes of Wada-type natural transformation $\mathcal{W}$ up to the swap-dual, backward-dual and inverse equivalences, listed below.*

1. $(W,V)_{1,m}(g_1,g_2) = \left(g_2, g_2^m g_1 g_2^{-m}\right)$ *where* $m \in \mathbb{Z}$;
2. $(W,V)_2(g_1,g_2) = (g_1,g_2)$;
3. $(W,V)_3(g_1,g_2) = \left(g_2, g_1^{-1}\right)$;
4. $(W,V)_4(g_1,g_2) = \left(g_2, g_2 g_1 g_2^{-1}\right)$;
5. $(W,V)_5(g_1,g_2) = \left(g_1^{-1}, g_2^{-1}\right)$;
6. $(W,V)_6(g_1,g_2) = \left(g_2^{-1}, g_2 g_1 g_2\right)$;
7. $(W,V)_7(g_1,g_2) = \left(g_1 g_2^{-1} g_1^{-1}, g_1 g_2^2\right)$.

Remark 1.2.41. Note that the action given by the first Wada representation with $m = 1$ is a generalization of the Artin representation.

Notation 1.2.42. The actions given by the k-th Wada-type natural transformation will be denoted by $a_{n,k} : \mathbf{B}_n \hookrightarrow Aut(\mathbf{F}_n)$. In particular, for $k = 1$ with $m = 1$, we recover the Artin representation (see Example 1.2.15).

For all $1 \leq k \leq 8$, it clearly follows from their definitions that the families of morphisms $\left\{a_{n,k} : \mathbf{B}_n \to Aut(\mathbf{F}_n)\right\}_{n\in\mathbb{N}}$ satisfy Condition 1.2.10. Hence, for $1 \leq k \leq 8$, we consider a family of morphisms $\left\{\varsigma_{n,k} : \mathbf{F}_n \to \mathbf{B}_{n+1}\right\}$ assumed to be coherent with respect to the morphisms $\left\{a_{n,k} : \mathbf{B}_n \hookrightarrow Aut(\mathbf{F}_n)\right\}_{n\in\mathbb{N}}$ (in the sense of Definition 1.2.14). Such morphisms $\varsigma_{n,k}$ always exist because we could at least take the family of morphisms $\left\{\varsigma_{n,*} : \mathbf{F}_n \to \mathbf{B}_{n+1}\right\}$ (see Example 1.2.17). We denote by $\mathbf{LM}_k : \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ the corresponding Long-Moody functor defined in Theorem 2.2.30 for $k \in \{1, \dots, 8\}$.

Let us imitate the procedure of Section 1.2.3.1. We assume that $\mathbb{K} = \mathbb{C}\left[t^{\pm 1}\right]$. Let n be a fixed natural number. Let us consider the case of $k = 2$. Using the isomorphism Λ_n of Remark 1.2.24, we obtain the functor $\mathbf{LM}_2(\mathfrak{X}) : \boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$, defined for $\sigma_i \in \mathbf{B}_n$ by:

$$\mathbf{LM}_2(F)(\sigma_i) = (F(\sigma_i))^{\oplus n}.$$

For $k = 3$, using Λ_n, we compute that the functor $t^{-1}\mathbf{LM}_3(t\mathfrak{X}) : \boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ is defined for $\sigma_i \in \mathbf{B}_n$ by:

$$t^{-1}\mathbf{LM}_3(t\mathfrak{X})(\sigma_i) = Id_{i-1} \oplus \begin{bmatrix} 0 & -\varsigma_{n,3}(g_i) \\ 1 & 0 \end{bmatrix} \oplus Id_{n-i-1}.$$

Hence, the functor $t^{-1}\mathbf{LM}_3(t\mathfrak{X})$ is very similar to the one associated with the Tong-Yang-Ma representations (recall Definition 1.1.2). We deduce that the identity natural equivalence gives $t^{-1}\mathbf{LM}_3(t\mathfrak{X}) \cong \mathfrak{TYM}_{-\varsigma_{n,3}(g_i)}$ as objects of $\mathbf{Fct}(\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.

For the actions given by the Wada-type natural transformation 4, 5, 6 and 7 in Theorem 1.2.40, the produced functors $t^{-1}\mathbf{LM}_i(t\mathfrak{X}) : \boldsymbol{\beta} \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ are mild variants of what is given by the case $i = 1$.

1.3 Strong polynomial functors

We deal here with the concept of a strong polynomial functor. This type of functor will be the core of our work in Section 4. We review (and actually extend) the definition and properties of a strong polynomial functor due to Djament and Vespa in [DV17] and also a particular case of coefficient systems of finite degree used by Randal-Williams and Wahl in [RWW17].

In [DV17, Section 1], Djament and Vespa construct a framework to define strong polynomial functors in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, where $\mathfrak{M}$ is a symmetric monoidal category, the unit is an initial object and $\mathcal{A}$ is an abelian category. Here, we generalize this definition for functors from pre-braided monoidal categories having the same additional property. In particular, the notion of strong polynomial functor will be defined for the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. The keypoint of this section is Proposition 1.3.2, in so far as it constitutes the crucial property necessary and sufficient to extend the definition of strong polynomial functor to the pre-braided case.

1.3.1 Strong polynomiality

We first introduce the translation functor, which plays the central role in the definition of strong polynomiality.

Definition 1.3.1. Let $(\mathfrak{M}, \natural, 0)$ be a strict monoidal small category, let $\mathfrak{D}$ be a category and let x be an object of $\mathfrak{M}$. The monoidal structure defines the endofunctor $x\natural - : \mathfrak{M} \longrightarrow \mathfrak{M}$. We define the translation by x functor $\tau_x : \mathbf{Fct}(\mathfrak{M}, \mathfrak{D}) \to \mathbf{Fct}(\mathfrak{M}, \mathfrak{D})$ to be the endofunctor obtained by precomposition by the functor $x\natural -$.

The following proposition establishes the commutation of two translation functors associated with two objects of $\mathfrak{M}$. It is the keystone property to define strong polynomial functors.

Proposition 1.3.2. *Let $(\mathfrak{M}, \natural, 0)$ be a pre-braided strict monoidal small category (see Definition 1.1.13) and $\mathfrak{D}$ be a category. Let x and y be two objects of $\mathfrak{M}$. Then, there exists a natural isomorphism between functors from* $\mathbf{Fct}(\mathfrak{M}, \mathfrak{D})$ *to* $\mathbf{Fct}(\mathfrak{M}, \mathfrak{D})$*:*

$$\tau_x \circ \tau_y \cong \tau_y \circ \tau_x.$$

Proof. First, because of the associativity of the monoidal product $\natural$ and the strictness of $\mathfrak{M}$, we have that $\tau_x \circ \tau_y = \tau_{x\natural y}$ and $\tau_y \circ \tau_x = \tau_{y\natural x}$. We denote by $b^{\mathfrak{M}}_{-,-}$ the pre-braiding of $\mathfrak{M}$. The key point is the fact that as $b^{\mathfrak{M}}_{-,-}$ is a braiding on the maximal subgroupoid of $\mathfrak{M}$ (see Definition 1.1.13), $b^{\mathfrak{M}}_{x,y} : x\natural y \xrightarrow{\cong} y\natural x$ defines an isomorphism. Hence, precomposition by $b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}$ defines a natural transformation $\left(b^{\mathfrak{M}}_{x,y}\natural id_{\mathfrak{M}}\right)^* : \tau_{x\natural y} \to \tau_{y\natural x}$. It is an isomorphism since we analogously construct an inverse natural transformation $\left(\left(b^{\mathfrak{M}}_{x,y}\right)^{-1}\natural id_{\mathfrak{M}}\right)^* : \tau_{y\natural x} \to \tau_{x\natural y}$. □

Remark 1.3.3. In Proposition 1.3.2, the natural isomorphism is not unique: as the proof shows, we could have used the morphism $\left(b^{\mathfrak{M}}_{y,x}\right)^{-1}\natural id_{\mathfrak{M}}$ instead to define an isomorphism between $\tau_{x\natural y}(F)$ and $\tau_{y\natural x}(F)$. In fact, a category only needs to be equipped with natural (in x and y) isomorphisms $x\natural y \cong y\natural x$ to satisfy the conclusion of Proposition 1.3.2.

Let us move on to the introduction of the evanescence and difference functors, which will characterize the (very) strong polynomiality of a functor in $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$. Recall that, if $\mathfrak{M}$ is a small category and $\mathcal{A}$ is an abelian category, then the functor category $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ is an abelian category (see [ML13, Chapter VIII]).

From now until the end of Section 1.3, we fix $(\mathfrak{M},\natural,0)$ a pre-braided strict monoidal category such that the monoidal unit 0 is an initial object, $\mathcal{A}$ an abelian category and x denotes an object of $\mathfrak{M}$.

Definition 1.3.4. For all objects F of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$, we denote by $i_x(F):\tau_0(F)\to\tau_x(F)$ the natural transformation induced by the unique morphism $\iota_x:0\to x$ of $\mathfrak{M}$. This induces $i_x: Id_{\mathbf{Fct}(\mathfrak{M},\mathcal{A})}\to\tau_x$ a natural transformation of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$. Since the category $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ is abelian, the kernel and cokernel of the natural transformation i_x exist. We define the functors $\kappa_x=\ker(i_x)$ and $\delta_x=\operatorname{coker}(i_x)$. The endofunctors κ_x and δ_x of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ are called respectively evanescence and difference functor associated with x.

The following proposition presents elementary properties of the translation, evanescence and difference functors. They are either consequences of the definitions, or direct generalizations of the framework considered in [DV17] where $\mathfrak{M}$ is symmetric monoidal.

Proposition 1.3.5. *Let y be an object of $\mathfrak{M}$. Then the translation functor τ_x is exact and we have the following exact sequence in the category of endofunctors of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$:*

$$0\longrightarrow\kappa_x\xrightarrow{\Omega_x}Id\xrightarrow{i_x}\tau_x\xrightarrow{\Delta_x}\delta_x\longrightarrow 0. \tag{1.3.1}$$

Moreover, for a short exact sequence $0\longrightarrow F\longrightarrow G\longrightarrow H\longrightarrow 0$ in the category $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$, there is a natural exact sequence in the category $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$:

$$0\longrightarrow\kappa_x(F)\longrightarrow\kappa_x(G)\longrightarrow\kappa_x(H)\longrightarrow\delta_x(F)\longrightarrow\delta_x(G)\longrightarrow\delta_x(H)\longrightarrow 0. \tag{1.3.2}$$

In addition:

1. *The translation endofunctor τ_x of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ commutes with limits and colimits.*
2. *The difference endofunctors δ_x and δ_y of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ commute up to natural isomorphism. They commute with colimits.*
3. *The endofunctors κ_x and κ_y of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ commute up to natural isomorphism. They commute with limits.*
4. *The natural inclusion $\kappa_x\circ\kappa_x\hookrightarrow\kappa_x$ is an isomorphism.*
5. *The translation endofunctor τ_x and the difference endofunctor δ_y commute up to natural isomorphism.*
6. *The translation endofunctor τ_x and the endofunctor κ_y commute up to natural isomorphism.*
7. *We have the following natural exact sequence in the category of endofunctors of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$:*

$$0\longrightarrow\kappa_y\longrightarrow\kappa_{x\natural y}\longrightarrow\tau_x\kappa_y\longrightarrow\delta_y\longrightarrow\delta_{x\natural y}\longrightarrow\tau_y\delta_x\longrightarrow 0. \tag{1.3.3}$$

Proof. In the symmetric monoidal case, this is [DV17, Proposition 1.4]: the numbered properties are formal consequences of the commutation property of the translation endofunctors given by Proposition 1.3.2. Hence, the proofs carry over mutatis mutandis to the pre-braided setting. □

Using Proposition 1.3.5, we can define strong polynomial functors.

Definition 1.3.6. We recursively define on $n\in\mathbb{N}$ the category $\mathcal{P}ol^{strong}_n(\mathfrak{M},\mathcal{A})$ of strong polynomial functors of degree less than or equal to n to be the full subcategory of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ as follows:

1. If $n<0$, $\mathcal{P}ol^{strong}_n(\mathfrak{M},\mathcal{A})=\{0\}$;

2. if $n \geq 0$, the objects of $\mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$ are the functors F such that for all objects x of $\mathfrak{M}$, the functor $\delta_x(F)$ is an object of $\mathcal{P}ol_{n-1}^{strong}(\mathfrak{M}, \mathcal{A})$.

For an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ which is strong polynomial of degree less than or equal to $n \in \mathbb{N}$, the smallest $d \in \mathbb{N}$ ($d \leq n$) for which F is an object of $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ is called the strong degree of F.

Remark 1.3.7. By Proposition 1.1.14, the category $(\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$ is a pre-braided monoidal category such that 0 is initial object. This example is the first one which led us to extend the definition of [DV17]. Thus, we have a well-defined notion of strong polynomial functor for the category $\mathfrak{U}\boldsymbol{\beta}$.

The following three propositions are important properties of the framework in [DV17] adapted to the pre-braided case. Their proofs follow directly from those of their analogues in [DV17, Propositions 1.7, 1.8 and 1.9].

Proposition 1.3.8. *[DV17, Proposition 1.7] Let $\mathfrak{M}'$ be another pre-braided strict monoidal category and $\alpha : \mathfrak{M} \longrightarrow \mathfrak{M}'$ be a strong monoidal functor. Then, the precomposition by α restricts to a functor from $\mathcal{P}ol_n^{strong}(\mathfrak{M}', \mathcal{A})$ to $\mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$.*

Proposition 1.3.9. *[DV17, Proposition 1.8] The category $\mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$ is closed under the translation endofunctor τ_x, under quotient, under extension and under colimits. Moreover, assuming that there exists a set $\mathfrak{E}$ of objects of $\mathfrak{M}$ such that:*

$$\forall m \in Obj(\mathfrak{M}),\ \exists \{e_i\}_{i \in I} \in Obj(\mathfrak{E}) \text{ where } I \text{ is finite},\ m \cong \underset{i \in I}{\natural} e_i,$$

then, an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ belongs to $\mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$ if and only if $\delta_e(F)$ is an object of $\mathcal{P}ol_{n-1}^{strong}(\mathfrak{M}, \mathcal{A})$ for all objects e of $\mathfrak{E}$.

Corollary 1.3.10. *Let n be a natural number. Let F be a strong polynomial functor of degree n in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Then a direct summand of F is necessarily an object of the category $\mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$.*

Proof. According to Proposition 1.3.9, the category $\mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$ is closed under quotients. □

Remark 1.3.11. The category $\mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$ is not necessarily closed under subobjects. For example, we will see in Section 1.3.3 that for $\mathfrak{M} = \mathfrak{U}\boldsymbol{\beta}$ and $\mathcal{A} = \mathbb{C}\left[t^{\pm 1}\right]$-$\mathfrak{Mod}$, the functor $\overline{\mathfrak{Bur}}_t$ is a subobject of $\tau_1 \overline{\mathfrak{Bur}}_t$ (see Proposition 1.3.28), $\overline{\mathfrak{Bur}}_t$ is strong polynomial of degree 2 (see Proposition 1.3.28) whereas $\tau_1 \overline{\mathfrak{Bur}}_t$ is strong polynomial of degree 1 (see Proposition 1.3.29). If we assume that the unit 0 is also a terminal object of $\mathfrak{M}$, then κ_x is the null endofunctor, δ_x is exact and commutes with all limits. In this case, the category $\mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$ is closed under subobjects.

Remark 1.3.12. If we consider $\mathfrak{M} = \mathfrak{U}\boldsymbol{\beta}$, then each object n (ie a natural number) is clearly $1^{\natural n}$. Hence, because of the last statement of Proposition 1.3.9, when we will deal with strong polynomiality of objects in $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathcal{A})$, it will suffice to consider τ_1.

Proposition 1.3.13. *[DV17, Proposition 1.9] Let F be an object of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Then, the functor F is an object of $\mathcal{P}ol_0^{strong}(\mathfrak{M}, \mathcal{A})$ if and only if it the quotient of a constant functor of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$.*

Finally, let us point out the following property of the strong polynomial degree with respect to the translation functor.

Lemma 1.3.14. *Let d and k be natural numbers and F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ such that $\tau_k(F)$ is an object of $\mathcal{P}ol_d^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Then, F is an object of $\mathcal{P}ol_{d+k}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.*

Proof. We proceed by induction on the degree of polynomiality of $\tau_k(F)$. First, assuming that $\tau_k(F)$ belongs to $\mathcal{P}ol_0^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, we deduce from the commutation property 6 of Proposition 2.4.2 that $\tau_k(\delta_1 F) = 0$. It follows from the definition of $\tau_k(F)$ (see Definition 1.3.1) that for all $n \geq 2$, $\delta_1(F)(n) = 0$. Hence

$$\underbrace{\delta_1 \cdots \delta_1 \delta_1}_{k+1 \text{ times}}(F) \cong 0$$

and therefore F is an object of $\mathcal{P}ol_k(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Now, assume that $\tau_k(F)$ is a strong polynomial functor of degree $d \geq 0$. Since $(\tau_k \circ \delta_1)(F) \cong (\delta_1 \circ \tau_k)(F)$ by the commutation property 6 of Proposition 2.4.2, $(\tau_k \circ \delta_1)(F)$ is an object of $\mathcal{P}ol_{d-1}^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. The inductive hypothesis implies that $\delta_1(F)$ is an object of $\mathcal{P}ol_{d+k}^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. □

Remark 1.3.15. Let us consider the atomic functor $\mathfrak{A}_n$ (with $n > 0$), which is strong polynomial of degree n (see Example 1.3.21). Then $\tau_k(\mathfrak{A}_n) \cong \mathfrak{A}_{n-k}^{\oplus n}$ is strong polynomial of degree $n-k$, for k a natural number such that $k \leq n$. This illustrates the fact that $d+k$ is the best boundary for the degree of polynomiality in Lemma 1.3.14.

1.3.2 Very strong polynomial functors

Let us introduce a particular type of strong polynomial functor, related to coefficient systems of finite degree (see Remark 1.3.17 below). We recall that we consider a pre-braided strict monoidal category $(\mathfrak{M}, \natural, 0)$ such that the monoidal unit 0 is an initial object and an abelian category $\mathcal{A}$.

Definition 1.3.16. We recursively define the category $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ of very strong polynomial functors of degree less than or equal to n to be the full subcategory of $\mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$ as follows:

1. If $n < 0$, $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A}) = \{0\}$;
2. if $n \geq 0$, a functor $F \in \mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$ is an object of $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ if for all objects x of $\mathfrak{M}$, $\kappa_x(F) = 0$ and the functor $\delta_x(F)$ is an object of $\mathcal{VP}ol_{n-1}(\mathfrak{M}, \mathcal{A})$.

For an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ which is very strong polynomial of degree less than or equal to $n \in \mathbb{N}$, the smallest $d \in \mathbb{N}$ ($d \leq n$) for which F is an object of $\mathcal{VP}ol_d(\mathfrak{M}, \mathcal{A})$ is called the very strong degree of F.

Remark 1.3.17. A certain type of functor, called a coefficient system of finite degree, closely related to the strong polynomial one, is used by Randal-Williams and Wahl in [RWW17, Definition 4.10] for their homological stability theorems, generalizing the concept introduced by van der Kallen for general linear groups [vdK80]. Using the framework introduced by Randal-Williams and Wahl, a coefficient system in every object x of $\mathfrak{M}$ of degree n at $N = 0$ is a very strong polynomial functor.

Remark 1.3.18. As we force κ_x to be null for all objects x of $\mathfrak{M}$, the category $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ is closed under kernel functors of the epimorphisms. In particular, this category is closed under direct summands. However, $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ is not necessarily closed under subobjects. For instance, as for Remark 1.3.11, we have that the functor $\overline{\mathfrak{Bur}}_t$ is strong polynomial of degree 2 (see Proposition 1.3.28), the functor $\tau_1\overline{\mathfrak{Bur}}_t$ is very strong polynomial of degree 1 (see Proposition 1.3.29), but $\overline{\mathfrak{Bur}}_t$ is a subobject of $\tau_1\overline{\mathfrak{Bur}}_t$ (see Proposition 1.3.28).

Proposition 1.3.19. *The category $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ is closed under the translation endofunctor τ_x, under kernel of epimorphism and under extension. Moreover, assuming that there exists a set $\mathfrak{E}$ of objects of $\mathfrak{M}$ such that:*

$$\forall m \in Obj(\mathfrak{M}),\ \exists \{e_i\}_{i\in I} \in Obj(\mathfrak{E})\ (\text{where } I \text{ is finite}),\ m \cong \underset{i\in I}{\natural} e_i,$$

then, an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ belongs to $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ if and only if $\kappa_e(F) = 0$ and $\delta_e(F)$ is an object of $\mathcal{VP}ol_{n-1}(\mathfrak{M}, \mathcal{A})$ for all objects e of $\mathfrak{E}$.

Proof. The first assertion follows from the fact that for all objects x of $\mathfrak{M}$, the endofunctor τ_x commutes with the endofunctors δ_x and κ_x (see Proposition 1.3.5). For the second and third assertions, let us consider two short exact sequences of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$: $0 \longrightarrow G \longrightarrow F_1 \longrightarrow F_2 \longrightarrow 0$ and $0 \longrightarrow F_3 \longrightarrow H \longrightarrow F_4 \longrightarrow 0$ with F_i a very strong polynomial functor of degree n for all i. Let x be an object of $\mathfrak{M}$. We use the exact sequence (1.3.2) of Proposition 1.3.5 to obtain the two following exact sequences in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \kappa_x(G) \longrightarrow 0 \longrightarrow 0 \longrightarrow \delta_x(G) \longrightarrow \delta_x(F_1) \longrightarrow \delta_x(F_2) \longrightarrow 0;$$

$$0 \longrightarrow 0 \longrightarrow \kappa_x(H) \longrightarrow 0 \longrightarrow \delta_x(F_3) \longrightarrow \delta_x(H) \longrightarrow \delta_x(F_4) \longrightarrow 0.$$

Therefore, $\kappa_x(G) = \kappa_x(H) = 0$ and the result follows directly by induction on the degree of polynomiality. For the last point, we consider the long exact sequence (1.3.3) of Proposition 1.3.5 applied to an object F of $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ to obtain the following exact sequence in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \kappa_y(F) \longrightarrow \kappa_{x\natural y}(F) \longrightarrow \tau_x\kappa_y(F) \longrightarrow \delta_y(F) \longrightarrow \delta_{x\natural y}(F) \longrightarrow \tau_y\delta_x(F) \longrightarrow 0.$$

Hence, by induction on the length of objects as monoidal product of $\{e_i\}_{i\in I}$, we deduce that $\kappa_m(F) = 0$ for all objects m of $\mathfrak{M}$ if and only if $\kappa_e(F) = 0$ for all objects e of $\mathfrak{E}$. Moreover, since $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ is closed under extension and by the translation endofunctor τ_y, the result follows by induction on the degree of polynomiality n. $\square$

Proposition 1.3.20. *Let F be an object of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. The functor F is an object of $\mathcal{VP}ol_0(\mathfrak{M}, \mathcal{A})$ if and only if it is isomorphic to $\tau_k F$ for all natural numbers k.*

Proof. The result follows using the long exact sequence (2.4.3) of Proposition 2.4.2 applied to F. □

The following example show that there exist strong polynomial functors which are not very strong polynomial in any degree.

Example 1.3.21. Let us consider the categories $\mathfrak{U}\boldsymbol{\beta}$ and $\mathbb{K}$-$\mathfrak{Mod}$, and n a natural number. Let $\mathbb{K}$ be considered as an object of $\mathbb{K}$-$\mathfrak{Mod}$ and 0 be the trivial $\mathbb{K}$-module. Let $\mathfrak{A}_n$ be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, defined by:

- Objects: $\forall m \in \mathbb{N}, \mathfrak{A}_n(m) = \begin{cases} \mathbb{K} & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$.
- Morphisms: let $[j-i, f]$ with $f \in \mathbf{B}_n$ be a morphism from i to j in the category $\mathfrak{U}\boldsymbol{\beta}$. Then:

$$\mathfrak{A}_n(f) = \begin{cases} id_{\mathbb{K}} & \text{if } i = j = n \\ 0 & \text{otherwise.} \end{cases}$$

The functor $\mathfrak{A}_n$ is called an atomic functor in $\mathbb{K}$ of degree n. For coherence, we fix $\mathfrak{A}_{-1}$ to be the null functor of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Then, it is clear that $i_p(\mathfrak{A}_n)$ is the zero natural transformation. On the one hand, we deduce the following natural equivalence $\kappa_1(\mathfrak{A}_n) \cong \mathfrak{A}_n$ and a fortiori $\mathfrak{A}_n$ is not a very strong polynomial functor. On the other hand, it is worth noting the natural equivalence $\delta_1(\mathfrak{A}_n) \cong \tau_1(\mathfrak{A}_n)$ and the fact that $\tau_1(\mathfrak{A}_n) \cong \mathfrak{A}_{n-1}$. Therefore, we recursively prove that $\mathfrak{A}_n$ is a strong polynomial functor of degree n.

Remark 1.3.22. Contrary to $\mathcal{P}ol_n^{strong}(\mathfrak{M}, \mathcal{A})$, a quotient of an object F of $\mathcal{VP}ol_n(\mathfrak{M}, \mathcal{A})$ is not necessarily a very strong polynomial functor. For example, for $\mathfrak{M} = \mathfrak{U}\boldsymbol{\beta}$ and $\mathcal{A} = \mathbb{K}$-$\mathfrak{Mod}$, let us consider the functor $\mathfrak{A}_0$ defined in Example 1.3.21, which we proved to be a strong polynomial functor of degree 0. Let $\mathfrak{A}$ be the constant object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ equal to $\mathbb{K}$. Then, we define a natural transformation $\alpha : \mathfrak{A} \to \mathfrak{A}_0$ assigning:

$$\forall n \in \mathbb{N}, \ \alpha_n = \begin{cases} id_{\mathbb{K}} & \text{if } n = 0 \\ t_{\mathbb{K}} & \text{otherwise.} \end{cases}$$

Moreover, it is an epimorphism in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ since for all natural numbers n, $coker(\alpha_n) = 0_{\mathbb{K}\text{-}\mathfrak{Mod}}$. We proved in Example 1.3.21 that $\mathfrak{A}_0$ is not a very strong polynomial functor of degree 0 whereas $\mathfrak{A}$ is a very strong polynomial functor of degree 0 by Proposition 1.3.20.

Finally, let us remark the following behaviour of the translation functor with respect to very strong polynomial degree.

Lemma 1.3.23. *Let d and k be a natural numbers and F be an object of $\mathcal{VP}ol_d(\mathfrak{M}, \mathbb{K}\text{-}\mathfrak{Mod})$. Then the functor $\tau_k(F)$ is very strong polynomial of degree equal to that of F.*

Proof. We proceed by induction on the degree of polynomiality of F. First, if we assume that F belongs to $\mathcal{VP}ol_0(\mathfrak{M}, \mathbb{K}\text{-}\mathfrak{Mod})$, then according to Proposition 1.3.20, $\tau_k(F) \cong F$ is a degree 0 very strong polynomial functor. Now, assume that F is a very strong polynomial functor of degree $n \geq 0$. Using the commutation properties 5 and 6 of Proposition 2.4.2, we deduce that $(\kappa_1 \circ \tau_k)(F) \cong (\tau_k \circ \kappa_1)(F) = 0$ and $(\delta_1 \circ \tau_k)(F) \cong (\tau_k \circ \delta_1)(F)$. Since the functor $\delta_1(F)$ is a degree $n-1$ very strong polynomial functor, the result follows from the inductive hypothesis. □

Remark 1.3.24. The previous proof does not work for strong polynomial functors since the initial step fails. Indeed, considering the atomic functor $\mathfrak{A}_1$, which is strong polynomial of degree 1 (see Example 1.3.21), then $\tau_2(\mathfrak{A}_0) = 0$.

1.3.3 Examples of polynomial functors over $\mathfrak{U}\boldsymbol{\beta}$

The different functors introduced in Section 1.1.2 are strong polynomial functors.

Very strong polynomial functors of degree one: Let us first investigate the polynomiality of the functors $\mathfrak{Bur}_t$ and $\mathfrak{TYM}_t$.

Proposition 1.3.25. *The functors $\mathfrak{Bur}_t$ and $\mathfrak{TYM}_t$ are very strong polynomial functors of degree* 1.

Proof. For the functor $\mathfrak{Bur}_t$, this is a consequence of [RWW17, Example 4.15]. We will thus focus on the case of the functor $\mathfrak{TYM}_t$. Let n be a natural number. By Remark 1.3.12, it is enough to consider the application $i_1\mathfrak{TYM}_t\left([0,id_n]\right) = \iota_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'-n}} \oplus id_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}}$. This map is a monomorphism and its cokernel is $\mathbb{C}\left[t^{\pm1}\right]$. Hence $\kappa_1\mathfrak{TYM}_t$ is the null functor of $\mathbf{Fct}\left(\mathfrak{U}\beta, \mathbb{C}\left[t^{\pm1}\right]\text{-}\mathfrak{Mod}\right)$. Let n' be a natural number such that $n' \geq n$ and let $[n'-n,\sigma] \in Hom_{\mathfrak{U}\beta}\left(n,n'\right)$. By naturality and the universal property of the cokernel, there exists a unique endomorphism of $\mathbb{C}\left[t^{\pm1}\right]$ such that the following diagram commutes, where the lines are exact. It is exactly the definition of $\delta_1\mathfrak{TYM}_t\left([n'-n,\sigma]\right)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}\left[t^{\pm1}\right]^{\oplus n} & \xrightarrow{\iota_{\mathbb{C}\left[t^{\pm1}\right]}\oplus id_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n}}} & \mathbb{C}\left[t^{\pm1}\right]^{\oplus n+1} & \xrightarrow{\pi_{n+1}} & \mathbb{C}\left[t^{\pm1}\right] & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \mathfrak{TYM}([n'-n,\sigma])} & & \Big\downarrow {\scriptstyle \tau_1(\mathfrak{TYM})([n'-n,\sigma])} & & \Big\downarrow {\scriptstyle \exists!} & & \\
0 & \longrightarrow & \mathbb{C}\left[t^{\pm1}\right]^{\oplus n'} & \xrightarrow[\iota_{\mathbb{C}\left[t^{\pm1}\right]}\oplus id_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'}}]{} & \mathbb{C}\left[t^{\pm1}\right]^{\oplus n'+1} & \xrightarrow[\pi_{n'+1}]{} & \mathbb{C}\left[t^{\pm1}\right] & \longrightarrow & 0.
\end{array}$$

For all $(a,b) \in \mathbb{C}\left[t^{\pm1}\right] \oplus \mathbb{C}\left[t^{\pm1}\right]^{\oplus n} = \mathbb{C}\left[t^{\pm1}\right]^{\oplus n+1}$, $\tau_1\left(\mathfrak{TYM}_t\right)\left([n'-n,\sigma]\right)(a,b) = \left(a, \mathfrak{TYM}_t\left([n'-n,\sigma]\right)(b)\right)$. Therefore, $\left(\pi_{n'+1} \circ \tau_1\left(\mathfrak{TYM}_t\right)\left([n'-n,\sigma]\right)\right)(a,b) = a = \pi_{n+1}(a,b)$. Hence, $id_{\mathbb{C}\left[t^{\pm1}\right]}$ also makes the diagram commutative and thus $\delta_1\mathfrak{TYM}_t\left([n'-n,\sigma]\right) = id_{\mathbb{C}\left[t^{\pm1}\right]}$. Hence, $\delta_1\mathfrak{TYM}_t$ is the constant functor equal to $\mathbb{C}\left[t^{\pm1}\right]$. A fortiori, because of Proposition 1.3.20, $\delta_1\mathfrak{TYM}_t$ is a very strong polynomial functor of degree 0. □

The particular case of $\overline{\mathfrak{Bur}}_t$:

Definition 1.3.26. Let $\mathcal{T}_1 : \mathfrak{U}\beta \longrightarrow \mathbb{C}\left[t^{\pm1}\right]\text{-}\mathfrak{Mod}$ be the subobject of the constant functor $\mathfrak{X}$ (see Notation 1.2.26) such that $\mathcal{T}_1(0) = 0$ and $\mathcal{T}_1(n) = \mathbb{C}\left[t^{\pm1}\right]$ for all non-zero natural numbers n.

Remark 1.3.27. It follows from Definition 1.3.26 that $\delta_1\mathcal{T}_1 \cong \mathfrak{A}_0$ (where $\mathfrak{A}_0$ is introduced in Example 1.3.21). Therefore, $\mathcal{T}_1$ is a strong polynomial functor of degree 1, but is not very strong polynomial. Nevertheless, it is worth noting that $\kappa_1\mathcal{T}_1 = 0$.

Proposition 1.3.28. *The functor $\overline{\mathfrak{Bur}}$ is a strong polynomial functor of degree* 2*. This functor is not very strong polynomial. More precisely, we have the following short exact sequence in* $\mathbf{Fct}\left(\mathfrak{U}\beta, \mathbb{C}\left[t^{\pm1}\right]\text{-}\mathfrak{Mod}\right)$*:*

$$0 \longrightarrow \overline{\mathfrak{Bur}}_t \longrightarrow \tau_1\overline{\mathfrak{Bur}}_t \longrightarrow \mathcal{T}_1 \longrightarrow 0\,.$$

Proof. The natural transformation $i_1\left(\overline{\mathfrak{Bur}}_t\right)_n : \overline{\mathfrak{Bur}}_t(n) \to \tau_1\overline{\mathfrak{Bur}}_t(n)$ (introduced in Definition 1.3.4) is defined to be $\iota_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n'-n}} \oplus id_{\mathbb{C}\left[t^{\pm1}\right]^{\oplus n-1}}$. Let $n \geq 2$ be a natural number. This map is a monomorphism (so $\kappa_1\overline{\mathfrak{Bur}}_t = 0$) and its cokernel is $\mathbb{C}\left[t^{\pm1}\right]$. Repeating mutatis mutandis the work done in the proof of Proposition 1.3.25, we deduce that for all $[n'-n,\sigma] \in Hom_{\mathfrak{U}\beta}\left(n,n'\right)$ (with $n' \geq n \geq 2$), $\delta_1\overline{\mathfrak{Bur}}_t\left([n'-n,\sigma]\right) = Id_{\mathbb{C}\left[t^{\pm1}\right]}$. In addition, since $\overline{\mathfrak{Bur}}_t(1) = 0$ and $\tau_1\overline{\mathfrak{Bur}}_t(1) = \mathbb{C}\left[t^{\pm1}\right]$, we deduce that $\delta_1\overline{\mathfrak{Bur}}_t(1) = \mathbb{C}\left[t^{\pm1}\right]$ and for all $n' \geq 1$, for all $[n'-1,\sigma] \in Hom_{\mathfrak{U}\beta}\left(1,n'\right)$, $\delta_1\overline{\mathfrak{Bur}}_t\left([n'-1,\sigma]\right) = Id_{\mathbb{C}\left[t^{\pm1}\right]}$. Hence, we prove that $\delta_1\overline{\mathfrak{Bur}}_t \cong \mathcal{T}_1$ where $\mathcal{T}_1$ is introduced in Definition 1.3.26. The results follow from the fact that $\delta_1\mathcal{T}_1 \cong \mathfrak{A}_0$ by Remark 1.3.27. □

For formal reasons (see Proposition 1.3.5), $\overline{\mathfrak{Bur}}_t$ is a subfunctor of $\tau_1\overline{\mathfrak{Bur}}_t$. The following proposition illustrates Remarks 1.3.11 and 1.3.18.

Proposition 1.3.29. *The functor $\tau_1\overline{\mathfrak{Bur}}_t$ is a very strong polynomial functor of degree* 1.

Proof. Repeating mutatis mutandis the work done in the proof of Proposition 1.3.28, we prove that $\delta_1\overline{\mathfrak{Bur}}_t$ is the constant functor equal to $\mathbb{C}\left[t^{\pm1}\right]$ (denoted by $\mathfrak{X}$ in Notation 1.2.26). Since $\mathfrak{X}$ is a constant functor, $\delta_1\overline{\mathfrak{Bur}}_t$ is by Proposition 1.3.20 a very strong polynomial functor of degree 0. □

A very strong polynomial functor of degree two: We could have defined the unreduced Burau functor of Example 1.1.2 assigning $\left(\left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]\right)^{\oplus n}$ to each object $n \in \mathbb{N}$.

Notation 1.3.30. Abusing the notation, $\left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right] : \mathfrak{U}\boldsymbol{\beta} \to \left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]\text{-}\mathfrak{Mod}$ denotes the constant functor at $\left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]$. The functor $\mathfrak{Bur}_t \underset{\mathbb{C}[t^{\pm1}]}{\otimes} \left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]$ is denoted by $\check{\mathfrak{Bur}}_t : \mathfrak{U}\boldsymbol{\beta} \to \left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]\text{-}\mathfrak{Mod}$.

Remark 1.3.31. These functors $\left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]$ and $\check{\mathfrak{Bur}}_t$ are also very strong polynomial of degree one (the proof is exactly the same as the one for $\mathfrak{Bur}_t$ in Proposition 1.3.27).

Lemma 1.3.32. *Considering the modified version of the unreduced Burau functor of Remark 1.3.30, then we have* $\delta_1\mathfrak{LK} = \check{\mathfrak{Bur}}_t$.

Proof. We consider the application $i_1\mathfrak{LK}([0, id_n])$. This map is a monomorphism and its cokernel is $\underset{1\leq i\leq n}{\bigoplus} V_{i,n+1}$. Let n and n' be two natural numbers such that $n' \geq n$. Let $[n'-n,\sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$. By naturality and because of the universal property of the cokernel, there exists a unique endomorphism of $\left(\mathbb{C}\left[t^{\pm1}\right]\right)\left[q^{\pm1}\right]$-modules such that the following diagram commutes, where the lines are exact. It is the definition of $\delta_1\mathfrak{LK}([n'-n,\sigma])$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underset{1\leq j<k\leq n}{\bigoplus} V_{j,k} & \xrightarrow{\mathfrak{LK}([1,id_{1+n}])} & \underset{1\leq i<l\leq n+1}{\bigoplus} V_{i,l} & \xrightarrow{\pi_n} & \underset{2\leq l\leq n+1}{\bigoplus} V_{1,l} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathfrak{LK}([n'-n,\sigma])} & & \downarrow{\scriptstyle \tau_1(\mathfrak{LK})([n'-n,\sigma])} & & \downarrow{\scriptstyle \exists!} & & \\
0 & \longrightarrow & \underset{1\leq j'<k'\leq n'}{\bigoplus} V_{j',k'} & \xrightarrow{\mathfrak{LK}([1,id_{1+n'}])} & \underset{1\leq l'\leq n'+1}{\bigoplus} V_{i',l'} & \xrightarrow{\pi_{n'}} & \underset{2\leq l'\leq n'+1}{\bigoplus} V_{1,l'} & \longrightarrow & 0.
\end{array}$$

Let $i \in \{1,\ldots,n-1\}$, $l \in \{2,\ldots,n+1\}$ and $v_{1,l}$ be an element of $V_{1,l}$. Then we compute:

$$\tau_1\mathfrak{LK}(\sigma_i)\,v_{1,l} = \mathfrak{LK}(\sigma_{1+i})\,(v_{1,l}) = \begin{cases} v_{1,l} & \text{if } i+1 \notin \{l-1,l\}, \\ tv_{1,i+1} + (1-t)\,v_{1,i+2} - \left(t^2-t\right)qv_{i+1,i+2} & \text{if } i+2 = l, \\ v_{1,i+2} & \text{if } i+1 = l. \end{cases}$$

We deduce that in the canonical basis $\{\mathbf{e}_{1,2}, \mathbf{e}_{1,3}, \ldots, \mathbf{e}_{1,n+1}\}$ of $\underset{2\leq l\leq n+1}{\bigoplus} V_{1,l}$:

$$\delta_1\mathfrak{LK}(\sigma_i) = Id_{i-1} \oplus \begin{bmatrix} 0 & t \\ 1 & 1-t \end{bmatrix} \oplus Id_{n-i-1} = r_n \circ \check{\mathfrak{Bur}}_t(\sigma_i) \circ r_n^{-1}.$$

So as to identify $\delta_1\mathfrak{LK}$, it remains to consider the action on morphisms of type $[1, id_n]$. According to the definition of the Lawrence-Krammer functor, we have $\tau_1(\mathfrak{LK})([1,id_n]) = \mathfrak{LK}\left(\sigma_1^{-1}\right) \circ \mathfrak{LK}([1,id_{n+2}])$ and:

$$\mathfrak{LK}(\sigma_1)\,(v_{1,k}) = \begin{cases} v_{2,k} & \text{if } k \in \{3,\ldots,n+2\}, \\ -qt^2v_{1,2} & \text{if } k = 2. \end{cases}$$

It follows that for all $v_{i,l} \in V_{i,l}$ with $1 \leq i < l \leq n+1$:

$$\pi_{n+1} \circ \tau_1(\mathfrak{LK})([1,id_n])\,(v_{i,l}) = \begin{cases} v_{i,l+1} & \text{if } i = 1 \text{ and } l \in \{2,\ldots,n+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we deduce that for all $2 \leq l \leq n+1$, $\delta_1\mathfrak{LK}([1,id_n])\,(v_{1,l}) = v_{1,l+1} = \check{\mathfrak{Bur}}_t([1,id_n])\,(v_{1,l})$. □

Proposition 1.3.33. *The functor $\mathfrak{LK}$ is a very strong polynomial functor of degree* 2.

Proof. Let n be a natural number. By Remark 1.3.12, we only have to consider the application $i_1\mathfrak{LK}([0,id_n])$. Since this map is a monomorphism with cokernel $\underset{1\leq i\leq n}{\bigoplus} V_{i,n+1}$, $\kappa_1\mathfrak{LK}$ is the null constant functor. Since the functor $\check{\mathfrak{Bur}}_t$ is very strong polynomial of degree one (following exactly the same proof as the one of Proposition 1.3.25), we deduce from Lemma 1.3.32 that $\mathfrak{LK}$ is very strong polynomial of degree two. □

1.4 The Long-Moody functor applied to polynomial functors

Let us move on to the effect of the Long-Moody functors on (very) strong polynomial functors. For this purpose, it is enough by Remark 1.3.12 to consider the cokernel of the map $i_1\mathbf{LM}$. First, we decompose the functor $\tau_1 \circ \mathbf{LM}$ (see Proposition 1.4.19) so as to understand the behaviour of the image of $i_1\mathbf{LM}$ through this decomposition. This allows us to prove a splitting decomposition of the difference functor (see Theorem 1.4.23). This is the key point to prove our main results, namely Corollary 1.4.27 and Theorem 1.4.28. Finally, we give some additional properties of Long-Moody functors with respect to polynomial functors.

Let $\{\varsigma_n : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ be coherent families of morphisms (see Definition 1.2.14), with associated Long-Moody functor $\mathbf{LM}_{a,\varsigma}$ (see Theorem 1.2.20), which we fix for all the work of this section (in particular, we omit the "a,ς" from the notation).

1.4.1 Decomposition of the translation functor

We introduce two functors which will play a key role in the main result. First, let us recall the following crucial property of the augmentation ideal of a free product of groups, which follows by combining [Coh72, Lemma 4.3] and [Coh72, Theorem 4.7].

Proposition 1.4.1. *Let G and H be groups. Then, there is a natural $\mathbb{K}[G*H]$-module isomorphism:*

$$\mathcal{I}_{\mathbb{K}[G*H]} \cong \left(\mathcal{I}_{\mathbb{K}[G]} \underset{\mathbb{K}[G]}{\otimes} \mathbb{K}[G*H]\right) \oplus \left(\mathcal{I}_{\mathbb{K}[H]} \underset{\mathbb{K}[H]}{\otimes} \mathbb{K}[G*H]\right).$$

Remark 1.4.2. In the statement of Proposition 1.4.1, recall that the augmentation ideal $\mathcal{I}_{\mathbb{K}[G]}$ (respectively $\mathcal{I}_{\mathbb{K}[H]}$) is a free right $\mathbb{K}[G]$-module (respectively $\mathbb{K}[H]$-module) by Proposition 1.2.22. Moreover, the group ring $\mathbb{K}[G*H]$ is a left $\mathbb{K}[G]$-module (respectively left $\mathbb{K}[H]$-module) via the morphism $id_G * \iota_H : G \to G*H$ (respectively $\iota_G * id_H : H \to G*H$).

Notation 1.4.3. Let n and n' be natural numbers such that $n' \geq n$. We consider the morphism $id_{\mathbf{F}_n} * \iota_{\mathbf{F}_{n'-n}} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$. This corresponds to the identification of $\mathbf{F}_n$ as the subgroup of $\mathbf{F}_{n'}$ generated by the n first copies of $\mathbf{F}_1$ in $\mathbf{F}_{n'}$.

In addition, the group morphism $id_{\mathbf{F}_n} * \iota_{\mathbf{F}_{n'-n}} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$ canonically induces a $\mathbb{K}$-module morphism $id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} * \iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \hookrightarrow \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}$.

For F an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, we consider the functor $(\tau_1 \circ \mathbf{LM})(F)$. For all natural numbers n, by Proposition 1.4.1, we have a $\mathbb{K}[\mathbf{F}_{1+n}]$-module isomorphism:

$$\begin{aligned}
&\mathcal{I}_{\mathbb{K}[\mathbf{F}_{1+n}]} \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} F(n+2)\\
&\cong \left(\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]} \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} \mathbb{K}[\mathbf{F}_{1+n}]\right) \oplus \left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} \mathbb{K}[\mathbf{F}_{1+n}]\right)\right) \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} F(n+2).
\end{aligned}$$

Now, by Remark 1.4.2, the $\mathbb{K}[\mathbf{F}_{n+1}]$-module $F(n+2)$ is a $\mathbb{K}[\mathbf{F}_1]$-module via

$$F\left(\varsigma_{1+n}\left(id_{\mathbf{F}_1} * \iota_{\mathbf{F}_n}\right)\right) : \mathbf{F}_1 \to Aut_{\mathbb{K}\text{-}\mathfrak{Mod}}(F(n+2))$$

and $\mathbb{K}[\mathbf{F}_n]$-module via

$$F\left(\varsigma_{1+n}\left(\iota_{\mathbf{F}_1} * id_{\mathbf{F}_n}\right)\right) : \mathbf{F}_n \to Aut_{\mathbb{K}\text{-}\mathfrak{Mod}}(F(n+2)).$$

Therefore, because of the distributivity of tensor product with respect to the direct sum, we have the following proposition.

Proposition 1.4.4. *Let $F \in Obj(\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}))$ and n be a natural number. Then, we have the following $\mathbb{K}$-module isomorphism:*

$$\tau_1\mathbf{LM}(F)(n) \cong \left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]} \underset{\mathbb{K}[\mathbf{F}_1]}{\otimes} F(n+2)\right) \oplus \left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(n+2)\right). \quad (1.4.1)$$

Definition 1.4.5. For all natural numbers n and $F \in Obj\left(\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}\right)\right)$, we denote by

- $\upsilon\left(F\right)_{n}$ the monomorphism of $\mathbb{K}$-modules $\left(id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{1}]}} * \iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n}]}}\right) \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} id_{F(n+2)} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_{1}]} \underset{\mathbb{K}[\mathbf{F}_{1}]}{\otimes} F\left(n+2\right) \hookrightarrow \tau_{1}\mathbf{LM}\left(F\right)\left(n\right)$,
- $\xi\left(F\right)_{n}$ the monomorphism of $\mathbb{K}$-modules $\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{1}]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n}]}}\right) \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} id_{F(n+2)} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n}]} \underset{\mathbb{K}[\mathbf{F}_{n}]}{\otimes} F\left(n+2\right) \hookrightarrow \tau_{1}\mathbf{LM}\left(F\right)\left(n\right)$,

associated with the direct sum of Proposition 1.4.4.

The aim of this section is in fact to show that this $\mathbb{K}$-module decomposition leads to a decomposition of $\tau_{1}\mathbf{LM}$ (see Theorem 1.4.23) as a functor.

1.4.1.1 Additional conditions

We need two additional conditions so as to make the decomposition of Proposition 1.4.4 functorial. First, we require the morphisms $\left\{a_{n} : \mathbf{B}_{n} \to Aut\left(\mathbf{F}_{n}\right)\right\}_{n\in\mathbb{N}}$ to satisfy the following property.

Condition 1.4.6. Let n and n' be natural numbers such that $n' \geq n$. We require $a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1} \natural id_{n}\right) \circ \left(\iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_{n+1}}\right) \circ \left(id_{\mathbf{F}_{1}} * \iota_{\mathbf{F}_{n}}\right) = id_{\mathbf{F}_{1}} * \iota_{\mathbf{F}_{n'}}$. In other words, the following diagram is commutative:

$$\begin{array}{ccc}
\mathbf{F}_{1} & \xrightarrow{id_{\mathbf{F}_{1}} * \iota_{\mathbf{F}_{n'}}} & \mathbf{F}_{1+n'} \\
{\scriptstyle id_{\mathbf{F}_{1}} * \iota_{\mathbf{F}_{n}}}\downarrow & & \uparrow{\scriptstyle a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1} \natural id_{n}\right)} \\
\mathbf{F}_{1+n} & \xrightarrow[\iota_{\mathbf{F}_{n'-n}} * id_{\mathbf{F}_{1+n}}]{} & \mathbf{F}_{n'-n} * \mathbf{F}_{1+n} \cong \mathbf{F}_{1+n'}.
\end{array}$$

Remark 1.4.7. Condition 1.4.6 will be used to define an intermediary functor (see Proposition 1.4.14).

In addition, we will assume that the morphisms $\left\{a_{n} : \mathbf{B}_{n} \to Aut\left(\mathbf{F}_{n}\right)\right\}_{n\in\mathbb{N}}$ satisfy the following condition.

Condition 1.4.8. Let n and n' be natural numbers such that $n' \geq n$. We require $a_{n'}\left(id_{n'-n}\natural -\right) : \mathbf{B}_{n} \to Aut\left(\mathbf{F}_{n'}\right)$ maps to the stabilizer of the homomorphism $id_{\mathbf{F}_{n'-n}} * \iota_{\mathbf{F}_{n}} : \mathbf{F}_{n'-n} \longrightarrow \mathbf{F}_{n'}$, ie for all element σ of $\mathbf{B}_{n}$ the following diagram is commutative:

$$\begin{array}{ccc}
\mathbf{F}_{n'-n} & \xrightarrow{id_{\mathbf{F}_{n'-n}} * \iota_{\mathbf{F}_{n}}} & \mathbf{F}_{n'} \\
& {\scriptstyle id_{\mathbf{F}_{n'-n}} * \iota_{\mathbf{F}_{n}}}\searrow \quad \nearrow{\scriptstyle a_{n'}\left(id_{n'-n}\natural\sigma\right)} & \\
& \mathbf{F}_{n'}. &
\end{array}$$

Remark 1.4.9. Condition 1.4.8 will be used in the proof of Propositions 1.4.14 and 1.4.15.

Remark 1.4.10. The relations of Conditions 1.4.6 and 1.4.8 remain true mutatis mutandis, for all natural numbers n, considering the induced morphisms $a_{n} : \mathbf{B}_{n} \to Aut\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n}]}\right)$ and $id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n}]}} * \iota_{\mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'-n}\right]}} : \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n}]} \hookrightarrow \mathcal{I}_{\mathbb{K}\left[\mathbf{F}_{n'}\right]}$.

Definition 1.4.11. If the morphisms $\left\{a_{n} : \mathbf{B}_{n} \to Aut\left(\mathbf{F}_{n}\right)\right\}_{n\in\mathbb{N}}$ also satisfy conditions 1.4.6 and 1.4.8, the coherent families of morphisms $\left\{\varsigma_{n} : \mathbf{F}_{n} \hookrightarrow \mathbf{B}_{n+1}\right\}_{n\in\mathbb{N}}$ and $\left\{a_{n} : \mathbf{B}_{n} \to Aut\left(\mathbf{F}_{n}\right)\right\}_{n\in\mathbb{N}}$ are said to be reliable.

Proposition 1.4.12. *The coherent families of morphisms* $\left\{a_{n,1} : \mathbf{B}_{n} \to Aut\left(\mathbf{F}_{n}\right)\right\}_{n\in\mathbb{N}}$ *and* $\left\{\varsigma_{n,1} : \mathbf{F}_{n} \hookrightarrow \mathbf{B}_{n+1}\right\}_{n\in\mathbb{N}}$ *of Examples 1.2.7 and 1.2.15 are reliable.*

Proof. Recall from Definition 1.1.4 that $\left(b^{\beta}_{1,n'-n}\right)^{-1} = \sigma_{1}^{-1} \circ \sigma_{2}^{-1} \circ \cdots \circ \sigma_{n'-n}^{-1}$. We consider the element $e_{\mathbf{F}_{n'-n}} * g_{1} * e_{\mathbf{F}_{n}} = g_{n'-n+1} \in \mathbf{F}_{(n'-n)+1+n}$. The definition of $a_{n,1}$ gives that $a_{1+n',1}\left(\sigma_{n'-n}\right)\left(g_{n'-n}\right) = g_{n'-n+1}$. Therefore, we have that:

$$a_{1+n',1}\left(\sigma_{n'-n}^{-1}\right)\left(g_{n'-n+1}\right) = g_{n'-n}.$$

Iterating this observation, we deduce that $a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_{n}\right)(g_{n'-n+1}) = g_1 \in \mathbf{F}_{1+n'}$. Hence, the family of morphisms $\{a_{n,1}:\mathbf{B}_n \to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ satisfies Condition 1.4.6.

Similarly to Example 1.2.15 earlier, for all $g\in\mathbf{F}_{n'-n}$ and each Artin generator $\sigma_i\in\mathbf{B}_n$, $a_{n'}(id_{n'-n}\natural\sigma_i)(g*e_{\mathbf{F}_n}) = g*e_{\mathbf{F}_n}$. Hence, the family of morphisms $\{a_{n,1}:\mathbf{B}_n\to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$ satisfies Condition 1.4.8. □

From now until the end of Section 1.4, we fix coherent reliable families of morphisms $\{\varsigma_n:\mathbf{F}_n\hookrightarrow\mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ and $\{a_n:\mathbf{B}_n\to Aut(\mathbf{F}_n)\}_{n\in\mathbb{N}}$.

1.4.1.2 The intermediary functors

The functor τ_2: Let us consider the factor $\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}\underset{\mathbb{K}[\mathbf{F}_1]}{\otimes}F(n+2)$ of $\tau_1\mathbf{LM}(F)(n)$ in the decomposition of Proposition 1.4.4.

Notation 1.4.13. For all objects F of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$, for all natural numbers n, we denote $\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}\underset{\mathbb{K}[\mathbf{F}_1]}{\otimes}F(n+2)$ by $\Upsilon(F)(n)$.

Recall the monomorphisms $\{v(F)_n:\Upsilon(F)(n)\hookrightarrow\tau_1\mathbf{LM}(F)(n)\}_{n\in\mathbb{N}}$ of Definition 1.4.5.

Proposition 1.4.14. *Let F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$. For all natural numbers n and n' such that $n'\geq n$, and for all $[n'-n,\sigma]\in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$, assign:*

$$\Upsilon(F)([n'-n,\sigma]) = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}}\underset{\mathbb{K}[\mathbf{F}_1]}{\otimes}F(id_2\natural[n'-n,\sigma]).$$

This defines a subfunctor $\Upsilon(F):\mathfrak{U}\boldsymbol{\beta}\to\mathbb{K}\text{-}\mathfrak{Mod}$ of $\tau_1\mathbf{LM}(F)$, using the monomorphisms $\{v(F)_n\}_{n\in\mathbb{N}}$.

Proof. Let us check that the assignment $\Upsilon(F)$ is well defined with respect to the tensor product. Let n and n' be natural numbers such that $n'\geq n$, and $[n'-n,\sigma]\in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n,n')$ with $\sigma\in\mathbf{B}_{n'}$. Recall from Proposition 1.1.14 that $id_2\natural[n'-n,\sigma] = \left[n'-n,(id_2\natural\sigma)\circ\left(\left(b^{\beta}_{2,n'-n}\right)^{-1}\natural id_n\right)\right]$. On the one hand, by Condition 1.2.12, we have:

$$(id_2\natural\sigma)\circ\varsigma_{1+n'}(g_1) = \varsigma_{1+n'}(a_{1+n'}(id_1\natural\sigma)(g_1))\circ(id_2\natural\sigma).$$

Hence, it follows from Condition 1.4.8 that

$$(id_2\natural\sigma)\circ\varsigma_{1+n'}(g_1) = \varsigma_{1+n'}(g_1)\circ(id_2\natural\sigma). \tag{1.4.2}$$

On the other hand, Condition 1.4.6 gives that

$$g_1 = a_{2+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_{n+1}\right)(g_{n'-n+1})$$

and by Condition 1.4.8 we have

$$g_1 = a_{2+n'}\left(id_1\natural\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n\right)(g_1).$$

By the definition of the braiding $b^{\beta}_{-,-}$ (see Definition 1.1.4), we deduce that:

$$\varsigma_{1+n'}(g_1) \quad = \quad \varsigma_{1+n'}\left(a_{2+n'}\left(\left(b^{\beta}_{2,n'-n}\right)^{-1}\natural id_n\right)(g_{n'-n+1})\right).$$

Then, it follows from the combination of Conditions 1.2.3 and 1.2.12 that as morphisms in $\mathfrak{U}\boldsymbol{\beta}$:

$$\begin{aligned}&\left[n'-n,\varsigma_{1+n'}(g_1)\circ\left(\left(b^{\beta}_{2,n'-n}\right)^{-1}\natural id_n\right)\right]\\ &=\left[n'-n,\left(\left(b^{\beta}_{2,n'-n}\right)^{-1}\natural id_n\right)\circ(id_{n'-n}\natural\varsigma_{1+n}(g_1))\right].\end{aligned} \tag{1.4.3}$$

Hence, we deduce from the relations (1.4.2) and (1.4.3) that:

$$\begin{aligned}&\left[n'-n,\left((id_2\natural\sigma)\circ\left(\left(b^{\beta}_{2,n'-n}\right)^{-1}\natural id_n\right)\right)\circ\left(id_{n'-n}\natural\varsigma_{1+n}(g_1)\right)\right]\\=&\left[n'-n,\varsigma_{1+n'}(g_1)\circ\left((id_2\natural\sigma)\circ\left(\left(b^{\beta}_{2,n'-n}\right)^{-1}\natural id_n\right)\right)\right].\end{aligned}$$

A fortiori, $F(id_2\natural[n'-n,\sigma])\circ F(\varsigma_{1+n}(g_1)) = F(\varsigma_{1+n'}(g_1))\circ F(id_2\natural[n'-n,\sigma])$. Hence, our assignment is well defined with respect to the tensor product.

Let us prove that the subspaces $\Upsilon(F)(n)$ are stable under the action of $\mathfrak{U}\boldsymbol{\beta}$. Let $i\in\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}$ and $v\in F(n+2)$. We deduce from the definition of the monoidal structure morphisms of $\mathfrak{U}\boldsymbol{\beta}$ (see Proposition 1.1.14) and from the definition of the Long-Moody functor (see Theorem 1.2.20) that, for all $i\in\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}$ and for all $v\in F(n+2)$:

$$\begin{aligned}&\left((\tau_1\mathbf{LM}(F)([n'-n,\sigma]))\circ v(F)_n\right)\left(i\underset{\mathbb{K}[\mathbf{F}_1]}{\otimes}v\right)\\=&a_{1+n'}(id_1\natural\sigma)\left(a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n\right)\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}}*id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}}*\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i)\right)\\&\underset{\mathbb{K}[\mathbf{F}_{n'+1}]}{\otimes}F\left(id_1\natural id_1\natural[n'-n,\sigma]\right)(v).\end{aligned}$$

It follows from Condition 1.4.6 that:

$$a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n\right)\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}}*id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}}*\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i)=\left(id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}}*\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}}\right)(i).$$

Since by Condition 1.4.8, $a_{1+n'}(id_1\natural\sigma)\left(id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}}*\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}}\right)(i)=\left(id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}}*\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}}\right)(i)$ for all elements σ of $\mathbf{B}_{n'}$, we deduce that:

$$\left(\tau_1\mathbf{LM}(F)([n'-n,\sigma])\circ v(F)_n\right)\left(i\underset{\mathbb{K}[\mathbf{F}_1]}{\otimes}v\right)=\left(v(F)_{n'}\circ\Upsilon(F)([n'-n,\sigma])\right)\left(i\underset{\mathbb{K}[\mathbf{F}_m]}{\otimes}v\right).$$

Therefore, the functorial structure of $\tau_1\mathbf{LM}(F)$ induces by restriction the one of $\Upsilon(F)$. □

Now, we can lift this link between $\Upsilon(F)$ *of* $\tau_1\mathbf{LM}(F)$ to endofunctors of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$.

Proposition 1.4.15. *Let F and G be two objects of* $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$, *and $\eta: F\to G$ be a natural transformation. For all natural numbers n, assign :*

$$(\Upsilon(\eta))_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}}\underset{\mathbb{K}[\mathbf{F}_1]}{\otimes}\eta_{n+2}.$$

Then we define a subfunctor $\Upsilon:\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})\to\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$ *of* $\tau_1\mathbf{LM}$ *using the monomorphisms* $\{v(F)_n\}_{n\in\mathbb{N}}$.

Proof. The consistency of our definition follows repeating mutatis mutandis point 4 of the proof of Theorem 1.2.20. It directly follows from the definitions of $(\Upsilon(\eta))_n$, $v(G)_n$ and $\tau_1\circ\mathbf{LM}$ (see Definition 1.2.2) that $v(G)_n\circ(\Upsilon)(\eta)_n=(\tau_1\circ\mathbf{LM})(\eta)_n\circ v(F)_n$. □

In fact, we have an easy description of the functor Υ.

Proposition 1.4.16. *There is a natural equivalence $\Upsilon\cong\tau_2$ where τ_2 is the translation functor introduced in Definition 1.3.1.*

Proof. Let F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta},\mathbb{K}\text{-}\mathfrak{Mod})$. By Proposition 1.2.22, for all natural numbers n, we have an isomorphism:

$$\begin{array}{rccc}\chi_{n,F}: & \mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}\underset{\mathbb{K}[\mathbf{F}_1]}{\otimes}F(n+2) & \overset{\cong}{\longrightarrow} & F(n+2).\\ & (g_1-1)\underset{\mathbb{K}[\mathbf{F}_n]}{\otimes}v & \longmapsto & v\end{array}$$

It follows from Definition 1.3.1 and Proposition 1.4.14 that the isomorphisms $\{\chi_{n,F}\}_{n\in\mathbb{N}}$ define the desired natural equivalence $\Upsilon\overset{\chi}{\to}\tau_2$. □

The functor $\mathbf{LM}\circ\tau_1$: Now, let us consider the part $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\underset{\mathbb{K}[\mathbf{F}_n]}{\otimes}F(n+2)$ of $\tau_1\circ\mathbf{LM}(F)(n)$ in the decomposition of Proposition 1.4.4. In fact, we are going to prove that these modules assemble to form a functor which identifies with $\mathbf{LM}(\tau_1 F)$. We recall from Theorem 1.2.20 and Definition 1.3.1 the following fact.

Remark 1.4.17. The functor $\mathbf{LM}\circ\tau_1:\mathbf{Fct}(\mathfrak{U}\beta,\mathbb{K}\text{-}\mathfrak{Mod})\to\mathbf{Fct}(\mathfrak{U}\beta,\mathbb{K}\text{-}\mathfrak{Mod})$ is defined by:

- for $F\in Obj(\mathbf{Fct}(\mathfrak{U}\beta,\mathbb{K}\text{-}\mathfrak{Mod}))$, $\forall n\in\mathbb{N}$, $(\mathbf{LM}\circ\tau_1)(F)(n)=\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}\underset{\mathbb{K}[\mathbf{F}_n]}{\otimes}F(n+2)$, where $F(n+2)$ is a left $\mathbb{K}[\mathbf{F}_n]$-module using $F(id_1\natural\varsigma_n(-)):\mathbf{F}_n\to Aut_{\mathbb{K}\text{-}\mathfrak{Mod}}(F(n+2))$. For $n,n'\in\mathbb{N}$, such that $n'\geq n$, and $[n'-n,\sigma]\in Hom_{\mathfrak{U}\beta}(n,n')$:
$$(\mathbf{LM}\circ\tau_1)(F)([n'-n,\sigma])=a_{n'}(\sigma)\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}}*id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)\underset{\mathbb{K}[\mathbf{F}_{n'}]}{\otimes}F(id_1\natural id_1\natural[n'-n,\sigma]).$$
- Morphisms: let F and G be two objects of $\mathbf{Fct}(\mathfrak{U}\beta,\mathbb{K}\text{-}\mathfrak{Mod})$, and $\eta:F\to G$ be a natural transformation. The natural transformation $(\mathbf{LM}\circ\tau_1)(\eta):(\mathbf{LM}\circ\tau_1)(F)\to(\mathbf{LM}\circ\tau_1)(G)$ for all natural numbers n is given by:
$$((\mathbf{LM}\circ\tau_1)(\eta))_n=id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\underset{\mathbb{K}[\mathbf{F}_n]}{\otimes}\eta_{n+2}.$$

Proposition 1.4.18. *For all $F\in Obj(\mathbf{Fct}(\mathfrak{U}\beta,\mathbb{K}\text{-}\mathfrak{Mod}))$, the monomorphisms $\{\xi(F)_n\}_{n\in\mathbb{N}}$ (see Definition 1.4.5) allow to define a natural transformation $\xi'(F):(\mathbf{LM}\circ\tau_1)(F)\to(\tau_1\circ\mathbf{LM})(F)$ where, for all natural numbers n:*
$$\xi'(F)_n=\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}}*id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)\underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes}F\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right).$$
This yields a natural transformation $\xi':\mathbf{LM}\circ\tau_1\to\tau_1\circ\mathbf{LM}$.

Proof. Let n and n' be natural numbers such that $n'\geq n$, and $[n'-n,\sigma]\in Hom_{\mathfrak{U}\beta}(n,n')$ with $\sigma\in\mathbf{B}_{n'}$. Let $i\in\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, $v\in F(n+2)$ and $g\in\mathbf{F}_n$. By Condition 1.2.3 (using Lemma 1.2.5 with $n'=n+1$) the following equality holds in $\mathbf{B}_{n+2}$:
$$\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right)\circ(id_1\natural\varsigma_n(g))=\varsigma_{1+n}(e_{\mathbf{F}_1}*g)\circ\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural id_n\right).$$
Recall that $F(n+2)$ is a $\mathbb{K}[\mathbf{F}_n]$-module via $F\left(\varsigma_{1+n}\circ\left(\iota_{\mathbf{F}_1}*id_{\mathbf{F}_n}\right)\right)$ and $\tau_1F(n+1)$ is a $\mathbb{K}[\mathbf{F}_n]$-module via $F\left(id_1\natural\left(\varsigma_n\circ id_{\mathbf{F}_n}\right)\right)$. Then it follows that the assignment $\xi'(F)_n$ is well-defined with respect to the tensor product structures of $(\mathbf{LM}\circ\tau_1)(F)(n)$ and $(\tau_1\circ\mathbf{LM})(F)(n)$. Moreover, we compute that:
$$\begin{aligned}&((\tau_1\circ\mathbf{LM})(F)([n'-n,\sigma]))\circ(\xi'(F)_n)\left(i\underset{\mathbb{K}[\mathbf{F}_n]}{\otimes}v\right)\\ =\;& a_{1+n'}(id_1\natural\sigma)\left(a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n\right)\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{1+n'-n}]}}*id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i)\right)\\ &\underset{\mathbb{K}[\mathbf{F}_{n'+1}]}{\otimes}F\left(\left(b^{\beta}_{1,1}\right)^{-1}\natural[n'-n,\sigma]\right)(v).\end{aligned}$$
It follows from Condition 1.2.10 that:
$$a_{1+n'}\left(\left(b^{\beta}_{1,n'-n}\right)^{-1}\natural id_n\right)\circ\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{1+n'-n}]}}*id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i)=\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{1+n'-n}]}}*id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i).$$
Again by Condition 1.2.10, we deduce that:
$$a_{1+n'}(id_1\natural\sigma)\circ\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{1+n'-n}]}}*id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i)=\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}}*a_{n'}(\sigma)\left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'-n}]}}*id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i).$$

Hence, we deduce that:

$$((\tau_1 \circ \mathbf{LM})(F)([n'-n,\sigma])) \circ (\xi'(F)_n) = (\xi'(F)_{n'}) \circ ((\mathbf{LM} \circ \tau_1)(F)([n'-n,\sigma])).$$

Let $\eta : F \to G$ be a natural transformation in the category $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$ and let n be a natural number. Since η is a natural transformation, we have:

$$G\left(\left(b_{1,1}^{\beta}\right)^{-1} \natural id_n\right) \circ \eta_{n+2} = \eta_{n+2} \circ F\left(\left(b_{1,1}^{\beta}\right)^{-1} \natural id_n\right).$$

Hence, we deduce from the definitions of $\tau_1 \circ \mathbf{LM}$ (see Theorem 1.2.20) and of $\mathbf{LM} \circ \tau_1$ (see Remark 1.4.17) that:

$$\xi'(G)_n \circ (\mathbf{LM} \circ \tau_1)(\eta)_n = (\tau_1 \circ \mathbf{LM})(\eta)_n \circ \xi'(F)_n.$$

□

1.4.1.3 Splitting of the translation functor

Now, we can establish a decomposition result for the translation functor applied to a Long-Moody functor.

Proposition 1.4.19. *There is a natural equivalence of endofunctors of* $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$*:*

$$\tau_1 \circ \mathbf{LM} \cong \tau_2 \oplus (\mathbf{LM} \circ \tau_1).$$

Proof. Recall the natural transformations $v : \Upsilon \to \tau_1 \circ \mathbf{LM}$ (introduced in Proposition 1.4.15) and $\xi' : \mathbf{LM} \circ \tau_1 \to \tau_1 \circ \mathbf{LM}$ (defined in Proposition 1.4.18). The direct sum in the category $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$ (induced by the direct sum in the category $\mathbb{K}\text{-}\mathfrak{Mod}$) allows us to define a natural transformation:

$$v \oplus \xi' : \Upsilon \oplus (\mathbf{LM} \circ \tau_1) \longrightarrow (\tau_1 \circ \mathbf{LM})(F).$$

This is a natural equivalence since for all natural numbers n, we have an isomorphism of $\mathbb{K}$-modules according to Proposition 1.4.4: $\Upsilon(F)(n) \oplus (\mathbf{LM} \circ \tau_1)(F)(n) \cong (\tau_1 \circ \mathbf{LM})(F)(n)$. We conclude using Proposition 1.4.16. □

1.4.2 Splitting of the difference functor

Recall the natural transformation $i_1 : Id_{\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})} \to \tau_1$ of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$. Our aim is to study the cokernel of $i_1 \circ \mathbf{LM}$. We recall that for F an object of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$, for all natural numbers n, $(i_1\mathbf{LM})(F)_n = \mathbf{LM}(F)([1, id_{1+n}])$ (see Definition 1.3.4).

Remark 1.4.20. Explicitly for all elements i of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, for all elements v of $F(n)$:

$$(i_1\mathbf{LM})(F)_n\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = \left(\iota_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_1]}} * id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}}\right)(i) \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} F(id_1 \natural \iota_1 \natural id_n)(v).$$

The natural transformation $\mathbf{LM} \circ i_1$: Let us consider the exact sequence (1.3.1) in the category of endofunctors of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$ of Proposition 1.3.5:

$$0 \longrightarrow \kappa_1 \xrightarrow{\Omega_1} Id \xrightarrow{i_1} \tau_1 \xrightarrow{\Delta_1} \delta_1 \longrightarrow 0.$$

Since the Long-Moody functor is exact (see Proposition 1.2.23), we have the following exact sequence:

$$0 \longrightarrow \mathbf{LM} \circ \kappa_1 \xrightarrow{\mathbf{LM}(\Omega_1)} \mathbf{LM} \xrightarrow{\mathbf{LM}(i_1)} \mathbf{LM} \circ \tau_1 \xrightarrow{\mathbf{LM}(\Delta_1)} \mathbf{LM} \circ \delta_1 \longrightarrow 0. \quad (1.4.4)$$

Remark 1.4.21. From the definition of $\mathbf{LM}$ (see Theorem 1.2.20), we deduce that for F an object of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod})$, for all natural numbers n, for all elements i of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, for all elements v of $F(n)$:

$$\mathbf{LM}(i_1)(F)_n\left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} F(\iota_1 \natural id_1 \natural id_n)(v).$$

Recall the natural transformation $\xi' : \mathbf{LM} \circ \tau_1 \to \tau_1 \circ \mathbf{LM}$ introduced in 1.4.18.

Lemma 1.4.22. *As natural transformations from* $\mathbf{LM}$ *to* $\tau_1 \circ \mathbf{LM}$*, which are endofunctors of the category* $\mathbf{Fct}\left(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}\right)$*, the following equality holds:*

$$\xi' \circ \left(\mathbf{LM}\left(i_1\right)\right) = i_1 \mathbf{LM}.$$

Proof. Let F be an object of $\mathbf{Fct}\left(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}\right)$. Let n be a natural number. Let i be an element of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and let v be an element of $F(n)$. Since $\left(b^{\beta}_{1,1}\right)^{-1} \circ \left(\iota_1 \natural id_1\right) = id_1 \natural \iota_1$ by Definition 1.1.13, we deduce from Proposition 1.4.18, Remark 1.4.21 and Remark 1.4.20, that:

$$\left(\xi' \circ \left(\mathbf{LM}\left(i_1\right)\right)\right)(F)_n \left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right) = \left(id_1 * i\right) \underset{\mathbb{K}[\mathbf{F}_{1+n}]}{\otimes} F\left(id_1 \natural \iota_1 \natural id_n\right)(v) = \left(i_1 \mathbf{LM}\right)(F)_n \left(i \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} v\right).$$

□

Decomposition results: Lemma 1.4.22 leads to the following key result.

Theorem 1.4.23. *There is a natural equivalence in the category* $\mathbf{Fct}\left(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{Mod}\right)$*:*

$$\delta_1 \circ \mathbf{LM} \cong \tau_2 \oplus \left(\mathbf{LM} \circ \delta_1\right).$$

Proof. It follows from the definition of i_1 (see Proposition 1.3.5) and from Lemma 1.4.22 that the following diagram is commutative and the row is an exact sequence:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \kappa_1 \circ \mathbf{LM} & \xrightarrow{\Omega_1 \mathbf{LM}} & \mathbf{LM} & \xrightarrow{i_1 \mathbf{LM}} & \tau_1 \circ \mathbf{LM} & \xrightarrow{\Delta_1 \mathbf{LM}} & \delta_1 \circ \mathbf{LM} & \longrightarrow 0 \\
 & & & \Vert & & \uparrow \xi' \text{ by Lemma 1.4.22} & & & \\
 & & & \mathbf{LM} & \xrightarrow{\mathbf{LM}(i_1)} & \mathbf{LM} \circ \tau_1. & & &
\end{array}$$

We denote by $i^{\oplus}_{\mathbf{LM}\circ\tau_1}$ the inclusion morphism $\mathbf{LM} \circ \tau_1 \hookrightarrow \tau_2 \oplus \left(\mathbf{LM} \circ \tau_1\right)$. Then, recalling the exact sequence (1.4.4), we obtain that the following diagram is commutative and that the two rows are exact:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \kappa_1 \circ \mathbf{LM} & \xrightarrow{\Omega_1 \circ \mathbf{LM}} & \mathbf{LM} & \xrightarrow{i_1 \circ \mathbf{LM}} & \tau_1 \mathbf{LM} & \xrightarrow{\Delta_1 \circ \mathbf{LM}} & \delta_1 \circ \mathbf{LM} & \longrightarrow 0 \\
 & & & \Vert & \cong \text{ by Proposition 1.4.19} & \uparrow v \oplus \xi' & & & \\
 & & & \mathbf{LM} & \xrightarrow[i^{\oplus}_{\mathbf{LM}\circ\tau_1} \circ (\mathbf{LM}(i_1))]{} & \tau_2 \oplus \left(\mathbf{LM} \circ \tau_1\right) & \xrightarrow[id_{\tau_2} \oplus (\mathbf{LM}(\Delta_1))]{} & \tau_2 \oplus \left(\mathbf{LM} \circ \delta_1\right) & \longrightarrow 0.
\end{array} \tag{1.4.5}$$

A fortiori, by definition of δ_1 (see Definition 1.3.4) and the universal property of the cokernel, we deduce that:

$$\tau_2 \oplus \left(\mathbf{LM} \circ \delta_1\right) \cong \delta_1 \circ \mathbf{LM}.$$

□

Furthermore, we can determine the behaviour of the evanescence functor.

Theorem 1.4.24. *The endofunctor* κ_1 *commutes with the endofunctor* $\mathbf{LM}$*. In other words, there is a natural equivalence* $\kappa_1 \circ \mathbf{LM} \cong \mathbf{LM} \circ \kappa_1$*.*

Proof. Recall the exact sequence (1.4.4). Since the inclusion morphism $i^{\oplus}_{\mathbf{LM}\circ\tau_1} : \mathbf{LM} \circ \tau_1 \hookrightarrow \tau_2 \oplus \left(\mathbf{LM} \circ \tau_1\right)$ is a monomorphism, we deduce that the functor $\mathbf{LM} \circ \kappa_1$ is also the kernel of the natural transformation $i^{\oplus}_{\mathbf{LM}\circ\tau_1} \circ \left(\mathbf{LM} \circ i_1\right)$. Hence, recalling the commutative diagram (1.4.5), we obtain the following commutative diagram, in which the two rows are exact sequences.

$$\begin{array}{ccccccccc}
0 \longrightarrow & \kappa_1 \circ \mathbf{LM} & \xrightarrow{\Omega_1 \mathbf{LM}} & \mathbf{LM} & \xrightarrow{i_1 \mathbf{LM}} & \tau_1 \circ \mathbf{LM} & \xrightarrow{\Delta_1 \mathbf{LM}} & \delta_1 \circ \mathbf{LM} & \longrightarrow 0 \\
 & & & \Vert & \cong \text{ by Proposition 1.4.19} & \uparrow v \oplus \xi' & & \uparrow \cong \text{ by Theorem 1.4.23} & \\
0 \longrightarrow & \mathbf{LM} \circ \kappa_1 & \xrightarrow[\mathbf{LM}(\Omega_1)]{} & \mathbf{LM} & \xrightarrow[i^{\oplus}_{\mathbf{LM}\circ\tau_1} \circ (\mathbf{LM}(i_1))]{} & \tau_2 \oplus \left(\mathbf{LM} \circ \tau_1\right) & \xrightarrow[id_{\tau_2} \oplus (\mathbf{LM}(\Delta_1))]{} & \tau_2 \oplus \left(\mathbf{LM} \circ \delta_1\right) & \longrightarrow 0
\end{array}$$

By the unicity up to isomorphism of the kernel, we conclude that $\kappa_1 \circ \mathbf{LM} \cong \mathbf{LM} \circ \kappa_1$. □

1.4.3 Increase of the polynomial degree

The results formulated in Theorems 1.4.23 and 1.4.24 allow us to understand the effect of the Long-Moody functors on (very) strong polynomial functors.

Proposition 1.4.25. *Let F be a non-null object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. If the functor F is strong polynomial of degree d, then:*

1. *the functor $\tau_2(F)$ belongs to $\mathcal{P}ol_d^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$;*
2. *the functor $\mathbf{LM}(F)$ belongs to $\mathcal{P}ol_{d+1}^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.*

Proof. We prove these two results by induction on the degree of polynomiality. For the first result, it follows from the commutation property 5 of Proposition 1.3.5 for τ_2. For the second result, let us first consider F a strong polynomial functor of degree 0. By Theorem 1.4.23, we obtain that $\delta_1 \mathbf{LM}(F) \cong \tau_2(F)$. Therefore $\mathbf{LM}(F)$ is a strong polynomial functor of degree less than or equal to 1. Now, assume that F is a strong polynomial functor of degree $n \geq 0$. By Theorem 1.4.23: $\delta_1 \mathbf{LM}(F) \cong \mathbf{LM}(\delta_1 F) \oplus \tau_2(F)$. By the inductive hypothesis and the result on τ_2, we deduce that $\mathbf{LM}(F)$ is a strong polynomial functor of degree less than or equal to $n+1$. □

Corollary 1.4.26. *For all natural numbers d, the endofunctor $\mathbf{LM}$ of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ restricts to a functor:*

$$\mathbf{LM} : \mathcal{P}ol_d^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \longrightarrow \mathcal{P}ol_{d+1}^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}).$$

Corollary 1.4.27. *Let d be a natural number and F be an object of $\mathcal{P}ol_d^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ such that the strong polynomial degree of $\tau_2(F)$ is equal to d. Then, the functor $\mathbf{LM}(F)$ is a strong polynomial functor of degree equal to $d+1$.*

Theorem 1.4.28. *Let d be a natural number and F be an object of $\mathcal{VP}ol_d(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ of degree equal to d. Then, the functor $\mathbf{LM}(F)$ is a very strong polynomial functor of degree equal to $d+1$.*

Proof. Using Lemma 1.3.23, it follows from Corollary 1.4.27 that $\mathbf{LM}(F)$ is a strong polynomial functor of degree equal to $n+1$. Since the functor $\mathbf{LM}$ commutes with the evanescence functor κ_1 by Theorem 1.4.24, we deduce that $(\kappa_1 \circ \mathbf{LM})(F) \cong (\mathbf{LM} \circ \kappa_1)(F) = 0$. Moreover, using Theorem 1.4.23, we have:

$$(\kappa_1 \circ (\delta_1 \circ \mathbf{LM}))(F) \cong (\kappa_1 \circ \tau_2)(F) \bigoplus (\kappa_1 \circ (\mathbf{LM} \circ \delta_1))(F).$$

Therefore, the fact that τ_2 commutes with the evanescence functor κ_1 (see the commutation property 6 of Proposition 1.3.5) and Theorem 1.4.24 together imply that:

$$(\kappa_1 \circ (\delta_1 \circ \mathbf{LM}))(F) \cong (\tau_2 \circ \kappa_1)(F) \bigoplus (\mathbf{LM} \circ (\kappa_1 \circ \delta_1))(F).$$

The result then follows from the fact that F is an object of $\mathcal{VP}ol_n(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ and τ_2 is a reduced endofunctor of the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. □

Example 1.4.29. By Proposition 1.3.20, $\mathfrak{X}$ is a very strong polynomial functor of degree 0. Now applying the Long-Moody functor $\mathbf{LM}_1$, we proved in Proposition 1.2.30 that $t^{-1}\mathbf{LM}_1(t\mathfrak{X})$ is naturally equivalent to $\mathfrak{Bur}_{t^2}$, which is very strong polynomial of degree 1 by Proposition 1.3.25.

1.4.4 Other properties of the Long-Moody functors

We have proven in the previous section that a Long-Moody functor sends (very) strong polynomial functors to (very) strong polynomial functors. We can also prove that a (very) strong polynomial functor in the essential image of a Long-Moody functor is necessarily the image of another strong polynomial functor.

Proposition 1.4.30. *Let d be a natural number. Let F be a strong polynomial functor of degree d in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. Assume that there exists an object G of the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ such that $\mathbf{LM}(G) = F$. Then, the functor G is a strong polynomial functor of degree less than or equal to $d+1$ in the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$.*

Proof. It follows from Theorem 1.4.23 that:

$$\delta_1 F \cong \tau_2 (G) \oplus (\mathbf{LM} \circ \delta_1)(G).$$

According to Corollary 1.3.10, the functor $\tau_2 (G)$ is an object of the category $\mathcal{P}ol_{d-1}^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$, and because of Lemma 1.3.14 the functor G is an object of the category $\mathcal{P}ol_{d+1}^{strong}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. □

Proposition 1.4.31. *The Long-Moody functor* $\mathbf{LM} : \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \longrightarrow \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ *is not essentially surjective.*

Proof. Let l be a natural number. Let $E_l : \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{K}\text{-}\mathfrak{Mod}$ be the functor which factorizes through the category $\mathbb{N}$, such that $E_l(n) = \mathbb{K}^{\oplus n^l}$ for all natural numbers n and for all $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}(n, n')$ (with n, n' natural numbers such that $n' \geq n$), $E_l([n'-n, \sigma]) = \iota_{\mathbb{C}[t^{\pm 1}]^{\oplus n'^l - n^l}} \oplus id_{\mathbb{C}[t^{\pm 1}]^{\oplus n^l}}$. In particular, for all natural numbers n, for every Artin generator σ_i of $\mathbf{B}_n$, $E_l(\sigma_i) = id_{\mathbb{K}^{\oplus n^l}}$. It inductively follows from this definition and direct computations that E_l is a very strong polynomial functor of degree l.

Let us assume that $\mathbf{LM}$ is essentially surjective. Hence, there exists an object F of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ such that $\mathbf{LM}(F) \cong E_l$. Because of the definition of $\mathbf{LM}(F)$ on morphisms (see Theorem 1.2.20), this implies that for all natural numbers n and for all $\sigma \in \mathbf{B}_n$, $a_n(\sigma) = id_n$. Also, if $\mathbf{LM}$ is essentially surjective, there exists an object T of the category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ such that we can recover the Burau functor from $\mathbf{LM}(T)$, ie something like $\alpha\mathbf{LM}(T)$ (see Notation 1.2.28) with $\alpha \in \mathbb{K}$. We deduce from the definition of $\mathbf{LM}(T)$ on objects and morphisms that for all $n \geq 1$, $T(n) = \mathbb{K}$ and for all generator σ_i of $\mathbf{B}_n$:

$$\mathbf{LM}(T)(\sigma_i) = T(\sigma_i) \cdot Id_n.$$

Then necessarily, for all $i \in \{1, \ldots, n\}$, $T(\sigma_i) = \delta$ such that $\delta^2 = t$ and we consider $\delta^{-1}\mathbf{LM}(T)$. We deduce that there exists a natural transformation $\omega : \delta^{-1}\mathbf{LM}(T) \overset{\cong}{\to} \mathfrak{Bur}_t$. This contradicts the fact that for all $\sigma \in \mathbf{B}_n$, $a_n(\sigma) = id_n$. □

Remark 1.4.32. The proof of Proposition 1.4.31 shows in particular that a Long-Moody functor $\mathbf{LM}$ is not essentially surjective on very strong polynomial functors in any degree.

In [BB05, Section 4.7, Open Problem 7], Birman and Brendle ask "whether all finite dimensional unitary matrix representations of $\mathbf{B}_n$ arise in a manner which is related to the construction" recalled in Theorem 1.2.20. Since the Tong-Yang-Ma and unreduced Burau representations recalled in Theorem 1.1.19 are unitary representations, the proof of Proposition 1.4.31 shows that any Long-Moody functor (and especially the one based on the version of the construction of Theorem 1.2.20) cannot provide all the functors encoding unitary representations. Therefore, we refine the problem asking whether all functors encoding families of finite dimensional unitary representations of braid groups lie in the image of a Long-Moody functor.

Remark 1.4.33. Another question is to ask whether we can directly obtain the reduced Burau functor $\overline{\mathfrak{Bur}}_t$ by a Long-Moody functor. Recall that for all natural numbers n, $\overline{\mathfrak{Bur}}_t(n) = \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n-1}$ and $\mathbf{LM}(F)(n) \cong (F(n+1))^{\oplus n}$ for any Long-Moody functor $\mathbf{LM}$ and any object F of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ (see Remark 1.2.24). Therefore, for dimensional considerations on the objects, it is clear that we have to consider a modified version of the Long-Moody construction. This modification would be to take the tensor product with $\mathcal{I}_{\mathbf{F}_{n-1}}$ on $\mathbf{F}_{n-1}$, the $\mathbb{K}$-module $F(n+1)$ being a $\mathbb{K}[\mathbf{F}_{n-1}]$-module using a morphism $\mathbf{F}_{n-1} \to \left(\mathbf{F}_{n-1} \underset{a'_n}{\rtimes} \mathbf{B}_{n+1}\right) \to \mathbf{B}_{n+1}$ for all natural numbers n, where $a'_n : \mathbf{B}_{n+1} \to Aut(\mathbf{F}_{n-1})$ is a group morphism.

Chapter 2

Generalised Long-Moody functors

Abstract: *In this paper, we generalise the Long-Moody construction for braid groups to other families of groups, such as mapping class groups of orientable and non-orientable surfaces or symmetric groups. Fixing an appropriate family of groups, we define endofunctors, called Long-Moody functors, between the category of functors from the homogeneous category associated with the family of groups to a module category. We prove that, under some additional assumptions, the Long-Moody functors increase by one the degree of (very) strong and weak polynomiality of functors. For the particular case of braid groups, we recover the results previously obtained by the author.*

Introduction

In 1994, as a result of a collaboration with Moody, Long [Lon94] gave a method to construct from a linear representation $\rho : \mathbf{B}_{n+1} \to GL(V)$ a new linear representation $lm(\rho) : \mathbf{B}_n \to GL(V^{\oplus n})$ of braid groups, where $\mathbf{B}_n$ denotes the braid group on n strands. Applying this construction to a one dimensional representation of $\mathbf{B}_{n+1}$, one obtains a mild variant of the unreduced Burau representation of $\mathbf{B}_n$. This construction depends on families of group morphisms $a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)$ and $\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}$, where $\mathbf{F}_n$ denotes the free group on n generators. Long and Moody fixed such a choice but a similar construction can be made for other choices (see [Sou17b]). In [Sou17b], it is proved that the Long-Moody construction and the ones obtained from other choices of a_n and ς_n are functorial; more precisely, we consider the category $\mathfrak{U}\boldsymbol{\beta}$ associated with braid groups, given by Quillen's bracket construction (see [Gra76, p.219]) applied to the braid groupoid $\boldsymbol{\beta}$, and the functor category $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, R\text{-}\mathfrak{Mod})$, where $R\text{-}\mathfrak{Mod}$ is the category of R-modules (with R a commutative ring). For choices of a_n and ς_n there is a functor $\mathbf{LM}_{a,\varsigma} : \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, R\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, R\text{-}\mathfrak{Mod})$, called the Long-Moody functor associated with the morphisms a_n and ς_n. These functors allow inter alia to recover functors encoding the well-known families of Burau and Tong-Yang-Ma representations, by applying appropriate Long-Moody functors to a constant functor (see [Sou17b, Section 2.3]).

Moreover, studying the behaviour of Long-Moody functors on a very strong polynomial of degree n functor (see [Sou17b, Section 2] for an introduction of this notion, inspired of [DV17, Section 1]), it is shown that $\mathbf{LM}(F)$ is a very strong polynomial functor of degree $n+1$ (see [Sou17b, Section 4]). Thus, the Long-Moody functors provide by iteration very strong polynomial functors of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, R\text{-}\mathfrak{Mod})$ in any degree. This type of functor turns out to be very useful for homological stability problems: in [RWW17], Randal-Williams and Wahl prove homological stability for different families of groups for coefficients given by a very strong polynomial functor. Besides braid groups, their results also hold among others for automorphism groups of free products of groups, mapping class groups of orientable and non-orientable surfaces or mapping class groups of 3-manifolds. As for braid groups, the representation theory of these groups is complicated and a current research topic (see for example [BB05, Section 4.6], [Fun99], [Kor02] or [Mas08]). A fortiori, the very strong polynomial functors associated with these groups are not well-known.

The aim of this paper is to extend the Long-Moody construction to other families of groups and the study of its behaviour on (very) strong polynomial functors. In addition, we are also interested in the effect of this construction on weak polynomial functors, a notion introduced by Djament and Vespa in [DV17, Section 3.1] for symmetric monoidal categories and extended in the pre-braided case in the present paper (see Section 2.5.6). This last notion is more appropriate for understanding the stable behaviour of a given functor.

For this, we consider a family of groups $\{H_m\}_{m\in\mathbb{N}}$, where H_m is the free product $H^{*m} * H_0$, with H and H_0 two given groups, and the groupoid $\mathcal{G}$ associated with a family of groups $\{G_n\}_{n\in\mathbb{N}}$. More precisely, the groupoid $\mathcal{G}$ is assumed to be a subgroupoid of a braided monoidal groupoid $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$ (see Section 2.2.1.1) such that the set of objects of $\mathcal{G}$ is isomorphic to the natural numbers, its objects are denoted by $\underline{n}$ (for n a natural number) and the automorphism group $Aut_{\mathcal{G}'}(\underline{n})$ is the group G_n. We denote by $\mathfrak{U}\mathcal{G}$ the full subcategory generated by $\mathcal{G}$ in the category $\mathfrak{U}\mathcal{G}'$ provided by Quillen's bracket construction (see Section 2.1). We denote by $\mathcal{I}_{R[H_n]}$ the augmentation ideal of the group H_n. For families of morphisms $G_n \to Aut(H_n)$ and $H_n \to G_{n+1}$ satisfying some coherence properties (see Sections 2.2.1.1 and 2.2.1.2), using the same principle as for braid groups in [Sou17b, Section 3], we prove:

***Theorem A* (Proposition 2.2.30, Theorem 2.5.29 and Theorem 2.5.36).** *For H and H_0 two groups, assume that the families of groups $\{H_m = H^{*m} * H_0\}_{m\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ satisfy Assumptions 2.2.1 and 2.2.13, and Conditions 2.2.17, 2.2.24. Then, there exists a functor $\mathbf{LM} : \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, called a Long-Moody functor, such that:*

$$\mathbf{LM}(F)(\underline{n}) = \mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} F(\underline{n+1})$$

for all objects F of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ and $\underline{n}$ objects of $\mathcal{G}$.

Moreover, if Assumption 2.5.16 and Condition 2.5.9 are satisfied, then:

- *The functor $\mathbf{LM}$ increases by one the very strong polynomial degree if H and H_0 are free.*
- *If H is free, then the functor $\mathbf{LM}$ increases by one the weak polynomial degree if H_0 is free or if the groupoid $\mathcal{G}'$ is symmetric monoidal.*

For the family of braid groups $\{\mathbf{B}_n\}_{n\in\mathbb{N}}$, the first statement of this theorem corresponds to [Sou17b, Theorem A] and the others recover [Sou17b, Theorem B]. Additionally, in this paper, we prove that the families of symmetric groups, automorphism groups of free products of groups, surface braid groups, mapping class groups of orientable and non-orientable surfaces or mapping class groups of 3-manifolds fit into this framework. We determine the effect of a Long-Moody functor on a constant functor, which is the most basic functor to study. As an example, for the family of mapping class group of compact orientable connected surfaces of genus g with one boundary component, from a constant functor, we recover a functor encoding the family of symplectic representations of mapping class groups, which is therefore very strong and weak polynomial of degree 1 (see Corollary 2.5.31).

When the groupoid $\mathcal{G}'$ is symmetric monoidal, the homogenous category $\mathfrak{U}\mathcal{G}'$ is also symmetric monoidal (see [RWW17, Proposition 1.8]). In this case, we extend a Long-Moody endofunctor from $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ to a category of functors from a symmetric monoidal category where the unit is a null object. More precisely, denoting by $\mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{ini}}$ (resp. $\mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{null}}$) the category of symmetric strict monoidal small categories $(\mathfrak{M}, \natural, 0)$ such that the unit 0 is an initial object (resp. a null object), we are interested in the left adjoint of the forgetful functor $\mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{null}} \hookrightarrow \mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{ini}}$. This functor was considered by Djament and Vespa in [DV17, Section 3] and is denoted by $\widetilde{-} : \mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{ini}} \to \mathfrak{Mon}^{\mathfrak{symm}}_{\mathfrak{null}}$. We prove that, under an additional assumption(see Section 2.6.2), a Long-Moody functor $\mathbf{LM} : \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ extends to a functor $\widetilde{\mathbf{LM}} : \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right) \to \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right)$. Explicitly:

***Theorem B* (Propositions 2.6.24 and 2.6.25).** *Assume that families of groups $\{H_m\}_{m\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ satisfy the same properties as in Theorem A as well as Condition 2.6.20. Then, there exists a functor $\widetilde{\mathbf{LM}} : \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right) \to \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right)$ such that the following diagram is commutative:*

$$\begin{array}{ccc}
\mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right) & \xrightarrow{\widetilde{\mathbf{LM}}} & \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right) \\
\left(incl^{\widetilde{\mathfrak{U}\mathcal{G}}}_{\mathfrak{U}\mathcal{G}}\right)^{*} \downarrow & & \downarrow \left(incl^{\widetilde{\mathfrak{U}\mathcal{G}}}_{\mathfrak{U}\mathcal{G}}\right)^{*} \\
\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) & \xrightarrow{\mathbf{LM}} & \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}).
\end{array}$$

Finally, the framework and definition of generalised Long-Moody functors in Section 2.2 leads to the wider notion of tensorial functors introduced in Section 2.7.

The paper is organized as follows. In Section 2.1, we recall necessary notions on Quillen's bracket construction. In Section 2.2, after setting up the general framework of the families of groups which we will deal with, in particular exposing the properties they have to satisfy, we define the generalisation and give some first properties of the Long-Moody functors. Section 2.3 is devoted to the application of Long-Moody functors for the mapping class groups of surfaces, recovering in particular the case of braid groups. In Section 2.4, we recall the notions of strong and very strong polynomial functors for our framework, and we adapt in this context the one of weak polynomial functors introduced in [DV17, Section 3.1]. Then, in Section 2.5, we are interested in the effect of Long-Moody functors on strong and weak polynomial functors, presenting in particular the keystone relations of the difference and evanescence functors with Long-Moody functors. In Section 2.6, we prove that in the situation where a symmetric monoidal category is considered, we can extend a Long-Moody functor taking as source the category modified by the construction $\widetilde{-}$. Finally, in Section 2.7, we introduce the notion of tensorial functors.

Notation 2.0.1. We fix a commutative ring R throughout this work. We denote by R-$\mathfrak{Mod}$ the category of R-modules. We denote by $\mathfrak{Gr}$ the category of groups and by $*$ the coproduct in this category.

Let $\mathfrak{Cat}$ denote the category of small categories. Let $\mathfrak{C}$ be an object of $\mathfrak{Cat}$. We use the abbreviation $Obj(\mathfrak{C})$ to denote the objects of $\mathfrak{C}$. If there exists an initial object $\varnothing$ in the category $\mathfrak{C}$, then we denote by $\iota_A : \varnothing \longrightarrow A$ the unique morphism from $\varnothing$ to A. If $*$ is terminal object in the category $\mathfrak{C}$, then we denote by $t_A : A \longrightarrow *$ the unique morphism from A to $*$.

The maximal subgroupoid $\mathscr{G}\mathfrak{r}(\mathfrak{C})$ is the subcategory of $\mathfrak{C}$ which has the same objects as $\mathfrak{C}$ and of which the morphisms are the isomorphisms of $\mathfrak{C}$. We denote by $\mathscr{G}\mathfrak{r} : \mathfrak{Cat} \longrightarrow \mathfrak{Cat}$ the functor which associates to a category its maximal subgroupoid.

For $\mathfrak{D}$ a category and $\mathfrak{C}$ a small category, we denote by $\mathbf{Fct}(\mathfrak{C}, \mathfrak{D})$ the category of functors from $\mathfrak{C}$ to $\mathfrak{D}$.

We denote by $\boldsymbol{\beta}$ the braid groupoid: its objects are the natural numbers $n \in \mathbb{N}$ and its morphisms are (for $n, m \in \mathbb{N}$):

$$Hom_{\boldsymbol{\beta}}(n,m) = \begin{cases} \mathbf{B}_n & \text{if } n = m \\ \varnothing & \text{if } n \neq m. \end{cases}$$

2.1 Recollections on Quillen's bracket construction

The aim of this section is to introduce the categorical framework necessary for our study. In particular, we recall notions and properties of Quillen's bracket construction introduced in [Gra76, p.219] for a monoidal category S acting on a category X, in the case $S = X = \mathfrak{G}$ where $\mathfrak{G}$ is a groupoid. Our review here is based on [RWW17, Section 1] to which we refer the reader for further details.

Beforehand, we take this opportunity to introduce or recall some terminology about strict monoidal categories. We refer to [ML13] for an introduction to (braided) strict monoidal categories and the skeleton of a category.

Notation 2.1.1. A strict monoidal category will be denoted by $(\mathfrak{C}, \natural, 0)$, where $\mathfrak{C}$ is the category, $\natural$ is the monoidal product and 0 is the monoidal unit. If it is braided, then its braiding is denoted by $b^{\mathfrak{C}}_{A,B} : A \natural B \overset{\sim}{\to} B \natural A$ for all objects A and B of $\mathfrak{C}$.

Definition 2.1.2. Let $(\mathfrak{C}, \natural, 0)$ be a strict monoidal category. A full subcategory $\mathfrak{D}$ of $\mathfrak{C}$ is said to be finitely generated by the monoidal structure if there exists a finite set E of objects of the category $\mathfrak{C}$ such that for all objects d of $\mathfrak{D}$, d is isomorphic to a finite monoidal product of objects of E.

We fix a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ throughout this section, so as to define Quillen's bracket construction $\mathfrak{U}$ following [Gra76].

Definition 2.1.3. [RWW17, Section 1.1] Quillen's bracket construction on the groupoid $\mathfrak{G}$, denoted by $\mathfrak{U}\mathfrak{G}$, is the category defined by:

- Objects: $Obj(\mathfrak{U}\mathfrak{G}) = Obj(\mathfrak{G})$;
- Morphisms: for A and B objects of $\mathfrak{G}$:

$$Hom_{\mathfrak{U}\mathfrak{G}}(A,B) = \underset{\mathfrak{G}}{colim}\left[Hom_{\mathfrak{G}}(-\natural A, B)\right].$$

Thus, a morphism from A to B in the category $\mathfrak{U}\mathfrak{G}$ is an equivalence class of pairs (X, f), where X is an object of $\mathfrak{G}$ and $f : X\natural A \to B$ is a morphism of $\mathfrak{G}$; this is denoted by $[X, f] : A \to B$.

- For all objects X of $\mathfrak{U}\mathfrak{G}$, the identity morphism in the category $\mathfrak{U}\mathfrak{G}$ is given by $[0, id_X] : X \longrightarrow X$.
- Let $[X, f] : A \longrightarrow B$ and $[Y, g] : B \longrightarrow C$ be two morphisms in the category $\mathfrak{U}\mathfrak{G}$. Then, the composition in the category $\mathfrak{U}\mathfrak{G}$ is defined by:
$$[Y, g] \circ [X, f] = [Y\natural X, g \circ (id_Y \natural f)] .$$

Proposition 2.1.4. *[RWW17, Proposition 1.8 (i)] The unit 0 of the monoidal structure of the groupoid $(\mathfrak{G}, \natural, 0)$ is an initial object in the category $\mathfrak{U}\mathfrak{G}$.*

Remark 2.1.5. Let X be an object of $\mathfrak{G}$. Let $\phi \in Aut_{\mathfrak{G}}(X)$. Then, as an element of $Hom_{\mathfrak{U}\mathfrak{G}}(X, X)$, we will abuse the notation and write ϕ for $[0, \phi]$. This comes from the canonical functor $\mathfrak{c}_{\mathfrak{U}\mathfrak{G}} : \mathfrak{G} \to \mathfrak{U}\mathfrak{G}$ defined as the identity on objects and sending $\phi \in Aut_{\mathfrak{G}}(X)$ to $[0, \phi]$.

A natural question to ask is the relationship between the automorphisms of the groupoid $\mathfrak{G}$ and those of its associated Quillen bracket construction $\mathfrak{U}\mathfrak{G}$. Recall the following notion.

Definition 2.1.6. The strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors if, for all objects A and B of $\mathfrak{G}$, $A\natural B \cong 0$ if and only if $A \cong B \cong 0$.

Then, recall the result:

Proposition 2.1.7. *[RWW17, Proposition 1.7] Assume that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $Aut_{\mathfrak{G}}(0) = \{id_0\}$. Then, the groupoid $\mathfrak{G}$ is the maximal subgroupoid of $\mathfrak{U}\mathfrak{G}$.*

Henceforth, we assume that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $Aut_{\mathfrak{G}}(0) = \{id_0\}$.

A natural question is to wonder when an object of $\mathbf{Fct}(\mathfrak{G}, \mathscr{C})$ extends to an object of $\mathbf{Fct}(\mathfrak{U}\mathfrak{G}, \mathscr{C})$ for a category $\mathscr{C}$, which is the aim of the following lemma. Analogous statements can be found in [RWW17, Proposition 2.4] and [Sou17b, Lemma 1.12] (for the category $\mathfrak{U}\boldsymbol{\beta}$ for this last reference).

Lemma 2.1.8. *Let $\mathscr{C}$ be a category and F an object of $\mathbf{Fct}(\mathfrak{G}, \mathscr{C})$. Assume that for $A, X, Y \in Obj(\mathfrak{G})$, there exist assignments $F([X, id_{X\natural A}]) : F(A) \to F(X\natural A)$ such that:*

$$F([Y, id_{Y\natural X\natural A}]) \circ F([X, id_{X\natural A}]) = F([Y\natural X, id_{Y\natural X\natural A}]) . \tag{2.1.1}$$

Then, the assignments $F([X, \gamma]) = F(\gamma) \circ F([X, id_{X\natural A}])$ for all $[X, \gamma] \in Hom_{\mathfrak{U}\mathfrak{G}}(A, id_{X\natural A})$ define a functor $F : \mathfrak{U}\mathfrak{G} \to \mathscr{C}$ if and only if for all $A, X \in Obj(\mathfrak{G})$, for all $\gamma'' \in Aut_{\mathfrak{G}}(A)$ and all $\gamma' \in Aut_{\mathfrak{G}}(X)$:

$$F([X, id_{X\natural A}]) \circ F(\gamma'') = F(\gamma'\natural\gamma'') \circ F([X, id_{X\natural A}]) . \tag{2.1.2}$$

Proof. Assume that relation (2.1.2) is satisfied. Note that (2.1.1) implies that $F([0, id_A]) = id_{F(A)}$ for all objects A. First, let us prove that our assignment conforms with the defining equivalence relation of $\mathfrak{U}\mathfrak{G}$. Let $A, X \in Obj(\mathfrak{G})$. Let $\gamma, \gamma' \in Aut_{\mathfrak{G}}(X\natural A)$ such that there exists $\psi \in Aut_{\mathfrak{G}}(X)$ so that $\gamma' \circ (\psi\natural id_A) = \gamma$. According to the relation (2.1.2) and since F is a functor over $\mathfrak{G}$, we deduce that $F([X, \gamma]) = F(\gamma') \circ F([X, id_{X\natural A}]) \circ F(id_A) = F([X, \gamma'])$. Now, let us check the composition axiom. Let $A, X, Y \in Obj(\mathfrak{G})$, let $([X, \gamma]) \in Hom_{\mathfrak{U}\mathfrak{G}}(A, X\natural A)$ and $([Y, \gamma']) \, Hom_{\mathfrak{U}\mathfrak{G}}(X\natural A, Y\natural X\natural A)$. We deduce from relation (2.1.2) that:

$$F([Y, \gamma']) \circ F([X, \gamma]) \quad = \quad F(\gamma') \circ (F(id_Y\natural\gamma) \circ F([Y, id_{Y\natural X\natural A}])) \circ F([X, id_{X\natural A}]) .$$

So, it follows from relation (2.1.1) that:

$$F([Y, \gamma']) \circ F([X, \gamma]) = F(\gamma' \circ (id_Y\natural\gamma)) \circ F([Y\natural X, id_{Y\natural X\natural A}]) = F([Y, \gamma'] \circ [X, \gamma]) .$$

Conversely, assume that the functor $F : \mathfrak{U}\mathfrak{G} \to \mathscr{C}$ is well-defined. In particular, the composition axiom in $\mathfrak{U}\mathfrak{G}$ is satisfied and implies that for all $A, X \in Obj(\mathfrak{G})$, for all $\gamma \in Aut_{\mathfrak{G}}(A)$, $F([X, id_{X\natural A}]) \circ F(\gamma) = F([X, id_X\natural\gamma])$. So it follows from the defining equivalence relation of $\mathfrak{U}\mathfrak{G}$ that relation (2.1.2) is satisfied. □

Similarly, we can find a criterion for extending a morphism in the category $\mathbf{Fct}(\mathfrak{G},\mathscr{C})$ to a morphism in the category $\mathbf{Fct}(\mathfrak{UG},\mathscr{C})$.

Lemma 2.1.9. *Let $\mathscr{C}$ be a category, F and G be objects of $\mathbf{Fct}(\mathfrak{UG},\mathscr{C})$ and $\eta : F \to G$ a natural transformation in $\mathbf{Fct}(\mathfrak{G},\mathscr{C})$. The restriction $\mathbf{Fct}(\mathfrak{UG},\mathscr{C}) \to \mathbf{Fct}(\mathfrak{G},\mathscr{C})$ is obtained by precomposing by the canonical inclusion $\mathfrak{c}_{\mathfrak{UG}}$ of Remark 2.1.5. Then, η is a natural transformation in the category $\mathbf{Fct}(\mathfrak{UG},\mathscr{C})$ if and only if for all $A, B \in Obj(\mathfrak{G})$ such that $B \cong X\natural A$ with $X \in Obj(\mathfrak{G})$:*

$$\eta_B \circ F([X, id_B]) = G([X, id_B]) \circ \eta_A. \tag{2.1.3}$$

Proof. The natural transformation η extends to the category $\mathbf{Fct}(\mathfrak{UG},\mathscr{C})$ if and only if for all $A, B \in Obj(\mathfrak{G})$ such that $B \cong X\natural A$ with $X \in Obj(\mathfrak{G})$, for all $[X,\gamma] \in Hom_{\mathfrak{UG}}(A,B)$:

$$\eta_B \circ F([X,\gamma]) = G([X,\gamma]) \circ \eta_A.$$

Since η is a natural transformation in the category $\mathbf{Fct}(\mathfrak{G},\mathscr{C})$, we already have $\eta_B \circ F(\gamma) = G(\gamma) \circ \eta_A$. So, η extends to the category $\mathbf{Fct}(\mathfrak{UG},\mathscr{C})$ if and only if relation (2.1.3) is satisfied. □

Pre-braided monoidal categories: If the strict monoidal groupoid $(\mathfrak{G},\natural,0)$ is braided, Quillen's bracket construction $\mathfrak{UG}$ inherits a strict monoidal structure (see Proposition 2.1.12). However, the braiding $b^{\mathfrak{G}}_{-,-}$ does not extend in general to $\mathfrak{UG}$. First recall the notion of a pre-braided monoidal category, generalising that of a braided strict monoidal category, introduced by Randal-Williams and Wahl in [RWW17].

Definition 2.1.10. [RWW17, Definition 1.5] Let $(\mathfrak{C},\natural,0)$ be a strict monoidal category such that the unit 0 is initial. We say that the monoidal category $(\mathfrak{C},\natural,0)$ is pre-braided if:

- The maximal subgroupoid $\mathscr{G}\mathfrak{r}(\mathfrak{C})$ (see Notation 2.0.1) is a braided monoidal category, where the monoidal structure is induced by that of $(\mathfrak{C},\natural,0)$.
- For all objects A and B of $\mathfrak{C}$, the braiding associated with the maximal subgroupoid $b^{\mathfrak{C}}_{A,B} : A\natural B \longrightarrow B\natural A$ satisfies:
$$b^{\mathfrak{C}}_{A,B} \circ (id_A \natural \iota_B) = \iota_B \natural id_A : A \longrightarrow B\natural A.$$
(The notation $\iota_B : 0 \to B$ was introduced in Notation 2.0.1).

Remark 2.1.11. A braided monoidal category is automatically pre-braided. However, a pre-braided monoidal category is not necessarily braided (see for example [Sou17b, Remark 1.15]).

Finally, let us give the following key property when Quillen's bracket construction is applied on a strict monoidal groupoid $(\mathfrak{G},\natural,0)$.

Proposition 2.1.12. *[RWW17, Proposition 1.8] Suppose that the strict monoidal groupoid $(\mathfrak{G},\natural,0)$ has no zero divisors and that $Aut_{\mathfrak{G}}(0) = \{id_0\}$. If the groupoid $(\mathfrak{G},\natural,0)$ is braided, then the category $(\mathfrak{UG},\natural,0)$ is pre-braided monoidal. If moreover $(\mathfrak{G},\natural,0)$ is symmetric monoidal, then the category $(\mathfrak{UG},\natural,0)$ is symmetric monoidal.*

Remark 2.1.13. The monoidal structure on the category $(\mathfrak{UG},\natural,0)$ is defined on objects by that of $(\mathfrak{G},\natural,0)$ and defined on morphisms by letting for $[X,f] \in Hom_{\mathfrak{UG}}(A,B)$ and $[Y,g] \in Hom_{\mathfrak{UG}}(C,D)$:

$$[X,f]\natural[Y,g] = \left[X\natural Y, (f\natural g) \circ \left(id_X \natural \left(b^{\mathfrak{G}}_{A,Y}\right)^{-1} \natural id_C\right)\right].$$

In particular, the canonical functor $\mathfrak{G} \to \mathfrak{UG}$ (see Remark 2.1.5) is monoidal.

2.2 The generalised Long-Moody functors

In this section, we develop a generalisation of the Long-Moody functors (see [Sou17b, Section 2]) inspired by the Long-Moody construction (see [Lon94, Theorem 2.1]). First, we introduce the general framework of our study (see Section 2.2.1). Then, we define the generalised Long-Moody functors and establish their first properties in Section 2.2.2.

The two first subsections are generalisations of [Sou17b, Section 2.1] and [Sou17b, Section 2.2]. We give a new approach to some tools and conditions previously considered in [Sou17b, Section 2], allowing a wider application of our constructions. We will emphasise the new aspects of this work, giving details only when necessary for the convenience of the reader or for the sake of completeness.

2.2.1 Framework of the construction

Throughout Section 2.2, we consider a groupoid $\mathcal{G}$ such that $Obj(\mathcal{G}) \cong \mathbb{N}$.

2.2.1.1 Monoidal properties of $\mathcal{G}$

First, we assume that a monoidal structure is induced on the groupoid $\mathcal{G}$. Namely:

Assumption 2.2.1. *There exists $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$ a braided monoidal groupoid with no zero divisors such that $Aut_{\mathcal{G}'}(0_{\mathcal{G}'}) = \left\{id_{0_{\mathcal{G}'}}\right\}$ and:*

- *$\mathcal{G}$ is a full subgroupoid of $\mathcal{G}'$;*
- *there exist two objects 0 and 1 of $\mathcal{G}'$ such that for all objects x of $\mathcal{G}$, there exists a unique $n \in \mathbb{N}$ such that $x = 1^{\natural n}\natural 0$.*

By hypothesis $\mathcal{G}$ is finitely generated by the monoidal structure of $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$ with $\{0, 1\}$ as generating set. The object 0 should not be confused with the unit $0_{\mathcal{G}'}$ of the monoidal structure $\natural$. For n a natural number, the objects $1^{\natural n}$ and $1^{\natural n}\natural 0$ of $\mathcal{G}'$are different. In particular, $1^{\natural n}\natural 0$ is an object of $\mathcal{G}$ whereas $1^{\natural n}$ is not. However, one could be tempted to denote both of them by "n". To avoid this confusion, we introduce the following notation:

Notation 2.2.2. For all natural numbers n, we denote the object $1^{\natural n}\natural 0$ of $\mathcal{G}$ by $\underline{n}$ and the object $1^{\natural n}$ of $\mathcal{G}'$ by n. Note that if $\mathcal{G}' = \mathcal{G}$, then $\underline{n} = n$.

Remark 2.2.3. Under the properties of Assumption 2.2.1, Quillen's bracket construction $(\mathfrak{U}\mathcal{G}', \natural, 0_{\mathcal{G}'})$ is a pre-braided monoidal category such that the unit $0_{\mathcal{G}'}$ is an initial object (see Proposition 2.1.12).

Definition 2.2.4. Let $\mathfrak{U}\mathcal{G}$ be the full subcategory of $\mathfrak{U}\mathcal{G}'$ on the objects of $\mathcal{G}$.

Remark 2.2.5. Let m, n and n' be natural numbers such that $n' \geq n$. Then:

- $m\natural\underline{n} = \underline{m+n}$;
- considering the morphism $\left[n'-n, id_{\underline{n'}}\right]$, then the "$n'-n$" in the notation is not an object of $\mathcal{G}$: it is the unique object of $\mathcal{G}'$ such that $(n'-n)\natural\underline{n} = \underline{n'}$ as objects of $\mathcal{G}$.

Warning: the category $\mathfrak{U}\mathcal{G}$ is not in general Quillen's bracket construction of Definition 2.1.3. However, assuming that $\mathcal{G}' = \mathcal{G}$, then $\mathfrak{U}\mathcal{G}$ is indeed Quillen's bracket construction bracket construction, $0 = 0_{\mathcal{G}'}$ and we have a is pre-braided monoidal structure $(\mathfrak{U}\mathcal{G}, \natural, 0)$. This is for instance the case in the previous work [Sou17b], where $\mathcal{G}' = \mathcal{G} = \boldsymbol{\beta}$ is the braid groupoid (see Notation 2.0.1).

The present framework allows us to work with more examples, such as mapping class groups of surfaces with non-zero (orientable or non-orientable) genus (see Section 2.3.3). For instance, in the various situations of Section 2.3, the groupoids $\mathfrak{M}_2^{+,s}$, $\mathfrak{M}_2^{-,s}$ and $\mathfrak{M}_2^{g,c}$ (see Sections 2.3.3 and 2.3.4) are full subgroupoids of the braided monoidal groupoid $\left(\mathfrak{M}_2, \natural, \Sigma_{0,0,1}^{0}\right)$ (see Proposition 2.3.4).

Finally, as the objects of $\mathfrak{U}\mathcal{G}$ are natural numbers, we consider:

Definition 2.2.6. Let $(\mathbb{N}, \leq)$ be the category of natural numbers (natural means non-negative) considered as a directed set.

Notation 2.2.7. For all natural numbers n, we denote by γ_n the unique element of $Hom_{(\mathbb{N},\leq)}(n, n+1)$. For all natural numbers n and n' such that $n' \geq n$, we denote by $\gamma_{n,n'} : n \to n'$ the unique element of $Hom_{(\mathbb{N},\leq)}(n, n')$, composition of the morphisms $\gamma_{n'-1} \circ \gamma_{n'-2} \circ \cdots \circ \gamma_{n+1} \circ \gamma_n$. The addition defines a strict monoidal structure on $(\mathbb{N}, \leq)$, denoted by $((\mathbb{N}, \leq), +, 0)$.

Definition 2.2.8. Let $\mathscr{O} : (\mathbb{N}, \leq) \to \mathfrak{U}\mathcal{G}$ be the faithful functor defined by $\mathscr{O}(n) = \underline{n}$ and $\mathscr{O}(\gamma_n) = \left[1, id_{\underline{1+n}}\right]$ for all natural numbers n.

2.2.1.2 Long-Moody triple

Let us fix H_0 and H two groups, with H non-trivial.

Notation 2.2.9. For all natural numbers m, we denote the free product $H^{*m} * H_0$ by H_m. We denote by e_H (resp. e_{H_0}) the identity element of the group H (resp. H_0).

Example 2.2.10. The classical example is the free group on m generators denoted by $\mathbf{F}_m = \langle f_1, \ldots, f_m \rangle$. Indeed, taking H to be $\mathbb{Z}$ and H_0 to be the trivial group $0_{\mathfrak{Gr}}$, one identifies $\mathbf{F}_m \cong \mathbb{Z}^{*m}$. The framework of [Sou17b] uses this example $H_m = \mathbf{F}_m$.

Remark 2.2.11. In many examples considered here, such as mapping class groups of surfaces with non-zero genus (see Section 2.3), H_0 is non-trivial, contrary to [Sou17b].

The object $0_{\mathfrak{Gr}}$ being null in the category of groups $\mathfrak{Gr}$, recall that $\iota_G : 0_{\mathfrak{Gr}} \to G$ introduced in Notation 2.0.1 denotes the unique morphism from $0_{\mathfrak{Gr}}$ to the group G. We consider $\iota_H * id_{H_m} : H_m \hookrightarrow H_{m+1}$ which corresponds to the identification of H_m as the subgroup of H_{m+1} generated by the m last copies of H in H_{m+1}. Iterating this morphism, we obtain for all natural numbers $m' \geq m$ the morphism $\iota_{H^{*(m'-m)}} * id_{H_m} : H_m \hookrightarrow H_{m'}$.

Notation 2.2.12. For all natural numbers n, we denote by G_n the automorphism group $Aut_{\mathcal{G}}(\underline{n})$.

We require the groups G_n to have an action on the groups H_n for all natural numbers n. More precisely:

Assumption 2.2.13. *There exists a functor $\mathcal{H} : \mathfrak{U}\mathcal{G} \to \mathfrak{Gr}$ such that:*

- *for all objects $\underline{n}$ of $\mathcal{G}$, $\mathcal{H}(\underline{n}) = H_n$. In other words, $\mathcal{H}\left(1^{\natural n} \natural 0\right) = H_n$ for all natural numbers n.*
- *$\mathcal{H}\left(\left[1, id_{\underline{n+1}}\right]\right) = \iota_H * id_{H_n}$ for all natural numbers n.*

Consequences of Assumption 2.2.13 will be heavily used in our study, for instance in the key results Theorem 2.2.30 and Proposition 2.5.12. The following lemma clarifies some subtleties of Assumption 2.2.13. It also implies that Condition 2.11 of [Sou17b] is equivalent to this previous assumption.

Lemma 2.2.14. *The functor $\mathcal{H}$ of Assumption 2.2.13 restricts to a functor $\mathscr{A} : \mathcal{G} \overset{\mathfrak{c}_{\mathfrak{U}\mathfrak{G}}}{\hookrightarrow} \mathfrak{U}\mathcal{G} \overset{\mathcal{H}}{\to} \mathfrak{Gr}$ such that for natural numbers n and n', for all elements g of G_n and g' of $G_{n'-n}$:*

$$\mathcal{H}\left(\gamma_{n,n'}\right) \circ \mathscr{A}(g) = \mathscr{A}\left(g' \natural g\right) \circ \mathcal{H}\left(\gamma_{n,n'}\right) \tag{2.2.1}$$

as morphisms $H_n \to H_{n'}$.

Proof. This is a straightforward consequence of Lemma 2.1.8. □

Remark 2.2.15. The functor $\mathcal{H}$ of Assumption 2.2.13 provides group morphisms:

$$G_n \to Aut_{\mathfrak{Gr}}(H_n)$$

for all natural numbers n, satisfying compatibility relations in $\mathfrak{U}\mathcal{G}$ given by relation (2.2.1) of Lemma 2.2.14.

Definition 2.2.16. The groupoid $\mathcal{G}$ and a fixed functor $\mathcal{H}$ form a Long-Moody triple if Assumptions 2.2.1 and 2.2.13 are satisfied. It is denoted by $(\mathcal{H}, \mathcal{G}, \mathcal{G}')$.

2.2.1.3 Coherence conditions

For our framework, we require two additional general conditions (see Conditions 2.2.17 and 2.2.24).

We fix a Long-Moody triple $(\mathcal{H}, \mathcal{G}, \mathcal{G}')$ throughout the remainder of this section.

Recall that we assume $Obj(\mathcal{G}) \cong \mathbb{N}$ so we denote objects of $\mathcal{G}$ by $\underline{n}$ with n a natural number and that the braiding associated with $(\mathcal{G}', \natural, 0)$ is denoted by $b^{\mathcal{G}'}_{-,-}$ (see Notation 2.1.1).

First, we need to consider particular group morphisms from the group H_n to $G_{n+1} = Aut_{\mathcal{G}}(\underline{n+1})$ for all natural numbers n. The condition that they have to satisfy will be used to prove that the generalised Long-Moody functor is well defined on morphisms with respect to the tensor product structure in Theorem 2.2.30. Moreover, it will also be used in the proof of Propositions 2.5.23 and 2.5.12.

Condition 2.2.17. There exist group morphisms $\{\varsigma_n : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$ such that for all elements $h \in H_n$, for all natural numbers n and n' such that $n' \geq n$, the following diagram is commutative in the category $\mathfrak{U}\mathcal{G}$:

$$\begin{array}{ccc}
1\natural\underline{n} & \xrightarrow{\varsigma_n(h)} & 1\natural\underline{n} \\
\downarrow{\scriptstyle id_1\natural[n'-n,id_{\underline{n'}}]} & & \downarrow{\scriptstyle id_1\natural[n'-n,id_{\underline{n'}}]} \\
1\natural\underline{n'} & \xrightarrow[\varsigma_{n'}\left(\left(\iota_{H^{*(n'-n)}}*id_{H_n}\right)(h)\right)]{} & 1\natural\underline{n'}.
\end{array}$$

Remark 2.2.18. By definition of the braiding $b^{\mathcal{G}'}_{-,-}$, we have:

$$\left(b^{\mathcal{G}'}_{1,n'-n}\right)^{-1}\natural id_n = \left(\left(b^{\mathcal{G}'}_{1,1}\right)^{-1}\natural id_n\right)\circ\left(id_1\natural\left(b^{\mathcal{G}'}_{1,(n'-n)-1}\right)^{-1}\natural id_{n-1}\right).$$

Hence, a straightforward recursion (see for example the proof of [Sou17b, Lemma 2.5]) proves that Condition 2.2.17 is equivalent to assuming that for all elements $h \in H_n$, for all natural numbers n, the morphisms $\{\varsigma_n\}_{Obj(\mathcal{G})}$ satisfy the following equality, as morphisms in the category $\mathfrak{U}\mathcal{G}$:

$$\left[1,\left(\left(b^{\mathcal{G}'}_{1,1}\right)^{-1}\natural id_{\underline{n}}\right)\circ\left(id_1\natural\varsigma_n(h)\right)\right] \quad = \quad \left[1,\varsigma_{n+1}\left(\left(\iota_H * id_{H_n}\right)(h)\right)\circ\left(\left(b^{\mathcal{G}'}_{1,1}\right)^{-1}\natural id_{\underline{n}}\right)\right].$$

It follows from Remark 2.2.18 that:

Proposition 2.2.19. *Assume that for all elements $h \in H_n$, for all natural numbers n, the following equality holds in G_{n+2}:*

$$\left(\left(b^{\mathcal{G}'}_{1,1}\right)^{-1}\natural id_{\underline{n}}\right)\circ\left(id_1\natural\varsigma_n(h)\right) \quad = \quad \varsigma_{n+1}\left(\mathcal{H}\left(\left[1,id_{\underline{n+1}}\right]\right)(h)\right)\circ\left(\left(b^{\mathcal{G}'}_{1,1}\right)^{-1}\natural id_{\underline{n}}\right). \tag{2.2.2}$$

Then, Condition 2.2.17 is satisfied.

Remark 2.2.20. If $Aut_{\mathfrak{U}\mathcal{G}}(1) = \{id_1\}$, Condition 2.2.17 is equivalent to the equality (2.2.2) for all elements $h \in H_n$, for all natural numbers n.

When $(\mathfrak{U}\mathcal{G}',\natural,0_{\mathcal{G}'}) = (\mathfrak{U}\mathcal{G},\natural,0_{\mathcal{G}}) = (\mathfrak{U}\boldsymbol{\beta},\natural,0)$, Condition 2.2.17 recovers [Sou17b, Conditions 2.3]. For this particular, the assumption $Aut_{\mathfrak{U}\boldsymbol{\beta}}(1) = \{id_1\}$ is satisfied and a fortiori equality (2.2.2). However, this is not necessarily the case for all the examples which fit into the present larger framework, such as mapping class groups of surfaces with non-zero genus (see Section 2.3.3). Nevertheless, in Section 2.5, we will have to assume that the stronger equality (2.2.2) holds (see Condition 2.5.9).

Notation 2.2.21. For all natural numbers n, we denote by $\varsigma_{n,t}$ the trivial morphism $H_n \to 0_{\mathfrak{Gr}} \to G_{n+1}$.

Example 2.2.22. The family of morphisms $\{\varsigma_{n,t} : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$ satisfies Condition 2.2.17.

We fix a family of morphisms $\{\varsigma_n : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$ satisfying the Condition 2.2.17. We require a compatibility relation between the morphisms $\{\varsigma_n : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$ and the functor $\mathcal{H}$. This is essential in the definition of the Long-Moody functor on objects in Theorem 2.2.30 (see Condition 2.2.24).

Notation 2.2.23. For all natural numbers n, we denote by $\mathscr{A}_n : G_n \to Aut_{\mathfrak{Gr}}(H_n)$ the group morphisms induced by the functor $\mathcal{H}$.

Condition 2.2.24. Let n be a natural number. We assume that the morphism given by the coproduct $\varsigma_n * id_1\natural- : H_n * G_n \to G_{n+1}$ factors across the canonical surjection to $H_n \underset{\mathscr{A}_n}{\rtimes} G_n$. In other words, the following diagram is commutative:

$$\begin{array}{ccccc}
H_n & \hookrightarrow & H_n \underset{\mathscr{A}_n}{\rtimes} G_n & \hookleftarrow & G_n \\
& {\scriptstyle\varsigma_n}\searrow & \downarrow & \swarrow{\scriptstyle id_1\natural-} & \\
& & G_{n+1}, & &
\end{array}$$

where the morphism $H_n \underset{\mathscr{A}_n}{\rtimes} G_n \to G_{n+1}$ is induced by the morphism $H_n * G_n \to G_{n+1}$ and the group morphism $id_1\natural- : G_n \to G_{n+1}$ is induced by the monoidal structure of Assumption 2.2.1.

Remark 2.2.25. When $(\mathfrak{U}\mathcal{G}', \natural, 0_{\mathcal{G}'}) = (\mathfrak{U}\mathcal{G}, \natural, 0_{\mathcal{G}}) = (\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$, Condition 2.2.24 recovers [Sou17b, Conditions 2.12].

Definition 2.2.26. With the previous notation, a coherent Long-Moody system, denoted by $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$, is a Long-Moody triple $(\mathcal{H}, \mathcal{G}, \mathcal{G}')$ equipped with group morphisms $\{\varsigma_n : H_n \to G_{n+1}\}_{n \in Obj(\mathcal{G})}$ satisfying Conditions 2.2.17 and 2.2.24.

Remark 2.2.27. Condition 2.2.24 is satisfied for the family of morphisms $\{\varsigma_{n,t} : H_n \to G_{n+1}\}_{n \in \mathbb{N}}$ of Example 2.2.22 and any functor $\mathcal{H} : \mathfrak{U}\mathcal{G}' \to \mathfrak{Gr}$. A fortiori, we define a coherent Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma_{-,t}\}$.

2.2.2 Definition of the generalised Long-Moody functors

This section deals with introducing generalised Long-Moody functors, inspired from the Long-Moody construction [Lon94]. It generalises and adapts [Sou17b, Section 2.2] to a larger setting. A large variety of groups falls within this framework (see Sections 2.3 and 2.6.3). Moreover, the new point of view on some tools detailed in Section 2.2.2.1 allows a clearer understanding of this construction.

2.2.2.1 Group ring and augmentation ideal functors

For all objects G of $\mathfrak{Gr}$, the group rings $R[G]$ (resp. augmentation ideals $\mathcal{I}_{R[G]}$) assemble to define the group ring functor $R[-] : \mathfrak{Gr} \to R\text{-}\mathfrak{Mod}$ (resp. the augmentation ideal functor $\mathcal{I}_{R[-]} : \mathfrak{Gr} \to R\text{-}\mathfrak{Mod}$). Let $(\mathcal{H}, \mathcal{G}, \mathcal{G}')$ a Long-Moody triple. Thanks to Assumption 2.2.13, we introduce the following two functors:

Definition 2.2.28. Let $R[\mathcal{H}] : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$ and $\mathcal{I} : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$ be the functors defined by the composites

$$\mathfrak{U}\mathcal{G} \xrightarrow{\mathcal{H}} \mathfrak{Gr} \xrightarrow{R[-]} R\text{-}\mathfrak{Mod}$$

and

$$\mathfrak{U}\mathcal{G} \xrightarrow{\mathcal{H}} \mathfrak{Gr} \xrightarrow{\mathcal{I}_{R[-]}} R\text{-}\mathfrak{Mod}\ .$$

We call $R[\mathcal{H}]$ (resp. $\mathcal{I}$) the group ring (resp. the augmentation ideal) functor induced by the Long-Moody triple $(\mathcal{H}, \mathcal{G}, \mathcal{G}')$.

Compared to the particular case of braid groups in [Sou17b], to introduce these functors for the generalisation of Long-Moody functors gives a more conceptual point of view on underlying structure of this construction. For the augmentation ideals, the functor $\mathcal{I}$ encodes the consequences of Condition 2.11 of [Sou17b] in the previous framework.

2.2.2.2 The Long-Moody functors

We fix a coherent Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ throughout this section.

Notation 2.2.29. When there is no ambiguity, once the Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ is fixed, we omit it from the notation.

Let F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ and n be a natural number. The R-module $F(\underline{n+1})$ is simultaneously endowed with a $R[G_{n+1}]$-module structure and a (left) $R[H_n]$-module structure via the morphism $\varsigma_n : H_n \to G_{n+1}$. As the augmentation ideal $\mathcal{I}_{R[H_n]}$ is a right $R[H_n]$-module, we can consider the tensor product $\mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} F(\underline{n+1})$. In the following theorem, using this tensor product, we define an endofunctor of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$.

Theorem 2.2.30. *The following assignment defines a functor $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_n\}} : \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. It is called the (generalised) Long-Moody functor associated with the coherent Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$.*

- *Objects: for $F \in Obj(\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}))$, $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(F) : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$ is defined by:*
  - *Objects: $\forall \underline{n} \in Obj(\mathcal{G})$, $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(F)(\underline{n}) = \mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} F(\underline{n+1})$.*

– *Morphisms: let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n'-n, g] \in Hom_{\mathfrak{U}\mathcal{G}}(\underline{n}, \underline{n'})$. We define*

$$\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(F)\left(\left[n'-n,g\right]\right) : \mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} F\left(\underline{n+1}\right) \to \mathcal{I}_{R\left[H_{n'}\right]} \underset{R\left[H_{n'}\right]}{\otimes} F\left(\underline{n'+1}\right)$$

to be the unique morphism, denoted by $\mathcal{I}([n'-n,g]) \underset{R\left[H_{n'}\right]}{\otimes} F(id_1 \natural [n'-n,g])$, induced by the universal property of the tensor product $\underset{R[H_n]}{\otimes}$ with respect to the $R[H_n]$-balanced map

$$\mathcal{I}_{R[H_n]} \times F\left(\underline{n+1}\right) \xrightarrow{\mathcal{I}([n'-n,g]) \times F(id_1 \natural [n'-n,g])} \mathcal{I}_{R\left[H_{n'}\right]} \times F\left(\underline{n'+1}\right) \xrightarrow{\underset{R\left[H_{n'}\right]}{\otimes}} \mathcal{I}_{R\left[H_{n'}\right]} \underset{R\left[H_{n'}\right]}{\otimes} F\left(\underline{n'+1}\right).$$

- *Morphisms: let F and G be two objects of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, and $\eta : F \to G$ be a natural transformation. We define $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(\eta) : \mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(F) \to \mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(G)$ for all objects $\underline{n}$ of $\mathcal{G}$ to be the unique morphism, denoted by $id_{\mathcal{I}_{R[H_n]}} \underset{R[H_n]}{\otimes} \eta_{\underline{n+1}}$, induced by the universal property of the tensor product $\underset{R[H_n]}{\otimes}$ with respect to the $R[H_n]$-balanced map*

$$\mathcal{I}_{R[H_n]} \times F\left(\underline{n+1}\right) \xrightarrow{id_{\mathcal{I}_{R[H_n]}} \times \eta_{\underline{n+1}}} \mathcal{I}_{R\left[H_{n'}\right]} \times G\left(\underline{n+1}\right) \xrightarrow{\underset{R[H_n]}{\otimes}} \mathcal{I}_{R\left[H_{n'}\right]} \underset{R[H_n]}{\otimes} G\left(\underline{n+1}\right).$$

Proof. For this proof, F, G and H are objects of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, n, n' and n'' are natural numbers such that $n'' \geq n' \geq n$. We have three points to prove.

1. First, let us show that the assignment of $\mathbf{LM}(F)$ on morphisms is well-defined. Consider $[n'-n, g]$ and $[n'-n, g']$ such that $[n'-n, g] = [n'-n, g']$. In other words, we assume that there exists $\psi \in G_{n'-n}$ so that $g' \circ (\psi \natural id_n) = g$. Since the monoidal product $\natural$ is well-defined on $\mathfrak{U}\mathcal{G}'$ by Proposition 2.1.12 and as $\mathfrak{U}\mathcal{G}$ is a full subcategory of $\mathfrak{U}\mathcal{G}'$, we deduce that $id_1 \natural [n'-n, g] = id_1 \natural [n'-n, g']$. So it follows from Definition 2.2.28 and Assumption 2.2.13 that:

$$\mathcal{I}\left(\left[n'-n,g\right]\right) \underset{R}{\otimes} F\left(id_1 \natural \left[n'-n,g\right]\right) \quad = \quad \mathcal{I}\left(\left[n'-n,g'\right]\right) \underset{R}{\otimes} F\left(id_1 \natural \left[n'-n,g'\right]\right).$$

After checking that $\mathcal{I}([n'-n,g]) \underset{R\left[H_{n'}\right]}{\otimes} F(id_1 \natural [n'-n,g])$ is a $R[H_n]$-balanced map, it will follow from this relation that $\mathbf{LM}(F)([n'-n,g]) = \mathbf{LM}(F)([n'-n,g'])$. Therefore, we will have proved that the assignment of $\mathbf{LM}(F)$ on morphisms respects the defining equivalence relation of $\mathfrak{U}\mathcal{G}$.
Proving that $\mathcal{I}([n'-n,g]) \underset{R\left[H_{n'}\right]}{\otimes} F(id_1 \natural [n'-n,g])$ is a $R[H_n]$-balanced map amounts to show that for all $h \in H_n$ and $i \in \mathcal{I}_{R[H_n]}$:

$$\mathcal{I}\left(\left[n'-n,g\right]\right)(i \cdot h) \underset{R\left[H_{n'}\right]}{\otimes} F\left(id_1 \natural \left[n'-n,g\right]\right) \quad = \quad \mathcal{I}\left(\left[n'-n,g\right]\right) \underset{R\left[H_{n'}\right]}{\otimes} F\left(id_1 \natural \left[n'-n,g\right]\right) \circ F\left(\varsigma_n(h)\right)$$

Recall from Definition 2.2.28 and Assumption 2.2.13 that:

$$\mathcal{I}\left(\left[n'-n, id_{\underline{n'}}\right]\right)(i \cdot h) = \mathcal{I}\left(\left[n'-n, id_{\underline{n'}}\right]\right)(i) \cdot \mathcal{H}\left(\left[n'-n, id_{\underline{n'}}\right]\right)(h)$$

and that the group morphism $\mathcal{I}_n : G_n \to Aut_{R\text{-}\mathfrak{Mod}}\left(\mathcal{I}_{R[H_n]}\right)$ defined by the functor $\mathcal{I}$ is canonically induced by $\mathscr{A}_n : G_n \to Aut_{\mathfrak{Gr}}(H_n)$ (see Notation 2.2.23). Therefore, for all $h \in H_n$ and $i \in \mathcal{I}_{R[H_n]}$:

$$\mathcal{I}\left(\left[n'-n,g\right]\right)(i \cdot h) = \mathcal{I}\left(\left[n'-n,g\right]\right)(i) \cdot \mathcal{H}\left(\left[n'-n,g\right]\right)(h).$$

Hence, proving the compatibility with respect to the tensor product amounts to proving that:

$$F\left(id_1 \natural \left[n'-n,g\right]\right) \circ F\left(\varsigma_n(h)\right) \quad = \quad F\left(\varsigma_{n'}\left(\mathcal{H}\left(\left[n'-n,g\right]\right)(h)\right)\right) \circ F\left(id_1 \natural \left[n'-n,g\right]\right). \tag{2.2.3}$$

Using Condition 2.2.17, we have:

$$\left[n'-n,\left(\left(b^{\mathcal{G}'}_{1,n'-n}\right)^{-1}\natural id_{\underline{n}}\right)\circ\left(id_{n'-n}\natural\varsigma_{n}\left(h\right)\right)\right]$$
$$=\left[n'-n,\varsigma_{n'}\left(\mathcal{H}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\left(h\right)\right)\circ\left(\left(b^{\mathcal{G}'}_{1,n'-n}\right)^{-1}\natural id_{\underline{n}}\right)\right].$$

Since $(id_1\natural g)\circ\varsigma_{n'}\left(\mathcal{H}\left(\left[n'-n,id_{\underline{n'}}\right]\right)(h)\right)=\mathcal{H}\left(\left[n'-n,g\right]\right)(h)\circ(id_1\natural g)$ by Condition 2.2.24, we have:

$$\left[n'-n,(id_1\natural g)\circ\left(\left(b^{\mathcal{G}'}_{1,n'-n}\right)^{-1}\natural id_{\underline{n}}\right)\circ\left(id_{n'-n}\natural\varsigma_{n}\left(h\right)\right)\right]$$
$$=\left[n'-n,\varsigma_{n'}\left(\mathcal{H}\left(\left[n'-n,g\right]\right)\left(h\right)\right)\circ(id_1\natural g)\circ\left(\left(b^{\mathcal{G}'}_{1,n'-n}\right)^{-1}\natural id_{\underline{n}}\right)\right].$$

The desired equality (2.2.3) follows from the functoriality of F.

2. Let us prove that the assignment $\mathbf{LM}(F)$ is a functor. Since $\mathcal{I}$ and F are functors and $id_1\natural-$ is a group morphism, it follows from the definition that $\mathbf{LM}(F)(id_{\underline{n}})=id_{\mathbf{LM}(F)(\underline{n})}$. The composition axiom follows from the functorialities of F and $\mathcal{I}$ over $\mathfrak{U}\mathcal{G}$ (see Proposition 2.2.28) and from the compatibility of the monoidal structure $\natural$ with composition.

3. The remaining point to check for $\mathbf{LM}$ to be a functor is the consistency of our definition on morphisms. For $\eta:F\to G$ a natural transformation, we have:

$$G\left(id_1\natural\left[n'-n,g\right]\right)\circ\eta_{\underline{n+1}}=\eta_{\underline{n'+1}}\circ F\left(id_1\natural\left[n'-n,g\right]\right).$$

Hence, it follows that:

$$\mathbf{LM}(G)\left(\left[n'-n,g\right]\right)\circ\mathbf{LM}(\eta)_{\underline{n}}=\mathbf{LM}(\eta)_{\underline{n'}}\circ\mathbf{LM}(F)\left(\left[n'-n,g\right]\right).$$

Therefore $\mathbf{LM}(\eta)$ is a morphism in the category $\mathbf{Fct}(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod})$. Denoting by $id_F:F\to F$ the identity natural transformation, it is clear that $\mathbf{LM}(id_F)=id_{\mathbf{LM}(F)}$. Finally, let us check the composition axiom. Let $\eta:F\to G$ and $\mu:G\to H$ be natural transformations. Let n be a natural number. Now, because μ and η are morphisms in the category $\mathbf{Fct}(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod})$:

$$\mathbf{LM}(\mu\circ\eta)_{\underline{n}}=id_{\mathcal{I}_{R[H_n]}}\underset{R[H_n]}{\otimes}\left(\mu_{\underline{n+1}}\circ\eta_{\underline{n+1}}\right)(v)=\mathbf{LM}(\mu)_{\underline{n}}\circ\mathbf{LM}(\eta)_{\underline{n}}.$$

□

Remark 2.2.31. When $(\mathfrak{U}\mathcal{G}',\natural,0_{\mathcal{G}'})=(\mathfrak{U}\mathcal{G},\natural,0_{\mathcal{G}})=(\mathfrak{U}\boldsymbol{\beta},\natural,0)$, Theorem 2.2.30 recovers [Sou17b, Theorem 2.19].

Remark 2.2.32. If we had considered a Long-Moody triple $(\mathcal{H},\mathcal{G},\mathcal{G}')$ together with group morphisms $\varsigma_n:H_n\to G_{n+1}$ for all natural numbers n such that only Condition 2.2.24 is satisfied (but not necessarily satisfying Condition 2.2.17), then the assignments of Theorem 2.2.30 defines a functor with $\mathcal{G}$ as source category $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_n\}}:\mathbf{Fct}(\mathcal{G},R\text{-}\mathfrak{Mod})\to\mathbf{Fct}(\mathcal{G},R\text{-}\mathfrak{Mod})$.

Let us give some immediate properties of a Long-Moody functor.

Proposition 2.2.33. *The functor* $\mathbf{LM}$ *associated with the coherent Long-Moody system* $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$ *is reduced, right exact and commutes with all colimits.*

Proof. Let $0_{\mathbf{Fct}(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod})}:\mathfrak{U}\mathcal{G}\to R\text{-}\mathfrak{Mod}$ denote the null functor. It follows from the definition of the Long-Moody functor that $\mathbf{LM}\left(0_{\mathbf{Fct}(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod})}\right)=0_{\mathbf{Fct}(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod})}$, ie $\mathbf{LM}$ is reduced.
The right-exactness of the Long-Moody functor is a consequence of the well-known fact that the functor $\mathcal{I}_{R[H_n]}\underset{R[H_n]}{\otimes}-:R\text{-}\mathfrak{Mod}\longrightarrow R\text{-}\mathfrak{Mod}$ is right exact for all natural numbers n (see for example [Wei94, Application 2.6.2]), the naturality for morphisms following from the definition of the Long-Moody functor. Similarly, the commutation property with all colimits is a formal consequence of the commutation with all colimits of the tensor products $\mathcal{I}_{R[H_n]}\underset{R[H_n]}{\otimes}-$ for all natural numbers n. □

2.2.2.3 Case of free groups

Recall the following result.

Lemma 2.2.34. *Let G be a group. The augmentation ideal $\mathcal{I}_{R[G]}$ is a projective $R[G]$-module if and only if G is a free group.*

Proof. Let us assume that $\mathcal{I}_{R[G]}$ is a projective $R[G]$-module. The following short exact sequence is a projective resolution of R as a $R[G]$-module.

$$0 \longrightarrow \mathcal{I}_{R[G]} \longrightarrow R[G] \longrightarrow R \longrightarrow 0$$

Hence the homological dimension of G is one. Thus, according to a theorem due to Swan [Swa69, Theorem A], G is a free group. The converse is a classical result of homological algebra (see [Wei94, Corollary 6.2.7]). □

Corollary 2.2.35. *If H_0 and H are free groups, then the Long-Moody functor associated with the coherent Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ is exact and commutes with all finite limits.*

Proof. Let n be a natural number. Since the augmentation ideal $\mathcal{I}_{R[H_n]}$ is a projective $R[H_n]$-module (by Lemma 2.2.34), it is a flat $R[H_n]$-module. Then, the result follows from the fact that the functor $\mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} - : R\text{-}\mathfrak{Mod} \to R\text{-}\mathfrak{Mod}$ is an exact functor, the naturality for morphisms following from the definition of the Long-Moody functor (see Theorem 2.2.30). The commutation result for finite limits is a general property of exact functors (see for example [ML13, Chapter 8, section 3]). □

Remark 2.2.36. Assume that H is a free group. Let M be a $R[H]$-module. Since H is free, $\mathcal{I}_{R[H]}$ is a free $R[H]$-module of rank $rank(H)$, hence there are isomorphisms of R-modules:

$$\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} M \cong \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} R\right) \underset{R}{\otimes} M \cong M^{\oplus rank(H)}.$$

We denote by $\Lambda_{rank(H),M}$ the composition of these isomorphisms.

First homology of $\mathcal{H}$ functor: We denote by $R : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$ the constant functor at R. Assuming that H_0 and H are free groups, applying classical homological algebra (see [Wei94, Corollary 6.2.7]), we deduce that for all natural numbers n:

$$\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(R)(n) \cong H_1(H_n, R). \tag{2.2.4}$$

This isomorphism is functorial. Indeed, since the homology group $H_1(-, R)$ defines a functor from the category $\mathfrak{Gr}$ to the category $R\text{-}\mathfrak{Mod}$ (see for example [Bro12, Section 8]), we can introduce the following functor:

Definition 2.2.37. Let $(\mathcal{H}, \mathcal{G}, \mathcal{G}')$ be a Long-Moody triple. The homology groups $\{H_1(H_n, R)\}_{n\in\mathbb{N}}$ assemble to define a functor $H_1(H_-, R) : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$ by the composition:

$$\mathfrak{U}\mathcal{G} \xrightarrow{\mathcal{H}} \mathfrak{Gr} \xrightarrow{H_1(-,R)} R\text{-}\mathfrak{Mod}.$$

Then, the isomorphism (2.2.4) extends to define a natural equivalence:

Lemma 2.2.38. *If H_0 and H are free groups, then as functor $\mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$:*

$$\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(R) \cong H_1(H_-, R).$$

Proof. The naturality follows from the fact that the assignments of the functor $H_1(H_-, R)$ (see Definition 2.2.37) and $\mathcal{I}$ (see Definition 2.2.28) on morphisms of $\mathfrak{U}\mathcal{G}$ are both induced by the functor $\mathcal{H}$. □

2.2.2.4 Case of trivial ς

As stated in Remark 2.2.27, any functor $\mathcal{H} : \mathfrak{U}\mathcal{G} \to \mathfrak{Gr}$ gives a coherent Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma_{-,t}\}$. We have the following property:

Proposition 2.2.39. *Let F be an object of* $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. *Then, as objects of* $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$*:*

$$\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_t\}}(F) \cong \mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_t\}}(R) \underset{R}{\otimes} F(1\natural -).$$

Proof. Let n be a natural number. The action induced by $\varsigma_{n,t} : H_n \to G_{n+1}$ of Example 2.2.22 makes $F(\underline{1+n}) = F(1\natural\underline{n})$ a trivial $R[H_n]$-module. A fortiori, there is an R-module isomorphism:

$$\mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} F(\underline{n+1}) \cong \left(\mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} R\right) \underset{R}{\otimes} F(1\natural\underline{n}).$$

It is straightforward to check that this isomorphism is natural. □

2.3 Applications for mapping class groups of surfaces

In [Sou17b], Long-Moody functors were defined for braid groups $\mathbf{B}_n$, which are the mapping class groups of a n-punctured disc. Therefore, the groups $\{G_n\}_{n\in\mathbb{N}}$ for which it is natural to define the first generalised Long-Moody functors are mapping class groups of surfaces. In this section, we will focus on exhibiting the functors that we recover by applying the Long-Moody functors on the constant functor R. We are interested in these functors for two reasons. First, R is the most basic functor to study. Secondly, considering the particular case of the family of trivial morphisms $\{\varsigma_{n,t} : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$, understanding $\mathbf{LM}(R)$ allows us to describe completely $\mathbf{LM}(F)$ for all objects F of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ by Proposition 2.2.39.

2.3.1 The monoidal groupoid associated with surfaces

Let us first introduce a suitable category for our work, inspired by [RWW17, Section 5.6]. Namely:

Definition 2.3.1. The decorated surfaces groupoid $\mathcal{M}_2$ is the groupoid defined by:

- Objects: decorated surfaces (S, I), where S is a smooth connected compact surface with one boundary component denoted by $\partial_0 S$ with $I : [-1, 1] \hookrightarrow \partial S$ is a parametrised interval in the boundary and $p = 0 \in I$ a basepoint, where a finite number of points is removed from the interior of S (in other words with punctures);
- Morphisms: the isotopy classes of homeomorphisms restricting to the identity on a neighbourhood of the parametrised interval I, freely moving the punctures, denoted by $\pi_0 Homeo^I(S, \{\text{punctures}\})$.

Remark 2.3.2. A homeomorphism of a surface which fixes an interval in a boundary component is isotopic to a homeomorphism which fixes pointwise the boundary component of the surface. Denote by $\hat{S}$ the surface obtained from $S \in Obj(\mathcal{M}_2)$ removing a disc on a neighbourhood of each puncture. Note from [FM11, Section 1.4.2] that $\pi_0 Homeo^I(S, \{\text{punctures}\})$ identifies with the group $\pi_0 Diff^{\partial_0}(\hat{S})$ of isotopy classes of diffeomorphisms of $\hat{S}$ fixing the boundary component ∂_0 and moving freely the other boundary components.

When the surface S is orientable, the orientation on S is induced by the orientation of I. The isotopy classes of homeomorphisms then automatically preserve that orientation as they restrict to the identity on a neighbourhood of I.

Notation 2.3.3. When there is no ambiguity, we omit the parametrised interval I from the notation.

We denote by $\Sigma^0_{0,0,1}$ a disc. We fix a unit disc with one puncture denoted by $\Sigma^1_{0,0,1}$, a torus with one boundary component denoted by $\Sigma^0_{1,0,1}$ and a Möbius band denoted by $\Sigma^0_{0,1,1}$.

The groupoid $\mathcal{M}_2$ has a monoidal structure induced by gluing; for completeness, the definition is outlined below (see [RWW17, Section 5.6.1] for technical details). For two decorated surfaces (S_1, I_1) and (S_2, I_2), the boundary connected sum $(S_1, I_1) \natural (S_2, I_2) = (S_1 \natural S_2, I_1 \natural I_2)$ is defined with $S_1 \natural S_2$ the surface obtained from gluing S_1 and S_2 along the half-interval I_1^+ and the half-interval I_2^-, and $I_1 \natural I_2 = I_1^- \cup I_2^+$. The homeomorphisms being the identity on a neighbourhood of the parametrised intervals I_1 and I_2, we canonically extend the homeomorphisms of S_1 and S_2 to $S_1 \natural S_2$. Hence, we have:

Proposition 2.3.4. *[RWW17, Proposition 5.18] The boundary connected sum $\natural$ induces a strict braided monoidal structure $\left(\mathcal{M}_2, \natural, \left(\Sigma^0_{0,0,1}, I\right)\right)$. There are no zero divisors in the category $\mathcal{M}_2$ and $Aut_{\mathcal{M}_2}\left(\Sigma^0_{0,0,1}\right) = \left\{id_{\Sigma^0_{0,0,1}}\right\}$.*

The braiding of the monoidal structure $b^{\mathcal{M}_2}_{(S_1,I_1),(S_2,I_2)} : (S_1, I_1) \natural (S_2, I_2) \to (S_2, I_2) \natural (S_1, I_1)$ is given by doing half a Dehn twist in a pair of pants neighbourhood of ∂S_1 and ∂S_2 (see [RWW17, Section 5.6.1, Figure 2]).

Definition 2.3.5. Let $\hat{\mathfrak{M}}_2$ be the full subgroupoid of $\mathcal{M}_2$ of the boundary connected sum on the objects $\Sigma^0_{0,0,1}$, $\Sigma^1_{0,0,1}$, $\Sigma^0_{1,0,1}$ and $\Sigma^0_{0,1,1}$. Let $\mathfrak{M}_2$ be the skeleton of $\hat{\mathfrak{M}}_2$.

Remark 2.3.6. Let S be an object of the groupoid $\mathfrak{M}_2$. Then, there exist $g, s, c \in \mathbb{N}$ such that there is an homeomorphism:

$$S \cong \left(\underset{s}{\natural}\Sigma^1_{0,0,1}\right) \natural \left(\underset{g}{\natural}\Sigma^0_{1,0,1}\right) \natural \left(\underset{c}{\natural}\Sigma^0_{0,1,1}\right).$$

A fortiori, by Proposition 2.3.4:

Proposition 2.3.7. *The groupoid $\left(\mathfrak{M}_2, \natural, \Sigma^0_{0,0,1}\right)$ is small braided monoidal with no zero divisors and such that $Aut_{\mathfrak{M}_2}\left(\Sigma^0_{0,0,1}\right) = \left\{id_{\Sigma^0_{0,0,1}}\right\}$. The braiding of the monoidal structure is denoted by $b^{\mathfrak{M}_2}_{-,-}$.*

By Definition 2.1.3, we denote by $\mathfrak{UM}_2$ Quillen's bracket construction on the groupoid $\left(\mathfrak{M}_2, \natural, \Sigma^0_{0,0,1}\right)$; by Proposition 2.1.12 we obtain a pre-braided strict monoidal category $\left(\mathfrak{UM}_2, \natural, \Sigma^0_{0,0,1}\right)$.

2.3.2 Fundamental group functor

Let us introduce a non-trivial functor with $\mathfrak{UM}_2$ as source category. The isotopy classes of the homeomorphisms of a surface $S \in Obj\left(\mathfrak{M}_2\right)$ act on its fundamental group $\pi_1(S, p)$ (see for example [FM11, Chapter 4]).

Notation 2.3.8. We denote this action by $a_S : \pi_0 Homeo^I(S, \{\text{punctures}\}) \to Aut_{\mathfrak{Gr}}(\pi_1(S, p))$. So, we define a functor $\pi_1(-, p) : \left(\mathfrak{M}_2, \natural, \Sigma^0_{0,1}\right) \to \mathfrak{gr}$ assigning $\pi_1(-, p)(S) = \pi_1(S, p)$ on objects and for all $\varphi \in \pi_0 Diff^{\partial}(S)$, $\pi_1(-, p)(\varphi) = a_S(\varphi)$.

Remark 2.3.9. In Notation 2.3.8, we fix maps

$$\pi_0 Homeo^I(S, \{\text{punctures}\}) \to Aut_{\mathfrak{Gr}}(\pi_1(S, p)).$$

Note that we could make other choices of such morphisms so that the following study still works. We refer to Remark 2.3.42 for more details about this fact for the particular case of braid groups.

Notation 2.3.10. Let $\mathfrak{gr}$ denote the full subcategory of $\mathfrak{Gr}$ of finitely-generated free groups. The free product $* : \mathfrak{gr} \times \mathfrak{gr} \to \mathfrak{gr}$ defines a monoidal structure on $\mathfrak{gr}$, with 0 the unit, denoted by $(\mathfrak{gr}, *, 0)$.

Lemma 2.3.11. *The functor $\pi_1(-, p) : \left(\mathfrak{M}_2, \natural, \Sigma^0_{0,0,1}\right) \to (\mathfrak{gr}, *, 0_{\mathfrak{Gr}})$ is strict monoidal.*

Proof. By Van Kampen's theorem (see for example [Hat02, Section 1.2]), we have that, for $S, S' \in Obj\left(\mathfrak{M}_2\right)$:

$$\pi_1\left(S'\natural S, p\right) \cong \pi_1\left(S', p\right) * \pi_1(S, p).$$

It is clear that $\pi_0 Homeo^I(S, \{\text{punctures}\})$ (resp. $\pi_0 Homeo^I(S', \{\text{punctures}\})$) acts trivially on $\pi_1(S', p)$ (resp. $\pi_1(S, p)$) in $\pi_1(S'\natural S, p)$. Therefore, $id_{\pi_1(-,p)} * id_{\pi_1(-,p)}$ is a natural equivalence. □

Proposition 2.3.12. *The functor $\pi_1(-, p)$ of Lemma 2.3.11 extends to a functor $\pi_1(-, p) : \mathfrak{UM}_2 \to \mathfrak{gr}$ by assigning for all $S, S' \in Obj\left(\mathfrak{M}_2\right)$:*

$$\pi_1(-, p)\left(\left[S', id_{S'\natural S}\right]\right) = \iota_{\pi_1(S',p)} * id_{\pi_1(S,p)}.$$

Proof. It follows from the definitions that relation (2.1.1) of Lemma 2.1.8 is satisfied for

$$\pi_1(-,p)\left[\Sigma^1_{0,0,1}, id_{\Sigma^1_{0,0,1}\natural S}\right],\ \pi_1(-,p)\left[\Sigma^0_{1,0,1}, id_{\Sigma^0_{1,0,1}\natural S}\right] \text{ and } \pi_1(-,p)\left[\Sigma^0_{0,1,1}, id_{\mathcal{N}\Sigma^0_{1,1}\natural S}\right].$$

Let S and S' be objects of $\mathfrak{M}_2$. Let $\varphi \in \pi_0 Homeo^I(S,\{\text{punctures}\})$ and $\varphi' \in \pi_0 Homeo^I(S',\{\text{punctures}\})$. According to Lemma 2.3.11:

$$\pi_1(-,p)(\varphi'\natural\varphi)\circ\pi_1(-,p)([S', id_{S'\natural S}]) = (\pi_1(-,p)(\varphi')*\pi_1(-,p)(\varphi))\circ\pi_1(-,p)([S', id_{S'\natural S}]).$$

Hence, by definition of the morphism $\iota_{\pi_1(S',p)}$, we have:

$$\pi_1(-,p)(\varphi'\natural\varphi)\circ\pi_1(-,p)([S', id_{S'\natural S}]) = \pi_1(-,p)([S', id_{S'\natural S}])\circ\pi_1(-,p)(\varphi).$$

Relation (2.1.2) of Lemma 2.1.8 is thus satisfied, which implies the desired result. □

2.3.3 Modifying the orientable or non-orientable genus

We fix the number s of punctures throughout Section 2.3.3.

2.3.3.1 Orientable surfaces:

Let $\mathfrak{M}_2^{+,s}$ be the full subgroupoid of $\mathfrak{M}_2$ of orientable surfaces with s punctures. According to Proposition 2.3.6, the objects are $\left\{\Sigma^s_{n,0,1}\right\}_{n\in\mathbb{N}}$. Therefore, $Obj\left(\mathfrak{M}_2^{+,s}\right)\cong\mathbb{N}$ and the groupoid $\mathfrak{M}_2^{+,s}$ is finitely generated by the monoidal structure in $\left(\mathfrak{M}_2,\natural,\Sigma^0_{0,0,1}\right)$. Hence, by Proposition 2.3.7 Assumption 2.2.1 is satisfied for the groupoid $\left(\mathfrak{M}_2,\natural,\Sigma^0_{0,0,1}\right)$.

Notation 2.3.13. Denote the mapping class group $\pi_0 Homeo^I\left(\Sigma^s_{n,0,1},\{\text{punctures}\}\right)$ by $\mathbf{\Gamma}^s_{n,1}$, for all $n\in\mathbb{N}$.

Let H be the group $\pi_1\left(\Sigma^0_{1,0,1},p\right)\cong\mathbf{F}_2$ and H_0 be the group $\pi_1\left(\Sigma^s_{0,0,1},p\right)\cong\mathbf{F}_s$. and therefore, $H_n\cong\pi_1\left(\Sigma^s_{n,0,1},p\right)\cong\mathbf{F}_{2n+s}$ for all natural numbers n. We denote by $\pi_1\left(\Sigma^s_{-,0,1},p\right)$ the associated functor of Assumption 2.2.13.

Proposition 2.3.14. *$\left\{\pi_1\left(\Sigma^s_{-,0,1},p\right),\mathfrak{M}_2^{+,s},\mathfrak{M}_2,\varsigma_{-,t}\right\}$ is a coherent Long-Moody system, where $\varsigma_{n,t}:\pi_1\left(\Sigma^s_{n,0,1},p\right)\to\mathbf{\Gamma}^s_{n+1,1}$ is the trivial morphism for all natural numbers n (see Example 2.2.22).*

Proof. The functor $\mathcal{H}_g$ extends to give a functor $\pi_1\left(\Sigma^s_{-,0,1},p\right):\mathfrak{UM}_2^{+,s}\to\mathfrak{Gr}$ by Proposition 2.3.12, so that Assumption 2.2.13 is satisfied. Thus the result follows from Remark 2.2.27. □

Example 2.3.15. We denote by $H_1\left(\Sigma^s_{-,0,1},R\right)$ the functor induced by the functor $H_1(H_-,R)$ of Proposition 2.2.37. For all natural numbers n and $s=0$, the action of $\mathbf{\Gamma}_{n,1}$ on $H_1\left(\Sigma^0_{n,0,1},R\right)$ is the symplectic representation of the mapping class group $\mathbf{\Gamma}^s_{n,1}$. We deduce from Lemma 2.2.38 that:

$$H_1\left(\Sigma^s_{-,0,1},R\right)\cong\mathbf{LM}_{\left\{\pi_1\left(\Sigma^s_{-,0,1},p\right),\mathfrak{M}_2^{+,s},\mathfrak{M}_2,\varsigma_{-,t}\right\}}(R). \tag{2.3.1}$$

Remark 2.3.16. This functor was introduced by Cohen and Madsen in [CM09] and by Boldsen in [Bol12]. Furthermore, the homology of the mapping class groups $\mathbf{\Gamma}_{n,1}$ for a large natural number n with coefficients $H_1\left(\Sigma^0_{n,0,1},R\right)$ were computed by Harer in [Har91, Section 7] (see also the forthcoming work [Sou18]).

Assume that $R=\mathbb{C}$ and $s=0$. Since the morphisms $\mathbf{\Gamma}^0_{n+1,1}\to Aut\left(\pi_1\left(\Sigma^0_{n+1,0,1},p\right)\right)$ are non-trivial for natural numbers $n\geq 2$, the action of $\mathbf{\Gamma}^0_{n,1}$ on $\mathbf{LM}_{\left\{\pi_1\left(\Sigma^0_{-,0,1},p\right),\mathfrak{M}_2^{+,s},\mathfrak{M}_2,\varsigma_{n,t}\right\}}(R)(n)$ is not trivial for $n\geq 3$. So the result (2.3.1) is consistent with [Kor11, Theorem 1] asserting that for $n\geq 3$, a non-trivial linear representation of $\mathbf{\Gamma}_{n,1}$ of dimension $2n$ is equivalent to the symplectic representation.

2.3.3.2 Non-orientable surfaces:

Let $\mathfrak{M}_2^{-,s}$ be the full subgroupoid of $\mathfrak{M}_2$ on non-orientable surfaces with s punctures. According to Proposition 2.3.6, its objects are $\left\{\Sigma_{0,n,1}^{s}\right\}_{n\in\mathbb{N}}$. Therefore, $Obj\left(\mathfrak{M}_2^{-,s}\right)\cong\mathbb{N}$ and the groupoid $\mathfrak{M}_2^{-,s}$ is finitely generated by the monoidal structure of $\left(\mathfrak{M}_2,\natural,\Sigma_{0,0,1}^{0}\right)$. Thus, by Proposition 2.3.7, Assumption 2.2.1 is satisfied using the groupoid $\left(\mathfrak{M}_2,\natural,\Sigma_{0,0,1}^{0}\right)$.

Notation 2.3.17. Denote the mapping class group $\pi_0 Homeo^I\left(\Sigma_{0,n,1}^{s},\{\text{punctures}\}\right)$ by $\mathcal{N}_{n,1}^{s}$ for all $n\in\mathbb{N}$.

Let H be the group $\pi_1\left(\Sigma_{0,1,1}^{0},p\right)\cong\mathbf{F}_1$ and H_0 be $\pi_1\left(\Sigma_{0,0,1}^{s},p\right)\cong\mathbf{F}_s$, so that $H_n\cong\pi_1\left(\Sigma_{0,n,1}^{s},p\right)\cong\mathbf{F}_{n+s}$ for all natural numbers n. We denote by $\pi_1\left(\Sigma_{0,-,1}^{s},p\right)$ the associated functor of Assumption 2.2.13.

Proposition 2.3.18. *The setting* $\left\{\pi_1\left(\Sigma_{0,-,1}^{s},p\right),\mathfrak{M}_2^{-,s},\mathfrak{M}_2,\varsigma_{-,t}\right\}$ *is a coherent Long-Moody system, where* $\varsigma_{n,t}:\pi_1\left(\Sigma_{0,n,1}^{s},p\right)\to\mathcal{N}_{n+1,1}^{s}$ *is the trivial morphism for all natural numbers n (see Example 2.2.22).*

Proof. The functor $\pi_1\left(\Sigma_{0,-,1}^{s},p\right)$ extends to give a functor $\pi_1\left(\Sigma_{0,-,1}^{s},p\right):\mathfrak{U}\mathfrak{M}_2^{-,s}\to\mathfrak{Gr}$ by Proposition 2.3.12; so that Assumption 2.2.13 is satisfied. Hence the result follows from Remark 2.2.27. □

Example 2.3.19. We denote by $H_1\left(\Sigma_{0,-,1}^{s},R\right)$ the functor induced by the functor $H_1\left(H_-,R\right)$ of Proposition 2.2.37. We deduce from Lemma 2.2.38 that:

$$H_1\left(\Sigma_{0,-,1}^{s},R\right)\cong\mathbf{LM}_{\left\{\pi_1\left(\Sigma_{0,-,1}^{s},p\right),\mathfrak{M}_2^{-,s},\mathfrak{M}_2,\varsigma_{-,t}\right\}}\left(R\right).$$

Remark 2.3.20. Proposition 2.2.39 ensures that the functor $\mathbf{LM}_{\left\{\pi_1\left(\Sigma_{0,-,1}^{s},p\right),\mathfrak{M}_2^{-,s},\mathfrak{M}_2,\varsigma_{-,t}\right\}}$ is determined by $H_1\left(\Sigma_{0,-,1}^{s},R\right)$.

Remark 2.3.21. In [Stu09], Stukow computes the homology groups $H_1\left(\mathcal{N}_{n,1},H_1\left(\Sigma_{0,n,1}^{0},\mathbb{Z}\right)\right)$ for all natural numbers n.

2.3.4 Modifying the number of punctures

We fix a natural number g throughout Section 2.3.4, and let $\mathfrak{M}_2^{g,0}$ be the full subgroupoid of $\mathfrak{M}_2$ on surfaces with orientable genus g and non-orientable genus 0. According to Proposition 2.3.6, the objects of $\mathfrak{M}_2^{g,0}$ are $\left\{\Sigma_{g,0,1}^{n}\right\}_{n\in\mathbb{N}}$. Therefore, $Obj\left(\mathfrak{M}_2^{g,0}\right)\cong\mathbb{N}$ and the groupoid $\mathfrak{M}_2^{g,0}$ is finitely generated by the monoidal structure in $\left(\mathfrak{M}_2,\natural,\Sigma_{0,0,1}^{0}\right)$. By Proposition 2.3.7, the groupoid $\left(\mathfrak{M}_2,\natural,\Sigma_{0,0,1}^{0}\right)$ satisfies Assumption 2.2.1.

Let H be the group $\pi_1\left(\Sigma_{0,0,1}^{1},p\right)\cong\mathbf{F}_1$ and H_0 be the group $\pi_1\left(\Sigma_{g,0,1}^{0},p\right)\cong\mathbf{F}_{2g}$. Therefore, $H_n=\pi_1\left(\Sigma_{g,0,1}^{n},p\right)\cong\mathbf{F}_{n+2g}$ for all natural numbers n. We denote by $\pi_1\left(\Sigma_{g,0,1}^{-},p\right)$ the associated functor of Assumption 2.2.13 defined by $\pi_1\left(\Sigma_{g,0,1}^{-},p\right)(n)=\pi_1\left(\Sigma_{g,0,1}^{n},p\right)$ for all natural numbers n.

To define the group morphisms $\left\{\varsigma_n:\pi_1\left(\Sigma_{g,0,1}^{n},p\right)\to\mathbf{\Gamma}_{g,0,1}^{n}\right\}_{n\in\mathbb{N}}$ considered in this section, we first need to recall some classical facts about mapping class groups of surfaces.

Notation 2.3.22. For all natural numbers n, we denote by $\mathbf{\Gamma}_{g,0,1}^{[1],n}$ the subgroup of the mapping class group

$$\pi_0 Homeo^I\left(\Sigma_{0,0,1}^{1}\natural\Sigma_{g,0,1}^{n},\{\text{punctures}\}\right)\cong\mathbf{\Gamma}_{g,0,1}^{1+n}$$

where the puncture of the surface $\Sigma_{0,0,1}^{1}$ in $\Sigma_{0,0,1}^{1}\natural\Sigma_{g,0,1}^{n}$ is fixed. Hence, there is a canonical embedding $\mathscr{E}:\mathbf{\Gamma}_{g,0,1}^{[1],n}\hookrightarrow\mathbf{\Gamma}_{g,0,1}^{1+n}$.

Remark 2.3.23. Denoting by $\Sigma_{g,0,1}^{[x],n}$ the surface $\Sigma_{g,0,1}^{n}$ with a marked point denoted by x, the group $\mathbf{\Gamma}_{g,0,1}^{[1],n}$ is isomorphic to the isotopy classes of homeomorphisms of the surface $\Sigma_{g,0,1}^{[x],n}$ restricting to the identity on the boundary component, freely moving the punctures and fixing pointwise the marked point x (see [FM11, Section 4.1.2]).

Recall that for all natural numbers n and g, $\pi_1\left(\Sigma_{g,0,1}^{n},x\right)$ is a free group with generators $\{f_i\}_{i\in\{1,\ldots,n+2g\}}$. Each generator f_i is represented by a simple closed curve α_{f_i} in $\Sigma_{g,0,1}^{[x],n}$ based at x. Let $N(f)\overset{\phi}{\cong}\mathbb{S}^1\times[-1,1]$ be a tubular neighbourgood of the curve α_{f_i}. Denote by f_i^- and f_i^+ the isotopy classes of the curves $\phi^{-1}\left(\mathbb{S}^1\times\{-1\}\right)$ and $\phi^{-1}\left(\mathbb{S}^1\times\{1\}\right)$. The group morphism $Push:\pi_1\left(\Sigma_{g,0,1}^{n},x\right)\to\mathbf{\Gamma}_{g,0,1}^{[1],n}$ is defined by sending f_i to $\tau_{f_i^-}\circ\tau_{f_i^+}^{-1}$, where τ_α denotes the Dehn twist along a simple closed curve α (see [FM11, Fact 4.7]). The Birman exact sequence uses the map $Push$ to describe the effect of forgetting a marked point fixed by the mapping class group. Namely:

Theorem 2.3.24. *[FM11, Theorem 4.6] Let n be a natural number such that $2g+n\geq 2$. The following sequence is exact:*

$$1\longrightarrow\pi_1\left(\Sigma_{g,0,1}^{n},x\right)\xrightarrow{Push}\mathbf{\Gamma}_{g,0,1}^{[1],n}\xrightarrow{Forget}\mathbf{\Gamma}_{g,0,1}^{n}\longrightarrow 1 \tag{2.3.2}$$

where the map $Forget:\mathbf{\Gamma}_{g,0,1}^{[1],n}\to\mathbf{\Gamma}_{g,0,1}^{n}$ is induced by forgetting that the point x is marked.

Lemma 2.3.25. *Let n be a natural number such that $2g+n\geq 2$. The Birman exact sequence splits, hence induces an isomorphism $\mathbf{\Gamma}_{g,0,1}^{[1],n}\overset{\mathscr{B}}{\cong}\pi_1\left(\Sigma_{g,0,1}^{n},p\right)\underset{a_{\Sigma_{g,0,1}^{n}}}{\rtimes}\mathbf{\Gamma}_{g,0,1}^{n}$ (where $a_{\Sigma_{g,0,1}^{n}}$ is introduced in Notation 2.3.8).*

Proof. Recall that there is an homeomorphism $\Sigma_{g,0,1}^{[x],n}\cong\Sigma_{0,0,1}^{[x]}\natural\Sigma_{g,0,1}^{n}$. Hence, the embedding of $\Sigma_{g,0,1}^{n}$ into $\Sigma_{g,0,1}^{[x],n}$ as the complement of the disc $\Sigma_{0,0,1}^{[x]}$ with the marked point x induces an injective morphism $\mathbf{\Gamma}_{g,0,1}^{n}\hookrightarrow\mathbf{\Gamma}_{g,0,1}^{[1],n}$. The action of $\mathbf{\Gamma}_{g,0,1}^{n}$ on $\pi_1\left(\Sigma_{g,0,1}^{n},x\right)$ is denoted by $a_{\Sigma_{g,0,1}^{n}}^{x}$. This provides a splitting of the exact sequence (2.3.2) and hence we have an isomorphism:

$$\mathbf{\Gamma}_{g,0,1}^{[1],n}\cong\pi_1\left(\Sigma_{g,0,1}^{n},x\right)\underset{a_{\Sigma_{g,0,1}^{n}}^{x}}{\rtimes}\mathbf{\Gamma}_{g,0,1}^{n}.$$

Recall that the definition of boundary connected sum in $\mathfrak{M}_2$ (see Proposition 2.3.4) implies that the point $p\in\partial\Sigma_{g,0,1}^{[x],n}$ belongs to the intersection of $\partial\Sigma_{0,0,1}^{[x],0}$ and $\partial\Sigma_{g,0,1}^{n}$ in $\Sigma_{0,0,1}^{[x],0}\natural\Sigma_{g,0,1}^{n}$. Hence we can consider a path γ in $\Sigma_{0,0,1}^{[x],0}$ connecting the point p to x. Moving the point p to x along such a path γ induces an isomorphism:

$$\lambda_n:\pi_1\left(\Sigma_{g,0,1}^{n},p\right)\overset{\cong}{\to}\pi_1\left(\Sigma_{g,0,1}^{n},x\right).$$

Since $\mathbf{\Gamma}_{g,0,1}^{n}$ acts trivially on the disc $\Sigma_{0,0,1}^{[x]}$ with the marked point x in $\Sigma_{0,0,1}^{[x]}\natural\Sigma_{g,0,1}^{n}\cong\Sigma_{g,0,1}^{[x],n}$, we deduce that for all $\phi\in\mathbf{\Gamma}_{g,0,1}^{n}$:

$$\lambda_n\circ a_{\Sigma_{g,0,1}^{n}}(\phi)=a_{\Sigma_{g,0,1}^{n}}^{x}(\phi)\circ\lambda_n.$$

Hence, the following morphism is well-defined with respect to the semidirect product structure:

$$\begin{array}{ccc}\pi_1\left(\Sigma_{g,0,1}^{n},p\right)\underset{a_{\Sigma_{g,0,1}^{n}}}{\rtimes}\mathbf{\Gamma}_{g,0,1}^{n} & \longrightarrow & \pi_1\left(\Sigma_{g,0,1}^{n},x\right)\underset{a_{\Sigma_{g,0,1}^{n}}^{x}}{\rtimes}\mathbf{\Gamma}_{g,0,1}^{n}\\ (f,\phi) & \longmapsto & (\lambda_n(f),\phi).\end{array}$$

This is an isomorphism by the five lemma. □

Definition 2.3.26. Let n be a natural number such that $2g+n\geq 2$. We define the morphism $\varsigma_{n,1}:\pi_1\left(\Sigma_{g,0,1}^{n},p\right)\to\mathbf{\Gamma}_{g,0,1}^{1+n}$ to be the composition:

$$\pi_1\left(\Sigma_{g,0,1}^{n},p\right)\hookrightarrow\pi_1\left(\Sigma_{g,0,1}^{n},p\right)\underset{a_{\Sigma_{g,0,1}^{n}}}{\rtimes}\mathbf{\Gamma}_{g,0,1}^{n}\overset{\mathscr{B}}{\to}\mathbf{\Gamma}_{g,0,1}^{[1],n}\overset{\mathscr{E}}{\hookrightarrow}\mathbf{\Gamma}_{g,0,1}^{1+n}.$$

If $g=0$, we define $\varsigma_{0,1}:\pi_1\left(\Sigma_{0,0,1}^{0},p\right)\to 0_{\mathfrak{Gr}}$ to be the trivial morphism and $\varsigma_{1,1}:\pi_1\left(\Sigma_{0,0,1}^{1},p\right)\to\mathbf{B}_2$ to be the morphism sending the generator f_1 of $\pi_1\left(\Sigma_{0,0,1}^{1},p\right)$ to σ_1^2 (where σ_1 denotes the Artin generator of the braid group on two strands $\mathbf{B}_2$).

Remark 2.3.27. For $2g+n\geq 2$, a generator f_i of $\pi_1\left(\Sigma_{g,0,1}^{n},p\right)\cong\langle f_1,\ldots,f_{n+2g}\rangle$ is represented by a simple closed curve in $\Sigma_{g,0,1}^{n}$ based at p. Using a path γ in $\Sigma_{0,0,1}^{[x],0}$ connecting the point p to x, we thus associate to the generator f_i a simple closed curve α_{f_i} in $\Sigma_{g,0,1}^{[x],n}$ based at the marked point x filling in the additional puncture. By the definitions of the morphisms $\mathscr{B}$ (see Lemma 2.3.25) and $\mathscr{E}$ (see Notation 2.3.22), the generator f_i is sent by $\varsigma_{n,1}$ to $\tau_{f_i^-}\circ\tau_{f_i^+}^{-1}$, where f_i^- and f_i^+ are the isotopy classes of the simple closed curves in $\Sigma_{g,0,1}^{[x],n}$ encircling a tubular neighbourgood of the curve α_{f_i}.

Lemma 2.3.28. *The setting* $\left\{\pi_1\left(\Sigma_{g,0,1}^{-},p\right),\mathfrak{M}_2^{g,0},\mathfrak{M}_2,\varsigma_{n,1}\right\}$ *satisfies Condition 2.2.24.*

Proof. It is clear from our assignments that if $2g+n\geq 2$, then the composition $\mathbf{\Gamma}_{g,0,1}^{n}\hookrightarrow\pi_1\left(\Sigma_{g,0,1}^{n},p\right)\underset{a_{\Sigma_{g,0,1}^{n}}}{\rtimes}\mathbf{\Gamma}_{g,0,1}^{n}\overset{\mathscr{B}}{\to}\mathbf{\Gamma}_{g,0,1}^{[1],n}\overset{\mathscr{E}}{\hookrightarrow}\mathbf{\Gamma}_{g,0,1}^{1+n}$ is the morphism $id_{\Sigma_{0,0,1}^{1}}\natural-:\mathbf{\Gamma}_{g,0,1}^{n}\to\mathbf{\Gamma}_{g,0,1}^{1+n}$. Hence, the following diagram is commutative:

$$\begin{array}{ccccc}
\pi_1\left(\Sigma_{g,0,1}^{n},p\right) & \hookrightarrow & \pi_1\left(\Sigma_{g,0,1}^{n},p\right)\underset{a_{\Sigma_{g,0,1}^{n}}}{\rtimes}\mathbf{\Gamma}_{g,0,1}^{n} & \hookleftarrow & \mathbf{\Gamma}_{g,0,1}^{n}\\
 & \searrow^{\varsigma_{n,1}} & \downarrow{\scriptstyle \mathscr{E}\circ\mathscr{B}} & \swarrow^{id_{\Sigma_{0,0,1}^{1}}\natural-} & \\
 & & \mathbf{\Gamma}_{g,0,1}^{1+n}. & &
\end{array}$$

If $g=0$, the braid groups $\mathbf{B}_0$ and $\mathbf{B}_1$ being the trivial group, Condition 2.2.24 is easily checked. □

Furthermore, we have the property:

Lemma 2.3.29. *The morphism $\varsigma_{n,1}$ satisfies Condition 2.2.17 for any natural number n.*

Proof. If $g=0$ and $n\leq 1$, the result follows from [Sou17b, Proposition 2.8]. Assume that $2g+n\geq 2$. Let us fix basis $\langle f_1,\ldots,f_{n+2g}\rangle$ of $\mathbf{F}_{n+2g}\cong\pi_1\left(\Sigma_{g,0,1}^{n},p\right)$ and $\langle f_1,\ldots,f_{1+n+2g}\rangle$ of $\mathbf{F}_{1+n+2g}\cong\pi_1\left(\Sigma_{g,0,1}^{1+n},p\right)$. Namely, a generator f_i is represented by a simple loop either encircling a meridian or a parallel of one of the copies of the torus $\Sigma_{1,0,1}^{0}$, or else encircling a circle around one of the copies of the one punctured disc $\Sigma_{0,0,1}^{1}$. By our conventions, the generator f_i of $\pi_1\left(\Sigma_{g,0,1}^{n},p\right)$ is sent through

$$\mathcal{H}_s\left[1,id_{\underline{n+1}}\right]=\iota_{\pi_1\left(\Sigma_{0,0,1}^{1},p\right)}*id_{\pi_1\left(\Sigma_{g,0,1}^{n},p\right)}:\pi_1\left(\Sigma_{g,0,1}^{n},p\right)\hookrightarrow\pi_1\left(\Sigma_{g,0,1}^{1+n},p\right)$$

to the generator f_{1+i} of $\pi_1\left(\Sigma_{g,0,1}^{1+n},p\right)$. So according to Remark 2.2.18, it is enough to prove that, as elements of $\mathbf{\Gamma}_{g,0,1}^{2+n}$:

$$\varsigma_{n+1,1}\left(f_{1+i}\right)\circ\left(\left(b_{\Sigma_{0,0,1}^{1},\Sigma_{0,0,1}^{1}}^{\mathfrak{M}_2}\right)^{-1}\natural id_n\right)=\left(\left(b_{\Sigma_{0,0,1}^{1},\Sigma_{0,0,1}^{1}}^{\mathfrak{M}_2}\right)^{-1}\natural id_n\right)\circ\left(id_1\natural\varsigma_{n,1}\left(f_i\right)\right).\qquad(2.3.3)$$

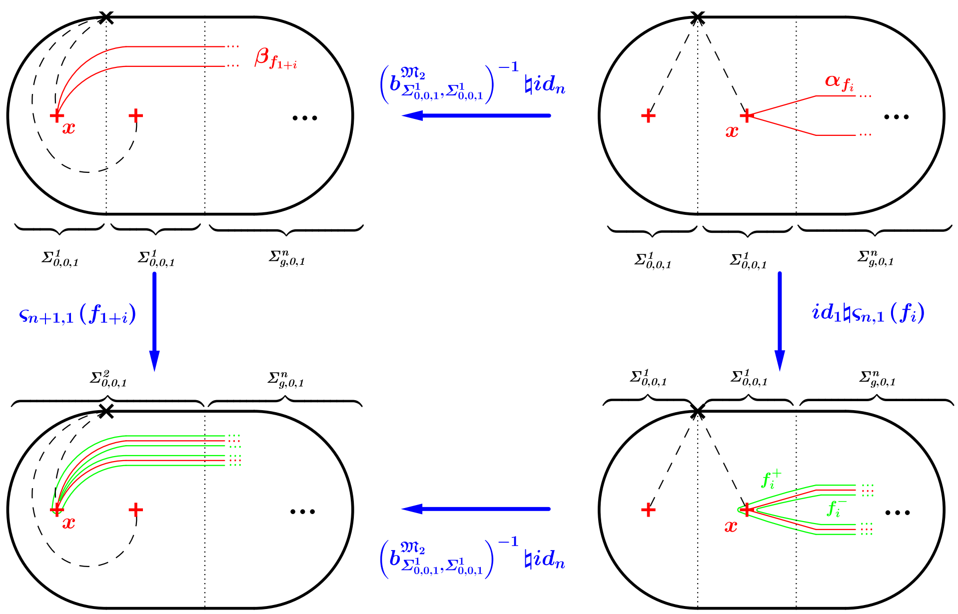


Figure 2.3.1: Proof of equality (2.3.3)

Let α_{f_i} (resp. $\beta_{f_{1+i}}$) be the simple loop associated to the generator f_i (resp. f_{1+i}) in $\Sigma_{g,0,1}^{[1],n}$ (resp. $\Sigma_{g,0,1}^{[1],1+n}$) based at the marked point filling in the additional puncture (see Remark 2.3.27). Recall that:

$$\varsigma_{n,1}(f_i) = \tau_{f_i^-} \circ \tau_{f_i^+}^{-1} \quad \text{and} \quad \varsigma_{n+1,1}(f_{1+i}) = \tau_{f_{1+i}^-} \circ \tau_{f_{1+i}^+}^{-1}$$

where f_i^- and f_i^+ (resp. f_{1+i}^- and f_{1+i}^+) are the isotopy classes of the simple closed curves in $\Sigma_{g,0,1}^{[1],n}$ (resp. $\Sigma_{g,0,1}^{[1],1+n}$) obtained by pushing α_{f_i} (resp. $\beta_{f_{1+i}}$) off itself to the left and right. Since $b_{\Sigma_{0,0,1}^1,\Sigma_{0,0,1}^1}^{\mathfrak{M}_2}$ is given by doing half a Dehn twist in a pair of pants neighbourhood of $\partial\Sigma_{0,0,1}^1$ and $\partial\Sigma_{0,0,1}^1$, applying $\left(b_{\Sigma_{0,0,1}^1,\Sigma_{0,0,1}^1}^{\mathfrak{M}_2}\right)^{-1} \natural id_n$ sends α_{f_i} to $\beta_{f_{1+i}}$, f_i^- to f_{1+i}^- and f_i^+ to f_{1+i}^+ (see Figure 2.3.1). Hence conjugating $id_{\Sigma_{0,0,1}^1} \natural \left(\tau_{f_i^-} \circ \tau_{f_i^+}^{-1}\right)$ by $\left(b_{\Sigma_{0,0,1}^1,\Sigma_{0,0,1}^1}^{\mathfrak{M}_2}\right)^{-1} \natural id_n$ gives $\tau_{f_{1+i}^-} \circ \tau_{f_{1+i}^+}^{-1}$. A fortiori the equality (2.3.3) is satisfied. □

Corollary 2.3.30. *With the previous assignments and notation, $\left\{\pi_1\left(\Sigma_{g,0,1}^-, p\right), \mathfrak{M}_2^{g,0}, \mathfrak{M}_2, \varsigma_{-,1}\right\}$ is a coherent Long-Moody system.*

Proof. We have already checked Assumption 2.2.1. Moreover, the functor $\pi_1\left(\Sigma_{g,0,1}^-, p\right)$ extends to give a functor $\pi_1\left(\Sigma_{g,0,1}^-, p\right) : \mathfrak{UM}_2^{g,c} \to \mathfrak{Gr}$ by Proposition 2.3.12 so that Assumption 2.2.13 is satisfied. Conditions 2.2.24 and 2.2.17 are checked in Lemmas 2.3.28 and 2.3.29. □

Example 2.3.31. We denote by $H_1\left(\Sigma_{g,0,1}^-, R\right)$ the functor induced by the functor $H_1(-, R)$ of Proposition 2.2.37. We deduce from Lemma 2.2.38 that:

$$H_1\left(\Sigma_{g,0,1}^-, R\right) \cong \mathbf{LM}_{\left\{\pi_1\left(\Sigma_{g,0,1}^-, p\right), \mathfrak{M}_2^{g,0}, \mathfrak{M}_2, \varsigma_{-,1}\right\}}(R).$$

Remark 2.3.32. Contrary to the cases of Section 2.3.3, since the morphisms $\varsigma_{n,1}$ are not trivial, the computation of $\mathbf{LM}_{\left\{\pi_1\left(\Sigma^-_{g,0,1},p\right),\mathfrak{M}_2^{g,0},\mathfrak{M}_2,\varsigma_{n,1}\right\}}$ on an object F of $\mathbf{Fct}\left(\mathfrak{UM}_2^{g,0}, R\text{-}\mathfrak{Mod}\right)$ is not given by Proposition 2.2.39. We thus obtain new families of representations of the mapping class groups $\left\{\mathbf{\Gamma}^n_{g,0,1}\right\}_{n\in\mathbb{N}}$.

Remark 2.3.33. Instead of modifying only the number of punctures or only the orientable or non-orientable genus, we can modify several of these parameters at the same time.
For instance, let $\mathfrak{M}^0_{2,g,p}$ be the full subgroupoid of $\mathfrak{M}_2$ on surfaces $\left\{\Sigma^n_{n,0,1}\right\}_{n\in\mathbb{N}}$ (see Proposition 2.3.6). Let H be the group $\pi_1\left(\Sigma^1_{1,0,1},p\right)\cong\mathbf{F}_3$ and H_0 be the trivial group $\pi_1\left(\Sigma^0_{0,0,1},p\right)$, and a fortiori $H_n=\pi_1\left(\Sigma^n_{n,0,1},p\right)=\mathbf{F}_{3n}$ for all natural numbers n. We denote by $\pi_1\left(\Sigma^-_{-,0,1},p\right)$ the associated functor of Assumption 2.2.13. Thanks to the canonical embedding (using Notation 2.3.22):

$$\mathscr{E}':\mathbf{\Gamma}^{[1],n}_{n,0,1}\overset{\mathscr{E}}{\hookrightarrow}\mathbf{\Gamma}^{1+n}_{n,0,1}\hookrightarrow\pi_0 Homeo^I\left(\Sigma^0_{1,0,1}\natural\Sigma^n_{n,0,1},\{\text{punctures}\}\right)\cong\mathbf{\Gamma}^{1+n}_{1+n,0,1},$$

we define the morphism $\varsigma^{g,p}_{n,1}:\pi_1\left(\Sigma^n_{n,0,1},p\right)\to\mathbf{\Gamma}^{1+n}_{1+n,0,1}$ to be the composition using Lemma 2.3.25:

$$\pi_1\left(\Sigma^n_{n,0,1},p\right)\hookrightarrow\pi_1\left(\Sigma^n_{n,0,1},p\right)\underset{a_S}{\rtimes}\mathbf{\Gamma}^n_{n,0,1}\overset{\mathscr{B}}{\cong}\mathbf{\Gamma}^{[1],n}_{n,0,1}\overset{\mathscr{E}'}{\hookrightarrow}\mathbf{\Gamma}^{1+n}_{1+n,0,1}.$$

Repeating mutatis mutandis the proofs of Lemmas 2.3.28 and 2.3.29 and of Corollary 2.3.30 shows that

$$\left\{\pi_1\left(\Sigma^-_{-,0,1},p\right),\mathfrak{M}^0_{2,g,p},\mathfrak{M}_2,\varsigma^{g,p}_{-,1}\right\}$$

is a coherent Long-Moody system. Denoting by $H_1\left(\Sigma^-_{-,0,1},R\right)$ the functor induced by $H_1(-,R)$ of Proposition 2.2.37, we deduce from Lemma 2.2.38 that:

$$H_1\left(\Sigma^-_{-,0,1},R\right)\cong\mathbf{LM}_{\left\{\pi_1\left(\Sigma^-_{-,0,1},p\right),\mathfrak{M}^0_{2,g,p},\mathfrak{M}_2,\varsigma^{g,p}_{-,1}\right\}}(R).$$

Since the morphisms $\varsigma^{g,p}_{n,1}$ are not trivial, we obtain families of representations of the mapping class groups $\left\{\mathbf{\Gamma}^n_{n,0,1}\right\}_{n\in\mathbb{N}}$ by iterating $\mathbf{LM}_{\left\{\pi_1\left(\Sigma^-_{-,0,1},p\right),\mathfrak{M}^0_{2,g,p},\mathfrak{M}_2,\varsigma^{g,p}_{n,1}\right\}}$ which are not determined by $H_1\left(\Sigma^-_{-,0,1},R\right)$ using Proposition 2.2.39.

2.3.5 Surface braid groups

We fix a natural number g throughout Section 2.3.5; let $\mathfrak{B}_2$ (resp. $\mathfrak{B}_2^{g,0}$) be the subgroupoid of $\mathfrak{M}_2$ (resp. $\mathfrak{M}_2^{g,0}$) with the same objects and with morphisms those that become trivial forgetting all the punctures. Namely, for all objects $\Sigma^n_{g,0,1}$ of $\mathfrak{M}_2$, we have the following short exact sequence (see for example [GJP, Section 2.4]):

$$1\longrightarrow\mathbf{B}^g_n\longrightarrow\mathbf{\Gamma}^n_{g,0,1}\longrightarrow\mathbf{\Gamma}^0_{g,0,1}\longrightarrow 1$$

where $\mathbf{B}^g_n=\mathbf{B}\left(\Sigma^n_{g,0,1}\right)$ denotes the braid group of the surface $\Sigma^n_{g,0,1}$. Note that $Obj\left(\mathfrak{B}_2^{g,0}\right)\cong\mathbb{N}$. The monoidal structure $(\mathfrak{M}_2,\natural,0)$ restricts to a braided monoidal structure on the subgroupoid $\mathfrak{B}_2$, denoted in the same way $(\mathfrak{B}_2,\natural,0)$. Remark that the groupoid $\mathfrak{B}_2^{g,0}$ is finitely generated by the monoidal structure in $\left(\mathfrak{B}_2,\natural,\Sigma^0_{0,0,1}\right)$. By Proposition 2.3.7, Assumption 2.2.1 is thus satisfied using the groupoid $\left(\mathfrak{B}_2,\natural,\Sigma^0_{0,0,1}\right)$.

Let H be the free group $\pi_1\left(\Sigma^1_{0,0,1},p\right)\cong\mathbf{F}_1$ and H_0 be the free group $\pi_1\left(\Sigma^0_{g,0,1},p\right)\cong\mathbf{F}_{2g}$. For all natural numbers n, we consider the restriction

$$\pi_1(-,p)_{\mathfrak{UB}_2}:\mathfrak{UB}_2\to\mathfrak{UM}_2\overset{\pi_1(-,p)}{\longrightarrow}\mathfrak{gr}$$

defined by the morphisms

$$a^{\mathfrak{b}}_{\Sigma^n_{g,0,1}} : \mathbf{B}^g_s \to \mathbf{\Gamma}^n_{g,0,1} \xrightarrow{a_{\Sigma^n_{g,0,1}}} \pi_1\left(\Sigma^n_{g,0,1}, p\right),$$

to obtain the associated functor $\pi_1\left(\Sigma^-_{g,0,1}, p\right)^{\mathfrak{b}} : \mathfrak{UB}^{g,0}_2 \to \mathfrak{Gr}$ of Assumption 2.2.13.

Notation 2.3.34. For all natural numbers n, we denote by $\mathbf{B}^g_{[1],n}$ the subgroup of $\mathbf{B}\left(\Sigma^1_{0,0,1}\natural\Sigma^n_{g,0,1}\right)$ where the puncture of the surface $\Sigma^1_{0,0,1}$ is fixed. This group $\mathbf{B}^g_{[1],n}$ is also known as the intertwining $(1,n)$-braid group on the surface $\Sigma^n_{g,1}$ (see for example [AK10, Section 1]) which is the kernel of the morphism:

$$\mathbf{\Gamma}^{[1],n}_{g,1} \to \mathbf{\Gamma}^{[1]}_{g,1} \to 1$$

defined by filling in the n punctures. Hence, there is a canonical embedding:

$$\mathscr{E}^{\mathfrak{b}} : \mathbf{B}^g_{[1],n} \hookrightarrow \ker\left(\mathbf{\Gamma}^{1+n}_{g,0,1} \to \mathbf{\Gamma}^0_{g,0,1}\right) \cong \mathbf{B}^g_{1+n}.$$

Lemma 2.3.35. *For all natural numbers n, there is an isomorphism:*

$$\mathbf{B}^g_{[1],n} \overset{\mathscr{B}'}{\cong} \pi_1\left(\Sigma^n_{g,0,1}, p\right) \underset{a^{\mathfrak{b}}_{\Sigma^n_{g,0,1}}}{\rtimes} \mathbf{B}^g_n.$$

Proof. Recall the isomorphism of Lemma 2.3.28:

$$\mathbf{\Gamma}^{[1],n}_{g,0,1} \overset{\mathscr{B}}{\cong} \pi_1\left(\Sigma^n_{g,0,1}, p\right) \underset{a_{\Sigma^n_{g,0,1}}}{\rtimes} \mathbf{\Gamma}^n_{g,0,1}.$$

The desired isomorphism is a consequence of the universal property of the kernel of the morphism $\mathbf{\Gamma}^{[1],n}_{g,0,1} \to \mathbf{\Gamma}^{[1]}_{g,0,1} \to 1$. □

Definition 2.3.36. Let n be a natural number such that $2g+n \geq 2$. We define the morphism $\varsigma^{\mathfrak{b}}_{n,1} : \pi_1\left(\Sigma^n_{g,0,1}, p\right) \to \mathbf{B}^g_{1+n}$ to be the composition:

$$\pi_1\left(\Sigma^n_{g,0,1}, p\right) \hookrightarrow \pi_1\left(\Sigma^n_{g,0,1}, p\right) \underset{a^{\mathfrak{b}}_{\Sigma^n_{g,0,1}}}{\rtimes} \mathbf{B}^g_n \overset{\mathscr{B}'}{\to} \mathbf{B}^g_{[1],n} \overset{\mathscr{E}^{\mathfrak{b}}}{\hookrightarrow} \mathbf{B}^g_{1+n}.$$

If $g=0$, we define $\varsigma_{0,1} : \pi_1\left(\Sigma^0_{0,0,1}, p\right) \to 0_{\mathfrak{Gr}}$ to be the trivial morphism and $\varsigma_{1,1} : \pi_1\left(\Sigma^1_{0,0,1}, p\right) \to \mathbf{B}_2$ to be the morphism sending the generator f_1 of $\pi_1\left(\Sigma^1_{0,0,1}, p\right)$ to σ_1^2 (where σ_1 denotes the Artin generator of the braid group on two strands $\mathbf{B}_2$).

Lemma 2.3.37. *The setting $\left\{\mathcal{H}^{\mathfrak{b}}_s, \mathfrak{B}^{g,0}_2, \mathfrak{B}_2, \varsigma^{\mathfrak{b}}_{n,1}\right\}$ satisfies Condition 2.2.24.*

Proof. The proof follows mutatis mutandis that of Lemma 2.3.28. □

Proposition 2.3.38. *With the previous assignments and notation, $\left\{\mathcal{H}^{\mathfrak{b}}_s, \mathfrak{B}^{g,0}_2, \mathfrak{B}_2, \varsigma^{\mathfrak{b}}_{-,1}\right\}$ is a coherent Long-Moody system.*

Proof. The functor $\pi_1\left(\Sigma^-_{g,0,1}, p\right)^{\mathfrak{b}}$ extends to give a functor $\pi_1\left(\Sigma^-_{g,0,1}, p\right)^{\mathfrak{b}} : \mathfrak{UB}^g_2 \to \mathfrak{Gr}$ by Proposition 2.3.12, so that Assumption 2.2.13 is satisfied. Condition 2.2.24 is checked in Lemma 2.3.37. Finally, as $\mathbf{B}^g_n$ is a subgroup of $\mathbf{\Gamma}^n_{g,0,1}$, repeating mutatis mutandis the proof of Lemma 2.3.29, the morphisms $\varsigma^{\mathfrak{b}}_{n,1}$ satisfy Condition 2.2.17 for all natural numbers n. □

Example 2.3.39. We denote by $H_1\left(\Sigma^{-}_{g,0,1}, R\right)_{\mathfrak{UB}_2}$ the restriction of the functor induced by the functor $H_1\left(\Sigma^{-}_{g,0,1}, R\right)$ of Example 2.3.31 to the subcategory $\mathfrak{UB}_2^{g,0}$ of $\mathfrak{UM}_2^{g,0}$. We deduce from Lemma 2.2.38 that:

$$H_1\left(\Sigma^{-}_{g,0,1}, R\right)_{\mathfrak{B}_2} \cong \mathbf{LM}_{\left\{\pi_1\left(\Sigma^{-}_{g,0,1},p\right)^{\mathfrak{b}},\mathfrak{B}_2^{g,0},\mathfrak{B}_2,\varsigma_{-,1}\right\}}(R).$$

Remark 2.3.40. As for Example 2.3.31, since the morphisms $\left\{\varsigma^{\mathfrak{b}}_{n,1}\right\}_{n\in\mathbb{N}}$ are not trivial, the computation of $\mathbf{LM}_{\left\{\pi_1\left(\Sigma^{-}_{g,0,1},p\right)^{\mathfrak{b}},\mathfrak{M}_2^{g,0},\mathfrak{B}_2,\right.}$ on an object F of $\mathbf{Fct}\left(\mathfrak{UB}_2^{g,0}, R\text{-}\mathfrak{Mod}\right)$ is not a priori determined by $H_1\left(\Sigma^{-}_{g,0,1}, R\right)_{\mathfrak{B}_2}$ using Proposition 2.2.39. Hence, the iterates of the Long-Moody functor $\mathbf{LM}_{\left\{\pi_1\left(\Sigma^{-}_{g,0,1},p\right)^{\mathfrak{b}},\mathfrak{M}_2^{g,0},\mathfrak{B}_2,\varsigma_{n,1}\right\}}$ define new representations for surface braid groups. As far as the author knows, there are very few explicit examples of representations of surfaces braid groups for $g \geq 1$.

The case of braid groups: Assuming that $g = 0$ and that $H_n = \pi_1\left(\Sigma^{n}_{0,0,1}, p\right) \cong \mathbf{F}_n$, we recover the results of [Sou17b]. Indeed, in this case we consider the category $\mathfrak{UB}_2^{0,0} = \mathfrak{U}\boldsymbol{\beta}$, which is Quillen's bracket construction on the braid groupoid $\boldsymbol{\beta}$ (see Notation 2.0.1). The choice $\varsigma^{\mathfrak{b}}_{n,1} : \mathbf{F}_n \to \mathbf{B}_{n+1}$ of Definition 2.3.36 corresponds to the morphism introduced in [Sou17b, Example 2.7]:

$$\begin{array}{rcl} \varsigma_{n,1} : \mathbf{F}_n & \longrightarrow & \mathbf{B}_{n+1} \\ g_i & \longmapsto & \begin{cases} \sigma_1^2 & \text{if } i = 1 \\ \sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{i-1}^{-1} \circ \sigma_i^2 \circ \sigma_{i-1} \circ \cdots \circ \sigma_2 \circ \sigma_1 & \text{if } i \in \{2, \ldots, n\}. \end{cases} \end{array}$$

In Notation 2.3.8, we fixed the actions

$$a_{\Sigma^{n}_{0,0,1}} : \mathbf{B}_n \cong \boldsymbol{\Gamma}^{n}_{0,0,1} \to Aut_{\mathfrak{Gr}}\left(\pi_1\left(\Sigma^{n}_{0,0,1}, p\right)\right),$$

which correspond to Artin's representations for all natural numbers n. By [Sou17b, Section 2.3.1] we obtain:

Proposition 2.3.41. $\mathbf{LM}_{\left\{\pi_1\left(\Sigma^{-}_{0,0,1},p\right)^{\mathfrak{b}},\mathfrak{B}_2^{0,0},\mathfrak{B}_2,\varsigma^{\mathfrak{b}}_{-,1}\right\}} = \mathbf{LM}_1$ *where* $\mathbf{LM}_1$ *denotes the Long-Moody functor of [Sou17b, Section 2.3.1]. In particular, if* $R = \mathbb{C}\left[t^{\pm 1}\right]$*, by [Sou17b, Proposition 2.31] we have:*

$$t^{-1} \cdot \mathbf{LM}_{\left\{\mathcal{H}^{\mathfrak{b}}_{s},\mathfrak{B}_2^{0,0},\mathfrak{B}_2,\varsigma^{\mathfrak{b}}_{-,1}\right\}}\left(t \cdot \mathbb{C}\left[t^{\pm 1}\right]\right) \cong \mathfrak{Bur}_{t^2},$$

where $\mathfrak{Bur}_{t^2} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ *denotes the functor associated with the family of unreduced Burau representations with parameter* t^2 *(see [Sou17b, Section 1.2]).*

Remark 2.3.42. As pointed out in Remark 2.3.9 and in [Sou17b, Section 2.3.2], we could have chosen other actions $a_n : \mathbf{B}_n \to Aut\left(\mathbf{F}_n\right)$ and morphisms $\varsigma_n : \mathbf{F}_n \to \mathbf{B}_{n+1}$. For instance, the functor associated with the family of Tong-Yang-Ma representations (see [Sou17b, Section 1.2]) is recovered by the Long-Moody functor defined using a Wada representation other than the Artin representation (see [Sou17b, Section 2.3.2]).

This way, we can recover all the Long-Moody functors introduced in [Sou17b]. In addition, the new framework developped in the present paper recovers even more families of representations of braid groups that we could not obtain with the work of [Sou17b]. Let us give an example illustrating this fact.

Example 2.3.43. Let n be a natural number. Using the terminology of [Waj99], there is a classical geometric embedding $\mathscr{W}_n : \mathbf{B}_{2n+1} \hookrightarrow \boldsymbol{\Gamma}^{0}_{n,0,1}$ that sends the standard generators of the braid group to Dehn twists around a fixed system of meridians and parallels on the surface $\Sigma^{0}_{n,0,1}$ (we refer to [BT12, Section 1] for more details about this embedding). Let $\mathcal{W}$ be the subgroupoid of $\mathfrak{M}_2^{+,0}$ defined by the embeddings $\{\mathscr{W}_n\}_{n\in\mathbb{N}}$. We assign H to be the group $\pi_1\left(\Sigma^{0}_{n,0,1}, p\right)$ and H_0 to be the trivial group.

Hence, the functor $\pi_1\left(\Sigma^0_{-,0,1},p\right)$ of Section 2.3.3.1 provides a functor $\pi_1\left(\Sigma^0_{-,0,1},p\right)^{\mathfrak{b},2}:\mathfrak{UW}\to\mathfrak{UM}_2^{+,0}\to\mathfrak{Gr}$ by restriction so that Assumption 2.2.13 is satisfied. We consider $\varsigma_{n,t}:\pi_1\left(\Sigma^0_{n,0,1},p\right)\to\mathbf{B}_{n+1}$ the trivial morphism (see Example 2.2.22). According to Remark 2.2.27, $\left\{\pi_1\left(\Sigma^0_{-,0,1},p\right)^{\mathfrak{b},2},\mathcal{W},\mathfrak{M}_2^{+,0},\varsigma_{-,t}\right\}$ is a coherent Long-Moody system. Then, it is clear from Lemma 2.2.38 that:

$$H_1\left(\Sigma^0_{-,0,1},R\right)_{\mathfrak{UW}}\cong\mathbf{LM}_{\left\{\pi_1\left(\Sigma^0_{-,0,1},p\right)^{\mathfrak{b},2},\mathcal{W},\mathfrak{M}_2,\varsigma_{n,t}\right\}}(R)$$

where $H_1\left(\Sigma^0_{-,0,1},R\right)_{\mathfrak{UW}}$ denotes the restriction of the functor $H_1\left(\Sigma^0_{-,0,1},R\right)$ to the category $\mathfrak{UW}$. In [CS17], Callegaro and Salvetti compute the homology of braid groups with twisted coefficients given by the functor $H_1\left(\Sigma^0_{-,0,1},\mathbb{Z}\right)_{\mathfrak{UW}}$.

Remark 2.3.44. In [Sou17b], H_n is the free group on n generators $\mathbf{F}_n$. A fortiori, for dimensional considerations on the objects, there was no way to directly recover the functor of Example 2.3.43 applying a Long-Moody functor with this setting.

Remark 2.3.45. Assume that $g\geq 1$ and consider the presentation of surface braid groups $\left\{\mathbf{B}_n^g\right\}_{n\in\mathbb{N}}$ introduced by Bellingeri in [Bel04]. Following the situation for braid groups, we can consider functors of type

$$t^{-1}\cdot\mathbf{LM}_{\left\{\mathcal{H}_g^{\mathfrak{b}},\mathfrak{B}_2^{g,0},\varsigma_{-,1}\right\}}\left(t\cdot\mathbb{C}\left[t^{\pm1}\right]\right)$$

where, for an object F of $\mathbf{Fct}\left(\mathfrak{UB}_2^{g,0},R\text{-}\mathfrak{Mod}\right)$, $t\cdot F:\mathfrak{UB}_2^{g,0}\to\mathbb{C}\left[t^{\pm1}\right]\text{-}\mathfrak{Mod}$ is the functor defined for all natural numbers n by $t\cdot F(n)=F(n)$ and such that:

- for braid generators $\{\sigma_i\}_{i\in\{1,\dots,n-1\}}$ of $\mathbf{B}_n^g$, $(t\cdot F)(\sigma_i)=F(\sigma_i)$;
- for generators $\{a_i\}_{i\in\{1,\dots,g\}}$ and $\{b_i\}_{i\in\{1,\dots,g\}}$ of $\mathbf{B}_n^g$, $(t\cdot F)(a_i)=t\cdot F(a_i)$ and $(t\cdot F)(b_i)=t\cdot F(b_i)$.

These functors induce new representations of surface braid groups. The analogous approach for braid groups (see Proposition 2.3.41) allowed us to recover inter alia the unreduced Burau representations. Note that the restriction of the linear representations induced by $t^{-1}\cdot\mathbf{LM}_{\left\{\mathcal{H},\mathfrak{B}_2^{g,0},\varsigma_{n,1}\right\}}\left(t\cdot\mathbb{C}\left[t^{\pm1}\right]\right)$ (or by any iteration of such type of functors) do not restrict to the unreduced Burau or Lawrence-Krammer-Bigelow representations for braid groups by [BGG17, Section 5].

2.4 Strong and weak polynomial functors

This section introduces the notions of (very) strong and weak polynomial functors with respect to the framework of the present paper. The first subsection presents the notions of strong and very strong polynomial functors and their first properties. We thus extend [Sou18, Section 3] to a slightly larger framework. In the second subsection, we introduce the notion weak polynomial functors for pre-braided monoidal categories with an initial object and study their basic properties, generalising the previous notion of [DV17, Section 1].

2.4.1 Strong and very strong polynomial functors

For the remainder of Section 2.4.1, $(\mathfrak{M}',\natural,0)$ is a pre-braided strict monoidal small category where the unit 0 is an initial object. We consider $\mathfrak{M}$ a full subcategory of $(\mathfrak{M}',\natural,0)$. Finally, we fix $\mathcal{A}$ an abelian category.

In this section, we introduce the notions of strong and very strong polynomiality for objects in the functor category $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$. In [Sou17b, Section 3], a framework is given for defining the notions of strong and very strong polynomial functors in the category $\mathbf{Fct}(M,\mathcal{A})$, where M is a pre-braided monoidal category where the unit is an initial object. It generalises the previous work of Djament and Vespa in [DV17, Section 1]. We also refer to [Pal17] for a comparison of the various instances of the notions of twisted coefficient system and polynomial functor. This section thus extends the definitions and properties of [Sou17b, Section 3] to the present larger framework, the various proofs being direct generalisations of this previous work.

Notation 2.4.1. We denote by $Obj(\mathfrak{M}')_{\natural}$ the set of objects m' of $\mathfrak{M}'$ such that $m'\natural m$ is an object of $\mathfrak{M}$ for all objects m of $\mathfrak{M}$.

Let $m \in Obj(\mathfrak{M}')_{\natural}$. We denote by $\tau_m : \mathbf{Fct}(\mathfrak{M}, \mathcal{A}) \to \mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ the translation functor defined by $\tau_m(F) = F(m\natural -)$, $i_m : Id \to \tau_m$ the natural transformation of $\mathbf{Fct}(\mathfrak{M}, R\text{-}\mathfrak{Mod})$ induced by the unique morphism $\iota_m : 0 \to m$. We define $\delta_m = \operatorname{coker}(i_m)$ the difference functor and $\kappa_m = \ker(i_m)$ the evanescence functor. The following basic properties are direct generalisations of [Sou17b, Propositions 3.2 and 3.5]:

Proposition 2.4.2. *Let $m, m' \in Obj(\mathfrak{M}')_{\natural}$. Then the translation functor τ_m is exact and we have the following exact sequence of endofunctors of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:*

$$0 \longrightarrow \kappa_m \xrightarrow{\Omega_m} Id \xrightarrow{i_m} \tau_m \xrightarrow{\Delta_m} \delta_m \longrightarrow 0. \tag{2.4.1}$$

Moreover, for a short exact sequence $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, there is a natural exact sequence in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \kappa_m(F) \longrightarrow \kappa_m(G) \longrightarrow \kappa_m(H) \longrightarrow \delta_m(F) \longrightarrow \delta_m(G) \longrightarrow \delta_m(H) \longrightarrow 0. \tag{2.4.2}$$

In addition, the functors τ_m and $\tau_{m'}$ commute up to natural isomorphism and they commute with limits and colimits; the difference functors δ_m and $\delta_{m'}$ commute up to natural isomorphism and they commute with colimits; the functors κ_m and $\kappa_{m'}$ commute up to natural isomorphism and they commute with limits; the functor τ_m commute with the functors δ_m and κ_m up to natural isomorphism.

We can define the notions of strong and very strong polynomial functors using Proposition 2.4.2. Namely:

Definition 2.4.3. We recursively define on $d \in \mathbb{N}$ the categories $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ and $\mathcal{VP}ol_d(\mathfrak{M}, \mathcal{A})$ of strong and very strong polynomial functors of degree less than or equal to d to be the full subcategories of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ as follows:

1. If $d < 0$, $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A}) = \mathcal{VP}ol_d(\mathfrak{M}, \mathcal{A}) = \{0\}$;

2. if $d \geq 0$, the objects of $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ are the functors F such that for all $m \in Obj(\mathfrak{M}')_{\natural}$, the functor $\delta_m(F)$ is an object of $\mathcal{P}ol_{d-1}^{strong}(\mathfrak{M}, \mathcal{A})$; the objects of $\mathcal{VP}ol_d(\mathfrak{M}, \mathcal{A})$ are the objects F of $\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$ such that $\kappa_m(F) = 0$ and the functor $\delta_m(F)$ is an object of $\mathcal{VP}ol_{d-1}(\mathfrak{M}, \mathcal{A})$ for all $m \in Obj(\mathfrak{M}')_{\natural}$.

For an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ which is strong (respectively very strong) polynomial of degree less than or equal to $n \in \mathbb{N}$, the smallest natural number $d \leq n$ for which F is an object of $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ (resp. $\mathcal{VP}ol_d(\mathfrak{M}, \mathcal{A})$) is called the strong (resp. very strong) degree of F.

Finally, let us recall the following useful properties of the categories associated with strong and very strong polynomial functors. They are direct generalisations of [Sou17b, Propositions 3.9 and 3.19].

Proposition 2.4.4. *We assume that the category $\mathfrak{M}$ is finitely generated by the monoidal structure in $(\mathfrak{M}', \natural, 0)$. We denote by E a finite generating set of $\mathfrak{M}$.*

Let d be a natural number. The category $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ is closed under the translation functor, under quotient, under extension and under colimits. The category $\mathcal{VP}ol_d(\mathfrak{M}, \mathcal{A})$ is closed under the translation functors, under normal subobjects and under extension.

Moreover, an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ belongs to $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathcal{A})$ (resp. $\mathcal{VP}ol_d(\mathfrak{M}, \mathcal{A})$) if and only if $\delta_e(F)$ is an object of $\mathcal{P}ol_{d-1}^{strong}(\mathfrak{M}, \mathcal{A})$ (resp. $\kappa_e(F) = 0$ and $\delta_e(F)$ is an object of $\mathcal{VP}ol_{n-1}(\mathfrak{M}, \mathcal{A})$), for all objects e of $E \cap Obj(\mathfrak{M}')_{\natural}$.

2.4.2 Weak polynomial functors

We deal here with the concept of weak polynomial functor. It is introduced by Djament and Vespa in [DV17, Section 1] in the category $\mathbf{Fct}(S, A)$ where S is a symmetric monoidal category where the unit is an initial object, and A is a Grothendieck category. Weak polynomial functors form a thick subcategory of $\mathbf{Fct}(S, A)$ (see Definition 2.4.13 and Proposition 2.4.16). In particular, this notion happens to be more appropriate to study the stable behaviour for objects of the category $\mathbf{Fct}(S, A)$ (see [DV17, Section 5], [Dja17] and Remark 2.5.39).

We adapt the definition and properties of weak polynomial functors in the present larger setting. In particular, the notion of weak polynomial functor will be well-defined for the category $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ where $\mathfrak{U}\mathcal{G}$ is Quillen's bracket construction applied to the groupoid $\mathcal{G}$ given by a reliable Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$. We refer the reader to [Gab62, Chapitres II et III] for general notions on abelian categories and quotient abelian category which will be necessary for this section. A Grothendieck category is a cocomplete abelian category which admits a generator and such that direct limits are exact.

For the remainder of Section 2.4.2, we assume that the abelian category $\mathcal{A}$ is a Grothendieck category. We recall that we consider $(\mathfrak{M}', \natural, 0)$ a pre-braided strict monoidal small category where the unit 0 is an initial object and $\mathfrak{M}$ a full subcategory of $(\mathfrak{M}', \natural, 0)$ finitely generated by the monoidal structure.

Remark 2.4.5. We recall that therefore the functor category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is a Grothendieck category (see [Gab62]).

Definition 2.4.6. [DV17, Definition 1.10] Let F be an object of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. The subfunctor $\sum_{m \in Obj(\mathfrak{M}')_{\natural}} \kappa_m F$ of F is denoted by $\kappa(F)$. The functor F is said to be stably null if $\kappa(F) = F$. Stably null objects of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ form a full subcategory of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, denoted by $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$.

We have the following basic properties:

Lemma 2.4.7. *The functor κ is left exact. Moreover, the functor $\kappa(F)$ is an object of $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ for all objects F of* $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$.

Proof. A filtration on the evanescence functors $\{\kappa_m\}_{m \in Obj(\mathfrak{M}')_{\natural}}$ is given by the inclusions $\kappa_{n'} \hookrightarrow \kappa_{n'\natural n}$ and $\kappa_n \hookrightarrow \kappa_{n'\natural n}$ induced by $n \to n'\natural n$ and $n' \to n'\natural n$. Hence, κ is left exact as the filtered colimit of the left exact functors $\{\kappa_m\}_{m \in Obj(\mathfrak{M}')_{\natural}}$. For F an object of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, $\kappa_m F$ is an object of $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ for all $m \in Obj(\mathfrak{M}')_{\natural}$ since filtered colimit commute with finite limits (see [ML13, Chapter IX, section 2, Theorem 1]). Hence, the second result follows from the commutation of κ with filtered colimits since it is a filtered colimit (see [ML13, Chapter IX, section 2]). $\square$

The following property is an extension of the result [DV17, Corollary 1.15].

Proposition 2.4.8. *The category $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is a thick subcategory of* $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ *and it is closed under colimits.*

Proof. Recall that the functor κ commutes with filtered colimits (see Remark 2.4.7). Hence, the category $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is closed under filtered colimits. As $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is a Grothendieck category, the category $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is closed under colimits (see [ML13, Chapters V and IX]).

Let us prove that $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is a thick subcategory of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Let G be a subfunctor of F. As $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is a Grothendieck category, we denote by F/G the quotient. Hence, since κ is left exact, the following diagram (where the lines are exact and the vertical arrows are the inclusions) is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \kappa(G) & \longrightarrow & \kappa(F) & \longrightarrow & \kappa(F/G) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & F & \longrightarrow & F/G & \longrightarrow & 0. \end{array}$$

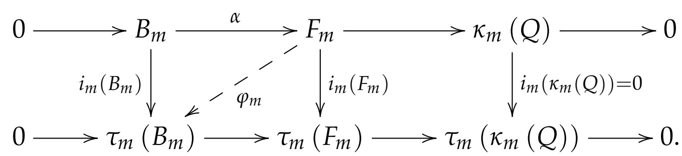
It follows from the five lemma that $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is closed under subobject.

Let $f : F \to Q \to 0$ be an epimorphism of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Consider the following commutative diagram where the vertical arrows are the inclusions:

$$\begin{array}{ccccc} \kappa(F) & \xrightarrow{\kappa(f)} & \kappa(Q) & & \\ \downarrow & & \downarrow & & \\ F & \xrightarrow{f} & Q & \longrightarrow & 0. \end{array}$$

Thus if $F \in Obj(\mathcal{S}n(\mathfrak{M}, \mathcal{A}))$, then the arrow $\kappa(Q) \hookrightarrow Q$ is an equality. Hence, $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is closed under quotient.

Finally, let $0 \to B \to F \to Q \to 0$ be a short exact sequence of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ with $B, Q \in Obj(\mathcal{S}n(\mathfrak{M}, \mathcal{A}))$. Let m be an object of $Obj(\mathfrak{M}')_{\natural}$. Let F_m be the pullback of the morphisms $F \twoheadrightarrow Q$ and $\kappa_m(Q) \hookrightarrow Q$: F is thus the filtered colimit (with respect to the inclusions) of the pullbacks $\{F_m\}_{m \in Obj(\mathfrak{M}')_{\natural}}$. Let B_m be the kernel of $F_m \twoheadrightarrow \kappa_m(Q)$. Recall that κ commutes with filtered colimits and that filtered colimits in $\mathcal{A}$ are exact (since it is a Grothendieck category). Hence, it is enough to prove that F_m is in $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ for all $m \in Obj(\mathfrak{M}')_{\natural}$ to show that $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is closed under extension. As κ_m is the kernel of a natural transformation between the identity functor and a left exact functor, $\kappa_m \circ \kappa_m$ is isomorphic to κ_m and therefore $i_m(\kappa_m(Q)) = 0$. By the universal property of the kernel, there exists a unique morphism φ_m such that the following diagram is commutative:



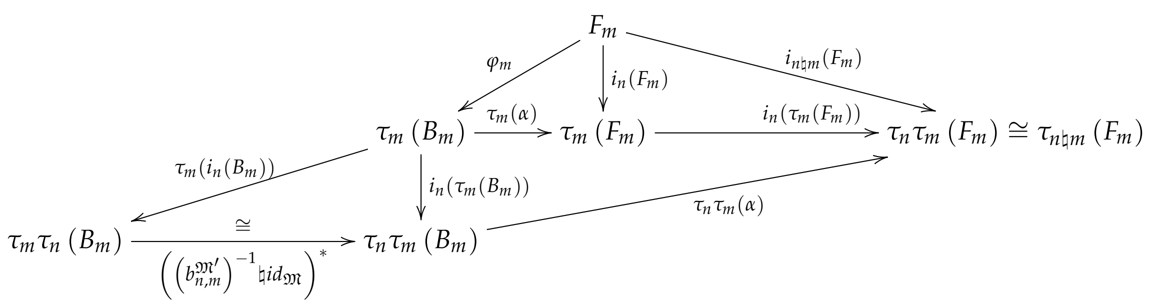
For all $n \in Obj(\mathfrak{M}')_{\natural}$, let $\varphi_m^{-1}(\tau_m(\kappa_n(B_m)))$ be the pullback of the morphisms $\varphi_m : F_m \to \tau_m(B_m)$ and $\tau_m(\kappa_n(B_m)) \hookrightarrow \tau_m(B_m)$. As a pullback commutes with a filtered colimit in an abelian category and since τ_m commutes with filtered colimits, we deduce that

$$\underset{n \in Obj(\mathfrak{M}')_{\natural}}{Colim} \left(\varphi_m^{-1}(\tau_m(\kappa_n(B_m)))\right) = F_m.$$

In addition, since $\mathfrak{M}'$ is pre-braided monoidal (see Definition 3.1.6), the precomposition by $\left(b_{n,m}^{\mathfrak{M}'}\right)^{-1} \natural id_{\mathfrak{M}}$ defines a natural isomorphism $\left(\left(b_{n,m}^{\mathfrak{M}'}\right)^{-1} \natural id_{\mathfrak{M}}\right)^* : \tau_m \circ \tau_n \xrightarrow{\sim} \tau_n \circ \tau_m$ for all $n \in Obj(\mathfrak{M}')_{\natural}$, such that

$$\left(\left(b_{n,m}^{\mathfrak{M}'}\right)^{-1} \natural id_{\mathfrak{M}}\right)^* \circ (\tau_m \circ i_n) = (i_n \circ \tau_m)$$

in the category of endofunctors of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Hence, the following diagram is commutative for all $n \in Obj(\mathfrak{M}')_{\natural}$:



We deduce from the previous commutative diagram and the universal property of the kernel that there exists an inclusion morphism $\varphi_m^{-1}(\tau_m(\kappa_n(B_m))) \hookrightarrow \kappa_{n\natural m}(F_m)$ for all $n \in Obj(\mathfrak{M}')_{\natural}$. Using the definition of κ as a filtered colimit (see Definition 2.4.6), we deduce that $\underset{n \in Obj(\mathfrak{M}')_{\natural}}{Colim}\left(\varphi_m^{-1}(\tau_m(\kappa_n(B_m)))\right)$ is a subobject of $\kappa(F_m)$. Hence, we have $\kappa(F_m) = F_m$ and thus $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ is closed under extension. □

Remark 2.4.9. We see here why we require the category $\mathcal{A}$ to have more properties than just being an abelian category: it is necessary for the proof of Proposition 2.4.8 to assume that the filtered colimits in the category $\mathcal{A}$ are exact, which is the case for a Grothendieck category.

The thickness property of Proposition 2.4.8 ensures that we can consider the quotient category of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ by $\mathcal{S}n(\mathfrak{M}, \mathcal{A})$ (see [Gab62, Chapter III]).

Definition 2.4.10. [DV17, Definition 1.16] Let $\mathbf{St}(\mathfrak{M}, \mathcal{A})$ be the quotient category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})/\mathcal{S}n(\mathfrak{M}, \mathcal{A})$. The canonical functor associated with this quotient is denoted by $\pi_{\mathfrak{M}} : \mathbf{Fct}(\mathfrak{M}, \mathcal{A}) \to \mathbf{Fct}(\mathfrak{M}, \mathcal{A})/\mathcal{S}n(\mathfrak{M}, \mathcal{A})$, the right adjoint functor of $\pi_{\mathfrak{M}}$ (see [Gab62, Section 3.1]) is denoted by $s_{\mathfrak{M}} : \mathbf{Fct}(\mathfrak{M}, \mathcal{A})/\mathcal{S}n(\mathfrak{M}, \mathcal{A}) \to \mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ and called the section functor.

Remark 2.4.11. The functor $\pi_{\mathfrak{M}}$ is exact, essentially surjective and commutes with all colimits (see [Gab62, Chapter 3]).

The following proposition introduces the induced translation and difference functors on the category $\mathbf{St}(\mathfrak{M},\mathcal{A})$. Its proof is analogous to that of [DV17, Proposition 1.19], using Proposition 2.4.2 which extends [DV17, Proposition 1.4].

Proposition 2.4.12. *[DV17, Proposition 1.19] Let $m \in Obj(\mathfrak{M}')_{\natural}$. The translation functor τ_m and the difference functor δ_m of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ respectively induce an exact endofunctor of $\mathbf{St}(\mathfrak{M},\mathcal{A})$ which commute with colimits, respectively again called the translation functor τ_m and the difference functor δ_m. In addition:*

1. *The following relations hold: $\delta_m \circ \pi_{\mathfrak{M}} = \pi_{\mathfrak{M}} \circ \delta_m$ and $\tau_m \circ \pi_{\mathfrak{M}} = \pi_{\mathfrak{M}} \circ \tau_m$.*
2. *The exact sequence (2.4.1) induces a short exact sequence of endofunctors of $\mathbf{St}(\mathfrak{M},\mathcal{A})$:*
$$0 \longrightarrow Id \xrightarrow{i_m} \tau_m \xrightarrow{\Delta_m} \delta_m \longrightarrow 0. \tag{2.4.3}$$
3. *For another object m' of $\mathfrak{M}$, the endofunctors δ_m, $\delta_{m'}$, τ_m and $\tau_{m'}$ of $\mathbf{St}(\mathfrak{M},\mathcal{A})$ pairwise commute up to natural isomorphism.*

We can now introduce the notion of a weak polynomial functor.

Definition 2.4.13. [DV17, Definition 1.22] We recursively define on $d \in \mathbb{N}$ the category $\mathcal{P}ol_d(\mathfrak{M},\mathcal{A})$ of polynomial functors of degree less than or equal to n to be the full subcategory of $\mathbf{St}(\mathfrak{M},\mathcal{A})$ as follows:

1. If $d<0$, $\mathcal{P}ol_d(\mathfrak{M},\mathcal{A}) = \{0\}$;
2. if $d \geq 0$, the objects of $\mathcal{P}ol_d(\mathfrak{M},\mathcal{A})$ are the functors F such that the functor $\delta_x(F)$ is an object of $\mathcal{P}ol_{d-1}(\mathfrak{M},\mathcal{A})$ for all $x \in Obj(\mathfrak{M}')_{\natural}$.

For an object F of $\mathbf{St}(\mathfrak{M},\mathcal{A})$ which is polynomial of degree less than or equal to $d \in \mathbb{N}$, the smallest natural number $n \leq d$ for which F is an object of $\mathcal{P}ol_d(\mathfrak{M},\mathcal{A})$ is called the degree of F. An object F of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ is weak polynomial of degree at most d if its image $\pi_{\mathfrak{M}}(F)$ is an object of $\mathcal{P}ol_d(\mathfrak{M},\mathcal{A})$. The degree of polynomiality of $\pi_{\mathfrak{M}}(F)$ is called the (weak) degree of F.

Remark 2.4.14. A strong polynomial functor of degree d is always weak polynomial of degree less than or equal to d by the first property of Proposition 2.4.12.

We conclude this subsection by giving some important properties of the categories of weak polynomial functors. Their proofs follow mutatis mutandis their analogues in [DV17, Section 1].

Proposition 2.4.15. *[DV17, Proposition 1.24] We assume that the category $\mathfrak{M}$ is finitely generated by the monoidal structure in $(\mathfrak{M}',\natural,0)$. We denote by E a finite generating set of $\mathfrak{M}$. Let F be an object of $\mathbf{St}(\mathfrak{M},\mathcal{A})$ and d be a natural number. Then, the functor F is an object of $\mathcal{P}ol_d(\mathfrak{M},\mathcal{A})$ if and only if the functor $\delta_e(F)$ is an object of $\mathcal{P}ol_{d-1}(\mathfrak{M},\mathcal{A})$ for all objects e of $E \cap Obj(\mathfrak{M}')_{\natural}$.*

Proposition 2.4.16. *[DV17, Proposition 1.25] Let d be a natural number. The subcategory $\mathcal{P}ol_d(\mathfrak{M},\mathcal{A})$ of $\mathbf{St}(\mathfrak{M},\mathcal{A})$ is thick and closed under limits and colimits.*

Proposition 2.4.17. *[DV17, Proposition 1.26] There is an equivalence of categories:*
$$\mathcal{A} \simeq \mathcal{P}ol_0(\mathfrak{M},\mathcal{A}).$$

Finally, if the category $(\mathfrak{M}',\natural,0)$ is symmetric monoidal as in [DV17], we have an equivalent definition of stably null functor of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$. Namely, following mutatis mutandis [DV17, Section 1.2] and the proof of [DV17, Proposition 1.13], we have:

Lemma 2.4.18. *We assume that the category $(\mathfrak{M}',\natural,0)$ is symmetric monoidal and that there exist two objects e and e' of $\mathfrak{M}'$ such that for all objects m of the category $\mathfrak{M}$, there exists a natural number n such that $m \cong e^{\natural n}\natural e'$. Then, an object F of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ is stably null if and only if $\underset{n\in(\mathbb{N},\leq)}{Colim}\left(F\left(e^{\natural n}\natural e'\right)\right) = 0$, where $(\mathbb{N},\leq)$ is considered as a subcategory of $\mathfrak{M}$ using the functor $(\mathbb{N},\leq) \to \mathfrak{M}$ sending a natural number n to $e^{\natural n}\natural e'$ and assigning $\iota_e \natural id_{e^{\natural n}\natural e'}$ to the unique morphism $\gamma_n : n \to n+1$.*

Remark 2.4.19. In some situations, this alternative definition is more convenient than the original one of Definition 2.4.6. This is the case for example for the proof of Lemma 2.5.33.

2.5 Behaviour of the generalised Long-Moody functors on polynomial functors

In this section, we study the effect of the generalised Long-Moody functors on (very) strong and weak polynomial functors. Indeed, they have the property to increase by one the degree of very strong and weak polynomiality, assuming that the groups H and H_0 are free (see Theorems 2.5.29 and 2.5.36). The five first subsections generalise [Sou17b, Part 4]. We will insist on the aspects which differ from this previous work. The last subsection gives new results on the effect of Long-Moody functors on weak polynomial functors.

Let $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$ be a coherent Long-Moody system (see Definition 2.2.26), which is fixed throughout this section. We consider the associated Long-Moody functor $\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_n\}}$ (see Theorem 2.2.30), which we fix for all the work of this section (in particular, we omit the "$\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_n\}$" from the notation most of the time).

Remark 2.5.1. If we consider $\mathfrak{M}$ to be the category $\mathfrak{U}\mathcal{G}$ associated with the coherent Long-Moody system $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$, it is enough to use the translation functor τ_1, as stated in Propositions 2.4.4 and 2.4.15. Indeed, the category $\mathfrak{U}\mathcal{G}$ is generated by 1 using the monoidal product $\natural$ (see Section 2.2.1.1).

2.5.1 Preliminaries

The observations of this first subsection are generalisations of [Sou17b, Section 4.1]. Recall the following crucial property of the augmentation ideal of a free product of groups.

Proposition 2.5.2. *[Coh72, Section 4, Lemma 4.3 and Theorem 4.7] Let G_1 and G_2 be groups. Then, there is a natural $R[G_1 * G_2]$-module isomorphism:*

$$\mathcal{I}_{R[G_1*G_2]} \cong \left(\mathcal{I}_{R[G_1]} \underset{R[G_1]}{\otimes} R[G_1 * G_2]\right) \oplus \left(\mathcal{I}_{R[G_2]} \underset{R[G_2]}{\otimes} R[G_1 * G_2]\right).$$

Remark 2.5.3. The augmentation ideal $\mathcal{I}_{R[G_1]}$ (respectively $\mathcal{I}_{R[G_2]}$) is a right $R[G_1]$-module (respectively $R[G_2]$-module). Moreover, the group ring $R[G_1 * G_2]$ is a left $R[G_1]$-module (respectively left $R[G_2]$-module) via the morphism $id_{G_1} * \iota_{G_2} : G_1 \to G_1 * G_2$ (respectively $\iota_{G_1} * id_{G_2} : G_2 \to G_1 * G_2$).

For F an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, recall that we introduced the augmentation ideal functor $\mathcal{I}$ in Definition 2.2.28. For all natural numbers n, by Proposition 2.5.2, we have a $R[H * H_n]$-module isomorphism:

$$\begin{aligned} &\mathcal{I}(\underline{n+1}) \underset{R[H_{1+n}]}{\otimes} F(\underline{n+2}) \\ &\cong \left(\left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} R[H_{1+n}]\right) \oplus \left(\mathcal{I}(\underline{n}) \underset{R[H_n]}{\otimes} R[H_{1+n}]\right)\right) \underset{R[H_{1+n}]}{\otimes} F(\underline{n+2}). \end{aligned}$$

Notation 2.5.4. Let n and n' be natural numbers such that $n' \geq n$. We denote by $\overline{\mathcal{I}}\left(\left[n'-n, id_{\underline{n'}}\right]\right) : \mathcal{I}_{R[H^{*n}]} \to \mathcal{I}(\underline{n'})$ the R-module morphism induced by the group morphism $id_{H^{*n}} * \iota_{H_{n'-n}} : H^{*n} \to H_{n'}$.

Recall that $t_G : G \to 0_{\mathfrak{Gr}}$ (see Notation 2.0.1) denotes the unique morphism from the group G to $0_{\mathfrak{Gr}}$. We denote by $\overline{\mathcal{I}}^{-1}\left(\left[n'-n, id_{\underline{n'}}\right]\right) : \mathcal{I}(\underline{n'}) \to \mathcal{I}_{R[H^{*n}]}$ the R-module morphism induced by the group morphism $id_{H^{*n}} * t_{H_{n'-n}} : H_{n'} \to H^{*n}$.

Remark 2.5.5. By Remark 2.5.3, the $R[H_{1+n}]$-module $F(\underline{n+2})$ is a $R[H]$-module via

$$F(\varsigma_{1+n} \circ (id_H * \iota_{H_n})) : H \to Aut_{R\text{-}\mathfrak{Mod}}(F(\underline{n+2}))$$

and $R[H_n]$-module via

$$F(\varsigma_{1+n} \circ (\iota_H * id_{H_n})) : H_n \to Aut_{R\text{-}\mathfrak{Mod}}(F(\underline{n+2})).$$

Then, the distributivity of the tensor product with respect to direct sum gives:

Lemma 2.5.6. *Let $F \in Obj(\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}))$ and n be a natural number. Then, we have the following R-module isomorphism:*

$$\tau_1 \mathbf{LM}(F)(n) \quad \cong \quad \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} F(\underline{n+2})\right) \oplus \left(\mathcal{I}(\underline{n}) \underset{R[H_n]}{\otimes} F(\underline{n+2})\right). \tag{2.5.1}$$

Definition 2.5.7. For all natural numbers n and $F \in Obj(\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}))$, we denote by

- $v(F)_{\underline{n}}$ the monomorphism of R-modules $\overline{\mathcal{I}}([n, id_{n+1}]) \underset{R[H_{1+n}]}{\otimes} id_{F(\underline{n+2})} : \mathcal{I}_{R[H]} \underset{R[H]}{\otimes} F(\underline{n+2}) \hookrightarrow \tau_1 \mathbf{LM}(F)(\underline{n})$,
- $\xi(F)_{\underline{n}}$ the monomorphism of R-modules $\mathcal{I}([1, id_{\underline{n+1}}]) \underset{R[H_{1+n}]}{\otimes} id_{F(\underline{n+2})} : \mathcal{I}(\underline{n}) \underset{R[H_n]}{\otimes} F(\underline{n+2}) \hookrightarrow \tau_1 \mathbf{LM}(F)(\underline{n})$,

associated with the direct sum of Lemma 2.5.6.

Similarly to [Sou17b, Section 4.1], this R-module decomposition will lead (under an additional assumption, see Section 2.5.3.1) to a decomposition of $\tau_1 \mathbf{LM}$ (see Corollary 2.5.25) as a functor.

2.5.2 Factorisation of the natural transformation $i_1 \mathbf{LM}$ by $\mathbf{LM}(i_1)$

Recall from Proposition 2.4.2 the exact sequence in the category of endofunctors of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, which defines the natural transformation i_1:

$$0 \longrightarrow \kappa_1 \xrightarrow{\Omega_1} Id \xrightarrow{i_1} \tau_1 \xrightarrow{\Delta_1} \delta_1 \longrightarrow 0. \tag{2.5.2}$$

Our objective is to study the cokernel of the natural transformation $i_1 \mathbf{LM} : \mathbf{LM} \to \tau_1 \circ \mathbf{LM}$. We recall that for F an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, for all natural numbers n, this is defined by the morphisms:

$$(i_1 \mathbf{LM})(F)_{\underline{n}} = \mathbf{LM}(F)(\iota_1 \natural id_{\underline{n}}) = \mathbf{LM}(F)([1, id_{\underline{1+n}}]) : \mathbf{LM}(F)(\underline{n}) \to \tau_1 \mathbf{LM}(F)(\underline{n}).$$

Since the generalised Long-Moody functor is right-exact (see Proposition 2.2.33), we have the following exact sequence:

$$\mathbf{LM} \xrightarrow{\mathbf{LM}(i_1)} \mathbf{LM} \circ \tau_1 \xrightarrow{\mathbf{LM}(\Delta_1)} \mathbf{LM} \circ \delta_1 \longrightarrow 0. \tag{2.5.3}$$

Remark 2.5.8. If the groups H_0 and H are free, since the generalised Long-Moody functor is then exact (see Corollary 2.2.35), the following sequence is exact:

$$0 \longrightarrow \mathbf{LM} \circ \kappa_1 \xrightarrow{\mathbf{LM}(\Omega_1)} \mathbf{LM} \xrightarrow{\mathbf{LM}(i_1)} \mathbf{LM} \circ \tau_1 \xrightarrow{\mathbf{LM}(\Delta_1)} \mathbf{LM} \circ \delta_1 \longrightarrow 0. \tag{2.5.4}$$

First of all, we impose an additional condition on the morphisms $\{\varsigma_n : H_n \to G_{n+1}\}_{n \in \mathbb{N}}$.

Condition 2.5.9. The group morphisms $\{\varsigma_n : H_n \to G_{n+1}\}_{n \in \mathbb{N}}$ of Condition 2.2.17 are such that for all elements $h \in H_n$, for all natural numbers n, the following equality holds in G_{n+2}:

$$\left(\left(b_{1,1}^{\mathcal{G}'}\right)^{-1} \natural id_{\underline{n}}\right) \circ (id_1 \natural \varsigma_n(h)) \quad = \quad \varsigma_{n+1}\left(\mathcal{H}\left([1, id_{\underline{n+1}}]\right)(h)\right) \circ \left(\left(b_{1,1}^{\mathcal{G}'}\right)^{-1} \natural id_{\underline{n}}\right).$$

Remark 2.5.10. As stated in Remark 2.2.18, Condition 2.5.9 implies Condition 2.2.17.

Remark 2.5.11. The family of trivial morphisms $\{\varsigma_{n,t} : H_n \to G_{n+1}\}_{n \in \mathbb{N}}$ satisfies Condition 2.2.17.

Henceforth, we assume that Condition 2.5.9 is satisfied by the coherent Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$.

We show in Lemma 2.5.13 that $i_1 \mathbf{LM}$ factors through $\mathbf{LM}(i_1)$. Beforehand, we remark that the R-modules $\left\{\mathcal{I}(\underline{n}) \underset{R[H_n]}{\otimes} F(\underline{n+2})\right\}_{n \in \mathbb{N}}$ in the decomposition of Proposition 2.5.6 assemble to form a functor which identifies with $\mathbf{LM}(\tau_1 F)$.

Proposition 2.5.12. *For all $F \in Obj\left(\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)\right)$, the monomorphisms $\{\xi(F)_n\}_{n\in\mathbb{N}}$ (see Definition 2.5.7), induce a natural transformation $\xi'(F) : (\mathbf{LM} \circ \tau_1)(F) \to (\tau_1 \circ \mathbf{LM})(F)$, assigning for all natural numbers n:*

$$\xi'(F)_{\underline{n}} = \left(id_{\mathcal{I}(\underline{1+n})} \underset{R[H_{1+n}]}{\otimes} F\left(\left(b^{\mathcal{G}'}_{1,1}\right)^{-1} \natural id_{\underline{n}}\right)\right) \circ \xi(F)_n .$$

This yields a natural transformation $\xi' : \mathbf{LM} \circ \tau_1 \to \tau_1 \circ \mathbf{LM}$.

Proof. Let n be a natural number. The fact that the assignment $\xi'(F)_n$ is well-defined with respect to the tensor product structures is a direct consequence of Condition 2.5.9 (see [Sou17b, Proposition 4.18]).

Let us show that $\xi'(F)$ is a natural transformation. Let n and n' be natural numbers such that $n' \geq n$, and $[n'-n, g] \in Hom_{\mathfrak{U}\mathcal{G}}(\underline{n}, \underline{n'})$. Since $\mathcal{I}$ is a functor and by the defining equivalence relation of $\mathfrak{U}\mathcal{G}'$ (see Definition 2.1.3) and since $\mathfrak{U}\mathcal{G}$ is a full subcategory of $\mathfrak{U}\mathcal{G}'$, we have:

$$\begin{aligned}
\mathcal{I}\left(id_1 \natural \left[n'-n, g\right]\right) \circ \mathcal{I}\left(\left[1, id_{\underline{n+1}}\right]\right) &= \mathcal{I}\left(\left[n'-n+1, (id_1 \natural g)\right]\right) \\
&= \mathcal{I}\left(\left[1, id_{\underline{n'+1}}\right]\right) \circ \mathcal{I}\left(\left[n'-n, g\right]\right).
\end{aligned}$$

So we deduce that:

$$\left((\tau_1 \circ \mathbf{LM})(F)\left(\left[n'-n, g\right]\right)\right) \circ \left(\xi'(F)_{\underline{n}}\right) = \left(\xi'(F)_{\underline{n'}}\right) \circ \left((\mathbf{LM} \circ \tau_1)(F)\left(\left[n'-n, g\right]\right)\right).$$

That ξ' is a natural transformation follows, mutatis mutandis, from the argument in the proof of [Sou17b, Proposition 4.18]. □

Now, using the natural transformation ξ', we can prove the desired following factorisation.

Lemma 2.5.13. *As natural transformations from $\mathbf{LM}$ to $\tau_1 \circ \mathbf{LM}$, the following equality holds:*

$$\xi' \circ (\mathbf{LM}(i_1)) = i_1 \mathbf{LM}.$$

Proof. Let F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. Let n be a natural number. Since $\left(b^{\mathcal{G}'}_{1,1}\right)^{-1} \circ (\iota_1 \natural id_1) = id_1 \natural \iota_1$ by Definition 2.1.10, we deduce from Proposition 2.5.12 that:

$$\left(\xi' \circ (\mathbf{LM}(i_1))\right)(F)_{\underline{n}} = \mathcal{I}\left(\left[1, id_{\underline{n+1}}\right]\right) \underset{R[H_{1+n}]}{\otimes} F\left(id_1 \natural \iota_1 \natural id_{\underline{n}}\right) = (i_1 \mathbf{LM})(F)_{\underline{n}}.$$

□

2.5.3 Study of Coker (ξ')

It follows from the definition of i_1 and from Lemma 2.5.13 that the following diagram is commutative and the rows are exact sequences in the category of endofunctors of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \kappa_1 \circ \mathbf{LM} & \xrightarrow{\Omega_1 \mathbf{LM}} & \mathbf{LM} & \xrightarrow{i_1 \mathbf{LM}} & \tau_1 \circ \mathbf{LM} & \xrightarrow{\Delta_1 \mathbf{LM}} & \delta_1 \circ \mathbf{LM} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \mathbf{LM}(i_1)} & & \| & & & & \\
 & & 0 & \longrightarrow & \mathbf{LM} \circ \tau_1 & \underset{\text{by Lemma 2.5.13}}{\overset{\xi'}{\hookrightarrow}} & \tau_1 \circ \mathbf{LM} & \longrightarrow & \mathrm{Coker}(\xi') & \longrightarrow & 0.
\end{array}$$

The universal property of the cokernel implies:

Proposition 2.5.14. *There exists a unique natural transformation $\mathbf{LM} \circ \delta_1 \to \delta_1 \circ \mathbf{LM}$ such that the following diagram is commutative and the rows are exact sequences in the category of endofunctors of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{LM} \circ \tau_1 & \xrightarrow{\xi'} & \tau_1 \circ \mathbf{LM} & \longrightarrow & Coker(\xi') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbf{LM} \circ \delta_1 & \longrightarrow & \delta_1 \circ \mathbf{LM} & \longrightarrow & Coker(\xi') & \longrightarrow & 0.
\end{array}$$

We denote by ϱ the natural transformation $\tau_1 \circ \mathbf{LM} \to \mathrm{Coker}(\xi')$ in the diagram of Proposition 2.5.14. By Lemma 2.5.6, it is clear that the R-modules $\mathrm{Coker}(\xi')(\underline{n})$ are isomorphic to the factor $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} F(\underline{n+2})$ of $\tau_1\mathbf{LM}(F)(\underline{n})$ for all natural numbers n. Recall the notation $\overline{\mathcal{I}}^{-1}\left(\left[n'-n, id_{\underline{1+n'}}\right]\right)$ introduced in Notation 2.5.4. It follows from Lemma 2.5.6 and Lemma 2.5.13 that for all $F \in Obj(\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}))$ and for all natural numbers n and n' such that $n' \geq n$:

$$\varrho(F)_{\underline{n}} = \overline{\mathcal{I}}^{-1}\left(\left[n, id_{\underline{n+1}}\right]\right) \underset{R[H_{1+n}]}{\otimes} id_{F(\underline{n+2})}.$$

This leads ineluctably to wonder if the decomposition of Lemma 2.5.6 is functorial. To prove this, we need a further assumption.

2.5.3.1 Additional assumption

Consider the functor $\mathcal{H} : \mathfrak{U}\mathcal{G}' \to \mathfrak{Gr}$ given by the coherent Long-Moody system. Recall the pre-braided monoidal category $(\mathfrak{U}\mathcal{G}', \natural, 0_{\mathcal{G}'})$ given by Assumption 2.2.1.

Notation 2.5.15. We denote by $\mathfrak{Gr}_{H,H_0}$ the full subcategory of $\mathfrak{Gr}$ of the finite free products on the objects $0_{\mathfrak{Gr}}$, H and H_0. The free product $*$ defines a symmetric strict monoidal product on $\mathfrak{Gr}_{H,H_0}$, with $0_{\mathfrak{Gr}}$ the unit. The symmetry of the monoidal structure is given by the canonical bijection $A * B \cong B * A$ which permutes the two terms of the free product, for A and B two objects of $\mathfrak{Gr}_{H,H_0}$.

Let $\mathcal{G}'_{(0,1)}$ be the full subgroupoid of $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$ of the finite monoidal products (ie using $\natural$) on the objects $0_{\mathcal{G}'}$, 0 and 1 of $\mathcal{G}'$. Note that the monoidal structure $\natural$ restricts to give a braided monoidal groupoid $\left(\mathcal{G}'_{(0,1)}, \natural, 0_{\mathcal{G}'}\right)$.

Under an additional assumption on the augmentation ideal functor $\mathcal{I}$, we have a enlightening description of the functor $\mathrm{Coker}(\xi')$. Namely, we assume:

Assumption 2.5.16. *The functor $\mathcal{H}$ of Assumption 2.2.13 defines a braided strict monoidal functor $\mathcal{H} : \left(\mathcal{G}'_{(0,1)}, \natural, 0_{\mathcal{G}'}\right) \to (\mathfrak{Gr}_{H,H_0}, *, 0_{\mathfrak{Gr}})$.*

Lemma 2.5.17. *Assumption 2.5.16 implies in particular that for all natural numbers m and n, for all $g \in G_n$:*

- $\mathcal{H}(id_m \natural g) \circ (id_{H^{*m}} * \iota_{H_n}) = (id_{H^{*m}} * \iota_{H_n})$;
- $\mathcal{H}\left(b^{\mathcal{G}'}_{m,n}\right) = b^{\mathfrak{Gr}_{H,H_0}}_{H^{*m},H^{*n}}$.

Proof. These relations are straightforward consequences of the definition of a braided strict monoidal functor. □

As the functor $\mathcal{I}$ is induced by $\mathcal{H}$ (see Definition 2.2.28) and the morphisms $\overline{\mathcal{I}}([n'-n, id_{n'}])$ are induced by the morphisms $id_{H^{*n}} * \iota_{H_{n'-n}} : H^{*n} \to H_{n'}$ (see Notation 2.5.4), we deduce from Assumption 2.5.16 the following relations, used in the proof of Proposition 2.5.23:

Corollary 2.5.18. *For all n and n' be natural numbers such that $n' \geq n$, for all $g \in G_n$:*

- $\mathcal{I}(id_{n'-n} \natural g) \circ \overline{\mathcal{I}}([n'-n, id_{\underline{n'}}]) = \overline{\mathcal{I}}([n'-n, id_{\underline{n'}}])$;
- $\mathcal{I}\left(\left(b^{\mathcal{G}'}_{1,n'-n}\right)^{-1} \natural id_{\underline{n}}\right) \circ \left(\mathcal{I}\left(\left[n'-n, id_{\underline{n'+1}}\right]\right) \circ \overline{\mathcal{I}}\left(\left[n, id_{\underline{n+1}}\right]\right)\right) = \overline{\mathcal{I}}\left(\left[n, id_{\underline{n+1}}\right]\right)$.

Remark 2.5.19. The relations of Corollary 2.5.18 will be used to prove Proposition 2.5.23.

Remark 2.5.20. When $(\mathfrak{U}\mathcal{G}', \natural, 0_{\mathcal{G}'}) = (\mathfrak{U}\mathcal{G}, \natural, 0_{\mathcal{G}}) = (\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$, Lemma 2.5.17 shows that Assumption 2.5.16 implies assuming [Sou17b, Condition 4.8] and [Sou17b, Condition 4.6].

Definition 2.5.21. A coherent Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ is said to be reliable if it satisfies Condition 2.5.9 and Assumption 2.5.16.

From now until the end of Section 2.5, we assume that the fixed coherent Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ is reliable.

2.5.3.2 Identification with a translation functor

Now, we can prove that the isomorphism $\mathrm{Coker}\left(\xi'\right)\left(\underline{n}\right)\cong\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\left(\tau_{2}F\right)\left(\underline{n}\right)$ is functorial.

Lemma 2.5.22. *For F an object of $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right)$, the R-modules $\left\{ \mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\left(\tau_{2}F\right)\left(\underline{n}\right)\right\} _{n\in\mathbb{N}}$ assemble to form a functor $\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\left(\tau_{2}F\right):\mathfrak{U}\mathcal{G}\rightarrow R\text{-}\mathfrak{Mod}$. Assigning $id_{\mathcal{I}_{R[H]}}\underset{R[H]}{\otimes}\tau_{2}\left(\eta\right)$ for any natural transformation η of $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right)$, we define an endofunctor:*

$$\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\tau_{2}:\mathbf{Fct}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right)\rightarrow\mathbf{Fct}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right).$$

Proof. The result is clear from the functoriality of F. □

Proposition 2.5.23. *Let F be an object of $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right)$. The monomorphisms $\left\{ \upsilon\left(F\right)_{n}\right\} _{n\in\mathbb{N}}$ (see Definition 2.5.7) define a natural transformation $\upsilon\left(F\right):\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\left(\tau_{2}F\right)\rightarrow\left(\tau_{1}\circ\mathbf{LM}\right)\left(F\right)$. This yields a natural transformation $\upsilon:\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\tau_{2}\rightarrow\tau_{1}\circ\mathbf{LM}$.*

Proof. This generalises [Sou17b, Proposition 4.14]; we give the key points for the convenience of the reader.Let n and n' be natural numbers such that $n'\geq n$, $\left[n'-n,g\right]\in Hom_{\mathfrak{U}\mathcal{G}}\left(\underline{n},\underline{n'}\right)$ and $h\in H$. Note that, by Lemma 2.5.17, as morphisms $H\rightarrow H_{2+n'}$:

$$\begin{aligned}\mathcal{H}\left(\left(b_{2,n'-n}^{\mathcal{G}'}\right)^{-1}\natural id_{\underline{n}}\right)\circ\left(\iota_{H^{n'-n}}*id_{1}*\iota_{H_{1+n}}\right) & = \mathcal{H}\left(id_{1}\natural\left(\left(b_{1,n'-n}^{\mathcal{G}'}\right)^{-1}\natural id_{\underline{n}}\right)\right)\left(id_{1}*\iota_{H_{1+n'}}\right)\\ & = id_{1}*\iota_{H_{1+n'}}.\end{aligned}$$

Hence, we deduce that:

$$\varsigma_{1+n'}\left(\left(id_{H}*\iota_{H_{n'-n}}\right)\left(h\right)\right)=\varsigma_{1+n'}\left(\mathcal{H}\left(\left(b_{2,n'-n}^{\mathcal{G}'}\right)^{-1}\natural id_{\underline{n}}\right)\left(\iota_{H^{n'-n}}*id_{1}*\iota_{H_{1+n}}\right)\left(h\right)\right).$$

Then, it follows from Conditions 2.5.9 and 2.2.24 that as morphisms in $\mathfrak{U}\mathcal{G}$:

$$\begin{aligned} & \left[n'-n,\varsigma_{1+n'}\left(\left(id_{H}*\iota_{H_{n'-n}}\right)\left(h\right)\right)\circ\left(\left(b_{2,n'-n}^{\mathcal{G}'}\right)^{-1}\natural id_{\underline{n}}\right)\right]\\ = & \left[n'-n,\left(\left(b_{2,n'-n}^{\mathcal{G}'}\right)^{-1}\natural id_{\underline{n}}\right)\circ\left(id_{n'-n}\natural\varsigma_{1+n}\left(\left(id_{H}*\iota_{H_{n}}\right)\left(h\right)\right)\right)\right].\end{aligned}\tag{2.5.5}$$

Since by Condition 2.2.24

$$\left(id_{2}\natural g\right)\circ\varsigma_{1+n'}\left(\left(id_{H}*\iota_{H_{n'-n}}\right)\left(h\right)\right)=\varsigma_{1+n'}\left(\mathcal{H}\left(id_{2}\natural g\right)\left(id_{H}*\iota_{H_{n'-n}}\right)\left(h\right)\right)\circ\left(id_{2}\natural g\right),$$

it follows from the first relation of Lemma 2.5.17 that

$$\left(id_{2}\natural g\right)\circ\varsigma_{1+n'}\left(\left(id_{H}*\iota_{H_{n'-n}}\right)\left(h\right)\right)=\varsigma_{1+n'}\left(\left(id_{H}*\iota_{H_{n'-n}}\right)\left(h\right)\right)\circ\left(id_{2}\natural g\right).\tag{2.5.6}$$

Hence, it follows from the combination of the relations (2.5.5) and (2.5.6) that:

$$\begin{aligned} & \left[n'-n,\left(id_{2}\natural g\right)\circ\left(\left(b_{2,n'-n}^{\mathcal{G}'}\right)^{-1}\natural id_{\underline{n}}\right)\circ\left(id_{n'-n}\natural\varsigma_{1+n}\left(\left(id_{H^{*n}}*\iota_{H_{n'-n}}\right)\left(h\right)\right)\right)\right]\\ = & \left[n'-n,\varsigma_{1+n'}\left(\left(id_{H^{*n}}*\iota_{H_{n'-n}}\right)\left(h\right)\right)\circ\left(id_{2}\natural g\right)\circ\left(\left(b_{2,n'-n}^{\mathcal{G}'}\right)^{-1}\natural id_{\underline{n}}\right)\right].\end{aligned}$$

A fortiori, $F\left(id_{2}\natural\left[n'-n,g\right]\right)\circ F\left(\varsigma_{1+n}\left(h\right)\right)=F\left(\varsigma_{1+n'}\left(h\right)\right)\circ F\left(id_{2}\natural\left[n'-n,g\right]\right)$. Hence, $\upsilon\left(F\right)$ is well defined with respect to the tensor product.

To prove that $v(F)$ is a natural transformation, remark that the relations of Corollary 2.5.18 imply that:

$$\mathcal{I}\left(id_1\natural g\right)\circ\mathcal{I}\left(\left(b^{\mathcal{G}'}_{1,n'-n}\right)^{-1}\natural id_{\underline{n}}\right)\circ\left(\mathcal{I}\left(\left[n'-n,id_{\underline{n'+1}}\right]\right)\circ\overline{\mathcal{I}}\left(\left[n,id_{\underline{n+1}}\right]\right)\right)=\overline{\mathcal{I}}\left(\left[n,id_{\underline{n+1}}\right]\right).$$

We then deduce from the definition of the generalised Long-Moody functor (see Theorem 2.2.30) that:

$$\left(\tau_1\mathbf{LM}\left(F\right)\left(\left[n'-n,g\right]\right)\circ v\left(F\right)_n\right) \quad = \quad v\left(F\right)_{n'}\circ\left(\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\tau_2\right)\left(F\right)\left(\left[n'-n,g\right]\right).$$

The proof that v is a natural transformation follows mutatis mutandis that of [Sou17b, Proposition 4.15]. □

Remark 2.5.24. Since it follows from Notation 2.5.4 that $\overline{\mathcal{I}}^{-1}\left(\left[n,id_{\underline{n+1}}\right]\right)\circ\overline{\mathcal{I}}\left(\left[n,id_{\underline{n+1}}\right]\right)=id_{\mathcal{I}_{R[H]}}$ for all natural numbers n, it is clear that $v:\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\tau_2\to\tau_1\circ\mathbf{LM}$ is a right inverse of the natural transformation $\varrho:\tau_1\circ\mathbf{LM}\to$ Coker$\left(\xi'\right)$.

Corollary 2.5.25. *For $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$ a reliable Long-Moody system, as endofunctors of* $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right)$*:*

$$Coker\left(\xi'\right)\cong\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\tau_2,$$

and there is a natural equivalence of endofunctors of $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right)$*:*

$$\tau_1\circ\mathbf{LM}\cong\left(\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\tau_2\right)\oplus\left(\mathbf{LM}\circ\tau_1\right). \tag{2.5.7}$$

Furthermore, if we assume that the groups H_0 and H are free, the isomorphisms $\Lambda_{rank(H),M}$ of Remark 2.2.36 provide a natural equivalence $\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\tau_2\cong\tau_2^{\oplus rank(H)}$.

2.5.4 Key relations with the difference and evanescence functors

This section presents the key relations of the generalised Long-Moody functors with the evanescence and difference functors. Lemma 2.5.13 and Corollary 2.5.25 lead to the following result.

Theorem 2.5.26. *Let $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$ be a reliable Long-Moody system. There is a natural equivalence in the category* $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right)$*:*

$$\delta_1\circ\mathbf{LM}\cong\left(\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\tau_2\right)\oplus\left(\mathbf{LM}\circ\delta_1\right). \tag{2.5.8}$$

Moreover, if we assume that the groups H_0 and H are free, then the evanescence endofunctor κ_1 commutes with the endofunctor $\mathbf{LM}$ *and the isomorphisms $\Lambda_{rank(H),M}$ of Remark 2.2.36 provide a natural equivalence:*

$$\delta_1\circ\mathbf{LM}\cong\tau_2^{\oplus rank(H)}\oplus\left(\mathbf{LM}\circ\delta_1\right). \tag{2.5.9}$$

Proof. This generalises [Sou17b, Theorems 4.23 and 4.24]. We denote by $i^{\oplus}_{\mathbf{LM}\circ\tau_1}$ the inclusion morphism $\mathbf{LM}\circ\tau_1\hookrightarrow\tau_2\oplus\left(\mathbf{LM}\circ\tau_1\right)$. Then, recalling the exact sequence (2.5.3), we obtain that the following diagram is commutative and that the two rows are exact:

$$\begin{array}{ccccccc}
\mathbf{LM} & \xrightarrow{i_1\circ\mathbf{LM}} & \tau_1\mathbf{LM} & \xrightarrow{\Delta_1\circ\mathbf{LM}} & \delta_1\circ\mathbf{LM} & \longrightarrow & 0\\
\Vert & & \uparrow{\scriptstyle v\oplus\xi'} \;\; {\scriptstyle\cong \text{ by Corollary 2.5.25}} & & & & \\
\mathbf{LM} & \xrightarrow[i^{\oplus}_{\mathbf{LM}\circ\tau_1}\circ(\mathbf{LM}\circ i_1)]{} & \left(\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\tau_2\right)\oplus\left(\mathbf{LM}\circ\tau_1\right) & \xrightarrow[id_{\tau_2}\oplus(\mathbf{LM}\circ\Delta_1)]{} & \left(\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\tau_2\right)\oplus\left(\mathbf{LM}\circ\delta_1\right) & \longrightarrow & 0.
\end{array}$$

A fortiori, by the universal property of the cokernel and 5-lemma, we deduce that $\tau_2 \oplus (\mathbf{LM} \circ \delta_1) \cong \delta_1 \circ \mathbf{LM}$.

Furthermore, assuming that the groups H_0 and H are free, so that we have the exact sequence (2.5.4) of Remark 2.5.8, we obtain the following commutative diagram, in which the two rows are exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \kappa_1 \circ \mathbf{LM} & \xrightarrow{\Omega_1 \mathbf{LM}} & \mathbf{LM} & \xrightarrow{i_1 \circ \mathbf{LM}} & \tau_1 \mathbf{LM} \\
 & & & & \Vert & & \uparrow{\scriptstyle \cong \, v \oplus \xi'} \\
0 & \longrightarrow & \mathbf{LM} \circ \kappa_1 & \xrightarrow[\mathbf{LM}(\Omega_1)]{} & \mathbf{LM} & \xrightarrow[i^{\oplus}_{\mathbf{LM}\circ\tau_1} \circ (\mathbf{LM} \circ i_1)]{} & \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 \right) \oplus (\mathbf{LM} \circ \tau_1) .
\end{array}$$

By the universal property of of the kernel, we conclude that $\kappa_1 \circ \mathbf{LM} \cong \mathbf{LM} \circ \kappa_1$. □

Remark 2.5.27. Let $m \geq 1$ be a natural number. Assume that the groups H_0 and H are free. Repeating mutatis mutandis the work of Sections 2.5.1, 2.5.2 and 2.5.3.1, we prove that:

$$\tau_m \circ \mathbf{LM} \cong \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_{m+1} \right) \oplus (\mathbf{LM} \circ \tau_m) .$$

Then, following the proof of Theorem 2.5.26, it follows from the exactness of the Long-Moody functor (see Corollary 2.2.35) that the evanescence endofunctor κ_m commutes with the Long-Moody functor.

2.5.5 Effect on strong polynomial functors

Here, we focus on the behaviour of the generalised Long-Moody functor on (very) strong polynomial functors, recovering the results of [Sou17b, Section 4.3]. Beforehand, remark the following property.

Lemma 2.5.28. *The functor $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2$ commutes with the difference functor δ_1. Moreover, if H is free, then $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 (F)$ commutes with the evanescence functor κ_m for all natural numbers $m \geq 1$.*

Proof. The commutation result with the difference functor δ_1 is a consequence of the right-exactness of the functor $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} - : R\text{-}\mathfrak{Mod} \to R\text{-}\mathfrak{Mod}$, of the exactness and the commutation property of the translation functor τ_2 (see Proposition 2.4.2). Assuming that the group H is free, the functor $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} - : R\text{-}\mathfrak{Mod} \to R\text{-}\mathfrak{Mod}$ is exact (as a consequence of Lemma 2.2.34). Hence, the claim follows from the commutation of the evanescence functor κ_m with the translation functor τ_2 (see Proposition 2.4.2). □

Theorem 2.5.29. *Let d be a natural number and F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. Recall that we consider a reliable Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$. If the functor F is strong polynomial of degree d, then:*

- *the functor $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 (F)$ belongs to $\mathcal{P}ol^{strong}_{d} (\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$;*
- *the functor $\mathbf{LM}(F)$ belongs to $\mathcal{P}ol^{strong}_{d+1} (\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$.*

Moreover, if the groups H_0 and H are free and F is very strong polynomial of degree d, then the functor $\mathbf{LM}(F)$ is a very strong polynomial functor of degree equal to $d+1$.

Proof. This generalises [Sou17b, Proposition 4.25, Theorems 4.28]. By induction on the polynomial degree, the result on $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 (F)$ follows from Lemma 2.5.28 and we deduce the first result on $\mathbf{LM}(F)$ using the relation (2.5.8) of Theorem 2.5.26.

Assume now that the groups H_0 and H are free groups. Recall that H is non-trivial. For a very strong polynomial functor F of degree d, an easy induction on the polynomial degree proves that $\tau_2^{rank(H)}(F)$ is very strong polynomial of degree d. A fortiori, the result follows from the relation (2.5.9) of Theorem 2.5.26. □

Applications:

Proposition 2.5.30. *The coherent Long-Moody systems $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$ of Sections 2.3.3, 2.3.4 and 2.3.5 are reliable (see Definition 2.5.21).*

Proof. Recall from Lemma 2.3.11 that the functor $\pi_1(-,p)$ is strict monoidal, it is clear that the symmetry $b^{\mathfrak{gr}}_{\pi_1(S,p),\pi_1(S',p)}$ is equal to $\pi_1\left(b^{\mathfrak{M}_2}_{S,S'},p\right)$. Hence the functor $\pi_1(-,p)$ is braided strict monoidal and a fortiori Assumption 2.5.16 is satisfied.

For the families of trivial $\{\varsigma_{n,t}\}_{n\in\mathbb{N}}$, we have already noted in Remark 2.5.11 that Condition 2.5.9 is automatically satisfied. For the family of morphisms $\{\varsigma_{n,1}\}_{n\in\mathbb{N}}$ and $\left\{\varsigma^{\mathfrak{b}}_{n,1}\right\}_{n\in\mathbb{N}}$, the equality (2.3.3) of Lemma 2.3.29 implies that Condition 2.5.9 is satisfied. □

Hence, applying a Long-Moody functor on the constant functor R, we prove:

Corollary 2.5.31. *The following functors are very strong polynomial of degree one:*

- $H_1\left(\Sigma^{s}_{-,0,1},R\right)$ *of Example 2.3.15;*
- $H_1\left(\Sigma^{s}_{0,-,1},R\right)$ *of Example 2.3.19;*
- $H_1\left(\Sigma^{-}_{g,c,1},R\right)$ *of Example 2.3.31;*
- $H_1\left(\Sigma^{0}_{-,0,1},R\right)_{\mathfrak{U}\mathcal{W}}$ *of Example 2.3.43;*
- $H_1\left(\Sigma^{-}_{g,c,1},R\right)_{\mathfrak{U}\mathfrak{B}_2}$ *of Example 2.3.39.*

In [RWW17, Section 5], Randal-Williams and Wahl prove homological stability for the families of mapping class groups of surfaces families considered in Section 2.3, with twisted coefficients given by very strong polynomial functors. This framework is generalised by Krannich to a topological setting in [Kra17]. Namely, for the coherent Long-Moody systems $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$ introduced in the examples of Section 2.3, they show:

Theorem 2.5.32. *[RWW17, Theorem A] If $F:\mathfrak{U}\mathcal{G}\to\mathbb{Z}\text{-}\mathfrak{Mod}$ is a very strong polynomial functor of degree d, then the canonical maps*

$$H_*\left(G_n,F\left(\underline{n}\right)\right)\to H_*\left(G_{n+1},F\left(\underline{n+1}\right)\right)$$

are isomorphisms for $N(*,r)\leq n$ with $N(*,r)\in\mathbb{N}$ depending on $*$ and r.

As representation theory of mapping class groups of surfaces is difficult and a current important research topic (see for example [BB05, Section 4.6], [Fun99] or [Kor02]), there are very few examples of very strong polynomial functors over $\mathfrak{U}\mathcal{G}$. Using Long-Moody functors (and in particular their iterates), we thus provide very strong polynomial functors in any degree for these families of groups.

2.5.6 Effect on weak polynomial functors

We investigate the effect on weak polynomial functors of the Long-Moody functor associated with the reliable Long-Moody system $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$. The first step of this study consists in defining the Long-Moody functor on the quotient category $\mathbf{St}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right)$. First, note the following property.

Lemma 2.5.33. *Let F be an object of $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right)$. Assume that the groups H_0 and H are free, or that the groupoid $(\mathcal{G}',\natural,0)$ is symmetric monoidal. If the functor F is in $\mathcal{S}n\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right)$, then the functors $\mathbf{LM}(F)$ and $\mathcal{I}_{R[H]}\underset{R[H]}{\otimes}\tau_2(F)$ are in $\mathcal{S}n\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right)$.*

Proof. Assume that H_0 and H are free. Recall from Remark 2.5.27 and Lemma 2.5.28 that the endofunctors **LM** and $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2$ commute with the evanescence functor κ_m for all natural numbers $m \geq 1$. It follows from Proposition 2.2.33 and the commutation with all colimits of $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} - : R\text{-}\mathfrak{Mod} \to R\text{-}\mathfrak{Mod}$ that if F is in $\mathcal{S}n\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$, then:

$$\kappa\left(\mathbf{LM}\left(F\right)\right) = \mathbf{LM}\left(\kappa\left(F\right)\right) = \mathbf{LM}\left(F\right) \text{ and } \kappa\left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} F\right) = \mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \kappa\left(F\right) = \mathcal{I}_{R[H]} \underset{R[H]}{\otimes} F.$$

If one of H_0 or H is not free, the hypothesis that $\mathcal{G}'$ is symmetric monoidal allows Lemma 2.4.18 to be applied. For all natural numbers n and n' such that $n' \geq n$, recall that $\mathbf{LM}\left(F\right)\left(\left[n'-n, id_{\underline{n'}}\right]\right)$ is the unique morphism induced by the universal property of the tensor product with respect to the map

$$\mathcal{I}_{R[H_n]} \times F\left(\underline{1+n}\right) \xrightarrow{\mathcal{I}\left(\left[n'-n, id_{\underline{n'}}\right]\right) \times F\left(id_1 \natural \left[n'-n, id_{\underline{n'}}\right]\right)} \mathcal{I}_{R[H_{n'}]} \times F\left(\underline{1+n'}\right) \xrightarrow{\underset{R[H_{n'}]}{\otimes}} \mathcal{I}_{R[H_{n'}]} \underset{R[H_{n'}]}{\otimes} F\left(\underline{1+n'}\right).$$

For a fixed natural number n, let $i \in \mathcal{I}_{R[H_n]}$ and let $x \in F\left(\underline{1+n}\right)$. We assume that F is in $\mathcal{S}n\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$. Since the translation functor τ_1 commutes with all the evanescence functors (see Proposition 2.4.2), $\tau_1 F$ is in $\mathcal{S}n\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$. Recall that by Lemma 2.4.18, $\underset{n\in(\mathbb{N},\leq)}{Colim}\left(\tau_1 F\left(\underline{n}\right)\right) = 0$, where $(\mathbb{N},\leq)$ is a subcategory of $\mathfrak{U}\mathcal{G}$ via the functor $\mathscr{O}$ of Definition 2.2.8. This is equivalent to the fact that for all natural numbers n, for all $x \in F\left(\underline{1+n}\right)$, there exists a natural number m_x such that $F\left(id_1 \natural \left[m_x - n, id_{\underline{m_x}}\right]\right)(x) = 0$ and a fortiori:

$$\mathbf{LM}\left(F\right)\left(\left[m_x - n, id_{\underline{m_x}}\right]\right)\left(i \underset{R[H_n]}{\otimes} x\right) = 0.$$

Hence, $\underset{n\in(\mathbb{N},\leq)}{Colim}\left(\mathbf{LM}\left(F\right)\left(\underline{n}\right)\right) = 0$. The result for $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2\left(F\right)$ follows using previous argument. □

From now until the end of Section 2.5.6, we assume that the groups H_0 and H are free, or that the groupoid $(\mathcal{G}', \natural, 0)$ is symmetric monoidal.

By Lemma 2.5.33, the endofunctors **LM** and $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2$ induce two functors on the quotient category $\mathbf{St}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$, denoted by

$$\mathbf{LM}_{\mathbf{St}} : \mathbf{St}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right) \to \mathbf{St}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right) \text{ and } \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2\right)_{\mathbf{St}} : \mathbf{St}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right) \to \mathbf{St}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right).$$

Remark 2.5.34. If H is a free group, the isomorphisms $\Lambda_{rank(H),M}$ of Remark 2.2.36 provide a natural equivalence:

$$\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 \cong \tau_2^{\oplus rank(H)}. \tag{2.5.10}$$

Thus, for F an object of $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$, if the functor $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2\left(F\right)$ is in $\mathcal{S}n\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$, then the functor F is in $\mathcal{S}n\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$. A fortiori, the induced functor $\left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2\right)_{\mathbf{St}}$ is equivalent to the functor $\tau_2^{\oplus rank(H)}$.

The behaviour of the Long-Moody functor of Theorem 2.5.26 and $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2$ of Lemma 2.5.28 with respect to the difference functor remain true for the induced functors in the category $\mathbf{St}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$.

Proposition 2.5.35. *Let F be an object of $\mathbf{St}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$. Then, as objects of $\mathbf{St}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$, there are natural equivalences:*

$$\delta_1 \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 \right)_{\mathbf{St}} (F) \quad \cong \quad \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 \right)_{\mathbf{St}} (\delta_1 F), \tag{2.5.11}$$

$$\delta_1 \mathbf{LM}_{\mathbf{St}} (F) \quad \cong \quad \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 \right)_{\mathbf{St}} (F) \oplus \mathbf{LM}_{\mathbf{St}} (\delta_1 F). \tag{2.5.12}$$

Proof. As a consequence of the definitions of the induced difference functor (see Proposition 2.4.12) and of the induced functors $\left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 \right)_{\mathbf{St}}$ and $\mathbf{LM}_{\mathbf{St}}$, we have natural equivalences:

$$\delta_1 \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 \right)_{\mathbf{St}} \cong \left(\delta_1 \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 \right) \right)_{\mathbf{St}} \quad \text{and} \quad \delta_1 \mathbf{LM}_{\mathbf{St}} \cong (\delta_1 \circ \mathbf{LM})_{\mathbf{St}}.$$

Hence, the result follows from Lemma 2.5.28 and Theorem 2.5.26. □

Theorem 2.5.36. *Let d be a natural number and F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. Assume that the groups H_0 and H are free, or that the groupoid $(\mathcal{G}', \natural, 0)$ is symmetric monoidal. Assume that F is weak polynomial of degree d. Then the functor $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 (F)$ is a weak polynomial functor of degree less than or equal to d and the functor $\mathbf{LM}(F)$ is a weak polynomial functor of degree less than or equal to $d+1$.*

Moreover, if H is free, then the functor $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 (F)$ is a weak polynomial functor of degree d and the functor $\mathbf{LM}(F)$ is a weak polynomial functor of degree $d+1$.

Proof. The first result for $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2$ is a direct consequence of the relation (2.5.11) of Proposition 2.5.35. If H is a free group, we proceed by induction on the degree of polynomiality of F. If F is weak polynomial of degree 0, then according to Proposition 2.4.17, there exists a constant functor C of $\mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ such that $\pi_{\mathfrak{U}\mathcal{G}}(F) \cong C$. By Remark 2.5.34, we have

$$\left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 \right)_{\mathbf{St}} (C) \cong C^{\oplus rank(H)}$$

which is a degree 0 weak polynomial functor. Now, assume that F is weak polynomial functor of degree $n \geq 0$. The result follows from the relation (2.5.11) of Proposition 2.5.35 and the inductive hypothesis.

For $\mathbf{LM}$, we also proceed by induction. Assume that F is a weak polynomial functor of degree 0. So $\pi_{\mathfrak{U}\mathcal{G}}(F)$ is a constant functor according to Proposition 2.4.17. By the equivalence (2.5.12) of Proposition 2.5.35, we have:

$$\delta_1 (\pi_{\mathfrak{U}\mathcal{G}} (\mathbf{LM}(F))) \cong \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 \right) (\pi_{\mathfrak{U}\mathcal{G}}(F)).$$

According to the result on $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2$, this is weak polynomial functor of degree less than or equal to 0, and if H is free the degree is exactly 0. Therefore, $\mathbf{LM}(F)$ is a weak polynomial functor of degree less than or equal to 1. Now, assume that F is a weak polynomial functor of degree $d \geq 1$. By the equivalence (2.5.12):

$$\delta_1 (\pi_{\mathfrak{U}\mathcal{G}} (\mathbf{LM}(F))) \cong \left(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 \right) (\pi_{\mathfrak{U}\mathcal{G}}(F)) \bigoplus \mathbf{LM}_{\mathbf{St}} (\delta_1 (\pi_{\mathfrak{U}\mathcal{G}}(F))).$$

The result follows from the inductive hypothesis and the result on $\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2$. □

Examples and applications: Examples of weak polynomial functors for mapping class groups of surfaces are given by Theorem 2.5.36. Indeed, the constant functor R being weak polynomial of degree 0 (according to Proposition 2.4.17 since $\pi_{\mathfrak{U}\mathcal{G}}(R) = R$), applying a Long-Moody functor to the constant functor R we obtain:

Proposition 2.5.37. *The following functors are weak polynomial of degree one:*

- $H_1\left(\Sigma^s_{-,0,1}, R\right)$ *of Example 2.3.15;*
- $H_1\left(\Sigma^s_{0,-,1}, R\right)$ *of Example 2.3.19;*
- $H_1\left(\Sigma^-_{g,c,1}, R\right)$ *of Example 2.3.31;*
- $H_1\left(\Sigma^0_{-,0,1}, R\right)_{\mathfrak{U}\mathcal{W}}$ *of Example 2.3.43;*
- $H_1\left(\Sigma^-_{g,c,1}, R\right)_{\mathfrak{U}\mathfrak{B}_2}$ *of Example 2.3.39.*

Recall from Remark 2.4.14 that a strong polynomial functor is always weak polynomial. The converse is false (see [DV17, Example 4.4] for a counterexample). The weak polynomial degree of a strong polynomial functor can be strictly smaller than its strong polynomial degree as the following example shows. Recall from [Sou17b, Section 1.3] the functor $\overline{\mathfrak{Bur}} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ which encodes the family of reduced Burau representations.

Proposition 2.5.38. *The functor* $\overline{\mathfrak{Bur}} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ *is a strong polynomial functor of degree* 2 *and weak polynomial of degree* 1.

Proof. The strong polynomial result is proved in [Sou17b, Proposition 3.28], using the following short exact sequence in $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}\right)$:

$$0 \longrightarrow \overline{\mathfrak{Bur}}_t \longrightarrow \tau_1\overline{\mathfrak{Bur}}_t \longrightarrow R_{\geq 1} \longrightarrow 0\,,$$

where $R_{\geq 1}$ is the subfunctor of R which is null at 0 and equal to R elsewhere. The functor $\pi_{\mathfrak{U}\mathcal{G}}$ being exact (see Remark 2.4.11), we deduce that:

$$\delta_1\left(\pi_{\mathfrak{U}\mathcal{G}}\left(\overline{\mathfrak{Bur}}_t\right)\right) \cong \pi_{\mathfrak{U}\mathcal{G}}\left(R_{\geq 1}\right).$$

The functor $R_{\geq 1}$ is a subfunctor of a weak polynomial functor of degree 0 and it is not stably null. So, we deduce from Proposition 2.4.16 that $R_{\geq 1}$ is weak polynomial of degree 0 and therefore the functor $\overline{\mathfrak{Bur}}_t$ is weak polynomial of degree 1. ☐

Remark 2.5.39. The fact that the reduced Burau functor is a strong polynomial functor of degree 2 is a consequence of an unstable phenomenon for the first values of this functor. Namely, this comes from the equivalence $\delta_1\overline{\mathfrak{Bur}}_t \cong R_{\geq 1}$ where $R_{\geq 1}$ is strong polynomial of degree 1 and note that however $R_{\geq 1}$ is constant for $n \geq 1$.

Another fundamental reason for the notion of of weak polynomial functors to be introduced in [DV17] is that, contrary to the category $\mathcal{P}ol^{strong}_d(\mathfrak{M}, \mathcal{A})$ (see [Sou17b, Remark 3.18]), the category $\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$ is localizing (see Proposition 2.4.16). This allows the quotient categories

$$\mathcal{P}ol_{d+1}(\mathfrak{M}, \mathcal{A})\,/\mathcal{P}ol_d(\mathfrak{M}, \mathcal{A})$$

to be considered. The main results of [DV17] concern the study of these quotient categories for $\mathfrak{M}$ the category of Hermitian objects in an additive category $\mathcal{C}$ equipped with a duality functor. Remark that as a consequence of Theorem 2.5.36, we obtain:

Proposition 2.5.40. *The Long-Moody functor defined by the reliable Long-Moody system* $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ *induces a functor:*

$$\mathcal{P}ol_d(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})\,/\mathcal{P}ol_{d-1}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \to \mathcal{P}ol_{d+1}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})\,/\mathcal{P}ol_d(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})\,,$$

if the groups H_0 *and* H *are free, or if the groupoid* $(\mathcal{G}', \natural, 0)$ *is symmetric monoidal.*

2.6 The case of symmetric monoidal categories

We fix $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma\}$ a reliable Long-Moody system (see Definition 2.5.21) throughout this section. For the work of this section, we make the following assumption.

Assumption 2.6.1. *We assume that the braided monoidal groupoid $(\mathcal{G}, \natural, 0_{\mathcal{G}})$ of Assumption 2.2.1.1 is symmetric monoidal.*

Remark 2.6.2. A fortiori, the pre-braided homogenous category $(\mathfrak{U}\mathcal{G}', \natural, 0_{\mathcal{G}'})$ is symmetric monoidal, using Proposition 2.1.12.

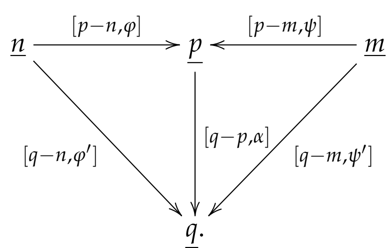
2.6.1 General constructions for symmetric monoidal categories

We present a category of generalised Cospan introduced in [Ves08, Ves06, Ves05], and inspired by the span category due to Bénabou in [Bén67]. The following definition is a direct extension of [Ves08, Definitions 2.5 and 2.6].

Definition 2.6.3. The category $\mathrm{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G})$ is the category which has the same objects as $\mathfrak{U}\mathcal{G}$ and for $n, m \in \mathbb{N}$, $Hom_{\mathrm{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G})}(\underline{n}, \underline{m})$ is the equivalence class of diagrams $\left\langle \underline{n} \xrightarrow{[p-n,\varphi]} \underline{p} \xleftarrow{[p-m,\psi]} \underline{m} \right\rangle$ where p is a natural number such that $p \geq n, m$, $[p-n,\varphi] \in Hom_{\mathfrak{U}\mathcal{G}}(\underline{n}, \underline{m})$ and $[p-m,\psi] \in Hom_{\mathfrak{U}\mathcal{G}}\left(\underline{m}, \underline{p}\right)$. The equivalence relation is the one generated on $Hom_{\mathrm{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G})}(\underline{n}, \underline{m})$ by the relation $\mathscr{R}$ defined by

$$\left\langle \underline{n} \xrightarrow{[p-n,\varphi]} \underline{p} \xleftarrow{[p-m,\psi]} \underline{m} \right\rangle \mathscr{R} \left\langle \underline{n} \xrightarrow{[q-n,\varphi']} \underline{q} \xleftarrow{[q-m,\psi']} \underline{m} \right\rangle$$

if and only if there exists a morphism $[q-p,\alpha] \in Hom_{\mathfrak{U}\mathcal{G}}\left(\underline{p}, \underline{q}\right)$ such that the following diagram commutes:



For all objects $n \in \mathbb{N}$, the identity morphism in the category $\mathrm{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G})$ is given by $\left\langle \underline{n} \xrightarrow{id_{\underline{n}}} \underline{n} \xleftarrow{id_{\underline{n}}} \underline{n} \right\rangle$.

Let $\varphi = \left\langle \underline{n} \xrightarrow{[p-n,\alpha]} \underline{p} \xleftarrow{[p-m,\beta]} \underline{m} \right\rangle$ and $\psi = \left\langle \underline{m} \xrightarrow{[k-m,\gamma]} \underline{k} \xleftarrow{[k-l,\delta]} \underline{l} \right\rangle$ be two morphisms in the category $\mathrm{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G})$. The composition in the category $\mathrm{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G})$ is defined by:

$$\psi \circ \varphi = \left\langle \underline{n} \xrightarrow{A} (k-m) \natural (p-m) \natural \underline{m} \xleftarrow{B} \underline{l} \right\rangle$$

where

$$A = \left[(k-m) + (p-n), id_{k-m} \natural \left(\beta^{-1} \circ \alpha \right) \right]$$

and

$$B = \left[(p-m) + (k-l), \left(\left(b^{\mathcal{G}'}_{k-m,p-m} \right)^{-1} \natural id_{\underline{m}} \right) \circ \left(id_{p-m} \natural \left(\gamma^{-1} \circ \delta \right) \right) \right].$$

Remark 2.6.4. The fact that the category is well-defined follows analogously to [Ves08, Lemme 2.7].

Notation 2.6.5. Let n and n' be natural numbers such that $n' \geq n$. In the category $\text{Cospan}^{\dagger}(\mathfrak{UG})$, a morphism of type $\left\langle \underline{n} \xrightarrow{[n'-n,\varphi]} \underline{n'} \xleftarrow{id_{\underline{n'}}} \underline{n'} \right\rangle$ is denoted by $\mathcal{L}([n'-n,\varphi])$ and a morphism of type $\left\langle \underline{n'} \xrightarrow{id_{\underline{n'}}} \underline{n'} \xleftarrow{[n'-n,\varphi]} \underline{n} \right\rangle$ is denoted by $\mathcal{R}([n'-n,\varphi])$, where $[n'-n,\varphi] \in Hom_{\mathfrak{UG}}(\underline{n},\underline{n'})$.

Remark 2.6.6. The morphisms of type $\mathcal{L}$ in notation 2.6.5 induce a canonical inclusion functor $incl_{\mathfrak{UG}}^{\text{Cospan}^{\dagger}(\mathfrak{UG})} : \mathfrak{UG} \hookrightarrow \text{Cospan}^{\dagger}(\mathfrak{UG})$ defined as the identity on objects and sending any $[n'-n,\varphi] \in Hom_{\mathfrak{UG}}(\underline{n},\underline{n'})$ to $\mathcal{L}([n'-n,\varphi])$. In the same way, the morphisms of type $\mathcal{R}$ in Notation 2.6.5 induce a canonical inclusion functor $incl_{\mathfrak{UG}^{op}}^{\text{Cospan}^{\dagger}(\mathfrak{UG})} : \mathfrak{UG}^{op} \hookrightarrow \text{Cospan}^{\dagger}(\mathfrak{UG})$ defined as the identity on objects and sending any $[n'-n,\varphi] \in Hom_{\mathfrak{UG}^{op}}(\underline{n'},\underline{n})$ to $\mathcal{R}([n'-n,\varphi])$.

The following property is a direct consequence from the composition in the category $\text{Cospan}^{\dagger}(\mathfrak{UG})$.

Lemma 2.6.7. *[Ves05, Proposition 2.2.10] Let p, n and m be natural numbers such that $p \geq n$ and $p \geq m$. Any morphisms $\left\langle \underline{n} \xrightarrow{[p-n,\varphi]} \underline{p} \xleftarrow{[p-m,\psi]} \underline{m} \right\rangle$ in the category $\text{Cospan}^{\dagger}(\mathfrak{UG})$ admits the following decomposition:*

$$\left\langle \underline{n} \xrightarrow{[p-n,\varphi]} \underline{p} \xleftarrow{[p-m,\psi]} \underline{m} \right\rangle = \mathcal{R}([p-m,\psi]) \circ \mathcal{L}([p-n,\varphi]).$$

Finally, we are interested in a way to extend an object of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ to an object of $\mathbf{Fct}\left(\text{Cospan}^{\dagger}(\mathfrak{UG}), R\text{-}\mathfrak{Mod}\right)$.

Proposition 2.6.8. *Let $\mathscr{C}$ be a category and M be an object of $\mathbf{Fct}(\mathfrak{UG},\mathscr{C})$. Assume that for all $n, n' \in \mathbb{N}$ such that $n' \geq n$, $M\left([n'-n, id_{\underline{n'}}]\right) : M(\underline{n}) \to M(\underline{n'})$ is left-invertible and we set $M\left(\mathcal{R}\left([n'-n, id_{\underline{n'}}]\right)\right)$ a left inverse.*

Then, assigning $M(\mathcal{R}([n'-n,\gamma])) = M\left(\mathcal{R}\left([n'-n, id_{\underline{n'}}]\right)\right) \circ M(\gamma)^{-1}$ (for all $[n'-n,\gamma] \in Hom_{\mathfrak{UG}^{op}}(\underline{n'},\underline{n})$) defines a functor $M : \text{Cospan}^{\dagger}(\mathfrak{UG}) \to \mathscr{C}$ if and only if for all natural numbers l,k,n,n',m such that $l \geq n$, $k \geq n$, $k \geq n'$ and $n \geq m$, for all $[k-n,\phi] \in Hom_{\mathfrak{UG}}(\underline{n},\underline{k})$, $[k-n',\psi] \in Hom_{\mathfrak{UG}}(\underline{n'},\underline{k})$, $[n-m,\chi] \in Hom_{\mathfrak{UG}}(\underline{m},\underline{n})$ and $[l-n,\omega] \in Hom_{\mathfrak{UG}}(\underline{n},\underline{l})$:

$$M\left(\mathcal{R}\left([k-n',\psi]\right) \circ \mathcal{L}([k-n,\phi])\right) = M\left(\mathcal{R}\left([k-n',\psi]\right)\right) \circ M\left(\mathcal{L}([k-n,\phi])\right), \tag{2.6.1}$$

$$M\left(\mathcal{R}([n-m,\chi]) \circ \mathcal{R}([k-n,\phi])\right) = M\left(\mathcal{R}([n-m,\chi])\right) \circ M\left(\mathcal{R}([k-n,\phi])\right), \tag{2.6.2}$$

$$M\left(\mathcal{L}([l-n,\omega]) \circ \mathcal{R}([k-n,\phi])\right) = M\left(\mathcal{L}([l-n,\omega])\right) \circ M\left(\mathcal{R}([k-n,\phi])\right). \tag{2.6.3}$$

Proof. First, we have to show that the assignment of M on morphisms is well-defined with respect to the defining equivalence relation of the category $\text{Cospan}^{\dagger}(\mathfrak{UG})$. Let $A = \mathcal{R}([p-m,\psi]) \circ \mathcal{L}([p-n,\varphi])$ and $B = \mathcal{R}([p'-m,\psi']) \circ \mathcal{L}([p'-n,\varphi'])$ be two morphisms such that $A\mathscr{R}B$. Denoting by $[p'-p,\alpha] \in Hom_{\mathfrak{UG}}\left(\underline{p},\underline{p'}\right)$ a morphism such that $A\mathscr{R}B$ (see Definition 2.6.3), since:

$$M\left(\mathcal{R}\left([p'-p,\alpha]\right)\right) \circ M\left(\mathcal{L}\left([p'-p,\alpha]\right)\right) = id_{M(\underline{p})},$$

we directly conclude from the compositions relations (2.6.1) and (2.6.3) that $M(B) = M(A)$.

Remark that since M is a functor over the category $\mathfrak{UG}$, the identity axiom and the composition axiom for morphisms of type $\mathcal{L}([k-n,\phi]) \circ \mathcal{L}([n-m,\chi])$ (see Notation 2.6.5) are already checked. Now, it is clear from the definition of morphisms in the category $\text{Cospan}^{\dagger}(\mathfrak{UG})$ that composition axiom for the category $\text{Cospan}^{\dagger}(\mathfrak{UG})$ is satisfied by M if and only if the three relations (2.6.1), (2.6.2) and (2.6.3) are checked. ☐

A criterion for extending natural transformations from $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ to $\mathbf{Fct}\left(\text{Cospan}^{\dagger}(\mathfrak{UG}), R\text{-}\mathfrak{Mod}\right)$ can similarly be given. As $\mathfrak{UG}$ is a subcategory of $\text{Cospan}^{\dagger}(\mathfrak{UG})$ by $incl_{\mathfrak{UG}}^{\text{Cospan}^{\dagger}(\mathfrak{UG})}$, an object of the functor category $\mathbf{Fct}\left(\text{Cospan}^{\dagger}(\mathfrak{UG}), R\text{-}\mathfrak{Mod}\right)$ is an object of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$. Abusing the notation, the restriction of an object of $\mathbf{Fct}\left(\text{Cospan}^{\dagger}(\mathfrak{UG}), R\text{-}\mathfrak{Mod}\right)$ to $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ is denoted in the same way.

Proposition 2.6.9. *Let $\mathscr{C}$ be a category, and M and M' be objects of* $\mathbf{Fct}\left(Cospan^{\dagger}(\mathfrak{U}\mathcal{G}),\mathscr{C}\right)$ *and $\eta : M \to M'$ a natural transformation in the category* $\mathbf{Fct}(\mathfrak{U}\mathcal{G},\mathscr{C})$*. Then, η is a natural transformation in the category* $\mathbf{Fct}\left(Cospan^{\dagger}(\mathfrak{U}\mathcal{G}),\mathscr{C}\right)$ *if and only if for all $n, n' \in \mathbb{N}$ such that $n' \geq n$:*

$$\eta_{\underline{n'}} \circ M\left(\mathcal{R}\left(\left[n'-n, id_{\underline{n'}}\right]\right)\right) = M'\left(\mathcal{R}\left(\left[n'-n, id_{\underline{n'}}\right]\right)\right) \circ \eta_{\underline{n}}. \tag{2.6.4}$$

Proof. The natural transformation η extends to the category $\mathbf{Fct}\left(\text{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G}),\mathscr{C}\right)$ if and only if for all $n, n' \in \mathbb{N}$ such that $n' \geq n$, for all $\mathcal{R}([p-m,\psi]) \circ \mathcal{L}([p-n,\varphi]) \in Hom_{\text{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G})}(\underline{n},\underline{n'})$:

$$M'\left(\mathcal{R}([p-m,\psi]) \circ \mathcal{L}([p-n,\varphi])\right) \circ \eta_{\underline{n}} = \eta_{\underline{n'}} \circ M\left(\mathcal{R}([p-m,\psi]) \circ \mathcal{L}([p-n,\varphi])\right).$$

Since η is a natural transformation in the category $\mathbf{Fct}(\mathfrak{U}\mathcal{G},\mathscr{C})$, we already have $\eta_{\underline{p}} \circ M(\mathcal{L}([p-n,\varphi])) = M'(\mathcal{L}([p-n,\varphi])) \circ \eta_{\underline{n'}}$. Hence, this implies that the necessary and sufficient relation to satisfy is relation (2.6.4). □

Equivalence with $\widetilde{-}$-construction: We denote by $\mathfrak{Mon}_{\mathrm{ini}}^{\mathfrak{symm}}$ (resp. $\mathfrak{Mon}_{\mathrm{null}}^{\mathfrak{symm}}$) the category of symmetric strict monoidal small categories $(\mathfrak{M},\natural,0)$ such that the unit 0 is an initial object (resp. a null object). We denote by $\widetilde{-} : \mathfrak{Mon}_{\mathrm{ini}}^{\mathfrak{symm}} \to \mathfrak{Mon}_{\mathrm{null}}^{\mathfrak{symm}}$ the left adjoint of the forgetful functor $\mathfrak{Mon}_{\mathrm{null}}^{\mathfrak{symm}} \hookrightarrow \mathfrak{Mon}_{\mathrm{ini}}^{\mathfrak{symm}}$, considered by Djament and Vespa in [DV17, Section 3]. This construction is notably used to classify weak polynomial objects of $\mathbf{Fct}(\mathfrak{M},\mathcal{A})$ for $\mathfrak{M} \in Obj(\mathfrak{Mon}_{\mathrm{ini}}^{\mathfrak{symm}})$ and $\mathcal{A}$ a Grothendieck category in [DV17, Theorem 3.8]. More precisely, for all natural numbers d, Djament and Vespa prove that

$$\mathcal{P}ol_d\left(\widetilde{\mathfrak{M}},\mathcal{A}\right)/\mathcal{P}ol_{d-1}\left(\widetilde{\mathfrak{M}},\mathcal{A}\right) \cong \mathcal{P}ol_d(\mathfrak{M},\mathcal{A})/\mathcal{P}ol_{d-1}(\mathfrak{M},\mathcal{A})$$

and for the category $\mathbf{H}(\mathcal{C})$ of Hermitian objects in an additive category $\mathcal{C}$ equipped with a duality functor $D : \mathcal{C}^{op} \to \mathcal{C}$, they prove that the forgetful functor $\widetilde{\mathbf{H}(\mathcal{C})} \to \mathcal{C}$ induces an equivalence

$$\mathcal{P}ol_d\left(\widetilde{\mathbf{H}(\mathcal{C})},\mathcal{A}\right) \cong \mathcal{P}ol_d(\mathcal{C},\mathcal{A}).$$

The functor $\widetilde{-}$ generalises constructions considered by other authors. For instance, let FI denote the category of finite sets and injections, which is equivalent to Quillen's construction $\mathfrak{U}\Sigma$ over the groupoid associated with symmetric groups (see Section 2.6.3.1). The category $\widetilde{FI}$ is equivalent to the category $FI\sharp$ considered by Church, Ellenberg and Farb in [CEF15], to study the projective objects of the category $\mathbf{Fct}(FI, R\text{-}\mathfrak{Mod})$.

We recall the explicit description of the functor $\widetilde{-}$ given in [DV17, Proposition 3.4].

Definition 2.6.10. [DV17, Section 3] Let $\mathfrak{M} \in Obj\left(\mathfrak{Mon}_{\mathrm{ini}}^{\mathfrak{symm}}\right)$. The category $\widetilde{\mathfrak{M}}$ is the category which has the same objects as $\mathfrak{M}$ and for all $m, m' \in Obj(\mathfrak{M})$, $Hom_{\widetilde{\mathfrak{M}}}(m,m') = \underset{\mathfrak{M}}{\mathrm{Colim}}\left(\tau_{m'} Hom_{\mathfrak{M}}(m,-)\right)$.

Remark 2.6.11. As stated in [DV17, Section 3], the symmetric strict monoidal structure $(\mathfrak{M},\natural,0)$ extends to a symmetric strict monoidal structure $\left(\widetilde{\mathfrak{M}},\natural,0\right)$, taking the colimit using the symmetry of the considered structure.

Notation 2.6.12. Considering the Long-Moody system $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$, we denote by $\widetilde{\mathfrak{U}\mathcal{G}}$ the full subcategory of $\widetilde{\mathfrak{U}\mathcal{G}'}$ on the objects of $\mathcal{G}$.

As suggested in [DV17, Exemple 3.3], the category $\widetilde{\mathfrak{U}\mathcal{G}}$ is equivalent to $\text{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G})$. Define the functor $\Xi : \widetilde{\mathfrak{U}\mathcal{G}} \to \text{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G})$ to be the identity on objects and sending any morphism $f \in Hom_{\widetilde{\mathfrak{U}\mathcal{G}}}(\underline{n},\underline{n'})$, represented by some $[p+n'-n,\varphi] \in Hom_{\mathfrak{U}\mathcal{G}}\left(\underline{n},\underline{p\natural n'}\right)$ to

$$\left\langle \underline{n} \xrightarrow{[p+n'-n,\varphi]} \underline{p\natural n'} \xleftarrow{\left[p, id_{\underline{p\natural n'}}\right]} \underline{n'} \right\rangle.$$

It is a direct consequence of the equivalence relations in $\text{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G})$ that this functor is well-defined.

Proposition 2.6.13. *The functor $\Xi : \widetilde{\mathfrak{U}\mathcal{G}} \to Cospan^{\dagger}(\mathfrak{U}\mathcal{G})$ is an equivalence of categories.*

Proof. We clearly define an inverse $\Xi^{-1} : \mathrm{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G}) \to \widetilde{\mathfrak{U}\mathcal{G}}$ assigning the identity on objects and sending any morphism of $\mathrm{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G})$

$$\left\langle \underline{n} \xrightarrow{[p-n,\varphi]} \underline{p} \xleftarrow{[p-m,\psi]} \underline{m} \right\rangle = \left\langle \underline{n} \xrightarrow{[p-n,\psi^{-1}\circ\varphi]} \underline{p} \xleftarrow{[p-m,id_{\underline{m}}]} \underline{m} \right\rangle$$

to the morphism of $Hom_{\widetilde{\mathfrak{U}\mathcal{G}}}(\underline{n},\underline{m})$ represented by $[p-n,\psi^{-1}\circ\varphi] \in Hom_{\mathfrak{U}\mathcal{G}}\left(\underline{n},\underline{p}\right)$. □

Remark 2.6.14. By Remark 2.6.11, we are given a symmetric strict monoidal structure $\left(\widetilde{\mathfrak{U}\mathcal{G}'},\natural,0\right)$ and a fortiori on $\mathrm{Cospan}^{\dagger}(\mathfrak{U}\mathcal{G}')$, induced by the one of $(\mathfrak{U}\mathcal{G}',\natural,0)$.

2.6.2 Lifted functors for symmetric monoidal categories

The aim of this section is to prove that, under an additional condition (see Condition 2.6.20), the generalised Long-Moody functor $\mathbf{LM} : \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ defined in Proposition 2.2.30 can be lifted to a functor $\widetilde{\mathbf{LM}} : \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right) \to \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right)$. Let introduce these additional properties. Recall that we introduced the augmentation ideal functor $\mathcal{I} : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$ in Definition 2.2.28.

Notation 2.6.15. Recall that $t_G : G \to 0_{\mathfrak{Gr}}$ denotes the unique morphism from the group G to $0_{\mathfrak{Gr}}$. Let $\mathcal{H}^{op} : (\mathbb{N}^{op},\leq) \to \mathfrak{Gr}$ be the family of groups defined on objects by $\mathcal{H}^{op}(m) = H_m$ for all natural numbers m, and for morphisms by $\mathcal{H}^{op}(\gamma_m) = t_H * id_{H_m}$. We denote for natural numbers $n' \geq n$ by $\mathcal{I}^{op}\left(\left[n'-n, id_{\underline{n'}}\right]\right) : \mathcal{I}(\underline{n'}) \to \mathcal{I}(\underline{n})$ the R-module morphism canonically induced for the augmentation ideals by the group morphism $t_{H^{*(n'-n)}} * id_{H_n} : H_{n'} \to H_n$.

This new functor $\mathcal{H}^{op}$ satisfies analogous properties to that of the functor $\mathcal{H}$ (see Assumption 2.2.13). Indeed, we have:

Proposition 2.6.16. *The functor $\mathcal{H}^{op}$ of Notation 2.6.15 defines a functor $\mathcal{H}^{op} : \mathfrak{U}\mathcal{G}^{op} \to \mathfrak{Gr}$ such that $\mathcal{H}^{op}\left(\left[1, id_{\underline{n+1}}\right]\right) = \mathcal{H}^{op}(\gamma_n)$ for all natural numbers n, and $\mathcal{H}^{op}(g) = \mathcal{H}\left(g^{-1}\right)$ for all $g \in G_n$ and all natural numbers n.*

Proof. By Lemma 2.1.8, it is enough to prove that for all natural numbers n and n' such that $n' \geq n$, for all $g' \in G_{n'-n}$ and $g \in G_n$:

$$\mathcal{H}^{op}\left(\left[n'-n, id_{\underline{n'}}\right]\right) \circ \mathcal{H}\left(g'\natural g\right)^{-1} = \mathcal{H}(g)^{-1} \circ \mathcal{H}^{op}\left(\left[n'-n, id_{\underline{n'}}\right]\right).$$

This follows from the definition of the morphism $t_{H_{n'-n}} * id_{H_n}$ and the fact that the functor $\mathcal{H} : \mathcal{G}'_{(0,1)} \to \mathfrak{Gr}_{H,H_0}$ is strict monoidal by Assumption 2.5.16. □

Remark 2.6.17. Similarly to Definition 2.2.28, the functor $\mathcal{H}^{op}$ induces a functor $\mathcal{I}^{op} : \mathfrak{U}\mathcal{G}^{op} \to R\text{-}\mathfrak{Mod}$ assigning $\underline{n} \mapsto \mathcal{I}_{R[H_n]}$ and the assignments for $[n'-n,g] \in Hom_{\mathfrak{U}\mathcal{G}}(\underline{n},\underline{n'})$ are the R-modules morphisms:

$$\mathcal{I}^{op}\left(\left[n'-n,g\right]\right) = \mathcal{I}^{op}\left(\left[n'-n, id_{\underline{n'}}\right]\right) \circ \mathcal{I}(g)^{-1}. \tag{2.6.5}$$

Also, we deduce from Lemma 2.1.8 for n and n' be natural numbers, for all elements g of G_n and g' of $G_{n'-n}$, as morphisms $H_{n'} \to H_n$:

$$\mathcal{I}^{op}\left(\left[n'-n, id_{\underline{n'}}\right]\right) \circ \mathcal{I}\left(g'\natural g\right)^{-1} = \mathcal{I}(g)^{-1} \circ \mathcal{I}^{op}\left(\left[n'-n, id_{\underline{n'}}\right]\right).$$

Moreover, note that the following property is satisfied. This will be used in the proof of Theorem 2.6.24.

Lemma 2.6.18. *Let k, l and n be natural numbers such that $k,l \geq n$. The following diagram is commutative with $\underline{q} = (l-n)\natural(k-n)\natural\underline{n}$:*

$$\begin{array}{ccccc}
H_k & \xrightarrow{\mathcal{H}\left(\left[l-n,id_{\underline{q}}\right]\right)} & H_q & \xrightarrow{\mathcal{H}\left(b^{\mathcal{G}}_{k-n,l-n}\natural id_{\underline{n}}\right)} & H_q \\
{\scriptstyle \mathcal{H}^{op}\left(\left[k-n,id_{\underline{k}}\right]\right)}\downarrow & & & & \downarrow{\scriptstyle \mathcal{H}^{op}\left(\left[k-n,id_{\underline{q}}\right]\right)} \\
H_n & & \xrightarrow[\mathcal{H}\left(\left[l-n,id_{\underline{l}}\right]\right)]{} & & H_l .
\end{array}$$

Proof. Recall that the functor $\mathcal{H} : \mathcal{G}'_{(0,1)} \to \mathfrak{Gr}_{H,H_0}$ is braided strict monoidal by Assumption 2.5.16. Hence this result is a direct consequence of the definitions of the morphisms $t_{H_{k-n}} * id_{H_n}$ and $t_{H_{k-n}} * id_{H_l}$. □

Remark 2.6.19. Lemma 2.6.18 remains true for the functors $\mathcal{I}$ and $\mathcal{I}^{op}$.

Finally, we require the following property for the morphisms $\{\varsigma_n : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$ of Condition 2.5.9.

Condition 2.6.20. For all elements $h \in H_{n'}$, for all natural numbers $n' \geq n$, the following diagram is commutative in the category $\widetilde{\mathfrak{U}\mathcal{G}}$:

$$\begin{array}{ccc}
1\natural\underline{n'} & \xrightarrow{\varsigma_{n'}(h)} & 1\natural\underline{n'} \\
{\scriptstyle id_1\natural\mathcal{R}\left(\left[n'-n,id_{\underline{n'}}\right]\right)}\downarrow & & \downarrow{\scriptstyle id_1\natural\mathcal{R}\left(\left[n'-n,id_{\underline{n'}}\right]\right)} \\
1\natural\underline{n} & \xrightarrow[\varsigma_n\left(\mathcal{H}^{op}\left(\left[n'-n,id_{\underline{n'}}\right]\right)(h)\right)]{} & 1\natural\underline{n}.
\end{array}$$

Remark 2.6.21. It follows from the equivalence relation of Definition 2.6.3 that Condition 2.2.17 is equivalent to assuming that for all natural numbers n, for all elements $h \in H_{n+1}$, the morphisms $\{\varsigma_n\}_{n\in\mathbb{N}}$ satisfy the following equality in G_{n+1}:

$$\varsigma_{n+1}(h) \circ \left(id_1\natural\left[1, id_{\underline{n+1}}\right]\right) = \left[1, \left(\left(b^{\mathcal{G}}_{1,1}\right)^{-1}\natural id_{\underline{n}}\right) \circ \left(id_1\natural\varsigma_n\left(\left(t_H * id_{H_n}\right)(h)\right)\right)\right].$$

Definition 2.6.22. A reliable Long-Moody system $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$ is said to be liftable if it satisfies Condition 2.6.20.

Remark 2.6.23. Consider the family of morphisms $\{\varsigma_{n,t} : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$ of Example 2.2.22. Then Condition 2.6.20 is always satisfied and therefore the reliable Long-Moody system $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_{-,t}\}$ is always liftable.

In the following theorem, we prove that a generalised Long-Moody functor associated with a liftable Long-Moody system $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$ defines an endofunctor of $\mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right)$. It will be called the lifted generalised Long-Moody functor.

Theorem 2.6.24. *Let $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}$ be a liftable Long-Moody system. The following assignment defines a functor* $\widetilde{\mathbf{LM}}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}} : \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right) \to \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right)$.

- *Objects: for* $F \in Obj\left(\mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}}, R\text{-}\mathfrak{Mod}\right)\right)$, $\widetilde{\mathbf{LM}}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(F) : \widetilde{\mathfrak{U}\mathcal{G}} \to R\text{-}\mathfrak{Mod}$ *is defined by:*
  - *Objects:* $\forall \underline{n} \in Obj(\mathcal{G})$, $\widetilde{\mathbf{LM}}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(F)(\underline{n}) = \mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} F(\underline{n+1})$.
  - *Morphisms: for* $n, n' \in \mathbb{N}$ *and* $\mathcal{R}([k-n',g']) \circ \mathcal{L}([k-n,g]) \in Hom_{\widetilde{\mathfrak{U}\mathcal{G}}}(\underline{n},\underline{n'})$. *We define*

    $$\widetilde{\mathbf{LM}}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}(F)\left(\mathcal{R}\left([k-n',g']\right) \circ \mathcal{L}([k-n,g])\right) : \mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} F(\underline{n+1}) \to \mathcal{I}_{R[H_{n'}]} \underset{R[H_{n'}]}{\otimes} F\left(\underline{n'+1}\right)$$

    to be the unique morphism induced by the universal property of the tensor product $\underset{R[H_n]}{\otimes}$ *with respect to the* $R[H_n]$*-balanced map*

    $$\mathcal{I}_{R[H_n]} \times F(\underline{n+1}) \xrightarrow{\Xi([n'-n,g])} \mathcal{I}_{R[H_{n'}]} \times F\left(\underline{n'+1}\right) \xrightarrow{\underset{R[H_{n'}]}{\otimes}} \mathcal{I}_{R[H_{n'}]} \underset{R[H_{n'}]}{\otimes} F\left(\underline{n'+1}\right),$$

with

$$\Xi\left(\left[n'-n,g\right]\right) = \left(\mathcal{I}^{op}\left(\left[k-n',g'\right]\right)\circ\mathcal{I}\left(\left[k-n,g\right]\right)\right) \times F\left(id_{1}\natural\left(\mathcal{R}\left(\left[k-n',g'\right]\right)\circ\mathcal{L}\left(\left[k-n,g\right]\right)\right)\right).$$

- *Morphisms: let F and G be two objects of* $\mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}},R\text{-}\mathfrak{Mod}\right)$*, and $\eta:F\rightarrow G$ be a natural transformation. We define* $\widetilde{\mathbf{LM}}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}\left(\eta\right):\widetilde{\mathbf{LM}}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}\left(F\right)\rightarrow\widetilde{\mathbf{LM}}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}\left(G\right)$ *for all natural numbers n by:*

$$\left(\widetilde{\mathbf{LM}}_{\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma\}}\left(\eta\right)\right)_{\underline{n}} = id_{\mathcal{I}_{R[H_{n}]}}\underset{R[H_{n}]}{\otimes}\eta_{\underline{n+1}}.$$

Proof. We have three points to prove. Let $F\in Obj\left(\mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}},R\text{-}\mathfrak{Mod}\right)\right)$.

1. First, let us check the compatibility of the assignment $\widetilde{\mathbf{LM}}\left(F\right)$ with respect to the tensor product. Recall that the R-module $F\left(\underline{n+1}\right)$ is endowed with a (left) $R\left[H_{n}\right]$-module structure using the morphism $\varsigma_{n}:H_{n}\rightarrow G_{n+1}$. This compatibility holds for morphisms of type $\mathcal{L}$ (see Notation 2.6.5) by the definition of $\mathbf{LM}\left(F\right)$, Lemma 2.6.7 ensures that the remaining point to prove is the compatibility of $\widetilde{\mathbf{LM}}\left(F\right)\left(\mathcal{R}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\right)$ with n and n' natural numbers such that $n'\geq n$. Let $h\in H_{n'}$ and $i\in\mathcal{I}_{R\left[H_{n'}\right]}$. It follows from Condition 2.6.20 that in G_{n+1}:

$$\left(id_{1}\natural\mathcal{R}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\right)\circ\varsigma_{n'}\left(h\right)=\varsigma_{n}\left(\mathcal{H}^{op}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\left(h\right)\right)\circ\left(id_{1}\natural\mathcal{R}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\right).$$

Since $\mathcal{I}^{op}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\left(i\cdot h\right)=\mathcal{I}^{op}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\left(i\right)\cdot\mathcal{H}^{op}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\left(h\right)$, we deduce that:

$$\begin{aligned} & \widetilde{\mathbf{LM}}\left(F\right)\left(\mathcal{R}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\right)\left(i\underset{R\left[H_{n'}\right]}{\otimes}F\left(\varsigma_{n}\left(h\right)\right)\left(v\right)\right)\\ = & \mathcal{I}^{op}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\left(i\right)\underset{R\left[H_{n}\right]}{\otimes}F\left(\varsigma_{n}\left(\mathcal{H}^{op}\left(h\right)\right)\right)F\left(id_{1}\natural\left[n'-n,id_{\underline{n'}}\right]\right)\left(v\right)\\ = & \mathcal{I}^{op}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\left(i\cdot h\right)\underset{R\left[H_{n}\right]}{\otimes}F\left(id_{1}\natural\left[n'-n,id_{\underline{n'}}\right]\right)\left(v\right)\\ = & \widetilde{\mathbf{LM}}\left(F\right)\left(\mathcal{R}\left(\left[n'-n,id_{\underline{n'}}\right]\right)\right)\left(i\cdot h\underset{R\left[H_{n'}\right]}{\otimes}v\right).\end{aligned}$$

2. Let us prove that the assignment $\widetilde{\mathbf{LM}}\left(F\right)$ is a functor. Recall that the functor $\mathbf{LM}\left(F\right)$ is well-defined by Theorem 2.2.30. Let $\left[k-n,\phi\right]\in Hom_{\mathfrak{U}\mathcal{G}}\left(\underline{n},\underline{k}\right)$, $\left[k-n',\psi\right]\in Hom_{\mathfrak{U}\mathcal{G}}\left(\underline{n'},\underline{k}\right)$, $\left[n-m,\chi\right]\in Hom_{\mathfrak{U}\mathcal{G}}\left(\underline{m},\underline{n}\right)$ and $\left[l-n,\omega\right]\in Hom_{\mathfrak{U}\mathcal{G}}\left(\underline{n},\underline{l}\right)$ with natural numbers l,k,n,n',m such that $l\geq n$, $k\geq n$, $k\geq n'$ and $n\geq m$. By relation (2.6.5) of Remark 2.6.17, we have:

$$\mathcal{I}^{op}\left(\left[k-n,\phi\right]\right)\circ\mathcal{I}\left(\left[k-n,\phi\right]\right)=id_{\mathcal{I}_{R\left[H_{n}\right]}},$$

thus it follows from the compatibility of the monoidal structure $\natural$ with composition and the functoriality of F that:

$$\widetilde{\mathbf{LM}}\left(F\right)\left(\mathcal{R}\left(\left[k-n,\phi\right]\right)\right)\circ\widetilde{\mathbf{LM}}\left(F\right)\left(\mathcal{L}\left(\left[k-n,\phi\right]\right)\right) = id_{\widetilde{\mathbf{LM}}\left(F\right)\left(\underline{n}\right)}.$$

Hence, according to Proposition 2.6.8, it is enough to check the relations (2.6.1), (2.6.2) and (2.6.3).
The relation (2.6.1) follows from the definition of $\widetilde{\mathbf{LM}}\left(F\right)$ on morphisms and the fact that F is a functor. Now, let us prove that $\widetilde{\mathbf{LM}}\left(F\right)$ satisfies the relation (2.6.2). The functoriality of $\mathcal{I}^{op}$ deduced from Proposition 2.6.16 ensures that:

$$\mathcal{I}^{op}\left(\left[k-m,id_{k-n}\natural\chi\right]\right)\circ\mathcal{I}\left(\phi^{-1}\right)=\mathcal{I}^{op}\left(\left[n-m,\chi\right]\right)\circ\mathcal{I}^{op}\left(\left[k-n,\phi\right]\right).$$

Then, the desired result follows from the compatibility of the monoidal structure $\natural$ with composition and the functoriality of F. Finally, let us prove that $\widetilde{\mathbf{LM}}(F)$ satisfies the relation (2.6.3). The functoriality of $\mathcal{I}$ by Definition 2.2.28 and the one of $\mathcal{I}^{op}$ deduced from Proposition 2.6.16 imply that:

$$\begin{aligned}&\mathcal{I}^{op}\left(\left[k-n,\left(\left(b^{\mathcal{G}'}_{k-n,l-n}\right)^{-1}\natural id_{\underline{n}}\right)\circ\left(id_{k-n}\natural\left(\omega^{-1}\right)\right)\right]\right)\circ\mathcal{I}\left(\left[l-n,id_{l-n}\natural\phi^{-1}\right]\right)\\ = \;&\mathcal{I}(\omega)\circ\left(\mathcal{I}^{op}\left(\left[k-n,id_{\underline{l+k-n}}\right]\right)\circ\mathcal{I}\left(b^{\mathcal{G}'}_{k-n,l-n}\natural id_{\underline{n}}\right)\circ\mathcal{I}\left(\left[l-n,id_{\underline{l+k-n}}\right]\right)\right)\circ\mathcal{I}^{op}(\phi)\,.\end{aligned}$$

By Remark (2.6.19), we have:

$$\mathcal{I}^{op}\left(\left[k-n,id_{\underline{l+k-n}}\right]\right)\circ\mathcal{I}\left(b^{\mathcal{G}'}_{k-n,l-n}\natural id_{\underline{n}}\right)\circ\mathcal{I}\left(\left[l-n,id_{\underline{l+k-n}}\right]\right)=\mathcal{I}\left(\left[l-n,id_{\underline{l}}\right]\right)\circ\mathcal{I}^{op}\left(\left[k-n,id_{\underline{k}}\right]\right).$$

Hence:

$$\begin{aligned}&\mathcal{I}^{op}\left(\left[k-n,\left(\left(b^{\mathcal{G}'}_{k-n,l-n}\right)^{-1}\natural id_{\underline{n}}\right)\circ\left(id_{l-n}\natural\left(\omega^{-1}\right)\right)\right]\right)\circ\mathcal{I}\left(\left[l-n,\phi^{-1}\right]\right)\\ = \;&\mathcal{I}\left(\left[l-n,\omega\right]\right)\circ\mathcal{I}^{op}\left(\left[k-n,\phi\right]\right).\end{aligned}$$

Again the compatibility of the monoidal structure $\natural$ with composition and the functoriality of F finally imply that the composition axiom is satisfied.

3. The remaining point to check for $\widetilde{\mathbf{LM}}$ to be a functor is the consistency of our definition on morphisms. Recalling that the functor $\mathbf{LM}$ is well-defined by Theorem 2.2.30, according to Proposition 2.6.9, we only have to check that for $\eta : F\to G$ natural transformation:

$$\widetilde{\mathbf{LM}}\left(\eta_{n'}\right)\circ\widetilde{\mathbf{LM}}(F)\left(R\left(\left[n'-n,id_{\underline{n'}}\right]\right)\right)=\widetilde{\mathbf{LM}}(G)\left(R\left(\left[n'-n,id_{\underline{n'}}\right]\right)\right)\circ\widetilde{\mathbf{LM}}\left(\eta_{\underline{n}}\right).$$

This is a consequence of the definition of the naturality of η and the assignment of $\widetilde{\mathbf{LM}}(\eta)$. The verification of the composition axiom repeats mutatis mutandis the one of Theorem 2.2.30.

□

Corollary 2.6.25. *The following diagram is commutative:*

$$\begin{array}{ccc}\mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}},R\text{-}\mathfrak{Mod}\right) & \xrightarrow{\widetilde{\mathbf{LM}}_{\{\mathcal{H},\mathcal{G},\varsigma_n\}}} & \mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}},R\text{-}\mathfrak{Mod}\right)\\ \downarrow{\left(incl^{\widetilde{\mathfrak{U}\mathcal{G}}}_{\mathfrak{U}\mathcal{G}}\right)^{*}} & & \downarrow{\left(incl^{\widetilde{\mathfrak{U}\mathcal{G}}}_{\mathfrak{U}\mathcal{G}}\right)^{*}}\\ \mathbf{Fct}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right) & \xrightarrow{\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\varsigma_n\}}} & \mathbf{Fct}\left(\mathfrak{U}\mathcal{G},R\text{-}\mathfrak{Mod}\right),\end{array}$$

where $\left(incl^{\widetilde{\mathfrak{U}\mathcal{G}}}_{\mathfrak{U}\mathcal{G}}\right)^{*}$ *denotes the precomposition by the functor* $incl^{\widetilde{\mathfrak{U}\mathcal{G}}}_{\mathfrak{U}\mathcal{G}}$ *introduced in Remark 2.6.6.*

Proof. Let F be an object of $\mathbf{Fct}\left(\widetilde{\mathfrak{U}\mathcal{G}},R\text{-}\mathfrak{Mod}\right)$. Recall that for n, $n'\in\mathbb{N}$ such that $n'\geq n$, $Hom_{\mathfrak{U}\mathcal{G}}\left(\underline{n},\underline{n'}\right)=\left\{\mathcal{L}\left(\left[n'-n,g\right]\right)\in Hom_{\widetilde{\mathfrak{U}\mathcal{G}}}\left(\underline{n},\underline{n'}\right)\right\}$. Hence, the commutativity of the diagram follows from the definition of $\widetilde{\mathbf{LM}}$ on morphisms of type $\mathcal{L}\left(\left[n'-n,g\right]\right)$ in Theorem 2.6.24. □

2.6.3 Examples

The homological stability with twisted coefficients result due to Randal-Williams and Wahl [RWW17, Theorem A] (recalled in Theorem 2.5.32) holds for families of groups other that mapping class groups of surfaces: it is also true for symmetric groups, automorphism groups of free products of groups and mapping class groups of compact, connected, oriented 3-manifolds with boundary (see [RWW17, Section 5]). The following work presents the use of Long-Moody functors in these situations and provides very strong polynomial functors in any degree for these families of groups. In particular, for automorphism groups of free products of groups and mapping class groups of 3-manifolds, Long-Moody functors come in handy in so far as there are very few examples of very strong polynomial functors associated with these families of groups.

2.6.3.1 Symmetric groups

Let Σ be the skeleton of the groupoid of finite sets and bijections. Note that $Obj\left(\Sigma\right)\cong\mathbb{N}$ and that the automorphism groups are the symmetric groups $\mathfrak{S}_{n}$. The disjoint union of finite sets $\sqcup$ induces a monoidal structure $\left(\Sigma,\sqcup,0\right)$, the unit 0 being the empty set. This groupoid is symmetric monoidal, the symmetry being given by the canonical bijection $b_{n_{1},n_{2}}^{\Sigma}:n_{1}\sqcup n_{2}\rightarrow n_{2}\sqcup n_{1}$ for all natural numbers n_{1} and n_{2}.

Remark 2.6.26. The category $\mathfrak{U}\Sigma$ is equivalent to the category of finite sets and injections FI studied in [CEF15]. Furthermore, the classical surjections $\left\{ \mathbf{B}_{n}\overset{\mathfrak{p}_{n}}{\twoheadrightarrow}\mathfrak{S}_{n}\right\} _{n\in\mathbb{N}}$, sending each Artin generator $\sigma_{i}\in\mathbf{B}_{n}$ to the transposition $\tau_{i}\in\mathfrak{S}_{n}$ for all $i\in\left\{ 1,\ldots,n-1\right\}$ and for all natural numbers n, assemble to define a functor $\mathfrak{P}:\mathfrak{U}\boldsymbol{\beta}\rightarrow\mathfrak{U}\Sigma$. In addition, it is clear that the functor $\mathfrak{P}$ is strict monoidal with respect to the monoidal structures $\left(\mathfrak{U}\boldsymbol{\beta},\natural,0\right)$ and $\left(\mathfrak{U}\Sigma,\sqcup,0\right)$.

Notation 2.6.27. For all natural numbers n, we denote by $a_{n}^{\mathfrak{S}}:\mathfrak{S}_{n}\rightarrow Aut\left(\mathbf{F}_{n}\right)$ the morphism defined by $a_{n}^{\mathfrak{S}}\left(\sigma\right)\left(f_{i}\right)=f_{\sigma(i)}$ for all $\sigma\in\mathfrak{S}_{n}$ and generator f_{i} of $\mathbf{F}_{n}$.

Let H be the free group $\mathbf{F}_{1}$ and H_{0} be the trivial group. Thus, we define functors $\mathcal{H}_{\mathfrak{S}}:\Sigma\rightarrow\mathfrak{gr}$ assigning $\mathcal{H}_{\mathfrak{S}}\left(n\right)=\mathbf{F}_{n}$ on objects and for all $\sigma\in\mathfrak{S}_{n}$, $\mathcal{H}_{\mathfrak{S}}\left(\sigma\right)=a_{n}^{\mathfrak{S}}\left(\sigma\right)$.

Lemma 2.6.28. *The functor $\mathcal{H}_{\mathfrak{S}}:\left(\Sigma,\sqcup,0\right)\rightarrow\left(\mathfrak{gr},*,0_{\mathfrak{Gr}}\right)$ is symmetric strict monoidal. Moreover, the functor $\mathcal{H}_{\mathfrak{S}}$ extends to define a functor $\mathcal{H}_{\mathfrak{S}}:\mathfrak{U}\Sigma\rightarrow\mathfrak{gr}$ assigning for all natural numbers n_{1} and n_{2}:*

$$\mathcal{H}_{\mathfrak{S}}\left[n_{1},id_{n_{1}\sqcup n_{2}}\right]=\iota_{n_{1}}\oplus id_{n_{2}}.$$

Proof. Fixing a basis for $\mathcal{H}_{\mathfrak{S}}\left(n\right)$ for any natural number n, we deduce that for $n_{1},n_{2}\in Obj\left(\Sigma\right)$:

$$\mathcal{H}_{\mathfrak{S}}\left(n_{1}\sqcup n_{2}\right)\cong\mathcal{H}_{\mathfrak{S}}\left(n_{1}\right)*\mathcal{H}_{\mathfrak{S}}\left(n_{2}\right).$$

It is clear that $\mathfrak{S}_{n_{1}}$ (resp. $\mathfrak{S}_{n_{2}}$) acts trivially on $\mathcal{H}_{\mathfrak{S}}\left(n_{2}\right)$ (resp. $\mathcal{H}_{\mathfrak{S}}\left(n_{1}\right)$) in $\mathcal{H}_{\mathfrak{S}}\left(n_{1}\sqcup n_{2}\right)$. Therefore, $id_{\mathcal{H}_{\mathfrak{S}}\left(n_{1}\right)}*id_{\mathcal{H}_{\mathfrak{S}}\left(n_{2}\right)}$ is a natural equivalence. Moreover, it is clear from the fact that the functor $\mathcal{H}_{\mathfrak{S}}$ is strict monoidal that the symmetry $b_{\mathcal{H}_{\mathfrak{S}}\left(n_{1}\right),\mathcal{H}_{\mathfrak{S}}\left(n_{2}\right)}^{\mathfrak{gr}}$ is equal to $\mathcal{H}_{\mathfrak{S}}\left(b_{n_{1},n_{2}}^{\Sigma}\right)$.

It follows from the assignments that relation (2.1.1) of Lemma 2.1.8 is satisfied by $\mathcal{H}_{\mathfrak{S}}\left[n_{1},id_{n_{1}\sqcup n_{2}}\right]$. Let $\sigma_{1}\in\mathfrak{S}_{n_{1}}$ and $\sigma_{2}\in\mathfrak{S}_{n_{2}}$. Then, it follows from the definition of $\iota_{n_{1}}$ that:

$$\begin{array}{ccc}
\mathcal{H}_{\mathfrak{S}}\left(\sigma_{1}\sqcup\sigma_{2}\right)\circ\mathcal{H}_{\mathfrak{S}}\left[n_{1},id_{n_{1}\sqcup n_{2}}\right] & = & \left(\mathcal{H}_{\mathfrak{S}}\left(\sigma_{1}\right)*\mathcal{H}_{\mathfrak{S}}\left(\sigma_{2}\right)\right)\circ\mathcal{H}_{\mathfrak{S}}\left[n_{1},id_{n_{1}\sqcup n_{2}}\right]\\
 & = & \mathcal{H}_{\mathfrak{S}}\left[n_{1},id_{n_{1}\sqcup n_{2}}\right]\circ\mathcal{H}_{\mathfrak{S}}\left(\sigma_{2}\right).
\end{array}$$

Relation (2.1.2) of Lemma 2.1.8 is thus satisfied, which implies the desired result. ☐

Corollary 2.6.29. *With the previous assignments and notations, $\left\{ \mathcal{H}_{\mathfrak{S}},\Sigma,\Sigma,\varsigma_{-,t}\right\}$ defines a liftable Long-Moody system, where $\varsigma_{n,t}:\mathbf{F}_{n}\rightarrow\mathfrak{S}_{n+1}$ is the trivial morphism for all natural numbers n (see Example 2.2.22).*

Proof. The Long-Moody system $\left\{ \mathcal{H}_{\mathfrak{S}},\Sigma,\Sigma,\varsigma_{-,t}\right\}$ is reliable by Remark 2.2.27, Assumptions 2.2.13 and 2.5.16 being satisfied by Lemma 2.6.28 and Assumption 2.2.1 being checked using the groupoid $\left(\Sigma,\sqcup,0\right)$ (noting this category has no zero divisors and that $Aut_{\Sigma}\left(0_{\Sigma}\right)=\left\{ id_{0_{\Sigma}}\right\}$). We conclude using Remark 2.6.23. ☐

The functor $\mathbf{LM}_{\left\{ \mathcal{H}_{\mathfrak{S}},\Sigma,\Sigma,\varsigma_{n,t}\right\}}$ defined by this liftable Long-Moody system is closely related to the functor $\mathbf{LM}_{\left\{ \mathcal{H}_{S}^{\mathfrak{b}},\mathfrak{B}_{2}^{0,0},\varsigma_{n,1}^{\mathfrak{b}}\right\}}=\mathbf{LM}_{1}$ for braid groups (see Proposition 2.3.41) introduced in [Sou17b, Section 1.3].

Proposition 2.6.30. *The following diagram is commutative:*

$$\begin{array}{ccc}
\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},R\text{-}\mathfrak{Mod}\right) & \overset{\mathbf{LM}_{1}}{\longrightarrow} & \mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta},R\text{-}\mathfrak{Mod}\right)\\
{\scriptstyle \left(\mathfrak{P}\right)^{*}}\uparrow & & \uparrow{\scriptstyle \left(\mathfrak{P}\right)^{*}}\\
\mathbf{Fct}\left(\mathfrak{U}\Sigma,R\text{-}\mathfrak{Mod}\right) & \overset{\mathbf{LM}_{\left\{ \mathcal{H}_{\mathfrak{S}},\Sigma,\Sigma,\varsigma_{n,t}\right\}}}{\longrightarrow} & \mathbf{Fct}\left(\mathfrak{U}\Sigma,R\text{-}\mathfrak{Mod}\right),
\end{array}$$

where $\left(\mathfrak{P}\right)^{*}$ denotes the precomposition by the functor $\mathfrak{P}$ introduced in Remark 2.6.26.

Proof. First, it follows from $\mathfrak{p}_{n+1}\left(\sigma_i^2\right) = 1_{\mathfrak{S}_n}$ (where $1_{\mathfrak{S}_n}$ is the neutral element of $\mathfrak{S}_n$) that $\mathfrak{p}_{n+1} \circ \varsigma_{n,1} = \varsigma_{n,t}$. A fortiori, the definition of a Long-Moody functor (see Theorem 2.2.30), the fact that $\mathfrak{P}$ is strict monoidal (see Remark 2.6.26) and that $\mathcal{H}_{s}^{\flat}\left(\left[n'-n, id_{n'}\right]\right) = \mathcal{H}_{\mathfrak{S}}\left(\left[n'-n, id_{n'}\right]\right)$ for all natural numbers $n' \geq n$, ensure that it is enough to prove that for all object F of $\mathbf{Fct}\left(\mathfrak{U}\Sigma, R\text{-}\mathfrak{Mod}\right)$, for all Artin generators $\sigma_i \in \mathbf{B}_n$ for n a natural number:

$$\mathcal{I}_1\left(\sigma_i\right) \underset{R[\mathbf{F}_n]}{\otimes} \left(F \circ \mathfrak{P}\right)\left(id_1 \natural \sigma_i\right) \quad = \quad \mathcal{I}_{\mathfrak{S}}\left(\mathfrak{P}\left(\sigma_i\right)\right) \underset{R[\mathbf{F}_n]}{\otimes} F\left(id_1 \sqcup \mathfrak{P}\left(\sigma_i\right)\right). \tag{2.6.6}$$

First, we deduce from the strict monoidal property of $\mathfrak{P}$ that $\left(F \circ \mathfrak{P}\right)\left(id_1 \natural \sigma_i\right) = F\left(id_1 \sqcup \mathfrak{P}\left(\sigma_i\right)\right)$. It follows from the definition of Artin representation (see [Sou17b, Section 2.3.1]) that:

$$\begin{array}{rcl} \mathcal{I}_1\left(\sigma_i\right) : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} & \longrightarrow & \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \\ f_j - 1 & \longmapsto & \begin{cases} f_{i+1} - 1 & \text{if } j = i \\ f_{i+1}^{-1} f_i f_{i+1} - 1 = \left[f_i - 1\right] f_{i+1} + \left[f_{i+1} - 1\right]\left(1 - f_{i+1}^{-1} f_i f_{i+1}\right) & \text{if } j = i+1 \\ f_j - 1 & \text{if } j \notin \{i, i+1\}. \end{cases} \end{array}$$

We deduce from the relations $\mathfrak{p}_{n+1} \circ \varsigma_{n,1} = \varsigma_{n,t}$ and $\mathcal{I}_{\mathfrak{S}}\left(\mathfrak{P}\left(\sigma_i\right)\right)\left(f_{i+1} - 1\right) = f_i - 1$ that if $j = i+1$:

$$\begin{aligned} \mathcal{I}_1\left(\sigma_i\right)\left(f_{i+1} - 1\right) \underset{R[\mathbf{F}_n]}{\otimes} \left(F \circ \mathfrak{P}\right)\left(id_1 \natural \sigma_i\right) &= \left[f_i - 1\right] \underset{R[\mathbf{F}_n]}{\otimes} F\left(\varsigma_{n,t}\left(f_{i+1}\right)\right)\left(F \circ \mathfrak{P}\right)\left(id_1 \natural \sigma_i\right) \\ &\quad + \left[f_{i+1} - 1\right] \underset{R[\mathbf{F}_n]}{\otimes} F\left(\varsigma_{n,t}(1) - \varsigma_{n,t}\left(f_{i+1}^{-1} f_i f_{i+1}\right)\right)\left(F \circ \mathfrak{P}\right)\left(id_1 \natural \sigma_i\right) \\ &= \mathcal{I}_{\mathfrak{S}}\left(\mathfrak{P}\left(\sigma_i\right)\right)\left(f_{i+1} - 1\right) \underset{R[\mathbf{F}_n]}{\otimes} F\left(id_1 \sqcup \mathfrak{P}\left(\sigma_i\right)\right). \end{aligned}$$

The others cases being clear, this proves that the relation (2.6.6) is true. □

Notation 2.6.31. For all natural numbers, we denote by $\mathfrak{Perm}_n$ the permutation representation of the symmetric group to $GL_n(R)$. Namely, it is defined assigning:

$$\mathfrak{Perm}_n\left(\sigma_i\right) = Id_{R^{\oplus i}} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus Id_{R^{\oplus n-i-1}}$$

for every transposition $\sigma_i \in \mathfrak{S}_n$ (with $i \in \{1, \ldots, n-1\}$).

It is a well-known fact (see for example [CEF15]) that the permutation representations $\{\mathfrak{Perm}_n\}_{n \in \mathbb{N}}$ assemble to form a functor $\mathfrak{Perm} : \mathfrak{U}\Sigma \to R\text{-}\mathfrak{Mod}$. Namely, for all natural numbers n and n' such that $n' \geq n$:

$$\mathfrak{Perm}\left(\left[n'-n, id_{n'}\right]\right) = \iota_{R^{\oplus(n'-n)}} \oplus id_{R^{\oplus n}}$$

and the relations (2.1.1) and (2.1.2) of Lemma 2.1.8 are easily checked. In particular, the functor $\mathfrak{Perm}$ can be seen as the restriction of the unreduced Burau functor $\mathfrak{Bur}_1$ to $\mathfrak{U}\Sigma$.

Corollary 2.6.32. *For $R : \mathfrak{U}\Sigma \to R\text{-}\mathfrak{Mod}$ the constant functor:*

$$\mathbf{LM}_{\{\mathcal{H}_{\mathfrak{S}}, \Sigma, \varsigma_{n,t}\}}(R) \cong \mathfrak{Perm}.$$

Remark 2.6.33. By Proposition 2.2.39, all the iterations of $\mathbf{LM}_{\{\mathcal{H}_{\mathfrak{S}}, \Sigma, \varsigma_{n,t}\}}$ on an object F of $\mathbf{Fct}\left(\mathfrak{U}\Sigma, R\text{-}\mathfrak{Mod}\right)$ are determined by $\mathfrak{Perm}$.

We conclude the study for symmetric groups giving the following result, obtained as a corollary of [Lon94, Theorem 4.3].

Proposition 2.6.34. *Let m be a natural number. Consider the iteration $\mathbf{LM}^{\circ(m+1)}_{\{\mathcal{H}_{\mathfrak{S}}, \Sigma, \varsigma_{n,t}\}}(R)$ of the Long-Moody functor $\mathbf{LM}_{\{\mathcal{H}_{\mathfrak{S}}, \Sigma, \varsigma_{n,t}\}}$. Then, all the irreducible representations of the symmetric group $\mathfrak{S}_m$ are subrepresentations of the induced representation*

$$\mathbf{LM}^{\circ(m+1)}_{\{\mathcal{H}_{\mathfrak{S}}, \Sigma, \varsigma_{n,t}\}}(R)_{|\mathfrak{S}_m} : \mathfrak{S}_m \to GL_M(R)$$

where $M = \frac{(2m+1)!}{m!}$.

2.6.3.2 Automorphisms of free products of groups

Let H and H_0 be two arbitrary groups. Recall from Notation 2.2.9 that H_m denotes the free product $H^{*m} * H_0$ for all natural numbers m.

Notation 2.6.35. We denote by $\mathcal{H}_{fp}$ the functor of Assumption 2.2.13 for this example. Namely, $\mathcal{H}_{fp} : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ is defined for all natural integers n assigning $\mathcal{H}_{fp}(n) = H_n$ and $\mathcal{H}_{fp}(\gamma_n) = \iota_H * id_{H_n}$.

Let $\mathfrak{fG}$ denote the skeleton of the groupoid $f\mathcal{G}$ of finitely-generated groups and their isomorphisms introduced in [RWW17, Section 5.2]. The free product of groups $*$ induces a strict symmetric monoidal structure $(\mathfrak{fG}, *, 0_{\mathfrak{Gr}})$. In particular, the symmetry of the monoidal structure $b^{\mathfrak{fG}}_{G_1,G_2}$ is given by the canonical permutation of the free product. Hence, we deduce:

Lemma 2.6.36. *The groupoid $(\mathfrak{fG}, *, 0_{\mathfrak{Gr}})$ is symmetric strict monoidal with no zero divisors and $Aut_{\mathfrak{fG}}(0_{\mathfrak{Gr}}) = \{id_{0_{\mathfrak{Gr}}}\}$.*

Notation 2.6.37. We denote by $Id_{\mathfrak{UfG}}$ the identity endofunctor of $(\mathfrak{UfG}, *, 0)$.

Let $\mathfrak{fG}_{H,H_0}$ be the full subgroupoid of $\mathfrak{fG}$ of the groups $\{H_m\}_{m\in\mathbb{N}}$. Note that $Obj(\mathfrak{fG}_{H,H_0}) \cong \mathbb{N}$ and that the groupoid $\mathfrak{fG}_{H,H_0}$ is finitely generated by the free product in $(\mathfrak{fG}, *, 0_{\mathfrak{Gr}})$.

Corollary 2.6.38. *With the previous notations, $\left\{\mathcal{H}_{fp}, \mathfrak{fG}_{H,H_0}, \mathfrak{fG}, \varsigma_{-,t}\right\}$ (where $\varsigma_{n,t} : H_n \to Aut(H_{n+1})$ is the trivial morphism for all natural numbers n) defines a reliable Long-Moody systems.*

Proof. By Lemma 2.6.36, Assumption 2.2.1 is satisfied using the groupoid $(\mathfrak{fG}, *, 0_{\mathfrak{Gr}})$. Remark 2.6.37 ensures that Assumptions 2.2.13 and 2.5.16 are satisfied using the identity functor $Id_{\mathfrak{UfG}}$. We conclude using Remark 2.6.23. ☐

Remark 2.6.39. Assume that $R = \mathbb{Z}$. Let $H = \pi_1(P)$, with P an orientable prime 3-manifold different from the 3-disc $\mathbb{D}^3$, whose diffeomorphism group surjects onto the automorphism group of its fundamental group. Let $H_0 = \pi_1(M)$, with M a finite connected sum of prime 3-manifolds different from the 3-disc $\mathbb{D}^3$, whose diffeomorphisms groups surject onto the automorphism groups of their fundamental groups. Under mild additional technical assumptions on P and M (see hypothesis (1), (2) and (3) of [RWW17, Theorem 5.2.2]), according to [RWW17, Theorem 5.7], there is homological stability for the automorphism groups $Aut(H^{*n} * H_0)$ with twisted coefficients given by all the iterations of $\mathbf{LM}_{\left\{\mathcal{H}_{fp}, \mathfrak{fG}_{H,H_0}, \varsigma_{n,conj}\right\}}$ and $\mathbf{LM}_{\left\{\mathcal{H}_{fp}, \mathfrak{fG}_{H,H_0}, \varsigma_{n,t}\right\}}$ on a very strong polynomial functor M of $\mathbf{Fct}\left(\mathfrak{UfG}_{\mathbb{Z},0_{\mathfrak{Gr}}}, \mathbb{Z}\text{-}\mathfrak{Mod}\right)$ using Theorem 2.5.29.

Assume now that H_0 is the trivial group and $H = \mathbb{Z}$. By Theorem 2.5.29, [RWW17, Theorem 5.4] ensures that there is homological stability for the automorphism groups of free groups $Aut(\mathbf{F}_n)$ with twisted coefficients given by all the iterations of $\mathbf{LM}_{\left\{\mathcal{H}_{fp}, \mathfrak{fG}_{H,H_0}, \mathfrak{fG}, \varsigma_{n,t}\right\}}$ on a very strong polynomial functor M of $\mathbf{Fct}\left(\mathfrak{UfG}_{\mathbb{Z},0_{\mathfrak{Gr}}}, \mathbb{Z}\text{-}\mathfrak{Mod}\right)$.

Example 2.6.40. Let $\mathfrak{a}_R : \mathfrak{gr} \to R\text{-}\mathfrak{Mod}$ denote the abelianisation functor tensorized by R, with $\mathfrak{gr}$ the category introduced in Notation 2.3.10. This functor is a fundamental object in the category $\mathbf{Fct}(\mathfrak{gr}, R\text{-}\mathfrak{Mod})$. Indeed, the stable homology computations for automorphism groups $Aut(\mathbf{F}_n)$ with twisted coeffcients of [DV15] rely heavily on the study of the functor $\mathfrak{a}_R$: a polynomial functor in the category $\mathbf{Fct}(\mathfrak{gr}, R\text{-}\mathfrak{Mod})$ can be obtained by extensions of functors factoring through $\mathfrak{a}_R$ (see [DV15, Section 2]) and a general cancellation criterion for functor homology groups can be drawn from an explicit projective resolution of $\mathfrak{a}_R$ (see [DV15, Section 4]). Furthermore, Satoh computes the homology groups $H_1(Aut(\mathbf{F}_n), \mathfrak{a}_{\mathbb{Z}}(n))$ for $n \geq 2$ in [Sat06].

We consider the functor $i : \mathfrak{UfG}_{\mathbb{Z},0_{\mathfrak{Gr}}} \to \mathfrak{gr}$ defined in [DV15, Definition 4.2]. More precisely, it is the identity on objects and it sends a morphism $[n_2 - n_1, g] : \mathbb{Z}^{*n_1} \to \mathbb{Z}^{*n_2}$ of $\mathfrak{UfG}_{\mathbb{Z},0_{\mathfrak{Gr}}}$ (where $g \in Aut_{\mathfrak{Gr}}(\mathbb{Z}^{*n_2})$) to the morphism $g \circ \left(\iota_{\mathbb{Z}^{*(n_2-n_1)}} * id_{\mathbb{Z}^{*n_1}}\right) : \mathbb{Z}^{*n_1} \hookrightarrow \mathbb{Z}^{*n_2}$ of $\mathfrak{gr}$. Hence, we define a functor $\mathfrak{a}_R \circ i : \mathfrak{UfG}_{\mathbb{Z},0_{\mathfrak{Gr}}} \to R\text{-}\mathfrak{Mod}$. We deduce from Lemma 2.2.38 that:

$$\mathfrak{a}_R \circ i \cong \mathbf{LM}_{\left\{\mathcal{H}_{fp}, \mathfrak{fG}_{H,H_0}, \mathfrak{fG}, \varsigma_{n,t}\right\}}(R).$$

Remark 2.6.41. As pointed out in Proposition 2.2.39, $\mathfrak{a}_R \circ i$ is enough to determine all the iterations of $\mathbf{LM}_{\left\{\mathcal{H}_{fp}, \mathfrak{fG}_{H,H_0}, \mathfrak{fG}, \varsigma_{n,t}\right\}}$ on an object F of $\mathbf{Fct}\left(\mathfrak{UfG}_{\mathbb{Z},0_{\mathfrak{Gr}}}, R\text{-}\mathfrak{Mod}\right)$.

2.6.3.3 Mapping class groups of 3-manifolds

Let us introduce from [RWW17, Section 5.6] the suitable category to work with. Namely:

Definition 2.6.42. The oriented 3-manifold groupoid $\mathfrak{M}_3^+$ and $\overline{\mathfrak{M}_3}^+$ are the groupoids defined by:

- Objects: compact, connected, oriented 3-manifold M with boundary, equipped with a marked disc $\mathbb{D}^2 \hookrightarrow \partial M$ in its boundary;
- Morphisms: :
  - for $\mathfrak{M}_3^+$: the isotopy classes of orientation preserving diffeomorphisms restricting to the identity on the disc, denoted by $\pi_0\left(Diff\left(M \text{ rel } \mathbb{D}^2\right)\right)$;
  - for $\overline{\mathfrak{M}_3}^+$: $\pi_0\left(Diff\left(M \text{ rel } \mathbb{D}^2\right)\right)$ modulo Dehn twists along spheres and discs, denoted by

$$\pi_0\left(Diff\left(M \text{ rel } \mathbb{D}^2\right)\right)/_{twists}.$$

Henceforth, we fix two objects M_0 and M of $\mathfrak{M}_3^+$.

Recall from [RWW17, Section 5.7] that the boundary connected sum along marked half-discs $\natural$ defines a monoidal product for $\mathfrak{M}_3^+$ and $\overline{\mathfrak{M}_3}^+$, and the unit disc $\mathbb{D}^3$ is the unit. The braiding of the monoidal structure is given by doing half a Dehn twist in a neighbourhood of the marked half-discs $b_{M_1,M_2}^{\mathfrak{M}_3^+} : M_1\natural M_2 \to M_2\natural M_1$ and it is a symmetry. By the Poincaré conjecture, there are no zero divisors in $\mathfrak{M}_3^+$ and $\overline{\mathfrak{M}_3}^+$. We refer to [RWW17, Section 5.7] for more technical details on this operation.

Notation 2.6.43. We denote by $\mathfrak{M}_{3,(M,M_0)}^+$ (resp. $\overline{\mathfrak{M}}_{3,(M,M_0)}^+$) the full subgroupoid of $\mathfrak{M}_3^+$ (resp. $\overline{\mathfrak{M}_3}^+$) generated by the boundary connected sums of $\mathbb{D}^3$, M_0 and M.

Corollary 2.6.44. *The groupoids* $\left(\mathfrak{M}_{3,(M,M_0)}^+, \natural, \mathbb{D}^3\right)$ *and* $\left(\overline{\mathfrak{M}}_{3,(M,M_0)}^+, \natural, \mathbb{D}^3\right)$ *are small symmetric strict monoidal with no zero divisors and automorphisms group of* $\mathbb{D}^3$ *in each of these groupoids is trivial.*

For N an object of $\mathfrak{M}_3^+$, the mapping class group $\pi_0\left(Diff\left(N \text{ rel } \mathbb{D}^2\right)\right)$ (and a fortiori $\pi_0\left(Diff\left(N \text{ rel } \mathbb{D}^2\right)\right)/_{twists}$) acts on the fundamental group $\pi_1(N)$. We define functors:

$$\pi_1 : \left(\mathfrak{M}_{3,(M,M_0)}^+, \natural, \mathbb{D}^3\right) \to \left(\mathfrak{fG}, *, 0_{\mathfrak{Gr}}\right) \quad \text{and} \quad \pi_1^{/twists} : \left(\overline{\mathfrak{M}}_{3,(M,M_0)}^+, \natural, \mathbb{D}^3\right) \to \left(\mathfrak{fG}, *, 0_{\mathfrak{Gr}}\right).$$

Recall that by Van Kampen's theorem $\pi_1\left(M_1\natural M_2\right) \cong \pi_1\left(M_1\right) * \pi_1\left(M_2\right)$ for two objects M_1 and M_2 of $\mathfrak{M}_3^+$. Recall the symmetric symmetric monoidal category $\left(\mathfrak{Gr}_{\pi_1(M),\pi_1(M_0)}, *, 0_{\mathfrak{Gr}}\right)$ introduced in Notation 2.5.15 and the groupoid $\mathfrak{fG}$ defined in Section 2.6.3.2. Repeating mutatis mutandis the work of Lemma 2.3.11 and Proposition 2.3.12, we deduce that:

Proposition 2.6.45. *The following functors are symmetric strict monoidal:*

$$\pi_1 : \left(\mathfrak{M}_{3,(M,M_0)}^+, \natural, \mathbb{D}^3\right) \to \left(\mathfrak{Gr}_{\pi_1(M),\pi_1(M_0)}, *, 0_{\mathfrak{Gr}}\right) \quad and \quad \pi_1^{/twists} : \left(\overline{\mathfrak{M}}_{3,(M,M_0)}^+, \natural, \mathbb{D}^3\right) \to \left(\mathfrak{Gr}_{\pi_1(M),\pi_1(M_0)}, *, 0_{\mathfrak{Gr}}\right).$$

They define functors $\pi_1 : \mathfrak{UM}_{3,(M,M_0)}^+ \to \mathfrak{Gr}$ *and* $\pi_1^{/twists} : \overline{\mathfrak{UM}}_{3,(M,M_0)}^+ \to \mathfrak{Gr}$ *assigning for* $M_1, M_2 \in \{M, M_0\}$*:*

$$\pi_1\left(\left[M_1, id_{M_1\natural M_2}\right]\right) = \iota_{\pi_1(M_1)} \oplus id_{\pi_1(M_2)} \quad and \quad \pi_1^{/twists}\left(\left[M_1, id_{M_1\natural M_2}\right]\right) = \iota_{\pi_1^{/twists}(M_1)} \oplus id_{\pi_1^{/twists}(M_2)}.$$

Let $\mathfrak{M}_{3,M,M_0}^+$ (resp. $\overline{\mathfrak{M}}_{3,M,M_0}^+$) be the full subgroupoid of $\mathfrak{M}_{3,(M,M_0)}^+$ (resp. $\overline{\mathfrak{M}}_{3,(M,M_0)}^+$) of the groups $\left\{M^{\natural m}\natural M_0\right\}_{m\in\mathbb{N}}$. Note that $Obj\left(\mathfrak{M}_{3,M,M_0}^+\right) = Obj\left(\overline{\mathfrak{M}}_{3,M,M_0}^+\right) \cong \mathbb{N}$ and that the groupoid $\mathfrak{M}_{3,M,M_0}^+$ (resp. $\overline{\mathfrak{M}}_{3,M,M_0}^+$) is finitely generated by the free product in $\left(\mathfrak{M}_{3,(M,M_0)}^+, \natural, \mathbb{D}^3\right)$ (resp. $\left(\overline{\mathfrak{M}}_{3,(M,M_0)}^+, \natural, \mathbb{D}^3\right)$).

Proposition 2.6.46. *With the previous notations,* $\left\{\pi_1, \mathfrak{M}^+_{3,M,M_0}, \mathfrak{M}^+_{3,(M,M_0)}, \varsigma_{n,t}\right\}$ *and* $\left\{\pi_1^{/twists}, \overline{\mathfrak{M}}^+_{3,M,M_0}, \overline{\mathfrak{M}}^+_{3,(M,M_0)}, \varsigma_{-,t}\right\}$ *define liftable Long-Moody systems, where* $\varsigma_{n,t}$ *is the trivial morphism in both cases for all natural numbers n (see Example 2.2.22).*

Proof. These Long-Moody systems are reliable by Remark 2.2.27, Assumptions 2.2.13 and 2.5.16 being satisfied by Proposition 2.6.45 and Assumption 2.2.1 being checked using the groupoids $\mathfrak{M}^+_{3,(M,M_0)}$ and $\overline{\mathfrak{M}}^+_{3,(M,M_0)}$ (by Corollary 2.6.44). We conclude using Remark 2.6.23. □

Handlebody mapping class groups: Take $M_0 = \mathbb{D}^2$ and $M = \mathbb{S}^1 \times \mathbb{D}^2$, with $\mathbb{S}^1$ the 1-sphere and $\mathbb{D}^2$ the 2-disc. Then, for all natural numbers n:

$$Aut_{\mathfrak{M}^+_{3,\mathbb{S}^1\times\mathbb{D}^2,\mathbb{D}^2}}(n) = \mathscr{H}_{n,1}$$

is the handlebody mapping class group of a surface of genus n fixing a disc on the boundary pointwise.

Example 2.6.47. Consider the liftable Long-Moody system $\left\{\pi_1, \mathfrak{M}^+_{3,\mathbb{S}^1\times\mathbb{D}^2,\mathbb{D}^2}, \mathfrak{M}^+_{3,(\mathbb{S}^1\times\mathbb{D}^2,\mathbb{D}^2)}, \varsigma_{-,t}\right\}$ of Proposition 2.6.46. In this case:

$$\pi_1\left(\mathbb{S}^1 \times \mathbb{D}^2\right)^{*n} = \langle f_1, \ldots, f_n\rangle \cong \mathbf{F}_n.$$

We denote by $H_1\left(\left(\mathbb{S}^1 \times \mathbb{D}^2\right)^{\natural -}, R\right)$ the functor induced by the functor $H_1(H_-, R)$ of Proposition 2.2.37. For all natural numbers n, the action of $\mathscr{H}_{n,1}$ on $H_1\left(\left(\mathbb{S}^1 \times \mathbb{D}^2\right)^{\natural n}, R\right)$ is the natural representation of the handlebody mapping class group $\mathscr{H}_{n,1}$. We deduce from Lemma 2.2.38 that:

$$H_1\left(\left(\mathbb{S}^1 \times \mathbb{D}^2\right)^{\natural -}, R\right) \cong \mathbf{LM}_{\left\{\pi_1, \mathfrak{M}^+_{3,\mathbb{S}^1\times\mathbb{D}^2,\mathbb{D}^2}, \varsigma_{n,t}\right\}}(R).$$

Furthermore, recall that the handlebody mapping class group $\mathscr{H}_{n,1}$ is a subgroup of the mapping class group $\mathbf{\Gamma}_{n,1}$ of the surface $\Sigma^0_{n,0,1}$ for all natural numbers n. Hence, we can define another Long-Moody system associated with handlebody mapping class groups. Recall the reliable Long-Moody $\left\{\mathcal{H}_g, \mathfrak{M}_2^{+,s}, \mathfrak{M}_2, \varsigma_{n,t}\right\}$ system of Section 2.3.3.1.

Proposition 2.6.48. *The setting* $\left\{\mathcal{H}_g, \mathfrak{M}^+_{3,\mathbb{S}^1\times\mathbb{D}^2,\mathbb{D}^2}, \mathfrak{M}^+_{3,(\mathbb{S}^1\times\mathbb{D}^2,\mathbb{D}^2)}, \varsigma_{-,t}\right\}$ *(with* $\varsigma_{n,t} : \pi_1\left(\Sigma^0_{n,0,1}, p\right) \to \mathscr{H}_{n+1,1}$ *the trivial morphism for all natural numbers n) is a liftable Long-Moody system.*

Proof. Since we consider the family of trivial morphisms, by Remarks 2.2.27 and 2.6.23, it is enough to check that Assumptions 2.2.13 , 2.5.16 and 2.2.1 are satisfied.

Assumption 2.2.1 is checked using the groupoid $\mathfrak{M}_2$ (see Section 2.3.2). The groupoid $\mathfrak{M}^+_{3,\mathbb{S}^1\times\mathbb{D}^2,\mathbb{D}^2}$ is a subgroupoid of $\mathfrak{M}_2$ using the embeddings $\mathscr{H}_{n,1} \hookrightarrow \mathbf{\Gamma}_{n,1}$ for all natural numbers n. Therefore Assumptions 2.2.13 and 2.5.16 are satisfied repeating mutatis mutandis the work of Lemma 2.3.11 and Proposition 2.3.12. □

Example 2.6.49. Assume that $\pi_1\left(\Sigma^0_{1,0,1}, p\right)$ acts trivially on the commutative ring R. We denote by $H_1(\Sigma_{-,1}, R)_{\mathbb{S}^1\times\mathbb{D}^2}$ the functor induced by the functor $H_1(H_-, R)$ of Proposition 2.2.37. We deduce from Lemma 2.2.38 that:

$$H_1(\Sigma_{-,1}, R)_{\mathbb{S}^1\times\mathbb{D}^2} \cong \mathbf{LM}_{\left\{\mathcal{H}_g, \mathfrak{M}^+_{3,\mathbb{S}^1\times\mathbb{D}^2,\mathbb{D}^2}, \varsigma_{n,t}\right\}}(R).$$

Remark 2.6.50. In [IS17], Ishida and Sato compute the homology groups $H_1\left(\mathscr{H}_{n,1}, H_1(\Sigma_{-,1}, R)_{\mathbb{S}^1\times\mathbb{D}^2}\right)$ for all natural numbers n.

2.7 Tensorial functors

This last section introduces a new construction of which generalised Long-Moody functors are particular cases. It is inter alia useful for the forthcoming work [Sou18]. We fix a groupoid $\mathcal{G}$ such that $Obj(\mathcal{G}) \cong \mathbb{N}$ such that Assumption 2.2.1 is satisfied. Recall that for all natural numbers n, the automorphism groups $Aut_{\mathcal{G}}(\underline{n})$ are denoted by G_n.

2.7.1 Tools and framework

Let us first generalise the framework of Section 2.2.1.

2.7.1.1 Tensorial framework

Throughout this section, we fix two groups H and H_0, λ a natural number and an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$.

Assumption 2.7.1. *There exists a functor $\mathcal{H}_{\varphi} : \mathfrak{U}\mathcal{G} \to \mathfrak{Gr}$ such that:*

- *for all objects $\underline{n}$ of $\mathcal{G}$, $\mathcal{H}\left(\underline{n}\right) = H^{*\varphi(n)} * H_0$;*
- *$\mathcal{H}\left(\left[1, id_{\underline{n+1}}\right]\right) = \iota_{H^{*(\varphi(m+1)-\varphi(m))}} * id_{\mathcal{H}_{\varphi}(m)}$ for all natural numbers n.*

Remark 2.7.2. Assigning $\varphi = id_{\mathbb{N}}$, we recover the situation of Assumption 2.2.13.

Notation 2.7.3. For all natural numbers, we denote by $\mathscr{A}_{\varphi(n)} : G_n \to Aut_{\mathcal{G}}\left(H_{\varphi(n)}\right)$ the morphism induced by the functor $\mathcal{H}_{\varphi}$ by Assumption 2.7.1.

Generalising Section 2.2.1.3, we need two additional conditions for our framework. First, we require:

Condition 2.7.4. There exist group morphisms $\left\{\varsigma_n : H_{\varphi(n)} \to G_{\lambda+n}\right\}_{n\in\mathbb{N}}$ such that for all elements $h \in H_{\varphi(n)}$, for all natural numbers n and n' such that $n' \geq n$, the following diagram is commutative in the category $\mathfrak{U}\mathcal{G}$:

$$\begin{array}{ccc}
\lambda\natural\underline{n} & \xrightarrow{\varsigma_n(h)} & \lambda\natural\underline{n} \\
{\scriptstyle id_{\lambda}\natural\left[n'-n, id_{\underline{n'}}\right]}\downarrow & & \downarrow{\scriptstyle id_{\lambda}\natural\left[n'-n, id_{\underline{n'}}\right]} \\
\lambda\natural\underline{n'} & \xrightarrow[\varsigma_{n'}\left(\mathcal{H}\left(\left[n'-n, id_{\underline{n'}}\right]\right)(h)\right)]{} & \lambda\natural\underline{n'}.
\end{array}$$

Once a choice of morphisms $\left\{\varsigma_n : H_{\varphi(n)} \to G_{\lambda+n}\right\}_{n\in\mathbb{N}}$ satisfying the Condition 2.7.4 is made, we require:

Condition 2.7.5. Let n be a natural number. We assume that the morphism given by the coproduct $H_{\varphi(n)} * G_n \to G_{\lambda+n}$ factors across the canonical surjection to $H_{\varphi(n)} \underset{\mathscr{A}_{\varphi(n)}}{\rtimes} G_n$. In other words, the following diagram is commutative:

$$\begin{array}{ccccc}
H_{\varphi(n)} & \hookrightarrow & H_{\varphi(n)} \underset{\mathscr{A}_{\varphi(n)}}{\rtimes} G_n & \hookleftarrow & G_n \\
& {\scriptstyle \varsigma_n}\searrow & \downarrow & \swarrow{\scriptstyle id_{\lambda}\natural-} & \\
& & G_{\lambda+n}. & &
\end{array}$$

where the morphism $H_{\varphi(n)} \underset{\mathscr{A}_{\varphi(n)}}{\rtimes} G_n \to G_{\lambda+n}$ is induced by the morphism $H_{\varphi(n)} * G_n \to G_{\lambda+n}$ and the group morphism $id_{\lambda}\natural- : G_n \to G_{\lambda+n}$ is induced by the monoidal structure of Assumption 2.2.1.

Definition 2.7.6. We say that the groupoid $\mathcal{G}$, $\lambda \in \mathbb{N}$, $\varphi : \mathbb{N} \to \mathbb{N}$, the functor $\mathcal{H}$ of Assumption 2.7.1 and morphisms $\left\{\varsigma_n : H_{\varphi(n)} \to G_{\lambda+n}\right\}_{n\in\mathbb{N}}$ form a tensorial framework, denoted by $\left\{\lambda, \varphi, \mathcal{H}_{\varphi}, \mathcal{G}, \varsigma_n\right\}$, if Assumptions 2.7.1 and Conditions 2.7.4 and 2.7.5 are satisfied.

Example 2.7.7. Assigning $\lambda = 1$ and $\varphi = id_{\mathbb{N}}$, we recover the definition of a Long-Moody system of Definition 2.2.26.

2.7.1.2 Tensorial functor category

Fix a tensorial framework $\{\lambda, \varphi, \mathcal{H}_\varphi, \mathcal{G}, \varsigma_n\}$. The work of Section 2.7.2 requires introducing a subcategory of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$.

Definition 2.7.8. Let $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}}$ be the full subcategory of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ on objects I such that the R-module $I(\underline{n})$ has a right $R\left[H_{\varphi(n)}\right]$-module structure for all natural numbers n, given by:

$$I\left(\left[n'-n,g\right]\right)(i\cdot h) = I\left(\left[n'-n,g\right]\right)(i)\cdot\mathcal{H}_\varphi\left(\left[n'-n,g\right]\right)(h), \tag{2.7.1}$$

for all $h \in H_{\varphi(n)}$, $i \in \mathcal{I}(\underline{n})$ and $[n'-n,g] \in Hom_{\mathfrak{U}\mathcal{G}}(\underline{n},\underline{n'})$. It is called the tensorial functor category associated with the tensorial framework $\{\lambda, \varphi, \mathcal{H}_\varphi, \mathcal{G}, \varsigma_n\}$.

Example 2.7.9. Considering a Long-Moody system $\{\mathcal{H}, \mathcal{G}, \mathcal{G}', \varsigma_n\}$, the augmentation ideal functor of Definition 2.2.28 is a functor in the category $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})_{\{1,id_{\mathbb{N}},\mathcal{H},\mathcal{G},\varsigma_n\}}$.

2.7.2 Definition of tensorial right functors

We fix a tensorial framework $\{\lambda, \varphi, \mathcal{H}_\varphi, \mathcal{G}, \varsigma_n\}$ throughout this section.

Notation 2.7.10. When there is no ambiguity, we omit the notation for the tensorial framework $\{\lambda, \varphi, \mathcal{H}_\varphi, \mathcal{G}, \varsigma_n\}$.

Theorem 2.7.11. *The following assignment defines a bifunctor*

$$\mathfrak{T}_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}} : \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}} \times \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}),$$

called the tensorial functor associated with the coherent Long-Moody system $\{\lambda, \varphi, \mathcal{H}_\varphi, \mathcal{G}, \mathcal{G}', \varsigma_n\}$.

- *Objects: for* $I \in Obj\left(\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}}\right)$ *and* $F \in Obj(\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}))$, $\mathfrak{T}_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}}(I,F) : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$ *is defined by:*
  - *Objects:* $\forall \underline{n} \in Obj(\mathcal{G})$, $\mathfrak{T}_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}}(I,F)(\underline{n}) = I(\underline{n}) \underset{R\left[H_{\varphi(n)}\right]}{\otimes} F(\underline{\lambda+n})$.
  - *Morphisms: for* $n, n' \in \mathbb{N}$, *such that* $n' \geq n$, *and* $[n'-n,g] \in Hom_{\mathfrak{U}\mathcal{G}}(\underline{n},\underline{n'})$, *assign:*
  $$\mathfrak{T}_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}}(I,F)\left(\left[n'-n,g\right]\right) = I\left(\left[n'-n,g\right]\right) \underset{R\left[H_{\varphi(n')}\right]}{\otimes} F\left(id_\lambda \natural \left[n'-n,g\right]\right).$$
- *Morphisms: let* F *and* G *be objects of* $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, I *and* J *be objects of* $Obj\left(\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}}\right)$, $\eta : F \to G$ *and* $\mu : I \to J$ *be natural transformations. For all objects* $\underline{n}$ *of* $\mathcal{G}$, *we define* $\mathfrak{T}_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}}(\mu,\eta) : \mathfrak{T}_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}}(I,F) \to \mathfrak{T}_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}}(J,G)$ *by:*
$$\left(\mathfrak{T}_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}}(F)(\eta)\right)_{\underline{n}} = \mu_{\underline{n}} \underset{R\left[H_{\varphi(n)}\right]}{\otimes} \eta_{\underline{\lambda+n}}.$$

Proof. The proof generalises the one of Theorem 2.2.30. For this proof, F, G and H are objects of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, I, J and K are two objects of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})_{\{\lambda,\varphi,\mathcal{H}_\varphi,\mathcal{G},\varsigma_n\}}$, n, n' and n'' are natural numbers such that $n'' \geq n' \geq n$. There are three points to check.

1. First, we have to preove that the assignment of $\mathfrak{T}(I,F)$ on morphisms makes sense. Considering $[n'-n,g]$ and $[n'-n,g']$ such that $[n'-n,g] = [n'-n,g']$, ie assume that there exists $\psi \in G_{n'-n}$ so that $g' \circ (\psi \natural id_n) = g$. Since the monoidal product $\natural$ is well-defined on $\mathfrak{U}\mathcal{G}'$ (see Proposition 2.1.12), recalling that I and F are functors on $\mathfrak{U}\mathcal{G}$, we deduce that:
$$I\left(\left[n'-n,g\right]\right) \underset{R}{\otimes} F\left(id_\lambda \natural \left[n'-n,g\right]\right) = I\left(\left[n'-n,g'\right]\right) \underset{R}{\otimes} F\left(id_\lambda \natural \left[n'-n,g'\right]\right).$$

So it remains to check the compatibility of the assignment $\mathfrak{T}(I,F)$ with respect to the tensor products, ie to show that for all $h \in H_{\varphi(n)}$ and $i \in \mathcal{I}(\underline{n})$:

$$I\left(\left[n'-n,g\right]\right)(i\cdot h)\underset{R\left[H_{\varphi(n')}\right]}{\otimes}F\left(id_{\lambda}\natural\left[n'-n,g\right]\right) = I\left(\left[n'-n,g\right]\right)\underset{R\left[H_{\varphi(n')}\right]}{\otimes}F\left(id_{\lambda}\natural\left[n'-n,g\right]\right)\circ F\left(\varsigma_{n}(h)\right)$$

Recalling the equality 2.7.1, the compatibility with respect to the tensor product amounts to proving that:

$$F\left(id_{\lambda}\natural\left[n'-n,g\right]\right)\circ F\left(\varsigma_{n}(h)\right) = F\left(\varsigma_{n'}\left(\mathcal{H}_{\varphi}\left(\left[n'-n,g\right]\right)(h)\right)\right)\circ F\left(id_{\lambda}\natural\left[n'-n,g\right]\right).$$

This is a direct consequence of Conditions 2.7.4 and 2.7.5.

2. Let us prove that the assignment $\mathfrak{T}(I,F)$ is a functor. The functorialities of I and F over $\mathfrak{U}\mathcal{G}$ and from the compatibility of the monoidal structure $\natural$ with composition imply the composition axiom and that:

$$\mathfrak{T}(I,F)\left(id_{G_n}\right) = id_{\mathfrak{T}(I,F)(\underline{n})}.$$

3. The remaining point to check for $\mathfrak{T}$ to be a functor is the consistency of our definition on morphisms. For $\eta : F \to G$ and $\mu : I \to J$ natural transformations, it is clear to check that:

$$\mathfrak{T}(J,G)\left(\left[n'-n,g\right]\right)\circ\mathfrak{T}(\mu,\eta)_{\underline{n}} = \mathfrak{T}(\mu,\eta)_{\underline{n'}}\circ\mathfrak{T}(I,F)\left(\left[n'-n,g\right]\right).$$

Therefore $\mathfrak{T}(\mu,\eta)$ is a morphism in the category $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$. It is clear that $\mathfrak{T}(id_I, id_F) = id_{\mathfrak{T}(I,F)}$. Finally, let $\eta : F \to G$, $\eta' : G \to H$, $\mu : I \to J$ and $\mu : J \to K$ be natural transformations. Let n be a natural number. Now, because η, η', μ and μ' are morphisms in the category $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}\right)$, we deduce that:

$$\mathfrak{T}\left(\left(\mu',\eta'\right)\circ(\mu,\eta)\right)_{\underline{n}} = \left(\mu'_{\underline{n}}\circ\mu_{\underline{n}}\right)\underset{R\left[H_{\varphi(n)}\right]}{\otimes}\left(\eta'_{\underline{\lambda+n}}\circ\eta_{\underline{\lambda+n}}\right)(v) = \mathfrak{T}\left(\mu',\eta'\right)_{\underline{n}}\circ\mathfrak{T}(\mu,\eta)_{\underline{n}}.$$

□

Example 2.7.12. Consider a Long-Moody system $\{\mathcal{H},\mathcal{G},\mathcal{G}',\varsigma_n\}$ and the augmentation ideal functor $\mathcal{I}$ of Definition 2.2.28. Then:

$$\mathbf{LM}_{\{\mathcal{H},\mathcal{G},\varsigma_n\}} = \mathfrak{T}_{\{1,id_{\mathbb{N}},\mathcal{H},\mathcal{G},\varsigma_n\}}\left(\mathcal{I},-\right).$$

Chapter 3

Computations of stable homology with twisted coefficients for mapping class groups

Abstract: *In this paper, we compute the stable homology of braid groups, mapping class groups of surfaces and of automorphism groups of certain right-angled Artin groups with twisted coefficients. On the one hand, the computations are led using semidirect product structures arising naturally from these groups. On the one hand, we compute the stable homology with twisted coefficients by FI-modules. This notably uses a decomposition result of the stable homology with twisted coefficients due to Djament and Vespa for symmetric monoidal categories, and we take this opportunity to extend this result to pre-braided monoidal categories.*

Introduction

In [RWW17], Randal-Williams and Wahl prove homological stability for some families of mapping class groups of surfaces and 3-manifolds, with twisted coefficients given by particular kind of functors. Namely, they consider a set of groups $\{G_n\}_{n\in\mathbb{N}}$ such that there exists a canonical injection $G_n \hookrightarrow G_{n+1}$ for all natural numbers n. We denote by $\mathcal{G}$ the groupoid with natural numbers as objects and the groups $\{G_n\}_{n\in\mathbb{N}}$ as automorphism groups, by $\mathfrak{U}\mathcal{G}$ the Quillen's bracket construction on $\mathcal{G}$ (see Section 3.1), and by $\mathbf{Ab}$ the category of abelian groups. Randal-Williams and Wahl show that if $F : \mathfrak{U}\mathcal{G} \to \mathbf{Ab}$ is a very strong polynomial functor of degree d (see [Sou17a, Section 4] for this notion), then the canonical induced maps

$$H_*(G_n, F(n)) \to H_*(G_{n+1}, F(n+1))$$

are isomorphisms for $N(*, d) \leq n$ with some $N(*, d) \in \mathbb{N}$ depending on $*$ and d. The value of the homology for $n \geq N(*, d)$ is called the stable homology of the family of groups $\{G_n\}_{n\in\mathbb{N}}$ and denoted by $H_*(G_\infty, F_\infty)$.

In this paper, we are interested in explicit computations of the stable homology with twisted coefficients for mapping class groups of surfaces and 3-manifolds. On the one hand, we use semidirect product structures naturally arising from mapping class groups to compute their stable homology with particular twisted coefficients. Namely, on the strength of Lyndon-Hochschild-Serre spectral sequence, we prove:

Theorem A **(Proposition 3.2.18, Theorems 3.2.17, 3.2.29 and 3.2.45).** *We have:*

1. *For n a natural number, we denote by $\mathbf{B}_n$ the braid group on n strands, by $\mathfrak{Cor}(n)$ the complex Coxeter representation* and by $\mathfrak{Bur}_t(n)$ the unreduced Burau representation of $\mathbf{B}_n$. *From the stability result [Gor78, Theorem C], we deduce that for all natural numbers $n \geq q+2$:*

$$H_q(\mathbf{B}_n, \mathfrak{Cor}(n)) \cong \begin{cases} \mathbb{C}^{\oplus 2} & \text{if } q \geq 2, \\ \mathbb{C} & \text{if } q = 0, 1. \end{cases}$$

Hence, we recover a result of [Vas92, Chapter II, Section 5]. Moreover, for all natural numbers $n \geq 3$ and $q \geq 3$:

$$H_q\left(\mathbf{B}_n, \mathfrak{Bur}_t(n)\right) \cong \begin{cases} \mathbb{C}\left[t^{\pm 1}\right]/(1-t) & \text{if } 3 \leq q < n-2, \\ \mathbb{C}\left[t^{\pm 1}\right]/(1-t) & \text{if } q = n-2 \text{ and } n \text{ is odd}, \\ \mathbb{C}\left[t^{\pm 1}\right]/\left(1-t^2\right) & \text{if } q = n-2 \text{ and } n \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

2. *We denote by $\Gamma_{g,1}$ the isotopy classes of diffeomorphisms restricting to the identity on the boundary component of a compact connected orientable surface with one boundary component and genus $g \geq 0$. Then, from the stability results of [Bol12, CM09], for m, n and q natural numbers such that $2n \geq 3q + m$, there is an isomorphism:*

$$H_q\left(\Gamma_{n,1}, H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m}\right) \cong \bigoplus_{\left\lfloor \frac{q-1}{2} \right\rfloor \geq k \geq 0} H_{q-(2k+1)}\left(\Gamma_{n,1}, H_1\left(\Sigma_{n,1}, \mathbb{Z}\right)^{\otimes m-1}\right).$$

Hence, we recover inter alia results of [Har91] and [Kaw08].

3. *We denote by $A^s_{n,k}$ the group of path-components of the space of homotopy equivalences of the space $\mathscr{G}^s_{n,k}$ with $n \in \mathbb{N}$ circles, $k \in \mathbb{N}$ distinguished circles and $s \in \mathbb{N}$ basepoints (we refer the reader to Section 3.2.2.4 for an introduction to these groups). Let $s \geq 2$ and $q \geq 1$ be natural numbers and $F : \mathfrak{gr} \to \mathbf{Ab}$ a reduced polynomial functor where $\mathfrak{gr}$ denotes the category of finitely generated free groups. Then, from the stability results of [HW10], for all natural numbers $n \geq 2q+1$:*

$$H_q\left(A^s_{n,0}, F(n)\right) = 0.$$

Moreover, $H_q\left(A^s_{n,k}, \mathbb{Q}\right) = 0$ for all natural numbers $n \geq 2q+2$ and $k \geq 0$. We thus recover the results of [Jen04] for holomorphs of free groups.

On the other hand, we deal with stable homology for mapping class groups with twisted coefficients factoring through some finite groups. Let $(\Sigma, \sqcup, 0)$ (resp. $(W\Sigma, \sqcup, 0)$) be the symmetric monoidal groupoid with objects the natural numbers and automorphism groups the symmetric groups (resp. hyperoctahedral groups). Note that Quillen's bracket construction $\mathfrak{U}\Sigma$ (see Section 3.1) is equivalent to the category FI of finite sets and injections used inter alia in [CEF15]. For R a commutative ring, R-$\mathfrak{Mod}$ denotes the category of R-modules. We prove the following results.

***Theorem B* (Proposition 3.4.14, Proposition 3.4.22, Proposition 3.4.26, Corollary 3.4.30).** *Let $\mathbb{K}$ be a field of characteristic zero and d be a natural number. Considering functors $F : FI \to \mathbb{K}$-$\mathfrak{Mod}$ and $G : \mathfrak{U}(W\Sigma) \to \mathbb{K}$-$\mathfrak{Mod}$, we have:*

1. $H_d\left(\mathbf{B}_\infty, F_\infty\right) \cong \underset{n \in FI}{Colim}\left(H_d\left(\mathbf{PB}_n, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} F(n)\right)$ *where $\mathbf{B}_n$ (respectively $\mathbf{PB}_n$) denotes the braid (respectively pure braid) group on n strands.*

2. $H_d\left(\mathscr{S}_\infty, G_\infty\right) \cong \underset{n \in \mathfrak{U}(W\Sigma)}{Colim}\left(H_d\left(\mathscr{PS}_n, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} G(n)\right)$ *where $\mathscr{S}_n$ (respectively $\mathscr{PS}_n$) denotes the symmetric (respectively pure symmetric) automorphisms group of free group on n strands (we refer the reader to Section 3.4.2.2 for the definitions of these groups).*

3. $H_d\left(\Gamma^\infty_{\infty,1}, F_\infty\right) \cong \underset{n \in FI}{Colim}\left[\bigoplus_{k+l=d}\left(H_k\left(\Gamma_{n,1}, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} H_l\left(\left(\mathbb{C}P^\infty\right)^{\times n}, \mathbb{K}\right)\right) \underset{\mathbb{K}}{\otimes} F(n)\right]$, *where $\Gamma^s_{g,1}$ denotes the isotopy classes of diffeomorphisms permuting the marked points and restricting to the identity on the boundary component of a compact connected orientable surface with one boundary component, genus $g \geq 0$ and $s \geq 0$ marked points. In particular, $H_{2k+1}\left(\Gamma^\infty_{\infty,1}, F_\infty\right) = 0$ for all natural numbers k.*

4. $H_d\left(Aut\left(\left(\mathbb{Z}^{*k}\right)^{\times\infty}\right), F_\infty\right) = 0$ *for a fixed natural number $k \geq 2d+1$.*

The proof of Theorem B requires a splitting result for the twisted stable homology for some families of groups: this decomposition consists in the graded direct sum of tensor products of the homology of an associated category with the stable homology with constant coefficients. Namely, we consider a pre-braided locally homogeneous category $(\mathfrak{U}\mathcal{G}, \natural, 0)$ (we refer the reader to Section 3.1 for an introduction to these notions) such that the unit 0 is an initial object. We denote by $\mathfrak{U}\mathcal{G}_{(A,X)}$ the full subcategory of $\mathfrak{U}\mathcal{G}$ on the objects $\left\{A\natural X^{\natural n}\right\}_{n\in\mathbb{N}}$ and by $H_*\left(\mathfrak{U}\mathcal{G}_{(A,X)}, F\right)$ the homology of the category $\mathfrak{U}\mathcal{G}_{(A,X)}$ (we refer the reader to the papers [FP03, Section 2] and [DV10, Appendice A] for an introduction to this last notion). We prove the following statement.

***Theorem C* (Proposition 3.3.7) .** *Let $\mathbb{K}$ be a field. For all functors $F : \mathfrak{U}\mathcal{G}_{(A,X)} \to \mathbb{K}\text{-}\mathfrak{Mod}$, we have a natural isomorphism of $\mathbb{K}$-modules:*

$$H_*\left(G_\infty, F_\infty\right) \cong \bigoplus_{k+l=*}\left(H_k\left(G_\infty, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} H_l\left(\mathfrak{U}\mathcal{G}_{(A,X)}, F\right)\right).$$

If the groupoid $\mathcal{G}$ is symmetric monoidal, then Theorem C recovers the previous analogous results [DV10, Propositions 2.22, 2.26].

For sake of completeness, we finally recall that for braid groups, the homology with coefficients in the ring of Laurent polynomials $\mathbb{Z}\left[t^{\pm 1}\right]$ is computed by Callegaro in [Cal06], the one with coefficients in the Tong-Yang-Ma representations (see [TYM96]) is obtained by Callegaro, Moroni and Salvetti in [CMS08] and the one with coefficients in the reduced Burau representations is computed by Chen in [Che17]. Futhermore, the first stable homology group of compact connected non-orientable surfaces with one boundary component with coefficients in the first homology group of the considered surface is computed by Stukow in [Stu14].

The paper is organized as follows. In Section 3.1, we recall necessary notions on Quillen's bracket construction, pre-braided monoidal categories and locally homogeneous categories. In Section 3.2, after setting up the general framework for the families of groups we will deal with and applying Lyndon-Hochschild-Serre spectral sequence, we prove the various results of Theorem A. Section 3.3 is devoted to the proof of the splitting general result Theorem C for the stable homology. Finally, in Section 3.4, we deal with the twisted stable homology for mapping class groups with non-trivial finite quotient groups and prove Theorem B.

Notation 3.0.1. We fix R a commutative ring and $\mathbb{K}$ a field throughout this work. We denote by $R\text{-}\mathfrak{Mod}$ and $\mathbb{K}\text{-}\mathfrak{Mod}$ the categories of R-modules and $\mathbb{K}$-vector spaces.

We denote by $(\mathbb{N}, \leq)$ the category of natural numbers (natural means non-negative) considered as a directed set. For all natural numbers n, we denote by γ_n the unique element of $Hom_{(\mathbb{N},\leq)}(n, n+1)$. For all natural numbers n and n' such that $n' \geq n$, we denote by $\gamma_{n,n'} : n \to n'$ the unique element of $Hom_{(\mathbb{N},\leq)}(n, n')$, composition of the morphisms $\gamma_{n'-1} \circ \gamma_{n'-2} \circ \cdots \circ \gamma_{n+1} \circ \gamma_n$. The addition defines a strict monoidal structure on $(\mathbb{N}, \leq)$, denoted by $((\mathbb{N}, \leq), +, 0)$.

We denote by $\mathfrak{Gr}$ the category of groups and by $*$ the coproduct in this category. We denote by **Ab** the full subcategory of $\mathfrak{Gr}$ of abelian groups. We denote by $\mathfrak{gr}$ the full subcategory of $\mathfrak{Gr}$ of finitely generated free groups. The fee product of groups is denoted by $*$ and defines a monoidal structure over $\mathfrak{gr}$, with the trivial group $0_{\mathfrak{Gr}}$ the unit, denoted by $(\mathfrak{gr}, *, 0_{\mathfrak{Gr}})$. We denote by $\times$ the direct product of groups and by $Aut_{\mathfrak{Gr}}(G)$ (or $Aut(G)$) the automorphism group of a group G.

Let $\mathfrak{Cat}$ denote the category of small categories. Let $\mathfrak{C}$ be an object of $\mathfrak{Cat}$. We use the abbreviation $Obj(\mathfrak{C})$ to denote the objects of $\mathfrak{C}$. For $\mathfrak{D}$ a category, we denote by $\mathbf{Fct}(\mathfrak{C}, \mathfrak{D})$ the category of functors from $\mathfrak{C}$ to $\mathfrak{D}$. If 0 is initial object in the category $\mathfrak{C}$, then we denote by $\iota_A : 0 \to A$ the unique morphism from 0 to A. The maximal subgroupoid $\mathscr{G}\mathfrak{r}(\mathfrak{C})$ is the subcategory of $\mathfrak{C}$ which has the same objects as $\mathfrak{C}$ and of which the morphisms are the isomorphisms of $\mathfrak{C}$. We denote by $\mathscr{G}\mathfrak{r} : \mathfrak{Cat} \to \mathfrak{Cat}$ the functor which associates to a category its maximal subgroupoid.

Definition 3.0.2. A family of groups is a functor $\mathbf{G}_- : (\mathbb{N}, \leq) \longrightarrow \mathfrak{Gr}$ such that for all natural numbers n, $\mathbf{G}_-(\gamma_n) : \mathbf{G}_n \hookrightarrow \mathbf{G}_{n+1}$ is an injective group morphism.

For an introduction to braided monoidal categories, we refer to [ML13, Section XI]. Standardly, a strict monoidal category will be denoted by $(\mathfrak{C}, \natural, 0)$, where $\natural : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ is the monoidal structure and 0 is the monoidal unit. If the category is braided, we denote by $b^{\mathfrak{C}}_{-,-}$ its braiding.

3.1 Categorical framework

This section recollects Quillen's bracket construction, pre-braided monoidal categories and locally homogeneous categories for the convenience of the reader. It takes up the framework of [RWW17, Section 1].

3.1.1 Quillen's bracket construction

We fix a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$.

Definition 3.1.1. [RWW17, Section 1.1] Quillen's construction on the groupoid $\mathfrak{G}$, denoted by $\mathfrak{U}\mathfrak{G}$ is the category defined by:

- Objects: $Obj(\mathfrak{U}\mathfrak{G}) = Obj(\mathfrak{G})$;
- Morphisms: for A and B objects of $\mathfrak{G}$,

$$Hom_{\mathfrak{U}\mathfrak{G}}(A,B) = \underset{\mathfrak{G}}{colim}\left[Hom_{\mathfrak{G}}(-\natural A, B)\right].$$

  A morphism from A to B in the category $\mathfrak{U}\mathfrak{G}$ is an equivalence class of pairs (X, f), where X is an object of $\mathfrak{G}$ and $f : X\natural A \to B$ is a morphism of $\mathfrak{G}$; this is denoted by $[X, f] : A \to B$.

- For all objects X of $\mathfrak{U}\mathfrak{G}$, the identity morphism in the category $\mathfrak{U}\mathfrak{G}$ is given by $[0, id_X] : X \longrightarrow X$.
- Let $[X, f] : A \longrightarrow B$ and $[Y, g] : B \longrightarrow C$ be morphisms in the category $\mathfrak{U}\mathfrak{G}$. Then, the composition in the category $\mathfrak{U}\mathfrak{G}$ is defined by:

$$[Y, g] \circ [X, f] = [Y\natural X, g \circ (id_Y \natural f)].$$

Ìt is clear that the unit 0 of the monoidal structure of the groupoid $(\mathfrak{G}, \natural, 0)$ is an initial object in the category $\mathfrak{U}\mathfrak{G}$ (see [RWW17, Proposition 1.8 (i)]).

Definition 3.1.2. The strict monoidal category $(\mathfrak{G}, \natural, 0)$ is said to have no zero divisors if for all objects A and B of $\mathfrak{G}$, $A\natural B \cong 0$ if and only if $A \cong B \cong 0$.

Proposition 3.1.3. *[RWW17, Proposition 1.7] Assume that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $Aut_{\mathfrak{G}}(0) = \{id_0\}$. Then, the groupoid $\mathfrak{G}$ is the maximal subgroupoid of $\mathfrak{U}\mathfrak{G}$.*

Henceforth, we assume that the groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $Aut_{\mathfrak{G}}(0) = \{id_0\}$.

Remark 3.1.4. Let X be an object of $\mathfrak{G}$. Let $\phi \in Aut_{\mathfrak{G}}(X)$. Then, as an element of $Hom_{\mathfrak{U}\mathfrak{G}}(X, X)$, we will abuse the notation and write ϕ for $[0, \phi]$.

Finally, we recall the following lemma.

Lemma 3.1.5. *[Sou17a, Lemma 1.8]Let $\mathscr{C}$ be a category and F an object of $\mathbf{Fct}(\mathfrak{G}, \mathscr{C})$. Assume that for $A, X, Y \in Obj(\mathfrak{G})$, there exist assignments $F([X, id_{X\natural A}]) : F(A) \to F(X\natural A)$ such that:*

$$F\left(\left[Y, id_{Y\natural X\natural A}\right]\right) \circ F\left(\left[X, id_{X\natural A}\right]\right) = F\left(\left[Y\natural X, id_{Y\natural X\natural A}\right]\right). \tag{3.1.1}$$

Then, the assignment $F([X, g]) = F(g) \circ F([X, id_{X\natural A}])$ for $[X, g] \in Hom_{\mathfrak{U}\mathfrak{G}}(A, id_{X\natural A})$ defines a functor $F : \mathfrak{U}\mathfrak{G} \to \mathscr{C}$ if and only if for all $A, X \in Obj(\mathfrak{G})$, for all $g'' \in Aut_{\mathfrak{G}}(A)$ and all $g' \in Aut_{\mathfrak{G}}(X)$:

$$F\left(\left[X, id_{X\natural A}\right]\right) \circ F(g'') = F(g'\natural g'') \circ F\left(\left[X, id_{X\natural A}\right]\right). \tag{3.1.2}$$

3.1.2 Pre-braided monoidal categories:

Assuming that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ is braided, Quillen's construction $\mathfrak{U}\mathfrak{G}$ will also inherit a strict monoidal structure (see Proposition 3.1.8). Beforehand, we recall the notion of pre-braided category, introduced by Randal-Williams and Wahl in [RWW17, Section 1].

Definition 3.1.6. [RWW17, Definition 1.5] Let $(\mathfrak{C}, \natural, 0)$ be a strict monoidal category such that the unit 0 is initial. We say that the monoidal category $(\mathfrak{C}, \natural, 0)$ is pre-braided if:

- The maximal subgroupoid $\mathscr{G}\mathfrak{r}(\mathfrak{C}, \natural, 0)$ (see Notation 3.0.1) is a braided monoidal category, where the monoidal structure is induced by that of $(\mathfrak{C}, \natural, 0)$.
- For all objects A and B of $\mathfrak{C}$, the braiding associated with the maximal subgroupoid $b^{\mathfrak{C}}_{A,B} : A\natural B \longrightarrow B\natural A$ satisfies:
$$b^{\mathfrak{C}}_{A,B} \circ (id_A \natural \iota_B) = \iota_B \natural id_A : A \longrightarrow B\natural A. \tag{3.1.3}$$

Remark 3.1.7. A braided monoidal category is automatically pre-braided. However, a pre-braided monoidal category is not necessarily braided (see for example [Sou17b, Remark 1.15]).

Finally, recall the remarkable behaviour of Quillen's bracket construction over the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$.

Proposition 3.1.8. *[RWW17, Proposition 1.8] Suppose that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $Aut_{\mathfrak{G}}(0) = \{id_0\}$. If the groupoid $(\mathfrak{G}, \natural, 0)$ is braided, then the category $(\mathfrak{U}\mathfrak{G}, \natural, 0)$ is pre-braided monoidal. If the groupoid $(\mathfrak{G}, \natural, 0)$ is symmetric, then the category $(\mathfrak{U}\mathfrak{G}, \natural, 0)$ is symmetric monoidal.*

The monoidal structure on the category $(\mathfrak{U}\mathfrak{G}, \natural, 0)$ is defined on objects as for $(\mathfrak{G}, \natural, 0)$ and defined on morphisms by letting, for $[X, f] \in Hom_{\mathfrak{U}\mathfrak{G}}(A, B)$ and $[Y, g] \in Hom_{\mathfrak{U}\mathfrak{G}}(C, D)$:

$$[X, f] \natural [Y, g] = \left[X\natural Y, (f\natural g) \circ \left(id_X \natural \left(b^{\mathfrak{G}}_{A,Y}\right)^{-1} \natural id_C\right)\right].$$

In particular, the canonical functor $\mathfrak{G} \to \mathfrak{U}\mathfrak{G}$ (see Remark 3.1.4) is monoidal.

3.1.3 Locally homogeneous categories

The notion of homogeneous category is introduced by Randal-Williams and Wahl in [RWW17, Section 1], inspired by the set-up of Djament and Vespa in [DV10, Section 1.2]. With two additional assumptions, Quillen's construction $\mathfrak{U}\mathfrak{G}$ from a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ is endowed with an homogeneous category structure. This type of category are very useful to deal with homological stability with twisted coefficients questions (see [RWW17]) or to work on the stable homology with twisted coefficient (see [DV10], [DV15] and Section 3.3).

Let $(\mathfrak{C}, \natural, 0)$ be a small strict monoidal category in which the unit 0 is also initial. For all objects A and B of $\mathfrak{C}$, we consider the morphism $\iota_A \natural id_B : 0\natural B \longrightarrow A\natural B$ and a set of morphisms characterised by this morphism:

$$Fix_A(B) = \{\phi \in Aut(A\natural B) \mid \phi \circ (\iota_A \natural id_B) = \iota_A \natural id_B\}.$$

Remark 3.1.9. Since $(\mathfrak{C}, \natural, 0)$ is assumed to be small, $Hom_{\mathfrak{C}}(A, B)$ is a set and $Aut_{\mathfrak{C}}(B)$ defines a group (with composition of morphisms as the group product). The group $Aut_{\mathfrak{C}}(B)$ acts by post-composition on $Hom_{\mathfrak{C}}(A, B)$:

$$\begin{array}{ccc} Aut_{\mathfrak{C}}(B) \times Hom_{\mathfrak{C}}(A, B) & \longrightarrow & Hom_{\mathfrak{C}}(A, B). \\ (\phi, f) & \longmapsto & \phi \circ f \end{array}$$

Definition 3.1.10. Let $(\mathfrak{C}, \natural, 0)$ be a small strict monoidal category where the unit 0 is initial. We consider the following axioms:

- (**H1**): for all objects A and B of the category $\mathfrak{C}$, the action by post-composition of $Aut_{\mathfrak{C}}(B)$ on $Hom_{\mathfrak{C}}(A, B)$ is transitive.
- (**LH1**): for a pair of objects (A, X): for all natural numbers $0 \leq p < n$, the action by post-composition of $Aut_{\mathfrak{C}}(A\natural X^{\natural n})$ on $Hom_{\mathfrak{C}}\left((X^{\natural(p+1)}, A\natural X^{\natural n}\right)$ is transitive.
- (**H2**): for all objects A and B of the category $\mathfrak{C}$, the map
$$\begin{array}{ccc} Aut_{\mathfrak{C}}(A) & \longrightarrow & Aut_{\mathfrak{C}}(A\natural B) \\ f & \longmapsto & f\natural id_B \end{array}$$
is injective with image $Fix_A(B)$.

- (**LH2**): for a pair of objects (A, X): for all natural numbers $0 \leq p < n$, the map

$$\begin{array}{ccc} Aut_{\mathfrak{C}}(A\natural X^{\natural(n-p-1)}) & \longrightarrow & Aut_{\mathfrak{C}}(A\natural X^{\natural n}). \\ f & \longmapsto & f\natural id_{X^{\natural(p+1)}} \end{array}$$

  is injective with image $Fix_{A\natural X^{\natural(n-p-1)}}\left(X^{\natural(p+1)}\right)$.

The category $(\mathfrak{C}, \natural, 0)$ is locally homogeneous at a pair of objects (A, X) (respectively homogeneous) if it satisfies the axioms (**LH1**) and (**LH2**) at (A, X) (respectively the axioms (**H1**) and (**H2**)).

Remark 3.1.11. If $(\mathfrak{C}, \natural, 0)$ is a homogeneous category, then it follows form axioms (**H1**) and (**H2**) that for all objects A and B:

$$Hom_{\mathfrak{C}}(B, A\natural B) \cong Aut_{\mathfrak{C}}(A\natural B) / Aut_{\mathfrak{C}}(A),$$

where $Aut_{\mathfrak{C}}(A)$ acts on $Aut_{\mathfrak{C}}(A\natural B)$ by precomposition.

We now give the two additional properties so that if a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ satisfy them, then Quillen's bracket construction $\mathfrak{U}\mathfrak{G}$ is (locally) homogeneous.

Definition 3.1.12. Let $(\mathfrak{C}, \natural, 0)$ be a strict monoidal category. We define two assumptions.

- (**C**): for all objects A, B and C of $\mathfrak{C}$, if $A\natural C \cong B\natural C$ then $A \cong B$.
- (**LC**): for a pair of objects (A, X): for all natural numbers $0 \leq p < n$, if $Y \in Obj(\mathfrak{C})$ is such that $Y\natural X^{\natural(p+1)} \cong A\natural X^{\natural n}$ then $Y \cong A\natural X^{\natural(n-p-1)}$.
- (**I**): for all objects A, B of $\mathfrak{C}$, the following morphism is injective:

$$\begin{array}{ccc} Aut_{\mathfrak{C}}(A) & \longrightarrow & Aut_{\mathfrak{C}}(A\natural B). \\ f & \longmapsto & f\natural id_B \end{array}$$

- (**LI**): for a pair of objects (A, X): for all natural numbers $0 \leq p < n$, the following morphism is injective:

$$\begin{array}{ccc} Aut_{\mathfrak{C}}(A\natural X^{\natural(n-p-1)}) & \longrightarrow & Aut_{\mathfrak{C}}(A\natural X^{\natural n}). \\ f & \longmapsto & f\natural id_{X^{\natural(p+1)}} \end{array}$$

Theorem 3.1.13. *[RWW17, Theorem 1.10] Let $(\mathfrak{G}, \natural, 0)$ be a braided monoidal groupoid with no zero divisors. If the groupoid $\mathfrak{G}$ satisfies (**C**) and (**I**), then $\mathfrak{U}\mathfrak{G}$ is homogeneous. If the groupoid $\mathfrak{G}$ satisfies (**LC**) and (**LI**) for a pair of objects (A, X), then $\mathfrak{U}\mathfrak{G}$ is locally homogeneous at (A, X).*

3.2 Twisted stable homologies of semidirect products

This section introduces a general method to compute the stable homology with twisted coefficients using semidirect product structures arising naturally from the families of mapping class groups. We first establish the general result of Corollary 3.2.5 for the homology of semidirect products with twisted coefficients. This result is then applied in Section 3.2.2 to compute explicitly some homology groups with twisted coefficients for braid groups, mapping class groups of orientable and non-orientable surfaces and automorphisms of free groups with boundaries.

3.2.1 Framework of the study

3.2.1.1 A general result for the homology of semidirect products

First, we present some properties for the homology with twisted coefficients for a semidirect product using Lyndon-Hochschild-Serre spectral sequence and prove the general statement of Corollary 3.2.5. Let $\mathcal{Q}$ be a groupoid with natural numbers as objects and denote by $Aut_{\mathcal{Q}}(n) = \mathcal{Q}_n$ the automorphism groups.

Assumption 3.2.1. *We assume that there exists a family of free groups $K_{-} : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ and a functor $\mathcal{A}_{\mathcal{Q}} : \mathcal{Q} \to \mathfrak{Gr}$ such that $\mathcal{A}_{\mathcal{Q}}(n) = K_n$ for all natural numbers n.*

Notation 3.2.2. For all natural numbers n, we denote by $\mathcal{A}_{\mathcal{Q},n} : Q_n \to Aut_{\mathfrak{Gr}}(K_n)$ the group morphisms induced by the functor $\mathcal{A}_{\mathcal{Q}}$.

Using Assumption 3.2.1, we form the split short exact sequence:

$$1 \longrightarrow K_n \xrightarrow{k_n} K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n \xrightarrow{q_n} Q_n \longrightarrow 1 \tag{3.2.1}$$

and we denote by $s_n : Q_n \to K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n$ the splitting of q_n. For all natural numbers n, we fix M_n a $R\left[K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n\right]$-module.

Notation 3.2.3. We abuse the notation and write M_n for $\mathrm{Res}_{K_n}^{K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n}(M_n)$, where $\mathrm{Res}_{K_n}^{K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n}$ denotes the restriction functor.

Proposition 3.2.4. *For M_n an $R\left[K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n\right]$-module, the short exact sequence (3.2.1) induces a long exact sequence:*

$$\begin{array}{c} \cdots \longrightarrow H_{*+1}(Q_n, H_0(K_n, M_n)) \\ \downarrow d^2_{*+1,0} \\ H_{*-1}(Q_n, H_1(K_n, M_n)) \xrightarrow{\varphi_*} H_*\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right) \xrightarrow{\psi_*} H_*(Q_n, H_0(K_n, M_n)) \\ \downarrow d^2_{*,0} \\ H_{*-2}(Q_n, H_1(K_n, M_n)) \longrightarrow \cdots \end{array} \tag{3.2.2}$$

where $\left\{d^2_{p,q}\right\}_{p,q\in\mathbb{N}}$ denote the differentials of the second page of the Lyndon-Hochschild-Serre spectral sequence associated with the short exact sequence (3.2.1).

Proof. Applying the Lyndon-Hochschild-Serre spectral sequence (see for instance [Wei94, Proposition 6.8.2]) to the short exact sequence (3.2.1), we obtain the following convergent first quadrant spectral sequence:

$$E^2_{pq} : H_p(Q_n, H_q(K_n, M_n)) \Longrightarrow H_{p+q}\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right). \tag{3.2.3}$$

Since K_n is a free group, $H_q(K_n, M_n) = 0$ for $q \geq 2$ (see for instance [Wei94, Proposition 6.2.7]). The result is a classical consequence of the fact that the spectral sequence (3.2.3) has only two rows (see for instance [Wei94, Exercise 5.2.2]). In particular, the map φ_* is defined by the composition:

$$\begin{array}{ccc} H_{*-1}(Q_n, H_1(K_n, M_n)) & \overset{\varphi_*}{\dashrightarrow} & \\ \downarrow & & \\ H_{*-1}(Q_n, H_1(K_n, M_n)) / \mathrm{Im}\left(d^2_{*+1,0}\right) & \hookrightarrow & H_*\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right); \end{array}$$

the map ψ_* is the coinflation map $\mathrm{Coinf}_{K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n}^{Q_n}(M_n)$, induced by the composition:

$$\begin{array}{ccc} H_*\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right) & \twoheadrightarrow & Ker\left(d^2_{*,0}\right) \\ & \overset{\psi_*}{\searrow} & \downarrow \\ & & H_*\left(Q_n, H_0\left(K_n, M_n\right)\right). \end{array}$$

□

Corollary 3.2.5. *Let n be a natural number. Assume that the free group K_n acts trivially on the R-module M_n. Then, for all natural numbers $q \geq 1$:*

$$H_q\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right) \cong H_{q-1}\left(Q_n, H_1\left(K_n, R\right) \underset{R}{\otimes} M_n\right) \oplus H_q\left(Q_n, M_n\right). \tag{3.2.4}$$

Proof. First, as M_n is a trivial K_n-module:

$$H_1\left(K_n, M_n\right) \cong H_1\left(K_n, R\right) \underset{R}{\otimes} M_n \text{ and } H_0\left(K_n, M_n\right) \cong M_n,$$

and the coinflation map $\psi_* = \mathrm{Coinf}_{K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n}^{Q_n}(M_n)$ is equal to the corestriction maps $\mathrm{Cores}_{K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n}^{Q_n}(M_n)$ (see for example [Wei94, Section 6.7.3]). Hence, denoting by $H_*\left(p_n, M_n\right)$ the map induced in homology by $p_n : K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n \to Q_n$ (see [Wei94, Section 6.7.5]), we deduce that:

$$\psi_* = H_*\left(p_n, M_n\right).$$

By the functoriality of the homology (see [Wei94, Section 6.7.5]), the splitting $s_n : Q_n \to K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n$ of p_n induces a splitting in homology $H_*\left(s_n, M_n\right)$ of $H_*\left(p_n, M_n\right)$. Hence, $H_*\left(p_n, M_n\right)$ is an epimorphism and a fortiori $Ker\left(d^2_{*,0}\right) \cong H_*\left(Q_n, M_n\right)$. Therefore, $d^2_{*,0} = 0$ and the exact sequence (3.2.2) gives a split short exact sequence of abelian groups for every $q \geq 1$:

$$1 \longrightarrow H_{q-1}\left(Q_n, H_1\left(K_n, R\right) \underset{R}{\otimes} M_n\right) \xrightarrow{\varphi_*} H_q\left(K_n \underset{\mathcal{A}_{\mathcal{Q},n}}{\rtimes} Q_n, M_n\right) \xrightarrow{H_*(p_n, M_n)} H_q\left(Q_n, M_n\right) \longrightarrow 1.$$

□

3.2.1.2 Relation to Long-Moody functors:

In [Sou17b, Sou17a], the notion of Long-Moody functor is set for the groupoid $(\mathcal{Q}, \natural, 0)$ and the family of groups K_-. Our aim here is to outline the relation between the twisted coefficients $H_1\left(K_n, R\right) \underset{R}{\otimes} M_n$ appearing in Corollary 3.2.5 and the notion of Long-Moody functor so as to prove Corollary 3.2.10. This last result will be useful to prove Theorem 3.2.45.

Assumption 3.2.6. *We assume that the groupoid $\mathcal{Q}$ is a braided strict monoidal category (we denote by $(\mathcal{Q}, \natural, 0)$ the monoidal structure) and that there exists a free group K such that $K_n \cong K^{*n}$ for all natural numbers n. Moreover, we assume that $K_-\left(\gamma_n\right) = \iota_K * id_{K_n}$ for all natural numbers n (we recall that γ_n is the unique element of $Aut_{(\mathbb{N},\leq)}\left(n, n+1\right)$ see Notation 3.0.1) and that the functor $\mathcal{A}_{\mathcal{Q}}$ of Assumption 3.2.1 defines a strict monoidal functor $(\mathcal{Q}, \natural, 0) \to (\mathfrak{gr}, *, 0)$.*

Throughout the remainder of Section 3.2.1, we assume that Assumption 3.2.6 is satisfied. This allows to define the functor K_- on the category $\mathfrak{U}\mathcal{Q}$:

Lemma 3.2.7. *Assigning $\mathcal{A}_{\mathcal{Q}}([1, id_{n+1}]) = K_{-}(\gamma_n)$ for all natural numbers n, we define a functor $\mathcal{A}_{\mathcal{Q}} : \mathfrak{U}\mathcal{Q} \to \mathfrak{Gr}$.*

Proof. We use Lemma 3.1.5 to prove this result: namely, we show that relations (3.1.1) and (3.1.2) of this lemma are satisfied. It follows from the fact that K_{-} is a functor on $(\mathbb{N}, \leq)$, that the relation (3.1.1) of Lemma 3.1.5 is satisfied by $\mathcal{A}_{\mathcal{Q}}$. Let n and n' be natural numbers such that $n' \geq n$, let $q \in Q_n$ and $q' \in Q_{n'}$. We denote by $e_{K_{n'}}$ the neutral element of $K_{n'}$. Since $\mathcal{A}_{\mathcal{Q}}$ is monoidal, we compute for all $k \in K_n$:

$$\begin{aligned}\left(\mathcal{A}_{\mathcal{Q}}(q' \natural q) \circ \mathcal{A}_{\mathcal{Q}}\left(\left[n', id_{n'+n}\right]\right)\right)(k) &= \left(\mathcal{A}_{\mathcal{Q}}(q') * \mathcal{A}_{\mathcal{Q}}(q)\right)\left(e_{K_{n'}} * k\right)\\ &= e_{K_{n'}} * \mathcal{A}_{\mathcal{Q}}(q)(k)\\ &= \left(\mathcal{A}_{\mathcal{Q}}\left(\left[n', id_{n'+n}\right]\right) \circ \mathcal{A}_{\mathcal{Q}}(q)\right)(k).\end{aligned}$$

Hence, the relation (3.1.2) of Lemma (3.1.5) is satisfied by $\mathcal{A}_{\mathcal{Q}}$. □

Let F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{Q}, R\text{-}\mathfrak{Mod})$ and n be a natural number.

Notation 3.2.8. We denote by $\varsigma_{n,t}$ the trivial morphisms $K_n \to 0_{\mathfrak{Gr}} \to Q_{n+1}$.

Hence, the Q_{n+1}-module $F(n+1)$ is both a Q_n-module using precomposition by the morphism $id_1 \natural - : Q_n \to Q_{n+1}$ and a trivial K_n-module using precomposition by $\varsigma_{n,t}$. Recall that the homology group $H_1(-, R)$ defines a functor from the category $\mathfrak{Gr}$ to the category $R\text{-}\mathfrak{Mod}$ (see for example [Bro12, Section 8]). Hence, we define a functor $H_1(\mathcal{A}_{\mathcal{Q}}, R) : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$ by the composition:

$$\mathfrak{U}\mathcal{Q} \xrightarrow{\mathcal{A}_{\mathcal{Q}}} \mathfrak{Gr} \xrightarrow{H_1(-,R)} R\text{-}\mathfrak{Mod}.$$

In addition, the pointwise tensor product of two objects of $\mathbf{Fct}(\mathfrak{U}\mathcal{Q}, R\text{-}\mathfrak{Mod})$ defines an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{Q}, R\text{-}\mathfrak{Mod})$, assigning

$$\left(F \underset{R}{\otimes} F'\right)(n) = F(n) \underset{R}{\otimes} F'(n)$$

for $F, F' \in \mathbf{Fct}(\mathfrak{U}\mathcal{Q}, R\text{-}\mathfrak{Mod})$ and for all objects n of $\mathfrak{U}\mathcal{Q}$.

Taking up the notations and framework of [Sou17a], we deduce that:

Lemma 3.2.9. *[Sou17a, Lemma 2.37 and Proposition 2.38] If Assumption 3.2.6 is satisfied, then for all objects F of $\mathbf{Fct}(\mathfrak{U}\mathcal{Q}, R\text{-}\mathfrak{Mod})$, there is a isomorphism in the category $\mathbf{Fct}(\mathfrak{U}\mathcal{Q}, R\text{-}\mathfrak{Mod})$:*

$$H_1(K_{-}, R) \underset{R}{\otimes} F(1 \natural -) \cong \mathbf{LM}_{\{\mathcal{A}_{\mathcal{Q}}, \mathcal{Q}, \mathcal{Q}, \varsigma_{n,t}\}}(F),$$

where $\mathbf{LM}_{\{\mathcal{A}_{\mathcal{Q}}, \mathcal{Q}, \mathcal{Q}, \varsigma_{n,t}\}}$ denotes the Long-Moody functor associated with the functor $\mathcal{A}_{\mathcal{Q}}$, the braided monoidal groupoid $(\mathcal{Q}, \natural, 0)$ and the family of trivial morphisms $\{\varsigma_{n,t}\}_{n \in \mathbb{N}}$.

We refer the reader to [Sou17a, Section 3] for an introduction to the notion of strong polynomial functors. It thus follows from Lemma 3.2.9 and [Sou17a, Theorem A] that:

Corollary 3.2.10. *Let F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{Q}, R\text{-}\mathfrak{Mod})$. If F is a strong polynomial functor of degree equal to d, then $H_1(K_{-}, R) \underset{R}{\otimes} F(1 \natural -)$ is a strong polynomial functor of degree less than or equal to d.*

3.2.2 Applications

Many families of mapping class groups fit into the framework of Section 3.2.1. Proposition 3.2.4 and Corollary 3.2.5 are key results to compute the homology with twisted coefficients for these families of groups.

3.2.2.1 Braid groups

We denote by $\mathbf{B}_n$ the braid group on n strands and by $\mathbf{F}_n$ the free group on n generators. The braid groupoid $\boldsymbol{\beta}$ is the groupoid with objects the natural numbers $n \in \mathbb{N}$ and braid groups as automorphism groups. It is endowed with a strict braided monoidal product $\natural : \boldsymbol{\beta} \times \boldsymbol{\beta} \longrightarrow \boldsymbol{\beta}$, defined by the usual addition for the objects and laying two braids side by side for the morphisms. The object 0 is the unit of this monoidal product. The braiding of the strict monoidal groupoid $(\boldsymbol{\beta}, \natural, 0)$ is defined for all natural numbers n and m by:

$$b_{n,m}^{\boldsymbol{\beta}} = (\sigma_m \circ \cdots \circ \sigma_2 \circ \sigma_1) \circ \cdots \circ (\sigma_{n+m-2} \circ \cdots \circ \sigma_n \circ \sigma_{n-1}) \circ (\sigma_{n+m-1} \circ \cdots \circ \sigma_{n+1} \circ \sigma_n)$$

where $\{\sigma_i\}_{i \in \{1,\ldots,n+m-1\}}$ denote the Artin generators of the braid group $\mathbf{B}_{n+m}$. We refer the reader to [ML13, Chapter XI, Section 4] for more details.

For all natural numbers n, Artin representations $a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)$ are defined for all elementary braids σ_i where $i \in \{1, \ldots, n-1\}$ by:

$$\begin{array}{rcl} a_n(\sigma_i) : \mathbf{F}_n & \longrightarrow & \mathbf{F}_n \\ g_j & \longmapsto & \begin{cases} g_{i+1} & \text{if } j = i \\ g_{i+1}^{-1} g_i g_{i+1} & \text{if } j = i+1 \\ g_j & \text{if } j \notin \{i, i+1\}. \end{cases} \end{array}$$

Identifying $\mathbf{B}_n$ as the mapping class group of a n-punctured disc, a_n is the induced action on the fundamental group of the n-punctured disc. Artin representations thus provide a functor $\mathcal{A}_{\boldsymbol{\beta}} : \boldsymbol{\beta} \longrightarrow \mathfrak{Gr}$ and Assumption 3.2.6 is satisfied.

Remark 3.2.11. For all natural numbers n, the semidirect product $\mathbf{F}_n \underset{\mathcal{A}_{\beta,n}}{\rtimes} \mathbf{B}_n$ identifies with the annular braid group, also known as circular braid group or Artin group of type B_n (see [CMS08]), denoted by $C\mathbf{B}_n$: this is the subgroup of $\mathbf{B}_{n+1}$ that leaves the first puncture invariant. We refer the reader to [CMS08, Section 2] for more details. In [Gor78, Theorem C], Gorjunov computes the cohomology groups $H^q(C\mathbf{B}_n, \mathbb{Q})$. Therefore, using the universal coefficient theorem for cohomology (see for example [Wei94, Theorem 3.6.5]), it follows that for $n \geq q+2$:

$$H_q\left(\mathbf{F}_n \underset{\mathcal{A}_{\beta,n}}{\rtimes} \mathbf{B}_n, \mathbb{C}\right) \cong \begin{cases} \mathbb{C}^{\oplus 2} & \text{if } q \geq 1 \\ \mathbb{C} & \text{if } q = 0. \end{cases}$$

Computation of $H_*(\mathbf{B}_n, \mathfrak{Bur}_t)$: For this paragraph, we fix $R = \mathbb{C}\left[t^{\pm 1}\right]$, the ring of Laurent polynomials in one variable. In [Sou17b, Section 1.2], we prove that the unreduced Burau (respectively reduced Burau) representations of braid groups assemble to form a functor $\mathfrak{Bur}_t : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ (respectively $\overline{\mathfrak{Bur}}_t : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$).

In [Che17], Chen computes the homology groups of braid groups with coefficients in the reduced Burau functor. We briefly review here the work led for this computation.

Notation 3.2.12. We denote by $\mathbb{C}\left[t^{\pm 1}\right]$ the object of $\mathbf{Fct}\left(\mathfrak{U}\boldsymbol{\beta}, \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}\right)$ which is constant at $\mathbb{C}\left[t^{\pm 1}\right]$.

Theorem 3.2.13. *[Che17, Theorem 1] For $n \geq 3$, we have:*

$$H_q\left(\mathbf{B}_n, \overline{\mathfrak{Bur}}_t(n)\right) \cong \begin{cases} 0 & \text{if } q = 0, \\ \mathbb{C}\left[t^{\pm 1}\right] / (1-t) & \text{if } 1 \leq q < n-2, \\ \mathbb{C}\left[t^{\pm 1}\right] / (1-t) & \text{if } q = n-2 \text{ and } n \text{ is odd}, \\ \mathbb{C}\left[t^{\pm 1}\right] / (1-t^2) & \text{if } q = n-2 \text{ and } n \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. For all natural numbers n, we consider $\mathbb{C}\left[t^{\pm 1}\right]$ as a trivial $\mathbf{B}_n$-module and assume that each generator of the free group $\mathbf{F}_n$ acts on $\mathbb{C}\left[t^{\pm 1}\right]$ by multiplying by t. It follows from the definition of Artin representation that these actions induce a well-defined action of the semi-direct product $\mathbf{F}_n \underset{\mathcal{A}_{\beta,n}}{\rtimes} \mathbf{B}_n$ on $\mathbb{C}\left[t^{\pm 1}\right]$. Then, [Che17, Lemma 3] proves that for q and $n \geq 3$ natural numbers, there is an isomorphism of $\mathbb{C}\left[t^{\pm 1}\right]$-modules:

$$H_q\left(\mathbf{B}_n, \overline{\mathfrak{Bur}}_t(n)\right) \cong H_q\left(\mathbf{B}_n, H_1\left(\mathbf{F}_n, \mathbb{C}\left[t^{\pm 1}\right]\right)\right).$$

Note that $H_0\left(\mathbf{F}_n, \mathbb{C}\left[t^{\pm 1}\right]\right) \cong \mathbb{C}$ and recall that $H_*\left(\mathbf{B}_n, \mathbb{C}\right) = 0$ if $* \geq 2$ (see for example [Ver98, Section 4]). Hence, the result follows from Proposition 3.2.4 and the computation in [CMS08, Theorem 4.2] of

$$H_q\left(\mathbf{F}_n \underset{\mathcal{A}_{\beta,n}}{\rtimes} \mathbf{B}_n, \mathbb{C}\left[t^{\pm 1}\right]\right).$$

□

Remark 3.2.14. In [Sou17b, Section 1.2], we also prove that the family of Tong-Yang-Ma representations (see [TYM96]) of braid groups assemble to form a functor $\mathfrak{TYM}_t : \mathfrak{U}\beta \to \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$. As in [CMS08, Proposition 4.3], we can prove using Schapiro's lemma that for $n \geq 3$ and $q \geq 2$:

$$H_q\left(\mathbf{B}_n, \mathfrak{TYM}_t(n)\right) \cong H_{q-1}\left(\mathbf{B}_n, \overline{\mathfrak{Bur}}_t(n)\right).$$

The following proposition relates the reduced and unreduced Burau functors.

Notation 3.2.15. For an object F of $\mathbf{Fct}\left(\mathfrak{U}\beta, \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}\right)$, for all natural numbers k, we denote by $F_{\geq k} : \mathfrak{U}\beta \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ the subfunctor of F which is null on the objects such that $n < k$ and equal to F for $n \geq k$.

Proposition 3.2.16. *We have the following short exact sequence in* $\mathbf{Fct}\left(\mathfrak{U}\beta, \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}\right)$*:*

$$0 \longrightarrow \left(\mathbb{C}\left[t^{\pm 1}\right]\right)_{\geq 3} \longrightarrow \left(\mathfrak{Bur}_t\right)_{\geq 3} \overset{p}{\longrightarrow} \left(\overline{\mathfrak{Bur}}_t\right)_{\geq 3} \longrightarrow 0\,. \tag{3.2.5}$$

Proof. For all natural numbers n, we fix:

$$r_n = \begin{bmatrix} \overbrace{\begin{matrix} 1 & -1 & 0 & \cdots & 0 \end{matrix}}^{n} \\ \begin{matrix} 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & -1 \\ 0 & \cdots & \cdots & 0 & 1 \end{matrix} \end{bmatrix}.$$

Then, for all $i \in \{1, \ldots, n-1\}$ and $n \geq 3$, $r_n \circ \mathfrak{Bur}_t(\sigma_i) \circ r_n^{-1} = \begin{bmatrix} \overbrace{\begin{matrix}\overline{\mathfrak{Bur}}_t(\sigma_i) & 0\end{matrix}}^{n} \\ \begin{matrix} L_i & 1 \end{matrix} \end{bmatrix}$ where $L_i = \left[\overbrace{\begin{matrix} 0 & \cdots & 0 & \delta_{i,n-1} & 1 \end{matrix}}^{n}\right]$ and $\delta_{i,n-1}$ denotes the Kronecker delta. Hence, the functor $\left(\mathfrak{Bur}_t\right)_{\geq 3}$ is equivalent to the functor $\widetilde{\mathfrak{Bur}_t} : \mathfrak{U}\beta \longrightarrow \mathbb{C}\left[t^{\pm 1}\right]\text{-}\mathfrak{Mod}$ which assigns $\mathbb{C}\left[t^{\pm 1}\right]^{\oplus n}$ for all objects, the matrix $\begin{bmatrix} \overline{\mathfrak{Bur}}_t(\sigma_i) & 0 \\ L_i & 1 \end{bmatrix}$ for all Artin generator σ_i of $\mathbf{B}_n$ and $\widetilde{\mathfrak{Bur}_t}\left([1, id_n]\right) : \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n} \hookrightarrow \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n+1}$ is the embedding $\iota_{\mathbb{C}[t^{\pm 1}]} \oplus id_{\mathbb{C}[t^{\pm 1}]^{\oplus n}}$ for all natural numbers n (recall from Notation 3.0.1 that $\iota_{\mathbb{C}[t^{\pm 1}]}$ denotes the unique group morphism $0_{\mathfrak{Gr}} \to \mathbb{C}\left[t^{\pm 1}\right]$). For all natural numbers $n \geq 3$, the projections $p_n : \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n} \twoheadrightarrow \mathbb{C}\left[t^{\pm 1}\right]^{\oplus n-1}$ on the $n-1$ first copies of $\mathbb{C}\left[t^{\pm 1}\right]$ determine the natural transformation p of the short exact sequence (3.2.5). It is clear that the kernel of this natural transformation is $\left(\mathbb{C}\left[t^{\pm 1}\right]\right)_{\geq 3}$. □

Hence, we can prove:

Theorem 3.2.17. *For all natural numbers $n \geq 3$ and $q \geq 3$:*

$$H_q\left(\mathbf{B}_n, \mathfrak{Bur}_t(n)\right) \cong \begin{cases} \mathbb{C}\left[t^{\pm 1}\right]/(1-t) & \text{if } 3 \leq q < n-2, \\ \mathbb{C}\left[t^{\pm 1}\right]/(1-t) & \text{if } q = n-2 \text{ and } n \text{ is odd}, \\ \mathbb{C}\left[t^{\pm 1}\right]/(1-t^2) & \text{if } q = n-2 \text{ and } n \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let $n \geq 3$ be a fixed natural number. From the short exact sequence of Proposition 3.2.16, we deduce the long exact sequence in homology:

$$\cdots \longrightarrow H_*\left(\mathbf{B}_n, \mathbb{C}\left[t^{\pm 1}\right]\right) \longrightarrow H_*\left(\mathbf{B}_n, \mathfrak{Bur}_t(n)\right) \longrightarrow H_*\left(\mathbf{B}_n, \overline{\mathfrak{Bur}_t}(n)\right) \longrightarrow H_{*-1}\left(\mathbf{B}_n, \mathbb{C}\left[t^{\pm 1}\right]\right) \longrightarrow \cdots \tag{3.2.6}$$

Since $H_k\left(\mathbf{B}_n, \mathbb{C}\left[t^{\pm 1}\right]\right) = \begin{cases} \mathbb{C}\left[t^{\pm 1}\right] & \text{if } k = 0,1 \\ 0 & \text{otherwise} \end{cases}$ (see for example [Ver98, Section 4]), the result follows from Theorem 3.2.13. □

Computation of $H_*\left(\mathbf{B}_n, \mathfrak{Cox}\right)$: The unreduced (respectively reduced) Coxeter representations of braid groups (see [Vas92, Chapter II, Section 5] for this terminology) is given by specializing the unreduced (respectively reduced) Burau representations at $t = 1$. Namely, the Coxeter functor $\mathfrak{Cox} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\text{-}\mathfrak{Mod}$ and reduced Coxeter functor $\overline{\mathfrak{Cox}} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}\text{-}\mathfrak{Mod}$ are defined by $\mathfrak{Cox}(n) = \mathbb{C}^{\oplus n}$ and $\overline{\mathfrak{Cox}}(n) = \mathbb{C}^{\oplus n-1}$ for all natural numbers $n \geq 1$, $\mathfrak{Cox}\left([1, id_{n+1}]\right) = \iota_{\mathbb{C}} \oplus id_{\mathbb{C}^{\oplus n}}$ and $\overline{\mathfrak{Cox}}\left([1, id_{n+1}]\right) = \iota_{\mathbb{C}} \oplus id_{\mathbb{C}^{\oplus n-1}}$, and for all Artin generator σ_i of $\mathbf{B}_n$:

$$\mathfrak{Cox}(\sigma_i) = Id_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus Id_{n-i-1} \text{ and } \overline{\mathfrak{Cox}}(\sigma_i) = Id_{i-2} \oplus \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \oplus Id_{n-i-2}.$$

The unreduced (respectively reduced) Coxeter representation corresponds for each natural number n to the representation of $\mathbf{B}_n$ factoring through the permutation (respectively standard) representation of symmetric group on n elements.

Proposition 3.2.18. *For all natural numbers $n \geq q + 2$:*

$$H_q\left(\mathbf{B}_n, \mathfrak{Cox}(n)\right) \cong \begin{cases} \mathbb{C}^{\oplus 2} & \text{if } q \geq 2, \\ \mathbb{C} & \text{if } q = 0,1. \end{cases}$$

Proof. Let n be a natural number. Note that the free group $\mathbf{F}_n$ acts trivially on $\mathfrak{Cox}(n)$ and therefore Assumption 3.2.6 is satisfied. In addition, we have

$$\mathfrak{Cox}(n) \cong H_1\left(\mathbf{F}_n, \mathbb{C}\right),$$

and the actions of $\mathbf{B}_n$ on $\mathfrak{Cox}(n)$ and $H_1\left(\mathbf{F}_n, \mathbb{C}\right)$ are the same: it is given by the permutation of the copies of $\mathbb{C}$. Since $H_k\left(\mathbf{B}_n, \mathbb{C}\right) = \begin{cases} \mathbb{C} & \text{if } k = 0,1 \\ 0 & \text{otherwise} \end{cases}$ (see for example [Ver98, Section 4]), the result follows from Corollary 3.2.5. □

It follows from the long exact sequence analogous to (3.2.6) with t specialized at 1, that:

Corollary 3.2.19. *For all natural numbers n and q such that $n \geq 3$ and $n \geq q + 2$:*

$$H_q\left(\mathbf{B}_n, \overline{\mathfrak{Cox}}(n)\right) \cong \begin{cases} 0 & \text{if } q = 0, \\ \mathbb{C} & \text{if } q = 1, \\ \mathbb{C}^{\oplus 2} & \text{if } q \geq 2, \end{cases}$$

Remark 3.2.20. In [Vas92, Chapter II, Section 5], using analogous methods, Vassiliev computes the cohomology groups $H^q\left(\mathbf{B}_n, \mathfrak{Cox}(n)\right)$ and $H^q\left(\mathbf{B}_n, \overline{\mathfrak{Cox}}(n)\right)$. Using the universal coefficient theorem for twisted coefficients (see for example [God58, Théorème I.5.5.2]), Proposition 3.2.18 and Corollary 3.2.19 recover these results.

3.2.2.2 Mapping class groups of orientable surfaces

Let $\Sigma_{g,i}^s$ denote a smooth compact connected orientable surface with (orientable) genus $g \in \mathbb{N}$, $s \in \mathbb{N}$ marked points and $i \in \{1,2\}$ boundary components with $I : [-1,1] \to \partial\Sigma_{g,i}^s$ a parametrised interval in the boundary. We denote by $\mathbf{\Gamma}_{g,1}^s$ (resp. $\mathbf{\Gamma}_{g,1}^{[s]}$) the isotopy classes of diffeomorphisms of $\Sigma_{g,1}^s$ preserving the orientation, restricting to the identity on a neighbourhood of the parametrised interval I and permuting (resp. fixing) the marked points (if $s = 0$, we omit it from the notation). Recall that fixing the interval I is the same as fixing the whole boundary component pointwise. When there is no ambiguity, we omit the parametrised interval I from the notation.

We denote by $\mathbf{\Gamma}_{g,2}$ the isotopy classes of diffeomorphisms of $\Sigma_{g,2}^{0}$ preserving the orientation and restricting to the identity on a neighbourhood of the parametrised interval I and fixing the other boundary component pointwise. Recall that R is a commutative ring and we assume that the various mapping class groups act trivially on it.

The following result is an essential tool for our work:

Theorem 3.2.21. *[Bir74b] Let $g \geq 1$, $s \geq 0$ be natural numbers and x be a marked point in the interior of $\Sigma_{g,1}^{s}$. Deleting x induces a map $\mathbf{\Gamma}_{g,1}^{[s+1]} \to \mathbf{\Gamma}_{g,1}^{[s]}$ which defines the following short exact sequence:*

$$1 \longrightarrow \pi_1\left(\Sigma_{g,1}^{s}, x\right) \longrightarrow \mathbf{\Gamma}_{g,1}^{[s+1]} \longrightarrow \mathbf{\Gamma}_{g,1}^{[s]} \longrightarrow 1. \tag{3.2.7}$$

Gluing a disc with a marked point disc $\Sigma_{0,1}^{1}$ on the boundary component without I induces the following short exact sequence:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathbf{\Gamma}_{g,2} \longrightarrow \mathbf{\Gamma}_{g,1}^{1} \longrightarrow 1. \tag{3.2.8}$$

Notation 3.2.22. For all natural numbers g and s, we denote by $a_{\Sigma_{g,1}^{s}}^{x}$ the action of the mapping class group $\mathbf{\Gamma}_{g,1}^{[s]}$ on the fundamental group of the surface $\pi_1\left(\Sigma_{g,1}^{s}, x\right)$.

Lemma 3.2.23. *The short exact sequence (3.2.7) splits.*

Proof. The embedding of $\Sigma_{g,1}^{s}$ into $\Sigma_{g,1}^{s+1}$ as the complement of the disc $\Sigma_{0,1}^{1}$ with the marked point x induces an injective morphism $\mathbf{\Gamma}_{g,1}^{[s]} \hookrightarrow \mathbf{\Gamma}_{g,1}^{[s+1]}$. This provides a splitting of the exact sequence (3.2.7) and we have an isomorphism:

$$\mathbf{\Gamma}_{g,1}^{[s+1]} \cong \pi_1\left(\Sigma_{g,1}^{s}, x\right) \underset{a_{\Sigma_{g,1}^{s}}^{x}}{\rtimes} \mathbf{\Gamma}_{g,1}^{[s]}.$$

□

Let us introduce a suitable groupoid for our work, inspired by [RWW17, Section 5.6].

Definition 3.2.24. Let s be a fixed natural number. Let $\mathfrak{M}_2^s$ be the skeleton of the groupoid defined by:

- Objects: the smooth compact connected orientable surfaces $\Sigma_{n,1}^{s}$ for all natural numbers n with x a basepoint in the interior;
- Morphisms: $Aut_{\mathfrak{M}_2^s}\left(\Sigma_{n,1}^{s}\right) = \mathbf{\Gamma}_{n,1}^{s}$ for all natural numbers n.

Remark 3.2.25. The morphisms $\left\{a_{\Sigma_{n,1}^{s}}^{x}\right\}_{n\in\mathbb{N}}$ of Notation 3.2.22 assemble to define a functor $\mathcal{A}_{\mathfrak{M}_2^s} : \mathfrak{M}_2^s \to \mathfrak{Gr}$ such that $\mathcal{A}(n) = \pi_1\left(\Sigma_{n,1}^{s}, x\right)$ for all natural numbers n. Hence, recalling that $\pi_1\left(\Sigma_{n,1}^{s}, x\right)$ is a free group of rank $2n + s$, Assumption 3.2.1 is satisfied.

Proposition 3.2.26. *Let n, s and $q \geq 1$ be natural numbers. Let M_n be a $R\left[\mathbf{\Gamma}_{n,1}^{[s+1]}\right]$-module (and a fortiori a $R\left[\mathbf{\Gamma}_{n,1}^{[s]}\right]$-module using the surjection $\mathbf{\Gamma}_{g,1}^{[s+1]} \twoheadrightarrow \mathbf{\Gamma}_{g,1}^{[s]}$) on which $\pi_1\left(\Sigma_{n,1}^{s}, x\right)$ acts trivially. Then:*

$$H_q\left(\mathbf{\Gamma}_{n,1}^{[s+1]}, M_n\right) \cong H_{q-1}\left(\mathbf{\Gamma}_{n,1}^{[s]}, H_1\left(\Sigma_{n,1}^{s}, R\right) \underset{R}{\otimes} M_n\right) \oplus H_q\left(\mathbf{\Gamma}_{n,1}^{[s]}, M_n\right). \tag{3.2.9}$$

Proof. The result follows from Corollary 3.2.5 and Lemma 3.2.23. □

Computation of $H_d\left(\mathbf{\Gamma}_{\infty,1}, H_1\left(\Sigma_{\infty,1},\mathbb{Z}\right)^{\otimes m}\right)$: A first application of Proposition 3.2.26 is to compute the stable homology groups $H_d\left(\mathbf{\Gamma}_{\infty,1}, H_1\left(\Sigma_{\infty,1},\mathbb{Z}\right)^{\otimes m}\right)$ for all natural numbers m and d. We consider the groupoid $\mathfrak{M}_2^0$ (see Definition 3.2.24). By [RWW17, Proposition 5.18], the boundary connected sum $\natural$ induces a strict braided monoidal structure $\left(\mathfrak{M}_2^0, \natural, \left(\Sigma_{0,1}^0, I\right)\right)$. By Van Kampen's theorem, the fundamental group functor $\pi_1\left(-,x\right)$: $\left(\mathfrak{M}_2, \natural, \Sigma_{0,1}^0\right) \to \left(\mathfrak{gr}, *, 0_{\mathfrak{Gr}}\right)$ is strict monoidal and assigning for all $n, n' \in \mathbb{N}$

$$\pi_1\left(-,x\right)\left(\left[\Sigma_{n',1}, id_{\Sigma_{n'+n,1}}\right]\right) = \iota_{\pi_1\left(\Sigma_{n',1},x\right)} * id_{\pi_1\left(\Sigma_{n,1},x\right)},$$

defines a functor $\pi_1\left(-,x\right) : \mathfrak{UM}_2^0 \to \mathfrak{gr}$. We refer to [Sou17a, Section 3.2] for more details. Hence Assumption (3.2.6) is satisfied. Recall that the homology group $H_1\left(-,\mathbb{Z}\right)$ defines a functor from the category $\mathfrak{Gr}$ to the category **Ab** (see for example [Bro12, Section 8]). As $\pi_1\left(\Sigma_{n,1}^0, x\right)$ is finitely generated for all natural numbers n, the target category of the composition $H_1\left(-,\mathbb{Z}\right) \circ \pi_1\left(-,x\right)$ is the full subcategory of **Ab** of finitely generated abelian groups, denoted by **ab**. Hence, for m a natural number, we define a functor $H_1\left(\Sigma_{-,1},\mathbb{Z}\right)^{\otimes m} : \mathfrak{UM}_2^0 \to \mathbf{ab}$ by the composition:

$$\mathfrak{UM}_2^0 \xrightarrow{\pi_1(-,x)} \mathfrak{Gr} \xrightarrow{H_1(-,\mathbb{Z})} \mathbf{ab} \xrightarrow{-^{\otimes m}} \mathbf{ab}\ ,$$

where $-^{\otimes m} : \mathbf{ab} \to \mathbf{ab}$ sends an object G to $G^{\otimes m}$.

Remark 3.2.27. Let m be a natural number. We refer to [Sou17b, Section 4] for the notion of very strong polynomial functors. As $\pi_1\left(-,x\right)$ is a strict monoidal functor, we deduce from [Sou17b, Proposition 3.8] that $H_1\left(\Sigma_{-,1},\mathbb{Z}\right)$ is very strong polynomial of degree 1. Using again [Sou17b, Proposition 3.8], we deduce that $H_1\left(\Sigma_{-,1},\mathbb{Z}\right)^{\otimes m}$ is a very strong polynomial functor of degree m.

Note that for all natural numbers n, since the free group $\pi_1\left(\Sigma_{n,1},x\right)$ acts trivially on the homology group $H_1\left(\Sigma_{n,1},\mathbb{Z}\right)$, we have an isomorphism:

$$H_1\left(\pi_1\left(\Sigma_{n,1},x\right), H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right) \cong H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes(m+1)}.$$

For all natural numbers n, the action of $\mathbf{\Gamma}_{n,2}$ on $H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}$ is induced by the one of $\mathbf{\Gamma}_{n,1}$ via the surjections $\mathbf{\Gamma}_{n,2} \twoheadrightarrow \mathbf{\Gamma}_{n,1}^1 \twoheadrightarrow \mathbf{\Gamma}_{n,1}$. Using the terminology of [Bol12] and [CM09], $H_1\left(\Sigma_{-,1},\mathbb{Z}\right)^{\otimes m}$ is thus a coefficient system of degree m. Hence, it follows from the stability results of Boldsen [Bol12] or Cohen and Madsen [CM09] that:

Theorem 3.2.28. *[Bol12, Theorem 4.17][CM09, Theorem 0.4] Let m, n and q be natural numbers such that $2n \geq 3q + m$:*

$$H_q\left(\mathbf{\Gamma}_{n,2}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right) \cong H_q\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right).$$

Then, we can prove:

Theorem 3.2.29. *Let m, n and q be natural numbers such that $2n \geq 3q + m$. Then, there is an isomorphism:*

$$H_q\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right) \cong \bigoplus_{\left\lfloor\frac{q-1}{2}\right\rfloor \geq k \geq 0} H_{q-(2k+1)}\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m-1}\right).$$

Proof. The Lyndon-Hochschild-Serre spectral sequence with coefficients given by $H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}$ associated with the short exact sequence (3.2.8) has only two non-trivial rows. Hence, for all natural numbers $n \geq 1$, we obtain the following long exact sequence.

$$\cdots \xrightarrow{d^2_{q+1,0}} H_{q-1}\left(\mathbf{\Gamma}_{n,1}^1, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes(m+1)}\right) \longrightarrow H_q\left(\mathbf{\Gamma}_{n,2}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right) \xrightarrow{\varphi_q} H_q\left(\mathbf{\Gamma}_{n,1}^1, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right) \xrightarrow{d^2_{q,0}} \cdots \quad (3.2.10)$$

We fix a natural number n such that $2n \geq 3q + m$. Using Theorem 3.2.28 and Proposition 3.2.26, the projection

$$H_q\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right) \oplus H_{q-1}\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m+1}\right) \twoheadrightarrow H_q\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right)$$

defines a splitting of φ_q in the long exact sequence (3.2.10):

$$\begin{array}{ccc}
\cdots \longrightarrow H_q\left(\mathbf{\Gamma}_{n,2}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right) & \xrightarrow{\quad\varphi_q\quad} & H_q\left(\mathbf{\Gamma}^1_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right) \xrightarrow{d^2_{q,0}} \cdots \\
\downarrow \cong \text{ by Theorem 3.2.28} & & \downarrow \cong \text{ by Proposition 3.2.26} \\
H_q\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right) & \xleftarrow{\text{splitting}} & H_q\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right) \oplus H_{q-1}\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m+1}\right).
\end{array}$$

Hence, using again Proposition 3.2.26, we have the following isomorphism:

$$H_{q-1}\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m+1}\right) \cong H_{q-2}\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m}\right) \oplus H_{q-3}\left(\mathbf{\Gamma}_{n,1}, H_1\left(\Sigma_{n,1},\mathbb{Z}\right)^{\otimes m+1}\right).$$

The result thus follows by induction on q. □

Remark 3.2.30. For $m = 1$ and rational coefficients, the result of Theorem 3.2.29 recovers the computation due to Harer [Har91, Theorem 7.1.(b)], where the index of the direct sum in this reference should start at $i = 1$.

In [Kaw08], Kawazumi leads the analogous computation for cohomology: namely, [Kaw08, Theorem 1.B.] gives the stable cohomology value

$$H^q\left(\mathbf{\Gamma}_{\infty,1}, H_1\left(\Sigma_{\infty,1},\mathbb{Z}\right)^{\otimes m}\right)$$

for all natural numbers q and m. The method and techniques used in [Kaw08] are different from the ones presented here. Using the universal coefficient theorem for twisted coefficients (see for example [God58, Théorème I.5.5.2]), Theorem 3.2.29 recovers the computation of [Kaw08, Theorem 1.B.].

Computation of $H_d\left(\mathbf{\Gamma}^{[s]}_{\infty,1},\mathbb{Z}\right)$: Another application of Proposition 3.2.26 is to compute the stable homology groups $H_d\left(\mathbf{\Gamma}^{[s]}_{\infty,1},\mathbb{Z}\right)$ for all natural numbers d. Using Proposition 3.2.26 with constant module $\mathbb{Z}$ and Theorem 3.2.29 with $m = 1$, we prove:

Theorem 3.2.31. *Let n and q be natural numbers such that $2n \geq 3q$. Then, there is an isomorphism:*

$$H_q\left(\mathbf{\Gamma}^{[s+1]}_{n,1},\mathbb{Z}\right) \cong \bigoplus_{\lfloor \frac{q}{2} \rfloor \geq k \geq 0} H_{q-2k}\left(\mathbf{\Gamma}^{[s]}_{n,1},\mathbb{Z}\right).$$

Remark 3.2.32. The analogous isomorphism for the rational homology is obtained by Harer in [Har91, Theorem 7.1.(a)].

Furthermore, using other techniques (namely an equivalence of classifying spaces), Bödigheimer and Tillmann prove the equivalent result:

Theorem 3.2.33. *[BT01, Corollary 1.2] Let q and n be natural numbers such that $n \geq 2q$. For all natural numbers s:*

$$\begin{aligned}
H_q\left(\mathbf{\Gamma}^{[s]}_{n,1},\mathbb{Z}\right) &\cong \bigoplus_{k+l=q}\left(H_k\left(\mathbf{\Gamma}_{n,1},\mathbb{Z}\right) \underset{\mathbb{Z}}{\otimes} H_l\left(\left(\mathbb{C}P^\infty\right)^{\times s},\mathbb{Z}\right)\right) \\
&\cong H_k\left(\mathbf{\Gamma}_{n,1},\mathbb{Z}\right) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}\left[x_1,\ldots,x_s\right],
\end{aligned}$$

where $\mathbb{C}P^\infty$ denotes the infinite dimensional complex projective space and each x_i has degree two.

3.2.2.3 Mapping class groups of non-orientable surfaces

Let $\mathcal{N}\Sigma_{g,1}$ denote a smooth compact connected non-orientable surface with non-orientable genus $g \in \mathbb{N}$ and one boundary component with $I : [-1,1] \to \partial\mathcal{N}\Sigma_{g,1}$ a parametrised interval in the boundary. We denote by $\mathcal{N}_{g,1}$ the isotopy classes of diffeomorphisms restricting to the identity on a neighbourhood of the parametrised interval I. Recall from [FM11, Section 1.4.2] that the mapping class group $\mathcal{N}_{g,1}$ identifies with the group of isotopy classes of homeomorphisms of $\mathcal{N}\Sigma_{g,1}$. Recall that fixing the interval I is the same as fixing the whole boundary component pointwise. When there is no ambiguity, we omit the parametrised interval I from the notation.

Notation 3.2.34. For all natural numbers g, we denote by $a^{x}_{\mathcal{N}\Sigma_{g,1}}$ the action of the mapping class group $\mathcal{N}_{g,1}$ on the fundamental group of the surface $\pi_1\left(\mathcal{N}\Sigma_{g,1}, x\right)$ with x a basepoint in the interior of $\mathcal{N}\Sigma_{g,1}$. We denote by $\mathcal{N}^{1}_{g,1}$ the isotopy classes of diffeomorphisms restricting to the identity on a neighbourhood of the parametrised interval I and fixing the marked point.

Let us establish the following result, analogous to the short exact sequence (3.2.7) of Theorem 3.2.21:

Proposition 3.2.35. *Let $g \geq 3$ be a natural number and x be a marked point in the interior of $\mathcal{N}\Sigma_{g,1}$. Deleting x induces a map $\mathcal{N}^{1}_{g,1} \to \mathcal{N}_{g,1}$ which defines the following split short exact sequence:*

$$1 \longrightarrow \pi_1\left(\mathcal{N}\Sigma_{g,1}, x\right) \longrightarrow \mathcal{N}^{1}_{g,1} \longrightarrow \mathcal{N}_{g,1} \longrightarrow 1. \tag{3.2.11}$$

A fortiori, there is an isomorphism:

$$\mathcal{N}^{1}_{g,1} \cong \pi_1\left(\mathcal{N}\Sigma_{g,1}, x\right) \underset{a^{x}_{\mathcal{N}\Sigma_{g,1}}}{\rtimes} \mathcal{N}_{g,1}.$$

Proof. We denote by $Homeo^{\partial}\left(\mathcal{N}\Sigma_{g,1}\right)$ the group of self-homeomorphisms preserving the boundary of $\mathcal{N}\Sigma_{g,1}$ and by $Homeo^{\partial}\left(\mathcal{N}\Sigma_{g,1}, x\right)$ the group of self-homeomorphisms of $\mathcal{N}\Sigma_{g,1}$ preserving the boundary which fix the point x. We have a surjective map $Homeo\left(\mathcal{N}\Sigma_{g,1}\right) \twoheadrightarrow Int\left(\mathcal{N}\Sigma_{g,1}\right)$ (where $Int\left(\mathcal{N}\Sigma_{g,1}\right)$ denotes the interior of the surface $\mathcal{N}\Sigma_{g,1}$) defined by $\varphi \mapsto \varphi(x)$. This is a fiber bundle with fiber $Homeo\left(\mathcal{N}\Sigma_{g,1}, x\right)$. Hamstrom proves in [Ham66, Theorem 5.3] that $Homeo^{\partial}\left(\mathcal{N}\Sigma_{g,1}\right)$ is contractible. Similar results can be found in [Gra73, Théorème 2] or [ES70]. Hence, the short exact sequence (3.2.11) is induced by the homotopy long exact sequence associated with the fibration $Homeo\left(\mathcal{N}\Sigma_{g,1}\right) \twoheadrightarrow Int\left(\mathcal{N}\Sigma_{g,1}\right)$.

The embedding of $\mathcal{N}\Sigma_{g,1}$ into $\mathcal{N}\Sigma^{1}_{g,1}$ as the complement of the disc $\Sigma^{1}_{0,1}$ with the marked point x induces an injective morphism $\mathcal{N}_{g,1} \hookrightarrow \mathcal{N}^{1}_{g,1}$. This provides a splitting of the exact sequence (3.2.11). □

Let us introduce a suitable groupoid for our work, inspired by [RWW17, Section 5.6].

Definition 3.2.36. Let $\mathfrak{M}_2^{-}$ be the skeleton of the groupoid defined by:

- Objects: the smooth compact connected non-orientable surfaces $\mathcal{N}\Sigma_{n,1}$ for all natural numbers n with $x \in int\left(\mathcal{N}\Sigma_{n,1}\right)$ a basepoint;
- Morphisms: $Aut_{\mathfrak{M}_2^{-}}\left(\mathcal{N}\Sigma_{n,1}\right) = \mathcal{N}_{n,1}$ for all natural numbers n.

Remark 3.2.37. The morphisms $\left\{a^{x}_{\mathcal{N}\Sigma_{n,1}}\right\}_{n\in\mathbb{N}}$ of Notation 3.2.34 assemble to define a functor $\mathcal{A}_{\mathfrak{M}_2^{-}} : \mathfrak{M}_2^{-} \to \mathfrak{Gr}$ such that $\mathcal{A}(n) = \pi_1\left(\mathcal{N}\Sigma_{n,1}, x\right)$ for all natural numbers n. Hence, recalling that $\pi_1\left(\mathcal{N}\Sigma_{n,1}, x\right)$ is a free group of rank n, Assumption 3.2.1 is satisfied.

Hence, we deduce from Corollary 3.2.5 and Proposition 3.2.35 that:

Corollary 3.2.38. *Let $n \geq 3$ and $q \geq 1$ be natural numbers. Let M_n be a $R\left[\mathcal{N}^{1}_{n,1}\right]$-module (and a fortiori a $R\left[\mathcal{N}_{n,1}\right]$-module using the surjection $\mathcal{N}^{1}_{g,1} \twoheadrightarrow \mathcal{N}_{g,1}$) on which $\pi_1\left(\mathcal{N}\Sigma_{n,1}, x\right)$ acts trivially. Then:*

$$H_q\left(\mathcal{N}^{1}_{n,1}, M_n\right) \cong H_{q-1}\left(\mathcal{N}_{n,1}, H_1\left(\Sigma_{n,1}, R\right) \underset{R}{\otimes} M_n\right) \oplus H_q\left(\mathcal{N}_{n,1}, M_n\right). \tag{3.2.12}$$

Remark 3.2.39. Assigning $M_n = \mathbb{Z}$ for all natural numbers n, the isomorphism (3.2.12) thus allows to compute the twisted stable homology with twisted coefficient $H_{q-1}(\mathcal{N}_{\infty,1}, H_1(\mathcal{N}\Sigma_{\infty,1}, \mathbb{Z}))$ from the stable homologies with rational coefficient $H_q\left(\mathcal{N}^1_{\infty,1}, \mathbb{Z}\right)$ and $H_q(\mathcal{N}_{\infty,1}, \mathbb{Z})$ for all natural numbers q. Unfortunately, as far as the author knows, even if $H_q(\mathcal{N}_{\infty,1}, \mathbb{Z})$ is computed in [RW08] for $1 \leq q \leq 6$, the stable homology $H_q\left(\mathcal{N}^1_{\infty,1}, \mathbb{Z}\right)$ is computed only for $q = 1$. Namely, it follows by the computations of Korkmaz in [Kor98] and the stability result with respect to marked points of Hanbury in [Han09] that $H_1\left(\mathcal{N}^1_{\infty,1}, \mathbb{Z}\right) = \mathbb{Z}_2$ and a fortiori $H_0(\mathcal{N}_{\infty,1}, H_1(\mathcal{N}\Sigma_{\infty,1}, \mathbb{Z})) = 0$ by [RW08].

Also, note that Stukow computes $H_1(\mathcal{N}_{n,1}, H_1(\mathcal{N}\Sigma_{n,1}, \mathbb{Z}))$ in [Stu14, Theorem 1.1] for $n \geq 3$. Using [RW08], we deduce that $H_2\left(\mathcal{N}^1_{\infty,1}, \mathbb{Z}\right) \cong \mathbb{Z}_2^{\oplus 5}$.

3.2.2.4 Automorphisms of free groups with boundaries

Let $\mathscr{G}_{n,k}$ denote the topological space consisting of the wedge of $n \in \mathbb{N}$ circles together with k distinguished circles joined by arcs to the basepoint. For $s \in \mathbb{N}$, let $\mathscr{G}^s_{n,k}$ be the space obtained from $\mathscr{G}_{n,k}$ by wedging $s-1$ edges at the basepoint. We denote by $A^s_{n,k}$ the group of path-components of the space of homotopy equivalences of $\mathscr{G}^s_{n,k}$ which fix the k distinguished circles and the s basepoints. For instance, for n a natural number and denoting by $\mathbf{F}_n$ the free group of rank n, then $A^1_{n,0}$ is isomorphic to the automorphism group of $\mathbf{F}_n$ denoted by $Aut(\mathbf{F}_n)$ and $A^2_{n,0}$ is isomorphic to the holomorph of the free group $\mathbf{F}_n$. We refer the reader to [HW05] and [Jen04] for more details on these groups.

For $k, n \in \mathbb{N}$, we denote by $Aut_{n,k}$ the subgroup of $Aut(\mathbf{F}_{n+k})$ of automorphisms that take each of the last k generators to a conjugate of itself. We recall that the homotopy long exact sequence associated with the fibration induced by restricting the homotopy equivalences of $\mathscr{G}_{n,k}$ to their rotations of the k distinguished circles provides a surjective map $A_{n,k} \twoheadrightarrow Aut_{n,k}$.

Notation 3.2.40. For $k, n \in \mathbb{N}$, we denote by $a_{A_{n,k}}$ the composition $A^1_{n,k} \to A_{n,k} \twoheadrightarrow Aut_{n,k} \hookrightarrow Aut(\mathbf{F}_{n+k})$ where the map $A^1_{n,k} \to A_{n,k}$ forgets the basepoint.

We recall the following useful result:

Lemma 3.2.41. *[HW05] Let n, k and $s \geq 2$ be natural numbers. There is a split short exact sequence*

$$1 \longrightarrow \mathbf{F}_{n+k} \longrightarrow A^s_{n,k} \longrightarrow A^{s-1}_{n,k} \longrightarrow 1, \tag{3.2.13}$$

where the map $A^s_{n,k} \to A^{s-1}_{n,k}$ forgets the last basepoint and a fortiori

$$A^s_{n,k} \cong (\mathbf{F}_{n+k})^{s-1} \rtimes A^1_{n,k}$$

where $A^1_{n,k}$ acts diagonally on $(\mathbf{F}_{n+k})^{s-1}$ via the map $a_{A_{n,k}} : A^1_{n,k} \to Aut(\mathbf{F}_{n+k})$.

We introduce the suitable groupoid to work with the automorphisms of free groups with boundaries.

Definition 3.2.42. Let k and s be fixed natural numbers. Let $\mathfrak{A}_{s,k}$ be the groupoid with the topological spaces $\mathscr{G}^s_{n,k}$ as objects and $A^s_{n,k}$ as automorphism groups for all natural numbers.

Remark 3.2.43. In particular, for $s = 1$ and $k = 0$, $\mathfrak{A}_{1,0}$ is the maximal subgroupoid of the category $\mathfrak{gr}$ of finitely generated free groups. The coproduct $*$ thus induces a strict symmetric monoidal structure $(\mathfrak{A}_{1,0}, *, 0_{\mathfrak{Gr}})$. Moreover, we define a functor $i : \mathfrak{U}\mathfrak{A}_{1,0} \to \mathfrak{gr}$ by the identity on objects and sending a morphism $[n_2 - n_1, g] : \mathbf{F}_{n_1} \to \mathbf{F}_{n_2}$ of $\mathfrak{U}\mathfrak{A}_{1,0}$ (where $g \in Aut_{\mathfrak{Gr}}(\mathbf{F}_{n_2})$) to the morphism $g \circ \left(\iota_{\mathbf{F}_{n_2-n_1}} * id_{\mathbf{F}_{n_1}}\right) : \mathbf{F}_{n_1} \hookrightarrow \mathbf{F}_{n_2}$ of $\mathfrak{gr}$.

Let k and $s \geq 1$ be natural numbers. Precomposing by the surjection

$$A^s_{n,k} \twoheadrightarrow A^{s-1}_{n,k} \twoheadrightarrow \cdots \twoheadrightarrow A^1_{n,k}$$

(see Lemma 3.2.41), the morphisms $\left\{a_{A_{n,k}}\right\}_{n \in \mathbb{N}}$ of Notation 3.2.40 assemble to define a functor $\mathcal{A}_{\mathfrak{A}_{s,k}} : \mathfrak{A}_{s,k} \to \mathfrak{Gr}$ such that $\mathcal{A}_{\mathfrak{A}_{s,k}}(n) = \mathbf{F}_{n+k}$ for all natural numbers n. Hence, Assumption 3.2.1 is satisfied.

Furthermore, we recall the stable homology result for automorphism groups of free groups due to Galatius for constant coefficients and Djament and Vespa for twisted coefficients:

Theorem 3.2.44. *Let $q \geq 1$ be a natural number. Then:*

- *[Gal11] for $n \geq 2q+1$, $H_q(Aut(\mathbf{F}_n), \mathbb{Q}) = 0$;*
- *[DV15, Théorème 1] for $F : \mathfrak{gr} \to \mathbf{Ab}$ a reduced (ie null on the trivial group) polynomial functor such that $F(0) = 0$, then $H_q(Aut(\mathbf{F}_n), F(n)) = 0$ for $n \geq 2q+1$.*

Hence, we can establish the main result of Section 3.2.2.4.

Theorem 3.2.45. *Let $s \geq 2$ and $q \geq 1$ be natural numbers.*

1. *Let $F : \mathfrak{gr} \to \mathbf{Ab}$ be a reduced polynomial functor such that $F(0) = 0$. The action of $A^s_{n,0}$ on $F(n)$ is induced by the surjections*
$$A^s_{n,0} \twoheadrightarrow A^{s-1}_{n,0} \twoheadrightarrow \cdots \twoheadrightarrow A^1_{n,0}.$$
For all natural numbers $n \geq 2q+1$:
$$H_q\left(A^s_{n,0}, F(n)\right) = 0.$$
2. *For all natural numbers $n \geq 3q+3$ and $k \geq 0$, $H_q\left(A^s_{n,k}, \mathbb{Q}\right) = 0$.*

Proof. We consider the functor $F \circ i : \mathfrak{U}\mathfrak{A}_{1,0} \to \mathbf{Ab}$. As i is a strict monoidal functor, we deduce from [Sou17b, Proposition 3.8] that $F \circ i$ is strong polynomial. It follows from Corollary 3.2.10 that $H_1(\mathbf{F}_n, F(-)) : \mathfrak{U}\mathfrak{A}_{1,0} \xrightarrow{i} \mathfrak{gr} \to \mathbf{Ab}$ is a strong polynomial functor. Hence, the first result follows from Corollary 3.2.5 and Theorem 3.2.44.

In [HW10, Theorem 1.1], Hatcher and Wahl prove that the stabilization morphism $A^s_{n,k} \to A^s_{n,k+1}$ induces an isomorphism for the rational homology $H_q\left(A^s_{n,k}, \mathbb{Q}\right) \xrightarrow{\sim} H_q\left(A^s_{n,k+1}, \mathbb{Q}\right)$ if $n \geq 3q+3$. The second result thus follows from the previous statement. □

Remark 3.2.46. For $k = 0$ and $s = 2$, Theorem 3.2.45 recovers the results [Jen04, Theorem 1.2 (b) and (c)] due to Jensen.

3.3 A general result for twisted stable homology

In this section, we prove a decomposition result for the stable homology with twisted coefficients for families of groups whose associated groupoid is a full subcategory of a pre-braided locally homogeneous groupoid (see Theorem 3.3.7). It extends a previous analogous result due Djament and Vespa in [DV10, Section 1 and 2] when the ambient monoidal structure is symmetric.

We refer the reader to the papers [FP03, Section 2] and [DV10, Appendice A] for an introduction to homological algebra in functor categories and we assume that all the definitions, properties and results there are known.

3.3.1 General framework

Throughout Section 3.3, we consider $(\mathcal{G}, \natural, 0)$ a small braided strict monoidal groupoid with no zero divisors (see Definition 3.1.2), such that $Aut_{\mathcal{G}}(0) = \{id_0\}$ and $\mathbb{K}$ is a field. We fix a pair of objects (A, X) of $\mathcal{G}$ and assume that $\mathcal{G}$ satisfies the properties (**LC**) and (**LI**) of Definition 3.1.12 at (A, X).

Remark 3.3.1. By Theorem 3.1.13, Quillen's bracket construction $\mathfrak{U}\mathcal{G}$ is locally homogeneous at (A, X).

Definition 3.3.2. Let $\mathcal{G}_{(A,X)}$ (respectively $\mathfrak{U}\mathcal{G}_{(A,X)}$) be the full subgroupoid of $\mathcal{G}$ (respectively $\mathfrak{U}\mathcal{G}$) on the objects $\left\{A \natural X^{\natural n}\right\}_{n \in \mathbb{N}}$.

Remark 3.3.3. If $A = 0$, then the braided monoidal structure $(\mathcal{G}, \natural, 0)$ induces a small braided strict monoidal structure on $\mathcal{G}_{(0,X)}$, denoted by $\left(\mathcal{G}_{(0,X)}, \natural, 0\right)$. Moreover, Quillen's bracket construction $\mathfrak{U}\mathcal{G}_{(0,X)}$ is homogeneous by Theorem 3.1.13.

Definition 3.3.4. Let $\mathscr{O}_{\mathcal{G}_{(A,X)}} : (\mathbb{N}, \leq) \to \mathfrak{U}\mathcal{G}_{(A,X)}$ be the faithful and essentially surjective functor assigning $\mathscr{O}(n) = A\natural X^{\natural n}$ and $\mathscr{O}(\gamma_n) = id_{A\natural X^{\natural n}}\natural\iota_X$ for all natural numbers n.

Notation 3.3.5. For all natural numbers n, we denote the automorphism group $Aut_{\mathcal{G}}(A\natural X^{\natural n})$ by G_n and the group morphism $G_n \to G_{n+1}$ taking φ to $\varphi\natural id_X$ by g_n. Hence, we define a family of groups $G_- : (\mathbb{N}, \leq) \longrightarrow \mathfrak{Gr}$ using the functor $\mathscr{O}$.

Since 0 is an initial object in $\mathfrak{U}\mathcal{G}$, we have canonical morphisms in $\mathfrak{U}\mathcal{G}$ for all natural numbers n and n' such that $n' \geq n$:

$$id_{A\natural X^{\natural n}}\natural\iota_{X^{\natural(n'-n)}} : A\natural X^{\natural n} \to A\natural X^{\natural n'}.$$

We fix F an object of $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}_{(A,X)}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$. Our goal is to compute the stable homology of the family of groups G_- with coefficients given by F.

Notation 3.3.6. We denote by G_∞ the colimit with respect to $(\mathbb{N}, \leq)$ of the family of groups G_- and by F_∞ the colimit of the G_n-modules $F\left(A\natural X^{\natural n}\right)$ with respect to the morphisms $F\left(id_{A\natural X^{\natural n}}\natural\iota_{X^{\natural(n'-n)}}\right)$. Recall from Definition 3.3.4 that there is a faithful functor $\mathscr{O}_{\mathcal{G}_{(A,X)}} : (\mathbb{N}, \leq) \to \mathfrak{U}\mathcal{G}_{(A,X)}$. Then we denote $H_*(G_\infty, F_\infty) = \underset{n\in(\mathbb{N},\leq)}{Colim}\left(H_*(G(n), F\left(A\natural X^{\natural n}\right))\right)$. This notation makes sense since group homology commutes with filtered colimits (see [Wei94, Theorem 2.6.10]).

3.3.2 Splitting result for stable homology

As categories with one object, the groups $\{G_n\}_{n\in\mathbb{N}}$ are subcategories of $\mathfrak{U}\mathcal{G}_{(A,X)}$. We denote by $\Pi : G_\infty \times \mathfrak{U}\mathcal{G}_{(A,X)} \to \mathfrak{U}\mathcal{G}_{(A,X)}$ the projection functor and by Π^* the precomposition by Π. Hence, for all natural numbers n, the canonical group morphism $G_n \to G_\infty$ and the faithful functors $G_n \to \mathfrak{U}\mathcal{G}_{(A,X)}$ induce a natural inclusion functor $\Psi_{F,n} : H_*\left(G_n, F\left(A\natural X^{\natural n}\right)\right) \to H_*\left(G_\infty \times \mathfrak{U}\mathcal{G}_{(A,X)}, \Pi^* F\right)$ by the functoriality of the homology of categories (see [DV10, Appendice A]).

Using the group morphisms g_n and the morphisms $id_{A\natural X^{\natural n}}\natural\iota_X$, by the functoriality in two variables of group homology (see for example [Bro12, Section III.8]), we define maps $H_*\left(G_n, F\left(A\natural X^{\natural n}\right)\right) \to H_*\left(G_{n+1}, F\left(A\natural X^{\natural(n+1)}\right)\right)$ such that the inclusion functors $\Psi_{F,n}$ are natural with respect to n. Hence, we form a morphism:

$$\Psi_F : H_*(G_\infty, F_\infty) \to H_*\left(G_\infty \times \mathfrak{U}\mathcal{G}_{(A,X)}, \Pi^* F\right).$$

Let us state the main result of this section.

Theorem 3.3.7. *Let $\mathbb{K}$ be a field. We consider a pre-braided locally homogeneous at (A, X) category $\left(\mathfrak{U}\mathcal{G}_{(A,X)}, \natural, 0\right)$ such that the unit 0 is an initial object, as detailed in Section 3.3.1. For all functors $F : \mathfrak{U}\mathcal{G}_{(A,X)} \to \mathbb{K}\text{-}\mathfrak{Mod}$, the morphism Ψ_F is a $\mathbb{K}$-modules isomorphism. Moreover, Ψ_F decomposes as a natural isomorphism:*

$$H_*(G_\infty, F_\infty) \cong \bigoplus_{k+l=*}\left(H_k(G_\infty, \mathbb{K}) \underset{\mathbb{K}}{\otimes} H_l\left(\mathfrak{U}\mathcal{G}_{(A,X)}, F\right)\right).$$

Proof. Let i and j be natural numbers. The morphism $G_j \to G_{i+j}$ defined by $\varphi \mapsto X^{\natural i}\natural\varphi$ is conjugated to the one defined by $\varphi \mapsto \varphi\natural X^{\natural i}$ using the braiding $\left(b^{\mathcal{G}}_{X^{\natural j}, X^{\natural i}}\right)^{-1} : X^{\natural i}\natural X^{\natural j} \to X^{\natural j}\natural X^{\natural i}$ of the pre-braided monoidal structure, recalling from the relation (3.1.3) of Definition (3.1.6) that:

$$\left(b^{\mathcal{G}}_{A\natural X^{\natural j}, X^{\natural i}}\right)^{-1} \circ \left(\iota_{X^{\natural i}}\natural id_{A\natural X^{\natural j}}\right) = id_{X^{\natural j}}\natural\iota_{X^{\natural i}} \text{ and } b^{\mathcal{G}}_{A\natural X^{\natural j}, X^{\natural i}} \circ \left(id_{A\natural X^{\natural j}}\natural\iota_{X^{\natural i}}\right) = \iota_{X^{\natural i}}\natural id_{A\natural X^{\natural j}}.$$

The result then follows mutatis mutandis from the proof of [DV10, Proposition 2.22]. As pointed out in [DV10, Remark 2.23], this is the only place in this framework where the symmetry of the monoidal structure is used: all the other constructions and proofs work exactly in the same way assuming the monoidal structure is pre-braided.

However, for the convenience of the reader, we detail here a proof assuming for simplicity that $A = 0$. Recall from Remark 3.3.3 that the pre-braided category $\left(\mathfrak{U}\mathcal{G}_{(0,X)}, \natural, 0\right)$ is thus homogeneous. Note that the morphism

Ψ_F is a morphism of δ-functors commuting with filtered colimits (see [Wei94, Section 2.1]). Since the category $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G}_{(0,X)}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$ has enough projectives, provided by direct sums of the standard projective generators functors $P_{X^{\natural n}}^{\mathfrak{U}\mathcal{G}_{(0,X)}} = \mathbb{K}\left[Hom_{\mathfrak{U}\mathcal{G}_{(0,X)}}\left(X^{\natural n}, -\right)\right]$ for all natural numbers n (see [DV10, Appendice A]), we only have to show that Ψ_F is an isomorphism when $F = P_{X^{\natural n}}^{\mathfrak{U}\mathcal{G}_{(0,X)}}$. Indeed, the result for an ordinary functor F thus follows from a resolution of F by direct sums of $P_{X^{\natural n}}^{\mathfrak{U}\mathcal{G}_{(0,X)}}$.

We deduce from Remark 3.1.11 that for all natural numbers $m \geq n$, we have the isomorphism of G_m-sets:

$$Hom_{\mathfrak{U}\mathcal{G}_{(0,X)}}\left(X^{\natural n}, X^{\natural m}\right) \cong G_m / G_{m-n}.$$

Hence, $P_{X^{\natural n}}^{\mathfrak{U}\mathcal{G}_{(0,X)}}\left(X^{\natural m}\right) \cong \mathbb{K}\left[G_m\right] \underset{\mathbb{K}[G_{m-n}]}{\otimes} \mathbb{K}$ as G_m-modules. Therefore, it follows from Schapiro's lemma (see [Wei94, Section 6.3]) that:

$$H_*\left(G_m, P_{X^{\natural n}}^{\mathfrak{U}\mathcal{G}_{(0,X)}}\left(X^{\natural m}\right)\right) \cong H_*\left(G_{m-n}, \mathbb{K}\right).$$

Taking the colimit with respect to m, we deduce the isomorphism $H_*\left(G_\infty, P_{X^{\natural n}}^{\mathfrak{U}\mathcal{G}_{(0,X)}}\left(X^{\natural \infty}\right)\right) \cong H_*\left(G_\infty, \mathbb{K}\right)$. We conclude using the first Künneth spectral sequence of [DV10, Proposition 2.16]:

$$H_*\left(G_\infty, \mathbb{K}\right) \cong H_*\left(G_\infty \times \mathfrak{U}\mathcal{G}_{(0,X)}, \Pi^*\left(P_{X^{\natural n}}^{\mathfrak{U}\mathcal{G}_{(0,X)}}\right)\right).$$

The second part of the statement follows applying Künneth formula for homology of categories (see [DV10, Proposition 2.27]). □

3.4 Twisted stable homologies for FI-modules and related functors

In this section, we present a general principle to compute the twisted stable homology for mapping class groups with non-trivial finite quotient groups. In Section 3.4.1, we establish in Corollary 3.4.11 a general formula to compute the stable homology with twisted coefficients given by functors over categories associated with the aforementionned finite quotient groups. This allows to set explicit formulas for the stable homology with coefficients given by FI-modules for braid groups, mapping class groups of orientable surfaces and some particular right-angled Artin groups in Section 3.4.2.

Throughout Section 3.4, we assume that the field $\mathbb{K}$ is of characteristic 0.

Let us consider three families of groups K_-, G_- and C_- (see Definition 3.0.2), such that the group C_n is finite for all natural numbers n, and which fit into the following short exact sequence in the category $\mathbf{Fct}\left(\left(\mathbb{N}, \leq\right), \mathfrak{Gr}\right)$:

$$0 \longrightarrow K_- \xrightarrow{k} G_- \xrightarrow{c} C_- \longrightarrow 0, \tag{3.4.1}$$

where $k : K_- \to G_-$ and $c : G_- \to C_-$ are natural transformations and 0 denotes the constant object of $\mathbf{Fct}\left(\left(\mathbb{N}, \leq\right), \mathfrak{Gr}\right)$ at $0_{\mathfrak{Gr}}$.

Notation 3.4.1. Let $\mathcal{K}$, $\mathcal{G}$ and $\mathcal{C}$ denote the groupoids with natural numbers as objects and $Aut_{\mathcal{K}}(n) = K_n$, $Aut_{\mathcal{G}}(n) = G_n$ and $Aut_{\mathcal{C}}(n) = C_n$ for morphisms.

We assume that these groupoids satisfy further properties:

Assumption 3.4.2. *The groupoids $\mathcal{G}$ and $\mathcal{C}$ are endowed with braided strict monoidal structures $(\mathcal{G}, \natural_{\mathcal{G}}, 0_{\mathcal{G}})$ and $(\mathcal{C}, \natural_{\mathcal{C}}, 0_{\mathcal{C}})$, where $\natural_{\mathcal{G}}$ and $\natural_{\mathcal{C}}$ are defined by the addition on objects, such that:*

- *the morphisms $\{c_n\}_{n\in\mathbb{N}}$ induce a strict monoidal functor $\mathfrak{c} : \mathcal{G} \to \mathcal{C}$ defined by the identity on objects;*
- *$G_-(\gamma_n) = id_1 \natural_{\mathcal{G}} - : G_n \hookrightarrow G_{n+1}$ and $C_-(\gamma_n) = id_1 \natural_{\mathcal{C}} - : C_n \hookrightarrow C_{n+1}$ for all natural numbers n.*

Definition 3.4.3. Let $\mathscr{O}'_{\mathcal{G}} : (\mathbb{N}, \leq) \to \mathfrak{U}\mathcal{G}$ and $\mathscr{O}'_{\mathcal{C}} : (\mathbb{N}, \leq) \to \mathfrak{U}\mathcal{C}$ be the faithful and essentially surjective functors assigning $\mathscr{O}'_{\mathcal{G}}(n) = \mathscr{O}'_{\mathcal{C}}(n) = n$ and $\mathscr{O}'_{\mathcal{G}}(n) = \mathscr{O}'_{\mathcal{C}}(n) = [1, id_{n+1}]$ for all natural numbers n.

Remark 3.4.4. Recall that the associated Quillen's bracket construction $(\mathfrak{U}\mathcal{G}, \natural_{\mathcal{G}}, 0)$ and $(\mathfrak{U}\mathcal{C}, \natural_{\mathcal{C}}, 0)$ are pre-braided strict monoidal by Proposition 3.1.8.

Using the functors $\mathscr{O}'_{\mathcal{G}}$ and $\mathscr{O}'_{\mathcal{C}}$ introduced in Definition 3.4.3, the natural transformation $c : G_- \to C_-$ identifies the morphisms $[n'-n, id_{n'}]$ (with natural numbers $n' \geq n$) of $\mathfrak{U}\mathcal{G}$ and $\mathfrak{U}\mathcal{C}$. The criteria (3.1.1) and (3.1.2) of Lemma 3.1.5 being trivially checked, we abuse the notation and write $\mathfrak{c}$ for the functor $\mathfrak{U}\mathcal{G} \to \mathfrak{U}\mathcal{C}$ induced by $\mathfrak{c} : \mathcal{G} \to \mathcal{C}$.

The short exact sequence (3.4.1) implies that the braided strict monoidal structure $(\mathcal{G}, \natural_{\mathcal{G}}, 0_{\mathcal{G}})$ induces a braided strict monoidal structure on $\mathcal{K}$, denoted by $(\mathcal{K}, \natural_{\mathcal{G}}, 0_{\mathcal{G}})$, such that:

$$K_-(\gamma_n) = id_1 \natural_{\mathcal{G}} - : K_n \hookrightarrow K_{n+1}$$

for all natural numbers n. As for the morphisms $\{c_n\}_{n\in\mathbb{N}}$, the morphisms $\{k_n\}_{n\in\mathbb{N}}$ induce a strict monoidal functor $\mathfrak{k} : \mathfrak{U}\mathcal{K} \to \mathfrak{U}\mathcal{G}$.

We fix F an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, \mathbb{K}\text{-}\mathfrak{Mod})$.

Notation 3.4.5. For all natural numbers n, we abuse the notation and write $F(n)$ for $\mathrm{Res}_{K_n}^{G_n}(F(n))$, where $\mathrm{Res}_{K_n}^{G_n}$ denotes the restriction functor.

Our aim is to compute the stable homology $H_*(G_\infty, F_\infty)$ of the family of groups G_-. A first step is given by the following result:

Proposition 3.4.6. *Let K_-, G_- and C_- be three families of groups fitting in the short exact sequence (3.4.1), such that the group C_n is finite for all natural numbers n and Assumption 3.4.2 is satisfied. Then, for all natural numbers q:*

$$H_q(G_n, F(n)) \cong H_0(C_n, H_q(K_n, F(n))). \tag{3.4.2}$$

Proof. Applying the Lyndon-Hochschild-Serre spectral sequence for the short exact sequence (3.4.1), we obtain the following convergent first quadrant spectral sequence:

$$E^2_{pq} : H_p(C_n, H_q(K_n, F(n))) \Longrightarrow H_{p+q}(G_n, F(n)). \tag{3.4.3}$$

Fixing n a natural number, we have for $p \neq 0$:

$$H_p(C_n, H_q(K_n, F(n))) = 0,$$

since C_n is a finite group (see for example [Wei94, Proposition 6.1.10]). Hence, the second page of the spectral sequence (3.4.3) has non-zero terms only on the 0-th column and zero differentials. A fortiori, the convergence gives that $E^2 = E^\infty$ and this gives the desired result. □

3.4.1 A general equivalence for stable homology

Let us focus on a key property for the homologies of the kernels $\{K_n\}_{n\in\mathbb{N}}$ which improves Proposition (3.4.6).

Remark 3.4.7. Let n be a natural number. As K_n is a normal subgroup of G_n, the map $conj_n : G_n \to Aut_{\mathfrak{Gr}}(K_n)$ sending an element $g \in G_n$ to the left conjugation by g is a group morphism.

Lemma 3.4.8. *We define a functor $\tilde{K}_- : \mathfrak{U}\mathcal{G} \to \mathfrak{Gr}$ assigning $\tilde{K}_-(n) = K_n$ for all natural numbers n and:*

1. *for all $g \in G_n$, $\tilde{K}_-(g) \in Aut_{\mathfrak{Gr}}(K_n)$ to be $conj_n(g) : k \mapsto gkg^{-1}$ for all $k \in K_n$,*
2. *$\tilde{K}_-([1, id_{n+1}]) = id_1 \natural_{\mathcal{G}} -$.*

Proof. It follows from the first assignment of Lemma 3.4.8 that we define a functor $\tilde{K}_- : \mathcal{G} \to \mathfrak{Gr}$. The relation (3.1.1) of Lemma 3.1.5 follows from the definition of the monoidal product $\natural_{\mathcal{G}}$.

Let n and n' be natural numbers such that $n' \geq n$, let $g \in G_n$ and $g' \in G_{n'}$. We compute for all $k \in K_n$:

$$\begin{aligned}\left(\tilde{K}_-(g' \natural_{\mathcal{G}} g) \circ \tilde{K}_-([n', id_{n'+n}])\right)(k) &= (g' \natural_{\mathcal{G}} g)(id_{n'} \natural_{\mathcal{G}} k)(g' \natural_{\mathcal{G}} g)^{-1} \\ &= id_{n'} \natural_{\mathcal{G}} \left(gkg^{-1}\right) \\ &= \left(\tilde{K}_-([n', id_{n'+n}]) \circ \tilde{K}_-(g)\right)(k).\end{aligned}$$

Hence, the relation (3.1.2) is satisfied a fortiori the result follows from Lemma 3.1.5. □

Lemma 3.4.8 is useful to prove the following key result.

Proposition 3.4.9. *Let K_-, G_- and C_- three families of groups fitting into the short exact sequence (3.4.1)such that Assumption 3.4.2 is satisfied and let F be an object of* $\mathbf{Fct}\left(\mathfrak{U}\mathcal{G},\mathbb{K}\text{-}\mathfrak{Mod}\right)$*. Then, for all natural numbers q, the homology groups $\left\{H_q\left(K_n,F\left(n\right)\right)\right\}_{n\in\mathbb{N}}$ define a functor $H_q\left(K_-,F\left(-\right)\right):\mathfrak{U}\mathcal{C}\to\mathbb{K}\text{-}\mathfrak{Mod}$.*

Proof. Let $\mathscr{P}$ be the category of pairs (G,M) where G is a group and M is a G-module for objects; for (G,M) and (G',M') objects of $\mathscr{P}$, a morphism from (G,M) to (G',M') is a pair (φ,α) where $\varphi\in Hom_{\mathfrak{Gr}}(G,G')$ and $\alpha:M\to M'$ is a G-module morphism, where M' is endowed with a G-module structure via φ. Using the functor $F:\mathfrak{U}\mathcal{G}\to\mathbb{K}\text{-}\mathfrak{Mod}$, by Lemma 3.4.8 $\tilde{K}_-$ defines a functor $\left(\tilde{K}_-,F\left(-\right)\right):\mathfrak{U}\mathcal{G}\to\mathscr{P}$. Recall from [Wei94, Section 6.7.5] or [Bro12, Section 8] that group homology defines a covariant functor $H_*:\mathscr{P}\to\mathbb{K}\text{-}\mathfrak{Mod}$ for all $q\in\mathbb{N}$. Hence the composition with the functor $\left(\tilde{K}_-,F\left(-\right)\right):\mathfrak{U}\mathcal{G}\to\mathscr{P}$ gives a functor:

$$H_q\left(K_-,F\left(-\right)\right):\mathfrak{U}\mathcal{G}\to\mathbb{K}\text{-}\mathfrak{Mod}.$$

Moreover, since inner automorphisms act trivially in homology, we deduce that for all natural numbers n, the conjugation action of G_n on $(K_n,F(n))$ induces an action of C_n on $H_*(K_n,F(n))$ (see for instance [Bro12, Section 8, Proposition 8.1 and Corollary 8.2] for more details). The monoidal structures $(\mathcal{G},\natural_{\mathcal{G}},0_{\mathcal{G}})$ and $(\mathcal{C},\natural_{\mathcal{C}},0_{\mathcal{C}})$ being compatible by Assumption 3.4.2, we deduce that the functor $H_q\left(K_-,F\left(-\right)\right)$ factors through the category $\mathfrak{U}\mathcal{C}$ using the functor $\mathfrak{c}:\mathfrak{U}\mathcal{G}\to\mathfrak{U}\mathcal{C}$. □

Finally, recall the following property for the homology of a category:

Proposition 3.4.10. *[FP03, Example 2.5] Let $\mathfrak{C}$ be an object of $\mathfrak{Cat}$ and let F be an object of* $\mathbf{Fct}\left(\mathfrak{C},R\text{-}\mathfrak{Mod}\right)$*. Then, $H_0\left(\mathfrak{C},F\right)$ is isomorphic to the colimit over $\mathfrak{C}$ of the functor $F:\mathfrak{C}\to R\text{-}\mathfrak{Mod}$.*

We thus deduce from Proposition 3.4.9:

Corollary 3.4.11. *Let K_-, G_- and C_- three families of groups fitting in the short exact sequence (3.4.1), such that the group C_n is finite for all natural numbers n and Assumption 3.4.2 is satisfied. Then, for all natural numbers q:*

$$H_q\left(G_\infty,F_\infty\right)\quad\cong\quad\underset{l\in\mathfrak{U}\mathcal{C}}{Colim}\left(H_q\left(K_l,F\left(l\right)\right)\right).$$

Moreover, if F factors through the category $\mathfrak{U}\mathcal{C}$ (in other words, $F:\mathfrak{U}\mathcal{G}\overset{\mathfrak{c}}{\to}\mathfrak{U}\mathcal{C}\to\mathbb{K}\text{-}\mathfrak{Mod}$), then:

$$H_q\left(G_\infty,F_\infty\right)\cong\underset{l\in\mathfrak{U}\mathcal{C}}{Colim}\left(H_q\left(K_l,\mathbb{K}\right)\underset{\mathbb{K}}{\otimes}F\left(l\right)\right).$$

Proof. Applying Theorem 3.3.7 to Proposition 3.4.6, we obtain that:

$$\underset{n\in\mathbb{N}}{Colim}\left(H_0\left(C_n,H_q\left(K_n,F\left(n\right)\right)\right)\right)\cong\underset{n\in\mathbb{N}}{Colim}\left(H_0\left(C_n,\mathbb{K}\right)\underset{\mathbb{K}}{\otimes}H_0\left(\mathfrak{U}\mathcal{C},H_q\left(K_-,F\right)\right)\right).$$

The first result thus follows from the fact that $H_0\left(\mathfrak{U}\mathcal{C},H_q\left(K_-,F\right)\right)\cong\underset{l\in\mathfrak{U}\mathcal{C}}{Colim}\left(H_q\left(K_l,F\left(l\right)\right)\right)$ by Proposition 3.4.10 and as $H_0\left(C_n,\mathbb{K}\right)\cong\mathbb{K}$. □

3.4.2 Applications

We present now how to apply the general result of Corollary 3.4.11 for various families of groups related to mapping class groups. Beforehand, we fix some notations.

Notation 3.4.12. We denote by $\mathfrak{S}_n$ the symmetric group on n elements.

Let Σ be the skeleton of the groupoid of finite sets and bijections. Note that $Obj\left(\Sigma\right)\cong\mathbb{N}$ and that the automorphism groups are the symmetric groups $\mathfrak{S}_n$. The disjoint union of finite sets $\sqcup$ induces a monoidal structure $(\Sigma,\sqcup,0)$, the unit 0 being the empty set. This groupoid is symmetric monoidal, the symmetry being given by the canonical bijection $n_1\sqcup n_2\overset{\sim}{\to}n_2\sqcup n_1$ for all natural numbers n_1 and n_2. The category $\mathfrak{U}\Sigma$ is equivalent to the category of finite sets and injections FI studied in [CEF15].

Notation 3.4.13. We denote by $\mathfrak{S}_-:(\mathbb{N},\leq)\to\mathfrak{Gr}$ the family of groups defined by $\mathfrak{S}_-(n)=\mathfrak{S}_n$ and $\mathfrak{S}_-(\gamma_n)=id_1\sqcup-$ for all natural numbers n.

3.4.2.1 Braid groups

We denote by $\mathbf{PB}_n$ the pure braid group on n strands. Recall from Section 3.2.2.1 that the braid groupoid $\boldsymbol{\beta}$ (which has natural numbers as objects and braid groups as automorphism groups) is endowed with a braided strict monoidal structure $(\boldsymbol{\beta}, \natural, 0)$.

The classical surjections $\left\{\mathbf{B}_n \overset{\mathfrak{p}_n}{\twoheadrightarrow} \mathfrak{S}_n\right\}_{n\in\mathbb{N}}$, sending each Artin generator $\sigma_i \in \mathbf{B}_n$ to the transposition $\tau_i \in \mathfrak{S}_n$ for all $i \in \{1, \dots, n-1\}$ and for all natural numbers n, assemble to define a functor $\mathfrak{P} : \mathfrak{U}\boldsymbol{\beta} \to FI$. In addition, it is clear that the functor $\mathfrak{P}$ is strict monoidal with respect to the monoidal structures $(\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$ and $(FI, \sqcup, 0)$. In addition, they define the following short exact sequence for all natural numbers n (see for example [Bir74a] or [KT08]):

$$1 \longrightarrow \mathbf{PB}_n \longrightarrow \mathbf{B}_n \xrightarrow{\mathfrak{p}_n} \mathfrak{S}_n \longrightarrow 1\,.$$

Let $\mathbf{PB}_- : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ and $\mathbf{B}_- : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ be the families of groups defined by $\mathbf{PB}_-(n) = \mathbf{PB}_n$, $\mathbf{B}_-(n) = \mathbf{B}_n$ and $\mathbf{B}_-(\gamma_n) = \mathbf{PB}_-(\gamma_n) = id_1 \natural -$ for all natural numbers n. Hence Assumption 3.4.2 is satisfied and therefore by Corollary 3.4.11:

Proposition 3.4.14. *Let F be an object of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$. For all natural numbers q, $H_q(\mathbf{B}_\infty, F_\infty) \cong \underset{n\in FI}{Colim}\left(H_q(\mathbf{PB}_n, F(n))\right)$, and if F factors through the category FI, then:*

$$H_q(\mathbf{B}_\infty, F_\infty) \cong \underset{n\in FI}{Colim}\left(H_q(\mathbf{PB}_n, \mathbb{K}) \underset{\mathbb{K}}{\otimes} F(n)\right).$$

Remark 3.4.15. The rational cohomology ring of the pure braid group on $n \in \mathbb{N}$ strands is computed by Arnol'd in [Arn69]. Namely, $H^*(\mathbf{PB}_n, \mathbb{Q})$ is the graded exterior algebra generated by the degree one classes $\omega_{i,j}$ for $i, j \in \{1, \dots, n\}$ and $i < j$, subject to the relations $\omega_{i,j}\omega_{j,k} + \omega_{j,k}\omega_{k,i} + \omega_{k,i}\omega_{i,j} = 0$. Note that using the universal coefficient theorem for cohomology (see for example [Wei94, Theorem 3.6.5]) and since $H^q(\mathbf{PB}_n, \mathbb{K})$ is a finite-dimensional vector space, the homology group $H_q(\mathbf{PB}_n, \mathbb{K}) \cong H^q(\mathbf{PB}_n, \mathbb{K})$.

We recall that by the universal coefficient theorem for cohomology and since $H^q(\mathbf{PB}_n, \mathbb{K})$ is a finite-dimensional vector space, $H^q(\mathbf{PB}_n, \mathbb{K}) \cong Hom_{\mathbb{K}\text{-}\mathfrak{Mod}}\left(H_q(\mathbf{PB}_n, \mathbb{K}), \mathbb{K}\right)$ and therefore $\underset{n\in FI}{Colim}\left(H_q(\mathbf{PB}_n, \mathbb{K})\right) \cong \underset{n\in FI}{Lim}\left(H_q(\mathbf{PB}_n, \mathbb{K})\right)$. Since $H_q(\mathbf{PB}_n, \mathbb{K})$ is a quotient of the exterior algebra $\Lambda^q\left[\omega_{i,j}\right]$, we deduce that for $q > n \geq 2$, $H_q(\mathbf{PB}_n, \mathbb{K}) = 0$. Hence, $\underset{n\in FI}{Colim}\left(H_q(\mathbf{PB}_n, \mathbb{K})\right) = 0$ for $q \geq 2$. Furthermore, by direct computations, we have that $H_0(\mathbf{PB}_n, \mathbb{K}) = \mathbb{K}$ and $\underset{n\in FI}{Colim}\left(H_1(\mathbf{PB}_n, \mathbb{K})\right) = \mathbb{K}$. Hence, we recover the classical result of the homology of braid groups with constant coefficients in a field of characteristic zero (see for example [Ver98, Section 4]).

3.4.2.2 Symmetric automorphisms groups of free groups

We focus on symmetric automorphisms groups of free groups, also known as string motion groups. We refer the reader to [Dam17] for a complete and unified presentation of the various definitions of this group. We recall here an algebraic definition of these groups.

Definition 3.4.16. Let n be a natural number. The symmetric automorphism group of free group of rank n, denoted by $\mathscr{S}_n$, is the group defined by a presentation given by generators $\left\{\sigma_i, \tau_i, \rho_j \mid i \in \{1, \dots, n-1\} \text{ and } j \in \{1, \dots, n\}\right\}$

together with relations:

$$\begin{cases}\sigma_i\sigma_k = \sigma_k\sigma_i & \text{if } \mid i-k \mid\geq 2,\\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{if } i\in\{1,\ldots,n-2\},\\ \tau_i\tau_k = \tau_k\tau_i & \text{if } \mid i-k \mid\geq 2,\\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} & \text{if } i\in\{1,\ldots,n-2\},\\ \tau_i^2 = 1 & \text{if } i\in\{1,\ldots,n-1\},\\ \sigma_i\tau_k = \tau_k\sigma_i & \text{if } \mid i-k \mid\geq 2,\\ \tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1} & \text{if } i\in\{1,\ldots,n-2\},\\ \sigma_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\sigma_{i+1} & \text{if } i\in\{1,\ldots,n-2\},\\ \rho_j\rho_k = \rho_k\rho_j & \text{if } \mid j-k \mid\geq 1,\\ \rho_j^2 = 1 & \text{if } j\in\{1,\ldots,n-1\},\\ \rho_j\sigma_i = \sigma_i\rho_j & \text{if } \mid i-j \mid\geq 2,\\ \rho_j\sigma_i = \sigma_i\rho_{j+1} & \text{if } i\in\{1,\ldots,n-1\},\\ \rho_j\tau_i = \tau_i\rho_j & \text{if } \mid i-j \mid\geq 2,\\ \rho_i\tau_i = \tau_i\rho_{i+1} & \text{if } i\in\{1,\ldots,n-1\},\\ \tau_i\sigma_i^{-1}\tau_i\rho_i = \rho_{i+1}\sigma_i & \text{if } i\in\{1,\ldots,n-1\}.\end{cases}$$

For all natural numbers $i, j \in \{1,\ldots,n\}$ such that $i \neq j$, we denote by $\alpha_{i,j}$ the composition:

$$\tau_i^{-1}\circ\tau_{i+1}^{-1}\circ\cdots\circ\tau_{j-1}^{-1}\circ\sigma_j\circ\tau_{j-1}\circ\cdots\circ\tau_{i+1}\circ\tau_i.$$

The pure string motion group of rank n, denoted by $\mathscr{PS}_n$ is the subgroup of $\mathscr{S}_n$ generated by the elements $\left\{\alpha_{i,j} \mid i,j\in\{1,\ldots,n\}, i\neq j\right\}$ with the relations:

$$\begin{cases}\alpha_{i,j}\alpha_{k,l} = \alpha_{k,l}\alpha_{i,j},\\ \alpha_{i,k}\alpha_{j,k} = \alpha_{j,k}\alpha_{i,k},\\ \alpha_{i,j}\left(\alpha_{i,k}\alpha_{j,k}\right) = \left(\alpha_{i,k}\alpha_{j,k}\right)\alpha_{i,j}.\end{cases}$$

Remark 3.4.17. Symmetric automorphisms groups of free group are a generalisation of braid groups. Let $C_n = c_1\sqcup c_2\sqcup\cdots\sqcup c_n$ be the disjoint union of n smoothly embedded, oriented, unlinked, unknotted circles c_i in the ball $\mathbb{D}^3$. Then, $\mathscr{S}_n$ is isomorphic to the group of isotopy classes of self-homeomorphisms of $\mathbb{D}^3$ that preserve its orientation, fixes its boundary pointwise, and globally fixes C_n (see [Dam17, Section 2]).

Denoting by $\mathbf{F}_n$ the free group of rank n, $\mathscr{S}_n$ identifies with the subgroup of the automorphism group $Aut(\mathbf{F}_n)$ of the automorphisms which map each generator of $\mathbf{F}_n$ to a conjugate of this generator or of the inverse of a generator (see [Dam17, Section 4]). This justifies the denomination of symmetric automorphisms groups of free group for $\mathscr{S}_n$.

Furthermore, to introduce a suitable categorical framework for symmetric automorphisms groups of free groups, we need another equivalent definition. Denoting by $\mathbb{D}^2$ the 2-disc, $\mathbb{D}^3$ the 3-disc and $\mathbb{S}^1$ the 1-sphere, we consider the compact, connected, oriented 3-manifold with boundary $\left(\mathbb{S}^1\times\mathbb{D}^2\right)\setminus\mathbb{D}^3$, equipped with a marked disc $\mathbb{D}^2\hookrightarrow\partial\left(\mathbb{S}^1\times\mathbb{D}^2\right)\setminus\mathbb{D}^3$ in its boundary. We denote by $\natural$ the boundary connected sum along marked half-discs between two compact, connected, oriented 3-manifolds with boundary.

Definition 3.4.18. Let $\mathfrak{SA}$ be the groupoid defined by:

- Objects: the finite boundary connected sums of $\left(\mathbb{S}^1\times\mathbb{D}^2\right)\setminus\mathbb{D}^3$; namely, $\left(\left(\mathbb{S}^1\times\mathbb{D}^2\right)\setminus\mathbb{D}^3\right)^{\natural n}$ for $n\in\mathbb{N}$;
- Morphisms: the isotopy classes of orientation preserving diffeomorphisms restricting to the identity on the marked disc modulo Dehn twists along embedded 2-spheres, denoted by

$$\pi_0\left(Diff\left(\left(\left(\mathbb{S}^1\times\mathbb{D}^2\right)\setminus\mathbb{D}^3\right)^{\natural n}\text{ rel }\mathbb{D}^2\right)\right)/_{twists}.$$

Remark 3.4.19. By [HW05, Theorem 1.1], the mapping class group $\pi_0\left(Diff\left(\left(\left(\mathbb{S}^1\times\mathbb{D}^2\right)\setminus\mathbb{D}^3\right)^{\natural n}\ \mathrm{rel}\ \mathbb{D}^2\right)\right)/_{twists}$ is isomorphic to $\mathscr{S}_n$ for all natural numbers n.

Recall from [RWW17, Section 5.7] that the boundary connected sum along marked half-discs $\natural$ defines a monoidal product on $\mathfrak{SA}$, and the 3-disc $\mathbb{D}^3$ is the unit. The braiding of the monoidal structure is given by doing half a Dehn twist in a neighbourhood of the marked half-disc and it is a symmetry. We refer to [RWW17, Section 5.7] for more technical details on this operation.

Notation 3.4.20. We denote by $\mathbf{W}_n$ the hyperoctahedral group, namely the wreath product $(\mathbb{Z}/2\mathbb{Z})\wr\mathfrak{S}_n$, where $\mathfrak{S}_n$ acts on $(\mathbb{Z}/2\mathbb{Z})^n$ permuting the copies of $\mathbb{Z}/2\mathbb{Z}$. Let $W\Sigma$ be the skeleton of the groupoid with finite sets as objects and hyperoctahedral groups automorphism group. As for the groupoid Σ, the disjoint union of finite sets $\sqcup$ induces a symmetric monoidal structure $(\Sigma,\sqcup,0)$, the unit 0 being the empty set. We denote by $\mathscr{S}_{-}:(\mathbb{N},\leq)\to\mathfrak{Gr}$ the family of groups defined by $\mathscr{S}_{-}(n)=\mathscr{S}_n$ and $\mathscr{S}_{-}(\gamma_n)=id_1\sqcup -$ for all natural numbers n.

Moreover, we have the following result:

Lemma 3.4.21. *[BL93] For all natural numbers, we have the following short exact sequence:*

$$1\longrightarrow\mathscr{PS}_n\longrightarrow\mathscr{S}_n\xrightarrow{\mathfrak{ps}_n}\mathbf{W}_n\longrightarrow 1\,.$$

It is clear that the surjections $\{\mathfrak{ps}_n\}_{n\in\mathbb{N}}$ define a strict monoidal functor $\mathfrak{PS}:\mathfrak{SA}\to W\Sigma$. Let $\mathscr{PS}_{-}:(\mathbb{N},\leq)\to\mathfrak{Gr}$ and $\mathscr{S}_{-}:(\mathbb{N},\leq)\to\mathfrak{Gr}$ be the families of groups defined by $\mathscr{PS}_{-}(n)=\mathscr{PS}_n$, $\mathscr{S}_{-}(n)=\mathscr{S}_n$ and $\mathscr{S}_{-}(\gamma_n)=\mathscr{PS}_{-}(\gamma_n)=id_1\natural -$ for all natural numbers n. Hence Assumption 3.4.2 is satisfied and therefore by Corollary 3.4.11:

Proposition 3.4.22. *Let F be an object of $\mathbf{Fct}(\mathfrak{USA},\mathbb{K}\text{-}\mathfrak{Mod})$. For all natural numbers q,*

$$H_q(\mathscr{S}_\infty,F_\infty)\cong\underset{n\in\mathfrak{U}(\mathfrak{SA})}{Colim}\left(H_q\left(\mathscr{PS}_n,F(n)\right)\right),$$

and if F factors through the category $\mathfrak{U}(W\Sigma)$, then:

$$H_q(\mathscr{S}_\infty,F_\infty)\cong\underset{n\in\mathfrak{U}(W\Sigma)}{Colim}\left(H_q\left(\mathscr{PS}_n,\mathbb{K}\right)\underset{\mathbb{K}}{\otimes}F(n)\right).$$

Remark 3.4.23. By [JMM06, Theorem 6.7], the cohomology ring $H^*(\mathscr{PS}_n,\mathbb{Z})$ is the exterior algebra generated by the degree-one classes $\alpha^*_{i,j}$ for $i,j\in\{1,\ldots,n\}$ and $i\neq j$, subject to the relations $\alpha^*_{i,j}\wedge\alpha^*_{j,i}=0$ and $\alpha^*_{k,j}\wedge\alpha^*_{j,i}=\alpha^*_{k,j}\wedge\alpha^*_{k,i}-\alpha^*_{i,j}\wedge\alpha^*_{k,i}$. A fortiori, we compute for all natural numbers q:

$$H_q(\mathscr{PS}_n,\mathbb{K})\cong H^q(\mathscr{PS}_n,\mathbb{Z})\underset{\mathbb{Z}}{\otimes}\mathbb{K}.$$

Using a combinatorial argument, Wilson proves in [Wil12, Sections 6 and 7] that the trivial W_n–representation does not occur in $H^q(\mathscr{PS}_n,\mathbb{K})$ for $q\geq 1$ and n large enough, and a fortiori,

$$H_q(\mathscr{S}_\infty,\mathbb{K})\cong\underset{n\in\mathfrak{U}(W\Sigma)}{Colim}\left(H_q\left(\mathscr{PS}_n,\mathbb{K}\right)\right)=0.$$

3.4.2.3 Mapping class group of orientable surfaces

We take the notations of Section 3.2.2.2.

Definition 3.4.24. Let $\mathfrak{M}_2$ be the skeleton of the groupoid defined by:

- Objects: the smooth compact connected orientable surfaces $\Sigma^n_{n,1}$ for all natural numbers n;
- Morphisms: $Aut_{\mathfrak{M}_2}\left(\Sigma^n_{n,1}\right)=\mathbf{\Gamma}^n_{n,1}$ for all natural numbers n.

By [RWW17, Proposition 5.18], the boundary connected sum $\natural$ induces a strict braided monoidal structure $\left(\mathfrak{M}_2, \natural, \left(\Sigma_{0,1}^{0}, I\right)\right)$. Moreover, the injections $\left\{\Gamma_{n,1}^{[n]} \overset{\mathfrak{i}_n}{\hookrightarrow} \Gamma_{n,1}^{n}\right\}_{n\in\mathbb{N}}$ induced by the inclusions

$$Homeo^{\partial}_{\text{fix the marked points}}\left(\Sigma_{n,1}^{n}\right) \hookrightarrow Homeo^{\partial}_{\text{permute the marked points}}\left(\Sigma_{n,1}^{n}\right)$$

provide the following short exact sequence for all natural numbers n (see for example [Bir69]):

$$1 \longrightarrow \Gamma_{n,1}^{[n]} \xrightarrow{\mathfrak{i}_n} \Gamma_{n,1}^{n} \xrightarrow{\mathfrak{pm}_n} \mathfrak{S}_n \longrightarrow 1\,.$$

It is clear that the surjections $\{\mathfrak{pm}_n\}_{n\in\mathbb{N}}$ define a strict monoidal functor $\mathfrak{M}_2 \to \Sigma$. Let $\Gamma_{-,1}^{[-]} : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ and $\Gamma_{-,1}^{-} : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ be the families of groups defined by $\Gamma_{-,1}^{[-]}(n) = \Gamma_{n,1}^{[n]}$, $\Gamma_{-,1}^{-}(n) = \Gamma_{n,1}^{n}$ and $\Gamma_{-,1}^{[-]}(\gamma_n) = \Gamma_{-,1}^{-}(\gamma_n) = id_1 \natural -$ for all natural numbers n. Hence Assumption 3.4.2 is satisfied.

Remark 3.4.25. Let n be a natural number. The action of the symmetric group $\mathfrak{S}_n$ on $\Gamma_{n,1}^{[n]}$ is induced by the natural action of $\mathfrak{S}_n$ on $\Sigma_{n,1}^{n}$ given by permuting the marked points. Recall from Theorems 3.2.31 and 3.2.33 that for all natural numbers q such that $n \geq 2q$:

$$H_q\left(\Gamma_{n,1}^{[n]}, \mathbb{K}\right) \cong H_q\left(\Gamma_{n,1}, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} \mathbb{K}\left[x_1, \ldots, x_n\right].$$

Hence, according to the decomposition of the classifying space associated with the pure mapping class groups in [BT01, Theorem 1], the action of $\mathfrak{S}_n$ on $H_q\left(\Gamma_{n,1}^{[n]}, \mathbb{K}\right)$ corresponds to permuting the variables $\{x_i\}_{i\in\{1,\ldots,n\}}$ on the right hand side.

A fortiori the homology group $H_*\left(\Gamma_{n,1}, \mathbb{K}\right)$ is a trivial $\mathfrak{S}_n$-module. Recall also from [MW07] that:

$$H_*\left(\Gamma_{\infty,1}, \mathbb{K}\right) \cong \mathbb{K}\left[\kappa_1, \kappa_2, \ldots\right]$$

where each κ_i has degree $2i$.

By Corollary 3.4.11, we have:

Proposition 3.4.26. *Let F be an object of* **Fct** $(\mathfrak{UM}_2, \mathbb{K}\text{-}\mathfrak{Mod})$. *For all natural numbers q,*

$$H_q\left(\Gamma_{\infty,1}^{\infty}, F_\infty\right) \cong \underset{n\in\mathfrak{UM}_2}{Colim}\left(H_q\left(\Gamma_{n,1}^{[n]}, F(n)\right)\right),$$

and if F factors through the category FI, then:

$$H_q\left(\Gamma_{\infty,1}^{\infty}, F_\infty\right) \cong \underset{n\in FI}{Colim}\left(\bigoplus_{k+l=q}\left(H_k\left(\Gamma_{n,1}, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} H_l\left(\left(\mathbb{C}P^\infty\right)^{\times n}, \mathbb{K}\right)\right) \underset{\mathbb{K}}{\otimes} F(n)\right).$$

In particular, if F factors through the category FI, then $H_{2k+1}\left(\Gamma_{\infty,1}^{\infty}, F_\infty\right) = 0$ for all natural numbers k.

3.4.2.4 Particular right-angled Artin groups

A right-angled Artin group (abbreviated RAAG) is a group with a finite set of generators $\{s_i\}_{1\leq i\leq k}$ with $k \in \mathbb{N}$ and relations $s_i s_j = s_j s_i$ for some $i, j \in \{1, \ldots, n\}$. For instance, the free group on k generators $\mathbf{F}_k$ is a RAAG. By [GW16, Proposition 3.1], any RAAG admits a maximal decomposition as a direct product of RAAGs, unique up to isomorphism and permutation of the factors. A RAAG is said to be unfactorizable if its maximal decomposition is itself. We refer to [Vog15] or [GW16, Section 3] for more details on these groups.

Let A be a fixed unfactorizable RAAG different from $\mathbb{Z}$. We have the following key property:

Proposition 3.4.27. *[GW16, Proposition 3.3] For all natural numbers n, we have the following split short exact sequence:*

$$1 \longrightarrow Aut(A)^{\times n} \xrightarrow{i_n} Aut\left(A^{\times n}\right) \xrightarrow{s_n} \mathfrak{S}_n \longrightarrow 1.$$

In other words, for all natural numbers n, denoting by $e_n : \mathfrak{S}_n \to Aut\left(Aut(A)^{\times n}\right)$ the permutation action of the symmetric group $\mathfrak{S}_n$ of factors $Aut(A)$ in $Aut(A)^{\times n}$, Proposition 3.4.27 is equivalent to the fact that:

$$Aut\left(A^{\times n}\right) \cong Aut(A)^{\times n} \underset{e_n}{\rtimes} \mathfrak{S}_n.$$

Definition 3.4.28. Let $\mathcal{R}_A$ be the groupoid with the groups $A^{\times n}$ for all natural numbers n as objects and $Aut\left(A^{\times n}\right)$ as automorphism groups.

Note that the groupoid $\mathcal{G} = \mathcal{R}_A$ is symmetric monoidal (see [GW16, Section 5]). The direct product $\times$ induces a strict symmetric monoidal structure $(\mathcal{R}_A, \times, 0_{\mathfrak{Gr}})$ (we refer the reader to [GW16, Section 1] if more details is needed). It is clear that the surjections $\{s_n\}_{n\in\mathbb{N}}$ define a strict monoidal functor $S : \mathcal{R}_A \to \Sigma$. Let $Aut\left(A^{\times -}\right) : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ and $Aut(A)^{\times -} : (\mathbb{N}, \leq) \to \mathfrak{Gr}$ be the families of groups defined by $Aut\left(A^{\times -}\right)(n) = Aut\left(A^{\times n}\right)$, $Aut(A)^{\times -}(n) = Aut(A)^{\times n}$ and $Aut\left(A^{\times -}\right)(\gamma_n) = Aut(A)^{\times -}(\gamma_n) = id_1 \times -$ for all natural numbers n. Hence Assumption 3.4.2 is satisfied and therefore by Corollary 3.4.11:

Proposition 3.4.29. *Let F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{R}_A, \mathbb{K}\text{-}\mathfrak{Mod})$ and A be a fixed unfactorizable right-angled Artin group different from $\mathbb{Z}$. For all natural numbers q, $H_q\left(Aut\left(A^{\times\infty}\right), F_\infty\right) \cong \underset{n\in\mathfrak{U}\mathcal{R}_A}{Colim}\left(H_q\left(Aut(A)^{\times n}, F(n)\right)\right)$, and if F factors through the category FI, then:*

$$H_q\left(Aut\left(A^{\times\infty}\right), F_\infty\right) \cong \underset{n\in FI}{Colim}\left(H_q\left(Aut(A)^{\times n}, \mathbb{K}\right) \underset{\mathbb{K}}{\otimes} F(n)\right). \tag{3.4.4}$$

Corollary 3.4.30. *Let A be a fixed unfactorizable right-angled Artin group different from $\mathbb{Z}$, such that there exists $N_A \in \mathbb{N}$ such that $H_q(Aut(A), \mathbb{K}) = 0$ for $1 \leq q \leq N_A$. Then for all objects F of $\mathbf{Fct}(\mathfrak{U}\mathcal{R}_A, \mathbb{K}\text{-}\mathfrak{Mod})$ factoring through the category FI, for all natural numbers q such that $1 \leq q \leq N_A$:*

$$H_q\left(Aut\left(A^{\times\infty}\right), F_\infty\right) = 0.$$

Proof. It follows from Künneth Theorem (see for example [Wei94, Exercise 6.1.7]) that for all natural numbers q such that $1 \leq q \leq N_A$, $H_q\left(Aut(A)^{\times n}, \mathbb{K}\right) = 0$. Then, the result follows from (3.4.4). □

Example 3.4.31. Recall that $\mathbf{F}_k$ denotes the free group on k generators for all natural numbers k. According to [Gal11, Corollary 1.2], for $k \geq 2q+1$ and $q \neq 0$, $H_q(Aut(\mathbf{F}_k), \mathbb{K}) = 0$. Let F be an object of $\mathbf{Fct}\left(\mathfrak{U}\mathcal{R}_{\mathbf{F}_k}, \mathbb{K}\text{-}\mathfrak{Mod}\right)$ factoring through the category FI. Hence, for all natural numbers q and k such that $1 \leq q \leq \frac{k-1}{2}$:

$$H_q\left(Aut\left((\mathbf{F}_k)^{\times\infty}\right), F_\infty\right) = 0.$$

In particular, $H_q\left(Aut\left((\mathbf{F}_\infty)^{\times\infty}\right), F_\infty\right) = 0$ for all objects F of $\mathbf{Fct}(\mathfrak{gr}, \mathbb{K}\text{-}\mathfrak{Mod})$ factoring through the category FI.

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Parmi les représentations linéaires des groupes de tresses, les représentations de Burau peuvent être construites à partir d'une représentation triviale via une construction introduite par Long en 1994, à l'issue d'une collaboration avec Moody. Cette construction, dite de Long-Moody, permet ainsi de construire des représentations de plus en plus complexes des groupes de tresses. Dans cette thèse, on adopte un point de vue fonctoriel sur cette construction, ce qui permet d'en dégager plus aisément des variantes. De plus, le degré de polynomialité d'un foncteur permet d'en mesurer la complexité. On montre ainsi que la construction Long-Moody définit un foncteur $\mathbf{LM}$, qui augmente le degré de très forte polynomialité. Par ailleurs, on définit des foncteurs analogues pour d'autres familles de groupes telles que les groupes de difféotopie des surfaces et des 3-variétés, les groupes symétriques ou les groupes d'automorphismes des groupes libres. Ils vérifient des propriétés similaires sur la polynomialité. Les foncteurs de Long-Moody fournissent ainsi des coefficients tordus entrant dans le cadre des résultats de stabilité homologique de Randal-Williams et Wahl pour les familles de groupes susmentionnées. On donne enfin un résultat de comparaison entre l'homologie stable à coefficient dans un foncteur F et celle à coefficient dans le foncteur $\mathbf{LM}(F)$ obtenu en appliquant un foncteur de Long-Moody.
Cette thèse se décompose en trois chapitres. Le premier introduit les foncteurs de Long-Moody pour les groupes de tresses et traite de leur effet sur la polynomialité. Le deuxième traite de la généralisation des foncteurs de Long-Moody pour d'autres familles de groupes. Le dernier chapitre concerne des calculs d'homologie stable pour les groupes de difféotopie.

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Constructions de Long-Moody et fonceurs polynomiaux

Parmi les représentations linéaires des groupes de tresses, les représentations de Burau peuvent être construites à partir d'une représentation triviale via une construction introduite par Long en 1994, à l'issue d'une collaboration avec Moody. Cette construction, dite de Long-Moody, permet ainsi de construire des représentations de plus en plus complexes des groupes de tresses. Dans cette thèse, on adopte un point de vue fonctoriel sur cette construction, ce qui permet d'en dégager plus aisément des variantes. De plus, le degré de polynomialité d'un foncteur permet d'en mesurer la complexité. On montre ainsi que la construction Long-Moody définit un foncteur **LM**, qui augmente le degré de très forte polynomialité. Par ailleurs, on définit des foncteurs analogues pour d'autres familles de groupes telles que les groupes de difféotopie des surfaces et des 3-variétés, les groupes symétriques ou les groupes d'automorphismes des groupes libres. Ils vérifient des propriétés similaires sur la polynomialité. Les foncteurs de Long-Moody fournissent ainsi des coefficients tordus entrant dans le cadre des résultats de stabilité homologique de Randal-Williams et Wahl pour les familles de groupes susmentionnées. On donne enfin un résultat de comparaison entre l'homologie stable à coefficient dans un foncteur **F** et celle à coefficient dans le foncteur **LM(F)** obtenu en appliquant un foncteur de Long-Moody.

Cette thèse se décompose en trois chapitres. Le premier introduit les foncteurs de Long-Moody pour les groupes de tresses et traite de leur effet sur la polynomialité. Le deuxième traite de la généralisation des foncteurs de Long-Moody pour d'autres familles de groupes. Le dernier chapitre concerne des calculs d'homologie stable pour les groupes de difféotopie.

Mots-Clés : groupe de difféotopie, foncteurs polynomiaux, construction de Long-Moody, homologie stable.

Among the linear representations of braid groups, Burau representations are recovered from a trivial representation using a construction introduced by Long in 1994, following a collaboration with Moody. This construction, called the Long-Moody construction, thus allows to construct more and more complex representations of braid groups. In this thesis, we have a functorial point of view on this construction, which allows find more easily some variants. Moreover, the degree of polynomiality of a functor measures its complexity. We thus show that the Long-Moody construction defines a functor **LM**, which increases the degree of polynomiality. Furthermore, we define analogous functors for other families of groups such as mapping class groups of surfaces and 3-manifolds, symmetric groups or automorphism groups of free groups. They satisfy similar properties on the polynomiality. Hence, Long-Moody functors provide twisted coefficients fitting into the framework of the homological stability results of Randal-Williams and Wahl for the aforementioned families of groups. Finally, we give a comparison result for the stable homology with coefficient given by a functor **F** and the one with coefficient given by the functor **LM(F)**, obtained applying a Long-Moody functor.

This thesis has three chapters. The first one introduces Long-Moody functors for braid groups and deals with their effect on the polynomiality. The first one deals with the generalisation of Long-Moody functors for other families of groups. The last chapter touches on stable homology computations for mapping class group.

Keywords : mapping class groups, polynomial functors, Long-Moody construction, stable homology.