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La contrôlabilité frontière exacte et la synchronisation frontière exacte pour un système couplé d'équations des ondes avec des contrôles frontières de Neumann et des contrôles frontières couplés de Robin

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Exact boundary controllability and exact boundary synchronization for a coupled system of wave equations with Neumann and coupled Robin boundary controls

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Contents

Exact boundary controllability and exact boundary synchronization for a coupled system of wave equations with Neumann and coupled Robin boundary controls

ABSTRACT

For a coupled system of wave equations, the exact boundary controllability and the exact boundary synchronization can be realized by different types of boundary controls, such as Dirichlet type, Neumann type and coupled Robin type and so on. Up to now, there are already fairly complete results for a coupled system of wave equations with Dirichlet boundary controls. Based on these results, in this thesis, we will further consider the exact boundary synchronization for a coupled system of wave equations with Neumann boundary controls and coupled Robin boundary controls.

This thesis is organised as follows:

In Chapter 1, we give a brief introduction to the history of the research of synchronization as a natural phenomenon, the significance of studying the synchronization, its research background and the present situation, including the relationship between synchronization and controllability.

In Chapter 2, for a coupled system of wave equations with Neumann boundary controls, on the basis of the results on the exact boundary controllability, we consider its exact boundary synchronization and its exact boundary synchronization by groups. Moreover, the determination of the state of synchronization is discussed in details for the exact boundary synchronization and the exact boundary synchronization by 2 and 3 groups, respectively.

In Chapter 3, we consider the exact boundary controllability and the exact boundary synchronization (by groups) for a coupled system of wave equations with coupled Robin boundary controls. Due to difficulties from the lack of regularity of the solution, we have to face a bigger challenge than that in the case with Dirichlet or Neumann boundary controls. In order to overcome this difficulty, we take full advantage of the regularity results for the mixed problem with Neumann boundary conditions (Lasiecka and Triggiani) to discuss the regularity of the mixed problem with coupled Robin boundary conditions. Then we prove the exact boundary controllability for the system, and by the method of compact perturbation, we obtain the non-exact controllability for the system with lack of boundary controls. Based on this, we further study the exact boundary synchronization (by groups) for the same system, the determination of the state of synchronization by groups, and also the necessity of the corresponding compatibility conditions of the coupling matrices.

Keywords: Exact boundary controllability, exact boundary synchronization, exact boundary synchronization by groups, coupled system of wave equations, Neumann boundary control, coupled Robin boundary control.

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La contrôlabilité frontière exacte et la synchronisation frontière exacte pour un système couplé d'équations des ondes avec des contrôles frontières de Neumann et des contrôles frontières couplés de Robin

RESUME

Pour un système couplé d'equations des ondes, la contrôlabilité frontière exacte et la synchronisation frontière exacte peuvent être réalisées par des contrôles frontières de Dirichlet, de Neumann ou de Robin, etc. A ce jour, il existe beaucoup de résultats pour le système couplé d'équations des ondes avec des contrôles frontières de Dirichlet. Sur la base de ces résultats, dans cette thèse, nous considérons la synchronisation frontière exacte pour le système couplé d'équations des ondes avec des contrôles frontières de Neumann et des contrôles frontières couplés de Robin.

Cette thèse est organisée comme suit:

Dans le premier chapitre, nous donnerons une courte introduction sur la synchronisation et l'état de l'art en recherche. Elle comprend la définition du sujet, la situation actuelle de la recherche, et la relation entre la synchronisation et la contrôlabilité.

Dans le deuxième chapitre, en se basant sur les résultats de la contrôlabilité frontière exacte, pour un système couplé d'équations des ondes avec des contrôles frontières de Neumann, nous considérons la synchronisation frontière exacte et la synchronisation frontière exacte par groupes. De plus, nous allons détailler la détermination de l'état de synchronisation exacte et synchronisation exacte par 2 et 3 groupes respectivement.

Dans le chapitre 3, nous considérons la contrôlabilité exacte et la synchronisation exacte (par groupes) pour le système couplé d'équations des ondes avec des contrôles frontières couplés de Robin. A cause du manque de régularité de la solution, nous rencontrons beaucoup plus de difficultés que dans les cas de Dirichlet ou de Neumann. Afin de surmonter ces difficultés, on transforme le problème de Robin en un problème de Neumann. Puis, grâce aux résultats de régularité optimale de Lasiecka-Triggiani sur le problème de Neumann, on obtient un résultat sur la trace de la solution faible du problème de Robin. Ceci nous a permis d'établir la contrôlabilité exacte du système s'il y a suffisamment de contrôles, et, par la méthode de la perturbation compacte, la non-contrôlabilité frontière exacte du système quand il n'y a pas assez de contrôle frontière. En se basant sur ces résultats, on étudie la synchronisation frontière exacte (par groupes) du même système, la détermination de l'état de synchronisation par groupes, ainsi que la nécessité des conditions de compatibilité des matrices de couplage.

Mots-clés: Contrôlabilité frontière exacte, synchronisation frontière exacte, synchronisation frontière exacte par groupes, système couplé d'équations des ondes, contrôle frontière de Neumann, contrôle frontière couplé de Robin.

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Résumé en français

0.1 Introduction

0.1.1 La contrôlabilité et la synchronisation

On considère un système dynamique d'une équation différentielle ordinaire:

$$\dot{x}(t) = f(x(t), h(t)), \tag{0.1.1}$$

où x est la variable d'état, et h désigne la fonction de contrôle. Le problème de contrôle consiste à trouver une fonction de contrôle h convenable telle que le comportement dynamique du système satisfait une condition finale sous l'action du contrôle h. La contrôlabilité est une propriété importante, elle joue un rôle crucial dans beaucoup de problèmes mathématiques.

Pour les systèmes gouvernés par les équations aux dérivées partielles, le problème de contrôle est un sujet très important. Pour un système hyperbolique, la **contrôlabilité exacte** signifie que pour toutes les données initiales et finales, il existe un $T > 0$ et un contrôle convenable sur l'intervalle $[0, T]$ (contrôle frontière ou contrôle interne), tels que le système satisfait exactement la condition initiale et la condition finale. Quand le contrôle est appliqué sur la frontière, on a la **contrôlabilité frontière exacte**. Quand le contrôle est appliqué sur le domaine, on a la **contrôlabilité exacte interne**. Pour le système hyperbolique quasi-linéaire, la contrôlabilité exacte ne peut être réalisée que pour les données initiales et données finales assez petites, on a la **contrôlabilité exacte locale**, sinon la **contrôlabilité exacte globale**. Dans cette thèse nous considérons principalement la contrôlabilité frontière exacte.

Pour le système hyperbolique étudié dans cette thèse, l'étude du problème de contrôle remonte au travail de David L. Russell dans les années 1960. Il a synthétisé les résultats sur la contrôlabilité et la stabilité pour l'équation aux dérivées partielles linéaire dans [42]. Il a également mentionné beaucoup de problèmes ouverts et intéressants. Dans les années 1980, J.-L. Lions a proposé la Hilbert Uniqueness Method (HUM). Cette méthode fournit un cadre pour étudier la contrôlabilité d'équation hyperbolique linéaire, en particulier l'équation des ondes linéaire ([36], [37]). Cette méthode permet d'établir la contrôlabilité du système grâce à une inégalité d'observabilité du système adjoint. En combinant la HUM et le théorème du point fixe de Schauder, on peut aussi étudier l'equation des ondes semi-linéaire ([45]-[46])

$$u_{tt} - \Delta u = f(u). \tag{0.1.2}$$

Pour le système hyperbolique quasi-linéaire, actuellement il y a peu de résultats sur la contrôlabilité (Cirinà [2]–[3]). Depuis 2002, Tatsien Li et Bopeng Rao ont proposé une méthode de construction directe avec une structure modulaire ([21]–[22]). Ils ont établi une théorie complète sur la contrôlabilité frontière exacte pour le système hyperbolique quasi-linéaire de premier ordre

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u) \tag{0.1.3}$$

avec des conditions frontières non-linéaires basé sur la théorie de la C^1 solution semi-globale ([15] et [17]).

Pour un système hyperbolique, sous la condition des multiplicateurs du domaine et avec suffisamment de contrôles frontières, la contrôlabilité frontière exacte peut être réalisée si le temps est suffisamment grand. Toutefois, quand l'une des conditions mentionnées ci-dessus n'est pas satisfaite, nous ne pouvons pas obtenir la contrôlabilité frontière exacte en général. Par exemple, dans des applications spécifiques, certaines conditions frontières ont des significations physiques clairement définies, sur lesquelles les contrôles frontières ne peuvent pas être effectuées. Cela signifie qu'il y aura un manque de contrôles frontières, de sorte que le système n'a pas de contrôlabilité frontière exacte. Dans ce cas, nous devons examiner si le système possède une sorte de contrôlabilité frontière dans un sens plus faible. Tatsien Li et Bopeng Rao ont apporté une contribution pionnière dans les deux aspects suivants. D'une part, par moins de contrôles frontières, ils ont prouvé que le système peut atteindre la **synchronisation frontière exacte** ([24]–[25]), dans lequel toutes les composantes de la variable d'état atteignent un même **état de synchronisation**, mais l'état de synchronisation est a priori inconnu. D'autre part, ils ont défini la **contrôlabilité frontière approchée** ([23]) pour le système hyperbolique: il existe une suite de contrôles frontières, telle que la suite des solutions correspondantes s'approche de zéro quand $n \to +\infty$. Mais la suite des contrôles frontières ne converge pas nécessairement. De même, la **synchronisation frontière approchée** a également été prise en compte. La recherche a conduit à une combinaison de synchronisation et de contrôlabilité, puis l'étude de la synchronisation a développé la théorie du contrôle pour les systèmes gouvernés par des équations aux dérivées partielles.

0.1.2 La synchronisation

La synchronisation est un phénomène bien répandu dans la nature. Elle a été observé pour la première fois par Huygens en 1665 ([7]), qui a trouvé que deux pendules sur le mur se synchronisent en phase. Ensuite, les scientifiques ont progressivement commencé l'observation et l'étude sur ces phénomènes intéressants, en particulier, la synchronisation de tuyaux d'orgue, le clignotement synchrone des lucioles dans les forêts tropicales, les cris synchrones des grenouilles après une pluie d'été, la synchronisation dans le système neuronal, et la synchronisation des pacemakers cardiaques, etc ([43]).

Dans les années 1950, N. Wiener a commencé une recherche théorique sur la synchronisation d'un point de vue mathématique ([44]). Les modèles biologiques ont été proposés par A.T. Winfree en 1967. Ce modèle ne considère que la phase plutôt que l'amplitude des oscillateurs, et ce travail a largement favorisé le

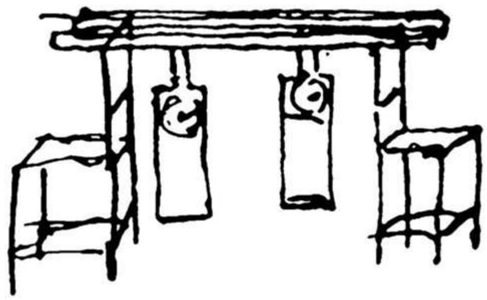

Figure 1: Manuscrit de Huygens, expérience des pendules.

développement de la recherche mathématique sur la synchronisation. En 1990, L. M.Pecora et T.L. Carroll ont constaté que la synchronisation se produit également dans les systèmes chaotiques. Toutefois, la forme du couplage entre les composants n'est pas toujours tout-à-tout, c'est-à-dire que les deux éléments ont un impact sur l'autre. Pour cette raison, les mathématiciens ont essayé d'utiliser la théorie des graphes pour résoudre les problèmes concernés, et en particulier, le graphe aléatoire a été introduit dans l'étude de la synchronisation. D. J. Watts et S. Strogatz ont proposé le réseau du petit monde en 1998, R. Albert et A.L. Barabasi ont presenté le réseau sans échelle en 1999, ce qui indique une importance croissante de la synchronisation dans les réseaux complexes.

Le modèle proposé par Kuramoto en 2003 est le plus réussi pour représenter la synchronisation, car un grand nombre de phénomènes peuvent être essentiellement décrits par ce modèle.

Ce modèle décrit les interactions dépendantes sinusoïdalement des différences de phase entre N oscillateurs couplés:

$$\frac{d\theta_i}{dt} = \omega_i + \sum_j K_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N,$$

où θ_i est la phase du i-ème oscillateur, ω_i est la fréquence propres du i-ème oscillateur, et K_{ij} indique la force de couplage. Des restrictions adéquates sur ces paramètres nous permettent d'utiliser ce modèle pour discuter divers phénomènes. On a une compréhension plus approfondie de la synchronisation à partir de ce modèle:

1. Ce n'est que lorsque le couplage K_{ij} est assez fort et que toutes les fréquences propres ω_i sont presque identiques, qu'une solution de synchronisation est possible. Sinon, lorsque le couplage est faible et que les fréquences propres sont clairement différentes, la synchronisation ne peut pas être réalisée.

2. Inspirée par la sociologie, la synchronisation complète ne peut pas être réalisée immédiatement, l'état de synchronisation se développe au fil du temps, par exemple, certaines parties des variables d'état peuvent être synchronisées avant la synchronisation complète entre toutes les variables d'état.

3. La structure topologique du système détermine les différents modes de développement de la synchroni-

sation. Au contraire, les différents modes de la synchronisation reflètent la structure topologique du système. Il est alors facile de voir l'application de la théorie des graphes.

4. D'ailleurs, en cas de coefficients variables: $\omega_i = \omega_i(t), K_{ij} = K_{ij}(t)$, le système est un réseau adaptatif, la structure se développe au fil du temps et forme une boucle de rétroaction, de sorte que le système peut observer et ajuster continuellement sa structure pour obtenir une synchronisation ou l'éviter.

D'un autre côté, non seulement la procédure de développement est fortement liée à la structure du réseau, mais la structure topologique du réseau détermine aussi la stabilité de l'état de synchronisation dans une large mesure.

Dans les problèmes pratiques, en particulier dans les systèmes chaotiques, les solutions sont sensibles aux données initiales, ce qui nous amène à poser la question suivante: après l'ajout de la perturbation au moment initial, le signal supplémentaire disparaît-il avec le temps?

En 1983, H. Fujisaka et T. Yamada ont proposé une méthode de matrice Lyapunov étendue pour étudier la stabilité de synchronisation pour les systèmes chaotiques. Ils supposent que le système part d'un état de synchronisation au moment initial, et que les états de synchronisation forment un ensemble invariant. Ils ont ajouté une perturbation au système, décomposé la solution en une partie synchronisée et une partie perturbée:

$$\theta_i(t) = s(t)(\text{partie synchronisée}) + \delta\theta_i(t)(\text{partie perturbée}), \qquad i = 1, \cdots, N,$$

puis étudié ensuite l'équation différentielle de la partie perturbée, et proposé les conditions nécessaires pour sa stabilité.

En 1998, L.M. Pecora et T.L.Carroll ont développé la méthode de matrice Lyapunov. Ils ont construit une fonction de stabilité maîtresse pour discuter la relation entre la structure topologique et la stabilité, non seulement le cas stable, mais aussi le cas instable ont été pris en considération. Cette méthode s'appelle la méthode de fonction de stabilité maîtresse. Cependant, cette méthode ne peut que traiter la stabilité linéaire et locale. En 2007, D. A. Wiley, S. H. Strogatz et M. Girvan ont proposé une nouvelle méthode, connue sous le nom de méthode de stabilité du bassin, pour étudier la stabilité globale non-linéaire. La méthode de stabilité du bassin peut obtenir le domaine d'attraction des attracteurs par une méthode de simulation numérique. Avec cette méthode, nous pouvons non seulement calculer la probabilité que la solution retourne à un état de synchronisation après une perturbation stochastique, mais aussi obtenir la dépendance de l'état de synchronisation à la structure topologique.

0.1.3 Synchronisation pour un système couplé d'équations aux dérivées partielles

Comme de plus en plus de résultats sur la synchronisation sont acquis pour les systèmes gouvernés par des équations différentielles ordinaires, Tatsien Li et Bopeng Rao ont pour la première fois étudié la synchronisation de systèmes décrits par des équations aux dérivées partielles en 2012-2013. Prenant un système couplé d'équations des ondes avec des contrôles frontières de Dirichlet comme exemple, ils ont proposé le concept de synchronisation frontière exacte dans le cadre de solutions faibles, à savoir, par des contrôles frontières, le système peut réaliser la synchronisation dans un temps fini ([21], [24], [25], [28], [34], [40]). Après cela, eux et leurs collaborateurs, ils ont successivement obtenu beaucoup de résultats. En 2014, Tatsien Li, Bopeng Rao et Long Hu ont étudié la synchronisation frontière exacte pour un système couplé d'équations des ondes avec divers contrôles frontières (type Dirichlet, Neumann, Robin couplé et dissipatif couplé) en cas unidimensionnel dans le cadre de solutions classiques, et profondément étudié la synchronisation par groupes ([27], [29]) et la détermination de l'état de synchronisation ([6], [34]). En outre, Tatsien Li, Bopeng Rao et Yimin Wei ont étendu la définition de la synchronisation dans un travail en 2014, dans lequel ils ont proposé le concept de synchronisation généralisée.

De plus, lorsque la condition des multiplicateurs échoue, et (ou) qu'il y a un manque de contrôles frontières, nous devrions considérer la synchronisation dans un sens plus faible, c'est-à-dire, lorsque les différences entre les composantes de la variable d'état tendent vers zéro, ce qui est appelé la synchronisation frontière approchée. Dans le travail de Tatsien Li et Bopeng Rao en 2014 ([26], [30]), ils ont également mentionné le critère de Kalman comme une condition nécessaire pour la synchronisation frontière approchée.

En se basant sur la recherche de la synchronisation frontière pour le système avec des contrôles frontières de Dirichlet, c'est une étape nécessaire et difficile pour étudier plus loin la synchronisation pour le système avec d'autres contrôles frontires. Différents types de contrôles frontières correspondent à différents modèles physiques et donnent des systèmes différents en substance. Nous devrions étudier si le changement de la solution résultant du changement des conditions frontières aura un impact sur la synchronisation, et si des résultats similaires peuvent être obtenus comme ceux avec des contrôles frontières de Dirichlet. Ce sera l'étude principal dans cette thèse.

0.1.4 Résultats principaux

Dans cette thèse, on considère la contrôlabilité frontière exacte et la synchronisation frontière exacte pour le système couplé d'équations des ondes

$$U'' - \Delta U + AU = 0 \qquad \text{dans} \quad (0, +\infty) \times \Omega \tag{0.1.4}$$

avec la condition aux bords de Dirichlet sur Γ_0:

$$U = 0 \qquad \text{sur} \quad (0, +\infty) \times \Gamma_0, \tag{0.1.5}$$

la condition aux bords de Neumann ou de Robin couplé sur Γ_1:

$$\partial_\nu U = DH \qquad \text{sur} \quad (0, +\infty) \times \Gamma_1, \tag{0.1.6}$$

$$\partial_\nu U + BU = DH \qquad \text{sur} \quad (0, +\infty) \times \Gamma_1, \tag{0.1.7}$$

et la condition initiale

$$t = 0: \quad U = U_0, \quad U' = U_1 \qquad \text{dans} \quad \Omega, \tag{0.1.8}$$

où $\Omega \subset \mathbb{R}^n$ est un ouvert borné avec la frontière régulière $\Gamma = \Gamma_1 \cup \Gamma_0$ $(\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset)$,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{0.1.9}$$

est le Laplacien n-dimensionnel, ∂_ν désigne la dérivée normale à l'extérieur sur la frontière, la matrice de couplage $A = (a_{ij})$ est une matrice d'ordre N avec les éléments constants, la $N \times M$ $(M \leq N)$ matrice D de contrôle est de rang plein avec les éléments constants, $U = (u^{(1)}, \cdots, u^{(N)})^T$ et $H = (h^{(1)}, \cdots, h^{(M)})^T$ sont la variable d'état et le contrôle frontière respectivement.

On donne la définition de la contrôlabilité frontière exacte et celle de la synchronisation frontière exacte pour le système ci-dessus avec des contrôles frontières de Neumann. La définition est similaire pour le système avec des contrôles frontières couplés de Robin.

On définit les espaces

$$\mathcal{H}_0 = L^2(\Omega), \qquad \mathcal{H}_1 = H^1_{\Gamma_0}(\Omega), \qquad \mathcal{L} = L^2_{loc}(0, +\infty; L^2(\Gamma_1)), \tag{0.1.10}$$

où $H^1_{\Gamma_0}(\Omega)$ est le sous-espace de $H^1(\Omega)$ composé des fonctions de $H^1(\Omega)$ avec la trace nulle sur Γ_0.

Definition 0.1. *S'il existe un $T > 0$, tel que pour toutes les données initiales $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$, on peut trouver une fonction $H \in \mathcal{L}^M$ à support compact dans $[0, T]$, telle que le problème* (1.4.1)–(1.4.4) *et* (1.4.6) *admet une solution $U = U(t, x) \in \big(C^0_{loc}([0, +\infty); \mathcal{H}_{1-s})\big)^N \cap \big(C^1_{loc}([0, +\infty); \mathcal{H}_{-s})\big)^N$ unique, où $s > \frac{1}{2}$, qui satisfait la condition finale*

$$t \geq T \quad U(t, x) \equiv 0, \quad x \in \Omega, \tag{0.1.11}$$

alors le système (1.4.1)–(1.4.4) *possède la* **contrôlabilité frontière exacte** *au moment $T > 0$.*

Si la solution $U = (u^{(1)}, \cdots, u^{(N)})^T$ satisfait

$$t \geq T: \; u^{(1)}(t, x) \equiv \cdots \equiv u^{(N)}(t, x) \overset{def.}{=} u(t, x), \quad x \in \Omega, \tag{0.1.12}$$

alors le système (1.4.1)–(1.4.4) *possède la* **synchronisation frontière exacte** *au moment* $T > 0$*, où* $u = u(t,x)$ *inconnu a priori, est* **l'état de synchronisation**.

Posant $0 = m_0 < m_1 < m_2 < \cdots < m_p = N$*, si la solution* $U = U(t,x)$ *satisfait*

$$t \geq T \quad u^{(k)} \equiv u^{(l)} \stackrel{def.}{=} u_s, \quad m_{s-1}+1 \leq k,l \leq m_s,\ 1 \leq s \leq p, \quad x \in \Omega, \tag{0.1.13}$$

alors le système (1.4.1)–(1.4.4) *possède la* **synchronisation frontière exacte par p-groupes** *au moment* $T > 0$*, où* $(u_1, \cdots, u_p)^T$*, inconnu a priori, est* **l'état de synchronisation par p-groupes**.

Il existe déjà des résultats complets sur la contrôlabilité frontière exacte et la synchronisation frontière exacte pour le système couplé d'équations des ondes avec des contrôles frontières de Dirichlet. Sur cette base, nous étudions la contrôlabilité frontière exacte et la synchronisation frontière exacte pour le système couplé d'équations des ondes avec d'autres types de contrôles frontières.

0.1.5 La synchronisation frontière exacte pour le système couplé d'équations des ondes avec des contrôles frontières de Neumann

Le deuxième chapitre concerne la synchronisation frontière exacte et la synchronisation frontière exacte par groupes pour un système couplé d'équations des ondes avec des contrôles frontières de Neumann. Dans le cas avec des contrôles frontières de Dirichlet, nous savons que le système est exactement contrôlable et exactement synchronisable pour toutes les données initiales dans $L^2(0,L) \times H^{-1}(0,L)$ par un contrôle frontière dans $L^2(0,T)$, cependant, avec des contrôles frontières de Neumann, l'inégalité d'observabilité pour le problème adjoint correspondant n'est valide que sous une norme plus faible, donc il faut une plus forte régularité pour l'espace contrôlable.

De plus, l'espace contrôlable ne peut pas être exactement décrit comme celui sous les contrôles frontières de Dirichlet.

Afin d'obtenir la synchronisation frontière exacte pour le système initial, nous définissons la matrice de synchronisation suivante:

$$C_1 = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \end{pmatrix}_{(N-1)\times N}. \tag{0.1.14}$$

Supposons que la matrice A de couplage satisfait la C_1-condition de compatibilité:

$$A\mathrm{Ker}(C_1) \subseteq \mathrm{Ker}(C_1), \tag{0.1.15}$$

ce qui équivaut au fait qu'il existe une unique matrice $\overline{A}$ d'ordre $(N-1)$, telle que

$$C_1 A = \overline{A} C_1. \tag{0.1.16}$$

Posant $W = C_1 U$, on obtient un système réduit:

$$\begin{cases} W'' - \Delta W + \overline{A} W = 0 & \text{dans} \quad (0, +\infty) \times \Omega, \\ W = 0 & \text{sur} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu W = \overline{D} H & \text{sur} \quad (0, +\infty) \times \Gamma_1 \end{cases} \tag{0.1.17}$$

avec la condition initiale

$$t = 0: \; W = W_0, \quad W' = W_1 \quad \text{dans } \Omega, \tag{0.1.18}$$

où $\overline{D} = C_1 D,\ W_0 = C_1 U_0,\ W_1 = C_1 U_1$. On transforme la synchronisation frontière exacte du système initial en la contrôlabilité frontière exacte du système réduit en W. On essaie de trouver un contrôle frontière adéquat pour obtenir la contrôlabilité frontière exacte du système réduit, et ce contrôle est exactement celui qui réalise la synchronisation frontière exacte du système initial. De cette manière, on peut étudier la synchronisation par la contrôlabilité. Par ailleurs, avec assez de contrôles frontières, la contrôlabilité frontière exacte du système réduit est réalisable, de même que la synchronisation frontière exacte du système initial.

On prouvera que la C_1-condition de compatibilité (1.4.13) est suffisante pour garantir la synchronisation frontière exacte. La nécessité de la C_1-condition de compatibilité est un autre sujet important dans la recherche de la synchronisation.

Théorème 0.1. *Supposons qu'on a la condition des multiplicateurs. Soit $x_0 \in \mathbb{R}^n$, tel que en posant $m = x - x_0$, on a*

$$(m, \nu) \leq 0, \quad \forall x \in \Gamma_0, \qquad (m, \nu) > 0, \quad \forall x \in \Gamma_1, \tag{0.1.19}$$

où $(\cdot, \cdot)$ désigne le produit scalaire dans $\mathbb{R}^n$. Sous la C_1-condition de compatibilité (1.4.13)*, si la matrice D du contrôle satisfait rang$(C_1 D) = N - 1$, alors il existe un $T > 0$, tel que pour toutes les données initiales $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$, on peut trouver un contrôle frontière $H \in \mathcal{L}^M$ à support compact dans $[0, T]$, et*

$$\|H\|_{\mathcal{L}^M} \leq c \|(U_0, U_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^N}, \tag{0.1.20}$$

où c est une constante positive, telle que le problème (1.4.1)–(1.4.4) *et* (1.4.6) *soit exactement synchronisable au moment $T > 0$.*

Réciproquement, si $M = N - 1$, si le système (1.4.1)–(1.4.4) *est exactement synchronisable, alors la matrice A satisfait la C_1-condition de compatibilité* (1.4.13).

En revanche, si $M < N - 1$, le système n'est pas exactement synchronisable.

Ensuite, on étudie la détermination de l'état de synchronisation. Si la matrice A possède certaines propriétés, l'état de synchronisation est indépendant du contrôle frontière. En général, l'état de synchronisation dépend de la donnée initiale (U_0, U_1) et aussi du contrôle H appliqué.

Théorème 0.2. *Sous la C_1-condition de compatibilité (1.4.13), il existe une constante a indépendante de $i = 1, \cdots, N$, telle que*

$$\sum_{j=1}^{N} a_{ij} \overset{\text{def.}}{=} a \quad (i = 1, \cdots, N). \tag{0.1.21}$$

Soit $e = (1, \cdots, 1)^T$ le vecteur propre de A correspondant à la valeur caractéristique a. On suppose que E est orthogonal à e. Alors, il existe une matrice D et une constante $c > 0$ indépendante de la donnée initiale, telles que l'état de synchronisation u satisfait l'estimation suivante:

$$t \geq T: \qquad \|(u, u')(t) - (\psi, \psi')(t)\|_{\mathcal{H}_{2-s} \times \mathcal{H}_{1-s}} \leq c \|C_1(U_0, U_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^{N-1}}, \tag{0.1.22}$$

où ψ est la solution du problème

$$\begin{cases} \psi'' - \Delta\psi + a\psi = 0 & \text{dans} \quad (0, +\infty) \times \Omega, \\ \psi = 0 & \text{sur} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \psi = 0 & \text{sur} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad \psi = (E, U_0), \quad \psi' = (E, U_1) & \text{dans} \quad \Omega, \end{cases} \tag{0.1.23}$$

et $s > \frac{1}{2}$.

Il est clair que la solution du problème, qui détermine l'état de synchronisation, possède une plus forte régularité, malgré que la solution du problème initial soit de régularité faible.

De plus, on peut aussi obtenir des résultats similaires pour la synchronisation frontière exacte par groupes de la même façon. A ce moment, la $(N - p) \times N$ matrice de synchronisation par p-groupes correspondante est:

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}, \tag{0.1.24}$$

où S_s est la $(m_s - m_{s-1} - 1) \times (m_s - m_{s-1})$ matrice de rang plein:

$$S_s = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \end{pmatrix}, \qquad 1 \leq s \leq p. \tag{0.1.25}$$

Posant $W = C_p U$, la synchronisation frontière exacte du système initial est transformée en la contrôlabilité frontière exacte du système réduit en W. Le reste est similaire, cependant, il faut au moins $(N-p)$ contrôles frontières pour réaliser la synchronisation frontière exacte par p-groupes.

0.1.6 La contrôlabilité frontière exacte et la synchronisation frontière exacte pour un système couplé d'équations des ondes avec des contrôles frontières couplés de Robin

Au chapitre 3, nous considérons la contrôlabilité frontière exacte et la synchronisation frontière exacte (par groupes) pour un système couplé d'équations des ondes avec des contrôles frontières couplés de Robin. Le problème est plus difficile à cause du manque de régularité de la solution faible. On transforme le problème aux conditions frontières de Robin en un problème aux conditions frontières de Neumann. Puis, on utilise les résultats de la régularité optimale du problème de Neumann de Lasiecka-Triggiani et la méthode de la perturbation compacte pour établir la contrôlabilité exacte et la non-contrôlabilité exacte selon le nombre de contrôles. D'ailleurs, dans cette situation, nous avons non seulement une matrice de couplage A dans le système, mais aussi une matrice de couplage B sur les conditions frontières couplées de Robin. L'interaction entre ces deux matrices complique le problème ([20]).

Ici et après, selon les cas différents, nous définissons α, β, respectivement, comme suit:

$$\begin{cases} \alpha = 3/5 - \epsilon, \ \beta = 3/5, & \text{si } \Omega \text{ est un ouvert borné général avec la frontière régulière,} \\ \alpha = \beta = 3/4 - \epsilon, & \text{si } \Omega \text{ est un parallélépipède,} \end{cases} \tag{0.1.26}$$

où $\epsilon > 0$ est un réel arbitrairement petit.

Théorème 0.3. *Pour tout* $H \in (L^2(0,T;L^2(\Gamma_1)))^M$ *et tous* $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$*, la solution faible* U *du problème* (1.4.1)–(1.4.2)*et* (1.4.5)–(1.4.6) *satisfait*

$$(U, U') \in C^0([0,T]; (H^\alpha(\Omega) \times H^{\alpha-1}(\Omega))^N) \tag{0.1.27}$$

et

$$U|_{\Gamma_1} \in (H^{2\alpha-1}(\Sigma_1))^N, \tag{0.1.28}$$

où $\Sigma_1 = (0,T) \times \Gamma_1$*. D'ailleurs, l'application*

$$(U_0, U_1, H) \to (U, U')$$

est continue pour les topologies correspondantes.

De même, nous avons la contrôlabilité frontière exacte pour le système couplé avec des contrôles frontières couplés de Robin par suffisamment de contrôles frontières.

Théorème 0.4. *On suppose que* $M = N$. *Alors, il existe un* $T > 0$, *tel que pour toutes les données initiales* $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$, *on peut trouver un contrôle frontière* $H \in (L^2_{loc}(0, +\infty; L^2(\Gamma_1)))^N$ *à support compact dans* $[0, T]$, *tel que le système* (1.4.1)–(1.4.2) *et* (1.4.5) *est exactement contrôlable au moment* T. *Le contrôle* H *dépend continûment de la donnée initiale:*

$$\|H\|_{(L^2(0,T;L^2(\Gamma_1)))^N} \leq c\|(U_0, U_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^N}. \tag{0.1.29}$$

où $c > 0$ *est une constante positive.*

Par la méthode de perturbation compacte, pour un domaine Ω parallélépipédique, on peut prouver la non-contrôlabilité exacte quand il n'y a pas assez de contrôles. Dans ce cas, la trace $U|_{\Gamma_1}$ a une régularité optimale $(H^{\frac{1}{2}}(\Sigma_1))^N$, ce qui est crucial pour la preuve.

Théorème 0.5. *On suppose que rang*$(D) = M < N$ *et que* $\Omega \subset \mathbb{R}^n$ *est un parallélépipède. Alors le système* (1.4.1)–(1.4.2) *et* (1.4.5) *est non exactement contrôlable pour toutes les données initiales* $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$.

En se basant sur ces résultats, par la méthode de perturbation compacte, on obtient

Théorème 0.6. *On suppose que* $\Omega \subset \mathbb{R}^n$ *est un parallélépipède. Si le système couplé d'équations des ondes* (1.4.1)–(1.4.2) *et* (1.4.5) *est exactement synchronisable, alors*

$$rang(C_1 D) = N - 1. \tag{0.1.30}$$

Sous certaines conditions, les matrices de couplage A et B satisfont les C_1-conditions de compatibilité:

Théorème 0.7. *On suppose que* $\Omega \subset \mathbb{R}^n$ *est un parallélépipède. On suppose que rang*$(D) = N - 1$. *Si le système couplé* (1.4.1)–(1.4.2) *et* (1.4.5) *est exactement synchronisable, alors on a les* C_1*-conditions de compatibilité suivantes:*

$$AKer(C_1) \subseteq Ker(C_1), \qquad BKer(C_1) \subseteq Ker(C_1). \tag{0.1.31}$$

De la même manière, on a la synchronisation frontière exacte du système avec des contrôles frontières couplés de Robin.

Théorème 0.8. *On suppose que* A *et* B *satisfont les* C_1*-conditions de compatibilité* (1.4.32). *Alors, on peut trouver une matrice* D *du contrôle vérifiant:*

$$rang(D) = rang(C_1 D) = N - 1, \tag{0.1.32}$$

telle que le système (1.4.1)–(1.4.2) *et* (1.4.5) *est exactement synchronisable, et le contrôle frontière* H *dépend continûment de la donnée initiale:*

$$\|H\|_{(L^2(0,T,L^2(\Gamma_1)))^{N-1}} \leq c\|C_1(U_0, U_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^{N-1}}. \tag{0.1.33}$$

Ensuite, on considère la détermination de l'état de synchronisation. Sous certaines conditions algébriques des matrices A et B, l'état de synchronisation est indépendant du contrôle appliqué. En général, l'état de synchronisation dépend non seulement de la donnée initiale (U_0, U_1), mais aussi du contrôle H appliqué.

Théorème 0.9. *On suppose que A et B satisfont les C_1-conditions de compatibilités* (1.4.32). *On suppose que A^T et B^T possèdent un vecteur propre commun $E \in \mathbb{R}^N$ avec $(E, e) = 1$, où $e = (1, \cdots, 1)^T$. Alors on peut trouver une matrice D du contrôle, $rang(D) = N - 1$, telle que le système* (1.4.1)–(1.4.2) *et* (1.4.5) *est exactement synchronisable, et l'état de synchronisation est indépendant du contrôle H appliqué.*

Réciproquement, on suppose que sous les C_1-conditions de compatibilité (1.4.32), *le système* (1.4.1)–(1.4.2) *et* (1.4.5) *est exactement synchronisable. On suppose qu'il existe un vecteur non-trivial $E \in \mathbb{R}^N$, tel que la projection $\phi = (E, U)$ est indépendante du contrôle, alors E est un vecteur propre commun de A^T et B^T et $E \in Ker(D^T)$, tel que $(E, e) = 1$.*

Théorème 0.10. *On suppose que A et B satisfont les C_1-conditions de compatibilité* (1.4.32). *Soit $E \in \mathbb{R}^N$ un vecteur propre du B^T, tel que $(E, e) = 1$. Alors il existe une matrice D du contrôle, telle que le système* (1.4.1)–(1.4.2) *et* (1.4.5) *est exactement synchronisable, et que l'état de synchronisation satisfait*

$$\|(u, u')(T) - (\phi, \phi')(T)\|_{H^{\alpha+1}(\Omega)\times H^{\alpha}(\Omega)} \leq c\|C_1(U_0, U_1)\|_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-1}}, \tag{0.1.34}$$

où ϕ est la solution du problème:

$$\begin{cases} \phi'' - \Delta\phi + \lambda\phi = 0 & dans \quad (0, +\infty) \times \Omega, \\ \phi = 0 & sur \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu\phi + \mu\phi = 0 & sur \quad (0, +\infty) \times \Gamma_0, \\ t = 0: \quad \phi = (U_0, E), \quad \phi' = (U_1, E) & dans \quad \Omega, \end{cases} \tag{0.1.35}$$

où λ et μ sont définis par $Ae = \lambda e, Be = \mu e$.

En projetant le système sur un sous-espace, on peut obtenir un système réduit, qui détermine l'état de synchronisation. De cette façon, on peut simplifier largement la détermination de l'état de synchronisation.

Dans la thèse, on a beaucoup développé la synchronisation frontière exacte par groupes pour le système couplé avec des contrôles frontières couplés de Robin. Cette partie est plus difficile à cause du manque de régularité de la solution faible. La nécessité des conditions de compatibilité des matrices de couplage a été établie seulement dans des domaines spécifiques, par exemple, parallélépipèdes. Il reste encore beaucoup de problèmes intéressants et prometteurs à considérer.

Chapter 1

Introduction

1.1 From controllability to synchronization

Consider a finite dimensional dynamical system

$$\dot{x}(t) = f(x(t), h(t)), \tag{1.1.1}$$

where $x(t)$ is the state variable of the system at the time t, and $h(t)$ denotes the control function. The corresponding control problem is to choose a proper control function h such that the dynamic behavior of the system satisfies a given requirement under the action of control h. Controllability is an important property of a system, and plays a vital role in many mathematical and practical problems, such as the stability of feed back systems, optimal control problems and so on.

For the **distributed parameter system** described by partial differential equations, the control problem is also an important research subject. For general hyperbolic systems, the **exact controllability** is defined as follows: for any given initial data φ and final data ψ, there exists a $T > 0$ and a suitable control (boundary control or internal control), such that the system can drive any given initial data φ at the time $t = t_0$ to any given final data ψ at the time $t = t_0 + T$ by means of the control. When the control is applied only on the boundary, we have the **exact boundary controllability**. When the control acts on a part of the domain, we have the **exact internal controllability**. In general, there is also the **exact controllability**, where the control acts on both the boundary and an internal part of the domain. Besides, for non-linear systems, such as quasi-linear hyperbolic systems, in the sense of classical solutions, if the exact controllability can be achieved only for sufficiently small initial data and final data, we have the **local exact controllability**, otherwise, the **global exact controllability**. In this thesis we mainly consider the exact boundary controllability.

For the hyperbolic system studied in this thesis, the research on its control problem traces back to the work of David L. Russell in the 1960s, who systematically summarized the results on controllability and stability for linear partial differential equations in [42], and mentioned many problems that are worth for further study. After that, in the 1980s, J.-L. Lions proposed the pioneering Hilbert Uniqueness Method (HUM), which provides a general framework on the study of controllability for linear hyperbolic equations, and in particular for the linear wave equations (see [36], [37]). For linear hyperbolic systems, controllability and observability are dual respects of the same problem, and the HUM obtains the controllability by proving the corresponding observability inequality. By combining HUM and Schauder fixed-point theorem, some results on the exact boundary controllability for semi-linear wave equations

$$u_{tt} - \Delta u = f(u) \tag{1.1.2}$$

have also been obtained ([45]-[46]).

For quasi-linear hyperbolic systems, there were only few work on the study of controllability (Cirinà [2]–[3]). Since 2002, Tatsien Li and Bopeng Rao proposed a direct constructive method with modular structure (see [21]–[22]), and established a complete theory on the exact boundary controllability for the general first order quasi-linear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u) \tag{1.1.3}$$

with general non-linear boundary conditions based on the semi-global C^1 solution theory (see [15] and [17]).

For a coupled hyperbolic system, under the usual multiplier geometric condition on the domain, the exact boundary controllability can be always realized by adequate boundary controls in a suitable large control time. However, when any one of the conditions mentioned above fails, we can not get the exact boundary controllability in general. For example, in practical applications, some boundary conditions have clear and definite physical meanings, on which boundary controls can not be acted. This means that there will be a lack of controls on the boundary, so that the system can not obtain the exact boundary controllability. In this case, we need to consider whether the system possesses some kind of boundary controllability in a weaker sense or not. The answer is positive, and according to different requirements, the corresponding problems and research methods will be different. In this direction, Tatsien Li and Bopeng Rao made a pioneering contribution in the following two aspects. On one hand, by fewer boundary controls, they proved that system may achieve the **exact boundary synchronization** (see [24]–[25]), in which all the components of the state variable reach a same **state of synchronization**, but the state of synchronization is a priori unknown. On the other hand, they defined the **approximate boundary null controllability** (see [23]) for the hyperbolic system: under suitable boundary controls, there exists a sequence of boundary control functions, such that the corresponding sequence of solutions approaches to zero as $n \to +\infty$, but the sequence of boundary control functions does not necessarily converge. Similarly, the **approximate boundary synchronization** was also taken into consideration for the system. Their research led to a combination of synchronization and controllability, then the study of synchronization became a part of the control theory, and at the same time the research of synchronization for systems governed by partial differential equations rose and developed.

1.2 Synchronization phenomena and their study

Synchronization is a widespread natural phenomenon. It was first observed by Huygens in 1665 ([7]), who found that two pendulum clocks hanging on the wall synchronized in phase. Then, scientists gradually started the observation and research on these interesting phenomena, including the synchronization of organ pipes, firefly synchronous flashing in tropical forests, crickets chirping synchronously, synchronization in the neural system, and synchronization of heart peacemaker cells and so on ([43]).

In the 1950s, N. Wiener first began a theoretical research on synchronization from a mathematical point of view ([44]). Biological models were then abstracted into a model of oscillators by A.T. Winfree in 1967. This model considers only the phase rather than the amplitude of the oscillators, and this milestone work largely promoted the development of research on synchronization in mathematics. In 1990, L. M.Pecora and T.L. Carroll from the US found that synchronization also occurs in chaotic systems. However, in nature the coupling form between components is not always all-to-all, that is to say, not every two elements have an impact on each other. For this reason, mathematicians tried to use the graph theory to solve related problems, and in particular, the random graph was introduced into the study of synchronization. D. J. Watts and S. Strogatz proposed the small-world network in 1998, R. Albert and A.L. Barabasi put forward the scale-free network in 1999, which indicates an increasing significance of synchronization in complex networks.

Kuramoto model proposed by Kuramoto in 2003 is the most successful one to depict synchronization, since a large number of phenomena can be essentially described by this model. This model deals with the interactions that depend sinusoidally on the phase difference between N coupled oscillators:

$$\frac{d\theta_i}{dt} = \omega_i + \sum_j K_{ij}\sin(\theta_j - \theta_i), \quad i = 1, \cdots, N,$$

where θ_i is the phase of the i-th oscillator, ω_i is the intrinsic natural frequency of the i-th oscillator, and K_{ij} denotes the coupling strength. Proper restrictions on these parameters allow us to use this model to discuss diverse phenomena. A deeper understanding on synchronization is gained from this model:

1. Only when the coupling coefficients K_{ij} are sufficiently strong and all the natural frequencies ω_i are almost uniform, a fully synchronized solution is possible. Otherwise, when the coupling is weak and the natural frequencies are obviously different, synchronization can not be realized.

2. Inspired by sociology, the full synchronization can not be achieved immediately, the state of synchronization develops as time goes on, for example, some parts of the state variables of the system may be synchronized before the full synchronization between all the state variables.

3. The topologic structure of the system determines different developing modes of synchronization. Conversely, different developing modes of synchronization reflects the topologic structure of the system. It is then easy to see the application of graph theory.

4. Furthermore, in the case of variable coefficients: $\omega_i = \omega_i(t), K_{ij} = K_{ij}(t)$, the system is an adaptive network, the structure of which develops with the time and forms a feed-back loop, so that it is possible for the system to continuously observe and adjust its structure to obtain synchronization or avoid it.

On the other hand, not only the developing procedure is strongly related to the network structure, but also the topologic structure of the network determines to a large degree the stability of the state of synchronization, which is a parallel research subject corresponding to synchronization. In practical problems, and especially in chaotic systems, solutions are very sensitive to the initial data, which prompts us to raise the question: after adding perturbation at the initial time, does the extra signal disappear with time or not?

In 1983, H. Fujisaka and T. Yamada proposed an extended Lyapunov matrix method to study the synchronization stability for chaotic systems. They assume that the system starts from a state of synchronization at the initial time, and the states of synchronization form an invariant set. They added perturbation to the system, decomposed the solution into a synchronized part and a perturbed part:

$$\theta_i(t) = s(t)(\text{synchronized part}) + \delta\theta_i(t)(\text{perturbed part}), \qquad i = 1, \cdots, N,$$

then studied the differential equation satisfied by the perturbed part of the solution, and put forward necessary conditions of stability.

In 1998, L.M. Pecora and T.L.Carroll further developed the Lyapunov matrix method. They constructed a Master Stability Function to discuss the relationship between the topologic structure and stability, in which not only the stable case, but also the unstable case were taken into consideration. This method is called the Master Stability Function Method. However, this method can only solves the problem of linear and local stability. In 2007, D. A. Wiley, S. H. Strogatz and M. Girvan suggested a new method, known as the Basin Stability Method, to study the non-linear global stability of the system. The Basin Stability Method can obtain the attraction domain of attractors by the numerical simulation method. With this method, we can not only calculate the probability of the solution returning back to a state of synchronization after a stochastic disturbance, but also obtain the dependance of the state of synchronization to the topologic structure.

1.3 Synchronization for a coupled system of partial differential equations

As more and more results on the research of synchronization are acquired for systems governed by ordinary differential equations, Tatsien Li and Bopeng Rao for the first time studied the synchronization for systems described by partial differential equations in 2012–2013. Taking a coupled system of wave equations with Dirichlet boundary controls as an example, they proposed the concept of exact boundary synchronization in the framework of weak solutions, namely, by boundary controls, the system can realize the synchronization in a finite time, and switching off the controls after, the state of synchronization remains. After that, they and their collaborators successively got quite a lot of results ([21], [24], [25], [28], [34], [40]). In 2014, Tatsien Li, Bopeng Rao and Long Hu discussed the exact boundary synchronization for a coupled system of wave equations with various boundary controls (Dirichlet type, Neumann type, coupled Robin type and coupled dissipative type) in one-space-dimensional case in the framework of classical solutions, and deeply studied the synchronization by groups ([27], [29]) and the determination of the state of synchronization ([6], [34]). Besides, Tatsien Li, Bopeng Rao and Yimin Wei expanded the definition of synchronization in a work of 2014, put forward the concept of generalize synchronization.

Furthermore, when the added assumptions are further weakened, namely, when the usual multiplier geometric condition on the domain fails, and (or) there is a further lack of boundary control functions, we should consider the synchronization in a much weaker sense, that is to say, we can find a sequence of solutions, the differences between components of the state variable tend to zero, which is called to be the approximate boundary synchronization, see the work of Tatsien Li and Bopeng Rao in 2014 ([26], [30]), where they also mentioned Kalman's criterion as a necessary condition for the approximate boundary synchronization.

Based on the research of boundary synchronization for the system with Dirichlet boundary controls, it is a necessary and challenging stage to further study the synchronization for system with other boundary controls. Different types of boundary controls correspond to different physical models, and give different systems in essence. We should figure out whether the change of the property of solution resulting from the change of boundary conditions will impact the synchronization of the system, and whether can similar results be obtained to that with Dirichlet boundary controls? These will be the key ingredients of the study in this thesis.

1.4 Main results

In this thesis, we consider the exact boundary controllability and the exact boundary synchronization for the following coupled system of wave equations with different types of boundary controls:

$$U'' - \Delta U + AU = 0 \qquad \text{in} \quad (0, +\infty) \times \Omega \tag{1.4.1}$$

with the homogeneous Dirichlet boundary condition on Γ_0:

$$U = 0 \qquad \text{on} \quad (0, +\infty) \times \Gamma_0, \tag{1.4.2}$$

the boundary condition of Dirichlet type, Neumann type and coupled Robin type, respectively, on Γ_1:

$$U = DH \qquad \text{on} \quad (0, +\infty) \times \Gamma_1, \tag{1.4.3}$$

$$\partial_\nu U = DH \qquad \text{on} \quad (0, +\infty) \times \Gamma_1, \tag{1.4.4}$$

$$\partial_\nu U + BU = DH \qquad \text{on} \quad (0, +\infty) \times \Gamma_1, \tag{1.4.5}$$

and the corresponding initial condition

$$t = 0: \quad U = U_0, \quad U' = U_1 \qquad \text{in} \quad \Omega, \tag{1.4.6}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ $(\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset)$,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{1.4.7}$$

is the n-dimensional Laplace operator, ∂_ν denotes the outward normal derivative on the boundary, the coupling matrix $A = (a_{ij})$ is a matrix of order N with constant elements, the boundary control matrix D is a full column-rank matrix of order $N \times M$ $(M \leq N)$ with constant elements, $U = (u^{(1)}, \cdots, u^{(N)})^T$ and $H = (h^{(1)}, \cdots, h^{(M)})^T$ are the state variable and the boundary control function, respectively.

As an example, we give the exact boundary controllability and the exact boundary synchronization for the above system with Neumann boundary controls. Definitions can be given in a similar way for the system with coupled Robin boundary controls.

Denote

$$\mathcal{H}_0 = L^2(\Omega), \qquad \mathcal{H}_1 = H^1_{\Gamma_0}(\Omega), \qquad \mathcal{L} = L^2_{loc}(0, +\infty; L^2(\Gamma_1)), \tag{1.4.8}$$

where $H^1_{\Gamma_0}(\Omega)$ is the subspace of $H^1(\Omega)$ composed of all the functions with the null trace on Γ_0.

Definition 1.1. *If there exists a $T > 0$, such that for any given initial data $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$, we can find a boundary control function $H \in \mathcal{L}^M$ with compact support on $[0, T]$, such that the mixed initial-boundary value problem (1.4.1)–(1.4.2), (1.4.4) and (1.4.6) admits a unique solution $U = U(t, x) \in \left(C^0_{loc}([0, +\infty); \mathcal{H}_{1-s})\right)^N \cap \left(C^1_{loc}([0, +\infty); \mathcal{H}_{-s})\right)^N$ on $t \geq 0$, where $s > \frac{1}{2}$, which satisfies*

$$t \geq T \quad U(t, x) \equiv 0, \quad x \in \Omega, \tag{1.4.9}$$

then system (1.4.1)–(1.4.2) and (1.4.4) is **exactly boundary null controllable** *at the time $T > 0$.*

If the solution $U = (u^{(1)}, \cdots, u^{(N)})^T$ satisfies

$$t \geq T: \ u^{(1)}(t, x) \equiv \cdots \equiv u^{(N)}(t, x) \overset{def.}{=} u(t, x), \quad x \in \Omega, \tag{1.4.10}$$

then system (1.4.1)–(1.4.2) and (1.4.4) is **exactly boundary synchronizable** *at the time $T > 0$, where $u = u(t, x)$, being a priori unknown, is called to be the* **state of synchronization**.

Let $0 = m_0 < m_1 < m_2 < \cdots < m_p = N$. If the solution $U = U(t, x)$ satisfies

$$t \geq T \ \ u^{(k)} \equiv u^{(l)} \overset{def.}{=} u_s, \quad m_{s-1} + 1 \leq k, l \leq m_s, \ 1 \leq s \leq p, \quad x \in \Omega, \tag{1.4.11}$$

then system(1.4.1)–(1.4.2) and (1.4.4) is **exactly boundary synchronizable by p groups** *at the time $T > 0$, and $(u_1, \cdots, u_p)^T$, being a priori unknown, is called to be the* **state of synchronization by p-groups**.

There are already complete results on the exact boundary controllability and the exact boundary synchronization for the coupled system of wave equations with Dirichlet boundary controls. Based on this, we study the exact boundary controllability and the exact boundary synchronization for the coupled system of wave equations with other types of boundary controls.

1.4.1 Exact boundary synchronization for a coupled system of wave equations with Neumann boundary controls

Chapter 2 concerns the exact boundary synchronization and the exact boundary synchronization by groups for a coupled system of wave equations with Nuemann boundary controls. In the case with Dirichlet boundary controls, we know that the system is exactly boundary controllable and exactly boundary synchronizable for all the initial data in $L^2(0,L) \times H^{-1}(0,L)$ by boundary control functions in $L^2(0,T)$, however, under Nuemann boundary controls, the observability inequality for the corresponding adjoint problem to this coupled system is valid only in a weaker norm, therefore, the controllable space of the system is required to have a higher regularity. Moreover, the controllable space can not be precisely described as in the case with Dirichlet boundary controls.

In order to get the exact boundary synchronization for the original system, we define the matrix of synchronization as

$$C_1 = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \end{pmatrix}_{(N-1)\times N}. \tag{1.4.12}$$

Assume that the coupling matrix A satisfies the following C_1-compatibility condition:

$$A\mathrm{Ker}(C_1) \subseteq \mathrm{Ker}(C_1), \tag{1.4.13}$$

which is equivalent to the fact that there exists a unique $(N-1)$ matrix $\overline{A}$, such that

$$C_1 A = \overline{A} C_1. \tag{1.4.14}$$

Let $W = C_1 U$. The original system of U can be reduced to a self-closed system of W:

$$\begin{cases} W'' - \Delta W + \overline{A} W = 0 & \text{in} \quad (0,+\infty) \times \Omega, \\ W = 0 & \text{on} \quad (0,+\infty) \times \Gamma_0, \\ \partial_\nu W = \overline{D} H & \text{on} \quad (0,+\infty) \times \Gamma_1 \end{cases} \tag{1.4.15}$$

with the corresponding initial condition

$$t = 0: \; W = W_0, \quad W' = W_1 \quad \text{in} \quad \Omega, \tag{1.4.16}$$

where $\overline{D} = C_1 D$, $W_0 = C_1 U_0$, $W_1 = C_1 U_1$. Thus, the exact boundary synchronization for the original system (1.4.1)–(1.4.2) and (1.4.4) of U is equivalent to the exact boundary null controllability of the reduced system (1.4.15) of W. Hence, if we can find a proper boundary control function such that the reduced system is exact boundary null controllable, we can correspondingly find the boundary control function that realizes the exact boundary synchronization for the original system. It then turns out that it is possible to study the synchronization problem by means of studying the controllability. We know that the reduced system is exactly boundary controllable with enough boundary controls, hence, the original system should be exactly boundary synchronizable.

The exact boundary synchronization by groups can be discussed in a similar method. The corresponding $(N-p) \times N$ matrix of synchronization by p-groups is given by

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}, \tag{1.4.17}$$

where S_s is the following $(m_s - m_{s-1} - 1) \times (m_s - m_{s-1})$ full row-rank matrix:

$$S_s = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \end{pmatrix}, \qquad 1 \le s \le p. \tag{1.4.18}$$

Correspondingly, we denote $W = C_p U$.

We can prove that the C_1-compatibility condition (1.4.13) is sufficient to gurantee the exact boundary synchronization. The necessity of the C_1-compatibility condition is another important subject in the research of synchronization.

Theorem 1.1. *Assume that Ω satisfies the usual multiplier geometric condition. Without loss of generality, we assume that there exists an $x_0 \in \mathbb{R}^n$, such that for $m = x - x_0$, we have*

$$(m, \nu) \le 0, \quad \forall x \in \Gamma_0, \qquad (m, \nu) > 0, \quad \forall x \in \Gamma_1, \tag{1.4.19}$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^n$. Assume furthermore that $M = N-1$. Under the C_1-compatibility condition (1.4.13)*, if the boundary control matrix D satisfies rank$(C_1 D) = N-1$, then there exists a constant*

$T>0$ so large that for any given initial data $(U_0,U_1)\in(\mathcal{H}_1\times\mathcal{H}_0)^N$, we can find a boundary control function $H\in\mathcal{L}^M$ with compact support in $[0,T]$, satisfying

$$\|H\|_{\mathcal{L}^M}\le c\|(U_0,U_1)\|_{(\mathcal{H}_1\times\mathcal{H}_0)^N},\tag{1.4.20}$$

where c is a positive constant, such that the mixed initial-boundary value problem (1.4.1)–(1.4.2), (1.4.4) and (1.4.6) admits a unique weak solution $U=U(t,x)\in\left(C^0_{loc}([0,+\infty);\mathcal{H}_{1-s})\right)^N\cap\left(C^1_{loc}([0,+\infty);\mathcal{H}_{-s})\right)^N$ on $t\ge 0$, where $s>\frac{1}{2}$, which is exactly boundary synchronizable at the time T.

On the contrary, as $M=N-1$, if system (1.4.1)–(1.4.2) and (1.4.4) is exactly boundary synchronizable, then the coupling matrix A should satisfy the C_1-compatibility condition (1.4.13).

In particular, as $M<N-1$, no matter how large $T>0$ is taken, the system is not exactly boundary synchronizable.

Similar results hold for the synchronization by groups, and in this situation, at least $(N-p)$ boundary controls are obligatory for the synchronization by p-groups.

Theorem 1.2. *Assume that system (1.4.1)–(1.4.2) and (1.4.4) is exactly boundary synchronizable by p-groups, then $M\ge N-p$. In particular, as $M=N-p$, the coupling matrix $A=(a_{ij})$ satisfies the following C_p-compatibility condition:*

$$AKer(C_p)\subseteq Ker(C_p),\tag{1.4.21}$$

where C_p is defined by (1.4.17).

Under the C_p-compatibility condition (1.4.21), if rank$(C_pD)=N-p$, then system (1.4.1)–(1.4.2) and (1.4.4) is exactly boundary synchronizable by p-groups under the action of boundary control function $H\in\mathcal{L}^{N-p}$.

The C_p-compatibility condition (1.4.21) is equivalent to the fact that there exists constants α_{rs} $(1\le r,s\le p)$ such that

$$\sum_{j=m_{s-1}+1}^{m_s}a_{ij}=\alpha_{rs},\quad m_{r-1}+1\le i\le m_r,\quad 1\le r,s\le p,\tag{1.4.22}$$

which indicates that the coupling matrix satisfies the row-sum condition by blocks.

Moreover, the determination of the state of synchronization is discussed with details for the exact boundary synchronization and the exact boundary synchronization by 2 and 3-groups, respectively. Generally speaking, the state of synchronization by p-groups depends on both the initial data (U_0,U_1) and the applied boundary control function H. But when the coupling matrix A possesses certain properties, the state of synchronization by p-groups is independent of the applied boundary control function and determined entirely by the solution to a coupled system of wave equations with homogeneous boundary conditions.

For $1 \leq s \leq p$, let e_s be vectors defined by

$$(e_s)_j = \begin{cases} 1, & m_{s-1}+1 \leq j \leq m_s, \\ 0, & \text{others}. \end{cases} \tag{1.4.23}$$

Theorem 1.3. *Under the C_p-compatibility condition* (1.4.21)*, assume that A^T possesses an invariant subspace Span$\{E_1, E_2, \cdots, E_p\}$ which is bi-orthonormal to Ker(C_p) = Span$\{e_1, \cdots, e_p\}$. Then there exists a boundary control matrix D, such that the state of synchronization by p-groups $u = (u_1, \cdots, u_p)^T$ is independent of the applied boundary controls, and can be determined as follows:*

$$t \geq T: \quad u = \psi, \tag{1.4.24}$$

where $\psi = (\psi_1, \cdots, \psi_p)^T$ is the solution to the following problem with homogeneous boundary conditions:

$$\begin{cases} \psi_r'' - \Delta\psi_r + \sum_{s=1}^{p} \alpha_{rs}\psi_s = 0 & in \quad (0, +\infty) \times \Omega, \\ \psi_r = 0 & on \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \psi_r = 0 & on \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad \psi_r = (E_r, U_0), \quad \psi_r' = (E_r, U_1) & in \quad \Omega, \end{cases} \tag{1.4.25}$$

where α_{rs} $(1 \leq r, s \leq p)$ are given by (1.4.22).

Even if the coupling matrix A does not satisfy the above assumption, the solution to problem (1.4.25) can be still used to estimate the state of synchronization by p-groups.

Theorem 1.4. *Under the C_p-compatibility condition* (1.4.21)*, assume that $\{E_1, E_2, \cdots, E_p\}$ is bi-orthonormal to $\{e_1, \cdots, e_p\}$. Then there exist a boundary control matrix D and a positive constant c independent of the initial data, such that the state of synchronization by p-groups $u = (u_1, \cdots, u_p)^T$ satisfies the following estimate:*

$$t \geq T: \qquad \|(u, u')(t) - (\psi, \psi')(t)\|_{(\mathcal{H}_{2-s} \times \mathcal{H}_{1-s})^p} \leq c\|C(U_0, U_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^{N-p}}, \tag{1.4.26}$$

where $\psi = (\psi_1, \cdots, \psi_p)$ is the solution to problem (1.4.25)*, and $s > \frac{1}{2}$.*

From above, it is clear that although the solution to the problem with Neumann boundary controls possesses a weaker regularity, the solution to the mixed problem which determines the state of synchronization by p-groups possesses a higher regularity than the original problem itself, thus the regularity of the state of synchronization by p-groups is relatively improved, which makes it possible to approach the state of synchronization by p-groups by a solution to a more regular problem.

1.4.2 Exact boundary controllability and exact boundary synchronization for a coupled system of wave equations with coupled Robin boundary controls

In Chapter 3, we consider the exact boundary controllability and the exact boundary synchronization (by groups) for a coupled system of wave equations with coupled Robin boundary controls. Due to difficulties from the lack of regularity of the solution, we have to confront a bigger challenge than that in the case with Dirichlet or Neumann boundary controls. In particular, in higher-space-dimensional case, the regularity of the solution in the general case is not enough to be used to prove the non exact boundary controllability for the system lacking boundary controls. In order to overcome this difficulty, we transform the mixed initial-boundary problem with coupled Robin boundary conditions into a mixed problem with Neumann boundary conditions, then by the results on the regularity of the solution to the mixed problem with Neumann boundary conditions (Lasiecka and Triggiani), as well as the method of compact perturbation, we can discuss the regularity of the solution to the mixed problem with coupled Robin boundary conditions. Moreover, in the present situation, we have not only a coupling matrix A in the coupled system of wave equations, but also a coupling matrix B on the coupled Robin boundary conditions. The interaction between these two coupling matrices makes the problem more complicated ([20]).

Here and hereafter, according to different situations, we define α, β, respectively, as

$$\begin{cases} \alpha = 3/5 - \epsilon,\ \beta = 3/5, & \Omega \text{ is a general bounded smooth domain,} \\ \alpha = \beta = 3/4 - \epsilon, & \Omega \text{ is a parallelepiped,} \end{cases} \tag{1.4.27}$$

where $\epsilon > 0$ is an arbitrarily given small constant.

Theorem 1.5. *For any given $H \in (L^2(0,T;L^2(\Gamma_1)))^M$ and any given $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$, the weak solution U to the problem* (1.4.1)–(1.4.2) *and* (1.4.5)–(1.4.6) *satisfies*

$$(U, U') \in C^0([0,T]; (H^\alpha(\Omega) \times H^{\alpha-1}(\Omega))^N) \tag{1.4.28}$$

and

$$U|_{\Gamma_1} \in (H^{2\alpha-1}(\Sigma_1))^N, \tag{1.4.29}$$

where $\Sigma_1 = (0,T) \times \Gamma_1$, α is defined by (1.4.27). *Furthermore, the mapping*

$$(U_0, U_1, H) \to (U, U')$$

is continuous respect to the corresponding topologies.

Similarly to proving the exact boundary controllability for the corresponding system with Neumann boundary controls, we have the exact boundary controllability for the coupled system with coupled Robin boundary controls with enough boundary controls.

Theorem 1.6. *Assume that $M = N$. Then there exists a $T > 0$, such that for any given initial data $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$, we can find a boundary function $H \in (L^2_{loc}(0, +\infty; L^2(\Gamma_1)))^N$ with compact support on $[0, T]$, such that system* (1.4.1)–(1.4.2) *and* (1.4.5) *is exactly boundary controllable at the time T, and the control function H continuously depends on the initial data:*

$$\|H\|_{(L^2(0,T;L^2(\Gamma_1)))^N} \leq c\|(U_0, U_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^N}, \tag{1.4.30}$$

where $c > 0$ is a positive constant.

To obtain the non exact boundary controllability for the system lacking boundary controls, we apply the method of compact perturbation. Nevertheless, only in the case that the domain Ω is a parallelepiped, the trace $U|_{\Gamma_1}$ can almost have an optimal regularity $(H^{\frac{1}{2}}(\Sigma_1))^N$, and the proof can be accomplished with such a regularity. In general, under fewer boundary controls, the non exact boundary controllability for the coupled system with coupled Robin boundary controls is still an open problem ([20]).

Theorem 1.7. *Assume that $rank(D) = M < N$ and $\Omega \subset \mathbb{R}^n$ is a parallelepiped. Then no matter how large $T > 0$ is, system* (1.4.1)–(1.4.2) *and* (1.4.5) *is not exactly boundary null controllable for any given initial data $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$.*

Based on this, by the method of compact perturbation, we get

Theorem 1.8. *Assume that $\Omega \subset \mathbb{R}^n$ is a parallelepiped. If the coupled system of wave equations with coupled Robin boundary controls* (1.4.1)–(1.4.2) *and* (1.4.5) *is exactly boundary synchronizable, then*

$$rank(C_1 D) = N - 1. \tag{1.4.31}$$

Under certain conditions, the coupling matrices A and B satisfy the corresponding C_1-compatibility conditions similar to (1.4.13). In fact, we have

Theorem 1.9. *Let $\Omega \subset \mathbb{R}^n$ be a parallelepiped. Assume that $rank(D) = N - 1$. If the coupled system* (1.4.1)–(1.4.2) *and* (1.4.5) *is exactly boundary synchronizable, then the following C_1-compatibility conditions hold:*

$$AKer(C_1) \subseteq Ker(C_1), \qquad BKer(C_1) \subseteq Ker(C_1). \tag{1.4.32}$$

Similarly to the case with Neumann boundary controls, the exact boundary synchronization (by groups) for the coupled system with coupled Robin boundary controls can be realized from the exact boundary null controllability of the corresponding reduced system.

Theorem 1.10. *Assume that Ω satisfies the usual multiplier geometric condition. Assume furthermore that both A and B satisfy the C_1-compatibility conditions* (1.4.32). *Then we can find a boundary control matrix*

D satisfying

$$rank(D) = rank(C_1 D) = N - 1, \tag{1.4.33}$$

such that system (1.4.1)–(1.4.2) *and* (1.4.5) *is exactly boundary synchronizable, and the applied boundary control function H continuously depends on the initial data:*

$$\|H\|_{(L^2(0,T,L^2(\Gamma_1)))^{N-1}} \leq c\|C_1(U_0, U_1)\|_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-1}}. \tag{1.4.34}$$

For the exact boundary synchronization by groups, we can get similar results. At least $(N-p)$ boundary controls are obligatory for the synchronization by p-groups (see §3.8). Here, the coupling matrix A in the coupled system of wave equations satisfies the C_p-compatibility condition (1.4.21). However, compared with the internal coupling matrix A, to study the necessity of the C_p-compatibility condition of the coupling matrix B on the boundary is more complicated. This concerns the regularity of the solution to the problem with coupled Robin boundary conditions (see §3.9). We have obtained the C_p-compatibility condition of B only under certain restricted conditions (see Theorem 3.17 and Theorem 3.18).

We also study the determination of the state of synchronization (see §3.7). Similarly, under certain algebraic conditions satisfied by the coupling matrices A and B, the state of synchronization is independent of boundary control function which realizes the synchronization. In general, the state of synchronization depends not only on the initial data, but also on the applied boundary control function, in this case, an estimate on the state of synchronization can be still given.

Theorem 1.11. *Assume that both A and B satisfy the C_1-compatibility condition* (1.4.32)*. Assume furthermore that A^T and B^T possess a common eigenvector $E \in \mathbb{R}^N$ with $(E, e) = 1$, where $e = (1, \cdots, 1)^T$. Then we can find a boundary control matrix D, rank$(D) = N - 1$, such that system* (1.4.1)–(1.4.2) *and* (1.4.5) *is exactly boundary synchronizable, and the state of synchronization is independent of the applied boundary control function.*

On the contrary, assume that under the C_1-compatibility condition (1.4.32)*, system* (1.4.1)–(1.4.2) *and* (1.4.5) *is exactly boundary synchronizable. Assume furthermore that there exists a non-trivial vector $E \in \mathbb{R}^N$, such that the project $\phi = (E, U)$ is independent of the applied boundary control function H in $(0, T) \times \Omega$, then E is a common eigenvector of A^T and B^T, $E \in Ker(D^T)$, and, without loss of generality, we may assume that $(E, e) = 1$.*

Theorem 1.12. *Assume that both A and B satisfy the C_1-compatibility condition* (1.4.32)*. Let $E \in \mathbb{R}^N$ be a eigenvector of B^T, satisfying $(E, e) = 1$. Then there exists a boundary control matrix D such that system* (1.4.1)–(1.4.2) *and* (1.4.5) *is exactly boundary synchronizable, and the state of synchronization u satisfies the following estimate:*

$$\|(u, u')(T) - (\phi, \phi')(T)\|_{H^{\alpha+1}(\Omega)\times H^{\alpha}(\Omega)} \leq c\|C_1(U_0, U_1)\|_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-1}}, \tag{1.4.35}$$

where ϕ is the solution to the following mixed initial-boundary value problem:

$$\begin{cases} \phi'' - \Delta\phi + \lambda\phi = 0 & \textit{in} \quad (0, +\infty) \times \Omega, \\ \phi = 0 & \textit{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \phi + \mu\phi = 0 & \textit{on} \quad (0, +\infty) \times \Gamma_0, \\ t = 0: \quad \phi = (U_0, E), \quad \phi' = (U_1, E) & \textit{in} \quad \Omega, \end{cases} \tag{1.4.36}$$

where λ and μ are defined by $Ae = \lambda e, Be = \mu e$.

Generally speaking, in order to determine the state of synchronization, denote V as the minimum invariant subspace of both A and B, containing e. Correspondingly, let W be an invariant subspace of both A^T and B^T, being bi-orthonormal to V. We can get a subsystem by projecting the system into the subspace W, so that the state of synchronization can be accurately determined (see Theorem 3.13).

Therefore, when the problem possesses a large scale, we can determine the state of synchronization by a problem with smaller dimension, so that we can reduce the computational complexity to a large degree. Nevertheless, the dimension of the small-scale problem is determined by the property of the coupling matrices A and B, namely, the dimension of their common invariant subspace.

For the determination of the state of synchronization by groups, we have similar results (see §3.10).

Chapter 2

Exact boundary synchronization for a coupled system of wave equations with Neumann boundary controls

2.1 Introduction

In this paper, we consider the following coupled system of wave equations with Neumann boundary controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ U = 0 & \text{on} \quad (0,+\infty)\times\Gamma_0, \\ \partial_\nu U = DH & \text{on} \quad (0,+\infty)\times\Gamma_1 \end{cases} \tag{2.1.1}$$

and the corresponding initial data

$$t=0: \quad U=U_0, \quad U'=U_1, \tag{2.1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset$ and $\text{mes}(\Gamma_0) > 0$, ∂_ν denotes the outward normal derivative on the boundary, the coupling matrix $A = (a_{ij})$ is of order N, the boundary control matrix D is a full column-rank matrix of order $N \times M$ $(M \leq N)$, both A and D have real constant elements, $U = (u^{(1)}, \cdots, u^{(N)})^T$ and $H = (h^{(1)}, \cdots, h^{(M)})^T$ denote the state variables and the boundary controls, respectively.

Denote

$$\mathcal{H}_0 = L^2(\Omega), \qquad \mathcal{H}_1 = H^1_{\Gamma_0}(\Omega), \qquad \mathcal{L} = L^2(0,+\infty; L^2(\Gamma_1)), \tag{2.1.3}$$

where $H^1_{\Gamma_0}(\Omega)$ is the subspace of $H^1(\Omega)$, composed of all the functions with the null trace on Γ_0, and $T > 0$ is a given constant.

We assume that Ω satisfies the usual multiplier geometric condition ([36]). Without loss of generality, we assume that there exists an $x_0 \in \mathbb{R}^n$, such that setting $m = x - x_0$, we have

$$(m,\nu) \leq 0, \quad \forall x \in \Gamma_0; \qquad (m,\nu) > 0, \quad \forall x \in \Gamma_1, \tag{2.1.4}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^n$.

Define the linear unbounded operator $-\Delta$ in $\mathcal{H}_0$ by

$$D(-\Delta) = \{\Phi \in H^2(\Omega): \quad \Phi|_{\Gamma_0} = 0, \;\; \partial_\nu \Phi|_{\Gamma_1} = 0\}.$$

Clearly, $-\Delta$ is a positively definite self-adjoint operator with a compact resolvent. Then, for any given $s \in \mathbb{R}$, we can define the operator $(-\Delta)^{\frac{s}{2}}$ with the domain

$$\mathcal{H}_s = D((-\Delta)^{\frac{s}{2}}),$$

which, endowed with the norm $\|\Phi\|_s = \|(-\Delta)^{\frac{s}{2}}\Phi\|_{L^2(\Omega)}$ constitutes a Hilbert space, and its dual space is $\mathcal{H}_s' = \mathcal{H}_{-s}$. In particular, we have (see [38])

$$\mathcal{H}_1 = D(\sqrt{-\Delta}) = \{\Phi \in H^1(\Omega): \quad \Phi|_{\Gamma_0} = 0\}.$$

Lemma 2.1 (See [31]). *For any given initial data* $(U_0, U_1) \in (\mathcal{H}_{1-s} \times \mathcal{H}_{-s})^N$ *with* $s > \frac{1}{2}$, *and any given boundary function* $H \in \mathcal{L}^M$, *the mixed initial-boundary value problem* (2.1.1)–(2.1.2) *admits a unique weak solution* $U \in (C_{loc}([0, +\infty); \mathcal{H}_{1-s}))^N \cap (C^1_{loc}([0, +\infty); \mathcal{H}_{-s}))^N$ *with continuous dependance.*

Definition 2.1. *System* (2.1.1) *is exactly null controllable at the time* $T > 0$ *in the space* $(\mathcal{H}_1 \times \mathcal{H}_0)^N$, *if for any given initial data* $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$, *there exists a boundary control* $H \in \mathcal{L}^M$ *with compact support in* $[0, T]$, *such that the corresponding mixed initial-boundary value problem* (2.1.1)–(2.1.2) *admits a unique weak solution* $U \in (C_{loc}([0, +\infty); \mathcal{H}_{1-s}))^N \cap (C^1_{loc}([0, +\infty); \mathcal{H}_{-s}))^N$ *with* $s > 1/2$, *satisfying*

$$t \geq T : \quad U = U' \equiv 0. \tag{2.1.5}$$

Moreover, we have the continuous dependence:

$$\|H\|_{\mathcal{L}^M} \leq C\|(U_0, U_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^N}, \tag{2.1.6}$$

where C *is a positive constant.*

For the exact null controllability and the non-exact null controllability of system (2.1.1), the following results have been proved in [31].

Lemma 2.2. *When* $M = N$, *there exists a constant* $T > 0$, *such that system* (2.1.1) *is exactly null controllable at the time* T *for any given initial data* $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$.

Remark 2.1. *In fact, the controllable space is not a usual function space. However, it contains the subspace* $(\mathcal{H}_1 \times \mathcal{H}_0)^N$. *The choice of* $(\mathcal{H}_1 \times \mathcal{H}_0)^N$ *as the controllable space is convenient for the application.*

However, if there is a lack of boundary controls, we have

Lemma 2.3. *When* $M < N$, *no matter how large* $T > 0$ *is, system* (2.1.1) *is not exactly null controllable at the time* T *for any given initial data* $(U_0, U_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$.

Therefore, it is necessary to discuss whether system (2.1.1) is controllable in some weaker senses when there is a lack of boundary controls, namely, when $M < N$. Although the results are similar to those for the coupled system of wave equations with Dirichlet boundary controls, since the solution to a coupled system of wave equations with Neumann boundary condition has a relatively weaker regularity, in order to realize the desired result, we need stronger function spaces, and the corresponding adjoint problem is also different.

Since Lemma 2.1 in [29] is independent of the type of boundary conditions, we still have

Lemma 2.4. *Assume that* U *is the solution to the mixed problem* (2.1.1)–(2.1.2). *Let* C *be a full row-rank* $(N - p) \times N$ *matrix (where* $p \geq 1$*) such that*

$$t \geq T : \qquad CU = 0 \quad \text{in} \quad \Omega. \tag{2.1.7}$$

Then we have either

$$AKer(C) \subseteq Ker(C) \tag{2.1.8}$$

or there exists a full row-rank $(N-p+1)\times N$ *matrix* $\hat{C}$ *such that*

$$t \geq T: \qquad \hat{C}U = 0 \quad in \quad \Omega. \tag{2.1.9}$$

2.2 Exact boundary synchronization

Definition 2.2. *System* (2.1.1) *is exactly synchronizable at the time* $T>0$ *in the space* $(\mathcal{H}_1\times\mathcal{H}_0)^N$*, if for any given initial data* $(U_0,U_1)\in(\mathcal{H}_1\times\mathcal{H}_0)^N$*, there exists a boundary control* $H\in\mathcal{L}^M$ *with compact support in* $[0,T]$*, such that the weak solution* $U=U(t,x)$ *to the mixed initial-boundary value problem* (2.1.1)–(2.1.2) *satisfies*

$$t \geq T: \quad u^{(1)}(t,x) \equiv \cdots \equiv u^{(N)}(t,x) \overset{def.}{=} u(t,x), \tag{2.2.1}$$

where, $u=u(t,x)$*, being unknown a priori, is called the corresponding state of exact synchronization.*

The above definition requires that system (2.1.1) maintains the state of synchronization even after canceling the boundary control since the time T.

Theorem 2.1. *Assume that* $M<N$*. If system* (2.1.1) *is exactly synchronizable in the space* $(\mathcal{H}_1\times\mathcal{H}_0)^N$*, then the coupling matrix* $A=(a_{ij})$ *should satisfy the following condition of compatibility (the sum of elements in every row is equal to each other):*

$$\sum_{j=1}^{N} a_{ij} \overset{\text{def.}}{=} a \quad (i=1,\cdots,N), \tag{2.2.2}$$

where a *is a constant independent of* $i=1,\cdots,N$*.*

Proof By Lemma 2.3, since $M<N$, system (2.1.1) is not exactly null controllable, then there exists an initial data $(U_0,U_1)\in(\mathcal{H}_1\times\mathcal{H}_0)^N$, such that for any given boundary control H, the corresponding state of synchronization $u(t,x)\not\equiv 0$. Then, noting (2.2.1), the solution to problem (2.1.1) corresponding to this initial data satisfies

$$u'' - \Delta u + \Big(\sum_{j=1}^{N} a_{ij}\Big)u = 0, \quad i=1,\cdots,N \quad \text{in} \quad \mathfrak{D}'((T,+\infty)\times\Omega). \tag{2.2.3}$$

Then, for $i,k=1,\cdots,N$, we have

$$\Big(\sum_{j=1}^{N} a_{kj} - \sum_{j=1}^{N} a_{ij}\Big)u = 0 \quad \text{in} \quad \mathfrak{D}'((T,+\infty)\times\Omega), \tag{2.2.4}$$

therefore

$$\sum_{j=1}^{N} a_{kj} = \sum_{j=1}^{N} a_{ij}, \quad i,k = 1,\cdots,N, \tag{2.2.5}$$

which is just the required condition of compatibility (2.2.2).

Now, let

$$C_1 = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}_{(N-1)\times N} \tag{2.2.6}$$

be the corresponding matrix of synchronization. C_1 is a full row-rank matrix, and $\mathrm{Ker}(C_1) = \mathrm{Span}\{e_1\}$, where $e_1 = (1,1,\cdots,1)^T$.

Clearly, the synchronization (2.2.1) can be equivalently written as

$$t \geq T: \quad C_1 U(t,x) \equiv 0 \quad \text{in} \quad \Omega. \tag{2.2.7}$$

By Lemma 4.7 in Appendix, we have

Lemma 2.5. *The following properties are equivalent:*

(1) *The condition of compatibility* (2.2.2) *holds;*

(2) $e_1 = (1,1,\cdots,1)^T$ *is a right eigenvector of A, corresponding to the eigenvalue a given by* (2.2.2)*;*

(3) *$Ker(C_1)$ is an one-dimensional invariant subspace of A:*

$$AKer(C_1) \subseteq Ker(C_1); \tag{2.2.8}$$

(4) *There exists a unique matrix $\overline{A}_1$ of order $(N-1)$, such that*

$$C_1 A = \overline{A}_1 C_1. \tag{2.2.9}$$

$\overline{A}_1 = (\bar{a}_{ij})$ is called the reduced matrix of A by C_1, where

$$\bar{a}_{ij} = \sum_{p=j+1}^{N} (a_{i+1,p} - a_{ip}) = \sum_{p=1}^{j} (a_{ip} - a_{i+1,p}), \quad i,j = 1,\cdots,N-1. \tag{2.2.10}$$

Theorem 2.2. *Assume that $M = N-1$. Under the condition of compatibility* (2.2.2)*, if the matrix $C_1 D$ is invertible, namely, $rank(C_1 D) = N-1$, then there exists a constant $T > 0$ so large that system* (2.1.1) *is exactly synchronizable at the time T in the space $(\mathcal{H}_1 \times \mathcal{H}_0)^N$, moreover, we have the continuous dependence:*

$$\|H\|_{\mathcal{L}^{N-1}} \leq C \|C_1(U_0,U_1)\|_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-1}}, \tag{2.2.11}$$

where C is a positive constant.

On the other hand, when rank$(C_1D) < N-1$ (especially, when $M < N-1$), no matter how large $T > 0$ is, system (2.1.1) *is not exactly synchronizable at the time T.*

Proof Under the condition of compatibility (2.2.2), let

$$W = C_1U, \qquad W_0 = C_1U_0, \qquad W_1 = C_1U_1.$$

Noting (2.2.9), it is easy to see that the original mixed problem (2.1.1)–(2.1.2) for U can be reduced to the following self-closed mixed problem for W:

$$\begin{cases} W'' - \Delta W + \overline{A}_1 W = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ W = 0 & \text{on} \quad (0,+\infty)\times\Gamma_0, \\ \partial_\nu W = \overline{D}H & \text{on} \quad (0,+\infty)\times\Gamma_1, \\ t=0: \ W = W_0, \quad W' = W_1 & \text{in} \qquad \Omega, \end{cases} \tag{2.2.12}$$

where $\overline{D} = C_1D$. Noting that C_1 is a surjection from $(\mathcal{H}_1\times\mathcal{H}_0)^N$ onto $(\mathcal{H}_1\times\mathcal{H}_0)^{N-1}$, we easily check that the exact boundary synchronization of system (2.1.1) for U is equivalent to the exact boundary null controllability of system (2.2.12) for W. Since rank$(\overline{D})$ = rank$(C_1D) = N-1$, by Lemma 2.2, for any given initial data $(W_0, W_1)\in(\mathcal{H}_1\times\mathcal{H}_0)^{N-1}$, system (2.2.12) is exactly null controllable by means of a boundary control $\overline{D}H \in \mathcal{L}^{N-1}$. By (2.1.6) in the Definition 2.1, we get the continuous dependence of (2.2.11). Since $\overline{D}$ is invertible matrix, there exists a corresponding boundary control $H\in\mathcal{L}^{N-1}$, such that system (2.1.1) is exactly synchronizable.

On the other hand, when rank$(C_1D) < N-1$, by Lemma 2.3, the reduced system (2.2.12) is not exactly null controllable, then system (2.1.1) is not exactly synchronizable.

2.3 Exact boundary synchronization by p-groups

When there is a further lack of boundary controls, we consider the exact boundary synchronization by p-groups ($p\ge 1$; when $p=1$, it becomes the exact boundary synchronization). This indicates that the components of U are divided into p groups:

$$(u^{(1)},\cdots,u^{(m_1)}), \quad (u^{(m_1+1)},\cdots,u^{(m_2)}),\cdots,(u^{(m_{p-1}+1)},\cdots,u^{(m_p)}), \tag{2.3.1}$$

where $0 = m_0 < m_1 < m_2 < \cdots < m_p = N$, and each group is required to possess the exact boundary synchronization, respectively, and, in the meantime, every group is independent of each other.

Definition 2.3. *System* (2.1.1) *is exactly synchronizable by p-groups at the time $T>0$ in the space $(\mathcal{H}_1\times\mathcal{H}_0)^N$, if for any given initial data $(U_0,U_1)\in(\mathcal{H}_1\times\mathcal{H}_0)^N$, there exists a boundary control $H\in\mathcal{L}^M$ with* *compact support in* $[0, T]$*, such that the weak solution* $U = U(t, x)$ *to the mixed initial-boundary value problem* (2.1.1)–(2.1.2) *satisfies*

$$t \geq T: \quad u^{(k)} \equiv u^{(l)} \stackrel{def.}{=} u_s, \quad m_{s-1} + 1 \leq k, l \leq m_s, \ 1 \leq s \leq p, \tag{2.3.2}$$

where, $(u_1, \cdots, u_p)^T$*, being unknown a prior, is called the corresponding state of synchronization by p-groups.*

Let S_s be a $(m_s - m_{s-1} - 1) \times (m_s - m_{s-1})$ full row-rank matrix:

$$S_s = \begin{pmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & & \ddots & & \\ & & & & 1 & -1 \end{pmatrix}, \qquad 1 \leq s \leq p, \tag{2.3.3}$$

and let C_p be the following $(N - p) \times N$ matrix of synchronization by p-groups:

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}. \tag{2.3.4}$$

Obviously, we have

$$\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \cdots, e_p\}, \tag{2.3.5}$$

where for $1 \leq s \leq p$,

$$(e_s)_j = \begin{cases} 1, & m_{s-1} + 1 \leq j \leq m_s, \\ 0, & \text{otherwise}. \end{cases} \tag{2.3.6}$$

Thus, (2.3.2) can be equivalently written as

$$t \geq T: \quad C_p U \equiv 0 \quad or \quad U = \sum_{s=1}^{p} u_s e_s \quad \text{in} \quad \Omega. \tag{2.3.7}$$

Theorem 2.3. *Assume that system* (2.1.1) *is exactly synchronizable by p-groups. Then we necessarily have* $M \geq N - p$*. Especially, when* $M = N - p$*, the coupling matrix* $A = (a_{ij})$ *should satisfy the following condition of compatibility:*

$$A Ker(C_p) \subseteq Ker(C_p). \tag{2.3.8}$$

Proof By (2.3.7), we have $C_pU = 0$ in Ω when $t \geq T$. If $A\text{Ker}(C_p) \not\subseteq \text{Ker}(C_p)$, by Lemma 2.4, we can construct a full row-rank $(N-p+1) \times N$ matrix $\tilde{C}_1$ such that $\tilde{C}_1 U = 0$ in Ω when $t \geq T$. If $A\text{Ker}(\tilde{C}_1) \not\subseteq \text{Ker}(\tilde{C}_1)$, still by Lemma 2.4, we can construct another full row-rank $(N-p+2) \times N$ matrix $\tilde{C}_2$ such that $\tilde{C}_2 U = 0$ in Ω when $t \geq T$, $\cdots$. This procedure should stop at the r^{th} step, where $0 \leq r \leq p$. Thus, we get a full row-rank $(N-p+r) \times N$ matrix $\tilde{C}_r$ such that

$$t \geq T: \quad \tilde{C}_r U = 0 \quad \text{in} \quad \Omega \tag{2.3.9}$$

and

$$A\text{Ker}(\tilde{C}_r) \subseteq \text{Ker}(\tilde{C}_r). \tag{2.3.10}$$

By Lemma 4.7 in Appendix, there exists a unique matrix $\tilde{A}$ of order $(N-p+r)$, such that

$$\tilde{C}_r A = \tilde{A}\tilde{C}_r.$$

Setting $W = \tilde{C}_r U$ in (2.1.1), we get the following reduced problem:

$$\begin{cases} W'' - \Delta W + \tilde{A}W = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ W = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu W = \tilde{D}H & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \ W = \tilde{C}_r U_0, \ W' = \tilde{C}_r U_1 & \text{in} \quad \Omega, \end{cases} \tag{2.3.11}$$

where $\tilde{D} = \tilde{C}_r D$. Moreover, by (2.3.9) we have

$$t \geq T: \quad W \equiv 0. \tag{2.3.12}$$

Noting that $\tilde{C}_r$ is a $(N-p+r) \times N$ full row-rank matrix, the linear mapping

$$(U_0, U_1) \to (\tilde{C}_r U_0, \tilde{C}_r U_1) \tag{2.3.13}$$

is a surjection from $(\mathcal{H}_1 \times \mathcal{H}_0)^N$ onto $(\mathcal{H}_1 \times \mathcal{H}_0)^{N-p+r}$, then, (2.3.11) is exactly null controllable at the time T in the space $(\mathcal{H}_1 \times \mathcal{H}_0)^{N-p+r}$. By Lemma 2.2 and Lemma 2.3, it is necessary that

$$\text{rank}(\tilde{C}_r D) = N - p + r,$$

then

$$M = \text{rank}(D) \geq \text{rank}(\tilde{C}_r D) = N - p + r \geq N - p. \tag{2.3.14}$$

In particular, when $M = N - p$, we have $r = 0$, namely, the condition of compatibility (2.3.8) holds.

Remark 2.2. *The condition of compatibility* (2.3.8) *is equivalent to the fact that there exist some constants* α_{rs} $(1 \le r, s \le p)$ *such that*

$$Ae_s = \sum_{r=1}^{p} \alpha_{rs} e_r, \quad 1 \le s \le p, \tag{2.3.15}$$

or, noting (2.3.6), A *satisfies the following row-sum condition by blocks:*

$$\sum_{j=m_{s-1}+1}^{m_s} a_{ij} = \alpha_{rs}, \quad m_{r-1}+1 \le i \le m_r, \quad 1 \le r, s \le p. \tag{2.3.16}$$

Especially, this condition of compatibility becomes (2.2.2) *when* $p = 1$.

Theorem 2.4. *Let* C_p *be the* $(N-p) \times N$ *matrix of synchronization by p-groups defined by* (2.3.3)–(2.3.4). *Under the condition of compatibility* (2.3.8), *assume that the* $N \times (N-p)$ *boundary control matrix* D *has full column-rank and satisfies rank*$(C_pD) = N - p$. *Then system* (2.1.1) *is exactly synchronizable by p-groups by means of boundary control* $H \in \mathcal{L}^{N-p}$, *moreover, we have the continuous dependence:*

$$\|H\|_{\mathcal{L}^{N-p}} \le C\|C_p(U_0, U_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^{N-p}}, \tag{2.3.17}$$

where C *is a positive constant.*

On the other hand, when rank$(C_pD) < N - p$ *(especially, when* $M < N - p$*), no matter how large* $T > 0$ *is, system* (2.1.1) *is not exactly synchronizable by p-groups at the time* T.

Proof Assume that the coupling matrix $A = (a_{ij})$ satisfies the condition of compatibility (2.3.8). By Lemma 4.7 in Appendix, there exists a unique matrix $\overline{A}_p$ of order $(N-p)$, such that

$$C_pA = \overline{A}_pC_p. \tag{2.3.18}$$

Setting

$$W = C_pU, \qquad \overline{D} = C_pD.$$

We can similarly get the following reduced system for W:

$$\begin{cases} W'' - \Delta W + \overline{A}_pW = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ W = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu W = \overline{D}H & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \ W = C_pU_0, \quad W' = C_pU_1 & \text{in} \quad \Omega, \end{cases} \tag{2.3.19}$$

where W is a vector valued function of $(N-p)$ components. By the assumption that rank$(\overline{D})$ = rank(C_pD) = $N - p$ and Lemma 2.2, system (2.2.12) is exactly null controllable. Also, by (2.1.6) in the Definition 2.1, we get the continuous dependence of (2.3.17). Then the original system (2.1.1) for U is exactly synchronizable by p-groups.

On the other hand, when rank$(C_pD) < N - p$, by Lemma 2.3, the reduced system (2.2.12) is not exactly null controllable, then system (2.1.1) is not exactly synchronizable by p-groups.

2.4 Determination of the state of synchronization by p-groups

Now, we are going to discuss the determination of the state of synchronization of system (2.1.1). Generally speaking, the state of synchronization should depend on the initial data (U_0, U_1) and the applied boundary control H. However, when the coupling matrix A possesses some good properties, the state of synchronization is independent of the applied boundary control, and can be determined entirely by the solution to a system of wave equations with homogeneous boundary condition.

First, by Lemma 2.1 and noting that the space $(\mathcal{H}_1 \times \mathcal{H}_0)^N$ given in Definition 2.3 is included in $(\mathcal{H}_{1-s} \times \mathcal{H}_{-s})^N$ $(s > \frac{1}{2})$, differently from the case of Dirichlet boundary controls, the attainable set of states of exact boundary synchronization by p-groups for the system with Neumann boundary controls is not the whole space $(\mathcal{H}_{1-s} \times \mathcal{H}_{-s})^p$. Besides, as in the case of Dirichlet boundary controls ([28]), the choice of boundary controls is not unique. We have the following

Theorem 2.5. *Let $\mathbb{H}$ denote the set of all the boundary controls H which can realize the exact boundary synchronization by p-groups at the time T for system* (2.1.1). *If the condition of compatibility* (2.3.8) *holds, then for $\epsilon > 0$ small enough, the value of $H \in \mathbb{H}$ on $(0, \epsilon) \times \Gamma_1$ can be arbitrarily chosen.*

Proof First of all, there exists a $T_0 > 0$ independent of the initial data, such that, when $T > T_0$, the reduced problem (2.2.12) is exactly null controllable at the time T. According to the proof of Theorem 2.2, the exact synchronization of system (2.1.1) is equivalent to the exact null controllability of the reduced system (2.2.12). Therefore, taking an $\epsilon > 0$ so small that $T - \epsilon > T_0$, system (2.1.1) is still exactly synchronizable at the time $T - \epsilon$.

Assuming firstly that $(U_0, U_1) \in (C_0^\infty(\Omega) \times C_0^\infty(\Omega))^N$, and choosing arbitrarily

$$\widehat{H}_\epsilon \in (C_0^\infty([0, \epsilon] \times \Gamma_1))^{N-p},$$

we solve the forward problem (2.1.1) on $[0, \epsilon]$ with $H = \widehat{H}_\epsilon$, and get the solution $(\widehat{U}_\epsilon, \widehat{U}'_\epsilon) \in C^0([0, \epsilon]; (\mathcal{H}_1 \times \mathcal{H}_0)^N)$. Taking $(\widehat{U}_\epsilon(\epsilon, \cdot), \widehat{U}'_\epsilon(\epsilon, \cdot)) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$ as initial data, by Theorem 2.2, for system (2.1.1), there exists a boundary control

$$\widetilde{H}_\epsilon \in \left(L^2(\epsilon, T; L^2(\Gamma_1))\right)^{N-p}$$

such that the corresponding solution $\widetilde{U}_\epsilon$ satisfies exactly the initial condition

$$t = \epsilon : \qquad \widetilde{U}_\epsilon = \widehat{U}_\epsilon(\epsilon, x), \qquad \widetilde{U}'_\epsilon = \widehat{U}'_\epsilon(\epsilon, x)$$

and realizes the synchronization at the time $t = T$. Let

$$H = \begin{cases} \widehat{H}_\epsilon & t \in (0, \epsilon), \\ \widetilde{H}_\epsilon & t \in (\epsilon, T), \end{cases} \qquad U = \begin{cases} \widehat{U}_\epsilon & t \in (0, \epsilon), \\ \widetilde{U}_\epsilon & t \in (\epsilon, T). \end{cases}$$

It can be verified that U is the solution to the mixed problem (2.1.1) with boundary control H, and it is exactly synchronizable at the time T. By this way, we get an infinity of boundary controls H, the values of which on $(0,\epsilon)\times\Gamma_1$ can be taken arbitrarily. Finally, by the denseness of $C_0^\infty(\Omega)$ in $\mathcal{H}_1$ and $\mathcal{H}_0$, we can get the desired result.

In fact, the state of synchronization is closely related to the properties of the coupling matrix A. When A^T possesses an invariant subspace $\text{Span}\{E_1, E_2, \cdots, E_p\}$ such that

$$(E_i, e_j) = \delta_{ij}, \quad 1 \le i, j \le p,$$

$\text{Span}\{E_1, E_2, \cdots, E_p\}$ and $\text{Ker}(C_p) = \text{Span}\{e_1, \cdots, e_p\}$ are called to be bi-orthonormal.

Let

$$\mathcal{D}_{N-p} = \{D \in \mathbb{M}^{N\times(N-p)}(\mathbb{R}) : \quad \text{rank}(D) = \text{rank}(C_p D) = N - p\}.$$

By [29], $D \in \mathcal{D}_{N-p}$ if and only if it can be expressed by

$$D = (C_p^T + (e_1, \cdots, e_p)D_0)\bar{D}, \tag{2.4.1}$$

where D_0 is a $p\times(N-p)$ matrix, and $\bar{D}$ is a reversible matrix of order $(N-p)$.

Theorem 2.6. *Under the condition of compatibility* (2.3.8)*, assume that A^T possesses an invariant subspace Span$\{E_1, E_2, \cdots, E_p\}$ which is bi-orthonormal to Ker(C_p) = Span$\{e_1, \cdots, e_p\}$. Then there exists a boundary control matrix $D \in \mathcal{D}_{N-p}$, such that the state of synchronization by p-groups $u = (u_1, \cdots, u_p)^T$ is independent of the applied boundary controls, and can be determined as follows:*

$$t \ge T: \quad u = \psi, \tag{2.4.2}$$

where $\psi = (\psi_1, \cdots, \psi_p)^T$ is the solution to the following problem with homogeneous boundary condition:

$$\begin{cases} \psi_r'' - \Delta\psi_r + \sum_{s=1}^{p} \alpha_{rs}\psi_s = 0 & \textit{in} \quad (0, +\infty)\times\Omega, \\ \psi_r = 0 & \textit{on} \quad (0, +\infty)\times\Gamma_0, \\ \partial_\nu \psi_r = 0 & \textit{on} \quad (0, +\infty)\times\Gamma_1, \\ t = 0: \quad \psi_r = (E_r, U_0), \quad \psi_r' = (E_r, U_1) & \textit{in} \quad \Omega, \end{cases} \tag{2.4.3}$$

where α_{rs} $(1 \le r, s \le p)$ are given by (2.3.16)*.*

Proof Noting that $\text{Span}\{E_1, E_2, \cdots, E_p\}$ is bi-orthonormal to $\text{Ker}(C_p) = \text{Span}\{e_1, \cdots, e_p\}$, and taking

$$D_0 = -E^T C_p^T, \quad E = (E_1, E_2, \cdots, E_p) \tag{2.4.4}$$

in (2.4.1), we get a boundary control matrix $D \in \mathcal{D}_{N-p}$, such that

$$E_r \in \mathrm{Ker}(D^T), \quad 1 \le r \le p. \tag{2.4.5}$$

On the other hand, since $\mathrm{Span}\{E_1, E_2, \cdots, E_p\}$ is an invariant subspace of A^T, we may denote

$$A^T E_r = \sum_{s=1}^{p} \beta_{sr} E_s, \quad 1 \le r \le p,$$

where β_{sr} are some constants. By

$$(A^T E_r, e_s) = (E_r, Ae_s), \quad 1 \le r, s \le p,$$

and noticing (3.9.10), we have

$$(\sum_{t=1}^{p} \beta_{tr} E_t, e_s) = (E_r, \sum_{t=1}^{p} \alpha_{ts} e_t), \quad 1 \le r, s \le p.$$

Then by bi-orthonormality, we get

$$\beta_{sr} = \alpha_{rs},$$

namely,

$$A^T E_r = \sum_{s=1}^{p} \alpha_{rs} E_s, \quad 1 \le r \le p. \tag{2.4.6}$$

Let $\psi_r = (E_r, U)$, taking the inner product with E_r on both sides of (2.1.1), we get (2.4.3). Finally, for the state of synchronization by p-groups, by (2.3.7) we have

$$t \ge T: \quad \psi_r(t) = (E_r, U) = \sum_{s=1}^{p} (E_r, e_s) u_s = u_r, \quad 1 \le r \le p. \tag{2.4.7}$$

When the assumptions in Theorem 2.6 fail, we can use the solution of (2.4.3) to give an estimate on the state of synchronization by p-groups.

Theorem 2.7. *Under the condition of compatibility* (2.3.8), *assume that* $\{E_1, E_2, \cdots, E_p\}$ *is bi-orthonormal to* $\{e_1, \cdots, e_p\}$. *Then there exist a boundary control matrix* $D \in \mathcal{D}_{N-p}$ *and a constant* c *independent of the initial data, such that the state of synchronization by p-groups* $u = (u_1, \cdots, u_p)$ *satisfies the following estimate:*

$$t \ge T: \qquad \|(u, u')(t) - (\psi, \psi')(t)\|_{(\mathcal{H}_{2-s} \times \mathcal{H}_{1-s})^p} \le c \|C_p(U_0, U_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^{N-p}}, \tag{2.4.8}$$

where $\psi = (\psi_1, \cdots, \psi_p)$ *is the solution to problem* (2.4.3), *and* $s > \frac{1}{2}$.

Proof Since $\{E_1, E_2, \cdots, E_p\}$ is bi-orthonormal to $\{e_1, \cdots, e_p\}$, similarly to (2.4.4), there exists a boundary control matrix $D \in \mathcal{D}_{N-p}$, such that (2.4.5) holds. Let $\phi_r = (E_r, U)$. Taking the inner product with E_r on both sides of (2.1.1)–(2.1.2), we get

$$\phi_r'' - \Delta\phi_r + (E_r, AU) = 0.$$

Since

$$\begin{aligned}(E_r, AU) = (A^T E_r, U) &= (\sum_{s=1}^{p} \alpha_{rs} E_s + A^T E_r - \sum_{s=1}^{p} \alpha_{rs} E_s, U) \\ &= \sum_{s=1}^{p} \alpha_{rs}(E_s, U) + (A^T E_r - \sum_{s=1}^{p} \alpha_{rs} E_s, U) \\ &= \sum_{s=1}^{p} \alpha_{rs}\phi_s + (A^T E_r - \sum_{s=1}^{p} \alpha_{rs} E_s, U),\end{aligned} \tag{2.4.9}$$

and for any given $k \in \{1, \cdots, p\}$, we have

$$\begin{aligned}(A^T E_r - \sum_{s=1}^{p} \alpha_{rs} E_s, e_k) &= (E_r, Ae_k) - \sum_{s=1}^{p} \alpha_{rs}(E_s, e_k) = (E_r, \sum_{s=1}^{p} \alpha_{sk} e_s) - \alpha_{rk} \\ &= \sum_{s=1}^{p} \alpha_{sk}(E_r, e_s) - \alpha_{rk} = \alpha_{rk} - \alpha_{rk} = 0,\end{aligned} \tag{2.4.10}$$

we get

$$A^T E_r - \sum_{s=1}^{p} \alpha_{rs} E_s \in \{Ker(e_1, \cdots, e_p)\}^{\perp} = \mathrm{Im}(C_p^T).$$

Therefore, there exists an vector $R_r \in \mathbb{R}^{N-p}$ such that

$$A^T E_r - \sum_{s=1}^{p} \alpha_{rs} E_s = C_p^T R_r. \tag{2.4.11}$$

Thus, for $r = 1, \cdots, p$, we have

$$\begin{cases} \phi_r'' - \Delta\phi_r + \sum_{s=1}^{p} \alpha_{rs}\phi_s = (R_r, C_p U) & \text{in} \quad (0, +\infty) \times \Omega, \\ \phi_r = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \phi_r = 0 & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0 : \quad \phi_r = (E_r, U_0), \quad \phi_r' = (E_r, U_1) & \text{in} \quad \Omega, \end{cases} \tag{2.4.12}$$

where α_{rs} $(1 \le r, s \le p)$ are defined by (2.3.16), and $U = U(t, x) \in C(0, T; (\mathcal{H}_{1-s})^N) \cap C^1(0, T; (\mathcal{H}_{-s})^N)$ is the solution to the mixed initial-boundary value problem (2.1.1)–(2.1.2). Moreover, we have

$$t \ge T: \quad \phi_r(t) = (E_r, U) = \sum_{s=1}^{p} (E_r, e_s) u_s = u_r, \quad r = 1, \cdots, p. \tag{2.4.13}$$

Noting that (2.4.3) and (2.4.12) possess the same initial data and boundary condition, by the well-posedness for a system of wave equations with Neumann boundary condition, we have (see [41]) that, when $t \geq 0$,

$$\|(\psi,\psi')(t)-(\phi,\phi')(t)\|^2_{(\mathcal{H}_{2-s}\times\mathcal{H}_{1-s})^p} \leq c\int_0^T \|C_pU\|^2_{(\mathcal{H}_{1-s})^{N-p}}ds, \tag{2.4.14}$$

where, c is a positive constant. Noting that $W = C_pU$, by well-posedness of the reduced problem (2.3.19)(see Lemma 2.1), we have

$$\int_0^T \|C_pU\|^2_{(\mathcal{H}_{1-s})^{N-p}}ds \leq c(\|C_p(U_0,U_1)\|^2_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-p}} + \|\overline{D}H\|_{\mathcal{L}^{N-p}}). \tag{2.4.15}$$

Moreover, by (2.3.17) we have

$$\|\overline{D}H\|_{\mathcal{L}^{N-p}} \leq c\|C_p(U_0,U_1)\|^2_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-p}}. \tag{2.4.16}$$

Substituting it into (2.4.15), we have

$$\int_0^T \|C_pU\|^2_{(\mathcal{H}_{1-s})^{N-p}}ds \leq c\|C_p(U_0,U_1)\|^2_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-p}}, \tag{2.4.17}$$

then, by (2.4.14) we get

$$\|(\psi,\psi')(t)-(\phi,\phi')(t)\|^2_{(\mathcal{H}_{2-s}\times\mathcal{H}_{1-s})^p} \leq c\|C_p(U_0,U_1)\|^2_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-p}}. \tag{2.4.18}$$

Substituting (2.4.13) into (2.4.18), we get (3.12).

Remark 2.3. *Differently from the case of Dirichlet boundary controls, although the solution to the problem with Neumann boundary controls* (2.1.1) *possesses a weaker regularity, the solution to the problem* (2.4.3), *which determines the state of synchronization by p-groups, possesses a higher regularity than the original problem* (2.1.1) *itself, then, this improved regularity makes it possible to approach the state of synchronization by p-groups by a solution to a relatively smoother problem.*

In order to exactly express the state of synchronization by p-groups, we can extend the subspace $\text{Span}\{e_1,\cdots,e_p\}$ to an invariant subspace $\text{Span}\{e_1,\cdots,e_p,\cdots,e_q\}$ of A, such that A^T possesses an invariant subspace $\text{Span}\{E_1,\cdots,E_p,\cdots,E_q\}$, which is bi-orthonormal to $\text{Span}\{e_1,\cdots,e_p,\cdots,e_q\}$. Let

$$P = \sum_{s=1}^{q} e_s \otimes E_s, \tag{2.4.19}$$

in which the tensor product is defined by

$$(e\otimes E)U = (E,U)e = E^TUe, \qquad \forall\, U \in \mathbb{R}^N.$$

P can be represented by a matrix of order N. It is easy to see that

$$\text{Im}(P) = \text{Span}\{e_1,e_2,\cdots,e_q\}, \qquad \text{Ker}(P) = \big(\text{Span}\{E_1,E_2,\cdots,E_q\}\big)^{\perp} \tag{2.4.20}$$

and

$$PA = AP. \tag{2.4.21}$$

Let $U = U(t, x)$ be the solution to the mixed initial-boundary value problem (2.1.1)–(2.1.2). We define its synchronizable part U_s and controllable part U_c, respectively, as follows:

$$U_s := PU, \qquad U_c := (I - P)U. \tag{2.4.22}$$

If system (2.1.1) is exactly synchronizable by p-groups, then

$$t \geq T: \qquad U \in \mathrm{Span}\{e_1, \cdots, e_p\} \subseteq \mathrm{Span}\{e_1, \cdots, e_p, \cdots, e_q\} = \mathrm{Im}(P), \tag{2.4.23}$$

then we have

$$t \geq T: \qquad U_s = PU = U, \qquad U_c = (I - P)U = 0.$$

Noting (2.4.21), multiplying P and $(I - P)$ from the left on both sides of (2.1.1) respectively, we see that the synchronizable part U_s of U satisfies the following system:

$$\begin{cases} U_s'' - \Delta U_s + AU_s = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ U_s = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu U_s = PDH & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \qquad U_s = PU_0, \quad U_s' = PU_1 & \text{in} \quad \Omega, \end{cases} \tag{2.4.24}$$

while, the controllable part U_c of U satisfies the following system:

$$\begin{cases} U_c'' - \Delta U_c + AU_c = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ U_c = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu U_c = (I - P)DH & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \qquad U_c = (I - P)U_0, \quad U_c' = (I - P)U_1 & \text{in} \quad \Omega. \end{cases} \tag{2.4.25}$$

In fact, under the boundary control H, U_c with the initial data $((I - P)U_0, (I - P)U_1) \in \mathrm{Ker}(P) \times \mathrm{Ker}(P)$ is exactly null controllable, while, U_s with the initial data $(PU_0, PU_1) \in \mathrm{Im}(P) \times \mathrm{Im}(P)$ is exactly synchronizable.

Theorem 2.8. *Let P be defined by* (2.4.19)*. If system* (2.1.1) *is exactly synchronizable by p-groups, and the synchronizable part U_s is independent of the applied boundary control H, then we have*

$$p = q \quad \textit{and} \quad PD = 0. \tag{2.4.26}$$

In particular, if $PU_0 = PU_1 = 0$, then, for such initial data (U_0, U_1), system (2.1.1) *is exactly null controllable.*

Proof By Theorem 2.5, the value of H on $(0,\epsilon)\times\Gamma_1$ can be arbitrarily taken. If the synchronizable part U_s is independent of the applied boundary control H, then we have

$$PD = 0,$$

hence

$$\mathrm{Im}(D) \subseteq \mathrm{Ker}(P).$$

Noting (2.4.20), we have

$$\dim \mathrm{Ker}(P) = N - q, \quad \dim \mathrm{Im}(D) = N - p,$$

then $p = q$.

2.5 State of exact boundary synchronization

By Lemma 2.5, $e = (1, 1, \cdots, 1)^T$ is a right eigenvector of A, corresponding to the eigenvalue a, defined by (2.2.2). Let $e_1, e_2, \cdots, e_r$ and $E_1, E_2, \cdots, E_r$ with $r \geq 1$ be the Jordan chains of A and A^T, respectively, corresponding to the eigenvalue a, and $Span\{e_1, e_2, \cdots, e_r\}$ is bi-orthonormal to $Span\{E_1, E_2, \cdots, E_r\}$. Thus we have

$$\begin{cases} Ae_l = ae_l + e_{l+1}, & 1 \leq l \leq r, \\ A^T E_k = aE_k + E_{k-1}, & 1 \leq k \leq r, \\ (E_k, e_l) = \delta_{kl}, & 1 \leq k, l \leq r, \\ e_r = (1, 1 \cdots, 1)^T, \quad e_{r+1} = 0, \quad E_0 = 0. \end{cases} \tag{2.5.1}$$

Let $U = U(t, x)$ be the solution to the mixed initial-boundary value problem (2.1.1)–(2.1.2). If system (2.1.1) is exactly synchronizable, then

$$t \geq T: \qquad U = ue_r, \tag{2.5.2}$$

where $u = u(t, x)$ is the corresponding state of synchronization. The synchronizable part and the controllable part are, respectively,

$$t \geq T: \qquad U_s = ue_r, \qquad U_c = 0.$$

If the synchronizable part is independent of the applied boundary control H, by Theorem 2.8, we have $r = 1$, then A possesses a left eigenvector E such that

$$(E, e) = 1.$$

Generally speaking, when $r \geq 1$, setting

$$\psi_k = (E_k, U), \qquad 1 \leq k \leq r,$$

noting (2.4.19) and (2.4.22), we have

$$U_s = \sum_{k=1}^{r} (E_k, U) e_k = \sum_{k=1}^{r} \psi_k e_k.$$

Thus, $(\psi_1, \cdots, \psi_r)$ are the coordinates of U_s under the basis $(e_1, e_2, \cdots, e_r)$.

Taking the inner product with E_k on both sides of (2.4.24), we get the following

Theorem 2.9. *Let $e_1, e_2, \cdots, e_r$ and $E_1, E_2, \cdots, E_r$ be the Jordan chains of A and A^T, respectively, corresponding to the eigenvalue a, in which $e_r = (1, \cdots, 1)^T$. Then the synchronizable part $U_s = (\psi_1, \cdots, \psi_r)$ is determined by the following system $(1 \leq k \leq r)$:*

$$\begin{cases} \psi_k'' - \Delta\psi_k + a\psi_k + \psi_{k-1} = 0 & in \quad (0, +\infty) \times \Omega, \\ \psi_k = 0 & on \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \psi_k = h_k & on \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \qquad \psi_k = (E_k, U_0), \quad \psi_k' = (E_k, U_1) & in \qquad \Omega, \end{cases} \tag{2.5.3}$$

where

$$\psi_0 = 0, \qquad h_k = E_k^T D H. \tag{2.5.4}$$

Noting (2.5.2), we have

$$t \geq T: \qquad \psi_k = (E_k, U) = (E_k, \tilde{\tilde{u}} e_r) = \tilde{\tilde{u}} \delta_{kr}, \qquad 1 \leq k \leq r.$$

Thus, the state of synchronization $\tilde{\tilde{u}}$ is determined by

$$t \geq T: \qquad u = u(t, x) = \psi_r(t, x).$$

However, in order to get the state of synchronization $\tilde{\tilde{u}}$, we must solve the whole coupled problem (2.5.3)–(2.5.4).

2.6 State of exact boundary synchronization by 2-groups

In this section, we will discuss the case $p = 2$ for the state of exact boundary synchronization by p-groups of system (2.1.1).

Assume that, when $t \geq T$, we have

$$u^{(1)}(t,x) \equiv \cdots \equiv u^{(m)}(t,x) \overset{\text{def.}}{=} u_1(t,x), \tag{2.6.1}$$

$$u^{(m+1)}(t,x) \equiv \cdots \equiv u^{(N)}(t,x) \overset{\text{def.}}{=} u_2(t,x). \tag{2.6.2}$$

Let C_2 be the matrix of synchronization by 2-groups, defined by (2.3.4). Obviously,

$$\mathrm{Ker}(C_2) = \mathrm{Span}\{\tilde{e}_1, \tilde{e}_2\}, \tag{2.6.3}$$

where

$$\tilde{e}_1 = (\underbrace{1,\cdots,1}_{m},\underbrace{0,\cdots,0}_{N-m})^T, \qquad \tilde{e}_2 = (\underbrace{0,\cdots,0}_{m},\underbrace{1,\cdots,1}_{N-m})^T,$$

and the state of synchronization (2.6.1)–(2.6.2) means that

$$t \geq T: \quad U = \tilde{\tilde{u}}_1\tilde{e}_1 + \tilde{\tilde{u}}_2\tilde{e}_2. \tag{2.6.4}$$

Assume that the subspace $\mathrm{Span}\{\tilde{e}_1, \tilde{e}_2\}$ contains two right eigenvectors e_r and f_s of A, corresponding to eigenvalues λ and μ, respectively. Let $e_1, e_2, \cdots, e_r$ and $f_1, f_2, \cdots, f_s$ be the Jordan chains corresponding to these two right eigenvectors, respectively:

$$\begin{cases} Ae_i = \lambda e_i + e_{i+1}, & 1 \leq i \leq r, \qquad e_{r+1} = 0, \\ Af_j = \mu f_j + f_{j+1}, & 1 \leq j \leq s, \qquad f_{s+1} = 0. \end{cases} \tag{2.6.5}$$

Accordingly, let $\xi_1, \xi_2, \cdots, \xi_r$ and $\eta_1, \eta_2, \cdots, \eta_s$ be the Jordan chains to the corresponding left eigenvectors, respectively:

$$\begin{cases} A^T\xi_i = \lambda\xi_i + \xi_{i-1}, & 1 \leq i \leq r, \qquad \xi_0 = 0, \\ A^T\eta_j = \mu\eta_j + \eta_{j-1}, & 1 \leq j \leq s, \qquad \eta_0 = 0, \end{cases} \tag{2.6.6}$$

such that

$$(e_i, \xi_l) = \delta_{il} \ (i,l = 1,\cdots r), \quad (f_j, \eta_m) = \delta_{jm} \ (j,m = 1,\cdots s) \tag{2.6.7}$$

and

$$(e_i, \eta_j) = (f_j, \xi_i) = 0 \ (i = 1,\cdots r; \ j = 1,\cdots s). \tag{2.6.8}$$

Let

$$\phi_i = (U, \xi_i), \quad \psi_j = (U, \eta_j).$$

Taking the inner product with ξ_i and η_j on both sides of (2.1.1)–(2.1.2), respectively, we get

$$\begin{cases} \phi_i'' - \Delta\phi_i + \lambda\phi_i + \phi_{i-1} = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ \phi_i = 0 & \text{on} \quad (0,+\infty)\times\Gamma_0, \\ \partial_\nu\phi_i = \xi_i^T DH & \text{on} \quad (0,+\infty)\times\Gamma_1, \\ t=0: \qquad \phi_i = (\xi_i, U_0), \quad \phi_i' = (\xi_i, U_1) & \text{in} \quad \Omega \end{cases} \tag{2.6.9}$$

and

$$\begin{cases} \psi_j'' - \Delta\psi_j + \mu\psi_j + \psi_{j-1} = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ \psi_j = 0 & \text{on} \quad (0,+\infty)\times\Gamma_0, \\ \partial_\nu\psi_j = \eta_j^T DH & \text{on} \quad (0,+\infty)\times\Gamma_1, \\ t=0: \qquad \psi_j = (\eta_j, U_0), \quad \psi_j' = (\eta_j, U_1) & \text{in} \quad \Omega, \end{cases} \tag{2.6.10}$$

where $i = 1, \cdots, r;\ j = 1, \cdots, s$, and

$$\phi_0 = \psi_0 = 0. \tag{2.6.11}$$

Once we get the solutions $\phi_1 \cdots, \phi_r$ and $\psi_1, \cdots, \psi_s$, the corresponding state of synchronization by 2-groups can be determined. Since the sum of the numbers of right eigenvectors is bigger than or equal to 2, we discuss the following two cases, respectively.

(1) When $r \geq 1$ and $s \geq 1$, we have

$$\begin{cases} \tilde{e}_1 = \alpha_1 e_r + \alpha_2 f_s, \\ \tilde{e}_2 = \beta_1 e_r + \beta_2 f_s. \end{cases} \tag{2.6.12}$$

By (2.6.4), when $t \geq T$, we have

$$U = (\tilde{\tilde{u}}_1\alpha_1 + \tilde{\tilde{u}}_2\beta_1)e_r + (\tilde{\tilde{u}}_1\alpha_2 + \tilde{\tilde{u}}_2\beta_2)f_s. \tag{2.6.13}$$

Noting (2.6.7)–(2.6.8), we have

$$t \geq T: \quad \begin{cases} \phi_r = \tilde{\tilde{u}}_1\alpha_1 + \tilde{\tilde{u}}_2\beta_1, \\ \psi_s = \tilde{\tilde{u}}_1\alpha_2 + \tilde{\tilde{u}}_2\beta_2. \end{cases} \tag{2.6.14}$$

Since $\tilde{e}_1, \tilde{e}_2$ are linearly independent, by (2.6.12) and noticing (2.6.7)–(2.6.8), we have

$$\det\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \neq 0. \tag{2.6.15}$$

Then, we can get the state $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2)^T$ of synchronization by 2-groups by solving the linear system (2.6.14).

When $r = 1$ and $s = 1$, there exists a boundary control matrix $D \in \mathcal{D}_{N-2}$, such that $\xi_1^T D = \eta_1^T D = 0$, hence the state $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2)^T$ of synchronization by 2-groups is independent of the applied boundary controls.

(2) When $r \geq 2$ and $s = 0$ (or $r = 0$ and $s \geq 2$), we have

$$\begin{cases} \tilde{e}_1 = \alpha_1 e_r + \alpha_2 e_{r-1}, \\ \tilde{e}_2 = \beta_1 e_r + \beta_2 e_{r-1}. \end{cases} \tag{2.6.16}$$

Similarly to (2.6.13), when $t \geq T$, we have

$$U = (\tilde{\tilde{u}}_1 \alpha_1 + \tilde{\tilde{u}}_2 \beta_1) e_r + (\tilde{\tilde{u}}_1 \alpha_2 + \tilde{\tilde{u}}_2 \beta_2) e_{r-1}. \tag{2.6.17}$$

Similarly, noting (2.6.7)–(2.6.8), (2.6.15) holds, thus, the state $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2)^T$ of synchronization by 2-groups can be determined by solving the following linear system:

$$\begin{cases} \phi_r = \tilde{\tilde{u}}_1 \alpha_1 + \tilde{\tilde{u}}_2 \beta_1, \\ \phi_{r-1} = \tilde{\tilde{u}}_1 \alpha_2 + \tilde{\tilde{u}}_2 \beta_2. \end{cases} \tag{2.6.18}$$

If $r = 2$, then $\mathrm{Ker}(C_2) = \mathrm{Span}\{e_1, e_2\}$, and $\{\xi_1, \xi_2\}$ is an invariant subspace of A^T, which is bi-orthonormal to $\mathrm{Ker}(C_2)$. By Theorem 2.6, similarly to (2.4.4), there exists a boundary control matrix $D \in \mathcal{D}_{N-2}$, such that $\xi_1^T D = \xi_2^T D = 0$, then the state $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2)^T$ of synchronization by 2-groups is independent of the applied boundary controls.

2.7 State of exact boundary synchronization by 3-groups

Finally, we discuss the case $p = 3$ for the state of exact boundary synchronization by p-groups of system (2.1.1).

Assume that, when $t \geq T$, we have

$$u^{(1)}(t,x) \equiv \cdots \equiv u^{(m_1)}(t,x) \overset{\text{def.}}{=} u_1(t,x), \tag{2.7.1}$$

$$u^{(m_1+1)}(t,x) \equiv \cdots \equiv u^{(m_2)}(t,x) \overset{\text{def.}}{=} u_2(t,x), \tag{2.7.2}$$

$$u^{(m_2+1)}(t,x) \equiv \cdots \equiv u^{(N)}(t,x) \overset{\text{def.}}{=} u_3(t,x). \tag{2.7.3}$$

Let C_3 be the matrix of synchronization by 3-groups, defined by (2.3.4). Obviously,

$$\mathrm{Ker}(C_3) = \mathrm{Span}\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}, \tag{2.7.4}$$

where

$$\tilde{e}_1 = (\underbrace{1, \cdots, 1}_{m_1}, \underbrace{0, \cdots, 0}_{m_2 - m_1}, \underbrace{0, \cdots, 0}_{N - m_2})^T, \ \tilde{e}_2 = (\underbrace{0, \cdots, 0}_{m_1}, \underbrace{1, \cdots, 1}_{m_2 - m_1}, \underbrace{0, \cdots, 0}_{N - m_2})^T,$$

$$\tilde{e}_3 = (\underbrace{0, \cdots, 0}_{m_1}, \underbrace{0, \cdots, 0}_{m_2-m_1}, \underbrace{1, \cdots, 1}_{N-m_2})^T,$$

and the state of synchronization (2.7.1)–(2.7.3) means that

$$t \geq T: \quad U = \tilde{\tilde{u}}_1\tilde{e}_1 + \tilde{\tilde{u}}_2\tilde{e}_2 + \tilde{\tilde{u}}_3\tilde{e}_3. \tag{2.7.5}$$

Assume that there exist three right eigenvectors $e_r,\ f_s$ and g_t of A in the invariant subspace $\text{Span}\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ of C_3, corresponding to eigenvalues λ, μ and ν, respectively. Let $e_1, e_2, \cdots, e_r$; $f_1, f_2, \cdots, f_s$ and $g_1, g_2, \cdots, g_t$ be the Jordan chains corresponding to these three right eigenvectors, respectively:

$$\begin{cases} Ae_i = \lambda e_i + e_{i+1}, & 1 \leq i \leq r, & e_{r+1} = 0, \\ Af_j = \mu f_j + f_{j+1}, & 1 \leq j \leq s, & f_{s+1} = 0, \\ Ag_k = \nu g_k + g_{k+1}, & 1 \leq k \leq t, & g_{t+1} = 0. \end{cases} \tag{2.7.6}$$

Correspondingly, let $\xi_1, \xi_2, \cdots, \xi_r$; $\eta_1, \eta_2, \cdots, \eta_s$ and $\zeta_1, \zeta_2, \cdots, \zeta_t$ be the Jordan chains to the corresponding left eigenvectors, respectively:

$$\begin{cases} A^T\xi_i = \lambda\xi_i + \xi_{i-1}, & 1 \leq i \leq r, & \xi_0 = 0, \\ A^T\eta_j = \mu\eta_j + \eta_{j-1}, & 1 \leq j \leq s, & \eta_0 = 0, \\ A^T\zeta_k = \nu\zeta_k + \zeta_{k-1}, & 1 \leq k \leq t, & \zeta_0 = 0, \end{cases} \tag{2.7.7}$$

such that

$$\begin{gathered} (e_i, \xi_l) = \delta_{il}, \quad (f_j, \eta_m) = \delta_{jm}, \quad (g_k, \zeta_n) = \delta_{kn} \\ (i, l = 1, \cdots r;\ j, m = 1, \cdots s;\ k, n = 1, \cdots t) \end{gathered} \tag{2.7.8}$$

and

$$\begin{gathered} (e_i, \eta_j) = (e_i, \zeta_k) = (f_j, \xi_i) = (f_j, \zeta_k) = (g_k, \xi_i) = (g_k, \eta_j) = 0 \\ (i = 1, \cdots r;\ j = 1, \cdots s;\ k = 1, \cdots t). \end{gathered} \tag{2.7.9}$$

Taking the inner product with $\xi_i,\ \eta_j\ \zeta_k$ on both sides of (2.1.1)–(2.1.2), respectively, and denoting

$$\phi_i = (U, \xi_i),\ \psi_j = (U, \eta_j),\ \theta_k = (U, \zeta_k),$$

we get

$$\begin{cases} \phi_i'' - \Delta\phi_i + \lambda\phi_i + \phi_{i-1} = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \phi_i = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu\phi_i = \xi_i^T DH & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad \phi_i = (\xi_i, U_0), \quad \phi_i' = (\xi_i, U_1) & \text{in} \quad \Omega, \end{cases} \tag{2.7.10}$$

$$\begin{cases} \psi_j'' - \Delta\psi_j + \mu\psi_j + \psi_{j-1} = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \psi_j = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \psi_j = \eta_j^T DH & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad \psi_j = (\eta_j, U_0), \quad \psi_j' = (\eta_j, U_1) & \text{in} \quad \Omega \end{cases} \tag{2.7.11}$$

and

$$\begin{cases} \theta_k'' - \Delta\theta_k + \nu\theta_k + \theta_{k-1} = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \theta_k = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \theta_k = \zeta_k^T DH & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad \theta_k = (\zeta_k, U_0), \quad \theta_k' = (\zeta_k, U_1) & \text{in} \quad \Omega, \end{cases} \tag{2.7.12}$$

where $i = 1, \cdots, r;\ j = 1, \cdots, s;\ k = 1, \cdots, t$, and

$$\phi_0 = \psi_0 = \theta_0 = 0. \tag{2.7.13}$$

Once we get the solutions $\phi_1 \cdots, \phi_r$; $\psi_1, \cdots, \psi_s$ and $\varphi_1, \cdots, \varphi_t$, the state of synchronization by 3-groups can be determined. Since the sum of the numbers of right eigenvectors is bigger than or equal to 3, we discuss the following three cases, respectively.

(1) None of r, s, t is equal to 0 ($r \geq 1,\ s \geq 1$ and $t \geq 1$). We have

$$\begin{cases} \tilde{e}_1 = \alpha_1 e_r + \alpha_2 f_s + \alpha_3 g_t, \\ \tilde{e}_2 = \beta_1 e_r + \beta_2 f_s + \beta_3 g_t, \\ \tilde{e}_3 = \gamma_1 e_r + \gamma_2 f_s + \gamma_3 g_t. \end{cases} \tag{2.7.14}$$

By (2.7.5), when $t \geq T$, we have

$$U = (\tilde{\tilde{u}}_1\alpha_1 + \tilde{\tilde{u}}_2\beta_1 + \tilde{\tilde{u}}_3\gamma_1)e_r + (\tilde{\tilde{u}}_1\alpha_2 + \tilde{\tilde{u}}_2\beta_2 + \tilde{\tilde{u}}_3\gamma_2)f_s + (\tilde{\tilde{u}}_1\alpha_3 + \tilde{\tilde{u}}_2\beta_3 + \tilde{\tilde{u}}_3\gamma_3)g_t. \tag{2.7.15}$$

Noting (2.7.8)–(2.7.9), we have

$$t \geq T: \quad \begin{cases} \phi_r = \tilde{\tilde{u}}_1\alpha_1 + \tilde{\tilde{u}}_2\beta_1 + \tilde{\tilde{u}}_3\gamma_1, \\ \psi_s = \tilde{\tilde{u}}_1\alpha_2 + \tilde{\tilde{u}}_2\beta_2 + \tilde{\tilde{u}}_3\gamma_2, \\ \theta_t = \tilde{\tilde{u}}_1\alpha_3 + \tilde{\tilde{u}}_2\beta_3 + \tilde{\tilde{u}}_3\gamma_3. \end{cases} \tag{2.7.16}$$

Since $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ are linear independent, by (2.7.14) and noting (2.7.8)–(2.7.9), we have

$$\det \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \neq 0. \tag{2.7.17}$$

Then, the state $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2, \tilde{\tilde{u}}_3)^T$ of synchronization by 3-groups can be determined by solving the linear system (2.7.16).

When $r = 1$, $s = 1$ and $t = 1$, there exists a boundary control matrix $D \in \mathcal{D}_{N-3}$, such that $\xi_1^T D = \eta_1^T D = \zeta_1^T D = 0$, then the state $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2, \tilde{\tilde{u}}_3)^T$ of synchronization by 3-groups is independent of the applied boundary controls.

(2) One of r, s, t is equal to 0. Without loss of generality, we may assume that $r \geq 2, t \geq 1$ and $s = 0$ and let

$$\begin{cases} \tilde{e}_1 = \alpha_1 e_r + \alpha_2 e_{r-1} + \alpha_3 g_t, \\ \tilde{e}_2 = \beta_1 e_r + \beta_2 e_{r-1} + \beta_3 g_t, \\ \tilde{e}_3 = \gamma_1 e_r + \gamma_2 e_{r-1} + \gamma_3 g_t. \end{cases} \tag{2.7.18}$$

Similarly, when $t \geq T$, we have

$$U = (\tilde{\tilde{u}}_1 \alpha_1 + \tilde{\tilde{u}}_2 \beta_1 + \tilde{\tilde{u}}_3 \gamma_1) e_r + (\tilde{\tilde{u}}_1 \alpha_2 + \tilde{\tilde{u}}_2 \beta_2 + \tilde{\tilde{u}}_3 \gamma_2) e_{r-1} + (\tilde{\tilde{u}}_1 \alpha_3 + \tilde{\tilde{u}}_2 \beta_3 + \tilde{\tilde{u}}_3 \gamma_3) g_t. \tag{2.7.19}$$

Noting (2.7.8)–(2.7.9), (2.7.17) holds. Thus, the state $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2, \tilde{\tilde{u}}_3)^T$ of synchronization by 3-groups can be determined by solving the following linear system:

$$\begin{cases} \phi_r = \tilde{\tilde{u}}_1 \alpha_1 + \tilde{\tilde{u}}_2 \beta_1 + \tilde{\tilde{u}}_3 \gamma_1, \\ \phi_{r-1} = \tilde{\tilde{u}}_1 \alpha_2 + \tilde{\tilde{u}}_2 \beta_2 + \tilde{\tilde{u}}_3 \gamma_2, \\ \theta_t = \tilde{\tilde{u}}_1 \alpha_3 + \tilde{\tilde{u}}_2 \beta_3 + \tilde{\tilde{u}}_3 \gamma_3. \end{cases} \tag{2.7.20}$$

If $r = 2$ and $t = 1$, there exists a boundary control matrix $D \in \mathcal{D}_{N-3}$, such that $\xi_1^T D = \xi_2^T D = \zeta_1^T D = 0$, then the state $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2, \tilde{\tilde{u}}_3)^T$ of synchronization by 3-groups is independent of the applied boundary controls.

(3) Two of r, s, t are equal to 0. Without loss of generality, we may assume that $r \geq 3, s = t = 0$. Then we have:

$$\begin{cases} \tilde{e}_1 = \alpha_1 e_r + \alpha_2 e_{r-1} + \alpha_3 e_{r-2}, \\ \tilde{e}_2 = \beta_1 e_r + \beta_2 e_{r-1} + \beta_3 e_{r-2}, \\ \tilde{e}_3 = \gamma_1 e_r + \gamma_2 e_{r-1} + \gamma_3 e_{r-2}. \end{cases} \tag{2.7.21}$$

Similarly, when $t \geq T$, we have

$$U = (\tilde{\tilde{u}}_1 \alpha_1 + \tilde{\tilde{u}}_2 \beta_1 + \tilde{\tilde{u}}_3 \gamma_1) e_r + (\tilde{\tilde{u}}_1 \alpha_2 + \tilde{\tilde{u}}_2 \beta_2 + \tilde{\tilde{u}}_3 \gamma_2) e_{r-1} + (\tilde{\tilde{u}}_1 \alpha_3 + \tilde{\tilde{u}}_2 \beta_3 + \tilde{\tilde{u}}_3 \gamma_3) e_{r-2}. \tag{2.7.22}$$

Noting (2.7.8)–(2.7.9), (2.7.17) holds. Thus, the state $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2, \tilde{\tilde{u}}_3)^T$ of synchronization by 3-groups can be determined by solving the following linear system:

$$\begin{cases}\phi_r = \tilde{\tilde{u}}_1\alpha_1 + \tilde{\tilde{u}}_2\beta_1 + \tilde{\tilde{u}}_3\gamma_1,\\ \phi_{r-1} = \tilde{\tilde{u}}_1\alpha_2 + \tilde{\tilde{u}}_2\beta_2 + \tilde{\tilde{u}}_3\gamma_2,\\ \phi_{r-2} = \tilde{\tilde{u}}_1\alpha_3 + \tilde{\tilde{u}}_2\beta_3 + \tilde{\tilde{u}}_3\gamma_3.\end{cases} \tag{2.7.23}$$

If $r = 3$, then $\mathrm{Ker}(C_3) = \mathrm{Span}\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$, and $\{\xi_1, \xi_2, \xi_3\}$ is an invariant subspace of A^T, which is bi-orthonormal to $\mathrm{Ker}(C_3)$. By Theorem 2.6, similarly to (2.4.4), there exists a boundary control matrix $D \in \mathcal{D}_{N-3}$, such that $\xi_1^T D = \xi_2^T D = \xi_3^T D = 0$, then the state $(\tilde{\tilde{u}}_1, \tilde{\tilde{u}}_2, \tilde{\tilde{u}}_3)^T$ of synchronization by 3-groups is independent of the applied boundary controls.

The state of synchronization by p-groups can be discussed in a similar way.

Chapter 3

Exact boundary controllability and exact boundary synchronization for a coupled system of wave equations with coupled Robin boundary controls

3.1 Introduction

Synchronization is a widespread natural phenomenon. It was first observed by Huygens in 1665 ([7]). The theoretical research on synchronization from mathematical point of view dates back to N. Wiener in 1950s ([44]). Since 2012, Li and Rao started the research on synchronization for coupled systems governed by PDEs, meanwhile, synchronization can be realized in a finite time by means of proper boundary controls. Consequently, the study of synchronization becomes a part of research in control theory. Precisely speaking, Li and Rao considered the exact boundary synchronization for a coupled system of wave equations with Dirichlet boundary controls in any given space dimensions in the framework of weak solutions ([24], [25], [28]) and in one-space-dimensional case in the framework of classical solutions ([6], [34], [40]). Corresponding results have been expanded to the exact boundary synchronization by $p(\geq 1)$ groups ([27], [29]). Moreover, Li and Rao proposed the concept of approximate boundary null controllability and approximate boundary synchronization in [26] and [30] and further studied them.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0 (\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset)$, ∂_ν denotes the outward normal derivative on the boundary, the coupling matrix $A = (a_{ij})$ is of order N, the boundary control matrix D is a full column-rank matrix of order $N \times M$ $(M \leq N)$, both A and D having real constant elements, $U = (u^{(1)}, \cdots, u^{(N)})^T$ and $H = (h^{(1)}, \cdots, h^{(M)})^T$ denote the state variables and the boundary controls, respectively. The discussion on the control problem will become more flexible because of the introduction of the boundary control matrix D.

In this paper, we always assume that Ω satisfies the usual multiplier geometric condition ([36]). Without loss of generality, we assume that there exists an $x_0 \in \mathbb{R}^n$, such that by setting $m = x - x_0$, we have

$$(m, \nu) \leq 0, \quad \forall x \in \Gamma_0; \qquad (m, \nu) > 0, \quad \forall x \in \Gamma_1, \tag{3.1.1}$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^n$.

Inspired by the synchronization of the system with Dirichlet boundary controls, Li, Lu and Rao studied the following coupled system of wave equations with Neumann boundary controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ U = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu U = DH & \text{on} \quad (0, +\infty) \times \Gamma_1, \end{cases} \tag{3.1.2}$$

and the corresponding results on the exact boundary synchronization and the approximate boundary synchronization have been obtained ([18], [19], [31]). In particular, we have

Lemma 3.1. *Assume that $M = N$. Then there exists a $T > 0$, for any given initial data $(U_0, U_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$, where*

$$\mathcal{H}_0 = L^2(\Omega), \qquad \mathcal{H}_1 = H^1_{\Gamma_0}(\Omega), \tag{3.1.3}$$

in which $H^1_{\Gamma_0}(\Omega)$ is the subspace of $H^1(\Omega)$, composed of all the functions with the null trace on Γ_0, there exists a boundary control function $H \in L^2_{loc}(0, +\infty; L^2(\Gamma_1))^N$ with compact support in $[0, T]$, such that system (3.1.2) *is exactly boundary null controllable at the time T, namely, the corresponding solution $U = U(t, x)$ satisfies*

$$t \geq T: \quad U(t, x) \equiv 0, \quad x \in \Omega. \tag{3.1.4}$$

Remark 3.1. *By the method given in [31], boundary control function H can be chosen to continuously depend on the initial data:*

$$\|H\|_{(L^2(0,T,L^2(\Gamma_1)))^N} \leq c\|(U_0, U_1)\|_{(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N}, \tag{3.1.5}$$

here and hereafter, c is a positive constant independent of the initial data.

On the other hand, when there is a lack of boundary controls, we have

Lemma 3.2. *When $M < N$, no matter how large $T > 0$ is, system* (3.1.2) *is not exactly boundary controllable at the time T for any given initial data $(U_0, U_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$.*

With more complicated boundary conditions, the study of synchronization will be more difficult. In this paper, we consider the following coupled system of wave equations with coupled Robin boundary controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ U = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu U + BU = DH & \text{on} \quad (0, +\infty) \times \Gamma_1 \end{cases} \tag{3.1.6}$$

and corresponding initial condition

$$t = 0: \quad U = U_0, \quad U' = U_1 \qquad \text{in} \quad \Omega, \tag{3.1.7}$$

where $B = (b_{ij})$ is the boundary coupling matrix of order N with constant elements.

3.2 Regularity of solutions with Neumann boundary conditions

As a problem with Neumann boundary conditions, a problem with Robin boundary conditions no longer enjoys the hidden regularity as in the case with Dirichlet boundary conditions. In particular, in higher dimensional space, the solution to the problem with Robin boundary condition is not smooth enough for the proof of the non-exact boundary controllability of the system, which makes a huge trouble.

Consider the following second order hyperbolic problem on a bounded domain $\Omega \subset R^n$ $(n \geq 2)$ with boundary Γ:

$$\begin{cases} y_{tt} + A(x,\partial)y = f & \text{in} \quad (0,T)\times\Omega = Q, \\ \frac{\partial y}{\partial \nu_A}|_\Sigma = g & \text{on} \quad (0,T)\times\Gamma = \Sigma, \\ t=0: \quad y = y_0, \quad y_t = y_1 & \text{in} \quad \Omega, \end{cases} \tag{3.2.1}$$

where

$$A(x,\partial) = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j} + c_0(x), \tag{3.2.2}$$

in which $a_{ij}(x)$ with $a_{ij}(x) = a_{ji}(x)$, $b_j(x)$ and $c_0(x)$ are smooth real coefficients, and the principal part of $A(x,\partial)$ is supposed to be uniformly strong elliptic in Ω:

$$\sum_{i,j=1}^{n} a_{ij}(x)\eta_i\eta_j \geq c\sum_{j=1}^{n}\eta_j^2, \quad \forall\, x \in \Omega, \quad \forall\, \eta = (\eta_1,\cdots,\eta_n) \in \mathbb{R}^n, \tag{3.2.3}$$

where, $c > 0$ is a positive constant, $\frac{\partial y}{\partial \nu_A}$ is the outward normal derivative of A:

$$\frac{\partial y}{\partial \nu_A} = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x)\frac{\partial y}{\partial x_i}\nu_j, \tag{3.2.4}$$

$\nu = (\nu_1,\cdots,\nu_n)^T$ being the unit normal vector.

Define the operator $\mathcal{A}$ by

$$\mathcal{A} = A(x,\partial), \quad \mathcal{D}(\mathcal{A}) = \big\{y \in H^2(\Omega) : \frac{\partial y}{\partial \nu_A}|_\Gamma = 0\big\}. \tag{3.2.5}$$

In [13], Lasiecka and Triggiani got the optimal regularity of the solution to problem (3.2.1) by means of the theory of cosine operator. In particular, more regularity results can be obtained when the domain is a parallelepiped. For conciseness and clarity, we list only those results needed in this paper.

Let $\epsilon > 0$ be an arbitrarily small number. Here and hereafter, we always assume that α, β are given respectively as follows:

$$\begin{cases} \alpha = 3/5 - \epsilon,\ \beta = 3/5, & \text{for a general smooth bounded domain } \Omega, \textit{and} \\ & A(x,\partial) \text{is defined by (3.2.2)}, \\ \alpha = \beta = 3/4 - \epsilon, & \text{for a parallelepiped } \Omega,\ A(x,\partial) = -\Delta. \end{cases} \tag{3.2.6}$$

Lemma 3.3. *Assume that $y_0 \equiv y_1 \equiv 0$ and $f \equiv 0$. For any given $g \in L^2(0,T;L^2(\Gamma))$, problem* (3.2.1) *with inhomogeneous Neumann boundary condition admits a unique solution y such that*

$$(y, y') \in C^0([0,T]; H^\alpha(\Omega) \times H^{\alpha-1}(\Omega)), \tag{3.2.7}$$

$$y'' \in L^2(0,T;(D(\mathcal{A}^{1-\alpha/2}))') \tag{3.2.8}$$

and

$$y|_\Gamma \in H^{2\alpha-1}(\Sigma) = L^2(0,T;H^{2\alpha-1}(\Gamma)) \cap H^{2\alpha-1}(0,T;L^2(\Gamma)), \tag{3.2.9}$$

where $\Sigma = (0,T) \times \Gamma$, $(D(A^\gamma))'(\gamma > 0)$ is the dual space of $D(A^\gamma)$ with respect to $L^2(\Omega)$.

Lemma 3.4. *Assume that $y_0 \equiv y_1 \equiv 0$ and $g \equiv 0$. For any given $f \in L^2(0,T;L^2(\Omega))$, the solution y to the problem* (3.2.1) *with inhomogeneous term f satisfies*

$$(y, y') \in C^0([0,T]; H^1(\Omega) \times L^2(\Omega)) \tag{3.2.10}$$

and

$$y|_\Gamma \in H^\beta(\Sigma). \tag{3.2.11}$$

Lemma 3.5. *Assume that $f \equiv 0$ and $g \equiv 0$.*

(1) *If $(y_0, y_1) \in H^1(\Omega) \times L^2(\Omega)$, then problem* (3.2.1) *admits a unique solution y such that*

$$(y, y') \in C^0([0,T]; H^1(\Omega) \times L^2(\Omega)) \tag{3.2.12}$$

and

$$y|_\Gamma \in H^\beta(\Sigma). \tag{3.2.13}$$

(2) *If $(y_0, y_1) \in L^2(\Omega) \times (H^1(\Omega))'$, where $(H^1(\Omega))'$ is the dual space of $H^1(\Omega)$ with respect to $L^2(\Omega)$, then problem* (3.2.1) *admits a unique solution y such that*

$$(y, y') \in C^0([0,T]; L^2(\Omega) \times (H^1(\Omega))') \tag{3.2.14}$$

and

$$y|_\Gamma \in H^{\alpha-1}(\Sigma). \tag{3.2.15}$$

Remark 3.2. *In the results mentioned above, the mappings of solutions are continuous with respect to the corresponding topologies.*

3.3 Existence and uniqueness of solutions to the mixed initial-boundary value problem with coupled Robin boundary conditions

Lemma 3.6 (Lax-Milgram Theorem[14]). *Assume that $a(u,v)$ is a bounded coercive bilinear functional on the Hilbert space X, namely, there exists an $M>0$, such that*

$$|a(u,v)| \le M\|u\|_X\|v\|_X, \quad \forall u,v \in X, \tag{3.3.1}$$

and there exists a $\delta>0$, such that

$$a(u,u) \ge \delta\|u\|_X^2, \quad \forall u \in X. \tag{3.3.2}$$

Then for any given $f \in X'$, there exists a unique $u \in X$, satisfying

$$a(u,v) =< f, v >, \quad \forall v \in X \tag{3.3.3}$$

with

$$\|u\|_X \le \frac{1}{\delta}\|f\|_{X'}, \tag{3.3.4}$$

where X' is the dual space of X, and $< \cdot, \cdot >$ is the dual product of X and X'.

Lemma 3.7 ([39]). *Assume that Ω is a bounded domain in $\mathbb{R}^n$ with C^1 boundary Γ. Then, for any given function $u \in H^1(\Omega)$, we have the following interpolation inequality:*

$$\|u\|_{L^2(\Gamma)} \le c\|u\|_{H^1(\Omega)}^{\frac{1}{2}}\|u\|_{L^2(\Omega)}^{\frac{1}{2}}, \tag{3.3.5}$$

where c is a positive constant independent of u.

Lemma 3.8 ([39]). *Let X be a Hilbert space and let $\mathcal{A}$ be an unbounded linear operator, the domain $\mathcal{D}(\mathcal{A})$ of which is dense in X. If $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, $\rho(\mathcal{A})$ being the resolvent of $\mathcal{A}$, then $\mathcal{A}$ is the infinitesimal generator of a C_0 contractive semigroup on X.*

Lemma 3.9 ([41]). *Let X be a Hilbert space and let $\mathcal{A}$ be the infinitesimal generator of a C_0 semigroup $S(t)$ on X, satisfying $\|S(t)\| \le Me^{\omega t}$. If $\mathcal{B}$ is a bounded linear operator on X, then $\mathcal{A}+\mathcal{B}$ is the infinitesimal generator of a C_0 semigroup $T(t)$ on X, satisfying $\|T(t)\| \le Me^{(\omega+M\|B\|)t}$, $\forall\, t \ge 0$.*

Theorem 3.1. *Assume that B is similar to a real symmetric matrix. Then for any given $(\Phi_0, \Phi_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$, the adjoint problem*

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \Phi = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \Phi + B^T\Phi = 0 & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad \Phi = \Phi_0, \quad \Phi' = \Phi_1 & \text{in} \quad \Omega \end{cases} \tag{3.3.6}$$

admits a unique weak solution

$$(\Phi, \Phi') \in C^0_{loc}([0, +\infty); (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N) \tag{3.3.7}$$

in the sense of C_0 semigroup, where $\mathcal{H}_1$ and $\mathcal{H}_0$ are defined by (3.1.3).

Proof. Since B is similar to a real symmetric matrix, there exists an invertible matrix P such that $\hat{B} = P^{-1}BP$ is symmetric. Let $\hat{\Phi} = P^T\Phi$ and $\hat{A} = P^{-1}AP$. Problem (3.3.6) can be transformed to

$$\begin{cases} \hat{\Phi}'' - \Delta\hat{\Phi} + \hat{A}^T\hat{\Phi} = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \hat{\Phi} = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \hat{\Phi} + \hat{B}\hat{\Phi} = 0 & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \;\; \hat{\Phi} = P^T\Phi_0 \triangleq \hat{\Phi}_0, \quad \hat{\Phi}' = P^T\Phi_1 \triangleq \hat{\Phi}_1 & \text{in} \quad \Omega, \end{cases} \tag{3.3.8}$$

therefore, it is only necessary to prove that problem (3.3.8) is well-posed.

Let $x_0 \in \mathbb{R}^n$ and λ be a constant large enough. Define

$$m(x) = x - x_0, \quad h(x) = e^{\lambda |m(x)|^2/2}, \quad \hat{\Phi} = h\tilde{\Phi}.$$

Let

$$\tilde{B} = \hat{B} + \lambda(m, \nu)I \tag{3.3.9}$$

and

$$F(\tilde{\Phi}) = A^T\tilde{\Phi} - \lambda(N + \lambda|m|^2)\tilde{\Phi} - 2\lambda m \otimes \nabla\tilde{\Phi}, \tag{3.3.10}$$

where $\otimes$ denotes the tensor product. A direct calculation leads to

$$\begin{cases} \tilde{\Phi}'' - \Delta\tilde{\Phi} + F(\tilde{\Phi}) = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \tilde{\Phi} = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \tilde{\Phi} + \tilde{B}\tilde{\Phi} = 0 & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \tilde{\Phi} = \frac{1}{h}\hat{\Phi}_0, \;\; \tilde{\Phi}' = \frac{1}{h}\hat{\Phi}_1 & \text{in} \quad \Omega. \end{cases} \tag{3.3.11}$$

By the multiplier geometric condition (3.1.1), for any given $\theta > 0$, there exists a positive constant $\lambda > 0$ so large that

$$(\tilde{B}\xi, \xi) \geq \theta|\xi|^2, \quad \forall x \in \Gamma_1,\ \forall \xi \in \mathbb{R}^n, \tag{3.3.12}$$

then $\tilde{B}$ is a positive definite symmetric matrix.

Let

$$X = (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N. \tag{3.3.13}$$

For any given $(\Phi_i, \Psi_i)^T \in X (i = 1, 2)$, we define the inner product in X as

$$\langle \begin{pmatrix} \Phi_1 \\ \Psi_1 \end{pmatrix}, \begin{pmatrix} \Phi_2 \\ \Psi_2 \end{pmatrix} \rangle_X = \int_\Omega \nabla\Phi_1 \cdot \nabla\Phi_2 dx + \int_{\Gamma_1} (\tilde{B}\Phi_1, \Phi_2) d_\Gamma + \int_\Omega \Psi_1 \Psi_2 dx. \tag{3.3.14}$$

Define the operator $\mathcal{A}$ by

$$\mathcal{A} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \begin{pmatrix} \Psi \\ \Delta\Phi \end{pmatrix} \tag{3.3.15}$$

and

$$\mathcal{D}(\mathcal{A}) = (H^2(\Omega) \cap \mathcal{H}_1)^N \times (\mathcal{H}_1)^N \subseteq X. \tag{3.3.16}$$

Obviously, $\mathcal{D}(\mathcal{A})$ is dense in X. We now prove that $\mathcal{A}$ is the infinitesimal generator of a C_0 contractive semigroup on X.

Firstly, we prove that $\mathcal{A}$ is dissipative. In fact, it is easy to see from (3.3.14) that

$$\begin{aligned}
\langle \mathcal{A} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \rangle_X &= \langle \begin{pmatrix} \Psi \\ \Delta\Phi \end{pmatrix}, \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \rangle_X \\
&= \int_\Omega \nabla\Phi \cdot \nabla\Psi dx + \int_{\Gamma_1} (\tilde{B}\Psi, \Phi) d_\Gamma + \int_\Omega \Delta\Phi \cdot \Psi dx \\
&= \int_\Omega \nabla\Phi \cdot \nabla\Psi dx + \int_{\Gamma_1} (\tilde{B}\Psi, \Phi) d_\Gamma + \int_{\Gamma_1} \partial_\nu\Phi \cdot \Psi d_\Gamma - \int_\Omega \nabla\Phi \cdot \nabla\Psi dx \\
&= \int_{\Gamma_1} (\Psi, \tilde{B}\Phi) d_\Gamma - \int_{\Gamma_1} (\tilde{B}\Phi, \Psi) d_\Gamma = 0, \quad \forall (\Phi, \Psi) \in \mathcal{D}(\mathcal{A}).
\end{aligned} \tag{3.3.17}$$

Then, we prove that $0 \in \rho(\mathcal{A})$. For any given $(\Phi_i, \Psi_i) \in X$ $(i = 1, 2)$, we define the following bilinear functional:

$$\begin{aligned}
a(\begin{pmatrix} \Phi_1 \\ \Psi_1 \end{pmatrix}, \begin{pmatrix} \Phi_2 \\ \Psi_2 \end{pmatrix}) &= \langle \mathcal{A} \begin{pmatrix} \Phi_1 \\ \Psi_1 \end{pmatrix}, \begin{pmatrix} \Phi_2 \\ \Psi_2 \end{pmatrix} \rangle \\
&\stackrel{\text{def}}{=} \int_\Omega \Psi_1 \cdot \Psi_2 dx + \int_{\Gamma_1} (\tilde{B}\Phi_1, \Phi_2) d_\Gamma + \int_\Omega \nabla\Phi_1 \cdot \nabla\Phi_2 dx.
\end{aligned} \tag{3.3.18}$$

By Lemma 3.7, (3.3.1) holds. Obviously, for any given $(\Phi, \Psi) \in X$, we have

$$a(\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}) = \int_\Omega |\nabla \Phi|^2 dx + \int_{\Gamma_1} (\tilde{B}\Phi, \Phi) d_\Gamma + \int_\Omega |\Psi|^2 dx = \|(\Phi, \Psi)\|_X^2. \tag{3.3.19}$$

Hence, it follows from Lemma 3.6 that for any given $F \in X'$, there exists a unique $(\Phi_1, \Psi_1) \in X$, such that

$$a(\begin{pmatrix} \Phi_1 \\ \Psi_1 \end{pmatrix}, \begin{pmatrix} \Phi_2 \\ \Psi_2 \end{pmatrix}) = \langle \mathcal{A} \begin{pmatrix} \Phi_1 \\ \Psi_1 \end{pmatrix}, \begin{pmatrix} \Phi_2 \\ \Psi_2 \end{pmatrix} \rangle = \langle F, \begin{pmatrix} \Phi_2 \\ \Psi_2 \end{pmatrix} \rangle, \quad \forall \begin{pmatrix} \Phi_2 \\ \Psi_2 \end{pmatrix} \in X \tag{3.3.20}$$

and

$$\|(\Phi_1, \Psi_1)\|_X = \|\mathcal{A}^{-1} F\|_X \le \frac{1}{\delta} \|F\|_{X'}, \tag{3.3.21}$$

where X' is the dual space of X. It shows that $\mathcal{A}^{-1}$ is a bounded linear operator from X' to X, namely $0 \in \rho(\mathcal{A})$. Thus, by Lemma 3.8, $\mathcal{A}$ is the infinitesimal generator of a C_0 contractive semigroup on X.

Finally, problem (3.3.11) can be written as

$$\begin{pmatrix} \tilde{\Phi} \\ \tilde{\Phi}' \end{pmatrix}' = \mathcal{A} \begin{pmatrix} \tilde{\Phi} \\ \tilde{\Phi}' \end{pmatrix} + \mathcal{B} \begin{pmatrix} \tilde{\Phi} \\ \tilde{\Phi}' \end{pmatrix}, \tag{3.3.22}$$

in which

$$\mathcal{B} \begin{pmatrix} \tilde{\Phi} \\ \tilde{\Phi}' \end{pmatrix} = \begin{pmatrix} 0 \\ F(\tilde{\Phi}) \end{pmatrix}. \tag{3.3.23}$$

Noting that $F(\tilde{\Phi})$ is a bounded linear mapping in X, $\mathcal{B}$ is a bounded linear operator in X, by Lemma 3.9, $\mathcal{A} + \mathcal{B}$ is the infinitesimal generator of a C_0 semigroup on X. The proof is complete. □

Remark 3.3. *From now on, in order to guarantee the well-posedness of the problem with coupled Robin boundary condition, we always assume that B is similar to a real symmetric matrix.*

Definition 3.1. *U is a weak solution to the mixed problem* (3.1.6)–(3.1.7)*, if*

$$(U, U') \in C^0([0,T]; (\mathcal{H}_0)^N \times (\mathcal{H}_1')^N), \tag{3.3.24}$$

such that for any given $(\Phi_0, \Phi_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ and t ($0 \le t \le T$), we have the following equality:

$$\begin{aligned} &\langle (U'(t), -U(t)), (\Phi(t), \Phi'(t)) \rangle_{(\mathcal{H}_1')^N \times (\mathcal{H}_0)^N; (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N} \\ &\quad = \langle (U_1, -U_0), (\Phi_0, \Phi_1) \rangle_{(\mathcal{H}_1')^N \times (\mathcal{H}_0)^N; (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N} + \int_0^t \int_{\Gamma_1} (DH(\tau), \Phi(\tau)) dx dt, \end{aligned} \tag{3.3.25}$$

in which $\Phi(t)$ is the solution to the adjoint problem (3.3.6).

Theorem 3.2. *For any given* $H \in (L^2(0,T;L^2(\Gamma_1)))^M$ *and* $(U_0,U_1) \in (\mathcal{H}_0)^N \times (\mathcal{H}_1')^N$, *problem* (3.1.6)–(3.1.7) *admits a unique weak solution*

$$(U,U') \in C^0([0,T];(H_0)^N \times (\mathcal{H}_1')^N), \tag{3.3.26}$$

moreover, the mapping

$$(U_0,U_1,H) \to (U,U')$$

is continuous with respect to the corresponding topology.

Proof. Taking the inner product with Φ on both sides of (3.1.6) and integrating by parts, we have

$$\begin{aligned}\int_\Omega (U'(t),\Phi(t))dx - \int_\Omega (U(t),\Phi'(t))dx \\ = \int_\Omega (U_1,\Phi_0)dx - \int_\Omega (U_0,\Phi_1)dx + \int_0^t \int_{\Gamma_1} (DH(\tau),\Phi(\tau))dxdt,\end{aligned} \tag{3.3.27}$$

which can be written as

$$\begin{aligned}&\langle (U'(t),-U(t)),(\Phi(t),\Phi'(t))\rangle_{(\mathcal{H}_1')^N\times(\mathcal{H}_0)^N;(\mathcal{H}_1)^N\times(\mathcal{H}_0)^N} \\ &= \langle (U_1,-U_0),(\Phi_0,\Phi_1)\rangle_{(\mathcal{H}_1')^N\times(\mathcal{H}_0)^N;(\mathcal{H}_1)^N\times(\mathcal{H}_0)^N} + \int_0^t \int_{\Gamma_1} (DH(\tau),\Phi(\tau))dxdt.\end{aligned} \tag{3.3.28}$$

Define a linear form as follows:

$$L_t(\Phi_0,\Phi_1) = \langle (U_1,-U_0),(\Phi_0,\Phi_1)\rangle_{(\mathcal{H}_1')^N\times(\mathcal{H}_0)^N;(\mathcal{H}_1)^N\times(\mathcal{H}_0)^N} + \int_0^t \int_{\Gamma_1} (DH(\tau),\Phi(\tau))dxdt. \tag{3.3.29}$$

It is easy to show that for any given t $(0 \le t \le T)$, L_t is bounded in $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$. Let S_t be the semigroup in $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$, corresponding to the adjoint problem (3.3.6). Then $L_t \circ S_t^{-1}$ is bounded in $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$. By Riesz-Frêchet representation theorem, for any given $(\Phi_0,\Phi_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$, there exists a unique $(U'(t),-U(t)) \in (\mathcal{H}_1')^N \times (\mathcal{H}_0)^N$, such that

$$L_t \circ S_t^{-1}(\Phi(t),\Phi'(t)) = \langle (U'(t),-U(t)),(\Phi(t),\Phi'(t))\rangle_{(\mathcal{H}_1')^N\times(\mathcal{H}_0)^N;(\mathcal{H}_1)^N\times(\mathcal{H}_0)^N}. \tag{3.3.30}$$

By

$$L_t \circ S_t^{-1}(\Phi(t),\Phi'(t)) = L_t(\Phi_0,\Phi_1), \tag{3.3.31}$$

for any given $(\Phi_0,\Phi_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$, (3.3.25) holds, and we have

$$\begin{aligned}&\|(U'(t),-U(t))\|_{(\mathcal{H}_1')^N\times(\mathcal{H}_0)^N} = \|L_t \circ S_t^{-1}\| \\ &\le c(\|(U_1,U_0)\|_{(\mathcal{H}_1')^N\times(\mathcal{H}_0)^N} + \|H\|_{(L^2(0,T;L^2(\Gamma_1)))^M}), \qquad \forall\, t \in [0,T],\end{aligned} \tag{3.3.32}$$

where $\|L_t \circ S_t^{-1}\|$ is the operator norm of $L_t \circ S_t^{-1}$.

At last, by the classical dense approximation method, we obtain the regularity desired by(3.3.26). □

3.4 Regularity of solutions to the mixed initial-boundary value problem with coupled Robin boundary conditions

Theorem 3.3. *For any given $H \in (L^2(0,T;L^2(\Gamma_1)))^M$ and any given $(U_0, U_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$, the weak solution U to problem* (3.1.6)–(3.1.7) *satisfies*

$$(U, U') \in C^0([0,T];(H^\alpha(\Omega))^N \times (H^{\alpha-1}(\Omega))^N) \tag{3.4.1}$$

and

$$U|_{\Gamma_1} \in (H^{2\alpha-1}(\Sigma_1))^N, \tag{3.4.2}$$

where $\Sigma_1 = (0,T) \times \Gamma_1$, α is defined by (3.2.6)*. Moreover, the mapping*

$$(U_0, U_1, H) \to (U, U')$$

is continuous with respect to the corresponding topology.

Proof. Noting (3.2.12) and (3.2.13) in Lemme 3.5, we need only to consider the case when $U_0 \equiv U_1 \equiv 0$. Assume that Ω is sufficiently smooth, for example, with C^3 boundary, then by the inverse trace theory ([4]), there exists a function $h \in C^2(\overline{\Omega})$ such that

$$\nabla h = \nu \quad \text{on} \quad \Gamma_1, \tag{3.4.3}$$

where ν is the unit outward normal vector on the boundary Γ_1. Let λ be a eigenvalue of B^T corresponding to eigenvector e:

$$B^T e = \lambda e.$$

Defining

$$\phi = (e, U), \tag{3.4.4}$$

we have

$$\begin{cases} \phi'' - \Delta\phi = -(e, AU) & \text{in} \quad (0,T) \times \Omega, \\ \phi = 0 & \text{on} \quad (0,T) \times \Gamma_0, \\ \partial_\nu \phi + \lambda\phi = (e, DH) & \text{on} \quad (0,T) \times \Gamma_1, \\ t = 0: \ \phi = 0, \ \phi' = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.4.5}$$

Let

$$\psi = e^{\lambda h}\phi. \tag{3.4.6}$$

Problem (3.1.6) can be rewritten to the following problem with Neumann boundary conditions:

$$\begin{cases} \psi'' - \Delta\psi + 2\lambda\nabla h\cdot\nabla\psi + \lambda(\Delta h - \lambda|\nabla h|^2)\psi = -e^{\lambda h}(e, AU) & \text{in} \quad (0,T)\times\Omega, \\ \psi = 0 & \text{on} \quad (0,T)\times\Gamma_0, \\ \partial_\nu\psi = e^{\lambda h}(e, DH) & \text{on} \quad (0,T)\times\Gamma_1, \\ t=0: \quad \psi = 0, \quad \psi' = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.4.7}$$

By Theorem 3.2, $U \in C^0([0,T]; \mathcal{H}_0^N)$. By (3.2.10) in Lemma 3.4, the solution to the following problem with homogeneous Neumann boundary conditions:

$$\begin{cases} \psi'' - \Delta\psi + 2\lambda\nabla h\cdot\nabla\psi + \lambda(\Delta h - \lambda|\nabla h|^2)\psi = -e^{\lambda h}(e, AU) & \text{in} \quad (0,T)\times\Omega, \\ \psi = 0 & \text{on} \quad (0,T)\times\Gamma_0, \\ \partial_\nu\psi = 0 & \text{on} \quad (0,T)\times\Gamma_1, \\ t=0: \quad \phi = 0, \quad \phi' = 0 & \text{in} \quad \Omega \end{cases} \tag{3.4.8}$$

satisfies

$$(\psi, \psi') \in C^0([0,T]; H^1(\Omega)\times L^2(\Omega)) \subset C^0([0,T]; H^\alpha(\Omega)\times H^{\alpha-1}(\Omega)). \tag{3.4.9}$$

Next, we consider the following problem with inhomogeneous Neumann boundary conditions but without internal force terms:

$$\begin{cases} \psi'' - \Delta\psi + 2\lambda\nabla h\cdot\nabla\psi + \lambda(\Delta h - \lambda|\nabla h|^2)\psi = 0 & \text{in} \quad (0,T)\times\Omega, \\ \psi = 0 & \text{on} \quad (0,T)\times\Gamma_0, \\ \partial_\nu\psi = e^{\lambda h}(e, DH) & \text{on} \quad (0,T)\times\Gamma_1, \\ t=0: \quad \phi = 0, \quad \phi' = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.4.10}$$

By (3.2.7) and (3.2.9) in Lemma 3.3, we have

$$(\psi, \psi') \in C^0([0,T]; H^\alpha(\Omega)\times H^{\alpha-1}(\Omega)) \tag{3.4.11}$$

and

$$\psi|_{\Gamma_1} \in H^{2\alpha-1}(\Sigma_1) = H^{2\alpha-1}(0,T; L^2(\Gamma_1)) \cap L^2(0,T; H^{2\alpha-1}(\Gamma_1)). \tag{3.4.12}$$

Since this regularity result holds for all the eigenvectors of B^T, and all the eigenvectors of B^T constitute a set of basis in $\mathbb{R}^N$, we get the desired (3.4.1) and (3.4.2). □

Remark 3.4. *By Lemma 3.4, it is easy to check that the solution U to the following problem with any given* *internal force term* $F \in L^2(0,T;L^2(\Omega))$*:*

$$\begin{cases} U'' - \Delta U + AU = F & in \quad (0,+\infty)\times\Omega, \\ U = 0 & on \quad (0,+\infty)\times\Gamma_0, \\ \partial_\nu U + BU = DH & on \quad (0,+\infty)\times\Gamma_1, \\ t=0: \quad U = U_0, \quad U' = U_1 & in \quad \Omega \end{cases} \tag{3.4.13}$$

satisfies (3.4.1)–(3.4.2)*, too.*

3.5 Exact boundary controllability and non-exact boundary controllability

In this section, we study the exact boundary controllability and non-exact boundary controllability for the coupled system (3.1.6) of wave equations with coupled Robin boundary controls. We will prove that when $M = N$, namely, when D is invertible, system (3.1.6) is exactly boundary controllable for any given initial data $(U_0, U_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$. However, when $M < N$ and Ω is a parallelepiped, system (3.1.6) is not exactly boundary controllable in $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$.

Theorem 3.4. *Assume that* $M = N$*. Then there exists a* $T > 0$*, such that for any given initial data* $(U_0, U_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$*, there exists a boundary control function* $H \in (L^2_{loc}(0,+\infty;L^2(\Gamma_1)))^N$ *with compact support in* $[0,T]$*, such that system* (3.1.6) *is exactly boundary controllable at the time* T*, and the control function continuously depends on the initial data:*

$$\|H\|_{(L^2(0,T;L^2(\Gamma_1)))^N} \leq c\|(U_0,U_1)\|_{(\mathcal{H}_1)^N\times(\mathcal{H}_0)^N}, \tag{3.5.1}$$

where, $c > 0$ *is a positive constant.*

Proof. By Lemma 3.1 and Remark 3.1, for any given initial data $(U_0, U_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$, there exists a boundary control function $H \in (L^2_{loc}(0,+\infty;L^2(\Gamma_1)))^N$ with compact support in $[0,T]$, such that problem (3.1.2) with Neumann boundary controls is exactly boundary controllable at the time T, and the control function H continuously depends on the initial data:

$$\|H\|_{(L^2(0,T;L^2(\Gamma_1)))^N} \leq c\|(U_0,U_1)\|_{(\mathcal{H}_1)^N\times(\mathcal{H}_0)^N}. \tag{3.5.2}$$

Noting that $M = N$, D is invertible, then the boundary condition in (3.1.2)

$$\partial_\nu U = DH \tag{3.5.3}$$

can be rewritten as

$$\partial_\nu U + BU = DH + BU = D(H + D^{-1}BU) \stackrel{\text{def.}}{=} D\widehat{H}. \tag{3.5.4}$$

By the regularity result in Theorem 3.3 (in which we take B=0), the trace $U|_{\Gamma_1} \in (H^{2\alpha-1}(\Sigma_1))^N$, where α is defined by (3.2.6), and we have

$$\|U\|_{(L^2(\Sigma_1))^N} \leq c\big(\|(U_0, U_1)\|_{(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N} + \|H\|_{(L^2(0,T;L^2(\Gamma_1)))^N}\big). \tag{3.5.5}$$

Noting that $2\alpha - 1 > 0$, the new boundary control function $\widehat{H} \in (L^2(0,T;L^2(\Gamma_1)))^N$, and by (3.5.2) and (3.5.5), (3.5.1) holds. By the well-posedness theorem given in Theorem 3.2, it is easy to see that system (3.1.6) is exactly boundary controllable by the boundary control function $\widehat{H}$. Thus, we get the exact boundary controllability for the problem with coupled Robin boundary controls from the exact boundary controllability for the problem with Neumann boundary controls. The proof is complete. □

Remark 3.5. *Differently from the case with Neumann boundary controls, when there is a lack of boundary controls, the non-exact boundary controllability for the coupled system with coupled Robin boundary controls in a general domain is still an open problem. Fortunately, by Theorem 3.3, for some special domains, the solution may possess better regularity. In particular, when Ω is a parallelepiped, the optimal regularity of trace $U|_{\Gamma_1}$ almost reaches $(H^{\frac{1}{2}}(\Sigma_1))^N$. This benefits a lot in the proof of the non-exact boundary controllability for the system with fewer boundary controls.*

When Ω is a parallelepiped with $\Gamma_0 = \{\emptyset\}$, the boundary Γ of Ω is piecewise smooth. By Fourier analysis, the solution to the dual problem of problem (3.1.2) is still sufficiently smooth. Moreover, Lemma 3.1 and Lemma 3.2 still hold, provided that the initial space $(\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ satisfies the additional condition:

$$\int_\Omega \Phi dx = \int_\Omega \Psi dx = 0, \ \forall(\Phi, \Psi) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N.$$

Theorem 3.5. *Let $\mathcal{L}$ be a compact linear mapping from $L^2(\Omega)$ to $L^2(0,T;L^2(\Omega))$, and let $\mathcal{R}$ be a compact linear mapping from $L^2(\Omega)$ to $L^2(0,T;H^{1-\alpha}(\Gamma_1))$, where α is defined by (3.2.6). Then we can not find a $T > 0$, such that for any given $\theta \in L^2(\Omega)$, the solution to the following mixed problem:*

$$\begin{cases} w'' - \Delta w = \mathcal{L}\theta & \textit{in} \quad (0,T) \times \Omega, \\ w = 0 & \textit{on} \quad (0,T) \times \Gamma_0, \\ \partial_\nu w = \mathcal{R}\theta & \textit{on} \quad (0,T) \times \Gamma_1, \\ t = 0: \quad w = 0, \quad w' = \theta & \textit{in} \quad \Omega \end{cases} \tag{3.5.6}$$

satisfies the final condition

$$w(T) = w'(T) = 0. \tag{3.5.7}$$

Proof. Let ϕ be the solution to the following problem:

$$\begin{cases} \phi'' - \Delta\phi = 0 & \text{in} \quad (0,T)\times\Omega, \\ \phi = 0 & \text{on} \quad (0,T)\times\Gamma_0, \\ \partial_\nu\phi = 0 & \text{on} \quad (0,T)\times\Gamma_1, \\ t=0: \quad \phi=\theta, \quad \phi'=0 & \text{in} \quad \Omega. \end{cases} \tag{3.5.8}$$

By (3.2.14) and (3.2.15) in Lemma 3.5, we have

$$\|\phi\|_{L^2(0,T;L^2(\Omega))} \le c\|\theta\|_{L^2(\Omega)} \tag{3.5.9}$$

and

$$\|\phi\|_{L^2(0,T;H^{\alpha-1}(\Gamma_1))} \le c\|\theta\|_{L^2(\Omega)}. \tag{3.5.10}$$

By (3.5.7), taking the inner product with ϕ on both sides of (3.5.8) and integrating by parts, we get

$$\|\theta\|^2_{L^2(\Omega)} = \int_0^T \int_\Omega \mathcal{L}\theta\phi dx + \int_0^T \int_{\Gamma_1} \mathcal{R}\theta\phi d\Gamma. \tag{3.5.11}$$

Noting (3.5.9)–(3.5.10), we then have

$$\|\theta\|_{L^2(\Omega)} \le c(\|\mathcal{L}\theta\|_{L^2(0,T;L^2(\Omega))} + \|\mathcal{R}\theta\|_{L^2(0,T;H^{1-\alpha}(\Gamma_1))}), \qquad \forall\theta\in L^2(\Omega), \tag{3.5.12}$$

which contradicts the compactness of $\mathcal{L}$ and $\mathcal{R}$. The proof is complete. □

Theorem 3.6. *Assume that rank*$(D) = M < N$, *and* $\Omega\subset\mathbb{R}^n$ *is a parallelepiped with* $\Gamma_0 = \{\emptyset\}$. *Then, no matter how large* $T>0$ *is, system* (3.1.6) *is not exactly boundary null controllable for any given initial data* $(U_0, U_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$.

Proof. Since $M < N$, there exists an $e\in\mathbb{R}^N$, such that $D^Te = 0$. Take the special initial data

$$t=0: \quad U=0, \quad U'=e\theta \tag{3.5.13}$$

for system (3.1.6). If the system is exactly boundary controllable for any given $\theta\in L^2(\Omega)$, then there exists a boundary control function $H\in(L^2_{loc}(0,+\infty, L^2(\Gamma_1)))^M$ with compact support in $[0,T]$, such that the corresponding solution satisfies

$$U(T) = U'(T) = 0. \tag{3.5.14}$$

Let

$$w = (e,U), \qquad \mathcal{L}\theta = -(e, AU), \qquad \mathcal{R}\theta = -(e,BU)|_{\Sigma_1}. \tag{3.5.15}$$

Thus w satisfies system (3.5.6) and the final condition (3.5.7).

We then prove that $\mathcal{L}$ and $\mathcal{R}$ are compact mappings, which contradicts Theorem 3.5. In fact, by Theorem 3.4, $\theta \to H$ is a continuous mapping from $L^2(\Omega)$ to $(L^2(0,T;L^2(\Gamma_1)))^M$. By Theorem 3.3, $(\theta, H) \to U$ is a continuous mapping from $L^2(\Omega) \times (L^2(0,T;L^2(\Gamma_1)))^M$ to $(C^0(0,T;H^\alpha(\Omega)) \cap C^1(0,T;H^{\alpha-1}(\Omega)))^N$. Besides, by Lions's compact embedding theorem ([38]), the mapping from $\{\psi \in (L^2(0,T;H^\alpha(\Omega)))^N,\ \partial_t\psi \in (L^2(0,T;H^{\alpha-1}(\Omega)))^N\}$ to $(L^2(0,T;L^2(\Omega)))^N$ is compact, hence $\mathcal{L}$ is a compact mapping from $L^2(\Omega)$ to $L^2(0,T;L^2(\Omega))$. By (3.4.2), $H \to U|_{\Sigma_1}$ is a continuous mapping from $(L^2(0,T;L^2(\Gamma_1)))^M$ to $(H^{2\alpha-1}(\Sigma_1))^N$, then, $\mathcal{R}:\ \theta \to -(e, BU)|_{\Sigma_1}$ is a continuous mapping from $L^2(\Omega)$ to $H^{2\alpha-1}(\Sigma_1)$. When Ω is a parallelepiped, $\alpha = 3/4 - \epsilon$, then $2\alpha - 1 > 1 - \alpha$, and $H^{2\alpha-1}(\Sigma_1) \hookrightarrow H^{1-\alpha}(\Sigma_1)$ is a compact embedding, therefore, $\mathcal{R}$ is a compact mapping from $L^2(\Omega)$ to $H^{1-\alpha}(\Sigma_1)$. The proof is complete. □

Remark 3.6. *We obtain the non-exact boundary controllability for system* (3.1.6) *with coupled Robin boundary controls on a parallelepiped* Ω *in the case of lack of boundary controls. The main idea is to use the compact perturbation theory which has a higher requirement on the regularity of the solution to the problem with coupled Robin boundary condition. How to generalize this result to the general domain is still an open problem.*

3.6 Exact boundary synchronization

Based on the results on the exact boundary controllability and the non-exact boundary controllability, we continue to study the exact boundary synchronization for system (3.1.6) under coupled Robin boundary controls.

Let

$$C_1 = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}_{(N-1)\times N} \tag{3.6.1}$$

be the corresponding full row-rank matrix of synchronization. We have

$$\mathrm{Ker}(C_1) = \mathrm{Span}\{e\}, \tag{3.6.2}$$

where $e = (1, 1, \cdots, 1)^T$. By the definition of synchronization (see [18] and [25]), the system is exactly boundary synchronizable at the time $T > 0$, which means that for any given initial data $(U_0, U_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$, there exists a boundary control function $H \in (L^2_{loc}(0, +\infty; L^2(\Gamma_1)))^N$ with compact support in $[0, T]$, such that the corresponding solution $U = U(t, x)$ to the mixed problem (3.1.6)–(3.1.7) satisfies

$$t \geq T: \quad C_1 U \equiv 0 \quad \text{in} \quad \Omega. \tag{3.6.3}$$

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^n$ be a parallelepiped. Assume that the coupled system* (3.1.6) *of wave equations with coupled Robin boundary controls is exactly boundary synchronizable. Then we have*

$$rank(C_1 D) = N - 1. \tag{3.6.4}$$

Proof. If $Ker(D^T) \cap Im(C_1^T) = \{0\}$, by Lemma 4.10 in Appendix, we get

$$\text{rank}(C_1 D) = \text{rank}(D^T C_1^T) = \text{rank}(C_1^T) = N - 1. \tag{3.6.5}$$

Otherwise, if $Ker(D^T) \cap Im(C_1^T) \neq \{0\}$, then there exists a vector $E \neq 0$, such that

$$D^T C_1^T E = 0. \tag{3.6.6}$$

Let

$$w = (E, C_1 U), \qquad \mathcal{L}\theta = -(E, C_1 AU), \qquad \mathcal{R}\theta = -(E, C_1 BU). \tag{3.6.7}$$

For any given initial data $(U, U') = (0, \theta e)$, where $\theta \in L^2(\Omega)$, we can still get system (3.5.6) of w. By the exact boundary synchronization of system (3.1.6), for this initial data, there exists a boundary control function $H \in (L^2(0, T, L^2(\Gamma_1)))^M$, such that the corresponding solution w satisfies (3.5.14), then satisfies (3.5.7). Similarly to the proof in Theorem 3.6, when Ω is a parallelepiped, we can prove that $\mathcal{L}$ is a compact linear mapping from $L^2(\Omega)$ to $L^2(0, T; L^2(\Omega))$, and $\mathcal{R}$ is a compact linear mapping from $L^2(\Omega)$ to $L^2(0, T; H^{1-\alpha}(\Gamma_1))$. It contradicts Theorem 3.5. The proof is complete. $\square$

Remark 3.7. *If $rank(D) = N$, by Theorem 3.4, system* (3.1.6) *is exactly boundary controllable, then exactly boundary synchronizable. In order to exclude this trivial situation, in what follows we always assume that $rank(D) = N - 1$.*

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^n$ be a parallelepiped. Assume that $rank(D) = N - 1$. If the coupled system* (3.1.6) *of wave equations with coupled Robin boundary controls is exactly boundary synchronizable, then we have the following C_1-conditions of compatibility:*

$$AKer(C_1) \subseteq Ker(C_1), \qquad BKer(C_1) \subseteq Ker(C_1). \tag{3.6.8}$$

Proof. Let $e = (1, \cdots, 1)^T$. Then as $t \geq T$, we have

$$U = ue, \tag{3.6.9}$$

where u is the corresponding state of synchronization.

Taking the inner product with C_1 on both sides of (3.1.6), we get

$$t \geq T: \qquad C_1 Aeu = 0 \quad \text{in} \ \ \Omega, \qquad C_1 Beu|_{\Gamma_1} = 0 \quad \text{on} \ \ \Gamma_1. \tag{3.6.10}$$

By Theorem 3.6, system (3.1.6) is not exactly boundary controllable when $\text{rank}(D) = N-1$, hence there exists at least an initial data (U_0, U_1) such that

$$t \geq T: \qquad u \not\equiv 0 \qquad \text{in} \quad \Omega. \tag{3.6.11}$$

Therefore, we get $C_1 A e = 0$. Noting that $\text{Ker}(C_1) = \text{Span}\{e\}$, we have

$$A\text{Ker}(C_1) \subseteq \text{Ker}(C_1). \tag{3.6.12}$$

We next prove $C_1 B e = 0$. Otherwise, by the second formula in (3.6.10) we get $u|_{\Gamma_1} \equiv 0$. By (3.6.9), as $t \geq T$, (3.1.6) becomes

$$t \geq T: \qquad \begin{cases} u'' - \Delta u + au = 0 & \text{in} \quad (0,T) \times \Omega, \\ u = 0 & \text{on} \quad (0,T) \times \Gamma_0, \\ \partial_\nu u = u = 0 & \text{on} \quad (0,T) \times \Gamma_1, \end{cases} \tag{3.6.13}$$

where a is defined by (4.1.19) in Lemma 4.7 in Appendix. By Holmgren's uniqueness theorem ([37]), we have $u \equiv 0$ as $t \geq T$. It contradicts the non-exact boundary controllability of system (3.1.6). The proof is complete. □

Theorem 3.9. *Assume that rank*$(D) = N-1$. *Assume that both A and B satisfy the C_1-conditions of compatibility* (3.6.8). *Then there exists a boundary control matrix D with rank*$(C_1 D) = N-1$, *such that system* (3.1.6) *is exact boundary synchronizable, and the boundary control function H possesses the following continuous dependence:*

$$\|H\|_{(L^2(0,T,L^2(\Gamma_1)))^{N-1}} \leq c\|C_1(U_0, U_1)\|_{(\mathcal{H}_1)^{N-1} \times (\mathcal{H}_0)^{N-1}}. \tag{3.6.14}$$

Proof. Since both A and B satisfy the C_1-conditions of compatibility (3.6.8), by Lemma 4.7 in Appendix, there exist $(N-1)$ matrices $\overline{A}_1$ and $\overline{B}_1$ such that

$$C_1 A = \overline{A}_1 C_1, \qquad C_1 B = \overline{B}_1 C_1. \tag{3.6.15}$$

Let

$$W = C_1 U, \qquad \overline{D}_1 = C_1 D. \tag{3.6.16}$$

W satisfies

$$\begin{cases} W'' - \Delta W + \overline{A}_1 W = 0 & \text{in} \quad (0,T) \times \Omega, \\ W = 0 & \text{on} \quad (0,T) \times \Gamma_0, \\ \partial_\nu W + \overline{B}_1 W = \overline{D}_1 H & \text{on} \quad (0,T) \times \Gamma_1, \\ t = 0: \quad W = C_1 U_0, \quad W' = C_1 U_1 & \text{in} \quad \Omega. \end{cases} \tag{3.6.17}$$

Noting that C_1 is a surjection from $\mathbb{R}^N$ to $\mathbb{R}^{N-1}$, any given initial data $(U_0, U_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ corresponds to a unique initial data (C_1U_0, C_1U_1) to the reduced system (3.6.17). Then, it follows that the exact boundary synchronization for system (3.1.6) is equivalent to the exact boundary controllability for the reduced system (3.6.17), and then the boundary control H, which realizes the exact boundary controllability for the reduced system (3.6.17), is just the boundary control which realizes the exact boundary synchronization for system (3.1.6).

By Lemma 4.9 in Appendix, when B is similar to a symmetric matrix, its reduced matrix $\overline{B}_1$ is also similar to a symmetric matrix, which guarantees the well-posedness of the reduced system (3.6.17).

Defining the boundary control matrix D as follows:

$$D = C_1^T.$$

$$\overline{D}_1 = C_1 D = C_1 C_1^T \tag{3.6.18}$$

is an invertible matrix of order $(N-1)$, by Theorem 3.4, the reduced system (3.6.17) is exactly boundary controllable, hence system (3.1.6) is exactly boundary synchronizable. Moreover, by (3.5.1), we get (3.6.14). □

3.7 Determination of the state of synchronization

When there is a lack of boundary controls, although under certain conditions, the system can realize exact boundary synchronization by fewer boundary controls, the state of synchronization is a priori unknown, which depends not only on the given initial data, but also on the applied boundary controls. In this section, we will discuss the determination and estimate of the state of synchronization.

Theorem 3.10. *Assume that both A and B satisfy the C_1-conditions of compatibility* (3.6.8)*, and A^T and B^T have a common eigenvector $E \in \mathbb{R}^N$, such that $(E, e) = 1$, where $e = (1, \cdots, 1)^T$. Then there exists a boundary control matrix D with rank$(D) = N-1$, such that system* (3.1.6) *is exactly boundary synchronizable, and the state of synchronization is independent of the applied boundary control function.*

Proof. Taking the boundary control matrix D such that

$$\mathrm{Ker}(D^T) = Span\{E\}. \tag{3.7.1}$$

It is evident that $\mathrm{rank}(D) = N - 1$. Noticing $(E, e) = 1$, we have

$$\mathrm{Ker}(C_1) \cap \mathrm{Im}(D) = \mathrm{Span}(e) \cap \{\mathrm{Span}(E)\}^{\perp} = \{0\}. \tag{3.7.2}$$

Thus, by Lemma 4.10 in Appendix, we have

$$\text{rank}(C_1D) = \text{rank}(D) = N - 1. \tag{3.7.3}$$

Then, by Theorem 3.9, system (3.1.6) is exactly boundary synchronizable.

Next, we prove that the state of synchronization is independent of the boundary control function which realizes the synchronization. Noting that E is a common eigenvector of A^T and B^T, and B is similar to a real symmetric matrix, there exist $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{R}$, such that

$$A^T E = \lambda E, \qquad B^T E = \mu E. \tag{3.7.4}$$

Noticing (3.7.1), we have $D^T E = 0$, then $\phi = (E, U)$ satisfies

$$\begin{cases} \phi'' - \Delta\phi + \lambda\phi = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \phi = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \phi + \mu\phi = 0 & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad \phi = (E, U_0), \quad \phi' = (E, U_1) & \text{in} \quad \Omega. \end{cases} \tag{3.7.5}$$

Obviously, the solution ϕ to this problem is independent of the boundary control H.

On the other hand, noting

$$t \geq T: \qquad \phi = (E, U) = (E, e)u = u, \tag{3.7.6}$$

the state of synchronization is determined by the solution to problem (3.7.5), and is independent of the boundary control H. The proof is complete. □

Remark 3.8. *By [29], a matrix D satisfying rank$(C_1D) =$ rank$(D) = N-1$ and $D^T E = 0$ has the following form:*

$$D = (I - eE^T)C_1^T \bar{D},$$

where $\bar{D}$ is an $(N-1)$ invertible matrix.

Theorem 3.11. *Assume that both A and B satisfy the C_1-conditions of compatibility* (3.6.8). *Assume furthermore that system* (3.1.6) *is exactly boundary synchronizable. Let $E \neq 0$ be a vector in $\mathbb{R}^N$. If the projection $\phi = (E, U)$ is independent of boundary control H in $(0, T) \times \Omega$, then E must be a common eigenvector of A^T and B^T, $E \in Ker(D^T)$ and, without loss of generality, we may assume that $(E, e) = 1$.*

Proof. Taking $(U_0, U_1) = (0, 0)$ in problem (3.1.6)–(3.1.7), by Theorem 3.3, the mapping

$$F: \quad H \to U$$

is a continuous linear mapping from $(L^2(0,T;L^2(\Gamma_1)))^M$ to $C^0([0,T];(H^\alpha(\Omega))^N \times (H^{\alpha-1}(\Omega))^N)$, where α is defined by (3.2.6). Let $\hat{U}$ be the Gâteaux derivative of U in the direction of $\hat{H}$:

$$\hat{U} = F'(0)\hat{H}. \tag{3.7.7}$$

By linearity, $\hat{U}$ satisfies the same problem as U:

$$\begin{cases} \hat{U}'' - \Delta\hat{U} + A\hat{U} = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ \hat{U} = 0 & \text{on} \quad (0,+\infty)\times\Gamma_0, \\ \partial_\nu\hat{U} + B\hat{U} = D\hat{H} & \text{on} \quad (0,+\infty)\times\Gamma_1 \\ t=0: \quad \hat{U} = \hat{U}' = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.7.8}$$

Since the projection $\phi = (E, U)$ is independent of boundary control H in $(0,T)\times\Omega$, we have

$$(E,\hat{U}) \equiv 0 \quad \text{in} \quad (0,T)\times\Omega, \qquad \forall\ \hat{H} \in (L^2(0,T;L^2(\Gamma_1)))^M. \tag{3.7.9}$$

We first prove that $E \notin \mathrm{Im}(C_1^T)$. Otherwise, there exists a vector $R \in \mathbb{R}^{N-1}$, such that $E = C_1^T R$, then

$$0 = (E,\hat{U}) = (R, C_1\hat{U}). \tag{3.7.10}$$

Noting that $C_1\hat{U}$ is the solution to the reduced system (3.6.17) with the zero initial data, by the exact boundary synchronization of system (3.1.6), the reduced system (3.6.17) has the exact boundary controllability, then, the value of $C_1\hat{U}$ at the time T can be arbitrarily chosen, as a result, from (3.7.10) we get

$$R = 0,$$

which contradicts the fact that $E \neq 0$, hence $E \notin \mathrm{Im}(C_1^T)$. Then, noting (3.6.2), we can choose E such that

$$(E, e) = 1.$$

Thus, $\{E, C_1^T\}$ consists of a basis in $\mathbb{R}^N$, hence there exist a $\lambda \in \mathbb{C}$ and a vector $Q \in \mathbb{R}^{N-1}$, such that

$$A^T E = \lambda E + C_1^T Q. \tag{3.7.11}$$

By (3.7.10), taking the inner product with E on both sides of (3.7.8), and noting (3.7.9), we get

$$0 = (A\hat{U}, E) = (\hat{U}, A^T E) = (\hat{U}, C_1^T Q) = (C_1\hat{U}, Q). \tag{3.7.12}$$

Similarly, by the exact boundary controllability for the reduced system (3.6.17), we can get $Q = 0$, then, it follows from (3.7.11) that

$$A^T E = \lambda E.$$

On the other hand, by (3.7.10), taking the inner product with E on both sides of the boundary condition on Γ_1 in (3.7.8), we get

$$(E, B\hat{U}) = (E, D\hat{H}) \quad \text{on} \quad \Gamma_1. \tag{3.7.13}$$

By Theorem 3.3, we have

$$\|(E, D\hat{H})\|_{H^{2\alpha-1}((0,T)\times\Gamma_1)} = \|(E, B\hat{U})\|_{H^{2\alpha-1}((0,T)\times\Gamma_1)} \leq c\|\hat{H}\|_{L^2(0,T;L^2(\Gamma_1))^M}. \tag{3.7.14}$$

We claim $D^T E = 0$, namely, $E \in \text{Ker}(D^T)$. Otherwise, taking $\hat{H} = D^T E v$, we get

$$\|D^T E\| \cdot \|v\|_{H^{2\alpha-1}(\Sigma_1)} \leq c\|v\|_{L^2(0,T;L^2(\Gamma_1))}, \quad \forall v \in L^2(0, T; L^2(\Gamma_1)). \tag{3.7.15}$$

Noting $2\alpha - 1 > 0$, it contradicts the compactness of the embedding $H^{2\alpha-1}(\Sigma_1) \hookrightarrow L^2(\Sigma_1)$.

Thus, we have

$$(E, B\hat{U}) = 0 \quad \text{on} \quad \Gamma_1. \tag{3.7.16}$$

Similarly, since $\{E, C_1^T\}$ consists of a basis in $\mathbb{R}^N$, there exist a $\mu \in \mathbb{R}$ and a vector $P \in \mathbb{R}^{N-1}$, such that

$$B^T E = \mu E + C_1^T P. \tag{3.7.17}$$

Substituting it into (3.7.16) and noting (3.7.9), we have

$$(\mu E, \hat{U}) + (C_1^T P, \hat{U}) = (P, C_1\hat{U}) = 0, \tag{3.7.18}$$

then by the exact boundary controllability for the reduced system (3.6.17), we get $P = 0$, hence we have

$$B^T E = \mu E,$$

which indicates that E is a common eigenvector of A^T and B^T. The proof is complete. □

Theorem 3.12. *Assume that both A and B satisfy the C_1-conditions of compatibility* (3.6.8). *Let $E \in \mathbb{R}^N$ be an eigenvector of B^T, satisfying $(E, e) = 1$. Then there exists a boundary control matrix D such that system* (3.1.6) *is exactly boundary synchronizable, and the state of synchronization u satisfies the following estimate:*

$$\|(u, u')(T) - (\phi, \phi')(T)\|_{H^{\alpha+1}(\Omega)\times H^\alpha(\Omega)} \leq c\|C_1(U_0, U_1)\|_{(\mathcal{H}_1)^{N-1}\times(\mathcal{H}_0)^{N-1}}, \tag{3.7.19}$$

where ϕ is the solution to the following problem:

$$\begin{cases} \phi'' - \Delta\phi + \lambda\phi = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \phi = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu\phi + \mu\phi = 0 & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad \phi = (U_0, E), \quad \phi' = (U_1, E) & \text{in} \quad \Omega, \end{cases} \tag{3.7.20}$$

where $Ae = \lambda e, Be = \mu e$.

Proof. By Lemma 4.11 in Appendix, B^T has an eigenvector E such that

$$B^T E = \mu E, \quad (E, e) = 1. \tag{3.7.21}$$

Define the boundary control matrix D such that

$$Ker(D^T) = Span\{E\}. \tag{3.7.22}$$

Noting $\mathrm{Ker}(C_1) = \mathrm{Span}\{e\}$ and $(E, e) = 1$, we have

$$\mathrm{Ker}(C_1) \cap \mathrm{Im}(D) = \{e\} \cap \{E\}^{\perp} = \{0\}, \tag{3.7.23}$$

then, by Lemma 4.10 in Appendix, we have

$$\mathrm{rank}(C_1 D) = \mathrm{rank}(D) = N - 1. \tag{3.7.24}$$

Thus, by Theorem 3.9, system (3.1.6) is exactly boundary synchronizable.

Taking the inner product with E on both sides of problem (3.1.6)–(3.1.7) and denoting $\psi = (E, U)$, we get

$$\begin{cases} \psi'' - \Delta\psi + \lambda\psi = (\lambda E - A^T E, U) & \text{in} \quad (0, +\infty) \times \Omega, \\ \psi = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \psi + \mu\psi = 0 & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad \psi = (U_0, E), \quad \psi' = (U_1, E) & \text{in} \quad \Omega. \end{cases} \tag{3.7.25}$$

Since

$$(\lambda E - A^T E, e) = (E, \lambda e - Ae) = 0, \tag{3.7.26}$$

we have

$$\lambda E - A^T E \in \{\mathrm{Span}(e)\}^{\perp} = \mathrm{Im}(C_1^T). \tag{3.7.27}$$

Therefore, there exists a vector $R \in \mathbb{R}^{N-1}$, such that

$$\lambda E - A^T E = C_1^T R. \tag{3.7.28}$$

Let $\varphi = \psi - \phi$, where ϕ is the solution to (3.7.20). By (3.7.20) and (3.7.25), we have

$$\begin{cases} \varphi'' - \Delta\varphi + \lambda\varphi = (R, C_1 U) & \text{in} \quad (0, +\infty) \times \Omega, \\ \varphi = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \varphi + \mu\varphi = 0 & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad \varphi = 0, \quad \varphi' = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.7.29}$$

By Remark 3.4, we get

$$\|(\varphi, \varphi')(T)\|_{H^{\alpha+1}(\Omega) \times H^{\alpha}(\Omega)} \le c\|C_1(U, U')\|_{L^2(0,T;(H^{\alpha}(\Omega))^{N-1} \times (H^{\alpha-1}(\Omega))^{N-1})}. \tag{3.7.30}$$

Noting then $W = C_1 U$, we use Theorem 3.3 for the reduced problem (3.6.17) to get

$$\|C_1(U,U')\|_{(L^2(0,T;(H^\alpha(\Omega))^{N-1}\times(H^{\alpha-1}(\Omega))^{N-1}))} \le c(\|C_1(U_0,U_1)\|_{(\mathcal{H}_1)^{N-1}\times(\mathcal{H}_0)^{N-1}} + \|\overline{D}_1 H\|_{(L^2(0,T;L^2(\Gamma_1)))^{N-1}}), \tag{3.7.31}$$

then, by (3.5.1) we have

$$\|C_1(U,U')\|_{(L^2(0,T;(H^\alpha(\Omega))^{N-1}\times(H^{\alpha-1}(\Omega))^{N-1}))} \le c\|C_1(U_0,U_1)\|_{(\mathcal{H}_1)^{N-1}\times(\mathcal{H}_0)^{N-1}}. \tag{3.7.32}$$

Thus, we get

$$\|(\varphi,\varphi')(T)\|_{H^{\alpha+1}(\Omega)\times H^\alpha(\Omega)} \le c\|C_1(U_0,U_1)\|_{(\mathcal{H}_1)^{N-1}\times(\mathcal{H}_0)^{N-1}}. \tag{3.7.33}$$

On the other hand, we have

$$t \ge T: \qquad \psi = (E,U) = (E,e)u = u, \tag{3.7.34}$$

thus $\varphi = u - \phi$ for $t \ge T$. By (3.7.33), we get (3.7.19). □

In the general situation, in order to efficiently determine the state of synchronization, combining Lemma 4.5 and Lemma 4.6 in Appendix, we get immediately

Lemma 3.10. *There exists a minimum invariant subspace V of A and B, V containing e, such that there exists a subspace W which is invariant for A^T and B^T, and biorthogonal to V (see Definition 4.1 in Appendix).*

Now let V and W be such subspaces given in Lemma 3.10. By Lemma 4.3 in Appendix, we take a set of basis $e_1, e_2, \cdots, e_r$ in V, in which $e_1 = e$; correspondingly, $E_1, E_2, \cdots, E_r$ is a set of basis in W, which are biorthogonal to $e_1, e_2, \cdots, e_r$. Since W is invariant for both A^T and B^T, there exist $\alpha_{jk}, \beta_{jk} (1 \le j,k \le r)$ such that

$$A^T E_k = \sum_{j=1}^r \alpha_{jk} E_j, \quad B^T E_k = \sum_{j=1}^r \beta_{jk} E_j.$$

Let

$$\psi_k = (E_k, U), \qquad 1 \le k \le r.$$

We have

$$\begin{cases} \psi_k'' - \Delta\psi_k + \sum_{j=1}^r \alpha_{jk}\psi_j = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ \psi_k = 0 & \text{on} \quad (0,+\infty)\times\Gamma_0, \\ \partial_\nu \psi_k + \sum_{j=1}^r \beta_{jk}\psi_j = v_k & \text{on} \quad (0,+\infty)\times\Gamma_1, \\ t=0: \quad \psi_k = (E_k,U_0), \quad \psi_k' = (E_k,U_1) & \text{in} \quad \Omega, \end{cases} \tag{3.7.35}$$

in which

$$v_k = E_k^T DV. \tag{3.7.36}$$

Theorem 3.13. *Assume that both A and B satisfy the C_1-conditions of compatibility* (3.6.8). *Assume furthermore that system* (3.1.6) *is exactly boundary synchronizable. Then the state of synchronization u is determined by the solution to problem* (3.7.35).

Proof. Noting $(E_1, e_1) = 1$ and

$$t \geq T: \qquad \psi_k = (E_k, U) = (E_k, ue_1) = u\delta_{k1}, \qquad 1 \leq k \leq r,$$

the state of synchronization u is determined by

$$t \geq T: \qquad u = u(t,x) = \psi_1(t,x). \tag{3.7.37}$$

However, in order to get the state of synchronization u, we have to solve the whole coupled problem (3.7.35).

□

Remark 3.9. *By Theorem 3.13, when system* (3.1.6) *contains a large number of equations, we can determine the state of synchronization u by problem* (3.7.35) *with a smaller dimension, which will significantly reduce the computational complexity. Nevertheless, the dimension of problem* (3.7.35) *depends on the property of the coupling matrices A and B, namely, the dimension of their common invariant subspace.*

3.8 Exact boundary synchronization by groups

When there is a further lack of boundary controls, similarly to the case with Dirichlet or Neumann boundary controls, we consider the exact boundary synchronization by p-groups for system (3.1.6) ($p \geq 1$; the special case $p = 1$ means nothing but the exact boundary synchronization). This indicates that the components of U are divided into p groups:

$$(u^{(1)}, \cdots, u^{(m_1)}), \quad (u^{(m_1+1)}, \cdots, u^{(m_2)}), \cdots, (u^{(m_{p-1}+1)}, \cdots, u^{(m_p)}), \tag{3.8.1}$$

where $0 = m_0 < m_1 < m_2 < \cdots < m_p = N$, and each group is required to possess the exact boundary synchronization, respectively, and the requirement of synchronization for every group is independent of each other:

$$t \geq T: \quad u^{(k)} \overset{\text{def.}}{=} u_s, \quad m_{s-1} + 1 \leq k \leq m_s, \ 1 \leq s \leq p, \tag{3.8.2}$$

where the corresponding state of synchronization by p-groups $(u_1, \cdots, u_p)^T$ is a priori unknown.

Let S_s be a $(m_s - m_{s-1} - 1) \times (m_s - m_{s-1})$ full row-rank matrix:

$$S_s = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}, \qquad 1 \le s \le p, \tag{3.8.3}$$

and let C_p be the following $(N-p) \times N$ matrix of synchronization by p-groups:

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}. \tag{3.8.4}$$

Evidently, we have

$$\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \cdots, e_p\}, \tag{3.8.5}$$

where for $1 \le s \le p$,

$$(e_s)_j = \begin{cases} 1, & m_{s-1} + 1 \le j \le m_s, \\ 0, & \text{others.} \end{cases}$$

Thus, (3.8.2) can be equivalently rewritten as

$$t \ge T: \quad C_p U \equiv 0 \tag{3.8.6}$$

or

$$t \ge T: \quad U = \sum_{s=1}^{p} u_s e_s. \tag{3.8.7}$$

Theorem 3.14. *Let $\Omega \subset \mathbb{R}^n$ be a parallelepiped. Assume that system* (3.1.6) *is exactly synchronizable by p-groups. Then we have*

$$rank(C_p D) = N - p. \tag{3.8.8}$$

In particular, we have

$$rank(D) \ge N - p. \tag{3.8.9}$$

Proof. If $Ker(D^T) \cap Im(C_p^T) = \{0\}$, by Lemma 4.10 in Appendix, we have

$$\mathrm{rank}(C_p D) = \mathrm{rank}(D^T C_p^T) = \mathrm{rank}(C_p^T) = N - p. \tag{3.8.10}$$

Next, we prove that it is impossible to have $Ker(D^T) \cap Im(C_p^T) \neq \{0\}$. Otherwise there exists a a vector $E \neq 0$, such that

$$D^T C_p^T E = 0. \tag{3.8.11}$$

Let

$$w = (E, C_p U), \qquad \mathcal{L}\theta = -(E, C_p AU), \qquad \mathcal{R}\theta = -(E, C_p BU). \tag{3.8.12}$$

We get again problem (3.5.6) for w. Besides, the exact boundary synchronization by p-groups for system (3.1.6) indicates that the final condition (3.5.7) holds. Similarly to the proof of Theorem 3.7, we get a contradiction to Theorem 3.5. □

Theorem 3.15. *Let C_p be the $(N-p)\times N$ matrix of synchronization by p-groups defined by* (3.8.3)–(3.8.4). *Assume that both A and B satisfy the following C_p-conditions of compatibility:*

$$AKer(C_p) \subseteq Ker(C_p), \qquad BKer(C_p) \subseteq Ker(C_p). \tag{3.8.13}$$

Then there exists a boundary control matrix D satisfying

$$rank(D) = rank(C_p D) = N - p, \tag{3.8.14}$$

such that system (3.1.6) *is exactly boundary synchronizable by p-groups, and the corresponding boundary control function H possesses the following continuous dependence:*

$$\|H\|_{(L^2(0,T,L^2(\Gamma_1)))^{N-p}} \leq c\|C_p(U_0, U_1)\|_{(\mathcal{H}_1)^{N-p}\times(\mathcal{H}_0)^{N-p}}. \tag{3.8.15}$$

Proof. Since both A and B satisfy the C_p-conditions of compatibility (3.8.13), by Lemma 4.7 in Appendix, there exist matrices $\overline{A}_p$ and $\overline{B}_p$ of order $(N-p)$, such that

$$C_p A = \overline{A}_p C_p, \qquad C_p B = \overline{B}_p C_p. \tag{3.8.16}$$

Let

$$W = C_p U, \qquad \overline{D}_p = C_p D. \tag{3.8.17}$$

We have

$$\begin{cases} W'' - \Delta W + \overline{A}_p W = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ W = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu W + \overline{B}_p W = \overline{D}_p H & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad W = C_p U_0, \quad W' = C_p U_1 & \text{in} \quad \Omega. \end{cases} \tag{3.8.18}$$

Noting that C_p is a surjection from $\mathbb{R}^N$ to $\mathbb{R}^{N-p}$, any given initial data $(U_0, U_1) \in (\mathcal{H}_1)^N \times (\mathcal{H}_0)^N$ corresponds to a unique initial data $(C_p U_0, C_p U_1)$ for the reduced system (3.8.18). Thus, the exact boundary snchronization by p-groups for system (3.1.6) is equivalent to the exact boundary controllability for the reduced system (3.8.18), and the boundary control H, which realizes the exact boundary controllability for the reduced system (3.8.18), must be the boundary control which realizes the exact boundary synchronization for system (3.1.6).

Let D be defined by

$$\mathrm{Ker}(D^T) = \mathrm{Span}\{e_1, \cdots, e_p\} = \mathrm{Ker}(C_p). \tag{3.8.19}$$

We have $\mathrm{rank}(D) = N - P$, and

$$\mathrm{Ker}(C_p) \cap \mathrm{Im}(D) = \mathrm{Ker}(C_p) \cap \{\mathrm{Ker}(C_p)\}^{\perp} = \{0\}. \tag{3.8.20}$$

By Lemma 4.10 in Appendix, we get $\mathrm{rank}(C_pD) = \mathrm{rank}(D) = N - p$, thus $\overline{D}_p$ is an invertible matrix of order $(N - p)$. By Theorem 3.4, the reduced system (3.8.18) is exactly boundary controllable, then system (3.1.6) is exactly boundary synchronizable by p-groups. By (3.5.1), we get (3.8.15). □

3.9 C_p-conditions of compatibility

In this section, we discuss the necessity of the C_p-conditions of compatibility. This is a problem closely related to the number of boundary control functions. We first study the C_p-condition of compatibility for the coupling matrix A.

Theorem 3.16. *Let $\Omega \subset \mathbb{R}^n$ be a parallelepiped. Assume that $M = rank(D) = N - p$. If system* (3.1.6) *is exactly boundary synchronizable by p-groups, then the coupling matrix $A = (a_{ij})$ satisfies the following C_p condition of compatibility:*

$$A Ker(C_p) \subseteq Ker(C_p). \tag{3.9.1}$$

Proof. It suffices to prove that

$$C_pAe_s = 0, \quad 1 \leq s \leq p.$$

By (3.8.7), taking the inner product with C_p on both sides of (3.1.6), we get

$$t \geq T: \qquad \sum_{s=1}^{p} C_pAe_su_s = 0 \qquad \text{in} \quad \Omega. \tag{3.9.2}$$

If there exists an s such that $C_pAe_s \neq 0$, then there exist constant coefficients $\alpha_s (1 \leq s \leq p)$ such that

$$\sum_{s=1}^{p} \alpha_s u_s = 0 \qquad \text{in} \quad \Omega, \tag{3.9.3}$$

in which not all the $\alpha_s (1 \leq s \leq p)$ are equal to zero. Let

$$c_{p+1} = \sum_{s=1}^{p} \frac{\alpha_s e_s^T}{\|e_s\|^2}. \tag{3.9.4}$$

Noting $(e_r, e_s) = \|e_r\|^2 \delta_{rs}$, we have

$$t \geq T: \qquad c_{p+1} U = \sum_{s=1}^{p} \alpha_s u_s = 0 \qquad \text{in} \quad \Omega. \tag{3.9.5}$$

Let

$$\widetilde{C}_{p-1} = \begin{pmatrix} C_p \\ c_{p+1} \end{pmatrix}. \tag{3.9.6}$$

Noting (3.8.19) and (3.9.4), it is easy to see that $c_{p+1}^T \notin \text{Im}(C_p^T)$, then, $\text{rank}(\widetilde{C}_{p-1}) = N - p + 1$. By $\text{rank}(D) = N - p$, we have $\text{Ker}(D^T) \cap \text{Im}(\widetilde{C}_{p-1}^T) \neq \{0\}$, then there exists a vector $E \neq 0$, such that

$$D^T \widetilde{C}_{p-1}^T E = 0. \tag{3.9.7}$$

Let

$$w = (E, \widetilde{C}_{p-1} U), \qquad \mathcal{L}\theta = -(E, \widetilde{C}_{p-1} A U), \qquad \mathcal{R}\theta = -(E, \widetilde{C}_{p-1} B U). \tag{3.9.8}$$

We get again problem (3.5.6) for w. Noting (3.8.6) and (3.9.5), we have

$$t = T: \quad w(T) = (E, \widetilde{C}_{p-1} U) = (E, \begin{pmatrix} C_p \\ c_{p+1} \end{pmatrix} U) = 0. \tag{3.9.9}$$

Similarly, we have $w'(T) = 0$, then (3.5.7) holds. Noting that $\Omega \subset \mathbb{R}^n$ is a parallelepiped, similarly to the proof of Theorem 3.7, we can get a conclusion that contradicts Theorem 3.5. $\square$

Remark 3.10. *The C_p-condition of compatibility* (3.9.1) *is equivalent to the fact that there exist constants α_{rs} $(1 \leq r, s \leq p)$ such that*

$$A e_s = \sum_{r=1}^{p} \alpha_{rs} e_r, \quad 1 \leq s \leq p, \tag{3.9.10}$$

or, A satisfies the following row-sum condition by blocks:

$$\sum_{j=m_{s-1}+1}^{m_s} a_{ij} = \alpha_{rs}, \quad m_{r-1} + 1 \leq i \leq m_r, \quad 1 \leq r, s \leq p. \tag{3.9.11}$$

When $p = 1$, this condition of compatibility becomes (4.1.19) *below.*

In particular, when

$$\alpha_{rs} = 0, \quad 1 \leq r, s \leq p, \tag{3.9.12}$$

we say that A satisfies the zero-sum condition by blocks. In this case, we have

$$A e_s = 0, \quad 1 \leq s \leq p. \tag{3.9.13}$$

Comparing with the internal coupling matrix A, the study on the necessity of the C_p-condition of compatibility for the boundary coupling matrix B is more complicated. It concerns the regularity of solution to the problem with coupled Robin boundary conditions.

Let

$$\varepsilon_i = (0, \cdots, \overset{(i)}{1}, \cdots, 0)^T, \qquad 1 \leq i \leq N \tag{3.9.14}$$

be a set of classical orthogonal basis in $\mathbb{R}^N$, and let

$$V_k = \text{Span}\{\varepsilon_{m_{k-1}+1}, \cdots, \varepsilon_{m_k}\}, \qquad 1 \leq k \leq p. \tag{3.9.15}$$

In what follows, we discuss the necessity of the C_p-condition of compatibility for the boundary coupling matrix B under the assumption that $Ae_k \in V_k,\ Be_k \in V_k\ (1 \leq k \leq p)$.

Theorem 3.17. *Let $\Omega \subset \mathbb{R}^n$ be a parallelepiped. Assume that $M = rank(D) = N - p$ and*

$$Ae_k \in V_k,\ Be_k \in V_k\ (1 \leq k \leq p). \tag{3.9.16}$$

If system (3.1.6) is exactly boundary synchronizable by p-groups, then the boundary coupling matrix B satisfies the following C_p-condition of compatibility:

$$BKer(C_p) \subseteq Ker(C_p). \tag{3.9.17}$$

Proof. Noting (3.8.7), as $t \geq T$ we have

$$\begin{cases} \displaystyle\sum_{k=1}^{p}(u_k''e_k - \Delta u_k e_k + Au_k e_k) = 0 & \text{in} \quad (T, +\infty) \times \Omega, \\ \displaystyle\sum_{k=1}^{p}(\partial_\nu u_k e_k + Bu_k e_k) = 0 & \text{on} \quad (T, +\infty) \times \Gamma_1. \end{cases} \tag{3.9.18}$$

Noticing (3.9.16) and the fact that subspaces $V_k (1 \leq k \leq p)$ are orthogonal to each other, for $1 \leq k \leq p$ we have

$$\begin{cases} u_k''e_k - \Delta u_k e_k + Ae_k u_k = 0 & \text{in} \quad (T, +\infty) \times \Omega, \\ \partial_\nu u_k e_k + Be_k u_k = 0 & \text{on} \quad (T, +\infty) \times \Gamma_1. \end{cases} \tag{3.9.19}$$

Taking the inner product with C_p on both sides of the boundary condition in (3.9.19), and noting (3.8.5), we get

$$C_p Be_k u_k \equiv 0 \quad \text{on} \quad (T, +\infty) \times \Gamma_1, \quad 1 \leq k \leq p. \tag{3.9.20}$$

We claim that $C_p Be_k = 0$ $(k = 1, \cdots, p)$, which just mean that B satisfies the corresponding C_p-condition of compatibility. Otherwise, we may assume that there exists a k $(1 \leq k \leq p)$, such that

$$u_k \equiv 0 \qquad \text{on} \quad (T, +\infty) \times \Gamma_1,$$

then, by the boundary condition in system (3.9.19), we get

$$\partial_\nu u_k \equiv 0 \qquad \text{on} \quad (T, +\infty) \times \Gamma_1. \tag{3.9.21}$$

Thus, applying Holmgren's uniqueness theorem to (3.9.19) yields that

$$u_k \equiv 0 \qquad \text{in} \quad (T, +\infty) \times \Omega, \tag{3.9.22}$$

then it is easy to check that

$$t \geq T: \quad e_k^T U \equiv 0 \qquad \text{in} \quad \Omega. \tag{3.9.23}$$

Let

$$\tilde{C}_{p-1} = \begin{pmatrix} C_p \\ e_k^T \end{pmatrix}. \tag{3.9.24}$$

Noting (3.8.19), we have $e_k^T \notin \text{Im}(C_p^T)$, then it is easy to show that $\text{rank}(\widetilde{C}_{p-1}) = N - p + 1$. Noting $\text{rank}(\text{Ker}(D^T)) = p$, we have $\text{Ker}(D^T) \cap \text{Im}(\widetilde{C}_{p-1}^T) \neq \{0\}$, then there exists a vector $E \neq 0$, such that

$$D^T \widetilde{C}_{p-1}^T E = 0. \tag{3.9.25}$$

Let

$$w = (E, \widetilde{C}_{p-1} U), \qquad \mathcal{L}\theta = -(E, \widetilde{C}_{p-1} A U), \qquad \mathcal{R}\theta = -(E, \widetilde{C}_{p-1} B U). \tag{3.9.26}$$

We get again problem (3.5.6) for w, and (3.5.7) holds. Therefore, noting that Ω is a parallelepiped, similarly to the proof of Theorem 3.7, we get a contradiction to Theorem 3.5. □

Remark 3.11. *The restricted conditions that $Ae_k \in V_k$ and $Be_k \in V_k$ $(1 \leq k \leq p)$ in Theorem 3.17 naturally hold as $p = 1$.*

Under the restricted conditions that $Ae_k \in V_k$ and $Be_k \in V_k (1 \leq k \leq p)$, we have already got the C_p-condition of compatibility for B. Next, we will remove these restricted conditions in the special case $p = 2$, and further study the necessity of the C_p-condition of compatibility for B.

Theorem 3.18. *Let $\Omega \subset \mathbb{R}^n$ be a parallelepiped. Assume that coupling matrix A satisfies the zero-sum condition by blocks* (3.9.13)*. Assume furthermore that system* (3.1.6) *is exactly boundary synchronizable by 2-groups with rank$(D) = N - 2$. Then B necessarily satisfies the C_2-condition of compatibility:*

$$BKer(C_2) \subseteq Ker(C_2). \tag{3.9.27}$$

Proof. First, since B is similar to a real symmetric matrix, there exists an invertible real matrix P and a real symmetric matrix $\hat{B}$, such that $B = P^{-1} \hat{B} P$.

By the exact boundary synchronization by 2-groups, we have

$$t \geq T: \qquad U = e_1 u_1 + e_2 u_2 \qquad \text{in} \quad \Omega. \tag{3.9.28}$$

Noting (3.9.13), as $t \geq T$ we have

$$AU = \sum_{i=1}^{2} A e_i u_i = 0,$$

then it is easy to see that

$$t \geq T: \quad \begin{cases} U'' - \Delta U = 0 & \text{in} \quad \Omega, \\ U = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu U + BU = 0 & \text{on} \quad \Gamma_1. \end{cases} \tag{3.9.29}$$

Let

$$u = (u_1, u_2)^T, \qquad \hat{e}_i = P^T P e_i \quad (i = 1, 2). \tag{3.9.30}$$

Taking the inner product on both sides of (3.9.29) with $\hat{e}_j$, we get

$$t \geq T: \quad \begin{cases} Lu'' - L\Delta u = 0 & \text{in} \quad \Omega, \\ Lu = 0 & \text{on} \quad \Gamma_0, \\ L\partial_\nu u + \Lambda u = 0 & \text{on} \quad \Gamma_1, \end{cases} \tag{3.9.31}$$

where $L = (e_i, \hat{e}_j)$ and $\Lambda = (Be_i, \hat{e}_j)$ are 2×2 matrices.

We can prove that L is a positive definite symmetric matrix and Λ is a symmetric matrix. In fact, since

$$(e_i, \hat{e}_j) = (e_i, P^T P e_j) = (Pe_i, Pe_j), \tag{3.9.32}$$

it is easy to see that L is symmetric. Besides, for any given non-zero vector $X = (x, y)^T \in \mathbb{R}^2$, we have

$$X^T L X = \|P(xe_1 + ye_2)\|^2 > 0, \tag{3.9.33}$$

thus L is positive definite. Besides, since

$$(Be_i, \hat{e}_j) = (Be_i, P^T P e_j) = (PBe_i, Pe_j) = (\hat{B} P e_i, P e_j), \tag{3.9.34}$$

Λ is symmetric.

Taking the inner product on both sides of (3.9.31) with $L^{-\frac{1}{2}}$ and denoting $w = L^{\frac{1}{2}} u$, we get

$$t \geq T: \quad \begin{cases} w'' - \Delta w = 0 & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu w + \hat{\Lambda} w = 0 & \text{on} \quad \Gamma_1, \end{cases} \tag{3.9.35}$$

where $\hat{\Lambda} = L^{-\frac{1}{2}} \Lambda L^{-\frac{1}{2}}$ is a symmetric matrix.

On the other hand, taking the inner product with C_2 on both sides of the boundary condition of (3.1.6) on Γ_1, we get

$$t \geq T: \qquad C_2Be_1u_1 + C_2Be_2u_2 \equiv 0 \qquad \text{on} \quad \Gamma_1. \tag{3.9.36}$$

If $\text{rank}(C_2Be_1, C_2Be_2) = 0$, then $C_2Be_1 = C_2Be_2 = 0$, namely, B satisfies the C_2-condition of compatibility.

If $\text{rank}(C_2Be_1, C_2Be_2) = 2$, then we have $u \equiv 0$ on Γ_1. By the boundary condition on Γ_1 in (3.9.31), we get

$$\partial_\nu u \equiv 0 \quad \text{on} \quad \Gamma_1.$$

By Holmgren's uniqueness theorem ([37]), $u \equiv 0$ on the whole domain Ω, then system (3.1.6) is exactly boundary null controllable. However, $\text{rank}(D) = N - 2$, and Ω is a parallelepiped, it contradicts Theorem 3.6.

We now prove that it is also impossible to have $\text{rank}(C_2Be_1, C_2Be_2) = 1$. Otherwise, since

$$C_2Be_1 \neq 0 \quad \text{or} \quad C_2Be_2 \neq 0,$$

it follows from (3.9.36) that there exists a non-zero vector $D_2 \in \mathbb{R}^2$, such that

$$t \geq T: \qquad D_2^T u \equiv 0 \qquad \text{on} \quad \Gamma_1. \tag{3.9.37}$$

Denoting

$$\hat{D}^T = D_2^T L^{-\frac{1}{2}},$$

we have

$$t \geq T: \qquad \hat{D}^T w = 0 \qquad \text{on} \quad \Gamma_1, \tag{3.9.38}$$

in which $w = L^{\frac{1}{2}}u$. By Theorem 3.22 and Remark 3.12, the following Hautus's criterion

$$\text{rank}(\hat{\Lambda} - \mu I_2, \hat{D}) = 2, \quad \forall \mu \in \mathbb{R} \tag{3.9.39}$$

guarantees the uniqueness of the trivial solution to system (3.9.35) in the infinite time interval $[T, \infty)$ under observation (3.9.38). Then by the non-exact boundary null controllability of system (3.1.6), condition (3.9.39) does not hold. Thus, there exists a vector $E \neq 0$ in $\mathbb{R}^2$, such that

$$\hat{\Lambda}^T E = \hat{\Lambda} E = \mu E, \qquad \hat{D}^T E = 0. \tag{3.9.40}$$

Noting (3.9.38) and the second formula of (3.9.40), we have E and $w|_{\Gamma_1} \in \text{Ker}(\hat{D})$. Since $\text{Dim Ker}(\hat{D}) = 1$, there exists a constant α such that we have $w = \alpha E$ on Γ_1. Therefore, noting the first formula of (3.9.40), we have

$$\hat{\Lambda} w = \hat{\Lambda} \alpha E = \mu \alpha E = \mu w \quad \text{on} \;\; \Gamma_1. \tag{3.9.41}$$

Thus (3.9.35) can be rewritten as

$$t \geq T: \qquad \begin{cases} w'' - \Delta w = 0 & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu w + \mu w = 0 & \text{on} \quad \Gamma_1. \end{cases} \tag{3.9.42}$$

Let $z = \hat{D}^T w$. Noting (3.9.38), we have

$$t \geq T: \qquad \begin{cases} z'' - \Delta z = 0 & \text{in} \quad \Omega, \\ z = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu z = z = 0 & \text{on} \quad \Gamma_1. \end{cases} \tag{3.9.43}$$

Then, by Holmgren's uniqueness theorem, we have

$$t \geq T: \qquad z = \hat{D}^T w = D_2^T u \equiv 0 \qquad \text{in} \quad \Omega. \tag{3.9.44}$$

Let $D_2^T = (\alpha_1, \alpha_2)$. Define the following row vector

$$c_3 = \frac{\alpha_1 e_1^T}{\|e_1\|^2} + \frac{\alpha_2 e_2^T}{\|e_2\|^2}. \tag{3.9.45}$$

Noting $(e_1, e_2) = 0$ and (3.9.44), we have

$$t \geq T: \qquad c_3 U = \alpha_1 u_1 + \alpha_2 u_2 = D_2^T u \equiv 0 \qquad \text{in} \quad \Omega. \tag{3.9.46}$$

Let

$$\widetilde{C}_1 = \begin{pmatrix} C_2 \\ c_3 \end{pmatrix}. \tag{3.9.47}$$

By $c_3^T \notin \mathrm{Im}(C_2^T)$, it is easy to see that $\mathrm{rank}(\widetilde{C}_1) = N - 1$, and $\mathrm{Ker}(D^T) \cap \mathrm{Im}(\widetilde{C}_1^T) \neq \{0\}$, thus, there exists a vector $\tilde{E} \neq 0$, such that

$$D^T \widetilde{C}_1^T \tilde{E} = 0. \tag{3.9.48}$$

Let

$$u = (\tilde{E}, \widetilde{C}_1 U), \qquad \mathcal{L}\theta = -(\tilde{E}, \widetilde{C}_1 A U), \qquad \mathcal{R}\theta = -(\tilde{E}, \widetilde{C}_1 B U). \tag{3.9.49}$$

We get again problem (3.5.6) for w, satisfying (3.5.7). Noting that Ω is a parallelepiped, similarly to the proof of Theorem 3.7, we get a contradiction to Theorem 3.5. □

3.10 Determination of the state of synchronization by groups

When the coupling matrices A and B satisfy certain algebraic conditions, the state of synchronization by groups will be independent of applied boundary control functions. In the general situation, the state of synchronization by groups depends not only on the initial data, but also on the applied boundary control functions. In this section, we first discuss the determination of the state of synchronization by p-groups in the former situation, then we present the estimate on the state of synchronization by p-groups in the latter situation.

Theorem 3.19. *Assume that both A and B satisfy the C_p-conditions of compatibility* (3.8.13). *Assume furthermore that A^T and B^T possess a common invariant subspace V, being biorthogonal to Ker(C_p). Then there exists a boundary control matrix D such that rank(D) = rank$(C_pD) = N-p$, system* (3.1.6) *is exactly boundary synchronizable by p-groups, and the state of synchronization by p-groups $(u_1, \cdots, u_p)^T$ is independent of the applied boundary control functions.*

Proof. Define the boundary control function D such that

$$\mathrm{Ker}(D^T) = V. \tag{3.10.1}$$

Since V is biorthogonal to $\mathrm{Ker}(C_p)$, by Lemma 4.3 in Appendix, we have

$$\mathrm{Ker}(C_p) \cap \mathrm{Im}(D) = \mathrm{Ker}(C_p) \cap V^{\perp} = \{0\}, \tag{3.10.2}$$

then, by Lemma 4.10 in Appendix, we have

$$\mathrm{rank}(C_pD) = \mathrm{rank}(D) = N - p. \tag{3.10.3}$$

Thus, by Theorem 3.15, system (3.1.6) is exactly boundary synchronizable by p-groups.

By (3.10.2), noting $\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \cdots, e_p\}$, we may write

$$V = \mathrm{Span}\{E_1, \cdots, E_p\} \quad \text{with } (e_i, E_j) = \delta_{ij}. \tag{3.10.4}$$

Since V is a common invariant subspace of A^T and B^T, there exist constants α_{ij}, β_{ij} such that

$$A^T E_i = \sum_{j=1}^{p} \alpha_{ij} E_j, \qquad B^T E_i = \sum_{j=1}^{p} \beta_{ij} E_j. \tag{3.10.5}$$

For $i = 1, \cdots, p$, let

$$\phi_i = (E_i, U). \tag{3.10.6}$$

By (3.1.6) and noting (3.10.1), we have

$$\begin{cases} \phi_i'' - \Delta\phi_i + \sum_{j=1}^{p} \alpha_{ij}\phi_j = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \phi_i = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \phi_i + \sum_{j=1}^{p} \beta_{ij}\phi_j = 0 & \text{on} \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \quad \phi_i = (E_i, U_0), \quad \phi_i' = (E_i, U_1) & \text{in} \quad \Omega. \end{cases} \tag{3.10.7}$$

On the other hand, we have

$$t \geq T: \qquad \phi_i = (E_i, U) = \sum_{j=1}^{p} (E_i, e_j) u_j = \sum_{j=1}^{p} \delta_{ij} u_j = u_i. \tag{3.10.8}$$

Thus, the state of synchronization by p-groups $(u_1, \cdots, u_p)^T$ is entirely determined by the solution to problem (3.10.7), which is independent of applied boundary control function H. □

Theorem 3.20. *Assume that both A and B satisfy the C_p-conditions of compatibility* (3.8.13)*. Assume furthermore that system* (3.1.6) *is exactly boundary synchronizable by p-groups. Let $V = \{E_1, \cdots, E_p\}$ be a subspace of dimension p. If the projection functions*

$$\phi_i = (E_i, U), \qquad i = 1, \cdots, p \tag{3.10.9}$$

are independent of applied boundary control function H in $(0,T) \times \Omega$, which realizes the exact boundary synchronization by p-groups, then V is a common invariant subspace of A^T and B^T, and biorthogonal to $Ker(C_p)$.

Proof. Similarly to the proof of Theorem 3.11, assuming that $(U_0, U_1) = (0, 0)$, by Theorem 3.3, we have that

$$H \to U$$

is a continuous linear mapping from $(L^2(0,T; L^2(\Gamma_1)))^M$ to $C^0([0,T]; (H^\alpha(\Omega))^N \times (H^{\alpha-1}(\Omega))^N)$, where α is defined by (3.2.6).

Let $\hat{U}$ be the Gâteaux derivative of U in the direction of $\hat{H}$, defined by (3.7.7). $\hat{U}$ satisfies a similar system to that of U:

$$\begin{cases} \hat{U}'' - \Delta \hat{U} + A\hat{U} = 0 & \text{in} \quad (0,+\infty) \times \Omega, \\ \hat{U} = 0 & \text{on} \quad (0,+\infty) \times \Gamma_0, \\ \partial_\nu \hat{U} + B\hat{U} = D\hat{H} & \text{on} \quad (0,+\infty) \times \Gamma_1 \\ t = 0: \quad \hat{U} = \hat{U}' = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.10.10}$$

Since the projection functions $\phi_i = (E_i, U)$ $(i = 1, \cdots, p)$ are independent of applied boundary control function H in $(0,T) \times \Omega$, we have

$$(E_i, \hat{U}) \equiv 0 \qquad \forall\, \hat{H} \in L^2(0,T; L^2(\Gamma_1))^M, \quad i = 1, \cdots, p. \tag{3.10.11}$$

First, we prove $E_i \notin \text{Im}(C_p^T)$ for $i = 1, \cdots, p$. Otherwise, there exist an i and a vector $R_i \in \mathbb{R}^{N-p}$, such that $E_i = C_p^T R_i$, then we have

$$0 = (E_i, \hat{U}) = (R_i, C_p \hat{U}), \qquad \forall\, \hat{H} \in (L^2(0,T; L^2(\Gamma_1)))^M. \tag{3.10.12}$$

Since $C_p\hat{U}$ is the solution to the reduced problem (3.8.18), noting the equivalence between the exact boundary synchronization by p-groups for the original system and the exact boundary controllability for the reduced system, by the exact boundary synchronization by p-groups for system (3.1.6), we know that the reduced system (3.8.18) is exactly boundary controllable, then the value of $C_p\hat{U}$ at the time T can be chosen arbitrarily, thus we get

$$R_i = 0.$$

It contracidts $E_i \neq 0$, then, we have $E_i \notin \mathrm{Im}(C_p^T)$ $(i = 1, \cdots, p)$. Thus $V \cap \{\mathrm{Ker}(C_p)\}^{\perp} = V \cap \mathrm{Im}(C_p^T) = \{0\}$, then, by Lemma 4.2 and Lemma 4.3 in Appendix, we get that V is biorthogonal to $\mathrm{Ker}(C_p)$, and (V, C_p^T) consists of a set of basis in $\mathbb{R}^N$. Hence there exist constant coefficients α_{ij} $(i, j = 1, \cdots, p)$ and vectors $Q_i \in \mathbb{R}^{N-p}$ $(i = 1, \cdots, p)$, such that

$$A^T E_i = \sum_{j=1}^{p} \alpha_{ij} E_j + C_p^T Q_i, \quad i = 1, \cdots, p. \tag{3.10.13}$$

Noting (3.10.12) and taking the inner product with E_i on both sides of (3.10.10), we get

$$0 = (A\hat{U}, E_i) = (\hat{U}, A^T E_i) = (\hat{U}, C_p^T Q_i) = (C_p\hat{U}, Q_i), \quad i = 1, \cdots, p. \tag{3.10.14}$$

Similarly, by the exact boundary controllability for the reduced system (3.8.18), we get $Q_i = 0$ $(i = 1, \cdots, p)$, thus we have

$$A^T E_i = \sum_{j=1}^{p} \alpha_{ij} E_j, \qquad i = 1, \cdots, p,$$

which means that V is an invariant subspace of A^T.

On the other hand, noting (3.10.12) and taking the inner product with E_i on both sides of the boundary condition on Γ_1 in (3.10.10), we get

$$(E_i, B\hat{U}) = (E_i, D\hat{H}) \quad \text{on} \quad \Gamma_1, \qquad i = 1, \cdots, p. \tag{3.10.15}$$

By Theorem 3.3, for $i = 1, \cdots, p$ we have

$$\|(E_i, D\hat{H})\|_{H^{2\alpha-1}(\Sigma_1)} = \|(E_i, B\hat{U})\|_{H^{2\alpha-1}(\Sigma_1)} \leq c\|\hat{H}\|_{(L^2(0,T;L^2(\Gamma_1)))^M}. \tag{3.10.16}$$

We claim that $D^T E_i = 0$ $(i = 1, \cdots, p)$. Otherwise, for $i = 1, \cdots, p$, let $\hat{H} = D^T E_i v$. We have

$$\|D^T E_i\| \cdot \|v\|_{H^{2\alpha-1}(\Sigma_1)} \leq c\|v\|_{L^2(0,T;L^2(\Gamma_1))}. \tag{3.10.17}$$

Since $2\alpha - 1 > 0$, it contradicts the compactness of $H^{2\alpha-1}(\Sigma_1) \hookrightarrow L^2(\Sigma_1)$.

Thus, by (3.10.15) we have

$$(E_i, B\hat{U}) = 0 \quad \text{on} \quad (0, T) \times \Gamma_1, \qquad i = 1, \cdots, p. \tag{3.10.18}$$

Similarly, there exist constants β_{ij} $(i, j = 1, \cdots, p)$ and vectors $P_i \in \mathbb{R}^{N-p}$ $(i = 1, \cdots, p)$, such that

$$B^T E_i = \sum_{j=1}^{p} \beta_{ij} E_j + C_p^T P_i, \quad i = 1, \cdots, p. \tag{3.10.19}$$

Substituting it into (3.10.18) and noting (3.10.11), we have

$$(\sum_{j=1}^{p} \beta_{ij} E_j, \hat{U}) + (C_p^T P_i, \hat{U}) = (P_i, C_p \hat{U}) = 0, \qquad i = 1, \cdots, p. \tag{3.10.20}$$

Similarly, by the exact boundary controllability for the reduced system (3.8.18), we get $P_i = 0$ $(i = 1, \cdots, p)$, then we have

$$B^T E_i = \sum_{j=1}^{p} \beta_{ij} E_j, \qquad i = 1, \cdots, p,$$

which indicates that V is also an invariant subspace of B^T. The proof is complete. □

When A and B do not satisfy all the conditions mentioned in Theorem 3.19, the state of synchronization by p-groups depends on applied boundary control functions. We have the following

Theorem 3.21. *Assume that both A and B satisfy the C_p-conditions of compatibility* (3.8.13). *Let $V =$ span$\{E_1, \cdots, E_p\}$ be an invariant subspace of B^T, being biorthogonal to Ker(C_p). Then there exists a boundary control matrix D such that system* (3.1.6) *is exactly boundary synchronizable by p-groups, and the state of synchronization by p-groups $u = (u_1, \cdots, u_p)^T$ satisfies the following estimate:*

$$\|(u, u')(T) - (\phi, \phi')(T)\|_{(H^{\alpha+1}(\Omega))^p \times (H^{\alpha}(\Omega))^p} \leq c\|C_p(U_0, U_1)\|_{(\mathcal{H}_1)^{N-p} \times (\mathcal{H}_0)^{N-p}}, \tag{3.10.21}$$

where α is defined by (3.2.6), *while $\phi = (\phi_1, \cdots, \phi_p)$ is the solution to the following problem* $(1 \leq i \leq p)$

$$\begin{cases} \phi_i'' - \Delta \phi_i + \sum_{j=1}^{p} \alpha_{ji} \phi_j = 0 & in \quad (0, +\infty) \times \Omega, \\ \phi_i = 0 & on \quad (0, +\infty) \times \Gamma_0, \\ \partial_\nu \phi_i + \sum_{j=1}^{p} \beta_{ji} \phi_j = 0 & on \quad (0, +\infty) \times \Gamma_1, \\ t = 0: \qquad \phi_i = (E_i, U_0), \quad \phi_i' = (E_i, U_1) & in \quad \Omega, \end{cases} \tag{3.10.22}$$

in which

$$Ae_i = \sum_{j=1}^{p} \alpha_{ij} e_j, \quad Be_i = \sum_{j=1}^{p} \beta_{ij} e_j, \quad 1 \leq i \leq p. \tag{3.10.23}$$

Proof. Noting that $V = \text{span}\{E_1, \cdots, E_p\}$ is an invariant subspace of B^T, being biorthogonal to $\text{Ker}(C_p)$, by

$$Be_i = \sum_{j=1}^{p} \beta_{ij} e_j, \qquad i = 1, \cdots, p, \tag{3.10.24}$$

we have (see Lemma 4.11 in Appendix)

$$B^T E_i = \sum_{j=1}^{p} \beta_{ji} E_j, \qquad i = 1, \cdots, p. \tag{3.10.25}$$

Define the boundary control matrix D such that

$$Ker(D^T) = V. \tag{3.10.26}$$

Noting (3.8.5), we have

$$\mathrm{Ker}(C_p) \cap \mathrm{Im}(D) = \mathrm{Ker}(C_p) \cap \{\mathrm{Ker}(D^T)\}^{\perp} = \mathrm{Ker}(C_p) \cap V^{\perp} = \{0\}, \tag{3.10.27}$$

then, by Lemma 4.10 in Appendix, we have

$$\mathrm{rank}(C_p D) = \mathrm{rank}(D) = N - p. \tag{3.10.28}$$

Therefore, by Theorem 3.15, system (3.1.6) is exactly boundary synchronizable by p-groups.

Denoting $\psi_i = (E_i, U)(i = 1, \cdots, p)$, we have

$$\begin{aligned}(E_i, AU) = (A^T E_i, U) &= (\sum_{j=1}^{p} \alpha_{ji} E_j + A^T E_i - \sum_{j=1}^{p} \alpha_{ji} E_j, U)\\ &= \sum_{j=1}^{p} \alpha_{ji} (E_j, U) + (A^T E_i - \sum_{j=1}^{p} \alpha_{ji} E_j, U)\\ &= \sum_{j=1}^{p} \alpha_{ji} \psi_j + (A^T E_i - \sum_{j=1}^{p} \alpha_{ji} E_j, U).\end{aligned} \tag{3.10.29}$$

By the first formula of (3.10.23), noting the assumption that V is biorthogonal to $\mathrm{Ker}(C_p)$, without loss of generality, we may assume that

$$(E_i, e_j) = \delta_{ij} \quad (i, j = 1, \cdots, p), \tag{3.10.30}$$

then for any given $k \in \{1, \cdots, p\}$, we get

$$\begin{aligned}(A^T E_i - \sum_{j=1}^{p} \alpha_{ji} E_j, e_k) = (E_i, Ae_k) - \sum_{j=1}^{p} \alpha_{ji} (E_j, e_k) &= \sum_{j=1}^{p} \alpha_{kj} (E_i, e_j) - \alpha_{ki}\\ &= \alpha_{ki} - \alpha_{ki} = 0,\end{aligned} \tag{3.10.31}$$

hence

$$A^T E_i - \sum_{j=1}^{p} \alpha_{ji} E_j \in \{Ker(e_1, \cdots, e_p)\}^{\perp} = \mathrm{Im}(C_p^T), \quad i = 1, \cdots, p.$$

Thus, there exist $R_i \in \mathbb{R}^{N-p} (i = 1, \cdots, p)$ such that

$$A^T E_i - \sum_{j=1}^{p} \alpha_{ji} E_j = C_p^T R_i, \quad i = 1, \cdots, p. \tag{3.10.32}$$

Taking the inner product on both sides of (3.1.6) with E_i, and noting (3.10.25), we have

$$\begin{cases} \psi_i'' - \Delta\psi_i + \sum_{j=1}^{p} \alpha_{ji}\psi_j = -(R_i, C_p U) & \text{in} \quad (0,+\infty)\times\Omega, \\ \psi_i = 0 & \text{on} \quad (0,+\infty)\times\Gamma_0, \\ \partial_\nu\psi_i + \sum_{j=1}^{p}\beta_{ji}\psi_j = 0 & \text{on} \quad (0,+\infty)\times\Gamma_1, \\ t=0: \quad \psi_i = (E_i, U_0), \quad \psi_i' = (E_i, U_1) & \text{in} \quad \Omega, \end{cases} \tag{3.10.33}$$

Similarly to the proof of Theorem 3.12, we get

$$\|(\psi,\psi')(T) - (\phi,\phi')(T)\|_{(H^{\alpha+1}(\Omega))^p\times(H^\alpha(\Omega))^p} \le c\|C_p(U_0,U_1)\|_{(H^1(\Omega))^{N-p}\times(L^2(\Omega))^{N-p}}. \tag{3.10.34}$$

On the other hand, noting (3.10.30), it is easy to see that

$$t \ge T: \qquad \psi_i = (E_i, U) = \sum_{j=1}^{p}(E_i, e_j)u_j = u_i. \tag{3.10.35}$$

Substituting it into (3.10.34), we get (3.10.21). □

3.11 Continuous uniqueness theorem

In the proof or Theorem 3.18, we claim that under Hautus's criterion (3.9.39), the observation (3.9.38) on the infinite time interval $[T,+\infty)$ guarantees a unique trivial solution to problem (3.9.35). In this section, we further discuss this continuous uniqueness problem.

Noting that $\hat{\Lambda}$ in problem (3.9.35) is a symmetric matrix, without loss of generality, we consider the following coupled system of wave equations with Robin boundary conditions:

$$\begin{cases} u'' - \Delta u = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ v'' - \Delta v = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ u = v = 0, & \text{on} \quad (0,+\infty)\times\Gamma_0, \\ \frac{\partial u}{\partial\nu} + au = 0 & \text{on} \quad (0,+\infty)\times\Gamma_1, \\ \frac{\partial v}{\partial\nu} + bv = 0 & \text{on} \quad (0,+\infty)\times\Gamma_1. \end{cases} \tag{3.11.1}$$

Theorem 3.22. *Assume that $a, b > 0$ and $a \ne b$. If the solution (u,v) to system* (3.11.1) *satisfies the following partial boundary observation on an infinite time interval:*

$$\alpha u + \beta v = 0 \qquad \text{on} \quad (0,+\infty)\times\Gamma_1, \tag{3.11.2}$$

where $\alpha\beta \ne 0$, then we have

$$u \equiv v \equiv 0 \quad \text{in} \quad (0,+\infty)\times\Omega. \tag{3.11.3}$$

Proof. First we recall Green's formula

$$\int_\Omega \Delta uvdx = -\int_\Omega \nabla u\cdot\nabla vdx + \int_\Gamma \frac{\partial u}{\partial\nu}vd_\Gamma, \quad \forall\ u\in H^2(\Omega),\ v\in H^1(\Omega) \tag{3.11.4}$$

and Rellich's identity ([9])

$$2\int_\Omega \Delta u(m\cdot\nabla u)dx = (n-2)\int_\Omega |\nabla u|^2dx + 2\int_\Gamma \frac{\partial u}{\partial\nu}(m\cdot\nabla u)d_\Gamma - \int_\Gamma (m,\nu)|\nabla u|^2d_\Gamma, \quad \forall u\in H^2(\Omega), \tag{3.11.5}$$

where $m = x - x_0$.

Denote the energy function of system (3.11.1) as

$$E(t) = \frac{1}{2}\int_\Omega (|u_t|^2 + |\nabla u|^2 + |v_t|^2 + |\nabla v|^2)dx + \frac{a}{2}\int_{\Gamma_1}|u|^2d_\Gamma + \frac{b}{2}\int_{\Gamma_1}|v|^2d_\Gamma. \tag{3.11.6}$$

By system (3.11.1) and integrating by parts, we have

$$\begin{aligned} E'(t) &= \int_\Omega (u_tu_{tt} + \nabla u\cdot\nabla u_t + v_tv_{tt} + \nabla v\cdot\nabla v_t)dx + a\int_{\Gamma_1} uu_td_\Gamma + b\int_{\Gamma_1} vv_td_\Gamma \\ &= \int_\Omega (u_t\Delta u + \nabla u\cdot\nabla u_t + v_t\Delta v + \nabla v\cdot\nabla v_t)dx + a\int_{\Gamma_1} uu_td_\Gamma + b\int_{\Gamma_1} vv_td_\Gamma = 0, \end{aligned} \tag{3.11.7}$$

therefore

$$E(t) = E(0) \qquad \forall\ t\geq 0. \tag{3.11.8}$$

Taking the inner product with $2m\cdot\nabla u$ on both sides of the first equation of (3.11.1) and integrating by parts, we get

$$0 = 2\int_\Omega [u_t(m\cdot\nabla u)]|_{t=0}^T dx - \int_0^T\int_\Omega m\cdot\nabla|u_t|^2dxdt - 2\int_0^T\int_\Omega \Delta u(m\cdot\nabla u)dxdt. \tag{3.11.9}$$

Integrating by parts the second term on the right-hand side of (3.11.9), and using Rellich's identity (3.11.5) to the third term on the right-hand side of (3.11.9), we get

$$\begin{aligned} 0 = {} & 2\int_\Omega [u_tm\cdot\nabla u]|_{t=0}^T dx - \int_0^T\int_\Gamma (m,\nu)|u_t|^2d_\Gamma dt + n\int_0^T\int_\Omega |u_t|^2dxdt \\ & -(n-2)\int_0^T\int_\Omega |\nabla u|^2dxdt - 2\int_0^T\int_\Gamma \frac{\partial u}{\partial\nu}(m\cdot\nabla u)d_\Gamma dt + \int_0^T\int_\Gamma (m,\nu)|\nabla u|^2d_\Gamma dt, \end{aligned} \tag{3.11.10}$$

then

$$\begin{aligned} n\int_0^T\int_\Omega |u_t|^2dxdt + (2-n)\int_0^T\int_\Omega |\nabla u|^2dxdt \leq {} & cE(0) + \int_0^T\int_\Gamma (m,\nu)|u_t|^2d_\Gamma dt \\ & +2\int_0^T\int_\Gamma \frac{\partial u}{\partial\nu}(m\cdot\nabla u)d_\Gamma dt - \int_0^T\int_\Gamma (m,\nu)|\nabla u|^2d_\Gamma dt. \end{aligned} \tag{3.11.11}$$

Noting the multiplier geometric condition (3.1.1), there exist R and $\delta > 0$, such that

$$(m,\nu) \geq \delta > 0, \qquad \|m\|_\infty = R < +\infty,$$

then on Γ_1 we have

$$\begin{aligned}|\frac{\partial u}{\partial \nu}(m\cdot\nabla u)| \leq |\frac{\partial u}{\partial \nu}|\cdot R\cdot|\nabla u| &\leq \frac{\delta}{2}|\nabla u|^2 + \frac{2R^2}{\delta}|\frac{\partial u}{\partial \nu}|^2\\ &= \frac{\delta}{2}|\nabla u|^2 + \frac{2R^2}{\delta}a^2|u|^2.\end{aligned} \tag{3.11.12}$$

While, on Γ_0, noting $u = 0$, we have

$$\nabla u = \frac{\partial u}{\partial \nu}\nu \qquad \text{on} \quad (0,+\infty)\times\Gamma_0 \tag{3.11.13}$$

and

$$(m,\nu)|\nabla u|^2 = (m,\nu)|\frac{\partial u}{\partial \nu}|^2. \tag{3.11.14}$$

Thus, noting the multiplier geometric condition (3.1.1), we have

$$\int_0^T\int_{\Gamma_0}\frac{\partial u}{\partial \nu}(m\cdot\nabla u)d_\Gamma = \int_0^T\int_{\Gamma_0}(m,\nu)|\frac{\partial u}{\partial \nu}|^2 d_\Gamma \leq 0. \tag{3.11.15}$$

Substituting (3.11.12), (3.11.14) and (3.11.15) into (3.11.11), we get

$$\begin{aligned}&n\int_0^T\int_\Omega|u_t|^2dxdt + (2-n)\int_0^T\int_\Omega|\nabla u|^2dxdt\\ &\leq cE(0) + \int_0^T\int_{\Gamma_1}(m,\nu)|u_t|^2d_\Gamma dt + \frac{4R^2a^2}{\delta}\int_0^T\int_{\Gamma_1}|u|^2d_\Gamma dt + \int_0^T\int_{\Gamma_0}(m,\nu)|\frac{\partial u}{\partial \nu}|^2d_\Gamma\\ &\leq cE(0) + \int_0^T\int_{\Gamma_1}(m,\nu)|u_t|^2d_\Gamma dt + \frac{4R^2a^2}{\delta}\int_0^T\int_{\Gamma_1}|u|^2d_\Gamma dt.\end{aligned} \tag{3.11.16}$$

Taking the inner product with u on both sides of the first equation of (3.11.1) and integrating by parts, we have

$$0 = \int_\Omega[u_tu]|_{t=0}^Tdx - \int_0^T\int_\Omega|u_t|^2dxdt - \int_0^T\int_{\Gamma_1}\frac{\partial u}{\partial \nu}ud_\Gamma dt + \int_0^T\int_\Omega|\nabla u|^2dxdt, \tag{3.11.17}$$

then

$$-\int_0^T\int_\Omega|u_t|^2dxdt + a\int_0^T\int_{\Gamma_1}|u|^2d_\Gamma dt + \int_0^T\int_\Omega|\nabla u|^2dxdt \leq cE(0). \tag{3.11.18}$$

By (3.11.16) $+(n-1)\times$(3.11.18), we get

$$\begin{aligned}&\int_0^T\int_\Omega(|u_t|^2+|\nabla u|^2)dxdt\\ &\quad\leq (\frac{4R^2a^2}{\delta} - a(n-1))\int_0^T\int_{\Gamma_1}|u|^2d_\Gamma dt + \int_0^T\int_{\Gamma_1}(m,\nu)|u_t|^2d_\Gamma dt + cE(0).\end{aligned} \tag{3.11.19}$$

Similarly, we have

$$\int_0^T \int_\Omega (|v_t|^2 + |\nabla v|^2) dxdt \\ \leq (\frac{4R^2b^2}{\delta} - b(n-1)) \int_0^T \int_{\Gamma_1} |v|^2 d_\Gamma dt + \int_0^T \int_{\Gamma_1} (m,\nu)|v_t|^2 d_\Gamma dt + cE(0). \tag{3.11.20}$$

Thus, we have

$$\int_0^T E(t)dt \leq c \int_0^T \int_{\Gamma_1} (|u|^2 + |v|^2 + |u_t|^2 + |v_t|^2) d_\Gamma dt + cE(0). \tag{3.11.21}$$

Taking the inner product with v on both sides of the first equation of (3.11.1) and integrating by parts, we get

$$0 = \int_\Omega [u_t v]|_{t=0}^T dx - \int_0^T \int_\Omega u_t v_t dxdt + a \int_0^T \int_{\Gamma_1} uv d_\Gamma dt + \int_0^T \int_\Omega \nabla u \cdot \nabla v dxdt. \tag{3.11.22}$$

Similarly, taking the inner product with u on both sides of the second equation of (3.11.1) and integrating by parts, we get

$$0 = \int_\Omega [v_t u]|_{t=0}^T dx - \int_0^T \int_\Omega u_t v_t dxdt + b \int_0^T \int_{\Gamma_1} uv d_\Gamma dt + \int_0^T \int_\Omega \nabla u \cdot \nabla v dxdt. \tag{3.11.23}$$

It easily follows from (3.11.22)− (3.11.23) that

$$0 = \int_\Omega [u_t v - v_t u]|_{t=0}^T dx + (a-b) \int_0^T \int_{\Gamma_1} uv d_\Gamma dt. \tag{3.11.24}$$

By the boundary observation (3.11.2), we have $v = -\frac{\alpha}{\beta} u$ on Γ_1, then it comes from (3.11.24) that

$$\int_0^T \int_{\Gamma_1} |u|^2 d_\Gamma dt \leq cE(0). \tag{3.11.25}$$

Similarly, we have

$$\int_0^T \int_{\Gamma_1} |v|^2 d_\Gamma dt \leq cE(0). \tag{3.11.26}$$

Noting that the equations in (3.11.1) do not change after having taken derivative with respect to t, we obtain the following estimates of u_t, v_t on Γ_1:

$$\int_0^T \int_{\Gamma_1} |u_t|^2 d_\Gamma dt \leq c\hat{E}(0) \tag{3.11.27}$$

and

$$\int_0^T \int_{\Gamma_1} |v_t|^2 d_\Gamma dt \leq c\hat{E}(0), \tag{3.11.28}$$

where

$$\hat{E}(t) = \frac{1}{2} \int_\Omega (|\Delta u|^2 + |\nabla u_t|^2 + |\Delta v|^2 + |\nabla v_t|^2) dx + \frac{a}{2} \int_{\Gamma_1} |u_t|^2 d_\Gamma + \frac{b}{2} \int_{\Gamma_1} |v_t|^2 d_\Gamma. \tag{3.11.29}$$

Noting (3.11.8), and substituting (3.11.25)–(3.11.28) into (3.11.21), we get

$$TE(0) \leq c(E(0) + \hat{E}(0)). \tag{3.11.30}$$

Taking $T \to +\infty$, we have

$$E(0) = 0. \tag{3.11.31}$$

Hence, by (3.11.8) we get

$$E(t) \equiv 0, \quad \forall t \geq 0. \tag{3.11.32}$$

The proof is complete. □

Remark 3.12. *Denote the boundary coupling matrix B and the boundary control matrix D as*

$$B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad D = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{3.11.33}$$

respectively. It is easy to verify that the following Hautus's criterion

$$rank(B - \mu I_2, D) = 2, \quad \forall \mu \in \mathbb{R} \tag{3.11.34}$$

is equivalent to the fact that $a \neq b$ and $\alpha\beta \neq 0$. Therefore, Theorem 3.22 implies that the partial observation (3.11.2) *satisfying Hautus's criterion on an infinite time interval is a necessary and sufficient condition for the continuous uniqueness of system* (3.11.1)*.*

In the one-space-dimensional case, the result mentioned above can be further improved. In fact, under Hautus's criterion (3.11.34), the solution to problem (3.11.1) possesses the global continuous uniqueness only if it satisfies the partial boundary observation on a finite time interval.

Theorem 3.23. *Let $a \neq b > 0$. If the solution (u, v) to the one-space-dimensional system*

$$\begin{cases} u'' - u_{xx} = 0, & 0 < x < 1, \\ v'' - v_{xx} = 0, & 0 < x < 1, \\ u(t,0) = v(t,0) = 0, & \\ u_x(t,1) + au(t,1) = 0, & \\ v_x(t,1) + bv(t,1) = 0 & \end{cases} \tag{3.11.35}$$

satisfies the following partial boundary observation on a finite time interval large enough:

$$\alpha u(t,1) + \beta v(t,1) = 0, \qquad t \in [0,T], \tag{3.11.36}$$

where $\alpha\beta \neq 0$, then we have

$$u(t,x) = v(t,x) \equiv 0 \qquad in \quad (0,+\infty) \times (0,1). \tag{3.11.37}$$

We first recall a generalized Ingham's Inequality ([30]). Let $\mathbb{Z}$ denote the set of all the integers, and let $\{\beta_n^{(l)}\}_{1\le l\le m, n\in\mathbb{Z}}$ be a strictly increasing sequence of real numbers:

$$\cdots\beta_{-1}^{(1)} < \cdots < \beta_{-1}^{(m)} < \beta_0^{(1)} < \cdots < \beta_0^{(m)} < \beta_1^{(1)} < \cdots < \beta_1^{(m)} < \cdots. \tag{3.11.38}$$

Definition 3.2. *Sequence $\{e^{i\beta_n^{(l)}t}\}_{1\le l\le m; n\in\mathbb{Z}}$ is ω-linearly independent in $L^2(0,T)$, if*

$$\sum_{n\in\mathbb{Z}}\sum_{l=1}^{m} a_n^{(l)} e^{i\beta_n^{(l)}t} = 0 \quad on \quad [0,T] \quad and \quad \sum_{n\in\mathbb{Z}}\sum_{l=1}^{m} |a_n^{(l)}|^2 < +\infty$$

lead to

$$a_n^{(l)} = 0, \qquad n\in\mathbb{Z}, \quad 1\le l\le m.$$

Lemma 3.11. *Assume that (3.11.38) holds, and there exist positive constants c, s and γ, such that for all $1\le l\le m$ and $n\in\mathbb{Z}$ with $|n|$ large enough, we have*

$$\beta_{n+1}^{(l)} - \beta_n^{(l)} \ge m\gamma, \tag{3.11.39}$$

$$\frac{c}{|n|^s} \le \beta_n^{(l+1)} - \beta_n^{(l)} \le \gamma, \tag{3.11.40}$$

then, when $T > 2\pi D^+$, the sequence $\{e^{i\beta_n^{(l)}t}\}_{1\le l\le m; n\in\mathbb{Z}}$ is ω-linearly independent in $L^2(0,T)$, where D^+ is the upper density of the sequence $\{\beta_n^{(l)}\}_{1\le l\le m; n\in\mathbb{Z}}$ defined by

$$D^+ = \lim_{R\to+\infty} sup\frac{N(R)}{2R}, \tag{3.11.41}$$

in which $N(R)$ denotes the number of $\{\beta_n^{(l)}\}$ contained in the interval $[-R,R]$.

Proof of Theorem 3.23. Consider the following eigenvalue problem:

$$\begin{cases} \lambda^2\phi + \phi_{xx} = 0, & 0<x<1,\\ \lambda^2\psi + \psi_{xx} = 0, & 0<x<1,\\ \phi(0) = \psi(0) = 0, & \\ \phi_x(1) + a\phi(1) = 0, & \\ \psi_x(1) + b\psi(1) = 0. & \end{cases} \tag{3.11.42}$$

Let

$$\phi = \sin\lambda x, \qquad \psi = \sin\lambda x. \tag{3.11.43}$$

By the last two formulas in (3.11.42), we have

$$\lambda\cos\lambda + a\sin\lambda = 0, \qquad \lambda\cos\lambda + b\sin\lambda = 0. \tag{3.11.44}$$

Rewrite the first formula of (3.11.44) as

$$\tan\lambda + \frac{\lambda}{a} = 0. \tag{3.11.45}$$

By

$$e^{2i\lambda} = \cos 2\lambda + i \sin 2\lambda = \cos^2 \lambda - \sin^2 \lambda + 2i \sin \lambda \cos \lambda, \tag{3.11.46}$$

and noting (3.11.45), we have

$$\cos^2 \lambda - \sin^2 \lambda = \frac{1 - \tan^2 \lambda}{1 + \tan^2 \lambda} = \frac{a^2 - \lambda^2}{a^2 + \lambda^2}, \tag{3.11.47}$$

$$\sin \lambda \cos \lambda = \frac{\tan \lambda}{1 + \tan^2 \lambda} = -\frac{a\lambda}{a^2 + \lambda^2}, \tag{3.11.48}$$

then

$$e^{2i\lambda} = \frac{a^2 - \lambda^2}{a^2 + \lambda^2} - \frac{2a\lambda i}{a^2 + \lambda^2} = -\frac{\lambda + ai}{\lambda - ai}. \tag{3.11.49}$$

The asymptotic expansion of $e^{2i\lambda}$ at $\lambda = \infty$ gives

$$e^{2i\lambda} = -1 - \frac{2ai}{\lambda} + \frac{O(1)}{\lambda^2}. \tag{3.11.50}$$

Taking the logarithm on both sides, we get

$$\begin{aligned} 2i\lambda &= \ln(-1 - \frac{2ai}{\lambda} + \frac{O(1)}{\lambda^2}) + 2n\pi i \\ &= i\pi + \ln\big(1 + \frac{2ai}{\lambda} + \frac{O(1)}{\lambda^2}\big) + 2n\pi i \\ &= i\pi + \frac{2ai}{\lambda} + \frac{O(1)}{\lambda^2} + 2n\pi i, \end{aligned} \tag{3.11.51}$$

then

$$\lambda \triangleq \lambda_n^a = (n + \frac{1}{2})\pi + \frac{2a}{\lambda} + \frac{O(1)}{\lambda^2}. \tag{3.11.52}$$

Noting $\lambda_n^a \sim n\pi$, we get

$$\lambda_n^a = (n + \frac{1}{2})\pi + \frac{a}{n\pi} + \frac{O(1)}{n^2}. \tag{3.11.53}$$

Similarly, by the second formula of (3.11.44), we have

$$\lambda \triangleq \lambda_n^b = (n + \frac{1}{2})\pi + \frac{b}{n\pi} + \frac{O(1)}{n^2}. \tag{3.11.54}$$

Noting $a \neq b$, it follows from (3.11.45) that

$$\lambda_m^a \neq \lambda_n^b, \quad \forall m, n \in \mathbb{Z}. \tag{3.11.55}$$

Besides, by the monotonicity of the function $\lambda \to \tan \lambda + \frac{\lambda}{a}$, we have

$$\lambda_m^a \neq \lambda_n^a, \quad \lambda_m^b \neq \lambda_n^b, \quad \forall m, n \in \mathbb{Z}. \tag{3.11.56}$$

On the other hand, we have

$$\lambda_n^a - \lambda_n^b = \frac{a - b}{n\pi} + \frac{O(1)}{n^2}. \tag{3.11.57}$$

Without loss of generality, assuming $a > b > 0$, we can arrange $\{\lambda_n^a\} \cup \{\lambda_n^b\}$ into a monotonic increasing sequence

$$\cdots < \lambda_{-n}^a < \lambda_{-n}^b < \cdots < \lambda_1^b < \lambda_1^a < \cdots < \lambda_n^b < \lambda_n^a < \cdots,$$

and, by $a \neq b$, there exist positive constants $c, \gamma > 0$, such that

$$\frac{c}{|n|} \leq \lambda_n^a - \lambda_n^b \leq \gamma. \tag{3.11.58}$$

Thus, we get the corresponding eigenvectors

$$E_n^a = \begin{pmatrix} \frac{\sin \lambda_n^a x}{\lambda_n^a} \\ \sin \lambda_n^a x \end{pmatrix}, \qquad E_n^b = \begin{pmatrix} \frac{\sin \lambda_n^b x}{\lambda_n^b} \\ \sin \lambda_n^b x \end{pmatrix} \qquad n \in \mathbb{Z}, \tag{3.11.59}$$

and $\{E_n^a, E_n^b\}_{n\in\mathbb{Z}}$ consists a set of Hilbert basis in $(H^1(\Omega))^N \times (L^2(\Omega))^N$ (see [5]), then any given solution to system (3.11.35) can be represented by

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \sum_{n\in\mathbb{Z}} c_n^a e^{i\lambda_n^a t} E_n^a, \qquad \begin{pmatrix} v \\ v' \end{pmatrix} = \sum_{n\in\mathbb{Z}} c_n^b e^{i\lambda_n^b t} E_n^b. \tag{3.11.60}$$

By the boundary observation (3.11.36), we have

$$\alpha u(t,1) + \beta v(t,1) = \sum_{n\in\mathbb{Z}} \alpha c_n^a e^{i\lambda_n^a t} \frac{\sin \lambda_n^a}{\lambda_n^a} + \sum_{n\in\mathbb{Z}} \beta c_n^b e^{i\lambda_n^b t} \frac{\sin \lambda_n^b}{\lambda_n^b} = 0. \tag{3.11.61}$$

Since

$$D^+ = \limsup_{R\to+\infty} \frac{N(R)}{2R} = 2,$$

by Lemma 3.11, as $T > 4\pi$ we have

$$\alpha c_n^a \frac{\sin \lambda_n^a}{\lambda_n^a} = 0, \qquad \beta c_n^b \frac{\sin \lambda_n^b}{\lambda_n^b} = 0, \qquad \forall\, n \in \mathbb{Z}, \tag{3.11.62}$$

namely,

$$c_n^a = 0, \qquad c_n^b = 0, \qquad \forall\, n \in \mathbb{Z}, \tag{3.11.63}$$

hence (3.11.37) holds. □

Chapter 4

Appendix

4.1 Some preliminary algebraic results

Lemma 4.1 ([32]). *A subspace V of $\mathbb{R}^N$ is the complement of a subspace W of $\mathbb{R}^N$, if and only if*

$$dim(W) + dim(V) = N, \quad W \cap V = \{0\}. \tag{4.1.1}$$

Especially, W and $V^\perp$ (or $W^\perp$ and V) have the same dimension.

Lemma 4.2 ([32]). *Let V and W be two subspaces of $\mathbb{R}^N$. Then*

$$dim(W \cap V^\perp) = dim(V \cap W^\perp) \tag{4.1.2}$$

holds, if and only if

$$dim(W) = dim(V). \tag{4.1.3}$$

Definition 4.1. *Two non-trivial subspaces V and W of $\mathbb{R}^N$ are biorthogonal, if for any given set of basis $e_1, e_2, \cdots, e_r$ $(0 < r < N)$ of V, we can construct a corresponding set of basis $E_1, E_2, \cdots, E_r$ of W, such that $E_1, E_2, \cdots, E_r$ and $e_1, e_2, \cdots, e_r$ satisfy*

$$(e_i, E_j) = \delta_{ij}, \quad 1 \leq i, j \leq r, \tag{4.1.4}$$

where δ_{ij} is the Kronecker symbol.

Lemma 4.3 ([32]). *Assume that V and W are non-trivial subspaces of $\mathbb{R}^N$, then V and W are biorthogonal if and only if*

$$W \cap V^\perp = \{0\} \tag{4.1.5}$$

and

$$W^\perp \cap V = \{0\}. \tag{4.1.6}$$

Lemma 4.4 ([32]). *A subspace V of $\mathbb{R}^N$ is invariant for A, if and only if its orthogonal complement $V^\perp$ is invariant for A^T.*

Lemma 4.5. *Assume that a non-trivial subspace V is invariant for A and B. Then V possesses a complement $W^\perp$ which is also invariant for A and B, if and only if V and W are biorthogonal, and W is invariant for A^T and B^T.*

Proof. Let W be an invariant subspace for A^T and B^T, which is biorthogonal to V. By Lemma 4.4 in Appendix, $W^\perp$ is invariant for A and B. By Lemma 4.3 in Appendix, (4.1.5)-(4.1.6) hold. By Lemma 4.1 and Lemma 4.2 in Appendix, V and W have the same dimension, then we have

$$\dim(W^\perp) + \dim(V) = N - \dim(W) + \dim(V) = N. \tag{4.1.7}$$

By Lemma 4.1 in Appendix, $W^\perp$ is the complement of V.

On the contrary, assume that $W^\perp$ is the complement of V, and $W^\perp$ is invariant for A and B. By Lemma 4.4 in Appendix, $(W^\perp)^\perp = W$ is invariant for A^T and B^T. Besides, since $W^\perp$ is the complement of V, by Lemma 4.1 in Appendix we have

$$N = \dim(W^\perp) + \dim(V) = N - \dim(W) + \dim(V), \tag{4.1.8}$$

then V and W have the same dimension, hence, by Lemma 4.2 in Appendix, (4.1.5)–(4.1.6) hold. Thus, by Lemma 4.3 in Appendix, V and W are biorthogonal. □

Lemma 4.6. *There exists a unique minimum subspace V_0 containing e, such that V_0 possesses a complement $W^\perp$, and V_0 and $W^\perp$ are invariant for A and B.*

Proof. Let $\mathbb{V}$ be the set of all the subspaces V containing e, and V possesses a complement $W^\perp$ such that V and $W^\perp$ are invariant for A and B.

Assuming that $V_1, V_2 \in \mathbb{V}$, by Lemma 4.5 in Appendix, there exist two subspaces W_1, W_2 biorthogonal to V_1 and V_2, respectively, and W_1, W_2 are invariant for A^T and B^T. By $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$, we have

$$(W_1 \cap W_2)^\perp \cap (V_1 \cap V_2) \subseteq W_1^\perp \cap V_1 + W_2^\perp \cap V_2. \tag{4.1.9}$$

Since W_1, W_2 are biorthogonal to V_1 and V_2, respectively, by Lemma 4.3 in Appendix we have

$$W_1^\perp \cap V_1 = \{0\}, \quad W_2^\perp \cap V_2 = \{0\}, \tag{4.1.10}$$

then

$$(W_1 \cap W_2)^\perp \cap (V_1 \cap V_2) = \{0\}. \tag{4.1.11}$$

Similarly, we have

$$(W_1 \cap W_2) \cap (V_1 \cap V_2)^\perp = \{0\}. \tag{4.1.12}$$

By Lemma 4.1 and Lemma 4.2 in Appendix, $W_1 \cap W_2$ and $V_1 \cap V_2$ have the same dimension. Besides, since V_1 and V_2 contain e, then $W_1 \cap W_2$ and $V_1 \cap V_2$ are also non-trivial. By Lemma 4.3 in Appendix, $W_1 \cap W_2$ and $V_1 \cap V_2$ are biorthogonal.

On the other hand, noting that $V_1 \cap V_2$ is invariant for A and B, $W_1 \cap W_2$ is invariant for A^T and B^T, then, by Lemma 4.4 in Appendix, $(W_1 \cap W_2)^\perp$ is invariant for A and B. By Lemma 4.5 in Appendix, $V_1 \cap V_2$ possesses a complement $(W_1 \cap W_2)^\perp$, which is invariant for A and B, and $V_1 \cap V_2$ contains e, then $V_1 \cap V_2 \in \mathbb{V}$.

At last, define V_0 by

$$V_0 = \bigcap_{V \in \mathbb{V}} V. \tag{4.1.13}$$

Obviously, V_0 is unique, namely, it is the desired unique minimum subspace. □

Lemma 4.7 ([35]). *For any given $N \times N$ matrix A and any given full-row rank $M \times N$ matrix C, where $M < N$, there exists a unique $M \times M$ matrix $\overline{A}$ such that*

$$CA = \overline{A}C \tag{4.1.14}$$

if and only if $Ker(C)$ is an invariant subspace of A:

$$AKer(C) \subseteq Ker(C). \tag{4.1.15}$$

Matrix $\overline{A}$ satisfying (4.1.14) *can be expressed as*

$$\overline{A} = CAC^+, \tag{4.1.16}$$

and it is called the reduced matrix of A, where C^+ denotes the Moore-Penrose inverse of C:

$$C^+ = C^T(CC^T)^{-1}. \tag{4.1.17}$$

In particular, if

$$C = C_1 = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}_{(N-1)\times N}, \tag{4.1.18}$$

then the condition mentioned above is equivalent to the fact that $e = (1, 1, \cdots, 1)^T$ is an eigenvector of A, corresponding to the eigenvalue a, which is a constant independent of $i = 1, \cdots, N$, and defined by

$$\sum_{j=1}^{N} a_{ij} \overset{\text{def.}}{=} a \quad (i = 1, \cdots, N). \tag{4.1.19}$$

Lemma 4.8. *Let C_p be the matrix of synchronization by p-groups defined by* (3.8.4)*. Then for any given $N \times N$ positive definite symmetric matrix L, we have*

$$Ker(C_p) \cap Im(LC_p^T) = \{0\}. \tag{4.1.20}$$

Proof. Noting (3.8.5), if (4.1.20) does not hold, then there exist an $i (1 \le i \le p)$ and an $e_i \in \mathrm{Im}(LC_p^T)$, namely, there exists an $R \in \mathbb{R}^{N-1}$, such that

$$e_i = LC_p^T R \in \mathbb{R}^N, \tag{4.1.21}$$

then

$$\begin{aligned} L^{-1} e_i &= C_p^T R, \\ e_i^T L^{-1} &= R^T C_p. \end{aligned} \tag{4.1.22}$$

Taking the inner product on both sides with e_i and noting (3.8.5), we get

$$e_i^T L^{-1} e_i = R^T C_p e_i = 0. \tag{4.1.23}$$

This contradicts the fact that L is positive definite. □

Lemma 4.9. *If B is similar to a symmetric matrix, and B satisfies the C_p-condition of compatibility, then the reduced matrix $\overline{B}$ of B is also similar to a symmetric matrix.*

Proof. Since B is similar to a symmetric matrix, there exists a symmetric matrix $\hat{B}$ and an invertible matrix P such that $B = P\hat{B}P^{-1}$. Noting that B satisfies the C_p-condition of compatibility (see (4.1.14)), there exists a matrix $\overline{B}$ such that

$$CB = \overline{B}C. \tag{4.1.24}$$

Noting Lemma 4.8 in Appendix, PP^T is an $N \times N$ positive definite symmetric matrix, then

$$\mathrm{Ker}(C_p) \cap \mathrm{Im}(PP^T C_p^T) = \{0\}, \tag{4.1.25}$$

so that $\{PP^T C_p^T, e_1, e_2, \cdots, e_p\}$ consists a set of basis in $\mathbb{R}^N$, and we have

$$(CB - \overline{B}C)(PP^T C_p^T, e_1, e_2, \cdots, e_p) = 0. \tag{4.1.26}$$

By (3.8.5), we have

$$(CB - \overline{B}C)(e_1, e_2, \cdots, e_p) = 0, \tag{4.1.27}$$

then

$$(CB - \overline{B}C)PP^T C_p^T = 0, \tag{4.1.28}$$

namely,

$$CBPP^T C^T = \overline{B}CPP^T C^T. \tag{4.1.29}$$

Hence

$$\overline{B} = CP\hat{B}P^TC^T(CPP^TC^T)^{-1}, \tag{4.1.30}$$

which is similar to a symmetric matrix

$$(CPP^TC^T)^{-\frac{1}{2}}CP\hat{B}P^TC^T(CPP^TC^T)^{-\frac{1}{2}}. \tag{4.1.31}$$

□

Lemma 4.10 (See [32] and [33])**.** *Let C be an $(N-p)\times N$ $(1 \le p < N)$ matrix, and let D be an $N \times K$ matrix. Then*

$$rank(CD) = rank(D) \tag{4.1.32}$$

holds if and only if we have

$$Ker(C) \cap Im(D) = \{0\}. \tag{4.1.33}$$

Lemma 4.11. *Assume that B is similar to a symmetric matrix, and $Ker(C_p)$ is an invariant subspace of B. Then B^T admits an invariant subspace V, which is biorthogonal to $Ker(C_p)$.*

In particular, denoting

$$Ker(C_p) = Span\{e_1, \cdots, e_p\}, \quad V = Span\{E_1, \cdots, E_p\}, \tag{4.1.34}$$

if

$$Be_i = \sum_{j=1}^{p} \beta_{ij}e_j, \quad 1 \le i \le p, \tag{4.1.35}$$

then we have

$$B^TE_i = \sum_{j=1}^{p} \beta_{ji}E_j, \quad 1 \le i \le p. \tag{4.1.36}$$

Proof. Let

$$B = P\Lambda P^{-1}, \tag{4.1.37}$$

where P is an invertible matrix, Λ is a symmetric matrix. Let $V = \text{Span}(E_1, \cdots, E_p\}$, in which

$$E_i = P^{-T}P^{-1}e_i, \qquad i = 1, \cdots, p. \tag{4.1.38}$$

Noting (3.8.5) and the fact that $\text{Ker}(C_p)$ is an invariant subspace of B, we get

$$B^TE_i = P^{-T}P^{-1}Be_i \subset P^{-T}P^{-1}\text{Ker}(C_p) \subset V \qquad i = 1, \cdots, p, \tag{4.1.39}$$

then V is invariant for B^T.

Besides, noting

$$(PP^T E_i, e_j) = (e_i, e_j) = \delta_{ij}, \qquad 1 \leq i, j \leq p, \tag{4.1.40}$$

we can find a set of basis such that V and $\mathrm{Ker}(C_p)$ are biorthogonal.

In particular, denoting

$$Be_i = \sum_{j=1}^{p} \beta_{ij} e_j, \quad B^T E_i = \sum_{j=1}^{p} \hat{\beta}_{ij} E_j, \quad 1 \leq i \leq p, \tag{4.1.41}$$

we have

$$(Be_i, E_l) = \sum_{j=1}^{p} \beta_{ij} (e_j, E_l) = \beta_{il} \tag{4.1.42}$$

and

$$(e_i, B^T E_l) = \sum_{j=1}^{p} \hat{\beta}_{lj} (e_i, E_j) = \hat{\beta}_{li}, \tag{4.1.43}$$

then

$$\beta_{il} = \hat{\beta}_{li}. \tag{4.1.44}$$

□

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Dans cette thèse, nous étudions la synchronisation, qui est un phénomène bien répandu dans la nature. Elle a été observé pour la première fois par Huygens en 1665. En se basant sur les résultats de la contrôlabilité frontière exacte, pour un système couplé d'équations des ondes avec des contrôles frontières de Neumann, nous considérons la synchronisation frontière exacte (par groupes), ainsi que la détermination de l'état de synchronisation.
Ensuite, nous considérons la contrôlabilité exacte et la synchronisation exacte (par groupes) pour le système couplé avec des contrôles frontières couplés de Robin. A cause du manque de régularité de la solution, nous rencontrons beaucoup plus de difficultés. Afin de surmonter ces difficultés, on obtient un résultat sur la trace de la solution faible du problème de Robin grâce aux résultats de régularité optimale de Lasiecka-Triggiani sur le problème de Neumann. Ceci nous a permis d'établir la contrôlabilité exacte, et, par la méthode de la perturbation compacte, la non-contrôlabilité exacte du système. De plus, nous allons étudier la détermination de l'état de synchronisation, ainsi que la nécessité des conditions de compatibilité des matrices de couplage.

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