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Rational Curves on Irreducible Symplectic Varieties of OG10 type

submitted by

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Résumé de thèse

Les variétés holomorphes symplectiques irréductibles sont l'analogue algébrique des variétés hyperkähler irréductibles en géométrie Riemannienne. Soit X une variété complexe compact et kählerienne; X est holomorphe symplectique irréductible si son groupe fondamental est trivial et $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \sigma$, avec σ une 2-forme holomorphe non dégénérée en tout point.

Des exemples de variétés holomorphes symplectiques irréductibles sont: le schéma de Hilbert $\mathrm{Hilb}^n(S)$ qui paramètre les sous-schemas $Z \subset S$ avec S une surface $K3$, $\dim(Z) = 0$ et $lg(\mathcal{O}_Z) = n$, $\forall n \geq 2$ (cf. [Bea83], 6), les variétés de Kummer généralisées $\mathrm{K}_n(A)$, où A est tore complexe de dimension 2 et $n > 2$ (cf. [Bea83], 7), la variété $\tilde{\mathrm{K}}_6$ de O'Grady connue sous le nom de OG 6 (cf. [O'Gb]) et la variété $\tilde{\mathrm{M}}_{10}$ de O'Grady connue sous le nom de OG 10 (cf. [O'Ga]). Tous les exemples connus à ce jour de variétés holomorphes symplectiques irréductibles sont déformation d'un de ces exemples.

Soit X une variété holomorphe symplectique irréductible avec $\dim(X) = 2$; alors X est une surface $K3$. Dans ce cas, si de plus X est projective, d'après d'un théorème de Bogomolov et Mumford (cf. [MM83]), chaque courbe ample sur X est linéairement équivalente à une somme de courbes rationnelles. La présence de nombreuses courbes rationnelles dans X simplifie la structure du 0-group de Chow $\mathrm{CH}_0(X)_\mathbb{Q}$ de X: les courbes rationnelles définissent le cycle canonique de Beauville-Voisin (cf. [BV04]), qui est la classe d'un point dans une courbe rationnelle dans X. L'existence de cette classe est très importante car le group $\mathrm{CH}_0(X)_\mathbb{Q}$ n'est pas représentable (c'est le célèbre théorème de Mumford, cf. [Voi03b], Chapitre 10), et le cycle canonique de Beauville-Voisin fournit un candidat naturel pour la filtration de Bloch-Beilinson du group $\mathrm{CH}_0(X)$ pour une surface $K3$; en général, l'existence de la filtration de Bloch-Beilinson de $\mathrm{CH}_0(X)_\mathbb{Q}$ pour une variété projective X est une conjecture ouverte et très difficile, cf. [Voi03b], Chapitre 11.

Une question intéressante est de savoir si on peut construire de manière géométrique un candidat pour la filtration de Bloch-Beilinson de $\mathrm{CH}_0(X)_{\mathbb{Q}}$ pour toute les variétés holomorphes symplectiques irréductibles X. À cet égard il y a une nouvelle approche conjecturale de Voisin (cf. [Voi16], Conjecture 0.8), et le premier pas de cette approche consiste à montrer l'existence de "nombreux" diviseurs uniréglés dans X; rappelons qu'un diviseur est uniréglé s'il est couvert par des courbes rationnelles. L'existence de diviseurs uniréglés dans tous les systèmes linéaires amples serait une généralisation du thèoréme de Bogomolov et Mumford sur les courbes rationnelles sur les surfaces $K3$.

Dans [CMP19], Charles, Mongardi et Pacienza ont démontré l'existence de diviseurs uniréglés dans (presque) tous les systèmes linéaires amples sur une variété holomorphe symplectique qui est déformation d'un schéma de Hilbert $\mathrm{Hilb}^n(S)$; dans [MP17] et [MP19], Mongardi et Pacienza ont démontré le même résultat pour une variété holomorphe symplectique irréductible qui est déformation d'une variété de Kummer generalisée $\mathrm{K}_n(A)$. La stratégie de la preuve a été la même dans les deux cas: le groupe de monodromie $\mathrm{Mon}^2(X)$ permet d'identifier des représentants spéciaux dans toute composante connexe de l'espace des modules des variétés holomorphes symplectiques irréductibles d'un type de déformation fixé; en raison d'un résultat de déformation des courbes rationnelles sur une variété holomorphe symplectique irréductible (cf. [CMP19], Corollaire 3.5), il suffit de trouver des diviseurs uniréglés sur les représentants "spéciaux" déterminés grâce au groupe de monodromie.

Dans ma thése, j'ai travaillé sur le cas OG 10. La variété $\tilde{\mathrm{M}}_{10}$ définie par O'Grady est une désingularisation symplectique d'un espace des modules des faisceaux semistables sur une surface $K3$, avec des invariants fixés. Les variétés OG 10 sont importantes car elles ont fourni le premier exemple de variétés irréductibles holomorphes symplectiques qui ne soit pas déformation d'un schéma de Hilbert ponctuel sur une $K3$ ou d'une Kummer généralisée. Elles sont encore aujourd'hui très activement étudiées, cf. par exemple [MZ16], [LSV17], [Ono18], [dCRS19].

Le résultat principal de ma thèse démontre l'existence de diviseurs uniréglés amples sur chaque variétés holomorphes symplectiques irréductibles projectives appartenant à trois composantes connexes de l'espace de modules des OG 10. La stratégie consiste à construire de tels diviseurs sur des OG 10 spéciales. S'il est facile de vérifier qu'ils sont uniréglés, il est beaucoup plus compliqué de vérifier qu'ils sont amples. Cette vérification est faite en calculant le carré de ces diviseurs par rapport à la forme d'intersection de Beauville-Bogomolov-Fujiki, qui est une forme d'intersection sur le groupe

$H^2(X, \mathbb{Z})$ définie grâce à la forme symplectique; cette forme d'intersection a été calculée par Rapagnetta (cf. [Rap08]) dans le cas d'une variété holomorphe symplectique irréductible de type OG 10. Le résultat d'existence s'étend ensuite à toute la composante connexe grâce à un argument de déformation raffinant celui de [CMP19].

J'ai également démontré, grâce au calcul de certains invariants de monodromie, que les trois composantes connexes pour lesquelles mon résultat précédent montre l'existence de diviseurs uniréglés amples sont à deux à deux distinctes.

Malheureusement le groupe de monodromie n'est pas entièrement connu dans le cas OG 10: il n'y a qu'une description partielle (cf. [Mon14] et [Ono18]). Cette description partielle a été suffisante pour obtenir mes résultats, mais à ce stade elle ne permet pas de savoir s'il existe d'autres composantes connexes de l'espace des modules des variétés holomorphes symplectiques irréductibles de type OG10 non couvertes par mon résultat d'existence de diviseurs uniréglés amples.

Une considération sur le cas de la variété OG 6. Ce cas n'est pas encore connu, et l'intention de l'auteur est de l'explorer: le groupe de monodromie d'une variété holomorphe symplectique irréductible qui est déformation de OG 6 a été très récemment calculé par Mongardi et Rapagnetta (cf. [MR19]). Ces résultats n'étaient pas disponibles lorsque nous avons commencé à travailler sur le cas OG 10, ce qui explique pourquoi nous n'avons pas décidé d'enquêter avec le cas OG 6, ce qui semble aujourd'hui le cas naturel pour commencer.

Principaux résultats

Nous listons ici les principaux résultats que nous présentons dans ce travail. Nous nous référons à la section suivante pour une description de ce travail chapitre par chapitre.

- Corollaire 2.2.4: il s'agit d'un résultat sur la déformation de courbes rationnelles réduites mais réductibles régissant un diviseur sur une variété holomorphe symplectique irréductible, qui généralise celle présentée par Charles, Mongardi et Pacienza (cf. [CMP19]) pour les courbes irréductibles et réduites.

- Théorème 5.3.1 et Corollaire 5.3.3: nous y affirmons l'existence d'un premier exemple de diviseur ample uniréglés sur une variété holomorphe symplectique irréductible OG 10. Ces résultats sont la conséquence de nombreux résultats intermédiaires, présentés dans les Chapitres 4 et 5.

- Théorème 6.1.5 et Conjecture 6.1.6: nous construisons deux nouveaux diviseurs uniréglés sur variétés OG 10; nous affirmons que l'un est ample, et nous conjecturons qu'il en va de même pour le second. L'amplitude du second n'est conjecturale que parce qu'il manque quelques petites vérifications; nous présentons ici toutes les étapes terminées dans cette direction.

- Corollaire 6.2.5: nous calculons ici la divisibilité dans le réseau $H^2(\mathrm{OG}\,10, \mathbb{Z})$ des trois diviseurs trouvés précédemment. Cela donne de nouveaux invariants de déformation des nombreuses classes que nous avons trouvées.

Description de la thèse chapitre par chapitre

Dans le Chapitre 1, nous commençons par une présentation générale rapide des variétés holomorphes symplectiques irréductibles, en introduisant les constructions et en énonçant quelques résultats fondamentaux que nous utiliserons fortement dans tout ce qui viendra plus tard. En particulier, dans la section 1.1, nous introduisons la structure de réseau de Beauville-Bogomolov-Fuijiki sur le groupe $H^2(X, \mathbb{Z})$ et nous listons les exemples connus de variétés holomorphes symplectiques irréductibles. Dans la section **??**, nous introduisons l'espace des modules des variétés IHS marquées (et polarisées) et dans la Section 1.3, nous définissons le groupe de monodromie (polarisé) d'une variété holomorphes symplectiques irréductibles, qui se révèle être un outil fondamental pour comprendre l'espace des modules des variétés holomorphes symplectiques irréductibles marquées (et polarisées), et pour résoudre le problème de trouver sur eux de nombreux diviseurs uniréglés.

Avec Chapter 2, nous nous concentrons sur le problème de la recherche de courbes rationnelles sur les variétés holomorphes symplectiques irréductibles. Nous motivons notre intérêt pour ce problème dans la section 2.1, en exposant certaines conséquences pertinentes d'une telle existence sur le 0-groupe de Chow de la variété, et en le reliant à la filtration conjecturale de Voisin du 0-groupe de Chow, qui réalise conjecturellement la filtration Bloch-Beilinson plus générale dans le cas des variétés holomorphes symplectiques irréductibles. Dans la Section 2.2 nous exposons le résultat fondamental de Charles et Pacienza sur la déformation des courbes rationnelles sur les diviseurs sur les variétés holomorphes symplectiques irréductibles, qui motive la stratégie de résolution du problème, présentée dans la Section **??**. Avec Corollaire 2.2.4 nous présentons une légère généralisation du résultat original de Charles et Pacienza à des courbes non nécessairement irréductibles; nous avons besoin de cette généralisation pour les chapitres suivants. L'état de l'art concernant la recherche de courbes rationnelles sur les variétés holomorphes symplectiques irréductibles est présenté dans la section 2.3, où la stratégie de solution est également illustrée dans le cas OG 10, qui est celui qui nous intéresse dans ce thèse.

Historiquement, les variétés holomorphes symplectiques irréductibles de type OG 10 sont produites comme désingularisation des espaces de modules des faisceaux semi-stables sur les surfaces $K3$, avec quelques invariants fixes; ceci est présenté dans le Chapitre 3. Dans la Section 3.3, nous présentons la structure lagrangienne des espaces de modules qui nous intéressent, ce qui aide beaucoup à comprendre la géométrie de ces variétés et sera utilisé pour définir les exemples diviseurs amples uniréglés dans les chapitres suivants.

Dans le Chapitre 4, nous abordons le problème réel de trouver des courbes rationnelles sur les variétés holomorphes symplectiques irréductibles de type OG 10. Dans la Section 4.1, nous présentons une stratégie pour vérifier l'amplitude d'un diviseur sur une variété holomorphe symplectique irréductible, en calculant le carré Beauville-Bogomolov-Fujiki du diviseur; cette stratégie sera appliquée aux exemples de diviseurs uniréglés que nous présenterons en suite. La stratégie introduite commence par la définition de deux courbes sur la variété holomorphe symplectique irréductible X, définies dans la Section 4.2, et par le calcul de l'intersection de l'image des générateurs du réseau de Mukai avec ces courbes. Le résultat de ce calcul est indiqué dans la Proposition 4.2.3, et nous dédions la Section 4.3 à la preuve de ce résultat.

Le premier exemple de diviseur uniréglé est finalement introduit dans le Chapitre 5. Dans la Section 5.2, nous calculons son intersection avec les courbes introduites dans le chapitre précédent, ce qui est une étape dans la stratégie de calcul du carré du diviseur. Le carré est calculé dans le Théorème 5.3.1 et le Corollaire 6.1.9. La conclusion sur tout diviseur dans le même composant connecté de l'éspace de module est énoncée dans le Corollaire 5.3.4.

Dans le Chapitre 6, nous appliquons la même stratégie pour vérifier l'amplitude de deux nouveaux diviseurs uniréglés dans une variété holomorphe symplectique irréductible de type OG 10; dans le Théorème 6.1.5

et le Corollaire refconj.q.generalization, nous calculons le carré de Beauville-Bogomolov-Fujiki du premier diviseur, et nous conjecturons le carré du second. Le calcul du carré du premier diviseur présenté dans le Chapitre 6 est plus délicat que le cas présenté dans le Chapitre 5, car ce diviseur peut être non Cartier; cela dépend du modèle de OG 10-variété que nous devons choisir pour définir le diviseur. Étant donné que la stratégie présentée dans le chapitre 4 pour calculer le carré Beauville-Bogomolov-Fujiki d'un diviseur n'est valable que pour les diviseurs Cartier, nous commençons à prouver que le diviseur introduit est Cartier. Dans la section 6.2, le Corollaire 6.2.5, nous concluons le calcul d'un autre invariant de monodromie: la divisibilité des diviseurs.

À la fin, nous avons inséré deux appendices. Dans l'Appendice A, nous avons collecté des théorèmes sur la représentabilité des groupes de Chow et les filtrations conjecturales de Bloch-Beilinson et Voisin; ce motive notre intérêt pour les courbes rationnelles sur les variétés holomorphes symplectiques irréductibles. Dans l'Appendice B, nous avons rappelé la définition des transformées de Fourier-Mukai, qui donnent des morphismes birationnels parmi les variétés IHS de type OG 10 que nous avons utilisées dans la Section 6.2 pour calculer la divisibilité des diviseurs amples uniréglés trouvés précédemment.

Introduction

Irreducible holomorphic symplectic varieties are the algebraic analogous of irreducible hyperkähler manifolds in Riemannian geometry. A compact and connected Riemannian manifold (M, g) of dimension $4n$ is said to be hyperkähler if its holonomy group H is contained in the symplectic group $\mathrm{Sp}(n)$; it is said to be irreducible hyperkähler if $H = \mathrm{Sp}(n)$. In general, a manifold (M, g) is said to be irreducible if its holonomy representation is irreducible. By a theorem of de Rham, any complete and simply connected Riemmanian manifold decomposes as product of irreducible manifolds; furthermore, a theorem of Berger classifies all possible holonomy groups for an irreducible Riemannian manifold (M, g) which is not locally symmetric. When M is Kähler, the only possibilities are the unitary group $\mathrm{U}(m)$, the special unitary group $\mathrm{SU}(m)$ and the symplectic group $\mathrm{Sp}(n)$, with $\dim(M) = 2m = 4n$ (see e.g. [Bea83]). The locally symmetric case was already known by a result of Cartan (see Section 3 in [GHJ03]).

The holonomy group H is a powerful tool to study the geometry of the manifold (M, g). For example, if $H \subset \mathrm{U}(m)$, then M is Kähler; if $H \subset \mathrm{SU}(m)$, then (M, g) is Ricci flat and M is called a Calabi-Yau manifold. When $H \subset \mathrm{Sp}(n)$, which is the case we are interested in, as a consequence of the holonomy principle one gets an action of the quaternions $\mathbb{H}$ on the tangent bundle of M, and then an almost complex structure on M for any $h \in \mathbb{H}$ with $h^2 = -1$. It turns out that all those complex structures are integrable, and once fixed a basis $\{I, J, K\}$ of $\mathbb{H}$ any such an h can be written as $aI + bJ + cK$, with $a^2 + b^2 + c^2 = 1$. In this way one gets a S^2-family of complex structures on M, and g is Kähler with respect to any such $h \in S^2 \cong \mathbb{P}^1$. If M is compact and irreducible, all those Kähler structures give to M a structure of irreducible holomorphic symplectic variety, cf. Definition 1.1.1; see Remark 1.1.2 and [Huy99] for more details on the link between hyperkähler manifolds and irreducible holomorphic symplectic varieties. Irreducible holomorphic symplectic (IHS) varieties are the objects we are interested in, and most of the time will be interested on projective ones.

Several examples of IHS varieties are known; anyway, up to deformations, they reduce to the folliwing ones: two families of examples introduced by Beauville in [Bea83], for any possible dimension; they are the Hilbert scheme of 0-dimensional subschemes of length n on a $K3$ surface, for any $n \geq 1$ (see Example 1.1.9) and the generalized Kummer variety (see Example 1.1.10). The only two other examples are due to O'Grady (cf [O'Ga] and [O'Gb]); they are two special examples in dimension 6 and 10 respectively, see Example 1.1.11. Note that IHS varieties have always even complex dimension, because of the existence of a symplectic form, see Remark 1.1.3. We will refer to an IHS variety deformation equivalent to one of those listed above as to an IHS variety of $K3^{[n]}/K_n(A)/\operatorname{OG}6/\operatorname{OG}10$-type respectively.

When $\dim X = 2$, with X an IHS variety, then X is a $K3$ surface. In fact, IHS varieties of higher dimension share many properties with $K3$ surfaces. The intersection pairing gives to the second integral cohomology group of a $K3$ surfaces a lattice structure, which determines the $K3$ surface itself: the Torelli theorem for a $K3$ surface states that $K3$ surfaces with Hodge-isometric second integral cohomology groups are actually isomorphic. Thanks to the work of Beauville, Bogomolov and Fuijiki, it is possible to give a lattice structure to the group $H^2(X, \mathbb{Z})$ for any IHS variety X (see Fact 1.1.6), and a weaker version of the Torelli theorem is known thanks to Huybrechts (in [Huy99]), Markman (in [Mar11]) and Verbitsky (in [Ver13]), see Theorem 1.2.4.

A relevant geometrical property of $K3$ surfaces is that they contain many rational curves. This is the content of Bogomolov-Mumford theorem (cf. Theorem 2.1.1), which states that any ample linear system on a projective $K3$ surface contains an element which is sum of rational curves. Rational curves simplify the structure of the rational 0-Chow group $\mathrm{CH}_0(X)_{\mathbb{Q}}$ of a variety X: in the $K3$-case, they individuate a special class in $\mathrm{CH}_0(X)_{\mathbb{Q}}$, known as the Beauville-Voisin canonical 0-cycle, which is the class of any point on a rational curve. This is particularly relevant because the 0-Chow group of a $K3$ surface is known to be non-representable, as consequence of the Mumford theorem (see Theorem A.1.5), and the Beauville-Voisin canonical 0-cycle gives the Bloch-Beilinson filtration for the group $\mathrm{CH}_0(X)_{\mathbb{Q}}$, see Conjecture A.2.3.

Mumford theorem implies the non-representability of the group $\mathrm{CH}_0(X)_{\mathbb{Q}}$ not only for X a $K3$ surface, but for any IHS variety X. An interesting question is then weather it is possible to find a geometrical Bloch-Beilinson filtration for any IHS variety X, once proved the existence of enough rational curves in X; this is the content of Voisin's Conjecture 2.1.4 (cf [Voi16]). A first step in this direction would be the existence of rational curves ruling divisors on IHS varieties in any ample linear system; a divisor ruled by rational curves is called uniruled (see Definition 2.1.6). The existence of uniruled divisors in any ample system would be a generalization of the Bogomolov-Mumford theorem in the $K3$-case. By a theorem of Charles, Mongardi and Pacienza (see Theorem 2.1.7 and [CMP19]), the existence of uniruled divisors in any ample linear system would imply the existence of a canonical subgroup $S_1 \operatorname{CH}_0(X)_{\mathbb{Q}} \subset \operatorname{CH}_0(X)_{\mathbb{Q}}$, which is the group generated by points on an uniruled divisor; this would generalize the Beauville-Voisin canonical 0-cycle of the $K3$-case, realizing the first subgroup of the conjectural Voisin's filtration mentioned above.

The existence of uniruled divisors on IHS varieties have been proved for most of the IHS varieties of $K3^{[n]}/K_n(A)$-type by Charles, Mongardi, Pacienza in [CMP19] and Mongardi, Pacienza in [MP17] and [MP19] respectively, see Section 2.3 for the precise statement. The technique used is the same in both cases, and it is the following: they consider the moduli spaces $\mathfrak{M}^{pol}_{K3^{[n]}}$ and $\mathfrak{M}^{pol}_{K_n(A)}$ of marked and polarized IHS varieties of $K3^{[n]}$ and $K_n(A)$-type respectively, whose number of connected components equals the index of the monodromy group in the group of isometries of $H^2(X,\mathbb{Z})$, for X an IHS variety of $K3^{[n]}/K_n(A)$-type respectively (we refer to Section 1.2 and Section 1.3 for the fundamental definitions and results about that). In particular, thanks to the knowledge of the monodromy group in the $K3^{[n]}/K_n(A)$-case, they find an explicit representative for each connected component of $\mathfrak{M}^{pol}_{K3^{[n]}}$ and $\mathfrak{M}^{pol}_{K_n(A)}$ respectively, and they prove the existence of uniruled divisors for almost all those representatives. Thanks to a result of deformation of reduced and irreducible rational curves ruling divisors on IHS varieties presented in [CMP19] (see also Corollary 2.2.4), they conclude the existence of uniruled divisors in any class on a connected component of $\mathfrak{M}^{pol}_{K3^{[n]}}$ and $\mathfrak{M}^{pol}_{K_n(A)}$ where they could prove the existence of uniruled divisors for the respective representative.

The main goal of this work is to explore the OG 10-case. In this regard, a remark is necessary: the monodromy of an IHS variety of OG 10-type is not known yet. There are some partial results by Mongardi (cf. [Mon14]) and Onorati (cf. [Ono18]) in this direction (see Section 2.3), but they are not enough to estimate the number of connected components of the moduli space $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$ of marked and polarized IHS varieties of OG 10-type.

Nevertheless, one can still start finding explicit examples of ample uniruled divisors on IHS varieties of OG 10-type, and compute some monodromy invariants to check weather they are or not in the same connected component of $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$. We proved the existence of two ample uniruled divisors and we conjecture the existence of a third one, and we compute their monodromy invariants, see Chapter 5 and Chapter 6. The definition of these divisors is done using the rich geometry of some moduli spaces of semistable sheaves on $K3$ surfaces, which have a desingularization that is an IHS variety of OG 10-type containing, as dense open subset, a relative Jacobian on some linear system on the $K3$ surface. As consequence of the result by Charles, Mongardi and Pacienza about deformations of rational curves presented above, we can conclude the existence of ample uniruled divisors for any element of $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$ in the connected component of one of the three divisors found.

We conclude this section with a consideration on the OG 6-case. This case is not know yet, and it is intention of the author to explore it: the monodromy group of an IHS variety deformation equivalent to OG 6 has been very recently computed by Mongardi and Rapagnetta in [MR19]. This results was not available when we started working on the OG 10-case, which motivates why we didn't decide to investigate the OG 6-case, which seems today the natural case for starting with.

Main results

We list here the main results we present in this work. We refer to the next section for a description of this work chapter by chapter.

- Corollary 2.2.4: this is a result about deformation of reduced but reducible rational curves ruling a divisor on an IHS variety, that generalizes the one presented by Charles, Mongardi and Pacienza in [CMP19] for irreducible and reduced curves.
- Theorem 5.3.1 and Corollary 5.3.3: we state there the existence of a first example of ample uniruled divisor on a IHS variety of OG 10-type. These results are consequence of many intermediates results, presented in Chapter 4 and Chapter 5.
- Theorem 6.1.5 and Conjecture 6.1.6: we construct two new uniruled divisors on IHS varieties of OG 10; we state that one is ample, and we conjecture that the same holds true for the second one. The ampleness

of the second one is only conjectured because there are some small verification missing; we present here all the completed steps in this direction.

- Corollary 6.2.5: we compute here the divisibility in the lattice $H^2(\mathrm{OG}\,10, \mathbb{Z})$ of the three divisors found previously. This gives new deformation invariants of the ample classes that we have found.

Description of the thesis chapter by chapter

In Chapter 1 we start with a quick general presentation of IHS varieties, introducing the constructions and stating some fundamental results that we will strongly use in everything that will come later. In particular, in Section 1.1 we introduce the Beauville-Bogomolov-Fuijiki lattice structure on the group $H^2(X, \mathbb{Z})$ and we list the known examples of IHS varieties. In Section 1.2 we introduce the moduli space of marked (and polarized) IHS varieties and in Section 1.3 we define the (polarized) monodromy group of an IHS variety, which turns out to be a fundamental tool for understanding the moduli space of marked (and polarized) IHS varieties, and to solve the problem of finding ample uniruled divisors on them.

With Chapter 2 we focus on the problem of finding rational curves on IHS varieties. We motivate our interest in this problem in Section 2.1, stating some relevant consequence of such existence on the 0-Chow group of the variety, and relating it to the conjectural Voisin's filtration of the 0-Chow group, which conjecturally realizes the more general Bloch-Beilinson filtration in the case of IHS varieties. In Section 2.2 we state the fundamental result by Charles and Pacienza on deformation of rational curves on divisors on IHS varieties, which motivates the strategy of solution of the problem, presented in Section 2.3. With Corollary 2.2.4 we present a slight generalization of the original result by Charles and Pacienza to curves non necessarily irreducible; we need this generalization for the following chapters. The state of art about the research of rational curves on IHS varieties is presented in Section 2.3, where the strategy of solution is also illustrated in the OG 10-case, which is the one of interest in this thesis.

IHS varieties of OG 10-type are historically produced as desingularization of moduli spaces of semistable sheaves on $K3$ surfaces, with some fixed invariants; this is presented in Chapter 3. In Section 3.3 we present the Lagrangian structure of the moduli spaces we are interested in, which helps a lot to understand the geometry of these varieties and will be used to define the examples of ample uniruled divisors in the following chapters.

In Chapter 4 we enter into the actual problem of finding rational curves on IHS varieties of OG 10-type. In Section 4.1 we present a strategy to check the ampleness of a divisor on an IHS variety, done computing the Beauville-Bogomolov-Fujiki square of the divisor; this strategy will be applied to the examples of uniruled divisors that we will introduce in the sequel. The strategy introduced start with the definition of two curves on the IHS variety X, defined in Section 4.2, and with the computation of the intersection of the image of the generators of the Mukai lattice with those curves. The result of this computation is stated in Proposition 4.2.3, and we dedicate Section 4.3 to the proof of this result.

The first example of uniruled divisor is finally introduced in Chapter 5. In Section 5.2 we compute its intersection with the curves introduced in the chapter before, which is a step in the strategy to compute the square of the divisor. The square is computed in Theorem 5.3.1 and Corollary 6.1.9. The conclusion on any divisor in the same connected component of the ample uniruled found is stated in Corollary 5.3.4.

In Chapter 6 we apply the same strategy to check the ampleness of two new uniruled divisors in an IHS variety of OG 10-type; in Theorem 6.1.5 and Corollary 6.1.6 we compute the Beauville-Bogomolov-Fujiki square of the first one and we conjecture the square of the second one. The computation of the square of the first divisor presented in Chapter 6 is more delicate than the case presented in Chapter 5, since this divisor can happen to be non Cartier; this depends on the model of OG 10-variety that we need to chose in order to define it. Since the strategy presented in Chapter 4 to compute the Beauville-Bogomolov-Fujiki square of a divisor is valid only for Cartier divisors, we start proving that the divisor introduced is Cartier. In Section 6.2, Corollary 6.2.5 we conclude computing a further monodromy invariant: the divisibility of the divisors.

At the end we inserted two appendices. In Appendix A we collected theorems about representability of Chow groups and the conjectural filtrations by Bloch-Beilinson and Voisin, which motivates our interest in rational curves on IHS varieties. In Appendix B we recalled the definition of Fourier-Mukai transforms, which give birational morphisms among IHS varieties of OG 10-type that we used in Section 6.2 to compute the divisibiliy of the ample uniruled divisors found previously.

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Chapter 1

Generalities on irreducible holomorphic symplectic varieties

We start with a general introduction on irreducible holomorphic symplectic varieties, where we will give the basic definitions and we will collect some very useful and general results that we will need in the next chapters. In particular, we will introduce irreducible holomorphic symplectic varieties and give the fundamental examples, and we will present the moduli space of irreducible holomorphic symplectic varieties, stating some Torelli type theorems and introducing the monodromy group of these varieties.

Most of the results we are going to state can be found in Huybrechts' notes [Huy99]. We will always work over the field of complex numbers.

1.1 Definitions and examples

Definition 1.1.1. Let X a compact Kähler manifold. X is an irreducible holomorphic symplectic (IHS) variety if the following conditions hold true:

- $\pi_1(X) = \{e\}$
- $H^0(X, \Omega^2_X) = \mathbb{C} \cdot \sigma$, with σ everywhere non-degenerate holomorphic 2-form; σ is called the symplectic form of X.

Remark 1.1.2. IHS varieties are the algebraic analogous of hyperkähler manifolds in Riemannian geometry. A hyperkähler manifold (HK) is a Riemannian manifold (M, g) such that its holonomy group is contained in $\mathrm{Sp}(n)$.

IHS and HK are analogous in the sense that any IHS variety X with a fixed Kähler class α determines a HK manifold (M, g), where M is the real manifold underlying X and g is the unique Ricci-flat Kähler metric with Kähler class α. Conversely, any HK manifold (M, g) carries an integrable almost complex structure defining a complex Kähler variety X which turns out to be IHS. For more details, see [Bea83].

Remark 1.1.3. An immediate consequence of the existence of the symplectic form σ on an IHS variety X is that $\dim(X)$ is even. This just follows from the existence of a non-degenerate alternating form on the tangent space T_xX for any $x \in X$. In what follows, we will always denote $\dim(X) = 2n$.

Remark 1.1.4. Given an IHS variety X, the symplectic form σ gives an isomorphism $\mathcal{T}_X \cong \Omega^1_X$, and then $H^0(X, \mathcal{T}_X) \cong H^0(X, \Omega^1_X) \cong H^{1,0}(X) \cong H^{0,1}(X) = 0$: by definition X is simply connected, then $H_1(X, \mathbb{Z}) = 0$ and $0 = H^1(X, \mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X)$. More in general, $H^0(X, \Omega^p_X) = 0$ for any odd p, see Section 1.7 in [Huy99].

The top power σ^n of the symplectic form gives a nowhere vanishing global section of the canonical bundle ω_X; hence $\omega_X \cong \mathcal{O}_X$ and $c_1(X) = 0$. One of the main motivation behind the interest in IHS varieties is that they are actually building blocks of varieties with trivial first Chern class, as stated in the following Beauville-Bogomolov decomposition theorem (see [Bea83]):

Theorem 1.1.5. *Let X be a smooth projective variety with $c_1(X) = 0$. Then there exists a finite étale covering $X' \to X$ such that X' decomposes as $X' = T \times \Pi_i Y_i \times \Pi_j Z_j$ with T complex torus, Y_i Calabi-Yau varieties and Z_j IHS varieties.*

A generalization of Theorem 1.1.5 has been recently proved in [HP19] in the case of normal projective varieties with at most klt singularities with trivial first Chern class.

IHS varieties are the higher dimensional analogous of $K3$ surfaces. Indeed, not only any 2-dimensional IHS variety is a $K3$ surface (see Example 1.1.8), but it is also possible to define a symmetric form on the group $H^2(X, \mathbb{Z})$ for any IHS variety X, which generalizes the intersection product on a $K3$ surface and which permits to obtain very powerful consequences on the moduli space of IHS varieties, as we will see in Section 1.2. The existence of such a symmetric form is due to Beauville (see [Bea83]); we will quickly sketch his construction here.

Fact 1.1.6. Let X be an IHS variety and $\sigma \in H^0(X, \Omega_X^2)$ s.t. $\int_X (\sigma\bar{\sigma})^n = 1$. Given $\alpha \in H^2(X, \mathbb{Q})$,

$$f_X(\alpha) := \frac{n}{2}\int_X (\sigma\bar{\sigma})^{n-1}\alpha^2 + (1-n)\Big(\int_X \sigma^{n-1}\bar{\sigma}^n\alpha\Big)\cdot\Big(\int_X \sigma^n\bar{\sigma}^{n-1}\alpha\Big)$$

defines a quadratic form on $H^2(X, \mathbb{Q})$. Furthermore, there exists a positive constant $c \in \mathbb{R}$ s.t. $q_X := c \cdot f_X$ is a primitive integral quadratic form on $H^2(X, \mathbb{Z})$, known as the Beauville-Bogomolov-Fujiki form of X. $(H^2(X, \mathbb{Z}), q_X)$ is a lattice of index $(3, b_2(X) - 3)$. Furthermore, $q_X(\sigma) = 0$ and $q_X(\sigma + \bar{\sigma}) > 0$. We will denote by $q_X(\cdot, \cdot)$ the associated bilinear form.

Fact 1.1.7. In [Fuj87], Fujiki showed that there exists a positive constant $c \in \mathbb{Q}$, known as Fujiki constant, such that for any $\alpha \in H^2(X, \mathbb{Z})$

$$\int_X \alpha^{2n} = c \cdot q_X(\alpha)^n.$$

We end this section giving some example of IHS varieties. This list of examples is particularly relevant since any known example of IHS varieties is deformation equivalent to one of the examples we are going to present. It is not known whether this list of examples is complete or not.

Example 1.1.8. As fist example, we look to the case $\dim(X) = 2$. Since $\pi_1(X) = \{e\}$ one has $H_1(X, \mathbb{Z}) = 0$, and as we have already noticed the existence of the symplectic form implies that the canonical bundle of X is trivial. We get that X is a projective $K3$ surface. In this case, q_X is just the intersection product.

Example 1.1.9. Given S projective $K3$ surface, one can consider the Hilbert schemes of 0-dimensional subschemes of S of length n, that we will denote here and in the next by $S^{[n]}$. In [Bea83], Beauville showed that $S^{[n]}$ is an IHS variety; for $n = 2$ the result was already obtained by Fujiki.

Beauville also showed $b_2(S^{[n]}) = 23$ for $n > 1$, and he computed the Beauville-Bogomolov-Fujiki form of $S^{[n]}$:

$$(H^2(S^{[n]}, \mathbb{Z}), q_{S^{[n]}}) \cong H^2(S, \mathbb{Z}) \oplus \mathbb{Z} \cdot \delta$$

where $H^2(S, \mathbb{Z})$ is equipped with the intersection pairing, the direct sum is orthogonal and $\delta^2 = -2(n-1)$. Geometrically, δ is a primitive class s.t. $2\delta = [E]$ in $H^2(S^{[n]}, \mathbb{Z})$, where E is the exceptional divisor of the Hilbert-Chow morphism $S^{[n]} \to \mathrm{Sym}^n(S)$.

Example 1.1.10. Given A abelian surface, one can also consider the Hilbert scheme $A^{[n]}$ of 0-dimensional subschemes of lenght n, but it fails both to be simply connected and to have a unique symplectic form. In [Bea83], Beauville proved that a fiber of the composition of the Hilbert-Chow morphism and the sum map of the abelian surface $A^{[n+1]} \to A$ is an IHS variety of dimension $2n$, known as the generalized Kummer variety $K_n(A)$, $n \geq 1$.

Beauville also proved that $b_2(K_n(A)) = 7$ for $n > 1$ and he computed the Beauville-Bogomolov-Fujiki form of $K_n(A)$:

$$(H^2(K_n(A), \mathbb{Z}), q_{K_n(A)}) \cong H^2(A, \mathbb{Z}) \oplus \mathbb{Z} \cdot \delta$$

where as before $H^2(A, \mathbb{Z})$ is equipped with the intersection pairing, the sum is orthogonal and now $\delta^2 = -2(n+1)$. Geometrically, δ is a primitive class s.t. 2δ is the class of the restriction of the exceptional divisor of the Hilbert-Chow morphism $A^{[n+1]} \to \mathrm{Sym}^{n+1}(A)$ to $K_n(A)$.

Note that these families of examples given by Beauville actually give two examples of IHS varieties in each possible dimension, and for a long time they have been the only known examples up to deformations. Furthermore, these two families of examples are not deformation equivalent since they have different Betti numbers.

Example 1.1.11. In [O'Gb] and [O'Ga] O'Grady gave two examples of IHS varieties of new deformation types; one is known as $\tilde{\mathrm{K}}_6$ and has dimension 6, and another one is known as $\widetilde{\mathrm{M}}_{10}$ and has dimension 10. They are obtained as desingularization of moduli spaces of sheaves on an abelian variety or a projective $K3$ surface respectively, with some fixed invariants.

Very briefly, $\tilde{\mathrm{K}}_6$ is a Beauville-Bogomolov building block (cf. Theorem 1.1.5) of a desingularization of the moduli space of semistable sheaves $\mathcal{F}$ on the Jacobian J of a smooth projective genus two curve, with $rk(\mathcal{F}) = 2$, $c_1(\mathcal{F}) = 0$ and $ch_2(\mathcal{F}) = -2$; $\widetilde{\mathrm{M}}_{10}$ is a symplectic desingularization of the moduli space of semistable sheaves $\mathcal{F}$ on a $K3$ surface with $rk(\mathcal{F}) = 2$, $c_1(\mathcal{F}) = 0$ and $c_2(\mathcal{F}) = 4$. We will be particularly interested in $\widetilde{\mathrm{M}}_{10}$, that we will introduce in Chapter 3. O'Grady also showed that $b_2(\tilde{\mathrm{K}}_6) = 8$ and $b_2(\widetilde{\mathrm{M}}_{10}) \geq 24$, which proves that $\tilde{\mathrm{K}}_6$ and $\widetilde{\mathrm{M}}_{10}$ are not deformation equivalent to the Beauville's examples.

The Beauville-Bogomolov-Fujiki quadratic forms of $\tilde{\mathrm{K}}_6$ and $\widetilde{\mathrm{M}}_{10}$ have been computed by Rapagnetta in [Rap07] and [Rap08] respectively; in [Rap08], Rapagnetta also proved $b_2(\widetilde{\mathrm{M}}_{10}) = 24$. If we set $q_6 := q_{\tilde{\mathrm{K}}_6}$,

$$(H^2(\tilde{\mathrm{K}}_6, \mathbb{Z}), q_6) \cong H^2(J, \mathbb{Z}) \oplus \Lambda$$

where J is the Jacobian of a genus 2 curve where sheaves in $\tilde{\mathrm{K}}_6$ are supported, the direct sum is orthogonal and Λ is the lattice generated by the divisors A and $\tilde{\Sigma}$, where A is a primitive divisor s.t. $2A$ is the pullback through the desingularization of the locus of non-locally free sheaves in K_6, and $\tilde{\Sigma}$ is the strict transform of the singular locus $\Sigma \subset \mathrm{K}_6$ via the desingularization morphism; $q_6(A) = -2$, $q_6(\tilde{\Sigma}) = -4$ and $q_6(A, \tilde{\Sigma}) = 2$.

If we set $q_{10} := q_{\widetilde{\mathrm{M}}_{10}}$,

$$(H^2(\widetilde{\mathrm{M}}_{10}, \mathbb{Z}), q_{10}) \cong H^2(S, \mathbb{Z}) \oplus \Lambda$$

where S is the $K3$ surface where sheaves in $\widetilde{\mathrm{M}}_{10}$ are supported, the direct sum is orthogonal and Λ is a lattice generated by the divisors $\tilde{B}$ and $\tilde{\Sigma}$, where $\tilde{B}$ is the pullback through the desingularization of the locus of non-locally free sheaves in M_{10} and $\tilde{\Sigma}$ is the strict transform of the singular locus $\Sigma \subset \mathrm{M}_{10}$ via the desingularization morphism; $q_{10}(\tilde{B}) = -2$, $q_{10}(\tilde{\Sigma}) = -6$ and $q_{10}(\tilde{B}, \tilde{\Sigma}) = 3$.

Definition 1.1.12. A deformation of an IHS variety X is a smooth and proper holomorphic map $\mathcal{X} \to B$, where B analytic space and the fiber over a distinguished point $0 \in B$ is $\mathcal{X}_0 \cong X$.

Remark 1.1.13. Any small deformation of an IHS variety is again an IHS variety, see [Bea83]. As consequence, locally around $0 \in B$ all the fibers $\mathcal{X}_b$ are IHS varieties.

Terminology. We will refer to an IHS variety deformation equivalent to a $K3^{[n]}$/ $K_n(A)$/ $\tilde{\mathrm{K}}_6$/ $\widetilde{\mathrm{M}}_{10}$ as to an IHS of $K3^{[n]}$/ $K_n(A)$/ OG 6/ OG 10-type respectively.

Remark 1.1.14. In all the examples of IHS variety above, the Beauville-Bogomolov-Fujiki form is even, but up to now there is no general reason to expect it to hold true in general. Furthermore, we do not know IHS varieties with the same Beauville-Bogomolov-Fujiki form but not deformation equivalent, but again this has not been proved in general.

1.2 Moduli space of IHS varieties

In the next we will be interested in the moduli space of IHS varieties, that we are going to introduce in this section.

Fix a lattice Λ with signature $(3, k-3)$ with $k \geq 3$. We are interested in the moduli space

$$\mathfrak{M}_\Lambda := \{(X, \phi)\}/_\sim$$

where X is an IHS variety of fixed deformation type and ϕ a marking, i.e. an isometry of lattices $\phi : H^2(X, \mathbb{Z}) \tilde{\to} \Lambda$. We say that $(X, \phi) \sim (X', \phi')$ if and only if there exists an isomorphism $f : X \to X'$ s.t. $f^* = \pm\phi^{-1} \circ \phi'$; such an isomorphism is said to be an isomorphism of marked IHS varieties. The plus-minus sign in the equality is meant to identify the points (X, ϕ) and $(X, -\phi)$ in the construction of the moduli space, since they do not carry different geometrical information. Note that the notation $\mathfrak{M}_\Lambda$ is slightly misleading, since the deformation type is fixed but it does not appear in the notation; nevertheless, we decided to keep this notation here since it is the standard one used in literature. Only when the deformation type will be relevant we will denote by $\mathfrak{M}_{\mathbf{Y}}$ the moduli space of marked IHS varieties of deformation type $\mathbf{Y}$.

$\mathfrak{M}_\Lambda$ turns out to be a non-Hausdorff complex manifold of dimension $k-2$, where k is the rank of Λ. This is consequence of the work of several authors; we will briefly sketch here its proof, listing the main steps to reach it.

Fact 1.2.1. A well known result by Kuranishi (see [Kur65]) states that for any compact Kähler manifold X there exists a semiuniversal deformation $\mathcal{X} \to \mathrm{Def}(X)$; furthermore, the Zariski tangent space of $\mathrm{Def}(X)$ is isomorphic to $H^1(X, \mathcal{T}_X)$. If $H^0(X, \mathcal{T}_X) = 0$ then $\mathcal{X} \to \mathrm{Def}(X)$ is universal; if $\omega_X \cong \mathcal{O}_X$ then $\mathrm{Def}(X)$ is smooth, and one says that the deformations of X are unobstructed; this is due to Bogomolov, Tian and Todorov (see e.g. Theorem VII.1 in [Man05], and the same reference for more details on deformation theory).

We noticed in Remark 1.1.4 that if X is an IHS variety then $H^0(X, \mathcal{T}_X) = 0$. We conclude that for X an IHS variety there exists an universal deformation $\mathcal{X} \to \mathrm{Def}(X)$, where $\mathrm{Def}(X)$ is smooth of dimension

$$\dim(\mathrm{Def}(X)) = h^1(X, \mathcal{T}_X) = h^1(X, \Omega^1_X) = h^{1,1}(X) = b_2(X) - 2.$$

Note that $\mathcal{X} \to \mathrm{Def}(X)$ is universal for any fiber $\mathcal{X}_b$ for b close to 0 in B.

Let (X, ϕ) be a marked IHS variety. The marking extends to an isometry $\phi_\mathbb{C} : H^2(X, \mathbb{C}) \to \Lambda \otimes \mathbb{C} =: \Lambda_\mathbb{C}$; in the next, we will call $\phi_\mathbb{C}$ just ϕ. We define the period of X to be the point $\mathbb{P}(\phi(H^{2,0}(X))) \in \mathbb{P}(\Lambda_\mathbb{C})$. If $p : \mathcal{X} \to \mathrm{Def}(X)$ is the universal deformation of X, then $\phi : H^2(X, \mathbb{Z}) \to \Lambda$ defines a marking $\phi_b : H^2(\mathcal{X}_b, \mathbb{Z}) \to \Lambda$ on each fiber $\mathcal{X}_b$: up to restricting $\mathrm{Def}(X)$ to a simply connected space, $R^2p_*\mathbb{Z}$ becomes a constant sheaf on $\mathrm{Def}(X)$, then there exists a canonical isomorphism $\psi_b : H^2(X,\mathbb{Z}) \to H^2(\mathcal{X}_b,\mathbb{Z})$ for any $b \in B$, and $\phi_b := \phi \circ \psi_b^{-1}$.

One defines the period map to be

$$\begin{aligned} \mathcal{P} : \mathrm{Def}(X) &\to \mathbb{P}(\Lambda_{\mathbb{C}}) \\ b &\mapsto \mathbb{P}(\phi(H^{2,0}(\mathcal{X}_b))) \end{aligned}$$

The period map is holomorphic, see e.g. in [Voi03a]; this result is due to Griffiths, who also computed the differential of the period map. By Fact 1.1.6, the image $\mathrm{Im}(\mathcal{P})$ is contained in the period domain $Q_\Lambda = \{\lambda \in \mathbb{P}(\Lambda_{\mathbb{C}}) | q_\Lambda(\lambda) = 0,\ q_\Lambda(\lambda + \bar{\lambda}) > 0\}$.

In [Bea83], Beauville proved the following Local Torelli Theorem:

Theorem 1.2.2 (**Local Torelli for IHS varieties**). *Let (X,ϕ) be a marked IHS variety. Then the period map $\mathcal{P} : \mathrm{Def}(X) \to Q_\Lambda$ is a local isomorphism.*

The Local Torelli Theorem gives to $\mathfrak{M}_\Lambda$ a structure of non-Hausdorff complex manifold patching the charts $\mathrm{Def}(X)$. This allows to consider the period map as a holomorphic map

$$\mathcal{P} : \mathfrak{M}_\Lambda \to Q_\Lambda.$$

Remark 1.2.3. The period map $\mathcal{P}$ fails to be an isomorphism, conversely to the case of $K3$ surfaces. The first example of non-isomorphic IHS varieties with the isometric weight-two Hodge structure was found by Debarre in [Deb84]. Since the non-isomorphic IHS varieties found by Debarre are birational, the period map was hoped to determine the birational type of the IHS variety; this would have also been a nice generalization of the $K3$ case, since birational $K3$ surfaces are always isomorphic. In [Nam02], Namikawa showed that elements on a fiber of the period map are not necessarily birational.

The surjectivity of the period map has been proved by Huybrechts in [Huy99], while a description of the fibers of the period map has been done by Markman in [Mar11] and Verbitsky in [Ver13]:

Theorem 1.2.4 (**Global Torelli for IHS varieties**). *Let be $\mathfrak{M}^0_\Lambda \subset \mathfrak{M}_\Lambda$ a connected component and $\mathcal{P}^0 := \mathcal{P}|_{\mathfrak{M}^0_\Lambda}$ the restriction. Then $\mathcal{P}^0$ is surjective onto Q_Λ and two points on the same fiber are birational varieties.*

A nice reference for Global Torelli Theorem for IHS varieties is [Huy11].

A polarized IHS variety is an IHS variety X together with a polarization, i.e. an ample line bundle H on X. Since the first Chern class map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective for an IHS variety, we will often refer to a polarization as to its class $h := c_1(H) \in H^2(X, \mathbb{Z})$. Since in the following we will be interested in polarized IHS varieties, we want to consider the moduli space of marked and polarized IHS varieties.

Definition 1.2.5. Let (X, h) and (X', h') be polarized IHS varieties. (X, h) and (X', h') are deformation equivalent as polarized varieties if there exists a deformation of IHS varieties $\mathcal{X} \to B$, two points $b, b' \in B$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{X}_b \cong X$, $\mathcal{X}_{b'} \cong X'$, $c_1(\mathcal{L}|_{\mathcal{X}_b}) = h$ and $c_1(\mathcal{L}|_{\mathcal{X}'_b}) = h'$.

Analogously to the non polarized case, one has the following:

Fact 1.2.6. Given an IHS variety X, there exists a universal polarized deformation $(\mathcal{X}, \mathcal{L}) \to \mathrm{Def}(X, h)$ with $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ and $(\mathcal{X}_0, c_1(\mathcal{L}|_{\mathcal{X}_0})) \cong (X, h)$, for $0 \in \mathrm{Def}(X, h)$ distinguished point. The Zariski tangent space of $\mathrm{Def}(X, h)$ is isomorphic to $\ker(h : H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{O}_X)) \cong \mathbb{C} \cdot \bar{\sigma}$, where the map h is the contraction with h given by the natural pairing $H^1(X, \mathcal{T}_X) \otimes H^1(X, \Omega^1_X) \to H^2(X, \mathcal{O}_X)$; this contraction is in fact surjective (cf. 1.8 in [Huy99]). Furthermore, $\mathrm{Def}(X, h)$ is a smooth hypersurface of $\mathrm{Def}(X)$ (cf. 1.14 in [Huy99]).

The costruction of the moduli space of marked and polarized IHS varieties is more subtle than the non-polarized case, we refer to Section 8 in [Mar11] for a construction of it. In the next, we will denote by $\mathfrak{M}^{pol}_{\Lambda}$ the moduli space of marked and polarized IHS varieties with lattice Λ, or by $\mathfrak{M}^{pol}_{\mathbf{Y}}$ when we will want to emphasize that we are considering IHS varieties of deformation type $\mathbf{Y}$.

1.3 Monodromy and connected components

We are going to introduce the monodromy group of an IHS variety, since it is a fundamental tool for the study of the moduli space of marked (and polarized) IHS varieties. A very complete reference for what we are going to sketch is Markman's survey [Mar11].

Definition 1.3.1. Let X_1 and X_2 be IHS varieties. An isomorphism $g : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ is a parallel transport operator if there exists a deformation family $p : \mathcal{X} \to B$, two points $b_1, b_2 \in B$ and isomorphisms $\psi_1 : X_1 \xrightarrow{\sim} \mathcal{X}_{b_1}$, $\psi_2 : X_2 \xrightarrow{\sim} \mathcal{X}_{b_2}$ such that $(\psi_2^{-1})^* \circ g \circ \psi_1^* : H^2(\mathcal{X}_{b_1}) \to H^2(\mathcal{X}_{b_2})$ is the parallel transport (see Definition I.2.1 [GHJ03]) inside the local system $R^2 p_* \mathbb{Z}$ along a path γ in B from b_1 to b_2.

Definition 1.3.2. Let X be an IHS variety. A parallel transport operator $g : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ along a loop γ is called a monodromy operator.

Remark 1.3.3. One can define monodromy operators for any degree k, considering the parallel transport inside the local system $R^k p_* \mathbb{Z}$. Nevertheless, we will talk about monodromy operators always referring to the $k = 2$ case, which is the most interesting one for example because of the Torelli type theorems stated in Section 1.2.

Remark 1.3.4. By the deformation invariance of the Beauville-Bogomolov-Fujiki form (see e.g. Lemma 5.5 in [BL18]), a monodromy operator is in fact an isometry of the lattice $(H^2(X, \mathbb{Z}), q_X)$. We will denote by $\mathrm{Mon}^2(X)$ the subgroup of $O(H^2(X, \mathbb{Z}))$ generated by monodromy operators.

Note that by definition the groups $\mathrm{Mon}^2(X)$ and $\mathrm{Mon}^2(X')$ are isomorphic if X and X' are deformation equivalent, where the isomorphism is given by conjugation with a parallel transport operator from X to X'. Fixed a deformation type $\mathbf{Y}$ of IHS varieties, will denote by $\mathrm{Mon}^2(\mathbf{Y})$ the group of monodromy operators of IHS varities of $\mathbf{Y}$ type.

Let $\mathfrak{M}^0_\Lambda$ be a connected component of the moduli space $\mathfrak{M}_\Lambda$. Given $(X, \phi) \in \mathfrak{M}^0_\Lambda$, the subgroup $\mathrm{Mon}^2(\mathfrak{M}^0_\Lambda) := \phi \circ \mathrm{Mon}^2(X) \circ \phi^{-1} \subset O(\Lambda)$ is independent, up to conjugation, of the choice of the point in $\mathfrak{M}^0_\Lambda$. Furthermore, if τ_Λ is the set of connected components of $\mathfrak{M}_\Lambda$, then $O(\Lambda)$ acts on τ_Λ: a point $t \in \tau_\Lambda$ corresponds to a component $\mathfrak{M}^t_\Lambda$ of $\mathfrak{M}_\Lambda$, and given $(X, \phi) \in \mathfrak{M}^t_\Lambda$, $f \in O(\Lambda)$ sends t to the component of $(X, f \circ \phi)$. Note that this action is well defined, and it is transitive. Furthermore, the stabilizer of $t \in \tau_\Lambda$ is by definition the group $\mathrm{Mon}^2(\mathfrak{M}^t_\Lambda)$, since monodromy operators are those isometries coming from deformations of (X, ϕ). Summarizing, one has the following result:

Fact 1.3.5. Given a deformation type $\mathbf{Y}$ of IHS varieties, the cardinality $\tau_{\mathbf{Y}}$ of the set of connected components of $\mathfrak{M}_{\mathbf{Y}}$ equals the index $[\mathrm{Mon}^2(X) : O(H^2(X, \mathbb{Z}))]$, for any X IHS variety of $\mathbf{Y}$ type.

Remark 1.3.6. The cardinality of τ_Λ is actually finite. This result is due to the contribution of many authors, as Huybrechts in [Huy03], Markman in

[Mar11], Sullivan in [Sul77], Verbitsky in [Ver13]. We refer to [Mar11] for a presentation of this result.

In what follows, we will be interested to the polarized case.

Definition 1.3.7. Let (X_1, h_1) and (X_2, h_2) be polarized IHS varieties. An isomorphism $g : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ is a polarized parallel transport operator if there exists a deformation family $p : \mathcal{X} \to B$ such that g is the parallel trasport operator associated to this family, and if there exists a flat section h of $R^2 p_* \mathbb{Z}$ such that $h(b)$ is a polarization on $\mathcal{X}_b$ for any $b \in B$ and $\psi_{i,*}(h) = h_i$ for $i = 1, 2$.

Definition 1.3.8. Let (X, h) be a polarized IHS variety. A polarized parallel transport operator along a loop γ is called polarized monodromy operator. We denote by $\mathrm{Mon}^2(X, h)$ the group generated by polarized monodromy operators.

By definition, a polarized monodromy operator in $\mathrm{Mon}^2(X, h)$ needs to fix the polarization h. In [Mar11], Proposition 7.4, Markman proved that also the viceversa holds true:

Theorem 1.3.9. $\mathrm{Mon}^2(X, h)$ *equals the stabilizers of* h *in* $\mathrm{Mon}^2(X)$*.*

Let $O(H^2(X, \mathbb{Z}))_h$ be the group consisting of the isometries of $H^2(X, \mathbb{Z})$ stabilizing the class h; observe that any isometry of $H^2(X, \mathbb{Z})$ preserves the degree of h, that is its square with respect to the Beauville-Bogomolov-Fujiki form. By Theorem 1.3.9, we have $\mathrm{Mon}^2(X, h) = \mathrm{Mon}^2(X) \cap O(H^2(X, \mathbb{Z}))_h$.

As consequence of the discussion about the non-polarized case, we get the following:

Fact 1.3.10. The index $[\mathrm{Mon}^2(X, h) : O(H^2(X, \mathbb{Z}))_h]$ equals the number of connected components of $\mathfrak{M}^{pol}_{\mathbf{Y},d}$, where X is an IHS variety of deformation type $\mathbf{Y}$ and $\mathfrak{M}^{pol}_{\mathbf{Y},d}$ is the connected component of $\mathfrak{M}^{pol}_{\mathbf{Y}}$ of fixed polarization degree equal to d.

The monodromy group has been described in almost all the known examples of IHS varieties:

- $\mathrm{Mon}^2(K3^{[n]})$ has been described by Markman in [Mar10], Theorem 1.2 and Lemma 4.2.
- $\mathrm{Mon}^2(K_n(A))$ has been described by Mongardi in [Mon14], Theorem 2.3.

- $\mathrm{Mon}^2(\mathrm{OG}\,6)$ has been described by Mongardi and Rapagnetta in [MR19], Theorem 5.4(1).
- The OG 10 case is still open, but some partial results have been obtained by Mongardi in [Mon14], Theorem 3.3, and by Onorati in [Ono18], Theorem 5.1.12 and Theorem 5.3.2.

We will see in Chapter 2 how the monodromy group will play a central role in the research of rational curves on IHS varieties.

Chapter 2

Rational curves on IHS varieties

In this chapter we will try to motivate our interest in rational curves on IHS varieties. We will also present the state of the art of the problem, stating the known results and presenting a strategy by Charles, Mongardi and Pacienza to solve the problem in the $K3^{[n]}$ and $K_n(A)$-type case, and from which we will take inspiration to investigate the OG 10-type case.

2.1 Why rational curves

The existence of rational curves on IHS varieties is not known in general. The first known case was the 2-dimensional one, thanks to the following theorem due to Bogomolov and Mumford (see [MM83]).

Theorem 2.1.1. *Let S be a projective $K3$ surface. Any ample system on S contains an element which is sum of rational curves.*

A very interesting consequence of this theorem has been pointed out by Beauville and Voisin in [BV04]. Their result is about the 0-Chow group of a $K3$ surface; for an overview on Chow groups, see Appendix A.

Theorem 2.1.2. *Let S be a projective $K3$ surface and R a rational curve on S.*

1. *The subgroup $\mathrm{Im}(i_* : \mathrm{CH}_0(R)_{\mathbb{Q}} \to \mathrm{CH}_0(S)_{\mathbb{Q}}) \subset \mathrm{CH}_0(S)_{\mathbb{Q}}$ is independent of the rational curve R; here $i : R \hookrightarrow S$ is the inclusion. Furthermore, $\mathrm{Im}(i_* : \mathrm{CH}_0(R)_{\mathbb{Q}} \to \mathrm{CH}_0(S)_{\mathbb{Q}}) = \mathbb{Q} \cdot c_S$, where c_S is the class of*

any point on a rational curve in S; c_S is known as the Beauville-Voisin canonical 0-cycle of degree 1 on S.

2. *The image of the intersection product*

$$\mathrm{Pic}(S)_{\mathbb{Q}} \otimes \mathrm{Pic}(S)_{\mathbb{Q}} \to \mathrm{CH}_0(S)_{\mathbb{Q}}$$

is generated by c_S.

Remark 2.1.3. Theorem 2.1.2 is actually an immediate consequence of Theorem 2.1.1. Indeed, if two rational curves intersect, then any point on the first one is rationally equivalent to any point on the second one, since points on rational curves are movable up to rational equivalence. If two rational curves do not intersect, they both intersect any ample divisor, which by Theorem 2.1.1 is sum of rational curves up to linear equivalence, then again points are movable through those curves. Finally, point 2 follows from the fact that $\mathrm{Pic}(S)$ is generated by ample curves (see Chapter 8 in [Huy16]).

Theorem 2.1.1 is particularly relevant since it individuates a canonical class in the 0-Chow group of any projective $K3$ surface, which is well know to be not representable (see Corollary A.1.6). Furthermore, the Beauville-Voisin canonical class realizes the (conjectural) Bloch-Beilinson filtration in the case of a projective $K3$ surface (see Conjecture A.2.3 and Remark A.2.4).

In [Voi16], Voisin conjectured a geometrical shape of Bloch-Beilinson filtration of the 0-Chow group of an IHS variety, which would be a generalization of the one obtained in the $K3$ case; we briefly recall Voisin's conjecture here.

If X is an IHS variety and $x \in X$, let O_x be the orbit of rational equivalence of x, which is proved to be a countable union of closed algebraic subsets in X; therefore it makes sense to talk about the dimension of O_x, which is the supremum of the dimensions of the algebraic sets appearing in that union.

Voisin defines

$$S_iX := \{x \in X \,|\, \dim(O_x) \geq i\},$$

which is a coutable union of closed algebraic subsets in X; as before, its dimension is defined as the maximum of the dimensions of the components, and $\dim S_iX \leq 2n - i$.

In [Voi16], Conjecture 0.4, Voisin conjectured:

Conjecture 2.1.4. Let X be a projective IHS variety of dimension $2n$; then $\dim S_iX = 2n - i$ for any $i \leq n$.

Let $F^{\bullet}_{BB}\operatorname{CH}_0(X)_{\mathbb{Q}}$ be the Bloch-Beilinson filtration of $\operatorname{CH}_0(X)_{\mathbb{Q}}$ (see Conjecture A.2.3). In Remark 1.1.4 we noticed that if X is a projective IHS variety, then $h^{i,0}(X) = 0$ for i odd; it follows $F^{i}_{BB}\operatorname{CH}_0(X)_{\mathbb{Q}} = F^{i+1}_{BB}\operatorname{CH}_0(X)_{\mathbb{Q}}$ for i odd. As consequence we are just interested in $F^{2i}_{BB}\operatorname{CH}_0(X)_{\mathbb{Q}}$; we denote by $F'^{\bullet}_{BB}\operatorname{CH}_0(X)_{\mathbb{Q}}$ the filtration defined by $F'^{k}_{BB}\operatorname{CH}_0(X)_{\mathbb{Q}} := F^{2k}_{BB}\operatorname{CH}_0(X)_{\mathbb{Q}}$.

Voisin defines a filtration $S_{\bullet}\operatorname{CH}_0(X)_{\mathbb{Q}}$ of $\operatorname{CH}_0(X)_{\mathbb{Q}}$, where $S_i\operatorname{CH}_0(X)_{\mathbb{Q}}$ is the subgroup of $\operatorname{CH}_0(X)_{\mathbb{Q}}$ generated by classes of points in S_iX. Conjecture 2.1.4 would imply that the natural map

$$f : S_i\operatorname{CH}_0(X)_{\mathbb{Q}} \to \operatorname{CH}_0(X)/F'^{n-i+1}_{BB}\operatorname{CH}_0(X)_{\mathbb{Q}} \tag{2.1.1}$$

is surjective (see Lemma 3.9 in [Voi16]). Then Voisin conjectured

Conjecture 2.1.5. Let X be a projective IHS variety. Then the map f in (2.1.1) is an isomorphism.

For $i = 1$, S_1X is the set of points of X with orbit the union of subvarieties of dimension greater or equal to 1. This leads us to the following crucial definition:

Definition 2.1.6. A divisor $D \subset X$ is uniruled if there exists a variety Y with $\dim(Y) = \dim(D) - 1$ and a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow D$.

Note that $\operatorname{Im}(i_* : \operatorname{CH}_0(D) \to \operatorname{CH}_0(X))_{\mathbb{Q}} \subset S_1\operatorname{CH}_0(X)_{\mathbb{Q}}$ for $i : D \hookrightarrow X$ uniruled. A first evidence of Conjecture 2.1.4 is given by the following theorem, due to Charles, Mongardi and Pacienza (see Theorem 1.5 and Theorem 1.6 in [CMP19]):

Theorem 2.1.7. *Let X be a projective IHS variety such that there exists an ample divisor on X which is sum of irreducible uniruled divisors. Then:*

1. *The subgroup*

$$\operatorname{Im}(i_* : \operatorname{CH}_0(D)_{\mathbb{Q}} \to \operatorname{CH}_0(X)_{\mathbb{Q}}) = S_1\operatorname{CH}_0(X)_{\mathbb{Q}} \subset \operatorname{CH}_0(X)_{\mathbb{Q}}$$

is independent of the uniruled divisor D; here $i : D \hookrightarrow X$ is the inclusion.

2. *Suppose that $\operatorname{Pic}(X)_{\mathbb{Q}}$ is generated by classes of uniruled divisors. Then for any non-torsion $L \in \operatorname{Pic}(X)$ one has*

$$L \cdot \operatorname{CH}_1(X) = S_1\operatorname{CH}_0(X),$$

where the product is the intersection product.

Note that Theorem 2.1.7 is the higher-dimensional version of Theorem 2.1.2. As Theorem 2.1.2 was a consequence of the existence of rational curves on $K3$ surfaces, we would like to have a result of existence of uniruled divisors on IHS varieties, to verify the hypothesis of Theorem 2.1.7. We will investigate on the existence of uniruled divisors of IHS varieties in the next sections and in the rest of this work.

2.2 Deformation of rational curves

The results we will present in this section are a slight modification of some results proved by Charles, Mongardi and Pacienza in [CMP19]. We start by introducing some notations.

Let $p : \mathcal{X} \to B$ be a smooth projective morphism among quasi-projective varieties of relative dimension $2n$, and let $\alpha \in \Gamma(R^{4n-2}p_*\mathbb{Z}, B)$ be a class of type $(2n-1, 2n-1)$. Under these hypothesis one can consider the relative Kontsevich moduli stack $\overline{\mathcal{M}_0}(\mathcal{X}/B, \alpha)$ of genus zero stable curves, whose points parametrize maps $f : C \to X$ with C a stable curve of genus 0 and $X = \mathcal{X}_b$ a fiber of π, such that $f_*[C] = \alpha_b$; we will denote such a point by $[f]$. Note that the natural map $\overline{\mathcal{M}_0}(\mathcal{X}/B, \alpha) \to B$ is proper.

Let now X be a projective IHS variety of dimension $2n$ and $f : C \to X$ a fixed map from a stable genus 0 curve C; we also assume that f is unramified along the generic point of any irreducible component of C. Let $\mathcal{X} \to B$ as above, with central fiber $\mathcal{X}_0 = X$, $0 \in B$; let $\alpha \in \Gamma(R^{4n-2}p_*\mathbb{Z}, B)$ as above, with $\alpha_0 = f_*[C]$ in $H^{4n-2}(X, \mathbb{Z})$.

Under these notations, we state the following results, which are Proposition 3.1 and Proposition 3.2 of [CMP19]. Our goal is to arrive to Corollary 2.2.4, whose proof will be a consequence of the propositions we are going to state.

Proposition 2.2.1. *Let $M \subset \overline{M_0}(X, f_*[C])$ be an irreducible component containing the point $[f]$. Then $\dim M \geq 2n-2$, and if $\dim M = 2n-2$ then any irreducible component $\mathcal{M} \subset \overline{\mathcal{M}_0}(\mathcal{X}/B, \alpha)$ containing $[f]$ dominates B.*

Remark 2.2.2. The result above holds true for any X smooth projective with trivial canonical bundle.

Proposition 2.2.3. *Let X be a projective $2n$-dimensional manifold endowed with a symplectic form and let $Y \subset X$ be subvariety of codimension k. If $W \subset X$ is a subvariety such that any point of Y is rationally equivalent to a point in W, then the codimension of W in X is at most $2k$.*

From these results, one can conclude that it is possible to deform rational curves ruling a divisor along the base B, and that the deformations still rule a divisor. They proved this result in the case of irreducible curves (see Corollary 3.5 of [CMP19], whose statement is analogous to the one of Corollay 2.2.4 here below), but in the next we will need it also for non irreducible curves. For this reason we will prove here the reducible case, even if its proof is just a slight modification of the one given in [CMP19].

Corollary 2.2.4. *Let $f : C \to X$ be a non constant map from a possibly reducible stable genus zero curve C, and let M be an irreducible component of $\overline{M_0}(X, f_*[C])$ containing $[f]$. Let $D \subset X$ be the subscheme covered by the deformations of f parametrized by M. If D is a divisor, then any irreducible component of $\overline{\mathcal{M}_0}(\mathcal{X}/B, \alpha)$ containing $[f]$ dominates B. Furthermore, $\mathcal{X}_b$ contains a uniruled divisor D_b for any point $b \in B$.*

Proof. By Proposition 2.2.1, $\dim M \geq 2n-2$; we want to prove that the equality holds, so that Proposition 2.2.1 will imply that under our hypothesis any irreducible component of $\overline{\mathcal{M}_0}(\mathcal{X}/B, \alpha)$ containing $[f]$ dominates B.

Let us assume $\dim M \geq 2n-1$, and let $\mathcal{C} \to M$ the universal curve. Let us consider the evaluation map $\mathcal{C} \to D \subset X$, and let $D_0 \subset D$ an irreducible component of D ruled by the deformations of $f|_{C_0} : C_0 \to X$, with C_0 rational irreducible component of C. Since we are assuming $\dim M \geq 2n-1$, the fibers of the evaluation map on D_0 have dimension at least 1, which means that there exists a subvariety $W_0 \subset X$ with $\dim W_0 \leq \dim D - 2 = 2n-3$ such that any point of D_0 is rationally equivalent to a point of W_0. Choosing $Y = D_0$ and $W = W_0$ in Proposition 2.2.3, this gives a contradiction.

The last part of the statement follows as in the proof of Corollary 3.5 in [CMP19]; for completeness, we repeat here their argument. Let $\mathcal{M} \subset \overline{\mathcal{M}_0}(\mathcal{X}/B, \alpha)$ be an irreducible component containing M, and let $\mathcal{D} \subset \mathcal{X} \to B$ be the locus covered by deformations of f parametrized by $\mathcal{M}$. Any irreducible component of $\mathcal{D}$ dominates B, because from what we said before $\mathcal{M}$ dominates B. The central fiber of $\mathcal{D} \to B$ is by construction D, which is a divisor in $\mathcal{X}_0$; as consequence, the fiber of $\mathcal{D} \to B$ is a divisor $D_b \subset \mathcal{X}_b$ at any point $b \in B$, which is uniruled by construction.

□

Remark 2.2.5. We want to emphasize that the argument of the proof of Corollary 2.2.4 can not be used to prove that, given a rational curve whose deformations cover a subvariety of codimension $k > 1$, the deformations of the curve in another fiber $\mathcal{X}_b$ of $\mathcal{X} \to B$ still cover a subvariety of codimension k. Indeed the request that the deformations of the curve still cover a divisor is translated to an open condition, because it is the condition that the evaluation map $\mathcal{C} \to \mathcal{X}$ has maximal rank. The same would not hold true for a codimension $k > 1$.

2.3 A possible approach to the problem

We will present here a possible approach to the problem of finding uniruled divisors on any ample system on an IHS variety X of fixed deformation type. This has been introduced by Charles, Mongardi and Pacienza in [CMP19], where they used it in the case of IHS variety of $K3^{[n]}$-type; in [MP17] and [MP19], Mongardi and Pacienza used the same approach in the case of IHS varieties of $K_n(A)$-type. The reference we are giving for the $K3^{[n]}$-case is an amend of a previous work of Charles and Pacienza; for this reason it is more recent then the article presenting the $K_n(A)$-case.

The approach we are going to present will motivate the results of the next chapters, since they are part of the first step here below.

The approach can be divided three steps. Let $\mathbf{Y}$ be a fixed deformation type of IHS varieties.

First step: The moduli space $\mathfrak{M}_{\mathbf{Y}}^{pol}$ of polarized IHS varieties of $\mathbf{Y}$-type can have many connected components, counted by the polarized monodromy group $\mathrm{Mon}^2(\mathbf{Y})$ (see Section 1.3). Thanks to the knowledge of $\mathrm{Mon}^2(\mathbf{Y})$ one can try to write explicit representatives for each connected component of $\mathfrak{M}_{\mathbf{Y}}^{pol}$, as done in the $K3^{[n]}$-case in Section 2.2 of [CMP19] and in the $K_n(A)$-case in [MP17], Section 4.1 and in [MP19].

Second step: Find an example of ample uniruled divisor D in each connected component of $\mathfrak{M}_{\mathbf{Y}}^{pol}$ found in the first step thanks to $\mathrm{Mon}^2(\mathbf{Y})$. More precisely, one can try to find a pair $(X, c_1(D))$ with X an IHS variety of $\mathbf{Y}$-type and D ample and uniruled divisor for each representative of a connected component of $\mathfrak{M}_{\mathbf{Y}}^{pol}$. It turns out that it is not always possible to find uniruled divisors ruled by primitive curves in each connected component of $\mathfrak{M}_{\mathbf{Y}}^{pol}$, then the goal becomes to find explicit examples of such uniruled divisors in as many connected components as possible.

In the $K3^{[n]}$-case explicit examples of ample uniruled divisors have been presented in Section 4 of [CMP19]. Furthermore, they proved in Section 5 of the same work that there exists at most finitely many connected components of $\mathfrak{M}^{pol}_{K3^{[n]}}$ where ample uniruled divisors ruled by primitive curves do not exist, and they give conditions ensuring their existence (see also Corollary A.3 in [OSY18]).

Explicit examples of ample uniruled divisors are presented in Section 4.2 of [MP17] and in [MP19] in the $K_n(A)$-case.

Third step: This is the conclusion following by the previous steps, and the result one would like to reach. By the results about deformations of rational curves ruling divisors on IHS varieties (see Section 2.2), one can conclude that there exist ample uniruled divisors in each element of a connected component of $\mathfrak{M}^{pol}_{\mathbf{Y}}$ where explicit examples of uniruled divisors have be found in the second step.

We recap here the situation for the OG 10-case, since it is the case we are going to discuss in the rest of this work.

First step - OG10: A description of the monodromy group $\mathrm{Mon}^2(\mathrm{OG}\,10)$ is known just partially thanks to the work of Mongardi in [Mon14] and Onorati in [Ono18], as discussed at the end of Section 1.3. As consequence, we can not conclude yet what is the number of connected components the moduli space $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$.

Second step - OG10: Finding examples of ample uniruled divisors on IHS varieties of OG 10-type is the main goal of this work. Original results in this direction will be presented in Chapter 5 and Chapter 6.

Third step - OG10: Since in Chapter 5 and Chapter 6 we will prove the existence of three ample and uniruled divisors on IHS varieties of OG 10-type, we will conclude that the existence of ample uniruled divisors holds true for any element in the connected component of $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$ of those special examples.

Our contribution to the second and the third step will be discussed in detail along Chapter 4, 5 and 6. In order to do that, from the next chapter on we will focus on the 10-dimensional example of O'Grady, concluding here our discussion about IHS varieties in general.

Chapter 3

Moduli spaces of sheaves

Moduli spaces of semistable sheaves are one of the main tool to produce examples of IHS varieties. In this chapter we will introduce them, we will see how we can recover Beauville's examples of IHS varieties as moduli space of sheaves and we will explain how O'Grady produces two new examples of IHS varieties out of them. Finally, we will focus on some particular moduli spaces, which have a powerful geometric structure which makes them easier to handle with. Indeed, in the next chapters we will consider just those particular moduli spaces, where it will be easier to construct examples of ample uniruled divisors.

3.1 Some generalities about semistable sheaves

In this section X will be a smooth projective variety over $\mathbb{C}$ with a polarization H. All the notions we are introducing and all the fact we are stating here can be found in [HL10].

Given a coherent sheaf $E \in \mathrm{Coh}(X)$ on X, the Hilbert polynomial of E with respect to the polarization H is the polynomial $P_H(E,m) := \chi(E(m)) = \sum_{i=0}^{d} \alpha_i(E)\frac{m^i}{i!}$, with $E(m) := E \otimes H^{\otimes m}$ and $d := \dim(supp(E))$; d is called the dimension of the sheaf E and will be denoted $\dim(E)$.

For any $E \in \mathrm{Coh}(X)$, we define:

- the rank of E as $rk(E) := \frac{\alpha_d(E)}{\alpha_d(\mathcal{O}_X)}$; in case E is locally free, this is just the usual definition of rank for vector bundles;
- the degree of E as $\deg(E) := \alpha_{d-1}(E) - rk(E) \cdot \alpha_{d-1}(\mathcal{O}_X)$.

Definition 3.1.1. The reduced Hilbert polynomial of a coherent sheaf E of dimension d over X is the polynomial

$$p_H(E,m) := \frac{P_H(E,m)}{\alpha_d(E)}.$$

For simplicity, we will denote by $P(E)$ and $p(E)$ the Hilbert polynomial and the reduced Hilbert polynomial of a sheaf E, when the polarization will be clear from the context. We recall that there is an ordering of polynomials given by the lexicographic order of their coefficients, and that we will denote with the symbol $\leq$.

We say that $E \in \mathrm{Coh}(X)$ is pure if $\dim(F) = \dim(E)$ for any $F \subset E$ non trivial coherent subsheaf; for example, any torsion-free sheaf is pure.

Definition 3.1.2. A pure sheaf $E \in \mathrm{Coh}(X)$ is semistable if $p(F) \leq p(E)$ for any $F \subset E$ proper. E is called stable if it is semistable and $p(F) < p(E)$ for any $F \subset E$ proper.

Remark 3.1.3. The definition of (semi)stability really depends on the choice of the polarization.

In the following, it will be useful the following characterization of stability (for a reference see Proposition 1.2.6 in [HL10]):

Proposition 3.1.4. *Let $E \in \mathrm{Coh}(X)$ be a pure sheaf of dimension d. Then E is (semi)stable if an only if for any proper purely d-dimensional sheaf $E \to F \to 0$ one has $p(E)(\leq) < p(F)$.*

The following fact (see Corollary 1.2.8 in [HL10]) will be useful in the next:

Proposition 3.1.5. *If $E \in \mathrm{Coh}(X)$ is stable, then $End(E) \cong \mathbb{C}$, i.e. E is a simple sheaf.*

Given $E \in \mathrm{Coh}(X)$, a Jordan-Hölder filtration for E is a filtration in coherent sheaves:

$$0 = E_0 \subset E_1 \subset ... \subset E_k = E$$

such that E_i/E_{i-1} is stable with reduced polynomial $p(E)$ for any $i = 1, ..., k$. The sheaf $\mathrm{JH}(E) := \oplus_{i=1}^k E_i/E_{i-1}$ is called the Jordan-Hölder sheaf associated to E. The Jordan-Hölder sheaf is well defined for semistable sheaves, thanks to the following proposition (see Proposition 1.5.2 in [HL10]):

Proposition 3.1.6. *For $E \in \mathrm{Coh}(X)$ semistable there always exists a Jordan-Hölder filtration, which may not be unique. Up to isomorphism, the sheaf* $\mathrm{JH}(E)$ *does not depend on the choice of the filtration.*

Definition 3.1.7. Two semistable sheaves $E, F \in \mathrm{Coh}(X)$ with the same reduced Hilbert polynomial are called S-equivalent if $\mathrm{JH}(E) \cong \mathrm{JH}(F)$. A semistable sheaf $E \in \mathrm{Coh}(X)$ is called polystable if it is a direct sum of stable sheaves.

The S-equivalence relation will be crucial to have a moduli space of semistable sheaves with good properties.

Remark 3.1.8. Stable sheaves are S-equivalent if and only if they are isomorphic, since a Jordan-Hölder filtration is given just by the stable sheaf itself. Nevertheless, on semistable sheaves the S-equivalent relation is in general weaker than the equivalence relation given by isomorphisms.

Note also that every S-equivalence class of semistable sheaves contains, up to isomorphisms, only one polystable sheaf: the Jordan-Hölder sheaf of any element in the S-equivalence class.

We can finally introduce the moduli functor we are interested in. Fixed a Hilbert polynomial P, we can define a contravariant functor

$$\mathcal{M}_P : Sch/\mathbb{C} \to Sets$$

as follows:

- Given $T \in Ob(Sch/\mathbb{C})$, $\mathcal{M}_P(T) := \{E \in \mathrm{Coh}(X \times T) \mid E \; is \; T-flat, \; E_t \; semistable \; and \; P(E_t) = P \; \forall t \in T\}_{/\sim}$, where $E \sim E'$ if and only if $E \cong E' \otimes p^*L$ with $L \in \mathrm{Pic}(T)$ and $p : X \times T \to T$ the projection. Note that $E_t \cong (E \otimes p^*L)_t$ for every $t \in T$, which makes reasonable to consider elements up to the relation introduced.

- Given $f : T' \to T$ morphism in $Sch/\mathbb{C}$, $\mathcal{M}_P(f) : \mathcal{M}_P(T) \to \mathcal{M}_P(T')$ is the map $[E] \mapsto [(f \times id_X)^*E]$. For simplicity, if $f : T' \to T$ is a morphism we will call $F^* := \mathcal{M}_P(f)$.

We call $\mathcal{M}^s_P : Sch/\mathbb{C} \to Sets$ the analogous functor obtained considering stable sheaves instead of semistable ones. Obviously, $\mathcal{M}^s_P \subset \mathcal{M}_P$.

Definition 3.1.9. A contravariant functor $\mathcal{F} : Sch/\mathbb{C} \to Sets$ is corepresented by $F \in Ob(Sch/\mathbb{C})$ if there exists a natural transformation $\alpha : \mathcal{F} \to Mor_{Sch/\mathbb{C}}(\cdot, F)$ such that any other natural transformation $\alpha' : \mathcal{F} \to Mor_{Sch/\mathbb{C}}(\cdot, F')$ factorises through a unique $\beta : Mor_{Sch/\mathbb{C}}(\cdot, F) \to Mor_{Sch/\mathbb{C}}(\cdot, F')$.

$\mathcal{F}$ is represented by $F \in Ob(Sch/\mathbb{C})$ if there exists a natural isomorphism $\alpha : \mathcal{F} \tilde{\to} Mor_{Sch/\mathbb{C}}(\cdot, F)$.

We are interested in the representability of the functor $\mathcal{M}_P$. If a projective scheme M_P represents $\mathcal{M}_P$, we call it a fine moduli space; if M_P corepresents $\mathcal{M}_P$, we call it just a moduli space.

Remark 3.1.10. Representability of a functor can be read in terms of the existence of universal families. Let M_P be a fine moduli space for $\mathcal{M}_P$; then there exists a sheaf $\mathcal{E} \in \mathrm{Coh}(X \times \mathrm{M}_P)$ corresponding to the identity map $id_{\mathrm{M}_P} \in \mathrm{Hom}(\mathrm{M}_P, \mathrm{M}_P)$. By representability, any $F \in \mathrm{Coh}(X \times T)$ with $[F]_\sim \in \mathcal{M}_P(T)$ corresponds to a morphism $\phi_F \in \mathrm{Hom}(T, \mathrm{M}_P)$ such that the following diagram

$$\begin{array}{ccc} \mathcal{M}_p(M_P) & \xrightarrow{\mathcal{M}_P(\phi_F)=:\Phi_F^*} & \mathcal{M}_P(T) \\ \big\downarrow = & & \big\downarrow = \\ \mathrm{Hom}(\mathrm{M}_P, \mathrm{M}_P) & \xrightarrow{-\circ\phi_F} & \mathrm{Hom}(T, \mathrm{M}_P) \end{array}$$

is commutative, which means in particular that $\Phi_F^*(\mathcal{E}) \sim F$ since $(-\circ \phi_F)(id_{\mathrm{M}_P}) = \phi_F \in \mathrm{Hom}(T, \mathrm{M}_P)$ corresponds to $F \in \mathcal{M}_P(T)$. In other words, $\mathcal{E}$ satisfies the following property: for any $[F]_\sim \in \mathcal{M}_P(T)$ there exists $L \in \mathrm{Pic}(T)$ such that $\Phi_F^*(\mathcal{E}) = p^*L \otimes F$, where $p : X \times T \to T$ is the projection and $\Phi_F^* = (id_X \times \phi_F)^*$ by the very definition of $\mathcal{M}_P$.

A sheaf $\mathcal{E}$ in $\mathcal{M}_P(\mathrm{M}_P)$ satisfying this property is called a universal family. Note that the existence of a universal family is actually equivalent to the representability of the functor $\mathcal{M}_P$.

The following fundamental result is due to Maruyama and Simpson.

Theorem 3.1.11. *Given a Hilbert polynomial P, the functor $\mathcal{M}_P$ is corepresented by a projective scheme M_P. Closed points of M_P parametrize S-equivalence classes of semistable sheaves with Hilbert polynomial P.*

We will call $\mathrm{M}_P^s \subset \mathrm{M}_P$ the open subset parametrizing stable sheaves.

Remark 3.1.12. We want to remark that, since S-equivalent sheaves correspond to the same point in M_P, the functor $\mathcal{M}_P$ is for sure non representable every time there exists a properly semistable sheaf E on X with Hilbert polynomial P.

Universal families do not always exist. We define here a weaker notion of "universal" families that will be useful in the next.

Definition 3.1.13. A family of semistable sheaves $\mathcal{E} \in \mathrm{Coh}(X \times \mathrm{M}_P)$ flat over M_P is called quasi-universal of similitude ρ if for any T-flat family

$F \in \mathrm{Coh}(X \times T)$ of semistable sheaves with Hilbert polynomial P there exists a $\mathcal{O}_T$-locally free module V of rank ρ such that $F \otimes p^*V \cong \Phi_F^*\mathcal{E}$, where $p : X \times T \to T$ is the projection and $\Phi_F^* = \mathcal{M}_P(\phi_F)$ as in Remark 3.1.10.

We state here some results about the local structure of M_P.

Proposition 3.1.14. *Let M_P be the moduli space of Theorem 3.1.11, and let $x \in \mathrm{M}_P$ be a point corresponding to a stable sheaf E.*

1. *There exists an isomorphism $T_x \mathrm{M}_P \cong \mathrm{Ext}^1(E, E)$.*
2. *If the trace map $\mathrm{Ext}^2(E, E) \to H^2(X, \mathcal{O}_X)$ is injective and the Picard scheme Pic_X is smooth at the point corresponding to $\det(E)$, then M_P is smooth at x.*

For the definition of the Picard scheme Pic_X we refer to Chapter 10.1 of [Huy16].

In the following we will be interested in moduli spaces of sheaves on projective $K3$ surface, since they will give new examples of IHS varieties. A similar theory could be done for abelian surfaces, leading to examples deformation equivalent to the generalized Kummer varieties or to the O'Grady 6-dimensional example $\tilde{\mathrm{K}}_6$, see Example 1.1.11.

3.2 Moduli spaces of sheaves on $K3$ surfaces

From now on we will assume $X = S$ projective $K3$ surface with polarization H. In this case, instead of the Hilbert polynomial of a semistable sheaf, it is more convenient to fix a different invariant of the sheaf: its Mukai vector.

Definition 3.2.1. Let be $E \in \mathrm{Coh}(S)$. The Mukai vector of E is

$$\mathfrak{v}(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}(S)} = (rk(E), c_1(E), rk(E) + ch_2(E)) \in H^*(S, \mathbb{Z}).$$

Note that the equality above follows from $\sqrt{\mathrm{td}(S)} = (1, 0, 1)$ for S a $K3$ surface. Note also that, since S is $K3$, the cohomology ring $H^*(S, \mathbb{Z})$ is non trivial only in even degrees.

Remark 3.2.2. The Hirzebruch-Riemann-Roch formula

$$\chi(E) = \int_S \mathrm{ch}(E)\,\mathrm{td}(S) = \int_S \mathfrak{v}(E)\sqrt{\mathrm{td}(S)}$$

implies that given the Mukai vector of a sheaf on a $K3$ surface, one can get its Hilbert polynomial. Notice that the converse does not hold true: the Hilbert polynomial of a sheaf E fixes its rank and its second Chern class, but not its first Chern class: it only fixes $c_1(E) \cdot c_1(H)$.

Anyway, we can (and we will) consider the functor $\mathcal{M}_v$ defined as $\mathcal{M}_P$ but fixing a Mukai vector v instead of a Hilbert polynomial P. By Theorem 3.1.11 the functor $\mathcal{M}_v$ is corepresented by a projective scheme M_v, and we will denote by $\mathrm{M}_v^s \subset \mathrm{M}_v$ the open subset of stable sheaves.

Definition 3.2.3. Let $v, w \in H^*(S, \mathbb{Z})$. The Mukai pairing on $H^*(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ is

$$< v, w > := -\int_S v \wedge w^\vee$$

where, given $w = (w_0, w_1, w_2)$, we define $w^\vee := (w_0, -w_1, w_2)$.

The Mukai pairing makes $(H^*(S, \mathbb{Z}), <, >)$ a lattice, known as the Mukai lattice associated to the $K3$ surface. The minus sign in the definition of the Mukai pairing is just a convention, whose reason is to have a nicer formula in Corollary 3.2.4.

The Mukai lattice $H^*(S, \mathbb{Z})$ can be equipped by a pure Hodge structure of weight 2, defined as follows:

$$\begin{aligned}
(H^*(S))^{2,0} &:= H^{2,0}(S) \\
(H^*(S))^{1,1} &:= H^0(S, \mathbb{C}) \oplus H^{1,1}(S) \oplus H^4(S, \mathbb{C}) \\
(H^*(S))^{0,2} &:= H^{0,2}(S).
\end{aligned}$$

The Euler characteristic defines a pairing on the Groethendieck group $K(S)$, called Euler pairing:

$$\chi(E, F) := \sum_{i=0}^{2} (-1)^i \dim \mathrm{Ext}^i(E, F).$$

By Serre duality, the Euler pairing is symmetric. The Hirzebruch-Riemann-Roch theorem generalizes to

$$\chi(E, F) = \int_S \mathrm{ch}(E)^\vee \, \mathrm{ch}(F) \, \mathrm{td}(S) = - < \mathfrak{v}(E), \mathfrak{v}(F) > .$$

By Proposition 3.1.5 every stable sheaf is simple, i.e. $\mathrm{End}(E) \cong \mathbb{C}$. Furthermore, Serre duality and $S = K3$ give $\mathrm{Ext}^2(E, E) \cong \mathrm{Ext}^0(E, E \otimes \omega_S)^* \cong$

$\mathrm{End}(E) \cong \mathbb{C}$, and in this case that the trace map $\mathbb{C} \cong \mathrm{Ext}^2(E,E) \to H^2(X,\mathcal{O}_X) \cong \mathbb{C}\sigma$ is injective (see Section 2.1 in Chapter 10 of [Huy16] for more details). Furthermore, $H^1(S,\mathcal{O}_S) = 0$ gives that the Picard scheme Pic_S of the $K3$ surface consists of reduced isolated points (see again Chapter 10 in [Huy16]), hence in particular Pic_S is smooth. We get, as Corollary of Proposition 3.1.14:

Corollary 3.2.4. *If M_v^s is not empty, then it is a smooth quasi-projective variety of dimension $2+v^2$, where v^2 is the square wrt the Mukai pairing.*

As consequence, if there are not strictly semistable sheaves on S with Mukai vector v then the moduli space M_v is smooth. This happens under some condition, but in order to state the main result in this direction we need to introduce the notion of v-genericity.

Definition 3.2.5. An element $v = (r,l,s) \in H^*(S,\mathbb{Z})$ is called Mukai vector if $r \geq 0$ and $l \in NS(S)$, and if $r = 0$ then either l is the first Chern class of an effective divisor, or $l = 0$ and $s > 0$.

Note that, if we want M_v to be not empty, we need to ask that v is a Mukai vector. The other notion we need to introduce is the notion of v-genericity for a polarization H. Here we will just give the definition, for a more complete discussion on v-genericity see [PR13], section 2.1.

Let $v = (r,l,s)$ be a Mukai vector. We need to consider two cases separately:

1. $r \geq 1$: set $|v| := \frac{r^2}{4} < v,v > + \frac{r^4}{2}$; we define
$$W_v := \{D \in NS(S) \mid -|v| \leq D^2 < 0\}.$$

2. $r = 0$: let E be a pure sheaf with $\mathfrak{v}(E) = v$, and $F \subset E$ subsheaf with $\mathfrak{v}(F) = (0,l',s')$. The divisor associated to the pair (E,F) is by definition $D := ls' - l's$. We define W_v to be the set of all non-zero divisors associated to all the possible pairs (E,F).

Definition 3.2.6. Given $D \in W_v$, the v-wall associated to D is
$$W^D := \{a \in Amp(S)_{\mathbb{R}} \mid \alpha \cdot D = 0\}.$$

A v-chamber is a connected component of $Amp(S)_{\mathbb{R}} \smallsetminus \bigcup_{D \in W_v} W^D$.

Notice that W^D is an hyperplane in the ample cone $Amp(S)_{\mathbb{R}}$; furthermore, $\bigcup_{D \in W_v} W^D \subset Amp(S)_{\mathbb{R}}$ is locally finite.

Definition 3.2.7. Let H be a polarization on S. H is v-generic if it belongs to some v-chamber.

Notice that for a polarization H the condition of being v-generic is a generic condition. Very roughly speaking, it is useful to assume that a polarization H is v-generic because then it minimizes the amount of strictly semistable sheaves with respect to H.

Remark 3.2.8. There is a more general definition of v-genericity, see Definition 2.1 in [PR13]; usually, to belong to a v-chamber is given only as a consequence of this more general definition. Anyway, since some of the results we are going to state do not hold true for v-generic polarization according to this more general definition, we decided to give the definition above.

The key point is the following observation, whose proof can be found in [Yos01a]:

Proposition 3.2.9. *Let v a Mukai vector which is primitive in the lattice $H^*(S,\mathbb{Z})$. Then, given any v-generic polarization H, any H-semistable sheaf E is H-stable, i.e. $\mathrm{M}_v = \mathrm{M}_v^s$ if not empty is a smooth projective variety of dimension $2 + v^2$.*

We are interested in moduli spaces of sheaves on $K3$ surfaces because they turn out to be IHS varieties. Indeed, the moduli spaces M_v^s carry on a symplectic structure, as observed first by Mukai in [Muk84]:

Proposition 3.2.10. *The moduli space M_v^s is endowed with a regular and everywhere non degenerate 2-form $\sigma \in H^0(\mathrm{M}_v^s, \Omega^2_{\mathrm{M}_v^s})$.*

The symplectic form σ is just a globalization of the following pairing $T_x \mathrm{M}_v \times T_x \mathrm{M}_v \to \mathbb{C}$, that we can define thanks to Proposition 3.1.14: if x corresponds to the stable sheaf E, then Serre duality $\mathrm{Ext}^1(E,E) \cong \mathrm{Ext}^1(E,E)^*$ gives a non-degenerate pairing

$$\mathrm{Ext}^1(E,E) \times \mathrm{Ext}^1(E,E) \to \mathbb{C},$$

which is actually just $(\alpha,\beta) \mapsto \mathrm{tr}(\alpha \cup \beta) \in H^2(S,\mathcal{O}_S) \cong \mathbb{C}$, where $\cup$ denotes the Yoneda product and $\mathrm{tr} : \mathrm{Ext}^2(E,E) \to H^2(X,\mathcal{O}_X)$ is the trace map. Note that the isomorphism $H^2(S,\mathcal{O}_S) \cong \mathbb{C}$ is not canonical, but it is given by the choice of a global section of ω_S.

Example 3.2.11. Let $v = (0, 0, n)$. Then M_v parametrizes sheaves with constant Hilbert polynomial $P \equiv n$ and $\mathrm{M}_v \cong S^{(n)}$. When $n > 1$ one has $\mathrm{M}_v^s = \emptyset$, but M_v has a symplectic desingularization given by the Hilbert scheme $S^{[n]}$.

Example 3.2.12. Let $v = (1, 0, 1 - n)$. Then M_v parametrizes torsion free sheaves of rank one, i.e. sheaves of the form $E = L \otimes I_Z$, where $L \in \mathrm{Pic}(S)$ and Z is a 0-dimensional subscheme of S. Note that such a E fits in the short exact sequence

$$0 \to E \to L \to \mathcal{O}_Z \to 0$$

and then

$$(1, 0, 1 - n) = \mathfrak{v}(E) = \mathfrak{v}(L) - \mathfrak{v}(\mathcal{O}_Z) = \Big(1, c_1(L), \frac{c_1(L)^2}{2}\Big) - (0, 0, 1 - h^0(\mathcal{O}_Z))$$

which implies $L \cong \mathcal{O}_S$ and $h^0(\mathcal{O}_Z) = n$. In other words, a sheaf E in M_v gives a 0-dimensional subscheme of S of length n, and actually there exists an isomorphism $\mathrm{M}_v \cong S^{[n]}$, i.e. we get back Beauville's examples obtained starting from a $K3$ surface.

Thanks to the work of Beauville, Mukai, O'Grady and Yoshioka (see [Bea83], [Muk87], [o'G95], [Yos01a] and [Yos99]), it is known that M_v is indeed always an IHS variety for v primitive and H v-generic. We collect all their results in the following

Theorem 3.2.13. *Let v be a primitive Mukai vector on the projective $K3$ surface S and let H be a v-generic polarization.*

1. *If $v^2 = -2$, then $\mathrm{M}_v = \mathrm{M}_v^s$ is a single point.*
2. *If $v^2 = 0$, then $\mathrm{M}_v = \mathrm{M}_v^s$ is a projective $K3$ surface. Moreover, there exists a Hodge isometry $v^\perp/\mathbb{Z} \cdot v \to H^2(\mathrm{M}_v, \mathbb{Z})$ wrt the Mukai pairing and the intersection pairing of the $K3$ surface M_v.*
3. *If $v^2 \geq 2$ then $\mathrm{M}_v = \mathrm{M}_v^s$ is an IHS variety, which is deformation equivalent to $S^{[\frac{2+v^2}{2}]}$, the Hilbert scheme of $\frac{2+v^2}{2}$ points on S. Moreover, there exists a Hodge isometry $v^\perp \to H^2(\mathrm{M}_v, \mathbb{Z})$ wrt the Mukai pairing and the Beauville-Bolomolov-Fujiki form.*

Remark 3.2.14. The Hodge isometry of Theorem 3.2.13 comes from the existence of quasi-universal families on $\mathrm{M}_v^s = \mathrm{M}_v$. Indeed, even if in general an universal family does not exist even for v primitive (see [Yos98]), Mukai noticed that quasi-universal families always exist (see [Muk87] and Section 4.6 of [HL10]). If $\mathcal{E}$ is a quasi-universal family on $S \times \mathrm{M}_v^s$ of similitude ρ, one can define:

$$\begin{aligned} H^*(S,\mathbb{Z}) &\to H^2(\mathrm{M}_v^s,\mathbb{Q}) \\ \alpha &\mapsto \frac{1}{\rho}\big[p_{\mathrm{M}_v,*}\big(\mathrm{ch}(\mathcal{E})p_S^*(\alpha^\vee\sqrt{\mathrm{td}(S)})\big)\big]_1. \end{aligned}$$

Here by $[-]_1$ we mean that we consider the part of degree 1 of the expression above. Restricting the morphism to $v^\perp \subset H^*(S,\mathbb{Z})$, this does not depend on the choice of the quasi-universal family; furthermore, one can verify that it takes values in $H^2(\mathrm{M}_v^s,\mathbb{Z})$.

Theorem 3.2.13 says that moduli spaces of sheaves with primitive Mukai vector give examples of IHS varieties which are deformation equivalent to the Beauville's examples. The next case to explore is the case of non-primitive Mukai vector. The question in this case is whether the singular moduli spaces M_v admit a symplectic desingularization $\tilde{\pi} : \widetilde{\mathrm{M}}_v \to \mathrm{M}_v$, i.e. a desingularization extending the symplectic structure on M_v^s to $\widetilde{\mathrm{M}}_v$.

For a long time moduli spaces of sheaves have been expected to give new examples of IHS varieties, but actually this is the case only for very special non-primitive Mukai vectors. The first result in this direction has been obtained by O'Grady in [O'Ga]:

Theorem 3.2.15. *Let be $v = (2, 0, -2)$ and H a v-generic polarization on the projective $K3$ surface S. Then $\mathrm{M}_{10} := \mathrm{M}_v$ admits a symplectic desingularization $\tilde{\pi} : \widetilde{\mathrm{M}}_{10} \to \mathrm{M}_{10}$, which is an IHS variety of dimension 10 and second Betti number 24.*

Remark 3.2.16. In [O'Ga], O'Grady proved $b_2(\widetilde{\mathrm{M}}_{10}) \geq 24$; since deformation equivalent IHS varieties have the same Betti numbers, $b_2(\widetilde{\mathrm{M}}_{10}) \geq 24$ implies that $\widetilde{\mathrm{M}}_{10}$ is not deformation equivalent to any of the Beauville examples. Then $b_2(\widetilde{\mathrm{M}}_{10}) = 24$ has been proved by Rapagnetta in [Rap08].

$\widetilde{\mathrm{M}}_{10}$ is actually the only (up to deformation) new example of IHS variety that can be found out as symplectic resolution of a moduli space of sheaves on a projective $K3$ surface:

Theorem 3.2.17. *Let v be a Mukai vector of the form $v = mw$, with w primitive, $m \geq 2$ and $w^2 > 0$. Let H be a v-generic polarization on the projective $K3$ surface S.*

1. *If $m = 2$ and $w^2 = 2$, there exists a symplectic desingularization $\tilde{\pi} : \widetilde{\mathrm{M}}_v \to \mathrm{M}_v$ obtained as blow-up of M_v along the singular locus $\Sigma := \mathrm{M}_v \setminus \mathrm{M}_v^s$, taken with reduced structure. $\widetilde{\mathrm{M}}_v$ is deformation equivalent to $\widetilde{\mathrm{M}}_{10}$.*

2. *If $m \geq 3$ or $w^2 \geq 4$, then M_v does not admit any symplectic resolution.*

The existence of the symplectic resolution in point 1 of the theorem has been proved by Lehn and Sorger in [LS06], and the deformation type by Perego and Rapagnetta in [PR13]. The second point of the theorem as been proved by Kaledin, Lehn and Sorger in [KLS06].

Also in the case of non primitive Mukai vectors such that there exists a symplectic desingularization, there exists the following Hodge isometry of lattices, shown by Perego and Rapagnetta in [PR13]:

Theorem 3.2.18. *Under the hypothesis of Theorem 3.2.17 and $m = 2$, $w^2 = 2$, the pullback $\tilde{\pi}^* : H^2(\mathrm{M}_v, \mathbb{Z}) \to H^2(\widetilde{\mathrm{M}}_v, \mathbb{Z})$ is injective. $H^2(\mathrm{M}_v, \mathbb{Z})$ has a pure weight-two Hodge structure and lattice structure given by the restriction of the pure weight-two Hodge structure of $H^2(\widetilde{\mathrm{M}}_v, \mathbb{Z})$ and of its Beauville-Bogomolov-Fujiki form. Furthermore, there exists a Hodge isometry $\lambda_v : v^\perp \to H^2(\mathrm{M}_v, \mathbb{Z})$.*

Remark 3.2.19. The isometry λ_v is an extension of the morphism defined in Remark 3.2.14, i.e. given the inclusion $i : \mathrm{M}_v^s \hookrightarrow \mathrm{M}_v$ one has that $i^* \circ \lambda_v : v^\perp \to H^2(\mathrm{M}_v^s, \mathbb{Z})$ equals map in Remark 3.2.14 (see Section 3.2 in [PR13]).

Remark 3.2.20. A very similar theory can be done starting from an abelian surface instead of a $K3$ surface, see for example in [HL10]. In this case, when v is primitive then the moduli space M_v has a decomposition, with respect to the decomposition Theorem 1.1.5, such that one factor is an IHS variety deformation equivalent to a generalized Kummer variety. Regarding the non-primitive case, also for A abelian surface some non-primitive Mukai vectors give moduli spaces admitting a symplectic desingularization, that IHS varieties deformation equivalent to the 6-dimensional example $\tilde{\mathrm{K}}_6$ introduced by O'Grady in [O'Gb].

3.3 Lagrangian structure

In this section we want to focus on moduli spaces with some particular Mukai vectors, which are the ones we will use in the next chapters.

These moduli spaces are particularly interesting because they turn out to contain as dense open set a relative Jacobian over the smooth locus of a certain linear system of curves on S, which is very useful to understand the geometry of the moduli space itself.

Let S be a projective $K3$ surface with $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$, with H ample and $H^2 = 2$; put $h := c_1(H)$.

Remark 3.3.1. Under the hypotheses above, the polarization H induces a surjective morphism $f : S \to |H|^\vee \cong \mathbb{P}^2$ which has degree 2 and ramifies along a sextic. Since f has degree 2, it induces an involution on S, that we will denote ι. Furthermore, we will use many times in the next chapter that f^* induces a bijection among lines of $\mathbb{P}^2$ and curves of $|H|$ on S, and a bijection among conics in $\mathbb{P}^2$ and curves of $|2H|$ on S.

If the Mukai vector has the form $v = (0, bh, c)$ with $b > 0$, then there exists a regular morphism $p_{b,c} : \mathrm{M}_{(0,bh,c)} \to |bH|$ sending a sheaf in $\mathrm{M}_{(0,bh,c)}$ to its Fitting scheme, that is the support of the sheaf endowed with a schematic structure, see [LP93]. When $v = (0, bh, c)$ is not primitive, we call $\tilde{p}_{b,c} := p_{b,c} \circ \tilde{\pi} : \widetilde{\mathrm{M}}_{(0,bh,c)} \to |bH|$, where $\tilde{\pi} : \widetilde{\mathrm{M}}_{(0,bh,c)} \to \mathrm{M}_{(0,bh,c)}$ is the symplectic desingularization. Thanks to Matsushita's theorem (see [Mat99]), $\tilde{p}_{b,c}$ is a Lagrangian fibrations.

The following proposition will be crucial in the next. We decided to give a proof of it, since we could not find a detailed reference in literature.

Proposition 3.3.2. *If $C \in |bH|$ is a smooth curve then $p_{b,c}^{-1}(C) \cong J^{b^2+c}(C)$, where $J^{b^2+c}(C)$ is the Jacobian of degree $b^2 + c$ over the curve C. It follows, in particular, that a sheaf in $\mathrm{M}_{(0,bh,c)}$ supported on a smooth curve is always stable.*

Proof. Let $C \in |bH|$ be a smooth curve, and let $i : C \hookrightarrow S$ the inclusion of C in S. If $[F] \in p_{b,c}^{-1}(C)$, then $\mathfrak{v}(F) = (0, bh, c)$ implies $F = i_*G$, with $G \in \mathrm{Coh}(C)$. Under these hypotheses we can apply the Grothendieck-Riemann-Roch theorem on i, getting

$$ch_1(i_*G) = i_*(ch_0(G)td_0(T_{rel})) = i_*(rk(G)) = rk(G)[C]$$

where the rank of a coherent sheaf is the rank of the fiber over the generic point, and T_{rel} is the relative tangent bundle with respect to the inclusion $i : C \hookrightarrow S$. Again by $\mathfrak{v}(i_*G) = (0, bh, c)$, we get $ch_1(i_*G) \in |bH|$, and then by the equation above $C \in |bH|$ implies $rk(G) = 1$.

Since $[F] \in \mathrm{M}_{(0,bh,c)}$, we have $\dim(F) = 1$, and we call $P_H(F, m) = \alpha_0 + \alpha_1 m$ its Hilbert polynomial with respect to the polarization H. By definition

$$\deg(G) = \alpha_0(G) - \frac{\alpha_1(G)}{\alpha_1(\mathcal{O}_C)}\alpha_0(\mathcal{O}_C) = \chi(G) - \frac{\chi(G|_H)}{\chi(\mathcal{O}_C|_H)}\chi(\mathcal{O}_C)$$

where:

- $\chi(G) = \chi(F) = c$ because $[F] \in \mathrm{M}_{(0,bh,c)}$
- $\chi(\mathcal{O}_C|_H) = bH \cdot H = 2b$
- $\chi(G|_H) = 2b \cdot rk(G|_C) = 2b$
- $\chi(\mathcal{O}_C) = 1 - g(C) = -b^2$.

It follows that $\deg(G) = b^2 + c$. In other words, an element in the fiber of $p_{b,c}$ over a smooth curve $C \in |2H|$ is the push-forward of a coherent sheaf of rank 1 and degree $b^2 + c$ over C. Finally, a pure sheaf has no torsion on its support, and a torsion free sheaf of a smooth curve is locally free. □

Remark 3.3.3. Note that a similar result holds true on a curve $C \in |bH|$ that is union of k smooth curves $C_1, ..., C_k$, with $C_i \in |m_i H|$ and $m_1 + ... + m_k = b$: one can apply the Grothendieck-Riemann-Roch theorem on the smooth components of C and sum up the results. As consequence, we can say more in general that an element in the fiber of $p_{b,c}$ over a curve $C \in |2H|$ which is an union of the form of above is the push-forward of a coherent sheaf of rank 1 and degree $b^2 + c$ over C.

Note that in $\mathrm{M}_{(0,bh,c)}$ there are also sheaves of higher rank, e.g. sheaves of rank b supported on a single curve in H. As consequence, a similar description is not possible on non reduced curves.

Notation 1. By $\mathcal{J}^d_{|bH|^{sm}}$ we will denote the relative Jacobian of degree d on the smooth locus $|bH|^{sm}$ of the linear system $|bH|$, i.e. $\left(\mathcal{J}^d_{|bH|^{sm}}\right)|_C \cong J^d(C)$ for every $C \in |2H|^{sm}$. For a definition of relative Jacobian on $|2H|^{sm}$, see [ACGH85], Chapter XXI.2.

Corollary 3.3.4. *Using the notation just introduced,* $\mathcal{J}^{b^2+c}_{|bH|^{sm}} \subset \mathrm{M}_{(0,bh,c)}$*, and* $\mathcal{J}^{b^2+c}_{|bH|^{sm}}$ *is a dense open set in* $\mathrm{M}_{(0,bh,c)}$ *since* $|bH|^{sm}$ *is a dense open set of* $|bH|$.

In other words, we can look at $\mathrm{M}_{(0,bh,c)}$ as a compactification of $\mathcal{J}^{b^2+c}_{|bH|^{sm}}$. This point of view will always be adopted in the next.

Chapter 4

Rational curves on the O'Grady 10-dimensional example

Let X be a projective IHS variety of OG 10-type. The (optimistic) goal is to find ample uniruled divisors for any ample class on any such X, following the strategy presented in Section 2.3.

In this chapter we will deal with a preliminary problem: given an uniruled divisor $D \subset X$, how to check that it is ample. The strategy that we will present will be used in the next chapters, and we think it can be considered as a general strategy to check the ampleness of an (uniruled) divisors on an IHS variety of OG 10-type.

4.1 The ampleness of a divisor

Let X be an IHS variety, $D \subset X$ an uniruled divisor and q_X the Beauville-Bogomolov-Fujiki form of X. The following easy observation is a crucial point in the strategy we are going to present.

Remark 4.1.1. If $q_X(D) > 0$ then there exists a deformation of (X, D) where the divisor (or its dual) is ample. Indeed, we can deform (X, D) to a couple (X', D') where $\mathrm{Pic}(X') \cong \mathbb{Z}$; since $q_{X'}(D') = q_X(D) > 0$, by Huybrechts projectivity criterion (see [Huy99], Theorem 3.11) this implies that X' is projective, hence D' or $(D')^\vee$ is ample (this argument is also presented in Remark 3.12(i) of [Huy99]).

By the proof of Corollary 2.2.4, the ample deformation of D (or of its dual) is still uniruled. Furthermore, D and its the ample deformations obviously belong to the same connected component of $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$, the moduli space of marked and polarized IHS varieties of OG 10-type (see Section 1.2 for the notations).

Summarizing, the new goal is to answer to the following question:

Question 4.1.2. Let D be a (uniruled) divisor in X, with X an IHS variety of OG 10-type. How to compute the square $q_X(D)$?

An answer to this question is given by Theorem 3.2.18: given $X = \mathrm{M}_v$ with v Mukai vector as in the hypothesis of Theorem 3.2.17 part 1, there exists a Hodge isometry $\lambda_v : v^\perp \to H^2(\mathrm{M}_v, \mathbb{Z})$; furthermore, the desingularization $\pi : \tilde{\mathrm{M}}_v \to \mathrm{M}_v$ gives an inclusion of lattices $\pi^* : H^2(\mathrm{M}_v, \mathbb{Z}) \to H^2(\widetilde{\mathrm{M}}_v, \mathbb{Z})$. It follows that, given $D \subset \mathrm{M}_{10}$ divisor, one has $q_{10}(\pi^* D) = (\lambda_v^{-1} D)^2$, where the square on the right is with respect to the Mukai pairing introduced on the sublattice $v^\perp \subset H^*(S, \mathbb{Z})$ in Section 3.2.

As consequence, the new goal is to understand the class $\lambda_v^{-1} D$ inside the Mukai lattice; once this is done, the Mukai square $(\lambda_v^{-1} D)^2$ will follow from an easy computation. We will use the following Corollary of Theorem 3.2.18, proved by Perego and Rapagnetta in [PR14]:

Corollary 4.1.3. *Let X, H and v as in the hypothesis of Theorem 3.2.18. Then the isometry λ_v restricts to an isometry $(v^\perp)^{1,1} := v^\perp \cap (H^*(S))^{1,1} \to$ $\mathrm{Pic}(\mathrm{M}_v)$.*

Remark 4.1.4. In the corollary above we are identifying line bundles with their class in $H^2(\mathrm{M}_v, \mathbb{Z})$ through the first Chern class c_1. We have already pointed out that this can be done because for IHS varieties the first Chern class homomorphism is an injection.

For S with $\mathrm{Pic}(S) \cong \mathbb{Z}$, $\dim(v^\perp)^{1,1} = \dim \mathrm{Pic}(\mathrm{M}_v) = 2$; in the next, we will call $\{e, f\}$ a basis of $(v^\perp)^{1,1}$. We present here the strategy we will follow in order to answer Question 4.1.2:

Strategy 4.1.5. Let us assume the hypothesis of Theorem 3.2.18, and let $D \in \mathrm{Pic}(\mathrm{M}_v)$. In order to compute $q_{10}(D)$, we will follow the following steps.

1. We will define two "independent" curves in M_v, see the next remark for the precise meaning of the word "independent".

2. We will compute the geometrical intersection in M_v of the divisors $\lambda_v(e)$, $\lambda_v(f)$ with the curves defined in point 1.

3. We will compute the geometric intersection inside M_v of D with the curves defined in point 1.

4. We will compare what computed in points 2 and 3 to write $\lambda_v^* D$ in terms of the basis $\{e, f\}$ of $(v^\perp)^{1,1}$, and we will compute $(\lambda_v^* D)^2 = q_{10}(D)$.

Remark 4.1.6. By the word "independent" in point 1 of the Strategy 4.1.5 we mean that we will need two curves such that the system of two equations and two variables that we will get comparing the intersections in 2 and 3 will have a unique solution.

Remark 4.1.7. We will see in the next (cf. Chapter 6) that M_v is not always locally factorial; this means that a Weil divisor $D \subset \mathrm{M}_v$ can happen to be non Cartier. As consequence, once defined a Weil divisor in a moduli space M_v the first step will actually be to check that $D \in \mathrm{Pic}(\mathrm{M}_v)$.

4.2 Generators of the Mukai lattice

In this section we will deal with point 1 and 2 of the Strategy 4.1.5. First of all, we want to fix a moduli space:

Notation 2. From now on S will be a general $K3$ surface with $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$ with H ample and $H^2 = 2$; we will call $c_1(H) =: h$. We also fix the Mukai vector

$$v := (0, 2h, 4) \in H^*(S, \mathbb{Z}).$$

Note that v is not primitive in the Mukai lattice, and it satisfies the hypotheses of Theorems 3.2.17 and 3.2.18; it follows that there exists a symplectic desingularization $\tilde{\pi} : \widetilde{\mathrm{M}}_v \to \mathrm{M}_v$ s.t. $\widetilde{\mathrm{M}}_v$ is of OG 10-type. Notice also that v is chosen as in Section 3.3, then M_v carries a Lagrangian structure: M_v contains the relative Jacobian $\mathcal{J}^8_{|2H|^{sm}}$.

We start with point 1 of Strategy 4.1.5: we define two curves in M_v, that we will use in all our next computations. Consider:

- a smooth curve $\gamma \in |2H|$, a fixed point $p_0 \in \gamma$, the inclusion $i : \gamma \times \gamma \hookrightarrow S \times \gamma$ and the diagonal $\Delta \subset \gamma \times \gamma$.

- four generic points $q_1, ..., q_4 \in S$, their dual points $q_{i+4} := \iota(q_i)$ for $i = 1, ...4$ (cfr. Remark 3.3.1), the pencil $\tau \subset |2H| \cong \mathbb{P}^5$ defined by taking $\{q_1, ..., q_8\}$ as base points, the inclusion $j : \mathcal{C} \hookrightarrow S \times \tau$ of the universal curve of τ.

Definition 4.2.1. Using the notation of above, we define

$$\begin{aligned}\mathcal{E}_\Gamma &:= i_*\mathcal{O}_{\gamma\times\gamma}(7p_0\times\gamma+\Delta)\ \in \mathrm{Coh}(S\times\gamma)\\ \mathcal{E}_\mathcal{T} &:= j_*\mathcal{O}_{\mathcal{C}}\big(2(q_1\times\tau+q_2\times\tau+q_3\times\tau+q_4\times\tau)\big)\in \mathrm{Coh}(S\times\tau).\end{aligned}$$

The families $\mathcal{E}_\Gamma$ and $\mathcal{E}_\mathcal{T}$ are meant to define curves in the moduli space M_v, see Lemma 4.2.2 here below. They have been chosen in order to parametrize a "vertical" curve contained in $J^8(\gamma)\subset \mathrm{M}_v$, that is a copy of γ inside its Jacobian of degree 8, and an "horizontal" curve in M_v, that is a section of the pencil $\tau\subset|2H|$ obtained using the base points of the pencil.

Lemma 4.2.2. *The families $\mathcal{E}_\Gamma$ and $\mathcal{E}_\mathcal{T}$ of Definition 4.2.1 define two curves in M_v, that we will call Γ and $\mathcal{T}$ respectively.*

Proof. This is just consequence of the fact that M_v is the object corepresenting the functor $\mathcal{M}_v$, as explained in Section 3.1. The families $\mathcal{E}_\Gamma$ and $\mathcal{E}_\mathcal{T}$ define curves in the moduli space we are interested in because they parametrize families of line bundles of degree 8 on the curves of $|2H|$, and $\mathcal{J}^8_{|2H|^{sm}}\subset \mathrm{M}_v$. Note that these families are flat because their bases are reduced and all the fibers have the same Hilbert polynomial, cf. Proposition 2.1.2 in [HL10]. □

In order to pass to point 2 of Strategy 4.1.5, we start choosing a basis for $(v^\perp)^{1,1}$, where as before $v=(0,2h,4)\in H^*(S,\mathbb{Z})$.

$$\begin{aligned}(v^\perp)^{1,1} &:= \{(a,bh,c)\mid\ <(0,h,2),(a,bh,c)>=0\}=\{(a,ah,c),\ a,c\in\mathbb{Z}\}\\ &=<(1,h,0),(0,0,1)>_\mathbb{Z}\ .\end{aligned}$$

We set $e:=(1,h,0)$ and $f:=(0,0,1)$. An answer to point 2 of Strategy 4.1.5 is given by the following result:

Proposition 4.2.3. *Let e,f be the generators of $(v^\perp)^{1,1}$ just defined above, $\Gamma,\mathcal{T}\subset \mathrm{M}_v$ the curves corresponding to the universal families of Definition 4.2.1 and $\lambda_v:(v^\perp)^{1,1}\xrightarrow{\sim}\mathrm{Pic}(\mathrm{M}_v)$ the isometry of Corollary 4.1.3. One has the following intersections:*

1. $\lambda_v(e)\cdot\Gamma=-5,\ \ \lambda_v(f)\cdot\Gamma=0.$
2. $\lambda_v(e)\cdot\mathcal{T}=-5,\ \ \lambda_v(f)\cdot\mathcal{T}=1.$

We will end this section explaining the strategy we will use to compute the intersection in Proposition 4.2.3, leaving to the next section the actual computations. The fundamental tool is Theorem 4.2.7 below; in a particular case this is Proposition 2.3.7 in [Per08], and more in general it is Theorem

8.1.5 in [HL10]. But before stating the theorem, we need to translate some results obtained for M_v in terms of classes in the Grothendieck group $K(S)$ of S.

We have seen in Section 3.2 that the Mukai vector defines an homomorphism $\mathfrak{v} : K(S) \to H^*(S, \mathbb{Z})$ sending the class of a sheaf to its Mukai vector; it follows that one can define the usual moduli spaces starting from one class $c \in K(S)$: we will call $\mathrm{M}_c := \mathrm{M}_{\mathfrak{v}(c)}$. Since $\chi(E, F) = - < \mathfrak{v}(E), \mathfrak{v}(F) >$, given a class $c \in K(S)$ the Mukai vector homomorphism restricts to the orthogonal of c with respect to the Euler pairing: we get $c^\perp \to \mathfrak{v}(c)^\perp \subset H^*(S, \mathbb{Z})$. If $\mathfrak{v}^\vee : K(S) \to H^*(S, \mathbb{Z})$ is the dual Mukai vector homomorphism $c \mapsto \mathfrak{v}(c)^\vee$, we will call $\lambda_c : c^\perp \to H^2(\mathrm{M}_v, \mathbb{Z})$ the composition of $\mathfrak{v}^\vee$ restricted to $c^\perp$ and the isometry $\lambda_{\mathfrak{v}(c)}$ of Theorem 3.2.18. The next step is to understand what corresponds to $(v^\perp)^{1,1}$ in $c^\perp$.

Lemma 4.2.4. *Following the notation above, let $\alpha \in K(S)$; then $\alpha \in \{1, h, h^2\}^{\perp\perp}$ if and only if $c_1(\alpha) \in NS(S)$.*

Proof. Let $\beta \in K(S)$; then $\beta \in \{1, h, h^2\}^\perp$ if and only if:

- $\chi(\beta, 1) = \int \mathrm{ch}(\beta)\, \mathrm{ch}(\mathcal{O}_S)\, \mathrm{td}(S) = 2ch_0(\beta) + ch_2(\beta) = 0$, i.e. $ch_2(\beta) = -2ch_0(\beta)$;
- $\chi(\beta, h) = \int \mathrm{ch}(\beta)\, \mathrm{ch}(\mathcal{O}_S(H))\, \mathrm{td}(S) = ch_0(\beta) + ch_1(\beta) \cdot h = 0$, i.e. $ch_1(\beta) \cdot h = -ch_0(\beta)$;
- $\chi(\beta, h^2) = \int \mathrm{ch}(\beta)\, \mathrm{ch}(\mathcal{O}_S(2H))\, \mathrm{td}(S) = 4ch_0(\beta) - 2ch_1(\beta) \cdot h = 0$, i.e. $ch_1(\beta) \cdot h = -2ch_0(\beta)$.

Combining the last two points, we get $\mathrm{ch}(\beta) = (0, c_1(\beta), 0)$ with $c_1(\beta) \cdot h = 0$, i.e. $c_1(\beta) \in H^2(S, \mathbb{Z}) \cap (H^{2,0}(S) \oplus H^{0,2}(S))$. It follows that $\alpha \in \{1, h, h^2\}^{\perp\perp}$ if for any $b \in H^2(S, \mathbb{Z}) \cap (H^{2,0}(S) \oplus H^{0,2}(S))$ one has

$$\int_S (0, b, 0)\, \mathrm{ch}(\alpha)\, \mathrm{td}(S) = c_1(\alpha) \cdot b = 0,$$

meaning $c_1(\alpha) \in NS(S)$. $\square$

Remark 4.2.5. By definition $(v^\perp)^{1,1} = v^\perp \cap (H^0(S, \mathbb{C}) \oplus NS(S)_\mathbb{C} \oplus H^2(S, \mathbb{C}))$, i.e. $a = (a_0, a_1, a_2) \in v^\perp$ belongs to $(v^\perp)^{1,1}$ if and only if a_1 is the first Chern class of a line bundle. In other words, Lemma 4.2.4 says that the Mukai vector homomorphism restricts to an homomorphism $\lambda_c : c_h^\perp := c^\perp \cap \{1, h, h^2\}^{\perp\perp} \to (v^\perp)^{1,1}$.

Definition 4.2.6. Let C be a curve, $\mathcal{E}_C \in \mathrm{Coh}(S \times C)$ and $\pi_s : S \times C \to S$, $\pi_C : S \times C \to C$ the projections. We define the homomorphism $\lambda_{\mathcal{E}_C} : K(S) \to \mathrm{Pic}(C)$ to be the composition of the following homomorphisms:

$$K(S) \xrightarrow{\pi_S^*} K(S \times C) \xrightarrow{[\mathcal{E}_C]\cdot} K(S \times C) \xrightarrow{\pi_{C,!}} K(C) \xrightarrow{det} \mathrm{Pic}(C).$$

Theorem 4.2.7. *Following the notations above, let $\mathcal{E}_C$ be a flat family of semistable sheaves of class c parametrized by a curve C, with classifying morphism $\phi_{\mathcal{E}_C} : C \to \mathrm{M}_c$. Then the following diagram commutes:*

$$\begin{tikzcd}
c_h^{\perp} \arrow[r, hook, "\lambda_c"] \arrow[d, hook] & \mathrm{Pic}(\mathrm{M}_c) \arrow[d, "\phi_{\mathcal{E}_C}^*"] \\
K(S) \arrow[r, "\lambda_{\mathcal{E}_C}"] & \mathrm{Pic}(C)
\end{tikzcd}$$

In other words, given $\alpha \in c_h^{\perp}$ and a curve $C \subset \mathrm{M}_c$ with universal family $\mathcal{E}_C$, Theorem 4.2.7 gives a way to compute the intersection of $\lambda_c(\alpha)$ with the curve as follows:

$$\lambda_c(\alpha) \cdot C = deg\big(\lambda_{\mathcal{E}_C}(\alpha)\big).$$

Remark 4.2.8. We want to apply Theorem 4.2.7 to compute the intersection of $\lambda_v(e)$ and $\lambda_v(f)$ with the curves Γ and $\mathcal{T}$. Since the morphism λ_c is the composition of $\mathfrak{v}^{\vee}$ and $\lambda_{\mathfrak{v}(c)}$, we need to choose generators $[E],\ [F] \in c_h^{\perp}$ such that $\mathfrak{v}(E) = e^{\vee} = (1, -h, 1)$ and $\mathfrak{v}(F) = f^{\vee} = (0, 0, 1)$.

4.3 Proof of Proposition 4.2.3

In this section we will prove Proposition 4.2.3, applying Theorem 4.2.7 for $[E], [F] \in K(S)$ as in Remark 4.2.8 and $\mathcal{E}_\Gamma$, $\mathcal{E}_\mathcal{T}$ as in Definition 4.2.1.

4.3.1 Intersections 1 of Proposition 4.2.3

Following the notations above, in this section we want to compute $\lambda_c(e)\cdot\Gamma$ and $\lambda_c(f)\cdot\Gamma$. Let $\pi_S : S\times\gamma \to S$ and $\pi_\gamma : S\times\gamma \to \gamma$ be the two projections. Thanks to Theorem 4.2.7, we need to compute

$$c_1\big(\pi_{\gamma,!}(\mathcal{E}_\Gamma \otimes \pi_S^* E)\big) = \mathrm{ch}_1\big(\pi_{\gamma,!}(\mathcal{E}_\Gamma \otimes \pi_S^* E)\big)$$

and

$$c_1\big(\pi_{\gamma,!}(\mathcal{E}_\Gamma \otimes \pi_S^* F)\big) = \mathrm{ch}_1\big(\pi_{\gamma,!}(\mathcal{E}_\Gamma \otimes \pi_S^* F)\big).$$

The Grothendieck-Riemann-Roch theorem on the projection $\pi_\gamma : S \times \gamma \to \gamma$ states that the following diagram commutes:

$$\begin{tikzcd} K(S\times\gamma) \arrow[r, "\pi_{\gamma,!}"] \arrow[d, "\mathrm{ch}(\cdot)\,\mathrm{td}(T_{\pi_\gamma})"'] & K(\gamma) \arrow[d, "\mathrm{ch}(\cdot)"] \\ H^*(S\times\gamma,\mathbb{Q}) \arrow[r, "\pi_{\gamma,*}"] & H^*(\gamma,\mathbb{Q}) \end{tikzcd}$$

where $T_{\pi_\gamma} := T_{S\times\gamma} - \pi_\gamma^* T_\gamma \in K(S\times\gamma)$. In other words:

$$\mathrm{ch}_1\big(\pi_{\gamma,!}(\mathcal{E}_\Gamma \otimes \pi_S^* E)\big) = \pi_{\gamma,*}\big[\mathrm{ch}(\pi_S^* E \otimes \mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma})\big]_3 \tag{4.3.1}$$

and

$$\mathrm{ch}_1\big(\pi_{\gamma,!}(\mathcal{E}_\Gamma \otimes \pi_S^* F)\big) = \pi_{\gamma,*}\big[\mathrm{ch}(\pi_S^* F \otimes \mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma})\big]_3. \tag{4.3.2}$$

We start by computing the first expression.

Since $T_{S\times\gamma} - \pi_\gamma^* T_\gamma = \pi_S^* T_S$ in $K(S\times\gamma)$, then $\mathrm{td}(T_{\pi_\gamma}) = \mathrm{td}(\pi_S^* T_S) = \pi_S^*\,\mathrm{td}(S)$; it follows that:

$$\mathrm{ch}(\pi_S^* E \otimes \mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma}) = \mathrm{ch}(\mathcal{E}_\Gamma)\pi_S^*\big(\mathrm{ch}(E)\,\mathrm{td}(S)\big),$$

where $\mathrm{ch}(E)\,\mathrm{td}(S) = \mathfrak{v}(E)\sqrt{\mathrm{td}(S)} = (1,-h,1)$ because by definition $\mathfrak{v}(E) = (1,-h,0)$, and $\sqrt{\mathrm{td}(S)} = (1,0,1)$ since S is a $K3$ surface. It follows that:

$$\pi_S^*\big(\mathrm{ch}(E)\,\mathrm{td}(S)\big) = (1, -[H\times\gamma], [pt\times\gamma], 0). \tag{4.3.3}$$

It remains to compute the Chern character of $\mathcal{E}_\Gamma = i_*\mathcal{O}_{\gamma\times\gamma}(7(p_0\times\gamma)+\Delta)$; set $G := \mathcal{O}_{\gamma\times\gamma}(7(p_0\times\gamma)+\Delta)$. We use the Grothendieck-Riemann-Roch theorem on the inclusion $i : \gamma\times\gamma \hookrightarrow S\times\gamma$, i.e. the commutativity of the following diagram:

$$\begin{tikzcd} K(\gamma\times\gamma) \arrow[r, "i_!"] \arrow[d, "\mathrm{ch}(\cdot)\,\mathrm{td}(T_i)"'] & K(S\times\gamma) \arrow[d, "\mathrm{ch}(\cdot)"] \\ H^*(\gamma\times\gamma,\mathbb{Q}) \arrow[r, "i_*"] & H^*(S\times\gamma,\mathbb{Q}) \end{tikzcd}$$

where $T_i := T_{\gamma\times\gamma} - i^* T_{S\times\gamma} \in K(\gamma\times\gamma)$.

Since $i_! G = i_* G = \mathcal{E}_\Gamma$ (i is a closed inclusion), we get:

$$\mathrm{ch}_k\big(\mathcal{E}_\Gamma\big) = i_*\Big(\big[\mathrm{ch}(G)\,\mathrm{td}(T_i)\big]_{k-1}\Big).$$

G is a line bundle on $\gamma\times\gamma$, then:

$$\mathrm{ch}(G) = \Big(1, c_1(G), \frac{(c_1(G))^2}{2}\Big) = \Big(1, 7[p_0\times\gamma]+\Delta, 3\Big),$$

where we used the following intersections:

- $\Delta^2 = \deg(N_{\Delta/\gamma\times\gamma}) = \deg(T_\gamma) = -\deg(\omega_\gamma) = -8$, since a curve $\gamma \in |2H|$ has $g(\gamma) = 5$;
- $(p_0 \times \gamma)^2 = 0$;
- $\Delta \cdot (p_0 \times \gamma) = 1$.

To compute $\mathrm{td}(T_i)$ we argue as follows: by the very definition of T_i and by the following short exact sequence

$$0 \to T_{\gamma\times\gamma} \to i^* T_{S\times\gamma} \to N_{\gamma\times\gamma/S\times\gamma} \to 0$$

it follows that $T_i = N^{-1}_{\gamma\times\gamma/S\times\gamma} \in K(\gamma\times\gamma)$. If $\pi_1 : \gamma\times\gamma \to \gamma$ is the projection on the first factor, then $N^{-1}_{\gamma\times\gamma/S\times\gamma} = \pi_1^* N^{-1}_{\gamma/S} = \pi_1^*\omega_\gamma^{-1}$ where the last equality follows from the adjunction formula on the K3 surface S. Since $\mathrm{td}(\omega_\gamma^{-1}) = \big(1, \frac{c_1(\omega_\gamma^{-1})}{2}\big) = (1,-4)$, it follows that:

$$\mathrm{td}(T_i) = (1, -4[pt\times\gamma], 0).$$

Therefore, we get:

$$\mathrm{ch}(G)\,\mathrm{td}(T_i) = (1, 7[p_0\times\gamma] + \Delta, 3)(1, -4[pt\times\gamma], 0) = (1, 3[p_0\times\gamma] + \Delta, -1)$$

and then:

$$\mathrm{ch}(\mathcal{E}_\Gamma) = \big(0, [\gamma\times\gamma], 3[p_0\times\gamma] + [i_*\Delta], -1\big). \tag{4.3.4}$$

We can finally compute $\big[\mathrm{ch}(\pi_S^* E \otimes \mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma})\big]_3$, inserting in (4.3.1) what we have computed in (4.3.3) and (4.3.4):

$$\begin{aligned}
\big[\mathrm{ch}(\pi_S^* E \otimes \mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma})\big]_3 &= \big[\mathrm{ch}(\mathcal{E}_\Gamma)\pi_S^*\big(\mathrm{ch}(E)\,\mathrm{td}(S)\big)\big]_3 = \\
&= \Big[\big(0, [\gamma\times\gamma], 3[p_0\times\gamma] + [i_*\Delta], -1\big)\big(1, -[H\times\gamma], [pt\times\gamma], 0\big)\Big]_3 \\
&= [\gamma\times\gamma][pt\times\gamma] + [3(p_0\times\gamma) + i_*\Delta][-(H\times\gamma)] - 1 \\
&= -5,
\end{aligned}$$

where we used the following intersections:

- $(\gamma\times\gamma)(pt\times\gamma) = deg\big(N_{\gamma\times\gamma/S\times\gamma}\big)|_{pt\times\gamma} = 0$
- $i_*\Delta \cdot (H\times\gamma) = 4$
- $(p_0\times\gamma)(H\times\gamma) = 0.$

This means that $\lambda_v(e) \cdot \Gamma = \pi_{\gamma,*}(-10) = -5$.

The computation of $\lambda_c(f) \cdot \Gamma$ is similar; we start from 4.3.2, where now $\mathfrak{v}(F) = (0,0,1)$. We get:

$$\begin{aligned}\mathrm{ch}(\pi_S^* F \otimes \mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma}) &= \mathrm{ch}(\mathcal{E}_\Gamma)\pi_S^*\big(\mathrm{ch}(F)\,\mathrm{td}(S)\big) = \mathrm{ch}(\mathcal{E}_\Gamma)\pi_S^*\big(\mathfrak{v}(F)\sqrt{\mathrm{td}(S)}\big) \\ &= \mathrm{ch}(\mathcal{E}_\Gamma)\pi_S^*(0,0,1) = \big(0, [\gamma \times \gamma], 3[p_0 \times \gamma] + [i_*\Delta], -1\big)\big(0,0,[pt \times \gamma], 0\big)\end{aligned}$$

and then:

$$\big[\mathrm{ch}(\pi_S^* F \otimes \mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma})\big]_3 = [\gamma \times \gamma][pt \times \gamma] = 0,$$

i.e. $\lambda_v(f) \cdot \Gamma = 0$.

4.3.2 Intersections 2 of Proposition 4.2.3

Let $\pi_S : S \times \tau \to S$ and $\pi_\tau : S \times \tau \to \tau$ be the projections. We need to compute

$$c_1\big(\pi_{\tau,!}(\mathcal{E}_\mathcal{T} \otimes \pi_S^* E)\big) = \mathrm{ch}_1\big(\pi_{\tau,!}(\mathcal{E}_\mathcal{T} \otimes \pi_S^* E)\big)$$

and

$$c_1\big(\pi_{\tau,!}(\mathcal{E}_\mathcal{T} \otimes \pi_S^* F)\big) = \mathrm{ch}_1\big(\pi_{\tau,!}(\mathcal{E}_\mathcal{T} \otimes \pi_S^* F)\big).$$

By the Grothendieck-Riemann-Roch theorem on the projection π_τ, we get

$$c_1\big(\pi_{\tau,!}(\mathcal{E}_\mathcal{T} \otimes \pi_S^* E)\big) = \pi_{\tau,*}\big[\mathrm{ch}(\pi_S^* E \otimes \mathcal{E}_\mathcal{T})\,\mathrm{td}(T_{\pi_\tau})\big]_3$$

and

$$c_1\big(\pi_{\tau,!}(\mathcal{E}_\mathcal{T} \otimes \pi_S^* F)\big) = \pi_{\tau,*}\big[\mathrm{ch}(\pi_S^* F \otimes \mathcal{E}_\mathcal{T})\,\mathrm{td}(T_{\pi_\tau})\big]_3,$$

where $T_{\pi_\tau} := T_{S\times\tau} - \pi_\tau^* T_\tau = \pi_S^* T_S$ in $K(S \times \tau)$. Also in this section, we start from $\lambda_v(e) \cdot \mathcal{T}$.

Since $\mathrm{td}(T_{\pi_\tau}) = \pi_S^*\,\mathrm{td}(S)$, similarly to the previous section the following holds:

$$\mathrm{ch}(\pi_S^* E \otimes \mathcal{E}_\mathcal{T})\,\mathrm{td}(T_{\pi_\tau}) = \mathrm{ch}(\mathcal{E}_\mathcal{T})\big(1, -[H \times \tau], [pt \times \tau], 0\big). \tag{4.3.5}$$

It remains to compute the Chern characters of $\mathcal{E}_\mathcal{T}$.

Set $T := \mathcal{O}_\mathcal{C}\big(2(q_1 \times \tau + ... + q_4 \times \tau)\big)$, so that $\mathcal{E}_\mathcal{T} = j_* T$. Since $j : \mathcal{C} \hookrightarrow S \times \tau$ is a closed immersion, $j_* T = j_! T$, as in the section above; using the Grothendieck-Riemann-Roch theorem on j, we get:

$$\mathrm{ch}(\mathcal{E}_\mathcal{T})_k = j_*\big([\mathrm{ch}(T)\,\mathrm{td}(T_j)]_{k-1}\big),$$

where $T_j := T_\mathcal{C} - j^* T_{S\times\tau} \in K(\mathcal{C})$.

Remark 4.3.1. There exists an isomorphism $\mathcal{C} \cong \tilde{S}$, under which $q_i \times \tau$ gets mapped into the exceptional divisor E_i over the point q_i, $i = 1, ..., 8$. Now on, we will use the blow-up notations.

It follows:

$$\mathrm{ch}(T) = \Big(1, c_1(T), \frac{c_1(T)^2}{2}\Big) = (1, 2(E_1 + E_2 + E_3 + E_4), -8).$$

In order to compute $\mathrm{ch}(\mathcal{E}_{\mathcal{T}})$ it remains to compute $\mathrm{td}(T_j) = \Big(1, \frac{c_1(T_j)}{2}, \frac{c_1(T_j)^2}{12}\Big)$. By definition of T_j, we need to compute:

$$det(T_{\mathcal{C}}) \otimes det(T_{S\times\tau}|_{\mathcal{C}})^{-1} = \omega_{\mathcal{C}}^{-1} \otimes \pi_S^*(\omega_S)|_{\mathcal{C}} \otimes \pi_\tau^*(\omega_\tau)|_{\mathcal{C}}.$$

We will use the following facts:

- $\omega_S = 0$ since S is $K3$, and then $\pi_S^*(\omega_S)|_{\mathcal{C}} = 0$;
- since $\tau \cong \mathbb{P}^1$, $\omega_\tau = \mathcal{O}_\tau(-2)$. Furthermore, for $t \in \tau$, $\pi_\tau^{-1}(t)|_{\mathcal{C}} \cong C_t \times t$, with C_t the curve corresponding to t, and then $\pi_\tau^*(\omega_\tau)|_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(-2(C_t \times t))$;
- since $\mathcal{C} \cong \tilde{S}$, if $\pi : \mathcal{C} \to S$ is the blow-up map then $\omega_{\mathcal{C}}^{-1} = \pi^*\omega_S^{-1} - \sum_{i=1}^8 E_i = -\sum_{i=1}^8 E_i$.

Then:

$$\begin{aligned}\mathrm{td}(T_j) &= \Big(1, -[C_t \times t] - \sum_{i=1}^{8} \frac{[E_i]}{2}, \frac{\big(-2(C_t \times t) - \sum_{i=1}^8 E_i\big)^2}{12}\Big)\\ &= \Big(1, -[C_t \times t] - \sum_{i=1}^{8} \frac{[E_i]}{2}, 2\Big),\end{aligned}$$

where the last equality follows from $(C_t \times t)^2 = 0$ and $(C_t \times t) \cdot E_i = 1$, $\forall i = 1, ..., 8$.

In conclusion, we have that:

$$\begin{aligned}\mathrm{ch}(T)\,\mathrm{td}(T_j) &= \Big(1, 2[E_1 + E_2 + E_3 + E_4], -8\Big)\Big(1, -[C_t \times t] - \sum_{i=1}^{8} \frac{[E_i]}{2}, 2\Big)\\ &= \Big(1, 2[E_1 + E_2 + E_3 + E_4] - [C_t \times t] - \sum_{i=1}^{8} \frac{[E_i]}{2}, -10\Big)\end{aligned}$$

and we get:

$$\mathrm{ch}(\mathcal{E}_{\mathcal{T}}) = \Big(0, [\mathfrak{C}], 2[E_1 + E_2 + E_3 + E_4] - [C_t \times t] - \sum_{i=1}^{8} \frac{[E_i]}{2}, -10\Big). \quad (4.3.6)$$

Using the expression (4.3.6) just computed, we obtain that the expression (4.3.5) equals to:

$$[\mathfrak{C}][pt \times \tau] - 2[\sum_{i=1}^{4} E_i][H \times \tau] + [C_t \times t][H \times \tau] + \sum_{i=1}^{8} \frac{[E_i]}{2}[H \times \tau] - 10.$$

Notice that:

- $(\mathfrak{C})(pt \times \tau) = |\{(pt, t) \in S \times \tau \mid pt \in C_t\}|$; we can assume $pt = p \notin \{q_1, ..., q_8\}$, and such a p determines an unique $C_p \in \tau$. It follows $(\mathfrak{C})(pt \times \tau) = 1$;
- $E_i \cdot (H \times \tau) = (p_i \times \tau)(H \times \tau) = 0 \ \forall i = 1, ..., 8$;
- $(C_t \times t)(H \times \tau) = C_t \cdot H = 4$;

It follows that:

$$\big[\mathrm{ch}(\pi_S^* E \otimes \mathcal{E}_{\mathcal{T}}) \, \mathrm{td}(T_{\pi_\tau})\big]_3 = 1 + 4 - 10 = -5,$$

i.e. $\lambda_v(e) \cdot \mathcal{T} = -5$.

In order to compute $\lambda_v(f) \cdot \mathcal{T}$, we still need to compute only

$$c_1\big(\pi_{\tau,!}(\mathcal{E}_{\mathcal{T}} \otimes \pi_S^* F)\big) = j_*\big[\mathrm{ch}(\pi_S^* F \otimes \mathcal{E}_{\mathcal{T}}) \, \mathrm{td}(T_{\pi_\tau})\big]_3.$$

where now

$$\pi_S^*(\mathrm{ch}(F) \, \mathrm{td}(S)) = \pi_S^*(0, 0, 1) = (0, 0, [pt \times \tau], 0).$$

Using (4.3.6) of the previous section we get:

$$\lambda_v(f) \cdot \mathcal{T} = j_*\Big(\big[\mathrm{ch}(\pi_S^* F \otimes \mathcal{E}_{\mathcal{T}}) \, \mathrm{td}(T_{\pi_\tau})\big]_3\Big) = j_*\Big([\mathfrak{C}][pt \times \tau]\Big) = 1.$$

This completes the proof of Proposition 4.2.3.

Chapter 5

A first example of ample uniruled divisor

In this chapter we will define a divisor on an IHS variety of OG 10-type, and we will show that it has an ample and uniruled deformation, following Strategy 4.1.5 introduced in Section 4.1.

Along this chapter we will use the notations introduced in Notation 2.

5.1 Definition of the divisor

Fix $\rho \in |H|$ a rational curve; we ask ρ to satisfy the following conditions:

(a) Consider the surjective morphism $f : S \to \mathbb{P}^2$ discussed in Remark 3.3.1, and call τ' the pencil of conics in $\mathbb{P}^2$ corresponding to the pencil $\tau \subset |2H|$. We fix $\rho \in |H|$ such that the line $f(\rho) \subset \mathbb{P}^2$ is not tangent to any of the three singular conics in the pencil τ'.

(b) The line $f(\rho) \subset \mathbb{P}^1$ is not tangent to the conics of τ' which are tangent to the sextic in $\mathbb{P}^2$ that is the branch locus of $f : S \to \mathbb{P}^2$.

We will define a divisor in M_v using the open inclusion $\mathcal{J}^8_{|2H|^{sm}} \subset \mathrm{M}_v$ of Corollary 3.3.4.

Definition 5.1.1. Let $D \subset \mathrm{M}_v$ be the divisor defined as the closure in M_v of the locus in $\mathcal{J}^8_{|2H|^{sm}}$ consisting of sheaves of the form $i_*\mathcal{O}_C(4r + p_1 + p_2 + p_3 + p_4)$, where $C \in |2H|^{sm}$, $i : C \hookrightarrow S$, $p_1, .., p_4 \in C$ and $r \in C \cap \rho$.

Note that the multiplicities of the points in the definition of D are just chosen in order to have line bundles of degree 8 over the curves of $|2H|^{sm}$. The geometrical intuition behind the definition of D is given by the following:

Lemma 5.1.2. *The divisor D of Definition 5.1.1 is uniruled.*

Proof. There exists a rational dominant map

$$\phi : \rho \times Sym^4(S) \dashrightarrow D$$

defined as follows: $\phi : \xi := \big(r, (p_1, ..., p_4)\big) \mapsto i_*\mathcal{O}_{C_\xi}(4r + p_1 + ... + p_4)$, where C_ξ is the unique curve of $|2H|$ passing through ξ and $i_* : C_\xi \hookrightarrow S$ its embedding in S. Note that such a curve is actually unique for a generic choice of $r, p_1, ..., p_4$: it is the pullback through $f : S \to |H|^\vee \cong \mathbb{P}^2$ of the unique conic passing through the five generic points $f(r), f(p_1), ..., f(p_4) \in \mathbb{P}^2$, see Remark 3.3.1. $\square$

Remark 5.1.3. The schematic structure of D is given by the Brill-Noether theory on curves in the linear system $|2H|$, see [ACGH11], Section 3 of Chapter XXI. Consider $U := |2H|^{sm}$, its universal curve $\mathcal{U} \to U$ and a Poincaré line bundle $\mathcal{L}_8 \in \mathrm{Pic}(\mathcal{U} \times_U \mathcal{J}^8_U)$ of the relative Jacobian $\mathcal{J}^8_U$; we call

$$q : \mathcal{U} \times_U \mathcal{J}^8_U \to \mathcal{J}^8_U$$

the projection on the second factor. Furthemore, consider the universal curve $\mathcal{C} \subset \rho \times U$, i.e. $\mathcal{C} := \{(r, u) \in \rho \times U \mid r \in C_u := p^{-1}(u)\}$, and its projection to the second factor $p_U : \mathcal{C} \to U$. Finally, let $\mathcal{U} \times_U \mathcal{J}^8_U \to U$ be the natural map induced by the fiber product on U, and consider the fiber product:

$$\begin{array}{ccc} \mathcal{C} \times_U \mathcal{U} \times_U \mathcal{J}^8_U & \xrightarrow{q_U} & \mathcal{U} \times_U \mathcal{J}^8_U \\ \downarrow{\scriptstyle\alpha} & & \downarrow \\ \mathcal{C} & \xrightarrow{p_U} & U \end{array}$$

Let $s : \mathcal{C} \to \mathcal{C} \times_U \mathcal{U} \times_U \mathcal{J}^8_U$ be a section of α, and call $C := s(\mathcal{C})$. Given the sheaf

$$R^1 q_*\big(\mathcal{L}_8 \otimes q_{U,*}\mathcal{O}_{\mathcal{C}\times\mathcal{U}\times\mathcal{J}^8_U}(-4C)\big) \in \mathrm{Coh}(\mathcal{J}^8_U),$$

the scheme $\mathcal{W}^8_\rho \subset \mathcal{J}^8_U$ is defined to be its Fitting scheme. Notice that:

$$Supp(\mathcal{W}^8_\rho) = \{(u, L) \in U \times J^8(C_u) \mid \exists r \in C_u \cap \rho \; s.th. \; h^0(C_u, L(-4r)) > 0\}.$$

The divisor D is the closure of $\mathcal{W}^8_\rho$ in M_v. Notice that, as closure of the image of the map given in Lemma 5.1.2, D is irreducible. Furthermore, it is reduced, since it is irreducible and $\mathcal{W}^8_\rho|_{C_u}$ is reduced for any $u \in U$, see [ACGH85], Proposition 4.4 in Chapter IV.

Given $v \in H^*(S, \mathbb{Z})$ non primitive, the moduli spaces M_v can happen to be non locally factorial, i.e. there could be divisors which are Weil but non Cartier. We recall here the definition of locally and k-factorial.

Definition 5.1.4. Let X be a normal projective variety, and let $A^1(X)$ be the group of Weil divisors of X, up to linear equivalence; consider the natural inclusion $d : \mathrm{Pic}(X) \to A^1(X)$ associating to any line bundle its associated Weil divisor. X is said to be k-factorial if the cokerne of d is k-torsion, and it is said to be locally factorial if d is an isomorphism.

Perego and Rapagnetta showed in [PR14] that the moduli spaces M_v are either locally factorial or 2-factorial. In the same paper, they gave the following criterion:

Theorem 5.1.5. *Let us consider a Mukai vector of the form $v = 2w \in H^*(S, \mathbb{Z})$. Then*

- M_v *is 2-factorial if and only if it exists $\gamma \in (H^*(S))^{1,1}$ such that $< \gamma, w >= 1$.*
- M_v *is locally factorial if and only if for any $\gamma \in (H^*(S))^{1,1}$ one has $< \gamma, w > \in 2\mathbb{Z}$.*

Lemma 5.1.6. *The divisor $D \subset \mathrm{M}_v$ introduced in Definition 5.1.1 is a Cartier divisor.*

Proof. We just need to apply Theorem 5.1.5 to our case: $v = (0, 2h, 4)$, then $w = (0, h, 2)$; an element $\gamma \in (H^*(S))^{1,1}$ is a vector of the form (a, bh, c), then

$$\gamma \cdot (0, h, 2) = 2b - 2a \in 2\mathbb{Z}$$

for any $\gamma \in (H^*(S))^{1,1}$. It follows that $\mathrm{M}_{(0,2h,4)}$ is locally factorial and the divisor D is a Carter divisor. $\square$

5.2 Intersection of the divisor and the curves

We start this section with a very easy remark, that we want to write in detail since it will be useful in the next.

Remark 5.2.1. Let C be a smooth curve. $J^k(C)$ is a torsor on the abelian variety $J^0(C)$, i.e. by definition there is a free and transitive action of the group $J^0(C)$ on the space $J^k(C)$. A theta divisor on $J^k(C)$ is an effective divisor whose translate in $J^0(C)$ is a principal polarization. If g is the genus of C, then the subtorsor of $J^{g-1}(C)$ consisting of effective divisors up to linear equivalence is a theta divisor in $J^{g-1}(C)$ (this can be found for example in [BL13]). It follows that, for $r \in J^{k-g+1}(C)$,

$$\theta_r := \{r + \alpha |\ \alpha \text{ is effective}\}$$

is a theta divisor in $J^k(C)$, since it is a translate of the previous one. Furthermore, θ_r and θ_s do not coincide if $r \neq s$ in $J^{k-g+1}(C)$. Indeed $J^k(C)$ is a torsor on $J^0(C)$, then the morphism

$$\begin{aligned} \phi_\theta : J^0(C) &\to (J^k(C))^\vee \\ d &\mapsto \theta_d \otimes \theta^{-1} \end{aligned}$$

is an isomorphism for any theta divisor θ in $J^k(C)$; choosing the theta divisor $\theta_r \subset J^k(C)$ as before and $d = s - r \in J^0(C)$, we get $\theta_r \neq \theta_s$ if $r \neq s$ in $J^0(C)$.

Proposition 5.2.2. *Let D be the divisor introduced in Definition 5.1.1 and Γ be the curve defined by Lemma 4.2.2. Then $D \cdot \Gamma = 20$.*

Proof. Let $p : \mathrm{M}_v \to |2H|$ the map associating to a S-equivalence class of sheaves its Fitting scheme. Γ is supported inside the fiber $p^{-1}(\gamma)$, which is $J^8(\gamma)$ by Proposition 3.3.2. It follows that we need to compute the intersection of D and Γ just inside $J^8(\gamma)$; for simplicity, we set $D_\gamma := D|_{J^8(\gamma)}$.

If $AJ : \gamma \to J^8(\gamma)$ is the Abel-Jacobi map translated by $7p_0$, then

$$[\Gamma] = AJ_*[\gamma] = \left[\frac{\theta^4}{4!}\right] \in H^2(J^8(\gamma), \mathbb{Z}),$$

where $[\theta]$ is the class of a theta divisor on $J^8(\gamma)$. If we denote by $\{r_1, r_2, r_3, r_4\} = \rho \cap \gamma$, then $D_\gamma = \sum_{i=1}^4 D_{r_i}$, where D_{r_i} is the component of D_γ obtained by fixing the point r_i in the intersection $\rho \cap \gamma$; by Remark 5.2.1, $[D_{r_i}] = [\theta]$.

Using the Poincaré formula, it follows that:

$$\Gamma \cdot D = 4\left[\frac{\theta^4}{4!}\right] \cdot [\theta] = 4\left[\frac{\theta^5}{4!}\right] = 20.$$

□

Proposition 5.2.3. *Let D be the divisor introduced in Definition 5.1.1 and $\mathcal{T}$ the curve defined by Lemma 4.2.2. Then $D \cdot \mathcal{T} = 84$.*

Proof. We divide the proof in two parts. In the first one, we will give a strategy to compute the intersection $D \cdot \mathcal{T}$; in the second one, we will proceed with the computations.

1. Reformulation of the intersection $D \cdot \mathcal{T}$. We noticed in the proof of Proposition 4.2.3 that the universal curve $\mathcal{C} \subset S \times \tau$ parametrizing $\mathcal{T}$ is isomorphic to $\tilde{S} := Bl_{q_1,...,q_8}S$, where $q_1, ..., q_8$ are the base points of τ. In this setting, the universal family of τ is the sheaf

$$\mathcal{O}_{\tilde{S}}\big(2(E_1 + E_2 + E_3 + E_4)\big) \in \mathrm{Coh}(\tilde{S}),$$

where $E_1, ..., E_4$ are the exceptional divisors of the blow-up $\tilde{S}$ over the points $q_1, ..., q_4$ respectively. Notice that, in this setting, the flat morphism $\tilde{S} \xrightarrow{g} \tau$ is the morphism that at any point $s \in \tilde{S}$ associates the point in τ corresponding to the unique curve in τ individuated by $s, q_1, .., q_8$.

In order to compute the intersection of $\mathcal{T}$ with D, let us consider the following fiber product:

$$\begin{tikzcd} X := \hat{\rho} \times_\tau \tilde{S} \arrow[r, "\hat{f}"] \arrow[d, "\hat{g}"] & \tilde{S} \arrow[d, "g"] \\ \hat{\rho} \arrow[r, "f"] & \tau \end{tikzcd}$$

where $\hat{\rho} \xrightarrow{\nu} \rho$ is the normalization of ρ and $f : \hat{\rho} \to \tau$ is defined as follows: the inclusion of ρ in $\tilde{S}$ defines a map $h : \rho \to \tau$ of degree 4, since for any $\alpha \in \tau$ one has $\rho \cdot \alpha = 4$; let $f := h \circ \nu$. Note that $\hat{f} : X \to \tilde{S}$ is a covering of degree 4.

Let us consider the identity morphism $id : \hat{\rho} \to \hat{\rho}$ and the morphism $\hat{\rho} \to \tilde{S}$ given by the composition of the inclusion of $\rho \hookrightarrow \tilde{S}$ and the normalization morphism ν; by the universal property of the fiber product, that there exists a section $\sigma : \hat{\rho} \to X$ of $\hat{g}$. We call $C_\rho := \sigma(\hat{\rho})$ its image. Note that it is a divisor on the surface X. Finally, set $\hat{E}_i := \hat{f}^* E_i$ for $i = 1, ..., 8$. We finally define

$$L := \mathcal{O}_X\big(2(\hat{E}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4) - 4C_\rho\big) \in \mathrm{Pic}(X).$$

By the very definition of D (see also Remark 5.1.3), in order to compute the intersection $D \cdot \mathcal{T}$ we need to check when $h^0(L|_r) = 0$, for $r \in \hat{\rho}$. This will be make precise in what follows.

Since the family L is flat over $\hat{\rho}$, it follows that $\chi(L_r)$ is constant in $r \in \hat{\rho}$. Let $r \in \hat{\rho}$ be such that C_r is a smooth curve; thanks to the Riemann-Roch theorem:

$$\chi(L_r) = \deg(L_r) - g(C_r) + 1 = 0$$

since $g(C_r) = 5$ for $C_r \in |2H|$. It follows $h^0(L_r) = h^1(L_r)$ for any $r \in \hat{\rho}$. Furthermore, note that

$$H^1(\hat{g}^{-1}(r), L_r) \cong (R^1\hat{g}_*L)_r$$

since $H^2(\hat{g}^{-1}(r), L_r) = (R^2\hat{g}_*L)_r = 0$. Because of the schematic structure given to D in Remark 5.1.3, we conclude that

$$D \cdot \mathcal{T} = c_1(D|_{\mathcal{T}}) = c_1(R^1\hat{g}_*L).$$

2. Computation of the intersection. We need to compute $c_1(R^1\hat{g}_*L)$. In the Grothendieck group $K(S)$ one has

$$\hat{g}_!L = \hat{g}_*L - R^1\hat{g}_*L.$$

Notice that $\hat{g}_*L = 0$ because it is torsion free (as push forward of a line bundle) and with generic fiber equal to 0; it follows

$$\mathrm{ch}(R^1\hat{g}_*L) = -\mathrm{ch}(\hat{g}_!L).$$

We compute $\mathrm{ch}(\hat{g}_!L)$ using the Grothendieck-Riemann-Roch theorem on $\hat{g} : X \to \hat{\rho}$:

$$\mathrm{ch}(\hat{g}_!L) = \hat{g}_*\big(\mathrm{ch}(L)\,\mathrm{td}(X)\big)\mathrm{td}(\hat{\rho})^{-1}. \tag{5.2.1}$$

Note that X is smooth (which is a necessary condition to apply the Grothendieck-Riemann-Roch theorem). Indeed, X is a fiber product of smooth varieties, hence it is smooth where the morphisms f and g are smooth; g is not smooth on points lying on a singular curve of the pencil τ, while f is not smooth on ramifications points, that are the points of $\hat{\rho}$ where the image of $\hat{\rho}$ in S is tangent to a curve in τ. We conclude that X is smooth if the singular conincs in τ are not tangent to the rational curve ρ, which is true thanks to conditions (a) and (b) on the rational curve ρ.

In the following, we will compute the factors of equation (5.2.1).

1. $\mathrm{td}(\hat{\rho}) = \left(1, \frac{c_1(\omega_\rho^\vee)}{2}\right) = (1,1)$ since $\hat{\rho} \cong \mathbb{P}^1$; it follows that $\mathrm{td}(\hat{\rho})^{-1} = (1,-1)$.

2. $\mathrm{td}(X) = \mathrm{td}(T_X) = \left(1, \frac{c_1}{2}, \frac{c_1^2 - c_2}{12}\right)$, where $c_i := c_i(\omega_X^\vee)$ for $i = 1, 2$. Because of the short exact sequences

$$0 \to \hat{g}^*\Omega^1_{\hat{\rho}} \to \Omega^1_X \to \Omega^1_{\hat{\rho}/X} \cong \hat{f}^*\Omega^1_{\tau/S} \to 0$$

and

$$0 \to g^*\Omega^1_\tau \to \Omega^1_{\tilde{S}} \to \Omega^1_{\tau/\tilde{S}} \to 0$$

one has

$$\begin{aligned} c_1(\omega_X^\vee) &= \hat{g}^* c_1(\omega_{\hat{\rho}}^\vee) + \hat{f}^*[c_1(\omega_{\tilde{S}}^\vee) + g^* c_1(\omega_\tau)] \\ &= 2\xi + \hat{f}^*[-E_1 - \ldots - E_8 - 2\tilde{C}_t] = 2\xi + [-\hat{E}_1 - \ldots - \hat{E}_8] - 8\xi \\ &= -6\xi + [-\hat{E}_1 - \ldots - \hat{E}_8]. \end{aligned}$$

In the above computations ξ is the class of a fiber of $X \to \hat{\rho}$, $\tilde{C}_t = g^{-1}(t)$ for $t \in \tau$, and the intersections follow from: $\tilde{S}$ is the blow-up of a $K3$ surfaces in 8 points, $\tau \cong \mathbb{P}^1$ and $\hat{f} : X \to \tilde{S}$ is a covering of degree 4. It follows that:

$$c_1^2 = (-6\xi)^2 + 12(\xi \cdot [\hat{E}_1] + \ldots + \xi \cdot [\hat{E}_8]) + [\hat{E}_1 + \ldots + \hat{E}_8]^2.$$

Because of the following intersections:

- $\xi^2 = 0$ since ξ is a fiber;
- $\xi \cdot \hat{E}_i = 1$ because it is a product of a fiber and a section;
- $\hat{E}_i \cdot \hat{E}_j = 4E_i \cdot E_j = -4\delta_{ij}$

we get

$$c_1^2 = 12 \cdot 8 - 4 \cdot 8 = 64.$$

It remains to compute $c_2(T_X)$. We recall that, for a smooth projective complex manifold of dimension d, $c_d(T_X) = \chi_{top}(X)$; this is a combination of Hirzebruch-Riemann-Roch theorem, Borel-Serre identity and the Hodge decomposition theorem. Given a ramified covering $\phi : X \to Y$ of degree d with branch locus B_ϕ and ramification locus R_ϕ, one has $\chi(X \setminus R_\phi) = d \cdot \chi(Y \setminus B_\phi)$, which implies

$$\chi(X) = d \cdot \chi(Y) + \chi(R_\phi) - d \cdot \chi(B_\phi).$$

In our case: X is a covering of degree 4 of a blow-up of a $K3$ surface in 8 points; the branch locus of $f : \hat{\rho} \to \tau$ are the points of τ parametrizing curves of the pencil which are tangent in S to the rational curve ρ; translating the problem in $\mathbb{P}^2$ via the morphism $S \to \mathbb{P}^2$ associated to the linear system $|H|$ (cf. Remark 3.3.1), we need to count how many conics in a pencil are tangent to a line. This happens for 2 conics, which correspond to 2 curves on $\mathbb{P}^2$ and then 2 curves of $|2H|$ pulled-back on $\hat{S}$; it follows that the branch locus $B_{\hat{f}}$ of $\hat{f}$ corresponds to 2 curves of genus 5. Regarding the ramification locus of $\hat{f}$, observe the following:

each conic in $\mathbb{P}^2$ tangent to the line has one point of intersection with the line, which corresponds on S to 2 points of tangence among the curve of $|2H|$ and the rational curve ρ; this happens for 2 branch curves parametrized by τ, which means that the ramification locus R_f of f consists of 4 points, and then the ramification locus $R_{\hat{f}}$ of $\hat{f}$ consists of 4 curves of genus 5. Then:

$$\begin{aligned}\chi(X) &= 4\chi(\tilde{S}) + \chi(R_{\hat{f}}) - 4\chi(B_{\hat{f}}) \\ &= 4\chi(\tilde{S}) + 4\chi(\alpha) - 4 \cdot 2\chi(\alpha) \\ &= 4\chi(\tilde{S}) - 4\chi(\alpha)\end{aligned}$$

where α is a curve of genus 5. Because of the following expressions:

- $\chi(\tilde{S}) = \chi(S) + 8(\chi(\mathbb{P}^1) - \chi(pt)) = 24 + 8 \cdot 1 = 32$ since $\tilde{S}$ is the blow up of S along 8 points
- $\chi(\alpha) = 1 - 10 + 1 = -8$

it follows that $\chi(X) = 4 \cdot 32 + 32 = 5 \cdot 32$. Summing up, we obtain

$$\mathrm{td}(X) = \Big(1, \frac{-6\xi + [-\hat{E}_1 - ... - \hat{E}_8]}{2}, -8\Big).$$

3. $\mathrm{ch}(L) = \Big(1, c_1(L), \frac{c_1(L)^2}{2}\Big)$ with $c_1(L) = 2(\hat{E}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4) - 4C_\rho$ and $c_1(L)^2 = 4(\hat{E}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4)^2 + 16C_\rho^2 - 16C_\rho(\hat{E}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4)$. Notice that:

   - By the adjunction formula:

     $$C_\rho^2 = deg(K_{C_\rho}) - \deg(K_X|_{C_\rho}).$$

     Note that $\deg(K_{C_\rho}) = -2$ because $C_\rho \cong \mathbb{P}^1$. On the other hand

     $$K_X = \hat{f}^* K_{\tilde{S}} + R_{\hat{f}}$$

     and

     $$\begin{aligned}\deg(\hat{f}^* K_{\tilde{S}}|_{C_\rho}) &= \hat{f}^* K_{\tilde{S}} \cdot C_\rho = K_{\tilde{S}} \cdot \hat{f}_* C_\rho \\ &= (K_S + \sum E_i) \cdot \hat{f}_* C\rho.\end{aligned}$$

     Since S is a $K3$, $K_S = 0$. Furthermore, $\hat{f}_* C_\rho = \nu(\hat{\rho})$: by the universal property of the fiber product, $\hat{f} \circ \sigma$ is the inclusion of

$\nu(\hat{\rho})$ in $\tilde{S}$, with $\nu : \hat{\rho} \to \rho$ desingularization. The curve $\nu(\hat{\rho})$ does not intersect $E_1, ..., E_8$, since ρ does not pass through $\{q_1, ..., q_8\}$; it follows $(K_S + \sum E_i) \cdot \hat{f}_* C\rho = 0$.

Finally, $R_{\hat{f}} \cdot C_\rho = 4$, since the map f has 4 ramification points, as noticed before. It follows $C_\rho^2 = -6$.

- $C_\rho \cdot \hat{E}_i = \hat{f}_* C_\rho \cdot E_i = 0$ for all $i = 1, ..., 8$.

Then:

$$\mathrm{ch}(L) = \Big(1, 2[\hat{E}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4] - 4[C_\rho], -80\Big).$$

Combining all the above computations with equation (5.2.1), we finally get:

$$\begin{aligned} ch(\hat{g}_! L) &= \hat{g}_*\big(1, td_1(X) + ch_1(L), -88 + td_1(X) \cdot ch_1(L)\big)\big(1, -1\big) \\ &= \hat{g}_*\Big(1, \frac{-6\xi - \sum_{i=1}^{8}[\hat{E}_i]}{2} + 2[\hat{E}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4] - 4[C_\rho], -84\Big)(1, -1) \end{aligned}$$

where we used $\xi \cdot C_\rho = 1$ since ξ is a fiber and C_ρ is a section. Since $\hat{g}_*(\xi) = 0$, $\hat{g}_*(\hat{E}_i) = 1$ and $\hat{g}_*(C_\rho) = 1$, we get

$$\mathrm{ch}(\hat{g}_! L) = (0, -84)(1, -1) = (0, -84)$$

i.e. $\mathrm{ch}(R^1 \hat{g}_* L) = (0, 84)$ and $D \cdot \mathcal{T} = 84$.

□

Thanks to Proposition 5.2.2 and 5.2.3 we have now completed the Step (2) of Strategy 4.1.5.

5.3 The Beauville-Bogomolov-Fujiki square of the divisor

We are ready to compute the square with respect to the Beauville-Bogomolov-Fujiki form of the divisor D introduced in Definition 5.1.1.

Theorem 5.3.1. *Let $D \subset \mathrm{M}_v$ be the divisor defined in Definition 5.1.1 and $D^* := \tilde{\pi}^* D$, where $\tilde{\pi} : \widetilde{\mathrm{M}}_v \to \mathrm{M}_v$ the symplectic desingularization. Then*

$$q_{10}(D^*) = 544,$$

where q_{10} is the Beauville-Bogomolov-Fujiki pairing of $\widetilde{\mathrm{M}}_v$.

Proof. This is a straightforward consequences of the computations done along the previous sections. Indeed, write $D = a\lambda_v(e) + b\lambda_v(f)$, with $e = (1, h, 0)$ and $f = (0, 0, 1)$ as in Section 4.2. Intersecting D with Γ, combining Proposition 4.2.3 and Proposition 5.2.2 we get that $20 = D \cdot \Gamma = -5a$, which implies $a = -4$. Intersecting D with $\mathcal{T}$, combing Proposition 4.2.3 and Proposition 5.2.3 we get that $84 = D \cdot \mathcal{T} = 20 + b$, which means $b = 64$; hence

$$D = -4\lambda_v(e) + 64\lambda_v(f). \tag{5.3.1}$$

It follows

$$q_{10}(D^*) = q_{10}(D) = (-4e)^2 + 2 < -4e, 64f > +(64f)^2 = 544,$$

since $e^2 = 2$, $< e, f >= -1$ and $f^2 = 0$ in the Mukai lattice $H^*(S, \mathbb{Z})$. $\square$

Remark 5.3.2. We want to stress here that in (5.3.1) we have computed the class of D^* in $H^2(\tilde{M}_v, \mathbb{Z})$:

$$D^* = -4\tilde{\pi}^*\big(\lambda_v(e)\big) + 64\tilde{\pi}^*\big(\lambda_v(f)\big) \text{ in } H^2(\widetilde{\mathrm{M}}_v, \mathbb{Z})$$

with $e = (1, h, 0)$, $f = (0, 0, 1) \in v^\perp \subset H^*(S, \mathbb{Z})$.

Corollary 5.3.3. *Let $D^* \subset \tilde{\mathrm{M}}_v$ as in Theorem 5.3.1. Then D^* (or its dual) has a deformation which is ample and uniruled.*

Proof. This is consequence of Theorem 5.3.1 and of the discussion in Section 2.3. $\square$

Corollary 5.3.4. *For any $(X, c_1(H)) \in \mathfrak{M}^{pol}_{\mathrm{OG}\,10}$ in the connected component of $(\widetilde{\mathrm{M}}_v, c_1(D^*))$ (or of $(\widetilde{\mathrm{M}}_v, -c_1(D^*))$), a multiple of the ample linear system $|H|$ contains a divisor whose irreducible components are uniruled. As consequence, it satisfies the hypothesis of Theorem 2.1.7.*

Proof. This is consequence of Corollary 5.3.3 and of the strategy presented in Section 2.3. Note that D^* could be ruled by irreducible but not reduced curves, since D could contain the singular locus Σ; in this case, D^* would have a component ruled by (some) rational curve ruling the exceptional divisor $\tilde{\Sigma}$. We can anyway conclude thanks to the hypothesis of Corollary 5.3.3. $\square$

Chapter 6

Further developments

There are many natural modifications of the construction of the ample uniruled divisor presented in the previous section, that could give ample uniruled divisors in other connected components of $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$. In this chapter we will present some possible modifications, which give two new examples of ample uniruled divisors. Furthermore, we will compute some monodromy invariants of the examples found, which will ensure us that the divisors are in different connected components of $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$.

6.1 Natural modifications of the divisor

From Chapter 4 on we have fixed the Mukai vector $v = (0, 2h, 4) \in H^*(S, \mathbb{Z})$ and we have defined a uniruled divisor $D \subset \mathrm{M}_v$ using the inclusion $\mathcal{J}^8_{|2H|^{sm}} \subset \mathrm{M}_v$. It is very natural to define divisors similar to the one introduced in Definition 5.1.1 changing the Mukai vector to any $v = (0, 2h, 2a)$, using that $\mathcal{J}^{4+2a}_{|2H|^{sm}} \subset \mathrm{M}_{(0,2h,2a)}$, see Proposition 3.3.2. From now on, we will call $v_4 := (0, 2h, 4)$ and D_4 the divisor introduced in Definition 5.1.1.

In this section we will define two new divisors D_2 and D_6, and we will compute their Beauville-Bogomolov-Fuijiki squares. The will live in the moduli spaces $\mathrm{M}_{(0,2h,2)}$ and $\mathrm{M}_{(0,2h,6)}$ respectively, which are the cases where less modifications to the divisor D_4 are needed.

We want to emphasize that we are not going to define divisors D_{2a} in any $\mathrm{M}_{(0,2h,2a)}$ since, by the Strategy presented in Section 2.3, it is necessary to find only one ample uniruled divisor in each connected component determined by the polarized monodromy group. Since the polarized monodromy group of a OG 10-type IHS variety is not known yet, and then number of connected components of $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$ is not determined yet, we decided to stop here, for the moment, with the definition and the computation of the square of uniruled divisors in $\mathrm{M}_{(0,2h,2a)}$ similar to $D_4 \subset \mathrm{M}_{(0,2h,4)}$.

Set $v_2 := (0, 2h, 2)$, $v_6 := (0, 2h, 6) \in H^*(S, \mathbb{Z})$.

Definition 6.1.1. Let $D_2 \subset \mathrm{M}_{v_2}$ be the divisor defined as the closure in M_{v_2} of the locus in $\mathcal{J}^6_{|2H|^{sm}}$ consisting of sheaves of the form $i_*\mathcal{O}_C(2r+p_1+...+p_4)$, with $C \in |2H|$, $i : C \hookrightarrow S$, $p_1, ..., p_4 \in C$ and $r \in C \cap \rho$.

Definition 6.1.2. Let $D_6 \subset \mathrm{M}_{v_6}$ be the divisor defined as the closure in M_{v_6} of the locus in $\mathcal{J}^{10}_{|2H|^{sm}}$ consisting of sheaves of the form $i_*\mathcal{O}_C(6r+p_1+...+p_4)$, with $C \in |2H|$, $i : C \hookrightarrow S$, $p_1, ..., p_4 \in C$ and $r \in C \cap \rho$.

Remark 6.1.3. The schematic structure of D_2 and D_6 is given by the relative Brill-Nother theory on $|2H|$, cf. Remark 5.1.3: one can easily adapt the Remark to D_2 and D_4, changing the coefficients as in the definition of the divisors. Furthermore, adapting Lemma 5.1.2 to these new cases one gets immediately that D_2 and D_6 are uniruled.

Remark 6.1.4. Note that a similar definition in $\mathrm{M}_{(0,2h,8)}$ of a divisor consisting of sheaves of the form $i_*\mathcal{O}_C(8r+p_1+...+p_4)$ as in the definitions above would give a divisor not well defined on curves on the type $C_1 \cup C_2 \in |2H|$, since the associated sheaves on such curves would be unstable. For this reason, a generalization of the divisor D_4 to the moduli spaces $\mathrm{M}_{(0,2h,2a)}$ with $a \geq 4$ is less immediate, and we decided to stop with $a = 3$ since we don't know how many examples of uniruled divisors we actually need.

We state here the results about the Beauville-Bogomolov-Fujiki squares of the pullback in $\widetilde{\mathrm{M}}_{v_2}$ and $\widetilde{\mathrm{M}}_{v_6}$ of the divisors D_2 and D_6 respectively. The square of the second divisor will be stated as a conjecture: some details are still to check, but we have good evidence for thinking that the square of the divisor is as conjectured.

Theorem 6.1.5. *Let D_2 be the divisors introduced in Definition 6.1.1, and let $\tilde{\pi}_2 : \tilde{\mathrm{M}}_{v_2} \to \mathrm{M}_{v_2}$ be the symplectic resolution of M_{v_2}. Then:*

$$q_{10}(\tilde{\pi}_2^* D_2) = 232.$$

Conjecture 6.1.6. Let D_6 be the divisor introduced in Definition 6.1.2, and let $\tilde{\pi}_6 : \tilde{\mathrm{M}}_{v_6} \to \mathrm{M}_{v_6}$ be the symplectic resolution of M_{v_6} respectively. Then:

$$q_{10}(\tilde{\pi}_6^* D_6) = 1064.$$

The following Corollary assumes both the Theorem and the Conjecture stated above.

Corollary 6.1.7. *Assume that $\tilde{\pi}_2^* D_2$ and $\tilde{\pi}_6^* D_6$ are ample (this is true up to pass to their duals). For any $(X, c_1(H)) \in \mathfrak{M}^{pol}_{\mathrm{OG}\,10}$ in the connected component of $(\widetilde{\mathrm{M}}_{v_2}, c_1(\tilde{\pi}_2^* D_2))$ or of $(\widetilde{\mathrm{M}}_{v_6}, c_1(\tilde{\pi}_6^* D_6))$, a multiple of the ample linear system $|H|$ contains a divisor whose irreducible components are uniruled. As consequence, it satisfies the hypothesis of Theorem 2.1.7.*

Proof. This is consequence of Theorem 6.1.5, Conjecture 6.1.6, the discussion in Section 2.3 and the fact that in Corollary 2.2.4 we did not require the curves to be irreducible. □

Theorem 6.1.5 and the computations behind Conjecture 6.1.6 will be proved following Strastegy 4.1.5, as done for D_4; Nevertheless, the case of the divisors D_2 and D_6 is a bit more delicate, because the spaces M_{v_2} and M_{v_6} are not locally factorial, but only 2-factorial; this is an immediate consequence of Theorem 5.1.5, as done in Section 5.1 for D_4. As consequence, we will have to check that the divisors D_2 and D_6 are Cartier divisors, in order to apply Strategy 4.1.5.

Remark 6.1.8. In the next section we will compute the class of D_2^* in $H^2(\tilde{\mathrm{M}}_{v_2}, \mathbb{Z})$, and we will see that it contains the class of the exceptional divisor $\tilde{\Sigma}_2$ with multiplicity 2 (see (6.1.5)). As consequence, it is very natural to wonder weather the strict transform $\tilde{D}_2$ of D_2 also has a positive Beauville-Bogomolov-Fujiki square, since it would give a new class of ample uniruled divisor ruled by reduced and irreducible curves; this is indeed the case.

Corollary 6.1.9. *Let $\tilde{D}_2 \subset \tilde{\mathrm{M}}_v$ be the strict transform through $\tilde{\pi} : \tilde{M}_v \to \mathrm{M}_v$ of the divisor D_2. Then $q_{10}(\tilde{D}_2) = 208$.*

As consequence, for any $(X, c_1(H)) \in \mathfrak{M}^{pol}_{\mathrm{OG}\,10}$ in the connected component of $(\widetilde{\mathrm{M}}_{v_2}, c_1(\tilde{D}_2))$ (or of $(\widetilde{\mathrm{M}}_{v_2}, -c_1(\tilde{D}_2))$), a multiple of the ample linear system $|H|$ contains a divisor whose irreducible components are uniruled.

Proof. The expression in (6.1.5) gives $\tilde{D}_2 = D_2^* - 2\tilde{\Sigma}_2$. Then

$$q_{10}(\tilde{D}_2) = q_{10}(D_2^*) + 4q_{10}(\tilde{\Sigma}_2) - 4q_{10}(D_2^*, \tilde{\Sigma}_2)$$

where $q_{10}(D_2^*) = 232$ by Theorem 5.3.1 and $q_{10}(\tilde{\Sigma}_2) = -6$, as said in Example 1.1.11. The thesis follows from $q_{10}(D_2^*, \tilde{\Sigma}_2) = 0$, for which we refer the reader to Lemma B.2.3 □

Remark 6.1.10. In the moduli space $\mathrm{M}_{(0,2h,1)}$ we can consider the divisor D_1 similar to D_2, D_4 and D_6, i.e. D_1 is the closure in $\mathrm{M}_{(0,2h,1)}$ of the locus in $\mathcal{J}^5_{|2H|^{sm}}$ consisting of sheaves of the form $i_*\mathcal{O}_C(r + p_1 + ... + p_4)$, with $C \in |2H|$ and $r \in C \cap \rho$. The moduli space $\mathrm{M}_{(0,2h,1)}$ is birational to the Hilbert scheme $S^{[5]}$, and through this birational morphism the divisor D_1 corresponds to the first uniruled divisor found by Charles, Mongardi and Pacienza in [CMP19] (i.e. the divisor D_1 of Section 4). To complete our work, it would be natural to try to find new example of uniruled divisors writing all the divisors in [CMP19] in the moduli space $\mathrm{M}_{(0,2h,1)}$ through the birational morphism, and adapting their definition to moduli spaces M_v with v non primitive Mukai vector. For this a complete description of $\mathrm{Mon}^2(\mathrm{OG}\,10)$ is necessary, in order to obtain a numerical characterization of the connected components of $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$.

6.1.1 Proof of Theorem 6.1.5

The proof of Theorem 6.1.5 goes exactly as the proof of Theorem 5.3.1. We will present here all the steps, but we will skip most of the computations, since they are just slight modifications of the D_4 case presented in details in the previous chapters.

Let us assume the notations introduced right before Definition 4.2.1.

Definition 6.1.11. We define

$$\mathcal{E}_\Gamma := i_*\mathcal{O}_{\gamma\times\gamma}(5p_0 \times \gamma + \Delta) \ \in \mathrm{Coh}(S \times \gamma).$$
$$\mathcal{E}_\mathcal{T} := j_*\mathcal{O}_\mathcal{C}(3q_1 \times \tau + q_2 \times \tau + q_3 \times \tau + q_4 \times \tau) \ \in \mathrm{Coh}(S \times \tau).$$

Along this section we will call Γ and $\mathcal{T}$ the curves in M_{v_2} defined by $\mathcal{E}_\Gamma$ and $\mathcal{E}_\mathcal{T}$ respectively.

Remark 6.1.12. Notice that the coefficients in the definition of $\mathcal{E}_\mathcal{T}$ are chosen differently from the ones in Definition 4.2.1. Our idea behind the choice of the coefficients defining $\mathcal{E}_\mathcal{T}$ is the different cases is to make the induced line bundles as much balanced as possible on the points $q_1, ..., q_4$, in order to avoid unstable sheaves on the singular curves of τ.

Given $v_2 = (0, 2h, 2)$, one has $(v_2^\perp)^{1,1} = \{(2a, ah, b)|\ a, b \in \mathbb{Z}\}$; we choose as generators $e := (2, h, 0)$ and $f := (0, 0, 1)$. Let

$$\lambda_{v_2} : (v_2^\perp)^{1,1} \xrightarrow{\sim} \mathrm{Pic}(\mathrm{M}_{v_2})$$

be the isometry of Corollary 4.1.3.

Proposition 6.1.13. *Following the notations introduced above, one has:*

1. $\lambda_{v_2}(e) \cdot \Gamma = -10$, $\lambda_{v_2}(f) \cdot \Gamma = 0$.
2. $\lambda_{v_2}(e) \cdot \mathcal{T} = -8$, $\lambda_{v_2}(f) \cdot \mathcal{T} = 1$.

Proof. The proof goes exactly as the proof of Proposition 4.2.3. Let us fix $[E]$, $[F] \in K(S)$ with $\mathfrak{v}(E) = e^\vee = (2, -h, 0)$ and $\mathfrak{v}(F) = f^\vee = (0, 0, 1)$. We will use notations analogous to the ones in the proof of Proposition 4.2.3.

- We need to compute

$$\lambda_{v_2}(e) \cdot \Gamma = \big[\mathrm{ch}(\pi_S^* E \otimes \mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma})\big]_3 \tag{6.1.1}$$

  where

$$\begin{aligned}\mathrm{ch}(\pi_S^* E \otimes \mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma}) &= \pi_S^*\big(\mathrm{ch}(E)\,\mathrm{td}(S)\big)\mathrm{ch}(\mathcal{E}_\Gamma)\\ &= \pi_S^*(2, -h, 2)\,\mathrm{ch}(\mathcal{E}_\Gamma)\\ &= (2, -[H \times \gamma], 2[pt \times \gamma], 0)\,\mathrm{ch}(\mathcal{E}_\Gamma).\end{aligned}$$

  Since $\mathrm{ch}(\mathcal{E}_\Gamma) = i_*\big(\mathrm{ch}(G)\,\mathrm{td}(T_i)\big)$, with $G := \mathcal{O}_{\gamma\times\gamma}(5p_0 \times \gamma + \Delta)$,

$$\mathrm{ch}(G) = (1, 5[pt \times \gamma] + \Delta, 1)$$

  and

$$\mathrm{td}(T_i) = (1, -4[pt \times \gamma], 0)$$

  one gets

$$\mathrm{ch}(\mathcal{E}_\Gamma) = (0, [\gamma \times \gamma], [pt \times \gamma] + [i_*\Delta], -3).$$

  Combining everything in equation (6.1.1) one gets $\lambda_{v_4}(e) \cdot \Gamma = -10$.

- $\lambda_{v_4}(f) \cdot \Gamma = 0$ follows from

$$\mathrm{ch}(\pi_S^* F \otimes \mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma}) = (0, 0, [pt \times \gamma], 0)\,\mathrm{ch}(\mathcal{E}_\Gamma).$$

- We now need to compute $\big[\mathrm{ch}(\mathcal{E}_\mathcal{T})(2, -[H \times \tau], 2[pt \times \tau], 0)\big]_3$, where $\mathrm{ch}(\mathcal{E}_\mathcal{T}) = j_*\big(\mathrm{ch}(T)\,\mathrm{td}(T_j)\big)$, with $T := \mathcal{O}_\mathfrak{C}(3q_1 \times \tau + ... + q_4 \times \tau)$. One has

$$\mathrm{ch}(T) = (1, [3E_1 + ... + E_4], -6)$$

  and

$$\mathrm{td}(T_j) = \Big(1, -[C_t \times t] - [\sum_{i=1}^{8} \frac{E_i}{2}], 2\Big).$$

Therefore

$$\mathrm{ch}(\mathcal{E}_{\mathcal{T}}) = \Big(1, [\mathcal{C}], [3E_1 + ... + E_4] - \Big[\sum_{i=1}^{8} \frac{E_i}{2}\Big] - [C_t \times t], -7\Big)$$

which implies $\lambda_{v_4}(e) \cdot \mathcal{T} = -8$.

- $\lambda_{v_4}(f) \cdot \mathcal{T} = 1$ follows from

$$\mathrm{ch}(\pi_S^* F \otimes \mathcal{E}_{\mathcal{T}})\,\mathrm{td}(T_{\pi_\tau}) = (0, 0, [pt \times \tau], 0)\,\mathrm{ch}(\mathcal{E}_{\mathcal{T}}).$$

□

Before passing to the intersections of the curves with the divisor D_2, we need to check that D_2 is a Cartier divisor. This will be done in two steps.

Lemma 6.1.14. *The divisor D_2 contains the singular locus $\Sigma_2 := \mathrm{M}_{v_2} \setminus \mathrm{M}^s_{v_2}$.*

Proof. Let $R \subset |2H|$ be the locus of curves $C = C_1 \cup C_2$, with $C_1 \neq C_2$ and C_i smooth, $i = 1, 2$. Since $C_i \in |H|$, we have $C_1 \cdot C_2 = 2$. We assume that $C_1 \cap C_2 = \{n_1, n_2\}$ with $n_1 \neq n_2$.

A generic point $p \in \Sigma_2$ corresponds to a sheaf with support on a curve $C \in R$. Furthermore, O'Grady showed in [O'Ga] that a generic singular point in the moduli space M_{2w} with w primitive in $H^*(S, \mathbb{Z})$ is a S-equivalence class with polystable representative $F = F_1 \oplus F_2$, with F_1,F_2 non isomorphic stable sheaves with Mukai vectors $\mathfrak{v}(F_1) = \mathfrak{v}(F_2) = v$ (cf. proof of Proposition 5.2 in [LS06]). In our case, we can assume $p \in \Sigma_2$ generic to be of the form $[F_1 \oplus F_2]$, with F_1 sheaf supported on C_1 and F_2 on C_2, $\mathfrak{v}(F_1) = \mathfrak{v}(F_2) = (0, h, 1)$; it follows that each F_i is the push-forward of a torsion free sheaf on C_i of rank 1, hence a line bundle because of the smoothness of C_i. Summing up, we obtain that a generic point $p \in \Sigma_2$ is a class $[F_1 \oplus F_2]$ with F_i push-forward of a (generic) line bundle L_i of degree 2 on C_i, $i = 1, 2$. We want to prove that such a generic $p \in \Sigma_2$ belongs to D_2.

By generality of L_2, we can assume that it is effective, since $g(C_2) = 2$. Regarding L_1: fixed $r \in C_1 \cap \rho$, any line bundle of degree 2 on C_1 can be written as $x + 2r - n_1 - n_2$, with $x \in J^2(C_1)$; this is true because $x \mapsto x + 2r - n_1 - n_2$ is an automorphism of $J^2(C_1)$. Furthermore, we can assume that x is effective because $g(C_1) = 2$. Summing up, we got that we can write a generic $p \in \Sigma_2$ as

$$p = [j_{1,*}\mathcal{O}_{C_1}(2r + q_1 + q_2 - n_1 - n_2) \oplus j_{2,*}\mathcal{O}_{C_2}(q_3 + q_4)], \qquad (6.1.2)$$

for some $q_1, q_2 \in C_1$ and $q_3, q_4 \in C_2$, where $j_i : C_i \hookrightarrow S$ is the inclusion, $i = 1, 2$; we want to prove that such a point belongs to D_2. By generality of $p \in \Sigma$, the points $q_1, ..., q_4$ that we got in the expression (6.1.2) determine a pencil $Q \subset |2H|$, and a point in ρ determines one curve of the pencil. Let us consider the incidence variety $\mathcal{C} \subset S \times \rho$, $\mathcal{C} := \{(q, x) \in S \times \rho \mid q \in C_x\}$, with C_x the unique curve of Q determined by $x \in \rho$; let $i : \mathcal{C} \hookrightarrow S \times \rho$ be the inclusion and $\Delta \subset S \times \rho$ the diagonal. The curve parametrized by the family $i_* \mathcal{O}_{\mathcal{C}}(2\Delta + q_1 \times \rho + ... + q_4 \times \rho)$ is contained in the divisor D_2 (on the smooth curves of Q it consists of line bundles as in the definition of D_2), and choosing $r \in \rho$ as in (6.1.2) we get back the class of the point p. We conclude that $p \in D_2$ and then $\Sigma_2 \subset D_2$. □

The fact that the singular locus Σ_2 is contained in the divisor D_2 implies that D_2 can actually happen to be non Cartier. Luckily this is not the case, as stated in the following proposition.

Proposition 6.1.15. *D_2 is a Cartier divisor.*

Proof. Let $\tilde{D}_2$ be the strict transform of D_2 via the symplectic resolution $\tilde{\pi}_2 : \widetilde{\mathrm{M}}_{v_2} \to \mathrm{M}_{v_2}$, and let $\delta \subset \widetilde{\mathrm{M}}_{v_2}$ be the fiber of $\tilde{\pi}_2$ over a generic point $p \in \Sigma_2$. The first observation is that a divisor $D_2 \subset \mathrm{M}_{v_2}$ is Cartier if and only if the intersection $\tilde{D}_{v_2} \cdot \delta$ is even; we start proving this assertion. Let n be the smallest positive integer such that nD_2 is Cartier, and let m be the multiplicity of Σ_2 in nD_2, i.e. the multiplicity of a generic $p \in \Sigma_2$ in D_2; then

$$\tilde{\pi}^*(nD_2) = n\tilde{D}_2 + m\tilde{\Sigma}_2.$$

Intersecting the expression above with δ: $\tilde{\pi}^*(nD_2) \cdot \delta = 0$ by the projection formula, because δ gets contracted by $\tilde{\pi}$, and $\tilde{\Sigma}_2 \cdot \delta = -2$ because $\widetilde{\mathrm{M}}_{v_2}$ has trivial canonical bundle (cf. proof of Theorem 2.0.8 in [Rap08]). We get:

$$n\tilde{D}_2 \cdot \delta = 2m.$$

It follows that if the intersection $\tilde{D}_2 \cdot \delta$ is odd n needs to be an even number, ore more precisely $n = 2$, since as already noticed n can be just 1 or 2, see Theorem 5.1.5; in other words, if $\tilde{D}_2 \cdot \delta$ is odd then D_{v_2} is not a Cartier divisor. On the other hand, if $\tilde{D}_2 \cdot \delta = 2a$ then $m = an$ and

$$\tilde{\pi}^*(nD_2) = n\tilde{D}_2 + an\tilde{\Sigma}_2 = n(\tilde{D}_2 + a\tilde{\Sigma}_2)$$

where $(\tilde{D}_2 + a\tilde{\Sigma}_2) \cdot \delta = 0$. This last equality implies that $\tilde{D}_2 + a\tilde{\Sigma}_2$ is the pullback of some Cartier divisor in M_{v_2}, then D_2 is Cartier.

It follows that we need to compute the intersection $\tilde{D}_2 \cdot \delta$. In order to do that, we will use a modular interpretation of δ introduced in [O'Gb], that we are going to recall. Fix $p \in \Sigma_2$ generic; as already noticed at the beginning of the proof of 6.1.14, we can assume that p is the S-equivalence class of sheaves supported on a curve $C \in R \subset |2H|$, $C = C_1 \cup C_2$, $C_1 \cap C_2 = \{n_1, n_2\}$. Fix $L_j \in \mathrm{Pic}(C_j)$ for $j = 1, 2$, with $\deg(L_1) = 2$ and $\deg(L_2) = 4$; we will call $F_j := i_{j,*}L_j$, where as usual $i_j : C_j \hookrightarrow S$ is the inclusion, $j = 1, 2$. Fix a surjective morphism

$$F_1 \oplus F_2 \xrightarrow{\phi} \mathbb{C}_{n_1} \to 0$$

that is not null along C_1 or C_2, and consider the set

$$\alpha := \{\psi : F_1 \oplus F_2 \twoheadrightarrow \mathbb{C}_{n_2}\}/\mathbb{C}^*.$$

Notice that $\alpha \cong \mathbb{P}^1$. Consider the following surjective morphism of sheaves over $S \times \alpha$:

$$\pi_S^* F_1 \oplus \pi_S^* F_2 \xrightarrow{(\Phi,\Psi)} \pi^* \mathbb{C}_{n_1} \oplus (\pi_S^* \mathbb{C}_{n_2} \otimes \pi_\alpha^* \mathcal{O}_\alpha(1)) \to 0$$

where:

- $\pi_S : S \times \alpha \to S$ and $\pi_\alpha : S \times \alpha \to \alpha$ are the projections;
- $\Phi : \pi_S^* F_1 \oplus \pi_S^* F_2 \to \pi_S^* \mathbb{C}_{n_1}$ is the morphism induced by ϕ;
- $\Psi : \pi_S^* F_1 \oplus \pi_S^* F_2 \to \pi_S^* \mathbb{C}_{n_2} \otimes \pi_\alpha^* \mathcal{O}_\alpha(1)$ is the morphism induced by the dual of the tautological morphism over α.

Finally, we define:

$$\mathcal{K} := \ker(\Phi, \Psi) \in \mathrm{Coh}(S \times \alpha).$$

Notice that $[\mathcal{K}_\psi] = p$ for any $\psi \in \alpha$; in fact, the family $\mathcal{K}$ parametrizes the curve δ. In order to compute the intersection of δ with $\tilde{D}_2$, we proceed as follows: let $r \in \rho \cap C$, and consider the short exact sequence

$$0 \to \mathcal{H} \to \mathcal{K} \xrightarrow{\mathfrak{r}} j_* \mathcal{K}_{\{r\} \times \alpha} \cong j_* \mathcal{O}_{\{r\} \times \alpha} \to 0$$

where $\mathfrak{r} : \mathcal{K} \to \mathcal{K}_{r \times \alpha}$ is the restriction morphism and $j : \{r\} \times \alpha \hookrightarrow S \times \alpha$ is the inclusion; we do the same for $\mathcal{H}$:

$$0 \to \mathcal{G} \to \mathcal{H} \xrightarrow{\mathfrak{r}} j_* \mathcal{H}_{\{r\} \times \alpha} \cong j_* \mathcal{O}_{\{r\} \times \alpha} \to 0.$$

By Remark 5.1.3 and since $\rho \cdot C = 4$, we get

$$\tilde{D}_2 \cdot \delta = 4c_1(R^1 \pi_{\alpha,*} \mathcal{G}) = -4c_1(\pi_{\alpha,!} \mathcal{G}) \tag{6.1.3}$$

where the last equality follows from $\pi_{\alpha,*}\mathcal{G} = 0$: it is a torsion free sheaf (push-forward via a dominant map of a torsion free sheaf), whose generic fiber is equal to 0.

Notice that from equation (6.1.3) it follows that $\tilde{D}_2 \cdot \delta$ is even, then we can already conclude that $\tilde{D}_2$ is a Cartier divisor. Nevertheless, we want to compute the intersection $\tilde{D}_2 \cdot \delta$, because we will be interested in determining the multiplicity of $\tilde{\Sigma}_2$ in D_2.

From the Grothendieck-Riemann-Roch theorem on the projection $\pi_{\alpha,*} : S \times \alpha \to \alpha$ it follows that:

$$\operatorname{ch}(\pi_{\alpha,!}\mathcal{G}) = \pi_{\alpha,*}\big(\operatorname{ch}(\mathcal{G})\operatorname{td}(S \times \alpha)\big)\operatorname{td}(\alpha)^{-1}. \tag{6.1.4}$$

In the next we will compute all the blocks in the equation above.

1. $\operatorname{ch}(\mathcal{G}) = \operatorname{ch}(\mathcal{H}) - \operatorname{ch}(\mathcal{O}_{\{r\}\times\alpha}) = \operatorname{ch}(\mathcal{K}) - 2\operatorname{ch}(j_*\mathcal{O}_{\{r\}\times\alpha})$, where:

$$\begin{aligned}\operatorname{ch}(\mathcal{K}) &= \pi_S^*\operatorname{ch}(F_1) + \pi_S^*\operatorname{ch}(F_2) - \pi_S^*\operatorname{ch}(\mathbb{C}_{n_1}) - \pi_S^*\operatorname{ch}(\mathbb{C}_{n_2})\pi_\alpha^*(\mathcal{O}_\alpha(1))\\ &= \pi_S^*\big((0,[H],1) + (0,[H],3) - (0,0,1)\big) - \pi_S^*(0,0,1)\pi_\alpha^*(1,1)\\ &= (0, 2[H\times\alpha], 3[pt\times\alpha], 0) - (0,0,[pt\times\alpha],0)(1,[S\times pt],0,0)\\ &= (0, 2[H\times\alpha], 3[pt\times\alpha], 0) - (0,0,[pt\times\alpha],1)\\ &= (0, 2[H\times\alpha], 2[pt\times\alpha], -1)\end{aligned}$$

and

$$\operatorname{ch}(j_*\mathcal{O}_{\{r\}\times\alpha}) = (0,0,[r\times\alpha],0).$$

It follows that:

$$\operatorname{ch}(\mathcal{G}) = (0, 2[H\times\alpha], 0, -1).$$

2. $\operatorname{td}(S\times\alpha) = \pi_S^*\operatorname{td}(S)\pi_\alpha^*\operatorname{td}(\alpha)$, where $\operatorname{td}(S) = (1,0,2)$ because S is a $K3$ surface and $\operatorname{td}(\alpha) = (1,1)$ because $\alpha \cong \mathbb{P}^1$. It follows that:

$$\operatorname{td}(S\times\alpha) = (1,0,2[pt\times\alpha],0)(1,[S\times pt],0,0) = (1,[S\times pt],2[pt\times\alpha],2).$$

3. $\operatorname{td}(\alpha)^{-1} = (1,1)^{-1} = (1,-1)$.

It follows, by (6.1.4), that:

$$\begin{aligned}\operatorname{ch}(\pi_{\alpha,!}\mathcal{G}) &= \pi_{\alpha,*}\big((0,2[H\times\alpha],0,-1)(1,[S\times pt],2[pt\times\alpha],2)\big)(1,-1)\\ &= \pi_{\alpha,*}\big(0,2[H\times\alpha],2[H\times pt],-1\big)(1,-1)\\ &= (0,-1)(1,-1)\\ &= (0,-1).\end{aligned}$$

By the expression in (6.1.3), we conclude that $\tilde{D}_2 \cdot \delta = 4$. $\square$

Remark 6.1.16. Let m be the multiplicity of Σ_2 in the Cartier divisor D_2, i.e. the multiplicity of a generic $p \in \Sigma_2$ in D_2. Following the notations used in the proof of Proposition 6.1.15, $\tilde{\pi}^* D_2 = \tilde{D} + m\tilde{\Sigma}_2$, and we proved $4 = \tilde{D}_2 \cdot \delta = 2m$. It follows $m = 2$, i.e.

$$\tilde{\pi}^* D_2 = \tilde{D}_2 + 2\tilde{\Sigma}_2 \quad \text{in } H^2(\widetilde{\mathrm{M}}_{v_2}, \mathbb{Z}). \tag{6.1.5}$$

This expression has been used in Remark 6.1.8.

Proposition 6.1.17. *Following the notations above, one has:*

1. $D_2 \cdot \Gamma = 20$.
2. $D_2 \cdot \mathcal{T} = 44$.

Proof.
1. The proof is almost identical to the proof of Proposition 5.2.2: one only needs to consider the Jacobian of degree 6 instead of the one of degree 6 on the curve $\gamma \subset S$.

2. Using the notations of the proof of Proposition 5.2.3, one needs to compute $\mathrm{ch}(R^1\hat{g}_* L) = -\mathrm{ch}(\hat{g}_! L)$, where now

$$L := \mathcal{O}_X(3\hat{E}_1 + ... + \hat{E}_4 - 2C_\rho).$$

By the Grothendieck-Riemann-Roch theorem, one has

$$\mathrm{ch}(\hat{g}_! L) = \hat{g}_*\big(\mathrm{ch}(L)\,\mathrm{td}(X)\big)\mathrm{td}(\hat{\rho})^{-1}$$

where:

- $\mathrm{td}(\hat{\rho})^{-1} = (1, -1)$
- $\mathrm{td}(X) = \big(1, -3\xi + \frac{[-\hat{E}_1 - ... - \hat{E}_8]}{2}, -8\big)$
- $\mathrm{ch}(L) = (1, [3\hat{E}_1 + ... + \hat{E}_4] - 2[C_\rho], -36)$.

It follows that:

$$\mathrm{ch}(L)\,\mathrm{td}(X) = \Big(1, [3\hat{E}_1 + ... + \hat{E}_4] - 2[C_\rho] - 3\xi + \frac{[-\hat{E}_1 + ... - \hat{E}_8]}{2}, -44\Big)$$

and then $\hat{g}_*\big(\mathrm{ch}(L)\,\mathrm{td}(X)\big) = (0, -44)$ and $D_2 \cdot \mathcal{T} = 44$.

□

Combining Propositions 6.1.13 and 6.1.17, we obtain that:

$$D_2 = -2\lambda_{v_2}(e) + 28\lambda_{v_2}(f)$$

which implies

$$q_{10}(D_2) = q_{10}(\tilde{\pi}_2^* D_2) = 232.$$

6.1.2 Evidence for Conjecture 6.1.6

The structure of this section we will be the same of the previous one. We start by defining two curves in M_{v_6}, assuming as usual the notations introduced before Definition 4.2.1.

Definition 6.1.18. We define

$$\begin{aligned}
&\mathcal{E}_\Gamma := i_*\mathcal{O}_{\gamma\times\gamma}(9p_0\times\gamma+\Delta) \ \in \mathrm{Coh}(S\times\gamma).\\
&\mathcal{E}_\mathcal{T} := j_*\mathcal{O}_\mathfrak{C}\big(2(2q_1\times\tau+q_2\times\tau+q_3\times\tau+q_4\times\tau)\big) \in \mathrm{Coh}(S\times\tau).
\end{aligned}$$

Along this section we will call Γ and $\mathcal{T}$ the curves in M_{v_6} defined by $\mathcal{E}_\Gamma$ and $\mathcal{E}_\mathcal{T}$ respectively.

We now have $v_6=(0,2h,6)$, and then $(v_6^\perp)^{1,1}=\{(2a,3ah,b)|\ a,b\in\mathbb{Z}\}$; we choose as generators $e:=(2,3h,0)$ and $f:=(0,0,1)$. As usual, we call

$$\lambda_{v_6}:(v_6^\perp)^{1,1}\xrightarrow{\sim}\mathrm{Pic}(\mathrm{M}_{v_6})$$

the isometry of Corollary 4.1.3.

Proposition 6.1.19. *Following the notations introduced above, one has:*

1. $\lambda_{v_6}(e)\cdot\Gamma=-10$, $\lambda_{v_6}(f)\cdot\Gamma=0$.
2. $\lambda_{v_6}(e)\cdot\mathcal{T}=-20$, $\lambda_{v_6}(f)\cdot\mathcal{T}=1$.

Proof. We proceed as in the proof of Proposition 4.2.3, using the same notations; we fix here $[E]$, $[F]\in K(S)$ with $\mathfrak{v}(E)=e^\vee=(2,-3h,0)$ and $\mathfrak{v}(F)=f^\vee=(0,0,1)$. We follow word by word the proof of Proposition 6.1.13, with some modifications due to having a different Mukai vector and different curves.

- We need to compute

$$\lambda_{v_6}(e)\cdot\Gamma=\big[\mathrm{ch}(\pi_S^*E\otimes\mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma})\big]_3 \tag{6.1.6}$$

where

$$\begin{aligned}
\mathrm{ch}(\pi_S^*E\otimes\mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma}) &= \pi_S^*\big(\mathrm{ch}(E)\,\mathrm{td}(S)\big)\mathrm{ch}(\mathcal{E}_\Gamma)\\
&=\pi_S^*(2,-3h,2)\,\mathrm{ch}(\mathcal{E}_\Gamma)\\
&=(2,-3[H\times\gamma],2[pt\times\gamma],0)\,\mathrm{ch}(\mathcal{E}_\Gamma).
\end{aligned}$$

Since $\mathrm{ch}(\mathcal{E}_\Gamma) = i_*\big(\mathrm{ch}(G)\,\mathrm{td}(T_i)\big)$, with $G := \mathcal{O}_{\gamma\times\gamma}(9p_0 \times \gamma + \Delta)$, and using

$$\mathrm{ch}(G) = (1, 9[pt \times \gamma] + \Delta, 5)$$

and

$$\mathrm{td}(T_i) = (1, -4[pt \times \gamma], 0)$$

one gets

$$\mathrm{ch}(\mathcal{E}_\Gamma) = (0, [\gamma \times \gamma], 5[pt \times \gamma] + [i_*\Delta], 1).$$

Combining everything in equation (6.1.6) one gets $\lambda_{v_6}(e) \cdot \Gamma = -10$.

- $\lambda_{v_6}(f) \cdot \Gamma = 0$ follows from

$$\mathrm{ch}(\pi_S^* F \otimes \mathcal{E}_\Gamma)\,\mathrm{td}(T_{\pi_\gamma}) = (0, 0, [pt \times \gamma], 0)\,\mathrm{ch}(\mathcal{E}_\Gamma).$$

- We now need to compute

$$\big[\mathrm{ch}(\mathcal{E}_\mathcal{T})(2, -3[H \times \tau], 2[pt \times \tau], 0)\big]_3$$

where $\mathrm{ch}(\mathcal{E}_\mathcal{T}) = j_*\big(\mathrm{ch}(T)\,\mathrm{td}(T_j)\big)$, with $T := \mathcal{O}_\mathcal{C}\big(2(2q_1 \times \tau + ... + q_4 \times \tau)\big)$. One has

$$\mathrm{ch}(T) = (1, 2[2E_1 + ... + E_4], -14)$$

and

$$\mathrm{td}(T_j) = \Big(1, -[C_t \times t] - [\sum_{i=1}^{8} \frac{E_i}{2}], 2\Big).$$

Then

$$\mathrm{ch}(\mathcal{E}_\mathcal{T}) = \Big(1, [\mathcal{C}], 2[2E_1 + ... + E_4] - \Big[\sum_{i=1}^{8} \frac{E_i}{2}\Big] - [C_t \times t], -17\Big)$$

which implies $\lambda_{v_6}(e) \cdot \mathcal{T} = -20$.

- $\lambda_{v_6}(f) \cdot \mathcal{T} = 1$ follows from

$$\mathrm{ch}(\pi_S^* F \otimes \mathcal{E}_\mathcal{T})\,\mathrm{td}(T_{\pi_\tau}) = (0, 0, [pt \times \tau], 0)\,\mathrm{ch}(\mathcal{E}_\mathcal{T}).$$

$\square$

We now pass to the intersections of the curves with the divisor D_6. As in the case of D_2 the space M_{v_6} is 2-factorial; thus we need to check weather D_6 is Cartier or not.

Conjecture 6.1.20. Let D_6 be the divisor introduced in Definition 6.1.2. D_6 is a Cartier divisor in M_{v_6}.

Proof. The proof that we will present here is not complete; this is the reason why we are stating the result about the Beauville-Bogomolov-Fujiki square of $\tilde{\pi}_6^* D_6$ as a conjecture.

As in the case of $D_2 \subset \mathrm{M}_{(0,2h,2)}$, we check that D_6 is Cartier computing the multiplicity of the singular locus $\Sigma_6 := \mathrm{M}_{v_6} \setminus \mathrm{M}^s_{v_6}$ in D_{v_6}, cfr. proof of Proposition 6.1.15. We show below that the singular locus is not contained in D_6, hence D_6 is Cartier.

We proceed as in the proof of Lemma 6.1.14. A generic point $p \in \Sigma_6$ is a sheaf with support on a curve $C \in R$, i.e. $C = C_1 \cup C_2$ with $C_i \in |H|$ smooth and $C_1 \cap C_2 = \{n_1, n_2\}$. It follows that $p = [F_1 \oplus F_2]$ with F_i torsion free sheaves on C_i; let us assume that F_i is locally free on C_i. It follows that it is a line bundle of degree 4 on C_i, $i = 1, 2$.

On the other hand, regarding D_6: a strictly semistable sheaf

$$L = \mathcal{O}_C(6r + p_1 + \dots + p_4)$$

on a curve $C \in R$ needs to be such that $r \in C_1$ and $p_1, \dots, p_4 \in C_2$ (or viceversa), i.e. the polystable class of L is $[L_1 \oplus L_2]$ with

$$L_1 := \mathcal{O}_{C_1}(6r - n_1 - n_2)$$

and

$$L_2 := \mathcal{O}_{C_2}(p_1 + \dots + p_4).$$

If L_2 is a generic line bundle of degree 4 on C_2, the line bundle L_1 is not a generic line bundle on C_1, which means that a point in Σ_6 represented by $[F_2 \oplus F_2]$ with F_i locally free on C_i generically can not have polystable class $[L_1 \oplus L_2]$.

By Lemma 1.0.7 in [Rap08] in the class of p there are two sheaves such that their restriction to C_i is not locally free. This case is still to check. □

Now on we will assume that D_6 is a Cartier divisor, in order to show how Conjecture 6.1.6 would follow.

Proposition 6.1.21. *Following the notations above, one has:*

1. $D_6 \cdot \Gamma = 20$.
2. $D_6 \cdot \mathcal{T} = 164$.

Proof. 1. Again, the proof is almost identical to the one of Proposition 5.2.2: one only needs to consider the Jacobian of degree 10 instead of the one of degree 6 on the curve $\gamma \subset S$.

2. We proceed as in the proof on Proposition 6.1.17, with some small modifications due to the different definitions of the divisor and the curves in this case. Using the notations of the proof of Proposition 5.2.3, one needs to compute $\mathrm{ch}(R^1\hat{g}_*L) = -\mathrm{ch}(\hat{g}_!L)$, where now

$$L := \mathcal{O}_X(2(2\hat{E}_1 + ... + \hat{E}_4) - 6C_\rho).$$

By the Grothendieck-Riemann-Roch theorem, one has

$$\mathrm{ch}(\hat{g}_!L) = \hat{g}_*\big(\mathrm{ch}(L)\,\mathrm{td}(X)\big)\mathrm{td}(\hat{\rho})^{-1}$$

where:

- $\mathrm{td}(\hat{\rho})^{-1} = (1, -1)$
- $\mathrm{td}(X) = \big(1, -3\xi + \frac{[-\hat{E}_1 - ... - \hat{E}_8]}{2}, -8\big)$
- $\mathrm{ch}(L) = (1, 2[2\hat{E}_1 + ... + \hat{E}_4] - 6[C_\rho], -164)$.

It follows

$$\mathrm{ch}(L)\,\mathrm{td}(X) = \Big(1, 2[2\hat{E}_1 + ... + \hat{E}_4] - 6[C_\rho] - 3\xi + \frac{[-\hat{E}_1 + ... - \hat{E}_8]}{2}, -164\Big)$$

and then $\hat{g}_*\big(\mathrm{ch}(L)\,\mathrm{td}(X)\big) = (0, -164)$ and $D_4 \cdot \mathcal{T} = 164$. □

Combining Propositions 6.1.19 and 6.1.21 we get

$$D_6 = -2\lambda_{v_6}(e) + 124\lambda_{v_6}(f)$$

which implies $q_{10}(D_6) = q_{10}(\tilde{\pi}_6^* D_6) = 1064$.

6.2 Monodromy invariants

Even if we do not have a characterization of the monodromy orbits of the moduli space $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$, there are some monodromy invariants that ensure that two polarized IHS varieties of OG 10-type (X, h) and (X', h') are not one deformation of the other. We are interested in the following two invariants of (X, h):

- the Beauville-Bogomolov-Fujiki square of h;
- the divisibility of h.

Definition 6.2.1. Let Λ be a lattice, $f(\cdot,\cdot)$ its symmetric bilinear form and and let $l \in \Lambda$. The divisibilty of l is the nonnegative integer $div(l)$ that generates the ideal $\{f(l,l')\}_{l' \in \Lambda} \subseteq \mathbb{Z}$.

Given $(X,h) \in \mathfrak{M}_{\Lambda}^{pol}$, the divisibility of h is its divisibility in the lattice $H^2(X,\mathbb{Z})$ endowed with the Beauville-Bogomolov-Fujiki form.

Remark 6.2.2. Given (S,h) and (S',h') primitively polarized $K3$ surface, the Global Torelli Theorem implies that they are deformation equivalent if and only if $h^2 = h'^2$. This is no longer the case in higher dimension: as shown in [Apo14], the Beauville-Bogomolov-Fujiki square and the divisibility together are not enough to determine the deformation orbit of (X,h). Notice that, by Eichler's criterion (see [Eic52]), if a lattice Λ contains two copies of the hyperbolic lattice U, then the divisibility and the square of an element determine its orbit in Λ under the action of $O^+(\Lambda)$. For all known examples of IHS varieties X, the lattice $H^2(X,\mathbb{Z})$ contains $U^{\oplus 2}$, but in general $\mathrm{Mon}^2(X) \subset O^+(H^2(X,\mathbb{Z}))$.

As last remark, note that for (S,h) polarized $K3$ surface one always has $div(h) = 1$, since the $K3$ lattice is unimodular (see for example in [Huy16]).

In this section we want to compute the monodromy invariants of the classes of the three divisors D_2^*, $D_4^* := \tilde{\pi}_4^* D_4$ and $D_6^* := \tilde{\pi}_6^* D_6$.

We start recalling their Beauville-Bogomolov-Fujiki squares, computed in Theorem 5.3.1, Theorem 6.1.5 and Conjecture 6.1.6:

1. $q_{10}(D_2^*) = 232$
2. $q_{10}(D_4^*) = 544$
3. $q_{10}(D_6^*) = 1064$.

In what follows we will compute the divisibility of all the three divisors. Note that all the computations on D_6^* are only conjectural.

Lemma 6.2.3. *Keeping the notations introduced in this chapter, the classes* $[D_2^*] \in H^2(\widetilde{\mathrm{M}}_{v_2},\mathbb{Z})$, $[D_4^*] \in H^2(\widetilde{\mathrm{M}}_{v_4},\mathbb{Z})$ *and* $[D_6^*] \in H^2(\widetilde{\mathrm{M}}_{v_6},\mathbb{Z})$ *correspond via the Fourier-Mukai transforms introduced in Appendix B to the following classes in* $H^2(\widetilde{\mathrm{M}}_{10},\mathbb{Z})$*:*

1. $[D_2^*]$*:* $30[H] - 56[\tilde{B}] - 28[\tilde{\Sigma}]$

2. $[D_4^*]$*:* $72[H]-140[\tilde{B}]-68[\tilde{\Sigma}]$

3. $[D_6^*]$*:* $134[H]-264[\tilde{B}]-132[\tilde{\Sigma}]$.

Proof. We recall the following relations:

$$\lambda_{v_2}^* D_2 = -2e+28f$$

$$\lambda_{v_4}^* D_4 = -4e+64f$$

$$\lambda_{v_6}^* D_6 = -2e+124f.$$

Applying the isometries $\tilde{\pi}_{10}^* \circ f_2 : H^2(\mathrm{M}_2,\mathbb{Z}) \to H^2(\widetilde{\mathrm{M}}_{10},\mathbb{Z})$ and $\tilde{\pi}_{10}^* \circ f_6 : H^2(\mathrm{M}_6,\mathbb{Z}) \to H^2(\widetilde{\mathrm{M}}_{10},\mathbb{Z})$, where f_2 and f_6 are the Fourier-Mukai isometries introduced in Section B.2, one gets the expressions 1 and 3 in the statement: the action of $\tilde{\pi}_{10}^* \circ f_2$ and $\tilde{\pi}_{10}^* \circ f_6$ on the usual generators of the lattices have been explicitly computed at the end of Section B.2, where we obtained the following expressions:

$$\begin{aligned}\tilde{\pi}_{10}^* \circ f_2 :& e \mapsto -[H],\ f \mapsto [H]-2[\tilde{B}]-[\tilde{\Sigma}]\\ \tilde{\pi}_{10}^* \circ f_6 :& e \mapsto -5[H]+8[\tilde{B}]+4[\tilde{\Sigma}],\ f \mapsto [H]-2[\tilde{B}]-[\tilde{\Sigma}].\end{aligned}$$

Regarding point 2, we need one step more. Since $|2H|^{sm}$ consists of hyperelliptic curves, there exists a canonical isomorphism $\mathcal{J}^2_{|2H|^{sm}} \to \mathcal{J}^0_{|2H|^{sm}}$, which extends to a birational map $\psi : \mathrm{M}_{(0,2h,-2)} \dashrightarrow \mathrm{M}_{(0,2h,-4)}$; we denote by $\tilde{\psi}$ the extension of ψ to the desingularized moduli spaces. The pullback morphism $\tilde{\psi}^* : \mathrm{Pic}(\widetilde{\mathrm{M}}_{(0,2h,-4)}) \to \mathrm{Pic}(\widetilde{\mathrm{M}}_{(0,2h,-2)})$ has been explicitly computed by Claudio Onorati, we thank him for having shared with us the following still unpublished result:

$$\begin{aligned}\tilde{\psi}^* : \mathrm{Pic}(\widetilde{\mathrm{M}}_{(0,2h,-4)}) &\longrightarrow \mathrm{Pic}(\widetilde{\mathrm{M}}_{(0,2h,-2)})\\ (1,-h,0) &\mapsto \left(\left(1,-\frac{h}{2},+\frac{3}{2}\right),\frac{\sigma}{2}\right)\\ (0,0,1) &\mapsto (0,0,1)\end{aligned}$$

where $(1,-h,0)$ and $(0,0,1)$ are generators of $\mathrm{Pic}(\widetilde{\mathrm{M}}_{(0,2h,-4)})$ and σ is the class of the singular locus, see [PR14] for a description of $\mathrm{Pic}(\widetilde{\mathrm{M}}_v)$ when M_v is 2-factorial (as in the case of $\mathrm{M}_{(0,2h,-2)}$). Finally we consider the birational map

$$\mathcal{F}_4 := \mathcal{F}_{2H^\vee} \circ \psi \circ \mathcal{F}_H \circ \mathcal{F}_2 : \mathrm{M}_{v_2} \dashrightarrow \mathrm{M}_{10}\,.$$

An explicit computation shows that the map $\mathcal{F}_4$ induces in cohomology a morphism f_4 such that

$$\begin{aligned}\tilde{\pi}_{10} \circ f_4 : \operatorname{Pic}(\mathrm{M}_{v_2}) &\to \operatorname{Pic}(\widetilde{\mathrm{M}}_{10}) \\ (1,h,0) &\mapsto -2[H] + 3[\tilde{B}] + [\tilde{\Sigma}] \\ (0,0,1) &\mapsto [H] - 2[\tilde{B}] - [\tilde{\Sigma}],\end{aligned}$$

from which the statement in point 2 follows immediately. □

Remark 6.2.4. The classes of the divisors D_2^*, D_4^* and D_6^* are not primitive in their respective cohomology lattices. We notice here that we could have stated Corollary 5.3.4 and Corollary 6.1.7 for the primitive classes associated to these divisors.

Corollary 6.2.5. *Keeping the notations introduced in this chapter, one has:*

1. $div(D_2^*) = 2$
2. $div(D_4^*) = 4$
3. $div(D_6^*) = 2.$

Proof. This is simply a consequence of Lemma 6.2.3. Recall that $H^2(\widetilde{\mathrm{M}}_{10}, \mathbb{Z}) \cong H^2(S,\mathbb{Z}) \oplus \mathbb{Z}[\tilde{B}] \oplus \mathbb{Z}[\tilde{\Sigma}]$, where the isomorphism is actually an isometry of lattices, see Example 1.1.11. Let $\xi + a[\tilde{B}] + b[\tilde{\Sigma}]$ be an element of $H^2(\widetilde{\mathrm{M}}_{10}, \mathbb{Z})$, with $\xi \in H^2(S,\mathbb{Z})$ and a, $b \in \mathbb{Z}$.

1. By Lemma 6.2.3, D_2^* corresponds to the class $30[H] - 56[\tilde{B}] - 28[\tilde{\Sigma}] = 2(15[H] - 28[\tilde{B}] - 14[\tilde{\Sigma}])$ in $H^2(\widetilde{\mathrm{M}}_{10}, \mathbb{Z})$. One has

$$q_{10}(15[H] - 28[\tilde{B}] - 14[\tilde{\Sigma}], \xi + a[\tilde{B}] + b[\tilde{\Sigma}]) = 15H \cdot \xi + 14a$$

   where $H \cdot \xi$ is the product inside the lattice $H^2(S,\mathbb{Z})$, which is unimodular; as consequence, there exists $\xi \in H^2(S,\mathbb{Z})$ such that $H \cdot \xi = 1$. Since one can choose $a = -1$, we conclude that $div(15[H] - 28[\tilde{B}] - 14[\tilde{\Sigma}]) = 1$ and $div(D_2^*) = 2$.

2. By Lemma 6.2.3, D_4^* corresponds to the class $72[H] - 140[\tilde{B}] - 68[\tilde{\Sigma}] = 4(18[H] - 35[\tilde{B}] - 17[\tilde{\Sigma}])$ in $H^2(\widetilde{\mathrm{M}}_{10}, \mathbb{Z})$. One has

$$q_{10}(18[H] - 35[\tilde{B}] - 17[\tilde{\Sigma}], \xi + a[\tilde{B}] + b[\tilde{\Sigma}]) = 3(6H \cdot \xi - b) + 19a$$

   which equals to 1 for $a = -1$, $b = 0$ and ξ such that $H \cdot \xi = 1$. It follows $div(18[H] - 35[\tilde{B}] - 17[\tilde{\Sigma}]) = 1$ and $div(D_4^*) = 4$.

3. By Lemma 6.2.3, D_6^* corresponds to the class $134[H]-264[\tilde{B}]-132[\tilde{\Sigma}] = 2(67[H]-132[\tilde{B}]-66[\tilde{\Sigma}])$ in $H^2(\widetilde{\mathrm{M}}_{10}, \mathbb{Z})$. One has

$$q_{10}(67[H]-132[\tilde{B}]-66[\tilde{\Sigma}], \xi + a[\tilde{B}] + b[\tilde{\Sigma}]) = 67H \cdot \xi + 66a$$

which equals to 1 for $a = 1$ and ξ such that $H \cdot \xi = -1$. It follows $div(67[H]-132[\tilde{B}]-66[\tilde{\Sigma}]) = 1$ and $div(D_6^*) = 2$.

□

Thanks to the computation of the monodromy invariants of the divisors introduced, we can conclude our work with the following:

Corollary 6.2.6. *The classes $[D_2^*]$, $[D_4^*]$ and $[D_6^*]$ are not one multiple of the other up to monodromy. As consequence, they give uniruled divisors inside three different connected components of $\mathfrak{M}^{pol}_{\mathrm{OG}\,10}$.*

Proof. This is consequence of the fact that they have monodromy invariants that are not the ones of proportional classes. □

Appendix A

Chow groups

In this appendix we recall some important results about Chow groups that are behind Chapter 2.

Here, as always, varieties are considered over the field of complex numbers $\mathbb{C}$.

A.1 Chow groups and representability

We start recalling the definition and the basic properties of Chow groups; for a detailed reference, see [Ful13].

Let X be a normal projective variety. If $D \subset X$ is an integral subscheme of codimension 1 (prime divisor), the local ring $\mathcal{O}_{X,D}$ is a DVR; as consequence, given $f \in k(X)^*$, one can define $ord_D(f)$, the order of vanishing of f along D. Given $f \in k(X)^*$ one defines

$$div(f) := \sum_{D \subset X} ord_D(f) D$$

with D any prime divisor in X. Let $Z^1(X)$ be the free abelian group generated by prime divisors; for any $f \in k(X)^*$ one has $div(f) \in Z^1(X)$, and such a divisor is called a principal divisor. Since the map $k(X)^* \to Z^1(X)$, $f \mapsto div(f)$ is a homomorpshim of groups, the subset of principal divisors $B^1(X) \subset Z^1(X)$ is a subgroup of $Z^1(X)$.

In the same way one can consider the free abelian group generated by codimension k-subschemes $W \subset X$, and one can define $div(f)$ for any $f \in K(W)$ going to the normalization of W. In this way one gets the groups $B^k(X) \subset Z^k(X)$.

Definition A.1.1. The k-Chow group of X is the quotient group $\mathrm{CH}^k(X) := Z^k(X)/B^k(X)$. Furthermore, $\mathrm{CH}_k(X) := \mathrm{CH}^{\dim(X)-k}(X)$.

Note that $\mathrm{CH}^1(X) = \mathrm{Pic}(X)$ when X is smooth.

Remark A.1.2. The definition of $\mathrm{CH}^k(X)$ is functorial. Indeed, given a proper morphism $f : X \to Y$ the push-forward

$$f_* : \mathrm{CH}_k(X) \to \mathrm{CH}_k(Y)$$

is well defined. Moreover, given a flat morphism $g : X \to Y$ the pull-back

$$g^* : \mathrm{CH}^k(Y) \to \mathrm{CH}^k(X)$$

is well defined. See [Ful13] for more details.

When X is smooth, a cycle class map

$$cl : \mathrm{CH}^k(X) \to H^{k,k}(X, \mathbb{Z})$$

is well defined. Its definition uses the fact that a smooth projective variety of complex dimension n admits a triangularization, then to a subvariety $W \subset X$ one can associate a cycle in $H_{2n-2k}(X, \mathbb{Z})$ and its Poincaré dual in $H^{2k}(X, \mathbb{Z})$ (see in [Ful13]). One defines

$$\mathrm{CH}^k(X)_{hom} := \ker\big(cl : \mathrm{CH}^k(X) \to H^{k,k}(X, \mathbb{Z})\big).$$

Finally, in the case of the 0-Chow group $\mathrm{CH}_0(X)$ we will use the Albanese map

$$alb_X : \mathrm{CH}_0(X)_{hom} \to \mathrm{Alb}(X) := H^0(\Omega^1_X)^\vee / H_1(X, \mathbb{Z})$$
$$[Z] \mapsto \Big(\alpha \mapsto \int_\gamma \alpha\Big)$$

where γ is a 1-chain such that $\partial\gamma = Z$. When $\dim(X) = 1$ the map alb_X is an isomorphism, otherwise in general it is not.

In the rest of this section we will be interested in the group $\mathrm{CH}_0(X)_{hom}$. A very complete reference for what we are going to sketch is [Voi03b].

Let $X^{(d)}$ be the d^{th}-symmetric product of X. One can define a map

$$\sigma_d : X^{(d)} \times X^{(d)} \to \mathrm{CH}_0(X)_{hom}$$
$$\big((x_1, ..., x_d), (y_1, ..., y_d)\big) \mapsto \sum_{i=1}^{d} [x_i] - [y_i].$$

Note that $\mathrm{Im}(\sigma_d) \subseteq \mathrm{Im}(\sigma_{d+1})$ and $\bigcup_{d\in\mathbb{N}} \mathrm{Im}(\sigma_d) = \mathrm{CH}_0(X)_{hom}$.

Definition A.1.3. The group $\mathrm{CH}_0(X)$ is said to be representable if there exists $d_0 \in \mathbb{N}$ such that $\mathrm{Im}(\sigma_{d_0}) = \mathrm{CH}_0(X)_{hom}$.

Representability is meant to measure the size of the group $\mathrm{CH}_0(X)_{hom}$. It turns out that in many cases the group $\mathrm{CH}_0(X)$ is actually non-representable.

The theorems we are going to state are consequence of the work of many authors (Mumford in [Mum69], Roitman in [Roi72], Bloch and Srinivas in [BS83], Voisin in [Voi12]). We collect here only two key results, that will imply the non representability of $\mathrm{CH}_0(X)$ when X is an IHS variety.

Theorem A.1.4. *Let X be a smooth projective variety with $\mathrm{CH}_0(X)$ representable. Then:*

1. *The Albanese map alb_X is an isomorphism.*
2. *There exists a smooth curve $i : C \hookrightarrow X$ such that $i_* : \mathrm{CH}_0(C) \to \mathrm{CH}_0(X)$ is surjective.*

Theorem A.1.5. *Let X be a smooth projective variety, and let $i : W \hookrightarrow X$ a closed subset such that $i_* : \mathrm{CH}_0(W) \to \mathrm{CH}_0(X)$ is surjective. Then $H^0(X, \Omega^p_X) = 0$ for all $p > \dim W$.*

Corollary A.1.6. *Let X be an IHS variety. The group $\mathrm{CH}_0(X)$ is non representable.*

Proof. This is an immediate consequences of Theorem A.1.4 and Theorem A.1.5, as $H^0(X, \Omega^2_X) \cong \mathbb{C}$. $\square$

A.2 The Bloch-Beilinson filtration

Definition A.2.1. Let X and Y be smooth projective varieties. A correspondence between X and Y is a class $\Gamma \in \mathrm{CH}^k(X \times Y)$.

Remark A.2.2. Correspondences are generalizations of regular morphisms: given any regular morphism $f : X \to Y$ one can consider the correspondence $[\Gamma_f] \in \mathrm{CH}^{\dim(Y)}(X \times Y)$, where Γ_f is the graph of f.

Let $\Gamma \in \mathrm{CH}^k(X \times Y)$ be a correspondence and let $\pi_X : X \times Y \to Y$, $\pi_Y : X \times Y \to Y$ be the projections. We define

$$\begin{aligned} \Gamma_* : \mathrm{CH}_r(X) &\to \mathrm{CH}^{k-r}(Y) \\ [Z] &\mapsto \pi_{Y,*}(\Gamma \cdot \pi_X^*[Z]) \end{aligned}$$

where the product is the intersection product defined on Chow groups (see [Ful13]). The cohomology cycle class $cl(\Gamma) \in H^{k,k}(X, \mathbb{Z})$ also defines a map

$$\begin{aligned} cl(\Gamma)_* : H^r(X, \mathbb{Z}) &\to H^{r+2k-2\dim(X)}(Y, \mathbb{Z}) \\ \alpha &\mapsto \pi_{Y,*}(cl(\Gamma) \cup \pi_X^* \alpha) \end{aligned}$$

Note that Γ_* and $cl(\Gamma)_*$ commute with the cycle-class map, and that they are the usual pushforwards in the case $\Gamma = [\Gamma_f]$ is the graph of a regular morphism $f : X \to Y$.

The Bloch and Beilinson conjecture we are going to state is about a filtration of the k^{th}-Chow group of any smooth projective variety; such a filtration also needs to satisfy some functorial properties. In [Jan94] there is a formulation of the Bloch-Beilinson filtration over an arbitrary field; here we will restrict ourselves, as always, to the case of the field of complex numbers $\mathbb{C}$.

Let $\mathrm{CH}^k(X)_{\mathbb{Q}} := \mathrm{CH}^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the k^{th}-rational Chow group of X. Given a correspondence $\Gamma \in \mathrm{CH}^k(X \times Y)$ one gets a $\mathbb{Q}$-linear map $\Gamma_* : \mathrm{CH}_r(X)_{\mathbb{Q}} \to \mathrm{CH}^{k-r}(X)_{\mathbb{Q}}$ by tensoring the definition above by $\mathbb{Q}$. We will call $\mathrm{CH}^k(X)_{\mathbb{Q},hom}$ the kernel of the rational cycle class map.

Conjecture A.2.3 (Bloch, Beilinson)**.** For any smooth projective variety X there exists a decreasing filtration of any k^{th}-rational Chow group

$$F^{\bullet}_{BB} \mathrm{CH}^k(X)_{\mathbb{Q}} : \ F^0_{BB} \mathrm{CH}^k(X)_{\mathbb{Q}} = \mathrm{CH}^k(X)_{\mathbb{Q}} \supset \ldots \supset F^l_{BB} \mathrm{CH}^k(X)_{\mathbb{Q}} \supset \ldots$$

with $F^i_{BB} \mathrm{CH}^k(X)_{\mathbb{Q}}$ $\mathbb{Q}$-subspaces, satisfying the following properties:

(BB1) $F^1_{BB} \mathrm{CH}^k(X)_{\mathbb{Q}} = \mathrm{CH}^k(X)_{\mathbb{Q},hom}$.

(BB2) $F^{k+1}_{BB} \mathrm{CH}^k(X)_{\mathbb{Q}} = 0$.

(BB3) (Good behaviour of the filtration with respect to correspondences). If $\Gamma \in \mathrm{CH}^r(X \times Y)$, then $\Gamma_*(F^i_{BB} \mathrm{CH}_k(X)_{\mathbb{Q}}) \subseteq F^i_{BB} \mathrm{CH}^{r-k}(Y)_{\mathbb{Q}}$.

(BB4) Given $\Gamma \in \mathrm{CH}^r(X \times Y)$ the map induced by (BB3):

$$(F^i_{BB} \mathrm{CH}_k(X)_{\mathbb{Q}})/(F^{i+1}_{BB} \mathrm{CH}_k(X)_{\mathbb{Q}}) \to (F^i_{BB} \mathrm{CH}^{r-k}(Y)_{\mathbb{Q}})/(F^{i+1}_{BB} \mathrm{CH}^{r-k}(Y)_{\mathbb{Q}})$$

vanishes if the restriction of $cl(\Gamma)_*$ to the following

$$F^{\dim(X)-k} H^{2\dim(X)-2k-i}(X, \mathbb{C}) \to H^{2r-2k-i}(Y, \mathbb{C})$$

vanishes. Here, as usual:

$$F^{\dim(X)-k}H^{2\dim(X)-2k-i}(X,\mathbb{C}) := \bigoplus_{\substack{p\geq \dim(X)-k \\ p+q=2\dim(X)-2k-i}} H^{p,q}.$$

(BB5) (Compatibility of the filtration with the intersection product).

$$(F^i_{BB}\operatorname{CH}^k(X)_{\mathbb{Q}})\cdot(F^j_{BB}\operatorname{CH}^l(X)_{\mathbb{Q}}) \subseteq F^{i+j}_{BB}\operatorname{CH}^{k+l}(X)_{\mathbb{Q}}.$$

We will call this filtration the Bloch-Beilinson filtration of $\operatorname{CH}^k(X)_{\mathbb{Q}}$.

Remark A.2.4. In the case of the 0-Chow group, something more is known about the Bloch-Beilinson conjecture: if the Bloch-Beilinson filtration exists, then $F^2_{BB}\operatorname{CH}_0(X)_{\mathbb{Q}} = \ker alb_X$, see Lemma 2.10 in [Jan94]. This result, together with the properties (BB1) and (BB3) of the conjecture, determines completely the filtration $F^{\bullet}_{BB}\operatorname{CH}_0(X)_{\mathbb{Q}}$ when X is a surface.

Appendix B

Fourier-Mukai transforms

In this appendix we want to sketch some results about Fourier-Mukai transforms between moduli spaces of sheaves, which are a very useful tool to construct birational morphisms among IHS varieties. Fourier-Mukai transforms have been used mainly in Chapter 6. For this reason we have decided to collect the main useful results in an appendix.

B.1 Definition and generalities

We start by introducing the Fourier-Mukai transforms, giving some examples of them and stating some general results. For a reference on them, see for example [Huy06].

Along this section X and Y will be projective varieties and

$$p_X : X \times Y \to X, \quad p_Y : X \times Y \to Y$$

the projections; let $D^b(X)$, $D^b(Y)$ and $D^b(X \times Y)$ be the bounded derived categories of $\mathrm{Coh}(X)$, $\mathrm{Coh}(Y)$ and $\mathrm{Coh}(X \times Y)$ respectively.

Definition B.1.1. Given $\mathcal{P} \in D^b(X \times Y)$, a Fourier-Mukai transform with kernel $\mathcal{P}$ is a functor equivalent to

$$\begin{aligned} \mathcal{F}_{\mathcal{P}} : D^b(X) &\to D^b(Y) \\ \mathcal{E} &\mapsto p_{Y,*}(\mathcal{P} \otimes p_X^* \mathcal{E}). \end{aligned}$$

Remark B.1.2. In the definition above, $p_{Y,*}$, p_X^* and $\otimes$ are the derived functors. However, p_X^* is simply the usual pullback since p_X^* is flat, and $\otimes$ is the usual tensor product when $\mathcal{E}$ is locally free.

Example B.1.3. Let S be a $K3$ surface with $\mathrm{Pic}(S) = \mathbb{Z}[H]$ and $H^2 = 2$.

1. Consider the diagonal inclusion $i : S \hookrightarrow S \times S$. We define $\mathcal{F}_H : D^b(S) \to D^b(S)$ to be the Fourier-Mukai transform with kernel $i_* H \in D^b(S \times S)$, and $\mathcal{F}_{H^\vee}$ the one with kernel $i_*(-H)$.

2. Let $\Delta \subset S \times S$ be the diagonal and I_Δ its ideal sheaf. We call $\mathcal{F}_\Delta : D^b(S) \to D^b(S)$ the Fourier-Mukai transform with kernel I_Δ.

It is very natural to ask when the Fourier-Mukai transform is an equivalence of categories. There are many results in this direction, see for example [Bri99] and [BO95]. It turns out that both the Fourier-Mukai transforms of Example B.1.3 are equivalence of categories, see again in [Huy06].

From now on, we will always assume that X and Y are $K3$ surfaces. A Fourier-Mukai transform $\mathcal{F}_\mathcal{P} : D^b(X) \to D^b(Y)$ induces an homomorphism

$$\begin{aligned} f_\mathcal{P} : H^*(X, \mathbb{Z}) &\to H^*(Y, \mathbb{Z}) \\ \alpha &\mapsto p_{Y,*}\big(\mathfrak{v}(\mathcal{P}) p_X^* \alpha\big) \end{aligned}$$

where $\mathfrak{v}(\mathcal{P})$ is the Mukai vector of the object $\mathcal{P} \in D^b(X \times Y)$, defined analogously to the Mukai vector of a coherent sheaf on a $K3$ surface (cfr. with Section 3.2). The following result is due to Mukai (see [Muk87]):

Theorem B.1.4. *If the Fourier-Mukai trasform $\mathcal{F}_\mathcal{P} : D^b(X) \to D^b(Y)$ is an equivalence among the derived categories of two $K3$ surfaces, then the induced homomorphism $f_\mathcal{P} : H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$ is a Hodge isometry.*

Example B.1.5. Let $X = Y = S$ be a $K3$ surface.

1. The Fourier-Mukai transforms $\mathcal{F}_H$ and $\mathcal{F}_{H^\vee}$ of Example B.1.3,1 are autoequivalences of the category $D^b(S)$, and the induced isometries f_H and $f_{H^\vee}$ are simply the multiplication by the Chern characters $\mathrm{ch}(\mathcal{O}_S(H))$ and $\mathrm{ch}(\mathcal{O}_S(H)^\vee)$ respectively.

2. The Fourier-Mukai transform $\mathcal{F}_\Delta$ of Example B.1.3,2 is an autoequivalence of the category $D^b(S)$, and it induces an isometry

   $$f_\Delta : H^*(S, \mathbb{Z}) \to H^*(S, \mathbb{Z})$$

   see Example 10.9 part (ii) in [Huy06].

B.2 Fourier-Mukai transforms between moduli spaces

We are intersted in Fourier-Mukai transforms because they sometimes happen to restrict to isomorphisms between the moduli spaces $\mathrm{M}_v \to \mathrm{M}_{v'}$. Roughly speaking, it sometimes happens that the Fourier-Mukai transform maps a sheaf in M_v to a sheaf (and not just to a complex in the derived category), and that it preserves semistability. This happens in our case; the following lemma is a particular case of Lemma 1.1 of [Yos01b]:

Lemma B.2.1. *Given a Mukai vector v, the Fourier-Mukai transforms $\mathcal{F}_H$ and $\mathcal{F}_{H^\vee}$ of Example B.1.3.1 give isomorphisms*

$$\mathcal{F}_H : \mathrm{M}_v \longrightarrow \mathrm{M}_{v\wedge \mathrm{ch}(\mathcal{O}_S(H))}$$

$$\mathcal{F}_{H^\vee} : \mathrm{M}_v \longrightarrow \mathrm{M}_{v\wedge \mathrm{ch}(\mathcal{O}_S(H)^\vee)}$$

sending the exceptional locus of the first moduli space to the exceptional locus of the second one.

The following result is due to Yoshioka and can be found again in [Yos01b]:

Lemma B.2.2. *Given a Mukai vector of the form $v = 2(0, \xi, a)$, the Fourier-Mukai transform $\mathcal{F}_\Delta$ of Example B.1.3.2 induces an isomorphism*

$$\begin{aligned} \mathcal{F}_\Delta : \mathrm{M}_{2(0,\xi,a)} &\longrightarrow \mathrm{M}_{2(a,\xi,0)} \\ [E] &\mapsto [\mathcal{F}_\Delta(E)^\vee] \end{aligned}$$

sending the exceptional locus of the first moduli space to the exceptional locus of the second one.

From now on we will assume that S is a $K3$ surface with $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$ and $H^2 = 2$ Set $h := c_1(H)$. We want to focus on the following compositions of Fourier-Mukai transforms:

- $\mathcal{F}_2 : \mathrm{M}_{(0,2h,2)} \xrightarrow{\mathcal{F}_\Delta} \mathrm{M}_{(2,2h,0)} \xrightarrow{\mathcal{F}_{H^\vee}} \mathrm{M}_{(2,0,-2)} = \mathrm{M}_{10}$, inducing the following map in cohomology:

$$\begin{aligned} f_2 : H^2(\mathrm{M}_{(0,2h,2)}, \mathbb{Z}) &\longrightarrow H^2(\mathrm{M}_{10}, \mathbb{Z}) \\ (2,h,0) &\xmapsto{f_\Delta} (0,-h,-2) \xmapsto{\wedge\,\mathrm{ch}(-h)} (0,-h,0) \\ (0,0,1) &\xmapsto{f_\Delta} (-1,0,0) \xmapsto{\wedge\,\mathrm{ch}(-h)} (-1,h,-1). \end{aligned}$$

- $\mathcal{F}_6 : \mathrm{M}_{(0,2h,6)} \xrightarrow{\mathcal{F}_{H^\vee}} \mathrm{M}_{(0,2h,2)} \xrightarrow{\mathcal{F}_2} \mathrm{M}_{10}$, inducing the following map in cohomology:

$$\begin{aligned} f_6 : H^2(\mathrm{M}_{(0,2h,6)}, \mathbb{Z}) &\longrightarrow H^2(\mathrm{M}_{10}, \mathbb{Z}) \\ (2,3h,0) \xmapsto{\wedge \operatorname{ch}(-h)} (2,h,-4) &\xmapsto{f_\Delta} (4,-h,-2) \xmapsto{\wedge \operatorname{ch}(-h)} (4,-5h,4) \\ (0,0,1) \xmapsto{\wedge \operatorname{ch}(-h)} (0,0,1) &\xmapsto{f_\Delta} (-1,0,0) \xmapsto{\wedge \operatorname{ch}(-h)} (-1,h,-1). \end{aligned}$$

By [Per10], given $(2,0,-2)^\perp = \{(n,\xi,n) | \ n \in \mathbb{Z}, \ \xi \in H^2(S,\mathbb{Z})\}$, one has

$$\begin{aligned} \lambda_{(2,0,-2)} : (2,0,-2)^\perp &\to H^2(\widetilde{\mathrm{M}}_{10}, \mathbb{Z}) \\ (n,\xi,n) &\mapsto \xi + 2n[\tilde{B}] + n[\tilde{\Sigma}] \end{aligned}$$

which gives

$$\begin{aligned} \tilde{\pi}_{10}^* \circ f_2 :& (2,h,0) \mapsto -[H], \ (0,0,1) \mapsto [H] - 2[\tilde{B}] - [\tilde{\Sigma}] \\ \tilde{\pi}_{10}^* \circ f_6 :& (2,3h,0) \mapsto -5[H] + 8[\tilde{B}] + 4[\tilde{\Sigma}], \ (0,0,1) \mapsto [H] - 2[\tilde{B}] - [\tilde{\Sigma}]. \end{aligned}$$

These Fourier-Mukai transforms have been used in computations in Chapter 6.

We conclude with the following observation, that have been used in the proof of Corollary 6.1.9:

Lemma B.2.3. *For any $v = (0, 2h, 2(2k+1))$ and any D divisor in M_v one has $q_{10}(\tilde{\pi}^* D, \tilde{\Sigma}) = 0$, where $\tilde{\pi} : \widetilde{\mathrm{M}}_v \to \mathrm{M}_v$ is the symplectic desingularization.*

Proof. Applying $\mathcal{F}_{H^\vee}$ a suitable number of times in order to arrive to $\mathrm{M}_{(0,2h,2)}$, and then composing with $\mathcal{F}_2$, one gets a composition of Fourier-Mukai transforms from M_v to M_{10}, fixing the singular locus. In M_{10} the relation in the statement is trivial: indeed, following [Per10], a class in $H^2(\mathrm{M}_{10}, \mathbb{Z})$ has the form $\xi + 2n[B]$ with $\xi \in H^2(S,\mathbb{Z})$, with pullback $\xi + 2n[\tilde{B}] + n[\tilde{\Sigma}]$ in $H^2(\widetilde{\mathrm{M}}_{10}, \mathbb{Z})$. By the explicit description of q_{10} in Example 1.1.11, we obtain $q_{10}(\xi + 2n\tilde{B} + n\tilde{\Sigma}, \tilde{\Sigma}) = 0$. □

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