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On the homotopy groups of the K(2)-localisation of a 2-local finite spectrum

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Introduction

A central problem in stable homotopy theory is to understand the homotopy groups of the sphere spectrum localised at each prime p, $\pi_*(S^0_{(p)})$. A powerful tool for computing the latter is the Adams spectral sequence, whose E_2-term is given by $\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$, the extension groups over the Steenrod algebra $\mathcal{A}_p$. However, this method only allows one to compute $\pi_*(S^0_{(p)})$ stem by stem. In the 60's, Frank Adams in his study of the image of J, [**Ada66**] showed the existence of an infinite family of elements of $\pi_*(S^0)$ living in arbitrarily large stems. This was the first periodic family discovered, known as the α-family, of the stable homotopy groups of the sphere. Adam's work and subsequent work by L. Smith, Toda and Miller-Mahowald-Wilson and others motivated and marked the beginning of chromatic homotopy theory.

In the 70's, Ravenel published a series of conjectures which described the global structure of the stable homotopy category. Most of the conjectures were then resolved by Hopkins and his collaborators. In fact, the chromatic point of view offers a promising tool to analyse $\pi_*(S^0_{(p)})$ in a systematic way by decomposing it into smaller pieces. More precisely, let L_n and $L_{K(n)}$ denote the Bousfield localisations with respect to the n^{th} Johnson-Wilson theory $E(n)$ and n^{th}-Morava K-theory, respectively (here the prime p is implicit in the notation). We have the chromatic convergence theorem.

Theorem 1 (Hopkins-Ravenel, [**Rav92**])**.** *Let X be a p-local finite spectrum. There is a tower*

$$... \to L_n X \to L_{n-1} X \to ... \to L_0 X \cong L_{\mathrm{H}\mathbb{Q}} X,$$

such that X is homotopy equivalent to its homotopy limit.

Furthermore, the chromatic fracture square asserts that L_n can be inductively determined from the Bousfield localisation $L_{K(m)}$ with respect to the m^{th} Morava K-theory for $0 \leq m \leq n$, via the homotopy pull-back squares

Theorem 2. [**Rav92**]. *For any spectrum X and all positive integers n, the following square is a homotopy pullback square*

$$\begin{array}{ccc} L_nX & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X. \end{array}$$

Therefore, in the chromatic approach to stable homotopy theory, it is crucial to understand the $K(n)$-local homotopy category at all primes and all natural numbers n, referred to as the chromatic level. For this purpose, a general strategy is to study the homotopy type of the $K(n)$-localisation of various finite spectra. A central result of the theory is the work of Devinatz and Hopkins [**DH04**] which expresses the $K(n)$-localisation of a finite spectrum X as the continuous homotopy fixed point spectrum

$$L_{K(n)}X = E_n^{h\mathbb{G}_n} \wedge X$$

where $\mathbb{G}_n$ is the extended Morava stabiliser group, which is profinite, and E_n is the n^{th} Morava E-theory, see Section 1 for more details. Moreover, for any closed subgroup F of $\mathbb{G}_n$, there is a $K(n)$-local E_n-based spectral sequence or homotopy fixed point spectral sequence converging to $\pi_*(E_n^{hF} \wedge X)$, with the E_2-term being the continuous cohomology of F with coefficients in $(E_n)_*(X)$:

$$\mathrm{H}_c^*(F, (E_n)_*(X)) \Longrightarrow \pi_*(E_n^{hF} \wedge X) \quad (1)$$

The study of chromatic level one was a great success: the homotopy groups of $L_{K(1)}S^0$ have been completely computed at all primes and, at the prime 2, $L_{K(1)}S^0$ detects essentially the image of J. Chromatic level two has also been thoroughly investigated at odd primes. It started with the computation by Shimomura and his collaborators of the L_2 localisation of various finite spectra (see [**SY95**], [**Shi97**], [**Shi00**], [**SW02**]). Later Goerss-Henn-Mahowald-Rezk in [**GHMR05**] proposed a conceptual framework to organise the $K(2)$-local homotopy category at the prime 3, in which the authors constructed a finite resolution of the $K(2)$-local sphere using higher real K-theories. See [**GHM04**], [**HKM13**], [**GH16**] for further investigations at $n = 2$ and $p = 3$ and [**Beh12**] for an exposition of L_2S^0 at $p \geq 5$.

The situation of chromatic level two at the prime 2 turns out to be much more complicated and we are only beginning to understand it better. Considerable effort has recently been made to understand the $K(2)$-local homotopy category at the prime 2 by the community. In [**BG18**], Bobkova and Goerss established a finite resolution of a spectrum related to the $K(2)$-local sphere at the prime 2 analogous to that of [**GHMR05**], which realised an algebraic resolution of $\mathbb{S}_2^1$, a certain closed subgroup of the second Morava stabiliser group, constructed by Beaudry [**Bea15**]. Based on her own work, Beaudry carried out a computation of $\mathrm{H}^*(\mathbb{S}_2^1, (E_2)_*(V(0)))$, the E_2-term of the spectral sequence associated to (1).

Motivated both by the limited state of the art and recent progress on the subject, this thesis aims to push further our knowledge of the $K(2)$-local homotopy category at the prime 2. One reason why the latter is hard to deal with lies largely in the fact that the cohomological properties of the group $\mathbb{G}_2$ are much more complicated at the prime 2. However, one exciting feature of chromatic level 2 is its close relationship with the theory of elliptic curves and modular forms, see Section 1. At chromatic level 2 and at the prime 2, we can choose the Morava E-theory to be the Lubin-Tate theory associated to the formal group law of the elliptic curve $C : y^2 + y = x^3$ over $\mathbb{F}_4$. We denote by E_C and $\mathbb{G}_C$ the corresponding Morava E-theory and Morava stabiliser group. Then the $K(2)$-localisation of any finite spectrum X can also be described as

$$L_{K(2)}X \cong E_C^{h\mathbb{G}_C} \wedge X.$$

One of the main tools used in this thesis is a certain finite resolution in the $K(2)$-local homotopy category. Let $\mathbb{S}_C^1$ be a closed subgroup of $\mathbb{G}_C$, G_{24} be the automorphism group of C and C_6 be a cyclic subgroup of order 6 of G_{24} (see Section 1 for details).

Theorem 3. [**BG18**] *There exists the following resolution of $E_C^{h\mathbb{S}_C^1}$ in the K(2)-local homotopy category at the prime* 2

$$E_C^{h\mathbb{S}_C^1} \xrightarrow{\delta_0} \mathcal{E}_0 \xrightarrow{\delta_1} \mathcal{E}_1 \xrightarrow{\delta_2} \mathcal{E}_2 \xrightarrow{\delta_3} \mathcal{E}_3$$

where $\mathcal{E}_0 = E_C^{hG_{24}}$, $\mathcal{E}_1 = \mathcal{E}_2 = E_C^{hC_6}$ and $\mathcal{E}_3 = \Sigma^{48} E_C^{hG_{24}}$.

This resolution is commonly called the topological duality resolution. The spectrum $E_C^{h\mathbb{S}_C^1}$ is used to build the spectrum $E_C^{h\mathbb{S}_C}$, where $\mathbb{S}_C$ is the Morava stabiliser group, via a certain cofiber sequence

$$E_C^{h\mathbb{S}_C} \to E_C^{h\mathbb{S}_C^1} \xrightarrow{1-\pi} E_C^{h\mathbb{S}_C^1},$$

and $E_C^{h\mathbb{S}_C}$ only differs from $L_{K(2)}S^0$ by the Galois action, i.e., there is a homotopy equivalence

$$L_{K(2)}S^0 \cong (E_C^{h\mathbb{S}_C})^{h\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)}.$$

Thus this theorem offers a useful instrument to study the homotopy type of $L_{K(2)}X$ for finite spectra X at the prime 2. In particular, it produces a spectral sequence, known as the topological duality spectral sequence, abbreviated by TDSS, converging to $\pi_*(E_C^{h\mathbb{S}_C^1} \wedge X)$

$$\mathrm{E}_1^{p,q} \cong \pi_q(\mathcal{E}_p \wedge X) \Longrightarrow \pi_{q-p}(E_C^{h\mathbb{S}_C^1} \wedge X). \tag{2}$$

By now, it should be clear that judicious choices of finite spectra become important. We refer to Section 1.7 for more information on the category of finite spectra. The latter has a stratification whose strata consists of finite spectra of type n for $0 \leq n \leq \infty$ and relevant for the study of the $K(n)$-local category are those of type at most n because finite spectra of type greater than n are $K(n)$-acyclic. As a general principle, the bigger the type of the spectrum, the harder the study of its Morava K-theory localisation. Therefore, at chromatic level n, it is preferable to start with the $K(n)$-localisation of certain type n complexes, then go down to type 0 complexes, notably the sphere spectrum.

Main players in this thesis are finite spectra constructed by Davis and Mahowald in [**DM81**]. Let A_1 denote a class of finite spectra whose cohomology is isomorphic, as a module over the subalgebra $\mathcal{A}(1)$ generated by $\langle Sq^1, Sq^2\rangle$ of the Steenrod algebra $\mathcal{A}$, to $\mathcal{A}(1)$. As shown in [**DM81**], the class A_1 contains four different homotopy types of finite spectra of type 2 which are distinguished by the structure of their mod-2 cohomology as modules over the Steenrod algebra. They are successively denoted by $A_1[00]$, $A_1[01]$, $A_1[10]$, $A_1[11]$, see Definition 3.2.1. The spectra $A_1[01]$ and $A_1[10]$ are Spanier-Whitehead self-dual, i.e., $D(A_1[01]) \simeq \Sigma^{-6}A_1[01]$ and $D(A_1[10]) \simeq \Sigma^{-6}A_1[10]$ and the spectra $A_1[00]$ and $A_1[11]$ are Spanier-Whitehead dual to each other, i.e., $D(A_1[00]) \simeq \Sigma^{-6}A_1[11]$ (here $D(-)$ denotes the function spectra $F(-, S^0)$). By an abuse of language, we write A_1 to refer to any of these four spectra and refer to any of them as a version of A_1. In particular, we use this notation in the statement of results that are true for all versions. We emphasis, however, that all results are *a priori* dependent on the version of A_1 and this is the case. The spectrum A_1 is constructed via three cofiber sequences starting from the sphere spectrum. First, let $V(0)$ be the mod 2 Moore spectrum, i.e., the cofiber of multiplication by 2 on the sphere. Next let Y be the cofiber of multiplication by η, the first Hopf element, on $V(0)$. Davis and Mahowald show that Y admits v_1-self maps, $v_1 : \Sigma^2 Y \to Y$. Then A_1 is the cofiber of any of these v_1-self maps of Y. We note also that even though Y admits eight v_1-self maps, the associated cofibers only have four different homotopy types.

One reason for working with A_1 is the fact that it is the cofiber of a v_1-self map of periodicity 1, making a few computations simpler; this is in contrast with the generalised Moore spectrum $M(2, v_1^4)$ which is the cofiber of a v_1-self map of periodicity 4 on the Moore spectrum $V(0)$. The second one is that a sufficient understanding of the homotopy type of $L_{K(2)}A_1$ might allow us to determine the Gross-Hopkins duality formula for the $K(2)$-local homotopy category at the prime 2. In fact, the spectrum A_1 can be considered as an analog of the Toda-Smith complex $V(1)$ at the prime 3 and as demonstrated in [**GH16**], computations of the homotopy groups of $L_{K(2)}V(1)$ allows one to characterise the Gross-Hopkins formula for the $K(2)$-local homotopy category at the prime 3. The third reason is that A_1 is a "small" finite spectrum of type 2 having only eight cells with the top cell being in dimension 6, hence it is reasonable to expect that a study of the homotopy type of A_1 gives us valuable information about the homotopy groups of S^0, at least about the v_2-periodic families of S^0. Let us expand this thought. The authors of [**BEM17**] show that A_1 admits a v_2^{32}-self map. Let $[(v_2^{32})^{-1}]A_1$ denote the associated telescope, i.e.,

$$[(v_2^{32})^{-1}]A_1 = \mathrm{hocolim}(A_1 \to \Sigma^{-192}A_1 \to \ldots \to \Sigma^{-192k}A_1 \to \ldots).$$

We note that the homotopy type of this telescope is independent on the choice of v_2-self map of A_1 by Nilpotence and Periodicity Technology, see [**Rav92**]. Suppose that $x \in \pi_t([(v_2^{32})^{-1}]A_1)$ is a nontrivial element. This means that the composite

$$S^{t+192k} \to \Sigma^{192k}A_1 \xrightarrow{v_2^{32k}} A_1$$

is essential for $k \in \mathbb{N}$. This gives rise to a nontrivial element of π_*S^0 in one of the stems $\{192k + t - i | 0 \leq i \leq 6\}$.

Moreover, the $K(2)$-localisation of A_1 might be used to detect nontrivial elements of the homotopy groups of $[(v_2^{32})^{-1}]A_1$. In fact, the $K(2)$-localisation map $A_1 \to L_{K(2)}A_1$ factors through $[(v_2^{32})^{-1}]A_1 \to L_{K(2)}A_1$. Ravenel's Telescope Conjecture predicts that the latter is a homotopy equivalence. As a key step towards the study of $\pi_*(L_{K(2)}A_1)$, as explained in the discussion following Theorem 3, we study in this thesis the Topological Duality spectral sequence for $E_C^{h\mathbb{S}_C^1} \wedge A_1$. We motivate, further, that the calculation of the TDSS for $E_C^{h\mathbb{S}_C^1} \wedge A_1$ will be useful to study the spectral sequence for Y, then that for $V(0)$ and finally that for S^0.

Results and summarise of the thesis

We summarise progress made in this thesis. Firstly, we compute the E_1-term of the TDSS for $E_C^{h\mathbb{S}_C^1} \wedge A_1$. More precisely, we compute completely the homotopy fixed point spectral sequences for $F = G_{24}, C_6$

$$\mathrm{H}^*(F, (E_C)_*(A_1)) \Longrightarrow \pi_*(E_C^{hF} \wedge A_1). \tag{3}$$

Here are qualitative versions of the results; see Theorem 5.3.19, 5.3.20 and Theorem 7.2.4 for more precise statements.

There are classes

$$\Delta^8 \in \mathrm{H}^0(G_{24}, (E_C)_{192}), \overline{\kappa} \in \mathrm{H}^4(G_{24}, (E_C)_{24}), \nu \in \mathrm{H}^1(G_{24}, (E_C)_4),$$

such that

Theorem 4. *As a module over the ring $\mathbb{F}_4[\Delta^{\pm 8}, \overline{\kappa}, \nu]/(\nu\overline{\kappa})$, the E_∞-term of the HFPSS for $E_C^{hG_{24}} \wedge A_1[01]$ and $E_C^{hG_{24}} \wedge A_1[10]$ is a direct sum of 46 explicitly known cyclic modules.*

Theorem 5. *As a module over the ring $\mathbb{F}_4[\Delta^{\pm 8}, \overline{\kappa}, \nu]/(\nu\overline{\kappa})$, the E_∞-term of the HFPSS for $E_C^{hG_{24}} \wedge A_1[00]$ and $E_C^{hG_{24}} \wedge A_1[11]$ is a direct sum of 48 explicitly known cyclic modules.*

There are classes $\Delta^2 \in \mathrm{H}^0(C_6, (E_C)_{48})$ and $x_{17} \in \mathrm{H}^1(C_6, (E_C)_{18})$ such that

Theorem 6. *As a module over the ring $\mathbb{F}_4[\Delta^{\pm 2}, x_{17}]$, the E_∞-term of the HFPSS for $E_C^{hC_6} \wedge A_1$ is a direct sum of eight explicitly known cyclic modules.*

Secondly, we prove that the edge homomorphism of the TDSS is an epimorphism, meaning that all differentials starting from the 0-line, consisting of $\pi_*(E_C^{hG_{24}} \wedge A_1)$, are trivial.

Theorem 7. *The induced homomorphism in homotopy of $\delta_0 : E_C^{h\mathbb{S}_C^1} \wedge A_1 \to E_C^{hG_{24}} \wedge A_1$ is surjective.*

Finally, we analyse the differentials $d_1 : \mathrm{E}_1^{1,q} \to \mathrm{E}_1^{2,q}$ and $d_1 : \mathrm{E}_1^{2,q} \to \mathrm{E}_1^{3,q}$. We prove that the latter is trivial and the former is potentially non-trivial only in two stems.

Theorem 8. *The induced homomorphism in homotopy of $\delta_2 : E_C^{hC_6} \wedge A_1 \to E_C^{hC_6} \wedge A_1$ is trivial except possibly on two families.*

Theorem 9. *The induced homomorphism in homotopy of $\delta_3 : E_C^{hC_6} \wedge A_1 \to \Sigma^{48} E_C^{hG_{48}} \wedge A_1$ is trivial.*

The thesis consists of five chapters. Here is a brief summarise of the contents of each chapter.

In Chapter I, we review some background and tools used in the computation of the Topological Duality spectral sequence. In particular, we sketch a proof of the relationship between topological modular forms with level structures and higher real K-theories. We also give a generalisation of the Davis-Mahowald spectral sequence which is an important tool to analyse the cohomology of various Hopf algebras.

In Chapter II, we analyse in detail the homotopy fixed point spectral sequence for $E_C^{hG_{24}} \wedge A_1$. We emphasise that there are two different outcomes for the E_∞-term of the homotopy fixed point spectral sequence, depending on the version of A_1, see Theorem 5.3.19 and 5.3.20 and figures II.22 to II.29. As a main tool, we apply the Davis-Mahowald spectral sequence to compute the E_2-term of the Adams spectral sequence for $tmf \wedge A_1$. We discuss some differentials in the latter, from which we obtain necessary homotopical input to run the homotopy fixed point spectral sequence.

In Chapter III, we compute the homotopy fixed point spectral sequence for $E_C^{hC_2} \wedge X$ and $E_C^{hC_6} \wedge X$ for $X = S^0, V(0), Y$ and A_1. It turns out, however, that the outcome does not depend on the version of A_1, see Theorem 7.2.3 and 7.2.4 and Figure III.10. Firstly, we compute the latter for $E_C^{hC_2} \wedge X$ and $E_C^{hC_6} \wedge X$ for $X = S^0, V(0), Y$ using the cofiber sequences defining these. Finally, we discuss A_1: similarly to the case of $E_C^{hG_{24}} \wedge A_1$, we need information coming from $tmf_0(3) \wedge A_1$ in order to complete the calculation of the homotopy fixed point spectral sequence.

In Chapter IV, we study the edge homomorphism of the topological duality spectral sequence. We reduce to the study of the induced map in homotopy of the tmf-Hurewicz map $A_1 \to tmf \wedge A_1$. We prove that the latter is surjective for all versions of A_1, see Theorem 7.2.18 and 9.0.5.

In Chapter V, we analyse the induced homomorphisms in homotopy of the maps δ_2 and δ_3 of the topological duality resolution. We start by analysing the mapping spectra between various higher real K-theories. Based on this analysis, we describe the maps δ_2 and δ_3 and then the induced maps in homotopy, see Theorem 11.1.5 and 11.2.6.

Convention and Notation. Unless otherwise stated, all spectra are localised at the prime 2. $\mathrm{H}^*(X)$ and $\mathrm{H}_*(X)$ denote the mod-2 cohomology and homology of the spectrum X, respectively. We also write $\mathcal{A}$ for the Steenrod algebra at the prime 2. Given a Hopf algebra A over a field k and M a A-comodule, we will often abbreviate $\mathrm{Ext}_A^*(k, M)$ by $\mathrm{Ext}_A^*(M)$. In general, we will write C_f for the cofiber of a map $f : X \to Y$ except that we will write $V(0)$ for the Moore spectrum which is the cofiber of the multiplication by 2 on the sphere. We reserve the notation C_2 for the cyclic group of order 2.

Résumé en français

L'un des problèmes centraux en théorie de l'homotopie stable est le calcul des groupes d'homotopie des spectres finis, notamment ceux de la sphère S^0. D'après les travaux de Serre dans les années 1950, les groupes d'homotopie de la sphère sont finiment engendrés. Par conséquent, il suffit de comprendre la localisation des spectres en chaque nombre premier à la fois.

Le début de la théorie de l'homotopie chromatique a l'origine dans les travaux d'Adams sur l'image de l'homomorphisme de J, qui constitue une première famille infinie des éléments nontriviaux des groupes d'homotopie de la sphère; et puis de Miller, Ravenel, Wilson qui en ont construit d'autres. Ces familles sont connues sous le nom des familles périodiques. A l'issue de leurs travaux, Ravenel a publié une série de conjectures qui décrivaient certaines structures globales de la catégorie des spectres finis localisés au nombre premier p, par conséquent, prédisaient des comportements de $\pi_*(S^0_{(p)})$, les groupes d'homotopie de sphère localisée au nombre premier p. Par la suite, la plupart de ces conjectures ont été démontrées par Hopkins et ses collaborateurs dans les années 1980, posant des premiers jalons pour cette approche chromatique. Nous en citons deux qui ont motivé cette thèse. Notons L_n pour la localisation de Bousfield par rapport à la n-ième E-théorie de Morava. On peut aisément démontrer qu'il existe une transformation naturelle $L_n \to L_{n-1}$ pour tout nombre naturel n. Le premier théorème est la convergence chromatique qui dit que la sphère localisée en p peut être reconstruit comme une limite homotopique de la localisation de Bousfield par rapport à des E-théories de Morava.

Theorem 0.0.1 (Hopkins-Ravenel)**.** *Soit X un spectre fini p-local. L'application naturelle $X \xrightarrow{\cong} \operatorname{holim}_n L_n X$, induite par la transformation naturelle $L_n \to L_{n-1}$, est une équivalence homotopique.*

Le deuxième théorème vise à décrire L_n de manière récurrente à partir des localisations par rapport aux K-théorie de Morava. Notons $K(n)$ la n-ième K-théorie de Morava.

Theorem 0.0.2. *Le carré commutatif suivant est une tirée-en-arrière homotopique*

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{n-1}X \\ \downarrow & & \downarrow \\ L_{K(n)}X & \longrightarrow & L_{n-1}L_{K(n)}X. \end{array}$$

Au vu de ces résultats, les catégories des spectres $K(n)$-locaux sont considérées comme les briques élémentaires qui constituent la catégorie des spectres. En particulier, une bonne compréhension des localisations $L_{K(n)}X$ pour différents nombres naturels n permet d'avoir des informations subtantielles sur X lui-même. Il est non moins important de souligner que l'étude de l'interaction entre les niveaux chromatiques successifs constitue un pilier de la théorie d'homotopie chromatique. Une conjecture prépondérante dans cette direction est celle de scission chromatique qui prédit que l'application $L_{n-1}S^0_p \to L_{n-1}L_{K(n)}S^0_p$ au-dessus est scindée injectivement. Il s'ensuit de cette conjecture, si vérifiée, de nombreuses conséquences significatives qui décrivent de manière plus précise comment reconstituer la catégorie des spectres finis à partir de ces $K(n)$-localisations.

Cette thèse s'intéresse à la catégorie des spectres $K(2)$-locaux au nombre premier 2, qui se situe à la pointe de la recherche actuelle. Il est nécessaire de donner quelques justifications pour ce choix qui semble, de prime abord, très spécifique. Tout d'abord, la théorie d'homotopie chromatique offre un program prometteur pour analyser le type d'homotopie des spectres finis. Ainsi, il est judicieux d'avoir de bons échantillons de calculs explicites pour tester les conjectures, en formuler des nouvelles ainsi que pour acquérir des connaissances effectives. A titre d'exemple, le premier niveau chromatique est très bien compris : les groupes d'homotopie de $L_{K(1)}S^0_{(p)}$ sont explicitement calculés pour tous les nombres premiers p et pour $p = 2$, les groupes d'homotopie de $L_{K(1)}S^0$ détectent essentiellement l'image de l'homomorphism J, c'est-à-dire, l'homomorphisme naturel $\pi_*(S^0_{(2)}) \to \pi_*(L_{K(1)}S^0_{(2)})$ envoie les générateurs de l'image de J de manière nontriviale dans le but. Au deuxième niveau chromatique, aux nombres premiers au moins 5, le problème, quoique ne pas facile, demeure algébrique. En générale, fixant un niveau chromatique n, plus le nombre premier sous-jacent est petit, plus les phénomènes topologiques apparaissent, donc plus le problème est complexe. Certes, la catégorie $K(2)$-locale au nombre premier 2 représente les complexités caractéristiques des catégories $K(n)$-locaux, mais s'avère accessible aux calculs explicites, ce qui est dû principalement à deux causes. La première est que plus grand est le niveau chromatique, plus la complexité liée aux calculs augmente. La deuxième est qu'il existe un lien propice entre la théorie des courbes elliptiques et la catégorie $K(2)$-locale - ce qui sera précisé en dessous - permettant d'utiliser la géométrie des dernières pour faciliter des calculs. Ainsi, la complexité caractéristique et l'accessibilité font de la catégorie $K(2)$-locale au nombre premier 2 un endroit favorable pour tester les conjectures. Par exemple, Beaudry, Goerss et Henn a récemment démontré la conjecture de scission chromatique pour $n = p = 2$. L'objectif premier de cette thèse est de réaliser certains calculs explicites, qui seront décrits dans la suite, visant à améliorer notre connaissance sur les groupes d'homotopie de $L_{K(2)}S^0_{(2)}$.

Afin d'introduire plus précisément l'objet de cette thèse, nous introduisons quelques notations.

Spectre de Lubin-Tate. D'une importance prépondérante dans l'analyse de la catégorie $K(n)$-locale est la paire constituée du groupe de stabilisateur de Morava $\mathbb{G}_n$ agissant sur l'E-théorie de Morava E_n. La dernière est construite à partir d'un groupe formel de hauteur n défini sur un corps parfait de caractéristique p. Bien que les choix différents du groupe formel conduise au même résultat final, certains choix sont plus avantageux que les autres en vue des calculs explicites. Pour $n = p = 2$, c'est le groupe formel associé d'une courbe elliptique supersingulière. Soit C la courbe elliptique d'équation de Weierstrass $y^2 + y = x^3$ définie sur la corps $\mathbb{F}_4$. La complétion formelle de C à son origine est le groupe formel F_C de hauteur 2. Notons par $\mathbb{S}_C$ le groupe d'automorphismes de F_C, connu sous le nom du groupe de stabilisateur de Morava. Puisque tous les automorphismes de F_C sont définis sur $\mathbb{F}_4$, $\mathbb{S}_C$ admets une action du groupe de Galois de l'extension de $\mathbb{F}_4$ sur $\mathbb{F}_2$, Gal. Notons par $\mathbb{G}_C$ le produit semi-direct $\mathbb{S}_C \rtimes \mathrm{Gal}$, appelé le groupe de stabilisateur de Morava étendu. On rappelle que l'anneau de Lubin-Tate de déformation universelle de F_C est isomorphe à $\mathbb{W}(\mathbb{F}_4)[[u_1]]$. En utilisant le théorème du functor exact de Landweber, on peut construire une théorie de cohomologie généralisée E_C telle que

$$\pi_* E_C \cong \mathbb{W}(\mathbb{F}_4)[[u_1]][u^{\pm 1}]$$

avec $|u_1| = 0$ and $|u| = -2$. De plus, E_C admet une action à homotopie près du groupe de stabilisateur de Morava étendu $\mathbb{G}_C = \mathbb{S}_C \rtimes \mathrm{Gal}$ où Gal est le groupe de Galois de l'extension de $\mathbb{F}_4$ sur $\mathbb{F}_2$. La théorie d'obstruction de Goerss-Hopkins-Miller affirme que cette action peut être rigidifiée en une action stricte, c'est-à-dire, une action dans certaine catégorie de model des spectres.

Spectres de point fixe homotopique. En utilisant l'action de G_C sur E_C, Devinatz et Hopkins a construit, pour tout sous-groupe fermé F de $\mathbb{G}_C$, le spectre de point fixe homotopique continu E_C^{hF} . Ils ont montré, entre autres, que

$$L_{K(2)}S^0 \cong E_C^{h\mathbb{G}_C}.$$

Notons par $\mathbb{S}_C^1$ le noyau du déterminant réduit $\mathbb{S}_C \to \mathbb{Z}_2$. Il est une conséquence du travail de Devinatz-Hopkins qu'il y a une suite de cofibration

$$E_C^{h\mathbb{S}_C} \to E_C^{h\mathbb{S}_C^1} \to E_C^{h\mathbb{S}_C^1}.$$

Par conséquent, une étude de $E_C^{h\mathbb{S}_C^1}$ est un premier pas crucial vers une meilleure connaissance de $L_{K(2)}S^0$. Un calcul direct des groupes d'homotopie $E_C^{h\mathbb{S}_C^1}$ s'avérant complexe, la stratégie générale est d'analyser le produit smash de $E_C^{h\mathbb{S}_C^1}$ avec des spectres finis, notamment des spectres de types 2, i.e., ceux dont la $K(1)$-homologie est nul. En principe, ils sont tous utiles. Dans cette thèse, nous nous intéressons aux spectres A_1 construits par Davis et Mahowald.

Les spectres finis A_1. Les spectres A_1 sont construits via des suites de cofibration successives. D'abord, soit $V(0)$ le cofibre de la multiplication avec 2 sur le spectre des sphères.

$$S^0 \xrightarrow{\times 2} S^0 \to V(0). \tag{4}$$

Ensuite, la composition $S^1 \xrightarrow{\eta} S^0 \xrightarrow{\iota} V(0)$ où η est un élément de Hopf et ι inclusion de la cellule la plus basse dans $V(0)$, s'étend en un morphism $\Sigma V(0) \to V(0)$, dont le cofibre est appelé Y

$$\Sigma V(0) \xrightarrow{\times \eta} V(0) \to Y. \tag{5}$$

Le spectre Y est un spectre de type 1. Davis et Mahowald a montré qu'il admettait un v_1-self map : $v_1 : \Sigma^2 Y \to Y$ dont le cofibre est noté par A_1. En fait, il y a 8 choix différents pour v_1, qui induisent quatre types d'homotopie différents pour A_1. Notons les par $A_1[00]$, $A_1[11]$, $A_1[01]$, $A_1[10]$. Quand nous désignons ces quatre spectres en même temps, nous utilisons la notation A_1.

Le choix de travailler avec A_1 mérite une justification. D'abord, les spectres A_1 sont les plus *petits* spectres qui sont les cofibres d'une v_1-*self map* de **période** 1, ce qui rend les calculs plus maniables. Ensuite, les spectres A_1 sont construits à partir de trois cofibrations, donc sont *proches* de la sphère, ce qui signifie qu'il est raisonnable de pouvoir obtenir des informations sur $L_{K(2)}S^0$ à partir des ceux sur $L_{K(2)}A_1$. Finalement, les spectres A_1 ne possédant chacun que huit cellules, le calcul de groupes d'homotopie de $L_{K(2)}A_1$ nous permet aisément d'obtenir des informations sur les familles v_2-périodiques des groupes d'homotopie de S^0.

Bhattacharya, Egger, Mahowald ont démontré que A_1 admets un $(v_2)^{32}$-*self map*, c'est-à-dire, une application de A_1 à lui-même qui induit la multiplication par v_2^{32} en $K(2)_*$-homologie, la seconde K-théorie d'homologie de Morava.

Résolution et suite spectrale de dualité. Un des outils puissants pour analyser des groupes d'homotopie de $E_C^{h\mathbb{S}_C^1}$ sont des résolutions finies du dernier par des spectres de K-théories réelles supérieures, c'est-à-dire, des spectres de point fixe homotopique de E_C par l'action des sous-groupes finis de $\mathbb{G}_C$. Cette philosophie a émané des travaux de Goerss-Henn-Mahowald-Rezk qui ont construit une résolution finie pour $L_{K(2)}S^0$ au $p = 3$. Une telle résolution au $p = 2$ que nous allons utiliser dans cette thèse est la résolution de dualité due à Bobkova-Goerss. Il existe une suite de morphismes de spectres

$$E_C^{h\mathbb{S}_C^1} \to \mathcal{E}_0 \xrightarrow{\delta_1} \mathcal{E}_1 \xrightarrow{\delta_2} \mathcal{E}_2 \xrightarrow{\delta_3} \mathcal{E}_3 \tag{6}$$

où $\mathcal{E}_0 = E_C^{hG_{24}}$, $\mathcal{E}_3 \cong \Sigma^{48} E_C^{hG_{24}}$, $\mathcal{E}_1 = \mathcal{E}_2 = E^{hC_6}$ qui peut être raffinée dans une tour de fibrations. Cela résulte en une suite spectrale de dualité:

$$\mathrm{E}_1^{p,q} = \pi_q(\mathcal{E}_p \wedge A_1) \Longrightarrow \pi_{q-p}(E^{h\mathbb{S}_C^1} \wedge A_1). \tag{7}$$

C'est une petite suite spectrale qui a quatre lignes, donc a trois différentiels d_1, deux différentiels d_2 et un différentiel d_3.

Le corps de cette thèse consiste en cinq chapitres. Dans le premier chapitre nous révisons, entre autres, la suite spectrale de Davis-Mahowald en en donnant une légère généralisation. Dans le deuxième et troisième chapitres, nous étudions les suites spectrales de point fixe homotopique pour $E_C^{hG_{24}} \wedge A_1$ et pour $E_C^{hC_6} \wedge A_1$. Cela constitue la page E_1 de la suite spectrale de dualité. Dans le quatrième chapitre, nous montrons que l'homomorphisme du bord de la suite spectrale de dualité (7) est surjectif. Il s'ensuit que les différentiels d_1, d_2, d_3 partant de la ligne zéro sont triviaux. Finalement, dans le cinquième chapitre, nous analysons les deux autres différentielles d_1 de la suite spectrale de dualité en montrant que le différentiel $d_1 : \mathrm{E}_1^{2,p} \to \mathrm{E}_1^{3,p}$ est trivial et que $d_1 : \mathrm{E}_1^{1,p} \to \mathrm{E}_1^{2,p}$ est trivial sauf potentiellement sur deux familles.

Calcul du term E_1 de la suite spectrale de dualité. Le calcul du term E_1 consiste à déterminer les groupes d'homotopie du $E_C^{hG_{24}} \wedge A_1$ et du $E_C^{hC_6} \wedge A_1$. Un outil prominent est la suite spectrale de point fixe homotopique:

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(F, E_t A_1) \Longrightarrow \pi_{t-s}(E^{hF} \wedge A_1)$$

où $F = G_{24}$ ou C_6. Si le calcul du term E_2 de ces suites spectrales étant des calculs de cohomologie des groupes finis est assez élémentaire, l'analyse des différentielles n'en est moins.

La suite spectrale de point fixe homotopique pour $E_C^{hG_{24}} \wedge A_1$. Cette suite spectrale est celle de module sur la suite spectrale de point fixe homotopique pour $E_C^{hG_{24}}$. La dernière a la propriété remarquable suivante. Il y une classe, appelée $\overline{\kappa}$ de $\mathrm{H}^4(G_{24}, E_{24})$. Celle-ci possède trois propriétés suivantes. Premièrement, elle est un cycle permanent. Deuxièmement, sa sixième puissance $\overline{\kappa}^6$ est le cible d'une différentielle. Troisièmement, $\overline{\kappa}$ est une classe de périodicité cohomologique de $\mathrm{H}^*(G_{24}, E_*)$, ce qui implique que pour un spectre fini X, la multiplication par $\overline{\kappa}$ induit un homomorphism $\mathrm{H}^s(G_{24}, E_tX) \to \mathrm{H}^{s+24}(G_{24}, E_{t+24}X)$, qui est surjectif si $* = 0$ et bijectif si $* > 0$. Cette propriété efforce que cette suite spectrale de s'organise de manière suivante. Si x est un cycle permanent qui est $\overline{\kappa}$-libre, alors il existe un nombre naturel k au plus 6 et une class y tel que pour un r approprié, on ait

$$d_r(y) = \overline{\kappa}^k x.$$

Cependant, pour localiser les cycles permanents et donc les cycles non-permanents, il faudrait une connaissance préalable sur les groupes d'homotopie de $E_C^{hG_{24}} \wedge A_1$. Cette connaissance est acquise en comparant $E_C^{hG_{24}} \wedge A_1$ avec le spectre connectif des formes modulaires topologiques tmf. Il existe un morphisme de spectres en anneau $tmf \to E_C^{hG_{24}}$ qui induit une équivalence $\mathrm{Gal}_+ \wedge L_{K(2)}tmf \to E_C^{hG_{24}}$. Par conséquent, il y a une équivalence d'homotopie

$$\mathrm{Gal}_+ \wedge [v_2^{-32}]tmf \wedge A_1 \to E_C^{hG_{24}} \wedge A_1.$$

Ainsi, nous nous ramenons à étudier les groupes d'homotopie $tmf \wedge A_1$ par la suite spectrale d'Adams

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathrm{H}_*A_1) \Longrightarrow \pi_*(tmf \wedge A_1)$$

Puis en tirer des informations nécessaires pour analyser la suite spectrale de point fixe homotopique pour $E_C^{hG_{24}} \wedge A_1$.

Theorem 0.0.3. *En tant que module sur l'algèbre $\mathbb{F}_4[\nu, \overline{\kappa}, (\Delta^8)^{\pm 1}]$, la page E_∞ de la suite spectrale de point fixe homotopique pour $E_C^{hG_{24}} \wedge A_1$ est une somme directe de 46 (respectivement 48) modules cycliques pour $A_1 = A_1[01]$ et $A_1[10]$ (respectivement $A_1 = A_1[00]$ et $A_1[11]$).*

La suite spectrale de point fixe homotopique pour $E_C^{hC_6} \wedge A_1$. Cette suite spectrale est étudiée par de différents moyens. Tout d'abord, nous calculons complètement la suite spectrale de point fixe homotopique pour E^{hC_2}. En particulier, nous identifions une class $t \in \mathrm{H}^1(C_2, E_2)$ qui joue un rôle similaire à $\overline{\kappa}$ dans la suite spectrale de point fixe homotopique pour $E_C^{hG_{24}}$. Plus précisément, t est un cycle permanent; t est une classe de périodicité cohomologique et t^7 est le but d'une différentielle d_7. Cette propriété permet de calculer complètement la suite spectrale de point fixe homotopique pour $E^{hC_2} \wedge V(0)$ et $E^{hC_2} \wedge Y$ via les cofibrations (III.5) et (III.12). Nous poursuivons cette approche pour étudier la suite spectrale pour $E^{hC_2} \wedge A_1$. Cependant, nous ne pouvons en déduire qu'une partie des différentielles. Finalement, nous recourons à des formes modulaires topologiques avec structure de niveau. A été construit un spectre en anneau $tmf_0(3)$ tel qu'il y ait une équivalence d'homotopie

$$\mathrm{Gal}_+ \wedge [v_2^{-32}](tmf_0(3) \wedge A_1) \cong E_C^{hC_6} \wedge A_1.$$

Nous analysons une partie de la suite spectrale d'Adams pour $tmf_0(3) \wedge A_1$, ce qui nous permet d'obtenir des informations nécessaires sur les groupes d'homotopie de $E^{hC_2} \wedge A_1$ pour déterminer le reste des différentielles.

Theorem 0.0.4. *En tant que module sur l'algèbre $\mathbb{F}_4[x_{17}, (\Delta^2)^{\pm 1}]$, la page E_∞ de la suite spectrale de point fixe homotopique pour $E_C^{hC_6}$ est une somme directe de huit modules cycliques.*

Différentielles de la suite spectrale de dualité. Une fois la page E_1 calculée, nous continuons avec l'analyse des différentielles. Dans la limite de cette thèse, nous n'aborderons pas les différentielles $d_2 : \mathrm{E}_2^{1,q} \to \mathrm{E}_2^{3,q}$ entre la deuxième ligne et quatrième ligne. Nous discutons dans la suite les autres différentielles.

L'homomorphism du bord. L'étude des différentielles qui partent de la ligne la plus basse de la suite spectrale de dualité peut être ramenée à étudier la surjectivité de l'homomorphisme de bord. Le dernier est celle induite en homotopie du morphisme de restriction

$$E_C^{h\mathbb{S}_C^1} \wedge A_1 \to E_C^{hG_{24}} \wedge A_1$$

qui est induite par l'inclusion de sous-groupe $G_{24} \to \mathbb{S}_C^1$. Celle-ci peut être analysée en la comparant avec l'application $A_1 \to tmf \wedge A_1$.

Proposition 0.0.5. *Si l'application induite en homotopie de $A_1 \to tmf \wedge A_1$ est surjective, alors il en va de même pour $E_C^{h\mathbb{S}_C^1} \wedge A_1 \to E_C^{hG_{24}} \wedge A_1$*

Ensuite, nous étudions la surjectivité de l'application $\pi_*(A_1) \to \pi_*(tmf \wedge A_1)$ par le moyen des suites spectrales d'Adams. En effet, il y a une application de suites spectrale d'Adams

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathrm{H}_*(A_1)) & \Longrightarrow & \pi_*(A_1) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{\mathcal{A}(2)}^{s,t}(\mathbb{F}_2, \mathrm{H}_* A_1) & \Longrightarrow & \pi_*(tmf \wedge A_1) \end{array}$$

où l'homomorphisme des termes E_2 est induit par l'inclusion de subalgebra de Hopf $\mathcal{A}(2) \to \mathcal{A}$.

Dans un premier temps, nous montrons que l'homomorphism

$$\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_2, \mathrm{H}_*(A_1)) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathrm{H}_*A_1)$$

est surjectif. Dans un second temps, nous montrons que tous les cycles permanents de la suite spectrale d'Adams pour $tmf \wedge A_1$ se relèvent en un cycle permanent de celle pour A_1. Nous concluons que l'homomorphism induit en homotopie $\pi_*A_1 \to \pi_*(tmf \wedge A_1)$ est surjectif for les modèles $A_1[00]$ et $A_1[11]$.

Theorem 0.0.6. *L'homomorphisme de bord de la suite spectrale de dualité*

$$\pi_*(E_C^{h\mathbb{S}_C^1} \wedge A_1) \to \pi_*(E_C^{hG_{24}} \wedge A_1)$$

est surjective.

Différentielles $d_1 : \mathrm{E}_1^{1,p} \to \mathrm{E}_1^{2,p}$**.** En clair, ces différentielles s'identifient avec l'homomorphisme $\delta_2 : \pi_*(E_C^{hC_6} \wedge A_1) \to \pi_*(E_C^{hC_6} \wedge A_1)$. La multiplication avec un élément $\Delta^2 \in \pi_{48}(E_C^{hC_6})$ induit un isomorphisme

$$\pi_*(E_C^{hC_6} \wedge A_1) \to \pi_{*+48}(E_C^{hC_6} \wedge A_1).$$

Nous démontrons que ces différentielles sont linéaires par rapport à Δ^2, i.e.,

$$\delta_2(\Delta^2 x) = \Delta^2 \delta_2(x).$$

Nous réduisons à analyser le comportement de ces différentielles sur $\pi_*(E_C^{hC_6} \wedge A_1)$ avec $0 \leq * < 48$. Une analyse plus fine montre que ils ne peuvent être non-triviaux que sur deux familles.

Différentielles $d_1 : \mathrm{E}_1^{2,p} \to \mathrm{E}_1^{3,p}$**.** Ces différentielles s'identifient avec un homomorphisme $\delta_3 : \pi_*E_C^{hC_6} \wedge A_1 \to \pi_{*-48}(E_C^{hG_{24}} \wedge A_1)$. Nous montrons d'abord que ce dernier est le composite des morphismes suivants $\pi_*E_C^{hC_6} \wedge A_1 \xrightarrow{\times\Delta^{-1}} \pi_{*-48}E_C^{hC_6} \wedge A_1 \xrightarrow{1-\psi_\alpha} \pi_{*-48}E_C^{hC_6} \wedge A_1 \xrightarrow{Tr} \pi_{*-48}(E_C^{hG_{24}} \wedge A_1)$ où ψ_α désigne le morphisme induit par l'élément α du $\mathbb{G}_2$ et Tr le transfert $E_C^{hC_6} \to E_C^{hG_{24}}$. Delà, nous déduisons que ces différentielles d_1 sont triviaux.

Chapter I

Preliminaries

1 Recollection on chromatic homotopy theory

1.1 Transfer and Restriction

We discuss, in this section, the transfer and restriction maps between homotopy fixed point spectra. Let us denote by $\mathcal{S}p$ the category of spectra, see for example [**Ada74**] Chapter III. The objects of $\mathcal{S}p$ are sequences of compactly generated weak Hausdorff pointed topological spaces $(X_i)_{i\geq 0}$, together with the structure map $\epsilon_i^X : \Sigma X_i \to X_{i+1}$ which is pointed continuous map, for $i \geq 0$, where ΣX_i is the reduced suspension of X_i. A morphism between $X = (X_i, \epsilon_i^X)_{i\geq 0}$ and $Y = (Y_i, \epsilon_i^Y)_{i\geq 0}$ consists of continuous maps $f_i : X_i \to Y_i$ such that $f_{i+1} \circ \epsilon_i^X = \epsilon_i^Y \circ \Sigma f_i$. The k-th homotopy group of X is defined as $\pi_k X := \operatorname{colim}_n \pi_{k+n}(X_n)$, in which the transition maps are defined using the structure maps. Let $\eta_i^X : X_i \to \Omega X_{i+1}$ the adjoint of ϵ_i^X. If η_i^X is a weak equivalence for all $i \geq 0$, then X is called a Ω-spectrum. The category of spectra can be equipped with the stable model category whose weak equivalences are stable equivalences, i.e., maps $f : X \to Y$ such that $\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for all $k \in \mathbb{Z}$, see for example [**BF78**]. The fibrant objects are precisely Ω-spectra. Let $\mathrm{Ho}(\mathcal{S}p)$ denote the stable homotopy category of spectra (with respect to the stable equivalences).

Let G be a finite group, which stays like so in this section. Denote by $\mathcal{S}p^G$ the category of functors $G \to \mathcal{S}p$. In this thesis, we call $\mathcal{S}p^G$ the category of (naive) G-spectra. It is the category of spectra with a G-action and G-equivariant maps. As a diagram category, $\mathcal{S}p^G$ inherits a model structure category from $\mathcal{S}p$ such that the weak equivalences are G-equivariant maps $X \to Y$ whose induced map in homotopy is an isomorphism.

Induced and Coinduced functors. Let H is a subgroup of G. The restriction functor $\iota_H^G : \mathcal{S}p^G \to \mathcal{S}p^H$ has both a left adjoint and a right adjoint, namely, the induced and coinduced functors, respectively. For X an H-(pointed) space, the space $G_+ \wedge_H X$, the quotient space of $G_+ \wedge X$ by the relation $(gh, x) \sim (g, hx)$, is a G-space with the obvious G-action given by the left multiplication on G and $F_H(G_+, X)$, the space of H-map, is a G-space with the G-action given by the right multiplication on G. Now if X is an H-spectrum, the induced functor $G_+ \wedge_H X$ is defined by the formula

$$(G_+ \wedge_H X)_n = G_+ \wedge_H X_n$$

and the coinduced functor $F_H(G_+, X)$ by

$$F_H(G_+, X)_n = F_H(G_+, X_n).$$

We have then that

$$\mathcal{S}p^G(G_+ \wedge_H X, Y) \cong \mathcal{S}p^H(X, \iota_H^G Y), (f \mapsto (x \mapsto f(1 \wedge x))) \tag{I.1}$$

$$\mathcal{S}p^G(X, F_H(G_+, Y)) \cong \mathcal{S}p^H(\iota_H^G X, Y), (f \mapsto (x \mapsto f(x)(1))) \tag{I.2}$$

The H-map $G_+ \to H_+$ given by the identity on H and sending the complement $G \backslash H$ to the basepoint gives rise to a H-map $G_+ \wedge_H X \to X$, and so by the adjunction (I.2) a G-map

$$\Theta_H^G : G_+ \wedge_H X \to F_H(G_+, X).$$

In formula, the later is given by

$$\left((g, x) \mapsto \left(k \mapsto \begin{cases} kgx & if\ k \in Hg^{-1} \\ * & else \end{cases}\right)\right)$$

From this, it is straightforward to see that Θ_H^G - or Θ for short, if the context is clear - is a weak equivalence of G-spectra if $|G/H| < +\infty$.

Homotopy fixed points and homotopy orbits. Let X be an object of $\mathcal{S}p^G$. Fix a model of EG, a contractible space with a free G-action. Define the homotopy fixed point spectrum of X with respect to G by

$$X^{hG} = F(EG_+, (X)_f)^G,$$

where $(X)_f$ is a fibrant replacement of X and the homotopy orbit spectrum is

$$X_{hG} = EG_+ \wedge_G (X)_c,$$

where $(X)_c$ is a cofibrant replacement of X. Note that the fibrant and cofibrant replacements can be made functorial and will be implicit in the sequel. The homotopy fixed points and homotopy orbits descend to functors between homotopy categories $\mathrm{Ho}(\mathcal{S}p^G) \to \mathrm{Ho}\mathcal{S}p$. We denote by δ_G the diagonal functor, i.e., the functor that sends a spectrum to a G-spectrum with trivial action. Then $()_{hG}$ and $()^{hG}$ are the left and right adjoint to δ_G, i.e.,

$$\mathrm{Ho}\mathcal{S}p^G(X, \delta_G Y) \cong \mathrm{Ho}\mathcal{S}p(X_{hG}, Y) \tag{I.3}$$

$$\mathrm{Ho}\mathcal{S}p^G(\delta_G X, Y) \cong \mathrm{Ho}\mathcal{S}p(X, Y^{hG}). \tag{I.4}$$

Transfer and restriction. Let H be a subgroup of G. If X is a H-spectrum, then X^{hH} is naturally isomorphic to $F_H(G_+, X)^{hG}$ by the following chain of isomorphisms.

$$X^{hH} = F(EH_+, X)^H \cong F(EG_+, X)^H \xrightarrow{\cong} F_H(G_+, F(EG_+, X))^G$$

$$\cong F(EG_+, F_H(G_+, X))^G = F_H(G_+, X)^{hG},$$

in which the first equivalence uses the natural H-map $EH \to EG$, the second uses the natural homeomorphism of spaces $Y^H \simeq F_H(G_+, Y)^G$ and the last the usual adjunction between smash products and function spaces. Let $\Psi_H^G : X^{hH} \simeq F_H(G_+, X)^{hG}$, or Ψ if the context makes it clear which groups are concerned, denote this natural equivalence.

Let X be a G-spectrum. The transfer

$$\mathrm{Tr}_H^G : (\iota_H^G X)^{hH} \to X^{hG}$$

is constructed as the composite

$$(\iota_H^G X)^{hH} \xrightarrow{\Psi} F_H(G_+, \iota_H^G X)^{hG} \xrightarrow{\Theta^{-1}} (G_+ \wedge_H \iota_H^G X)^{hG} \xrightarrow{\epsilon^{hG}} X^{hG}, \tag{I.5}$$

where ϵ denotes the counit of the adjunction (I.1). Apply the restriction functor ι_H^G to the co-unit of the adjunction (I.4), we get a map $\delta_H X^{hG} \to \iota_H^G X$ in $\mathrm{Ho}(\mathcal{S}\mathrm{p}^{\mathrm{H}})$. The latter is adjoint, by the adjunction (I.4), to the restriction map

$$\mathrm{Res}_H^G : X^{hG} \to (\iota_H^G X)^{hH}.$$

We will abbreviate the transfer and the restriction maps by $\mathrm{Tr} : X^{hH} \to X^{hG}$ and $\mathrm{Res} : X^{hG} \to X^{hH}$, respectively, wherever the context gives no place for confusion.

Lemma 1.1.1. *Let X be a G-spectrum and H be a subgroup of G. If $G/H_+ \wedge X$ is equipped with the diagonal G-action, then there is a natural isomorphism of G-spectra, usually called the shearing isomorphism,*

$$tSh : G_+ \wedge_H X \xrightarrow{\simeq} G/H_+ \wedge X,$$

given by $(g, x) \mapsto (gH, gx)$.

Remark 1.1.2. The notation tSh is for topological shearing isomorphism to distinguish with the algebraic shearing isomorphism to be defined in Section 3.2.

Proof. It is straightforward to check that the following is the well-defined inverse of the given map

$$G/H_+ \wedge X \to G_+ \wedge_H X, (gH, x) \mapsto (g, g^{-1}x).$$

□

Lemma 1.1.3. *Let X be a G-spectrum and $H \leq K$ be subgroups of G. Then the following diagram is commutative*

$$\begin{array}{ccc} (G/H_+ \wedge X)^{hG} & \longrightarrow & (G/K_+ \wedge X)^{hG} \\ \simeq \uparrow & & \simeq \uparrow \\ X^{hH} & \xrightarrow{Tr} & X^{hK}, \end{array}$$

where the upper horizontal map is induced by the canonical G-map $G/H \to G/K$ and the vertical equivalences are the composites $(tSh)^{hG} \circ \Theta^{-1} \circ \Psi$ for appropriate inclusions of subgroups.

Proof. By construction, $Tr : X^{hH} \to X^{hK}$ fits into the commutative diagram

$$\begin{array}{ccc} X^{hH} & \xrightarrow{(\Theta_H^K)^{-1} \circ \Psi_H^K} & (K_+ \wedge_H X)^{hK} \\ \downarrow Tr & \swarrow \epsilon^{hK} & \\ X^{hK} & & \end{array}$$

By applying the naturality of $(\Theta_K^G)^{-1} \circ \Psi_K^G$ to the K-map $\epsilon : K_+ \wedge_H X \to X$, we obtain the commutative diagram

$$\begin{array}{ccc} (K_+ \wedge_H X)^{hK} & \xrightarrow{(\Theta_K^G)^{-1} \circ \Psi_K^G} & (G_+ \wedge_K \wedge K_+ \wedge_H X)^{hG} \\ \downarrow \epsilon & & \downarrow \\ X^{hK} & \xrightarrow{(\Theta_K^G)^{-1} \circ \Psi_K^G} & (G_+ \wedge_K X)^{hG}. \end{array}$$

Using the evident isomorphism of G-map $G_+ \wedge_K \wedge K_+ \wedge_H X \simeq G_+ \wedge_H X$, we see that the composite

$$X^{hH} \xrightarrow{(\Theta_H^K)^{-1} \circ \Psi_H^K} (K_+ \wedge_H X)^{hK} \xrightarrow{(\Theta_K^G)^{-1} \circ \Psi_K^G} (G_+ \wedge_H X)^{hG}$$

is homotopic to $(\Theta_H^G)^{-1} \circ \Psi_H^G$. Thus we obtain the following commutative diagram

$$\begin{array}{ccc} X^{hH} & \xrightarrow{(\Theta_H^G)^{-1} \circ \Psi_H^G} & (G_+ \wedge_H X)^{hG} \\ \Big\downarrow{\scriptstyle Tr} & & \Big\downarrow \\ X^{hK} & \xrightarrow{(\Theta_K^G)^{-1} \circ \Psi_K^G} & (G_+ \wedge_K X)^{hG}. \end{array}$$

Finally, by the definition of the shearing isomorphism, the right vertical map fits into the commutative diagram

$$\begin{array}{ccc} (G_+ \wedge_H X)^{hG} & \xrightarrow{tSh} & (G/H_+ \wedge X)^{hG} \\ \Big\downarrow & & \Big\downarrow \\ (G_+ \wedge_K X)^{hG} & \xrightarrow{tSh} & (G/K_+ \wedge X)^{hG}, \end{array}$$

where the right vertical map is induced by the natural projection $G/H \to G/K$, from which the lemma follows. □

1.2 Lubin-Tate theories

We recall some generalities on the deformation theory of formal group laws and Goerss-Hopkins-Miller theory. Let $\mathcal{FGL}$ be the category whose objects are pairs (k, Γ) where k is a perfect field of characteristic p and Γ is a formal group law over k and morphisms between (k, Γ) and (k', Γ') are pairs (i, ϕ) where $i : k' \to k$ is a homomorphism of fields and $\phi : \Gamma \xrightarrow{\cong} i^*\Gamma'$ is a morphism of formal group laws.

Let $(k, \Gamma) \in \mathcal{FGL}$ with Γ of height n. A deformation of (k, Γ) to a complete local ring R with maximal ideal m is a pair (F, ι) where F is a formal group law over R and $\iota : k \to R/m$ is a map of fields such that $p^*F = \iota^*\Gamma$ with p the canonical projection $R \to R/m$. A $\star$-isomorphism ϕ between two deformations to R is an isomorphism between the underlying formal group laws which reduces to the identity over R/m, i.e, $\phi \cong x \bmod (m)$. This defines a functor from the category of complete local rings $\mathfrak{Ring}_{c,l}$ to small groupoids $\mathfrak{Groupoid}$

$$\mathrm{Def}_\Gamma : \mathfrak{Ring}_{c,l} \to \mathfrak{Groupoid}$$

which associates to every complete local ring R the category of deformations of (k, Γ) over R and $\star$-isomorphisms between them. By Lubin-Tate deformation theory, Def_Γ is co-representable, see [LT66]. That is, there exists a complete local ring $E_{k,\Gamma}$, non-canonically isomorphic to $\mathbb{W}(k)[[u_1, u_2, ..., u_{n-1}]]$, such that

$$\mathrm{Def}_\Gamma(R) \cong \mathrm{Hom}_{\mathfrak{Ring}_{c,l}}(E_{k,\Gamma}, R).$$

Here $\mathbb{W}(k)$ denotes the ring of Witt vectors on k. Over $E_{k,\Gamma}$ lives a universal deformation $\tilde{\Gamma}$ of Γ. Consider the graded ring $E_{k,\Gamma}[u^{\pm 1}]$ where $|u_i| = 0$ for $1 \leq i \leq n-1$ and $|u| = -2$. Let MU be the cobordism spectrum. A Quillen's famous theorem asserts that the coefficient rings MU_* of MU supports the universal group law. Thus, the formal group law $u^{-1}\tilde{\Gamma}(ux, uy)$ is classified by a map of graded rings $MU_* \to E_{k,\Gamma}[u^{\pm 1}]$. Define a functor from the category of pointed spaces to that of graded abelian groups:

$$X \mapsto MU_*(X) \otimes_{MU_*} E_{k,\Gamma}[u^{\pm 1}].$$

The formal group $u^{-1}\tilde{\Gamma}(ux, uy)$ satisfies the Landweber exact functor criterion, see [Rez98]. By the Landweber exact functor theorem, the above functor is a homology functor. Thus, it is represented by a ring spectrum $E(k, \Gamma)$ with

$$(E(k, \Gamma))_* \cong \mathbb{W}(k)[[u_1, u_2, ..., u_{n-1}]][u^{\pm 1}].$$

The latter is known as a n^{th} Morava E-theory or Lubin-Tate theory.

Example 1. Let $(k, \Gamma) = (\mathbb{F}_2, \mathbb{G}_m)$ where $\mathbb{G}_m$ is the multiplicative formal group law, i.e, $\mathbb{G}_m(x, y) = x + y + xy$. Then $E(\mathbb{F}_2, \mathbb{G}_m) \simeq K\mathbb{Z}_2$, the 2-completed complex K-theory and $\mathbb{G}(\mathbb{F}_2, \mathbb{G}_m) = \mathbb{Z}_2^\times$, the unit of the 2-adic integers. Furthermore, the action of $\mathbb{G}(\mathbb{F}_2, \mathbb{G}_m)$ on $E(\mathbb{F}_2, \mathbb{G}_m)_*$ coincides with Adams operations.

The construction that associates to a formal group law (k, Γ) the Morava E-theory $E(k, \Gamma)$ defines a functor from $\mathcal{FGL}$ to $\mathrm{Ho}(Sp)$, the stable homotopy category. Let us denote by $\mathbb{G}(k, \Gamma)$ the automorphism group of the pair (k, Γ). We note that $\mathbb{G}(k, \Gamma)$ is a profinite group, see [Goe08], Section 7.2. By functoriality, the group $\mathbb{G}(k, \Gamma)$ acts on $E(k, \Gamma)$. This action is, however, defined only up to homotopy. The Goerss-Hopkins-Miller obstruction theory lifts this action to structured ring spectra.

Theorem 1.2.1. [GH04] *The spectrum $E(k, \Gamma)$ has an essentially unique structure of E_∞-ring. Furthermore, $\mathbb{G}(k, \Gamma)$ acts on $E(k, \Gamma)$ via E_∞-ring maps.*

1.3 Continuous homotopy fixed point spectra

Based on Theorem 1.2.1, Devinatz and Hopkins constructed a continuous homotopy fixed point spectrum $E(k, \Gamma)^{hK}$ for any closed subgroup K of $\mathbb{G}(k, \Gamma)$.

We point out that in the preprint [**BBGS18**], the authors give a pleasant expository account of the Devinatz-Hopkins construction.

Movara modules. A Morava module is a complete $E(k,\Gamma)_*$-module M with a twisted continuous $\mathbb{G}(k,\Gamma)$-action, i.e., for $g \in \mathbb{G}$, $a \in E_*$, $m \in M$

$$g(am) = g(a)g(m).$$

Let us denote by $\mathcal{EG}$ the category whose objects are Morava modules and morphisms are continuous maps of $E(k,\Gamma)_*$-modules which are $\mathbb{G}$-equivariant. Typical examples of Morava modules are $\pi_* L_{K(n)}(E(k,\Gamma) \wedge X)$ for any spectrum X, where the action of $\mathbb{G}(k,\Gamma)$ is induced by its action on the left hand factor of $E(k,\Gamma) \wedge X$.

Convention. Unless otherwise stated, smash products are taken in the $K(n)$-local homotopy category, i.e., $X \wedge Y := L_{K(n)}(X \wedge Y)$. Likewise, $E(k,\Gamma)_* X$ means $\pi_*(L_{K(n)}(E(k,\Gamma) \wedge X))$.

Theorem 1.3.1. [**DH04**] *For any closed subgroup K of $\mathbb{G}(k,\Gamma)$, there is a homotopy fixed point spectrum $E(k,\Gamma)^{hK}$. Furthermore, there is a spectral sequence whose E_2-term is the continuous cohomology group $\mathrm{H}^s_c(K, E(k,\Gamma)_t)$ converging to $\pi_{t-s}E(k,\Gamma)^{hK}$, i.e.,*

$$\mathrm{H}^s_c(K, E(k,\Gamma)_t) \Longrightarrow \pi_{t-s}E(k,\Gamma)^{hK}.$$

The group $\mathbb{G}(k,\Gamma)$ is a virtual p-adic Lie group, so are its closed subgroups. The notion of continuous cohomology is very well treated in [**Laz65**], [**SW00**]. When K is a finite subgroup, the construction of the continuous homotopy fixed point spectrum coincides with the usual homotopy fixed point spectrum with respect to the action of a finite group.

Example 2. Consider the case $(k,\Gamma) = (\mathbb{F}_2, \mathbb{G}_m)$. The group $\mathbb{G}(\mathbb{F}_2, \mathbb{G}_m) \cong \mathbb{Z}_2^\times$ contains C_2, the cyclic group of order 2 as a subgroup. Then, $E(\mathbb{F}_2, \mathbb{G}_m)^{hC_2}$ is homotopy equivalent to $KO\mathbb{Z}_2$, the 2-completed real K-theory.

This example motivates the following definition.

Definition 1.3.2. (Higher real K-theory) If G is a finite subgroup of $\mathbb{G}(k,\Gamma)$, the homotopy fixed point spectrum $E(k,\Gamma)^{hG}$ is called a higher real K-theory.

Higher real K-theories play an important role due essentially to the two following reasons. Firstly, they are more tractable in general than homotopy fixed point spectra with respect to infinite closed subgroups of $\mathbb{G}(k,\Gamma)$. Secondly, it is believed (or at least hoped) that homotopy fixed point spectra with respect to infinite closed subgroups can be built from higher real K-theories via finite resolutions.

Suppose, from now on, that all automorphisms of Γ are defined over k, i.e., if $\overline{k}$ denotes the algebraic closure of k, then

$$\mathrm{Aut}(\Gamma) := \mathrm{Aut}_{\overline{k}}(\Gamma) = \mathrm{Aut}_k(\Gamma). \tag{I.6}$$

Under this assumption, Devinatz and Hopkins proved the following result:

Theorem 1.3.3. *There is a homotopy equivalence*

$$E(k,\Gamma)^{h\mathbb{G}(k,\Gamma)} \cong L_{K(n)}S^0.$$

This theorem is a central result of chromatic homotopy theory in the sense that it provides new computational tools as well as conceptual interpretations of $L_{K(n)}S^0$. It is important to note that this theorem is a topological incarnation of Morava's change of rings theorem, (see [**Dev95**]).

Theorem 1.3.4 (Morava's change of rings)**.** *There is an isomorphism*

$$\mathrm{Ext}^{s,t}_{E(k,\Gamma)_*E(k,\Gamma)}(E(k,\Gamma)_*, E(k,\Gamma)_*) \cong \mathrm{H}^s_c(\mathbb{G}(k,\Gamma), E(k,\Gamma)_t).$$

The left hand side of the above isomorphism is the E_2-term of the $K(n)$-local $E(k,\Gamma)$-based spectral sequence. We refer to the appendix A of [**DH04**] for a discussion of this spectral sequence. We note that the E_2-term of the latter can be always expressed as Ext-groups in the category of comodules over the completed Hopf algebroid $(E(k,\Gamma)_*, E(k,\Gamma)_*E(k,\Gamma))$. The assumption (I.6) allows us to identify it with the continuous cohomology groups as in Theorem 1.3.4.

The computation of the homotopy fixed point spectral sequence is in general a hard problem. Already the calculation of the E_2-term is challenging. This is due to the fact that, in general, the action of $\mathbb{G}(k,\Gamma)$ on $E(k,\Gamma)_*$ can only be computed approximately and that $\mathbb{G}(k,\Gamma)$ is a cohomologically complicated group.

1.4 Topological modular forms

The main theme of this thesis is a computation in the $K(2)$-local homotopy category at the prime 2. An astute choice of Morava E-theory or equivalently a choice of formal group law of height 2 will make the calculation easier. Let C be the supersingular elliptic curve over $\mathbb{F}_4$ given by the Weierstrass equation $y^2+y=x^3$. Denote by F_C the formal completion of C at the origin. The latter is a formal group law of height 2. Let us denote

$$E_C = E(\mathbb{F}_4, F_C) \text{ and } \mathbb{G}_C = \mathbb{G}(\mathbb{F}_4, F_C).$$

One can check that

$$\mathrm{Aut}_{\mathbb{F}_4}(F_C) = \mathrm{Aut}_{\mathbb{F}_2}(F_C) =: \mathbb{S}_C.$$

Let Gal denote the Galois group of $\mathbb{F}_4$ over $\mathbb{F}_2$, i.e., $\mathrm{Gal} \cong C_2$. There is a short exact sequence

$$1 \to \mathbb{S}_C \to \mathbb{G}_C \to \mathrm{Gal} \to 1.$$

The image of $\mathbb{S}_C$ in $\mathbb{G}_C$ corresponds to the automorphisms of $(\mathbb{F}_4, F_C)$ fixing $\mathbb{F}_4$. Since F_C is defined over $\mathbb{F}_2$, Gal fixes F_C, the above short exact sequence splits, i.e., $\mathbb{G}_C \cong \mathbb{S}_C \rtimes \mathrm{Gal}$. The automorphism group of C has order 24 and these are all defined over $\mathbb{F}_4$, more precisely,

$$\mathrm{Aut}(C) = \mathrm{Aut}_{\mathbb{F}_4}(C) \cong SL_2(\mathbb{Z}/3) \cong Q_8 \rtimes C_3 =: G_{24},$$

where Q_8 is the quaternion group and $C_3 = \langle \omega \rangle$ is a cyclic group of order 3, see [Sil09]. The group Q_8 has a representation $\langle i, j | i^4 = 1, i^2 = j^2, iji^{-1} = j^{-1} \rangle$. The latter has 8 elements $\{1, i, j, k, -1, -i, -j, -k\}$ where -1 denotes $i^2 = j^2 = k^2$. The group C_3 acts on Q_8 by permuting i, j and $k := ij$

$$\omega i \omega^2 = j, \quad \omega j \omega^2 = k.$$

The elements ω and i correspond to the automorphisms $\omega(x, y) = (\zeta x, \zeta^2 y)$ and $i(x, y) = (x + 1, y + x + \zeta^2)$, respectively, where ζ is a primitive third root of the unity.

Since C is already defined over $\mathbb{F}_2$, Gal acts on $\mathrm{Aut}(C)$. Denote by G_{48} the semi-direct product $G_{24} \rtimes \mathrm{Gal}$. Moreover, the automorphism group $\mathrm{Aut}(C)$ of C maps injectively to $\mathbb{S}_C$, the automorphism group of F_C, and G_{48} maps injectively to $\mathbb{G}_C$. We view G_{24} and G_{48} as subgroups of $\mathbb{S}_C$ and $\mathbb{G}_C$, respectively.

We see that $C_2 = \langle -1 \rangle \leq Q_8$ is invariant under the action of C_3, and so $C_6 := C_2 \times C_3$ is a subgroup of G_{24}. As an automorphism of C, -1 is given by $(x, y) \mapsto (x, y + 1)$. We see immediately that both C_2 and C_3 are invariant by Gal, and hence $G_{12} := C_6 \rtimes \mathrm{Gal}$ is a subgroup of G_{48}

The homotopy fixed point spectra $E_C^{hG_{24}}$ and $E_C^{hC_6}$ as well as $E_C^{hG_{48}}$ and $E_C^{hG_{12}}$ will play a central role in this thesis, as already mentioned in the introduction.

The reasons for choosing the formal group law of the supersingular elliptic curve C are two-fold. First, the geometric origin of G_{48} allows one to have an explicit description of its action on $\pi_*(E_C)$, see [Bea17] for more details and further references. Thus, it allows us to adequately compute the E_2-term of various homotopy fixed point spectral sequences. Second, this choice of the Morava E-theory allows us to compare associated higher real K-theories with the spectrum of topological modular forms and topological modular forms with level structures, hence providing us with more tools to understand the formers.

Next, we recall the construction of the spectrum of topological modular forms (with level structures) and show their closed relationship with higher real K-theories. Let $\mathcal{M}$, $\mathcal{M}_0(3)$ and $\mathcal{M}(3)$ be the moduli stack of elliptic curves, elliptic curves with a level 3 structure and elliptic curves with a full level 3 structure over $\mathbb{Z}_{(2)}$. As functors of points on $\mathbb{Z}_{(2)}$-algebras, the latter are described as follows. If R is a $\mathbb{Z}_{(2)}$-algebra, then

- $\mathcal{M}(\mathrm{spec}(R))$ is the groupoid of elliptic curves over $\mathrm{spec}(R)$ and isomorphisms between them.

- $\mathcal{M}_0(3)(\mathrm{spec}(R))$ is the groupoid of pairs (E, H) consisting of an elliptic curve E with a subgroup H of order 3 and isomorphisms between them.

- $\mathcal{M}(3)(\mathrm{spec}(R))$ is the groupoid of pairs (E, ϕ) consisting of an elliptic curve E with an isomorphism of group schemes $\phi : \mathbb{Z}/3 \times \mathbb{Z}/3 \to E[3]$ over $\mathrm{spec}(R)$ where $E[3]$ is the subscheme of 3-torsion points of E and isomorphisms between them.

Theorem 1.4.1 (Goerss-Hopkins-Miller, see [**DFHH14**]). *There is an E_∞-ring spectra-valued sheaf $\mathcal{O}^{top}$ on the étale site $Aff_{\mathcal{M}}^{ét}$ of $\mathcal{M}$ such that*

1. *The sheafification of $\pi_0\mathcal{O}^{top}$ is the structure sheaf of $\mathcal{M}$.*
2. *If $E : \mathrm{spec}(R) \to \mathcal{M}$ is an étale morphism, then $\mathcal{O}^{top}(\mathrm{spec}(R))$ is a spectrum associated to the formal completion of E at its origin via the Landweber exact functor theorem.*

Remark 1.4.2. The spectra constructed by point 2. of the previous theorem are called elliptic spectra. They are even periodic spectra R whose formal group law on $\pi_0(R)$ is the completion of an elliptic curve. These are $E(2)$-local, see [**DFHH14**], Chapter 6, Lemma 4.2.

Let $G := GL_2(\mathbb{Z}/3)$ denote the automorphism group of the constant group scheme $\mathbb{Z}/3 \times \mathbb{Z}/3$ over $\mathbb{Z}_{(2)}$. Then G acts on $\mathcal{M}(3)$ by precomposition with the level structure. Also let $G_0(3)$ denote the subgroup of upper triangular matrices of $GL_2(\mathbb{Z}/3)$. The obvious forgetful functors give rise to finite étale morphisms of stacks (because 3 is invertible in $\mathbb{Z}_{(2)}$):

$$\mathcal{M}(3) \to \mathcal{M}_0(3) \to \mathcal{M}. \tag{I.7}$$

Thus, one can evaluate $\mathcal{O}^{top}$ at $\mathcal{M}$, $\mathcal{M}_0(3)$ and $\mathcal{M}(3)$. Define

$$TMF = \mathcal{O}^{top}(\mathcal{M}) := \operatorname*{holim}_{U \in Aff_{\mathcal{M}}^{ét}} \mathcal{O}^{top}(U),$$

$$TMF_0(3) = \mathcal{O}^{top}(\mathcal{M}_0(3)) := \underset{U \in Aff^{\acute{e}t}_{\mathcal{M}_0(3)}}{\operatorname{holim}} \mathcal{O}^{top}(U),$$

$$TMF(3) = \mathcal{O}^{top}(\mathcal{M}(3)) := \underset{U \in Aff^{\acute{e}t}_{\mathcal{M}(3)}}{\operatorname{holim}} \mathcal{O}^{top}(U).$$

Both morphisms of (I.7) are Galois covers. The Galois group of the composite is isomorphic to G and that of the first morphism is isomorphic to $G_0(3)$. As a consequence of the fact that $\mathcal{O}^{top}$ satisfies descent, one obtains that

$$TMF \cong TMF(3)^{hG} \tag{I.8}$$

and

$$TMF_0(3) \cong TMF(3)^{hG_0(3)} \tag{I.9}$$

It is known that $\mathcal{M}(3)$ is affine over the ring $\mathbb{Z}_{(2)}[\zeta]$ where ζ is a primitive third root of unity, see [**DR73**], also [**Sto14**]. Furthermore, up to isomorphism, there is a unique supersingular elliptic curve with a full level structure over $\mathbb{F}_4$. This follows from the fact that there is a unique supersingular elliptic curve over $\mathbb{F}_4$ (up to isomorphism) and that the automorphism group of the supersingular elliptic curve C has order 48, which is equal to that of G, the automorphism group of $\mathbb{Z}/3 \times \mathbb{Z}/3$. In other words, the fiber of the morphism $\mathcal{M}(3) \to \mathcal{M}$ over the supersingular locus of $\mathcal{M}$ is isomorphic to $\operatorname{spec}(\mathbb{F}_4)$, i.e., the following square is a pullback of stacks

$$\begin{array}{ccc} \operatorname{spec}(\mathbb{F}_4) & \longrightarrow & \mathcal{M}(3) \\ \downarrow & & \downarrow \\ \operatorname{spec}(\mathbb{F}_4)//G_{48} & \longrightarrow & \mathcal{M} \end{array}$$

where the bottom is given by specifying a supersingular elliptic curve, for example C. Therefore, by the construction of $\mathcal{O}^{top}$, $L_{K(2)}\mathcal{O}^{top}(\mathcal{M}(3))$ is the Lubin-Tate theory associated to the paire $(\mathbb{F}_4, F_C)$, see [**DFHH14**], Chapter 12. This means that there is a homotopy equivalence

$$L_{K(2)}TMF(3) \xrightarrow{\cong} E_C. \tag{I.10}$$

Note that G can be identified with $\operatorname{Aut}(C) = G_{48}$, such that the equivalence (I.10) is equivariant with respect to the action of G on the source and of G_{48} on the target, as follows. Suppose the the map $\operatorname{spec}(\mathbb{F}_4) \to \mathcal{M}(3)$ specifies the elliptic curve C and a 3 level structure $\mathbb{Z}/3^{\times 2} \xrightarrow{\Gamma} C$. Then for any $g \in G$, there is a unique

$\phi(g) \in G_{48}$ making the following diagram commute

$$\begin{array}{ccc} \mathbb{Z}/3^{\times 2} & \xrightarrow{\Gamma} & C \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle \phi(g)} \\ \mathbb{Z}/3^{\times 2} & \xrightarrow{\Gamma} & C. \end{array}$$

Via this identification ϕ, $G_0(3)$ is, up to a conjugation, equal to G_{12} in G_{48}. Thus, we have

Theorem 1.4.3. *There are homotopy equivalences*

$$L_{K(2)}TMF \cong E_C^{hG_{48}} \tag{I.11}$$

and

$$L_{K(2)}TMF_0(3) \cong E_C^{hG_{12}}. \tag{I.12}$$

Proof. Since an elliptic spectrum is $E(2)$-local, $TMF(3)$ is $E(2)$-local, being a homotopy limit of $E(2)$-local spectra. Using equivalences (I.8), (I.9), (I.12) and the fact that $K(2)$-localisation commutes with homotopy limit in the category of $E(2)$-local spectra, we obtain that

$$L_{K(2)}TMF \cong L_{K(2)}(TMF(3)^{hG}) \cong (L_{K(2)}TMF(3))^{hG} \cong E_C^{hG_{48}}$$

and

$$L_{K(2)}TMF_0(3) \cong L_{K(2)}(TMF(3)^{hG_0(3)}) \cong (L_{K(2)}TMF(3))^{hG_0(3)} \cong E_C^{hG_{12}}.$$

□

A connective model of TMF. In [DFHH14] a connective ring spectrum tmf was constructed together with a map of ring spectra

$$tmf \to TMF. \tag{I.13}$$

There is an element $\Delta^8 \in \pi_{192}tmf$ such that the latter map extends to a homotopy equivalence

$$[(\Delta^8)^{-1}]tmf \xrightarrow{\simeq} TMF, \tag{I.14}$$

see [DFHH14], hence (I.13) induces a $K(2)$-local equivalence

$$L_{K(2)}tmf \xrightarrow{\simeq} L_{K(2)}TMF. \tag{I.15}$$

An advantage of tmf is that the singular homology of tmf is explicitly known as a module over the Steenrod algebra $\mathcal{A}$ (c.f Section 2 for a recollection on the Steenrod algebra), hence the Adams spectral sequence gives a powerful tool to understand its homotopy groups. The following had been known by Hopkins and Mahowald and was shown by Mathew in [Mat16]:

Theorem 1.4.4. *There is an isomorphism of algebras over the Steenrod algebra:*

$$\mathrm{H}^*(tmf) \cong \mathcal{A}//\mathcal{A}(2),$$

where $\mathcal{A}(2)$ is the subalgebra of $\mathcal{A}$ generated by Sq^1, Sq^2, Sq^4. Equivalently, there is an isomorphism of comodule algebras over the dual of Steenrod algebra $\mathcal{A}_*$

$$\mathrm{H}_*(tmf) \cong \mathcal{A}_* \square_{\mathcal{A}(2)_*} \mathbb{F}_2,$$

where $\mathcal{A}(2)_*$ is the dual of $\mathcal{A}(2)$.

A connective model of $TMF_0(3)$. Similarly, there is a connective ring spectrum $tmf_0(3)$ constructed in [**DM10**], which enjoys the following properties. We refer to [**DM10**] for details and proofs.

1. There exists a finite spectrum X such that $tmf \wedge X \cong tmf_0(3)$.
2. As a $\mathbb{F}_2$-vector space, H^*X is 10-dimensional with generators x_i in dimension i for $i = 0, 4, 6, 7, 8, 10, 11, 12, 13, 14$, respectively. Let T denote the sub-$\mathcal{A}$-module

$$T = \mathbb{F}_2\{x_i | i = 4, 6, 7, 8, 10, 11, 12, 13, 14\}.$$

The structure of module over $\mathcal{A}$ of T is determined by the Adem relations and the following

$$Sq^2 x_4 = x_6, Sq^4 x_4 = x_8, Sq^8 x_4 = x_{12}, Sq^1 x_6 = x_7,$$
$$Sq^4 x_6 = x_{10}, Sq^4 x_7 = x_{11}, Sq^4 x_8 = x_{12},$$
$$Sq^1 x_{10} = x_{11}, Sq^2 x_{10} = x_{12}, Sq^4 x_{10} = x_{14},$$
$$Sq^2 x_{11} = x_{13}, Sq^2 x_{12} = x_{14}, Sq^1 x_{13} = x_{14}.$$

The subspace $\mathbb{F}_2\{x_0\}$ is a direct factor of H^*X as an $\mathcal{A}(2)$-module.
3. There is an element $\Delta^2 \in \pi_{48}(tmf_0(3))$ such that there is a homotopy equivalence (Corollary 3.11 of [**DM10**]):

$$[(\Delta^2)^{-1}]tmf_0(3) \cong TMF_0(3). \tag{I.16}$$

4. The spectrum $tmf_0(3)$ is a ring spectrum in the homotopy category of spectra whose unit $S^0 \to tmf \wedge X$ is the smash product of $S^0 \to tmf$, the unit of tmf and $S^0 \to X$, the inclusion of the bottom cell of X.

Remark 1.4.5. i) Although $tmf_0(3)$ is a ring spectrum, the equivalence (I.16) was not shown to be an equivalence of ring spectra, but this does not matter for our purposes.

ii) There is a more recent construction of $tmf_0(3)$ given by Hill and Lawson in [**HL16**]. To the best of the author's knowledge, it is not known if the two constructions give the same spectrum. But this seems plausible because they have the same homotopy groups.

1.5 Action of the Morava stabiliser group

We recall that the group $\mathbb{S}_C$ is the units of the endomorphism ring $\mathrm{End}(F_C)$ of F_C. One has that

$$\mathrm{End}(F_C) \cong \mathbb{W}\langle\langle T\rangle\rangle/(T^2 = -2, T\omega = \omega^2 T),$$

where $\mathbb{W} := \mathbb{W}(\mathbb{F}_4) \cong \mathbb{Z}_2[\omega]/(1 + \omega + \omega^2)$, T corresponds to the endomorphism $T(x) = x^2$ and ω the isomorphism $\omega(x) = \zeta x$, with ζ a primitive third root of unity. The group $\mathbb{S}_C$ has the following filtration by open subgroups

$$F_{n/2}\mathbb{S}_C = \{\gamma \in \mathbb{S}_C | \gamma \cong 1 \text{ modulo } (T^n)\}$$

refining the 2-adic filtration of $\mathbb{W}^\times$. As a module over $\mathbb{W}$, $\mathrm{End}(F_C)$ is free of rank 2, hence every element of $\mathrm{End}(F_C)$ can be written uniquely as $a + bT$ where $a, b \in \mathbb{W}$. Right multiplication of $\mathbb{S}_C$ on $\mathrm{End}(F_C)$ induces a 2-dimensional $\mathbb{W}$-representation of $\mathbb{S}_C$:

$$\mathbb{S}_C \to GL_2(\mathbb{W}), a + bT \mapsto \begin{pmatrix} a & b \\ -2b^\sigma & a^\sigma \end{pmatrix}. \tag{I.17}$$

Here a^σ denotes the action of the Frobenius on a, i.e., if $a = x + \omega y$ where $x, y \in \mathbb{Z}_2$, then $a^\sigma = x + \omega^2 y$. Post-composing the latter with the determinant $GL_2(\mathbb{W}) \to \mathbb{W}^\times$, one obtains a homomorphism $\mathbb{S}_C \to \mathbb{W}^\times$, which, as can be seen from (I.17), factors through $\mathbb{S}_C \to \mathbb{Z}_2^\times$. The latter is called the determinant or the norm homomorphism. The following composite is called the reduced determinant, in which the second map is the quotient of $\mathbb{Z}_2^\times$ by its finite subgroup C_2.

$$\mathbb{S}_C \xrightarrow{det} \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/C_2 \cong \mathbb{Z}_2 \tag{I.18}$$

The kernel of the reduced determinant is denoted by $\mathbb{S}_C^1$. The reduced determinant is a split surjection, i.e.,

$$\mathbb{S}_C \cong \mathbb{S}_C^1 \rtimes \mathbb{Z}_2.$$

Fix $\pi \in \mathbb{S}_C$, an element whose reduced determinant equals 3 which is a topological generator of $\mathbb{Z}_2$, see [**Bea17**], 3.2. Thus, π provides a section of the reduced determinant.

Finite subgroups of $\mathbb{S}_C$. We have discussed some finite subgroups of $\mathbb{S}_C$ which are G_{24}, C_6 and C_2. Another one which will play a role is the conjugate of G_{24} by π: define G'_{24} to be $\pi G_{24}\pi^{-1}$. While $\mathbb{S}_C$ has a unique conjugacy class of maximal finite subgroup isomorphic to G_{24}, $\mathbb{S}_C^1$ has two, isomorphic to G_{24} and to G'_{24}.

Action of $\mathbb{S}_C$ on $(E_C)_*$. The geometric origin of G_{24} was used to explicitly describe its action on $(E_C)_*$ see ([**Bea17**], 2.4). We record these formulae here for later reference.

Theorem 1.5.1. *Let $v_1 := u^{-1}u_1 \in (E_C)_2$ and ω, i, j, k generators of G_{24} defined in Section 1.4 The action of G_{24} on $\mathbb{W}(\mathbb{F}_4)[[u_1]][u^{\pm 1}]$ is given by*

$$\omega(u^{-1}) = \zeta^2 u^{-1} \qquad \omega(v_1) = v_1$$

$$i(u^{-1}) = \frac{-u^{-1} + v_1}{\zeta^2 - \zeta} \qquad i(v_1) = \frac{v_1 + 2u^{-1}}{\zeta^2 - \zeta}$$

$$j(u^{-1}) = \frac{-u^{-1} + \zeta^2 v_1}{\zeta^2 - \zeta} \qquad j(v_1) = \frac{v_1 + 2\zeta^2 u^{-1}}{\zeta^2 - \zeta}$$

$$k(u^{-1}) = \frac{-u^{-1} + \zeta v_1}{\zeta^2 - \zeta} \qquad k(v_1) = \frac{v_1 + 2\zeta u^{-1}}{\zeta^2 - \zeta}$$

In particular, one can check that the element $\Delta := u^{-12}(u_1^3 - 1) \in (E_C)_{24}$ is G_{24}-invariant. For our purposes, we need to know a sufficiently good approximation of the action of $\mathbb{S}_C$ on Δ. Each element $\gamma \in \mathbb{S}_C$ can be written uniquely as

$$\gamma = \sum_{i=0}^{\infty} a_i T^i$$

where the a_i are solutions to the equation $x^4 - x = 0$. By [**Bea17**], there are functions $t_0, t_1 : \mathbb{S}_C \to (E_C)_0$ such that

$$\gamma(u_1) = t_0(\gamma)u_1 + \frac{2t_1(\gamma)}{3t_0(\gamma)},$$
$$\gamma(u) = t_0(\gamma)u.$$

By ([**Bea17**], Prop. 6.3.9, Prop. 6.3.10), if $\gamma \in F_{2/2}\mathbb{S}_C$ then

$$t_0(\gamma) \equiv 1 + 2a_2 + (a_2 + a_2^2)u_1^3 \text{ modulo } (4, 2u_1, u_1^4), \tag{I.19}$$
$$t_1(\gamma) \equiv a_2^2 u_1 \text{ modulo } (2, u_1^3). \tag{I.20}$$

Therefore, we have:

Lemma 1.5.2. *For all $\gamma \in \mathbb{S}_C$,*

$$\gamma(\Delta) \equiv \Delta \textit{ modulo } (4, 2u_1, u_1^4).$$

Proof. Equations (I.19) and (I.20) imply that, if $\gamma \in F_{2/2}\mathbb{S}_C$, then

$$\gamma(u) \equiv (1 + 2a_2 + (a_2 + a_2^2)u_1^3)u \text{ modulo } (4, 2u_1, u_1^4),$$
$$\gamma(u_1) \equiv u_1 \text{ modulo } (4, 2u_1, u_1^4).$$

It follows that

$$\begin{aligned}\gamma(\Delta) &\equiv 27[1+2a_2+(a_2+a_2^2)u_1^3]^{-12}u^{-12}(u_1^3-1) \text{ modulo}(4,2u_1,u_1^4)\\ &\equiv 27u^{-12}(u_1^3-1)=\Delta \text{ modulo}(4,2u_1,u_1^4).\end{aligned}$$

Next, consider $\gamma\in\mathbb{S}_C$. Notice that

$$G_{24}/(G_{24}\cap F_{1/2}\mathbb{S}_C)\cong\mathbb{S}_C/F_{1/2}\mathbb{S}_C$$

and that

$$(F_{1/2}\mathbb{S}_C\cap G_{24})/(G_{24}\cap F_{2/2}\mathbb{S}_C)\cong F_{1/2}\mathbb{S}_C/F_{2/2}\mathbb{S}_C.$$

As a consequence, if $\gamma\in\mathbb{S}_C$, then there exists $g\in G_{24}$ such that $\gamma g\in F_{2/2}\mathbb{S}_C$, and so

$$\gamma(\Delta)=\gamma g g^{-1}(\Delta)=\gamma g(\Delta)\equiv\Delta \text{ modulo}(4,2u_1,u_1^4).$$

where the second equality is because Δ is G_{24}-invariant. □

1.6 Topological finite resolutions

Finite resolutions in the sense of [**Hen07**] have proved to be key tools for $K(2)$-local calculations at $p=2$ and $p=3$, see [**HKM13**], [**GHMR15**], [**GH16**], [**BGH17**], where finite resolutions are used in an essential way. Let us begin by recalling the definition from [**Hen07**]:

Definition 1.6.1. Let X be a spectrum and

$$X_0\xrightarrow{d_1}X_1\to\dots\xrightarrow{d_n}X_n \tag{I.21}$$

be a complex of maps, i.e., sequence of maps with

$$d_{i+1}\circ d_i\simeq 0,\ \forall\, 1\le i\le n-1.$$

This complex is said to be a finite resolution for X, with X on the bottom, if there is also a map $d_0: X:=X_{-1}\to X_0$ satisfying the condition that each d_i is decomposed as $X_{i-1}\xrightarrow{k_{2i-1}}C_i\xrightarrow{k_{2i}}X_i$ such that

i. $C_i\xrightarrow{k_{2i}}X_i\xrightarrow{k_{2i+1}}C_{i+1}$ is a cofibration
ii. k_{-1} and k_{2n} are equivalences.

Dually, this complex is a resolution for X, with X on the top, if there is also a map $d_{n+1}: X_n\to X_{n+1}:=X$ satisfying the condition that each d_i is decomposed as $X_{i-1}\xrightarrow{k_{2i-1}}C_i\xrightarrow{k_{2i}}X_i$ such that

i. $C_i\xrightarrow{k_{2i}}X_i\xrightarrow{k_{2i+1}}C_{i+1}$ is a cofibration
ii. k_1 and k_{2n+2} are equivalences.

Remark 1.6.2. If the sequence (I.21) is a resolution for X, with X on the bottom, it can be refined to a tower of cofibrations with X on the bottom

$$\begin{array}{ccccccc} X & \longleftarrow & \Sigma^{-1}C_1 & \longleftarrow & \Sigma^{-2}C_2 & \longleftarrow \dots \longleftarrow & \Sigma^{-n}C_n \\ \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle \Sigma^{-1}k_2} & & \downarrow{\scriptstyle \Sigma^{-2}k_4} & & \cong\downarrow{\scriptstyle \Sigma^{-n}k_{2n}} \\ X_0 & & \Sigma^{-1}X_1 & & \Sigma^{-2}X_2 & & \Sigma^{-n}X_n \end{array}$$

If it is a resolution for X, with X on the top, it can be refined to a tower of cofibrations, with X on the top

$$\begin{array}{ccccccc} X & \longrightarrow & \Sigma C_n & \longrightarrow & \Sigma^2 C_{n-1} & \longrightarrow \dots \longrightarrow & \Sigma^n C_1 \\ {\scriptstyle d_n}\uparrow & & {\scriptstyle k_{2n-1}}\uparrow & & {\scriptstyle \Sigma^2 k_{2n-3}}\uparrow & & {\scriptstyle \Sigma^n k_1}\uparrow\cong \\ X_n & & \Sigma^1 X_{n-1} & & \Sigma^2 X_{n-2} & & \Sigma^n X_0. \end{array}$$

Applied to the $K(n)$-local homotopy category, one wants to resolve the $K(n)$-local sphere and homotopy fixed point spectra of E with respect to closed subgroup of $\mathbb{G}$ by higher real K-theories. The typical example is the case $n = 1$ and $p = 2$. Recall from *Example* 2 that, in this case, $E(\mathbb{F}_2, \mathbb{G}_m)^{hC_2} \simeq KO\mathbb{Z}_2$. Then due to Adams-Baird [**Bou79**] and Ravenel [**Rav84**], there is a cofiber sequence:

$$L_{K(1)}S^0 \to E(\mathbb{F}_2, \mathbb{G}_m)^{hC_2} \to E(\mathbb{F}_2, \mathbb{G}_m)^{hC_2}.$$

At height 2 and prime 2, the construction of a finite resolution, known as the topological duality resolution, plays an important role in recent progress towards understanding the $K(2)$-local category at the prime 2; for example, in the chromatic splitting conjecture, see [**BGH17**]. Now we review the construction of the topological duality resolution for $E_C^{h\mathbb{S}_C^1}$.

Topological duality resolution for $E_C^{h\mathbb{S}_2^1}$. The construction of the topological duality resolution starts from an algebraic version. In [**Bea17**], Beaudry established a resolution of the trivial $\mathbb{Z}_2[[\mathbb{S}_2^1]]$-module $\mathbb{Z}_2$ by permutation modules. There is an exact sequence of $\mathbb{Z}_2[[\mathbb{S}_2^1]]$-modules.

$$\mathbb{Z}_2 \xleftarrow{\epsilon} \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \xleftarrow{\partial_1} \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \xleftarrow{\partial_2} \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \xleftarrow{\partial_3} \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}']] \leftarrow 0 \tag{I.22}$$

where ϵ is the augmentation sending each coset of $\mathbb{S}_2^1/G_{24}$ to 1. The other maps are given by

- $\partial_1 = 1 - \alpha$
- $\partial_2 = 1 + \alpha$ modulo $(2, (IS_C^1)^2)$
- $\partial_3 = \pi(1 + i + j + k)(1 - \alpha^{-1})\pi^{-1}$,

where α is a certain element of $\mathbb{S}_C^1$ whose determinant is equal to -1 and $S_C^1 = \mathbb{S}_C^1 \cap F_{1/2}\mathbb{S}_C$.

Remark 1.6.3. Rigorously speaking, the maps ∂_i are induced by multiplication by the respective elements of $\mathbb{Z}_2[[\mathbb{S}_2^1]]$ given above. While the formulae of ∂_1 and ∂_3 are explicitly given, an explicit formula for ∂_2 is only known modulo an ideal of $\mathbb{Z}_2[[\mathbb{S}_2^1]]$, see Chapter V for more details. Beaudry also worked out a better approximation of ∂_2, but the formula above suffices for our work.

By tensoring this resolution with $\mathbb{Z}_2[[\mathbb{G}_2]] \otimes_{\mathbb{Z}_2[[\mathbb{S}_2^1]]} -$, one obtains a resolution of the $\mathbb{Z}_2[[\mathbb{G}_2]]$-module $\mathbb{Z}_2[[\mathbb{G}_2/\mathbb{S}_2^1]]$

$$0 \leftarrow \mathbb{Z}_2[[\mathbb{G}_2/\mathbb{S}_2^1]] \xleftarrow{\epsilon} \mathbb{Z}_2[[\mathbb{G}_2/G_{24}]] \xleftarrow{\partial_1} \mathbb{Z}_2[[\mathbb{G}_2/C_6]] \xleftarrow{\partial_2} \mathbb{Z}_2[[\mathbb{G}_2/C_6]] \\ \xleftarrow{\partial_3} \mathbb{Z}_2[[\mathbb{G}_2/G_{24}]] \leftarrow 0$$

The last term is replaced by $\mathbb{Z}_2[[\mathbb{G}_2/G_{24}]]$ using the isomorphism of $\mathbb{G}_2$-modules induced by the multiplication with π^{-1}: $\mathbb{Z}_2[[\mathbb{G}_2/G_{24}]] \xrightarrow{\pi^{-1}} \mathbb{Z}_2[[\mathbb{G}_2/G_{24}']]$, and so ∂_3 becomes $(1 \mapsto (1+i+j+k)(1-\alpha^{-1})\pi^{-1})$. The authors of [**BG18**] showed that this resolution can be topologically realized as a resolution of $E^{h\mathbb{S}_2^1}$.

Theorem 1.6.4. *The following is a resolution of* $E^{h\mathbb{S}_2^1}$

$$E_C^{h\mathbb{S}_2^1} \xrightarrow{\delta_0} E_C^{hG_{24}} \xrightarrow{\delta_1} E_C^{hC_6} \xrightarrow{\delta_2} E_C^{hC_6} \xrightarrow{\delta_3} \Sigma^{48} E^{hG_{24}} \tag{I.23}$$

where the maps δ_0, δ_1, δ_2 *are the lifts of* $\epsilon, \partial_1, \partial_2$, *respectively.*

We note that the topological duality resolution was first announced in [**Hen07**], where the last spectrum in the resolution was not identified. Bobkova and Goerss in [**BG18**] identified the last spectrum.

Remark 1.6.5. In Chapter V, we explain what is meant by saying that δ_0, δ_1, δ_2 are the lifts of $\epsilon, \partial_1, \partial_2$, respectively and analyse in more details these maps as well as δ_3.

1.7 Finite spectra

We recollect here some materials on finite spectra which are used in this thesis. These materials can be found in [**Rav92**] or in [**HS99**]. Recall that the coefficient rings of, $K(n)$, the n^{th} Morava K-theory at the prime p, is isomorphic to

$$K(n)_* \cong \mathbb{F}_{p^n}[v_n^{\pm 1}]$$

where $|v_n| = 2p^n - 2$. Let $\mathrm{FH}_{(\mathrm{p})}$ denote the category of p-local finite spectra.

Definition 1.7.1. Let p be a prime number. For any non-negative integer n, let $K(n)$ be the n^{th} Morava K-theory at p. A p-local finite spectrum X has type n if $K(n)_*(X)$ is nontrivial and $K(m)_*(X) = 0$ for $m \leq n$. A contractible finite spectrum has type ∞.

Remark 1.7.2. In fact, to see that X is of type n, it is sufficient to check that $K(n)_*(X)$ is nontrivial and $K(n-1)_*(X) = 0$ because $K(m)_*(X) = 0$ implies that $K(m-1)_*(X) = 0$, see [**Rav84**].

Definition 1.7.3. Let X be a finite spectrum. A self map $f : \Sigma^l X \to X$ of X is said to be a v_n-self map if $K(n)_*(f)$ is given by multiplication by v_n^k for some integer k and $K(m)_*(f)$ is trivial for $m \neq n$. In this case, f is denoted by v_n^k and k is called the periodicity of the v_n-self map.

Remark 1.7.4. It is clear from the definitions that the cofiber of a v_n-self map is a finite spectra of type $n+1$.

Definition 1.7.5. A full subcategory $\mathcal{C}$ of $\mathrm{FP}_{(p)}$ is thick if the three following conditions are satisfied

(i) An object which is homotopy equivalent to an object of $\mathcal{C}$ is in $\mathcal{C}$.

(ii) If two out of three spectra in the cofibration $X \to Y \to Z$ are in $\mathcal{C}$, then the third is also in $\mathcal{C}$.

(iii) If $X \vee Y$ is in $\mathcal{C}$, then both X and Y are in $\mathcal{C}$.

Let $\mathcal{C}_n$ denote the full subcategory of finite spectra of type at least n, so that

$$\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \ldots \supseteq \mathcal{C}_\infty.$$

It is proved by Steve Mitchell in [**Mit85**] that each of these inclusions is proper. In [**HS98**], the authors show that a type n finite spectrum admits v_n-self maps and that if $\mathcal{C}$ is a thick subcategory of $\mathrm{FH}_{(p)}$, then $\mathcal{C} = \mathcal{C}_n$ for some $0 \leq n \leq \infty$.

Typical examples of finite spectra of type n are generalised Moore spectra. They are constructed by successively taking the cofiber of v_n-self maps starting from the sphere spectrum. More precisely, we have the following cofibrations

$$S^0 \xrightarrow{p^{i_0}} S^0 \to M(p^{i_0}),$$

$$\Sigma^{2i_1(p-1)} M(p^{i_0}) \xrightarrow{v_1^{i_1}} M(p^{i_0}) \to M(p^{i_0}, v_1^{i_1}),$$

$$\ldots$$

$$\Sigma^{2i_{n-1}(p^{n-1}-1)} M(p^{i_0}) \xrightarrow{v_{n-1}^{i_{n-1}}} M(p^{i_0}) \to M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}),$$

where $v_k^{i_k}$ is an appropriate v_k-self map of $M(p^{i_0}, v_1^{i_1}, ..., v_{k-1}^{i_{k-1}})$. Thus, for an appropriate n-tuple of integers $(i_0, i_1, ..., i_{n-1})$, the spectrum $M(p^{i_0}, v_1^{i_1}, ..., v_{n-1}^{i_{n-1}})$ constructed as above is called a generalised Moore spectrum. It is of practical importance to be able to construct generalised Moore spectra with the integers $i_0, i_1, ..., i_{n-1}$ as small as possible.

Example 3. (1) If $n = 1$, then $M(p^{i_0})$ exists for all positive integers i_0. When $i_0 = 1$, $M(p)$, also denoted by $V(0)$, is usually called the Moore spectrum.

(2) If $n = 2$ and p is odd, then there is a v_1-self map $v_1 : \Sigma^{2p-2}M(p) \to M(p)$ and $M(p, v_1)$ is also denoted by $V(1)$ and called the Toda-Smith complex. In constrast, when $p = 2$, the least exponent of a v_1-self map on $V(0)$ is 4, i.e., there is a v_1-self map $v_1^4 : \Sigma^8 V(0) \to V(0)$.

Fix a positive integer n. In [**HS99**], the authors construct, using results of Hopkins and Smith, a sequence of ideals $J(i)$ of $\mathbb{Z}_{(p)}[v_1, v_2, ..., v_{n-1}]$ for $i \geq 0$ and the associated generalised Moore spectrum $M(J(i))$ such that

1. $J(i+1) \subset J(i)$ and $\bigcap_i J(i) = 0$.
2. There are maps of spectra $M(J(i+1)) \to M(J(i))$ such that for any $K(n)$-local spectrum Y, there is a homotopy equivalence

$$Y \simeq \operatorname{holim}_i Y \wedge M(J(i)).$$

Such a tower of generalised Moore spectra is called a cofinal tower.

1.8 Gross-Hopkins duality

This notion of duality is a version of the Brown-Comenetz duality for the $K(n)$-local homotopy category. It was introduced by Gross and Hopkins in [**HG94**]. We will need to compute the Gross-Hopkins dual of some spectra in Chapter III, as an important step in the computation of a related homotopy fixed point spectral sequence. Brown-Comenetz duality was introduced in the 1970's to study the duality phenomena of spaces [**BC74**]. Because $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group, the functor

$$X \mapsto \operatorname{Hom}_{Ab}(\pi_0 X, \mathbb{Q}/\mathbb{Z})$$

is cohomological, so is represented by a spectrum, denoted by $I_{\mathbb{Q}/\mathbb{Z}}$. The Brown-Comenetz dual of a spectrum X is defined to be the function spectrum from X to $I_{\mathbb{Q}/\mathbb{Z}}$

$$I_{\mathbb{Q}/\mathbb{Z}}X := F(X, I_{\mathbb{Q}/\mathbb{Z}}),$$

so that

$$\pi_*(I_{\mathbb{Q}/\mathbb{Z}}X) \cong \operatorname{Hom}_{Ab}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z}).$$

Through the lens of chromatic stable homotopy theory, the spectrum $I_{\mathbb{Q}/\mathbb{Z}}$ can be approximated by the Gross-Hopkins dual (see [**HG94**]). Fix a prime number p and a chromatic height n. For each $n \geq 1$, there is a natural transformation $L_n \to L_{n-1}$. The fiber M_nX of $L_nX \to L_{n-1}X$ is referred to as the n^{th} monochromatic layer. The Gross-Hopkins dual of X is defined to be the Brown-Comenetz dual of M_nX, i.e.,

$$I_nX := F(M_nX, I_{\mathbb{Q}/\mathbb{Z}}).$$

We abbreviate I_nS^0 by I_n. Work of Gross-Hopkins [**HG94**] relates this duality to the Grothendieck-Serre duality on the Lubin-Tate space of the universal deformation of a height n formal group law. As a result, there exists a spectrum P_n representing an element in the exotic Picard group of $Sp_{K(n)}$ such that

$$I_n \cong \Sigma^{n^2-n} P_n \wedge S^0\langle det \rangle$$

where $S^0\langle det \rangle$ is the determinant sphere, see [**BBGS18**] for a construction of the latter. It is then of particular importance to determine, or at least characterise, the homotopy type of P_n.

2 The Davis-Mahowald spectral sequence

We introduce a generalisation of the Davis-Mahowald spectral sequence, which is an useful tool, in this thesis, for analysing Ext-groups over various Hopf algebras. Initially, this spectral sequence was used by Davis and Mahowald in [**DM82**] to compute Ext-groups over the subalgebra $\mathcal{A}(2)$ of the Steenrod algebra.

2.1 Construction of the Davis-Mahowald spectral sequence

Let k be a field of characteristic 2. We will later specialise to the case $k = \mathbb{F}_2$, the field of two elements. Let $(A, \Delta, \mu, \epsilon, \eta, \chi)$ be a commutative Hopf algebra over k with $\Delta, \mu, \epsilon, \eta, \chi$ being coproduct, product, counit, unit, the conjugation, respectively.

Definition 2.1.1. Let E be the graded exterior algebra on a finite dimensional k-vector space V with all elements of V having degree 1. An A-comodule algebra structure on E is called almost graded if the natural embedding $k \oplus V \to E$ is a map of A-comodules.

This definition is motivated by the following examples which are of main interest in this thesis. Recall that the Steenrod algebra $\mathcal{A}$ is generated by the Steenrod squares Sq^i for $i \geq 0$, subject to the Adem relations

$$Sq^a Sq^b = \sum_{i=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i$$

for all $a, b > 0$ and $a < 2b$. Let $\mathcal{A}_*$ denote the dual of the Steenrod algebra. In [**Mil58**], Milnor determines the Hopf algebra structure of $\mathcal{A}_*$. As a graded algebra, $\mathcal{A}_* = \mathbb{F}_2[\xi_i | i \geq 1]$ where ξ_i is in degree $|\xi_i| = 2^i - 1$. The coproduct is given by the formula

$$\Delta(\xi_k) = \sum_{i=0}^{k} \xi_i^{2^{k-i}} \otimes \xi_{k-i},$$

where $\xi_0 = 1$. Let us denote by ζ_i the conjugate ζ_i of ξ_i. Then we have

$$\Delta(\zeta_k) = \sum_{i+j=k} \zeta_i \otimes \zeta_j^{2^i}. \tag{I.24}$$

An Hopf ideal of a Hopf algebra A is an ideal I such that $\Delta(I) \subset I \otimes A + A \otimes I$. If I is a Hopf ideal of A, then A/I inherits a structure of Hopf algebra from A such that the natural projection $A \to A/I$ is a map of Hopf algebras.

Example 4. Let $\mathcal{A}(n)_*$ be the quotient of $\mathcal{A}_*$ by the Hopf ideal I_n generated by $(\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, ..., \zeta_{n+1}^2, \zeta_{n+2}, ...)$. As an algebra,

$$\mathcal{A}(n)_* = \mathbb{F}_2[\zeta_1, \zeta_2, ..., \zeta_{n+1}]/(\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, ..., \zeta_{n+1}^2).$$

It is dual to the subalgebra $\mathcal{A}(n) = \langle Sq^1, Sq^2, ..., Sq^{2^n} \rangle$ of the Steenrod algebra A. The canonical projection $\pi : \mathcal{A}(n)_* \to \mathcal{A}(n-1)_*$ induced by the inclusion $I_n \subset I_{n-1}$ of Hopf ideals is a map of Hopf algebras, hence induces on $\mathcal{A}(n)_*$ a structure of right $\mathcal{A}(n-1)_*$-comodule algebra:

$$(id \otimes \pi)\Delta : \mathcal{A}(n)_* \to \mathcal{A}(n)_* \otimes \mathcal{A}(n)_* \to \mathcal{A}(n)_* \otimes \mathcal{A}(n-1)_*.$$

An easy computation shows that the group of primitives $\mathcal{A}(n)_* \square_{\mathcal{A}(n-1)_*} \mathbb{F}_2$ of this coaction is given by

$$\mathcal{A}(n)_* \square_{\mathcal{A}(n-1)_*} \mathbb{F}_2 = E(\zeta_1^{2^n}, \zeta_2^{2^{n-1}}, ..., \zeta_{n+1})$$

which is abstractly isomorphic to $E_n = E(x_1, ..., x_{n+1})$ where x_i stands for $\zeta_i^{2^{n+1-i}}$. Here and elsewhere in this paper, $E(X)$ denotes the exterior algebra on the k-vector space spanned by the set X. We see that the algebra $E(x_1, x_2, ..., x_{n+1})$ inherits a left $\mathcal{A}(n)_*$-comodule algebra structure from $\mathcal{A}(n)_*$, namely,

$$\Delta(x_k) = \sum_{i=0}^{k} \zeta_i^{2^{n+1-k}} \otimes x_{k-i},\ 1 \leq k \leq n+1$$

where $x_0 = 1$ by convention. This means that E_n is an almost graded $\mathcal{A}(n)_*$-comodule.

Example 5. Let $B(n)_*$ be the quotient of $\mathcal{A}_*$ by the Hopf ideal J_n generated by $(\zeta_1^{2^n}, \zeta_2^{2^n}, \zeta_3^{2^{n-1}}, ..., \zeta_{n+1}^2, \zeta_{n+2}, ...)$, so that

$$B(n)_* = \mathbb{F}_2[\zeta_1, \zeta_2, ..., \zeta_{n+1}]/(\zeta_1^{2^n}, \zeta_2^{2^n}, \zeta_3^{2^{n-1}}, ..., \zeta_{n+1}^2).$$

Similarly to *Example* 4, the projection $B(n)_* \to \mathcal{A}(n-1)_*$ induced by the inclusion of Hopf ideals $J_n \subset I_{n-1}$ defines a structure of right $\mathcal{A}(n-1)_*$-comodule algebra on $B(n)_*$. A calculation shows that

$$B(n)_* \square_{\mathcal{A}(n-1)_*} \mathbb{F}_2 = E(\zeta_2^{2^{n-1}}, \zeta_3^{2^{n-2}}, ..., \zeta_{n+1}),$$

which is abstractly isomorphic to $F_n := E(x_2, ..., x_{n+1})$. The notation is chosen to be coherent with that of *Example* 4. We see that F_n inherits a structure of left $B(n)_*$- comodule algebra from that of $B(n)_*$, namely,

$$\Delta(x_k) = \sum_{i=0, i\neq 1}^{k} \zeta_i^{2^{n+1-k}} \otimes x_{k-i}, \; 2 \leq k \leq n+1$$

where $x_0 = 1$. Thus, F_n is a almost graded $B(n)_*$-comodule.

Let E be an almost graded A-comodule exterior algebra on a finite dimensional k-vector space V. We will construct an A-comodule polynomial algebra, called the Koszul dual of E as follows. Let P be the graded polynomial algebra of V with all elements of V having degree 1. Let us denote by E_i and P_i the subspace of elements of homogeneous degree i for $i \geq 0$ of E and P, respectively. Let us also denote by $E_{\leq i}$ the direct sum $\bigoplus_{j=0}^{j=i} E_j$. Notice that P_1 sits in a short exact sequence:

$$0 \to k \to k \oplus E_1 \xrightarrow{p} P_1 \to 0. \tag{I.25}$$

The embedding $k \to k \oplus E_1$ is clearly a map of left A-comodules. Thus P_1 admits a (unique) structure of left A-comodule such that $p : k \oplus E_1 \to P_1$ is a map of A-comodules.

Lemma 2.1.2. *If $P_1^{\otimes n}$ is equipped with the usual structure of A-comodule of a tensor product. Then P_n admits a unique structure of A-comodule making the multiplication $P_1^{\otimes n} \to P_n$ a map of A-comodules.*

Proof. This map is surjective and its kernel is spanned by elements of the form $y_1 \otimes ... \otimes y_n - y_{\sigma(1)} \otimes ... \otimes y_{\sigma(n)*}$ where σ is a permutation of the set $\{1, 2, ..., n\}$. Then, since A is commutative, we see that the kernel is stable under the coaction of A. The lemma follows. □

This lemma shows that $P = \bigoplus_{i\geq 0} P_i$ admits a left A-comodule algebra structure.

Now, let us define a cochain complex, called the Koszul complex,

$$(E \otimes P, d) \tag{I.26}$$

with

i) $(E \otimes P)_{-1} = k$

ii) $(E \otimes P)_m = E \otimes P_m$ for $m \geq 0$

iii) $d : k = (E \otimes P)_{-1} \to E = (E \otimes P)_0$ being the unit of E

iv) $d(\prod_{j=1}^{n} x_{i_j} \otimes z) = \sum_{t=1}^{n} \prod_{j\neq t} x_{i_j} \otimes p(x_{i_t})z$ where $x_{i_j} \in E_1$, $z \in P_m$ and p is the projection of (I.25).

Remark 2.1.3. In other words, $d : E_{\leq n} \otimes P_m \to E_{\leq n-1} \otimes P_{m+1}$ is the unique homomorphism making the following diagram commute

$$\begin{array}{ccc} E_{\leq 1}^{\otimes n} \otimes P_{\leq m} & \xrightarrow{(\sum_{\sigma}(Id^{\otimes(n-1)}\otimes p)\circ\sigma)\otimes Id} & E_{\leq 1}^{\otimes(n-1)} \otimes P_1 \otimes P_m \\ \Big\downarrow{\scriptstyle \mu\otimes Id} & & \Big\downarrow{\scriptstyle \mu\otimes\mu} \\ E_{\leq n} \otimes P_m & \xrightarrow{\quad d \quad} & E_{\leq n-1} \otimes P_{m+1}, \end{array} \tag{I.27}$$

where in the upper horizontal map, the sum is taken over all cyclic permutations on n factors of E_1 in the tensor product $E_1^{\otimes n}$ and p is the restriction on E_1 of the map of $(I.25)$.

Proposition 2.1.4. *The complex $(E \otimes P, d)$ is an exact sequence of A-comodules. Furthermore, $(E \otimes P, d)$ has a structure of differential graded algebra induced from the algebra structure of E and P.*

Proof. Let $x_1, ..., x_n$ be a basis of E_1. As a cochain complex over k, $(E \otimes P, d)$ is isomorphic to the tensor product of $(E(x_i) \otimes k[y_i], d_i)$ where $y_i = p(x_i)$ for $1 \leq i \leq n$. Here, each $(E(x_i) \otimes k[y_i], d_i)$ is defined in the same manner as $(E \otimes P, d)$ is. It is not hard to see that the cochain complex $(E(x_i) \otimes k[y_i], d_i)$ is exact. Hence, $(E \otimes P, d)$ is exact by the Künneth theorem. This proves the first part.

Let us check that d is a map of A-comodules. In the diagram (I.27), the two vertical maps are ones of A-comodules because E and P are A-comodule algebras. In addition, they are surjective. It remains to check that the upper horizontal map is a map of A-comodules. Or equivalently, each map $E_{\leq 1}^{\otimes n} \xrightarrow{(Id^{\otimes(n-1)}\otimes p)\circ\sigma} E_{\leq 1}^{\otimes(n-1)} \otimes P_1$ is a map of A-comodules where σ is a cyclic permutation on n elements. This is true because σ is a map of A-comodules as A is commutative and p is a map of A-comodules by definition. The second part follows.

Finally, it is straightforward from the formula of d in (I.26.iv) that d satisfies the Leibniz rule. □

This lemma allows us to construct a spectral sequence of algebras converging to $\mathrm{Ext}_A^s(k)$ see ([**Rav86**], Theorem A1.3.2).

Proposition 2.1.5. *(1) There is a spectral sequence of algebras converging to* $\mathrm{Ext}_A^s(k)$ *:*

$$\mathrm{E}_1^{s,t} = \mathrm{Ext}_A^s(k, E \otimes P_t) \Longrightarrow \mathrm{Ext}_A^{s+t}(k, k)\ . \tag{I.28}$$

(2) If M is a A-comodule, then there is a spectral sequence converging to $\mathrm{Ext}_A^s(M)$

$$\mathrm{E}_1^{s,t} = \mathrm{Ext}_A^s(k, E \otimes P_t \otimes M) \Longrightarrow \mathrm{Ext}_A^{s+t}(k, M).$$

Furthermore, this spectral sequence is a spectral sequence of modules over that of (I.28).

Terminology. We will call these spectral sequences the Davis-Mahowald spectral sequences or DMSS for short, associated to the almost graded A-module algebra E. The first grading s of the E_n-term is referred to as the cohomological grading or degree and the second grading t is referred to as the Davis-Mahowald grading or degree (or DM grading or degree for short).

In view of carrying out explicit computations of products in $\mathrm{Ext}_A^*(k)$ and the action of $\mathrm{Ext}_A^*(k)$ on $\mathrm{Ext}_A^*(M)$, we recall a double complex from which the above spectral sequence is derived.

For each $t \geq 0$, let $(C^s(A, E \otimes P_t), d_v)_{s \geq 0}$ be the cobar complex whose cohomology is $\mathrm{Ext}_A^*(E \otimes P_t)$, i.e.,

$$C^s(A, E \otimes P_t) = A^{\otimes s} \otimes E \otimes P_t$$

and $d_v : A^{\otimes s} \otimes E \otimes P_t \to A^{\otimes s+1} \otimes E \otimes P_t$ is given by

$$d_v(a_1 \otimes ... \otimes a_s \otimes m) = 1 \otimes a_1 \otimes ... \otimes a_s \otimes m + \sum_{i=1}^{s} a_1 \otimes ... \otimes a_{i-1} \otimes \Delta(a_i) \otimes ... \otimes a_s \otimes m \\ + a_1 \otimes ... \otimes a_s \otimes \Delta(m),$$

where $a_i \in A$ for $1 \leq i \leq s$ and $m \in E \otimes P_t$.

Notation. We will shorten $m_1 \otimes ... \otimes m_s \in M_1 \otimes ...M_s$ by $[m_1|...|m_s]$.

By an abuse of notation, we will denote by d_v the differentials in the cobar complexes associated to $E \otimes P_t$ for different t. The fact that $d : E \otimes P_t \to E \otimes P_{t+1}$ is a map of A-comodules implies that the maps $d_h = Id^{\otimes s} \otimes d : C^s(A, E \otimes P_t) \to C^s(A, E \otimes P_{t+1})$ assemble to give a map of cochain complexes $d_h : (C^s(A, E \otimes P_t), d_v)_{s\geq 0} \to (C^s(A, E \otimes P_{t+1}), d_v)_{s\geq 0}$. Finally, it is easily seen that the maps of cochain complexes assemble to form a double complex $(C^s(A, E \otimes P_t), d_v, d_h)_{s,t\geq 0}$

$$\begin{array}{ccccccccc}
E & \xrightarrow{d_h} & E \otimes P_1 & \xrightarrow{d_h} & E \otimes P_2 & \xrightarrow{d_h} & E \otimes P_3 & \xrightarrow{d_h} & \dots \\
\downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \\
A \otimes E & \xrightarrow{d_h} & A \otimes E \otimes P_1 & \xrightarrow{d_h} & A \otimes E \otimes P_2 & \xrightarrow{d_h} & A \otimes E \otimes P_3 & \xrightarrow{d_h} & \dots \\
\downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \\
A^{\otimes 2} \otimes E & \xrightarrow{d_h} & A^{\otimes 2} \otimes E \otimes P_1 & \xrightarrow{d_h} & A^{\otimes 2} \otimes E \otimes P_2 & \xrightarrow{d_h} & A^{\otimes 2} \otimes E \otimes P_3 & \xrightarrow{d_h} & \dots \\
\downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \\
\dots & & \dots & & \dots & & \dots & & \dots
\end{array}$$

We can see that the spectral sequence associated to the horizontal filtration has E_1-term isomorphic to $(A^s \otimes k, d_v)_{s\geq 0}$ which identifies with the cobar complex of the trivial A-comodule k. Thus this spectral sequence degenerates at the E_2-term and the $\mathrm{E}_\infty = \mathrm{E}_2$-term identifies with $\mathrm{Ext}^s_A(k)$. Since there are no possible extension problems, the cohomology of the total complex is isomorphic to $\mathrm{Ext}^s_A(k)$. Now, the spectral sequence associated to the vertical filtration has E_1-term isomorphic to $\mathrm{Ext}^s_A(E \otimes P_t)$. This spectral sequence is exactly the one appearing in Proposition 2.1.5.

Remark 2.1.6. The differential $d_1 : \mathrm{Ext}^0_A(E \otimes P_t) \to \mathrm{Ext}^0_A(E \otimes P_{t+1})$ is the restriction of the derivation d in (I.26) on the A-primitives of $E \otimes P_t$.

2.2 Naturality of the Davis-Mahowald spectral sequence

We notice that the above construction is natural in pairs (A, E) where A is a commutative Hopf algebra and E is an almost graded left A-comodule exterior algebra. This allows us to compare Davis-Mahowald spectral sequences associated to different pairs (A, E). We will make use of this property to reduce computations in a crucial way. Let us first define morphisms between such pairs.

Definition 2.2.1. Let (A, E) and (B, F) be such that A and B are commutative Hopf algebras, E and F are almost graded exterior comodule algebras over A and

B, respectively. A morphism between (A, E) and (B, F) consists of $f_1 : A \to B$ and $f_2 : E \to F$ where f_1 is a map of Hopf algebras and f_2 is a map of B-comodule graded algebras with the B-comodule structure on E being induced from f_1.

Remark 2.2.2. The map $f_2 : E \to F$ is determined by a map of B-comodules $k \oplus E_1 \to k \oplus F_1$.

Proposition 2.2.3. *A morphism between (A, E) and (B, F) induces a map between the associated Davis-Mahowald spectral sequences.*

Proof. Let P and Q be the Koszul dual of E and F, respectively. The map of B-comodule algebras $f_2 : E \to F$ induces a map of graded B-comodule algebras $P \to Q$ such that the following diagram is commutative

$$\begin{array}{ccc} k \oplus E_1 & \xrightarrow{p} & P_1 \\ \downarrow{\scriptstyle f_2} & & \downarrow \\ k \oplus F_1 & \xrightarrow{p} & Q_1. \end{array}$$

Then one can check that the induced map $E \otimes P \to F \otimes Q$ is a map of Koszul complexes. Therefore one obtains a map of double complexes $(A^{\otimes s} \otimes E \otimes P_t) \to (B^{\otimes s} \otimes F \otimes Q_t)$, hence a map of spectral sequences. □

Remark 2.2.4. Although we have only treated the ungraded situation so far, the construction carries over verbatim to the graded one. More precisely, suppose that A and E are graded algebras. We refer to this grading as the internal degree. We require that the structural maps in the A-comodule structure of E to preserve the internal degree. Then we see that the Koszul dual P of E is also internally graded and the Koszul complex is a graded cochain complex with respect to the internal degree. It follows that the associated DMSS is tri-graded with the third grading associated to the internal grading and the differentials preserve the internal degree.

We continue with *Example* 4 and 5.

Example 6. Recall that E_n is an almost graded $\mathcal{A}(n)_*$-comodule. Let R_n denote the Koszul dual of E_n. In particular, it follows from Proposition 2.1.5 that for any graded left $\mathcal{A}(n)_*$-comodule M, the DMSS converging to $\mathrm{Ext}^{*,*}_{\mathcal{A}(n)_*}(\mathbb{F}_2, M)$ has E_1-term isomorphic to

$$\mathrm{E}_1^{s,t,\sigma} \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(E_n \otimes R_n^{\sigma} \otimes M),$$

where s is the cohomological grading, t is the internal grading and σ is the Davis-Mahowald grading. The change-of-rings isomorphism tells us that

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(E_n \otimes R_n^{\sigma} \otimes M) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(n-1)_*}(R_n^{\sigma} \otimes M),$$

see [**Rav84**], Appendix A1.3.13 for the change-of-rings isomorphism. That means that the problem of computing $\mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(-)$ can be reduced to two steps: first computing $\mathrm{Ext}^{s,t}_{\mathcal{A}(n-1)_*}(-)$, then studying the corresponding Davis-Mahowald spectral sequence. We will demonstrate the efficiency of this method by carrying out explicit computations in the case $n = 2$ and some relevant M.

Example 7. Recall that F_n is an almost graded $B(n)_*$-comodule. Let S_n denote the Koszul dual of F_n. The DMSS is the spectral sequence of algebras

$$\mathrm{E}_1^{s,t,\sigma} = \mathrm{Ext}^{s,t}_{B(n)_*}(F_n \otimes S_n^\sigma) \Longrightarrow \mathrm{Ext}^{s+\sigma,t}_{B(n)_*}(\mathbb{F}_2)\ .$$

By the change-of-rings theorem, the E_1-term is isomorphic to $\mathrm{Ext}^{s,t}_{\mathcal{A}(n-1)_*}(S_n^\sigma)$, because $F_n = B(n)_*\square_{\mathcal{A}(n-1)_*}\mathbb{F}_2$. Moreover, for any graded left $B(n)_*$-comodule M, the DMSS for $\mathrm{Ext}^{s+\sigma,t}_{B(n)_*}(\mathbb{F}_2)$ is a spectral sequence of modules over the above spectral sequence

$$\mathrm{Ext}^{s,t}_{B(n)_*}(F_n \otimes S_n^\sigma \otimes M) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(n-1)_*}(S_n^\sigma \otimes M) \Longrightarrow \mathrm{Ext}^{s+\sigma,t}_{B(n)_*}(\mathbb{F}_2)\ .$$

Comparison of DMSS. There is a morphism between $(\mathcal{A}(n)_*, E_n)$ and $(B(n)_*, F_n)$ given by the two projections

$$\mathcal{A}(n)_* \to B(n)_*; \zeta_i \mapsto \zeta_i$$

$$E_n \to F_n; x_1 \mapsto 0, x_i \mapsto x_i \text{ for } i \geq 2.$$

This induces a map of spectral sequences

$$\begin{array}{ccc}
\mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(E_n \otimes R_n^\sigma \otimes M) & \longrightarrow & \mathrm{Ext}^{s,t}_{B(n)_*}(F_n \otimes S_n^\sigma \otimes M) \\
\Downarrow & & \Downarrow \\
\mathrm{Ext}^{s+\sigma,t}_{\mathcal{A}(n)_*}(M) & \longrightarrow & \mathrm{Ext}^{s+\sigma,t}_{B(n)_*}(M).
\end{array}$$

As was mentioned earlier, this comparison allows us to transfer some computations in the former SS to the latter which are simpler because all modules involved in the latter are smaller. This observation will be made concrete in Section 3.

Chapter II

Homotopy groups of $E_C^{hG_{24}} \wedge A_1$

In this chapter we give a detailed computation of the homotopy fixed point spectral sequence - which is abbreviated by HFPSS, for $E_C^{hG_{24}} \wedge A_1$. One of the key step to this end is a comparison between $tmf \wedge A_1$ and $E_C^{hG_{24}} \wedge A_1$. In fact, we prove that there is a homotopy equivalence (Theorem 5.1.1):

$$(\Delta^8)^{-1} tmf \wedge A_1 \cong (E_C^{hG_{24}})^{h\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \wedge A_1$$

where Δ^8 is the periodicity generator of $\pi_* tmf$. Thus we first analyse the homotopy groups of $tmf \wedge A_1$, then invert Δ^8 to get information about the homotopy groups of $E_C^{hG_{24}} \wedge A_1$. The homotopy groups of $tmf \wedge A_1$ are accessible through the classical Adams spectral sequence - which is also abbreviated by ASS,

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1)) \Longrightarrow \pi_{t-s}(tmf \wedge A_1).$$

We notice that in [**BEM17**], Batacharya, Egger, Mahowald briefly discussed this Adams spectral sequence. Our approach is however different and contains more details - we give an explicit description of the E_2-term of the Adams spectral sequence using the Davis-Mahowald spectral sequence and determine some differentials (compare [**BEM17**]).

In Section 3, we recollect certain information of the Davis-Mahowald spectral sequence for the $\mathcal{A}(2)$-comodule $\mathbb{F}_2$. Then we come to discuss the Davis-Mahowald spectral sequence for A_1 and obtain the E_2-term of the Adams spectral sequence. We compute two products one of which is exotic, i.e., the one that is not be detected by a product in the E_∞-term of the Davis-Mahowald spectral sequence. These products allow us to determine some differentials in the Adams spectral sequence for A_1. In Section 4, we discuss some differentials in the later and then extract some suitable information about $\pi_*(tmf \wedge A_1)$. In Section 5, we finally study the homotopy fixed point spectral sequence for $E_C^{hG_{24}} \wedge A_1$.

3 The Davis-Mahowald spectral sequence for the $\mathcal{A}(2)_*$-comodule A_1

The goal of this section is to describe the structure of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(A_1)$ as a module over $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ for different $\mathcal{A}(2)_*$-comodules A_1 that will be recalled in Subsection 3.2. To achieve a part of this goal, we will study the DMSS

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(E_2 \otimes S_2^\sigma \otimes A_1) \Longrightarrow \mathrm{Ext}^{s+\sigma,t}_{\mathcal{A}(2)_*}(A_1)$$

as a spectral sequence of modules over the spectral sequence of algebras

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(E_2 \otimes S_2^\sigma) \Longrightarrow \mathrm{Ext}^{s+\sigma,t}_{\mathcal{A}(2)_*}(\mathbb{F}_2).$$

We obtain then the structure of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(A_1)$ as a graded abelian group and a partial action of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ on it. However, there is an important action of an element of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ on some elements of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(A_1)$ that cannot be seen at the E_1-term of the DMSS. One way of understanding these exotic products is to carry out computations at the level of double complexes: find representatives of the cohomological classes in question in the double complexes from which the DMSS is derived and carry out products at that level. It turns out that a brute-force attack is messy. Instead, computations are simplified drastically by comparing the DMSS associated to $(\mathcal{A}(2)_*, E_2)$ to that of $(B(2)_*, F_2)$:

$$\begin{array}{ccc} \mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(E_n \otimes R_2^\sigma \otimes A_1) & \longrightarrow & \mathrm{Ext}^{s,t}_{B(2)_*}(F_n \otimes S_2^\sigma \otimes (1)) \\ \Downarrow & & \Downarrow \\ \mathrm{Ext}^{s+\sigma,t}_{\mathcal{A}(2)_*}(A_1) & \longrightarrow & \mathrm{Ext}^{s+\sigma,t}_{B(2)_*}(A_1). \end{array}$$

3.1 Recollections on the Davis-Mahowald spectral sequence for the $\mathcal{A}(2)_*$-comodule $\mathbb{F}_2$

To fix notation, we recollect some information relevant for our purposes. This material was originally treated in [**DM82**] and reviewed in unpublished course notes of Rognes [**Rog12**]. As we will specialise to the case $n = 2$, we will simplify the notation by writing R, R_σ, S, S_σ for $R_2, R_2^\sigma, S_2, S_2^\sigma$ from *Example* 4 and 5, respectively.

Recall that R is a homogenous graded polynomial algebra on three generators, say y_1, y_2, y_3 and R_σ is its subspace of homogeneous elements of degree σ for $\sigma \geq 0$. Let us first explicitly give the coaction of $\mathcal{A}(2)_*$ on $R = \mathbb{F}_2[y_1, y_2, y_3]$ with $|y_1| = 4$, $|y_2| = 6$, $|y_3| = 7$. From *Example* 6, we have

$$\begin{aligned}\Delta(y_1) &= 1 \otimes y_1\\ \Delta(y_2) &= \xi_1^2 \otimes y_1 + 1 \otimes y_2\\ \Delta(y_3) &= \zeta_2 \otimes y_1 + \xi_1 \otimes y_2 + 1 \otimes y_3.\end{aligned}$$

By the change-of-rings theorem, the E_1-term of the DMSS for $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ is isomorphic to $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\bigoplus_{\sigma\geq 0} R_\sigma)$. The coaction of $\mathcal{A}(1)_*$ on R_1 is induced from that of $\mathcal{A}(2)_*$ and hence is given by

$$\begin{aligned}\Delta(y_1) &= 1 \otimes y_1\\ \Delta(y_2) &= \xi_1^2 \otimes y_1 + 1 \otimes y_2\\ \Delta(y_3) &= \zeta_2 \otimes y_1 + \xi_1 \otimes y_2 + 1 \otimes y_3.\end{aligned}$$

In particular, y_1, y_2^2, y_3^4 are $\mathcal{A}(1)_*$-primitives of R. Let R'_σ denote the $\mathcal{A}(1)_*$-subcomodule $\{y_1^i y_2^j y_3^k \in R_\sigma | k \leq 3\}$ of R_σ.

Lemma 3.1.1. *As an* $\mathcal{A}(1)_*$*-comodule,* R_σ *can be decomposed as*

$$R_\sigma \cong \bigoplus_{i\equiv\sigma(mod 4), i\leq\sigma} R'_i \otimes \mathbb{F}_2\{y_3^{\sigma-i}\}.$$

Therefore,

$$\bigoplus_{\sigma\geq 0} R_\sigma = (\bigoplus_{\sigma\geq 0} R'_\sigma) \otimes \mathbb{F}_2[y_3^4].$$

Proof. If one views $\mathbb{F}_2\{y_3^{\sigma-i}\}$ as a subvector space of $R_{\sigma-i}$, then the product of R produces an isomorphism of vector spaces

$$\bigoplus_{i\equiv\sigma(mod 4), i\leq\sigma} R'_i \otimes \mathbb{F}_2\{y_3^{\sigma-i}\} \xrightarrow{\cong} R_\sigma.$$

Since y_3^4 is a $\mathcal{A}(1)_*$-primitive of R_σ, this map is also a map of $\mathcal{A}(1)_*$-comodules. The lemma follows. □

Let us denote $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R'_\sigma)$ by G_σ, so that

$$\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R) \cong (\bigoplus_{\sigma\geq 0} G_\sigma) \otimes \mathbb{F}_2[v_2^4],$$

where $v_2^4 \in \mathrm{Ext}^{0,24}_{\mathcal{A}(1)_*}(R_4)$ represented by $y_3^4 \in R_4$. Determining the full multiplicative structure of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R)$ is quite involved. Instead, we will work modulo (v_2^4). This will suffice for us to obtain a set of algebra generators of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R)$. More precisely, since the product $R'_\sigma \otimes R'_\tau \to R_{\sigma+\tau}$ factorises through $R'_{\sigma+\tau} \oplus (R_{\sigma+\tau-4} \otimes \mathbb{F}_2\{y_3^4\})$, we obtain a map

$$G_\sigma \otimes G_\tau \to G_{\sigma+\tau} \oplus (G_{\sigma+\tau-4} \otimes \mathbb{F}_2\{v_2^4\}).$$

We will analyse the map $G_\sigma \otimes G_\tau \to G_{\sigma+\tau}$ which is the composite

$$G_\sigma \otimes G_\tau \to G_{\sigma+\tau} \oplus (G_{\sigma+\tau-4} \otimes \mathbb{F}_2\{v_2^4\}) \to G_{\sigma+\tau}$$

where the second map is the projection on the first factor.

In what follows, we compute G_i for $i \geq 0$ as modules over G_0. For this, we decompose R'_i into smaller pieces, compute the Ext groups over $\mathcal{A}(1)_*$ of these pieces, then determine G_i via long exact sequences. Next, we study the pairings

$$G_\sigma \otimes G_\tau \to G_{\sigma+\tau},$$

which allows us to determine a set of algebra generators of the E_1-term. Finally, we compute d_1-differentials on this set of algebra generators. We do not intend to describe completely the $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ but only a subalgebra in which we are interested.

Since y_1 is primitive, multiplication by y_1 induces injections of $\mathcal{A}(1)_*$-comodules

$$\Sigma^4 R'_\sigma \to R'_{\sigma+1}.$$

Lemma 3.1.2. *There are short exact sequences of $\mathcal{A}(1)_*$-comodules*

(a)

$$0 \to \mathrm{H}_*(\Sigma^{12} C_\eta) \to R'_2 \to \Sigma^8(\mathcal{A}(1)_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \to 0$$

where $\eta : S^1 \to S^0$ is the Hopf map and the map $\mathrm{H}_*(\Sigma^{12}C_\eta) \to R'_2$ sends the generators of $\mathrm{H}_{12}(\Sigma^{12}C_\eta)$ and $\mathrm{H}_{14}(\Sigma^{12}C_\eta)$ to y_2^2 and y_3^2, respectively.

(b)

$$0 \to \Sigma^4 R'_1 \to R'_2 \to \Sigma^{12} V_3 \to 0$$

where V_3 denotes $\mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2)$.

Proof. For part (a), the map $\Sigma^{12}\mathrm{H}_*(C_\eta) \to R'_2$ described in the statement of the Lemma 3.1.2 is a map of $\mathcal{A}(1)_*$-comodules. Its quotient is isomorphic to

$\mathbb{F}_2\{y_1^2, y_1y_2, y_1y_3, y_2y_3\}$ with the $\mathcal{A}(1)_*$-comodule structure given by

$$\begin{aligned}
\Delta(y_2y_3) &= 1 \otimes y_2y_3 + \xi_1^2 \otimes y_1y_3 + \xi_2 \otimes y_1y_2 + \zeta_2\xi_1^2 \otimes y_1^2 \\
\Delta(y_1y_3) &= 1 \otimes y_1y_3 + \xi_1 \otimes y_1y_2 + \zeta_2 \otimes y_1^2 \\
\Delta(y_1y_2) &= 1 \otimes y_1y_2 + \xi_1^2 \otimes y_1^2 \\
\Delta(y_1^2) &= 1 \otimes y_1^2.
\end{aligned}$$

We can check that this module is isomorphic to $\Sigma^8(\mathcal{A}(1)_*\Box_{\mathcal{A}(0)_*}\mathbb{F}_2)$ as $\mathcal{A}(1)_*$-comodules.

For part (b), the quotient of R_2' by $\Sigma^4 R_1'$ is isomorphic to $\mathbb{F}_2\{y_2^2, y_2y_3, y_3^2\}$ with $\mathcal{A}(1)_*$-comodule structure given by

$$\begin{aligned}
\Delta(y_2^2) &= 1 \otimes y_2^2 \\
\Delta(y_2y_3) &= \xi_1 \otimes y_2^2 + 1 \otimes y_2y_3 \\
\Delta(y_3^2) &= \xi_1^2 \otimes y_2^2 + 1 \otimes y_3^2.
\end{aligned}$$

One can check that this quotient is isomorphic to $\Sigma^{12}V_3$. $\square$

Lemma 3.1.3. *For every $\sigma \geq 3$, there is a short exact sequence of $\mathcal{A}(1)_*$-comodules*

$$0 \to \Sigma^4 R_{\sigma-1}' \xrightarrow{\times y_1} R_\sigma' \to \Sigma^{6\sigma}V_4 \to 0$$

where V_4 denotes $\mathrm{H}_*(V(0) \wedge C_\eta)$.

Remark 3.1.4. The spectrum $V(0) \wedge C_\eta$ is homotopy equivalent to Y, introduced in the Introduction (c.f Section 3.2 for a presentation of $\mathrm{H}^*(Y)$.)

Proof. The quotient of R_σ' by $\Sigma^4 R_{\sigma-1}'$ is isomorphic to $\mathbb{F}_2\{y_2^\sigma, y_2^{\sigma-1}y_3, y_2^{\sigma-2}y_3^2, y_2^{\sigma-3}y_3^3\}$ with $\mathcal{A}(1)_*$-comodule structure given by

$$\begin{aligned}
\Delta(y_2^\sigma) &= 1 \otimes y_2^\sigma \\
\Delta(y_2^{\sigma-1}y_3) &= \xi_1 \otimes y_2^\sigma + 1 \otimes y_2^{\sigma-1}y_3 \\
\Delta(y_2^{\sigma-2}y_3^2) &= \xi_1^2 \otimes y_2^\sigma + 1 \otimes y_2^{\sigma-2}y_3^2 \\
\Delta(y_2^{\sigma-3}y_3^3) &= \xi_1^3 \otimes y_2^\sigma + \xi_1^2 \otimes y_2^{\sigma-1}y_3 + \xi_1 \otimes y_2^{\sigma-2}y_3^2 + 1 \otimes y_2^{\sigma-3}y_3^3.
\end{aligned}$$

It can be easily seen that this quotient is isomorphic to $\Sigma^{6\sigma}V_4$. $\square$

Next we describe the Ext groups of some $\mathcal{A}(1)_*$-comodules as basic steps towards computing G_σ. These calculations are elementary and classical.

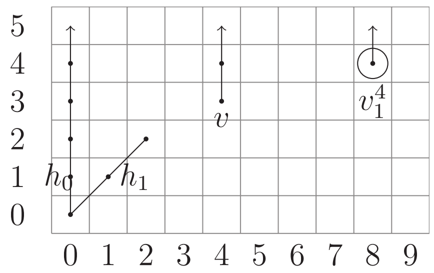

Figure II.1 – $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathbb{F}_2, \mathbb{F}_2)$ in the range $0 \leq t-s \leq 8$.

Proposition 3.1.5. *There are classes $h_0 \in \mathrm{Ext}^{1,1}$, $h_1 \in \mathrm{Ext}^{1,2}$, $v \in \mathrm{Ext}^{3,7}$, $v_1^4 \in \mathrm{Ext}^{4,12}$ such that there is an isomorphism of algebras*

$$G_0 := \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathbb{F}_2) \cong \mathbb{F}_2[h_0, h_1, v, v_1^4]/(h_1^3, h_0h_1, h_1v, v^2 - h_0^2v_1^4).$$

See for example ([**Rav86**], Theorem 3.1.25).

Lemma 3.1.6. *As a module over $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathbb{F}_2)$,*

(1) $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(V(0)))$ is generated by $h^0 \in \mathrm{Ext}^{0,0}$, $h^1 \in \mathrm{Ext}^{1,3}$ with the following relations $h_0h^0 = vh^0 = vh^1 = 0$ and $h_1^2.h^0 = h_0h^1$.

(2) $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(C_\eta))$ is generated by $\{h^i \in \mathrm{Ext}^{i,3i} | \; 0 \leq i \leq 3\}$ with $h_1h^i = 0$, $vh^0 = h_0h^2$, $vh^1 = h_0h^3$.

(3) $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2))$ is generated by $h^0 \in \mathrm{Ext}^{0,0}$, $h^1 \in \mathrm{Ext}^{1,3}$, $a^1 \in \mathrm{Ext}^{1,3}$, $h^2 \in \mathrm{Ext}^{2,6}$, $h^3 \in \mathrm{Ext}^{3,9}$ with $h_0h^0 = h_1h^0 = h_1h^1 = h_0a^1 = va^1 = h_1h^2 = vh^2 = h_1h^3 = vh^3 = 0$ and $h_0h^2 = h_1^2a^1$.

(4) $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ is generated by $\{h^i | \; 0 \leq i \leq 3\}$ with $h_0h^i = h_1h^i = vh^i = 0$.

See [**Rav86**], Theorem 3.1.27 for (1) and (4). The calculations for (2) and (3) are also elementary, so that we omit the detail.

Remark 3.1.7. We use the same notation h^i for $i = 0, 1, 2, 3$ to denote certain generators of the above groups. This abuse of notation is justified by the fact that these generators have close relationships which are described in the next lemma. The context will clarify the use of the notation.

Consider cell inclusions $V(0) \to Y$ and $S^0 \cup_2 e^1 \cup_\eta e^2 \to Y$. The induced homomorphisms in Ext over $\mathcal{A}(1)_*$ are described as follows.

Lemma 3.1.8. (i)*The homomorphism $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(V(0))) \to \mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ sends the classes h^0 and h^1 to the non-trivial classes of the same name.*

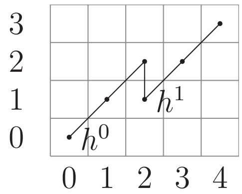

Figure II.2 – $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(V(0)))$ in the range $0 \le t-s \le 4$.

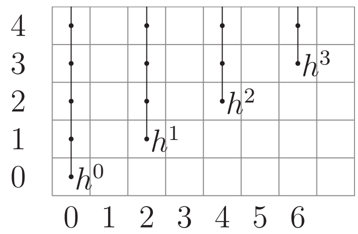

Figure II.3 – $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(C_\eta))$ in the range $0 \le t-s \le 6$.

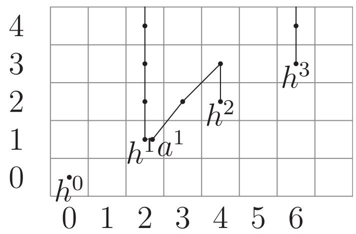

Figure II.4 – $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2))$ in the range $0 \le t-s \le 6$.

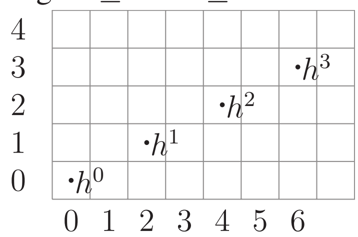

Figure II.5 – $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ in the range $0 \le t-s \le 6$.

(ii) *The homomorphism* $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2)) \to \mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ *sends the classes* h^0, h^1, h^2, h^3 *to the non-trivial classes of the same name.*

Proof. For part (i), consider the short exact sequence of $\mathcal{A}(1)_*$-comodules

$$0 \to \mathrm{H}_*(V(0)) \to \mathrm{H}_*(Y) \to \mathrm{H}_*(\Sigma^2 V(0)) \to 0$$

For degree reasons, the classes h^0 and h^1 of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(V(0)))$ do not belong to the image of the connecting homomorphism

$$\mathrm{Ext}^{s-1,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(\Sigma^2 V(0))) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(V(0))).$$

Therefore, they are sent to nontrivial classes of the same name in $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$. For part (ii), consider the short exact sequence of $\mathcal{A}(1)_*$-comodules

$$0 \to \mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2) \to \mathrm{H}_*(Y) \to \Sigma^3 \mathbb{F}_2 \to 0$$

and the resulting long exact sequence

$$\mathrm{Ext}^{s-1,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(\Sigma^3 \mathbb{F}_2)) \xrightarrow{\partial} \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2)) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y)).$$

For degree reasons, the classes h^0, h^2, h^3 of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2))$ are not in the image of the connecting homomorphism, and thus are sent to h^0, h^2, h^3 in $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y)$, respectively. Next, for degree reasons, the classes $h_0 h^1$ and $h_1 a^1$ are sent to $0 \in \mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$. The only way for this to happen is that the connecting homomorphism sends $\Sigma^3 1 \in \mathrm{Ext}^{0,3}_{\mathcal{A}(1)_*}(\mathbb{F}_2, \mathrm{H}_*(\Sigma^3\mathbb{F}_2))$ to the sum $h^1 + a^1$. It follows that h^1 is not in the image of the connecting homomorphism, and therefore is sent to $h^1 \in \mathrm{Ext}^{1,3}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ □

Lemma 3.1.9. $\mathrm{H}_*(Y)$ *has a structure of a* $\mathcal{A}(1)_*$*-comodule algebra. The resulting structure on* $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ *is that of a polynomial algebra.*

Proof. It is not hard to see that $\mathrm{H}_*(Y)$ is isomorphic to $\mathcal{A}(1)_* \square_{E(1)_*} \mathbb{F}_2$ as $\mathcal{A}(1)_*$-comodules, where $E(1)_*$ is the Hopf quotient of $\mathcal{A}(1)_*$ by the Hopf ideal (ζ_1), i.e., $E(1)_* \cong \mathbb{F}_2[\zeta_2]/(\zeta_2^2)$. In particular, $\mathrm{H}_*(Y)$ has the structure of an $\mathcal{A}(1)_*$-comodule algebra. As a consequence, $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ is an algebra and is furthermore isomorphic to $\mathrm{Ext}^{*,*}_{E(1)_*}(\mathbb{F}_2)$ by the change-of-rings isomorphism. It is well-known that the latter is a polynomial algebra on one variable. □

We now compute $G_\sigma = \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R'_\sigma)$. We denote by $\alpha_{s,t,\sigma}$ the non-trivial class of $\mathrm{Ext}^{s,s+t}_{\mathcal{A}(1)_*}(R'_\sigma)$ whenever there is a unique such one.

Proposition 3.1.10. *As a module over* G_0*,* $G_1 = \mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R'_1)$ *is generated by* $\alpha_{0,4,1} \in \mathrm{Ext}^{0,4}_{\mathcal{A}(1)_*}(R'_1)$ *and* $\alpha_{1,8,1} \in \mathrm{Ext}^{1,9}_{\mathcal{A}(1)_*}(R'_1)$ *with the relations* $h_1\alpha_{0,4,1} = 0$ *and* $v\alpha_{0,4,1} = h_0^2\alpha_{1,8,1}$.

Proof. Consider the short exact sequence of $\mathcal{A}(1)_*$-comodules

$$0 \to \Sigma^4\mathbb{F}_2 \to R'_1 \to \Sigma^6\mathrm{H}_*(V(0)) \to 0.$$

The connecting homomorphism

$$\partial : \mathrm{Ext}^{s,t-6}_{\mathcal{A}(1)_*}(V(0)) \to \mathrm{Ext}^{s+1,t-4}_{\mathcal{A}(1)_*}(\mathbb{F}_2)$$

of the resulting long exact sequence sends h^0 to h_1 and h^1 to 0. The latter follows from degree reasons and the former from the following map of short exact sequences of $\mathcal{A}(1)_*$-comodules and the naturality of the connecting homomorphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^4\mathbb{F}_2 & \longrightarrow & R_1 & \longrightarrow & \Sigma^6\mathrm{H}_*(V(0)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Sigma^4\mathbb{F}_2 & \longrightarrow & \mathrm{H}_*(\Sigma^4 C_\eta) & \longrightarrow & \Sigma^6\mathbb{F}_2 & \longrightarrow & 0.
\end{array}$$

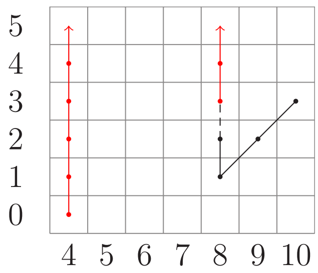

Figure II.6 – G_1 - The red part is the contribution of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\Sigma^4\mathbb{F}_2)$ and the black part from $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\Sigma^6\mathrm{H}_*(V(0)))$.

It follows that G_1 is v_1^4-periodic on the following generators (Figure II.6)

What remains to be established is the multiplication by h_0 on the generator of bidegree $(s, t-s) = (2, 8)$. This is done by a similar consideration of the connecting homomorphism associated to the short exact sequence of $\mathcal{A}(1)_*$-comodules

$$0 \to \mathrm{H}_*(\Sigma^4 C_\eta) \to R_1^{'} \to \Sigma^7\mathbb{F}_2 \to 0.$$

□

Proposition 3.1.11. *As a module over G_0, $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R_2^{'}) = G_2$ is generated by $\alpha_{s,t,2} \in \mathrm{Ext}^{s,s+t}$ where $(s,t) \in \{(0,8),(0,12),(1,14),(2,16),(3,18)\}$ with*

$$h_1\alpha_{s,t,2} = 0, v\alpha_{0,8,2} = h_0^3\alpha_{0,12,2}$$

$$v\alpha_{0,12,2} = h_0\alpha_{2,16,2}, v\alpha_{1,14,2} = h_0\alpha_{3,18,2}.$$

Proof. The short exact sequence in part (a) of Lemma 3.1.2 gives rise to the long exact sequence

$$\to \mathrm{Ext}^{s,t-12}_{\mathcal{A}(1)_*}(\mathrm{H}^*(C_\eta)) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R_2^{'}) \to \mathrm{Ext}^{s,t-8}_{\mathcal{A}(0)_*}(\mathbb{F}_2) \to \mathrm{Ext}^{s+1,t-12}_{\mathcal{A}(1)_*}(\mathrm{H}^*(C_\eta)) \to$$

Combining that $\mathrm{Ext}^{s,t}_{\mathcal{A}(0)_*}(\mathbb{F}_2) \cong \mathbb{F}_2[h_0]$ and the description of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}^*(C_\eta))$, we see that the connecting homomorphism is trivial for degree reasons.
What remains is to establish the v_1^4-multiplication on the class $\alpha_{0,8,2}$ of bidegree $(0,8)$. Consider the long exact sequence associated to the short exact sequence in part (b) of Lemma 3.1.2

$$\mathrm{Ext}^{s-1,t}_{\mathcal{A}(1)_*}(\Sigma^{12}V_3) \xrightarrow{\partial} \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^4 R_1^{'}) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R_2^{'}). \qquad \text{(II.1)}$$

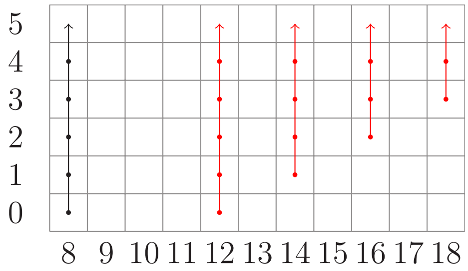

Figure II.7 – G_2 - The black part is the contribution of $\mathrm{Ext}^{s,t}_{\mathcal{A}(0)_*}(\mathbb{F}_2, \mathbb{F}_2)$ and the red one of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(C_\eta))$

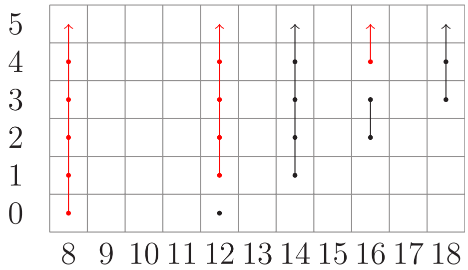

Figure II.8 – G_2 -The red part is the contribution of G_1 and the black one of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(V_3)$.

One can check that the class $\Sigma^4\alpha_{0,4,1} \in \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^4 R_1')$ is not in the image of ∂, and so is sent to $\alpha_{0,8,2} \in \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R_2')$. For degree reasons, we see that $v_1^4\Sigma^4\alpha_{0,4,1}$ is not in the image of ∂, thus $v_1^4\alpha_{0,8,2}$ is nontrivial in G_2. This completes the proof. □

Remark 3.1.12. We can make a complete calculation of the connecting homomorphism of (II.1), which results to the chart Figure-II.8.

Lemma 3.1.13. *As a module over* G_0, $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R_3') = G_3$ *is generated by* $\alpha_{s,t,3}$ *of* $\mathrm{Ext}^{s,s+t}$ *where* $(s,t) \in \{(0,12),(0,16),(0,18),(1,20),(2,22),(3,24)\}$ *with* $h_1\alpha_{s,t,3} = 0$, $v\alpha_{0,12,3} = h_0^3\alpha_{0,16,3}$, $v\alpha_{0,16,3} = h_0^2\alpha_{1,20,3}$, $v\alpha_{0,18,3} = h_0\alpha_{2,22,3}$, $v\alpha_{1,20,3} = h_0\alpha_{3,24,3}$.

Proof. The short exact sequence in Lemma 3.1.3 gives the long exact sequence

$$\to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^4 R_2') \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R_3') \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^{18} V_4) \to \mathrm{Ext}^{s+1,t}_{\mathcal{A}(1)_*}(\Sigma^4 R_2') \to$$

For degree reasons, the connecting homomorphism is trivial, hence we obtain the additive structure of G_3 as in Figure II.11. We need to establish the non-trivial h_0-multiplication on the generators $\{\alpha_{s,18+2s,3} |\ s \geq 0\}$. Taking the v_1^4-periodicity into account, we reduce to show this property for the generators of

$$\alpha_{0,18,3}, \alpha_{1,20,3}, \alpha_{2,22,3}, \alpha_{3,24,3}.$$

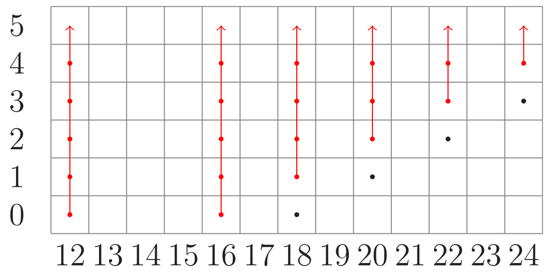

Figure II.9 – G_3 - The red part is the contribution of G_2 and the black one of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^{18} V_4)$.

For this, we can check that there are the following short exact sequences:

$$0 \to \Sigma^{18}\mathrm{H}_*(C_\eta) \to R_3 \to R_3/\Sigma^{18}\mathrm{H}_*(C_\eta) \to 0$$

and

$$0 \to \Sigma^4 R_2 \to R_3/\Sigma^{18}\mathrm{H}_*(C_\eta) \to \Sigma^{19}\mathrm{H}_*(C_\eta) \to 0$$

where, as a sub $\mathcal{A}(1)_*$-comodule of R_3, $\Sigma^{18}\mathrm{H}_*(C_\eta)$ is equal to $\mathbb{F}_2\{y_1y_3^2+y_2^3, y_2y_3^2\}$ and the map $\Sigma^4 R_2 \to R_3/\Sigma^{18}\mathrm{C}_\eta$ is the composite $\Sigma^4 R_2 \xrightarrow{\times y_1} R_3 \to R_3/\Sigma^{18}\mathrm{H}_*(C_\eta)$. As a consequence, $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R_3/\Sigma^{18}\mathrm{H}_*(C_\eta))$ sits in a long exact sequence

$$\mathrm{Ext}^{s-1,t}_{\mathcal{A}(1)_*}(\Sigma^{19}\mathrm{H}_*(C_\eta)) \xrightarrow{\partial} \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^4 R_2) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{R}_3/\Sigma^{18}\mathrm{H}_*(C_\eta)) \to .$$

Since ∂ is G_0-linear, one only needs to compute ∂ on the two generators of $\mathrm{Ext}^{0,19}_{\mathcal{A}(1)_*}(\Sigma^{19}\mathrm{H}_*(C_\eta))$ and $\mathrm{Ext}^{1,21}_{\mathcal{A}(1)_*}(\mathbb{F}_2, \Sigma^{19}\mathrm{H}_*(C_\eta))$. Direct computations show that ∂ act non-trivially on these classes. It follows that ∂ is a monomorphism and so $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{R}_3/\Sigma^{18}\mathrm{H}_*(C_\eta))$ is v_1-free on the generators depicted in Figure II.10. It follows immediately from the exact sequence

$$0 \to \Sigma^{18}\mathrm{H}_*(C_\eta) \to R_3 \to R_3/\Sigma^{18}\mathrm{H}_*(C_\eta) \to 0$$

that $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R_3)$ is as depicted in Figure II.11. In particular, missing h_0-extensions are established.

□

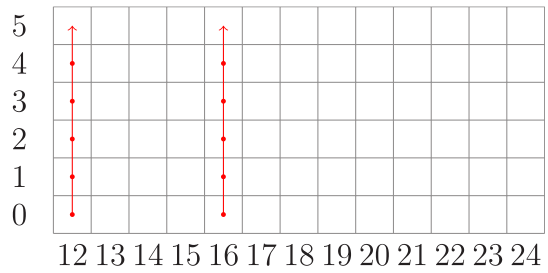

Figure II.10 – $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{R}_3/\Sigma^{18}\mathrm{H}_*(C_\eta))$.

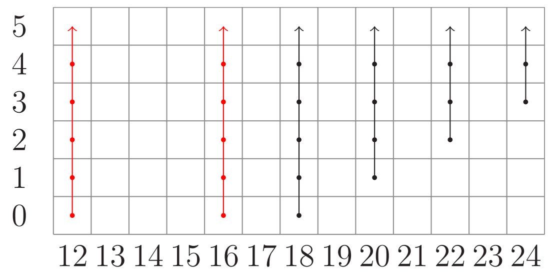

Figure II.11 – G_3 -The red part is the contribution of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{R}_3/\Sigma^{18}\mathrm{H}_*(C_\eta))$ and the black one of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^{18}\mathrm{H}_*(C_\eta))$.

Theorem 3.1.14. *As a module over G_0, we have*

(a) For every $\sigma \geq 2$, $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R'_\sigma) = G_\sigma$ is generated by $\alpha_{s,t,\sigma} \in \mathrm{Ext}^{s,t+s}_{\mathcal{A}(1)_*}(R'_\sigma)$ where $(s,t) \in \{(0,4\sigma),(0,2j+4\sigma)|2 \leq j \leq \sigma,(j,6\sigma+2j)|1 \leq j \leq 3\}$ with $h_1\alpha_{s,t,\sigma} = 0$.

(b) For all pairs of triples (s_1,t_1,σ_1) and (s_2,t_2,σ_2) with $\sigma_1 \geq 1$ and $\sigma_2 \geq 1$ except for $(2,9,1)$ and $(3,10,1)$, $(2,9,1)$ and $(2,9,1)$, $(3,10,1)$ and $(3,10,1)$we have that

$$\alpha_{s_1,t_1,\sigma_1}\alpha_{s_2,t_2,\sigma_2} = \alpha_{s_1+s_2,t_1+t_2,\sigma_1+\sigma_2}.$$

Proof. (a) The statement for $\sigma = 2$ is Lemma 3.1.11. Let us prove the claim for $\sigma \geq 3$ by induction. The base case is Lemma 3.1.13.

Suppose the claim is true for some $\sigma \geq 3$. The long exact sequence associated to the short exact sequence in Lemma 3.1.3 reads

$$\to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R'_{\sigma+1}) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^{6\sigma+6}V_4) \to \mathrm{Ext}^{s+1,t}_{\mathcal{A}(1)_*}(\Sigma^4 R'_\sigma) \to .$$

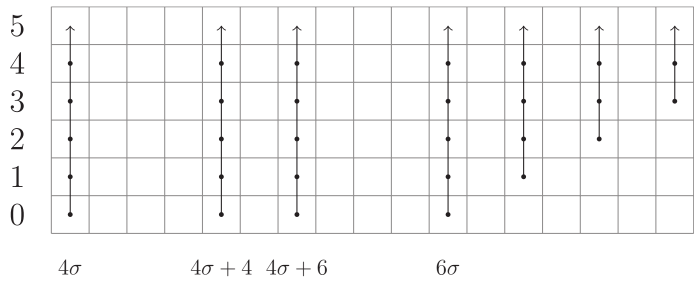

Figure II.12 – G_σ for $\sigma \geq 2$

Combining the additive structure of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^4 R'_\sigma)$ and that

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^{6\sigma+6}V_4) \cong \Sigma^{6\sigma+6}\mathbb{F}_2[v_1],$$

we obtain the additive structure of $G_{\sigma+1}$ as described in the lemma because the connecting homomorphism vanishes for degree reasons. To establish the non-trivial h_0-multiplication on the generators $\{\alpha_{s,2s+6\sigma+6,\sigma+1} | \; s \geq 0\}$, we use the following identities

(i) $G_{\sigma+1} \ni \alpha_{0,4,1}\alpha_{s,2s+6t,\sigma} \neq 0 \; \forall \sigma \geq 1$
(ii) $\alpha_{1,8,1}\alpha_{s,2s+6\sigma-6,\sigma-1} = \alpha_{s+1,2s+6\sigma+2,\sigma} \; \forall \sigma \geq 2$
(iii) $\alpha_{0,12,2}\alpha_{s,2s+6\sigma-6,\sigma-1} = \alpha_{s,2s+6\sigma+6,\sigma+1} \; \forall \sigma \geq 3$.

These identities are the content of part (b). For the sake of the presentation, we postpone the proof to (b); this is legitimate because, as we will see, the proof of (b) only uses the additive structure of $G'_\sigma s$. Let us show how these identities allow us to conclude the proof of (a). Indeed, the classes $\alpha_{s,2s+6\sigma-6,\sigma-1}$ exist (non-trivial) for all $\sigma \geq 3$ and $s \geq 0$. Therefore, we have that, for all $\sigma \geq 3$,

$$\begin{aligned} h_0\alpha_{s,2s+6\sigma+6,\sigma+1} &= h_0\alpha_{0,12,2}\alpha_{s,2s+6\sigma-6,\sigma-1} \text{ (multiplying both sides of (iii) by } h_0) \\ &= \alpha_{0,4,1}\alpha_{1,8,1}\alpha_{s,2s+6\sigma-6,\sigma-1} \text{ (because of (i))} \\ &= \alpha_{0,4,1}\alpha_{s+1,2s+2+6\sigma,\sigma} \text{ (because of (ii))} \\ &\neq 0 \text{ (because of (i))}. \end{aligned}$$

(b) For every $\sigma, \tau \geq 1$, there is a commutative diagram of $\mathcal{A}(1)_*$-comodules

$$\begin{array}{ccccc} R'_\sigma \otimes R'_\tau & \xrightarrow{\mu} & R_{\sigma+\tau} & \longrightarrow & R'_{\sigma+\tau} \\ \downarrow & & & & \downarrow \\ \mathrm{H}_*(\Sigma^{6\sigma}X_\sigma) \otimes \mathrm{H}_*(\Sigma^{6\tau}X_\tau) & & \xrightarrow{\mu} & & \mathrm{H}_*(\Sigma^{6\sigma+6\tau}X_{\sigma+\tau}) \\ \downarrow & & & & \downarrow \\ \mathrm{H}_*(\Sigma^{6\sigma}Y) \otimes \mathrm{H}_*(\Sigma^{6\tau}Y) & & \xrightarrow{\mu} & & \mathrm{H}_*(\Sigma^{6\sigma+6\tau}Y) \end{array}.$$

Let us explain the maps in this diagram. The spectrum X_σ is $V(0), S^0 \cup_2 e^1 \cup_\eta e^2$ or Y if $\sigma = 1, 2$ or $\sigma > 2$ respectively; and in each case the map $R'_\sigma \to \mathrm{H}_*(X_\sigma)$ is the projection appearing in the proof of Lemma 3.1.10, Lemma 3.1.2 or Lemma 3.1.3, respectively. The other vertical arrows are the inclusions of X_σ into Y. The bottom horizontal arrow is the multiplication on $\mathrm{H}_*(Y)$, described in Lemma 3.1.9, and the middle one is induced by the latter. The second upper arrow is the projection on the factor $R'_{\sigma+\tau}$ of the decomposition in Lemma 3.1.1.

The induced homomorphisms in Ext over $\mathcal{A}(1)_*$ of all vertical arrows are studied in the proof of Lemmas 3.1.8, 3.1.10, 3.1.11 and Theorem 3.1.14, according to which the classes $\alpha_{s,t,\sigma}$, where $\sigma \geq 1$ and $(s,t,\sigma) \notin \{(2,9,1),(3,10,1)\}$, are sent non-trivially in a unique way to $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$, hence their products are non-trivial by Lemma 3.1.9. This proves (b). □

Remark 3.1.15. Let us summarise what has been done so far. First, Lemma 3.1.1 implies that

$$\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R) \cong (\bigoplus_{i\geq 0} G_i) \otimes \mathbb{F}_2[v_2^4]$$

where $v_2^4 \in \mathrm{Ext}^{4,28}(\mathbb{F}_2, R_4)$ represented by y_3^4. Next, Lemma 3.1.14 describes completely the products between G_i's modulo the ideal generated by (v_2^4). It is then straightforward to verify that $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R)$ is generated by the classes of

$$h_0, h_1, v, v_1^4, \alpha_{0,4,1}, \alpha_{1,8,1}, \alpha_{0,12,2}, \alpha_{1,14,2}, \alpha_{3,18,2}, \alpha_{0,18,3}, v_2^4. \qquad \text{(II.2)}$$

Let us describe the subalgebra of primitives.

Corollary 3.1.16. There is the following isomorphism of graded algebras

$$\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R) \cong \mathbb{F}_2[\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4, \alpha_{0,18,3}]/(\alpha_{0,18,3}^2 = \alpha_{0,12,2}^3 + \alpha_{0,4,1}^2 v_2^4).$$

Proof. The $\mathbb{F}_2$-algebra $\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(\mathbb{F}_2, R)$ is naturally identified with a subalgebra of $R = \mathbb{F}_2[y_1, y_2, y_3]$. Through this identification, $\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4, \alpha_{0,18,3}$ identify with $y_1, y_2^2, y_3^4, y_2^3+y_1y_3^2$, respectively. Thus $\mathbb{F}_2[\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4, \alpha_{0,18,3}]/(\alpha_{0,18,3}^2 = \alpha_{0,12,2}^3 + \alpha_{0,4,1}^2 v_2^4)$ is isomorphic to the subalgebra of $\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(\mathbb{F}_2, R)$ generated by $\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4, \alpha_{0,18,3}$. On the other hand, it follows from Remark (3.1.15) that $\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4, \alpha_{0,18,3}$ generate the whole subalgebra of primitives of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R)$. This concludes the proof of the lemma. □

The differentials d_1. Since the DMSS for $\mathbb{F}_2$ is a spectral sequence of algebras, all d_1-differentials can be determined on the set of algebra generators of (II.2).

Proposition 3.1.17. *The d_1-differential is multiplicative and on generators, it is given as follows:*

1) $d_1(h_0) = 0$
2) $d_1(h_1) = 0$
3) $d_1(\alpha_{0,4,1}) = 0$
4) $d_1(\alpha_{1,14,2}) = 0$
5) $d_1(\alpha_{0,18,3}) = 0$
6) $d_1(v_1^4) = 0$
7) $d_1(\alpha_{0,12,2}) = \alpha_{0,4,1}^3$
8) $d_1(\alpha_{1,8,1}) = h_0\alpha_{0,4,1}^2$
9) $d_1(v) = h_0^3\alpha_{0,4,1}$
10) $d_1(\alpha_{3,18,2}) = h_0^3\alpha_{0,18,3}$
11) $d_1(v_2^4) = \alpha_{0,4,1}\alpha_{0,12,2}^2$.

Proof. 1), 2), 4) For degree reasons, there is no room for a non-trivial d_1-differential on $h_0, h_1, \alpha_{1,14,2}$

3) It is easy to see that $\mathrm{Ext}^{1,4}_{\mathcal{A}(2)_*}(\mathbb{F}_2, \mathbb{F}_2)$ is non-trivial and that $\alpha_{0,4,1}$ is the only class in the E_1-term that can contribute to it. Therefore $\alpha_{0,4,1}$ is a permanent cycle.

5) We see that $h_0\alpha_{0,18,3} = \alpha_{0,4,1}\alpha_{1,14,2}$. By the Leibniz rule, $h_0 d_1(\alpha_{0,18,3}) = 0$. As h_0 acts injectively on G_3, it follows that $d_1(\alpha_{0,18,3}) = 0$.

6) Since $h_0^2 v_1^4 = v^2$, $h_0^2 d_1(v_1^4) = 0$. This follows because $d_1(v_1^4)$ takes values in $\mathrm{Ext}^{4,8}_{\mathcal{A}(1)_*}(\mathbb{F}_2, R_1')$ on which h_0 acts injectively.

7) We have that $\alpha_{0,12,2}$ is represented by the $\mathcal{A}(2)$-primitive $[1|y_2^2]+[x_1|y_1^2] \in E\otimes R_2$. By Remark 2.1.6, $d_1(\alpha_{0,12,2})$ is represented by $d([1|y_2^2]+[x_1|y_1^2]) = [1|y_1^3] \in E \otimes R_3$, hence is equal to $\alpha_{0,4,1}^3$.

8) Because $\alpha_{0,4,1}\alpha_{1,8,1} = h_0\alpha_{0,12,2}$, the Leibniz rule implies that

$$\alpha_{0,12,2} d_1(\alpha_{1,8,1}) = h_0 d_1(\alpha_{0,12,2}) = h_0\alpha_{0,4,1}^3.$$

That $\alpha_{0,4,1}$ acts injectively on the E_1-term implies that $d_1(\alpha_{1,8,1}) = h_0\alpha_{0,4,1}^2$.
9) The relation $\alpha_{0,4,1}v = h_0^2\alpha_{1,8,1}$ implies that

$$\alpha_{0,4,1}d_1(v) = h_0^2 d_1(\alpha_{1,8,1}) = h_0^3\alpha_{0,4,1}^2$$

As $\alpha_{0,4,1}$ acts injectively on the E_1-term, we obtain that $d_1(v) = h_0^3\alpha_{0,4,1}$.

10) The relation $v\alpha_{1,14,2} = h_0\alpha_{3,18,2}$ shows that

$$h_0 d_1(\alpha_{3,18,2}) = \alpha_{1,14,2}d_1(v) = \alpha_{1,14,2}h_0^3\alpha_{0,4,1} = h_0^4\alpha_{0,18,3}$$

Therefore, $d_1(\alpha_{3,18,2}) = h_0^3\alpha_{0,18,3}$.

11) We check that v_2^4 is represented by the $\mathcal{A}(2)$-primitive $[1|y_3^4] + [x_1|y_2^4]$ in $E \otimes R_4$. By Remark 2.1.6, $d_1(v_2^4)$ is represented by $[1|y_1y_2^4]$, hence is equal to $\alpha_{0,4,1}\alpha_{0,12,2}^2$.

□

It turns out that the DMSS collapses at the E_2-term because there is no room for higher differentials. In particular, the classes $\alpha_{1,14,2}, \alpha_{0,4,1}, \alpha_{0,12,2}^2, v_2^8, \alpha_{0,18,3}$ survive the spectral sequence converging to elements of $\mathrm{Ext}_{\mathcal{A}(2)_*}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ in appropriate bidegrees. Following [**DFHH14**], those elements are denoted by $\alpha, h_2, g, w_2, \beta$, respectively. Furthermore, h_2, g, w_2, β generate a subalgebra of $\mathrm{Ext}_{\mathcal{A}(2)_*}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ isomorphic to $\mathbb{F}_2[h_2, g, w_2, \beta]/(h_2^3, h_2 g, \beta^4 - g^3)$. The relation $\beta^4 = g^3$ is a consequence of a d_1-differential. In effect, the relation $\alpha_{0,18,3}^2 = \alpha_{0,12,2}^3 + \alpha_{0,4,1}^2 v_2^4$ implies the relation $\beta^4 - g^3 - h_2^4 w_2 = 0$ in $\mathrm{Ext}_{\mathcal{A}(2)_*}^{*,*}(\mathbb{F}_2)$. But $\alpha_{0,4,1}^4 v_2^8$ gets hit by the differential

$$d_1(v_2^8\alpha_{0,4,1}\alpha_{0,12,2}) = v_2^8\alpha_{0,4,1}d_1(\alpha_{0,12,2}) = v_2^8\alpha_{0,4,1}^4.$$

Thus the relation $\beta^4 = g^3 + h_2^4 w_2$ becomes $\beta^4 = g^3$.

3.2 The Davis-Mahowald spectral sequence for A_1

The $\mathcal{A}(2)_*$-comodule structure of A_1. In [**DM81**], Davis and Mahowald constructed four type 2 finite spectra whose mod 2 cohomology are isomorphic to a free module of rank one over the subalgebra $\mathcal{A}(1) = \langle Sq^1, Sq^2\rangle$ of the Steenrod algebra $\mathcal{A}$. Let us review the construction of these spectra and their module structure over the subalgebra $\mathcal{A}(2) = \langle Sq^1, Sq^2, Sq^4\rangle$ of $\mathcal{A}$. Recall that Y is $V(0) \wedge C_\eta$. The $\mathcal{A}$-module structure of $\mathrm{H}^*(Y)$ is depicted in Figure II.13. An element of $\mathrm{Ext}_{\mathcal{A}(1)}^{1,3}(\mathrm{H}^*(Y), \mathrm{H}^*(Y))$ can be represented by an $\mathcal{A}(1)$-module M sitting in a short exact sequence of $\mathcal{A}(1)$-modules

$$0 \to \mathrm{H}^*(\Sigma^3 Y) \to M \to \mathrm{H}^*(Y) \to 0.$$

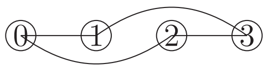

Figure II.13 – Diagram of $\mathrm{H}^*(Y)$: the straight lines represent Sq^1 and the curved lines represent Sq^2, the numbers represent the degree of the cell.

It can be checked that M must be isomorphic either to $\mathrm{H}^*(\Sigma^3 Y) \oplus \mathrm{H}^*(Y)$ or to $\mathcal{A}(1)$ as an $\mathcal{A}(1)$-module. This means that

$$\mathrm{Ext}^{1,3}_{\mathcal{A}(1)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y)) \cong \mathbb{F}_2. \tag{II.3}$$

The $\mathcal{A}(1)$-module structure of $\mathcal{A}(1)$ is depicted in Figure II.14. One can ask

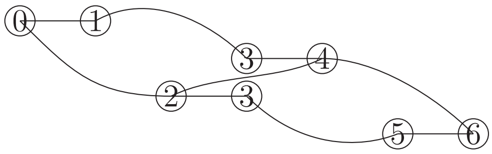

Figure II.14 – Diagram of $\mathcal{A}(1)$.

whether $\mathcal{A}(1)$ admits a structure of $\mathcal{A}(2)$-module. If such a structure exists, then according to the Adem relations $Sq^2Sq^1Sq^2 = Sq^4Sq^1 + Sq^1Sq^4$, there must be a nontrivial action of Sq^4 on the nontrivial class of degree 1. It is straightforward to verify that the latter is the only constraint to put an $\mathcal{A}(2)$-module structure on $\mathcal{A}(1)$. There are also possibilities for Sq^4 to act nontrivially on the classes of degree 0 and 2. These give in total four different $\mathcal{A}(2)$-module structures on A_1. In other words, the inclusion of Hopf algebras $\mathcal{A}(1) \hookrightarrow \mathcal{A}(2)$ induces a surjective homomorphism

$$\mathrm{Ext}^{1,3}_{\mathcal{A}(2)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y)) \to \mathrm{Ext}^{1,3}_{\mathcal{A}(1)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y))$$

whose kernel contains 4 element. Therefore,

$$\mathrm{Ext}^{1,3}_{\mathcal{A}(2)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y)) \cong \mathbb{F}_2^{\oplus 3}.$$

Next, one observes that restriction along $\mathcal{A}(2) \subset \mathcal{A}$ induces an isomorphism

$$\mathrm{Ext}^{1,3}_{\mathcal{A}}(\mathrm{H}^*(Y), \mathrm{H}^*(Y)) \cong \mathrm{Ext}^{1,3}_{\mathcal{A}(2)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y)),$$

because for any $\mathcal{A}$-module M sitting in a short exact sequence

$$0 \to \mathrm{H}^*(\Sigma^3 Y) \to M \to \mathrm{H}^*(Y) \to 0$$

there can not be any non-trivial Sq^k for $k \geq 8$ on M. It is proved in [**DM81**] that the four classes of $\mathrm{Ext}^{1,3}_{\mathcal{A}}(\mathrm{H}^*(Y), \mathrm{H}^*(Y))$ that are sent to the unique non-trivial class of $\mathrm{Ext}^{1,3}_{\mathcal{A}(1)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y))$ are permanent cycles in the Adams spectral sequence and converge to four v_1- self-maps of Y, i.e., the maps $\Sigma^2 Y \to Y$ inducing isomorphisms in $K(1)$-homology theory. As a consequence, the cofibers of these v_1-self-maps realise the four different $\mathcal{A}$-module structures on $\mathcal{A}(1)$. We will write A_1 to refer to any of these four finite spectra. Following [**BEM17**], we make the following definition.

Definition 3.2.1. We define by $A_1[i,j],\ i,j \in \{0,1\}$ the version of A_1 having the non-trivial Sq^4 on the generator of degree 0 respectively 2 if and only if $i = 1$ respectively $j = 1$.

As a $\mathbb{F}_2$-vector spaces,

$$\mathrm{H}_*(A_1[ij]) \cong \mathbb{F}_2\{a_0, a_1, a_2, a_3, \overline{a_3}, a_4, a_5, a_6\}, \tag{II.4}$$

where $a_0, a_1, a_2, a_4, a_5, a_6$ are duals to the generators of degree $0, 1, 2, 4, 5, 6$ of $\mathrm{H}^*(A_1[ij])$, respectively and $a_3, \overline{a_3}$ are duals to the images of the generator of degree 0 by $Sq^3, Sq^3 + Sq^2Sq^1$, respectively. From now on, we also denote by $A_1[ij]$ the mod 2 homology of $A_1[ij]$ and A_1 the mod 2 homology of A_1. By taking duals to the action of $\mathcal{A}(2)$ on $\mathrm{H}^*(A_1[ij])$, we obtain

Proposition 3.2.2. *The left coaction of $\mathcal{A}(2)_*$ on $A_1[ij]$ is given by*

$\Delta(a_1) = [1|a_1] + [\xi_1|a_0]$
$\Delta(a_2) = [1|a_2] + [\xi_1^2|a_2]$
$\Delta(a_3) = [1|a_3] + [\xi_1|a_2] + [\xi_1^2|a_1] + [\xi_1^3|a_0]$
$\Delta(\overline{a_3}) = [1|\overline{a_3}] + [\xi_1^2|a_1] + [\xi_2|a_0]$
$\Delta(a_4) = [1|a_4] + [\xi_1|\overline{a_3}] + [\xi_1^2|a_2] + [\xi_1^3|a_1] + [\xi_2|a_1] + [\xi_2\xi_1|a_0] + \alpha_{i,j}[\xi_1^4|a_0]$
$\Delta(a_5) = [1|a_5] + [\xi_1^2|\overline{a_3}] + [\xi_1^2|a_3] + [\xi_2|a_2] + [\xi_1^4|a_1] + [\xi_2\xi_1^2|a_0]$
$\Delta(a_6) = [1|a_6] + [\xi_1|a_5] + [\xi_1^2|a_4] + [\xi_1^3|\overline{a_3}] + [\xi_1^3|a_3] + [\xi_2|a_3] + [\xi_2\xi_1|a_2] + \beta_{i,j}[\xi_1^4|a_2] + [\xi_2\xi_1^2|a_1] + [\xi_1^5|a_1] + \gamma_{i,j}[\xi_1^6|a_0] + [\xi_2\xi_1^3|a_0] + \lambda_{i,j}[\xi_2^2|a_0]$, *where*

$$\alpha_{i,j} = \begin{cases} 0 & \text{if } (i,j) \in \{(0,0),(0,1)\} \\ 1 & \text{if } (i,j) \in \{(1,0),(1,1)\} \end{cases}$$

$$\beta_{i,j} = \begin{cases} 0 & \text{if } (i,j) \in \{(0,0),(1,0)\} \\ 1 & \text{if } (i,j) \in \{(0,1),(1,1)\} \end{cases}$$

$$\gamma_{i,j} = 1 + \alpha_{i,j}$$

and

$$\lambda_{i,j} = \alpha_{i,j} + \beta_{i,j}$$

Proof. The proof is a straightforward translation from $\mathcal{A}(2)$-module structure to $\mathcal{A}(2)_*$-comodule structure using the formula of the duals of the Milnor basis in [**Mil58**]. □

DMSS for A_1. In what follows, we will apply in many places the shearing homomorphism to find primitives representing certain cohomology classes,, see [**ABP69**], Theorem 3.1. It is useful to recall it here. In general, let C be a Hopf algebra with conjugation χ and B be Hopf-algebra quotient of C. Given a C-comodule M, consider the composite

$$C \otimes M \xrightarrow{id\otimes\Delta} C \otimes C \otimes M \xrightarrow{id\otimes\chi\otimes id} C \otimes C \otimes M \xrightarrow{\mu\otimes id} C \otimes M.$$

When restricting to $C\Box_B M$, this composite factors through $(C\Box_B k) \otimes M$ inducing the shearing isomorphism of C-comodules

$$Sh : C\Box_B M \to (C\Box_B k) \otimes M,$$

where C coacts on $C\Box_B M$ via the left factor and on $(C\Box_B k) \otimes M$ diagonally. Combined with the change-of-rings isomorphism, we have the following isomorphisms:

$$\mathrm{Ext}_B^*(k, M) \cong \mathrm{Ext}_C^*(k, C\Box_B M) \cong \mathrm{Ext}_C^*(k, (C\Box_B k) \otimes M).$$

In particular, via these isomorphisms, a class $x \in \mathrm{Ext}_B^0(k, M)$ is sent to $Sh(1\otimes x)$.

Proposition 3.2.3. *The E_1-term of the Davis-Mahowald spectral sequence converging to $\mathrm{Ext}_{\mathcal{A}(2)_*}^{s,t}(A_1)$ is given by*

$$\mathrm{E}_1^{s,\sigma,*} \cong \begin{cases} 0 & \text{if } s > 0 \\ R_\sigma & \text{if } s = 0. \end{cases}$$

As a module over $\mathbb{F}_2[\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4] (\subset \mathrm{Ext}_{\mathcal{A}(1)_*}^{*,*}(R))$, $\mathrm{E}_1^{*,*,*}$ is free of rank eight on the following generators of

$$1, y_3, y_3^2, y_3^3, y_2, y_2y_3, y_2y_3^2, y_2y_3^3. \tag{II.5}$$

Proof. In effect, $\mathrm{E}_1^{s,\sigma,t}$ is equal to $\mathrm{Ext}_{\mathcal{A}(1)_*}^{s,t}(R^\sigma \otimes A_1)$ by definition. The coaction of $\mathcal{A}(1)_*$ on $R^\sigma \otimes A_1$ is the usual diagonal coaction on tensor products. In addition, A_1 is isomorphic to $\mathcal{A}(1)_*$ as $\mathcal{A}(1)_*$-comodules. By the change-of-rings isomorphism, we obtain that

$$\mathrm{Ext}_{\mathcal{A}(1)_*}^{s,t}(R^\sigma \otimes A_1) \cong \mathrm{Ext}_{\mathbb{F}_2}^{s,t}(R^\sigma) \cong R^\sigma. \tag{II.6}$$

The first part of the proposition follows.

For the second part, the action of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R)$ on $\mathrm{E}_1^{s,t,\sigma}$

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R) \otimes \mathrm{Ext}^{s',t'}_{\mathcal{A}(1)_*}(R \otimes A_1) \longrightarrow \mathrm{Ext}^{s+s',t+t'}_{\mathcal{A}(1)_*}(R \otimes A_1)$$

is induced by the multiplication on R:

$$R \otimes (R \otimes A_1) \to R \otimes A_1.$$

Now let $r \in \mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R) \subset R$ and $s \in R \cong \mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R \otimes A_1)$. By applying the shearing isomorphism, the class s is represented by a unique element of the form $s \otimes a_0 + \sum s_i \otimes a_i \in R \otimes A_1$ where the a_i are in positive degrees. The action of r on s is then represented by $rs \otimes a_0 + \sum rs_i \otimes a_i$ which represents $rs \in R \cong \mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R \otimes A_1)$ via (II.6). In other words, the action of $\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R)$ on $\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R \otimes A_1)$ is given by the multiplication of the polynomial algebra R. The proof follows from the fact that $\mathbb{F}_2[\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4]$ is identified with the subalgebra of R generated by y_1, y_2^2, y_3^4. $\square$

Let us analyse the differentials in this spectral sequence. As the d_r-differentials decrease s-filtration by $r-1$, i.e., $d_r : \mathrm{E}_r^{s,\sigma,t} \to \mathrm{E}_r^{s-r+1,\sigma+r,t}$ and $\mathrm{E}_1^{s,\sigma,t} = 0$ if $s > 0$, the spectral sequence collapses at the E_2-term and there are no extension problems. Therefore,

$$\mathrm{E}_2^{0,\sigma,t} \cong \mathrm{Ext}^{\sigma,t}_{\mathcal{A}(2)_*}(A_1).$$

We now turn our attention to the d_1-differentials. As all elements of the E_1-term are in $\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R \otimes A_1)$, we can apply the remark after Proposition 2.1.5. We have determined the d_1-differential on the classes $\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4$ in Proposition 3.1.17. By the Leibniz rule, it remains to determine the d_1-differential on the classes of (II.5).

Proposition 3.2.4. *There are the following d_1-differentials*

1) $d_1(1) = 0$,
2) $d_1(y_2) = 0$,
3) $d_1(y_3) = 0$,
4) $d_1(y_2 y_3) = 0$,
5) $d_1(y_2 y_3^2) = 0$,
6) $d_1(y_2 y_3^3) = 0$,
7) $d_1(y_3^2) = \alpha_{0,4,1}^2 y_2$,
8) $d_1(y_3^3) = \alpha_{0,4,1}^2 y_2 y_3$.

Proof. Parts $1-4$ follow from the sparseness of the E_1-term.

5) The only nontrivial d_1-differential that $y_2y_3^2$ can support is

$$d_1(y_2y_3^2) = \alpha_{0,4,1}^2\alpha_{0,12,2}1.$$

However,

$$d_1(\alpha_{0,4,1}^2\alpha_{0,12,2}) = \alpha_{0,4,1}^2 d_1(\alpha_{0,12,2}) = \alpha_{0,4,1}^5 1 \neq 0.$$

This means that $\alpha_{0,4,1}^2\alpha_{0,12,2}$ is not a d_1-cycle, and so cannot be hit by a d_1-differential. Therefore, $y_2y_3^2$ is a d_1-cycle.

6) Similarly, a nontrivial d_1-differential on $y_2y_3^3$ would be

$$d_1(y_2y_3^3) = \alpha_{0,4,1}^2\alpha_{0,12,2}y_3.$$

However,

$$d_1(\alpha_{0,4,1}^2\alpha_{0,12,2}y_3) = \alpha_{0,4,1}^5 y_3 \neq 0$$

by the Leibniz rule. Thus, $y_2y_3^3$ is a d_1-cycle.

7-8) It suffices to prove that $\nu^2 y_2 = 0$ and $\nu^2 y_2 y_3 = 0$ in $\mathrm{Ext}_{\mathcal{A}(2)_*}^{*,*}(A_1)$ because the differentials in part 7) and 8) are the only possibilities for the latter to occur. We will proceed using juggling formulas for Massey products, see [Rav86], Section 4 of Appendix A1. In effect, the classes 1 and y_3 being permanent cycles by part 1) and part 3), they converge to classes in $\mathrm{Ext}_{\mathcal{A}(2)_*}^{0,0}(A_1)$ and $\mathrm{Ext}_{\mathcal{A}(2)_*}^{1,6}(A_1)$, respectively. By sparseness even at the level of the E_1-term of the DMSS, $\eta 1 = \eta y_3 = 0$. Hence the Massey product $\langle \nu, \eta, y_3^i\rangle$ with $i \in \{0,1\}$ can be formed. We have that

$$\nu^2 y_3^i = \langle \eta, \nu, \eta\rangle y_3^i = \eta\langle \nu, \eta, y_3^i\rangle.$$

By sparseness of the DMSS, $\alpha_{0,4,1}^2 y_3^i$ survives the DMSS and so $\nu^2 y_3^i \neq 0$. It follows that $\langle \nu, \eta, y_3^i\rangle$ is nontrivial and must be equal to $y_2y_3^i$. The fact that $\nu^3 = 0 \in \mathrm{Ext}_{\mathcal{A}(2)_*}^{3,12}(\mathbb{F}_2)$ allows us to do the following juggling

$$\nu^2 y_2 y_3^i = \nu^2\langle \nu, \eta, y_3^i\rangle = \langle \nu^2, \nu, \eta\rangle y_3^i.$$

However, the Massey product $\langle \nu^2, \nu, \eta\rangle$ lives in the group $\mathrm{Ext}_{\mathcal{A}(2)_*}^{3,14}(\mathbb{F}_2)$, which vanishes by Theorem 3.1.14. This concludes the proof of parts 7) and 8). □

E_2**-term of the Adams SS.** We describe $\mathrm{Ext}_{\mathcal{A}(2)_*}^{*,*}(A_1)$ as a module over

$$\mathbb{F}_2[h_2, g, v_2^8]/(h_2^3, h_2 g) \subset \mathrm{Ext}_{\mathcal{A}(2)_*}^{*,*}(\mathbb{F}_2).$$

We recall that g is represented by $\alpha_{0,12,2}^2$ in the DMSS for $\mathbb{F}_2$ We will denote by $e[s,t]$ where $s, t \in \mathbb{N}$ the unique non-trivial class belonging to $\mathrm{Ext}_{\mathcal{A}(2)_*}^{s,s+t}(A_1)$.

Proposition 3.2.5. *As a module over $\mathbb{F}_2[h_2, g, v_2^8]/(h_2^3, h_2 g)$, $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(A_1)$ is a direct sum of cyclic modules generated by the following elements*

e[0,0]	e[1,5]	e[1,6]	e[2,11]
1	y_2	y_3	y_2y_3
(0)	(h_2^2)	(0)	(h_2^2)

e[3,15]	e[3,17]	e[4,21]	e[4,23]
$y_2^3 + y_1y_3^2$	$y_2y_3^2$	$y_1y_3^3 + y_2^3y_3$	$y_2y_3^3$
(h_2^2)	(0)	(h_2^2)	(0)

e[6,30]	e[6,32]	e[7,36]	e[7,38]
$y_2^6 + y_1^2y_3^4$	$y_2^4y_3^2 + y_1y_2y_3^4$	$y_2^6y_3 + y_1^2y_3^5$	$y_2^4y_3^3 + y_1y_2y_3^5$
(h_2)	(h_2)	(h_2)	(h_2)

e[8,42]	e[9,47]	e[9,48]	e[10,53]
$y_2^6y_3^2 + y_1^2y_3^6 + y_1y_2^3y_3^4$	$y_2^7y_3^2 + y_1^2y_2y_3^6$	$y_2^6y_3^3 + y_1^2y_3^7 + y_1y_2^3y_3^5$	$y_2^7y_3^3 + y_1^2y_2y_3^7$
(h_2)	(h_2)	(h_2)	(h_2)

The second row in the table indicates a representative in the DMSS and the third row the annihilator ideal of the corresponding generator.

Proof. As a corollary of Proposition 3.2.3, the E_1-term of the DMSS for A_1 is isomorphic to a free module of rank 32 over $\mathbb{F}_2[h_2, g, v_2^8]$. In particular, these 32 generators are h_2-free. It turns out that one can choose these 32-generators in such a way that there are exactly 16 h_2-free towers that truncate 16 others by d_1-differentials. The question is how one can identify these 16 d_1-cycles. For this, we compute the d_1-differentials on the following 32 generators of the E_1-term: $\{y_2^i y_3^j | 0 \leq i \leq 3, 0 \leq j \leq 7\}$. Some of them are d_1-cycles, for example y_2, y_3. Whereas, some of them are not d_1-cycle at first, but become so after adding a multiple of h_2, for example $\alpha_{0,12,2}y_2 + h_2y_3^2 = y_2^3 + y_1y_3^2$. This procedure is straightforward but lengthy, so we omit details here. It can be checked that the generators listed in the table are d_1-cycles. Finally, since g and v_2^8 are d_1-cycles, Proposition 3.2.5 follows. □

3.3 Two products

Now we turn our attention to the product between $\alpha \in \mathrm{Ext}^{3,15}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ and $e[4,23] \in \mathrm{Ext}^{4,27}_{\mathcal{A}(2)_*}(A_1)$. This product is not detected in the DMSS because α has σ-filtration 1 in the DMSS whereas all non-trivial groups in the E_∞-term of the DMSS converging to $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(A_1)$ are in σ-filtration 0. Therefore, we need first to find a representative of α in the total cochain complex of the double complex $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R$ and that of $e[4,23]$ in $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R \otimes A_1$, then take the product at the level of cochain complexes and finally check if this product is a coboundary. It is tedious to carry out this procedure because any representative of $e[4,23]$ contains many terms, and so it is not easy to check if the product is a coboundary. In the sequel, we will express elements of $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R_*$, $B(2)_*^{\otimes *} \otimes E_2 \otimes R_*$) and $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R_* \otimes A_1$, $B(2)_*^{\otimes *} \otimes E_2 \otimes R_* \otimes A_1$ in terms of the obvious monomial basis formed by the tensor products of the monomial basis of $\mathcal{A}(2)_*$, E_2, R_*, $B(2)_*$, F_2, S_* and the one of A_1 given in (II.4). We recall from Section 2 that there is a map of pairs $(\mathcal{A}(2)_*, E_2) \to (B(2)_*, F_2)$ given by

$$\mathcal{A}(2)_* = \mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3]/(\zeta_1^8, \zeta_2^4, \zeta_3^2) \to B(2)_* = \mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3]/(\zeta_1^4, \zeta_2^4, \zeta_3^2)$$

$$\zeta_i \mapsto \zeta_i \;\; i \in \{1,2,3\}$$

$$E_2 = E(x_1, x_2, x_3) \to F_2 = E(x_2, x_3)$$

$$x_1 \mapsto 0, x_2 \mapsto x_2, x_2 \mapsto x_2.$$

The induced map on their Koszul duals is

$$R = \mathbb{F}_2[y_1, y_2, y_3] \to S = \mathbb{F}_2[y_2, y_3]$$

$$y_1 \mapsto 0, y_2 \mapsto y_2, y_3 \mapsto y_3.$$

By an abuse of notation, we will denote by p these projection maps. The context will make it clear which map is referred to.

The following two lemmas simplify computations.

Lemma 3.3.1. *The product of* α *and* $e[4,23]$ *is equal either to* 0 *or to* $ge[3,15]$.

Proof. This is trivial because $ge[3,15]$ is the only non-trivial class in the appropriate bidegree. □

Lemma 3.3.2. *The map* $p_* = \mathrm{Ext}^{7,42}_{\mathcal{A}(2)_*}(A_1) \to \mathrm{Ext}^{7,42}_{B(2)_*}(A_1)$ *induced by the projection* $\mathcal{A}(2)_* \to B(2)_*$ *sends* $ge[3,15]$ *to a non-trivial element.*

Proof. The projection $\mathcal{A}(2)_* \to B(2)_*$ induces a morphism of the DMSSs. The morphism of the E_1-terms reads

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(E_2 \otimes R \otimes A_1) \to \mathrm{Ext}^{s,t}_{B(2)_*}(F_2 \otimes S \otimes A_1).$$

By the change-of-rings isomorphism, this morphism identifies with the projection $p: R \to S$, which is surjective. The class $ge[3,15]$ is detected by $y_2^4(y_2^3 + y_1 y_3^2) \in R^7$, which maps to $y_2^7 \in S^7$ via p. By naturality, y_2^7 is a permanent cycle in the target DMSS. The only class in the E_1-term which can support a differential hitting y_2^7 is y_3^6. The class y_3^6 admits $v_2^4 y_3^2$ as a lift in the source DMSS. We have that

$$d_1(v_2^4 y_3^2) = d_1(v_2^4)y_3^2 + v_2^4 d_1(y_3^2) = (\alpha_{0,4,1}\alpha_{0,12,2}^2)y_3^2 + v_2^4(\alpha_{0,4,1}^2 y_2) = y_1 y_2^4 y_3^2 + y_3^4 y_1^2 y_2.$$

This uses the Leibniz rule, Proposition 3.1.17 part 11), Proposition 3.2.4 part 7). By naturality, the d_1-differential in the target DMSS is equal to $p(y_1 y_2^4 y_3^2 + y_3^4 y_1 y_2)$, which is equal to 0. Therefore, the image of $ge[3,15]$ is non-trivial.

□

Lemma 3.3.3. *The product of α and $e[4,23]$ is non-trivial, hence equal to $ge[3,15]$ if and only if the product of $p_*(\alpha)$ and $p_*(e[4,23])$ is non-trivial.*

Proof. The map $p : \mathcal{A}(2)_* \to B(2)_*$ induces the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{3,15}_{\mathcal{A}(2)_*}(\mathbb{F}_2) \otimes \mathrm{Ext}^{4,27}_{\mathcal{A}(2)_*}(A_1) & \longrightarrow & \mathrm{Ext}^{7,42}_{\mathcal{A}(2)_*}(A_1) \\ \downarrow p_* & & \downarrow p_* \\ \mathrm{Ext}^{3,15}_{B(2)_*}(\mathbb{F}_2) \otimes \mathrm{Ext}^{4,27}_{B(2)_*}(A_1) & \longrightarrow & \mathrm{Ext}^{7,42}_{B(2)_*}(A_1), \end{array}$$

where the horizontal maps are the respective multiplications. The result follows from the fact that $p_*(ge[3,15])$ is non-trivial by Lemma 3.3.2.

□

In view of Lemma 3.3.3, let us compute the product of $p_*(\alpha)$ and $p_*(e[4,23])$.

Lemma 3.3.4. *In the total cochain complexes of $B(2)_*^{\otimes *} \otimes F_2 \otimes S$ and of $B(2)_*^{\otimes *} \otimes F_2 \otimes S \otimes A_1$, respectively :*

i) $p_*(\alpha)$ *is represented by* $[\xi_2|1|y_2^2] + [\xi_1^3|1|y_2^2] + [\xi_1|1|y_3^2] \in B(2) \otimes F_2 \otimes S^2$;

ii) $p_*(e[4,23])$ *is represented by* $[1|y_2 y_3^3|a_0] + [1|y_2^2 y_3^2|a_1] + [1|y_2^3 y_3|a_2] + [1|y_2^4|a_3] \in F_2 \otimes S^4 \otimes A_1$.

Proof. A direct computation shows that these elements are cocycles of the total differentials, which are not coboundaries. One way to prove that they represent the right classes is to prove that they lift to cocycles in the total cochain complexes of $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R$ and of $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R \otimes A_1$, respectively.

It is easy to check that $[\xi_2|1|y_2^2] + [\xi_1^3|1|y_2^2] + [\xi_1|1|y_3^2] + [\xi_2|x_1|y_1^2] + [\xi_1^3|x_1|y_1^2] + [\xi_1|x_2|y_1^2] + [1|y_1^2 y_3] \in (\mathcal{A}(2)_* \otimes E_2 \otimes R^2) \oplus (E_2 \otimes R^3)$ is a cocycle in the total complex and is a lift of $[\xi_2|1|y_1^2] + [\xi_1^3|1|y_1^2] + [\xi_1|1|y_2^2]$.

For the other element, instead of finding a lift it suffices to show that p_* induces an isomorphism $\mathrm{Ext}^{4,27}_{\mathcal{A}(2)_*}(A_1) \xrightarrow{\cong} \mathrm{Ext}^{4,27}_{B(2)_*}(A_1)$, so that both are isomorphic to

$\mathbb{F}_2$. This can be proved by a similar argument to that used in the proof of Lemma 3.3.2. In effect, the non-trivial class of $\mathrm{Ext}^{4,27}_{\mathcal{A}(2)_*}(A_1)$ is detected by $y_2y_3^3$ in the DMSS. Via p_*, the latter is sent to $y_2y_3^3$ which is the unique non-trivial element of the E_1-term of the target DMSS in the appropriate tridegree. For degree reasons, $y_2y_3^3$ is not hit by any differential. Therefore, $y_2y_3^3$ survives the target DMSS and it follows that $\mathrm{Ext}^{4,27}_{\mathcal{A}(2)_*}(A_1) \xrightarrow{\cong} \mathrm{Ext}^{4,27}_{B(2)_*}(A_1) \cong \mathbb{F}_2$.

□

Set $M = [\xi_2|1|y_2^2] + [\xi_1^3|1|y_2^2] + [\xi_1|1|y_3^2]$ and $N = [1|y_2y_3^3|a_0] + [1|y_2^2y_3^2|a_1] + [1|y_2^3y_3|a_2] + [1|y_2^4|a_3]$. We need to show that MN, which is a $(d_v + d_h)$-cocycle, represents a non-trivial class in $\mathrm{Ext}^{7,42}_{B(2)_*}(A_1)$. We see that MN is an element in $B(2)_* \otimes F_2 \otimes S^6 \otimes A_1$ and $d_v(MN) = 0$. This means that MN represents a class in $\mathrm{Ext}^{1,42}_{B(2)_*}(F_2 \otimes S^6 \otimes A_1)$. However, the latter group is trivial because by the change-of-rings theorem, $\mathrm{Ext}^{*,*}_{B(2)_*}(\mathbb{F}_2, F_2 \otimes S \otimes A_1)$ is isomorphic to S which is concentrated only in cohomological degree 0. There must be an element $P \in F_2 \otimes S^6 \otimes A_1$ such that $d_v(P) = MN$, and so $d_h(P)$ represents the same class in $\mathrm{Ext}^{7,42}_{B(2)_*}(\mathbb{F}_2, A_1)$ as MN does.

We recall the values of $\lambda_{i,j}$ as introduced in Proposition 3.2.2: $\lambda_{1,0} = \lambda_{0,1} = 1$ and $\lambda_{0,0} = \lambda_{1,1} = 0$.

Lemma 3.3.5. *If we write* P *in the obvious monomial basis of* $B(2)\otimes F_2 \otimes S^6 \otimes A_1$,

$$P = \lambda_{i,j}[1|x_2|y_2^6|a_0] + ...,$$

where $\lambda_{i,j}$ *is as in Proposition 3.2.2, i.e.,* $\lambda_{1,0} = \lambda_{0,1} = 1$ *whereas* $\lambda_{0,0} = \lambda_{1,1} = 0$.

Proof. The product MN contains the term $[\xi_2|1|y_2^6|a_3]$. One can check that P must contain the term $[1|y_2^6|a_6]$, so that $d_v(P)$ contains the term $[\xi_2|1|y_2^6|a_3]$. Using the formula for the coaction of $\mathcal{A}(2)_*$ on a_6, one sees that $d_v(P)$ contains the term $\lambda_{i,j}[\xi_2^2|1|y_2^6|a_0]$, which is not a term of MN. In order to compensate this term, P must contain the term $\lambda_{i,j}[1|x_2|y_2^6|a_0]$. □

Lemma 3.3.6. *A* $(d_h + d_v)$*-cycle in* $F_2 \otimes S^7 \otimes A_1$ *gives rise to a non-trivial class in* $\mathrm{Ext}^{7,42}_{B(2)_*}(A_1)$ *if and only if it contains the term* $[1|y_2^7|a_0]$.

Proof. It is shown in the proof of Lemma 3.3.2 that

$$\mathrm{Ext}^{7,42}_{B(2)_*}(A_1) \cong \mathbb{F}_2$$

and that this group arises from

$$\mathrm{Ext}^{0,42}_{B(2)}(F_2 \otimes S^7 \otimes A_1) \cong \mathbb{F}_2\{y_2^7\} \subset S^7.$$

Therefore, by the shearing homomorphism, the only element in $F_2 \otimes S^7 \otimes A_1$ that represents the non-trivial class of $\mathrm{Ext}^{7,42}_{B(2)_*}(A_1)$ must contain the term $[1|y_2^7|a_0]$. □

Proposition 3.3.7. *The product* $\alpha e[4, 23]$ *is equal to* $\lambda_{i,j} g e[3, 15]$.

Proof. $\alpha e[4, 23]$ is non-trivial if and only if $d_h(P)$ represents a non-trivial class in $\mathrm{Ext}^{7,42}_{B(2)_*}(A_1)$. Lemma 3.3.5 shows that $d_h(P)$ contains the term $\lambda_{i,j}[1|y_1^7|a_0]$. Hence, Lemma 3.3.6 concludes the proof. □

The product between $\beta \in \mathrm{Ext}^{3,18}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ and $e[3, 15] \in \mathrm{Ext}^{3,18}_{\mathcal{A}(2)_*}(A_1)$ is easier because both have σ-filtration 0 in the Davis-Mahowald spectral sequence.

Proposition 3.3.8. $\beta e[3, 15] = e[6, 30]$.

Proof. The class β is represented by $y_2^3 + y_1 y_3^2$ in R^3 and $e[3, 15]$ is represented by $[y_2^3 + y_1 y_3^2|a_0]$ in $R^3 \otimes A_1$, both in the E_1-term of the respective DMSS. So the product $\beta e[3, 15]$ is represented by $[y_2^6 + y_1^2 y_3^4|a_0]$ in the E_1-term of the DMSS, which represents $e[6, 30]$ by Proposition 3.2.5. □

4 Partial study of the Adams spectral sequence for $tmf \wedge A_1$

In this section, we establish some differentials as well as some structures of the ASS for A_1. These are essential bits of information allowing us to run the homotopy fixed point spectral sequence in the next section.

Recall that the ASS for $tmf \wedge A_1$ which has E_2-term isomorphic to $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(A_1)$ is a spectral sequence of modules over that for tmf, whose E_2-term is isomorphic to $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$. We first recollect some known properties of the ASS for tmf, see [**DFHH14**], Chapter 13.

Theorem 4.0.1. *(i) The class* $g \in \mathrm{Ext}^{4,24}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ *is a permanent cycle detecting the image of* $\overline{\kappa} \in \pi_{20}(S^0)$ *via the Hurewicz map* $S^0 \to tmf$.

ii) There is the following d_2*-differential in the Adams spectral sequence for* tmf

$$d_2(w_2) = g\beta\alpha.$$

(iii) There is the following d_3*-differential in the Adams spectral sequence for* tmf

$$d_3(w_2^2(v_2^4\eta)) = g^6.$$

(iv) The class $\Delta^8 := w_2^4$ *survives the Adams spectral sequence.*

Proposition 4.0.2. *In the ASS for* $tmf \wedge A_1$*, there exists* $\lambda \in \mathbb{F}_2$ *such that the following statements are equivalent:*

i) $d_2(w_2e[4,23]) = \lambda g^2 e[6,30]$,
ii) $d_2(w_2e[9,48]) = \lambda g^4 e[3,15]$,
iii) $d_2(w_2e[10,53]) = \lambda g^5 e[0,0]$,
iv) $d_2(w_2e[7,38]) = \lambda g^4 e[1,5]$.

Proof. We will prove that $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv) \Rightarrow i)$. The charts of Figures (II.16) and (II.17) will make the proof easier to follow. First, we observe that all of the classes $e[4,23]$, $e[7,38]$, $e[9,48]$, $e[10,53]$ are permanent cycles, by sparseness.

$i) \Rightarrow ii)$ Suppose $d_2(w_2e[4,23]) = g^2e[6,30]$. Then $d_2(g^2w_2e[4,23]) = g^4e[6,30]$ by g-linearity. It follows that there is no room for a non-trivial differential on $w_2^2e[3,15]$. In order words, $w_2^2e[3,15]$ is a permanent cycle. Because of part $iii)$ of Theorem 4.0.1, a g^k-multiple of $w_2^2e[3,15]$ must be hit by a differential for some k less than 7. One can check that the only possibility is that $d_2(w_2^3e[9,48]) = g^4w_2^2e[3,15]$. Since w_2^2 is a d_2-cycle in the ASS for tmf, this differential implies that $d_2(w_2e[9,48]) = g^4e[3,15]$.

$ii) \Rightarrow iii)$ Suppose $d_2(w_2e[9,48]) = g^4e[3,15]$. Then the class $w_2^2e[0,0]$ is a permanent cycle, by sparseness. Again, a g^k-multiple of $w_2^2e[0,0]$ for some k smaller than 7 must be hit by a differential. Inspection shows that the classes $w_2^3e[10,53]$ and $w_2^4e[1,5]$ are the only ones that have the appropriate bidegree to support such a differential. However, $w_2^4e[1,5]$ is a permanent cycle, because w_2^4 and $e[1,5]$ are permanent cycles in their respective ASS. Thus, we have that $d_2(w_2e[10,53]) = g^5e[0,0]$.

$iii \Rightarrow iv)$ Suppose $d_2(w_2e[10,53]) = g^5e[0,0]$. Then the class $w_2^2e[1,5]$ is a permanent cycle, as there is no room for a non-trivial differential on it. Then $g^kw_2^2e[1,5]$ must be hit by a differential for some k less than 7. Inspection shows that the only possibility is that $d_2(w_2^3e[7,38]) = g^4w_2^2e[1,5]$. As w_2^2 is a d_2-cycle, it follows that $d_2(w_2e[7,38]) = g^4e[1,5]$.

$iv) \Rightarrow i)$ Suppose $d_2(w_2e[7,38]) = g^4e[1,5]$. By g-linearity, we get that $d_2(gw_2e[7,38]) = g^5e[1,5]$. It follows by sparseness that $w_2^2e[6,30]$ is a permanent cycle. Then the class $g^kw_2^2e[6,30]$ is hit by a differential for some k less than 7. Inspection shows that the only possibility is that $d_2(w_2^3e[4,23]) = g^2w_2^2e[6,30]$. Therefore, $d_2(w_2e[4,23]) = g^2e[6,30]$ by w_2^2-linearity.

□

Theorem 4.0.3. *In the Adams spectral sequence for* $tmf \wedge A_1[ij]$*, there are the following differential* d_2*:*

i) $d_2(w_2e[4,23]) = \lambda_{i,j} g^2 e[6,30]$,
ii) $d_2(w_2e[9,48]) = \lambda_{i,j} g^4 e[3,15]$,
iii) $d_2(w_2e[10,53]) = \lambda_{i,j} g^5 e[0,0]$,
iv) $d_2(w_2e[7,38]) = \lambda_{i,j} g^4 e[1,5]$.

Proof. By the Leibniz rule and part (ii) of Theorem 4.0.1,

$$d_2(w_2e[4,23]) = d_2(w_2)e[4,23] = g\beta\alpha e[4,23] = \lambda_{i,j}g^2e[6,30],$$

where the last equality follows from Proposition 3.3.7 and Proposition 3.3.8. Thus, the theorem follows from Proposition 4.0.2. □

Proposition 4.0.4. *There are the following* d_3*-differentials in the Adams spectral sequence for* $tmf \wedge A_1$

$$d_3(w_2^2e[10,53]) = g^5e[9,48]$$

$$d_3(w_2^3e[1,5]) = g^5w_2e[0,0].$$

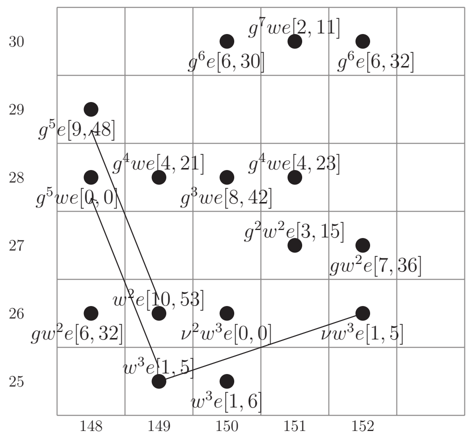

Figure II.15 – The Adams spectral sequence in the range $148 \leq t-s \leq 152$

Proof. We can check from the chart that $e[9,48]$ and $we[0,0]$ are permanent cycles. Then $g^le[9,48]$ and $g^kwe[0,0]$ must be targets of some differentials for some l and k less than 7. Inspection of the E_2-term shows that either

$$d_2(w_2^2e[10,53]) = g^5we[0,0] \text{ and } d_4(w_2^3e[1,5]) = g^5e[9,48]$$

or

$$d_3(w_2^2e[10,53]) = g^5e[9,48] \text{ and } d_3(w_2^3e[1,5]) = g^5w_2e[0,0].$$

However, the former possibility is ruled out because of the Leibniz rule:

$$d_2(w_2^2 e[10,53]) = d_2(w_2^2)e[10,53] = 2w_2 d_2(w_2)e[10,53] = 0,$$

where the first equality follows from the fact that $e[10,53]$ is a permanent cycle, by spareness. □

Corollary 4.0.5. The Toda bracket $\langle g^5, e[9,48], \nu\rangle$ can be formed and contains only elements which are divisible by g.

For references on Toda bracket, see [**Tod62**], [**Koc90**].

Proof. In the E_4-term of the ASS, the Massey product $\langle g^5, e[9,48], \nu\rangle$ has cohomological filtration 27 and is equal to zero with zero indeterminacy. On the other hand the corresponding Toda bracket can be formed with indeterminacy containing only multiples of g. We can check that all conditions of Moss's convergence theorem [**Mos70**] are met. This implies that the Toda bracket $\langle g^5, e[9,48], \nu\rangle$ contains an element detected in filtration 27 by 0, thus is a multiple of g. Therefore, this Toda bracket contains only multiples of g. □

Finally, we need to have control of the action of the class $\Delta^8 = w_2^4 \in \mathrm{Ext}^{32,224}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ on the E_∞-term of the ASS for $tmf \wedge A_1$. This will allow us to compare $\pi_*(tmf \wedge A_1)$ with $\pi_*(E_C^{hG_{24}} \wedge A_1)$ (see Corollary 5.1.3) and hence to discuss higher differentials in the HFPSS for $E_C^{hG_{24}} \wedge A_1$.

Proposition 4.0.6. *The class w_2^4 acts freely on the E_∞-term of the ASS for $tmf \wedge A_1$. As a consequence, the element $\Delta^8 \in \pi_{192}(tmf)$ acts freely on the homotopy groups of $tmf \wedge A_1$.*

Proof. Using the description of the E_2-term of the ASS for $tmf \wedge A_1$ in Theorem 3.2.5 and an elementary bidegree inspection, we can see that, if a class y is in an appropriate bidegree to support a differential hitting a class of the form $w_2^4 x$ for some class x, then y is divisible by w_2^4. Knowing that w_2^4 is a permanent cycle in the ASS for tmf, we conclude that, if a class x survives the E_r-term, then the multiple of x by all powers of w_2^4 also survive that term. Therefore, the Proposition follows by induction. □

Proposition 4.0.7. *For every element $x \in \pi_*(tmf \wedge A_1)$, the element $\Delta^8 x$ is divisible by $\overline{\kappa}$ (resp. ν) if and only if x is divisible by $\overline{\kappa}$ (resp. ν).*

Proof. The argument is similar to that used in the proof of Proposition 4.0.6. A bidegree inspection shows that, if a class $y \in \mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(A_1)$ is in an appropriate bidegree whose (exotic) product with g (resp. ν) might detect $\Delta^8 x$, then y is divisible by w_2^4. We conclude the proof by using the fact that the class w_2^4 acts freely on the ASS for $tmf \wedge A_1$, by Proposition 4.0.6. □

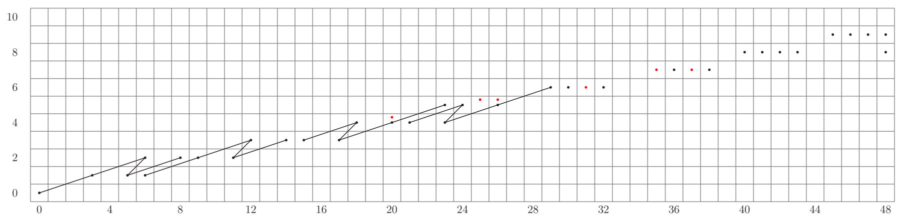

Figure II.16 – Adams spectral sequence for A_1 in the range $0 \leq t-s \leq 48$

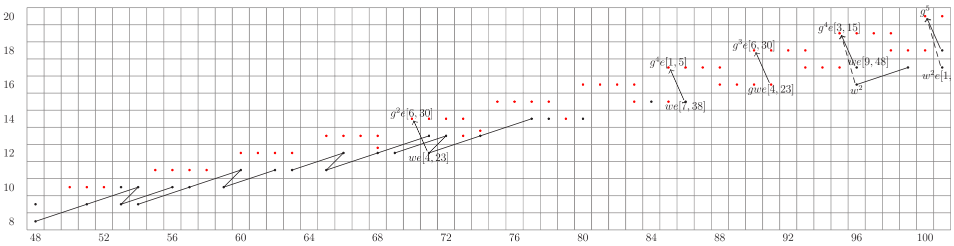

Figure II.17 – Adams spectral sequence for A_1 in the range $48 \leq t-s \leq 101$. The arrows in bold are differentials for the models $A_1[10]$ and $A_1[01]$ and the dashed arrows for the models $A_1[00]$ and $A_1[11]$

5 The homotopy fixed point spectral sequence for $E_C^{hG_{24}} \wedge A_1$

5.1 Preliminaries and recollection on cohomology of G_{24}

We recall the action of G_{24} on $(E_C)_*$ from Theorem 1.5.1. The action of G_{24} on $\mathbb{W}(\mathbb{F}_4)[[u_1]][u^{\pm 1}]$ is given by

$$\omega(u^{-1}) = \zeta^2 u^{-1} \qquad \omega(v_1) = v_1$$
$$i(u^{-1}) = \frac{-u^{-1} + v_1}{\zeta^2 - \zeta} \qquad i(v_1) = \frac{v_1 + 2u^{-1}}{\zeta^2 - \zeta}$$
$$j(u^{-1}) = \frac{-u^{-1} + \zeta^2 v_1}{\zeta^2 - \zeta} \qquad j(v_1) = \frac{v_1 + 2\zeta^2 u^{-1}}{\zeta^2 - \zeta}$$
$$k(u^{-1}) = \frac{-u^{-1} + \zeta v_1}{\zeta^2 - \zeta} \qquad k(v_1) = \frac{v_1 + 2\zeta u^{-1}}{\zeta^2 - \zeta}.$$

Equations I.11 and I.14 give us a way to get access to the homotopy groups of $E_C^{hG_{24}} \wedge A_1$.

Theorem 5.1.1. *There is a homotopy equivalence*

$$[(\Delta^8)^{-1}]tmf \wedge A_1 \simeq (E_C^{hG_{24}})^{h\mathrm{Gal}} \wedge A_1,$$

where Gal *denotes the Galois group* $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$.

Proof. We have

$$\begin{aligned}[(\Delta^8)^{-1}]tmf \wedge A_1 &\simeq TMF \wedge A_1 \text{ (Equation } I.14)\\ &\simeq L_2(TMF) \wedge A_1 \text{ (}TMF \text{ is } E(2)\text{-local)}\\ &\simeq L_2(TMF \wedge A_1) \text{ (}L_2 \text{ is smashing)}\\ &\simeq L_{K(2)}(TMF) \wedge A_1\\ &\simeq (E_C^{hG_{24}})^{h\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \wedge A_1 \text{ (Equation } I.11).\end{aligned}$$

The fourth equivalence is Lemma 7.2 of [HS99] applied to the $K(2)$-localisation and A_1, which is finite spectrum of type 2. □

Corollary 5.1.2. There is a homotopy equivalence

$$\mathrm{Gal}_+ \wedge [(\Delta^8)^{-1}]tmf \wedge A_1 \simeq E_C^{hG_{24}} \wedge A_1.$$

Therefore,

$$\mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} (\Delta^8)^{-1}(\pi_*(tmf \wedge A_1)) \cong \pi_*(E_C^{hG_{24}} \wedge A_1).$$

Proof. This is a consequence of Theorem 5.1.1 and Lemma 1.37 of [BG18]. □

Let us denote by

$$\Theta : \mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_*(tmf \wedge A_1) \to \pi_*(E_C^{hG_{24}} \wedge A_1), \tag{II.7}$$

given by pre-composing the isomorphism of Corollary 5.1.2 with the natural homomorphism $\pi_*(tmf \wedge A_1) \to \pi_*([(\Delta^8)^{-1}]tmf \wedge A_1)$.

Corollary 5.1.3. The homomorphism Θ is injective. Moreover, it remains injective after quotienting out by the ideal of $\pi_*(S^0)$ generated by $(\overline{\kappa}, \nu)$.

Proof. This follows from Theorem 5.1.1, Proposition 4.0.6 and Proposition 4.0.7. □

We continue to recollect some necessary information about the HFPSS converging to $\pi_*(E_C^{hG_{24}})$:

$$\mathrm{H}^s(G_{24}, (E_C)_t) \Longrightarrow \pi_{t-s}(E_C^{hG_{24}}). \tag{II.8}$$

The elements $\eta \in \pi_1(S^0)$, $\nu \in \pi_3(S^0)$, $\overline{\kappa} \in \pi_{20}(S^0)$ are sent non-trivially to elements of the same name in $\pi_*(E_C^{hG_{24}})$ via the Hurewicz map $S^0 \to E_C^{hG_{24}}$. As the latter factors through the unit map of tmf, the element $\overline{\kappa}^6 = 0$ in $\pi_*(E_C^{hG_{24}})$ because $\overline{\kappa}^6 = 0$ in $\pi_*(tmf)$ (see [Bau08]). These elements are detected by $\eta \in \mathrm{H}^1(G_{24}, (E_C)_2)$, $\nu \in \mathrm{H}^1(G_{24}, (E_C)_4)$, $\overline{\kappa} \in \mathrm{H}^4(G_{24}, (E_C)_{24})$, respectively. Furthermore, there is a class $\Delta \in \mathrm{H}^0(G_{24}, (E_C)_{24})$ such that Δ^8 is a permanent cycle detecting the periodicity of $E_C^{hG_{24}}$.

The HFPSS for $E_C^{hG_{24}} \wedge A_1$ is a spectral sequence of modules over that of (II.8):

$$\mathrm{H}^s(G_{24}, (E_C)_t A_1) \Longrightarrow \pi_{t-s}(E_C^{hG_{24}} \wedge A_1). \tag{II.9}$$

In Section 5.2, we will compute $\mathrm{H}^*(G_{24}, (E_C)_* A_1)$ as a module over a certain subalgebra of $\mathrm{H}^*(G_{24}, (E_C)_*)$. Let $\pi : (E_C)_* \to \mathbb{F}_4[u^{\pm 1}]$ be the quotient of $(E_C)_*$ by the maximal ideal $(2, u_1)$. As the ideal $(2, u_1)$ is preserved by the action of $\mathbb{S}_C$, the ring $\mathbb{F}_4[u^{\pm 1}]$ inherits an action of $\mathbb{S}_C$, and so of its subgroup G_{24}. We need the computation of the ring structure of $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm 1}])$, which is due to Hans-Werner Henn, see [Bea17], Appendix A.

Proposition 5.1.4. *There are classes $z \in \mathrm{H}^4(G_{24}, \mathbb{F}_4[u^{\pm 1}]_0)$, $a \in \mathrm{H}^1(G_{24}, \mathbb{F}_4[u^{\pm 1}]_2)$, $b \in \mathrm{H}^1(G_{24}, \mathbb{F}_4[u^{\pm 1}]_4)$, $v_2 \in \mathrm{H}^0(G_{24}, (\mathbb{F}_4[u^{\pm 1}])_6)$ such that there is an isomorphism of graded algebras*

$$\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm 1}]) \cong \mathbb{F}_4[v_2^{\pm 1}, z, a, b]/(ab, b^3 = v_2 a^3).$$

Proposition 5.1.5. *The homomorphism of graded algebras*

$$\mathrm{H}^*(G_{24}, E_{C*}) \to \mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm 1}])$$

induced by the projection $(E_C)_* \to \mathbb{F}_4[u^{\pm 1}]$ sends η to a, ν to b, $\overline{\kappa}$ to $v_2^4 z$, and Δ to v_2^4.

5.2 On the cohomology groups $\mathrm{H}^*(G_{24}, (E_C)_*(A_1))$

We first determine $(E_C)_*(A_1)$ using the cofiber sequences through which A_1 are defined. The cofiber sequence $\Sigma S^0 \xrightarrow{\eta} S^0 \to C_\eta$ gives rise to a short exact sequence of E_C-homology

$$0 \to (E_C)_* \to (E_C)_*(C_\eta) \to (E_C)_*(S^2) \to 0,$$

since $(E_C)_*$ is concentrated in even degrees. Hence, as an $(E_C)_*$-module

$$(E_C)_*(C_\eta) \cong \mathbb{W}(\mathbb{F}_4)[[u_1]][u^{\pm 1}]\{e_0, e_2\},$$

where e_0 is the image of $1 \in (E_C)_0$ and e_2 is a lift of $\Sigma^2 1 \in (E_C)_2(S^2)$. Next, the long exact sequence in E_C-homology associated to $C_\eta \xrightarrow{2} C_\eta \to Y$ is the short exact sequence

$$0 \to (E_C)_*(C_\eta) \xrightarrow{\times 2} (E_C)_*(C_\eta) \to (E_C)_*(Y) \to 0$$

since multiplication by 2 on $(E_C)_*(C_\eta) \cong \mathbb{W}(\mathbb{F}_4)[[u_1]][u^{\pm 1}]\{e_0, e_2\}$ is injective. Therefore

$$(E_C)_*(Y) \cong \mathbb{F}_4[[u_1]][u^{\pm 1}]\{e_0, e_2\}.$$

Now A_1 is the cofiber of some v_1-self map of Y: $\Sigma^2 Y \xrightarrow{v_1} Y \to A_1$. The following lemma describe the induced homomorphism in E_C-homology of these v_1-self maps.

Lemma 5.2.1. *The homomorphism* $(E_C)_*(v_1)$ *is given by multiplication by* $u_1 u^{-1}$. *Therefore,*

$$(E_C)_*(A_1) \cong \mathbb{F}_4[u^{\pm 1}]\{e_0, e_2\}.$$

Proof. Let $K(1)$ be the first Morava K-theory at the prime 2 such that $K(1)_* \cong \mathbb{F}_2[v_1^{\pm 1}]$ where $|v_1| = 2$ and BP be the Brown-Peterson spectrum at the prime 2. There is a map of ring spectra $BP \to K(1)$ that classifies the complex orientation of $K(1)$. Recall that the coefficient ring of BP is given by

$$BP_* \cong \mathbb{Z}_{(2)}[v_1, v_2, ...],$$

where $|v_i| = 2(2^i - 1)$, see [**Ada74**], Part II. The induced homomorphism of coefficient rings sends v_1 to v_1. The map $BP \to K(1)$ gives rise to the commutative diagram

$$\begin{array}{ccc} BP_*(\Sigma^2 Y) & \xrightarrow{BP_*(v_1)} & BP_*(Y) \\ \downarrow & & \downarrow \\ K(1)_*(\Sigma^2 Y) & \xrightarrow{K(1)_*(v_1)} & K(1)_*(Y) \end{array}$$

By definition, a v_1-self-map of Y induces in $K(1)$-homology multiplication by v_1. The above diagram forces, then for degree reasons, that $BP_*(v_1)$ is given by multiplication by $v_1 \in BP_2$. Now, let $c : BP \to E_C$ be the map of ring spectra that classifies the 2-typification of the formal group law of E_C. One can show that the 2-series of the latter has leading term $u_1 u^{-1} x^2$ modulo (2), see [**Bea17**], Proposition 6.1.1. This implies that the induced homomorphism $c_* : BP_* \to (E_C)_*$ sends v_1 to $u_1 u^{-1}$ modulo 2. By naturality, $(E_C)_*(v_1)$ is also given by multiplication by $u_1 u^{-1}$. □

We now describe the action of G_{24} on $(E_C)_*(A_1)$. For any 2-local finite spectrum X, the map c, introduced in the proof of Lemma 5.2.1, induces a map of ANSS

$$\begin{array}{ccc} \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*X) & \longrightarrow & \mathrm{Ext}^{s,t}_{(E_C)_*E_C}((E_C)_*, (E_C)_*X) \\ \Downarrow & & \Downarrow \\ \pi_{t-s}(X) & \longrightarrow & \pi_{t-s}(L_{K(2)}X) \end{array}$$

where $(E_C)_*E_C$ stands for $\pi_*(L_{K(2)}(E_C \wedge E_C))$. By Morava's change-of-ring theorem (see [**Dev95**]), one has

$$\mathrm{Ext}^{s,t}_{(E_C)_*E_C}((E_C)_*, (E_C)_*) \cong \mathrm{H}^s_c(\mathbb{G}_C, (E_C)_t).$$

Now the map c induces a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & BP_* & \xrightarrow{\times 2} & BP_* & \longrightarrow & BP_*/(2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle c_*} & & \\ 0 & \longrightarrow & E_{C*} & \xrightarrow{\times 2} & E_{C*} & \longrightarrow & E_{C*}/(2) & \longrightarrow & 0. \end{array}$$

Therefore, we obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*/2) & \xrightarrow{\delta_{BP}} & \mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*) \\ \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle c_*} \\ \mathrm{H}^0_c(\mathbb{G}_C, E_{C*}/2) & \xrightarrow{\delta_{E_C}} & \mathrm{H}^1_c(\mathbb{G}_C, E_{C*}), \end{array}$$

where δ_{BP} and δ_{E_C} denote the respective connecting homomorphisms. By [**Rav86**], Theorem 4.3.6, one has that

$$\mathrm{Ext}^{0,2}_{BP_*BP}(BP_*, BP_*/2) = \mathbb{Z}_{(2)}\{v_1\} \text{ and } \delta_{BP}(v_1) = \eta \in \mathrm{Ext}^{1,2}_{BP_*BP}(BP_*, BP_*),$$

where η is a permanent cycle representing the Hopf element $\eta \in \pi_1(S^0)$. By naturality, $\delta_{E_C}(v_1) = c_*(\eta)$. Therefore, as a cocycle in $Map_c(\mathbb{G}_C, (E_C)_2)$, $c_*(\eta)$ is given by

$$\mathbb{G}_C \to (E_C)_2, \quad g \mapsto \frac{g(v_1) - v_1}{2}$$

On the other hand, let us consider the short exact sequence

$$0 \to E_{C*} \to E_{C*}(C_\eta) \to E_{C*}(S^2) \to 0$$

representing the class $c_*(\eta)$, so that the connecting homomorphism sends $\Sigma^2 1$ to $c_*(\eta)$. Thus, if e_2 is a lift of $\Sigma^2 1$ in $E_{C*}(C_\eta)$, then $c_*(\eta)$ is represented by the cocycle

$$\mathbb{G}_C \to (E_C)_2, \;\; g \mapsto g(e_2) - e_2.$$

This implies that one can modify e_2 so that

$$\frac{g(v_1) - v_1}{2} = g(e_2) - e_2 \; \forall \; g \in \mathbb{G}_C.$$

With this choice of e_2, we see that $E_{C*}(C_\eta) = E_{C*}\{e_0, e_2\}$ and the action of $\mathbb{G}_C$ on e_2 is given by the formula

$$g(e_2) = e_2 + \frac{g(v_1) - v_1}{2} e_0 \tag{II.10}$$

Note that when determining $(E_C)_*(A_1)$, we did not specify any lift e_2 of $\Sigma^2 1$. From now on, we will fix e_2 such that the formula of (II.10) holds.

Proposition 5.2.2. *As an* $(E_C)_*$*-module,* $(E_C)_*(A_1)$ *is isomorphic to* $\mathbb{F}_4[u^{\pm 1}]\{e_0, e_2\}$ *and the action of* G_{24} *is given by*

$$\omega(u^{-1}) = \zeta^2 u^{-1}, \quad \omega(e_0) = e_0, \quad \omega(e_2) = e_2$$

$$i(u^{-1}) = u^{-1}, \quad i(e_0) = e_0, \quad i(e_2) = e_2 + u^{-1} e_0$$

$$j(u^{-1}) = u^{-1}, \quad j(e_0) = e_0, \quad j(e_2) = e_2 + \zeta^2 u^{-1} e_0$$

$$k(u^{-1}) = u^{-1}, \quad k(e_0) = e_0, \quad k(e_2) = e_2 + \zeta u^{-1} e_0$$

Proof. The first part of the statement is the content of Lemma 5.2.1. The second part follows from the action of G_{24} on v_1 given in Theorem 1.5.1 and the formula (II.10). □

Corollary 5.2.3. $E_{C*}(A_1)$ sits in a non-split short exact sequence of G_{24}-modules

$$0 \to \mathbb{F}_4[u^{\pm 1}]\{e_0\} \to E_{C*}(A_1) \to \mathbb{F}_4[u^{\pm 1}]\{e_2\} \to 0. \tag{II.11}$$

Proof. This is immediate in view of the explicite description of the action of G_{24} on $(E_C)_* A_1$. □

The cohomology group $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm 1}]\{e_i\})$ $i \in \{0, 2\}$ is free of rank one as a module over $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm 1}])$. For $i \in \{0, 2\}$, we choose the generators $e[0, i] \in \mathrm{H}^0(G_{24}, (\mathbb{F}_4[u^{\pm 1}]\{e_i\})_i)$ of these modules.

Corollary 5.2.4. The connecting homomorphism induced from the short exact sequence (II.11) in Corollary 5.2.3

$$\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm 1}]\{e_2\}) \xrightarrow{\delta} \mathrm{H}^{*+1}(G_{24}, \mathbb{F}_4[u^{\pm 1}]\{e_0\})$$

is $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm 1}])$-linear and sends $e[0,2]$ to $ae[0,0]$ up to a unit of $\mathbb{F}_4$, where, as a reminder, $a \in \mathrm{H}^1(G_{24}, (\mathbb{F}_4[u^{\pm 1}])_2)$.

Proof. That δ is $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm 1}])$-linear is a well-known property of the connecting homomorphism (See [**Bro82**], V.3). Next, since the short exact sequence in Corollary 5.2.3 does not split, the connecting homomorphism δ sends $e[2,0]$ to a non-trivial class and hence to $ae[0,0]$ up to a unit of $\mathbb{F}_4$. □

Using the description of $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm 1}])$ and the long exact sequence associated to the short exact sequence of Corollary 5.2.3, we obtain the following description of $\mathrm{H}^*(G_{24}, (E_C)_*(A_1))$:

Proposition 5.2.5. *As a module over* $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm}])$*, there is an isomorphism*

$$\mathrm{H}^*(G_{24}, (E_C)_*(A_1)) = \mathbb{F}_4[v_2^{\pm 1}, z, a, b]/(a, b^3)\{e[0,0], e[1,5]\}$$

where $e[0,0] \in \mathrm{H}^0(G_{24}, (E_C)_0(A_1))$ *and* $e[1,5] \in \mathrm{H}^1(G_{24}, (E_C)_6(A_1))$.

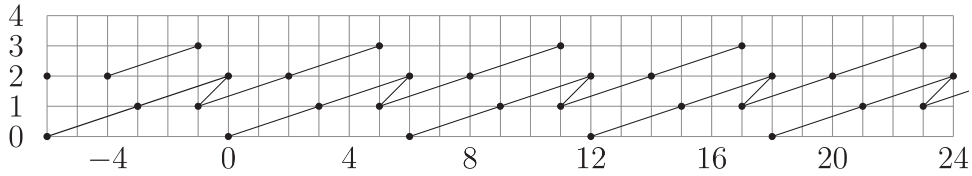

Figure II.18 – $\mathrm{H}^s(G_{24}, (E_C)_t(A_1))$ depicted in the coordinate (s, t-s))

The above proposition also gives the action of $\mathrm{H}^*(G_{24}, (E_C)_*)$ on $\mathrm{H}^*(G_{24}, (E_C)_*A_1)$. In effect, the action of E_{C*} on $E_{C*}(A_1)$ factors though $\mathbb{F}_4[u^{\pm 1}]$ via $E_{C*} \xrightarrow{\pi} \mathbb{F}_4[u^{\pm 1}]$. As a consequence, the action of $\mathrm{H}^*(G_{24}, E_{C*})$ on $\mathrm{H}^*(G_{24}, E_{C*}(A_1))$ factors through the induced homomorphism in cohomology of G_{24}. In particular, il follows from Proposition 5.1.5 that the classes $\Delta, \overline{\kappa}, \nu$ act on $\mathrm{H}^*(G_{24}, E_{C*}(A_1))$ as $v_2^4, v_2^4 z, b$ do, respectively. Because $\Delta, \overline{\kappa}, \nu$ are classes of the E_2-term of the HFPSS for $E_C^{hG_{24}}$, we will describe the HFPSS for $E_C^{hG_{24}} \wedge A_1$ in terms of the latter. We use the notation $e[s,t]$ to denote the unique non-trivial class, up to a non-zero element of $\mathbb{F}_4$, of $\mathrm{E}_r^{s,s+t}$. Thus, Proposition 5.2.5 can be rewritten as follows. As a module over $\mathbb{F}_4[\Delta^{\pm 1}, \overline{\kappa}, \nu]/(\nu^3)$, $\mathrm{H}^*(G_{24}, (E_C)_*A_1)$ is free on the following classes:

$$e[0,0], e[1,5], e[0,6], e[1,11], e[0,12], e[1,17], e[0,18], e[1,23].$$

5.3 Differentials of the homotopy fixed point spectral sequence for $E_C^{hG_{24}} \wedge A_1$

The HFPSS for $E_C^{hG_{24}} \wedge A_1$ has the following features. The spectrum $E_C \wedge A_1$ is a G_{24}-E_C-module in the sense that $E_C \wedge A_1$ is an E_C-module and the structure maps are G_{24}-equivariant. This guarantees that the HFPSS for $E_C^{hG_{24}} \wedge A_1$ is a module over that for $E_C^{hG_{24}}$. In particular, all differentials are $\overline{\kappa}$-linear. This element plays a central role here: the group G_{24} is a group with periodic cohomology (see [**Bro82**], Chapter VI) and $\overline{\kappa} \in \mathrm{H}^4(G_{24}, (E_C)_*)$ is a cohomological periodicity class. These features induce more structure on the HFPSS.

Definition 5.3.1. Let R be a ring spectrum and G be a finite group acting on R by maps of ring spectra. The pair (G, R) is said to be regular if G is a group with periodic cohomology and there exists a cohomological periodicity class $u \in \mathrm{H}^*(G, R_*)$ which is a permanent cycle in the HFPSS for R^{hG}.

Lemma 5.3.2. *Let (G, R) be a regular pair as in Definition 5.3.1 and X be a G-R spectrum. Suppose $u \in \mathrm{H}^k(G, R_*)$ is a cohomological periodicity class which is a permanent cycle in the HFPSS for R^{hG}. Then the E_r-term of the HFPSS for X^{hG} has the following properties:*

(i) All classes of cohomological filtration at least k are divisible by u;

(ii) All classes of cohomological filtration at least r are u-free.

Proof. We will prove by induction on r that the E_r-term of the HFPSS for X^{hG} has the properties (i) and (ii). The E_2-term is isomorphic to $\mathrm{H}^*(G, \pi_*(X))$. We recall that the natural map from the cohomology to the Tate cohomology $\iota : \mathrm{H}^s(G, \pi_t X) \to \hat{\mathrm{H}}^s(G, \pi_t(X))$ is an epimorphism and is an isomorphism when $s > 0$, see [**Bro82**], Chapter VI. Because G has periodic cohomology, we have

$$\hat{\mathrm{H}}^s(G, \pi_t X) \cong \hat{\mathrm{H}}^s(G, \pi_t X)[u^{-1}],$$

which means that the group $\hat{\mathrm{H}}^s(G, \pi_t X)$ is u-free and is divisible by u. Since $\iota : \mathrm{H}^s(G, \pi_t X) \to \hat{\mathrm{H}}^s(G, \pi_t(X))$ is an isomorphism when $s > 0$, all classes of positive cohomological degree of $\mathrm{H}^s(G, \pi_t X)$ are u-free.

Now suppose x is a class of $\mathrm{H}^s(G, \pi_t X)$ with $s \geq k$. Then the class $u^{-1}\iota(x) \in \hat{\mathrm{H}}^{s-k}(G, \pi_t X)$ has a pre-image $y \in \mathrm{H}^{s-k}(G, \pi_t X)$ (because $s - k \geq 0$), i.e.

$$\iota(y) = u^{-1}\iota(x).$$

This implies that

$$\iota(uy) = u\iota(y) = \iota(x),$$

and thus since $s > 0$,

$$uy = x.$$

Thus, the E_2-term has the properties (i) and (ii). Suppose that the E_r-term satisfies (i) and (ii). Let $[x] \in \mathrm{E}_{r+1}$ be a non-trivial class represented by $x \in \mathrm{E}_r$. Suppose that x has its cohomological filtration $s \geq k$. By the induction hypothesis, there exists $y \in \mathrm{E}_r^{s-k,*}$ such that $uy = x$. We show that y is a d_r-cycle. Because x is a d_r-cycle, we have by u-linearity that $ud_r(y) = d_r(uy) = d_r(x) = 0$. However, the cohomological filtration of $d_r(y)$ is at least r, and so it is u-free by the induction hypothesis, and so $d_r(y) = 0$. Therefore, $[x]$ is divisible by u.

Now we prove that E_{r+1} has the property (ii). Suppose that $[x]$ is u-torsion and has its cohomological filtration at least $r+1$. Without loss of generality, we can assume that $u[x] = 0$. Then there exists $y \in \mathrm{E}_r$ such that $d_r(y) = ux$. The cohomological filtration of y is at least to $r+1+k-r = k+1$, hence y is divisible by u, i.e., there exists $z \in \mathrm{E}_r$ such that $uz = y$, and then by u-linearity,

$$ud_r(z) = d_r(uz) = d_r(y) = ux.$$

However, $d_r(z) - x$ has cohomolgical filtration at least $r+1$, it must be u-free by hypothesis (ii), hence is equal to zero, i.e., $[x]$ is trivial in E_{r+1}.

We conclude that the E_{r+1}-term satisfies (i) and (ii), thus finishing the proof by induction. □

Corollary 5.3.3. Let (G, R) be a regular pair and X be a G-R spectrum. Suppose $u \in \mathrm{H}^k(G, R_*)$ is a cohomological periodicity class which is a permanent cycle in the HFPSS for R^{hG}. Then we have, in the HFPSS for X^{hG},

1. At the E_r-term, u-torsion classes are permanent cycles.
2. Any u-free tower is truncated by at most one other u-free tower by the same differential. More precisely, if x is a class of cohomological filtration less than k, then there exists at most one class y of cohomological filtration less than k such that there exists an unique integer l and a unique integer r such that $d_r(u^m y) = u^{m+l}x$ for all non-negative integers m. Moreover, all classes $u^i x$ for $i \in \{0, 1, ..., m-1\}$ survive the spectral sequence.
3. Suppose some power of u is hit by a differential in the HFPSS for R^{hG}. Then any u-free tower consisting of permanent cycles is truncated by a unique u-free tower. Moreover, the HFPSS has a horizontal vanishing line.
4. Every element of $\pi_*(X^{hG})$ that is detected in filtration at least k is divisible by $\overline{u}$ where $\overline{u}$ is an element of $\pi_*(R^{hG})$ detected by u.

Remark 5.3.4. This situation turns out to be abundant once the group in question is a group with periodic cohomology. For example, all finite subgroups of $\mathbb{G}_C$ have these properties.

We return to the HFPSS for $E_C^{hG_{24}} \wedge A_1$. We will call the set $\{\overline{\kappa}^l x | l \in \mathbb{N}\}$ associated to a class x in some page of the HFPSS the $\overline{\kappa}$-family of that class. We note that all classes of $\mathrm{E}_2^{s,t}$ with $t-s$ odd are trivial, so that all differentials of length even are trivial, i.e., we have that $\mathrm{E}_r = \mathrm{E}_{r+1}$ if r is even; this is called the checkerboard phenomenon. The following proposition gives us the horizontal vanishing line of the HFPSS for $E_C^{hG_{24}} \wedge A_1$.

Proposition 5.3.5. *The HFPSS for* $E_C^{hG_{24}} \wedge A_1$ *has a horizontal vanishing line of height* 23*, i.e.,* $\mathrm{E}_{24}^{s,t} = 0$ *if* $s > 23$*. As a consequence, it collapses at the* E_{24}*-term.*

Proof. As $\overline{\kappa}^6 = 0$ in $\pi_*(E_C^{hG_{24}})$, the class $\overline{\kappa}^6$ must be hit by a differential which is of length at most 23. This is because $\overline{\kappa}^6$ has cohomological filtration 24 and all even differentials are trivial. Hence $\overline{\kappa}^6$ is trivial in the E_{24}-term of the HFPSS for $E_C^{hG_{24}}$. Next, because the E_{24}-term of the HFPSS for $E_C^{hG_{24}} \wedge A_1$ is a module over that for $E_C^{hG_{24}}$, the class $\overline{\kappa}^6$ acts trivially on the E_{24}-term of the HFPSS for $E_C^{hG_{24}} \wedge A_1$. Since all classes which are not a multiple of $\overline{\kappa}$ have cohomological filtration at most 3, the HFPSS has the horizontal vanishing line of height 23. □

Proposition 5.3.6. *The following classes are permanent cycles*

$$e[0,0], e[1,5], e[0,6], e[1,11], e[1,15], e[1,17], e[1,21], e[1,23].$$

Proof. Firstly, the class $e[0,0]$ is a permanent cycle because it detects the inclusion $S^0 \to A_1$ into the bottom cell of A_1. Next, we recapitulate, in the following table, the associated graded object with respect to the induced Adams filtration on the groups $\pi_*(tmf \wedge A_1)/(\overline{\kappa})$ in the following stems.

Dim	6	15	17	21	23
Value	$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2 \oplus \mathbb{F}_2$

By Corollary 5.1.3, the groups $\pi_*(E_C^{hG_{24}} \wedge A_1)/(\overline{\kappa})$ in these dimensions must have order twice as big as the respective groups. Inspection in the E_2-term of the HFPSS through dimensions from 0 to 23 and in cohomological filtration less than 4 show that the classes $e[0,6], e[1,15], e[1,21], e[1,23]$ are permanent cycles.

Note that the groups $\pi_0(tmf \wedge A_1)$ and $\pi_6(tmf \wedge A_1)$ are annihilated by η. This means that $e[0,0]$ and $e[0,6]$ detects two elements which are annihilated by η. It follows that the Toda brackets $\langle \nu, \eta, e[0,0] \rangle$ and $\langle \nu, \eta, e[0,6] \rangle$ can be formed. By juggling,

$$\eta \langle \nu, \eta, e[0,0] \rangle = \langle \eta, \nu, \eta \rangle e[0,0] = \nu^2 e[0,0]$$

and

$$\eta \langle \nu, \eta, e[0,6] \rangle = \langle \eta, \nu, \eta \rangle e[0,6] = \nu^2 e[0,6].$$

Observe that $\nu^2 e[0,0]$ and $\nu^2 e[0,6]$ are nontrivial and are detected in cohomological filtration 2. Consequently, both $\langle \nu, \eta, e[0,0] \rangle$ and $\langle \nu, \eta, e[0,6] \rangle$ are nontrivial and are represented by classes in cohomological filtration at most 1. Therefore $e[1,5]$ and $e[1,11]$ are permanent cycles.

The unique nontrivial element of $\pi_{11}(tmf \wedge A_1)/(\overline{\kappa})$ is annihilated by ν^2. This implies that the class $\nu^2 e[1,11]$ is the target of some differential. Since $\pi_{17}(E_C^{hG_{24}} \wedge A_1)/(\overline{\kappa})$ has order at least equal to 4, the class $e[1,17]$ must be a permanent cycle representing the only element in dimension 17 of $\pi_*(E_C^{hG_{24}} \wedge A_1)/(\overline{\kappa})$. □

$d_3 -$ differentials

Proposition 5.3.7. *As a module over* $\mathbb{F}_4[\Delta^{\pm 1}, \overline{\kappa}, \nu]/(\nu^3)$*, the term* $\mathrm{E}_2 = \mathrm{E}_3$ *is free on the generators*

$$e[0,0], e[1,5], e[0,6], e[1,11], e[0,12], e[1,17], e[0,18], e[1,23]. \tag{II.12}$$

Proposition 5.3.8. *The* d_3*-differential in the HFPSS for* $E_C^{hG_{24}} \wedge A_1$ *is trivial on all of the generators of (II.12) with the exception of*

i) $d_3(e[0,12]) = \nu^2 e[1,5]$

ii) $d_3(e[0,18]) = \nu^2 e[1,11]$.

Proof. That $e[0,0], e[1,5], e[0,6], e[1,11], e[1,17], e[1,23]$ are d_3-cycles follows from Proposition 5.3.6. For the two other classes, the proof of Proposition 5.3.6 implies that the elements $\Theta(e[1,5])$ and $\Theta(e[2,11])$ are detected by $e[1,5]$ and $e[1,11]$, respectively. Moreover, the elements $e[1,5]$ and $e[2,11]$ are annihilated by ν^2 in $\pi_*(tmf \wedge A_1)$. It follows that, in the HFPSS, the classes $\nu^2 e[1,5]$ and $\nu^2 e[1,11]$ must be hit by some differentials. The only possibilities are $d_3(e[0,12]) = \nu^2 e[1,5]$ and $d_3(e[0,18]) = \nu^2 e[1,11]$. □

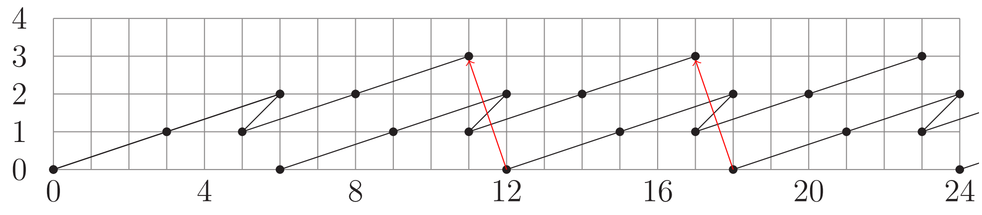

Figure II.19 – Differentials d_3

Corollary 5.3.9. As a module over $\mathbb{F}_4[\Delta^{\pm 1}, \overline{\kappa}, \nu]/(\nu^3)$, the term $\mathrm{E}_4 = \mathrm{E}_5$ is a direct sum of cyclic modules generated by the classes

$$e[0,0], e[1,5], e[0,6], e[1,11], e[1,15], e[1,17], e[1,21], e[1,23] \tag{II.13}$$

with the relations

$$\nu^2 e[1,5] = \nu^2 e[1,11] = \nu^2 e[1,15] = \nu^2 e[1,21] = 0. \qquad \text{(II.14)}$$

Proof. This is straightforward from Proposition 5.3.8 and from the fact that $\Delta, \overline{\kappa}, \nu$ are d_3-cycles in the HFPSS for $E_C^{hG_{24}}$. □

d_5-differentials. We need the d_5-differential, in the HFPSS for $E_C^{hG_{24}}$, $d_5(\Delta) = \overline{\kappa}\nu$ (see [**Bau08**], Section 8.3).

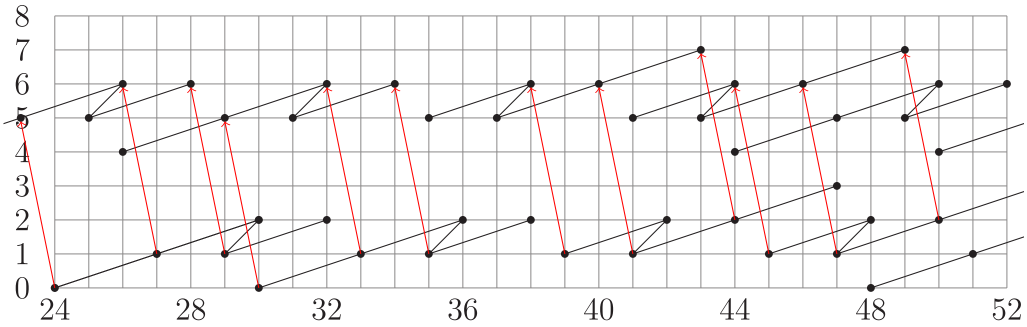

Figure II.20 – Differentials d_5

Proposition 5.3.10. *The* $\mathrm{E}_6 = \mathrm{E}_7$*-term is a module over* $\mathbb{F}_4[(\Delta^8)^{\pm 1}, \overline{\kappa}, \nu]/(\overline{\kappa}\nu)$ *and, as such,* $\mathrm{E}_6 = \mathrm{E}_7$ *is a direct sum of cyclic modules generated by the following classes for* $i \in 0, 2, 4, 6$ *with the respective annihilator ideal:*

generator	$\Delta^i e[0,0]$	$\Delta^i e[1,5]$	$\Delta^i e[0,6]$	$\Delta^i e[1,11]$
ideal	(ν^3)	(ν^2)	(ν^3)	(ν^2)
generator	$\Delta^i e[1,15]$	$\Delta^i e[1,17]$	$\Delta^i e[1,21]$	$\Delta^i e[1,23]$
ideal	(ν^2)	(ν^3)	(ν^2)	(ν^3)
generator	$\Delta^i e[2,30]$	$\Delta^i e[2,32]$	$\Delta^i e[2,36]$	$\Delta^i e[2,38]$
ideal	(ν)	(ν)	(ν)	(ν)
generator	$\Delta^i e[2,42]$	$\Delta^i e[3,47]$	$\Delta^i e[2,48]$	$\Delta^i e[3,53]$
ideal	(ν)	(ν)	(ν)	(ν).

Proof. The classes $\Delta^8, \overline{\kappa}, \nu$ acts on the HFPSS for $E_C^{hG_{24}} \wedge A_1$, since they are permanent cycles in the HFPSS for $E_C^{hG_{24}}$. Notice that, if x is a class in the E_5-term, then $d_5(\Delta^{2k}x) = \Delta^{2k}d_5(x)\ \forall k \in \mathbb{Z}$. This says in particular that the E_6-term is Δ^2-periodic. Next, if x is a d_5-cycle and is annihilated by ν^i, then $d_5(\Delta x) = \overline{\kappa}\nu x$ and $d_5(\Delta\nu^{i-1}x) = 0$. Together with the fact that all of the generators of (II.13) are permanent cycles (Proposition 5.3.6), it is straightforward to verify that the classes together with their annihilation ideal given in the statement of the Proposition generate the E_6-term as a module over $\mathbb{F}_4[(\Delta^8)^{\pm 1}, \overline{\kappa}, \nu]/(\overline{\kappa}\nu)$. □

Remark 5.3.11. Since Δ^8 is a permanent cycle in the HFPSS for $E_C^{hG_{24}}$, the HFPSS for $E_C^{hG_{24}} \wedge A_1$ is linear with respect to Δ^8. Note that all $\overline{\kappa}$-free generators in the E_7-term are of the form $(\Delta^8)^k x$ where $k \in \mathbb{Z}$ and x is one of the generators listed in Proposition 5.3.10. Then, by Corollary 5.3.3, these free $\overline{\kappa}$-families pair up so that each non-permanent $\overline{\kappa}$-family truncates one and only one permanent $\overline{\kappa}$-family. By Δ^8-linearity, among these 64 generators, only half of them are permanent cycles and the others support a differential. It reduces the problem into two steps: first identify all permanent $\overline{\kappa}$-families, then identify by which $\overline{\kappa}$-family they are truncated.

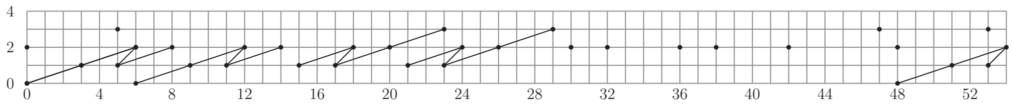

Figure II.21 – The E_7-term for $s \leq 3$ and $t - s \leq 54$

Proposition 5.3.12. *The generators*

$$e[2,30], e[2,32], e[2,36], e[2,38], e[2,42], e[3,47], e[2,48], e[3,53]$$

are permanent cycles.

Proof. We give the proof for $e[2,30]$ and the other generators are proven in a similar manner. In the E_6-term, the Massey product $\langle \overline{\kappa}, \nu, \nu^2 e[0,0] \rangle$ can be formed. Since $d_5(\Delta) = \overline{\kappa}\nu$ and $\nu^3 e[0,0] = 0 \in \mathrm{E}_5$, we see that

$$e[2,30] = \Delta \nu^2 e[0,0] \in \langle \overline{\kappa}, \nu, \nu^2 e[0,0] \rangle.$$

The indeterminacy consists of $\overline{\kappa}\mathrm{E}_6^{-2,8} + \mathrm{E}_6^{0,26}\nu^2 e[0,0]$, where $\mathrm{E}_6^{-2,8}$ is in the E_6-term of the HFPSS for $E_C^{hG_{24}} \wedge A_1$ and $\mathrm{E}_6^{0,26}$ for $E_C^{hG_{24}}$. The latter are zero groups, hence the indeterminacy is zero. Thus,

$$\langle \overline{\kappa}, \nu, \nu^2 e[0,0] \rangle = e[2,30].$$

At the level of the homotopy groups of $\pi_*(E_C^{hG_{24}} \wedge A_1)$ one can form the corresponding Toda bracket $\langle \overline{\kappa}, \nu, \nu^2 e[0,0] \rangle$ because $\nu\overline{\kappa} = 0$ in $\pi_*(E_C^{hG_{24}})$ and inspection in $\pi_*(tmf \wedge A_1)$ tells us that $\nu^3 e[0,0] = 0$. Furthermore, all hypotheses of Moss's convergence theorem are verified. Therefore, $e[2,30]$ is a permanent cycle representing the Toda bracket $\langle e[0,0], \nu^3, \overline{\kappa} \rangle$. For the sake of completeness, we record the Toda bracket expressions for the other elements

$$\langle \overline{\kappa}, \nu, \nu e[1,5] \rangle = e[2,32], \ \langle \overline{\kappa}, \nu, \nu^2 e[0,6] \rangle = e[2,36],$$

$$\langle \overline{\kappa}, \nu, \nu e[1,11]\rangle = e[2,38],\ \langle \overline{\kappa}, \nu, \nu e[1,15]\rangle = e[2,42],$$
$$\langle \overline{\kappa}, \nu, \nu^2 e[1,17]\rangle = e[3,47],\ \langle \overline{\kappa}, \nu, \nu e[1,21]\rangle = e[2,48],$$
$$\langle \overline{\kappa}, \nu, \nu^2 e[2,23]\rangle = e[3,53].$$

□

We have already identified 16 out of 32 permanent cycles. The next 16 ones are not the same for different versions of A_1. The difference reflects the different behavior of the d_2-differential in the ASS for different models of A_1 (see Proposition 4.0.3).

Proposition 5.3.13. *In the HFPSS for all four versions of* A_1*, the following* 12 *generators are permanent cycles :*

$$\Delta^2 e[0,0], \Delta^2 e[1,5], \Delta^2 e[0,6], \Delta^2 e[1,11], \Delta^2 e[1,15], \Delta^2 e[1,17]$$

$$\Delta^2 e[1,21], \Delta^2 e[2,30], \Delta^2 e[2,32], \Delta^2 e[2,36], , \Delta^2 e[2,42], \Delta^2 e[3,47].$$

The remaining four permanent cycles for $A_1[00]$ *and* $A_1[11]$ *are*

$$\Delta^2 e[1,23], \Delta^2 e[2,38], \Delta^2 e[2,48], \Delta^2 e[3,53],$$

whereas the remaining four permanent cycles for $A_1[10]$ *and* $A_1[01]$ *are*

$$\Delta^4 e[1,15], \Delta^4 e[0,0], \Delta^4 e[1,5], \Delta^4 e[2,30].$$

Proof. The graded associated object of the groups $\pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$, with respect to the Adams filtration, in the following stems are given in the following table:

Stem	48	53	54	59	63	65	69	78	80	84	90	95
Value	$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$

In view of Corollary 5.1.3 and Corollary 5.3.3, inspection in the E_7-term shows that the following 12 classes are permanent cycles in the HFPSS for all four versions of A_1.

$$\Delta^2 e[0,0], \Delta^2 e[1,5], \Delta^2 e[0,6], \Delta^2 e[1,11], \Delta^2 e[1,15], \Delta^2 e[1,17],$$

$$\Delta^2 e[1,21], \Delta^2 e[2,30], \Delta^2 e[2,32], \Delta^2 e[2,36], , \Delta^2 e[2,42], \Delta^2 e[3,47].$$

Next, in the ASS for $tmf \wedge A_1[00]$ and $tmf \wedge A_1[11]$, there are no non-trivial differentials before stem 96, by sparseness and Theorem 4.0.3. Inspection in the E_2-term then shows that

$$\pi_{71}(tmf \wedge A_1[00])/(\overline{\kappa}, \nu) = \pi_{71}(tmf \wedge A_1[11])/(\overline{\kappa}, \nu) \cong \mathbb{F}_2$$

and

$$\pi_{86}(tmf \wedge A_1[00])/(\overline{\kappa}, \nu) = \pi_{86}(tmf \wedge A_1[11])/(\overline{\kappa}, \nu) \cong \mathbb{F}_2$$

It follows that the classes $\Delta^2 e[1,23]$ and $\Delta^2 e[2,38]$ are permanent cycles in the HFPSS for $E_C^{hG_{24}} \wedge A_1[00]$ and $E_C^{hG_{24}} \wedge A_1[11]$.

On the other hand, in the ASS for $tmf \wedge A_1[10]$ and $tmf \wedge A_1[01]$, Lemma 4.0.3 and g-linearity imply that $d_2(g^2 w_2 e[4,23]) = g^4 e[6,30]$ and $d_2(g^2 w_2 e[7,38]) = g^6 e[1,5]$. Hence, $w_2^2 e[3,15]$ and $w_2^2 e[6,30]$ survive to the E$_\infty$-term, by sparseness. It then follows that $\Delta^4 e[1,15]$ and $\Delta^4 e[2,30]$ are permanent cycles in the HFPSS for $A_1[10]$ and $A_1[01]$.

For $A_1[00]$ and $A_1[11]$, the classes $w_2 e[9,48]$ and $w_2 e[10,53]$ do not support differentials, by Lemma 4.0.3, hence persist to the E_∞-term, by sparseness. They are also not divisible neither by $\overline{\kappa}$ nor by ν. Lastly, both $w_2 e[9,48]$ and $w_2 e[10,53]$ are annihilated by ν. The only classes in the HFPSS that match those properties are $\Delta^2 e[2,48]$ and $\Delta^2 e[3,53]$, respectively. Thus, the latter are the last two of the 32 permanent cycles in the HFPSS for $A_1[00]$ and $A_1[11]$.

For $A_1[10]$ and $A_1[01]$, the classes $w_2 e[9,48]$ and $w_2 e[10,53]$ support nontrivial d_2 differentials. Thus $w_2^2 e[0,0]$ and $w_2^2 e[1,5]$ survive to the E_∞-term. For degree reasons, both $w_2^2 e[0,0]$ and $w_2^2 e[1,5]$ are not divisible either by $\overline{\kappa}$ or by ν, and moreover their multiples by ν are not divisible by $\overline{\kappa}$. In the HFPSS for $E_C^{hG_{24}} \wedge A_1[10]$ and $E_C^{hG_{24}} \wedge A_1[10]$, $\Delta^4 e[0,0]$ and $\Delta^4 e[1,5]$ are the only classes verifying the respective properties, hence are permanent cycles.

□

Having determined all permanent $\overline{\kappa}$-families, we consider differentials. We recall, from Remark 5.3.11, that each permanent $\overline{\kappa}$-family is truncated by one and only one non-permanent $\overline{\kappa}$-family. We can proceed as follows: take a permanent cycle, say x; then locate all non-permanent classes that can support a differential killing $\overline{\kappa}^n x$ for some $n \leq 6$. Precisely, one of the following situations will happen:

1) There is no ambiguity: i.e., there is only one generator that can support a differential killing $\overline{\kappa}^n x$ for some $n \leq 6$, so this differential occurs.

2) There are two generators that can support a differential killing multiples of x by different powers of $\overline{\kappa}$. In order to decide, we inspect the $\overline{\kappa}$-exponent of x using the ASS.

3) There are two generators that can support a differential killing the multiple of x by the same power of $\overline{\kappa}$. In this case, inspection on the $\overline{\kappa}$-exponent of x does not help. We will treat each of the particularity case by case. Some Toda brackets will be involved to resolve these cases.

A permanent cycle is said to be of type 1, 2, 3 respectively if its $\overline{\kappa}$-family is as in the situation 1, 2, 3 above respectively. The HFPSS for different versions of A_1 do not behave in the same manner. It turns out the HFPSS for the versions $A_1[10]$ and $A_1[01]$ behave in the same way and $A_1[00]$ and $A_1[11]$ in the same way. We will treat the HFPSS for $A_1[10]$ and $A_1[01]$ in detail and then point out the changes needed for $A_1[00]$ and $A_1[11]$.

Differentials (continued) for $A_1[01]$ and $A_1[10]$. The reader is invited to follow the discussion of the differentials using Figures (II.22) to (II.25) below.

The d_9-differentials

Proposition 5.3.14. *There are the following d_9-differentials:*

(1) $d_9(\Delta^2 e[1,23]) = \overline{\kappa}^2 e[2,30]$

(2) $d_9(\Delta^6 e[1,23]) = \overline{\kappa}^2 \Delta^4 e[2,30]$.

Proof. The classes $e[2,30]$ and $\Delta^4 e[2,30]$ are of type 1 and the only possibilities are $d_9(\Delta^2 e[1,23]) = \overline{\kappa}^2 e[2,30]$ and $d_9(\Delta^6 e[1,23]) = \overline{\kappa}^2 \Delta^4 e[2,30]$, respectively. □

The d_{15}-differentials

Proposition 5.3.15. *There are the following d_{15}-differentials:*

(1) $d_{15}(\Delta^2 e[2,38]) = \overline{\kappa}^4 e[1,5]$

(2) $d_{15}(\Delta^2 e[2,48]) = \overline{\kappa}^4 e[1,15]$

(3) $d_{15}(\Delta^6 e[2,38]) = \overline{\kappa}^4 \Delta^4 e[1,5]$

(4) $d_{15}(\Delta^6 e[2,48]) = \overline{\kappa}^4 \Delta^4 e[1,15]$.

Proof. It is readily checked from the chart that all $e[1,5]$, $e[1,15]$, $\Delta^4 e[1,5]$, $\Delta^4 e[1,15]$ are of type 1 and their $\overline{\kappa}$-family is truncated as indicated in the proposition. □

The d_{17}-differentials

Proposition 5.3.16. *There are the following d_{17}-differentials:*

(1) $d_{17}(\Delta^2 e[3,53]) = \overline{\kappa}^5 e[0,0]$

(2) $d_{17}(\Delta^4 e[0,6]) = \overline{\kappa}^4 e[1,21]$

(3) $d_{17}(\Delta^4 e[1,17]) = \overline{\kappa}^4 e[2,32]$

(4) $d_{17}(\Delta^4 e[1,21]) = \overline{\kappa}^4 e[2,36]$

(5) $d_{17}(\Delta^4 e[2,32]) = \overline{\kappa}^4 e[3,47]$

(6) $d_{17}(\Delta^6 e[0,6]) = \overline{\kappa}^4 \Delta^2 e[1,21]$

(7) $d_{17}(\Delta^6 e[1,17]) = \overline{\kappa}^4 \Delta^2 e[2,32]$

(8) $d_{17}(\Delta^6 e[1,21]) = \overline{\kappa}^4 \Delta^2 e[2,36]$

(9) $d_{17}(\Delta^6 e[2,32]) = \overline{\kappa}^4 \Delta^2 e[3,47]$

(10) $d_{17}(\Delta^6 e[3,53]) = \overline{\kappa}^5 \Delta^4 e[0,0]$
(11) $d_{17}(\Delta^4 e[1,23]) = \overline{\kappa}^4 e[2,38]$
(12) $d_{17}(\Delta^4 e[2,38]) = \overline{\kappa}^4 e[3,53]$
(13) $d_{17}(\Delta^6 e[0,0]) = \overline{\kappa}^4 \Delta^2 e[1,15]$
(14) $d_{17}(\Delta^6 e[1,15]) = \overline{\kappa}^4 \Delta^2 e[2,30]$.

Proof. (1)-(10) All of the generators of

$$e[0,0], e[1,21], e[2,32], e[2,36], e[3,47]$$
$$\Delta^2 e[1,21], \Delta^2 e[2,32], \Delta^2 e[2,36], \Delta^2 e[3,47], \Delta^4 e[0,0]$$

are of type 1.

(11) $e[2,38]$ is of type 2. The differentials that can truncate its $\overline{\kappa}$-family are $d_{17}(\Delta^4 e[1,23]) = \overline{\kappa}^4 e[2,38]$ and $d_{25}(\Delta^6 e[1,15]) = \overline{\kappa}^6 e[2,38]$. The latter can not happen because the spectral sequence collapses at the E_{24}-term. Therefore, one must have that $d_{17}(\Delta^4 e[1,23]) = \overline{\kappa}^4 e[2,38]$.

(12) $e[3,53]$ is of type 2. Its $\overline{\kappa}$-family can be truncated by $d_{17}(\Delta^4 e[2,38]) = \overline{\kappa}^4 e[3,53]$ or $d_{25}(\Delta^6 e[2,30]) = \overline{\kappa}^6 e[3,53]$. As above, there can not be any d_{25}-differential in the spectral sequence. Hence, one must have that $d_{17}(\Delta^4 e[2,38]) = \overline{\kappa}^4 e[3,53]$.

(13) $\Delta^2 e[1,15]$ is of type 3. In its $\overline{\kappa}$-family, only $\overline{\kappa}^4 \Delta^2 e[1,15]$ can be a target of differentials, $d_{17}(\Delta^6 e[0,0]) = \overline{\kappa}^4 \Delta^2 e[1,15]$ and $d_{15}(\Delta^4 e[2,48]) = \overline{\kappa}^4 \Delta^2 e[1,15]$. However, if $d_{15}(\Delta^4 e[2,48]) = \overline{\kappa}^4 \Delta^2 e[1,15]$ then the only class that can truncate the $\overline{\kappa}$-family of $e[1,23]$ is $\Delta^6 e[0,0]$ and by a d_{25}-differential: $d_{25}(\Delta^6 e[0,0]) = \overline{\kappa}^6 e[1,23]$. This contradicts the fact that the spectral sequence collapses at the E_{24}-term. Thus, one must have that $d_{17}(\Delta^6 e[0,0]) = \overline{\kappa}^4 \Delta^2 e[1,15]$.

(14) $\Delta^2 e[2,30]$ is of type 2. Its $\overline{\kappa}$-family can be truncated by a d_9-differential on $\Delta^4 e[1,23]$ or by a d_{17}-differential on $\Delta^6 e[1,15]$. However, the former possibility can not occur because of part (11). Therefore, $d_{17}(\Delta^6 e[1,15]) = \overline{\kappa}^4 \Delta^2 e[2,30]$.

□

The d_{19}-differentials

Proposition 5.3.17. *There are the following* d_{19}*-differentials:*
(1) $d_{19}(\Delta^4 e[1,11]) = \overline{\kappa}^5 e[0,6]$
(2) $d_{19}(\Delta^4 e[3,47]) = \overline{\kappa}^5 e[2,42]$
(3) $d_{19}(\Delta^6 e[1,11]) = \overline{\kappa}^5 \Delta^2 e[0,6]$
(4) $d_{19}(\Delta^6 e[3,47]) = \overline{\kappa}^5 \Delta^2 e[2,42]$
(5) $d_{19}(\Delta^6 e[1,5]) = \overline{\kappa}^5 \Delta^2 e[0,0]$
(6) $d_{19}(\Delta^4 e[3,53]) = \overline{\kappa}^5 e[2,48]$.

Proof. (1)-(4) All of the classes

$$e[0,6], e[2,42], \Delta^2 e[0,6], \Delta^2 e[2,42]$$

are of type 1.

(5) The class $\Delta^2 e[0,0]$ is of type 3 and its $\overline{\kappa}$-family can be truncated either by $d_{17}(\Delta^4 e[3,53]) = \overline{\kappa}^5 \Delta^2 e[0,0]$ or by $d_{19}(\Delta^6 e[1,5]) = \overline{\kappa}^5 \Delta^2 e[0,0]$. Suppose $d_{17}(\Delta^4 e[3,53]) = \overline{\kappa}^5 \Delta^2 e[0,0]$. This would leave us with the differential $d_{21}(\Delta^6 e[1,5]) = \overline{\kappa}^5 e[2,48]$. It would imply the Massey product in the E_{22}-term

$$\langle \overline{\kappa}^5, e[2,48], \nu \rangle = \nu \Delta^6 e[1,5]$$

with zero indeterminacy in the E_{22}-term. All conditions of Moss's convergence theorem are met, the Toda bracket $\langle \overline{\kappa}^5, e[2,48], \nu \rangle$ could then be formed and would contain an element represented by $\nu \Delta^6 e[1,5]$. This contradicts Corollary 4.0.5. This contradiction proves that

$$d_{19}(\Delta^6 e[1,5]) = \overline{\kappa}^5 \Delta^2 e[0,0].$$

(6) The class $e[2,48]$ is of type 2 and its $\overline{\kappa}$-family is truncated either by $d_{19}(\Delta^4 e[3,53]) = \overline{\kappa}^5 e[2,48]$ or by $d_{21}(\Delta^6 e[1,5]) = \overline{\kappa}^5 e[2,48]$. However, part (5) of Proposition 5.3.17 rules out the latter.

□

The d_{23}-differentials

Proposition 5.3.18. *There are the following* d_{23}*-differentials:*

(1) $d_{23}(\Delta^4 e[2,36]) = \overline{\kappa}^6 e[1,11]$
(2) $d_{23}(\Delta^4 e[2,42]) = \overline{\kappa}^6 e[1,17]$
(3) $d_{23}(\Delta^4 e[2,48]) = \overline{\kappa}^6 e[1,23]$
(4) $d_{23}(\Delta^6 e[2,36]) = \overline{\kappa}^6 \Delta^2 e[1,11]$
(5) $d_{23}(\Delta^6 e[2,42]) = \overline{\kappa}^6 \Delta^2 e[1,17]$
(6) $d_{23}(\Delta^6 e[2,30]) = \overline{\kappa}^6 \Delta^2 e[1,5]$.

Proof. (1)-(5) All of the classes

$$e[1,11], e[1,17], e[1,23], \Delta^2 e[1,11], \Delta^2 e[1,17]$$

are of type 1.

(6) The class $\Delta^2 e[1,5]$ is of type 2. The two possibilities are $d_{15}(\Delta^4 e[2,38]) = \overline{\kappa}^4 \Delta^2 e[1,5]$ and $d_{23}(\Delta^6 e[2,30]) = \overline{\kappa}^6 \Delta^2 e[1,5]$. However, part (12) of Proposition 5.3.16 rules out the former because the class $\Delta^4 e[2,38]$ must pair up with the class $e[3,38]$, by a d_{17}-differential $d_{17}(\Delta^4 e[2,38]) = \overline{\kappa}^4 e[3,53]$.

□

The above differentials from d_9 to d_{23}, together with the $\overline{\kappa}$- and Δ^8-linearity exhaust all differentials. In the statement of Theorem 5.3.19 and 5.3.20, we write e_{t-s} for the permanent cycle $e[s, t-s]$ in bidegree (s,t) listed in Proposition 5.3.10, for the sake of presentation.

Theorem 5.3.19. *As a module over* $\mathbb{F}_4[\Delta^{\pm 8}, \overline{\kappa}, \nu]/(\overline{\kappa}\nu)$*, the* E_∞*-term of the HFPSS for* $E_C^{hG_{24}} \wedge A_1$ *for* $A_1 = A_1[10]$ *and* $A_1[01]$ *is a direct sum of cyclic modules generated by the following elements and with the respective annihilator ideal:*

$(0,0)$	$(1,5)$	$(0,6)$	$(1,11)$	$(1,15)$	$(1,17)$	$(1,21)$	$(1,23)$
$e[0,0]$	$e[1,5]$	$e[0,6]$	$e[1,11]$	$e[1,15]$	$e[1,17]$	$e[1,21]$	$e[1,23]$
$(\overline{\kappa}^5, \nu^3)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}^5, \nu^3)$	$(\overline{\kappa}^6, \nu^2)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}^6, \nu^3)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}^6, \nu^3)$
$(2,30)$	$(2,32)$	$(2,36)$	$(2,38)$	$(2,42)$	$(3,47)$	$(2,48)$	$(3,53)$
$e[2,30]$	$e[2,32]$	$e[2,36]$	$e[2,38]$	$e[2,42]$	$e[3,47]$	$e[2,48]$	$e[3,53]$
$(\overline{\kappa}^2, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^5, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^5, \nu)$	$(\overline{\kappa}^4, \nu)$

$(0,0)$	$(1,5)$	$(0,6)$	$(1,11)$	$(1,15)$	$(1,17)$	$(1,21)$	$(1,23)$
e_0	e_5	e_6	e_{11}	e_{15}	e_{17}	e_{21}	e_{23}
$(\overline{\kappa}^5, \nu^3)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}^5, \nu^3)$	$(\overline{\kappa}^6, \nu^2)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}^6, \nu^3)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}^6, \nu^3)$
$(2,30)$	$(2,32)$	$(2,36)$	$(2,38)$	$(2,42)$	$(3,47)$	$(2,48)$	$(3,53)$
e_{30}	e_{32}	e_{36}	e_{38}	e_{42}	e_{47}	e_{48}	e_{53}
$(\overline{\kappa}^2, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^5, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^5, \nu)$	$(\overline{\kappa}^4, \nu)$
$(0,48)$	$(1,53)$	$(0,54)$	$(1,59)$	$(1,63)$	$(1,65)$	$(1,69)$	$(2,74)$
$\Delta^2 e_0$	$\Delta^2 e_5$	$\Delta^2 e_6$	$\Delta^2 e_{11}$	$\Delta^2 e_{15}$	$\Delta^2 e_{17}$	$\Delta^2 e_{21}$	$\Delta^2 \nu e_{23}$
$(\overline{\kappa}^5, \nu^3)$	$(\overline{\kappa}^6, \nu^2)$	$(\overline{\kappa}^5, \nu^3)$	$(\overline{\kappa}^6, \nu^2)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}^6, \nu^3)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}, \nu^2)$
$(2,78)$	$(2,80)$	$(2,84)$	$(2,90)$	$(3,95)$	$(0,96)$	$(1,101)$	
$\Delta^2 e_{30}$	$\Delta^2 e_{32}$	$\Delta^2 e_{36}$	$\Delta^2 e_{42}$	$\Delta^2 e_{47}$	$\Delta^4 e_0$	$\Delta^4 e_5$	
$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^5, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^5, \nu^3)$	$(\overline{\kappa}^4, \nu^2)$	
$(1,105)$	$(2,110)$	$(1,111)$	$(2,116)$	$(2,120)$	$(2,122)$	$(2,126)$	
$\Delta^4 \nu e_6$	$\Delta^4 \nu e_{11}$	$\Delta^4 e_{15}$	$\Delta^4 \nu e_{17}$	$\Delta^4 \nu e_{21}$	$\Delta^4 \nu e_{23}$	$(\Delta^4 e_{30})$	
$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}^2, \nu)$	
$(1,147)$	$(2,152)$	$(1,153)$	$(2,158)$	$(2,162)$	$(2,164)$	$(2,168)$	$(2,170)$
$\Delta^6 \nu e_0$	$\Delta^6 \nu e_5$	$\Delta^6 \nu e_6$	$\Delta^6 \nu e_{11}$	$\Delta^6 \nu e_{15}$	$\Delta^6 \nu e_{17}$	$\Delta^6 \nu e_{21}$	$\Delta^6 \nu e_{23}$
$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu^2)$.

The case of $A_1[00]$ **and** $A_1[11]$**.** The analysis of the HFPSS for $A_1[00]$ and $A_1[11]$ can be done in the same manner as that for $A_1[10]$ and $A_1[01]$. All differentials are identical except for 8 ones involving 16 of the generators of Proposition 5.3.10. We will be content to point out all modifications, see Figures from II.26 to II.29.

$$d_{17}(\Delta^4 e[1,15]) = \overline{\kappa}^4 e[2,30] \text{ instead of } d_9(\Delta^2 e[1,23]) = \overline{\kappa}^2 e[2,30],$$
$$d_{17}(\Delta^6 e[1,23]) = \overline{\kappa}^4 \Delta^2 e[2,38] \text{ instead of } d_9(\Delta^6 e[1,23]) = \overline{\kappa}^2 \Delta^4 e[2,30],$$
$$d_{17}(\Delta^4 e[0,0]) = \overline{\kappa}^4 e[1,15] \text{ instead of } d_{15}(\Delta^2 e[2,48]) = \overline{\kappa}^4 e[1,15],$$
$$d_{17}(\Delta^6 e[2,38]) = \overline{\kappa}^4 \Delta^2 e[3,53] \text{ instead of } d_{15}(\Delta^6 e[2,38]) = \overline{\kappa}^4 \Delta^2 e[1,5],$$
$$d_{19}(\Delta^4 e[1,5]) = \overline{\kappa}^5 e[0,0] \text{ instead of } d_{17}(\Delta^2 e[3,53]) = \overline{\kappa}^5 e[0,0],$$
$$d_{19}(\Delta^6 e[3,53]) = \overline{\kappa}^5 \Delta^2 e[2,48] \text{ instead of } d_{17}(\Delta^6 e[3,53]) = \overline{\kappa}^5 \Delta^4 e[0,0],$$
$$d_{23}(\Delta^6 e[2,48]) = \overline{\kappa}^6 \Delta^2 e[1,23] \text{ instead of } d_{15}(\Delta^6 e[2,48]) = \overline{\kappa}^4 \Delta^4 e[1,15],$$
$$d_{23}(\Delta^4 e[2,30]) = \overline{\kappa}^6 e[1,5] \text{ instead of } d_{15}(\Delta^2 e[2,38]) = \overline{\kappa}^4 e[1,5].$$

Theorem 5.3.20. *As a module over* $\mathbb{F}_4[\Delta^{\pm 8}, \overline{\kappa}, \nu]/(\overline{\kappa}\nu)$, *the* E_∞*-term of the HF-PSS for* $E_C^{hG_{24}} \wedge A_1$ *for* $A_1 = A_1[00]$ *and* $A_1[11]$ *is a direct sum of cyclic modules generated by the following elements and with the respective annihilator ideals:*

$(0,0)$	$(1,5)$	$(0,6)$	$(1,11)$	$(1,15)$	$(1,17)$	$(1,21)$	$(1,23)$
e_0	e_5	e_6	e_{11}	e_{15}	e_{17}	e_{21}	e_{23}
$(\overline{\kappa}^5, \nu^3)$	$(\overline{\kappa}^6, \nu^2)$	$(\overline{\kappa}^5, \nu^3)$	$(\overline{\kappa}^6, \nu^2)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}^6, \nu^3)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}^6, \nu^3)$
$(2,30)$	$(2,32)$	$(2,36)$	$(2,38)$	$(2,42)$	$(3,47)$	$(2,48)$	$(3,53)$
e_{30}	e_{32}	e_{36}	e_{38}	e_{42}	e_{47}	e_{48}	e_{53}
$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^5, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^5, \nu)$	$(\overline{\kappa}^4, \nu)$
$(0,48)$	$(1,53)$	$(0,54)$	$(1,59)$	$(1,63)$	$(1,65)$	$(1,69)$	$(1,71)$
$\Delta^2 e_0$	$\Delta^2 e_5$	$\Delta^2 e_6$	$\Delta^2 e_{11}$	$\Delta^2 e_{15}$	$\Delta^2 e_{17}$	$\Delta^2 e_{21}$	$\Delta^2 e_{23}$
$(\overline{\kappa}^5, \nu^3)$	$(\overline{\kappa}^6, \nu^2)$	$(\overline{\kappa}^5, \nu^3)$	$(\overline{\kappa}^6, \nu^2)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}^6, \nu^3)$	$(\overline{\kappa}^4, \nu^2)$	$(\overline{\kappa}^6, \nu^3)$
$(2,78)$	$(2,80)$	$(2,84)$	$(2,86)$	$(2,90)$	$(3,95)$	$(2,96)$	$(3,101)$
$\Delta^2 e_{30}$	$\Delta^2 e_{32}$	$\Delta^2 e_{36}$	$\Delta^2 e_{38}$	$\Delta^2 e_{42}$	$\Delta^2 e_{47}$	$\Delta^2 e_{48}$	$\Delta^2 e_{53}$
$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^5, \nu)$	$(\overline{\kappa}^4, \nu)$	$(\overline{\kappa}^5, \nu)$	$(\overline{\kappa}^4, \nu)$
$(1,99)$	$(2,104)$	$(1,105)$	$(2,110)$	$(2,114)$	$(2,116)$	$(2,120)$	$(2,122)$
$\Delta^4 \nu e_0$	$\Delta^4 \nu e_5$	$\Delta^4 \nu e_6$	$\Delta^4 \nu e_{11}$	$\Delta^4 \nu e_{15}$	$\Delta^4 \nu e_{17}$	$\Delta^4 \nu e_{21}$	$\Delta^4 \nu e_{23}$
$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu^2)$
$(1,147)$	$(2,152)$	$(1,153)$	$(2,158)$	$(2,162)$	$(2,164)$	$(2,168)$	$(2,170)$
$\Delta^6 \nu e_0$	$\Delta^6 \nu e_5$	$\Delta^6 \nu e_6$	$\Delta^6 \nu e_{11}$	$\Delta^6 \nu e_{15}$	$\Delta^6 \nu e_{17}$	$\Delta^6 \nu e_{21}$	$\Delta^6 \nu e_{23}$
$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu^2)$	$(\overline{\kappa}, \nu)$	$(\overline{\kappa}, \nu^2)$.

Remark 5.3.21. We emphasise that the relations given in Theorem 5.3.19 and 5.3.20 are only the relations in the E_∞-term. In fact, we can see by sparseness that, the annihilator exponents of $\overline{\kappa}$ are still true in $\pi_*(E_C^{hG_{24}} \wedge A_1)$. Whereas, there are exotic extensions by ν, i.e., multiplications by ν that are not detected in the E_∞-term. These can be determined by two different methods: by using the Tate spectral sequence as in [**BO16**], Section 2.3 or by computing the Gross-Hopkins dual of $E_C^{hG_{24}} \wedge A_1$; however, we do not discuss this point here.

Using the structure of the E_∞-term, we can read off the action of the ideal $(\overline{\kappa}, \nu)$ on $\pi_*(E_C^{hG_{24}} \wedge A_1)$. From this, we obtain the following Corollary.

Corollary 5.3.22. We have

a) The map

$$\Theta' : \mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_k(tmf \wedge A_1)/(\overline{\kappa}, \nu) \to \pi_k(E_C^{hG_{24}} \wedge A_1)/(\overline{\kappa}, \nu),$$

induced by Θ in II.7, is an isomorphism for $k \geq 0$, independent of the version of A_1.

b) The map

$$\Theta : \mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_k(tmf \wedge A_1) \to \pi_k(E_C^{hG_{24}} \wedge A_1)$$

is also an isomorphism for $k \geq 0$, independent of the version of A_1.

Proof. For part a), Corollary 5.1.3 asserts that Θ' is injective. To show that the latter is surjective, it suffices to show that its source and target have the same order. The order of the target can be seen from Theorem 5.3.19 and 5.3.20; in particular, it has order 0 or 4 in all stems, except for the stems 48 and 53 modulo 192, in which it has order 8. The remaining part of the proof is an inspection of the ASS for $tmf \wedge A_1$, together with the fact that Θ is injective, by Corollary 5.1.3, and is linear with respect to $\overline{\kappa}$ and ν, to show that $\mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_*(tmf \wedge A_1)$ has the same order as of $\pi_*(E_C^{hG_{24}} \wedge A_1)$, in non-negative stems. Because of the dependance of the structure of $\pi_*(E_C^{hG_{24}} \wedge A_1)$ on the version of A_1, we consider them separately: we only give a detailed treatment for $A_1[00]$ and $A_1[11]$ and claim that the treatment for $A_1[01]$ and $A_1[10]$ is completely similar. For the remaining part of the proof, A_1 will be $A_1[00]$ or $A_1[11]$.

By sparseness and part (i) of Theorem 4.0.3, all classes $w_2^l e[i,j]$ for $l = 0, 1$ and $e[i,j]$, the classes in the table of Proposition 3.2.5 survive to the E_∞-term of the ASS for $tmf \wedge A_1$. Moreover, for degree reasons, these classes must converge to non-trivial elements of $\pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$ in the appropriate stems. Therefore, $\mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$ has the same order as of $\pi_*(E_C^{hG_{24}} \wedge A_1)$ up to stem 96 and in stem 101.

All of the classes

$$w_2^2e[0,0], w_2^2e[1,5], w_2^2e[1,6], w_2^2e[2,11],$$

$$w_2^2e[3,15], w_2^2e[3,17], w_2^2e[4,21], w_2^2e[4,23]$$

are d_2-cycles in the ASS and the d_3-differentials on them can only hit g-multiple classes. Thus, by ν-linearity and the fact that $g\nu = 0$ in $\mathrm{Ext}^{5,28}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$, the classes

$$\nu w_2^2e[0,0], \nu w_2^2e[1,5], \nu w_2^2e[1,6], \nu w_2^2e[2,11],$$

$$\nu w_2^2e[3,15], \nu w_2^2e[3,17], \nu w_2^2e[4,21], \nu w_2^2e[4,23]$$

are d_3-cycles and hence survive to the E_∞-term, by sparseness. As above, these classes must converge to non-trivial elements of $\pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$ in the appropriate stems. It follows that $\mathbb{W} \otimes_{\mathbb{Z}_2} \pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$ has the same order as of $\pi_*(E_C^{hG_{24}} \wedge A_1)$ for stems from 96 to 144.

Consider the classes

$$\begin{aligned}\nu w_2^3e[0,0], \nu w_2^3e[1,5], \nu w_2^3e[1,6], \nu w_2^3e[2,11],\\ \nu w_2^3e[3,15], \nu w_2^3e[3,17], \nu w_2^3e[4,21], \nu w_2^3e[4,23].\end{aligned} \tag{II.15}$$

As above, these classes survive to the E_4-term of the ASS for $tmf \wedge A_1$. By sparseness, $\nu w_2^3e[4,23]$ survives to the E_∞-term and converges to a non-trivial element of $\pi_{170}(tmf \wedge A_1)/(\overline{\kappa}, \nu)$. By sparseness, the other classes can only support d_4-differentials hitting the classes

$$g^7e[1,6], g^7e[2,11], g^6e[6,32], g^7e[3,17], g^7e[4,21], g^7e[4,23], g^6e[9,47],$$

respectively. However, the class $g^ke[i,j]$ for $(i,j) \in \{(1,6),(2,11),(6,32),(3,17),(4,21),(4,23),(9,47)\}$ is killed by a differential for a certain integer k less than 7, hence $g^7e[i,j]$ for $(i,j) \in \{(1,6),(2,11),(6,32),(3,17),(4,21),(4,23),(9,47)\}$ is killed by a differential on a certain g-multiple class. This means that

$$\nu w_2^3e[0,0], \nu w_2^3e[1,5], \nu w_2^3e[2,11], \nu w_2^3e[3,15], \nu w_2^3e[3,17]$$

survive to the E_∞-term, hence, as above, to non-trivial elements of $\pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$. Next, the map Θ sends $e[6,32]$ and $e[9,47]$ to $e[2,32]$ and $e[3,47]$, respectively. The latter are both annihilated by $\overline{\kappa}^4$, so that $g^4e[6,32]$ and $g^4e[9,47]$ are hit by certain differentials in the ASS, hence $g^6e[6,32]$ and $g^6e[9,47]$ are hit by differentials supported on g-multiple classes. As above, this implies that $\nu w_2^3e[1,6]$ and $\nu w_2^3e[4,23]$ survive to non-trivial elements of $\pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$.

In total, we have proved that all classes of II.15 converge to non-trivial elements of $\pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$; as a consequence, $\mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$ has the same order as of $\pi_*(E_C^{hG_{24}} \wedge A_1)/(\overline{\kappa}, \nu)$ in stems from 144 to 192.

Together with the fact that $\pi_*(E_C^{hG_{24}} \wedge A_1)/(\overline{\kappa}, \nu)$ is Δ^8-periodic, we conclude that $\Theta^{'}$ is a surjection, hence is an isomorphism.

For part b), there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_*(tmf \wedge A_1) & \xrightarrow{\Theta} & \pi_*(E_C^{hG_{24}} \wedge A_1) \\
\downarrow & & \downarrow \\
\mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu) & \xrightarrow{\Theta^{'}} & \pi_*(E_C^{hG_{24}} \wedge A_1)/(\overline{\kappa}, \nu).
\end{array}$$

Part b) then follows from part a) and the fact that $\pi_*(tmf \wedge A_1)$ is bounded below.

□

The Figures (II.22) to (II.25) represent the HFPSS for $E_C^{hG_{24}} \wedge A_1[10]$ and $E_C^{hG_{24}} \wedge A_1[01]$ from the E_7-term on. Each black dot • represents a class generating a group $\mathbb{F}_4$ which survives to the E_∞-term. Each circle ∘ represent a class which either is hit by a differential or supports a differential higher than d_5. We only represent the differentials on generators listed in Proposition 5.3.10 but not those generated by $\overline{\kappa}$-linearity.

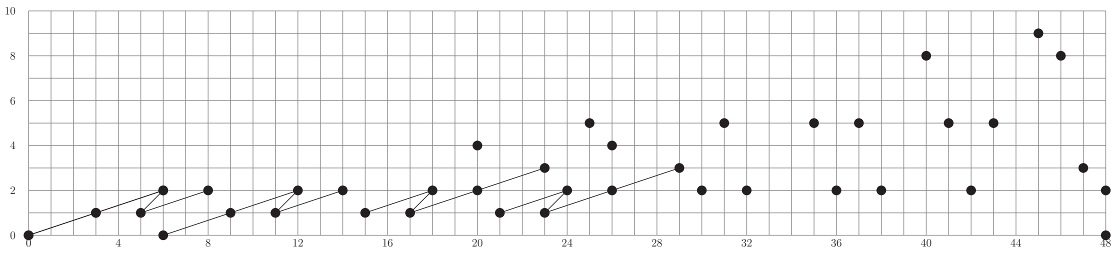

Figure II.22 – HFPSS for $A_1[10]$ and $A_1[01]$ from E_7-term with $0 \leq t-s \leq 48$

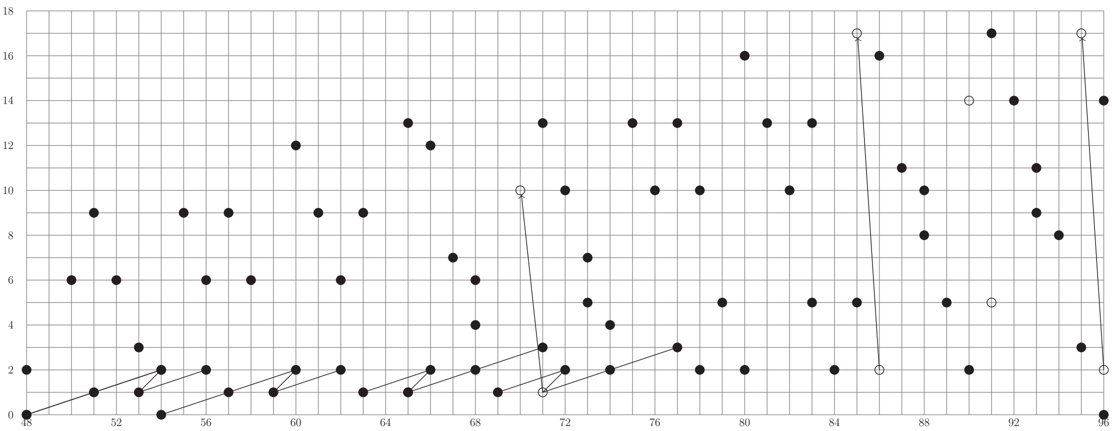

Figure II.23 – HFPSS for $A_1[10]$ and$A_1[01]$ from E_7-term with $48 \leq t-s \leq 96$

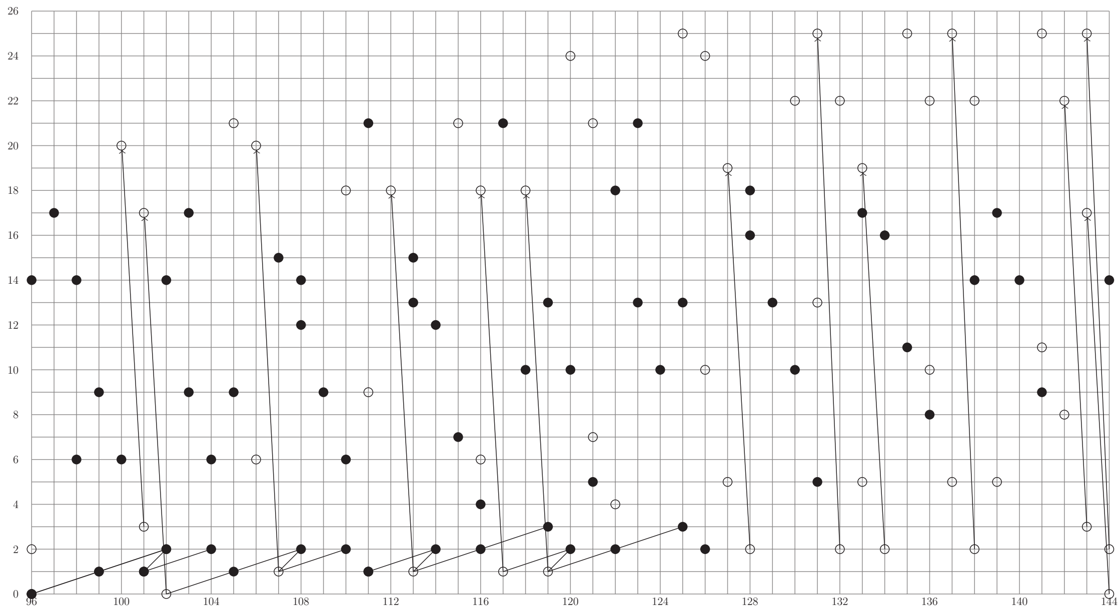

Figure II.24 – HFPSS for $A_1[10]$ and$A_1[01]$ from E_7-term with $96 \leq t - s \leq 144$

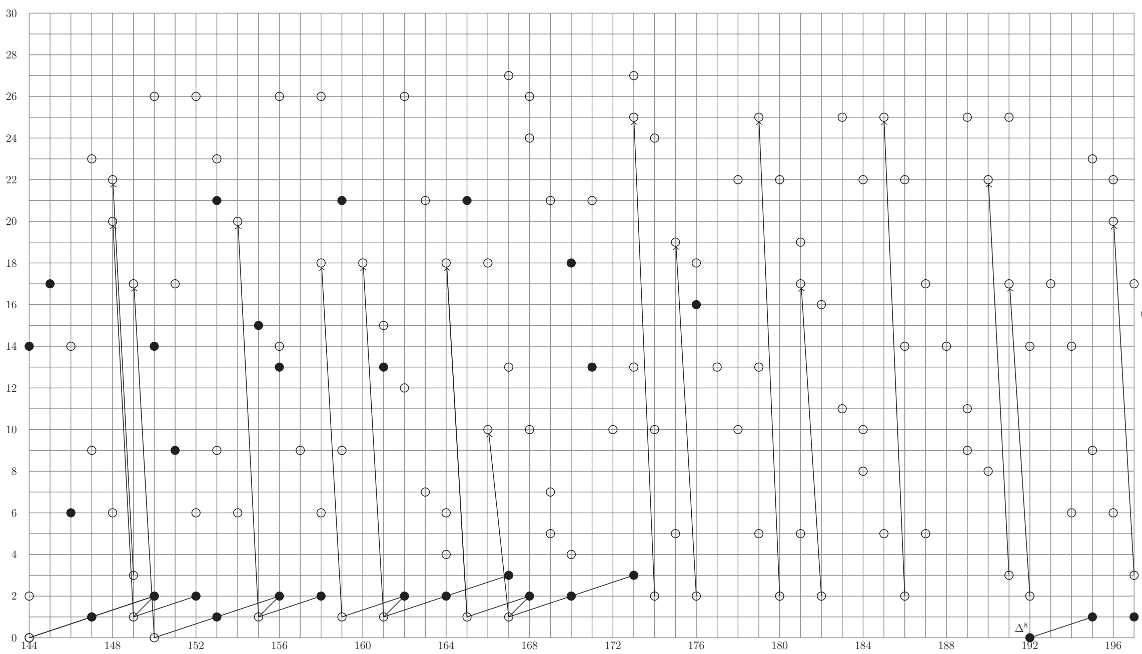

Figure II.25 – HFPSS for $A_1[10]$ and $A_1[01]$ from E_7-term with $144 \leq t-s \leq 197$

The Figures (II.26) to (II.29) represent the HFPSS for $E_C^{hG_{24}} \wedge A_1[00]$ and $E_C^{hG_{24}} \wedge A_1[11]$ from the E_7-term on. Each black dot $\bullet$ represents a class generating a group $\mathbb{F}_4$ which survives to the E_∞-term. Each circle $\circ$ represent a class which either is hit by a differential or supports a differential higher than d_5. We only represent the differentials on generators listed in Proposition 5.3.10 but not those generated by $\overline{\kappa}$-linearity.

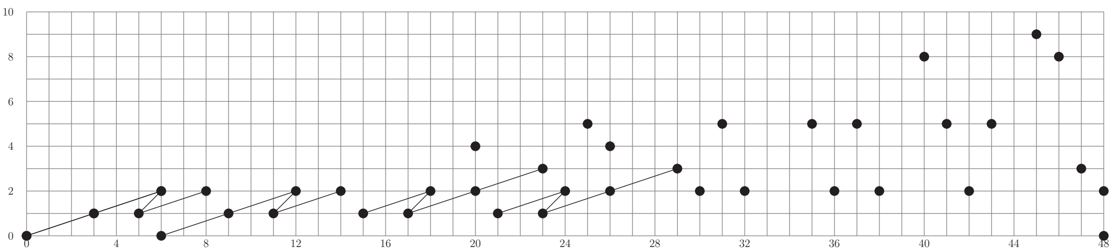

Figure II.26 – HFPSS for $A_1[00]$ and $A_1[11]$ from E_7-term with $0 \leq t - s \leq 48$

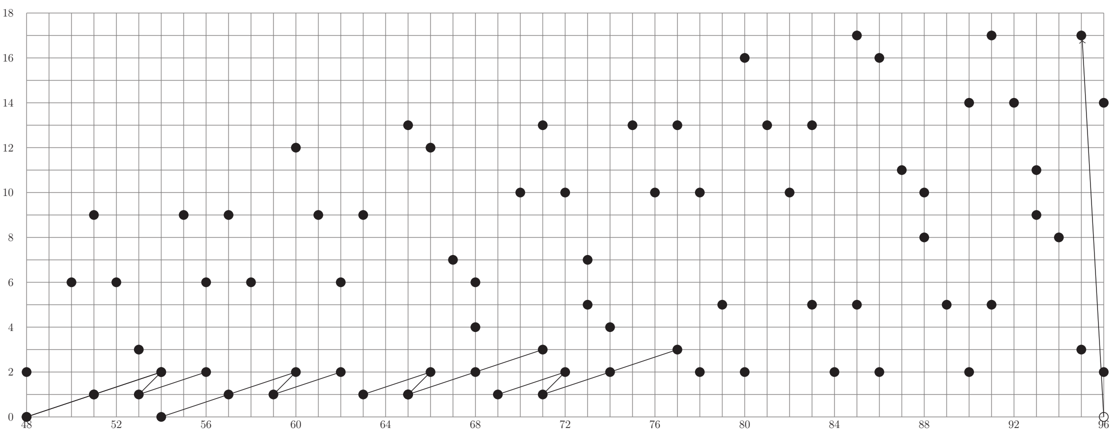

Figure II.27 – HFPSS for $A_1[00]$ and$A_1[11]$ from E$_7$-term with $48 \leq t-s \leq 96$

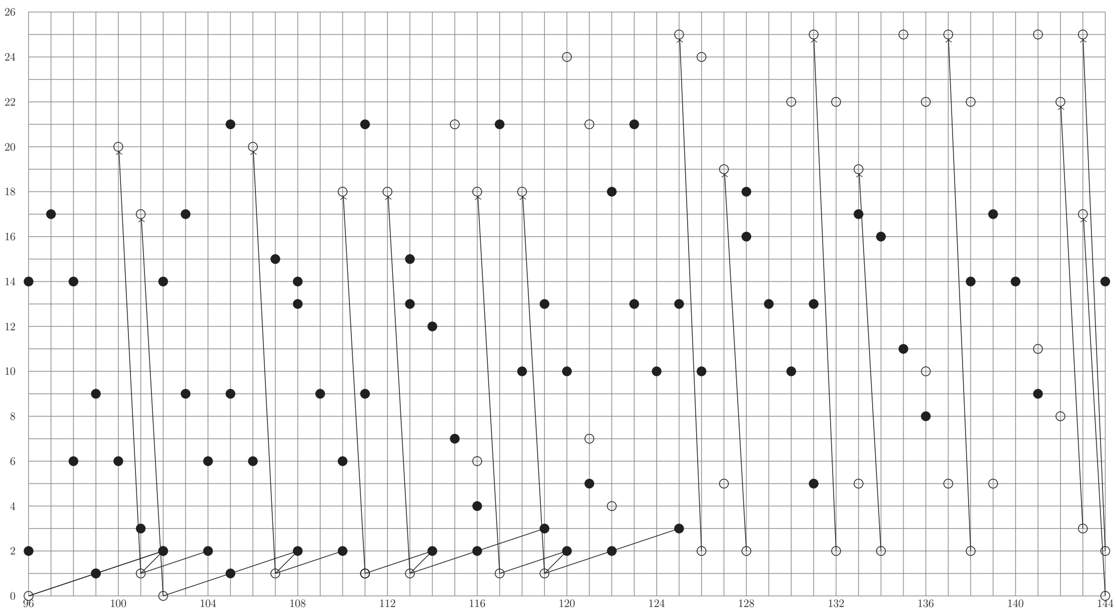

Figure II.28 – HFPSS for $A_1[00]$ and$A_1[11]$ from E_7-term with $96 \leq t-s \leq 144$

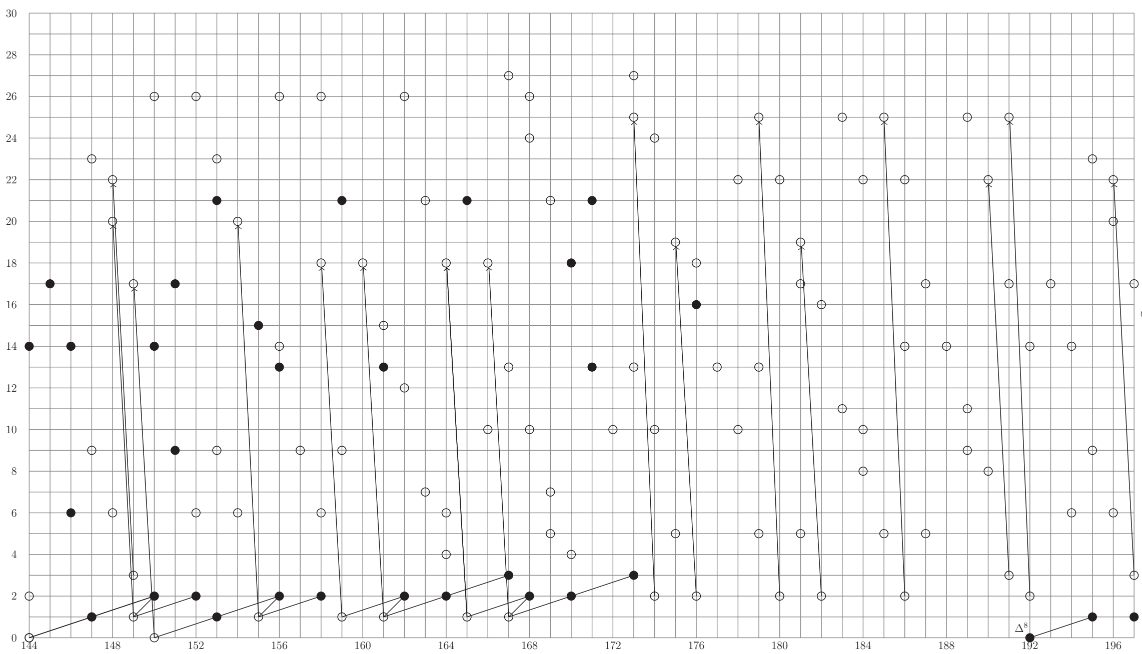

Figure II.29 – HFPSS for $A_1[00]$ and $A_1[11]$ from E_7-term with $144 \leq t - s \leq 197$

Chapter III

Homotopy groups of $E_C^{hC_6} \wedge A_1$

In this chapter, we will compute the HFPSS for $E_C^{hC_2}$ and $E_C^{hC_6}$ smashed with $S^0, V(0), Y$ and A_1. Although, our main objective (to which a large part of this chapter is devoted) is the HFPSS for $E_C^{hC_2} \wedge A_1$ and $E_C^{hC_6} \wedge A_1$, the calculation for the other spectra will give some input for computing the latter and will serve for future reference. The calculations for the group C_6 can be deduced immediately from those for C_2 by taking C_3-fixed points. To be more precise, since $E_C^{hC_6} \simeq (E_C^{hC_2})^{hC_3}$, the group C_3 acts on the HFPSS for C_2. Since C_3 has order prime to 2, the C_3-fixed points of the HFPSS for C_2 is isomorphic to the HFPSS for C_6. For this reason, we will mainly present a calculation for C_2 and indicate the C_3-action on the spectral sequences. We give the final result (i.e., the E_∞-term for the group C_6) in charts.

6 The homotopy fixed point spectral sequences for $E_C^{hC_6}, E_C^{hC_6} \wedge V(0), E_C^{hC_6} \wedge Y$

6.1 The homotopy fixed point spectral sequence for $E_C^{hC_6}$

Recall that $C_6 = C_2 \times C_3 = \langle -1 \rangle \times \langle \omega \rangle$ is a subgroup of G_{24}. The action of C_6 on $(E_C)_* \cong \mathbb{W}[[u_1]][u^{\pm 1}]$ is then deduced from Theorem 1.5.1, hence is given by

$$(-1)u = -u, \quad (-1)u_1 = u_1,$$

$$\omega(u) = \zeta u, \quad \omega(u_1) = \zeta u_1.$$

The computation of the ring of group cohomology $\mathrm{H}^*(C_2, (E_C)_*)$ is elementary.

Lemma 6.1.1. *a) There is a class $t \in \mathrm{H}^1(C_2, (E_C)_2)$ and an isomorphism of $\mathbb{W}$-algebras*

$$\mathrm{H}^*(C_2, (E_C)_*) \cong \mathbb{W}[[u_1]][u^{\pm 2}, t]/(2t).$$

b) The class $t \in \mathrm{H}^1(C_2, (E_C)_2)$ can be represented by the cocyle

$$t : C_2 \to (E_C)_2 = \mathbb{W}[[u_1]]\{u^{-1}\}, 1 \mapsto 0, -1 \mapsto u^{-1}.$$

c) The action of C_3 on $\mathrm{H}^*(C_2, \mathbb{W}[[u_1]][u^{\pm 1}])$ is given by

$$\omega(u_1) = \zeta u_1, \quad \omega(u^{-2}) = \zeta u^{-2}, \quad \omega(t) = \zeta^2 t.$$

Proof. a) We have that $\mathbb{W}[[u_1]][u^{\pm 1}] \cong \mathbb{Z}_2[u^{\pm 1}] \otimes_{\mathbb{Z}_2} \mathbb{W}[[u_1]]$ as C_2-modules. Thus, we have

$$\mathrm{H}^*(C_2, \mathbb{W}[[u_1]][u^{\pm 1}]) \cong \mathrm{H}^*(C_2, \mathbb{Z}_2[u^{\pm 1}]) \otimes_{\mathbb{Z}_2} \mathbb{W}[[u_1]]$$

as $\mathbb{W}[[u_1]]$ has a trivial C_2-action and it is flat over $\mathbb{Z}_2$. By an elementary calculation, we have

$$\mathrm{H}^*(C_2, \mathbb{Z}_2[u^{\pm 1}]) \cong \mathbb{Z}_2[u^{\pm 2}, \chi]/(2\chi) \oplus \mathbb{Z}_2[u^{\pm 2}, \chi]/(2)\{t\}$$

as modules over $\mathbb{Z}_2[u^{\pm 2}, \chi]/(2\chi)$, where $\chi \in \mathrm{H}^2(C_2, \mathbb{Z}_2)$ is the cohomological periodicity class of C_2 and $t \in \mathrm{H}^1(C_2, \mathbb{Z}_2\{u^{-1}\})$. To obtain the multiplicative structure of $\mathrm{H}^*(C_2, \mathbb{Z}_2[u^{\pm 1}])$, it remains to prove that $t^2 = \chi u^{-2}$.

Consider the following exact sequence of $\mathbb{Z}_2[u^{\pm 1}][C_2]$-modules

$$0 \to \mathbb{Z}_2[u^{\pm 1}] \xrightarrow{\times 2} \mathbb{Z}_2[u^{\pm 1}] \xrightarrow{p} \mathbb{F}_2[u^{\pm 1}] \to 0, \tag{III.1}$$

where C_2 acts on $\mathbb{F}_2[u^{\pm 1}]$ trivially. First, there is an isomorphism of algebras

$$\mathrm{H}^*(C_2, \mathbb{F}_2[u^{\pm 1}]) \cong \mathbb{F}_2[u^{\pm 1}, \tilde{\chi}],$$

where $\tilde{\chi}$ is the unique nontrivial class of $\mathrm{H}^1(C_2, \mathbb{F}_2)$. The sequence (III.1) induces a long exact sequence of $\mathrm{H}^*(C_2, \mathbb{Z}_2[u^{\pm 1}])$-modules:

$$\mathrm{H}^*(C_2, \mathbb{Z}_2[u^{\pm 1}]) \xrightarrow{\times 2} \mathrm{H}^*(C_2, \mathbb{Z}_2[u^{\pm 1}]) \xrightarrow{p_*} \mathrm{H}^*(C_2, \mathbb{F}_2[u^{\pm 1}]) \xrightarrow{\delta} \mathrm{H}^{*+1}(C_2, \mathbb{Z}_2[u^{\pm 1}]).$$

Since u^{-2} and t are not divisible by 2,

$$p_*(u^{-2}) = u^{-2} \quad \text{and} \quad p_*(t) = \tilde{\chi} u^{-1}. \tag{III.2}$$

It follows that

$$p_*(t) u^{-1} = \tilde{\chi} u^{-1} u^{-1} = u^{-2} \tilde{\chi}. \tag{III.3}$$

Similarly, since $\mathrm{H}^1(C_2, \mathbb{Z}_2) = \mathrm{H}^0(C_2, \mathbb{Z}_2\{u^{-1}\}) = 0$,

$$\delta(\tilde{\chi}) = \chi \quad \text{and} \quad \delta(u^{-1}) = t. \tag{III.4}$$

It follows that

$$u^{-2}\chi = u^{-2}\delta(\tilde{\chi}) = \delta(u^{-2}\tilde{\chi}) = \delta(p_*(t)u^{-1}) = t\delta(u^{-1}) = t^2,$$

where the second and the fourth equalities use the fact that δ is $\mathrm{H}^*(C_2, \mathbb{Z}_2[u^{\pm 1}])$-linear.

b) It is straightforward to show that the cocycle $C_2 \to \mathbb{Z}_2\{u^{-1}\}, 1 \mapsto 0, -1 \mapsto u^{-1}$ represents the unique non-trivial class $t \in \mathrm{H}^1(C_2, \mathbb{Z}_2\{u^{-1}\})$.

Part c) follows from the action of C_3 on $(E_C)_*$ and the cocycle representation of t. □

Consider the cofiber sequence

$$S^0 \xrightarrow{\times 2} S^0 \xrightarrow{\iota} V(0). \tag{III.5}$$

It induces a short exact sequence of $(E_C)_*[C_2]$-modules

$$0 \to (E_C)_* \xrightarrow{\times 2} (E_C)_* \xrightarrow{\iota_*} (E_C)_*/2 \to 0 \tag{III.6}$$

By Equation (III.4), the connecting homomorphism

$$\delta : \mathrm{H}^s(C_2, (E_C)_*/2) \to \mathrm{H}^{s+1}(C_2, (E_C)_*)$$

sends $u^{-1} \in \mathrm{H}^0(C_2, (E_C)_2/2)$ to t, i.e.,

$$\delta(u^{-1}) = t. \tag{III.7}$$

Furthermore, δ is a map of $\mathrm{H}^*(C_2, (E_C)_*)$-modules. Let us denote by v_1 the class $u_1u^{-1} \in (E_C)_2/2$. It follows that

$$\delta(v_1) = \delta(u_1u^{-1}) = u_1\delta(u^{-1}) = u_1t, \tag{III.8}$$

$$\delta(v_1^3) = \delta(u_1^3u^{-2}u^{-1}) = u_1^3u^{-2}\delta(u^{-1}) = u_1^3u^{-2}t. \tag{III.9}$$

The above equations use the fact that $u_1, u^{-2} \in \mathrm{H}^0(C_2, (E_C)_*)$. We will also need the following:

Lemma 6.1.2. *The induced map in homotopy of the unit* $S^0 \to E_C^{hC_2}$ *sends* ν *to a nontrivial element detected in filtration at most* 3 *of the HFPSS for* $E_C^{hC_2}$.

Proof. Consider the restriction and the transfer $E_C^{hG_{24}} \xrightarrow{Res} E_C^{hC_2} \xrightarrow{Tr} E_C^{hG_{24}}$. The maps Res and Tr induce on the E_2-term of the HFPSS the usual restriction and transfer for cohomology of groups. Algebraically, Tr_* send $1 \in \mathrm{H}^0(C_2, (E_C)_0)$ to $|G_{24}/C_2| = 12 \in \mathrm{H}^0(G_{24}, E_0)$. Thus, in homotopy groups Tr_* sends $1 \in \pi_0(E_C^{hC_2})$ to an element detected by 12. Moreover, we know that $\pi_0(E_C^{hG_{24}})$ is non-torsion. This can be deduced from the fact the knowledge of $\pi_*(tmf)$ [**Bau08**] and that $L_{K(2)}tmf \simeq (E_C^{hG_{24}})^{\mathrm{Gal}}$. Thus $\pi_0(E_C^{hG_{24}})$ is concentrated in the filtration 0 of the HFPSS, see Proposition 10.0.7. This means that the composite $Tr_* \circ Res_*$: $\pi_0(E_C^{hG_{24}}) \to \pi_0(E_C^{hG_{24}})$ sends the unit to 12 times of the unit. Together with the fact that the restriction and the transfer are maps of $E_C^{hG_{24}}$-modules, we conclude that $Tr \circ Res \simeq 12Id$. As a consequence,

$$Tr_*(Res_*(\nu)) = 12\nu = \eta^3 \in \pi_3(E_C^{hG_{24}}),$$

the last equation holds in $\pi_3(S^0)$. Finally, η^3 is non-trivial and is detected in filtration 3 of the HFPSS for $E_C^{hG_{24}}$. This is because η is detected in filtration 1 (see the discussion succeeding Corollary 5.1.3) and η^3 is not hit by a differential d_3, fact that can be deduced from the structure of $\mathrm{H}^*(G_{48}, (E_C)_*)$ and the differential d_3 given in Lemma 2.21 of [**BG18**]. Therefore, $Res_*(\nu) \in \pi_3(E_C^{hC_2})$ is detected in a filtration at most 3. □

Now we discuss the differentials of the HFPSS for $E_C^{hC_6}$. We note here that by the checkerboard phenomenon, all even differentials are trivial.

Proposition 6.1.3. *a) In the HFPSS for $E_C^{hC_2}$, the classes u_1, t, u^{-4} are d_3-cycles. The differentials d_3 are determined by the multiplicative structure and the following d_3-differential:*

$$d_3(u^{-2}) = u_1 t^3.$$

b) As a module over $\mathbb{W}[[u_1]][u^{\pm 4}, t]/(2t)$,

$$\mathrm{E}_5 \cong \mathbb{W}[[u_1]][u^{-4}, t]/(2t, u_1 t^3)\{1\} \oplus \mathbb{W}[[u_1]][u^{-4}, t]/(t)\{2u^{-2}\}.$$

There is no d_5-differential; hence $\mathrm{E}_5 = \mathrm{E}_7$. c) The class t^3 survives to the E_∞-term detecting ν.

Proof. a) To deduce the d_3-differential on u^{-2}, we compare the ANSS for $S^0_{(2)}$ with the HFSS for $E_C^{hC_2}$; there are maps of spectral sequences of rings induced by the map of ring spectra $BP \to E_C$ classifying the 2-typification of the formal group law of E_C:

$$\begin{array}{ccccc}
\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) & \longrightarrow & \mathrm{H}^s(\mathbb{G}, E_t) & \longrightarrow & \mathrm{H}^s(C_2, (E_C)_t) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\pi_{t-s} S^0_{(2)} & \longrightarrow & \pi_{t-s} L_{K(2)} S^0 & \longrightarrow & E_C^{hC_2}.
\end{array}$$

Recall that the connecting homomorphism

$$\delta : \mathrm{Ext}^{0,*}_{BP_*BP}(BP_*/2) \to \mathrm{Ext}^{1,*}_{BP_*BP}(BP_*)$$

sends v_1 and v_1^3 to the Greek letter elements $\eta = \alpha_1$ and α_3, respectively. See [**Rav86**]. The ring homomorphism $BP_* \to (E_C)_*$ sends v_1 to u_1u^{-1} modulo 2, see [**Bea17**], Proposition 6.1.1. By the naturality of the connecting homomorphism, the map $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*) \to \mathrm{H}^s(C_2, (E_C)_t)$ of the above spectral sequence sends η to u_1t and α_3 to $u_1^3u^{-2}t$, by Equation (III.8) and (III.9). In particular, u_1t is a permanent cycle in the HFPSS for $E_C^{hC_2}$. We also write η for $u_1t \in \mathrm{H}^1(C_2, (E_C)_2)$. In the Adams-Novikov spectral sequence, there is a d_3-differential

$$d_3(\alpha_3) = \eta^4.$$

By naturality, it induces the following d_3-differential in the HFPSS for $E_C^{hC_2}$:

$$d_3(u_1^3u^{-2}t) = u_1^4t^4 = u_1^2\eta u_1t^3.$$

On the other hand, the class u_1^2 is a d_3-cycle because $d_3(u_1^2) = 2u_1d_3(u_1)$ and $d_3(u_1)$ is annihilated by 2, as it lives in a positive cohomology group of C_2. Then, by the Leibniz rule, we have that

$$d_3(u_1^2\eta u^{-2}) = u_1^2\eta d_3(u^{-2}).$$

Because multiplication by $u_1^2\eta$ is injective on $\mathrm{E}_3^{s\geq 1,*}$, which denotes the groups of filtration at least 1, we obtain that

$$d_3(u^{-2}) = u_1t^3. \tag{III.10}$$

Now we show that t^3 is a permanent cycle detecting the image of ν via the unit $S^0 \to E_C^{hC_2}$. By Lemma 6.1.2, ν is non-trivial in $\pi_3(E_C^{hC_2})$ and is detected in filtration at most 3. At stem $t-s=3$ of the E_2-term, all groups of filtration less than 3 are trivial. Therefore, ν is detected by a class of $\mathrm{H}^3(C_2, (E_C)_6)$, which has the form $pt^3 + qu_1t^3$ where $p, q \in \mathbb{W}[[u_1^2]]$. Using the d_3-differential (III.10) and the fact that u_1^2 is a d_3-cycle as proved above, we deduce that

$$d_3(qu^{-2}) = qu_1t^3.$$

This means that pt^3 is a nontrivial permanent cycle detecting ν in the E_∞-term. In particular,

$$0 = d_3(pt^3) = pd_3(t^3),$$

which implies that $d_3(t^3) = 0$ because multiplication by $p \neq 0$ is injective in the positive cohomology groups. This implies that t is a d_3-cycle. In effect, since t^2 is a d_3-cycle for the same reason as u_1^2 is a d_3-cycle, we obtain that

$$0 = d_3(t^3) = t^2d_3(t).$$

The claim follows as multiplication by t is injective in positive filtration.

Similarly, since $\eta = u_1 t$ is a permanent cycle, we have that

$$0 = d_3(u_1 t) = t d_3(u_1),$$

which implies that

$$d_3(u_1) = 0, \tag{III.11}$$

because multiplication by t induces an injective map on positive cohomology groups.

Finally, (III.10) and (III.11) imply that, for any non-negative integer k, $d_3(u_1^k u^{-2}) = u_1^{k+1} t^3$, which induces that $\mathrm{E}_\infty^{3,6} \leq \mathbb{F}_4\{t^3\}$. Since $\nu \in \pi_3(E_C^{hC_2})$ must be detected in $\mathrm{E}_\infty^{3,6}$, we conclude that t^3 survives to the E_∞-term, detecting ν.

b) The module structure of the E_5-term follows immediately from the differentials d_3. By sparseness, the next possible non-trivial differentials are d_7.

c) This is proved in part a). □

Proposition 6.1.4. *a) In the HFPSS for $E_C^{hC_2}$, the classes $u_1, t, 2u^{-2}, u^{-8}$ are d_7-cycles. There is the following d_7 differential:*

$$d_7(u^{-4}) = t^7.$$

The other d_7-differentials follows from the multiplicative structure and the Leibniz rule.

b) As a module over the algebra $\mathbb{W}[[u_1]][u^{\pm 8}, t]/(t^7, 2t)$, the E_8-term is generated by $1, 2u^{-2}, 2u^{-4}, u_1 u^{-4}, 2u^{-6}$ with the following relations

$$t^7 = t(2u^{-2}) = t(2u^{-4}) = t^3(u_1 u^{-4}) = t(2u^{-6}) = 0,$$

$$2(u_1 u^{-4}) = u_1(2u^{-4}).$$

The spectral sequence collapses at the E_8-term and $\mathrm{E}_8^{s,t} = 0$ if $s \geq 8$.

c) The pair (C_2, E_C) is a regular pair.

Proof. We prove that t is a d_7-cycle. In effect, t^2 is a d_7-cycle because

$$d_7(t^2) = 2t d_7(t) = 0.$$

Because t^3 is a permanent cycle by Proposition 6.1.3 part c), implies that

$$0 = d_7(t^3) = d_7(t^2 t) = t^2 d_7(t).$$

It follows that $d_7(t) = 0$, because multiplication by t induces an injective map on $\mathrm{E}_7^{s\geq 3,*}$. Similarly, since $u_1 t$ is a permanent cycle, we have that

$$0 = d_7(u_1 t) = t d_7(u_1).$$

It follows that u_1 is a d_7-cycle by the same reason as above.

Next, by the structure of the E_7-term described in Proposition 6.1.3, all classes of $\mathrm{E}_7^{s\geq 7,*}$ is t-free. Since $2u^{-2}$ is t-torsion, $2u^{-2}$ is a d_7-cycle.

Now because $\nu^4 = 0 \in \pi_*(S^0_{(2)})$, t^{12} detecting ν^4 by must be hit by a differential. Inspection shows that the only possibility is that

$$d_7(u^{-4}t^5) = t^{12}.$$

Because t is a d_7-cycle and t acts injectively on $\mathrm{E}_7^{s\geq 3,*}$, the last equation implies that

$$d_7(u^{-4}) = t^7.$$

b) The structure of the E_8-term follows easily from the description of the d_7-differentials. By sparseness, the spectral sequence collapses at the E_8-term.

c) By 6.1.1, t is a cohomological periodicity class for $\mathrm{H}^*(C_2, (E_C)_*)$ and by part a) and b), t is a permanent cycle in the HFPSS for $E_C^{hC_2}$. □

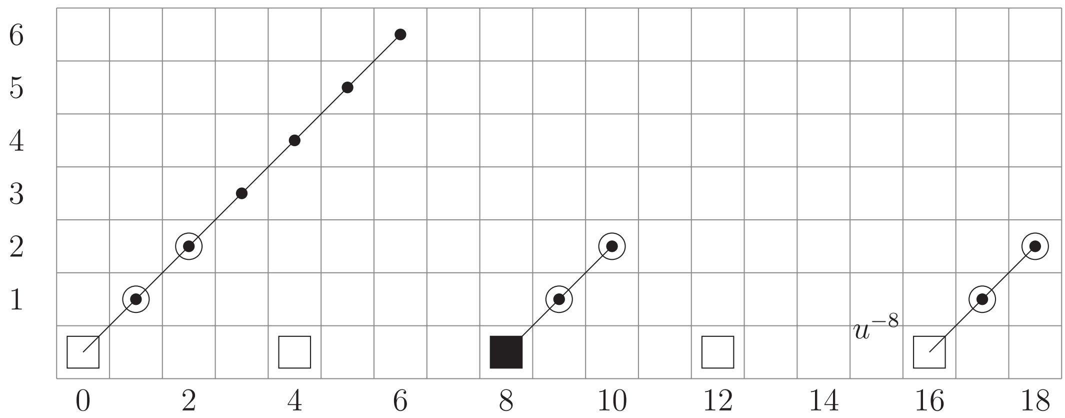

Figure III.1 – E_∞-term of the HFPSS for $E_C^{hC_2}$. A square $\square$ represents a copy of $\mathbb{W}[[u_1]]$, a black square the ideal $(2, u_1)$ of $\mathbb{W}[[u_1]]$, a circled black dot a copy of $\mathbb{F}_4[[u_1]]$, and a black dot a copy of $\mathbb{F}_4$. A line represents a multiplication by t. The E_∞-term is 16-periodic by multiplication by u^{-8}.d

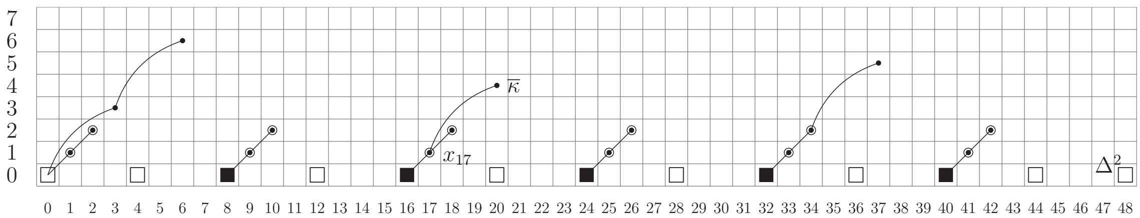

Figure III.2 – E_∞-term of the HFPSS for $E_C^{hC_6}$. A square $\square$ represents a copy of $\mathbb{W}[[u_1^3]]$, a black square represents a copy of the ideal $(2, u_1^3)$ of $\mathbb{W}[[u_1^3]]$, a circled black dot a copy of $\mathbb{F}_4[[u_1^3]]$ and a black dot a copy of $\mathbb{F}_4$. A curved line represents multiplication by ν, a straight line multiplication by η. The E_∞-term is 48-periodic by multiplication by Δ^2.

The E_∞-term of the HFPSS for $E_C^{hC_6}$ is deduced from that for $E_C^{hC_2}$ by taking C_3-fixed points; the action of C_3 is on the E_∞-term of the HFPSS for $E_C^{hC_2}$ is induced from the action on the E_2-term given in Lemma 6.1.1 Part c). The class $u^{-8}t \in E_\infty^{1,18}$ detects an element without ambiguity of $\pi_{17}(E_C^{hC_6})$. We write x_{17} for this element. We emphasise that the class x_{17} plays a role of a cohomological periodicity class which is a permanent cycle for the regular pair (C_6, E_C). The elements η et ν are represented by $u_1 t$ and t^3, respectively. The element $\overline{\kappa} \in \pi_{20}(E_C^{hG_{24}})$ has order 8. A similar argument as in the proof of Lemma 6.1.2 shows that $Res(\overline{\kappa})$ is nontrivial in $\pi_{20}E_C^{hC_6}$ and hence is detected in $E_\infty^{4,24}$. The class $\Delta^2 := u^{-24}$ detects the periodicity element of $\pi_*(E_C^{hC_6})$. From the multiplicative structure of $\pi_*(E_C^{hC_2})$, we have the following relations in $\pi_*(E_C^{hC_6})$

$$x_{17}^7 = 0, \nu x_{17} = \overline{\kappa}, x_{17}^3 = \Delta^2 \nu.$$

In what follows, we describe the HFPSS for $E_C^{hC_2} \wedge V(0)$, $E_C^{hC_2} \wedge Y$ or $E_C^{hC_2} \wedge A_1$. These spectral sequences are modules over that for $E_C^{hC_2}$. By Proposition 6.1.4, $(C_2, (E_C)_C)$ is a regular pair. Since the cohomological periodicity class t verifies that $t^7 = 0$, we have (see Corollary 5.3.3):

Proposition 6.1.5. *Let M be a $C_2 - E_C$-module spectrum. Then, in the HFPSS for M^{hC_2}, we have that*

a) At the E_r-term,

1. *All classes having positive cohomological filtration are divisible by t.*
2. *All classes having cohomological filtration greater than r are t-free.*
3. *All t-torsion classes are permanent cycles*

b) The spectral sequence collapses at the E_8-term and $\mathrm{E}_\infty^{s,t} = 0$ if $s \geq 7$.

6.2 The homotopy fixed point spectral sequence for $E_C^{hC_6} \wedge V(0)$

Lemma 6.2.1. *a) There is a class $t \in \mathrm{H}^1(C_2, \mathbb{F}_4[[u_1]]\{u^{-1}\})$ such that there is an isomorphism of algebras*

$$\mathrm{H}^*(C_2, (E_C)_*/2) \cong \mathbb{F}_4[[u_1]][u^{\pm 1}, t].$$

b) The map of algebras

$$\iota_* : \mathrm{H}^*(C_2, (E_C)_*) \to \mathrm{H}^*(C_2, (E_C)_*/2)$$

induced by the inclusion of the bottom cell $\iota : S^0 \to V(0)$ sends u_1 to u_1, u^{-2} to u^{-2} and t to t. As a module over $\mathrm{H}^*(C_2, (E_C)_*)$, $\mathrm{H}^*(C_2, (E_C)_*/2)$ is generated by $e_0 := 1$ and $u^{-1}e_0$.

c) The action of C_3 on $\mathrm{H}^*(C_2, \mathbb{F}_4[[u_1]][u^{\pm 1}])$ is given by

$$\omega(u_1) = \zeta u_1, \quad \omega(u^{-1}) = \zeta^2 u^{-1}, \quad \omega(t) = \zeta^2 t.$$

Proof. a) The map $\iota : S^0 \to V(0)$ induces, in E_C-homology, the mod-2 projection $(E_C)_* \to (E_C)_*/2$. It follows that C_2 acts trivially on $(E_C)_*V(0) = \mathbb{F}_4[[u_1]][u^{\pm 1}]$ by maps of $\mathbb{F}_4$-algebras. Thus by an elementary cohomology group calculation, we have

$$\mathrm{H}^*(C_2, (E_C)_*/2) \cong \mathrm{H}^*(C_2, \mathbb{F}_4) \otimes \mathbb{F}_4[[u_1]][u^{\pm 1}] \cong \mathbb{F}_4[[u_1]][u^{\pm 1}, t]$$

as $\mathbb{F}_4$-algebras with t the nontrivial class of $\mathrm{H}^1(C_2, \mathbb{F}_4\{u^{-1}\})$.

b) The map $\iota_* : (E_C)_* \to (E_C)_*/2$ is a map of C_2-algebras. It is immediate to see that the map of invariants $\mathrm{H}^0(C_2, (E_C)_*) \to \mathrm{H}^0(C_2, (E_C)_*/2)$ sends u_1 and u^{-2}

to classes of the same name. It remains to see that the map $\mathrm{H}^1(C_2, \mathbb{W}\{u^{-1}\}) \to \mathrm{H}^1(C_2, \mathbb{F}_4\{u^{-1}\})$ sends t to t. This is Equation (III.2) in the proof of Lemma 6.1.1.

c) This follows from part c) of Lemma 6.1.1 and part b) above. □

Proposition 6.2.2. *In the HFPSS for $E_C^{hC_2} \wedge V(0)$,*

- *a) The classes e_0 and $u^{-1}e_0$ are permanent cycles.*
- *b) All d_3-differentials are determined by the Leibniz rule and the following differentials*

$$d_3(u^{-2}e_0) = u_1 t^3 e_0, \quad d_3(u^{-3}e_0) = u_1 t^3 u^{-1} e_0.$$

- *c) The $\mathrm{E}_4 = \mathrm{E}_7$-term of the HFPSS for $E_C^{hC_2} \wedge V(0)$ is isomorphic to*

$$\mathrm{E}_4 = \mathrm{E}_7 \cong \mathbb{F}_4[[u_1]][u^{\pm 4}, t]/(u_1 t^3)\{e_0\} \oplus \mathbb{F}_4[[u_1]][u^{\pm 4}, t]/(u_1 t^3)\{u^{-1}e_0\}$$

with an evident structure of module over the sub-algebra $\mathbb{W}[[u_1]][u^{\pm 4}, t]$ of the $\mathrm{E}_4 = \mathrm{E}_7$-term of the HFPSS for $E_C^{hC_2}$.

Proof. a) By part b) of Lemma 6.2.1, the map ι_* sends 1 to e_0, so that e_0 is a permanent cycle, by naturality.

Next, by Equation (III.7), the connecting homomorphism

$$\delta : \mathrm{H}^0(C_2, (E_C)_2/2) \cong \mathbb{F}_4[[u_1]]\{u^{-1}e_0\} \to \mathrm{H}^1(C_2, (E_C)_2) \cong \mathbb{F}_4[[u_1]]\{t\}$$

is an isomorphism of $\mathbb{W}[[u_1]]$-modules sending $u^{-1}e_0$ to t. Since t is a non-trivial permanent cycle detecting an element of order 2 in $\pi_1(E_C^{hC_2})$, $u^{-1}e_0$ must also be a permanent cycle in the HFPSS for $E_C^{hC_2} \wedge V(0)$ by virtue of the Geometric Boundary Lemma (Proposition A.10 of [DH04]) applied to the cofiber sequence (III.5).

b) In the HFPSS for $E_C^{hC_2}$, there is a d_3-differential, by Proposition 6.1.3,

$$d_3(u^{-2}) = u_1 t^3.$$

By Part a) and the Leibniz rule, for $k = 0, -1$

$$d_3(u^{-2}(u^k e_0)) = u_1 t^3 u^k e_0.$$

The others d_3-differentials are generated by t- and u^{-4}-linearity.

c) The structure of the E_4-term of the HFPSS for $E_C^{hC_2} \wedge V(0)$ follows easily from the d_3-differentials (see Figure III.3, III.4). There is no d_5-differential because, on one hand the d_5-differentials are u_1, u^{-4}, t-linear, on the other hand, e_0 and $u^{-1}e_0$ are permanent cycles. □

Proposition 6.2.3. *In the HFPSS for $E_C^{hC_2} \wedge V(0)$,*

a) The differentials d_7 is $\mathbb{W}[[u_1]][u^{\pm 8}, t]$-linear and is determined by

$$d_7(u^{-4}e_0) = t^7 e_0, \quad d_7(u^{-4}u^{-1}e_0) = t^7 u^{-1} e_0.$$

b) As a module over the sub-algebra $\mathbb{W}[[u_1]][u^{\pm 8}, t]/(t^7, 2t)$ of the E_8-term of the HFPSS for $E_C^{hC_2}$, the $\mathrm{E}_8 = \mathrm{E}_\infty$-term is generated by the following classes, with the corresponding annihilation ideal:

Gen	e_0	$u^{-1}e_0$	$u_1 u^{-4} e_0$	$u_1 u^{-5} e_0$
Anni. Ideal	$(2, u_1 t^3)$	$(2, u_1 t^3)$	$(2, t^3)$	$(2, t^3)$

c) There are the following exotic extensions by 2, in the sense that they are not detected in the E_∞-term:

$$2(u^{-1}e_0) = u_1 t^2 e_0, \quad 2(u_1 u^{-5} e_0) = u_1^2 t^2 u^{-4} e_0.$$

Proof. a) In the HFPSS for $E_C^{hC_2}$, there is a d_7-differential

$$d_7(u^{-4}) = t^7.$$

The Leibniz rule and the fact that e_0 and $u^{-1}e_0$ are permanent cycles implies that, for $k = 0, -1$,

$$d_7(u^{-4}(u^k e_0)) = t^7 u^k e_0.$$

b) The module structure of the $\mathrm{E}_8 = \mathrm{E}_\infty$-term follows from a), see Figure III.5.

c) We can see from the E_∞-term that $2e_0 = 0$. Recall that there is a Toda bracket $\langle 2, \eta, 2 \rangle = \eta^2$ in $\pi_* S^0$. By juggling, we have that

$$\eta^2 e_0 = \langle 2, \eta, 2 \rangle e_0 = 2\langle \eta, 2, e_0 \rangle.$$

Because $\eta^2 e_0 \neq 0$, $\langle \eta, 2, e_0 \rangle$ is non-trivial and admits a non-trivial multiplication by 2. What is more, $\langle \eta, 2, e_0 \rangle$ is C_3-invariant, and so must be detected by $u_1 u^{-1} e_0 = v_1 e_0$ up to an invertible element of $\mathbb{F}_4[[u_1^3]]$. It follows that $2u^{-1}e_0 = u_1 t^2 e_0$ because u_1 acts injectively on $\mathrm{E}_\infty^{2,4}$. Multiplying the last equation by $u_1 u^{-4} \in \pi_8 E_C^{hC_2}$, we obtain the second equality of c). □

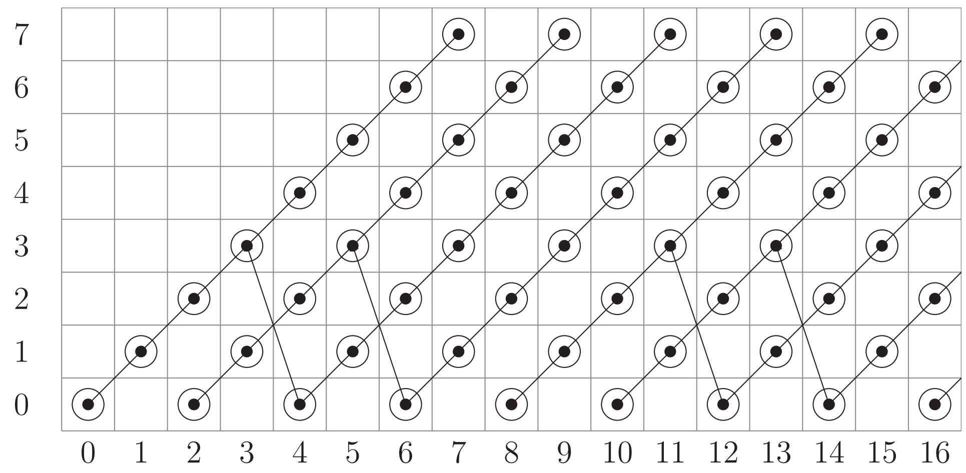

Figure III.3 – The E_2-term of the HFPSS for $E_C^{hC_2} \wedge V(0)$ and d_3-differentials. A circled black dot represents a copy of $\mathbb{F}_4[[u_1]]$.

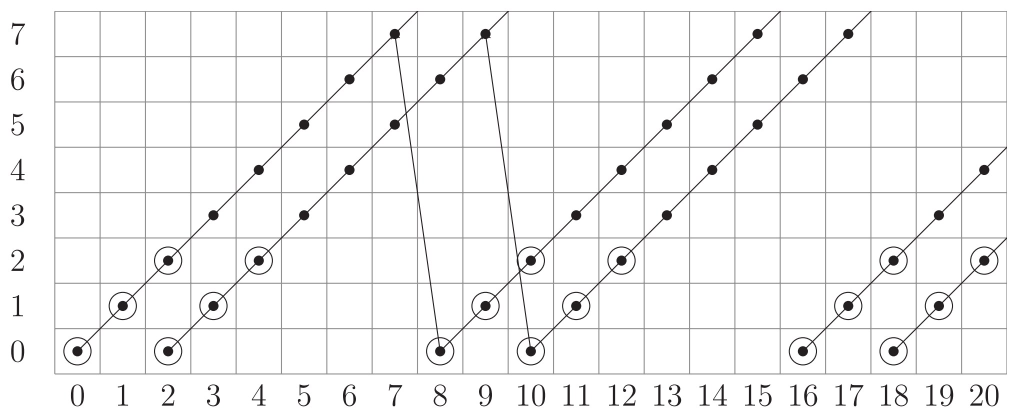

Figure III.4 – The $\mathrm{E}_4 = \mathrm{E}_7$-term of the HFPSS for $E_C^{hC_2} \wedge V(0)$ and d_7-differentials. A circled black dot represents a copy of $\mathbb{F}_4[[u_1]]$, a black dot a copy of $\mathbb{F}_4$.

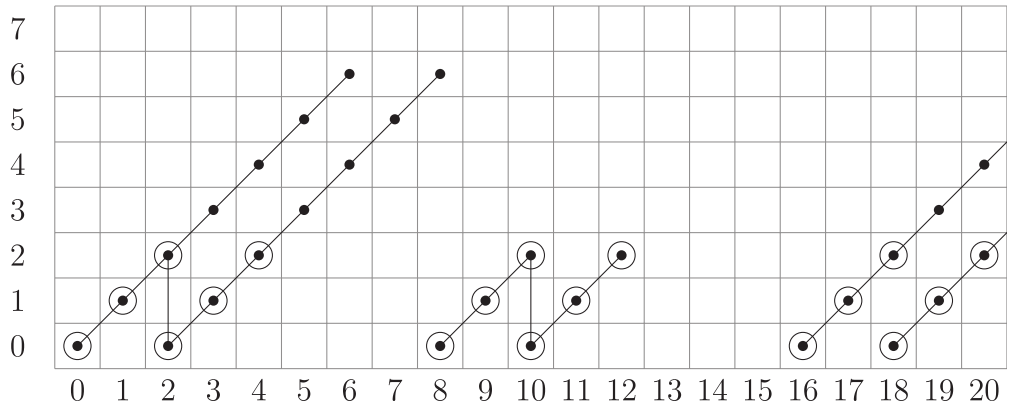

Figure III.5 – The E_8-term of the HFPSS for $E_C^{hC_2} \wedge V(0)$. A circled black dot represents a copy of $\mathbb{F}_4[[u_1]]$, a black dot a copy of $\mathbb{F}_4$, a vertical line represents extension by 2 and a line of slope 1 multiplication by t.

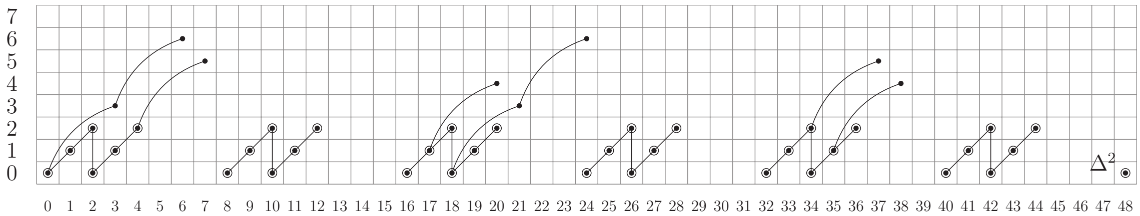

Figure III.6 – E_∞-term of the HFPSS for $E_C^{hC_6} \wedge V(0)$. A circled black dot represents a copy of $\mathbb{F}_4[[u_1^3]]$ and a black dot a copy of $\mathbb{F}_4$. A curved line represents multiplication by ν and a line of slope 1 multiplication by η, a vertical line multiplication with 2. The E_∞-term is 48-periodic by multiplication by Δ^2

6.3 The homotopy fixed point spectral sequence for $E_C^{hC_6} \wedge Y$

The cofibration

$$\Sigma V(0) \xrightarrow{\eta} V(0) \xrightarrow{\iota} Y \xrightarrow{p} \Sigma^2 V(0) \tag{III.12}$$

induces a short exact sequence of $(E_C)_*/2[C_2]$-modules:

$$0 \to (E_C)_*/2 \to (E_C)_*(Y) \to (E_C)_{*-2}/2 \to 0,$$

because $(E_C)_*/2$ is concentrated in even degrees. The associated connecting homomorphism $\delta : \mathrm{H}^s(C_2, (E_C)_*/2) \to \mathrm{H}^{s+1}(C_2, (E_C)_{*+2}/2)$ is $\mathrm{H}^s(C_2, (E_C)_*/2)$-linear.

Lemma 6.3.1. *a) δ sends $e_0 \in \mathrm{H}^0(C_2, (E_C)_0/2)$ to $\eta \in \mathrm{H}^1(C_2, (E_C)_2/2)$.*

b) There is an isomorphism of $\mathrm{H}^*(C_2, (E_C)_*/2)$-modules

$$\mathrm{H}^*(C_2, (E_C)_*(Y)) \cong \mathbb{F}_4[[u_1]][u^{\pm 1}, t]/(u_1 t)\{e_0\}.$$

c) The induced map on $\mathrm{H}^*(C_2, (E_C)_*(Y))$ by a v_1-self map of Y is injective on $\mathrm{H}^0(C_2, (E_C)_*Y)$ and is trivial on $\mathrm{H}^s(C_2, (E_C)_*Y)$ when $s > 0$.

d) The action of C_3 on $\mathrm{H}^*(C_2, (E_C)_*Y)$ is given by

$$\omega(e_0) = e_0, \quad \omega(u_1) = \zeta u_1, \quad \omega(u^{-1}) = \zeta^2 u^{-1}, \quad \omega(t) = \zeta^2 t.$$

Proof. Consider the evident map of cofiber sequences

$$\begin{array}{ccccccc} \Sigma S^0 & \xrightarrow{\eta} & S^0 & \longrightarrow & C_\eta & \longrightarrow & \Sigma^2 S^0 \\ \downarrow{\scriptstyle \Sigma\iota} & & \downarrow{\scriptstyle \iota} & & \downarrow & & \downarrow \\ \Sigma V(0) & \xrightarrow{\eta} & V(0) & \longrightarrow & Y & \longrightarrow & \Sigma^2 V(0) \end{array}$$

which induces a map of short exact sequences of C_2-$(E_C)_*$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (E_C)_* & \longrightarrow & (E_C)_*(C_\eta) & \longrightarrow & E_{*-2} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (E_C)_*V(0) & \longrightarrow & (E_C)_*Y & \longrightarrow & (E_C)_{*-2}V(0) & \longrightarrow & 0. \end{array}$$

By the naturality of the connecting homomorphism, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(C_2, (E_C)_0) & \xrightarrow{\partial} & \mathrm{H}^1(C_2, (E_C)_2) \\ \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\ \mathrm{H}^0(C_2, (E_C)_0(V(0))) & \xrightarrow{\partial} & \mathrm{H}^1(C_2, (E_C)_2 V(0)). \end{array}$$

The upper connecting homomorphism sends 1 to η, because the induced map on $\mathrm{H}^*(C_2, -)$ of $(E_C)_* \to (E_C)_*(C_\eta)$ sends η to 0, because of the fact that $\eta = 0 \in \pi_1(C_\eta)$. Since the map $\iota_* : \mathrm{H}^*(C_2, (E_C)_*) \to \mathrm{H}^*(C_2, (E_C)_*(V(0)))$ sends 1 to e_0 and η to ηe_0, part $a)$ follows.

Part b) is an immediate consequence of part a) together with the fact that δ is a map of $\mathrm{H}^*(C_2, (E_C)_*/2)$-modules. In effect, the connecting homomorphism is injective, there is an isomorphism of $\mathrm{H}^*(C_2, (E_C)_*/2)$-modules

$$\mathrm{coker}(\mathrm{H}^{s-1}(C_2, (E_C)_{t-2}/2) \xrightarrow{\eta} \mathrm{H}^s(C_2, (E_C)_t/2)) \cong \mathrm{H}^s(C_2, (E_C)_t Y).$$

c) By Lemma 5.2.1, a v_1-self map of Y induces on E_C-homology $(E_C)_*(\Sigma^2 Y) \to (E_C)_* Y$ multiplication by $v_1 \in (E_C)_*/2$ and it is also a map of $C_2 - (E_C)_*$-modules. It follows that the induced map on $\mathrm{H}^*(C_2, (E_C)_* Y)$ of a v_1-self map is given by multiplication by $v_1 \in \mathrm{H}^0(C_2, (E_C)_2/2)$ via the $\mathrm{H}^*(C_2, (E_C)_2/2)$-module structure of $\mathrm{H}^*(C_2, (E_C)_* Y)$. Part $c)$ now follows from the part $b)$ and relation $v_1 t = \eta u^{-1} e_0$ in $\mathrm{H}^*(C_2, (E_C)_*/2)$.

d) This follows from part c) of Lemma 6.2.1 and part b) of Lemma 6.3.1. □

Remark 6.3.2. We see that $\mathrm{H}^s(C_6, (E_C)_t Y) = 0$ if $t - s$ is odd, hence all even differentials are trivial.

Proposition 6.3.3. *a) There are no d_3- and d_5-differentials in the HFPSS. As a module over the subalgebra $\mathbb{W}[[u_1]][u^{\pm 8}, t]$ of the E_7-term of the HFPSS for $E_C^{hC_2}$, the E_7-term is isomorphic to*

$$\mathrm{E}_7 \cong \mathbb{F}_4[[u_1]][u^{\pm 8}, t]/(u_1 t)\{e_0, u^{-1}e_0, u^{-2}e_0, u^{-3}e_0, u^{-8}e_0, u^{-5}e_0, u^{-6}e_0, u^{-7}e_0\}.$$

b) The classes $e_0, u^{-1}e_0, u^{-2}e_0, u^{-3}e_0$ are permanent cycles.

c) The d_7-differentials are linear with respect to $\mathbb{W}[[u_1]][u^{\pm 8}, t]$ and are determined by

$$d_7(u^{-4}e_0) = t^7 e_0,\ d_7(u^{-5}e_0) = t^7 u^{-1} e_0,$$
$$d_7(u^{-6}e_0) = t^7 u^{-2} e_0,\ d_7(u^{-7}e_0) = t^7 u^{-3} e_0.$$

d) There are the following exotic multiplications by η:

$$\eta u^{-2} e_0 = t^5 e_0,\ \ \eta u^{-3} e_0 = t^5 u^{-1} e_0.$$

Proof. a) As a module over $\mathbb{W}[[u_1]][u^{\pm 4}, t]$, the E_3-term is generated by $u^k e_0$ for $k = -3, -2, -1, 0$. Since the d_3-differentials are C_3-equivariant, the d_3-differentials on $u^k e_0$ for $k = -3, -2, -1, 0$ are trivial in view of the C_3-action on the E_2-term. Since the d_3-differentials are $\mathbb{W}[[u_1]][u^{\pm 4}, t]$-linear, we conclude that the differentials d_3 are trivial. By the same reason, the differentials d_5 are trivial.

Part b). The map $\iota_* : \mathrm{H}^0(C_2, (E_C)_*V(0)) \to \mathrm{H}^0(C_2, (E_C)_*Y)$ sends e_0 and $u^{-1}e_0$ to the classes of the same name. By naturality, e_0 and $u^{-1}e_0$ are permanent cycles.

The cofibration (III.12) gives rise to an exact sequence

$$\pi_{*-1}(E_C^{hC_2} \wedge V(0) \xrightarrow{\times\eta} \pi_*(E_C^{hC_2} \wedge V(0)) \xrightarrow{\iota_*} \pi_{20}(E_C^{hC_2} \wedge Y).$$

This means that $\eta e_0 = \eta(u^{-1}e_0) = 0$ in $\pi_*(E_C^{hC_2} \wedge Y)$. We can form the Toda brackets $\langle \nu, \eta, u^k e_0 \rangle$ for $k = 0, -1$. By juggling,

$$\eta\langle \nu, \eta, u^k e_0 \rangle = \langle \eta, \nu, \eta \rangle u^k e_0 = \nu^2 u^k e_0.$$

Since the differentials d_3 and d_5 are trivial, the classes $t^6 u^k e_0$ for $k = 0, -1$ survive to the E_∞-term, and so $\nu^2 u^k e_0$ are nontrivial. It follows that $\langle \nu, \eta, u^k e_0 \rangle$ is non-trivial and is divisible by η. The structure of the E_2-term forces that $\langle \nu, \eta, u^k e_0 \rangle$ is represented by $tu^{k-2}e_0$. It means that $tu^{-2}e_0$ and $tu^{-3}e_0$, hence $u^{-2}e_0$ and $u^{-3}e_0$ are permanent cycles.

For part c), the fact that $t^7 = 0$ and that $u^k e_0$ for $k = 0, -1, -2, -3$ are permanent cycles force the indicated differentials d_7.

Part d) follows from the proof of part b). □

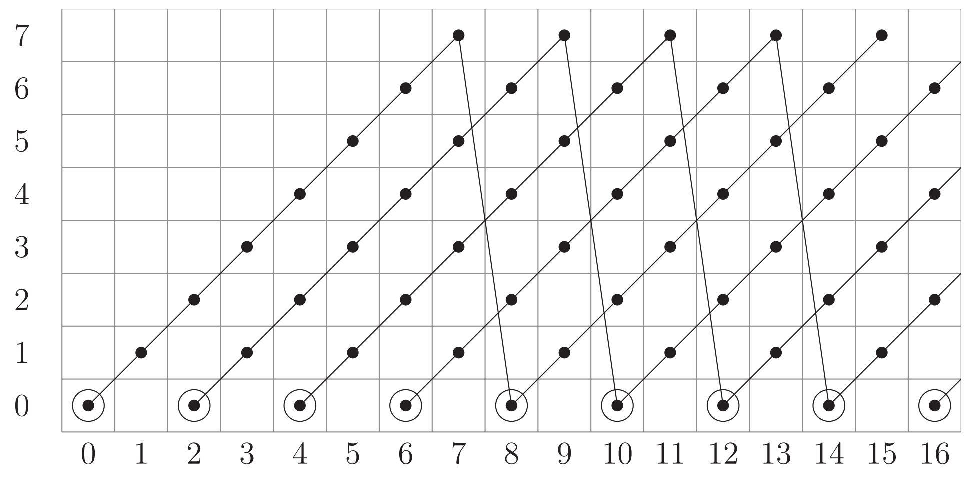

Figure III.7 – HFPSS for $E_C^{hC_2} \wedge Y$. A circled black dot represents a copy of $\mathbb{F}_4[[u_1]]$, a black dot a copy of $\mathbb{F}_4$.

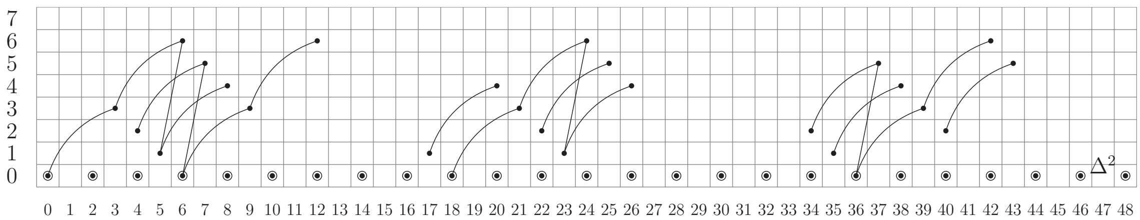

Figure III.8 – E_∞-term of the HFPSS for $E_C^{hC_6} \wedge Y$. A circled black dot represents a copy of $\mathbb{F}_4[[u_1^3]]$ and a black dot a copy of $\mathbb{F}_4$. A curved line represents multiplication by ν and a straight line multiplication by η. The E_∞-term is 48-periodic by multiplication by Δ^2.

7 The homotopy fixed point spectral sequence for $E_C^{hC_6} \wedge A_1$

Before starting to compute the HFPSS, we outline the main points in the arguments. After computing the E_2-term, we compare $E_C^{hC_2} \wedge A_1$ with $E_C^{hC_2} \wedge Y$ via the cofiber sequence

$$\Sigma^2 Y \xrightarrow{v_1} Y \xrightarrow{\iota} A_1 \xrightarrow{p} \Sigma^3 Y \tag{III.13}$$

and $E_C^{hC_2} \wedge A_1$ with $E_C^{hG_{24}} \wedge A_1$ via the restriction map $E_C^{hG_{24}} \wedge A_1 \to E_C^{hC_2} \wedge A_1$ to deduce some differentials. Next, a comparison of $E_C^{hC_6} \wedge A_1$ with a connective model $tmf_0(3) \wedge A_1$ allows us to deduce more differentials. Finally, by using the Gross-Hopkins dual of $E_C^{hC_6} \wedge A_1$, we deduce the rest of the differentials.

We begin with a calculation of the Gross-Hopkins dual of $E_C^{hC_6}$ and so of $E_C^{hC_6} \wedge A_1$. Note that in [**HLS18**], the authors computed the Gross-Hopkins dual of $E_n^{hC_2}$ for all heights n at the prime $p = 2$, and the Gross-Hopkins dual of $E_2^{hC_6}$ can be deduced from there. A calculation at height $n = 2$ is, however, elementary and instructive. We give here a complete proof with a quite different point of view.

7.1 The Gross-Hopkins dual of $E_C^{hC_6}$

We start with some generalities. We will denote by E the Morava E-theory associated to a height n formal group law, by $I()$ the Gross-Hopkins dual functor $I_n(-)$ and $\widehat{I}$ the Gross-Hopkins dual of the $K(n)$-local sphere. Let G be a finite subgroup of the Morava stabiliser group. Then the norm map $(E)_{hG} \to (E)^{hG}$ is a homotopy equivalence [1]. It follows that

$$I(E^{hG}) \cong F(E^{hG}, \widehat{I}) \cong F((E)_{hG}, \widehat{I}) \cong F(E, \widehat{I})^{hG} \cong (IE)^{hG}.$$

The spectrum IE is a G-E-module. Applying the G-homotopy fixed point functor to IE, we equip $(IE)^{hG}$ with a structure of module over E^{hG}. We will characterise $(IE)^{hG}$ using the HFPSS. First, we need to compute the Morava module of $(IE)^{hG}$. To this end, we will use the following lemma, see [**DH04**], Section 5 or [**BBGS18**], Proposition 2.1.

Lemma 7.1.1. *Let $W^\bullet$ be a cosimplicial spectrum. Suppose there exists an integer N and a finite spectrum V of type 0 so that, for all spectra Z, the Bousfield-Kan*

1. This fact is equivalent to that the Tate construction E^{tG} vanishes, which follows from the fact that the HFPSS for E^{hG} has a horizontal vanishing line.

spectral sequence

$$\pi^s \pi_t F(Z, V \wedge W^\bullet) \Rightarrow \pi_{t-s} F(Z, \text{holim}(V \wedge W^\bullet)) \tag{III.14}$$

has a horizontal vanishing line at $s = N$ at the E_∞-page. Then for any spectrum A and F there is a natural equivalence

$$L_F(A \wedge \text{holim} W^\bullet) \to \text{holim}(L_F(A \wedge W^\bullet)).$$

We have the following proposition whose proof is similar to that of Proposition 2.2 of [**BBGS18**].

Proposition 7.1.2. *There is an isomorphism of Morava modules*

$$(E)_*(F(E, \widehat{I})^{hG}) \cong \text{Map}_c(\mathbb{G}/G, E_{*-n}\langle det \rangle),$$

where $E_*\langle det \rangle = E_*(S^0\langle det \rangle)$.

Proof. We will first show that the natural map

$$E \wedge F(E, \widehat{I})^{hG} \to (E \wedge F(E, \widehat{I}))^{hG} \tag{III.15}$$

is a homotopy equivalence. We apply the previous lemma to $A = E$, $F = K(n)$ and

$$W^\bullet = F(G^{\times \bullet}, F(E, I))$$

so that holim $W^\bullet = F(E, I)^{hG}$. The E_2-term of the spectral sequence (III.14) is isomorphic to $\mathrm{H}^s(G, F(E, \widehat{I})^t(DV \wedge Z))$. By a construction of Jeff Smith, see [**Rav92**], there exists a finite spectrum Y of type 0 such that $E_*(V)$ is free as a $E_*[F]$-module for all cyclic subgroups F of $\mathbb{G}$. As a consequence, there exists an integer s_0 such that $\mathrm{H}^s(G, E_*V) = 0$ for $s \geq s_0$. By the Gross-Hopkins theorem,

$$F(E, \widehat{I}) \cong \Sigma^{-n} E \wedge S^0\langle det \rangle. \tag{III.16}$$

as $\mathbb{G}$-spectra in the homotopy category of spectra (where n is the chromatic height), see [**HG94**] or [**Str00**]. It follows that

$$F(E, \widehat{I})^t(DV \wedge Z) \cong E^{t-n}(DV \wedge Z)\langle det \rangle \cong E_*(V) \otimes_{E_*} E^{t-n}(Z)\langle det \rangle$$

as G-modules, where the last equivalence is because E_*V is free over E_*. It then follows that $\mathrm{H}^s(G, F(E, \widehat{I})^t(DV \wedge Z)) = 0$ for $s \geq s_0$. Therefore, the map (III.15) is an homotopy equivalence.

Now we calculate the homotopy groups of $(E \wedge F(E, \widehat{I}))^{hG}$ using the homotopy fixed point spectral sequence. This spectral sequence has E_2-term naturally isomorphic to $\mathrm{H}^s(G, (E)_*(F(E, \widehat{I}))$. By the equivalence (III.16), the latter is isomorphic to $\mathrm{H}^s(G, (E)_{*+n}(E\langle det\rangle)$. So the spectral sequence collapses at the E_2-term and we obtain an isomorphism of Morava modules

$$(E)_*(F(E, \widehat{I})^{hG}) \cong \mathrm{Map}_c(\mathbb{G}/G, \pi_*(\Sigma^{-n}E\langle det\rangle)).$$

□

In the statement of the following proposition, we use that fact that $\pi_* IE$ is of free of rank 1 over E_*, which is a direct consequence of the Gross-Hopkins formula III.16.

Proposition 7.1.3. *Suppose there exists $x \in \pi_k(IE)^G \subset \pi_k IE$ which generates $\pi_* IE$ as an E_*-module and which is a permanent cycle in the HFPSS for $(IE)^{hG}$. Then there is a homotopy equivalence of E^{hG}-modules $\Sigma^k E^{hG} \to (IE)^{hG}$.*

Proof. The element $x \in \pi_k(IE)$ extends to a map of E-modules $f_1 : \Sigma^k E \to IE$. By the assumption, x detects an element of $\pi_k(IE^{hG})$, which extends to a map of E^{hG}-modules $f_2 : \Sigma^k E^{hG} \to (IE)^{hG}$. In this case, there is a commutative diagram

$$\begin{array}{ccc} \Sigma^k E^{hG} & \xrightarrow{f_2} & (IE)^{hG} \\ \downarrow{\scriptstyle \Sigma^k Res} & & \downarrow{\scriptstyle Res} \\ \Sigma^k E & \xrightarrow{f_1} & IE. \end{array}$$

The induced map in homotopy of f_1 is given by multiplication with x, i.e.,

$$(f_1)_* : \pi_{*-k}E \to \pi_*(IE), a \mapsto ax.$$

Then we claim that the induced map in Morava modules of f_1 is given by [2]

$$E_*(f_1) : \mathrm{Map}(\mathbb{G}, E_{*-k}) \to \mathrm{Map}(\mathbb{G}, (IE)_*), (e \mapsto (g \mapsto e(g)g(x))).$$

The restrictions induce injections $E_*(Res) : \mathrm{Map}(\mathbb{G}/G, M) \to \mathrm{Map}(\mathbb{G}, M)$ induced by the projection $\mathbb{G} \to \mathbb{G}/G$ with $M = E_*$ or $(IE)_*$. It follows that $E_*(f_2)$ is given by

$$E_*(f_2) : \mathrm{Map}(\mathbb{G}/G, E_{*-k}) \to \mathrm{Map}(\mathbb{G}/G, (IE)_*), (e \mapsto (gG \mapsto e(gG)g(x))).$$

Since x generates $\pi_*(IE)$ as a E_*-module, $(f_1)_*$ is an isomorphism. It follows immediately that $E_*(f_2)$ is an isomorphism. Therefore, f_2 is an homotopy equivalence. □

2. We postpone the proof of this claim to Lemma 11.2.2, in which we only need to identify IE with $\Sigma^{-n}E$ by the Gross-Hopkins formula.

Let us turn back to the case of $n = p = 2$.

Theorem 7.1.4. *1) There is an integer* $k \equiv -2$ *mod* 12 *which is uniquely determined modulo* 48 *such that* $\Sigma^k E_C^{hC_6} \cong IE_C^{hC_6}$ *is a homotopy equivalence.*

2) The integer k *in part 1) is equal to* 22 *modulo* 48.

Proof. The HFPSS for $(IE_C)^{hC_6}$ is one of modules over that for $E_C^{hC_6}$. The E_2-term of the HFPSS for $(IE)^{hC_6}$ is isomorphic to $\mathrm{E}_2^{s,t} = \mathrm{H}^s(C_6, (E_C)_{t+2})$, with an obvious structure of module over $\mathrm{H}^*(C_6, (E_C)_*)$. In particular, the former is free of rank one over the latter. Let us denote by $\iota \in \mathrm{E}_2^{0,-2} = \mathrm{H}^0(C_6, (E_C)_0)$ a generator of the E_2-term of the HFPSS for $(IE_C)^{hC_6}$. We will show that there is an integer l such that $u^{-6l}\iota$ is a permanent cycle.

In the HFPSS for $E_C^{hC_6}$ there is a d_3-differential, by Proposition 6.1.3,:

$$d_3(u^{-6}) = u^{-4}d_3(u^{-2}) = u^{-4}u_1t^3.$$

A d_3-differential on ι is of the form

$$d_3(\iota) = t^3u^2u_1p\iota$$

with $p \in \mathbb{F}_4[[u_1^3]]$. Then, by the Leibniz rule,

$$d_3(u^{-6}\iota) = u^{-4}u_1t^3\iota + u^{-4}t^3u_1p\iota = u^{-4}u_1t^3(1+p)\iota.$$

It follows that

$$0 = d_3(u^{-4}u_1t^3(1+p)\iota) = u^{-4}u_1(1+p)t^3d_3(\iota) = u^{-2}t^3u_1^2(1+p)p\iota.$$

Therefore, $(1+p)p = 0$, which implies that either $p = 0$ or $1 + p = 0$. It means that either ι or $u^{-6}\iota$ is a d_3-cycle. As a consequence, the E_4-term of the HFPSS for $(IE)^{hC_6}$ is free of rank one over that for $E_C^{hC_6}$, generated by $u^{-6m}\iota$ where $m = 0$ or $m = 1$. By sparseness, there are no non-trivial d_5-differentials.

Next, in the HFPSS for $E_C^{hC_6}$, there is a d_7-differential

$$d_7(u^{-12}) = u^{-8}t^7.$$

If $u^{-6m}\iota$ is not a d_7-cycle then, by a similar argument for proving that $u^{-6m}\iota$ is a d_3-cycle, we see that $u^{-12-6m}\iota$ is a d_7-cycle. The spectral sequence collapses at the E_7-term. We conclude that there is an integer l such that $u^{-6l}\iota$ is a permanent cycle detecting a homotopy equivalence $\Sigma^{12l-2}E_C^{hC_6} \cong IE_C^{hC_6}$.

2) Choose A_1 be one of the Spanier-Whitehead self-dual versions of A_1, i.e., $DA_1 \simeq \Sigma^{-6}A_1$. We have that

$$I(E_C^{hC_6} \wedge A_1) \cong I(E_C^{hC_6}) \wedge DA_1 \cong \Sigma^{12l-2}E_C^{hC_6} \wedge \Sigma^{-6}A_1 \cong \Sigma^{12l-8}E_C^{hC_6} \wedge A_1.$$

As a consequence,

$$\pi_k(E_C^{hC_6} \wedge A_1) \cong \pi_{8-12l-k}(E_C^{hC_6} \wedge A_1). \tag{III.17}$$

Using the long exact sequence associated to the cofibration (III.13)

$$\pi_{k-2}(E_C^{hC_6} \wedge A_1) \xrightarrow{v_1} \pi_k(E_C^{hC_6} \wedge Y)) \to \pi_k(E_C^{hC_6} \wedge A_1) \to$$

$$\to \pi_{k-3}(E_C^{hC_6} \wedge Y) \xrightarrow{v_1} \pi_{k-1}(E_C^{hC_6} \wedge A_1)$$

and Proposition 6.3.3 (see Figure III.8), we check that

$$\pi_2(E_C^{hC_6} \wedge A_1) \cong 0,$$

$$\pi_6(E_C^{hC_6} \wedge A_1) \cong \mathbb{F}_4 \oplus \mathbb{F}_4 \text{ ou } \mathbb{W}/4,$$
$$\pi_{18}(E_C^{hC_6} \wedge A_1) = \mathbb{F}_4,$$
$$\pi_{42}(E_C^{hC_6} \wedge A_1) = \mathbb{F}_4.$$

This information together with Equation (III.17) rules out the possibility for $l = 0, 1, 3 \bmod 4$. Therefore, $l = 2 \bmod 4$ and

$$\Sigma^{22}E_C^{hC_6} \cong IE_C^{hC_6}.$$

□

7.2 The homotopy fixed point spectral sequence for $E_C^{hC_6} \wedge A_1$

Lemma 7.2.1. *We have*

a) There are classes

$$e_0 \in \mathrm{H}^0(C_2, (E_C)_0A_1),\ e_2 \in \mathrm{H}^0(C_2, (E_C)_2A_1)$$

such that there is an isomorphism of $\mathbb{F}_4[u^{\pm 1}, t]$*-modules*

$$\mathrm{H}^*(C_2, (E_C)_*A_1) \cong \mathbb{F}_4[u^{\pm 1}, t]\{e_0, e_2\}.$$

b) The action of C_3 *on* $\mathrm{H}^*(C_2, (E_C)_*(A_1))$ *is induced by*

$$\omega(e_0) = e_0, \quad \omega(e_2) = e_2, \quad \omega(u^{-1}) = \zeta^2u^{-1}, \quad \omega(t) = \zeta^2 t.$$

Proof. We have that

$$(E_C)_*A_1 \cong \mathbb{F}_4[u^{\pm 1}]\{e_0, e_2\}$$

on which C_2 acts trivially (see Proposition 5.2.2). The cohomology group calculation of $\mathrm{H}^*(C_2, (E_C)_*A_1)$ is now elementary. □

For the study of the differentials, we next describe the induced maps of spectral sequences associated to the cofiber sequence

$$\Sigma^2 Y \xrightarrow{v_1} Y \xrightarrow{\iota} A_1. \tag{III.18}$$

By Lemma 5.2.1, v_1 induces an injective map in E_C-homology. Thus, (III.18) gives rise to an exact sequence of $(E_C)_*/2[C_2]$-modules

$$0 \to (E_C)_*\Sigma^2 Y \to (E_C)_* Y \to (E_C)_* A_1 \to 0.$$

The associated long exact sequence reads

$$\mathrm{H}^s(C_2, (E_C)_*\Sigma^2 Y) \xrightarrow{v_1} \mathrm{H}^s(C_2, (E_C)_* Y) \xrightarrow{\iota_*} \mathrm{H}^s(C_2, (E_C)_* A_1)$$

$$\xrightarrow{\delta} \mathrm{H}^{s+1}(C_2, (E_C)_*\Sigma^2 Y).$$

By the description of the action of v_1 on $\mathrm{H}^*(C_2, (E_C)_* Y)$ in Lemma 6.3.1, we see that ι_* is a map of $\mathbb{F}_4[u^{\pm 1}, t]$-modules sending e_0 to e_0 and δ sending e_2 to ute_0.

Lemma 7.2.2. *In the HFPSS for $E_C^{hC_2} \wedge A_1$,*

- *a) The differentials d_3 are trivial.*
- *b) The classes $\{u^k e_0 | k \in \mathbb{Z}\}$ are d_5-cycles.*
- *c) The classes $e_0, u^{-1}e_0, u^{-2}e_0, u^{-3}e_0$ are permanent cycles.*
- *d) The class $u^{-1}e_2$ is a permanent cycle.*

Proof. Part a) The differentials d_3 are eliminated by C_3-equivariance as in the proof of part a) of Proposition 6.3.3.

Part b) follows from the naturality of the induced map of spectral sequences by ι of (III.18).

Part c) follows by the same reason as in Part b).

For Part d), we see that the connecting homomorphism δ sends $u^{-1}e_2$ to $te_0 \in H^1(C_2, (E_C)_2 Y)$. The class te_0 detects an element of $\pi_1(E_C^{hC_2} \wedge Y) \cong \pi_4(\Sigma^3 E_C^{hC_2} \wedge Y)$ that is annihilated by v_1, because v_1-multiplication is trivial in the E_2-term in filtration $s > 0$ and the only element in higher filtration in $\pi_3(Y)$ is the C_3-invariant class $t^3 e_0$, while $v_1 te_0$ can only be C_3-invariant if it is trivial. Thus, $u^{-1}e_2$ must be a permanent cycle by the Geometric Boundary Lemma (Proposition A.10 of [**DH04**]) □

Now neither the naturality nor the C_3-equivariance can rule out a non-trivial differential d_5 on the classes $u^k e_2$ for $k \equiv 0, 1, 2 \bmod (4)$. In fact, we are going to prove that the latter support a non-trivial differential d_5. We state the final result here

Theorem 7.2.3. *In the HFPSS for $E_C^{hC_2} \wedge A_1$, we have*

a) The differentials d_5 are $\mathbb{W}[u^{\pm 4}, t]$-linear and are determined by the following

$$d_5(u^k e_2) = t^5 u^{k+2} e_0 \text{ for } k \equiv 0, 1, 2 \bmod (4).$$

b) The differentials d_7 are $\mathbb{W}[u^{\pm 8}, t]$-linear and are determined by the following

$$d_7(u^{-7+8k} e_0) = t^7 u^{-3+8k} e_0, \quad d_7(u^{-5+8k} e_2) = t^7 u^{-1+8k} e_2.$$

c) The spectral sequence collapses at the E_8-term and there is an isomorphism of $\mathbb{F}_4[u^{\pm 8}, t]$-modules

$$\mathrm{E}_8^{*,*} \cong \mathbb{F}_4[u^{\pm 8}, t]/t^5\{e_0, u^{-1}e_0, u^{-2}e_0, u^{-4}e_0, u^{-5}e_0, u^{-6}e_0\}$$
$$\oplus \mathbb{F}_4[u^{\pm 8}, t]/(t^7)\{u^{-3}e_0, u^{-1}e_2\}.$$

By taking C_3-fixed points, we obtain the E_∞-term of the HFPSS for $E_C^{hC_6} \wedge A_1$.

Theorem 7.2.4. *As a module over $\mathbb{F}_4[\Delta^{\pm 2}, x_{17}]$, the E_∞-term of the HFPSS for $E_C^{hC_6} \wedge A_1$ is isomorphic to*

$$\mathrm{E}_\infty^{*,*} \cong \mathbb{F}_4[\Delta^{\pm 2}, x_{17}]/(x_{17}^5)\{e_0, u^{-6}e_0, u^{-9}e_0, u^{-12}e_0, u^{-18}e_0, u^{-21}e_0\}.$$
$$\oplus \mathbb{F}_4[\Delta^{\pm 2}, x_{17}]/(x_{17}^7)\{u^{-3}e_0, u^{-9}e_2\}.$$

We prove this theorem in Proposition 7.2.6, 7.2.7, 7.2.15 and 7.2.16.

Remark 7.2.5. We are going to settle the d_7-differentials before all. There is, however, no harm at all. In view of Proposition 6.1.5, we know that a permanent t-free tower is truncated by one and only one t-free tower. So, if a t-free tower involves in some differential, then none of the classes of that tower involves in any other differentials.

Proposition 7.2.6. *There are the following differentials d_7*

$$d_7(u^{-7+8k} e_0) = t^7 u^{-3+8k} e_0, \quad d_7(u^{-5+8k} e_2) = t^7 u^{-1+8k} e_2.$$

Proof. By part d) of Lemma 7.2.2, the class $u^{-1}e_2$ is a permanent cycle. Moreover, the class $t^5 u^{-1} e_2$ cannot be a target of any differential, by C_3-equivariance. This forces that $t^7 u^{-1} e_2$ is hit by the differential d_7:

$$d_7(u^{-5} e_2) = t^7 u^{-1} e_2.$$

This differential implies in particular that $u^{-5}e_2$ is a d_5-cycle. Furthermore, $u^{-6}e_0$ is also a d_5-cycle by part b) of Lemma 7.2.2. Thus, the group $\mathrm{H}^0(C_2, (E_C)_{12}A_1)$ consists only of d_5-cycles, and so there is no differential d_5 hitting the class $t^5u^{-3}e_0$, and so by naturality $t^7u^{-3}e_0$ is hit by the differential d_7:

$$d_7(u^{-7}e_0) = t^7u^{-3}e_0.$$

Finally, we deduce the other differentials d_7 by $\mathbb{W}[u^{\pm 8}, t]$-linearity. □

Proposition 7.2.7. *There are* d_5 *differentials*

$$d_5(u^{-3+4k}e_2) = t^5u^{-1+4k}e_0.$$

Proof. Let $u^{-3}e_0$ denote (by an abuse of language) an element of $\pi_6(E_C^{hC_2} \wedge A_1)$ detected by $u^{-3}e_0 \in \mathrm{H}^0(C_2, (E_C)_6A_1)$. The map

$$\iota_* : \pi_6(E_C^{hC_2}) \wedge Y \to \pi_6(E_C^{hC_2} \wedge A_1)$$

sends $u^{-3}e_0$ to $u^{-3}e_0$ up to an element detected in filtration at least 2, i.e.,

$$\iota_*(u^{-3}e_0) = u^{-3}e_0 + t^2a$$

for some $a \in \pi_4((E_C)^{hC_6} \wedge A_1)$. By Part e) of Proposition 6.3.3, $\eta(u^{-3}e_0) = t^5(u^{-1}e_0)$. It follows that

$$\iota_*(t^5u^{-1}e_0) = \iota_*(\eta u^{-3}e_0) = \eta u^{-3}e_0$$

because $\eta t^2 = 0 \in \pi_*(E_C^{hC_2})$. Consider the restriction map $Res : E_C^{hG_{24}} \wedge A_1 \to E_C^{hC_2} \wedge A_1$. We see that the induced map in cohomology $\mathrm{H}^0(G_{24}, E_6A_1) \to \mathrm{H}^0(C_2, (E_C)_6A_1)$ sends e_6 to $u^{-3}e_0$. Thus, in homotopy

$$Res_*(e_6) = u^{-3}e_0 + t^2b$$

for some $b \in \pi_4(E_C^{hC_2} \wedge A_1)$. In $\pi_*(E_C^{hG_{24}} \wedge A_1)$, we have that $\eta e_6 = 0$. It follows that $\eta u^{-3}e_0 = 0$, because $\eta t^2 = 0$ in $\pi_*(E_C^{hC_2})$. Together with

$$\iota_*(t^5u^{-1}e_0) = \eta u^{-3}e_0$$

we conclude that $\iota_*(t^5u^{-1}e_0) = 0$. It forces that, in the HFPSS for $E_C^{hC_2} \wedge A_1$, there is a d_5-differential hitting $t^5u^{-1}e_0$. By naturality, the source of the latter cannot be $u^{-4}e_0$. As a consequence,

$$d_5(u^{-3}e_2) = t^5u^{-1}e_0,$$

and so, by the Leibniz rule and the fact that u^{-4} is a d_5-cycle in the HFPSS for $E_C^{hC_2}$, we obtain that

$$d_5(u^{-3+8k}e_2) = t^5u^{-1+8k}e_0.$$

□

To determine the rest of the differentials d_5, we need some extra information about the homotopy groups of $E_C^{hC_6} \wedge A_1$. For this we will use the connective model $tmf_0(3)$ of $TMF_0(3)$ introduced Section 1.4. We recall that there exists a finite spectrum X such that $tmf_0(3) \simeq tmf \wedge X$. As a $\mathcal{A}(2)$-module, $\mathrm{H}^*X \cong \mathbb{F}_2 \oplus T$ where the $\mathcal{A}(2)$-module structure of T is given in Section 1.4.

Lemma 7.2.8. *a) The ring homomorphism*

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2) \to \mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(tmf_0(3)))$$

induced by the unit of $tmf_0(3)$ *sends* $g \in \mathrm{Ext}^{4,24}_{\mathcal{A}_*}(\mathbb{F}_2)$ *to a nontrivial class that we also call* g.

b) The image in $\pi_{20}(tmf_0(3))$ *of the element* $\overline{\kappa} \in \pi_{20}(S^0)$ *via the unit* $S^0 \to tmf_0(3)$*, denoted also by* $\overline{\kappa}$*, satisfies that*

$$\overline{\kappa}^2 = 0 \in \pi_{40}(tmf_0(3)).$$

Proof. a) The unit $S^0 \to tmf_0(3)$ of $tmf_0(3)$ factors through the unit of tmf. Therefore the map

$$\mathrm{Ext}^{4,24}_{\mathcal{A}_*}(\mathbb{F}_2) \to \mathrm{Ext}^{4,24}_{\mathcal{A}_*}(\mathrm{H}_*(tmf_0(3)))$$

factors through

$$\mathrm{Ext}^{4,24}_{\mathcal{A}_*}(\mathbb{F}_2) \to \mathrm{Ext}^{4,24}_{\mathcal{A}_*}(\mathrm{H}_*(tmf)) \to \mathrm{Ext}^{4,24}_{\mathcal{A}_*}(\mathrm{H}_*(tmf_0(3))).$$

By the change-of-rings theorem, the second map is identified with

$$\mathrm{Ext}^{4,24}_{\mathcal{A}(2)_*}(\mathbb{F}_2) \to \mathrm{Ext}^{4,24}_{\mathcal{A}(2)_*}(\mathrm{H}_*(X)).$$

By the $\mathcal{A}(2)$-module structure of $\mathrm{H}^*(X)$, the latter is a split injection. As a consequence, the class $g \in \mathrm{Ext}^{4,24}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ is sent non-trivially to $\mathrm{Ext}^{4,24}_{\mathcal{A}(2)_*}(\mathrm{H}_*(X))$. Because $g \in \mathrm{Ext}^{4,24}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ lifts to $\mathrm{Ext}^{4,24}_{\mathcal{A}_*}(\mathbb{F}_2)$, we are done.

b) By the description of the homotopy groups of $tmf_0(3)$ in Theorem 2.12 of [**DM10**], there is a unique non-trivial element of finite order in $\pi_{20}(tmf_0(3))$. Furthermore, this element is nilpotent of exponent 2. Thus because $\overline{\kappa} \in \pi_{20}S^0$ has finite order, if its image in $\pi_{20}(tmf_0(3))$ is non-trivial, then it must be of exponent 2. □

Lemma 7.2.9. *There is a homotopy equivalence*

$$[(\Delta^2)^{-1}](tmf_0(3) \wedge A_1) \cong (E_C^{hC_6})^{h\mathrm{Gal}} \wedge A_1.$$

Proof.

$$\begin{aligned}[(\Delta^2)^{-1}](tmf_0(3)\wedge A_1) &\cong TMF_0(3)\wedge A_1 \text{ (Equation } I.16)\\ &\cong L_2(TMF_0(3))\wedge A_1 \text{ } (TMF_0(3) \text{ is } E(2)-\text{local})\\ &\cong L_{K(2)}TMF_0(3)\wedge A_1 (A_1 \text{ is of type } 2)\\ &\cong (E_C^{hC_6})^{h\mathrm{Gal}}\wedge A_1 \text{ (Equation } I.12)\end{aligned}$$

□

Let us denote by f the following composite

$$tmf_0(3)\wedge A_1 \to (\Delta^2)^{-1}tmf_0(3)\wedge A_1 \xrightarrow{\cong} (E_C^{hC_6})^{h\mathrm{Gal}}\wedge A_1 \to E_C^{hC_6}\wedge A_1.$$

By Proposition 6.3.3, we inspect that the group $\pi_5(E_C^{hC_6}\wedge Y)\cong \mathbb{F}_4$ detected by $tu^{-2}e_0 \in \mathrm{E}_\infty^{1,6}$. The induced map at the E_2-term of the HFPSS of $\iota : Y \to A_1$ sends $tu^{-2}e_0$ to the class of the same name which must be a non-trivial permanent cycle, as it lives in filtration 1. So the induced map in homotopy $\pi_5(E_C^{hC_6}\wedge Y)\cong \mathbb{F}_4 \to \pi_5(E_C^{hC_6}\wedge A_1)$ is injective. We will write e_5 for the nontrivial element of $\pi_5((E_C^{hC_6})^{h\mathrm{Gal}}\wedge Y)\cong \mathbb{F}_2$ as well as its image via $\pi_5((E_C^{hC_6})^{h\mathrm{Gal}}\wedge Y)\to \pi_5(E_C^{hC_6}\wedge Y)$ and via the composite $\pi_5((E_C^{hC_6})^{h\mathrm{Gal}}\wedge Y)\to \pi_5(E_C^{hC_6}\wedge Y)\to \pi_5(E_C^{hC_6}\wedge A_1)$. In what follows, we will prove that $\overline{\kappa}e_5 = 0 \in \pi_{25}(E_C^{hC_6}\wedge A_1)$.

Some elements of $\pi_*(tmf_0(3)\wedge A_1)$. Consider the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(\mathrm{H}_*(X\wedge A_1)) \Longrightarrow \pi_{t-s}(tmf_0(3)\wedge A_1).$$

The E_2-term splits as

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(\mathrm{H}_*(X\wedge A_1)) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(\mathrm{H}_*(A_1)\oplus \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(T\otimes \mathrm{H}_*(A_1)).$$

Then we see immediately that the classes

$$e[0,0]\in \mathrm{Ext}^{0,0}_{\mathcal{A}(2)}(\mathrm{H}_*(A_1)),\ \ e[1,5]\in \mathrm{Ext}^{1,6}_{\mathcal{A}(2)}(\mathrm{H}_*(A_1))$$

are sent to non-trivial permanent cycles in the ASS for $tmf_0(3)\wedge A_1$. By an abuse of language, we also write $e[0,0]$ and $e[1,5]$ for the elements of $\pi_*(tmf_0(3)\wedge A_1)$ to which $e[0,0]$ and $e[1,5]$ converge, respectively (by sparseness of the E_2-term, there is no ambiguity to define $e[0,0]$ and $e[1,5]$).

Lemma 7.2.10. *In $\pi_*(tmf\wedge A_1)$, we have that*

$$e[1,5] = \langle \nu, \eta, e[0,0]\rangle.$$

Proof. In $\pi_*(tmf \wedge A_1)$, we have that $\eta e[0,0] = 0$ hence, by juggling

$$\nu^2 e[0,0] = \langle \eta, \nu, \eta \rangle e[0,0] = \eta \langle \nu, \eta, e[0,0] \rangle.$$

Since $\nu^2 e[0,0] \neq 0$, we have that $\langle \nu, \eta, e[0,0] \rangle = e[1,5]$. As a consequence, in $\pi_*(tmf_0(3) \wedge A_1)$, we have that

$$e[1,5] = \iota_*(e[1,5]) \in \langle \nu, \eta, \iota_*(e[0,0]) \rangle = \langle \nu, \eta, e[0,0] \rangle.$$

Moreover, the ambiguity of $\langle \nu, \eta, e[0,0] \rangle$ is equal to $\pi_5(S^0)e[0,0] + \nu\pi_2(tmf_0(3) \wedge A_1)$, which is equal to zero [3]. We conclude that, in $\pi_*(tmf_0(3) \wedge A_1)$, the Toda bracket $\langle \nu, \eta, e[0,0] \rangle$ is equal to $e[1,5]$. □

To prove Lemma 7.2.11 and Proposition 7.2.14, we need some knowledge on the structure of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(X \wedge A_1))$, the E_2-term of ASS for $tmf_0(3) \wedge A_1$. This is purely algebraic and, in principle, can be shown by hand. However, a by-hand computation should not be so easy because the $\mathcal{A}(2)$-module structure of $\mathrm{H}^*(X)$ is not simple. To overcome this difficulty, we use the Bruner's software, which is built to compute the Ext-group over subalgebras of the Steenrod algebra. This program takes as input the $\mathcal{A}(2)$ respectively $\mathcal{A}$-module structure of a module M and outputs the groups $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(M)$ respectively $\mathrm{Ext}^{s,t}_{\mathcal{A}}(M)$, up to a required stem. It can also give the action of classes of $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(\mathbb{F}_2)$ respectively $\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_2)$ on classes of $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(M)$ respectively $\mathrm{Ext}^{s,t}_{\mathcal{A}}(M)$. We refer to the Appendix of [**BEM17**] for an explanation on the use of this software.

Lemma 7.2.11. *The image of* $e[0,0] \in \pi_0(tmf_0(3) \wedge A_1)$ *by* f_* *is nontrivial in* $\pi_0(E_C^{hC_6} \wedge A_1)$. *The latter is detected by* $e_0 \in \mathrm{H}^0(C_6, (E_C)_0)$.

Proof. We need to prove that $v_2^{8l} e[0,0]$ survives to the E_∞-term of the Adams spectral sequence for $tmf_0(3) \wedge A_1$ for all non-negative integer l; then by Lemma 7.2.9, $f_*(e[0,0])$ must be non-trivial. We prove in fact that there is no potential source for any differential hitting $v_2^{8l} e[0,0]$. Suppose that x is a class in the E_2-term of the ASS that can support a differential hitting $v_2^{8l} e[0,0]$.

The Davis-Mahowald spectral sequence for $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)}(\mathrm{H}_*(X \wedge A_1))$ has the form

$$\mathbb{F}_2[y_1, y_2, y_3] \otimes \mathrm{H}_*(X) \Longrightarrow \mathrm{Ext}^{*,*}_{\mathcal{A}(2)}(\mathrm{H}_*(X \wedge A_1)).$$

Then x can be represented by a sum $\Sigma\, y_1^i y_2^j y_3^k a$ where $a \in \mathrm{H}_*(X)$. We see that i, j, k and the degree $|a|$ of a must satisfy that

$$i + j + k \leq 8l - 2, \quad 3i + 5j + 6k + |a| = 48l + 1$$

3. This is because $\pi_5(S^0) = 0$ and $\pi_2(tmf_0(3) \wedge A_1) = 0$. The latter is because $\mathrm{E}_2^{s,s+2} = 0$, which can be seen from the splitting of the E_2-term of the ASS

The only solutions to these equations are $(i,j,k,a) = (0,1,8l-3,x_{14}), (0,0,8l-2,x_{13})$ (here $x_{13}, x_{14} \in \mathrm{H}_*(X)$ are duals of the corresponding cohomology classes as introduced in the preliminaries). It means that x is of the form $v_2^{8(l-1)}y$ where $y \in \mathrm{Ext}_{\mathcal{A}}^{6,55}(\mathrm{H}_*(tmf_0(3) \wedge A_1))$. However, a calculation by the Bruner's software shows that the group $\mathrm{Ext}_{\mathcal{A}}^{6,55}(\mathrm{H}_*(tmf_0(3) \wedge A_1))$ is trivial.

For the second statement, by using the long exact sequence associated to the cofiber sequence (III.18), we see that $\pi_0(E_C^{hC_6} \wedge A_1)$ is isomorphic to $\mathbb{F}_4$ detected by $e_0 \in \mathrm{H}^0(C_6, (E_C)_*)$. Since $f_*(e[0,0])$ is non-trivial, it must be detected by e_0. □

Lemma 7.2.12. *a) The group $\pi_5(E_C^{hC_6} \wedge A_1)$ is isomorphic to $\mathbb{F}_4$ and is detected by the class $tu^{-2}e_0$ in the HFPSS.*

b) None of the non-trivial elements of $\pi_5(E_C^{hC_6} \wedge A_1)$ is divisible by ν.

Proof. a) The cofiber sequence (III.18) induces the following exact sequence

$$\pi_3(E_C^{hC_6} \wedge Y) \xrightarrow{v_1} \pi_5(E_C^{hC_6} \wedge Y) \to \pi_5(E_C^{hC_6} \wedge A_1) \to \pi_2(E_C^{hC_6} \wedge Y).$$

It is straightforward from Figure (III.8) and Part c) of Lemma 6.3.1 to show that multiplication by v_1 is injective on $\pi_2(E_C^{hC_6} \wedge Y)$ and that $\pi_5(E_C^{hC_6} \wedge Y) \cong \mathbb{F}_4\{tu^{-2}e_0\}$ is not in the image of v_1. Then it follows from the above exact sequence that

$$\pi_5(E_C^{hC_6} \wedge A_1) \cong \pi_5(E_C^{hC_6} \wedge Y) \cong \mathbb{F}_4.$$

Furthermore, the induced map in the E_2-term of $\iota : E_C^{hC_6} \wedge Y \to E_C^{hC_6} \wedge A_1$ sends the class $tu^{-2}e_0$ to a class of the same name. Therefore, $tu^{-2}e_0$ is a permanent cycle in the HFPSS for $E_C^{hC_6} \wedge A_1$ and survives to the E_∞-term, since it is in too low a filtration to be hit by a differential.

b) This is because $tu^{-2}e_0$ is in filtration 1 and ν is detected in filtration 3 in the HFPSS for $E_C^{hC_6}$. □

Lemma 7.2.13. *The image of $e_5 \in \pi_5(tmf_0(3) \wedge A_1)$ by f_* is equal to e_5 (up to a scalar of $\mathbb{F}_4^\times$).*

Proof. The element $e[1,5] \in \pi_5(tmf_0(3) \wedge A_1)$ is equal to the Toda bracket $\langle \nu, \eta, e[0,0] \rangle$, as explained before Lemma 7.2.11. It follows that

$$f_*(e[1,5]) \in \langle \nu, \eta, f_*(e[0,0]) \rangle = \langle \nu, \eta, e_0 \rangle.$$

It suffices to show that $\langle \nu, \eta, e_0 \rangle \subset \pi_5(E_C^{hC_6} \wedge A_1)$ contains only e_5.

In $\pi_*(E_C^{hC_6} \wedge Y)$, we have that $\langle \nu, \eta, e_0 \rangle$ does not contain zero and is detected by $tu^{-2}e_0$. By the proof Lemma 7.2.12, $\iota : \pi_5(E_C^{hC_6} \wedge Y) \to \pi_5(E_C^{hC_6} \wedge A_1)$

sends $tu^{-2}e_0$ to e_5. Thus $\langle \nu, \eta, e_0 \rangle$ contains the unique nontrivial element (up to a non-zero scalar of $\mathbb{F}_4$) of $\pi_5(E_C^{hC_6} \wedge A_1)$.

The ambiguity of the Toda bracket $\langle \nu, \eta, e_0 \rangle$ lives in $\pi_5(S^0)e_0 + \nu\pi_2(E_C^{hC_6} \wedge A_1)$ which is equal to zero because none of the nontrivial elements of $\pi_5(E_C^{hC_6} \wedge A_1)$ is divisible by ν. We conclude that $f_*(e[1,5])$ must be equal to e_5. □

Proposition 7.2.14. *The element* $\overline{\kappa}e[1,5] = 0 \in \pi_{25}(tmf_0(3) \wedge A_1)$.

Proof. Since $\overline{\kappa}^2 e[1,5] = 0 \in \pi_{45}(tmf_0(3) \wedge A_1)$, the class $g^2 e_5 \in \mathrm{Ext}^{9,54}_{A(2)}$ must be hit by a differential. The Ext-group calculation by Bruner's software shows by sparseness that the only possibility is given by a differential d_2 on the group $\mathrm{Ext}^{7,53}_{A(2)}(\mathrm{H}_*(X \wedge A_1))$. The latter is generated by two classes 7_6 and 7_7.[4] By a calculation of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(X \wedge A_1))$ by the Bruner's software, the class 7_6 is a ν-multiple, whereas 7_7 is not. The Bruner's software also tells us that $\mathrm{Ext}^{9,54}_{A(2)}(\mathrm{H}^*(X \wedge A_1)) \cong \mathbb{F}_2\{g^2 e[1,5]\}$ and that $g^2 e[1,5]$ is not divisible by ν. We must have then that

$$d_2(7_6) = 0, \quad d_2(7_7) = g^2 e[1,5].$$

We can also check by Bruner's software that $7_6 + 7_7$ is divisible by g, i.e., there is a class $a \in \mathrm{Ext}^{3,29}_{A(2)}(\mathrm{H}^*(X \wedge A_1))$ such that $ga = 7_6 + 7_7$. By the Leibniz rule, we obtain that $d_2(a) = ge[1,5]$. Since the groups $\mathrm{Ext}^{s,25+s}_{A(2)}(tmf_0(3) \wedge A_1) = 0$ for $s \geq 6$, we conclude that $\overline{\kappa}e[1,5] = 0 \in \pi_{25}(tmf_0(3) \wedge A_1)$. □

Proposition 7.2.15. *There are the following* d_5*-differentials in the HFPSS for* $E_C^{hC_2} \wedge A_1$

$$d_5(u^{-4+4k}e_2) = u^{-2+4k}t^5 e_0.$$

Proof. The map $f_* : \pi_*(tmf_0(3) \wedge A_1) \to \pi_*(E_C^{hC_6} \wedge A_1)$ is a map of π_*S^0-modules; in particular it is $\overline{\kappa}$-linear. Therefore, by Lemma 7.2.13 and Proposition 7.2.14, we have

$$\overline{\kappa}e_5 = 0 \in \pi_{25}(E_C^{hC_6} \wedge A_1) \subset \pi_{25}(E_C^{hC_2} \wedge A_1).$$

This means that the class $u^{-10}t^5 e_0 = (t^4 u^{-8})(tu^{-2}e_0)$ is hit by a differential d_5. By naturality, the source of this differential cannot be $u^{-13}e_0$. This forces that there is a differential d_5:

$$d_5(u^{-12}e_2) = u^{-10}t^5 e_0$$

or, equivalently, by the Leibniz rule and that u^{-4} is a d_5-cycle,

$$d_5(u^{-4}e_2) = u^{-2}t^5 e_0, \quad d_5(u^{-8}e_2) = u^{-6}t^5 e_0.$$

□

4. These are names given by Bruner's software the lower script indicates the order of the class in the respective filtration.

Proposition 7.2.16. *There are the following differentials* d_5

$$d_5(u^{-2+4k}e_2) = t^5u^{4k}e_0.$$

Proof. To deduce these d_5-differentials, we use the Gross-Hopkins dual of $E_C^{hC_2} \wedge A_1$. By Theorem 7.1.4, we know that

$$I_2(E_C^{hC_6} \wedge A_1) \cong \Sigma^{22}(E_C^{hC_6} \wedge DA_1).$$

It follows that

$$\pi_6(E_C^{hC_6} \wedge A_1) \cong \mathrm{Hom}(\pi_{-6}(E_C^{hC_6} \wedge A_1), \mathbb{Q}/\mathbb{Z}) \cong \pi_{-6}(E_C^{hC_6} \wedge A_1)$$
$$\cong \pi_{-28}(E_C^{hC_6} \wedge DA_1) \cong \pi_{20}(E_C^{hC_6} \wedge DA_1).$$

Using the known differentials in the HFPSS for $E_C^{hC_2} \wedge A_1$, we obtain by inspection that

$$(\pi_{20}(E_C^{hC_2} \wedge DA_1))^{C_3} \cong \mathbb{F}_4 \oplus \mathbb{F}_4 \text{ or } \mathbb{W}/4$$

for all versions of A_1. In the HFPSS for $E_C^{hC_2} \wedge A_1$, there are at least two non-trivial C_3-invariant permanent cycles in stem 6, namely $u^{-3}e_0$ and $t^2u^{-1}e_2$. Thus, t^6e_0 cannot survive to the E_∞-term; otherwise $\pi_6(E_C^{hC_2} \wedge A_1)$ would be too big as a set. This forces the differential d_5:

$$d_5(u^{-2}e_2) = t^5e_0$$

and, as u^{-4} is a d_5-cycle in the HFPSS for $E_C^{hC_2}$,

$$d_5(u^{-2+4k}e_2) = t^5u^{4k}e_0.$$

□

Propositions 7.2.6, 7.2.7, 7.2.15, 7.2.16 together determine all differentials in the HFPSS for $E_C^{hC_6} \wedge A_1$, see Figures III.9 and III.10.

Next, we come to study some exotic extensions in the E_∞-term.

Proposition 7.2.17. *In the* E_∞*-term of the HFPSS for* $E_C^{hC_6} \wedge A_1$*, we have*

a) There can only be exotic extension by 2 *in stems* 12 *and* 20*.*

b) There are only the following exotic extensions by η

$$\eta x_{17}u^3e_0 = \nu^2u^{-3}e_0, \quad \eta u^{-21}e_0 = \nu x_{17}^2u^{-3}e_0,$$

$$\eta u^{-9}e_2 = \nu u^{-9}e_0, \quad \eta x_{17}u^{-9}e_2 = \nu x_{17}u^{-9}e_0 \neq 0.\text{[5]}$$

5. Recall that $x_{17} \in \pi_{17}(E_C^{hC_6},)$ is detected by $u^{-8}t$, see the discussion following Figure III.2.

Proof. For part a), we just notice that if an element of $\pi_*(E_C^{hC_2} \wedge A_1)$ admits a lift to $\pi_*(E_C^{hC_2} \wedge Y)$ then so does twice that element. This remark together with an examination of the origin of the classes in the E_∞-term (meaning that if it lifts to $\pi_*(E_C^{hC_2} \wedge Y)$ or it maps nontrivially to $\pi_*(E_C^{hC_2} \wedge \Sigma^3 Y)$) rules out all potential exotic extension by 2, except for those in stems 12 and 20.

For part b). We see that any element detected by e_6 is annihilated by η. Using the Toda bracket $\langle \eta, \nu, \eta \rangle = \nu^2$ and juggling, we have

$$\nu^2 e_6 = \langle \eta, \nu, \eta \rangle e_6 = \eta \langle \nu, \eta, e_6 \rangle.$$

Since $\nu^2 e_6 \neq 0$, it must be divisible by η. Inspection shows that the only possibility is that $\langle \nu, \eta, e_6 \rangle = x_{17} \Delta^{-2} u^{-21} e_0$ and that

$$\eta x_{17} \Delta^{-2} u^{-21} e_0 = \nu^2 u^{-3} e_0. \tag{III.19}$$

As $\nu^2 u^{-3} e_0 = x_{17}^3 \nu \Delta^{-2} u^{-3} e_0$, Equation (III.19) implies that

$$x_{17}(\eta u^{-21} e_0 - x_{17}^2 \nu u^{-3} e_0) = 0.$$

Since multiplication by x_{17} is injective on $\pi_{43}(E_C^{hC_6} \wedge A_1)$, we obtain that

$$\eta u^{-21} e_0 = \nu x_{17}^2 u^{-3} e_0. \tag{III.20}$$

We see from the E_∞-term that $\nu^2 u^{-9} e_2$ is non-trivial and is not divisible by η, by sparseness. This forces, by the Toda bracket $\langle \eta, \nu, \eta \rangle = \nu^2$, that $u^{-9} e_2$ has a non-trivial multiple by η. The only possibility is that

$$\eta u^{-9} e_2 = \nu u^{-9} e_0. \tag{III.21}$$

Multiplying Equation (III.21) with x_{17}, we obtain that

$$\eta x_{17} u^{-9} e_2 = \nu x_{17} u^{-9} e_0 \neq 0. \tag{III.22}$$

□

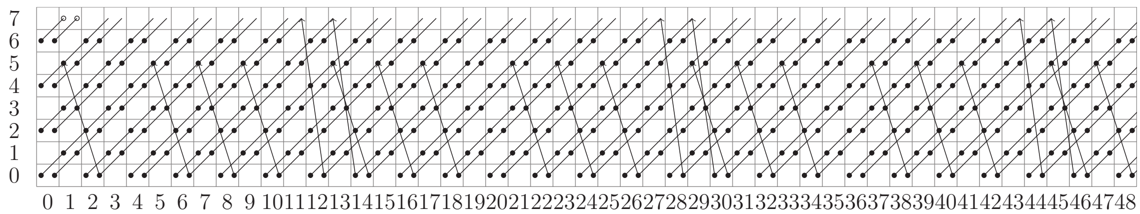

Figure III.9 – The E_2-term of the HFPSS for $E_C^{hC_2} \wedge A_1$ and the differentials d_5 and d_7. A black dot represents a copy of $\mathbb{F}_4$ and straight line a multiplication with t.

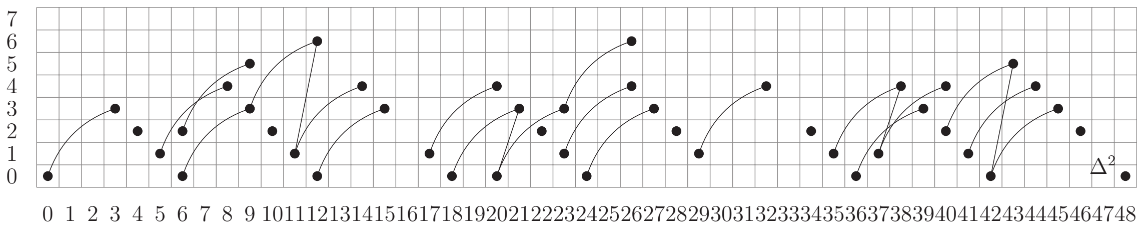

Figure III.10 – E_∞-term of the HFPSS for $E_C^{hC_6} \wedge A_1$. A black dot represents a copy of $\mathbb{F}_4$. A curved line represents multiplication by ν, a straight line multiplication by η. The E_∞-term is 48-periodic by multiplication by Δ^2.

Chapter IV

Surjectivity of the edge homomorphism

In this chapter, we will prove the following theorem

Theorem 7.2.18. *The edge homomorphism of the topological duality spectral sequence*

$$\pi_*(E_C^{h\mathbb{S}_C^1} \wedge A_1) \to \pi_*(E_C^{hG_{24}} \wedge A_1)$$

is surjective. Therefore, all differentials starting from the 0*-line of the topological duality spectral sequence are trivial.*

To this end, we study the induced map in homotopy of the Hurewicz map $H : A_1 \to tmf \wedge A_1$ and prove that the later is surjective for all versions of A_1. Let us see first how this allows us to deduce that the edge homomorphism of the TDSS is surjective.

The spectrum $tmf \wedge A_1$ supports at least two types of v_2-self maps: one comes from the periodicity of tmf and the other from a v_2-self map of A_1. The element $\Delta^8 \in \pi_{192}tmf$ extends to a map of tmf-modules $\Sigma^{192}tmf \to tmf$. Smashing with A_1 gives rise to a map of tmf-modules $\Delta^8 : \Sigma^{192}tmf \wedge A_1 \to tmf \wedge A_1$. By [**BEM17**], A_1 admits a v_2^{32}-self map $v_2^{32} : \Sigma^{192}A_1 \to A_1$, and smashing it with tmf gives rise to a map of tmf-modules $v_2^{32} : \Sigma^{192}tmf \wedge A_1 \to tmf \wedge A_1$. A priori, Δ^8 and v_2^{32} are not homotopic. However, Δ^8 and v_2^{32} induce the same localisation - there is a natural homotopy equivalence

$$[(v_2^{32})^{-1}](tmf \wedge A_1) \xrightarrow{\simeq} [(\Delta^8)^{-1}](tmf \wedge A_1). \tag{IV.1}$$

In effect, this follows from the following lemma

Lemma 7.2.19. *There is an positive integer k such that*

$$(\Delta^8)^k = (v_2^{32})^k \in \pi_{192k} F_{tmf}(tmf \wedge A_1, tmf \wedge A_1),$$

where $F_{tmf}(-,-)$ denotes the function spectrum in the homotopy category of tmf-modules.

Proof. It suffices to prove that some power of Δ^8 and v_2^{32} are the same in the ring

$$\pi_* F_{tmf}(tmf \wedge A_1, tmf \wedge A_1) \cong \pi_*(tmf \wedge A_1 \wedge DA_1).$$

Because the map $\mathbb{W} \otimes_{\mathbb{Z}_2} \pi_*(tmf \wedge A_1) \to \pi_*(E_C^{hG_{24}} \wedge A_1)$ is an isomorphism for $* \geq 0$ by Corollary 5.3.22 and linear with respect to Δ^8, multiplication by Δ^8 on $\pi_*(tmf \wedge A_1)$ induces an isomorphism $\pi_*(tmf \wedge A_1) \to \pi_{*+192}(tmf \wedge A_1)$ for $* \geq 0$, because $\pi_*(E_C^{hG_{24}} \wedge A_1)$ is Δ^8-periodic. It follows that multiplication by Δ^8 induces an isomorphism:

$$\pi_*(tmf \wedge A_1 \wedge DA_1) \to \pi_{*+192}(tmf \wedge A_1 \wedge DA_1) \text{ for } * \geq 0. \qquad \text{(IV.2)}$$

By the construction in [**BEM17**], a v_2^{32}-self map of A_1 is detected, in the E_2-term of the ASS for $A_1 \wedge DA_1$, by a class that is sent to a class detecting Δ^8 in the ASS for $tmf \wedge A_1 \wedge DA_1$. It means that the difference $\Delta^8 - v_2^{32}$ is detected in a filtration greater than 32, the Adams filtration of Δ^8 and of v_2^{32}. Together with the isomorphism (IV.2), the difference $\Delta^8 - v_2^{32}$ is equal to $\Delta^8 x$ for some element $x \in \pi_0(tmf \wedge A_1 \wedge DA_1)$ detected in a positive filtration of the ASS. As a consequence, x is nilpotent, as the ASS for $tmf \wedge A_1 \wedge DA_1$ has a vanishing line parallel to that for $tmf \wedge A_1$. Furthermore, $\Delta^8 - v_2^{32}$ has finite order and Δ^8 is in the center of $\pi_*(tmf \wedge A_1 \wedge DA_1)$. Therefore, by using the binomial formula, we see that $(\Delta^8 + v_2^{32} - \Delta^8)^{2^k}$ is equal to $(\Delta^8)^{2^k}$ for k large enough. □

The equivalence (IV.1) fits into the following commutative diagram:

$$\begin{array}{ccc} [(v_2^{32})^{-1}](tmf \wedge A_1) & \xrightarrow{\simeq} & [(\Delta^8)^{-1}](tmf \wedge A_1) \\ & \searrow \quad \swarrow & \\ & L_{K(2)}(tmf \wedge A_1), & \end{array}$$

where the unlabeled maps are natural maps from the respective telescope to the $K(2)$-localisation of $tmf \wedge A_1$. Moreover, the equivalences (I.14) and (I.15) implies that the natural map $[(\Delta^8)^{-1}](tmf \wedge A_1) \to L_{K(2)}(tmf \wedge A_1)$ is a homotopy equivalence. Thus, the map

$$[(v_2^{32})^{-1}](tmf \wedge A_1) \to L_{K(2)}(tmf \wedge A_1)$$

is a homotopy equivalence. If the Hurewicz map $A_1 \xrightarrow{H} tmf \wedge A_1$ induces a surjective map in homotopy groups, the same is true for its telescope localisation

$$[(v_2^{32})^{-1}]A_1 \to [(v_2^{32})^{-1}](tmf \wedge A_1).$$

Using the commutative diagram

$$\begin{array}{ccc} [(v_2^{32})^{-1}]A_1 & \xrightarrow{[(v_2^{32})^{-1}]H} & [(v_2^{32})^{-1}](tmf \wedge A_1) \\ \downarrow & & \downarrow \cong \\ L_{K(2)}A_1 & \xrightarrow{L_{K(2)}H} & L_{K(2)}(tmf \wedge A_1), \end{array}$$

we obtain that the induced map in homotopy groups of the $K(2)$-localisation of the Hurewicz map is also a surjection. Finally, by the equivalences (I.15) and I.11, the canonical map $L_{K(2)}tmf \to E_C^{hG_{48}}$ is a homotopy equivalence, hence, by smashing the latter with A_1 and pre-composing with $L_{K(2)}H$, the canonical map

$$L_{K(2)}A_1 \to E_C^{hG_{48}} \wedge A_1 \tag{IV.3}$$

induces a surjection in homotopy. By applying Lemma 1.37 of [**BG18**], there are homotopy equivalences

$$\mathrm{Gal}_+ \wedge L_{K(2)}S^0 \simeq E_C^{h\mathbb{S}_C}$$

and

$$\mathrm{Gal}_+ \wedge E_C^{h(G_{24} \rtimes \mathrm{Gal})} \simeq E_C^{hG_{24}},$$

which fit into a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}_+ \wedge L_{K(2)}S^0 & \longrightarrow & \mathrm{Gal}_+ \wedge E_C^{hG_{48}} \\ \downarrow \simeq & & \downarrow \simeq \\ E_C^{h\mathbb{S}_C} & \longrightarrow & E_C^{hG_{24}}, \end{array}$$

where the lower horizontal map is induced by the inclusion of subgroup. By smashing this diagram with A_1, we obtain that the map $E_C^{h\mathbb{S}_C} \wedge A_1 \to E_C^{hG_{24}} \wedge A_1$ induces a surjection in homotopy. Since the latter factors through $E_C^{h\mathbb{S}_C^1} \wedge A_1 \to E_C^{hG_{24}} \wedge A_1$, as G_{24} is a subgroup of $\mathbb{S}_C^1$, the edge homomorphism $\pi_*(E_C^{h\mathbb{S}_C^1} \wedge A_1) \to \pi_*(E_C^{hG_{24}} \wedge A_1)$ is also a surjection.

Now we explain the strategy to study the induced map in homotopy of the tmf-Hurewicz of A_1. Let $(\overline{\kappa}, \nu)$ be the ideal of $\pi_*(S^0_{(2)})$ generated by $\overline{\kappa}$ and ν. Consider the following commutative diagram

$$\begin{array}{ccc} \pi_*(A_1) & \xrightarrow{H_*} & \pi_*(tmf \wedge A_1) \\ \downarrow & & \downarrow \\ \pi_*(A_1)/(\overline{\kappa}, \nu) & \longrightarrow & \pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu). \end{array}$$

We see that the upper horizontal map is surjective if and only if the lower is surjective. In fact, this is an easy consequence of $\pi_*(A_1)$ being bounded below. To prove that the lower map is surjective we can proceed as follows. For all $\overline{x} \in \pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$, first, lift $\overline{x}$ to an element $x \in \pi_*(tmf \wedge A_1)$, then, find a class that detects x in the ASS for $tmf \wedge A_1$ and finally, show that class lifts to a permanent cycle in the ASS for A_1.

The first and the second steps are rather straightforward. It follows from the proof of Corollary 5.3.22, that we can identify a set of generators of $\pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$ as a $\mathbb{W}[\Delta^8]$-module. We give in Table (IV.1) and Table (IV.2), a list of generators of the non self-dual versions $A_1[00]$ and $A_1[11]$ and in Table (IV.3) and Table (IV.4), a list of generators of the self-dual versions $A_1[01]$ and $A_1[10]$. This distinction is because the proof that they lift to permanent cycles in the ASS for A_1 are different, see Proposition 9.0.6, 9.0.9, 9.0.10. We denote by M, N, P, Q the set of generators listed in Table (IV.1), Table (IV.2), Table (IV.3), Table (IV.4), respectively. In these tables, the pairs of integers indicate the bidegree $(t-s, s)$ of the corresponding generators and we switch to the notation e_{t-s} instead of $e[s, t-s]$ to denote a generator in bidegree $(t-s, s)$.

$(0,0)$	$(5,1)$	$(6,1)$	$(11,2)$	$(15,3)$	$(17,3)$	$(21,4)$	$(23,4)$
e_0	e_5	e_6	e_{11}	e_{15}	e_{17}	e_{21}	e_{23}
$(30,6)$	$(32,6)$	$(36,7)$	$(38,7)$	$(42,8)$	$(47,9)$	$(48,9)$	$(53,10)$
e_{30}	e_{32}	e_{36}	e_{38}	e_{42}	e_{47}	e_{48}	e_{53}
$(48,8)$	$(53,9)$	$(54,9)$	$(59,10)$	$(63,11)$	$(65,11)$	$(69,12)$	$(71,12)$
w_2e_0	w_2e_5	w_2e_6	w_2e_{11}	w_2e_{15}	w_2e_{17}	w_2e_{21}	w_2e_{23}
$(78,14)$	$(80,14)$	$(84,15)$	$(86,15)$	$(90,16)$	$(95,17)$	$(96,17)$	$(101,18)$
w_2e_{30}	w_2e_{32}	w_2e_{36}	w_2e_{38}	w_2e_{42}	w_2e_{47}	w_2e_{48}	w_2e_{53}

Table IV.1 – List M. Generators of $\pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$ as $\mathbb{F}_2[\Delta^8]$-module for the non-self dual versions $A_1[00]$ and $A_1[11]$.

$(99,17)$	$(104,18)$	$(105,18)$	$(110,19)$	$(114,20)$	$(116,20)$
$\nu w_2^2 e_0$	$\nu w_2^2 e_5$	$\nu w_2^2 e_6$	$\nu w_2^2 e_{11}$	$\nu w_2^2 e_{15}$	$\nu w_2^2 e_{17}$
$(120,21)$	$(122,21)$	$(147,25)$	$(152,26)$	$(153,26)$	
$\nu w_2^2 e_{21}$	$\nu w_2^2 e_{23}$	$\nu w_2^3 e_0$	$\nu w_2^3 e_5$	$\nu w_2^3 e_6$	
$(158,27)$	$(162,28)$	$(164,28)$	$(168,29)$	$(170,29)$	
$\nu w_2^3 e_{11}$	$\nu w_2^3 e_{15}$	$\nu w_2^3 e_{17}$	$\nu w_2^3 e_{21}$	$\nu w_2^3 e_{23}$	

Table IV.2 – List N. Generators of $\pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$ as $\mathbb{F}_2[\Delta^8]$-module for the non-self dual versions $A_1[00]$ and $A_1[11]$.

$(0,0)$	$(5,1)$	$(6,1)$	$(11,2)$	$(15,3)$	$(17,3)$	$(21,4)$	$(23,4)$
e_0	e_5	e_6	e_{11}	e_{15}	e_{17}	e_{21}	e_{23}
$(30,6)$	$(32,6)$	$(36,7)$	$(38,7)$	$(42,8)$	$(47,9)$	$(48,9)$	
e_{30}	e_{32}	e_{36}	e_{38}	e_{42}	e_{47}	e_{48}	
$(48,8)$	$(53,10)$	$(53,9)$	$(54,9)$	$(59,10)$	$(63,11)$	$(65,11)$	
$w_2 e_0$	e_{53}	$w_2 e_5$	$w_2 e_6$	$w_2 e_{11}$	$w_2 e_{15}$	$w_2 e_{17}$	
$(69,12)$	$(74,13)$	$(78,14)$	$(80,14)$	$(84,15)$	$(90,16)$	$(95,17)$	
$w_2 e_{21}$	$\nu w_2 e_{23}$	$w_2 e_{30}$	$w_2 e_{32}$	$w_2 e_{36}$	$w_2 e_{42}$	$w_2 e_{47}$	

Table IV.3 – List P. Generators of $\pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$ as $\mathbb{F}_2[\Delta^8]$-module for the self dual versions $A_1[01]$ and $A_1[10]$.

$(96,16)$	$(101,17)$	$(105,18)$	$(110,19)$	$(111,19)$	$(116,20)$
$w_2^2 e_0$	$w_2^2 e_5$	$\nu w_2^2 e_6$	$\nu w_2^2 e_{11}$	$w_2^2 e_{15}$	$\nu w_2^2 e_{17}$
$(120,21)$	$(122,21)$	$(126,22)$	$(147,25)$	$(152,26)$	$(153,26)$
$\nu w_2^2 e_{21}$	$\nu w_2^2 e_{23}$	$w_2^2 e_{30}$	$\nu w_2^3 e_0$	$\nu w_2^3 e_5$	$\nu w_2^3 e_6$
$(158,27)$	$(162,28)$	$(164,28)$	$(168,29)$	$(170,29)$	
$\nu w_2^3 e_{11}$	$\nu w_2^3 e_{15}$	$\nu w_2^3 e_{17}$	$\nu w_2^3 e_{21}$	$\nu w_2^3 e_{23}$	

Table IV.4 – List Q. Generators of $\pi_*(tmf \wedge A_1)/(\overline{\kappa}, \nu)$ as $\mathbb{F}_2[\Delta^8]$-module for the self dual versions $A_1[01]$ and $A_1[10]$.

In the next part of this Chapter, we proceed to prove that all classes in the above tables lift to permanent cycles in the Adams SS for A_1. There are two main steps to this end. In the first place, we show that the induced map on the E_2-terms of the Hurewicz map

$$H_* : \mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_* A_1) \to \mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_* A_1)$$

is surjective. This implies, in particular, that the classes in $M \cup N$ (respectively $P \cup Q$) lift to the E_2-term of the ASS for A_1. In the second place, we show that the $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*A_1)$ has a certain structure (see Theorem 9.0.1) that allows us to rule out non-trivial differentials on lifts of the classes in $M \cup N$ (respectively $P \cup Q$) in the ASS for A_1.

8 The algebraic tmf-Hurewicz homomorphism

Theorem 8.0.1. *(Vanishing line) For $n \geq 0$ or $n = \infty$, $\mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\mathrm{H}_*A_1)$ has vanishing line $t - s < f(s)$, where $f(s) = 5s - 4$ if $s \leq 6$ and $f(s) = 5s$ if $s > 6$, i.e.,*

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\mathrm{H}_*A_1) = 0 \text{ if } t - s < f(s).$$

Proof. The statement for $n = 0, 1$ follows from the fact that H_*A_1 is $\mathcal{A}(0)_*$- and $\mathcal{A}(1)_*$-cofree. The statement for $n = 2$ follows from the explicit structure of $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\mathrm{H}_*A_1)$ computed in Chapter II. Now suppose $n \geq 3$. Set $\Gamma = \mathcal{A}(n)_*\Box_{\mathcal{A}(2)_*}\mathbb{F}_2$: Γ is an $\mathcal{A}(n)$-comodule algebra. The unit $\mathbb{F}_2 \to \mathcal{A}(n)_*\Box_{\mathcal{A}(2)_*}\mathbb{F}_2$ is a map of Γ-comodules; denote by $\overline{\Gamma}$ the quotient of the latter; so that we have the short exact sequence

$$0 \to \Gamma^{\otimes r} \to \Gamma^{\otimes r+1} \to \Gamma^{\otimes r} \otimes \overline{\Gamma} \to 0,$$

for $r \geq 0$. Slicing these together, we get a long exact sequence of $\mathcal{A}(n)_*$-comodules

$$0 \to \mathbb{F}_2 \to \Gamma \to \Gamma \otimes \overline{\Gamma} \to \ldots \to \Gamma \otimes \overline{\Gamma}^{\otimes r} \to \ldots$$

which gives rise to a spectral sequence converging to $\mathrm{Ext}^{*,*}_{\mathcal{A}(n)_*}(\mathrm{H}_*A_1)$ with E_1-term isomorphic to $\mathrm{Ext}^{*,*}_{\mathcal{A}(n)_*}(\Gamma \otimes \overline{\Gamma}^{\otimes r} \otimes \mathrm{H}_*A_1)$:

$$\mathrm{E}_1^{s,t,r} = \mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\Gamma \otimes \overline{\Gamma}^{\otimes r} \otimes \mathrm{H}_*A_1) \Longrightarrow \mathrm{Ext}^{s+r,t}_{\mathcal{A}(n)_*}(\mathbb{F}_2, \mathrm{H}_*A_1). \qquad \text{(IV.4)}$$

By the change-of-rings isomorphism,

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\Gamma \otimes \overline{\Gamma}^{\otimes r} \otimes \mathrm{H}_*A_1) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\overline{\Gamma}^{\otimes r} \otimes \mathrm{H}_*A_1).$$

We see that $\overline{\Gamma}^{\otimes r}$ is $(8r - 1)$-connected because $\overline{\Gamma}$ is 7-connected. Together with the fact that

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\mathrm{H}_*A_1) = 0$$

if $t - s < f(s)$, we obtain that

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\overline{\Gamma}^{\otimes r} \otimes \mathrm{H}_*A_1) = 0$$

if $t - s < f(s) + 8r$ or equivalently if $t - (s + r) < f(s + r) + 2r$. We can now conclude by using the spectral sequence (IV.4). □

Theorem 8.0.2. *(Approximation Theorem) Let $m \geq n$ be two non-negative integers or $m = \infty$, the restriction homomorphism*

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\mathrm{H}_*A_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\mathrm{H}_*A_1)$$

is an isomorphism if $t - s < f(s-1) + 2^{n+1} - 1$ and is an epimorphism if $t - s < f(s) + 2^{n+1}$, where $\mathcal{A}(\infty)_* := \mathcal{A}$ and $f(s)$ is as in Theorem 8.0.1.

Proof. Pose $\Gamma = \mathcal{A}(m)_* \square_{\mathcal{A}(n)_*} \mathbb{F}_2$ and $\overline{\Gamma} = coker(\mathbb{F}_2 \to \Gamma)$. The restriction homomorphism is the composite

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\mathrm{H}_*A_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\Gamma \otimes \mathrm{H}_*A_1) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\mathrm{H}_*A_1)$$

where the first map is induced by the unit $\mathbb{F}_2 \to \Gamma$ and the second is the change-of-rings isomorphism. The short exact sequence of $\mathcal{A}(m)_*$-comodules $\mathbb{F}_2 \to \Gamma \to \overline{\Gamma}$ gives rise to a long exact sequence

$$\mathrm{Ext}^{s-1,t}_{\mathcal{A}(m)_*}(\overline{\Gamma} \otimes \mathrm{H}_*A_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\mathrm{H}_*A_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\mathrm{H}_*A_1)$$

$$\to \mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\overline{\Gamma} \otimes \mathrm{H}_*A_1)$$

Because $\overline{\Gamma}$ is 2^{n+1}-connected and $\mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\mathbb{F}_2, \mathrm{H}_*A_1)$ has the vanishing line $t - s < f(s)$,

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\overline{\Gamma} \otimes \mathrm{H}_*A_1) = 0$$

if $t - s < f(s) + 2^{n+1}$, hence the surjectivity of the respective restriction homomorphism; and

$$\mathrm{Ext}^{s-1,t}_{\mathcal{A}(m)_*}(\overline{\Gamma} \otimes \mathrm{H}_*A_1) = \mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\overline{\Gamma} \otimes \mathrm{H}_*A_1) = 0$$

if $t - s < f(s-1) + 2^{n+1} - 1$, hence the bijectivity of the respective restriction homomorphism. □

Corollary 8.0.3. The restriction map $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathrm{H}_*A_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\mathrm{H}_*A_1)$ is an epimorphism if $t - s < 5s + 8$ and is an isomorphism if $t - s < 5s + 2$

Theorem 8.0.4. *The restriction map $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1)) \to \mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ is an epimorphism.*

Proof. The restriction map $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1)) \to \mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}^*(A_1))$ is a map of modules over $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1 \wedge DA_1))$; This module structure comes from the fact that A_1 is a module over the ring spectrum $A_1 \wedge DA_1$. It is proved in [**BEM17**], Corollary 3.8 that the class $v_2^8 \in \mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1 \wedge DA_1))$ lifts to $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathrm{H}_*(A_1 \wedge DA_1))$. In particular, the restriction is a map of modules over the sub-algebra R generated by g, ν, v_2^8. We know by Proposition 3.2.5 that the classes e_i where

$$i \in \{0, 5, 6, 11, 15, 17, 21, 23, 30, 32, 36, 38, 42, 47, 48, 53\}$$

are generators of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}^*(A_1)$ as a module over R. These classes live in the region $\{t - s < 5s + 8\}$, and hence lift to $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}^*(A_1)$ by the Approximation Theorem 8.0.2. □

This theorem shows that all the classes of $M \cup N$ (respectively $P \cup Q$) lift to $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1))$. In the next section, we prove that the latter lift to permanent cycles.

9 The topological tmf-Hurewicz homomorphism

The key step is the following observation on the structure of the $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1))$.

Theorem 9.0.1. *The group* $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}^*(A_1))$ *has the following properties*

(i) All classes of $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathrm{H}^*A_1)$ *in the region*

$$F = \{s \geq 18, 5s \leq t - s \leq 5s + 6\} \cup \{s \geq 27, 5s \leq t - s \leq 5s + 14\}$$

are g-free and are divisible by g.

(ii) Any class x of $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathrm{H}^*A_1)$ *in the region*

$$D = \{s \geq 21, 5s \leq t - s \leq 5s + 12\} \cup \{s \geq 30, 5s \leq t - s \leq 5s + 20\}$$

is weakly g-divisible, i.e., there is a class y and a non-negative integer n such that $g^{n+1}y = g^n x$.

Because the classes involved in the statement of this theorem live in the region where there is an isomorphism $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathrm{H}^*(A_1)) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(4)_*}(\mathrm{H}^*(A_1))$ by the Approximation Theorem, it suffices to prove that $\mathrm{Ext}^{*,*}_{\mathcal{A}(4)_*}(\mathrm{H}^*A_1)$ has the required properties. We prove a stronger statement:

Theorem 9.0.2. *The group* $\mathrm{Ext}^{*,*}_{\mathcal{A}(4)_*}(\mathrm{H}^*(A_1))$ *has the following properties*

(i) All classes in the region

$$S_1 = \{20 \leq s \leq 27, 5s \leq t{-}s \leq 7s{-}40\} \cup \{s \geq 27, 5s \leq t{-}s \leq 5s{+}14\}$$

are g-free and are divisible by g. All classes in the region

$$S_2 = \{27 \leq s \leq 30, 5s + 14 \leq t - s \leq 7s - 40\} \cup \{s \geq 30, 5s + 14 \leq t - s \leq 5s + 20\},$$

are weakly divisible by g.

(ii) All classes in the region

$$T_1 = \{15 \leq s \leq 18, 5s \leq t-s \leq 7s-30\} \cup \{s \geq 18, 5s \leq t-s \leq 5s+6\}$$

are g-free and are divisible by g. All classes in the region

$$T_2 = \{18 \leq s \leq 21, 5s+6 \leq t-s \leq 7s-30\} \cup$$
$$\{s \geq 21, 5s+6 \leq t-s \leq 5s+12\},$$

are weakly divisible by g.

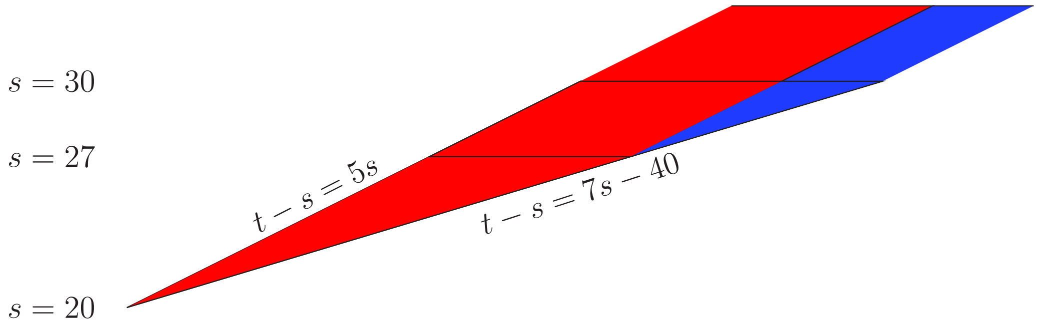

Figure IV.1 – The region in red is associated to S_1 or R_1, the blue to S_2 or R_2.

Before proving this theorem, let us explain the strategy of the proof. Observe that there is a sequence of extensions of commutative Hopf algebras

$$B_{i+1}\square_{B_i}\mathbb{F}_2 \to B_{i+1} \to B_i \text{ for } 0 \leq i \leq 8$$

in which each $B_{i+1}\square_{B_i}\mathbb{F}_2$ is isomorphic to an exterior algebra $\Lambda(h_i)$ on one generator h_i of degree at least 8 and $B_0 = \mathcal{A}(2)_*$, $B_9 = \mathcal{A}(4)_*$. We can then deduce information on $\mathrm{Ext}^{*,*}_{\mathcal{A}(4)_*}(\mathrm{H}_*A_1)$ from $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*A_1)$ by a sequence of Davis-Mahowald spectral sequences

$$\mathrm{E}_1^{s,t,\sigma} = \bigoplus_{\sigma\geq 0} \mathrm{Ext}^{s,t-|h_i|\sigma}_{B_i}(\mathrm{H}_*A_1 \otimes \mathbb{F}_2\{h_i^\sigma\}) \Longrightarrow \mathrm{Ext}^{s+\sigma,t}_{B_{i+1}}(\mathrm{H}_*A_1).$$

By the calculation of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*A_1)$, we see that the classes in the region S_1 and S_2 of the latter have the desired properties. Using this as the base case, we prove by induction that each $\mathrm{Ext}^{s+\sigma,t}_{B_{i+1}}(\mathrm{H}_*A_1)$ has the desired properties. To this end, we first prove, by induction again, that each term of the Davis-Mahowald spectral sequence has similar properties in the appropriate regions and then make sure that extensions cannot prevent the target of the spectral sequence from having the desired properties, where the fact that the degree of each h_i is at least 8 becomes crucial.

Proof. We have that

$$\mathcal{A}(4)_* = \mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]/(\zeta_1^{32}, \zeta_2^{16}, \zeta_3^8, \zeta_4^4, \zeta_5^2)$$

$$\mathcal{A}(2)_* = \mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3]/(\zeta_1^8, \zeta_2^4, \zeta_3^2).$$

From this, we can construct a sequence of maps of commutative Hopf algebras $(B_{i+1} \to B_i)$ with $0 \leq i \leq 9$, $B_0 = A(2)_*$ and $B_9 = A(4)_*$ such that for each i, $B_{i+1}\Box_{B_i}\mathbb{F}_2 = \Lambda(h_i)$ is an exterior algebra on one generator h_i of degree at least 8. Informally, we start with $B_0 = A(2)_*$ and successively "add" $\zeta_4, \zeta_3^2, \zeta_2^4, \zeta_1^8, \zeta_5, \zeta_4^2, \zeta_3^4, \zeta_2^8, \zeta_1^{16}$. For example, $B_1 = \mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \zeta_4]/(\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4^2)$. We will prove by induction on i that $\mathrm{Ext}_{B_i}^{*,*}(\mathbb{F}_2, \mathrm{H}_*A_1)$ has the property (i). The proof of (ii) works similarly, see Remark 9.0.3. First, we can directly check that

$$\mathrm{Ext}_{B_0}^{*,*}(\mathrm{H}_*A_1) = \mathrm{Ext}_{\mathcal{A}(2)_*}^{*,*}(\mathrm{H}_*A_1)$$

verifies (i), by inspecting its structure as shown in Proposition 3.2.5 of Chapter II. Suppose that $\mathrm{Ext}_{B_i}^{*,*}(\mathrm{H}_*A_1)$ verifies the properties (i). Consider the Davis-Mahowald spectral sequence

$$\mathrm{E}_1^{s,t,\sigma} = \bigoplus_{\sigma\geq 0} \mathrm{Ext}_{B_i}^{s,t}(\mathrm{H}_*A_1 \otimes \mathbb{F}_2\{h_i^\sigma\}) \Longrightarrow \mathrm{Ext}_{B_{i+1}}^{s+\sigma,t}(\mathrm{H}_*A_1) \tag{IV.5}$$

and the differential d_r goes from $E_r^{s,t,\sigma}$ to $E_r^{s-r+1,t,\sigma+r}$. Since h_i is a B_i-primitive, we have that

$$\mathrm{E}_1^{s,t,\sigma} = \bigoplus_{\sigma\geq 0} \mathrm{Ext}_{B_i}^{s,t-d\sigma}(\mathrm{H}_*A_1) \otimes \mathbb{F}_2\{h_i^\sigma\}$$

where $d = |h_i|$. We will prove by induction on $r \geq 1$ that each $\mathrm{E}_r^{s,t,\sigma}$-term of the Davis-Mahowald SS (IV.5) has the following properties

(a) All classes in the region

$$R_1 = \{20 \leq s+\sigma \leq 27, 5(s+\sigma) \leq t-s-\sigma \leq 7(s+\sigma) - 40\}$$

$$\cup\{s+\sigma \geq 27, 5(s+\sigma) \leq t-s-\sigma \leq 5(s+\sigma) + 14\}$$

are g-free and are divisible by g.

(b) All classes in the region

$$R_2 = \{27 \leq s+\sigma \leq 30, 5(s+\sigma) + 14 \leq t-s-\sigma \leq 7(s+\sigma) - 40\}$$

$$\cup\{s+\sigma \geq 30, 5(s+\sigma) + 14 \leq t-s-\sigma \leq 5(s+\sigma) + 20\}$$

are weakly divisible by g.

A similar proof as of Theorem 8.0.1 show that $\mathrm{Ext}^{s,t}_{B_i}(\mathrm{H}_*A_1)$ has the same vanishing line, and so

$$\mathrm{E}_1^{s,t,\sigma} = 0 \text{ if } s+\sigma > 6, t-(s+\sigma) < 5(s+\sigma) \text{ or } s+\sigma \leq 6, t-(s+\sigma) < 5(s+\sigma)-4. \tag{IV.6}$$

The $\mathrm{E}_1^{s,t,\sigma}$-term is spanned by classes $x \otimes h_i^\sigma$ with $x \in \mathrm{Ext}^{s,t-d\sigma}_{B_i}(\mathrm{H}_*A_1)$. For degree reasons ($d = |h_i| \geq 8$) and Equation (IV.6), a non-trivial class $x \otimes h_i^\sigma$ lies in R_1 and R_2 only if x lies in S_1 and $S_1 \cup S_2$, respectively. Then together with the induction hypothesis, it is straightforward to see that the E_1-term of the spectral sequence (IV.5) has the properties (a) and (b). Suppose that the E_r-term of (IV.5) has those properties. Let $x \in \mathrm{E}_r^{s,t,\sigma}$ represent a class $[x]$ of $\mathrm{E}_{r+1}^{s,t,\sigma}$.

Step 1. Suppose $[x]$ lives in R_1 and $[x]$ is g-torsion. Because R_1 is stable by multiplication by g, we can assume that $g[x] = 0$. Then there exists $y \in \mathrm{E}_r^{s+4+r-1,t+24,\sigma-r}$ such that $d_r(y) = gx$. We see that y belongs to the region $R_1 \cup R_2$. By the induction hypothesis, y is weakly divisible by g, i.e., there is a z and an integer n such that $g^{n+1}z = g^n y$. It follows that

$$g^{n+1}d_r(z) = d_r(g^{n+1}z) = d_r(g^n y) = g^n d_r(y) = g^n gx = g^{n+1}x.$$

However, $d_r(z)-x$ lies in R_1 which consists only of g-free classes, hence $d_r(z) = x$, and so $[x] = 0$. Therefore, all classes in R_1 of E_{r+1} are g-free.

Step 2. Suppose $[x]$ lies in R_1. By the induction hypothesis, there exists $y \in \mathrm{E}_r^{s-4,t-24,\sigma}$ such that $gy = x$. We claim that y is a d_r-cycle. We have that

$$gd_r(y) = d_r(gy) = d_r(x) = 0.$$

Moreover $d_r(y) \in E_r^{s-r-3,t-24,\sigma+r}$ living in R_1, hence is g-free. We conclude that $d_r(y) = 0$. Thus, $[x]$ is divisible by g.

Step 1 and Step 2 show that the E_{r+1}-term has the property (a).

Step 3. Now suppose that $[x]$ belongs to the region R_2. Then x is weakly divisible by g, i.e., there is a class $z \in \mathrm{E}_r^{s-4,t-24,\sigma}$ and an integer n such that $g^{n+1}z = g^n x$. We claim that z is a d_r-cycle. Since x is a d_r-cycle, we have

$$g^{n+1}d_r(z) = d_r(g^{n+1}z) = d_r(g^n x) = g^n d_r(x) = 0.$$

Moreover, $d_r(z) \in \mathrm{E}_r^{s-4-r+1,t-24,\sigma+r}$ which belongs to R_1, hence $d_r(z)$ is g-free, and so $d_r(z) = 0$. Therefore, we obtain that $g^{n+1}[z] = g^n[x]$, hence the E_{r+1}-term has the property (b).

Step 4. It is now straightforward to see that the E_∞-term also has the properties (a) and (b). To finish the proof, we will show that the target of the spectral sequence (IV.5) has the property (i). Let

$$... \subset F_\sigma \subset F_{\sigma-1} \subset ... \subset F_1 \subset F_0 = \mathrm{Ext}^{*,*}_{B_{i+1}}(\mathbb{F}_2, \mathrm{H}_*A_1)$$

be the filtration of $\mathrm{Ext}^{*,*}_{B_{i+1}}(\mathrm{H}_*A_1)$ associated to the Davis-Mahowald spectral sequence. A class belongs to F_σ only if it is represented in the E_1-term by a class of the form $x\otimes h_i^\sigma$ where $x\in \mathrm{Ext}^{s,t}_{B_i}(\mathrm{H}_*A_1)$ - here $t-s\geq 5s-4$ because of Equation IV.6. Such a class has bidegree $(s+\sigma, t+d\sigma)$, and so has the topological degree $t-s+(d-1)\sigma$. Because $d\geq 8$, the latter exceeds $5(s+\sigma)+20$ for σ sufficiently large. However, any class in $S_1\cup S_2$ has bidegree (t,s) satisfying $t-s\leq 5s+20$. This means that there is an integer m such that all classes of $S_1\cup S_2$ belongs to $F_0\backslash F_m\cup\{0\}$. From this, it is straightforward to verify that the properties (a) and (b) of the E_∞-term implies the property (i) of $\mathrm{Ext}^{*,*}_{B_{i+1}}(\mathbb{F}_2,\mathrm{H}_*A_1)$. □

Remark 9.0.3. The proof of (i) uses a double induction. What makes both base cases work is the fact that $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\mathrm{H}_*A_1)$ has the required properties and that the slope of each h_i is lower than $\frac{1}{7}$ which is exactly the slope of the lower line limiting the region in question. What makes the inductive step work is self-explained by the choice of the regions: relevant classes lie in the relevant regions. What makes the target of the DMSS have the required properties is that the slope of h_i is lower then the slope of the vanishing line. The regions T_1 and T_2 are chosen to have all of these features, hence the proof of (ii) is similar to that of (i).

We need the following property of the E_2-term of the Adams spectral sequence for S^0, which necessitates a calculation of the Ext group up to stem 43, see [**Tan70**], Theorem 4.42.

Lemma 9.0.4. *The class ν is annihilated by g^2, so is g-torsion in the E_2-term of the ASS for S^0.*

Theorem 9.0.5. *The induced map in homotopy of the Hurewicz map $A_1\to tmf\wedge A_1$ is surjective.*

Proof. The map $H_*:\pi_*(A_1)\to\pi_*(tmf\wedge A_1)$ is a map of $\pi_*(A_1\wedge DA_1)$-modules. In particular, it is a map of modules over the subalgebra R of $\pi_*(A_1\wedge A_1)$ generated by $\nu,\overline{\kappa},\Delta^8$. Therefore, we only need to prove that a set of generator of $\pi_*(tmf\wedge A_1)$ as a R-module belongs to the image of H.

Because of Theorem 8.0.4, we can choose a lift of classes of $M\cup N$ to $\mathrm{Ext}^{*,*}_{A}(\mathrm{H}_*A_1)$ such that classes which are divisible by ν lift to classes which are divisible by ν. We fix such a choice of lifting and call them also M and N, respectively. Let us prove that all classes of $M\cup N$ (respectively, $P\cup Q$) are permanent cycles in the ASS for A_1; then they must survive to the E_∞-term because their images in the ASS for $tmf\wedge A_1$ do. By comparing the bidegree of the classes of $M\cup N$ and the vanishing line of the E_2-term, we see that the differentials on classes of $M\cup N$ of length greater than 4 are trivial. The theorem now follows from Proposition 9.0.6, 9.0.9, 9.0.10 below. □

Proposition 9.0.6. *All classes in M of $A_1[00]$ and $A_1[11]$ and in P of $A_1[01]$ and $A_1[10]$ are permanent cycles in the respective ASS.*

Proof. Inspection of bidegrees together with the Vanishing Line theorem 8.0.1 show that there can only be nontrivial differentials d_2 on classes of M (respectively P) and moreover these differentials hit the region where there is an isomorphism between $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1))$ and $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$. However, all classes of M (respectively P) are permanent cycles in the ASS for $tmf \wedge A_1$. Therefore, the differentials d_2 on the classes in M (respectively P) in the ASS for A_1 are trivial. □

Lemma 9.0.7. *a) The target groups for d_3 on classes in N are g-free. More precisely, $\mathrm{E}_3^{s,t}$ is g-free if*

$$(s,t) \in F_3 := \{s \geq 23, 5s \leq t-s \leq 5s+1\} \cup \{s \geq 28, 5s \leq t-s \leq 5s+9\}.$$

b) Suppose $s \geq 30$ and $t-s \leq 5s+20$ and let $s \in \mathrm{E}_3^{s,t}$. Then x is weakly divisible by g, i.e., there exists an integer n and a class $y \in E_3^{s-4,t-24}$ such that $g^{n+1}y = g^n x$.

Proof. a) The differential d_2-arriving in $g\mathrm{E}_2^{s,t}$ with $(s,t) \in F_3$ starts in $\mathrm{E}_2^{s',t'}$ with

$$(s',t') = (s+2, t+23), i.e., (t'-s') = t-s+21.$$

Then we have

$$s' \geq 25, 5s' = 5s+10 \leq t-s+10 \leq 5s+11 = 5s'+1$$

respectively,

$$s' \geq 30, 5s' = 5s+10 \leq t-s+10 \leq 5s+19 = 5s'+9.$$

So (s',t') belongs to

$$\{s' \geq 25, 5s'+11 \leq t'-s' \leq 5s'+12\} \cup \{s' \geq 30, 5s'+11 \leq t'-s' \leq 5s'+20\}.$$

In this region, Theorem 9.0.1 guarantees that all classes are weakly divisible by g and this implies that $\mathrm{E}_3^{s,t}$ is g-free if $(s,t) \in F_3$. In effect, suppose x is a class which lies in F_3 and which is g-torsion. The region F_3 is stable by multiplication by g, so we can assume that $gx = 0$. Let a be a representative of x. Then there exists $b \in \mathrm{E}_2$ such that $d_2(b) = ga$. By the argument above, we know that a is weakly divisible by g, i.e., there exists an integer $n \geq 0$ and a class c at the E_2-term such that $g^n a = g^{n+1}c$. Then we have

$$g^{n+1}d_2(c) = d_2(g^{n+1}c) = d_2(g^n b) = g^n d_2(b) = g^n ga = g^{n+1}a.$$

However, F_3 belongs to the region which is g-free at the E_2-term, hence $d_2(c) = a$, which means that $x = 0$ at the E_3-term.

b) We represent x by a d_2-cycle a. By part (ii) of Theorem 9.0.1, there exists $b \in \mathrm{E}_2^{s-4,t-24}$ and an integer n such that $g^n a = g^{n+1} b$. It is enough to show that b is a d_2-cycle. We first note that, since a is a d_2-cycle, we have

$$g^{n+1} d_2(b) = d_2(g^{n+1} b) = d_2(g^n a) = 0.$$

Therefore it is enough to show that $d_2(b)$ is g-free. In fact, $d_2(b)$ is a class in $\mathrm{E}_2^{s',t'}$ with $s' = s - 2$ and

$$t' - s' = t - s - 19 \leq 5s + 20 - 19 = 5(s-2) + 11 = 5s' + 11,$$

so it is g-free by Theorem 9.0.1 part (ii) □

Lemma 9.0.8. *The target groups for the differential d_4 on classes in N are g-free. More precisely, $\mathrm{E}_4^{s,t}$ is g-free if*

$$(s,t) \in F_4 := \{s \geq 29, 5s \leq t - s \leq 5s + 4\}$$

Proof. The differential d_3 arriving in $g\mathrm{E}_3^{s,t}$ with $(s,t) \in F_4$ starts in $E_3^{s',t'}$ with

$$s' = s + 1, t' - s' = t - s + 21.$$

Then we have

$$s' \geq 30, 5s' + 16 \leq t' - s' \leq 5s' + 20.$$

By Lemma 9.0.7, all classes in such bidegrees are weakly divisible by g and this implies that $\mathrm{E}_4^{s,t}$ is g-free if $(s,t) \in F_4$. In effect, suppose x is a class which lies in F_4 and which is g-torsion. Because F_4 is stable by multiplication by g, we can assume that $gx = 0$. Let $a \in \mathrm{E}_3^{s,t}$ be a representative of x. Then there exists $b \in \mathrm{E}_3$ such that $d_3(b) = ga$. By the argument above, b is weakly divisible by g, i.e., there is an non-negative integer n and a class $c \in \mathrm{E}_3$ such that $g^{n+1} c = g^n b$. Then we have

$$g^{n+1} d_3(c) = d_3(g^{n+1} c) = d_3(g^n b) = g^n d_3(b) = g^n g a = g^{n+1} a.$$

However, F_4 belongs to the region where g acts freely at the E_3-term by Lemma 9.0.7 part (i), hence $d_3(c) = a$ and so $x = 0$ at the E_4-term. □

Proposition 9.0.9. *The differentials d_2, d_3, d_4 on the classes in N for $A_1[00]$ and $A_1[11]$ are trivial.*

Proof. All classes of N are divisible by ν, so are g-torsion in the E_2-term, hence are g-torsion at all terms. It is then enough to show that the target groups of differentials d_2, d_3, d_4 on the classes in N are g-free at the E_2, E_3, E_4-terms, respectively. In effect, the target groups for the differential d_2 on the classes in N lie in the region

$$\{s \geq 19, 5s \leq t-s \leq 5s+6\} \cup \{s \geq 27, 5s \leq t-s \leq 5s+14\}$$

consisting only of g-free classes, by Theorem $9.0.1(i)$. Next, a potential nontrivial differential d_3 or d_4 on the classes in N has its target in the region F_3 or F_4, respectively, which is g-free by Lemma 9.0.7 or Lemma 9.0.8, respectively. □

Proposition 9.0.10. *The differentials d_2, d_3, d_4 on the classes in Q for $A_1[10]$ and $A_1[01]$ are trivial.*

Proof. In this proof, A_1 denotes the self dual versions $A_1[10]$ and $A_1[01]$. The same argument as in the proof of Proposition 9.0.9 shows that the differentials d_2, d_3, d_4 on the classes in N which are divisible by ν are trivial. Consider the four other classes in N

$$w_2^2e_0, w_2^2e_5, w_2^2e_{15}, w_2^2e_{30}. \tag{IV.7}$$

These classes are g-free at the E_2-term. However, their g-multiple towers are truncated by differentials d_2 in the ASS for $tmf \wedge A_1$. In effect, since w_2^2 is a d_2-cycle in the ASS for tmf, the Leibniz rule and Theorem 4.0.3 show that

$$d_2(w_2^3e_{53}) = g^5w_2^2e_0,\ d_2(w_2^3e_{38}) = g^4w_2^2e_5,$$

$$d_2(w_2^3e_{48}) = g^4w_2^2e_{15},\ d_2(w_2^3e_{23}) = g^2w_2^2e_{30}.$$

It follows that the differentials d_2 on the classes of (IV.7) are g-torsion. Moreover, the differentials d_2 on the latter arrive in $\mathrm{E}_2^{s,t}$ with $s \geq 18, 5s \leq t-s \leq 5s+6$ which is g-free by Theorem 9.0.1. Thus, the classes of IV.7 are d_2-cycles and become g-torsions in the E_3-term.

The differentials d_3 on the classes of $(IV.7)$ arrive in $\mathrm{E}_3^{s,t}$ with $s \geq 19, t-s = 5s$. For these bidegrees, there is an isomorphism at the E_2-term of the ASS for A_1 and that for $tmf \wedge A_1$; in particular, the respective Ext-groups are isomorphic to $\mathbb{F}_2$. However, in the ASS for $tmf \wedge A_1$, the latter are hit by differentials d_2. Because of Theorem 8.0.4 and the naturality of the ASS, $\mathrm{E}_3^{s,t} = 0$ for $s \geq 19, t-s = 5s$ in the ASS for A_1. Thus, the differentials d_3 on the classes of (IV.7) are trivial.

Finally, the differential d_4 on the classes of (IV.7) land above the vanishing line, hence are trivial. □

Remark 9.0.11. To illustrate the proof of Proposition 9.0.10, we give some examples and more details.

a) First, a differential d_3 or d_4 on the first five classes in N listed in Table 2 has target living above the vanishing line, so it is trivial.

b) The other classes in N might support non-trivial differentials d_3. For example, a differential d_3 on the class $\nu w_2^2 e_{17}$ arrives in $\mathrm{E}_3^{s,t}$ with $s = 23, t - s = 115 = 5s$; the latter group is g-free by Lemma 9.0.7. The worst case is the class $\nu w_2^3 e_{23}$ on which a differential d_3 lives in $\mathrm{E}_3^{s,t}$ with $s = 32, t - s = 169 = 5s + 9$, which is g-free by Lemma 9.0.7.

c) Only the last eight classes in N as listed in Table 2 might support non-trivial differentials d_4. These classes lie in $\mathrm{E}_4^{s,t}$ with $s \geq 25, 5s \leq t - s \leq 5s + 25$. Then d_4 on these arrives in $\mathrm{E}_4^{s',t'}$ with $s' = s + 4$ and $t' = t + 3$, and so

$$s' \geq 29, 5s' \leq t' - s' \leq 5s' + 4.$$

This region consists only of g-free classes by Lemma 9.0.8

Chapter V

The differentials d_1 of the topological duality spectral sequence

Before starting the final Chapter, we recapitulate what has been shown. We study, in Chapter II and III, the E_1-term of the TDSS for A_1, which consists of

$$\mathrm{E}_1^{0,*} \cong \pi_*(E_C^{hG_{24}} \wedge A_1),$$
$$\mathrm{E}_1^{1,*} = \mathrm{E}_1^{2,*} \cong \pi_*(E_C^{hC_6} \wedge A_1),$$
$$\mathrm{E}_1^{3,*} \cong \pi_*(\Sigma^{48} E_C^{hG_{24}} \wedge A_1).$$

We describe the HFPSS for $E_C^{hG_{24}} \wedge A_1$ and $E_C^{hC_6} \wedge A_1$. From these results, $\mathrm{E}_1^{0,*}$ and $\mathrm{E}_1^{3,*}$ are 192-periodic and as modules over $\mathbb{W}(\mathbb{F}_4)[(\Delta^8)^{\pm 1}, \overline{\kappa}, \nu]/(\overline{\kappa}\nu)$, they are generated by 48 respectively 46 elements for the versions $A_1[00]$ and $A_1[11]$ respectively $A_1[01]$ and $A_1[10]$, while $\mathrm{E}_1^{1,*} = \mathrm{E}_1^{2,*}$ are 48-periodic and as modules over $\mathbb{W}(\mathbb{F}_4)[(\Delta^2)^{\pm 1}, x_{17}]$, they are generated by seven elements. Then we show, in Chapter IV, that the edge homomorphism of the TDSS is surjective, meaning that there are no non-trivial differentials on the filtration 0 of the TDSS.

In this Chapter, we study the differentials $d_1 : \mathrm{E}_1^{1,*} \to \mathrm{E}_1^{2,*}$ and $d_1 : \mathrm{E}_1^{2,*} \to \mathrm{E}_1^{3,*}$. We start by describing the maps appearing in the topological duality resolution, and from this study the induced maps in homotopy groups. In the first section, we analyse the groups $\pi_0 F(E^{hF}, E^{hK})$ and give a description of a set of elements of these groups. This description is general for all chromatic heights and all primes p. Some of the material at the beginning of this subsection is well-known in the literature; we hereby want, however, to give more details on how the transition maps of the homotopy limit in Proposition 10.0.4 are identified, compare [GHMR05], Section 2, [Hen07], Section 1.3.3.1 or [BG18], Section 1.4. New results are Theorem 10.0.8 and Proposition 10.0.14, where we observe that under some mild condition, there is a lift $s : \mathbb{Z}_p[[\mathbb{G}/H]]^K \to \pi_0 F(E^{hF}, E^{hK})$ which is continuous with respect to appropriate topologies. Then in the following section, we use this analysis to study maps in the topological duality resolution at $n = p = 2$ and the induced maps in homotopy.

10 Mapping spectra

We work with a height n formal group law (Γ, k) such that

$$\mathbb{S}(k, \Gamma) := \mathrm{Aut}_{\overline{k}}(\Gamma) \cong \mathrm{Aut}_k(\Gamma). \tag{V.1}$$

To simplify the notation, we will denote by $\mathbb{G}$ and E the corresponding Morava stabiliser group and Morava E-theory. Assumption (V.1) implies (e.g. see [**Hov04**]):

Theorem 10.0.1. *There is an isomorphism of Morava modules*

$$\pi_*(E \wedge E) \cong \mathrm{Map}_c(\mathbb{G}, E_*), \tag{V.2}$$

where $\mathbb{G}$ acts diagonally on the right hand term. This isomorphism of Morva modules is also $\mathbb{G}$-equivariant where the action of $\mathbb{G}$ on $\pi_*(E \wedge E)$ is induced by the action on the right hand factor of $E \wedge E$ and on $\mathrm{Map}^c(\mathbb{G}, E_*)$ by the right multiplication on $\mathbb{G}$, i.e., $(g.f)(h) = f(hg)$.

Let H be a closed subgroup of $\mathbb{G}$. There exists a nested sequence of open subgroups $(... \subset U_2 \subset U_1)$ of $\mathbb{G}$ such that $\bigcap\limits_i U_i = H$. For U an open subgroup of $\mathbb{G}$, Devinatz and Hopkins, in [**DH04**], construct the homotopy fixed point spectrum E^{hU} and then define

$$E^{hH} := L_{K(n)}\underset{i}{\mathrm{hocolim}} E^{hU_i}.$$

Let $N_{\mathbb{G}}(H)$ denote the normaliser of H and $W_{\mathbb{G}}(H) := N_{\mathbb{G}}(H)/H$ the Weyl group of H.

Theorem 10.0.2. [**DH04**] *Let H be a closed subgroup of $\mathbb{G}$. There is an isomorphism of Morava modules,*

$$\pi_*(E \wedge E^{hH}) \cong \mathrm{Map}_c(\mathbb{G}/H, E_*),$$

where $\mathbb{G}$ acts diagonally on the right hand term. This isomorphism is also $W_{\mathbb{G}}(H)$-equivariant where the action of $W_{\mathbb{G}}(H)$ on the left hand term is induced by its residual action on E^{hH} and on the right hand term by the right multiplication on $\mathbb{G}/H$, i.e., $(g.f)(hH) = f(hgH)$.

If U is an open subgroup of $\mathbb{G}$, then the construction of [**DH04**] provides us with a map

$$Act : \mathbb{G}/U_+ \wedge E^{hU} \to E.$$

Together with the multiplication of E, we obtain the composite

$$E \wedge \mathbb{G}/U_+ \wedge E^{hU} \to E \wedge E \to E$$

The adjoint of this map is a homotopy equivalence (see [**GHMR05**], Section 2)

$$E \wedge \mathbb{G}/U_+ \xrightarrow{\cong} F(E^{hU}, E). \tag{V.3}$$

The equivalence (V.3) is also $\mathbb{G}$-equivariant with $\mathbb{G}$ acting diagonally on the left hand side and on the second variable on the right hand side. If K a finite subgroup of $\mathbb{G}$, one has

$$F(E^{hH}, E^{hK}) \cong F(L_{K(n)}\underset{i}{\mathrm{hocolim}} E^{hU_i}, E^{hK}) \cong \underset{i}{\mathrm{holim}}\, F(E^{hU_i}, E)^{hK}. \tag{V.4}$$

Transition maps. Let $U_1 \subseteq U_2$ be open subgroups of $\mathbb{G}$. Let us denote by $\iota : E^{hU_2} \to E^{hU_1}$ the map induced by the inclusion of subgroups. Through the natural equivalence (V.3), the map

$$\iota^* : F(E^{hU_1}, E) \to F(E^{hU_2}, E)$$

is identified with

$$p \wedge Id : \mathbb{G}/U_{1+} \wedge E \to \mathbb{G}/U_{2+} \wedge E$$

where $p : \mathbb{G}/U_1 \to \mathbb{G}/U_2$ is the evident projection. In other words, there is a commutative diagram of $\mathbb{G}$-spectra

$$\begin{array}{ccc} E \wedge \mathbb{G}/U_{1+} & \xrightarrow{\cong} & F(E^{hU_1}, E) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \iota^*} \\ E \wedge \mathbb{G}/U_{2+} & \xrightarrow{\cong} & F(E^{hU_2}, E). \end{array}$$

Taking homotopy fixed points with respect to a finite subgroup K, one gets a commutative diagram:

$$\begin{array}{ccc} (E \wedge \mathbb{G}/U_{1+})^{hK} & \longrightarrow & F(E^{hU_1}, E)^{hK} \\ \downarrow & & \downarrow \\ (E \wedge \mathbb{G}/U_{2+})^{hK} & \longrightarrow & F(E^{hU_{2+}}, E)^{hK}. \end{array}$$

As a K-set, $\mathbb{G}/U_i$ decomposes as $\coprod_{x\in K\backslash\mathbb{G}/U_i} K/K_x$ where $K\backslash\mathbb{G}/U_i$ is the set of double cosets and $K_x = K\cap xU_ix^{-1}$, the stabiliser of the coset xU_i. This decomposition is explicitly given by the following formula

$$\begin{aligned}\coprod_{x\in K\backslash\mathbb{G}/U_i} K/K_x &\xrightarrow{\cong} \mathbb{G}/U_i\\ kK_x &\mapsto kxU_i.\end{aligned}$$

Moreover, we see that as i varies, these decompositions fit together in the following commutative diagram of K-sets

$$\begin{array}{ccc} \mathbb{G}/U_1 & \xrightarrow{\cong} & \coprod_{x\in K\backslash\mathbb{G}/U_1} K/K_x \\ \downarrow{\scriptstyle p_{1,2}} & & \downarrow \\ \mathbb{G}/U_2 & \xrightarrow{\cong} & \coprod_{y\in K\backslash\mathbb{G}/U_2} K/K_y \end{array} \tag{V.5}$$

where the right vertical map is given by the projection $K/K_x \to K/K_y$ induced by the inclusion of subgroup $K_x \subseteq K_y$ if y is the image of x by the evident projection $(K\backslash\mathbb{G}/U_1 \to K\backslash\mathbb{G}/U_2, KxU_1 \mapsto KxU_2)$. The projection $K/K_x \to K/K_y$ induces a obvious map of K-spectra $E\wedge(K/K_x)_+ \to E\wedge(K/K_y)_+$. These assemble to give a map

$$\bigvee_{x\in K\backslash\mathbb{G}/U_1} (E\wedge K/K_{x+})^{hK} \to \bigvee_{y\in K\backslash\mathbb{G}/U_2} (E\wedge K/K_{y+})^{hK}.$$

By the diagram (V.5), this map fits into a commutative diagram

$$\begin{array}{ccc} (E\wedge\mathbb{G}/U_{1+})^{hK} & \xrightarrow{\cong} & \bigvee_{x\in K\backslash\mathbb{G}/U_1} (E\wedge K/K_{x+})^{hK} \\ \downarrow & & \downarrow \\ (E\wedge\mathbb{G}/U_{2+})^{hK} & \xrightarrow{\cong} & \bigvee_{y\in K\backslash\mathbb{G}/U_2} (E\wedge K/K_{y+})^{hK}. \end{array}$$

Notice that if $K_x \subseteq K_y$, then the map $(E\wedge K/K_{x+})^{hK} \to (E\wedge K/K_{y+})^{hK}$ is naturally identified with the transfer $tr_{K_x}^{K_y}: E^{hK_x} \to E^{hK_y}$, by Lemma 1.1.3. In other words, there is a commutative diagram

$$\begin{array}{ccc} E^{hK_x} & \xrightarrow{\cong} & (E\wedge(K/K_x)_+)^{hK} \\ \downarrow{\scriptstyle Tr} & & \downarrow \\ E^{hK_y} & \xrightarrow{\cong} & (E\wedge(K/K_y)_+)^{hK}. \end{array}$$

Therefore, we obtain:

Lemma 10.0.3. *Suppose that K is a finite subgroup of $\mathbb{G}$ and $U_1 \subset U_2$ are open subgroups of $\mathbb{G}$. There is a commutative diagram*

$$\begin{array}{ccc} F(E^{hU_1}, E^{hK}) & \xrightarrow{\cong} & \bigvee_{x \in K\backslash\mathbb{G}/U_1} E^{hK_x} \\ \downarrow \iota^* & & \downarrow \\ F(E^{hU_2}, E^{hK}) & \xrightarrow{\cong} & \bigvee_{y \in K\backslash\mathbb{G}/U_2} E^{hK_y}, \end{array}$$

in which the right hand vertical map is induced by $tr_{K_x}^{K_y} : E^{hK_x} \to E^{hK_y}$ if y is the image of x by the evident projection $K\backslash\mathbb{G}/U_1 \to K\backslash\mathbb{G}/U_2$.

Proposition 10.0.4. *Let H be a closed subgroup and K be a finite subgroup of $\mathbb{G}$. Suppose that (U_i) is a decreasing sequence of open subgroups of $\mathbb{G}$ such that $\bigcap_i U_i = H$. Then*

$$F(E^{hH}, E^{hK}) = \operatorname{holim}_i \bigvee_{x \in K\backslash\mathbb{G}/U_i} E^{hK_{x,i}}$$

where $K_{x,i} = K \cap xU_ix^{-1}$ and the transition maps in the holimit *are described as in Lemma 10.0.3.*

The Hurewicz homomorphism. We introduce the following notations. Let $T = \lim_{\leftarrow} T_i$ be a profinite set, A be abelian group and M be a spectrum. Define

$$A[[T]] := \lim_{\leftarrow} A[T_i],$$

$$M[[T]] := \operatorname{holim}_{\leftarrow} M \wedge (T_i)_+.$$

For the rest of this section, H and K denote finite subgroups of $\mathbb{G}$, unless otherwise stated. Consider the E-Hurewicz homomorphism

$$h : \pi_0 F(E^{hH}, E^{hK}) \to \operatorname{Hom}_{\mathcal{EG}}(E_*E^{hH}, E_*E^{hK}).$$

By Theorem 10.0.2, if G is a closed subgroup of $\mathbb{G}$, then

$$E_*E^{hG} \cong \operatorname{Map}^c(\mathbb{G}/G, E_*) \cong \operatorname{Hom}^c_{E_*}(E_*[[\mathbb{G}/G]], E_*).$$

So there is a homomorphism, see [**GHMR05**], Proposition 2.7,

$$E_0[[\mathbb{G}/H]]^K \cong \operatorname{Hom}_{\mathcal{EG}}(E_*[[\mathbb{G}/K]], E_*[[\mathbb{G}/H]]) \to \operatorname{Hom}_{\mathcal{EG}}(E_*E^{hH}, E_*E^{hK}),$$

where $\mathcal{EG}$ denotes the category of Morava module (see Section 1.3), induced by applying $\mathrm{Hom}_{E_*}(-, E_*)$ to the first and the second variables of the left hand side. This is an isomorphism which makes the following diagram commutative,

$$\begin{array}{ccc} \pi_0(E[[\mathbb{G}/H]]^{hK}) & \xrightarrow{\cong} & [E^{hH}, E^{hK}] \\ \downarrow h & & \downarrow h \\ (E_0[[\mathbb{G}/H]])^K & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{EG}}(E_*E^{hH}, E_*E^{hK}), \end{array}$$

where the upper horizontal isomorphism follows from Proposition 10.0.4 and the equivalence $F(E^{hH}, E^{hK}) \simeq F(E^{hH}, E)^{hK}$.

In what follows, we analyse the image of the left hand Hurewicz homomorphism. Notice that the group $E_0[[\mathbb{G}/H]]^K$ contains $\mathbb{Z}_p[[\mathbb{G}/H]]^K$ as a subgroup. We describe the group $\mathbb{Z}_p[[\mathbb{G}/H]]^K$ and $E_0[[\mathbb{G}/H]]^K$ as inverse limits. With (U_i) a nested sequence of open subgroups of $\mathbb{G}$ satisfying $\bigcap_i U_i = H$ as before,

$$\mathbb{Z}_p[[\mathbb{G}/H]]^K \cong \lim_i \mathbb{Z}_p[\mathbb{G}/U_i]^K.$$

The transition map is simply induced by the projections $\mathbb{Z}_p[\mathbb{G}/U_{i+1}]^K \to \mathbb{Z}_p[\mathbb{G}/U_i]^K$. Using the diagram (V.5), the latter is identified as follows

$$\begin{array}{ccc} \mathbb{Z}_p[\mathbb{G}/U_{i+1}]^K & \xrightarrow{\cong} & \prod\limits_{x \in K\backslash\mathbb{G}/U_i} \mathbb{Z}_p[K/K_x]^K \\ \downarrow & & \downarrow \\ \mathbb{Z}_p[\mathbb{G}/U_i]^K & \xrightarrow{\cong} & \prod\limits_{y \in K\backslash\mathbb{G}/U_{i+1}} \mathbb{Z}_p[K/K_y]^K \end{array}$$

where the right vertical map is induced by the obvious projection $\mathbb{Z}_p[K/K_x]^K \to \mathbb{Z}_p[K/K_y]^K$ if y is the image of x by the map $K\backslash\mathbb{G}/U_{i+1} \to K\backslash\mathbb{G}/U_i$. Via the natural isomorphism $\mathbb{Z}_p[K/K_x]^K \cong \mathbb{Z}_p$ and $\mathbb{Z}_p[K/K_y]^K \cong \mathbb{Z}_p$, the map $\mathbb{Z}_p[K/K_x]^K \to \mathbb{Z}_p[K/K_y]^K$ above is identified with multiplication by $|K_y/K_x|$, i.e., there is a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_p[K/K_x]^K & \xrightarrow{\cong} & \mathbb{Z}_p \\ \downarrow & & \downarrow \times|K_y/K_x| \\ \mathbb{Z}_p[K/K_y]^K & \xrightarrow{\cong} & \mathbb{Z}_p. \end{array}$$

To sum up, we have

Lemma 10.0.5. *Let H and K be finite subgroups of $\mathbb{G}$. There is a an isomorphism*

$$\mathbb{Z}_p[[\mathbb{G}/H]]^K \cong \lim_i \prod_{x \in K\backslash\mathbb{G}/U_i} \mathbb{Z}_p,$$

where the transition map $\prod_{x\in K\backslash\mathbb{G}/U_{i+1}} \mathbb{Z}_p \to \prod_{y\in K\backslash\mathbb{G}/U_i} \mathbb{Z}_p$ is induced by multiplication by $|K_y/K_x|$ from the copy of $\mathbb{Z}_p$ at the coordinate $x \in K\backslash\mathbb{G}/U_{i+1}$ to the copy of $\mathbb{Z}_p$ at the coordinate $y \in K\backslash\mathbb{G}/U_i$ if y is the image of x by the map $K\backslash\mathbb{G}/U_{i+1} \to K\backslash\mathbb{G}/U_i$.

A similar argument shows that

Lemma 10.0.6. *Let H and K be finite subgroups of $\mathbb{G}$. There is an isomorphism*

$$E_0[[\mathbb{G}/H]]^K \cong \lim_i \prod_{x \in K\backslash\mathbb{G}/U_i} E_0^{K_x},$$

where the transition map $\prod_{x\in K\backslash\mathbb{G}/U_{i+1}} E_0^{K_x} \to \prod_{y\in K\backslash\mathbb{G}/U_i} E_0^{K_y}$ is induced by the transfer $E_0^{K_x} \to E_0^{K_y}$ if y is the image of x by the map $K\backslash\mathbb{G}/U_{i+1} \to K\backslash\mathbb{G}/U_i$.

Proposition 10.0.7. *Let F be a finite subgroup of $\mathbb{G}$. The following statements are equivalent:*

1) *The group $\pi_0(E^{hF})$ is torsion free.*
2) *The 0-stem of the E_∞-term of the HFPSS for E^{hF} is concentrated in the 0-filtration.*
3) *The edge homomorphism $\pi_0(E^{hF}) \to (E_0)^F$ of the HFPSS for E^{hF} is injective.*

Proof. It is clear that $2)$ is equivalent to $3)$. That $1)$ is equivalent to $2)$ is also straightforward by noting that the HFPSS has a horizontal vanishing line, (see Proposition 6.5 [HS99]) and that all groups living in positive filtration are torsion. □

Theorem 10.0.8. *Let H and K be finite subgroups of $\mathbb{G}$. We have the following*

(i) The subgroup $\mathbb{Z}_p[[\mathbb{G}/H]]^K$ of $E_0[[\mathbb{G}/H]]^K$ belongs to the image of the Hurewicz homomorphism.

(ii) Suppose that all subgroups of either H or of K verify the equivalent conditions of Proposition 10.0.7. Then there is a canonical lift of $\mathbb{Z}_p[[\mathbb{G}/H]]^K$ to $[E^{hH}, E^{hK}]$.

Denote by $s : \mathbb{Z}_p[[\mathbb{G}/H]]^K \to [E^{hH}, E^{hK}]$ this lift.

(iii) Suppose H, K, L are finite subgroups of $\mathbb{G}$ such that either all subgroups of K or all subgroups of H and of L verify the equivalent conditions of Proposition 10.0.7. Then there is a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_p[[\mathbb{G}/H]]^K \otimes \mathbb{Z}_p[[\mathbb{G}/K]]^L & \longrightarrow & \mathbb{Z}_p[[\mathbb{G}/H]]^L \\ \downarrow{\scriptstyle s\otimes s} & & \downarrow{\scriptstyle s} \\ \pi_0(E[[\mathbb{G}/H]]^{hK}) \otimes \pi_0(E[[\mathbb{G}/K]]^{hL}) & \longrightarrow & \pi_0(E[[\mathbb{G}/H]]^{hL}) \end{array}$$

where the horizontal arrows are compositions, using the identifications $\mathbb{Z}_p[[\mathbb{G}/H]]^K \cong \mathrm{Hom}_{\mathbb{G}}(\mathbb{Z}_p[[\mathbb{G}/K]], \mathbb{Z}_p[[\mathbb{G}/H]])$ and $\pi_0(E[[\mathbb{G}/H]]^{hK}) \cong \mathrm{Hom}_{\mathcal{EG}}(E_*[[\mathbb{G}/K]], E_*[[\mathbb{G}/H]])$.

Proof. For part (i), by the naturality of the Hurewicz homomorphism, we see that the following diagram is commutative

$$\begin{array}{ccc} \pi_0(E[[\mathbb{G}/H]]^{hK}) & \xrightarrow{\cong} & \lim_i \pi_0(E[\mathbb{G}/U_i]^{hK}) \cong \lim_i \prod_{x\in K\backslash\mathbb{G}/U_i} \pi_0 E^{hK_x} \; . \\ \downarrow & & \downarrow \\ (E_0[[\mathbb{G}/H]])^K & \xrightarrow{\cong} & \lim_i E_0[\mathbb{G}/U_i]^K \cong \lim_i \prod_{x\in K\backslash\mathbb{G}/U_i} E_0^{K_x} \end{array} \tag{V.6}$$

The upper horizontal map is an isomorphism due to the fact that $\lim^1$ of a system of profinite abelian groups is trivial, see Theorem 3, Chapitre IV of [**Gab62**]. The diagram (V.6) above factors through

$$\begin{array}{ccc} \pi_0(E[[\mathbb{G}/H]]^{hK}) & \xrightarrow{\cong} & \lim_i \prod_{x\in K\backslash\mathbb{G}/U_i} \pi_0 E^{hK_x} \\ \downarrow & & \downarrow \\ F_0/F_1 & \longrightarrow & \lim_i \prod_{x\in K\backslash\mathbb{G}/U_i} F_{0,x}/F_{1,x} \\ \hookdownarrow & & \hookdownarrow \\ (E_0[[\mathbb{G}/H]])^K & \xrightarrow{\cong} & \lim_i \prod_{x\in K\backslash\mathbb{G}/U_i} E_0^{K_x} . \end{array}$$

Here F_0 and F_1 denote the first two filtration groups in the filtration associated to the HFPSS and all surjections are the projections to the 0-filtration of the E_∞-term of the respective HFPSS and the middle horizontal map is induced by the naturality of the HFPSS. In particular, the transition maps of the middle inverse limit are determined by those of the lower inverse limit, which are described in

Lemma 10.0.6. As the upper and lower horizontal maps are isomorphisms, the middle horizontal map is also an isomorphism.

For any finite subgroup H of $\mathbb{G}$, $\mathbb{Z}_p \subset E_0$ are H-invariants and, further, $\mathbb{Z}_p$ consist only of permanent cycles. The latter can be seen as follows. Since E^{hH} is $K(n)$-local, for $(M(J(i)))_{i\geq 0}$ with $J(i) = (p^{j_{1,i}}, ..., v_{n-1}^{j_{n-1,i}})$, a cofinal tower of generalised Moore spectra of type n, there is a natural equivalence $E^{hH} \simeq \operatorname{holim} E^{hH} \wedge M(J(i))$, Section 1.7. Since E^{hH} is a ring spectrum, the unit map of E^{hH} gives rise to the maps $M(p^{j_{1,i}}) \to E^{hH} \wedge M(J(i))$. By taking homotopy limit of the latter, one obtain the map $S^0_p \to E^{hH}$, where S^0_p is the p-completed sphere spectrum, extending the unit map $S^0 \to E^{hH}$.

This means that the image of the Hurewicz homomorphism $\pi_0 E^{hH} \to E_0^H$ contains $\mathbb{Z}_p$. In other words, the inclusion $\prod_{x\in K\backslash\mathbb{G}/U_i} \mathbb{Z}_p \hookrightarrow \prod_{x\in K\backslash\mathbb{G}/U_i} E_0^{K_x}$ lifts to a monomorphism

$$\prod_{x\in K\backslash\mathbb{G}/U_i} \mathbb{Z}_p \hookrightarrow \prod_{x\in K\backslash\mathbb{G}/U_i} F_{0,x}/F_{0,x}. \tag{V.7}$$

These are compatible with the transition maps as i varies, because the inclusions $\prod_{x\in K\backslash\mathbb{G}/U_i} \mathbb{Z}_p \hookrightarrow \prod_{x\in K\backslash\mathbb{G}/U_i} E_0^{K_x}$ are compatible with the transition maps according to the description of these in Lemma 10.0.5 and Lemma 10.0.6. By taking the inverse limit over i, we obtain that the inclusion of $\mathbb{Z}_p[[\mathbb{G}/H]]^K$ into $E_0[[\mathbb{G}/H]]^K$ lifts to F_0/F_1.

For part (ii), let us first prove that there exists an open subgroup U of $\mathbb{G}$ containing H such that, for all $x \in \mathbb{G}$, there exists $g \in \mathbb{G}$ such that $K_x \leq K \cap gHg^{-1}$. In fact, for all $g \in \mathbb{G}$, there exists an open subgroup U_g of $\mathbb{G}$, containing H, such that $K \cap gU_gg^{-1} = K \cap gHg^{-1}$. Then $\forall x \in gU_g$, $K \cap xU_gx^{-1} = K \cap gU_gg^{-1} = K \cap gHg^{-1}$. Because $\cup_{g\in\mathbb{G}} gU_g = \mathbb{G}$ and $\mathbb{G}$ is compact, there exist $g_1, ..., g_n$ such that $\cup_{1\leq m\leq n} g_mU_{g_m} = \mathbb{G}$. Set $U = \cap_{1\leq i\leq n} U_{g_i}$; so that U is an open subgroup of $\mathbb{G}$. Then for all $x \in \mathbb{G}$, there exists m such that $x \in g_mU_m$ and so $K_x = K \cap xUx^{-1} \subset K \cap xU_{g_i}x^{-1} = K \cap g_mHg_m^{-1}$.

We can suppose, if needed, that there exists $i \geq 0$ such that $U_i \subset U$. This means that, if i is large enough, then, for any $x \in K\backslash\mathbb{G}/U_i$, there exists $g \in \mathbb{G}$ such that $K_x \leq K \cap gHg^{-1}$. The assumption on H and K implies then that, K_x satisfies the conditions of Proposition 10.0.7, hence the natural projection

$$\prod_{x\in K\backslash\mathbb{G}/U_i} \pi_0(E^{hK_x}) \twoheadrightarrow \prod_{x\in K\backslash\mathbb{G}/U_i} F_{0,x}/F_{1,x}$$

is an isomorphism. This implies that the map (V.7) lifts to

$$\prod_{x\in K\backslash\mathbb{G}/U_i}\mathbb{Z}_p \hookrightarrow \prod_{x\in K\backslash\mathbb{G}/U_i}\pi_0 E^{hK_x}$$

and that the lifts are compatible with the transition maps as i varies . We can conclude Part (ii) by taking the inverse limit of the above arrows.

For part (iii), consider the following diagram, where the horizontal maps are the obvious compositions.

$$\begin{array}{ccc}
\mathbb{Z}_p[[\mathbb{G}/H]]^K \otimes \mathbb{Z}_p[[\mathbb{G}/K]]^L & \longrightarrow & \mathbb{Z}_p[[\mathbb{G}/H]]^L \\
\downarrow{\scriptstyle s\otimes s} & & \downarrow{\scriptstyle s} \\
\pi_0(E[[\mathbb{G}/H]]^{hK}) \otimes \pi_0(E[[\mathbb{G}/K]]^{hL}) & \longrightarrow & \pi_0(E[[\mathbb{G}/H]]^{hL}) \\
\downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} \\
E_0[[\mathbb{G}/H]]^K \otimes E_0[[\mathbb{G}/K]]^L & \longrightarrow & E_0[[\mathbb{G}/H]]^L.
\end{array}$$

We see that the lower square is commutative, because of the naturality of the Hurewicz homomorphism, and the outer square is commutative because the composition of the vertical maps are inclusion of respective groups. Therefore the upper square is commutative, since the Hurewicz homomorphism is injective by assumption. □

Consider the case where $H = K$ are finite subgroups of $\mathbb{G}$. The residual action of $W_{\mathbb{G}}(K)$ on E^{hK} gives rise to a map of $\mathbb{Z}_p$-modules

$$A : \mathbb{Z}_p[W_{\mathbb{G}}(K)] \longrightarrow [E^{hK}, E^{hK}].$$

Moreover, $\mathbb{Z}_p[W_{\mathbb{G}}(K)]$ can be canonically identified with a submodule of $\mathbb{Z}_p[[\mathbb{G}/K]]^K$, because if $g \in N_{\mathbb{G}}(K)$, then the left coset $gK \in \mathbb{G}/K$ is fixed by K.

Proposition 10.0.9. *Let K be a finite subgroup of $\mathbb{G}$ and suppose that all subgroups of K verify the equivalent conditions of Proposition (10.0.7). Then there is a commutative diagram*

$$\begin{array}{ccc}
\mathbb{Z}_p[W_{\mathbb{G}}(K)] & \xrightarrow{A} & [E^{hK}, E^{hK}] \\
\big\downarrow & \nearrow{\scriptstyle s} & \\
\mathbb{Z}_p[[\mathbb{G}/K]]^K. & &
\end{array}$$

Proof. Let $gK \in W_{\mathbb{G}}(K)$ viewed also as an element of $\mathbb{Z}_p[[\mathbb{G}/K]]^K$. Tracing through the construction of s, we see that the induced map in Morava modules of $s(gK)$ is given by

$$E_*(s(gK)) : \mathrm{Map}^c(\mathbb{G}/K, E_*) \to \mathrm{Map}^c(\mathbb{G}/K, E_*), f \mapsto (f : hK \mapsto f(hgK)).$$

By Theorem 10.0.2, the latter is exactly the same as the induced map in Morava modules of $A(gK)$. Since the Hurewicz homomorphism is injective, we conclude that $A(gK) = s(gK)$. $\square$

Corollary 10.0.10. The lift by s of any element of $W_{\mathbb{G}}(K)$ is a map of ring spectra.

Proof. This is because the residual action of $W_{\mathbb{G}}(K)$ on E^{hK} is by maps of ring spectra, according to [**DH04**]. $\square$

We want to describe a subset of elements of $\pi_0 F(E^{hH}, E^{hK})$ which are in the image of the lift $s : \mathbb{Z}_p[[\mathbb{G}/H]]^K \to [E^{hH}, E^{hK}]$. To this end, we need to discuss topologies on $[E^{hH}, E^{hK}]$.

The natural topology. The group $[X, Y]$ can be equipped with the natural topology. This is a linear topology whose basic open neighbourhood of 0 are $U_f = ker(f^* : [X, Y] \to [F, Y])$ with f running through maps from a small spectrum F into X, see [**HS99**], Section 11 for more details. Let us recall the following:

Lemma 10.0.11. *([**HS99**], Lemma 11.5) If Y is such that $[F, Y]$ is finite for some small spectrum F, then $[X, Y]$ is compact Hausdorff for any $K(n)$-local spectrum X.*

Therefore, with the natural topology, $[E^{hH}, E^{hK}]$ is a compact Hausdorff topological group because $\pi_0(E^{hK} \wedge F)$ is finite for any type n spectrum F. The latter is due to the fact that $\pi_*(E \wedge F)$ is finite and the HFPSS for $E^{hK} \wedge F$ has a horizontal vanishing line, see [**HS99**] Proposition 6.5.

The inverse limit topology. The isomorphism

$$[E^{hH}, E^{hK}] \xrightarrow{\cong} \lim_i \, [E^{hU_i}, E^{hK}]$$

can be used to equip $[E^{hH}, E^{hK}]$ with the inverse limit topology in which each term of the limit has the natural topology. It turns out that these two topologies coincide.

Lemma 10.0.12. *The natural topology and the inverse limit topology on $[E^{hH}, E^{hK}]$ coincide.*

Proof. It is equivalent to show that the isomorphism

$$[E^{hH}, E^{hK}] \xrightarrow{\cong} \lim_i \, [E^{hU_i}, E^{hK}]$$

is a homeomorphism where the source has the natural topology and the target has the inverse limit topology. Note that the inverse limit topology is compact Hausdorff because each of the terms in the inverse limit is compact Hausdorff by Lemma 10.0.11. Since the homomorphism $[E^{hH}, E^{hK}] \to [E^{hU_i}, E^{hK}]$ is continuous for all i with respect to the natural topology, the map

$$[E^{hH}, E^{hK}] \to \lim_i [E^{hU_i}, E^{hK}]$$

is a continuous bijection. Since both the source and the target are compact Hausdorff, this isomorphism is a homeomorphism. □

Remark 10.0.13. By virtue of this lemma, we refer to either the natural topology or the inverse limit topology as the topology of $[E^{hH}, E^{hK}]$.

Proposition 10.0.14. *Let H and K be finite subgroups of $\mathbb{G}$ such that all subgroups of either H or K verify the equivalent conditions of Proposition 10.0.7. Then the lift $s : \mathbb{Z}_p[[\mathbb{G}/H]]^K \to [E^{hH}, E^{hK}]$ is continuous.*

Proof. The natural topology on each $[E^{hU_i}, E^{hK}]$ can be described as follows. Firstly, the isomorphism

$$[E^{hU_i}, E^{hK}] \cong \pi_0(\prod_{x \in K\backslash\mathbb{G}/U_i} E^{hK_x})$$

is a homeomorphism with respect to the natural topology on both sides. This is because $\pi_0 F(E^{hU_i}, E^{hK})$ is homeomorphic to $[E^{hU_i}, E^{hK}]$. Moreover, for a cofinal tower of generalised Moore spectra of type n, $(M(p^{i_1}, ..., v_{n-1}^{i_{n-1}}))$, there is an isomorphism, by the fact that $\lim^1$ of a system of finite abelian groups is trivial,

$$\pi_0 \prod_{x \in K\backslash\mathbb{G}/U_i} E^{hK_x} \xrightarrow{\cong} \lim_j \pi_0(\prod_{x \in K\backslash\mathbb{G}/U_i} E^{hK_x} \wedge M_j). \tag{V.8}$$

Since each $\pi_0(\prod_{x \in K\backslash\mathbb{G}/U_i} E^{hK_x} \wedge M_j)$ is finite and discrete (with respect to the natural topology), the inverse limit topology on the right hand side is compact Hausdorff. We see that this does not depend on the choice of the tower of generalised Moore spectra. We refer to this topology as the I_n-adic topology or simply adic topology. Now the same argument as in Lemma 10.0.12 shows that the isomorphism (V.8) is in fact a homeomorphism, meaning that the natural topology on $[E^{hU_i}, E^{hK}]$ coincides with its adic topology. Using the adic topology, we see that, for each i, the lift $\prod_{x \in K\backslash\mathbb{G}/U_i} \mathbb{Z}_p \to \prod_{x \in K\backslash\mathbb{G}/U_i} \pi_0 E^{hK_x}$ is continuous. Thus, the lift $s : \mathbb{Z}_p[[\mathbb{G}/H]]^K \to [E^{hH}, E^{hK}]$, being the inverse limit of continuous maps, is also continuous. □

11 Differentials d_1 in the topological duality spectral sequence

Now, we specialize to the case $n = p = 2$ and will study the maps δ_2 and δ_3 in the topological duality resolution, as well as their induced maps in homotopy groups.

11.1 The differential $d_1 : \mathrm{E}_1^{1,*} \to \mathrm{E}_1^{2,*}$

The differential $d_1 : \mathrm{E}_1^{2,*} \to \mathrm{E}_1^{3,*}$ is the induced map in homotopy of δ_2 of the topological duality resolution (I.23).

Lemma 11.1.1. *An element ϕ belonging to the image of the homomorphism $s : \mathbb{Z}_2[[\mathbb{G}_C/C_2]]^{C_2} \to [E_C^{hC_2}, E_C^{hC_2}]$ can be expressed as*

$$\phi = \lim_i \phi_i$$

where $\phi_i = \sum_{g \in S_i} g$ with S_i a finite subset of $\mathbb{G}_C/C_2$.

Proof. Because C_2 is a central subgroup of $\mathbb{G}_C$, $\mathbb{Z}_2[[\mathbb{G}_C/C_2]]^{C_2} = \mathbb{Z}_2[[\mathbb{G}_C/C_2]]$. Lemma 11.1.1 follows from Proposition 10.0.14 and the fact that every element of $\mathbb{Z}_2[[\mathbb{G}_C/C_2]]$ can be written as $\lim_i x_i$ where $x_i = \sum_{g \in S_i} g$ with S_i a finite subset of $\mathbb{G}_C/C_2$. □

Lemma 11.1.2. *There exists an element $\delta \in Im(s : \mathbb{Z}_2[[\mathbb{G}/C_2]]^{C_2} \to [E_C^{hC_2}, E_C^{hC_2}])$ making the following diagram commutative*

$$\begin{array}{ccc} E_C^{hC_6} & \xrightarrow{\delta_2} & E_C^{hC_6} \\ {\scriptstyle Res}\downarrow & & \downarrow{\scriptstyle Res} \\ E_C^{hC_2} & \xrightarrow{\delta} & E_C^{hC_2} \end{array}$$

where δ_2 is the middle map in the topological duality resolution and Res is the restriction map.

Proof. Note that $Res : E_C^{hC_6} \to E_C^{hC_2}$ is the image by s of the evident projection $Pr : \mathbb{Z}_2[[\mathbb{G}_C/C_2]] \to \mathbb{Z}_2[[\mathbb{G}_C/C_6]]$ induced by $gC_2 \mapsto gC_6$. It induces the map $Pr_* : \mathbb{Z}_2[[\mathbb{G}_C/C_2]]^{C_2} \to \mathbb{Z}_2[[\mathbb{G}_C/C_6]]^{C_2}$, obtained by post-composing an element of $\mathbb{Z}_2[[\mathbb{G}_C/C_2]]^{C_2} \cong \mathrm{Hom}_{\mathbb{Z}_2[[\mathbb{G}]]}(\mathbb{Z}_2[[\mathbb{G}_C/C_2]], \mathbb{Z}_2[[\mathbb{G}_C/C_2]])$ with Pr. By applying Theorem 10.0.8 Part (iii) to $H = K = C_2$, $L = C_6$ and noting that

$Res : E_C^{hC_6} \to E_C^{hC_2}$ is the image by s of the evident projection Pr, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_2[[\mathbb{G}_C/C_2]]^{C_2} & \xrightarrow{s} & [E_C^{hC_2}, E_C^{hC_2}] \\ \downarrow{\scriptstyle Pr_*} & & \downarrow{\scriptstyle Res^*} \\ \mathbb{Z}_2[[\mathbb{G}_C/C_6]]^{C_2} & \xrightarrow{s} & [E_C^{hC_6}, E_C^{hC_2}]. \end{array}$$

By the construction of δ_2, $Res \circ \delta_2$ is in the image of the lift $s : \mathbb{Z}_2[[\mathbb{G}_C/C_6]]^{C_2} \to [E_C^{hC_6}, E_C^{hC_2}]$. Because C_2 is central in $\mathbb{G}_C$, the left hand vertical map is surjective. Therefore, there is an $\delta \in \mathrm{Im}(s : \mathbb{Z}_2[[\mathbb{G}_C/C_2]] = \mathbb{Z}_2[[\mathbb{G}_C/C_2]]^{C_2} \to [E_C^{hC_2}, E_C^{hC_2}])$ making the diagram

$$\begin{array}{ccc} E_C^{hC_6} & \xrightarrow{\delta_2} & E_C^{hC_6} \\ {\scriptstyle Res}\downarrow & & \downarrow{\scriptstyle Res} \\ E_C^{hC_2} & \xrightarrow{\delta} & E_C^{hC_2} \end{array}$$

commute, as required. □

Theorem 11.1.3. *The induced map in homotopy of $\delta_2 : E_C^{hC_6} \wedge A_1 \to E_C^{hC_6} \wedge A_1$ commutes with multiplication by Δ^2.*

Proof. Since the restriction $E_C^{hC_6} \wedge A_1 \to E_C^{hC_2} \wedge A_1$ induces an injection in homotopy groups, it suffices to prove that $\delta_* : \pi_*(E_C^{hC_2} \wedge A_1) \to \pi_*(E_C^{hC_2} \wedge A_1)$ is Δ^2-linear, i.e, if $x \in \pi_*(E_C^{hC_2} \wedge A_1)$, then

$$\delta_*(\Delta^2 x) = \Delta^2 \delta_*(x).$$

According to Lemma 11.1.1, write $\delta = \lim_i \phi_i$ where $\phi_i = \sum_{g \in S_i} g$ with S_i a finite subset of $\mathbb{G}_C/C_2$. By [HS99], Corollary 11.2, for all k, the map

$$[E_C^{hC_2} \wedge A_1, E_C^{hC_2} \wedge A_1] \to \mathrm{Hom}(\pi_k(E_C^{hC_2} \wedge A_1), \pi_k(E_C^{hC_2} \wedge A_1))$$

is continuous where the target has the compact-open topology. Note that the target of this map is discrete, hence we have

$$\delta_*(\Delta^2 x) = \lim_i (\phi_i(\Delta^2 x)) = \lim_i \sum_{g \in S_i} g(\Delta^2 x)$$

$$= \lim_i \sum_{g \in S_i} g(\Delta^2) g(x).$$

For any $g \in \mathbb{G}_C$,

$$g(\Delta^2) = \Delta^2 \text{ modulo}(4, u_1^8).$$

This equation holds in $\pi_*(E_C^{hC_2})$ because the edge homomorphism $\pi_{48}(E_C^{hC_2}) \to (E_{48})^{C_2}$ is injective. It follows that

$$g(\Delta^2)g(x) = \Delta^2 g(x)$$

since the identity of $E_C^{hC_2} \wedge A_1$ is annihilated by $(4, u_1^8)$. Thus,

$$\begin{aligned}\lim_i \sum_{g \in S_i} g(\Delta^2)g(x) &= \lim_i (\Delta^2 \sum_{g \in S_i} g(x)) \\ &= \Delta^2(\lim_i \sum_{g \in S_i} g(x)) = \Delta^2 \delta_*(x).\end{aligned}$$

□

Proposition 11.1.4. *The induced map on the* E_∞*-term of the HFPSS for* $E_C^{hC_6} \wedge A_1$ *of* δ_2 *is trivial.*

Proof. The map $(\delta_2)_* : \mathrm{H}^*(C_6, (E_C)_*(A_1)) \to \mathrm{H}^*(C_6, (E_C)_*(A_1))$ is identified with the induced map in $\mathrm{Ext}^*_{\mathbb{Z}_2[[\mathbb{G}_C]]}(-, (E_C)_*(A_1))$ of the map

$$\partial_2 : \mathbb{Z}_2[[\mathbb{G}_C/C_6]] \to \mathbb{Z}_2[[\mathbb{G}_C/C_6]],$$

of the exact sequence (I.22). By the argument at the beginning of the proof of Lemma 11.1.2, there is a map of $\mathbb{G}_C$-modules $d : \mathbb{Z}_2[[\mathbb{G}_C/C_2]] \to \mathbb{Z}_2[[\mathbb{G}_C/C_2]]$ making the following diagram commutative

$$\begin{array}{ccc} \mathbb{Z}_2[[\mathbb{G}_C/C_2]] & \xrightarrow{d} & \mathbb{Z}_2[[\mathbb{G}_C/C_2]] \\ \downarrow{\scriptstyle Pr} & & \downarrow{\scriptstyle Pr} \\ \mathbb{Z}_2[[\mathbb{G}_C/C_6]] & \xrightarrow{\partial_2} & \mathbb{Z}_2[[\mathbb{G}_C/C_6]], \end{array}$$

where the vertical maps are the canonical projections. It induces the following commutative diagram, by applying $\mathrm{Ext}^*_{\mathbb{Z}_2[[\mathbb{G}_C]]}(-, (E_C)_*(A_1))$ and the Shapiro's lemma,

$$\begin{array}{ccc} \mathrm{H}^*(C_6, (E_C)_*(A_1)) & \xrightarrow{(\partial_2)^*} & \mathrm{H}^*(C_6, (E_C)_*(A_1)) \\ \downarrow{\scriptstyle Res} & & \downarrow{\scriptstyle Res} \\ \mathrm{H}^*(C_2, (E_C)_*(A_1)) & \xrightarrow{d^*} & \mathrm{H}^*(C_2, (E_C)_*(A_1)), \end{array}$$

where $(\partial_2)^*$ and d^* are induced maps by ∂_2 and d, respectively. Since the vertical maps are injective, it is enough to show that d^* is trivial. Let g be any element of

$\mathbb{G}_C$. Since C_2 is a central subgroup of $\mathbb{G}_C$, multiplication with g induces a $\mathbb{G}_C$-map from $\mathbb{Z}_2[[\mathbb{G}_C/C_2]]$ to itself. If M is a $\mathbb{G}_C$-module, then the induced map in cohomology $g^* : \mathrm{H}^*(C_2, M) \to \mathrm{H}^*(C_2, M)$ is induced by the action of g on M. It follows then that if $a \in \mathrm{H}^*(C_2, E_*)$ and $x \in \mathrm{H}^*(C_2, (E_C)_*(A_1))$, then

$$g(ax) = g(a)g(x).$$

Now let $g \in S^1_C$. It is straightforward to check, by using the cocycle representation of t given in Lemma 6.1.1, that

$$g(t) = t \text{ mod } (2, u_1),$$

where $t \in \mathrm{H}^1(C_2, (E_C)_2)$ the cohomological periodicity class defined in Lemma 6.1.1. It follows that of $x \in \mathrm{H}^*(C_2, (E_C)_*(A_1))$, then

$$g(tx) = g(t)g(x) = tg(x). \tag{V.9}$$

By Corollary 3.4.6 of [**Bea15**] ,

$$d = 1 + \alpha \text{ modulo } (2, (IS^1_2)^2).$$

Using the formula (II.10) (action of $\mathbb{G}_C$ on $(E_C)_*(A_1)$), we see that $1 + \alpha$ and $(IS^1_2)^2$ act trivially on $(E_C)_*(A_1) = \mathrm{H}^0(C_2, (E_C)_*(A_1))$. This means that d^* acts trivially on $(E_C)_*(A_1) = \mathrm{H}^0(C_2, (E_C)_*(A_1))$. Furthermore, by Equation (V.9), d^* is t-linear. Thus d^* acts trivially on $\mathrm{H}^*(C_2, (E_C)_*(A_1))$ since t is a cohomological periodicity class.

□

Theorem 11.1.5. *The map $(\delta_2)_* : \pi_*(E_C^{hC_6} \wedge A_1) \to \pi_*(E_C^{hC_6} \wedge A_1)$ can only be nontrivial on the elements $\Delta^{2k}e_{12}$ et $\Delta^{2k}e_{20}$.*

Proof. By Proposition 11.1.4, $(\delta_2)_*$ can only be nontrivial in stems where elements are detected in different filtrations. In such stems, the image of an element by $(\delta_2)_*$ is detected in a filtration higher than the filtration of that element. By inspecting the E_∞-term of the HFPSS for $E_C^{hC_6} \wedge A_1$ (c.f Figure III.10), we see that these stems are congruent to $6, 9, 12, 20, 23, 26, 40$ modulo 48.

The cofibration (III.18) induces a map of exact sequences

$$\begin{array}{ccccc} \pi_*(E_C^{hC_6} \wedge Y) & \longrightarrow & \pi_*(E_C^{hC_6} \wedge A_1) & \longrightarrow & \pi_*(E_C^{hC_6} \wedge \Sigma^3 Y) \\ \downarrow{\scriptstyle (\delta_2)_*} & & \downarrow{\scriptstyle (\delta_2)_*} & & \downarrow{\scriptstyle (\delta_2)_*} \\ \pi_*(E_C^{hC_6} \wedge Y) & \longrightarrow & \pi_*(E_C^{hC_6} \wedge A_1) & \longrightarrow & \pi_*(E_C^{hC_6} \wedge \Sigma^3 Y) \end{array}$$

It follows that if $x \in \pi_*(E_C^{hC_6} \wedge A_1)$ lifts to $\pi_*(E_C^{hC_6} \wedge Y)$, then $(\delta_2)_*(x)$ is sent trivially to $\pi_*(E_C^{hC_6} \wedge \Sigma^3 Y)$. This observation rules out the possibility for a nontrivial differential in stem $6, 9, 23, 26, 40$ modulo 48. □

Remark 11.1.6. Using the fact that $(\delta_2)_*$ is Δ^2-linear, we reduce to understanding the image of e_{12} and e_{20} by $(\delta_2)_*$.

11.2 The differential $d_1 : \mathrm{E}_1^{2,*} \to \mathrm{E}_1^{3,*}$

The differential $d_1 : \mathrm{E}_1^{2,*} \to \mathrm{E}_1^{3,*}$ is the induced map in homotopy of δ_3, which is the last map in the topological duality resolution (I.23). The identification of δ_3 is harder. Let us first introduce some notation. For any $a \in E_k^F$, define

$$E_a : \mathrm{Map}(\mathbb{G}/F, E_{*-k}) \to \mathrm{Map}(\mathbb{G}/F, E_*)$$

by the formula: for all $e \in \mathrm{Map}(\mathbb{G}/F, E_{*-k})$ and $g \in \mathbb{G}/F$,

$$E_a(e)(g) = g(a)e(g) \tag{V.10}$$

It is straightforward to see that E_a is a map of Morava modules. In particular, because Δ^2 is a G_{24}-invariant of $(E_C)_{48}$ (see the remark succeeding Theorem 1.5.1), $E_{\Delta^2} : (E_C)_*(\Sigma^{48} E_C^{hG_{24}}) \to (E_c)_* E_C^{hG_{24}}$ is a map of Morava modules.

Proposition 11.2.1. *The map δ_3 of (I.23) satisfies that*

$$(E_C)_{\Delta^2} \circ (E_C)_*(\delta_3) = \partial_3^*, \tag{V.11}$$

in other words, there is a commutative diagram

$$\begin{array}{ccc} (E_C)_* E_C^{hC_6} & \xrightarrow{\partial_3^*} & (E_C)_* E_C^{hG_{24}} \\ & {\scriptstyle E_*(\delta_3)} \searrow \quad \nearrow {\scriptstyle E_{\Delta^2}} & \\ & (E_C)_* \Sigma^{48} E_C^{hG_{24}} & \end{array}$$

Proof. For this we need to unwind the construction of the map δ_3 in [**BG18**]. Let us summarise it here. There is a $K(2)$-local spectrum Z together with a map $\partial : E_C^{hC_6} \to Z$ and an isomorphism of Morava modules $\phi : (E_C)_* Z \cong (E_C)_* E_C^{hG_{24}}$ such that

$$\phi \circ E_*(\partial) = \partial_3^*. \tag{V.12}$$

Then, there is a homotopy equivalence $f_1 \vee f_2 : \Sigma^{48} E_C^{hG_{48}} \vee \Sigma^{48} E_C^{hG_{48}} \to Z$ such that $f_i^* : E_C^*[[\mathbb{G}_C/G_{24}]] \to E_C^{*-48}[[\mathbb{G}_C/G_{48}]]$ is given by $(1 \mapsto \omega^i \Delta^2)$ for $i = 1, 2$ (see the proof of Theorem 5.8 of [**BG18**]). Here, one implicitly identifies $E_C^*(Z)$ with $\mathrm{Hom}_{(E_C)_*}((E_C)_* Z, (E_C)_*) \cong E_C^* E_C^{hG_{24}}$. Using the Galois decomposition

$$E_C^{hG_{48}} \vee E_C^{hG_{48}} \xrightarrow{\cong} E_C^{hG_{24}}, \tag{V.13}$$

one produces an equivalence $f : \Sigma^{48} E_C^{hG_{24}} \to Z$. Then put

$$\delta_3 = f^{-1} \circ \partial.$$

The decomposition (V.13) has the feature that the induced map in $E_C^*(-)$ is given by $(E_C^*[[\mathbb{G}_C/G_{24}]] \to E_C^*[[\mathbb{G}_C/G_{48}]] \oplus E_C^*[[\mathbb{G}_C/G_{48}]],\ 1 \mapsto (\omega, \omega^2))$. It follows immediately that the induced map in $E_C^*(-)$ of f is given by $(1 \mapsto \Delta^2)$. By taking the dual $\mathrm{Hom}_{(E_C)_*}(-, (E_C)_*)$, we can check that the induced map in $(E_C)_*$ of f followed by ϕ is equal to $(E_C)_{\Delta^2}$, i.e.,

$$\phi \circ (E_C)_*(f) = (E_C)_{\Delta^2}.$$

Together with the equation (V.12) and $(E_C)_*(\delta_3) = (E_C)_*(f^{-1}) \circ (E_C)_*(\partial)$, we obtain that

$$(E_C)_{\Delta^2} \circ (E_C)_*(\delta_3) = \partial_3^*.$$

□

Lemma 11.2.2. *Let F be a finite subgroup of $\mathbb{G}$. Suppose that $a \in E_k^F$ is a permanent cycle in the HFPSS that lifts to a map $\Sigma^k E^{hF} \to E^{hF}$ which is, by an abuse of notation, still denoted by a. Then $E_*(a) = E_a$.*

Proof. Since $E_*(a)$ and E_a are maps of $E_*(E^{hF})$-modules, we only need to check this identity on a generator of $E_*(\Sigma^k E^{hF})$. Let $\delta_{\Sigma^k 1} \in \mathrm{Map}(\mathbb{G}/F, \pi_*(\Sigma^k E))$, the constant map with value the element $\Sigma^k 1 \in \pi_k(\Sigma^k E)$, be such a generator. This function is the image of the k-fold suspension of the unit $S^0 \to E \wedge E^{hF}$ via the identification $\Phi : E_* E^{hF} \cong \mathrm{Map}_c(\mathbb{G}/F, E_*)$. The latter is sent to $S^k \xrightarrow{1\wedge a} E \wedge E^{hF}$ by a. And through Φ, it gets identified to the map $(\mathbb{G}/F \to E_*,\ g \mapsto g(a))$, which is equal to the image of $\delta_{\Sigma^k 1}$ by E_a. The conclusion of the lemma follows. □

Definition 11.2.3. Let I be an ideal of $[E_C^{hC_2}, E_C^{hC_2}]$ and X, Y be two spectra. We say that $f, g \in [X, Y]$ are congruent modulo I, denoted by $f \equiv g \bmod I$, if there exist elements $\epsilon, \delta \in [E_C^{hC_2}, E_C^{hC_2}]$ with $\epsilon \equiv \delta \bmod I$ such that

$$f = s\epsilon t \ \text{ and } \ g = s\delta t$$

for certain $t \in [X, E_C^{hC_2}]$ and $s \in [E_C^{hC_2}, Y]$.

Proposition 11.2.4. *The composite $\Sigma^{48} E_C^{hC_6} \xrightarrow{\pi(\Delta^2)} E_C^{hC_6} \xrightarrow{\delta_3} \Sigma^{48} E_C^{hG_{24}}$ is homotopic to $\Sigma^{48}((1+i+j+k)(1-\alpha^{-1})\pi^{-1} + (1+i+j+k)\alpha^{-1}\pi^{-1} \circ (1 - \frac{\pi\Delta^2}{\alpha(\pi\Delta^2)}))$ where the second term is the composite*

$$E_C^{hC_6} \xrightarrow{1-\frac{\pi\Delta^2}{\alpha(\pi\Delta^2)}} E_C^{hC_6} \xrightarrow{(1+i+j+k)\alpha^{-1}\pi^{-1}} E_C^{hG_{24}}.$$

As a consequence,

$$\delta_3 \equiv \Sigma^{48}(tr_{C_6}^{G_{24}} \circ (1-\alpha^{-1}) \circ \pi^{-1}) \circ \Delta^{-2} \textit{ modulo } (4, u_1^8),$$

see Definition 11.2.3.

Proof. Since C_6 satisfies the equivalent conditions of Proposition 10.0.7, following from the computation of the HFPSS for $E_C^{hC_6}$ in Section 6.1, the Hurewicz homomorphism

$$[\Sigma^{48}E_C^{hC_6}, \Sigma^{48}E_C^{hG_{24}}] \to \mathrm{Hom}_{\mathcal{EG}}((E_C)_*E_C^{hC_6}, (E_C)_*(E_C^{hG_{24}})$$

is injective. It is then enough to prove that the two maps in question induce the same maps of Morava modules. Now we compute the induced map of Morava modules of $\delta_3 \circ \pi(\Delta^2)$:

$$E_*(\delta_3 \circ \pi(\Delta^2)) : \mathrm{Hom}_c(\mathbb{Z}_2[[\mathbb{G}/C_6]], (E_C)_{*-48}) \to \mathrm{Hom}_c(\mathbb{Z}_2[[\mathbb{G}/G_{24}]], (E_C)_{*-48}).$$

For any $e \in \mathrm{Map}(\mathbb{G}/C_6, E_{*-48})$ and $g \in \mathbb{G}/G_{24}$, we have that

$$\begin{aligned}
&E_*(\delta_3 \circ \pi(\Delta^2))(e)(g) \\
&= E_{\Delta^2}^{-1} \circ \partial_3^* \circ E_{\pi(\Delta^2)}(e)(g) \ \text{ (because of the equation } (V.11) \text{ and Lemma 11.2.2)} \\
&= g(\Delta^{-2})(\partial_3^* \circ E_{\pi(\Delta^2)}(e)(g)) \\
&= g(\Delta^{-2})(E_{\pi(\Delta^2)}(e)(g(1+i+j+k)(1-\alpha^{-1})\pi^{-1})) \\
&= g(\Delta^{-2})\Big(\sum_{h \in G_{24}/C_6} gh\pi^{-1}\pi(\Delta^2)e(gh\pi^{-1}) \\
&- \sum_{h \in G_{24}/C_6} gh\alpha^{-1}\pi^{-1}\pi(\Delta^2)e(gh\alpha^{-1}\pi^{-1})\Big) \\
&= g(\Delta^{-2})\Big(g(\Delta^2)e(g(1+i+j+k)\pi^{-1}) - \sum_{h \in G_{24}/C_6} gh\alpha^{-1}(\Delta^2)e(gh\alpha^{-1}\pi^{-1})\Big) \\
&= e(g(1+i+j+k)(1-\alpha^{-1})\pi^{-1}) \\
&- \sum_{h \in G_{24}/C_6} gh\alpha^{-1}\pi^{-1}(\frac{\pi(\Delta^2)}{\pi\alpha\Delta^2} - 1)e(gh\alpha^{-1}\pi^{-1}) \\
&= \partial_3^*(e)(g) + ((1+i+j+k)\alpha^{-1}\pi^{-1})^* \circ E_{1-\frac{\pi(\Delta^2)}{\pi\alpha(\Delta^2)}}(e)(g)
\end{aligned}$$

Therefore,

$$E_*(\delta_3 \circ \Delta^{-2}) = \partial_3^* + ((1+i+j+k)\alpha^{-1}\pi^{-1})^* \circ E_{1-\frac{\pi(\Delta^2)}{\pi\alpha(\Delta^2)}}$$

and the right hand side is exactly the E_*-homology of

$$\Sigma^{48}\Big((1+i+j+k)(1-\alpha^{-1})\pi^{-1}+(1+i+j+k)\alpha^{-1}\pi^{-1}\circ(1-\frac{\pi\Delta^2}{\alpha(\pi\Delta^2)})\Big).$$

Thus,

$$\begin{aligned}\delta_3 &\equiv \Sigma^{48}((1-\alpha^{-1})\circ\pi^{-1}\circ tr_{C_6}^{G_{24}})\circ\pi(\Delta^{-2}) \text{ modulo } (8,4u_1^5,u_1^8)\\ &\equiv \Sigma^{48}((1-\alpha^{-1})\circ\pi^{-1}\circ tr_{C_6}^{G_{24}})\circ\Delta^{-2} \text{ modulo } (8,4u_1^5,u_1^8)\end{aligned}$$

The two reductions are because of the equations

$$1-\frac{\pi\Delta^2}{\alpha(\pi\Delta^2)}\equiv 0 \text{ modulo } (4,2u_1,u_1^4)^2\subset(8,4u_1^5,u_1^8)$$

and respectively

$$\pi(\Delta^{-2})\equiv\Delta^{-2} \text{ modulo } (8,4u_1^5,u_1^8),$$

which are in turn due to Lemma 1.5.2. □

Lemma 11.2.5. *The homomorphism*

$$Res:\mathrm{H}^*(G_{24},(E_C)_*(A_1))\to\mathrm{H}^*(C_6,(E_C)_*(A_1))$$

sends $u^{-3k}e_5$ to $u^{-2-3k}te_0$ for all $k\in\mathbb{Z}$.

Proof. We show that $Res(e_5)=u^{-2}te_0$. The class e_5 is represented by the cocycle $G_{24}\to(E_C)_6A_1$, determined by $i\mapsto u^{-3}e_0+u^{-2}e_2$, $\omega\mapsto 0$, where $e_2\in(E_C)_2(A_1)$ introduced in Formula II.10. The restriction of the latter to C_6 is determined by $-1\mapsto u^{-3}e_0$, $\omega\mapsto 0$, which is not a coboundary, hence represents the unique nontrivial class $u^{-2}te_0\in\mathrm{H}^1(C_6,(E_C)_6(A_1))$, up to a factor of $\mathbb{F}_4^\times$. Finally, the restriction map is linear with respect to $\mathbb{F}_4[u^{\pm3}]\subset\mathrm{H}^*(G_{24},\mathbb{F}_4[u^{\pm1}])$. □

Theorem 11.2.6. *The induced map in homotopy of $\delta_3:E_C^{hC_6}\wedge A_1\to E^{hG_{24}}\wedge A_1$ is trivial.*

Proof. Proposition 11.2.4 shows

$$(\delta_3\wedge Id_{A_1})_*=(\Sigma^{48}((1+i+j+k)\circ(1-\alpha^{-1})\circ\pi^{-1})\circ\Delta^{-2}\wedge Id_{A_1})_*.$$

Since multiplication by Δ^{-2} induces a bijection on $\pi_*(E_C^{hC_6}\wedge A_1)$, it is equivalent to show that the composite

$$E_C^{hC_6}\wedge A_1\xrightarrow{\pi^{-1}(1-\alpha^{-1})}E_C^{hC_6}\wedge A_1\xrightarrow{1+i+j+k}E_C^{hG_{24}}\wedge A_1$$

induces a trivial map in homotopy.

We write e_6, e_{12}, e_{20} for any element of $\pi_*(E_C^{hC_6} \wedge A_1)$ detected by $u^{-3}e_0$, $u^{-6}e_0$, $u^{-9}e_2 \in \mathrm{H}^*(C_6, (E_C)_*(A_1))$, respectively. By the same argument as in the proof of Theorem 11.1.5, the map $\pi^{-1} \circ (1 - \alpha^{-1}) : \pi_*(E_C^{hC_6} \wedge A_1) \to \pi_*(E_C^{hC_6} \wedge A_1)$ can only be nontrivial on the elements $\Delta^{2k}e_{12}$ and $\Delta^{2k}e_{20}$ and

$$\pi^{-1} \circ (1 - \alpha^{-1})(\Delta^{2k}e_{12}) = \lambda_1 \nu^2 \Delta^{2k} e_6$$

and

$$\pi^{-1} \circ (1 - \alpha^{-1})(\Delta^{2k}e_{20}) = \lambda_2 \nu \Delta^{2k} x_{17} e_0$$

for some $\lambda_1, \lambda_2 \in \mathbb{F}_4$.

Consider the map $(1 + i + j + k) : E_C^{hC_6} \wedge A_1 \to E^{hG_{24}} \wedge A_1$. Its induced map in cohomology $\mathrm{H}^s(C_6, (E_C)_t A_1) \to \mathrm{H}^s(G_{24}, (E_C)_t(A_1))$ is the cohomological transfer. It is elementary to check that the restriction $\mathrm{H}^s(G_{24}, (E_C)_t A_1) \to \mathrm{H}^s(C_6, (E_C)_t(A_1))$ sends the classes $\Delta^{2k}u^{-3}e_0$ and $\Delta^{2k}u^{-6}e_5$ to the classes detecting $\Delta^{2k}u^{-3}e_0$ and $\Delta^{2k}u^{-6}tu^{-2}e_0$, respectively by Lemma 11.2.5. Since,

$$tr(res(\Delta^{2k}u^{-3}e_0)) = 4\Delta^{2k}u^{-3}e_0 = 0$$

and

$$tr(res(\Delta^{2k}u^{-6}e_5)) = 4\Delta^{2k}u^{-6}e_5 = 0,$$

the induced map in homotopy of $(1 + i + j + k)$ must send $\Delta^{2k}e_6$ and $\Delta^{2k}x_{17}e_0$ to elements detected in filtration at least 1 and 2 in the HFPSS for $E_C^{hG_{24}} \wedge A_1$, respectively. Then, the latter must be detected in filtration at least 4 by sparseness in the E_∞-term of the HFPSS for $E_C^{hG_{24}} \wedge A_1$, hence $(1+i+j+k)_*(\nu\Delta^{2k}e_6)$ and $(1 + i + j + k)_*(\Delta^{2k}x_{17}e_0)$ are divisible by $\overline{\kappa}$. Therefore,

$$(1 + i + j + k)_*(\nu^2\Delta^{2k}e_6) = \nu(1 + i + j + k)_*(\nu\Delta^{2k}e_6) = 0$$

and

$$(1 + i + j + k)_*(\nu\Delta^{2k}x_{17}e_0) = \nu(1 + i + j + k)_*(\Delta^{2k}x_{17}e_0) = 0$$

because $\nu\overline{\kappa} = 0 \in \pi_*(E_C^{hG_{24}})$. □

Remark 11.2.7. In order to complete the analysis of the TDSS for A_1, it remains to study

1. The differential $d_1 : \mathrm{E}_1^{1,p} \to \mathrm{E}_1^{2,p}$ for $p = 12, 20$, see Remark 11.1.6,
2. The differential $d_2 : \mathrm{E}_2^{1,p} \to \mathrm{E}_2^{3,p-1}$ for $p \in \mathbb{Z}$.

A priori, there are many $p \in \mathbb{Z}$ for which the differential $d_2 : \mathrm{E}_2^{1,p} \to \mathrm{E}_2^{3,p-1}$ cannot be ruled out by sparseness. These differentials can be addressed by the following approach that was suggested to me by Agnès Beaudry. Let $E_C^{hG_{24}} \xrightarrow{f} F$ be the cofiber of $\delta_0 : E_C^{h\mathbb{S}_C^1} \to E_C^{hG_{24}}$ in the topological duality resolution I.23. Since $\delta_1 \circ \delta_0 \simeq 0$, the map δ_1 factorises as $E_C^{hG_{24}} \xrightarrow{f} F \xrightarrow{g} E_C^{hC_6}$. The study of the remaining differentials d_1 and d_2 is essentially equivalent to the study of the surjectivity of the induced map in homotopy of $F \wedge A_1 \xrightarrow{g \wedge Id_{A_1}} E_C^{hC_6} \wedge A_1$. For this, we need to understand sufficiently well the map g. However, we do not study this at all in the thesis.

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Let A_1 be any spectrum in a class of finite spectra whose mod 2 cohomology is isomorphic to a free module of rank one over the subalgebra $\mathcal{A}(1)$ of the Steenrod algebra. Let E_C be the second Morava-E theory associated to a universal deformation of the formal completion of the supersingular elliptic curve $(C) : y^2 + y = x^3$ defined over $\mathbb{F}_4$ and $\mathbb{S}_C^1$ the kernel of the reduced determinant $\mathbb{S}_C \to \mathbb{Z}_2$, where $\mathbb{S}_C$ is the Morava stabiliser group. As first steps towards understanding the homotopy type of the $K(2)$-localisation of the 2-local sphere spectrum, we analyse, in this thesis, the topological duality spectral sequence for $E_C^{h\mathbb{S}_C^1} \wedge A_1$, constructed by Irina Bobkova and Paul Goerss. In particular, we compute the E_1-term of the latter and prove that its edge homomorphism is surjective.

Soit A_1 un spectre dans une classe des spectres finis dont la cohomologie modulo 2 est isomorphe à un module libre de rang un sur la sous-algèbre $\mathcal{A}(1)$ de l'algèbre de Steenrod. Soit E_C la seconde E-théorie de Morava associée à une déformation universelle de la complétion formelle de la courbe elliptique supersingulière $(C) : y^2 + y = x^3$ définie sur $\mathbb{F}_4$ et $\mathbb{S}_C^1$ le noyau du morphism de déterminant réduit $\mathbb{S}_C \to \mathbb{Z}_2$, où $\mathbb{S}_C$ est le groupe des stabilisateurs de Morava. Dans la perspective de mieux comprendre le type d'homotopie de la localisation en $K(2)$ du spectre des sphères 2-local, nous analysons, dans cette thèse, la suite spectrale de dualité topologique pour $E_C^{h\mathbb{S}_C^1} \wedge A_1$, construite par Irina Bobkova and Paul Goerss. En particulier, nous calculons le term E_1 de la dernière et montrons que son application du bord est surjective.

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