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LOCALISATION DE REPRÉSENTATIONS LOCALEMENT ANALYTIQUES ADMISSIBLES

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This thesis is dedicated to the memory of:

Gabriel Sarrazola Zapata
Father: your memory will live eternally in the heart
of your wife, your children and grandchildren.

Université de Strasbourg

msii ÉCOLE DOCTORALE

UNIVERSITÉ DE RENNES 1

Andrés SARRAZOLA ALZATE

LOCALISATION DE REPRÉSENTATIONS LOCALEMENT ANALYTIQUES ADMISSIBLES

Résumé :

Soit $\mathbb{G}$ un schéma en groupes réductif, connexe et déployé sur l'anneau d'entiers d'une extension finie L du corps de nombres p-adiques $\mathbb{Q}_p$. Un théorème important dans la théorie des groupes c'est le théorème de localisation, ce qui a été démontré par A. Beilinson et J. Berstein, et par J.L. Brylinsky et M. Kashiwara. Il s'agit d'un résultat de $\mathcal{D}$-affinité pour la variété de drapeaux X_L du groupe algébrique $\mathbb{G}_L$ (la fibre générique de $\mathbb{G}$). En caractéristique mixte un progrès important se trouve dans les travaux de C. Huyghe et T. Schmidt. Ils donnent une réponse partielle en considérant des caractères algébriques. Les premières quatre chapitres de cette thèse sont consacrés à étendre cette correspondance (le théorème de localisation arithmétique) pour des caractères arbitraires.
Dans les chapitres cinq et six, nous traiterons l'objectif principal de cette thèse qui concerne les représentations localement analytiques. Nous montrerons que si λ est un caractère algébrique, tel que $\lambda + \rho$ est de plus dominant et régulier (ρ en étant le caractère de Weyl), alors la catégorie des représentations admissibles localement analytiques du groupe L-analytique $G := \mathbb{G}(L)$, à caractère central λ, c'est équivalente à la catégorie des $\mathscr{D}(\lambda)$-modules arithmétiques coadmissibles G-équivariants sur la famille des modèles formels de la variété de drapeaux rigide de $\mathbb{G}$.

Mots-clés : $\mathscr{D}$-modules arithmétiques, localisation, représentations localement analytiques admissibles, variétés de drapeaux, modèles formels, $\mathscr{D}(\lambda)$-modules arithmétiques coadmissibles G-équivariants.

Summary:

Let $\mathbb{G}$ be a split connected, reductive group scheme over the ring of integers $\mathfrak{o}$ of a finite extension L of the field of p-adic numbers $\mathbb{Q}_p$. An important theorem in group theory is the localization theorem, demonstrated by A. Beilinson and J. Berstein, and by J.L. Brylinsky and M. Kashiwara. This is a result about the $\mathcal{D}$-affinity of the flag variety X_L of the algebraic group $\mathbb{G}_L$ (the generic fiber of $\mathbb{G}$). In mixed characteristic an important progress is found in the work of C. Huyghe and T. Schmidt. They give a partial answer by considering algebraic characters. The first four chapters of this thesis are dedicated to extending this correspondence (the arithmetic localization theorem) for arbitrary characters.
In chapters five and six, we will treat the principal objective of this thesis, which concerns admissible locally analytic representations. We will show that if λ is an algebraic character, such that $\lambda + \rho$ is furthermore dominant and regular (ρ being the Weyl character), then the category of admissible locally analytic representations of the locally L-analytic group $G := \mathbb{G}(L)$, with central character λ, it is equivalent to the category of coadmissible G-equivariant arithmetic $\mathscr{D}(\lambda)$-modules over the family of formal models of the rigid flag variety of $\mathbb{G}$.

Key words: Arithmetic $\mathscr{D}$-modules, localization, admissible locally analytic representations, flag varieties, formal models, coadmissible G-equivariant arithmetic $\mathscr{D}(\lambda)$-modules.

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"The hardest arithmetic to master is that
which enables us to count our blessings."
Eric Hoffer

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Contents

Introduction

Version en Français

Un résultat important dans la théorie des représentations est le théorème de Beilinson-Bernstein [3]. Rappelons brièvement son énoncé : Soit G un groupe algébrique complexe semi-simple et $\mathfrak{g}_{\mathbb{C}} := \mathrm{Lie}(G)$ son algèbre de Lie. Soit $\mathfrak{t}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ une sous-algèbre de Cartan et $\mathfrak{z} \subseteq \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ le centre de l'algèbre enveloppante de $\mathfrak{g}_{\mathbb{C}}$. Pour chaque caractère $\lambda \in \mathfrak{t}_{\mathbb{C}}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ on note $\mathfrak{m}_\lambda \subseteq \mathfrak{z}$ l'idéal maximal correspondant, qui est induit via l'homomorphisme d'Harish-Chandra [22, Theorem 7.4.5]. On définit $\mathcal{U}_\lambda := \mathcal{U}(\mathfrak{g}_{\mathbb{C}})/\mathfrak{m}_\lambda$. Le théorème affirme que si X est la variété de drapeaux de G et $\mathcal{D}_{X,\lambda}$ le faisceau des opérateurs différentiels λ-tordus (cf. [3, Théorème principal]), alors on a une équivalence de catégories $\mathrm{Mod}_{\mathrm{qc}}(\mathcal{D}_{X,\lambda}) \simeq \mathrm{Mod}(\mathcal{U}_\lambda)$ à condition que λ soit un caractère dominant et régulier (2.5.3). Ici $\mathrm{Mod}_{\mathrm{qc}}(\mathcal{D}_{X,\lambda})$ est la catégorie des $\mathcal{D}_{X,\lambda}$-modules qui sont $\mathcal{O}_X$-quasi-cohérents. De plus, dans cette équivalence de catégories, les $\mathcal{D}_{X,\lambda}$-modules qui sont cohérents correspondent aux $\mathcal{U}_\lambda$-modules qui sont de type fini.

Le théorème de Beilinson-Bernstein a été démontré indépendamment par A. Beilinson et J. Bernstein dans [3], et par J-L. Brylinski et M. Kashiwara dans [17]. Il a été un outil important dans la preuve de la conjecture de la multiplicité de Kazhdan-Lusztig [42]. En caractéristique mixte, un progrès important se trouve dans les travaux de C. Huyghe [34, 35] et Huyghe-Schmidt [39]. Dans ce cas, si $\mathfrak{o}$ est l'anneau des entiers d'une extension finie L du corps des nombres p-adiques $\mathbb{Q}_p$ et $\mathbb{G}$ est un groupe réductif, connexe et déployé sur $\mathfrak{o}$, alors ils utilisent des opérateurs différentiels arithmétiques introduits par P. Berthelot dans [6] pour montrer une version arithmétique du théorème de Beilinson-Bernstein pour la variété de drapeaux formelle sur $\mathfrak{o}$. Dans ce contexte, les sections globales de ces opérateurs sont canoniquement isomorphes à une version cristalline de l'algèbre de distribution classique $\mathrm{Dist}(\mathbb{G})$ du schéma en groupes $\mathbb{G}$ (cf. [39, Théorème 3.2.3 (i)] et [38, Proposition 5.3.1]).

Avant de présenter les objets construits et les résultats montrés dans ce travail, nous remarquons tout au long de ce travail, si e est l'indice de ramification de L, alors $e \leq p-1$ (pour plus des détails sur cette condition technique le lecteur est invité à regarder l'exemple 1.1.1 et la proposition 5.3.1 de [38]). Soit $\mathbb{B} \subseteq \mathbb{G}$ un sous-groupe de Borel et $\mathbb{T} \subseteq \mathbb{B}$ un tore maximal et déployé de $\mathbb{G}$. Nous noterons $X := \mathbb{G}/\mathbb{B}$ le schéma de drapeaux associé à $\mathbb{G}$. Notre but sera d'introduire des faisceaux des *opérateurs différentielles tordus*[1] sur le $\mathfrak{o}$-schéma de drapeaux formel $\mathfrak{X}$ et nous montrerons un équivalent arithmétique du théorème de Beilinson-Bernstein, introduit dans le premier paragraphe. Ici le «twist» est fait par rapport à un morphisme d'algèbres $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$, où $\mathrm{Dist}(\mathbb{T})$ est l'algèbre de distribution au sens de [21]. Ces faisceaux sont notés $\mathscr{D}^\dagger_{\mathfrak{X},\lambda}$. En particulier, il existe une base $\mathcal{S}$ de $\mathfrak{X}$ constituée d'ouverts affines, tels que pour chaque $\mathcal{U} \in \mathcal{S}$ nous avons

$$\mathscr{D}^\dagger_{\mathfrak{X},\lambda}|_{\mathcal{U}} \simeq \mathscr{D}^\dagger_{\mathcal{U}}.$$

En d'autres termes, localement nous retrouvons le faisceau des opérateurs différentiels introduits par P. Berthelot [2]. Pour calculer ses sections globales, nous utiliserons la description de $\mathrm{Dist}(\mathbb{G})$, donnée par Huyghe-Schmidt dans [38], comme une limite inductive des $\mathfrak{o}$-algèbres noethériennes $\mathrm{Dist}(\mathbb{G}) = \varinjlim_{m\in\mathbb{N}} D^{(m)}(\mathbb{G})$, telle que pour tout $m \in \mathbb{N}$ nous avons

[1] Parfois on utilisera son équivalent en anglais «twist».

[2] Cette propriété clarifie pourquoi ils sont appelés «tordus».

$D^{(m)}(\mathbb{G}) \otimes_{\mathfrak{o}} L = \mathcal{U}(\mathrm{Lie}(\mathbb{G}) \otimes_{\mathfrak{o}} L)$, l'algèbre enveloppante de $\mathfrak{g}_L := \mathrm{Lie}(\mathbb{G}) \otimes_{\mathfrak{o}} L$. En particulier, chaque caractère $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ induit, via produit tensoriel avec L et l'homomorphisme d'Harish-Chandra, un caractère central $\chi_\lambda : \mathfrak{z} \to L$. Notons $\hat{D}^{(m)}(\mathbb{G})_\lambda$ la complétion p-adique de la réduction centrale $D^{(m)}(\mathbb{G})/(D^{(m)}(\mathbb{G}) \cap \mathrm{Ker}(\chi_{\lambda+\rho}))$ et $D^\dagger(\mathbb{G})_\lambda$ la limite inductive du système $\hat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathfrak{o}} L \to \hat{D}^{(m')}(\mathbb{G})_\lambda \otimes_{\mathfrak{o}} L$. Avant d'énoncer notre premier résultat, considérons le décalage suivant. Tout d'abord, la représentation adjointe [40, I, 7.18] induit une structure de $\mathbb{T}$-module sur $\mathfrak{g} := \mathrm{Lie}(\mathbb{G})$ telle que $\mathfrak{g}$ se décompose de la forme :

$$\mathfrak{g} = \mathrm{Lie}(\mathbb{T}) \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha.$$

Ici $\Lambda \subseteq X(\mathbb{T})$ représente les racines de $\mathbb{G}$ par rapport à $\mathbb{T}$. Nous choisissons un système positif des racines $\Lambda^+ \subseteq \Lambda$ et nous considérons le caractère de Weyl $\rho := \frac{1}{2} \sum_{\alpha \in \Lambda^+} \alpha$. Dans le chapitre 4 nous montrerons le théorème suivant (théorème 4.2.1).

Théorème 1. Soit $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ un caractère de l'algèbre de distribution $\mathrm{Dist}(\mathbb{T})$ tel que $\lambda + \rho \in \mathfrak{t}_L^* := (\mathrm{Lie}(\mathbb{T}) \otimes_{\mathfrak{o}} L)^*$[3] est un caractère dominant et régulier de $\mathfrak{t}_L := \mathrm{Lie}(\mathbb{T}) \otimes_{\mathfrak{o}} L$. Le foncteur sections globales induit une équivalence de catégories entre la catégorie des $\mathscr{D}^\dagger_{\mathfrak{X},\lambda}$-modules cohérents et la catégories des $D^\dagger(\mathbb{G})_\lambda$-modules de présentation finie.

Comme nous expliquerons dans la suite, le théorème est basé sur une version plus fine pour les faisceaux (des opérateurs différentiels tordus de niveau m) $\hat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$. Comme dans le cas classique, le foncteur inverse est déterminé par le foncteur de localisation

$$\mathscr{L}oc^\dagger_{\mathfrak{X},\lambda}(\bullet) := \mathscr{D}^\dagger_{\mathfrak{X},\lambda} \otimes_{D^\dagger(\mathbb{G})_\lambda} (\bullet)$$

avec une définition complètement analogue pour chaque $m \in \mathbb{N}$.

Le chapitre 1 à pour but de fixer quelques constructions arithmétiques (elles sont introduites dans [6], [34] et [38]). Dans le chapitre 2 nous construisons notre faisceau des opérateurs différentiels tordus de niveau m sur le schéma de drapeaux formel $\mathfrak{X}$ sur $\mathfrak{o}$. Pour cela, nous notons $\mathfrak{t} := \mathrm{Lie}(\mathbb{T})$ l'algèbre de Lie du tore $\mathbb{T}$ et $\mathfrak{t}_L := \mathfrak{t} \otimes_{\mathfrak{o}} L$. Ce sont des sous-algèbres de Cartan respectives de $\mathfrak{g}$ et de $\mathfrak{g}_L$. Soit $\mathbf{N}$ le radical unipotent du groupe de Borel $\mathbb{B}$ et considérons les $\mathfrak{o}$-schémas lisses et séparés $\widetilde{X} := \mathbb{G}/\mathbf{N}$ et $X := \mathbb{G}/\mathbb{B}$ (l'espace affine basique et le schéma de drapeaux). La projection canonique $\xi : \widetilde{X} \to X$ est un $\mathbb{T}$-torseur localement trivial pour la topologie de Zariski de X et, comme dans [12], nous considérons *l'algèbre enveloppante de niveau m* du torseur comme le sous-faisceau des $\mathbb{T}$-invariants de $\xi_* \mathcal{D}^{(m)}_{\widetilde{X}}$:

$$\widetilde{\mathcal{D}}^{(m)} := \left(\xi_* \mathcal{D}^{(m)}_{\tilde{X}}\right)^{\mathbb{T}}.$$

Comme nous l'expliquerons, c'est un faisceau de $D^{(m)}(\mathbb{T})$-modules qui localement, sur un ouvert affine $U \subset X$ qui trivialise le torseur, peut être décrit comme le produit tensoriel $\mathcal{D}^{(m)}_X|_U \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{T})$. D'autre part, si $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ est un morphisme de $\mathfrak{o}$-algèbres (qu'on appellera un caractère de $\mathrm{Dist}(\mathbb{T})$) alors, grâce aux propriétés juste annoncées, nous définirons un faisceau *d'opérateurs différentiels arithmétiques tordus sur X* par

$$\mathcal{D}^{(m)}_{X,\lambda} := \widetilde{\mathcal{D}^{(m)}} \otimes_{D^{(m)}(\mathbb{T})} \mathfrak{o}. \tag{1}$$

Ceci définit un modèle entier du faisceau des opérateurs différentiels tordus $\mathcal{D}_\lambda$ sur la variété de drapeaux $X_L := X \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(L)$. Dans la dernière partie du chapitre 2 nous allons explorer quelques propriétés de finitude de la cohomologie des $\mathcal{D}^{(m)}_{X,\lambda}$-modules cohérents. Un cas important est le cas où $\lambda + \rho \in \mathfrak{t}_L^*$ est un caractère dominant et régulier de $\mathfrak{t}_L$. Sous cette hypothèse, les groupes de cohomologie de tout $\mathcal{D}^{(m)}_{X,\lambda}$-module cohérent sont à p-torsion bornée, ce qui est un résultat central dans ce travail. Dans le chapitre 3 nous considérerons la complétion p-adique de (1) que nous désignerons par $\hat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}$, et

[3] Nous notons également par λ le caractère de l'algèbre de Lie $\mathrm{Lie}(\mathbb{T})$ induit par (2.26)

nous étudierons ses propriétés cohomologiques lorsque le caractère $\lambda + \rho \in \mathfrak{t}_L^*$ est dominant et régulier. Finalement, le chapitre 4 est consacré à l'étude du passage à la limite inductive

$$\mathscr{D}^\dagger_{\mathfrak{X},\lambda} := \varinjlim_{m\in\mathbb{N}} \hat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}, \qquad \hat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}} := \hat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda} \otimes_{\mathfrak{o}} L.$$

et à démontrer un théorème de Beilinson-Berstein pour les $\mathscr{D}^\dagger_{\mathfrak{X},\lambda}$-modules arithmétiques (Theorem 4.2.1).

Les travaux développés par C. Huyghe dans [35] et par D. Patel, T. Schmidt et M. Strauch dans [49], [50] et [36] montrent que le théorème arithmétique de Beilinson-Bernstein est un outil important dans le théorème de localisation suivant [36, Theorem 5.3.8] : si $\mathfrak{X}$ note le schéma de drapeaux formel du groupe $\mathbb{G}$, alors le théorème fournit une équivalence de catégories entre la catégorie des représentations admissibles localement analytiques de $G := \mathbb{G}(L)$ (à caractère trivial !) et la catégorie des $\mathscr{D}$-modules arithmétiques coadmissibles G-équivariants (sur la famille des modèles formels de la variété de drapeaux rigide de $\mathbb{G}$). Notre motivation a été d'étudier ce théorème de localisation dans le cas tordu. Pour cela, dans le chapitre 5, nous introduirons un faisceau d'opérateurs différentiels $\mathscr{D}^\dagger_{\mathfrak{X},k,\lambda}$ avec un niveau de congruence $k \in \mathbb{N}$ (définition 5.20). Moralement, nous suivrons la philosophie décrite dans [36] pour introduire un faisceau d'opérateurs différentiels sur chaque éclatement admissible de $\mathfrak{X}$. Plus précisément, si $\mathrm{pr} : \mathfrak{Y} \to \mathfrak{X}$ est un éclatement admissible de $\mathfrak{X}$ et $k >> 0$[4], alors

$$\mathscr{D}^\dagger_{\mathfrak{Y},k,\lambda} := \mathrm{pr}^*\mathscr{D}^\dagger_{\mathfrak{X},k,\lambda} = \mathcal{O}_{\mathfrak{Y}} \otimes_{\mathrm{pr}^{-1}\mathcal{O}_{\mathfrak{X}}} \mathrm{pr}^{-1}\mathscr{D}^\dagger_{\mathfrak{X},k,\lambda} \tag{2}$$

est un faisceau des anneaux sur $\mathfrak{Y}$. Dans ce travail nous considérerons le cas algébrique, c'est-à-dire, $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$. Dans cette situation, λ induit un faisceau inversible $\mathscr{L}(\lambda)$ sur $\mathfrak{Y}$ et $\mathscr{D}^\dagger_{\mathfrak{Y},k,\lambda}$ devient le faisceau des opérateurs différentiels qui agissent sur $\mathscr{L}(\lambda)$. À partir de maintenant nous noterons ce faisceau $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$ pour tenir compte de l'action sur $\mathscr{L}(\lambda)$, et nous supposerons que $\lambda + \rho \in \mathfrak{t}_L^*$ est un caractère dominant et régulier de $\mathfrak{t}_L$. Dans le chapitre 6 6.2 nous démontrerons que le foncteur pr_* induit une équivalence de catégories entre la catégorie des $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-modules cohérents et la catégorie des $\mathscr{D}^\dagger_{\mathfrak{X},k}(\lambda)$-modules cohérents. De plus, nous avons $\mathrm{pr}_*\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) = \mathscr{D}^\dagger_{\mathfrak{X},k}(\lambda)$, ce qui implique notamment que

$$H^0(\mathfrak{Y}, \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)) = H^0(\mathfrak{X}, \mathscr{D}^\dagger_{\mathfrak{X},k}(\lambda)) = D^\dagger(\mathbb{G}(k))_\lambda.$$

Ici, $\mathbb{G}(k)$ est le k-ième sous-groupe de congruence de $\mathbb{G}$. En particulier $H^0(\mathfrak{Y}, \bullet) = H^0(\mathfrak{X}, \bullet) \circ \mathrm{pr}_*$ est un foncteur exact et nous avons le théorème suivant.

Théorème 2. Soit $\mathrm{pr} : \mathfrak{Y} \to \mathfrak{X}$ un éclatement admissible. Supposons que $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ est un caractère algébrique tel que $\lambda+\rho \in \mathfrak{t}_L^*$ est un caractère dominant et régulier de $\mathfrak{t}_L$. Le foncteur $H^0(\mathfrak{Y}, \bullet)$ induit une équivalence entre les catégories des $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-modules cohérents et des $D^\dagger(\mathbb{G}(k))_\lambda$-modules de présentation finie.

Comme dans le théorème précédent, le foncteur inverse est déterminé par le foncteur de localisation suivant :

$$\mathscr{L}oc^\dagger_{\mathfrak{Y},k}(\lambda)(\bullet) := \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{D^\dagger(\mathbb{G}(k))_\lambda} (\bullet).$$

Décrivons maintenant les outils les plus importants dans notre théorème de localisation. Du côté algébrique, nous supposerons d'abord que $G_0 = \mathbb{G}(\mathfrak{o})$ et que $D(G_0, L)$ est l'algèbre de distribution du groupe analytique compact G_0. Le point clé sera de construire une structure d'algèbre de Fréchet-Stein faible sur $D(G_0, L)$ (au sens de [25, Definition 1.2.6]) qui nous permettra de localiser les $D(G_0, L)$-modules coadmissibles (par rapport à cette structure d'algèbre de Fréchet-Stein faible). Pour cela, nous commençons par remarquer que d'après les travaux de Huyghe-Schmidt dans [39] nous pouvons identifier l'algèbre $D^\dagger(\mathbb{G}(k))_\lambda$ avec la réduction centrale $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda$ de l'algèbre des distributions $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)$ (au sens

[4] Cette condition technique est clarifiée dans la proposition 6.1.2

d'Emerton [25]) du groupe rigide analytique $\mathbb{G}(k)^{\circ}$ («the wide open rigid-analytic k-th congruence subgroup» décrit dans la sous-section 6.4.2). On a donc un isomorphisme

$$D^{\dagger}(\mathbb{G}(k))_{\lambda} \xrightarrow{\simeq} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}.$$

De plus, d'après les travaux de Huyghe-Patel-Schmidt-Strauch dans [36], si $\mathcal{C}^{\mathrm{cts}}(G_0, L)_{\mathbb{G}(k)^{\circ}-\mathrm{an}}$ est l'espace des vecteurs localement $\mathbb{G}(k)^{\circ}$-analytiques de l'espace des fonctions continues à valeurs dans L et $D(\mathbb{G}(k)^{\circ}, G_0) := (\mathcal{C}^{\mathrm{cts}}(G_0, L)_{\mathbb{G}(k)^{\circ}-\mathrm{an}})'_b$ est son dual fort, alors nous avons un isomorphisme

$$D(G_0, L) \xrightarrow{\simeq} \varprojlim_{k\in\mathbb{N}} D(\mathbb{G}(k)^{\circ}, G_0)$$

qui définit une structure d'algèbre de Fréchet-Stein faible sur $D(G_0, L)$, telle que

$$D(\mathbb{G}(k)^{\circ}, G_0) = \bigoplus_{g\in G_0/G_k} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})\delta_g. \qquad (3)$$

Ici $G_k := \mathbb{G}(k)(\mathfrak{o})$ est un sous-groupe normal de G_0, la somme directe décrit un ensemble de représentants de la classe de G_k dans G_0 et δ_g est la distribution de Dirac supportée dans g. Nous noterons $\mathcal{C}_{G_0,\lambda}$ la catégorie des $D(G_0, L)$-modules coadmissibles à caractère central λ ($D(G_0, L)_{\lambda}$-modules coadmissibles, où $D(G_0, L)_{\lambda}$ est la réduction centrale).

Or, du côté géométrique, nous considérerons $\mathrm{pr} : \mathfrak{Y} \to \mathfrak{X}$ un éclatement admissible G_0-équivariant tel que le faisceau $\mathscr{L}(\lambda)$ est muni d'une G_0-action qui nous permet de définir une G_0-action à gauche $T_g : \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \to (\rho_g)_*\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$ sur $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$[5], au sens que pour chaque $g, h \in G_0$ nous avons la propriété de cocycle $T_{hg} = (\rho_g)_*T_h \circ T_g$. Nous dirons donc qu'un $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-module cohérent $\mathscr{M}$ est *fortement G_0-équivariant* s'il existe une famille $(\varphi_g)_{g\in G_0}$ d'isomorphismes $\varphi_g : \mathscr{M} \to (\rho_g)_*\mathscr{M}$ de faisceaux de L-espaces vectoriels, qui satisfont les propriétés suivantes (conditions (†)) :

- Pour tout $g, h \in G_0$, nous avons $(\rho_g)_*\varphi_h \circ \varphi_g = \varphi_{hg}$.
- Si $\mathcal{U} \subseteq \mathfrak{Y}$ est sous-ensemble ouvert, $P \in \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)(\mathcal{U})$ et $m \in \mathscr{M}(\mathcal{U})$ alors $\varphi_g(P \bullet m) = T_g(P) \bullet \varphi_g(m)$.
- [6] Pour tout $g \in G_{k+1}$ l'application $\varphi_g : \mathscr{M} \to (\rho_g)_*\mathscr{M}$ est égale à la multiplication par $\delta_g \in \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k))_{\lambda}$.

Un morphisme entre deux $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules fortement G_0-équivariants $(\mathscr{M}, (\varphi^{\mathscr{M}}_g)_{g\in G_0})$ et $(\mathscr{N}, (\varphi^{\mathscr{N}}_g)_{g\in G_0})$ est un morphisme $\psi : \mathscr{M} \to \mathscr{N}$ qui est $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-linéaire et tel que, pour tout $g \in G_0$, on a $\varphi^{\mathscr{N}}_g \circ \psi = (\rho_g)_*\psi \circ \varphi^{\mathscr{M}}_g$. Notons $\mathrm{Coh}(\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda), G_0)$ la catégorie des $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules cohérents qui sont fortement G_0-équivariants. Nous avons le résultat suivant [7] :

Théorème 3. Soit $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ un caractère algébrique tel que $\lambda + \rho \in \mathfrak{t}^*_L$ est un caractère dominant et régulier de $\mathfrak{t}_L$. Les foncteurs $\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)$ et $H^0(\mathfrak{Y}, \bullet)$ induisent des équivalences des catégories entre les catégories des $D(\mathbb{G}(k)^{\circ}, G_0)$-modules de présentation finie (à caractère central λ) et $\mathrm{Coh}(\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda), G_0)$.

Toujours du côté géométrique, considérons l'ensemble $\underline{\mathcal{F}}_{\mathfrak{X}}$ des couples $(\mathfrak{Y}, k)$ tels que $\mathfrak{Y}$ est un éclatement admissible de $\mathfrak{X}$ et $k \geq k_{\mathfrak{Y}}$, où

$$k_{\mathfrak{Y}} := \min_{\mathscr{I}} \min\{k \in \mathbb{N} \mid \varpi^k \in \mathscr{I}\}$$

et $\mathscr{I}$ est un faisceau d'idéaux de $\mathcal{O}_{\mathfrak{X}}$, tel que $\mathfrak{Y} \simeq V(\mathscr{I})$. Cet ensemble est ordonné par la relation $(\mathfrak{Y}', k') \geq (\mathfrak{Y}, k)$ si et seulement si $\mathfrak{Y}'$ est un éclatement admissible de $\mathfrak{Y}$ et $k' \geq k$. Comme il est montré dans [36] le groupe G_0-agit

[5] Ici $g \in G_0$ et $\rho_g : \mathfrak{Y} \to \mathfrak{Y}$ est le morphisme induit par G_0-équivariance

[6] Nous identifions ici $H^0(\mathfrak{Y}, \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda))$ avec $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}$ et nous utilisons le lemme 6.3.3 pour donner un sens à cette condition.

[7] Nous utilisons la relation (3) pour donner du sens à l'affirmation du théorème.

sur $\underline{\mathcal{F}}_{\mathfrak{X}}$ et cette action respecte le niveau de congruence. C'est-à-dire, pour tout couple $(\mathfrak{Y}, k) \in \underline{\mathcal{F}}_{\mathfrak{X}}$ il existe un couple $(\mathfrak{Y}.g, k_{\mathfrak{Y}.g}) \in \underline{\mathcal{F}}_{\mathfrak{X}}$ muni d'un isomorphisme $\rho_g : \mathfrak{Y} \to \mathfrak{Y}.g$ et tel que $k_{\mathfrak{Y}} = k_{\mathfrak{Y}.g}$. Nous dirons donc qu'une famille $\mathscr{M} := (\mathscr{M}_{\mathfrak{Y},k})_{(\mathfrak{Y},k)\in\underline{\mathcal{F}}_{\mathfrak{X}}}$ des $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules cohérentes est un $\mathscr{D}(\lambda)$-module coadmissible G_0-équivariant sur $\underline{\mathcal{F}}_{\mathfrak{X}}$ si pour tout $g \in G_0$, avec morphisme $\rho_g : \mathfrak{Y} \to \mathfrak{Y}.g$, il existe un isomorphisme

$$\varphi_g : \mathscr{M}_{\mathfrak{Y}.g,k} \to (\rho_g)_* \mathscr{M}_{\mathfrak{Y},k}$$

qui satisfait les conditions (†) et tel que, si $(\mathfrak{Y}', k') \geq (\mathfrak{Y}, k)$ avec $\pi : \mathfrak{Y}' \to \mathfrak{Y}$, alors il existe un morphisme de transition $\pi_* \mathscr{M}_{\mathfrak{Y}',k'} \to \mathscr{M}_{\mathfrak{Y},k}$, qui satisfait des conditions de transitivité évidentes. De plus, un morphisme $\mathscr{M} \to \mathscr{N}$ entre deux tels modules est un morphisme $\mathscr{M}_{\mathfrak{Y},k} \to \mathscr{N}_{\mathfrak{Y},k}$ de $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules qui est compatible avec les structures supplémentaires. Nous noterons cette catégorie $\mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$ et pour chaque objet $\mathcal{M} \in \mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$, nous considérerons la limite projective

$$\Gamma(\mathscr{M}) := \varprojlim_{(\mathfrak{Y},k)\in\underline{\mathcal{F}}_{\mathfrak{X}}} H^0(\mathfrak{Y}, \mathscr{M}_{\mathfrak{Y},k})$$

au sens des groupes abéliens.

Or, soit M un $D(G_0, L)_\lambda$-module coadmissible et $V := M'_b$ sa représentation localement analytique associée. L'espace des vecteurs $\mathbb{G}(k)^\circ$-analytiques $V_{\mathbb{G}(k)^\circ-\text{an}} \subseteq V$ est stable sous l'action de G_0 et son dual $M_k := (V_{\mathbb{G}(k)^\circ-\text{an}})'$ est un $D(\mathbb{G}(k)^\circ, G_0)$-module de présentation finie. Dans cette situation, le théorème 3 produit un $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-module cohérent

$$\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(M_k) := \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{\mathcal{D}^{\text{an}}(\mathbb{G}(k)^\circ)_\lambda} M_k$$

pour chaque élément $(\mathfrak{Y}, k) \in \underline{\mathcal{F}}_{\mathfrak{X}}$. Nous noterons cette famille

$$\mathscr{L}oc^{G_0}_{\lambda}(M) := \left(\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(M_k)\right)_{(\mathfrak{Y},k)\in\underline{\mathcal{F}}_{\mathfrak{X}}}.$$

Théorème 4. Soit $\lambda \in \text{Hom}(\mathbb{T}, \mathbb{G}_m)$ un caractère algébrique tel que $\lambda + \rho \in \mathfrak{t}^*_L$ est un caractère dominant et régulier de $\mathfrak{t}_L$. Les foncteurs $\mathscr{L}oc^{G_0}_{\lambda}(\bullet)$ et $\Gamma(\bullet)$ induisent des équivalences des catégories entre la catégorie $\mathcal{C}_{G_0,\lambda}$ (des $D(G_0, L)_\lambda$-modules coadmissibles) et la catégorie $\mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$.

Finalement, la dernière partie de ce travail est consacrée à l'étude de la catégorie des $D(G, L)_\lambda$-modules coadmissibles, où $G := \mathbb{G}(L)$[8]. Pour cela, nous considérerons l'immeuble de Bruhat-Tits $\mathcal{B}$ de G ([18] et [19]). Il s'agit d'un complexe simplicial équipé d'une action de G. Pour tout sommet spécial $v \in \mathcal{B}$, la théorie de Bruhat et Tits associe un groupe réductif $\mathbb{G}_v$ dont fibre générique est canoniquement isomorphe à $\mathbb{G} \times_{\text{Spec}(\mathfrak{o})} \text{Spec}(L)$. Soit X_v le schéma de drapeaux de $\mathbb{G}_v$, et $\mathfrak{X}_v$ sa complétée formelle le long de sa fibre spéciale. Nous considérons l'ensemble $\underline{\mathcal{F}}$ composé des triples $(\mathfrak{Y}_v, k, v)$ tels que v est un sommet spécial, $\mathfrak{Y}_v \to \mathfrak{X}_v$ est un éclatement admissible de $\mathfrak{X}_v$ et $k \geq k_{\mathfrak{Y}_v}$. D'après (6.6.2) $\underline{\mathcal{F}}$ est muni d'une relation d'ordre partiel. De plus, pour chaque sommet spécial $v \in \mathcal{B}$, chaque élément $g \in G$ induit un isomorphisme $\rho^v_g : \mathfrak{X}_v \to \mathfrak{X}_{vg}$, tel que si $(\rho^v_g)^\natural : \mathcal{O}_{\mathfrak{X}_{vg}} \to (\rho^v_g)_* \mathcal{O}_{\mathfrak{X}_v}$ est le comorphisme et $\pi : \mathfrak{Y}_v \to \mathfrak{X}_v$ est un éclatement admissible le long de $V(\mathscr{I})$, alors l'éclatement au long de $V((\rho^v_g)^{-1}(\rho^v_g)_*\mathscr{I})$ produit un schéma $\mathfrak{Y}_{vg}$ muni d'un isomorphisme $\rho^v_g : \mathfrak{Y}_v \to \mathfrak{Y}_{vg}$, tel que $k_{\mathfrak{Y}_v} = k_{\mathfrak{Y}_{vg}}$ et pour tout $g, h \in G$ nous avons $\rho^{vg}_h \circ \rho^v_g = \rho^v_{gh}$.

Un $\mathscr{D}(\lambda)$-module arithmétique coadmissible G-équivariant sur $\underline{\mathcal{F}}$, est une famille $(\mathscr{M}_{(\mathfrak{Y}_v,k,v)})_{(\mathfrak{Y}_v,k,v)\in\mathcal{F}}$ de $\mathscr{D}^{\dagger}_{\mathfrak{Y}_v,k}(\lambda)$-modules cohérents satisfaisant la condition (†) plus certaines propriétés de compatibilité (définition 6.6.4) permettant de former la

[8] Ici G_0 est un sous-goupe (maximal) compact de G. Cette propriété de compacité permet de définir la structure d'algèbre de Fréchet-Stein faible remarquée avant.

limite projective suivante :

$$\Gamma(\mathscr{M}) := \varprojlim_{(\mathfrak{Y}_v,k,v)\in\underline{\mathcal{F}}} H^0(\mathfrak{Y}_v, \mathscr{M}_{(\mathfrak{Y}_v,k,v)}).$$

Cette dernière, comme nous le montrerons, porte une structure de $D(G,L)_\lambda$-module coadmissible. D'autre part, étant donné un $D(G,L)_\lambda$-module coadmissible M, on considère $\mathrm{V} := M'_b$ son dual continu, qui est une représentation localement analytique de G. Soit ensuite $M_{v,k}$ l'espace dual du sous-espace $V_{\mathbb{G}_v(k)^\circ-\mathrm{an}} \subseteq V$ des vecteurs $\mathbb{G}_v(k)^\circ$-analytiques. Pour tout $(\mathfrak{Y}_v, k, v) \in \mathcal{F}$, nous avons le $\mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda)$-module cohérent

$$\mathscr{L}oc^\dagger_{\mathfrak{Y}_v,k}(\lambda)(M_{v,k}) = \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}_v(k)^\circ)_\lambda} M_{v,k}.$$

On note cette famille $\mathscr{L}oc^G_\lambda(M)$. Nous montrerons le résultat suivant (théorème 6.6.6) :

Théorème 5. Soit $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ un caractère algébrique tel que $\lambda + \rho \in \mathfrak{t}^*_L$ est un caractère dominant et régulier de $\mathfrak{t}_L$. Les foncteurs $\mathscr{L}oc^G_\lambda(\bullet)$ et $\Gamma(\bullet)$ donnent des équivalences quasi-inverses entre les catégories des $D(G,L)_\lambda$-modules coadmissibles et des $\mathscr{D}(\lambda)$- modules arithmétiques coadmissibles G-équivariants.

La dernière tâche a consisté à étudier la limite projective

$$X_\infty := \varprojlim_{(\mathfrak{Y}_v,k,v)} \mathfrak{Y}_v.$$

Il s'agit de l'espace de Zariski-Riemann associé à la variété de drapeau rigide $\mathbb{X}^{\mathrm{rig}}$. On peut aussi former la limite projective $\mathscr{D}(\lambda)$ des faisceaux $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$ qui est un faisceau des opérateurs G-équivariants des anneaux p-adiquement complètes sur $\mathfrak{X}_\infty$. De même, si $(\mathscr{M}_{(\mathfrak{Y}_v,k,v)})_{(\mathfrak{Y}_v,k,v)\in\mathcal{F}}$ est un $\mathscr{D}(\lambda)$-module arithmétique coadmissible G-équivariant, alors on peut former la limite projective $\mathscr{M}_\infty$. La donnée $\mathscr{M}_{(\mathfrak{Y}_v,k,v)\in\mathcal{F}} \rightsquigarrow \mathscr{M}_\infty$ induit un foncteur fidèle de la catégorie des $\mathscr{D}(\lambda)$- modules arithmétiques coadmissibles G-équivariants sur $\underline{\mathcal{F}}$ vers la catégorie des $\mathscr{D}(\lambda)$-modules G-équivariants sur $\mathfrak{X}_\infty$ (théorème 6.6.8).

Introduction

English version

An important result in representation theory is the so-called Beilinson-Bernstein theorem [3]. Let us briefly recall its statement. Let G be a semi-simple complex algebraic group and $\mathfrak{g}_{\mathbb{C}} := \text{Lie}(G)$ its Lie algebra. Let $\mathfrak{t}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ be a Cartan subalgebra and $\mathfrak{z} \subseteq \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. For each character $\lambda \in \mathfrak{t}_{\mathbb{C}}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ we denote by $\mathfrak{m}_\lambda \subseteq \mathfrak{z}$ the corresponding maximal ideal, which is induced via the homomorphism of Harish-Chandra [22, Theorem 7.4.5]. We define $\mathcal{U}_\lambda := \mathcal{U}(\mathfrak{g}_{\mathbb{C}})/\mathfrak{m}_\lambda$. The theorem states that if X is the flag variety of G and $\mathcal{D}_{X,\lambda}$ is the sheaf of λ-twisted differential operators [3, 2. Main theorem], then we have an equivalence of categories $\text{Mod}_{\text{qc}}(\mathcal{D}_{X,\lambda}) \simeq \text{Mod}(\mathcal{U}_\lambda)$, provided that λ is a dominant and regular character of $\mathfrak{t}_{\mathbb{C}}$ (2.5.3). Here $\text{Mod}_{\text{qc}}(\mathcal{D}_{X,\lambda})$ is the category of $\mathcal{D}_{X,\lambda}$-modules that are $\mathcal{O}_X$-quasi-coherent. In addition, in this equivalence of categories, coherent $\mathcal{D}_{X,\lambda}$-modules correspond to the $\mathcal{U}_\lambda$-modules that are of the finite type.

The Beilinson-Bernstein theorem was independently demonstrated by A. Beilinson and J. Bernstein in [3], and by J-L. Brylinski and M. Kashiwara in [17]. It has been an important tool in proving Kazhdan-Lusztig's multiplicity conjecture [42]. In mixed characteristic, an important progress can be found in the work of C. Huyghe [34, 35] and Huyghe-Schmidt [39]. In this situation, if $\mathfrak{o}$ is the ring of integers of a finite extension L of the field of p-adic numbers $\mathbb{Q}_p$ and $\mathbb{G}$ is a split conneccted, redactive group scheme over $\mathfrak{o}$, then they use the arithmetic differential operators introduced by P. Berthelot in [6] to show an arithmetic version of the Beilinson-Bernstein theorem for the formal flag $\mathfrak{o}$-scheme $\mathfrak{X}$. In this context, the global sections of these operators are canonically isomorphic to a crystalline version of the classical distribution algebra Dist($\mathbb{G}$) of the group scheme $\mathbb{G}$ (cf. [39, Theorem 3.2.3 (i)] and [38, Proposal 5.3.1]).

Before presenting the objects built and the results shown in this work, we remark for the reader that troughout this work, if e denotes the index of ramification of L, then $e \leq p-1$ (for more details about this technical condition the reader is invited to look at the example 1.1.1 and the proposition 5.3.1 of [38]). Let us take $\mathbb{B} \subseteq \mathbb{G}$ a Borel subgroup and $\mathbb{T} \subseteq \mathbb{B}$ a split maximal torus of $\mathbb{G}$. We will denote by $X := \mathbb{G}/\mathbb{B}$ the flag scheme associated to $\mathbb{G}$. Our major goal will be to introduce sheaves of twisted differential operators on the formal flag $\mathfrak{o}$-scheme $\mathfrak{X}$ and we will show an arithmetic equivalent of the Beilinson-Bernstein theorem, introduced in the first paragraph. Here the twist is made in relation to a morphism of $\mathfrak{o}$-algebras $\lambda : \text{Dist}(\mathbb{T}) \to \mathfrak{o}$, where Dist($\mathbb{T}$) is the sense of [21]. These sheaves are denoted by $\mathscr{D}^\dagger_{\mathfrak{X},\lambda}$. In particular, there is a base $\mathcal{S}$ of $\mathfrak{X}$ made up of open affine subsets, such that for each $\mathcal{U} \in \mathcal{S}$ we have

$$\mathscr{D}^\dagger_{\mathfrak{X},\lambda}|_{\mathcal{U}} \simeq \mathscr{D}^\dagger_{\mathcal{U}}.$$

In other words, locally we find the sheaf of differential operators introduced by P. Berthelot [9]. To calculate its global sections, we will use the description of Dist($\mathbb{G}$), given by Huyghe-Schmidt in [38], as an inductive limit of noetherian $\mathfrak{o}$-algebras $\text{Dist}(\mathbb{G}) = \varinjlim_{m\in\mathbb{N}} D^{(m)}(\mathbb{G})$, such that for every $m \in \mathbb{N}$ we have $D^{(m)}(\mathbb{G}) \otimes_{\mathfrak{o}} L = \mathcal{U}(\text{Lie}(\mathbb{G}) \otimes_{\mathfrak{o}} L)$, the universal enveloping algebra of $\mathfrak{g}_L := \text{Lie}(\mathbb{G}) \otimes_{\mathfrak{o}} L$. In particular, each character $\lambda : \text{Dist}(\mathbb{T}) \to \mathfrak{o}$ induces, via tensor product

[9] This property clarifies why they are called "twisted".

with L and the Harish-Chandra homomorphism, a central character $\chi_\lambda : \mathfrak{z} \to L$. We denote by $\hat{D}^{(m)}(\mathbb{G})_\lambda$ the p-adic completion of the central reduction $D^{(m)}(\mathbb{G})/(D^{(m)}(\mathbb{G}) \cap \mathrm{Ker}(\chi_{\lambda+\rho}))$ and by $D^{\dagger}(\mathbb{G})_\lambda$ the inductive limit of the system $\hat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathfrak{o}} L \to \hat{D}^{(m')}(\mathbb{G})_\lambda \otimes_{\mathfrak{o}} L$. Before stating our first result, let us consider the following shift. First, the adjoint representation [40, I, 7.18] induces a $\mathbb{T}$-module structure on $\mathfrak{g} := \mathrm{Lie}(\mathbb{G})$ such that $\mathfrak{g}$ breaks down as follows

$$\mathfrak{g} = \mathrm{Lie}(\mathbb{T}) \oplus \bigoplus_{\alpha\in\Lambda} \mathfrak{g}_\alpha.$$

Here $\Lambda \subseteq X(\mathbb{T})$ represents the roots of $\mathbb{G}$ respect to $\mathbb{T}$. We choose a positive root system $\Lambda^+ \subseteq \Lambda$ and we consider the Weyl character $\rho := \frac{1}{2}\sum_{\alpha\in\Lambda^+}\alpha$. In chapter 4 we will show the following theorem (theorem 4.2.1).

Theorem 1. Let $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ be a character of the distribution algebra $\mathrm{Dist}(\mathbb{T})$, such that $\lambda + \rho \in \mathfrak{t}_L^* := (\mathrm{Lie}(\mathbb{T}) \otimes_{\mathfrak{o}} L)^*$[10] is a dominant and regular character of $\mathfrak{t}_L := \mathrm{Lie}(\mathbb{T}) \otimes_{\mathfrak{o}} L$. The global sections functor induces an equivalence of categories between the category of coherent $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$ modules and the category of finitely presented $D^{\dagger}(\mathbb{G})_\lambda$-modules.

As we will explain later, the theorem is based on a finer version for the sheaves (of twisted differential operators of level m) $\hat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$. As in the classic case, the inverse functor is determined by the localization functor

$$\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}(\bullet) := \mathscr{D}^{\dagger}_{\mathfrak{X},\lambda} \otimes_{D^{\dagger}(\mathbb{G})_\lambda} (\bullet)$$

with a completely similar definition for each $m \in \mathbb{N}$.

Chapter 1 is dedicated to fixing some arithmetic constructions (they are introduced in [6], [34] and [38]). In Chapter 2 we construct our sheaf of twisted differential operators of level m on the formal flag $\mathfrak{o}$-scheme $\mathfrak{X}$. To do this, we will denote by $\mathfrak{t} := \mathrm{Lie}(\mathbb{T})$ the Lie algebra of the torus $\mathbb{T}$ and by $\mathfrak{t}_L := \mathfrak{t}\otimes_{\mathfrak{o}} L$. These are Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{g}_L$, respectively. Let us consider $\mathbf{N}$ the unipotent radical of the Borel subgroup $\mathbb{B}$ and consider the smooth and separated $\mathfrak{o}$-schems $\widetilde{X} := \mathbb{G}/\mathbf{N}$ and $X := \mathbb{G}/\mathbb{B}$ (the basic affine space and the flag scheme). The canonical projection $\xi : \widetilde{X} \to X$ is a locally trivial $\mathbb{T}$-torsor for the Zariski topology of X. As in [12], we will consider *the enveloping algebra of level m* of the torsor as the subsheaf of $\mathbb{T}$-invariants of $\xi_*\mathcal{D}^{(m)}_{\tilde{X}}$:

$$\widetilde{\mathcal{D}}^{(m)} := \left(\xi_*\mathcal{D}^{(m)}_{\tilde{X}}\right)^{\mathbb{T}}.$$

As we will explain, it is a sheaf of $D^{(m)}(\mathbb{T})$-modules which locally, over an open affine subset $U \subset X$ that trivializes the torsor, can be described as the tensor product $\mathcal{D}^{(m)}_X|_U \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{T})$. On the other hand, if $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ is a morphism of $\mathfrak{o}$-algebras (which we will call a character of $\mathrm{Dist}(\mathbb{T})$) then, thanks to the properties just announced, in section 2.5 we define a sheaf *of twisted arithmetic differential operators* on X by

$$\mathcal{D}^{(m)}_{X,\lambda} := \widetilde{\mathcal{D}^{(m)}} \otimes_{D^{(m)}(\mathbb{T})} \mathfrak{o}. \tag{4}$$

This defines an integer model of the sheaf of twisted differential operators $\mathcal{D}_\lambda$ on the flag variety $X_L := X\times_{\mathrm{Spec}(\mathfrak{o})}\mathrm{Spec}(L)$. The final part of chapter 2 is consecrated to exploring some finite properties of the cohomology of coherent $\mathcal{D}^{(m)}_{X,\lambda}$-modules. An important case is the one where $\lambda + \rho \in \mathfrak{t}_L^*$ is a dominant and regular character of $\mathfrak{t}_L$. Under this assumption, the cohomology groups of any coherent $\mathcal{D}^{(m)}_{X,\lambda}$-module are of bounded p-torsion, which is a central result in this work. In chapter 3 we will consider the p-adic completion of (4) which we will denote by $\hat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}$, and we will study its cohomological properties when the character $\lambda + \rho \in \mathfrak{t}_L^*$ is dominant and regular. Finally, chapter 4 is dedicated to the study of the inductive limit

$$\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda} := \varinjlim_{m\in\mathbb{N}} \hat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}, \qquad \hat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}} := \hat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda} \otimes_{\mathfrak{o}} L.$$

[10] We also denote by λ the character of the Lie algebra $\mathrm{Lie}(\mathbb{T})$ induced by (2.26)

and to demonstrate a Beilinson-Berstein theorem for arithmetic $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules (Theorem 4.2.1).

The work developed by C. Huyghe in [35] and by D. Patel, T. Schmidt and M. Strauch in [49], [50] and [36] shows that the arithmetic Beilinson-Bernstein theorem is an important tool in the following location theorem [36, Theorem 5.3.8]: if $\mathfrak{X}$ denotes the formal flag scheme of the group $\mathbb{G}$, then the theorem provides an equivalence of categories between the category of admissible locally analytic representations of $G := \mathbb{G}(L)$ (with trivial character!) and the category of admissible G-equivariant arithmetic $\mathscr{D}$-modules (on the family of formal models of the rigid flag variety). Our motivation was to study this localization theorem in the twisted case. To do this, in Chapter 5, we will introduce a set of differential operators $\mathscr{D}^{\dagger}_{\mathfrak{X},k,\lambda}$ with a congruence level $k \in \mathbb{N}$ (definition 5.20). Morally, we will follow the philosophy described in [36] to introduce a sheaf of differential operators on each admissible blow-up of $\mathfrak{X}$. More specifically, if $\mathrm{pr} : \mathfrak{Y} \to \mathfrak{X}$ is an admissible blow-up of $\mathfrak{X}$ and $k >> 0$[11], then

$$\mathscr{D}^{\dagger}_{\mathfrak{Y},k,\lambda} := \mathrm{pr}^{*}\mathscr{D}^{\dagger}_{\mathfrak{X},k,\lambda} = \mathcal{O}_{\mathfrak{Y}} \otimes_{\mathrm{pr}^{-1}\mathcal{O}_{\mathfrak{X}}} \mathrm{pr}^{-1}\mathscr{D}^{\dagger}_{\mathfrak{X},k,\lambda} \tag{5}$$

is a sheaf of rings on $\mathfrak{Y}$. In this work we will consider the algebraic case, i.e., $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$. In this situation, λ induces an invertible sheaf $\mathscr{L}(\lambda)$ on $\mathfrak{Y}$ and $\mathscr{D}^{\dagger}_{\mathfrak{Y},k,\lambda}$ becomes the sheaf of differential operators acting on $\mathscr{L}(\lambda)$. From now on, we will denote this sheaf by $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$ to take into account the action on $\mathscr{L}(\lambda)$, and we will assume that $\lambda+\rho \in \mathfrak{t}^{*}_{L}$ is a dominant and regular character of $\mathfrak{t}_L$. In chapter 6 we will demonstrate that the functor pr_* induces an equivalence of categories between the category of coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules and the category of coherent $\mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$-modules. In addition, we have $\mathrm{pr}_*\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) = \mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$, which implies that

$$H^0(\mathfrak{Y}, \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)) = H^0(\mathfrak{X}, \mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda)) = D^{\dagger}(\mathbb{G}(k))_{\lambda}.$$

Here, $\mathbb{G}(k)$ is the k-th congruence subgroup of $\mathbb{G}$. In particular $H^0(\mathfrak{Y}, \bullet) = H^0(\mathfrak{X}, \bullet) \circ \mathrm{pr}_*$ is an exact functor and we have the following theorem.

Theorem 2. Let $\mathrm{pr} : \mathfrak{Y} \to \mathfrak{X}$ be an admissible blow-up. Suppose that $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}^{*}_{L}$ is a dominant and regular character of $\mathfrak{t}_L$. The $H^0(\mathfrak{Y}, \bullet)$ induces an equivalence between the categories of coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules and finitely presented $D^{\dagger}(\mathbb{G}(k))_{\lambda}$-modules.

As in the previous theorem, the inverse functor is determined by the localization functor

$$\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(\bullet) := \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{D^{\dagger}(\mathbb{G}(k))_{\lambda}} (\bullet).$$

Let us now describe the most important tools in our localization theorem. On the algebraic side, we will first assume that $G_0 = \mathbb{G}(\mathfrak{o})$ and that $D(G_0, L)$ is the distribution algebra of the compact analytic group G_0. The key point will be to build a structure of weak Fréchet-Stein algebra on $D(G_0, L)$ (in the sense of [25, Definition 1.2.6]) that will allow us to localize the coadmissible $D(G_0, L)$-modules (relative to this weak Fréchet-Stein structure). To do this, we start by remarking that according to the work developped by Huyghe-Schmidt in [39], we can identify the algebra $D^{\dagger}(\mathbb{G}(k))_{\lambda}$ with the central reduction $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}$ of the algebra of analytic distributions $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})$ (in the sense of Emerton [25]) of the rigid analytic group $\mathbb{G}(k)^{\circ}$ (the wide open rigid-analytic k-th congruence subgroup described in subsection 6.4.2). So we have an isomorphism

$$D^{\dagger}(\mathbb{G}(k))_{\lambda} \xrightarrow{\simeq} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}.$$

Moreover, according to the work developped by Huyghe-Patel-Schmidt-Strauch in [36], if $\mathcal{C}^{\mathrm{cts}}(G_0, L)_{\mathbb{G}(k)^{\circ}-\mathrm{an}}$ is the vector space of locally analytic vectors of the space of continuous L-valued functions, and $D(\mathbb{G}(k)^{\circ}, G_0) := (\mathcal{C}^{\mathrm{cts}}(G_0, L)_{\mathbb{G}(k)^{\circ}-\mathrm{an}})'_b$

[11] This technical condition is clarified in the proposition 6.1.2

is its strong dual, then we have an isomorphism

$$D(G_0, L) \xrightarrow{\simeq} \varprojlim_{k\in\mathbb{N}} D(\mathbb{G}(k)^\circ, G_0)$$

which defines a structure on $D(G_0, L)$ of weak Fréchet-Stein algebra, such that

$$D(\mathbb{G}(k)^\circ, G_0) = \bigoplus_{g\in G_0/G_k} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)\delta_g. \tag{6}$$

Here $G_k := \mathbb{G}(k)(\mathfrak{o})$ is a normal subgroup of G_0, the direct sum runs through a set of representatives of the cosets of G_k in G_0 and δ_g is the Dirac distribution supported in g. We will denote by $\mathcal{C}_{G_0,\lambda}$ the category of coadmssible $D(G_0, L)$-modules with central character λ (coadmissible $D(G_0, L)_\lambda$-modules, where $D(G_0, L)_\lambda$ denotes the central reduction).

Now, on the geometric side, we will consider $\mathrm{pr} : \mathfrak{Y} \to \mathfrak{X}$ a G_0-equivariant admssible blow-up such that the invertible sheaf $\mathscr{L}(\lambda)$ is equipped with a G_0-action that allows us to define a left G_0-action $T_g : \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \to (\rho_g)_*\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$ on $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$[12], in the sense that for every $g, h \in G_0$ we have the cocycle condition $T_{hg} = (\rho_g)_* T_h \circ T_g$. So, we will say that a coherent $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-module $\mathscr{M}$ is *strongly G_0-equivariant* if there is a family $(\varphi_g)_{g\in G_0}$ of isomorphisms $\varphi_g : \mathscr{M} \to (\rho_g)_*\mathscr{M}$ of sheaves of L-vector spaces, which satisfy the following properties (conditions (†)) :

- For every $g, h \in G_0$ we have $(\rho_g)_*\varphi_h \circ \varphi_g = \varphi_{hg}$.
- If $\mathcal{U} \subseteq \mathfrak{Y}$ is an open subset, $P \in \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)(\mathcal{U})$ and $m \in \mathscr{M}(\mathcal{U})$ then $\varphi_g(P \bullet m) = T_g(P) \bullet \varphi_g(m)$.
- [13] For any $g \in G_{k+1}$ the application $\varphi_g : \mathscr{M} \to (\rho_g)_*\mathscr{M}$ is equal to the multiplication by $\delta_g \in \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k))_\lambda$.

A morphism between two strongly G_0-equivariant $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-modules $(\mathscr{M}, (\varphi^{\mathscr{M}}_g)_{g\in G_0})$ and $(\mathscr{N}, (\varphi^{\mathscr{N}}_g)_{g\in G_0})$ is a morphism $\psi : \mathscr{M} \to \mathscr{N}$ which is $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-linear and such that, for every $g \in G_0$, we have $\varphi^{\mathscr{N}}_g \circ \psi = (\rho_g)_*\psi \circ \varphi^{\mathscr{M}}_g$. We denote by $\mathrm{Coh}(\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda), G_0)$ the category of strongly G_0-equivariant $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-modules. We have the following result [14]

Theorem 3. Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$. The functors $\mathscr{L}oc^\dagger_{\mathfrak{Y},k}(\lambda)$ and $H^0(\mathfrak{Y}, \bullet)$ induce equivalences between the categories of finitely presented $D(\mathbb{G}(k)^\circ, G_0)$-modules (with central character λ) and $\mathrm{Coh}(\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda), G_0)$.

Still on the geometric side, let us consider the set $\underline{\mathcal{F}}_{\mathfrak{X}}$ of couples $(\mathfrak{Y}, k)$ such that $\mathfrak{Y}$ is an admissible blow-up of $\mathfrak{X}$ and $k \geq k_{\mathfrak{Y}}$, where

$$k_{\mathfrak{Y}} := \min_{\mathscr{I}} \min\{k \in \mathbb{N} \mid \varpi^k \in \mathscr{I}\}$$

and $\mathscr{I}$ is an ideal subsheaf of $\mathcal{O}_{\mathfrak{X}}$, such that $\mathfrak{Y} \simeq V(\mathscr{I})$. This set is ordered by the relationship $(\mathfrak{Y}', k') \geq (\mathfrak{Y}, k)$ if and only if $\mathfrak{Y}'$ is an admissible blow-up of $\mathfrak{Y}$ and $k' \geq k$. As shown in [36] the group G_0 acts on $\underline{\mathcal{F}}_{\mathfrak{X}}$ and this action respects the congruence level. This means that for every couple $(\mathfrak{Y}, k) \in \underline{\mathcal{F}}_{\mathfrak{X}}$ there is a couple $(\mathfrak{Y}.g, k_{\mathfrak{Y}.g}) \in \underline{\mathcal{F}}_{\mathfrak{X}}$ with an isomorphism $\rho_g : \mathfrak{Y} \to \mathfrak{Y}.g$ and such that $k_{\mathfrak{Y}} = k_{\mathfrak{Y}.g}$. So we will say that a family $\mathscr{M} := (\mathscr{M}_{\mathfrak{Y},k})_{(\mathfrak{Y},k)\in\underline{\mathcal{F}}_{\mathfrak{X}}}$ of coherent $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-modules is a coadmissible G_0-equivariant $\mathscr{D}(\lambda)$-module on $\underline{\mathcal{F}}_{\mathfrak{X}}$ if for any $g \in G_0$, with morphism $\rho_g : \mathfrak{Y} \to \mathfrak{Y}.g$, there is an isomorphism

$$\varphi : \mathscr{M}_{\mathfrak{Y}.g,k} \to (\rho_g)_*\mathscr{M}_{\mathfrak{Y}}$$

[12] Here $g \in G_0$ and $\rho_g : \mathfrak{Y} \to \mathfrak{Y}$ is the morphism induced by G_0-equivariance.

[13] We identify here $H^0(\mathfrak{Y}, \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda))$ with $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda$ and we use lemma 6.3.3 to give sense to this condition.

[14] We use the relationship (6) to give a sense to the assertion of the theorem.

that satisfies the conditions (†) and such that, if $(\mathfrak{Y}', k') \geq (\mathfrak{Y}, k)$ with $\pi : \mathfrak{Y}' \to \mathfrak{Y}$, then there is a transition morphism $\pi_* \mathscr{M}_{\mathfrak{Y}',k'} \to \mathscr{M}_{\mathfrak{Y},k}$ which satisfies obvious transitivity conditions. Moreover, a morphism $\mathscr{M} \to \mathscr{N}$ between two such a modules is a morphism $\mathscr{M}_{\mathfrak{Y},k} \to \mathscr{N}_{\mathfrak{Y},k}$ of $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-modules which is compatible with the additional structures. We will note this category $\mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$ and for every $\mathcal{M} \in \mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$, we will consider the projective limit

$$\Gamma(\mathscr{M}) := \varprojlim_{(\mathfrak{Y},k)\in\underline{\mathcal{F}}_{\mathfrak{X}}} H^0(\mathfrak{Y}, \mathscr{M}_{\mathfrak{Y},k})$$

in the sense of the Abelian groups

Now, let M be a coadmissible $D(G_0, L)_\lambda$-module and $V := M'_b$ its associated locally analytic representation. The vector space of $\mathbb{G}(k)^\circ$-analytic vectors $V_{\mathbb{G}(k)^\circ-\mathrm{an}} \subseteq V$ is stable under the action of G_0 and its dual $M_k := (V_{\mathbb{G}(k)^\circ-\mathrm{an}})'$ is a finitely presented $D(\mathbb{G}(k)^\circ, G_0)$-module. In this situation, theorem 3 produces a coherent $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-module

$$\mathscr{L}oc^\dagger_{\mathfrak{Y},k}(\lambda)(M_k) := \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda} M_k$$

for each element $(\mathfrak{Y}, k) \in \underline{\mathcal{F}}_{\mathfrak{X}}$. We will note this family

$$\mathscr{L}oc^{G_0}_\lambda(M) := \left(\mathscr{L}oc^\dagger_{\mathfrak{Y},k}(\lambda)(M_k)\right)_{(\mathfrak{Y},k)\in\underline{\mathcal{F}}_{\mathfrak{X}}}.$$

Theorem 4. Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$. The functors $\mathscr{L}oc^{G_0}_\lambda(\bullet)$ and $\Gamma(\bullet)$ induce equivalences of categories between the category $\mathcal{C}_{G_0,\lambda}$ (of coadmissible $D(G_0, L)_\lambda$-modules) and the category $\mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$.

Finally, the last part of this work is devoted to the study of coadmissible $D(G, L)_\lambda$-modules, where $G := \mathbb{G}(L)$[15]. To do this, we will consider the Bruhat-Tits building $\mathcal{B}$ of G ([18] and [19]). It is a simplicial complex equipped with a G-action. For any special vertex $v \in \mathcal{B}$, the theory of Bruhat and Tits associates a reductive group $\mathbb{G}_v$ whose generic fiber is canonically isomorphic to $\mathbb{G} \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(L)$. Let X_v be the flag scheme of $\mathbb{G}_v$, and $\mathfrak{X}_v$ its formal completion along its special fiber. We consider the set $\underline{\mathcal{F}}$ composed of triples $(\mathfrak{Y}_v, k, v)$ such that v is a special vertex, $\mathfrak{Y}_v \to \mathfrak{X}_v$ is an admissible blow-up of $\mathfrak{X}_v$ and $k \geq k_{\mathfrak{Y}_v}$. According to (6.6.2) $\underline{\mathcal{F}}$ is partially ordered. In addition, for each special vertex $v \in \mathcal{B}$, each element $g \in G$ induces an isomorphism $\rho^v_g : \mathfrak{X}_v \to \mathfrak{X}_{vg}$, such that if $(\rho^v_g)^\natural : \mathcal{O}_{\mathfrak{X}_{vg}} \to (\rho^v_g)_* \mathcal{O}_{\mathfrak{X}_v}$ is the comorphism map and $\pi : \mathfrak{Y}_v \to \mathfrak{X}_v$ is an admissible blow-up along $V(\mathscr{I})$, then the (admissible) blow-up along $V((\rho^v_g)^{-1}(\rho^v_g)_*\mathscr{I})$ produces a $\mathfrak{Y}_{vg}$ scheme with an isomorphism $\rho^v_g : \mathfrak{Y}_v \to \mathfrak{Y}_{vg}$, such that $k_{\mathfrak{Y}_v} = k_{\mathfrak{Y}_{vg}}$ and for every $g, h \in G$ we have $\rho^{vg}_h \circ \rho^v_g = \rho^v_{gh}$.

A coadmissible G-equivariant arithmetic $\mathscr{D}(\lambda)$-module on $\underline{\mathcal{F}}$, consists of a family $(\mathscr{M}_{(\mathfrak{Y}_v,k,v)})_{(\mathfrak{Y}_v,k,v)\in\mathcal{F}}$ of coherent $\mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda)$-modules satisfying the condition (†) plus some compatibility properties (definition 6.6.4) that allow us to form the projective limit

$$\Gamma(\mathscr{M}) := \varprojlim_{(\mathfrak{Y}_v,k,v)\in\underline{\mathcal{F}}} H^0(\mathfrak{Y}_v, \mathscr{M}_{(\mathfrak{Y}_v,k,v)}).$$

Which, as we will show, has a structure of coadmissible $D(G, L)_\lambda$-module. On the other hand, given a coadmissible $D(G, L)_\lambda$-module M, we consider $V := M'_b$ its continuous dual, which is a locally analytic representation of G. Then let $M_{v,k}$ be the dual space of the subspace $V_{\mathbb{G}_v(k)^\circ-\mathrm{an}} \subseteq V$ of $\mathbb{G}_v(k)^\circ$-analytic vectors. For every $(\mathfrak{Y}_v, k, v) \in \mathcal{F}$, we have a coherent $\mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda)$-module

$$\mathscr{L}oc^\dagger_{\mathfrak{Y}_v,k}(\lambda)(M_{v,k}) = \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}_v(k)^\circ)_\lambda} M_{v,k}.$$

[15]Here G_0 is a (maximal) compact subgroup of G. This compactness property allows to define the structure of weak Fréchet-Stein algebra.

We note this family $\mathscr{L}oc_{\lambda}^{G}(M)$. We will show the following result (theorem 6.6.6).

Theorem 5. Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}_L^*$ is a dominant and regular character of $\mathfrak{t}_L$. The functors $\mathscr{L}oc_{\lambda}^{G}(\bullet)$ and $\Gamma(\bullet)$ give an equivalence between the categories of coadmissible $D(G, L)_{\lambda}$-modules and coadmissible G-equivariant arithmetic $\mathscr{D}(\lambda)$-modules.

The last task was to study the projective limit

$$X_\infty := \varprojlim_{(\mathfrak{Y}_v,k,v)} \mathfrak{Y}_v.$$

This is the Zariski-Riemann space associated to the rigid flag variety $\mathbb{X}^{\mathrm{rig}}$. We can also form the projective limit $\mathscr{D}(\lambda)$ of the sheaves $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$ which is a sheaf of G-equivariant differential operators on $\mathfrak{X}_\infty$. Similarly, if $(\mathscr{M}_{(\mathfrak{Y}_v,k,v)})_{(\mathfrak{Y}_v,k,v)\in\mathcal{F}}$ is a coadmissible G-equivariant arithmetic $\mathscr{D}(\lambda)$-module, then we can form the projection limit $\mathscr{M}_\infty$. The data $\mathscr{M}_{(\mathfrak{Y}_v,k,v)\in\mathcal{F}} \rightsquigarrow \mathscr{M}_\infty$ induces a faithful functor from the category of coadmissible G-equivariant arithmetic $\mathscr{D}(\lambda)$-modules on $\underline{\mathcal{F}}$ to the category of G-equivariant $\mathscr{D}(\lambda)$-modules on $\mathfrak{X}_\infty$ (theorem 6.6.8).

Chapter 1

Arithmetic definitions

In this chapter we will describe the arithmetic objects on which the definitions and constructions of our work are based. We will give their functorial constructions and we will enunciate their most remarkable properties. For a more detailed approach, the reader is invited to take a look to the references [47], [34], [38] and [6].

1.1 Partial divided power structures of level m

Let $p \in \mathbb{Z}$ be a prime number. In this subsection $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ with respect to the prime ideal (p). We start recalling the following definition [8, Definition 3.1].

Definition 1.1.1. *Let A be a commutative ring and $I \subset A$ an ideal. By a structure of divided powers on I we mean a collection of maps $\gamma_i : I \to A$ for all integers $i \geq 0$, such that*

(i) For all $x \in I$, $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and $\gamma_i(x) \in I$ if $i \geq 2$.

(ii) For $x, y \in I$ and $k \geq 1$ we have $\gamma_k(x + y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$.

(iii) For $a \in A$ and $x \in I$ we have $\gamma_k(ax) = a^k\gamma_k(x)$.

(iv) For $x \in I$ we have $\gamma_i(x)\gamma_j(x) = ((i, j))\gamma_{i+j}(x)$, where $((i, j)) := (i + j)!(i!)^{-1}(j!)^{-1}$.

(v) We have $\gamma_p(\gamma_q(x)) = C_{p,q}\gamma_{pq}(x)$, where $C_{p,q} := (pq)!(p!)^{-1}(q!)^{-p}$.

Throughout this work we will use the terminology: "(I, γ) is a PD-ideal", "(A, I, γ) is a PD-ring" and "γ is a PD-structure on I". Moreover, we say that $\phi : (A, I, \gamma) \to (B, J, \delta)$ is a *PD-homomorphism* if $\phi : A \to B$ is a homorphism of rings such that $\phi(I) \subset J$ and $\delta_k \circ \phi|_I = \phi \circ \gamma_k$, for every $k \geq 0$.

Example 1.1.1. *[8, Section 3, Examples 3.2 (3)] Let $\mathfrak{o}$ be a discrete valuation ring of unequal characteristic $(0, p)$ and uniformizing parameter ϖ. Let us write $p = u\varpi^e$, with u a unit of $\mathfrak{o}$ and e a positive integer (called the absolute ramification index of $\mathfrak{o}$). Then $\gamma_k(x) := x^k/k!$ defines a PD-structure on (ϖ) if and only if $e \leq p - 1$. In particular, we dispose of PD-structure on $(p) \subset \mathbb{Z}_{(p)}$. We let $x^{[k]} := \gamma_k(x)$ and we denote by $((p), [\ ])$ this PD-ideal.*

Let us fix a positive integer $m \in \mathbb{Z}$. For the next terminology we will always suppose that (A, I, γ) is a $\mathbb{Z}_{(p)}$-PD-algebra whose PD-structure is compatible (in the sense of [6, subsection 1.2]) with the PD-structure induced by $((p), [\ ])$ (we recall to the reader that the PD-structure $((p), [\ ])$ always *extends* to a PD-structure on any $\mathbb{Z}_{(p)}$-algebra [8, proposition 3.15]).

Definition 1.1.2. *Let m be a positive integer. Let A be a $\mathbb{Z}_{(p)}$-algebra and $I \subset A$ an ideal. We call a m-PD-structure on I a PD-ideal $(J, \gamma) \subset A$ such that $I^{(p^m)} + pI \subset J$, where $I^{(p^m)}$ is the ideal generated by the powers x^{p^m} with $x \in I$.*

We will say that (I,J,γ) is a m-PD-ideal of A. Moreover, we say that $\phi : (A,I,J,\gamma) \to (A',I',J',\gamma')$ is a m-PD-morphism if $\phi : A \to A'$ is a ring morphism such that $\phi(I) \subset I'$, and such that $\phi : (A,J,\gamma) \to (A',J',\gamma')$ is a PD-morphism.
For every $k \in \mathbb{N}$ we denote by $k = p^m q + r$ the Euclidean division of k by p^m, and for every $x \in I$ we define $x^{\{k\}_{(m)}} := x^r(\gamma_q(x^{p^m}))$. We remark for the reader that the relation $q!\gamma_q(x) = x^q$ (which is an easy consequence of (i) and (iv) of definition 1.1.1) implies that $q!x^{\{k\}_{(m)}} = x^k$.
On the other side, the m-PD-structure (I,J,γ) allows us to define an increasing filtration $(I^{\{n\}})_{n\in\mathbb{N}}$ on the ring A called the *m-PD-filtration*. This is the finer filtration, which satisfies these following properties [7, 1.3]:

(i) $I^{\{0\}} = A$, $I^{\{1\}} = I$.

(ii) For every $n \geq 1$, $x \in I^{\{n\}}$ and $k \geq 0$ we have $x^{\{k\}} \in I^{\{kn\}}$.

(iii) For every $n \geq 0$, $(J + pA) \cap I^{\{n\}}$ is a PD-subideal of $(J + pA)$.

Proposition 1.1.3. *[8, proposition 1.4.1] Let us suppose that R is a $\mathbb{Z}_{(p)}$-algebra endowed with a m-PD-structure $(\mathfrak{a},\mathfrak{b},\alpha)$. Let A be a R-algebra and $I \subset A$ an ideal. There exists an R-algebra $P_{(m)}(I)$, an ideal $\overline{I} \subset P_{(m)}(I)$ endowed with a m-PD-structure $(\tilde{I},[\ ])$ compatible with $(\mathfrak{b},\alpha)$, and a ring homomorphism $\phi : A \to P_{(m)}(I)$ such that $\phi(I) \subset \overline{I}$. Moreover, $(P_{(m)}(I),\overline{I},\tilde{I},[\ ],\phi)$ satisfies the following universal property: for every R-homomorphism $f : A \to A'$ sending I to an ideal I' which is endowed with a m-PD-structure (J',γ') compatible with $(\mathfrak{b},\alpha)$, there exists a unique m-PD-morphism $g : (P_{(m)}(I),\overline{I},\tilde{I},[\ ]) \to (A',I',J',\gamma')$ such that $g\circ\phi = f$.*

Definition 1.1.4. *Under the hypothesis of the preceding proposition, we call the R-algebra $P_{(m)}(I)$, endowed with the m-PD-ideal $(\bar{I},\tilde{I},[\ ])$, the m-PD-envelope of (A,I).*

Finally, if we endow $P^n_{(m)}(I) := P_{(m)}(I)/\overline{I}^{\{n+1\}}$ with the quotient m-PD-structure [6, 1.3.4] we have

Corollary 1.1.5. *[6, Corollary 1.4.2] Under the hypothesis of the preceding proposition, there exists an R- algebra $P^n_{(m)}(I)$ endowed with a m-PD-structure $(\overline{I},\tilde{I},[\ ])$ compatible with $(\mathfrak{b},\alpha)$ and such that $\overline{I}^{\{n+1\}} = 0$. Moreover, there exists an R-homomorphism $\phi_n : A \to P^n_{(m)}(I)$ such that $\phi(I) \subset \overline{I}$, and universal for the R-homomorphisms $A \to (A',I',J',\gamma')$ sending I into a m-PD-ideal I' compatible with $(\mathfrak{b},\alpha)$ and such that $I'^{\{n+1\}} = 0$.*

1.2 Arithmetic differential operators

Let us suppose that $\mathfrak{o}$ is endowed with the m-PD-structure $(\mathfrak{a},\mathfrak{b},[\ ])$ defined in example 1.1.1. Let X be a smooth $\mathfrak{o}$-scheme and $\mathcal{I} \subset \mathcal{O}_X$ a quasi-coherent ideal. The presheaves (defined over a basis of affine open subsets of X)

$$U \subseteq X \mapsto P_{(m)}(\Gamma(U,\mathcal{I})) \quad \text{and} \quad U \subseteq X \mapsto P^n_{(m)}(\Gamma(U,\mathcal{I}))$$

are sheaves of quasi-coherent $\mathcal{O}_X$-modules which we denote by $\mathcal{P}_{(m)}(\mathcal{I})$ and $\mathcal{P}^n_{(m)}(\mathcal{I})$, respectively. In a completely analogous way, we can define a canonical ideal $\mathcal{I}$ of $\mathcal{P}_{(m)}(\mathcal{I})$, a sub-PD-ideal $(\tilde{\mathcal{I}},[\ ]) \subset \bar{\mathcal{I}}$, and the sequence of ideals $(\bar{\mathcal{I}}^{\{n\}})_{n\in\mathbb{N}}$ defining the m-PD-filtration. Those are also quasi-coherent sheaves on X [6, subsection 1.4].
Now, let us consider the diagonal embedding $\Delta : X \hookrightarrow X \times_{\mathfrak{o}} X$ and let $W \subset X \times_{\mathfrak{o}} X$ be an open subset such that $X \subset W$ is a closed subset defined by a quasi-coherent sheaf $\mathcal{I} \subset \mathcal{O}_W$. For every $n \in \mathbb{N}$, the algebra $\mathcal{P}^n_{X,(m)} := \mathcal{P}^n_{(m)}(\mathcal{I})$ is quasi-coherent and its support is contained in X. In particular, it is independent of the open subset W [6, 2.1]. Moreover, by proposition 1.1.3 the projections $p_1, p_2 : X \times_{\mathfrak{o}} X \to X$ induce two morphisms $d_1, d_2 : \mathcal{O}_X \to \mathcal{P}^n_{X,(m)}$ endowing $\mathcal{P}^n_{X,(m)}$ of a *left* and a *right* structure of $\mathcal{O}_X$-algebra, respectively.

Definition 1.2.1. *Let m, n be positive integers. The sheaf of differential operators of level m and order less or equal to n on X is defined by*

$$\mathcal{D}^{(m)}_{X,n} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X,(m)}, \mathcal{O}_X).$$

If $n \leq n'$ corollary 1.1.5 gives us a canonical surjection $\mathcal{P}^{n'}_{X,(m)} \to \mathcal{P}^n_{X,(m)}$ which induces the injection $\mathcal{D}^{(m)}_{X,n} \hookrightarrow \mathcal{D}^{(m)}_{X,n'}$ and the sheaf of *differential operators of level m* is defined by

$$\mathcal{D}^{(m)}_X := \bigcup_{n\in\mathbb{N}} \mathcal{D}^{(m)}_{X,n}.$$

We remark for the reader that by definition $\mathcal{D}^{(m)}_X$ is endowed with a natural filtration called the *order filtration*, and like the sheaves $\mathcal{P}^n_{X,(m)}$, the sheaves $\mathcal{D}^{(m)}_{X,n}$ are endowed with two natural structures of $\mathcal{O}_X$-modules. Moreover, the sheaf $\mathcal{D}^{(m)}_X$ acts on $\mathcal{O}_X$: if $P \in \mathcal{D}^{(m)}_{X,n}$, then this action is given by the composition $\mathcal{O}_X \xrightarrow{d_1} \mathcal{P}^n_{X,(m)} \xrightarrow{P} \mathcal{O}_X$.
Finally, let us give a local description of $\mathcal{D}^{(m)}_{X,n}$. Let U be a smooth open affine subset of X endowed with a family of local coordinates $x_1, \dots, x_N$. Let $dx_1, \dots, dx_N$ be a basis of $\Omega_X(U)$ and $\partial_{x_1}, \dots, \partial_{x_N}$ a basis of $\mathcal{T}_X(U)$ (as usual, $\mathcal{T}_X$ and Ω_X denote the tangent and cotangent sheaf on X, respectively). Let $\underline{k} \in \mathbb{N}^N$. Let us denote by $|\underline{k}| = \sum_{i=1}^N k_i$ and $\partial_i^{[k_i]} = \partial_{x_i}/k_i!$ for every $1 \leq i \leq N$. Then, using multi-index notation, we have $\underline{\partial}^{[\underline{k}]} = \prod_{i=1}^N \partial_i^{[k_i]}$ and $\underline{\partial}^{<\underline{k}>} = q_{\underline{k}}!\underline{\partial}^{[\underline{k}]}$. In this case, the sheaf $\mathcal{D}^{(m)}_{X,n}$ has the following description on U

$$\mathcal{D}^{(m)}_{X,n}(U) = \left\{ \sum_{|\underline{k}|\leq n} a_{\underline{k}}\underline{\partial}^{<\underline{k}>} \mid a_{\underline{k}} \in \mathcal{O}_X(U) \text{ and } \underline{k} \in \mathbb{N}^N \right\}. \tag{1.1}$$

1.3 Symmetric algebra of finite level

In this subsection we will focus on introducing the constructions in [34]. Let X be an $\mathfrak{o}$-scheme, $\mathcal{L}$ a locally free module of finite rank on X, $\mathbf{S}_X(\mathcal{L})$ the symmetric algebra associated to $\mathcal{L}$ and $\mathcal{I}$ the ideal of homogeneous elements of degree 1. Using the notation of the first section we define

$$\Gamma_{X,(m)}(\mathcal{L}) := \mathcal{P}_{\mathbf{S}_X(\mathcal{L}),(m)}(\mathcal{I}) \quad \text{and} \quad \Gamma^n_{X,(m)}(\mathcal{L}) := \Gamma_{X,(m)}(\mathcal{L})/\bar{\mathcal{I}}^{\{n+1\}}. \tag{1.2}$$

Those algebras are graded [34, Proposition 1.3.3], and if $\eta_1, ..., \eta_N$ is a local basis of $\mathcal{L}$, we have

$$\Gamma^n_{X,(m)}(\mathcal{L}) = \bigoplus_{|\underline{l}|\leq n} \mathcal{O}_X \underline{\eta}^{\{\underline{l}\}}.$$

As before $\underline{\eta}^{\{\underline{l}\}} = \prod_{i=1}^N \eta_i^{\{l_i\}}$ and $q_i!\eta_i^{\{l_i\}} = \eta^{l_i}$. We define by duality

$$\mathrm{Sym}^{(m)}(\mathcal{L}) := \bigcup_{k\in\mathbb{N}} \mathcal{H}om_{\mathcal{O}_X}\left(\Gamma^k_{X,(m)}, \mathcal{O}_X\right),$$

By [34, Propositions 1.3.1, 1.3.3 and 1.3.6] we know that $\mathrm{Sym}^{(m)}(\mathcal{L}) = \oplus_{n\in\mathbb{N}}\mathrm{Sym}^{(m)}_n(\mathcal{L})$ is a commutative graded algebra with noetherian sections over any open affine subset. Moreover, locally over a basis $\eta_1, ..., \eta_N$ we have the following description

$$\mathrm{Sym}_n^{(m)}(\mathcal{L}) = \bigoplus_{|\underline{l}|=n} \mathcal{O}_X \underline{\eta}^{<\underline{l}>}, \quad \text{where} \quad \frac{l_i}{q_i!}\eta_i^{<l_i>} = \eta_i^{l_i}.$$

Remark 1.3.1. *By [47, A.10] we have that Sym*$^{(0)}(\mathcal{L})$ *is the symmetric algebra of* $\mathcal{L}$*, which justifies the terminology.*

We end this subsection by remarking the following results from [35]. Let $\mathcal{I}$ be the kernel of the comorphism $\Delta^\sharp$ of the diagonal embedding $\Delta : X \to X \times_{\mathrm{Spec}(\mathfrak{o})} X$. In [34, Proposition 1.3.7.3] Huyghe shows that the graded algebra associated to the m-PD-adic filtration of $\mathcal{P}_{X,(m)}$ it is identified with the graded m-PD-algebra $\Gamma_{X,(m)}(\mathcal{I}/\mathcal{I}^2) = \Gamma_{X,(m)}(\Omega_X^1)$. More exactly, we dispose of a canonical morphism of $\mathcal{O}_X$-algebras

$$\mathbf{S}_X(\Omega_X) \to gr_\bullet \mathcal{P}_{X,(m)}$$

which extends, via universal property of the divided power, to a morphism $\Gamma^n_{X,(m)}(\Omega^1_X) \xrightarrow{\simeq} gr_\bullet(\mathcal{P}^n_{X,(m)})$. By definition, it induces a graded morphism

$$\mathrm{Sym}^{(m)}(\mathcal{T}_X) \xrightarrow{\simeq} gr_\bullet \mathcal{D}_X^{(m)} \tag{1.3}$$

which is in fact an isomorphism of $\mathcal{O}_X$-algebras.

1.4 Arithmetic distribution algebra of finite level

As in the introduction, let us consider $\mathbb{G}$ a split connected reductive group scheme over $\mathfrak{o}$ and $m \in \mathbb{N}$ fixed. We propose to give a description of the algebra of (arithmetic) distributions of level m introduced in [38]. Let I denote the kernel of the surjective morphism of $\mathfrak{o}$-algebras $\epsilon_{\mathbb{G}} : \mathfrak{o}[\mathbb{G}] \to \mathfrak{o}$, given by the identity element of $\mathbb{G}$. We know that I/I^2 is a free $\mathfrak{o} = \mathfrak{o}[\mathbb{G}]/I$-module of finite rank. Let $t_1, \ldots, t_l \in I$ such that modulo I^2, these elements form a basis of I/I^2. The m-divided power enveloping of $(I, \mathfrak{o}[\mathbb{G}])$ (proposition 1.1.3) denoted by $P_{(m)}(\mathbb{G})$, is a free $\mathfrak{o}$-module with basis the elements $\underline{t}^{\{\underline{k}\}} = t_1^{\{k_1\}} \ldots t_l^{\{k_l\}}$, where $q_i! t_i^{\{k_i\}} = t_i^{k_i}$, for every $k_i = p^m q_i + r_i$ and $0 \le r_i < p^m$ [6, 1.3.5.2]. These algebras are endowed with a decreasing filtration by ideals $I^{\{n\}}$ (subsection 1.1), such that $I^{\{n\}} = \oplus_{|\underline{k}| \ge n} \mathfrak{o}\ \underline{t}^{\{\underline{k}\}}$. The quotients $P^n_{(m)}(\mathbb{G}) := P_{(m)}(\mathbb{G})/I^{\{n+1\}}$ are therefore $\mathfrak{o}$-modules generated by the elements $\underline{t}^{\{\underline{k}\}}$ with $|\underline{k}| \le n$. Moreover, there exists an isomorphism of $\mathfrak{o}$-modules

$$P^n_{(m)}(\mathbb{G}) \simeq \bigoplus_{|\underline{k}| \le n} \mathfrak{o}\ \underline{t}^{\{\underline{k}\}}.$$

Corollary 1.1.5 gives us for any two integers n, n' such that $n \le n'$ a canonical surjection $\pi^{n',n} : P^{n'}_{(m)}(\mathbb{G}) \to P^n_{(m)}(\mathbb{G})$. Moreover, for every $m' \ge m$, the universal property of the divided powers gives us a unique morphism of filtered $\mathfrak{o}$-algebras $\psi_{m,m'} : P_{(m')}(\mathbb{G}) \to P_{(m)}(\mathbb{G})$ which induces a homomorphism of $\mathfrak{o}$-algebras $\psi^n_{m,m'} : P^n_{(m')}(\mathbb{G}) \to P^n_{(m)}(\mathbb{G})$. The module of distributions of level m and order n is $D^{(m)}_n(\mathbb{G}) := \mathrm{Hom}(P^n_{(m)}(\mathbb{G}), \mathfrak{o})$. *The algebra of distributions of level m* is

$$D^{(m)}(\mathbb{G}) := \varinjlim_n D^{(m)}_n(\mathbb{G}),$$

where the limit is formed respect to the maps $\mathrm{Hom}_{\mathfrak{o}}(\pi^{n',n}, \mathfrak{o})$. The multiplication is defined as follows. By universal property (Corollary 1.1.5) there exists a canonical application $\delta^{n,n'} : P^{n+n'}_{(m)}(\mathbb{G}) \to P^n_{(m)}\mathbb{G}) \otimes_{\mathfrak{o}} P^{n'}_{(m)}(\mathbb{G})$. If $(u, v) \in D^{(m)}_n(\mathbb{G}) \times$

$D^{(m)}_{n'}(\mathbb{G})$, we define $u.v$ as the composition

$$u.v\,:\,P^{n+n'}_{(m)}(\mathbb{G})\xrightarrow{\delta^{n,n'}}P^{n}_{(m)}(\mathbb{G})\otimes_{\mathfrak{o}}P^{n'}_{(m)}(\mathbb{G})\xrightarrow{u\otimes v}\mathfrak{o}.$$

Let us denote by $\mathfrak{g}\,:=\mathrm{Hom}_{\mathfrak{o}}(I/I^2,\mathfrak{o})$ the Lie algebra of $\mathbb{G}$. This is a free $\mathfrak{o}$-module with basis $\xi_1,\,\dots,\xi_l$ defined as the dual basis of the elements $t_1,\,\dots,t_l$. Moreover, if for every multi-index $\underline{k}\in\mathbb{N}^l$, $|\underline{k}|\le n$, we denote by $\underline{\xi}^{<\underline{k}>}$ the dual of the element $\underline{t}^{\{\underline{k}\}}\in P^n_{(m)}(\mathbb{G})$, then $D^{(m)}_n(\mathbb{G})$ is a free $\mathfrak{o}$-module of finite rank with a basis given by the elements $\underline{\xi}^{<\underline{k}>}$ with $|\underline{k}|\le n$ [38, proposition 4.1.6].

Remark 1.4.1. [1] *Let A be an $\mathfrak{o}$-algebra and E a free A-module of finite rank with base $(x_1,...,x_N)$. Let $(y_1,...,y_N)$ be the dual base of $E^\vee\,:=Hom_A(E,A)$. As in the preceding subsection, let $\boldsymbol{S}(E^\vee)$ be the symmetric algebra and $\boldsymbol{I}(E^\vee)$ the augmentation ideal. Let $\Gamma_{(m)}(E^\vee)$ be the m-PD-envelope of $(\boldsymbol{S}(E^\vee),\,\boldsymbol{I}(E^\vee))$. As usual we put $\Gamma^n_{(m)}(E^\vee)\,:=\Gamma_{(m)}(E^\vee)/\overline{I}^{\{n+1\}}$. These are free A-modules with base $y_1^{\{k1\}}\,\dots\,y_N^{\{k_N\}}$ with $\sum k_i\,\le\,n$ [35, 1.1.2]. Let $\{\underline{x}^{<\underline{k}>}\}_{|\underline{k}|\le n}$ be the dual base of $Hom_A(\Gamma^n_{(m)}(E^\vee),A)$. We define*

$$Sym^{(m)}(E)\,:=\bigcup_{n\in\mathbb{N}}Hom_A\left(\Gamma^n_{(m)}(E^\vee),A\right).$$

This is a free A-module with a base given by all the $\underline{x}^{<\underline{k}>}$. Moreover, it also has a structure of algebra defined as follows. By [35, Proposition 1.3.1] there exists an application $\Delta_{n,n'}\,:\,\Gamma^{n+n'}_{(m)}(E^\vee)\to\Gamma^n_{(m)}(E^\vee)\otimes_A\Gamma^{n'}_{(m)}(E^\vee)$, which allows to define the product of $u\in Hom_A(\Gamma^n_{(m)}(E^\vee),A)$ and $v\in Hom_A(\Gamma^{n'}_{(m)}(E^\vee),A)$ by the composition

$$u.v\,:\,\Gamma^{n+n'}_{(m)}(E^\vee)\xrightarrow{\Delta_{n,n'}}\Gamma^n_{(m)}(E^\vee)\otimes_A\Gamma^{n'}_{(m)}(E^\vee)\xrightarrow{u\otimes v}A.$$

This maps endows $Sym^{(m)}(E)$ of a structure of a graded noetherian A-algebra [35, Propositions 1.3.1, 1.3.3 and 1.3.6].

We have the following important properties [38, Proposition 4.1.15].

Proposition 1.4.2. *(i) There exists a canonical isomorphism of graded $\mathfrak{o}$-algebras $gr_\bullet(D^{(m)}(\mathbb{G}))\simeq Sym^{(m)}(\mathfrak{g})$.*

(ii) The $\mathfrak{o}$-algebras $gr_\bullet(D^{(m)}(\mathbb{G}))$ and $D^{(m)}(\mathbb{G})$ are noetherian.

1.5 Integral models

In this section we will assume that X is a smooth $\mathfrak{o}$-scheme endowed with a right $\mathbb{G}$-action.

Definition 1.5.1. *Let A be an L-algebra (resp. a sheaf of L-algebras). We say that an $\mathfrak{o}$-subalgebra A_0 (resp. a subsheaf of $\mathfrak{o}$-algebras) is an integral model of A if $A_0\otimes_{\mathfrak{o}}L=A$.*

Remark 1.5.2. *Let us recall that throughout this paper $\mathfrak{g}$ denotes the Lie algebra of a split connected reductive group $\mathfrak{o}$-scheme $\mathbb{G}$ and $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. As we have remarked in the introduction, if $\mathfrak{g}_L$ denotes the L-Lie algebra of the algebraic group $\mathbb{G}_L=\mathbb{G}\times_{Spec(\mathfrak{o})}Spec(L)$ and $\mathcal{U}(\mathfrak{g}_L)$ its universal enveloping algebra, then $\mathcal{U}(\mathfrak{g})$ is an integral model of $\mathcal{U}(\mathfrak{g}_L)$. Moreover, the algebra of distributions of level m, introduced in the preceding subsection, is also an integral model of $\mathcal{U}(\mathfrak{g}_L)$ [38, subsection 4.1]. This latest example will be specially important in this work.*

[1] This remark exemplifies the local situation when $X=\mathrm{Spec}(A)$ with A a $\mathbb{Z}_p$-algebra [35, Subsection 1.3.1].

Proposition 1.5.3. *The right $\mathbb{G}$-action induces a canonical homomorphism of filtered $\mathfrak{o}$-algebras*

$$\Phi^{(m)} : D^{(m)}(\mathbb{G}) \to H^0(X, \mathcal{D}_X^{(m)}).$$

Proof. The reader can find the proof of this proposition in [38, Proposition 4.4.1 (ii)], we will briefly discuss the construction of $\Phi^{(m)}$. The central idea in the construction is that if $\rho : X \times_{\mathfrak{o}} \mathbb{G} \to X$ denotes the action, then the comorphism $\rho^{\natural} : \mathcal{O}_X \to \mathcal{O}_X \otimes_{\mathfrak{o}} \mathfrak{o}[\mathbb{G}]$ induces a morphism

$$\rho_m^{(n)} : \mathcal{P}^n_{X,(m)} \to \mathcal{O}_X \otimes_{\mathfrak{o}} P^n_{(m)}(\mathbb{G})$$

for every $n \in \mathbb{N}$. Those applications are compatible when varying n. Let $u \in D_n^{(m)}(\mathbb{G})$ we define $\Phi^{(m)}(u)$ by

$$\Phi^{(m)}(u) : \mathcal{P}^n_{X,(m)} \xrightarrow{\rho_m^{(n)}} \mathcal{O}_X \otimes_{\mathfrak{o}} P^n_{(m)}(\mathbb{G}) \xrightarrow{id\otimes u} \mathcal{O}_X.$$

Again, those applications are compatible when varying n and we get the morphism of the proposition. □

Remark 1.5.4. *(i) If X is endowed with a left $\mathbb{G}$-action, then it turns out that $\Phi^{(m)}$ is an anti-homomorphism.*

(ii) In [38, Theorem 4.4.8.3] Huyghe and Schmidt have shown that if $X = \mathbb{G}$ and we consider the right (resp. left) regular action, then the morphism of the preceding proposition is in fact a canonical filtered isomorphism (resp. an anti-isomorphism) between $D^{(m)}(\mathbb{G})$ and $H^0(\mathbb{G}, \mathcal{D}_{\mathbb{G}}^{(m)})^{\mathbb{G}}$, the $\mathfrak{o}$-submodule of (left) $\mathbb{G}$-invariant global sections (cf. definition 2.2.7). This isomorphism induces a bijection between $D_n^{(m)}(\mathbb{G})$ and $H^0(\mathbb{G}, \mathcal{D}_{\mathbb{G},n}^{(m)})^{\mathbb{G}}$, and it is compatible when varying m.

We will denote by

$$\Phi_X^{(m)} : \mathcal{O}_X \otimes D^{(m)}(\mathbb{G}) \to \mathcal{D}_X^{(m)}$$

the morphism of sheaves (of $\mathfrak{o}$-modules) defined by: if $U \subset X$ is an open subset and $f \in \mathcal{O}_X(U)$, $u \in D^{(m)}(\mathbb{G})$, then

$$\Phi_{X,U}^{(m)}(f \otimes u) := f(\Phi^{(m)}(u)|_U).$$

Let us define $\mathcal{A}_X^{(m)} = \mathcal{O}_X \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G})$, and let us remark that we can endow this sheaf with the skew ring multiplication coming from the action of $D^{(m)}(\mathbb{G})$ on $\mathcal{O}_X$ via the morphism $\Phi_X^{(m)}$. This is

$$(f \otimes u) \cdot (g \otimes v) := f\Phi_X^{(m)}(u)g \otimes v + fg \otimes uv. \tag{1.4}$$

This multiplication defines over $\mathcal{A}_X^{(m)}$ a structure of a sheaf of associative $\mathfrak{o}$-algebras, such that it becomes an integral model of the sheaf of L-algebras $\mathcal{U}^\circ := \mathcal{O}_{X_L} \otimes_L \mathcal{U}(\mathfrak{g}_L)$. To see this, let us recall how the multiplicative structure of the sheaf $\mathcal{U}^\circ$ is defined (cf. [49, subsection 5.1] or [44, section 2]).

Differentiating the right action of $\mathbb{G}_L$ on X_L we get a morphism of Lie algebras

$$\tau : \mathfrak{g}_L \to H^0(X_L, \mathcal{T}_{X_L}). \tag{1.5}$$

This implies that $\mathfrak{g}_L$ acts on $\mathcal{O}_{X_L}$ by derivations and we can endow $\mathcal{U}^\circ$ with the skew ring multiplication

$$(f \otimes \eta)(g \otimes \zeta) = f\tau(\eta)g \otimes \zeta + fg \otimes \eta\zeta \tag{1.6}$$

for $\eta \in \mathfrak{g}_L$, $\zeta \in \mathcal{U}(\mathfrak{g}_L)$ and $f, g \in \mathcal{O}_{X_L}$. With this product the sheaf $\mathcal{U}^\circ$ becomes a sheaf of associative algebras [44, Section 2, page 11].

Remark 1.5.5. *As in (1.4) we can define a morphism (called the operator-representation)*

$$\Psi_{X_L} : \mathcal{O}_{X_L} \otimes_L \mathcal{U}(\mathfrak{g}_L) \to \mathcal{D}_{X_L}$$

of sheaves (of L-algebras) by the rule

$$\Psi_{X_L}(f \otimes \eta) := f\ \tau(\eta),$$

where $f \in \mathcal{O}_{X_L}$ (is a local section) and $\eta \in \mathfrak{g}_L$. We get the following commutative diagram

$$\begin{array}{ccc} D^{(m)}(\mathbb{G}) & \xrightarrow{\Phi^{(m)}} & H^0(X, \mathcal{D}_X^{(m)}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{U}(\mathfrak{g}_L) & \xrightarrow{\Psi_{X_L}} & H^0(X_L, \mathcal{D}_{X_L}). \end{array}$$

Given that $D^{(m)}(\mathbb{G})$ is an integral model of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_L)$, then by (1.4) and (1.6) we can conclude that $\mathcal{A}_X^{(m)}$ is also a sheaf of associative $\mathfrak{o}$-algebras being a subsheaf of $\mathcal{U}^{\circ}$.

Proposition 1.5.6. *[38, Corollary 4.4.6]*

(i) The sheaf $\mathcal{A}_X^{(m)}$ is a locally free $\mathcal{O}_X$-module.

(ii) There exists a unique structure over $\mathcal{A}_X^{(m)}$ of filtered $\mathcal{O}_X$-ring, compatible with the structure of algebra of $D^{(m)}(\mathbb{G})$. Moreover, there is a canonical isomorphism of graded $\mathcal{O}_X$-algebras $gr(\mathcal{A}_X^{(m)}) = \mathcal{O}_X \otimes_{\mathfrak{o}} Sym^{(m)}(\mathfrak{g})$.

(iii) The sheaf $\mathcal{A}_X^{(m)}$ (resp. $gr(\mathcal{A}_X^{(m)})$) is a coherent sheaf of $\mathcal{O}_X$-rings (resp. a coherent sheaf of $\mathcal{O}_X$-algebras), with noetherian sections over open affine subsets of X.

Chapter 2

Integral twisted arithmetic differential operators

In [35] Huyghe introduced sheaves of twisted differential operators on the smooth formal flag $\mathfrak{o}$-scheme $\mathfrak{X}$, which depends of an algebraic character of $\mathfrak{t}$. The objective of this chapter is to use the ideas of Borho-Brylinski in [11] to introduce (integral) twisted differential operators associated to an arbitrary character of $\mathfrak{t}$. In the next chapter we will discuss their properties when we pass to the formal completion.

2.1 Torsors

Let us suppose that $\mathbb{T}$ is a smooth affine algebraic group over $\mathfrak{o}$ with Lie algebra denoted by $\mathfrak{t}$, and that $\widetilde{X}$ and X are smooth separated schemes over $\mathfrak{o}$, such that $\widetilde{X}$ is endowed with a right $\mathbb{T}$-action $\sigma : \widetilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to \widetilde{X}$. We will also assume that $\mathbb{T}$ acts trivially on X. [1]

We say that a morphism $\xi : \widetilde{X} \to X$ is a $\mathbb{T}$-*torsor* for the Zariski topology, if ξ is a faithfully flat morphism such that the diagram

$$\begin{array}{ccc} \widetilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} & \xrightarrow{\sigma} & \widetilde{X} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \xi} \\ \widetilde{X} & \xrightarrow{\xi} & X \end{array}$$

is commutative and the map (induced by the previous diagram)

$$\widetilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to \widetilde{X} \times_X \widetilde{X}; \quad (x, h) \mapsto (x, xh) \tag{2.1}$$

is an isomorphism.

Let $U \subset X$ be an affine open subset and $\mathrm{pr}_1 : U \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to U$ the first projection. We say that U *trivializes the torsor* ξ if there is a $\mathbb{T}$-equivariant isomorphism $\alpha_U : U \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \xrightarrow{\simeq} \xi^{-1}(U)$, where $\mathbb{T}$ acts on $U \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T}$ by right translations on the second factor, and such that

[1] For example if $\mathbb{T} \subset \mathbb{B}$ and $X = \mathbb{G}/\mathbb{B}$ is the flag variety.

$$\begin{array}{ccc} U \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} & \xrightarrow{\alpha_U} & \xi^{-1}(U) \\ & \searrow^{\mathrm{pr}_1} \quad \swarrow_{\xi|_{\xi^{-1}(U)}} & \\ & U & \end{array} \qquad \mathrm{pr}_1 = \xi|_{\xi^{-1}(U)} \circ \alpha_U. \tag{2.2}$$

Remark 2.1.1. *As X is separated, the set $\mathcal{S}$ of open affine subschemes U of X that trivialises the torsor and such that $\mathcal{O}_X(U)$ is a finitely generated $\mathfrak{o}$-algebra, it is stable under intersections. Moreover, if $U \in \mathcal{S}$ and W is an open affine subscheme of U, then $W \in \mathcal{S}$.*

Definition 2.1.2. *We say that ξ is locally trivially for the Zariski topology if X can be covered by opens in $\mathcal{S}$.*

Notation. Let $\xi : \widetilde{X} \to X$ be a locally trivial $\mathbb{T}$-torsor. In what follows, we will always denote by $\mathcal{S}$ the basis for the Zariski topology on X consisting of open affine subschemes that trivializes the torsor.

Lemma 2.1.3. *Let $\xi : \widetilde{X} \to X$ be a locally trivial $\mathbb{T}$-torsor and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module. Then $R^1\xi_*\mathcal{M} = 0$.*

Proof. We recall for the reader that $\mathrm{R}^1\xi_*\mathcal{M}$ is the sheaf associated to the presheaf [30, chapter III, prop. 8.1]

$$U \subseteq X \mapsto H^1(\xi^{-1}(U), \mathcal{M}).$$

As ξ is locally trivial, the set $\mathcal{S}$ of affine open subsets of X that trivialises the torsor is a base for the Zariski topology of X. Moreover, if $U \in \mathcal{S}$ then by definition $\xi^{-1}(U)$ is an affine open subset of $\widetilde{X}$ and given that $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module, we can conclude that $H^1(\xi^{-1}(U), \mathcal{M}) = 0$. □

2.2 $\mathbb{T}$-equivariant sheaves and sheaves of $\mathbb{T}$-invariant sections

Throughout this section, we will keep the notation of the preceding section and we will denote by $\widetilde{X}$ a smooth and separated $\mathfrak{o}$-scheme endowed with a right $\mathbb{T}$-action $\sigma : \widetilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to \widetilde{X}$. Let us denote by $m : \mathbb{T} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to \mathbb{T}$ the group law of $\mathbb{T}$ and by

$$p_1 : \widetilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to \widetilde{X} \quad \text{and} \quad p_{1,2} : \widetilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to \widetilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T}$$

the respective projections. We will also denote by

$$f_1,\ f_2,\ f_3 : \widetilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to \widetilde{X}$$

the morphisms defined by $f_1(x, t_1, t_2) = x$, $f_2(x, t_1, t_2) = xt_1$ and $f_3(x, t_1, t_2) = xt_1t_2$. Following [46, chapter 0, section 3], we say that a sheaf $\mathcal{M}$ of $\mathcal{O}_{\widetilde{X}}$-modules is $\mathbb{T}$-*equivariant* if there exists an isomorphism

$$p_1^*\mathcal{M} \xrightarrow{\Psi} \sigma^*\mathcal{M} \tag{2.3}$$

such that the following diagram is commutative (cocycle condition [32, (9.10.10)])

$$\begin{array}{ccc} (id_{\widetilde{X}} \times m)^* p_1^*\mathcal{M} = f_1^*\mathcal{M} = p_{1,2}^* p_1^*\mathcal{M} & \xrightarrow{p_{1,2}^*\Psi} & p_{1,2}^*\sigma^*\mathcal{M} = f_2^*\mathcal{M} = (\sigma \times id_{\mathbb{T}})^* p_1^*\mathcal{M} \\ & \searrow^{(id_{\widetilde{X}} \times m)^*\Psi} & \downarrow (\sigma \times id_{\mathbb{T}})^*\Psi \\ & & (id_{\widetilde{X}} \times m)^*\sigma^*\mathcal{M} = f_3^*\mathcal{M} = (\sigma \times id_{\mathbb{T}})^*\sigma^*\mathcal{M}. \end{array} \tag{2.4}$$

From now on, we will say that a couple $(\mathcal{M}, \Psi)$ is a $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module, if $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module endowed with an isomorphism $\Psi : p_1^*\mathcal{M} \to \sigma^*\mathcal{M}$ making commutative the diagram (2.4).

We will need the following lemmas. First of all, we recall for the reader that the category of $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-modules is an abelian category [31, Lemma 2.17]. In particular it is complete and cocomplete.

Lemma 2.2.1. *Let $\mathcal{M}$ be an $\mathcal{O}_{\widetilde{X}}$-module filtered by a family $(\mathcal{M}_n)_{n\in\mathbb{N}}$ of $\mathbb{T}$-equivariant $\mathcal{O}_{\widetilde{X}}$-modules such that the inclusions $\mathcal{M}_n \hookrightarrow \mathcal{M}_{n+1}$ are $\mathbb{T}$-equivariant. Then $\mathcal{M}$ is also $\mathbb{T}$-equivariant.*

Proof. The exactness of the functors σ^* and p_1^* allows us to conclude that $\sigma^*(\mathcal{M})$ and $p_1^*(\mathcal{M})$ are endowed with canonical filtrations $(\sigma^*(\mathcal{M}_n))_{n\in\mathbb{N}}$ and $(p_1^*(\mathcal{M}))_{n\in\mathbb{N}}$, respectively. Since the components of this filtration have compatible $\mathbb{T}$-equivariant structures, we can conclude that $\mathcal{M}$ is also $\mathbb{T}$-equivariant via a filtered isomorphism. □

On the other hand, since the category of $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-modules is an abelian category, the cokernel of a $\mathbb{T}$-equivariant morphism $\mathcal{M} \to \mathcal{N}$ between two $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-modules is again a $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module. As we will demonstrate below, we can give an independent proof of this result. We will use the following lemma in subsection 5.1.1.

Lemma 2.2.2. *Let $(\mathcal{M}, \Phi_1)$ and $(\mathcal{N}, \Phi_2)$ be $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-modules. Let $\mathcal{L}$ be a quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module such that*

$$0 \to \mathcal{M} \xrightarrow{\phi} \mathcal{N} \xrightarrow{\psi} \mathcal{L} \to 0$$

is an exact sequence and ϕ is a $\mathbb{T}$-equivariant morphism. Then $\mathcal{L}$ is also $\mathbb{T}$-equivariant.

Proof. To define the right vertical morphism, we consider a basis $\mathscr{A}$ of $\widetilde{X}$ consisting of affine open subsets, which can be assumed stable under intersections because $\tilde{X}$ is separated. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_1^*\mathcal{M} & \xrightarrow{p_1^*\phi} & p_1^*\mathcal{N} & \xrightarrow{p_1^*\psi} & p_1^*\mathcal{L} & \longrightarrow & 0 \\ & & \downarrow{\Phi_1} & & \downarrow{\Phi_2} & & & & \\ 0 & \longrightarrow & \sigma^*\mathcal{M} & \xrightarrow{\sigma^*\phi} & \sigma^*\mathcal{N} & \xrightarrow{\sigma^*\psi} & \sigma^*\mathcal{L} & \longrightarrow & 0 \end{array}$$

Let us fix $U \in \mathscr{A}$. To soft the notation we will assume that $M(\sigma) := \sigma^*(\mathcal{M})(U \times_{\mathfrak{o}} \mathbb{T})$ (resp. $N(\sigma) := \sigma^*(\mathcal{N})(U \times_{\mathfrak{o}} \mathbb{T})$ and $L(\sigma) := \sigma^*(\mathcal{L})(U \times_{\mathfrak{o}} \mathbb{T})$) and $M(p_1) : p_1^*(\mathcal{M})(U \times_{\mathfrak{o}} \mathbb{T})$ (resp. $N(p_1) := p_1^*(\mathcal{N})(U \times_{\mathfrak{o}} \mathbb{T})$ and $L(p_1) := p_1^*(\mathcal{L})(U \times_{\mathfrak{o}} \mathbb{T})$). Also, we will suppose that $\phi_1 := p_1^*(\phi)(U \times_{\mathfrak{o}} \mathbb{T})$ (resp. $\psi_1 := p_1^*(\psi)(U \times_{\mathfrak{o}} \mathbb{T})$), $\phi_2 := \sigma^*(\phi)(U \times_{\mathfrak{o}} \mathbb{T})$ (resp. $\psi_2 := \sigma^*(\psi)(U \times_{\mathfrak{o}} \mathbb{T})$), $\Phi_{1,U} := \Phi_1(U \times_{\mathfrak{o}} \mathbb{T})$ and $\Phi_{2,U} := \Phi_2(U \times_{\mathfrak{o}} \mathbb{T})$. In such a way that we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(p_1) & \xrightarrow{\phi_1} & N(p_1) & \xrightarrow{\psi_1} & L(p_1) & \longrightarrow & 0 \\ & & \downarrow{\Phi_{1,U}} & & \downarrow{\Phi_{2,U}} & & & & \\ 0 & \longrightarrow & M(\sigma) & \xrightarrow{\phi_2} & N(\sigma) & \xrightarrow{\psi_2} & L(\sigma) & \longrightarrow & 0 \end{array}$$

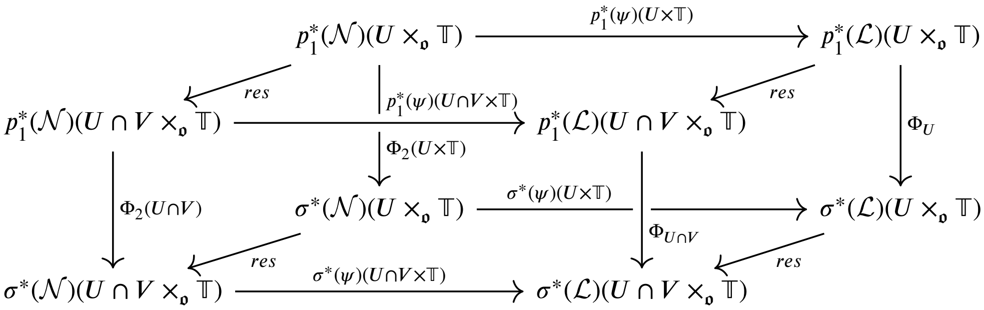
Let $x \in L(p_1)$. By surjectivity of ψ_1 we can find $y_1 \in N(p_1)$ such that $\psi_1(y_1) = x$. We define then $\Phi_U(x) := \psi_2(\Phi_{2,U}(y_1)) \in L(\sigma)$. Let us see that Φ_U is well-defined, this means that it does not depend of the choice of $y_1 \in N(p_1)$. Let $y_2 \in N(p_1)$ such that $\psi_1(y_2) = x$. We want to see $\Phi_U(x) := \psi_2(\Phi_{2,U}(y_1)) = \psi_2(\Phi_{2,U}(y_2))$. Let $y := y_1 - y_2 \in N(p_1)$. By definition $y \in \mathrm{Ker}(\psi_1) = \mathrm{Im}(\phi_1)$ and we can find $z \in M(p_1)$ such that $\phi_1(z) = y$. Let $z' = \Phi_{1,U}(z) \in M(\sigma)$. By commutative of the diagram we have

$$\phi_2(z') = \phi_2(\Phi_{1,U}(z)) = \Phi_{2,U}(\phi_1(z)) = \Phi_{2,U}(y) = \Phi_{2,U}(y_1) - \Phi_{2,U}(y_2)$$

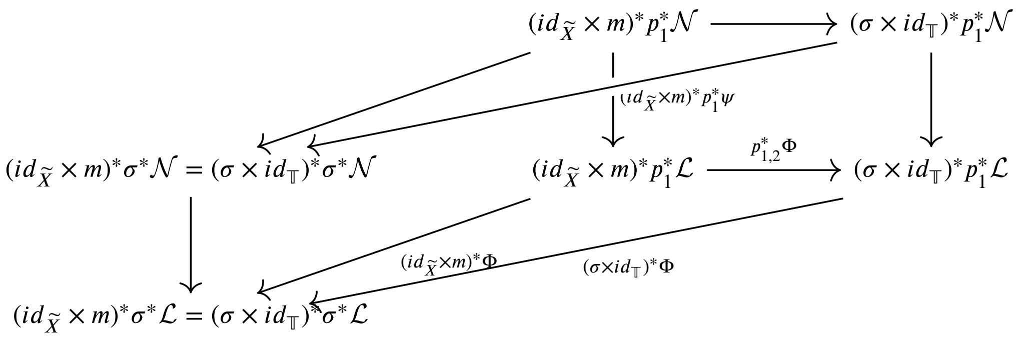
and therefore

$$0 = \psi_2(\phi_2(z')) = \psi_2(\Phi_{2,U}(y_1)) - \psi_2(\Phi_{2,U}(y_2)) = \Phi_U(x) - \psi_2(\Phi_{2,U}(y_2)).$$

Moreover it is straightforward to see that Φ_U is in fact a morphism of $\mathcal{O}_{\tilde{X}\times_{\mathfrak{o}}\mathbb{T}}(U \times_{\mathfrak{o}} \mathbb{T})$-modules, which by the well-known *five lemma* becomes an isomorphism. From the preceding reasoning, and the quasi-coherence of the sheaves involved we get an isomorphism $\tilde{\Phi}_U : p_1^*(\mathcal{N})|_{U\times\mathbb{T}} \to \sigma^*(\mathcal{L})|_{U\times\mathbb{T}}$ of sheaves of $\mathcal{O}_{U\times\mathbb{T}}$-modules, for every $U \in \mathscr{A}$.
To complete the construction of the right vertical isomorphism we need to globalize the preceding reasoning. This means that if we chose $U, V \in \mathscr{A}$ then we have $\tilde{\Phi}_U|_{U\cap V} = \tilde{\Phi}_V|_{U\cap V}$. As $\mathscr{A}$ is stable under intersections we can construct, in the same way as before, an isomorphism $\tilde{\Phi}_{U\cap V}$ over $U \cap V$. Let us see that $\tilde{\Phi}_U|_{U\cap V} = \tilde{\Phi}_{U\cap V}$ (the reader can fallow the same reasoning to proof that $\tilde{\Phi}_V|_{U\cap V} = \tilde{\Phi}_{U\cap V}$). We consider the following cube



Except for the right lateral face, all the other faces form, by construction or hypothesis, commutative diagrams which implies that also the right lateral face forms a commutative diagram. This shows that $\tilde{\Phi}_U|_{U\cap V} = \tilde{\Phi}_{U\cap V}$. We have constructed an isomorphism $\Phi : p_1^*(\mathcal{L}) \to \sigma^*(\mathcal{L})$ of quasi-coherent $\mathcal{O}_{\tilde{X}\times_{\mathfrak{o}}\mathbb{T}}$-modules. Let us show that Φ defines a $\mathbb{T}$-equivariant structure. To do that we consider the following diagram



The triangle on the top is commutative and by functoriality the lateral faces of the prism are also commutative. Its is straightforward to show from this that

$$(id_{\tilde{X}} \times m)^* \Phi \circ (id_{\tilde{X}} \times m)^* p_1^* \psi = (\sigma \times id_{\mathbb{T}})^* \Phi \circ p_{1,2}^* \Psi \circ (id_{\tilde{X}} \times m)^* p_1^* \psi,$$

and as $(id_{\tilde{X}} \times m)^* p_1^* \psi$ is surjective we can conclude that the triangle on the bottom is also commutative. Therefore Φ defines a $\mathbb{T}$-equivariant structure for $\mathcal{L}$. □

Lemma 2.2.3. *Let $(\mathcal{L}, \Psi)$ be a $\mathbb{T}$-equivariant locally free $\mathcal{O}_{\tilde{X}}$-module of finite rank. Then its dual $\mathcal{L}^\vee$ is also $\mathbb{T}$-equivariant.*

Proof. Given that $\mathcal{L}$ is a locally free sheaf of finite rank, we dispose of two canonical and functorial isomorphisms

$$\sigma^*\mathcal{L}^\vee \xrightarrow{\simeq} \mathscr{H}om_{\mathcal{O}_{\widetilde{X}\times_{\mathfrak{o}}\mathbb{T}}}(\sigma^*\mathcal{L}, \mathcal{O}_{\widetilde{X}\times_{\mathfrak{o}}\mathbb{T}}) \quad \text{and} \quad p_1^*\mathcal{L}^\vee \xrightarrow{\simeq} \mathscr{H}om_{\mathcal{O}_{\widetilde{X}\times_{\mathfrak{o}}\mathbb{T}}}(p_1^*\mathcal{L}, \mathcal{O}_{\widetilde{X}\times_{\mathfrak{o}}\mathbb{T}})$$

This implies that $(\Psi^{-1})^\vee$, the dual of Ψ^{-1}, defines the $\mathbb{T}$-equivariant structure on $\mathcal{L}^\vee$. □

Lemma 2.2.4. *Let $(\mathcal{L}, \Phi)$ and $(\mathcal{L}', \Phi')$ be two $\mathbb{T}$-invariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-modules, then $\mathcal{L}\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{L}'$ is a $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module.*

Proof. The functorial isomorphisms

$$p_1^*(\mathcal{L}\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{L}') \xrightarrow{\simeq} p_1^*(\mathcal{L})\otimes_{\mathcal{O}_{\widetilde{X}\times\mathbb{T}}} p_1^*(\mathcal{L}') \quad \text{and} \quad \sigma^*(\mathcal{L}\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{L}') \xrightarrow{\simeq} \sigma^*(\mathcal{L})\otimes_{\mathcal{O}_{\widetilde{X}\times\mathbb{T}}}\sigma^*(\mathcal{L}')$$

tell us that $\Phi\otimes\Phi'$ defines a $\mathbb{T}$-equivariant structure on $\mathcal{L}\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{L}'$. □

2.2.5. ***Equivariant sheaves of p-adic complete $\mathcal{O}_{\mathfrak{X}}$-modules.*** *We recall for the reader that we have denoted by $\widetilde{X}_i = \widetilde{X}\times_{Spec(\mathfrak{o})} Spec(\mathfrak{o}/\varpi^{i+1})$ the reduction module ϖ^{i+1}. Under the preceding hypothesis, the scheme $\widetilde{X}_i$ is endowed with a right action of the $(\mathfrak{o}/\varpi^{i+1})$-group scheme $\mathbb{T}_i := \mathbb{T}\times_{Spec(\mathfrak{o})} Spec(\mathfrak{o}/\varpi^{i+1})$. If $\widetilde{\mathfrak{X}}$ denote the completion of $\widetilde{X}$ along it special fiber, then we will denote by $\gamma_i : X_i \hookrightarrow \mathfrak{X}$ and $\theta_i : \widetilde{X}_i\times_{Spec(\mathfrak{o}/\varpi^{i+1})}\mathbb{T}_i \hookrightarrow \widetilde{X}_{i+1}\times_{Spec(\mathfrak{o}/\varpi^{i+2})}\mathbb{T}_{i+1}$ the closed embeddings, and by $\sigma_i : \widetilde{X}_i\times_{Spec(\mathfrak{o}/\varpi^{i+1})}\mathbb{T}_i \to \widetilde{X}_i$ the induced action. Let $\mathfrak{T}$ denote the formal completion of $\mathbb{T}$ along its special fiber.*

Definition 2.2.6. *Let $\mathscr{E}$ be a sheaf of complete $\mathcal{O}_{\widetilde{\mathfrak{X}}}$-modules for the p-adic topology. We will say that $\mathscr{E}$ is $\mathfrak{T}$-equivariant, if for every $i\in\mathbb{N}$, the sheaf $\mathscr{E}_i := \gamma_i^*(\mathscr{E})$ is a $\mathbb{T}_i$-equivariant $\mathcal{O}_{\widetilde{X}_i}$-module, and the following diagram is commutative*

$$\begin{array}{ccc} p_{1,i}^*(\mathscr{E}_i) & \xrightarrow{\Phi_i} & \sigma_i^*(\mathscr{E}_i) \\ \downarrow\wr & & \downarrow\wr \\ \theta_i^* p_{1,i+1}(\mathscr{E}_{i+1}) & \xrightarrow{\theta_i^*(\Phi_{i+1})} & \theta_i^*\sigma_{i+1}^*(\mathscr{E}_{i+1}). \end{array}$$

Let $(\mathcal{M}, \Psi)$ be a $\mathbb{T}$-equivariant $\mathcal{O}_{\widetilde{X}}$-module. By the Künneth formula [28, Theorem 6.7.8] we have a canonical isomorphism

$$H^0(\widetilde{X}\times_{\mathfrak{o}}\mathbb{T}, p_1^*\mathcal{M}) \simeq H^0(\widetilde{X}, \mathcal{M})\otimes_{\mathfrak{o}}\mathfrak{o}[\mathbb{T}]$$

which composing with the application

$$H^0(\widetilde{X}, \mathcal{M}) \longrightarrow H^0(\widetilde{X}\times_{\mathfrak{o}}\mathbb{T}, \sigma^*\mathcal{M}) \xrightarrow{H^0(\Psi)} H^0(\widetilde{X}\times_{\mathfrak{o}}\mathbb{T}, p_1^*\mathcal{M})$$

(the first application is induced via the canonical map $\mathcal{M}\to\sigma_*\sigma^*\mathcal{M}$) gives us a morphism

$$\Delta : H^0(\widetilde{X}, \mathcal{M}) \to H^0(\widetilde{X}, \mathcal{M})\otimes_{\mathfrak{o}}\mathfrak{o}[\mathbb{T}],$$

defining a structure of $\mathbb{T}$-comodule on $H^0(\widetilde{X}, \mathcal{M})$. The co-module relations are given by the cocycle condition [46, chapter 0, definition 1.6].

Definition 2.2.7. *The $\mathbb{T}$-invariant elements of $H^0(\widetilde{X}, \mathcal{M})$ are the elements $P\in H^0(\widetilde{X}, \mathcal{M})$ such that $\Delta(P) = P\otimes 1$. This subspace will be denoted by $H^0(\widetilde{X}, \mathcal{M})^{\mathbb{T}}$.*

Now, let us suppose that $\widetilde{X}$ can be covered by a family $\widetilde{\mathcal{S}}$ of affine open subsets, which are stable under finite intersection and invariant under the right action of $\mathbb{T}$. This means that for every $\tilde{U}\in\widetilde{\mathcal{S}}$ the morphism σ, inducing the right $\mathbb{T}$-action on

$\widetilde{X}$, induces a right $\mathbb{T}$-action $\tilde{\sigma} := \sigma|_{\tilde{U}\times_{\mathfrak{o}}\mathbb{T}} : \tilde{U}\times_{\mathfrak{o}}\mathbb{T} \to \tilde{U}$ on $\tilde{U}$. By pulling back Ψ under the inclusion $\tilde{U}\times_{\mathfrak{o}}\mathbb{T} \hookrightarrow \widetilde{X}\times_{\mathfrak{o}}\mathbb{T}$ we get an isomorphism $\Psi|_{\tilde{U}} : \tilde{\sigma}^*\mathcal{M}_{\tilde{U}} \to p_1^*\mathcal{M}_{\tilde{U}}$ which satisfies the respective cocycle condition (2.4), and, as before, we obtain a comodule map

$$\Delta_{\tilde{U}} : \Gamma(\tilde{U},\mathcal{M}) \to \Gamma(\tilde{U},\mathcal{M})\otimes_{\mathfrak{o}} \mathfrak{o}[\mathbb{T}].$$

As in definition 2.2.7, we can define the $\mathfrak{o}$-submodule of $\mathbb{T}$-*invariant sections* on $\tilde{U}$ by

$$\Gamma(\tilde{U},\mathcal{M})^{\mathbb{T}} := \left\{P\in\Gamma(\tilde{U},\mathcal{M})|\ \ \Delta_{\tilde{U}}(P) = P\otimes 1\right\}. \tag{2.5}$$

Finally, let us suppose that $\mathcal{M}$ is also quasi-coherent. By [30, Chapter II, Corollary 5.5] on every affine open subset $\tilde{U}\in\widetilde{\mathcal{S}}$ we can define a subsheaf

$$\left(\mathcal{M}|_{\tilde{U}}\right)^{\mathbb{T}} := \Gamma\widetilde{(\tilde{U},\mathcal{M})}^{\mathbb{T}} \subset \Gamma\widetilde{(\tilde{U},\mathcal{M})} = \mathcal{M}|_{\tilde{U}}.$$

By definition, and giving that $\widetilde{\mathcal{S}}$ was supposed to be stable under finite intersections, the preceding sheaves glue together to a subsheaf $(\mathcal{M})^{\mathbb{T}}\subset\mathcal{M}$ which does not depend of the covering $\widetilde{\mathcal{S}}$. We sum up the preceding construction in the next definition.

Definition 2.2.8. *Let $\widetilde{X}$ be a smooth separated $\mathfrak{o}$-scheme endowed with a right $\mathbb{T}$-action, and covered by a family of affine open subsets $\widetilde{\mathcal{S}}$ stable under finite intersections and the $\mathbb{T}$-action. For every $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module $\mathcal{M}$, the subsheaf $(\mathcal{M})^{\mathbb{T}}$ is called the subsheaf of $\mathbb{T}$-invariant sections of $\mathcal{M}$.*

As in section 2.1, let us suppose that X is another smooth and separated $\mathfrak{o}$-scheme, such that $\xi : \widetilde{X}\to X$ is a locally trivial $\mathbb{T}$-torsor. We recall for the reader that this means that ξ is a faithfully flat morphism, such that the diagram

$$\begin{array}{ccc} \widetilde{X}\times_{\mathrm{Spec}(\mathfrak{o})}\mathbb{T} & \xrightarrow{\sigma} & \widetilde{X} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \xi} \\ \widetilde{X} & \xrightarrow{\xi} & X \end{array}$$

is cartesian (cf. (2.1)), and there exists a covering $\mathcal{S}$ of X, consisting of affine open subsets that trivializes ξ (cf. (2.2)).

As an an application of the preceding construction let us point out that if $\xi : \widetilde{X}\to X$ is a locally trivial $\mathbb{T}$-torsor, then we actually dispose of a subsheaf of $\mathbb{T}$-invariant sections of the direct image sheaf $\xi_*\mathcal{M}$, with $\mathcal{M}$ a $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module. In fact, if $\mathcal{S}$ denotes the collection of all affine open subsets that trivialises the torsor ξ, then for every $U\in\mathcal{S}$ we know that $\xi^{-1}(U)$ is stable under the right $\mathbb{T}$-action and, as in (2.5), we can define

$$\left((\xi_*\mathcal{M})(U)\right)^{\mathbb{T}} := \left(\mathcal{M}(\xi^{-1}(U))\right)^{\mathbb{T}} \subset \mathcal{M}(\xi^{-1}(U)). \tag{2.6}$$

As $\widetilde{X}$ is noetherian $\xi_*\mathcal{M}$ is quasi-coherent and therefore from (2.6) we have a subsheaf

$$\left((\xi_*\mathcal{M})|_U\right)^{\mathbb{T}} := \left((\xi_*\widetilde{\mathcal{M})(U)}\right)^{\mathbb{T}} \subset \left(\xi_*\widetilde{\mathcal{M})}(U) = \left(\xi_*\mathcal{M}\right)|_U.$$

Since $\mathcal{S}$ is stable under finite intersections, those sheaves glue together to define a subsheaf

$$\left(\xi_*\mathcal{M}\right)^{\mathbb{T}} \subset \xi_*\mathcal{M}. \tag{2.7}$$

For the rest of this subsection we will always suppose that $\xi : \widetilde{X}\to X$ is a locally trivial $\mathbb{T}$-torsor.

Lemma 2.2.9. *The morphism $\xi : \widetilde{X}\to X$ induces an isomorphism $\xi^{\natural} : \mathcal{O}_X \to \left(\xi_*\mathcal{O}_{\widetilde{X}}\right)^{\mathbb{T}}$.*

Proof. As this is a local problem, we can take $U \in \mathcal{S}$ and suppose that $\xi : \xi^{-1}(U) = U \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to U$ is the first projection. Since rational cohomology commutes with direct limits [40, Part I, Lemma 4.17] and $\mathcal{O}_X(U)$ is a direct limit of free $\mathfrak{o}$-modules, we can conclude that $\left(\xi_* \mathcal{O}_{\widetilde{X}}\right)^{\mathbb{T}}(U) = \left(\mathcal{O}_X(U) \otimes_{\mathfrak{o}} \mathfrak{o}[\mathbb{T}]\right)^{\mathbb{T}} = \mathcal{O}_X(U)$. □

2.3 Relative enveloping algebras of finite level

Let us fix a positive integer $m \in \mathbb{Z}$. As in the preceding subsections $\widetilde{X}$ and X will denote smooth separated $\mathfrak{o}$-schemes, and $\mathbb{T}$ a smooth affine commutative algebraic group over $\mathfrak{o}$. We will also assume that $\xi : \widetilde{X} \to X$ is a locally trivial $\mathbb{T}$-torsor. We start this subsection recalling the construction of the $\mathbb{T}$-equivariant structures of the sheaf of level m differential operators $\mathcal{D}^{(m)}_{\widetilde{X}}$ (cf. [38, Proposition 3.4.1]).

Let $p_1 : \widetilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to \widetilde{X}$ and $p_2 : \widetilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to \mathbb{T}$ be the projections. For every $n \in \mathbb{N}$ the universal property of the m-PD-envelopes (proposition 1.1.5) gives us two canonical morphisms

$$d^n p_1 : p_1^* \mathcal{P}^n_{\widetilde{X},(m)} \to \mathcal{P}^n_{\widetilde{X}\times_{\mathrm{Spec}(\mathfrak{o})}\mathbb{T},(m)} \quad \text{and} \quad d^n p_2 : p_2^* \mathcal{P}^n_{\mathbb{T},(m)} \to \mathcal{P}^n_{\widetilde{X}\times_{\mathrm{Spec}(\mathfrak{o})}\mathbb{T},(m)}.$$

Let $\mathcal{J}$ be the m-PD-ideal of the m-PD-algebra $\mathcal{P}^n_{\mathbb{T},(m)}$. We have a canonical m-PD-morphism

$$s : \mathcal{P}^n_{\widetilde{X}\times_{\mathrm{Spec}(\mathfrak{o})}\mathbb{T},(m)} \to \mathcal{P}^n_{\widetilde{X}\times_{\mathrm{Spec}(\mathfrak{o})}\mathbb{T},(m)} / p_2^* \mathcal{J}$$

and $\rho := s \circ d^n p_1$ is a m-PD-isomorphism. Then we dispose of a canonical section of $d^n p_1$, named $q_1^n := \rho^{-1} \circ s$ [38, (14)]. On the other hand, by functionality, we obtain a morphism $d^n \sigma : \sigma^* \mathcal{P}^n_{\widetilde{X},(m)} \to \mathcal{P}^n_{\widetilde{X}\times_{\mathfrak{o}}\mathbb{T},(m)}$ and the $\mathbb{T}$-equivariant structure for $\mathcal{P}^n_{\widetilde{X}}$ is defined by $\Phi^n := q_1^n \circ d^n \sigma$ [38, Proposition 3.4.1]. Definition 1.2.1 and lemme 2.2.3 allow us to conclude that for every $n \in \mathbb{N}$ the sheaf $\mathcal{D}^{(m)}_{\widetilde{X},n}$ is $\mathbb{T}$-equivariant and the inclusions $\mathcal{D}^{(m)}_{\widetilde{X},n} \hookrightarrow \mathcal{D}^{(m)}_{\widetilde{X},n+1}$ are $\mathbb{T}$-equivariant morphisms. In particular, by lemma 2.2.1, the sheaf of level m differential operators is $\mathbb{T}$-equivariant.

Remark 2.3.1. *(Notation as at the end of subsection 1.3) Following the preceding lines of reasoning we can also show that, for every $n \in \mathbb{N}$, there exists a m-PD-morphism*

$$q_1^{\prime n} : \Gamma^n_{\widetilde{X}\times\mathbb{T},(m)} \to p_1^* \Gamma^n_{\widetilde{X},(m)}.$$

which is a section of the canonical m-PD-morphism $p_1^* \Gamma^n_{\widetilde{X},(m)} \to \Gamma^n_{\widetilde{X}\times\mathbb{T},(m)}$ induced by p_1 [38, Subsection 2.2.2]. Let $\Gamma^n(\sigma) : \sigma^* \Gamma^n_{\widetilde{X},(m)} \to \Gamma^n_{\widetilde{X}\times\mathbb{T},(m)}$ be the canonical m-PD-morphism induced by σ. Then $\Phi'^n := q_1^{\prime n} \circ \Gamma^n(\sigma)$ is a $\mathbb{T}$-equivariant structure for $\Gamma^n_{\widetilde{X},(m)}$. As before, this implies that $Sym^{(m)}(\mathcal{T}_{\widetilde{X}})$ is $\mathbb{T}$-equivariant.

Remark 2.3.2. *Although it is well-known that the tangent sheaf $\mathcal{T}_{\widetilde{X}}$ is a $\mathbb{T}$-equivariant quasi-coherent sheaf, we point out to the reader that this can be proven using the preceding discussion. In fact as $\mathcal{P}^0_{\widetilde{X},(m)} = \mathcal{O}_{\widetilde{X}}$ and $\mathcal{P}^1_{\widetilde{X},(m)} = \mathcal{O}_{\widetilde{X}} \oplus \Omega^1_{\widetilde{X}}$ [38, Subsection 2.2.3 (18)], we can apply the lemma 2.2.2 to the sequence*

$$0 \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}} \oplus \Omega^1_{\widetilde{X}} \to \Omega^1_{\widetilde{X}} \to 0$$

and lemma 2.2.3 gives us the $\mathbb{T}$-equivariance of $\mathcal{T}_{\widetilde{X}}$. In particular, we dispose of the sheaves $(\mathcal{T}_{\mathbb{T}})^{\mathbb{T}}$ and $(\xi_* \mathcal{T}_{\widetilde{X}})^{\mathbb{T}}$.

Let us recall the following discussion from [1, 4.4]. Let us suppose $U \in \mathcal{S}$ and $\tau \in \mathcal{T}_{\widetilde{X}}(\xi^{-1}(U))^{\mathbb{T}}$. This assumption in particular implies that τ is a $\mathbb{T}$-invariant vector field on $\xi^{-1}(U)$ and therefore a $\mathbb{T}$-invariant endomorphism of $\mathcal{O}_{\widetilde{X}}(\xi^{-1}(U))$. Hence it preserves $\mathcal{O}_X(\xi^{-1}(U))^{\mathbb{T}}$ and by lemma 2.2.9 it induces a vector field $\nu(\tau) \in \mathcal{T}_X(U)$. We get then a map of $\mathcal{O}_X$-modules

$$\nu : \left(\xi_* \mathcal{T}_{\widetilde{X}}\right)^{\mathbb{T}} \to \mathcal{T}_X.$$

On the other hand, differentiating the right $\mathbb{T}$-action on $\widetilde{X}$ we obtain an $\mathfrak{o}$-linear Lie homomorphism $\mathfrak{t} \to \mathcal{T}_{\widetilde{X}}$, which induces a map of $\mathcal{O}_X$-modules

$$\mathfrak{t} \otimes_{\mathfrak{o}} \mathcal{O}_X \to \left(\xi_* \mathcal{T}_{\widetilde{X}}\right)^{\mathbb{T}}.$$

We get a complex of $\mathcal{O}_X$-modules

$$\mathfrak{t} \otimes_{\mathfrak{o}} \mathcal{O}_X \to \left(\xi_* \mathcal{T}_{\widetilde{X}}\right)^{\mathbb{T}} \xrightarrow{\nu} \mathcal{T}_X$$

which is functorial in $\widetilde{X}$ [1, subsection 4.4].

Lemma 2.3.3. [2] *If $\xi : \widetilde{X} \to X$ is a locally trivial $\mathbb{T}$-torsor, then the preceding complex is in fact a short exact sequence*

$$0 \to \mathfrak{t} \otimes_{\mathfrak{o}} \mathcal{O}_X \to \left(\xi_* \mathcal{T}_{\widetilde{X}}\right)^{\mathbb{T}} \xrightarrow{\nu} \mathcal{T}_X \to 0.$$

Before starting the proof we recall for the reader the following relations, which come from the $\mathbb{T}$-equivariant structure of $\mathcal{T}_{\mathbb{T}}$ [12, Lemma 2]

$$H^0(\mathbb{T}, \mathcal{T}_{\mathbb{T}}) = \mathfrak{o}[\mathbb{T}] \otimes_{\mathfrak{o}} \mathfrak{t} \quad \text{and} \quad H^0(\mathbb{T}, \mathcal{T}_{\mathbb{T}})^{\mathbb{T}} = \mathfrak{t}. \tag{2.8}$$

Moreover, by [30, Section II, exercise 8.3] we also dispose of the local description

$$\mathcal{T}_{U \times_{\mathfrak{o}} \mathbb{T}} = \left(\mathcal{T}_U \otimes_{\mathfrak{o}} \mathcal{O}_{\mathbb{T}}\right) \oplus \left(\mathcal{O}_U \otimes_{\mathfrak{o}} \mathcal{T}_{\mathbb{T}}\right). \tag{2.9}$$

Proof of lemma 2.3.3. As the sheaves in the sequence are quasi-coherent it is enough to check exactness over an affine open subset $U \in \mathcal{S}$. First of all, since $\mathcal{T}_X(U)$ is a locally free $\mathcal{O}_X(U)$-module and $\mathcal{O}_X(U)$ is a flat $\mathfrak{o}$-algebra, we can conclude that $\mathcal{T}_X(U)$ is an inductive limit of free $\mathfrak{o}$-modules. Therefore

$$(\mathcal{T}_X(U) \otimes_{\mathfrak{o}} \mathfrak{o}[\mathbb{T}])^{\mathbb{T}} = \mathcal{T}_X(U) \otimes_{\mathfrak{o}} (\mathfrak{o}[\mathbb{T}])^{\mathbb{T}} = \mathcal{T}_X(U).$$

This relation, together with (2.8) and (2.9), allow us to conclude that

$$\mathcal{T}_{\widetilde{X}}(\xi^{-1}(U))^{\mathbb{T}} = \mathcal{T}_X(U) \oplus \left(\mathcal{O}_X(U) \otimes_{\mathfrak{o}} \mathfrak{t}\right). \tag{2.10}$$

□

Remark 2.3.4. *The preceding lemma shows that $\left(\xi_* \mathcal{T}_{\tilde{X}}\right)^{\mathbb{T}}$ is a locally free $\mathcal{O}_X$-modules of finite rank. In particular, $Sym^{(m)}((\xi_* \mathcal{T}_{\tilde{X}})^{\mathbb{T}})$ is well-defined.*

Definition 2.3.5. *Let $\xi : \widetilde{X} \to X$ be a locally trivial $\mathbb{T}$-torsor. Following [12, page 180] we define the level m relative enveloping algebra of the torsor to be the sheaf of $\mathbb{T}$-invariants of $\xi_* \mathcal{D}^{(m)}_{\widetilde{X}}$:*

$$\widetilde{\mathcal{D}^{(m)}} := \left(\xi_* \mathcal{D}^{(m)}_{\widetilde{X}}\right)^{\mathbb{T}}.$$

The preceding sheaf is endowed with a canonical filtration

$$\mathrm{Fil}_d\left(\widetilde{\mathcal{D}^{(m)}}\right) = \left(\xi_* \mathcal{D}^{(m)}_{\widetilde{X},d}\right)^{\mathbb{T}}, \quad (d \in \mathbb{N}). \tag{2.11}$$

[2] The reasoning is as in [1, Lemma 4.4].

Proposition 2.3.6. *For any $U \in \mathcal{S}$ there exists an isomorphism of sheaves of filtered $\mathfrak{o}$-algebras*

$$\widetilde{\mathcal{D}^{(m)}}|_U \xrightarrow{\simeq} \mathcal{D}^{(m)}_X|_U \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{T}).$$

Before starting the proof of the proposition let us consider the following facts. Let $n \in \mathbb{N}$ fix and $i \leq n$. For the next few lines we will suppose that X and Z are smooth $\mathfrak{o}$-schemes and that $Y = X \times_{\mathrm{Spec}(\mathfrak{o})} Z$. Let p_1 and p_2 be the projections. By following [38] we have defined in page 37 two canonical applications

$$q_1^i : \mathcal{P}^i_{Y,(m)} \to p_1^* \mathcal{P}^i_{X,(m)} \quad \text{and} \quad q_2^{n-1} : \mathcal{P}^{n-i}_{Y,(m)} \to p_2^* \mathcal{P}^{n-i}_{Z,(m)}.$$

Locally, if $(t_1, ..., t_N)$ and $(t'_1, ..., t'_{N'})$ are coordinated systems on X and Z, respectively, then we obtain a coordinated system on Y by putting $(p_1^*(t_1), ..., p_1^*(t_N), p_2^*(t'_1), ..., p_2^*(t'_{N'})$. We have

$$\mathcal{P}^i_{Y,(m)} \simeq \bigoplus_{|\underline{l_1}|+|\underline{l_2}|\leq i} \mathcal{O}_Y p_1^*(\underline{\tau}^{\{\underline{l_1}\}}) p_2^*(\underline{\tau'}^{\{\underline{l_2}\}}) \quad (\tau_i := 1 \otimes t_i - t_i \otimes 1 \text{ and } \tau'_i := 1 \otimes t'_1 - t'_i \otimes 1).$$

In this case [38, subsection 2.2.2]

$$q_1^i \left(\sum_{\underline{l_1},\underline{l_2}} a_{\underline{l_1},\underline{l_2}} p_1^*(\underline{\tau}^{\{\underline{l_1}\}}) p_2^*(\underline{\tau'}^{\{\underline{l_2}\}}) \right) = \sum_{\underline{l_1}} a_{\underline{l_1},0} p_1^*(\underline{\tau}^{\{\underline{l_1}\}})$$

(with a similar description for q_2^{n-i}) and we have an isomorphism

$$\mathcal{P}^n_{Y,(m)} \xrightarrow{\simeq} \bigoplus_{0\leq i\leq n} p_1^* \mathcal{P}^i_{X,(m)} \otimes_{\mathcal{O}_Y} p_2^* \mathcal{P}^{n-i}_{Z,(m)}. \tag{2.12}$$

Moreover, since $\mathcal{P}^i_{X,(m)}$ and $\mathcal{P}^{n-i}_{Z,(m)}$ are locally free $\mathcal{O}$-modules of finite rank, taking duals in (2.12) we get a canonical isomorphism

$$\mathcal{D}^{(m)}_{Y,n} \xrightarrow{\simeq} \bigoplus_{0\leq i\leq n} p_1^* \mathcal{D}^{(m)}_{X,i} \otimes_{\infty_Y} p_2^* \mathcal{D}^{(m)}_{Z,n-i} \tag{2.13}$$

Proof of proposition 2.3.6. Let $U \in \mathcal{S}$ and let $\xi^{-1}(U) \simeq U \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T}$ be a trivialization of ξ over U. We have the following isomorphisms of filtered $\mathfrak{o}$-algebras

$$\left(\xi_* \mathcal{D}^{(m)}_{\widetilde{X}}\right)^{\mathbb{T}} (U) = \mathcal{D}^{(m)}_{\widetilde{X}}(\xi^{-1}(U))^{\mathbb{T}} \simeq \mathcal{D}^{(m)}_{U\times\mathbb{T}}(U \times \mathbb{T})^{\mathbb{T}} \simeq \mathcal{D}^{(m)}_X(U) \otimes_{\mathfrak{o}} H^0(\mathbb{T}, \mathcal{D}^{(m)}_{\mathbb{T}})^{\mathbb{T}} \simeq \mathcal{D}^{(m)}_X(U) \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{T}),$$

where the first isomorphism follows from the fact that U trivializes the $\mathbb{T}$-torsor ξ, the second isomorphism becomes from (2.13) and the Kunneth formula [28, Theorem 6.7.8]), and the third isomorphism is given by (ii) in remark 1.5.4. Since the previous isomorphisms are compatible with restrictions to open affine subsets contained in U, we obtain the desired isomorphism of sheaves of filtered $\mathfrak{o}$-algebras. □

The inclusions $(\xi_* \mathcal{D}^{(m)}_{\widetilde{X},d})^{\mathbb{T}} \subset \xi_* \mathcal{D}^{(m)}_{\widetilde{X},d}$ induce a graded monomorphism $gr_\bullet(\widetilde{\mathcal{D}^{(m)}}) \hookrightarrow gr_\bullet(\xi_* \mathcal{D}^{(m)}_{\widetilde{X}})$.

Proposition 2.3.7. *If $\xi : \tilde{X} \to X$ is a locally trivial $\mathbb{T}$-torsor, then there exists a canonical and graded isomorphism*

$$Sym^{(m)}\left(\left(\xi_* \mathcal{T}_{\widetilde{X}}\right)^{\mathbb{T}}\right) \xrightarrow{\simeq} gr_\bullet\left(\widetilde{\mathcal{D}^{(m)}}\right).$$

Proof. We will divide the proof into two cases. We will first consider the case $m = 0$ and then we will generalize for all $m \in \mathbb{Z}_{>0}$.

Case 1. Let us suppose that $m = 0$. By the remark given after the proposition 1.2.2 in [34] we know that if $\mathcal{L}$ is a locally free $\mathcal{O}_{\widetilde{X}}$-module of finite rank then $\mathrm{Sym}^{(0)}(\mathcal{L}) = \mathbf{S}(\mathcal{L})$ is the symmetric algebra of $\mathcal{L}$. By (1.3), which is true for every $m \in \mathbb{N}$ (cf. [34, Proposition 1.3.7.3]), we have a canonical isomorphism of graded $\mathcal{O}_{\widetilde{X}}$-algebras

$$\mathbf{S}(\mathcal{T}_{\widetilde{X}}) \xrightarrow{\simeq} \mathrm{gr}_{\bullet}\left(\mathcal{D}^{(0)}_{\widetilde{X}}\right).$$

Applying the direct image functor ξ_* to the preceding isomorphism and then taking $\mathbb{T}$-invariants sections (both functors being exact by lemma 2.1.3 and the fact that $\mathbb{T}$ is diagonalisable [40, Part I, Lemma 4.3 (b)]) we get an isomorphism

$$\left(\xi_*\mathbf{S}\left(\mathcal{T}_{\widetilde{X}}\right)\right)^{\mathbb{T}} \xrightarrow{\simeq} \mathrm{gr}_{\bullet}\left(\left(\xi_*\mathcal{D}^{(0)}_{\widetilde{X}}\right)^{\mathbb{T}}\right).$$

We remark for the reader that the left-hand side of the previous isomorphism is well defined by remark 2.3.1. To complete the proof of the first case, we need to show that $(\xi_*\ \mathbf{S}(\mathcal{T}_{\widetilde{X}}))^{\mathbb{T}} = \mathbf{S}\left((\xi_*\mathcal{T}_{\widetilde{X}})^{\mathbb{T}}\right)$. To do that, we start by considering the canonical map of $\mathcal{O}_X$-modules

$$\left(\xi_*\mathcal{T}_{\widetilde{X}}\right)^{\mathbb{T}} \to \left(\xi_*\mathbf{S}\left(\mathcal{T}_{\widetilde{X}}\right)\right)^{\mathbb{T}}$$

which induces, by universal property of $\mathbf{S}(\bullet)$, a canonical morphism of graded $\mathcal{O}_X$-algebras

$$\mathbf{S}\left(\left(\xi_*\mathcal{T}_{\widetilde{X}}\right)^{\mathbb{T}}\right) \xrightarrow{\varphi} \left(\xi_*\mathbf{S}\left(\mathcal{T}_{\widetilde{X}}\right)\right)^{\mathbb{T}}.$$

Let us see that φ is indeed an isomorphism. Let us take $U \in \mathcal{S}$. We have a commutative diagram

$$\begin{array}{ccc} \widetilde{U} := \xi^{-1}(U) & \xrightarrow{\quad\simeq\quad} & U \times_{\mathfrak{o}} \mathbb{T} \\ & {\scriptstyle \xi}\searrow \quad \swarrow{\scriptstyle \mathrm{p}_1} & \\ & U & \end{array}$$

which tells us that (cf. [30, Section II, exercise 8.3])

$$\mathcal{T}_{\widetilde{U}} = \xi^*\mathcal{T}_U \oplus \mathrm{p}_2^*\mathcal{T}_{\mathbb{T}} = \xi^*\mathcal{T}_U \oplus \left(\mathcal{O}_{\widetilde{U}} \otimes_{\mathfrak{o}} \mathfrak{t}\right). \tag{2.14}$$

By (2.10), we have

$$\mathbf{S}\left(\left(\xi_*\mathcal{T}_{\widetilde{X}}\right)^{\mathbb{T}}\right)(U) = \mathbf{S}\left(\left(\xi_*\mathcal{T}_{\widetilde{X}}(U)\right)^{\mathbb{T}}\right) = \mathbf{S}\left(\mathcal{T}_U(U) \oplus \left(\mathcal{O}_U(U) \otimes_{\mathfrak{o}} \mathfrak{t}\right)\right).$$

On the other hand, by (2.14) we have the following relation

$$\mathbf{S}(\mathcal{T}_{\widetilde{U}}) = \mathbf{S}\left(\xi^*\mathcal{T}_U\right) \otimes_{\mathcal{O}_{\widetilde{U}}} \mathbf{S}\left(\mathcal{O}_{\widetilde{U}} \otimes_{\mathfrak{o}} \mathfrak{t}\right) = \xi^*\mathbf{S}\left(\mathcal{T}_U\right) \otimes_{\mathcal{O}_{\widetilde{U}}} \mathbf{S}\left(\mathcal{O}_{\widetilde{U}} \otimes_{\mathfrak{o}} \mathfrak{t}\right) \tag{2.15}$$

which implies, by the projection formula [30, Chapter II, section 5, exercise 5.1 (d)] that

$$\begin{aligned} \xi_*\mathbf{S}(\mathcal{T}_{\widetilde{U}}) &= \xi_*\left(\xi^*\mathbf{S}\left(\mathcal{T}_U\right) \otimes_{\mathcal{O}_{\widetilde{U}}} \mathbf{S}\left(\mathcal{O}_{\widetilde{U}} \otimes_{\mathfrak{o}} \mathfrak{t}\right)\right) \\ &= \mathbf{S}\left(\mathcal{T}_U\right) \otimes_{\mathcal{O}_U} \xi_*\mathbf{S}\left(\mathcal{O}_{\widetilde{U}} \otimes_{\mathfrak{o}} \mathfrak{t}\right). \end{aligned} \tag{2.16}$$

Taking $\mathbb{T}$-invariants and sections on U we get

$$\left(\xi_*\mathbf{S}(\mathcal{T}_{\widetilde{U}})\right)^{\mathbb{T}}(U) = \mathbf{S}\left(\mathcal{T}_U(U)\right) \otimes_{\mathcal{O}_U(U)} \mathbf{S}(\mathcal{O}_U(U) \otimes_{\mathfrak{o}} \mathfrak{t}).$$

Summing up, we have the following commutative diagram

$$\begin{array}{ccc}
\mathbf{S}\left(\left(\xi_*\mathcal{T}_{\widetilde{X}}\right)^{\mathbb{T}}\right)(U) & \xrightarrow{\quad\varphi_U\quad} & \left(\xi_*\mathbf{S}(\mathcal{T}_{\widetilde{X}})\right)^{\mathbb{T}}(U) \\
\downarrow\wr & & \downarrow\wr \\
\mathbf{S}\left(\mathcal{T}_U(U)\oplus\left(\mathcal{O}_U(U)\otimes_{\mathfrak{o}}\mathfrak{t}\right)\right) & \xrightarrow{\simeq} & \mathbf{S}\left(\mathcal{T}_U(U)\right)\otimes_{\mathcal{O}_U(U)}\mathbf{S}(\mathcal{O}_U(U)\otimes_{\mathfrak{o}}\mathfrak{t})
\end{array}$$

which ends the proof of the first case because $\mathcal{S}$ is a base for the Zariski topology of X.

Case 2. Let us suppose now that $m\in\mathbb{Z}_{>0}$. Exactly as we have done at the beginning of case 1, applying ξ_* to the isomorphism (1.3) and then taking $\mathbb{T}$-invariant sections, we get a canonical isomorphism of graded $\mathcal{O}_X$-algebras

$$\left(\xi_*\mathrm{Sym}^{(m)}\left(\mathcal{T}_{\widetilde{X}}\right)\right)^{\mathbb{T}}\xrightarrow{\simeq}\mathrm{gr}_{\bullet}\left(\left(\xi_*\mathcal{D}^{(m)}_{\widetilde{X}}\right)^{\mathbb{T}}\right).$$

We want to see that the map φ, built in case 1, induces an isomorphism

$$\mathrm{Sym}^{(m)}\left(\left(\xi_*\mathcal{T}_{\widetilde{X}}\right)^{\mathbb{T}}\right)\simeq\left(\xi_*\mathrm{Sym}^{(m)}\left(\mathcal{T}_{\widetilde{X}}\right)\right)^{\mathbb{T}}.$$

To do that, we take $U\in\mathcal{S}$ and we begin by noticing that analogously to case 1 the relation (2.10) gives us

$$\mathrm{Sym}^{(m)}\left(\left(\xi_*\mathcal{T}_{\widetilde{X}}\right)^{\mathbb{T}}\right)(U)=\mathrm{Sym}^{(m)}\left(\mathcal{T}_U(U)\oplus\left(\mathcal{O}_U(U)\otimes_{\mathfrak{o}}\mathfrak{t}\right)\right).\tag{2.17}$$

Moreover, the relation (2.10) and [34, Proposition 1.3.5] give us

$$\mathrm{Sym}^{(m)}\left(\mathcal{T}_{\widetilde{U}}\right)=\mathrm{Sym}^{(m)}\left(\xi^*\mathcal{T}_U\right)\otimes_{\mathcal{O}_{\widetilde{U}}}\mathrm{Sym}^{(m)}\left(\mathcal{O}_{\widetilde{U}}\otimes_{\mathfrak{o}}\mathfrak{t}\right)$$

which, following the same arguments that in (2.15) and (2.16), implies that

$$\left(\xi_*\mathrm{Sym}^{(m)}\left(\mathcal{T}_{\widetilde{X}}\right)\right)^{\mathbb{T}}(U)=\mathrm{Sym}^{(m)}\left(\mathcal{T}_U(U)\right)\otimes_{\mathcal{O}_U(U)}\mathrm{Sym}^{(m)}\left(\mathcal{O}_U(U)\otimes_{\mathfrak{o}}\mathfrak{t}\right).\tag{2.18}$$

Again, by [34, Proposition 1.3.5], we have that (2.17) and (2.18) are canonically isomorphic, so in order to globalize this map, which we denote by $\varphi_U^{(m)}$, we need to check that the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Sym}^{(m)}\left(\mathcal{T}_U(U)\oplus\left(\mathcal{O}_U(U)\otimes_{\mathfrak{o}}\mathfrak{t}\right)\right) & \hookrightarrow & \mathbf{S}\left(\mathcal{T}_U(U)\oplus\left(\mathcal{O}_U(U)\otimes_{\mathfrak{o}}\mathfrak{t}\right)\right)\otimes_{\mathfrak{o}}L \\
\downarrow\varphi_U^{(m)} & & \downarrow\varphi_U\otimes_{\mathfrak{o}}1_L \\
\mathrm{Sym}^{(m)}\left(\mathcal{T}_U(U)\right)\otimes_{\mathcal{O}_U(U)}\mathrm{Sym}^{(m)}\left(\mathcal{O}_U(U)\otimes_{\mathfrak{o}}\mathfrak{t}\right) & \hookrightarrow & \mathbf{S}\left(\mathcal{T}_U(U)\right)\otimes_{\mathcal{O}_U(U)}\mathbf{S}\left(\mathcal{O}_U(U)\otimes_{\mathfrak{o}}\mathfrak{t}\right)\otimes_{\mathfrak{o}}L.
\end{array}$$

Shrinking U if necessary, we can suppose that U is endowed with a set of local coordinates $x_1,\ \dots,\ x_N$, in such a way that if $\mathcal{T}_U(U)$ is generated as $\mathcal{O}_U(U)$-module by the derivations $\partial_{x_1},\ \dots,\ \partial_{x_N}$, and if $\zeta_1,\ \dots,\ \zeta_l$ denotes an $\mathfrak{o}$-basis of $\mathfrak{t}$, then $\mathrm{Sym}^{(m)}\left(\mathcal{T}_U(U)\oplus\left(\mathcal{O}_U(U)\otimes_{\mathfrak{o}}\mathfrak{t}\right)\right)$ is generated (as $\mathcal{O}_U(U)$-module) by all the elements of the form $\underline{\partial}^{<\underline{k}>}\cdot\underline{\zeta}^{<\underline{\upsilon}>}$ [3]. In particular,

$$\varphi_U^{(m)}\left(\underline{\partial}^{<\underline{k}>}\cdot\underline{\zeta}^{<\underline{\upsilon}>}\right)=\varphi_U\otimes_{\mathfrak{o}}1_L\left(\underline{\partial}^{<\underline{k}>}\cdot\underline{\zeta}^{<\underline{\upsilon}>}\right)=\frac{\underline{k}!}{q_{\underline{k}}!}\frac{\underline{\upsilon}!}{q_{\underline{\upsilon}}!}\underline{\partial}^{\underline{k}}\otimes_L\underline{\zeta}^{\underline{\upsilon}}.$$

which shows that the preceding diagram is commutative. This ends the proof of the proposition. □

[3] Here we use the multi-index notation introduced in sections 1.2 and 1.4.

2.4 Affine algebraic groups and homogeneous spaces

Let us suppose that $\mathbb{G}$ is a split connected reductive group scheme over $\mathfrak{o}$ and $\mathbb{T}$ is a split maximal torus in $\mathbb{G}$. As we know, the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})$ is a $\mathbb{T}$-module via the adjoint representation [40, I, 7.18] and the decomposition into weight spaces has the form

$$\mathrm{Lie}(\mathbb{G}) = \mathrm{Lie}(\mathbb{T}) \oplus \bigoplus_{\alpha\in\Lambda} (\mathrm{Lie}(\mathbb{G}))_{\alpha}.$$

Here Λ is the subset of $X(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ of non-zero weights of $\mathrm{Lie}(\mathbb{G})$, this means the roots of $\mathbb{G}$ with respect to $\mathbb{T}$. For each $\alpha \in \Lambda$ there exists a homomorphism $x_\alpha : \mathbb{G}_a \to \mathbb{G}$ satisfying

$$t\, x_\alpha(a)\, t^{-1} = x_\alpha(\alpha(t)\, a), \tag{2.19}$$

for any $\mathfrak{o}$-algebra A and all $t \in \mathbb{T}(A)$, and such that the tangent map $dx_\alpha : \mathrm{Lie}(\mathbb{G}_a) \to (\mathrm{Lie}(\mathbb{G}))_\alpha$ is an isomorphism [40, II, 1.2]. This homomorphism defines a functor $A \mapsto x_\alpha(\mathbb{G}_a(A))$ (A being an $\mathfrak{o}$-algebra) which is a closed subgroup of $\mathbb{G}$ and it is denoted by U_α. By definition we have $\mathrm{Lie}(U_\alpha) = (\mathrm{Lie}(\mathbb{G}))_\alpha$ and by (2.19) it is clear the $\mathbb{T}$ normalises U_α.
Now, let us choose a positive system $\Lambda^+ \subset \Lambda$. It is known that Λ^+ and $-\Lambda^+$ are unipotent and closed subsets of Λ[4] [40, II, 1.7]. Let $\mathbf{N}$ be the closed subgroup of $\mathbb{G}$ generated by all U_α with $\alpha \in \Lambda^+$. As we have remarked $\mathbb{T}$ normalises $\mathbf{N}$. We set

$$\mathbb{B} = \mathbf{N} \rtimes \mathbb{T} \tag{2.20}$$

a Borel subgroup of $\mathbb{G}$. With this terminology $\mathbf{N}$ is called the *unipotent radical* of $\mathbb{B}$. We put

$$\widetilde{X} := \mathbb{G}/\mathbf{N}, \quad X := \mathbb{G}/\mathbb{B}$$

for the corresponding quotients (the basic affine space and the flag scheme of $\mathbb{G}$ [1, subsection 4.7]). As $\mathfrak{o}$ is a discret valuation ring these are smooth and separated schemes over $\mathfrak{o}$ [1, Lemma 4.7 (a)].

Remark 2.4.1. *For technical reasons (cf. Proposition 1.5.3) in this work we will suppose that the group $\mathbb{G}$, and the schemes $\widetilde{X}$ and X are endowed with the right regular $\mathbb{G}$-action. This means that for any $\mathfrak{o}$-algebra A and $g_0, g \in \mathbb{G}(A)$ we have*

$$g_0 \bullet g = g^{-1} g_0, \quad g_0\, N(A) \bullet g = g^{-1} g_0\, N(A) \ \ \textit{and}\ \ g_0\, \mathbb{B}(A) \bullet g = g^{-1} g_0\, \mathbb{B}(A).$$

Under this actions, the canonical projections $\mathbb{G} \to \widetilde{X}$ and $\mathbb{G} \to X$ are clearly $\mathbb{G}$-equivariant.

Now, as $\mathbb{T}$ normalises $\mathbf{N}$ we have $\mathbf{N}\mathbb{T} = \mathbb{T}\mathbf{N}$ and therefore

$$(g\mathbf{N}(A)).t \subset g\mathbb{T}(A)\mathbf{N}(A),$$

for any $\mathfrak{o}$-algebra A, $g \in \mathbb{G}(A)$ and $t \in \mathbb{T}(A)$. This defines a right $\mathbb{T}$-action on $\widetilde{X}$ which clearly commutes with the right regular $\mathbb{G}$-action (cf. Remark 2.4.1). Moreover, this right $\mathbb{T}$-action makes the canonical projection $\xi : \widetilde{X} \to X$ a $\mathbb{T}$-torsor for the Zariski topology of X. To see this we recall first that from (2.20) the *abstract Cartan group* $\mathbf{H} := \mathbb{B}/\mathbf{N}$ is canonical isomorphic to $\mathbb{T}$. Let us consider the covering of X given by the open subschemes U_w, $w \in W := N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ (the Weyl group) where

$$U_w := \text{image of } w\mathbf{N}\mathbb{B}$$

under the canonical projection $\mathbb{G} \to X$. For every $w \in W$ we can find a morphism $\pi_w : U_w \to \mathbb{G}$ splitting the projection

[4] $\Lambda^+ \cap (-\Lambda^+) = \emptyset$ and $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Lambda \subset \Lambda^+$ for any $\alpha, \beta \in \Lambda^+$.

map $\mathbb{G} \to X$. These maps gives a map $\overline{\pi}_w : U_w \to \widetilde{X}$ such that $\xi\circ\overline{\pi}_w = id_{U_w}$. The map $(u, b\mathbf{N}) \mapsto \pi_w(u)b\mathbf{N}$ is the required $\mathbb{T}$-invariant isomorphism $U_w \times \mathbb{T} \xrightarrow{\simeq} U_w \times \mathbf{H} \xrightarrow{\simeq} \xi^{-1}(U_w)$. Now we can apply [45, Chaper III, Proposition 4.1 (b)]. As in definition 2.1.2 we denote by $\mathcal{S}$ the set of all affine open subsets of X that trivialise the torsor ξ. This forms a base for the Zariski topology of X.

2.5 Relative enveloping algebras of finite level on homogeneous spaces

In this section we adopt the notation of the preceding section. In particular, we recall for the reader that the set $\mathcal{S}$, of all affine open subsets of X that trivialise the torsor ξ forms a base for the Zariski topology of X. Let us recall that by proposition 1.5.3 and remark 2.4.1 the right regular $\mathbb{G}$-action on $\widetilde{X}$ (introduced in remark 2.4.1) induces a homomorphism $\Phi^{(m)} : D^{(m)}(\mathbb{G}) \to H^0\left(\widetilde{X}, \mathcal{D}^{(m)}_{\widetilde{X}}\right)$ which equals the operator-representation (notation in section 1.5 (remark 1.5.5))

$$\psi_{\widetilde{X}} : \mathcal{U}(\mathfrak{g}_L) \to H^0\left(\widetilde{X}_L, \mathcal{D}_{\widetilde{X}}\right)$$

if we tensor with L ($\mathcal{D}_{\widetilde{X}_L}$ denotes the usual sheaf of differential operators on $\widetilde{X}_L$). Let us consider the base change $\mathbb{T}_L := \mathbb{T} \times_{\mathfrak{o}} \mathrm{Spec}(L)$. We know by [40, Part I, 2.10 (3)] that

$$H^0\left(\widetilde{X}_L, \mathcal{D}_{\widetilde{X}_L}\right)^{\mathbb{T}_L} = H^0\left(\widetilde{X}, \mathcal{D}^{(m)}_{\widetilde{X}}\right)^{\mathbb{T}} \otimes_{\mathfrak{o}} L. \tag{2.21}$$

Given that the right regular $\mathbb{G}$-action on $\widetilde{X}$ commutes with the right action of the torus $\mathbb{T}$, the vector fields by which $\mathfrak{g}_L$ acts on $\widetilde{X}_L$ must be invariant under the $\mathbb{T}_L$-action [11, Lemma 4.5]. This means that the *operator-representation* $\psi_{\widetilde{X}}$ satisfies

$$\psi_{\widetilde{X}}(\mathfrak{g}_L) \subset H^0(\widetilde{X}_L, \mathcal{D}_{\widetilde{X}_L})^{\mathbb{T}_L}.$$

The relation (2.21) tells us that for every $x \in D^{(m)}(\mathbb{G})$ there exists $k(x) \in \mathbb{N}$ (a natural number that depends of x) such that

$$\varpi^{k(x)}\Phi^{(m)}(x) \subset H^0(\widetilde{X}, \mathcal{D}^{(m)}_{\widetilde{X}})^{\mathbb{T}}.$$

Since the $\mathbb{T}$-action on $H^0(\widetilde{X}, \mathcal{D}^{(m)}_{\widetilde{X}})$ is $\mathfrak{o}$-linear, for every $\mathfrak{o}$-algebra A and every $t \in \mathbb{T}(A)$ we have

$$\varpi^{k(x)}\Phi^{(m)}(x) = t.\left(\varpi^{k(x)}\Phi^{(m)}(x)\right) = \varpi^{k(x)}\left(t.\Phi^{(m)}(x)\right). \tag{2.22}$$

Since $\widetilde{X}$ is a smooth $\mathfrak{o}$-scheme, the local description (1.1) tells us that the sheaf $\mathcal{D}^{(m)}_{\widetilde{X}}$ is ϖ-torsion free. In particular, $H^0(\widetilde{X}, \mathcal{D}^{(m)}_{\widetilde{X}})$ is also ϖ-torsion free and therefore, from (2.22), we have that $\Phi^{(m)}$ induces the filtered morphism

$$\Phi^{(m)} : D^{(m)}(\mathbb{G}) \to H^0\left(\widetilde{X}, \mathcal{D}^{(m)}_{\widetilde{X}}\right)^{\mathbb{T}} = H^0\left(X, \xi_*\mathcal{D}^{(m)}_{\widetilde{X}}\right)^{\mathbb{T}}.$$

From the preceding reasoning we have an $\mathcal{O}_X$-morphism of sheaves of filtered $\mathfrak{o}$-algebras

$$\Phi^{(m)}_X : \mathcal{A}^{(m)}_X \to \widetilde{\mathcal{D}^{(m)}}. \tag{2.23}$$

The sheaf $\mathcal{A}^{(m)}_X := \mathcal{O}_X \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G})$ of associative $\mathfrak{o}$-algebras has been introduced in the subsection 2.5. We recall for the reader that this is an integral model of the sheaf $\mathcal{U}^{\circ} := \mathcal{O}_{X_L} \otimes_L \mathcal{U}(\mathfrak{g}_L)$.

To twist the sheaves $\widetilde{\mathcal{D}^{(m)}}$ introduced in section 2.3 (definition 2.3.5), we consider the classical distribution algebra as in [21, Chapter II, 4.6.1]. To define it, we suppose that $\varepsilon : \mathrm{Spec}(\mathfrak{o}) \to \mathbb{T}$ is the identity of $\mathbb{T}$ and we take $J := \{f \in$

$\mathfrak{o}[\mathbb{T}]|\ f(\varepsilon)=0\}$. Then $\mathfrak{o}[\mathbb{T}]=\mathfrak{o}\oplus J$. We put

$$\mathrm{Dist}_n(\mathbb{T}) := \left(\mathfrak{o}[\mathbb{T}]/J^{n+1}\right)^* = \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{o}[\mathbb{T}]/J^{n+1},\mathfrak{o}) \subset \mathfrak{o}[\mathbb{T}]^*$$

the space of distributions of order n, and then $\mathrm{Dist}(\mathbb{T}) := \varinjlim_{n\in\mathbb{N}} \mathrm{Dist}_n(\mathbb{T})$. Moreover, if $\Delta_{\mathbb{T}} : \mathfrak{o}[\mathbb{T}] \to \mathfrak{o}[\mathbb{T}]\otimes_{\mathfrak{o}}\mathfrak{o}[\mathbb{T}]$ denotes the comorphism associated to the multiplication of $\mathbb{T}$, $\varepsilon_{\mathbb{T}} : \mathfrak{o}[\mathbb{T}] \to \mathfrak{o}$ is the counit associated to the identity element and $i^*_{\mathbb{T}} : \mathfrak{o}[\mathbb{T}] \to \mathfrak{o}[\mathbb{T}]$ is the coinverse (these maps defining a structure of Hopf algebra on $\mathfrak{o}[\mathbb{T}]$), then the product

$$uv : \mathfrak{o}[\mathbb{T}] \xrightarrow{\Delta_{\mathbb{T}}} \mathfrak{o}[\mathbb{T}]\otimes_{\mathfrak{o}}\mathfrak{o}[\mathbb{T}] \xrightarrow{u\otimes v} \mathfrak{o} \qquad u,v\in\mathfrak{o}[\mathbb{T}]^*$$

defines a structure of algebra on $\mathfrak{o}[\mathbb{T}]^*$ and $\mathrm{Dist}(\mathbb{T})$ is a subalgebra with $\mathrm{Dist}_n(\mathbb{T}).\mathrm{Dist}_m(\mathbb{T}) \subset \mathrm{Dist}_{m+n}(\mathbb{T})$ [40, Part I, 7.7]. Furthermore, $\mathrm{Dist}_n(\mathbb{T}) \simeq \mathfrak{o}\oplus(J/J^{n+1})^*$.

Proposition 2.5.1. *[38, Subsection 4.1]*

(i) The applications $Hom_{\mathfrak{o}}(\psi_{m,m'},\mathfrak{o}) : D^{(m)}(\mathbb{T}) \to D^{(m')}(\mathbb{T})$, with $\psi_{m,m'}$ as in subsection 1.4, induce an isomorphism of filtered $\mathfrak{o}$-algebras $\varinjlim_{m\in\mathbb{N}} D^{(m)}(\mathbb{T}) \xrightarrow{\simeq} Dist(\mathbb{T})$.

(ii) The distribution algebra $Dist(\mathbb{T})$ is an integral model of $\mathcal{U}(\mathfrak{t}_L)$, this means that $Dist(\mathbb{T})\otimes_{\mathfrak{o}} L = \mathcal{U}(\mathfrak{t}_L)$.

Example 2.5.1. *Let us suppose that $\mathbb{T} = \mathbb{G}_m = Spec(\mathfrak{o}[T,T^{-1}])$. In this case J is generated by $T-1$ and the residue classes of $1,\ T-1,\ \dots,\ (T-1)^n$ form a basis of $\mathfrak{o}[\mathbb{T}]/J^{n+1}$. Let $\delta_n \in Dist(\mathbb{T})$ such that $\delta_n((T-1)^i)=\delta_{n,i}$ (the Kronecker delta). By [40, Part I, 7.8] all δ_n with $n\in\mathbb{N}$ form a basis of $Dist(\mathbb{T})$ and they satisfy the relation*

$$n!\delta_n = \delta_1(\delta_1-1)\dots(\delta_1-n+1). \tag{2.24}$$

Therefore $Dist(\mathbb{T})\otimes_{\mathfrak{o}} L = L[\delta_1]$. Since $\mathfrak{t} = (J/J^2)^*$ we can conclude that $Dist(\mathbb{T})\otimes_{\mathfrak{o}} L = \mathcal{U}(\mathfrak{t}_L)$.

The preceding proposition in particular implies that every morphism of $\mathfrak{o}$-algebras $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ induces for every $m\in\mathbb{N}$ a morphism of $\mathfrak{o}$-algebras $\lambda^{(m)} : D^{(m)}(\mathbb{T}) \to \mathfrak{o}$.

2.5.2. *Let us clarify the mysterious characters $\lambda : Dist(\mathbb{T}) \to \mathfrak{o}$. Let us suppose first that $L=\mathbb{Q}_p$ and that $\mathbb{T}=\mathbb{G}_m = Spec(\mathbb{Z}_p[T,T^{-1}])$. By the preceding example we know that the set of distributions $\{\delta_n\}_{n\in\mathbb{N}}$, where $\delta_n((T-1)^i)=0$ if $i<n$ and $\delta_n((T-1)^n)=1$, is a basis for $Dist(\mathbb{T})$. Moreover, $Dist(\mathbb{T})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = \mathbb{Q}_p[\delta_1]$. Now, let us take $\lambda\in\mathfrak{t}^*$, which induces a morphism of algebras $\lambda : \mathcal{U}(\mathfrak{t}) \to \mathbb{Z}_p$. Taking the tensor product with $\mathbb{Q}_p$ and using the canonical isomorphism $Dist(\mathbb{T})_{\mathbb{Q}_p} \simeq \mathcal{U}(\mathfrak{t}_{\mathbb{Q}_p})$ we obtain a character $\lambda : Dist(\mathbb{T}) \to \mathbb{Q}_p$ (of course, here we assume $Dist(\mathbb{T}) \subset Dist(\mathbb{T})_{\mathbb{Q}_p}$). To see that its image is contained in $\mathbb{Z}_p$, we need to check that $\lambda(\delta_n)\in\mathbb{Z}_p$. By (2.24) we have*

$$\lambda(\delta_n) = \lambda\left(\binom{\delta_1}{n}\right) = \binom{\lambda(\delta_1)}{n} \in \mathbb{Z}_p.$$

Where we have used the fact that the binomial coefficients extend to functions from $\mathbb{Z}_p$ to $\mathbb{Z}_p$ and the fact that $\delta_1\in\mathfrak{t}$. In the case of an arbitrary split maximal torus $\mathbb{T} = \mathbb{G}_m \times_{Spec(\mathbb{Z}_p)} \dots \times_{Spec(\mathbb{Z}_p)} \mathbb{G}_m$ (n-times), the reader can follow the same reasoning using the canonical isomorphism $Dist(\mathbb{T}) = Dist(\mathbb{G}_m)\otimes_{\mathbb{Z}_p}\dots\otimes_{\mathbb{Z}_p} Dist(\mathbb{G}_m)$ (n-times) [40, Part I, 7.9 (3)]. We have therefore, in the case $L=\mathbb{Q}_p$, a correspondence between the characters of $\mathfrak{t}$ (the Lie algebra of a split maximal torus $\mathbb{T}\subset\mathbb{G}$) and the characters of the distribution algebra studied in this text. Moreover, we have an isomorphism of $\mathbb{Z}_p$-modules

$$Hom_{\mathbb{Z}_p-mods}\left(\mathfrak{t},\mathbb{Z}_p\right) \xrightarrow{\simeq} Hom_{\mathbb{Z}_p-alg}\left(Dist(\mathbb{T}),\mathbb{Z}_p\right). \tag{2.25}$$

Now, let us take a finite extension $L|\mathbb{Q}_p$ and let us suppose that $\mathbb{T}$ is a split maximal torus of $\mathbb{G}$ (and therefore a group scheme over $\mathfrak{o}$). Let $\mathbb{T}'$ be a split model of $\mathbb{T}$ over $\mathbb{Z}_p$. If $\mathfrak{t}'$ denotes the $\mathbb{Z}_p$-Lie algebra of $\mathbb{T}'$, then the relation (2.25) gives us an isomorphism of $\mathfrak{o}$-modules

$$Hom_{\mathfrak{o}-mods}\left(\mathfrak{t}'\otimes_{\mathbb{Z}_p}\mathfrak{o},\mathfrak{o}\right)\xrightarrow{\simeq} Hom_{\mathfrak{o}-alg}\left(Dist\left(\mathbb{T}'\times_{Spec(\mathbb{Z}_p)}Spec(\mathfrak{o})\right),\mathfrak{o}\right) = Hom_{\mathfrak{o}-alg}\left(Dist(\mathbb{T}),\mathfrak{o}\right), \tag{2.26}$$

and we can conclude that we also have the stated correspondence for finite extensions of $\mathbb{Q}_p$.

2.5.3. *Let us consider the positive system $\Lambda^+\subset\Lambda\subset X(\mathbb{T})$ ($X(\mathbb{T})$ the group of algebraic characters) associated to the Borel subgroup scheme $\mathbb{B}\subset\mathbb{G}$ defined in the preceding subsection. The Weyl subgroup $W=N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ acts naturally on the space $\mathfrak{t}_L^*:=Hom_L(\mathfrak{t}_L,L)$, and via differentiation $d : X(\mathbb{T})\hookrightarrow\mathfrak{t}^*$ we may view $X(\mathbb{T})$ as a subgroup of $\mathfrak{t}^*$ in such a way that $X^*(\mathbb{T})\otimes_{\mathfrak{o}}L=\mathfrak{t}_L^*$. Let $\rho=\frac{1}{2}\sum_{\alpha\in\Lambda^+}\alpha$ be the so-called Weyl vector. Let $\check{\alpha}$ be a coroot of $\alpha\in\Lambda$ viewed as an element of $\mathfrak{t}_L$. An arbitrary weight $\lambda\in\mathfrak{t}_L^*$ is called dominant if $\lambda(\check{\alpha})\geq 0$ for all $\alpha\in\Lambda^+$. The weight λ is called regular if its stabilizer under the W-action is trivial.*

NOTATION: In the sequel we will refer to a character $\lambda\in\mathfrak{t}_L^*$ as an L-linear application induced, via base change, by a character $\lambda : \mathrm{Dist}(\mathbb{T})\to\mathfrak{o}$ of the distribution algebra of the torus $\mathbb{T}$.

We recall for the reader that $D^{(m)}(\mathbb{T})$ is also an integral model of the universal enveloping algebra $\mathcal{U}(\mathfrak{t}_L)$.

Definition 2.5.4. *We say that a morphism of $\mathfrak{o}$-algebras $\lambda : Dist(\mathbb{T})\to\mathfrak{o}$ is a character of the distribution algebra $Dist(\mathbb{T})$. We say that a character $\lambda : Dist(\mathbb{T})\to\mathfrak{o}$ is a dominant and regular character if the L-linear map induced by tensoring with L is a dominant and regular character of $\mathfrak{t}_L$.*

Let $\lambda : \mathrm{Dist}(\mathbb{T})\to\mathfrak{o}$ be a character of the distribution algebra of $\mathbb{T}$. For every $m\in\mathbb{N}$ we denote by $\mathfrak{o}_{\lambda^{(m)}}$ the ring $\mathfrak{o}$ considered as a $D^{(m)}(\mathbb{T})$-module via $\lambda^{(m)}$.

The reader can easily verify the following elementary lemma.

Lemma 2.5.5. *Let A be an L-algebra and $A_0\subset A$ an $\mathfrak{o}$-subalgebra such that $A_0\otimes_{\mathfrak{o}}L=A$. If $Z(A)$ denotes the center of A (resp. $Z(A_0)$ denotes the center of A_0), then $Z(A)=Z(A_0)\otimes_{\mathfrak{o}}L$.*

Let us consider $\mathcal{D}_{\tilde{X}_L}$ the usual sheaf of differential operators [29, 47] on $\tilde{X}_L := \tilde{X}\times_{\mathrm{Spec}(\mathfrak{o})}\mathrm{Spec}(L)$ (resp. $X_L := X\times_{\mathrm{Spec}(\mathfrak{o})}\mathrm{Spec}(L)$ and $\mathbb{T}_L=\mathbb{T}\times_{\mathrm{Spec}(\mathfrak{o})}\mathrm{Spec}(L)$). By [40, Part I, 2.10 (3)] we have

$$H^0\left(X_L,(\xi\times_{\mathfrak{o}}id_L)_*\mathcal{D}_{\tilde{X}_L}\right)^{\mathbb{T}_L}=H^0(\tilde{X}_L,\mathcal{D}_{\tilde{X}_L})^{\mathbb{T}_L}=H^0(\tilde{X},\mathcal{D}^{(m)}_{\tilde{X}})^{\mathbb{T}}\otimes_{\mathfrak{o}}L. \tag{2.27}$$

On the other hand, we know by 1.5.3 that the right $\mathbb{T}$-action on $\tilde{X}$ induces a canonical morphism of filtered $\mathfrak{o}$-algebras

$$\Phi^{(m)}_{\mathbb{T}} : D^{(m)}(\mathbb{T})\to H^0(\tilde{X},\mathcal{D}^{(m)}_{\tilde{X}})$$

and by [4, page 7] $\Phi^{(m)}_{\mathbb{T}}\otimes_{\mathfrak{o}}L$ factors through the center of $H^0(X_L,(\xi\times_{\mathfrak{o}}id_L)_*\mathcal{D}_{\tilde{X}_L})^{\mathbb{T}_L}$. By (2.27) and the preceding lemma we have the following morphism

$$D^{(m)}(\mathbb{T})\hookrightarrow\mathcal{U}(\mathfrak{t}_L)\xrightarrow{\Phi^{(m)}_{\mathbb{T}}\otimes_{\mathfrak{o}}L} Z\left(H^0(\tilde{X},\mathcal{D}^{(m)}_{\tilde{X}})^{\mathbb{T}}\right)\otimes_{\mathfrak{o}}L$$

(we recall for the reader that $\mathfrak{t}_L := \mathrm{Lie}(\mathbb{T})\otimes_{\mathfrak{o}}L$ and that $D^{(m)}(\mathbb{T})\otimes_{\mathfrak{o}}L=\mathcal{U}(\mathfrak{t}_L)$, for every $m\in\mathbb{N}$). Following the same lines of reasoning that in page 43 we can conclude that $\Phi^{(m)}_{\mathbb{T}}$ induce a morphism of filtered $\mathfrak{o}$-algebras

$$\Phi^{(m)}_{\mathbb{T}} : D^{(m)}(\mathbb{T})\to H^0(X,Z(\widetilde{\mathcal{D}^{(m)}})). \tag{2.28}$$

Here $Z(\widetilde{\mathcal{D}^{(m)}})$ is the center of $\widetilde{\mathcal{D}^{(m)}}$ and its filtration is the one induced by (2.11). We have the following definition.

Definition 2.5.6. *Let $\lambda^{(m)} : D^{(m)}(\mathbb{T}) \to \mathfrak{o}$ be an integral character. We define the sheaf of level m integral twisted arithmetic differential operators $\mathcal{D}^{(m)}_{X,\lambda}$ on the flag scheme X by*

$$\mathcal{D}^{(m)}_{X,\lambda} := \widetilde{\mathcal{D}^{(m)}} \otimes_{D^{(m)}(\mathbb{T})} \mathfrak{o}_{\lambda^{(m)}}.$$

Remark 2.5.7. *For every $m \in \mathbb{N}$, the sheaf $\mathcal{D}^{(m)}_{X,\lambda}$ has clearly a ring structure coming from the ring structure of $\widetilde{\mathcal{D}^{(m)}}$. Moreover, by (2.28) we know that $Ker(\lambda^{(m)})\widetilde{\mathcal{D}^{(m)}}$ is a two-sided ideal of $\widetilde{\mathcal{D}^{(m)}}$ and we have a canonical isomorphism of sheaves of rings*

$$\mathcal{D}^{(m)}_{X,\lambda} = \widetilde{\mathcal{D}^{(m)}} \big/ Ker(\lambda^{(m)})\widetilde{\mathcal{D}^{(m)}}.$$

Tensor product filtration. Let $\mathcal{A}$ be a filtered sheaf of commutative rings on a topological space Y [10, A: III. 2]. Let $\mathcal{M}$ and $\mathcal{N}$ be filtered $\mathcal{A}$-modules [10, A: III. 2.5]. The sheaf of $\mathcal{A}$-modules $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ carries a natural filtration called *the tensor product filtration* and it is defined as follows. Let $n \in \mathbb{N}$ fix. For every $U \subset Y$ we let $F_n(\mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U))$ be the abelian subgroup of $\mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U)$ generated by elements of type $x \otimes y$ with $x \in \mathcal{M}_l(U)$, $y \in \mathcal{N}_s(U)$, and such that $l + s \leq n$. This process defines a presheaf on Y and we let $F_n(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$ be its sheafification. The sheaf $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ becomes therefore a filtered sheaf of $\mathcal{A}$-modules

$$F_0(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}) \subseteq ... \subseteq F_n(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}) \subseteq ... \subseteq \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}.$$

Moreover, for every open subset $U \subset Y$ we have a canonical map

$$gr_{\bullet}\,(\mathcal{M}(U)) \otimes_{gr_{\bullet}(\mathcal{A}(U))} gr_{\bullet}\,(\mathcal{N}(U)) \to gr_{\bullet}(\mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U))$$

by putting $x_{(l)} \otimes y_{(s)} \to (x \otimes y)_{l+s}$, where $x \in F_l\mathcal{M}(U) - F_{l-1}\mathcal{M}(U)$, $y \in F_s\mathcal{N}(U) - F_{s-1}\mathcal{N}(U)$ and $x_{(l)} := x + F_{l-1}\mathcal{M}(U)$, $y_{(s)} := y + F_{s-1}\mathcal{N}(U)$. Furthermore, these morphisms are compatible under restrictions and therefore, by the universal property of the sheafification, we get a morphism of graded sheaves

$$gr_{\bullet}(\mathcal{M}) \otimes_{gr_{\bullet}(\mathcal{A})} gr_{\bullet}(\mathcal{N}) \to gr_{\bullet}\left(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}\right).$$

Taking stalks, we finally see that the previous morphism is surjective by [33, Section I, 6.13].

If we endow $\mathfrak{o}_{\lambda^{(m)}}$ with the *trivial filtration* as a $D^{(m)}(\mathbb{T})$-module, this means $0 =: \mathrm{F}_{-1}\mathfrak{o}_{\lambda^{(m)}}$ and $\mathrm{F}_i\mathfrak{o}_{\lambda^{(m)}} := \mathfrak{o}_{\lambda^{(m)}}$ for all $(i \geq 0)$, then using (2.11) we can view $\mathcal{D}^{(m)}_{X,\lambda}$ as a sheaf of filtered $\mathfrak{o}$-algebras, equipped with the tensor product filtration.

Proposition 2.5.8. *Let $U \in \mathcal{S}$. Then $\mathcal{D}^{(m)}_{X,\lambda}|_U$ is isomorphic to $\mathcal{D}^{(m)}_X|_U$ as a sheaf of filtered $\mathfrak{o}$-algebras.*

Proof. Let us recall that by proposition 2.3.6 for every $U \in \mathcal{S}$ we have an isomorphism of filtered $\mathfrak{o}$-algebras

$$\widetilde{\mathcal{D}^{(m)}}|_U \xrightarrow{\simeq} \mathcal{D}^{(m)}_X|_U \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{T})$$

which induces an isomorphism $\mathcal{D}^{(m)}_{X,\lambda}|_U \xrightarrow{\simeq} \mathcal{D}^{(m)}_X|_U$ of filtered $\mathfrak{o}$-algebras. □

Remark 2.5.9. *This proposition justifies the name of "twisted arithmetic differential operators".*

Let us recall that, as X is a smooth $\mathfrak{o}$-scheme, the sheaf of Berthelot's differential operators $\mathcal{D}^{(m)}_X$ is a sheaf of $\mathcal{O}_X$-rings with noetherian sections over all open affine subsets of X [6, corollary 2.2.5]. As ξ is locally trivial, the family $\mathcal{S}$ forms a base for the Zariski topology of X and therefore the preceding proposition implies the following meaningful result (cf. [37, Proposition 2.2.2 (iii)]).

Proposition 2.5.10. *The sheaf $\mathcal{D}^{(m)}_{X,\lambda}$ is a sheaf of $\mathcal{O}_X$-rings with noetherian sections over all open affine subsets of X.*

Proof. Let $U \subseteq X$ be an affine open subset. Let us denote by $\mathcal{D}_\lambda := \mathcal{D}^{(m)}_{X,\lambda}|_U$, $D_\lambda := \Gamma(U, \mathcal{D}_\lambda)$ and $R := \Gamma(U, \mathcal{O}_X)$. Let $U = \cup_{1 \leq l \leq s} U_l$ be a finite cover of U by open $U_l \in \mathcal{S}$. Since by propositions 2.3.6 and 2.5.8 the sheaf $\mathcal{D}_\lambda$ is an inductive limit of coherent $\mathcal{O}_U$-modules, we have

$$\mathcal{D}_\lambda = \mathcal{O}_U \otimes_R D_\lambda,$$

and $\mathcal{D}_\lambda$ is a flat D_λ-module. Moreover, the preceding proposition tells us that $\mathcal{D}_\lambda(U_l)$ is noetherian for each l. Let (J_i) be an increasing sequence of (left) ideals of D_λ, and let us consider

$$\mathcal{J}_i = \mathcal{D}_\lambda \otimes_{D_\lambda} J_i = \mathcal{O}_U \otimes_R J_i$$

which is an increasing sequence of sheaves of (left) ideals of $\mathcal{D}_\lambda$ by flatness of $\mathcal{D}_\lambda$ over D_λ. By noetherianess, the increasing sequence of ideals $\Gamma(U_l, \mathcal{J}_i)$ of $\Gamma(U_l, \mathcal{D}_\lambda)$ is stationary. Furthermore, given that J_i is an inductive limit of finite type R-modules, $\mathcal{J}_i$ is an inductive limit of coherent $\mathcal{O}_U$-modules, thus for every $1 \leq l \leq s$ we have

$$\mathcal{J}_i|_{U_l} = \mathcal{O}_{U_l} \otimes_{\Gamma(U_l, \mathcal{O}_U)} \Gamma(U_l, \mathcal{J}_i),$$

which implies that for each $1 \leq l \leq s$ there exists $k(l) \in \mathbb{N}$ such that $\mathcal{J}_i|_{U_l} = \mathcal{J}_{k(l)}|_{U_l}$, for every $i \geq k(l)$. Therefore, if $k := \max\{k(l) \in \mathbb{N} \| \ 1 \leq l \leq s\}$, we have that $\mathcal{J}_i = \mathcal{J}_k$ and $J_i = J_k$, for every $i \geq k$, and thus both sequence are stationary. This ends the proof of the proposition. □

Definition 2.5.11. *We will denote by*

$$\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda} := \varprojlim_j \mathcal{D}^{(m)}_{X,\lambda} / p^{j+1} \mathcal{D}^{(m)}_{X,\lambda} \tag{2.29}$$

the p-adic completion of $\mathcal{D}^{(m)}_{X,\lambda}$ and we consider it as a sheaf on $\mathfrak{X}$. Following the notation given at the beginning of this work, the sheaf $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ will denote our sheaf of level m twisted arithmetic differential operators on the formal flag scheme $\mathfrak{X}$.

Proposition 2.5.12. *(i) There exists a basis $\mathscr{B}$ of the topology of $\mathfrak{X}$, consisting of open affine subsets, such that for every $\mathfrak{U} \in \mathscr{B}$ the ring $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}(\mathfrak{U})$ is twosided noetherian.*

(ii) The sheaf of rings $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ is coherent.

Proof. To show (i) we can take an open affine subset $U \in \mathcal{S}$ and to consider $\mathfrak{U}$ its formal completion along the special fiber. We have

$$H^0(\mathfrak{U}, \widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}) = H^0\widehat{(U, \mathcal{D}^{(m)}_{X,\lambda})} = H^0\widehat{(U, \mathcal{D}^{(m)}_{X})} = H^0(\mathfrak{U}, \widehat{\mathscr{D}}^{(m)}_{\mathfrak{X}})$$

The first and third isomorphisms are given by [26, $(0_I$, 3.2.6)] and the second one arises from the preceding proposition. As we have have remarked the ring $H^0(\mathfrak{U}, \widehat{\mathscr{D}}^{(m)}_{\mathfrak{X}})$ is twosided noetherian. Therefore, we can take $\mathscr{B}$ as the set of affine open subsets of $\mathfrak{X}$ contained in the p-adic completion of an affine open subset $U \in \mathcal{S}$. This proves (i). By [6, proposition 3.3.4] we can conclude that (ii) is an immediately consequence of (i) because $H^0(\mathfrak{U}, \widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}) = H^0(\mathfrak{U}, \widehat{\mathscr{D}}_{\mathfrak{X},\lambda}) \otimes_{\mathfrak{o}} L$ [6, (3.4.0.1)]. □

Using the morphism $\Phi^{(m)}_X$ defined in (2.23) and the canonical projection from $\widetilde{\mathcal{D}^{(m)}}$ onto $\mathcal{D}^{(m)}_{X,\lambda}$ we can define a canonical map

$$\Phi^{(m)}_{X,\lambda} : \mathcal{A}^{(m)}_X \to \mathcal{D}^{(m)}_{X,\lambda}. \tag{2.30}$$

We recall for the reader that if $U \in \mathcal{S}$ then

$$\mathrm{Sym}^{(m)}\left(\left(\xi_*\mathcal{T}_{\tilde{X}}\right)^{\mathbb{T}}\right)(U) = \mathrm{Sym}^{(m)}\left(\mathcal{T}_X(U)\right) \otimes_{\mathfrak{o}} \mathrm{Sym}^{(m)}(\mathfrak{t}). \tag{2.31}$$

Proposition 2.5.13. *(i) There exists a canonical isomorphism $Sym^{(m)}(\mathcal{T}_X) \simeq gr_\bullet(\mathcal{D}^{(m)}_{X,\lambda})$.*

(ii) The canonical morphism $\Phi^{(m)}_{X,\lambda}$ is surjective.

(iii) The sheaf $\mathcal{D}^{(m)}_{X,\lambda}$ is a coherent $\mathcal{A}^{(m)}_X$-module.

Proof. In the preceding section we have constructed a canonical morphism

$$gr_\bullet\left(\widetilde{\mathcal{D}^{(m)}}\right) \otimes_{gr_\bullet(D^{(m)}(\mathbb{T}))} gr_\bullet(\mathfrak{o}_{\lambda^{(m)}}) \to gr_\bullet\left(\mathcal{D}^{(m)}_{X,\lambda}\right).$$

By proposition 2.3.7 we know that $gr_\bullet(\widetilde{\mathcal{D}^{(m)}}) \simeq Sym^{(m)}((\xi_*\mathcal{T}_{\tilde{X}})^{\mathbb{T}})$. Moreover, by definition, we know that $gr_\bullet(\mathfrak{o}_{\lambda^{(m)}}) = \mathfrak{o}$ as a $gr_\bullet(D^{(m)}(\mathbb{T}))$ ($= Sym^{(m)}(\mathfrak{t})$ proposition 1.4.2)-module. We obtain a morphism of sheaves of graded $\mathfrak{o}$-algebras

$$Sym^{(m)}\left((\xi_*\mathcal{T}_{\tilde{X}})^{\mathbb{T}}\right) \otimes_{Sym^{(m)}(\mathfrak{t})} \mathfrak{o} \to gr_\bullet\left(\mathcal{D}^{(m)}_{X,\lambda}\right)$$

(the structure of $\mathrm{Sym}^{(m)}(\mathfrak{t})$-module is guaranteed by (2.31)). Using the short exact sequence $0 \to \mathcal{O}\otimes_{\mathfrak{o}} \mathfrak{t} \to \left(\xi_*\mathcal{T}_{\tilde{X}}\right)^{\mathbb{T}} \xrightarrow{\nu} \mathcal{T}_X \to 0$ of lemma 2.3.3 we see that

$$Sym^{(m)}((\xi_*\mathcal{T}_{\tilde{X}})^{\mathbb{T}}) \otimes_{Sym^{(m)}(\mathfrak{t})} \mathfrak{o} \xrightarrow{Sym^{(m)}(\nu)\otimes 1} Sym^{(m)}(\mathcal{T}_X)$$

is an isomorphism and we get a canonical morphism of $\mathfrak{o}$-algebras

$$\varphi : \mathrm{Sym}^{(m)}(\mathcal{T}_X) \to gr_\bullet\left(\mathcal{D}^{(m)}_{X,\lambda}\right).$$

By proposition 2.5.8, we have a commutative diagram for any $U \in \mathcal{S}$

$$\begin{array}{ccc} \mathrm{Sym}^{(m)}\left(\mathcal{T}_X(U)\right) & \xrightarrow{\varphi_U} & \mathrm{gr}_\bullet\left(\mathcal{D}^{(m)}_{X,\lambda}(U)\right) \\ & \searrow \quad \swarrow & \\ & \mathrm{gr}_\bullet\left(\mathcal{D}^{(m)}_X(U)\right), & \end{array}$$

here the left diagonal arrow is given by (1.3). As $\mathcal{S}$ is a basis for the Zariski topology of X we can conclude that φ is an isomorphism.

For the second claim we can calculate $gr_\bullet(\Phi^{(m)}_{X,\lambda})$. By the first part of the proof and proposition 1.5.6 this morphism is identified with

$$\mathcal{O}_X \otimes_{\mathfrak{o}} Sym^{(m)}(\mathfrak{g}) \to Sym^{(m)}(\mathcal{T}_X)$$

which is surjective by [35, Proposition 1.6.1]. Finally, item (iii) follows from (ii) and proposition 1.5.6. □

Remark 2.5.14. *(a) By construction $\mathcal{D}^{(m)}_{X,\lambda,\mathbb{Q}} = \mathcal{D}_\lambda$ is the sheaf of usual twisted differential operators on the flag variety X_L [12, page 170].*

(b) Let us recall that the right regular action of $\mathbb{G}$ on X induces a natural map $\Phi_\lambda : \mathcal{U}(\mathfrak{g}_L) \to H^0(X_L, \mathcal{D}_\lambda)$. This

implies that if $\Phi_{\lambda}^{(m)}$ denotes the canonical map induced by $\Phi_{X,\lambda}^{(m)}$ by taking global sections, then $\Phi_{\lambda}^{(m)} \otimes L = \Phi_{\lambda}$ [12, Page 170 and 186].

The relation given in (2.26) tells us that, as in the classical case (see for example [2] or [11]), the sheaf $\widetilde{\mathcal{D}^{(m)}} := \left(\xi_* \mathcal{D}_{\widetilde{X}}^{(m)}\right)^{\mathbb{T}}$ can be regarded as a family of twisted differential operators on X parametrized by $\mathfrak{t}^* := \mathrm{Hom}(\mathfrak{t}, \mathfrak{o})$.

2.6 Finiteness properties

Let $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ be a character. In this section we start the study of the cohomological properties of coherent $\mathcal{D}_{X,\lambda}^{(m)}$-modules. We follow the arguments of [35] to show a technical important finiteness property about the p-torsion of the cohomology groups of coherent $\mathcal{D}_{X,\lambda}^{(m)}$-modules, when the character $\lambda + \rho \in \mathfrak{t}_L^*$ is dominant and regular (proposition 2.6.4). To start with, let us recall the twist by the sheaf $\mathcal{O}(1)$. As X is a projective $\mathfrak{o}$-scheme, there exists a very ample invertible sheaf $\mathcal{O}(1)$ on X [30, chapter II, remark 5.16.1]. Therefore, for any arbitrary $\mathcal{O}_X$-module $\mathcal{E}$ we can consider the twist

$$\mathcal{E}(r) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}(r),$$

where $r \in \mathbb{Z}$ and $\mathcal{O}(r)$ means the r-th tensor product of $\mathcal{O}(1)$ with itself. We recall to the reader that there exists $r_0 \in \mathbb{Z}$, depending of $\mathcal{O}(1)$, such that for every $k \in \mathbb{Z}_{>0}$ and for every $s \geq r_0$, $H^k(X, \mathcal{O}(s)) = 0$ [30, chapter II, theorem 5.2 (b)].

We start the results of this section with the following proposition which states three important properties of coherent $\mathcal{A}_X^{(m)}$-modules [38, proposition A.2.6.1]. This is a key result in this work. Let $\mathcal{E}$ be a coherent $\mathcal{A}_X^{(m)}$-module.

Proposition 2.6.1. *(i) $H^0(X, \mathcal{A}_X^{(m)}) = D^{(m)}(\mathbb{G})$ is a noetherian $\mathfrak{o}$-algebra.*

(ii) There exists a surjection of $\mathcal{A}_X^{(m)}$-modules $\left(\mathcal{A}_X^{(m)}(-r)\right)^{\oplus a} \to \mathcal{E} \to 0$ for suitable $r \in \mathbb{Z}$ and $a \in \mathbb{N}$.

(iii) For any $k \geq 0$ the group $H^k(X, \mathcal{E})$ is a finitely generated $D^{(m)}(\mathbb{G})$-module.

Inspired in proposition 2.5.13, in a first time we will be concentrated on coherent $\mathcal{A}_X^{(m)}$-modules. The next two results will play an important role when we consider formal completions.

Lemma 2.6.2. *For every coherent $\mathcal{A}_X^{(m)}$-module $\mathcal{E}$, there exists $r = r(\mathcal{E}) \in \mathbb{Z}$ such that $H^k(X, \mathcal{E}(s)) = 0$ for every $s \geq r$.*

Proof. Let us fix $r_0 \in \mathbb{Z}$ such that $H^k(X, \mathcal{O}(s)) = 0$ for every $k > 0$ and $s \geq r_0$. We have,

$$H^k(X, \mathcal{A}_X^{(m)}(s)) = H^k(X, \mathcal{O}(s)) \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}) = 0.$$

Now, by the second part of proposition 2.6.1 there exist $a_0 \in \mathbb{N}$ and $s_0 \in \mathbb{Z}$ together with an epimorphism of $\mathcal{A}_X^{(m)}$-modules

$$\mathcal{E}_0 := \left(\mathcal{A}_X^{(m)}(s_0)\right)^{\oplus a_0} \to \mathcal{E} \to 0.$$

If $r \geq r_0 - s_0$ we see that

$$H^k(X, \mathcal{E}_0(r)) = H^k(X, \mathcal{O}(r + s_0))^{\oplus a_0} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}) = 0.$$

We can now end the proof using a classical inductive argument as follows. Let $\mathcal{E}_1$ be the kernel of the epimorphism $\mathcal{E}_0 \to \mathcal{E} \to 0$ and let us consider the statement (a_i): for every coherent $\mathcal{A}_X^{(m)}$-module $\mathcal{F}$, there exists $r_i(\mathcal{F}) \in \mathbb{Z}$ such that for all $r \geq r_i(\mathcal{F})$ and all $i \leq k$ one has $H^k(X, \mathcal{F}) = 0$. For $i \geq \dim(X)$ the statement follows from Grothendieck's vanishing theorem [30, chapter III, theorem 2.7]. Now, let us suppose that (a_{i+1}) holds. Taking $r \geq \max\{r_0 - s_0, r_{i+1}(\mathcal{E}_1)\}$ and regarding the long exact sequence in cohomology we have

$$0 = H^i(X, \mathcal{E}_0(r)) \to H^i(X, \mathcal{E}(r)) \to H^{i+1}(X, \mathcal{E}_1(r)) = 0.$$

So we can take as $r_i(\mathcal{E})$ any of those r which are larger than the $\max\{r_0 - s_0, r_{i+1}(\mathcal{E}_1)\}$ to obtain the statement (a_i). The statement (a_1) shows the proposition. □

Lemma 2.6.3. *For every coherent $\mathcal{D}^{(m)}_{X,\lambda}$-module $\mathcal{E}$, there exist $r = r(\mathcal{E}) \in \mathbb{Z}$, a natural number $a \in \mathbb{N}$ and an epimorphism of $\mathcal{D}^{(m)}_{X,\lambda}$-modules*

$$\left(\mathcal{D}^{(m)}_{X,\lambda}(-r)\right)^{\oplus a} \to \mathcal{E} \to 0.$$

Proof. Using the epimorphism in proposition 2.5.13 we can suppose that $\mathcal{E}$ is also a coherent $\mathcal{A}^{(m)}_X$-module. In this case, by the second part of proposition 2.6.1, there exist $r = r(\mathcal{E}) \in \mathbb{Z}$, a natural number $a \in \mathbb{N}$ and an epimorphism of $\mathcal{A}^{(m)}_X$-modules

$$\left(\mathcal{A}^{(m)}_X(-r)\right)^{\oplus a} \to \mathcal{E} \to 0.$$

Taking the tensor product with $\mathcal{D}^{(m)}_{X,\lambda}$ we get the desired epimorphism of $\mathcal{D}^{(m)}_{X,\lambda}$-modules

$$\left(\mathcal{D}^{(m)}_{X,\lambda}(-r)\right)^{\oplus a} \simeq \mathcal{D}^{(m)}_{X,\lambda} \otimes_{\mathcal{A}^{(m)}_X} \left(\mathcal{A}^{(m)}_X(-r)\right)^{\oplus a} \to \mathcal{D}^{(m)}_{X,\lambda} \otimes_{\mathcal{A}^{(m)}_X} \mathcal{E} \simeq \mathcal{E} \to 0.$$

□

We recall for the reader that the distribution algebra of level m, which has been denoted by $D^{(m)}(\mathbb{G})$ in subsection 1.4, is a filtered noetherian $\mathfrak{o}$-algebra. Moreover $\lambda \in \mathfrak{t}^*_L$ is an L-linear application induced, via base change, by a character $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ of the distribution algebra of the torus $\mathbb{T}$.

Proposition 2.6.4. *Let us suppose that $\lambda + \rho \in \mathfrak{t}^*_L$ is a dominant and regular character (cf. 2.5.3).*

(i) Let us fix $r \in \mathbb{Z}$. For every positive integer $k \in \mathbb{Z}_{>0}$, the cohomology group $H^k(X, \mathcal{D}^{(m)}_{X,\lambda}(r))$ has bounded p-torsion.

(ii) For every coherent $\mathcal{D}^{(m)}_{X,\lambda}$-module $\mathcal{E}$, the cohomology group $H^k(X, \mathcal{E})$ has bounded p-torsion for all $k > 0$.

Proof. To show (i), we recall that $\mathcal{D}^{(m)}_{X,\lambda,\mathbb{Q}} = \mathcal{D}_\lambda$ is the usual sheaf of twisted differential operators on the flag variety X_L (remark 2.5.14). As $\mathcal{D}^{(m)}_{X,\lambda,\mathbb{Q}}(r)$ is a coherent $\mathcal{D}_\lambda$-module, the classical Beilinson-Bernstein theorem [3] allows us to conclude that $H^k(X, \mathcal{D}^{(m)}_{X,\lambda}(r)) \otimes_{\mathfrak{o}} L = 0$ for every positive integer $k \in \mathbb{Z}_{>0}$. This in particular implies that the sheaf $\mathcal{D}^{(m)}_{X,\lambda}(r)$ has p-torsion cohomology groups $H^k(X, \mathcal{D}^{(m)}_{X,\lambda}(r))$, for every $k > 0$ and $r \in \mathbb{Z}$.
Now, by proposition 2.5.13, we know that $\mathcal{D}^{(m)}_{X,\lambda}(r)$ is in particular a coherent $\mathcal{A}^{(m)}_X$-module and hence, by the third part of proposition 2.6.1 we get that for every $k \geq 0$ the cohomology groups $H^k(X, \mathcal{D}^{(m)}_{X,\lambda}(r))$ are finitely generated $D^{(m)}(\mathbb{G})$-modules. Consequently, of finite p-torsion for every integer $0 < k \leq \dim(X)$ and $r \in \mathbb{Z}$.
To show (ii) we may use before lemma 2.6.3 to obtain a surjective morphism of $\mathcal{D}^{(m)}_{X,\lambda}$-modules

$$\mathcal{C}_0 := \left(\mathcal{D}^{(m)}_{X,\lambda}(-r)\right)^{\oplus a} \xrightarrow{\alpha} \mathcal{E} \to 0$$

for suitable $r \in \mathbb{Z}$ and $a \in \mathbb{N}$. As in lemma 2.6.2, we will follow an inductive argument to end the proof. For every $i > 0$ we consider the statement (a_i): if $\mathcal{E}$ is a coherent $\mathcal{D}^{(m)}_{X,\lambda}$-module, there exists a positive integer $r_i = r_i(\mathcal{E}) \in \mathbb{Z}_{>0}$ such that for every $i \leq k$ the cohomology group $H^k(X, \mathcal{E})$ is annihilated by p^{r_i}. For $i \geq \dim(X)$ the statement follows from Grothendieck's vanishing theorem [30, chapter III theorem 2.7]. Now, let us suppose that (a_{i+1}) holds and let us denote by $\mathcal{C}_1$ the kernel of the morphism α. This is a coherent $\mathcal{D}^{(m)}_{X,\lambda}$-module by [6, proposition 3.1.3 (i)]. The long exact sequence in cohomology gives us the short exact sequence

$$H^i(X, \mathcal{C}_0) \xrightarrow{\beta} H^i(X, \mathcal{E}) \xrightarrow{\delta} H^{i+1}(X, \mathcal{C}_1). \tag{2.32}$$

Let $c \in \mathbb{N}$ such that p^c annihilates the image of β (of finite p-torsion by (i)) and, according to (a_{i+1}), let us take $r_{i+1}(\mathcal{C}_1) \in \mathbb{Z}$ such that $p^{r_{i+1}(\mathcal{C}_1)}$ annihilates the image of δ. So we may take $r_i(\mathcal{E}) := \max\{r_{i+1}(\mathcal{E}), c + r_{i+1}(\mathcal{C}_1)\}$ to obtain the statement (a_i). In particular (a_1) proves the proposition. $\square$

Chapter 3

Passing to formal completions

We recall for the reader that $X := \mathbb{G}/\mathbb{B}$ denotes the flag $\mathfrak{o}$-scheme of $\mathbb{G}$ and $\mathfrak{X}$ its completion along its special fiber. From now on, we consider $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ a character of $\mathrm{Dist}(\mathbb{T})$ such that $\lambda + \rho \in \mathfrak{t}_L^{*}$[1] is a dominant and regular character of $\mathfrak{t}_L$ (we point out to the reader that λ induces via tensor product with L a character of $\mathfrak{t}_L$ (2.5.3)). We have introduced the following sheaves of p-adically complete $\mathfrak{o}$-algebras on the formal p-adic completion $\mathfrak{X}$ of X

$$\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda} := \varprojlim_j \mathcal{D}^{(m)}_{X,\lambda}/p^{j+1}\mathcal{D}^{(m)}_{X,\lambda}.$$

The sheaf $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ is our sheaf of *level m twisted arithmetic differential operators* on the smooth formal flag scheme $\mathfrak{X}$.

3.1 Cohomological properties

Our objective in this subsection is to prove an analogue of proposition 2.6.4 for coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}$-modules and to conclude that $H^0(\mathfrak{X},\bullet)$ is an exact functor over the category of coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules.

Proposition 3.1.1. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{(m)}_{X,\lambda}$-module and $\widehat{\mathcal{E}} := \varprojlim_j \mathcal{E}/p^{j+1}\mathcal{E}$ its p-adic completion, which we consider as a sheaf on $\mathfrak{X}$.*

(i) For all $j \geq 0$ one has $H^j(\mathfrak{X},\widehat{\mathcal{E}}) = \varprojlim_k H^j(X,\mathcal{E}/p^{k+1}\mathcal{E})$.

(ii) For all $j > 0$ one has $H^j(\mathfrak{X},\widehat{\mathcal{E}}) = H^j(X,\mathcal{E})$.

(iii) The global section functor $H^0(\mathfrak{X},\bullet)$ satisfies $H^0(\mathfrak{X},\widehat{\mathcal{E}}) = \varprojlim_k H^0(X,\mathcal{E})/p^{k+1}H^0(X,\mathcal{E})$.

Proof. The arguments exhibit in this proof follow word for word the arguments given in [34, Proposition 3.2] and we do not claim any originality here. Let $\mathcal{E}_t$ denote the torsion subpresheaf of $\mathcal{E}$. This is, for any open subset $U \subseteq X$ we have $\mathcal{E}(U)_t := \mathcal{E}(U)_{\mathrm{tor}}$, where the right-hand side denotes the group of torsion elements of $\mathcal{E}(U)$. As X is a noetherian space, this is indeed a sheaf and furthermore a $\mathcal{D}^{(m)}_{X,\lambda}$-submodule of $\mathcal{E}$. Because the sheaf $\mathcal{D}^{(m)}_{X,\lambda}$ has noetherian rings of sections over open affine subsets of X (proposition 2.5.10), the submodule $\mathcal{E}_t$ is a coherent $\mathcal{D}^{(m)}_{X,\lambda}$-module. This submodule is thus generated by a coherent $\mathcal{O}_X$-module which is annihilated by a power p^c of p, and so is $\mathcal{E}_t$. The quotient $\mathcal{G} := \mathcal{E}/\mathcal{E}_t$ is again a coherent $\mathcal{D}^{(m)}_{X,\lambda}$-module and therefore we can assume, after possibly replacing c by a larger number, that $p^c\mathcal{E}_t = 0$ and $p^c H^j(X,\mathcal{E}) = p^c H^j(X,\mathcal{G}) = 0$, for all $j > 0$.

[1] We abuse of the notation and we denote again by λ the character of $\mathfrak{t}_L$ specified by (2.26).

Let k, l be integers which are great or equal to c and let $\upsilon_l : \mathcal{G} \to \mathcal{E}$ be the map induced by multiplication by p^{l+}. We have the following exact sequence

$$0 \to \mathcal{G} \xrightarrow{\upsilon_l} \mathcal{E} \to \mathcal{E}_l \to 0$$

where $\mathcal{E}_l := \mathcal{E}/p^{l+1}\mathcal{E}$. Now, let us consider the complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \xrightarrow{\upsilon_{l+k}} & \mathcal{E} & \longrightarrow & \mathcal{E}_{l+k} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p^k} & & \downarrow{\scriptstyle id} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G} & \xrightarrow{\upsilon_l} & \mathcal{E} & \longrightarrow & \mathcal{E}_l & \longrightarrow & 0 \end{array} \tag{3.1}$$

which induces a morphism of long exact sequences

$$\begin{array}{ccccccc} H^j(X,\mathcal{G}) & \xrightarrow{H^j(\upsilon_{l+k})} & H^j(X,\mathcal{E}) & \xrightarrow{\beta_{l+k}} & H^j(X,\mathcal{E}_{l+k}) & \longrightarrow & H^{j+1}(X,\mathcal{G}) \\ \downarrow{\scriptstyle p^k} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \alpha_{l+k,l}} & & \downarrow{\scriptstyle p^k} \\ H^j(X,\mathcal{G}) & \xrightarrow{H^j(\upsilon_l)} & H^j(X,\mathcal{E}) & \xrightarrow{\beta_l} & H^j(X,\mathcal{E}_l) & \xrightarrow{\tau_l} & H^j(X,\mathcal{G}). \end{array} \tag{3.2}$$

Given that $k \geq c$ the right-hand vertical map is zero, and hence $\tau_l \circ \alpha_{l+k,l} = 0$, which implies, by exactness, that $\mathrm{im}(\alpha_{l+k,l}) \subseteq \mathrm{im}(\upsilon_l)$. Since $\beta_{l+k} \circ \alpha_{l+k,l} = \upsilon_l$ we find that $\mathrm{im}(\alpha_{l+k,l}) = \mathrm{im}(\upsilon_l)$ for all $k \geq c$. In consequence, the projective system $(H^j(X,\mathcal{E}_k))_k$, with transition maps given by $\alpha_{k',k}$ with $k' \geq k$, satisfies the Mittag-Leffler conditions for any $j \geq 0$. Furthermore, the transition maps of the system $(\mathcal{E}_k)_k$ are clearly surjective and if $U \subset X$ is an affine open subset, then $H^j(U,\mathcal{E}_k) = 0$ for $j > 0$, because $\mathcal{E}_k$ is in particular a quasi-coherent $\mathcal{O}_X$-module. Hence, the exact sequence

$$0 \to \mathcal{E}_l \xrightarrow{p^k} \mathcal{E}_{l+k} \to \mathcal{E}_k \to 0$$

stays exact after taking sections over U, and therefore the projective system $(H^0(U,\mathcal{E}_k))_k$ satisfies the Mittag-Leffler conditions. The preceding lines imply that we are under the hypothesis of [28, Chapter 0, 13.3.1], which implies that for all $j \geq 0$

$$H^j(\mathfrak{X},\widehat{\mathcal{E}}) = \varprojlim_k H^j(X,\mathcal{E}/p^{k+1}\mathcal{E}).$$

We have proved the first assertion. For the second assertion we may consider the diagram (3.2) and the fact that $H^j(\upsilon_l) = 0$ for $j > 0$ and $l \geq 0$. In consequence, β_l is an isomorphism onto its image for these j and l. Therefore, the projective limit of the system $(H^j(X,\mathcal{E}_k))_k$ is equal to $H^j(X,\mathcal{E})$ when $j > 0$. This property together with (i) gives us (ii). Finally, to verify (i) we take two integers $l, k \geq c$. We consider the short exact sequence

$$0 \to \mathcal{E}_t \to \mathcal{E} \xrightarrow{p^{l+1}} \mathcal{E}_l \to 0$$

which splits into two exact sequences

$$0 \to \mathcal{E}_t \to \mathcal{E} \xrightarrow{\upsilon} \mathcal{G} \to 0 \quad \text{and} \quad 0 \to \mathcal{G} \xrightarrow{\upsilon_l} \mathcal{E} \to \mathcal{E}_l \to 0$$

inducing long exact sequences in cohomology

$$0 \to H^{(}X,\mathcal{E}_t) \to H^0(X,\mathcal{E}) \xrightarrow{\upsilon} H^0(X,\mathcal{G}) \to H^1(X,\mathcal{E}_t)$$
$$0 \to H^0(X,\mathcal{G}) \xrightarrow{\upsilon_l} H^0(X,\mathcal{E}) \to H^0(X,\mathcal{E}_l) \to H^1(X,\mathcal{G}).$$

From the second exact sequence and the morphism of exact sequences (3.1) we obtain the following morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X,\mathcal{E}_t) & \xrightarrow{v_{l+k}} & H^0(X,\mathcal{E}) & \xrightarrow{\beta_{l+k}} & H^0(X,\mathcal{E}_{l+k}) & \longrightarrow & H^0(X,\mathcal{G}) \\
 & & \downarrow{\scriptstyle p^k} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \alpha_{l+k,l}} & & \downarrow{\scriptstyle p^k} \\
0 & \longrightarrow & H^0(X,\mathcal{E}_t) & \xrightarrow{v_l} & H^0(X,\mathcal{E}) & \xrightarrow{\beta_l} & H^0(X,\mathcal{E}_l) & \xrightarrow{\tau_l} & H^0(X,\mathcal{G}).
\end{array} \tag{3.3}$$

Given that $v_l \circ v = p^{l+1}$, we get a canonical surjection

$$\gamma_l : H^0(X,\mathcal{E})/p^{l+1}H^0(X,\mathcal{E}) \to H^0(X,\mathcal{E})/v_l\left(H^0(X,\mathcal{G})\right).$$

These morphisms form a morphism of projective systems. Now, as v_l is injective, we have a canonical isomorphism

$$\begin{aligned}
\ker(\gamma_l) &= v_l\left(H^0(X,\mathcal{G})\right)/p^{l+1}H^{(}X,\mathcal{E}) \\
&= v_l\left(H^0(X,\mathcal{G})\right)/v_l\left(v\left(H^0(X,\mathcal{E})\right)\right) \\
&\simeq H^0(X,\mathcal{G})/v\left(H^0(X,\mathcal{E})\right) \\
&= \operatorname{coker}(H^0(u)),
\end{aligned}$$

and $\operatorname{coker}(H^0(u))$ embeds into $H^1(X,\mathcal{G})$ which is annihilated by p^c. Moreover, the morphism of exact sequences

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{E}_t & \longrightarrow & \mathcal{E} & \xrightarrow{p^{l+k+1}} & \mathcal{E} & \longrightarrow & \mathcal{E}_{l+k} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p^k} & & \downarrow{\scriptstyle p^k} & & \downarrow{\scriptstyle id} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}_t & \longrightarrow & \mathcal{E} & \xrightarrow{p^{l+1}} & \mathcal{E} & \longrightarrow & \mathcal{E}_{l+k} & \longrightarrow & 0
\end{array}$$

induces a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{coker}(H^0(v)) & \longrightarrow & H^0(X,\mathcal{E})/p^{l+k+1}H^0(X,\mathcal{E}) & \longrightarrow & H^0(X,\mathcal{E})/v_{l+k}\left(H^0(X,\mathcal{G})\right) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p^k} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{coker}(H^0(v)) & \longrightarrow & H^0(X,\mathcal{E})/p^{l+1}H^0(X,\mathcal{E}) & \longrightarrow & H^0(X,\mathcal{E})/v_l\left(H^0(X,\mathcal{G})\right) & \longrightarrow & 0.
\end{array}$$

Thus, the projective limit $\varprojlim_l \ker(\gamma_l)$ vanishes and the system $(\gamma_l)_l$ induces an isomorphism

$$\varprojlim_l H^0(X,\mathcal{E})/p^{l+1}H^0(X,\mathcal{E})/v_l\left(H^0(X,\mathcal{G})\right).$$

Looking at (3.3), we can conclude that right-hand side is canonically isomorphic to $\varprojlim_l H^0(X,\mathcal{E}_l) = H^0(\mathfrak{X},\widehat{\mathcal{E}})$, by the first assertion. $\square$

The next proposition is a natural result from lemmas 2.6.2 and 2.6.3. Except for some technical details, the proof is exactly the same that in [36, proposition 4.2.2].

Proposition 3.1.2. *Let $\mathscr{E}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}$-module.*

(i) There exists $r_2 = r_2(\mathscr{E}) \in \mathbb{Z}$ such that, for all $r \geq r_2$ there is $a \in \mathbb{Z}$ and an epimorphism of $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}$-modules

$$\left(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}(-r)\right)^{\oplus a} \to \mathscr{E} \to 0.$$

(ii) There exists $r_3 = r_3(\mathscr{E}) \in \mathbb{Z}$ such that, for all $r \geq r_3$ we have $H^i(\mathfrak{X}, \mathscr{E}(r)) = 0$, for all $i > 0$.

Proof. We start the proof of the part (i) by remarking that the torsion subsheaf $\mathscr{E}_t$ of $\mathscr{E}$ is a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}$-module. As $\mathfrak{X}$ is quasi-compact, there exists $c \in \mathbb{N}$ such that $p^c\mathscr{E}_t = 0$. Defining $\mathscr{G} := \mathscr{E}/\mathscr{E}_t$, $\mathscr{G}_0 := \mathscr{G}/\mathscr{G}_t$ and $\mathscr{E}_j := \mathscr{E}/p^{j+1}\mathscr{E}$, we have for every $j \geq c$ an exact sequence

$$0 \to \mathscr{G}_0 \xrightarrow{p^{j+1}} \mathscr{E}_{j+1} \to \mathscr{E}_j \to 0.$$

Viewing $\mathfrak{X}$ as a closed subset of X and denoting by ψ this topological embedding, we can suppose that $\psi_*\mathscr{G}_0$ is a coherent $\mathcal{D}^{(m)}_{X,\lambda}$-module via the canonical isomorphism of sheaves of rings $\mathcal{D}^{(m)}_{X,\lambda}/p\mathcal{D}^{(m)}_{X,\lambda} \simeq \psi_*\left(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}/p\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}\right)$ (similarly, we can consider $\mathscr{E}_c$ as a coherent $\mathcal{D}^{(m)}_{X,\lambda}$-module). By using the fact $\mathscr{G}_0$ is also a coherent $\mathcal{A}^{(m)}_X$-module, lemma 2.6.2 gives us an integers $r'_2(\mathscr{G}_0)$ such that the canonical maps

$$H^0(\mathfrak{X}, \mathscr{E}_{j+1}(r')) \to H^0(\mathfrak{X}, \mathscr{E}_j(r')) \tag{3.4}$$

are surjective for $r' \geq r'_2(\mathscr{G}_0)$ and $j \geq c$. Moreover, lemma 2.6.3 gives another integer $r'_3(\mathscr{E}_c)$ such that, for every $r'' \geq r'_3(\mathscr{E}_c)$ there exists $a \in \mathbb{N}$ and a surjection

$$\phi : \left(\mathcal{D}^{(m)}_{X,\lambda}/p^c\mathcal{D}^{(m)}_{X,\lambda}\right)^{\oplus a} \to \mathscr{E}_c(r'') \to 0.$$

Let us fix $r \geq r_2 := \max\{r'_2(\mathscr{E}_c), r'_3(\mathscr{G}_0)\}$ and let $e_1, \ ..., \ e_s$ be the standard basis of the domain of ϕ. We use (3.4) to lift each $\phi(e_l)$, $1 \leq l \leq s$, to an element of

$$\varprojlim_j H^0(\mathfrak{X}, \mathscr{E}_j(r)) \simeq H^0(\mathfrak{X}, \widehat{\mathscr{E}(r)}),$$

by the first assertion of the preceding proposition. By [6, 3.2.3 (v)] we have $\widehat{\mathscr{E}(r)} = \widehat{Ea}(r)$ and $\widehat{\mathscr{E}} = \mathscr{E}$, and therefore we have a morphism

$$\left(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}\right)^a \to \mathscr{E}(r) \to 0,$$

which is surjective because, modulo p^c, it is a surjective morphism of sheaves coming from coherent $\mathcal{D}^{(m)}_{X,\lambda}$-modules by redaction modulo p^c. To show the part (ii), we remark that if we fix $r_0 \in \mathbb{Z}$ such that $H^k(X, \mathcal{O}(r)) = 0$ for every $k > 0$ and $r \geq r_0$, exactly as we have done in lemma 2.6.2, then via the second part of proposition 3.1.2 we also have that $H^k(\mathfrak{X}, \widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}(r)) = 0$ and the rest of the proof can be deduced exactly as in the proof of lemma 2.6.2. □

Corollary 3.1.3. *Let $\mathscr{E}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}$-module. There exists $c = c(\mathscr{E}) \in \mathbb{N}$ such that for all $k > 0$ the cohomology group $H^k(\mathfrak{X}, \mathscr{E})$ is annihilated by p^c.*

Proof. Let $r \in \mathbb{Z}$. By the first assertion of proposition 3.1.1 we have for $k > 0$

$$H^k(\mathfrak{X}, \widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}(-r)) \simeq H^k(X, \mathcal{D}^{(m)}_{X,\lambda}(-r))$$

which is annihilated by a finite power of p, by part (i) of proposition 2.6.4. The proof now proceeds by descending induction exactly as we have done in the proof of part (ii) of proposition 2.6.4. □

Now, we want to extend the part (i) of the preceding proposition to the sheaves $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$. To do that, we need to show that the category of coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules admits integral models (definition 1.5.1).

Let $\mathrm{Coh}(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda})$ be the category of coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}$-modules and let $\mathrm{Coh}(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda})_{\mathbb{Q}}$ be the category of coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}$-modules up to isogeny. This means that $\mathrm{Coh}(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda})_{\mathbb{Q}}$ has the same class of objects as $\mathrm{Coh}(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda})$ and, for any two objects $\mathcal{M}$ and $\mathcal{N}$ in $\mathrm{Coh}(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda})_{\mathbb{Q}}$ one has

$$\mathrm{Hom}_{\mathrm{Coh}(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda})_{\mathbb{Q}}}(\mathcal{M},\mathcal{N}) = \mathrm{Hom}_{\mathrm{Coh}(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda})}(\mathcal{M},\mathcal{N}) \otimes_{\mathfrak{o}} L.$$

Proposition 3.1.4. *The functor $\mathcal{M} \mapsto \mathcal{M}\otimes_{\mathfrak{o}} L$ induces an equivalence of categories between $\mathrm{Coh}(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda})_{\mathbb{Q}}$ and $\mathrm{Coh}(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}})$.*

Proof. By definition, the sheaf $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ satisfies [6, conditions 3.4.1] and therefore [6, proposition 3.4.5] allows to conclude the proposition. □

The proof of the next theorem follows exactly the same lines than in [36, theorem 4.2.8]. We will reproduce the proof because it is a central result for our goal.

Theorem 3.1.5. *Let $\mathscr{E}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module.*

(i) There is $r(\mathscr{E}) \in \mathbb{Z}$ such that, for every $r \geq r(\mathscr{E})$ there exist $a \in \mathbb{N}$ and an epimorphism of $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules

$$\left(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}(-r)\right)^{\oplus a} \to \mathscr{E} \to 0.$$

(ii) For all $i > 0$ one has $H^i(\mathfrak{X},\mathscr{E}) = 0$.

Proof. By the preceding proposition, there exists a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}$-module $\mathscr{F}$ such that $\mathscr{F} \otimes_{\mathfrak{o}} L \simeq \mathscr{E}$. Using the first part of proposition 3.1.2 we can find $r(\mathscr{F}) \in \mathbb{Z}$, such that for all $r \geq r(\mathscr{F})$ there exist $a \in \mathbb{N}$ and a surjection

$$\left(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda}(-r)\right)^{\oplus a} \to \mathscr{F} \to 0.$$

Tensoring with L we get the desired surjection onto $\mathscr{E}$. Furthermore, as $\mathfrak{X}$ is a noetherian space, the corollary 3.1.3 allows us to conclude that

$$H^i(\mathfrak{X},\mathscr{E}) = H^i(\mathfrak{X},\mathscr{F}) \otimes_{\mathfrak{o}} L = 0$$

for every $k > 0$ [6, (3.4.0.1)]. □

3.2 Calculation of global sections

We recall for the reader that throughout this chapter $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ denotes a character of the distribution algebra $\mathrm{Dist}(\mathbb{T})$ such that $\lambda + \rho \in \mathfrak{t}^*_L$[2] is a dominant and regular character of $\mathfrak{t}_L$ (cf. (2.26)). In this subsection we propose to calculate the global sections of the sheaf $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$. Inspired in the arguments exhibited in [39], we will need the following lemma (cf. [39, lemma 3.3]) whose conclusion is an essential tool for our goal.

Lemma 3.2.1. *Let A be a noetherian $\mathfrak{o}$-algebra, M, N two A-modules of finite type, $\psi : M \to N$ an A-lineal application and $\widehat{\psi} : \widehat{M} \to \widehat{N}$ the morphism obtained after p-adic completion. If $\psi \otimes_{\mathfrak{o}} 1 : M \otimes_{\mathfrak{o}} L \to N \otimes_{\mathfrak{o}} L$ is an isomorphism, then $\widehat{\psi} \otimes_{\mathfrak{o}} 1 : \widehat{M} \otimes_{\mathfrak{o}} L \to \widehat{N} \otimes_{\mathfrak{o}} L$ is an isomorphism as well.*

Proof. Let K be the kernel (resp. the cokernel) of ψ. Since the ϖ-adic completion is an exact functor over the finitely generated A-modules [6, 3.2.3 (ii)], the ϖ-completion $\hat{K}$ is the kernel (resp. the cokernel) of $\hat{\psi}$. But $\hat{K} = K$ because K is of ϖ-torsion, and therefore $\hat{K} \otimes_{\mathfrak{o}} L = K \otimes_{\mathfrak{o}} L = 0$. □

[2] $\rho := \frac{1}{2}\sum_{\alpha\in\Lambda^*}\alpha$, cf. (2.5.3).

Let us identify the universal enveloping algebra $\mathcal{U}(\mathfrak{t}_L)$ of the Cartan subalgebra $\mathfrak{t}_L$ with the symmetric algebra $S(\mathfrak{t}_L)$, and let $Z(\mathfrak{g}_L)$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_L)$ of $\mathfrak{g}_L$. The classical Harish-Chandra isomorphism $Z(\mathfrak{g}_L) \simeq S(\mathfrak{t}_L)^W$ (the subalgebra of Weyl invariants) [22, theorem 7.4.5], allows us to define for every linear form $\lambda \in \mathfrak{t}_L^*$ a central character [22, 7.4.6]

$$\chi_\lambda : Z(\mathfrak{g}_L) \to L$$

which induces the central reduction $\mathcal{U}(\mathfrak{g}_L)_\lambda := \mathcal{U}(\mathfrak{g}_L) \otimes_{Z(\mathfrak{g}_L),\chi_\lambda+\rho} L$. If $\mathrm{Ker}(\chi_{\lambda+\rho})_{\mathfrak{o}} := D^{(m)}(\mathbb{G}) \cap \mathrm{Ker}(\chi_{\lambda+\rho})$, we can consider the central redaction

$$D^{(m)}(\mathbb{G})_\lambda := D^{(m)}(\mathbb{G})/D^{(m)}(\mathbb{G})\mathrm{Ker}(\chi_{\lambda+\rho})_{\mathfrak{o}}$$

and its p-adic completion $\widehat{D}^{(m)}(\mathbb{G})_\lambda$. It is clear that $D^{(m)}(\mathbb{G})_\lambda$ is an integral model of $\mathcal{U}(\mathfrak{g}_L)_\lambda$.

Theorem 3.2.2. *The homomorphism of $\mathfrak{o}$-algebras $\Phi_\lambda^{(m)} : D^{(m)}(\mathbb{G}) \to H^0(X, \mathcal{D}_{X,\lambda}^{(m)})$, defined by taking global sections in (2.30), induces an isomorphism of $\mathfrak{o}$-algebras*

$$\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathfrak{o}} L \xrightarrow{\simeq} H^0\left(\mathfrak{X}, \widehat{\mathscr{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}\right).$$

Proof. The key of the proof of the theorem is the following commutative diagram, which is an immediate consequence of remark 2.5.14

$$\begin{array}{ccc} D^{(m)}(\mathbb{G}) & \xrightarrow{\Phi_\lambda^{(m)}} & H^0(X, \mathcal{D}_{X,\lambda}^{(m)}) \\ \downarrow & & \downarrow \\ D^{(m)}(\mathbb{G}) \otimes_{\mathfrak{o}} L & \xrightarrow{\Phi_\lambda^{(m)}\otimes 1} & H^0(X, \mathcal{D}_{X,\lambda}^{(m)}) \otimes_{\mathfrak{o}} L \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{U}(\mathfrak{g}_L) & \xrightarrow{\Phi_\lambda} & H^0(X_L, \mathcal{D}_\lambda). \end{array}$$

By the classical Beilinson-Bernsein theorem [3] and the preceding commutative diagram, we have that $\Phi_\lambda^{(m)}$ factors through the morphism $\overline{\Phi}_\lambda^{(m)} : D^{(m)}(\mathbb{G})_\lambda \to H^0(X, \mathcal{D}_{X,\lambda}^{(m)})$ which becomes an isomorphism after tensoring with L. The preceding lemma implies therefore that $\overline{\Phi}_\lambda^{(m)}$ gives rise to an isomorphism

$$\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathfrak{o}} L \xrightarrow{\simeq} H^0\widehat{(X, \mathcal{D}_{X,\lambda}^{(m)})} \otimes_{\mathfrak{o}} L,$$

and proposition 3.1.1 together with the fact that $\mathfrak{X}$ is in particular a noetherian topological space end the proof of the theorem. □

3.3 The localization functor

In this section we will introduce the localization functor. For this, we will first fix the following notation which will make more pleasant the reading of the proof of our principal theorem. We will consider

$$\widehat{\mathrm{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)} := H^0\left(\mathfrak{X}, \widehat{\mathscr{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m)}\right).$$

Now, let E be a finitely generated $\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module. We define $\mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}(E)$ as the associated sheaf to the presheaf on $\mathfrak{X}$ defined by

$$\mathfrak{U} \subseteq \mathfrak{X} \mapsto \widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}(\mathfrak{U}) \otimes_{\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}} E.$$

It is clear that $\mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}$ is a functor from the category of finitely generated $\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules to the category of coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules.

3.4 The arithmetic Beilinson-Bernstein theorem

We are finally ready to prove one of the main results of this work. To start with, we will enunciate the following proposition. [3]

Proposition 3.4.1. *Let $\mathscr{E}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module. Then $\mathscr{E}$ is generated by its global sections as $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module. Furthermore, every coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module admits a resolution by finite free $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules.*

Proof. By theorem 3.1.5 we know that $\mathscr{E}$ is a quotient of a module $\left(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}(-r)\right)^{a}$ for some $r \in \mathbb{Z}$ and some $a \in \mathbb{N}$. We can therefore assume that $\mathscr{E} = \left(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}(-r)\right)$ for some $r \in \mathbb{Z}$. Let $F := H^0(X, \mathcal{D}^{(m)}_{X,\lambda}(-r))$, a finitely generated $D^{(m)}(\mathbb{G})$-module by proposition 2.6.1. Let us consider the linear map of $\mathcal{D}^{(m)}_{X,\lambda}$-modules equal to the composite

$$\mathcal{D}^{(m)}_{X,\lambda} \otimes_{D^{(m)}(\mathbb{G})} F \to \mathcal{D}^{(m)}_{X,\lambda} \otimes_{H^0(X,\mathcal{D}^{(m)}_{X,\lambda})} F \to \mathcal{D}^{(m)}_{X,\lambda}(-r) \tag{3.5}$$

where the first map is the surjection induced by the map $\Phi^{(m)}_{\lambda}$ of theorem 3.2.2. Let $\mathcal{F}$ be the cokernel of the composite map. Since $D^{(m)}(\mathbb{G})$ is noetherian, the source of this map is a coherent $\mathcal{D}^{(m)}_{X,\lambda}$-module and so is $\mathcal{F}$. Moreover, this module is of p-torsion because $\mathcal{D}^{(m)}_{X,\lambda}(-r) \otimes_{\mathfrak{o}} L$ is generated by its global sections [3]. Now, let us take a linear surjection $\left(D^{(m)}(\mathbb{G})\right)^{\oplus a} \to F$. By tensoring with $\mathcal{D}^{(m)}_{X,\lambda}$ we obtain the exact sequence of coherent modules

$$\left(\mathcal{D}^{(m)}_{X,\lambda}\right)^{\oplus a} \to \mathcal{D}^{(m)}_{X,\lambda}(-r) \to \mathcal{F} \to 0.$$

Passing to p-adic completions (which is exact in our situation [30, chapter II, proposition 9.1]) and inverting p yields the linear surjection. □

Theorem 3.4.2. *Let us suppose that $\lambda : Dist(\mathbb{T}) \to \mathfrak{o}$ is a character of $Dist(\mathbb{T})$ such that $\lambda + \rho \in \mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$. The functors $\mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}$ and $H^0(\mathfrak{X}, \bullet)$ are quasi-inverse equivalence of categories between the abelian categories of finitely generated $\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules and coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules.*

Proof. Let us take E a finite generated $\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module and $\mathscr{E}$ a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module. There exist canonical morphisms $E \to H^0(\mathfrak{X}, \mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}(E))$ and $\mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}(H^0(\mathfrak{X},\mathscr{E})) \to \mathscr{E}$. Moreover, given that E is a finitely generated $\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module, the third part of both propositions 2.6.1 and 3.1.1 allow us to find a resolution of

$$E : \left(\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}\right)^{\oplus b} \to \left(\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}\right)^{\oplus a} \to E \to 0$$

Therefore, by the preceding proposition, we get an exact sequence

$$\left(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}\right)^{\oplus b} \to \left(\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}\right)^{\oplus a} \to \mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}(E) \to 0.$$

[3] This proof is exactly as in [36, proposition 4.3.1].

From this resolution we derived the following diagram of short exact sequences

$$\begin{array}{ccccccc}
(\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}})^{\oplus b} & \longrightarrow & (\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}})^{\oplus a} & \longrightarrow & E & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
(\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}})^{\oplus b} & \longrightarrow & (\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}})^{\oplus a} & \longrightarrow & H^0(\mathfrak{X}, \mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}(E)) & \longrightarrow & 0.
\end{array}$$

By the preceding proposition, we know that the first two arrows are in fact isomorphisms thus, the canonical map $E \rightarrow H^0(\mathfrak{X}, \mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}(E))$ is an isomorphism as well. To see that the another canonical morphism is an isomorphism, the reader can follow a completely analogous reasoning (we recall for the reader that theorem 3.1.5 tells us that $H^0(\mathfrak{X}, \bullet)$ is an exact functor, over the category of coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules).

Finally, let us show that the functor $\mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}$ is fully and faithful. It is enough to show that it is faithful. Let $\psi : E \rightarrow F$ be an injective morphism between two finitely generated $\widehat{\mathrm{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules and let $\mathscr{K}$ be the coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module which is the kernel of the morphism $\mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}(\psi)$. By the preceding proposition, we know that $H^0(\mathfrak{X}, \mathscr{K})$ is the kernel of the application $H^0(\mathfrak{X}, \mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}(\psi))$ which is zero. But $\mathscr{K}$ is generated by global sections and therefore $\mathscr{K} = 0$. □

Given that any equivalence between abelian categories is exact, theorems 3.2.2 and 3.4.2 clearly imply

Theorem 3.4.3. ***(Principal theorem)*** *Let us suppose that $\lambda : Dist(\mathbb{T}) \rightarrow \mathfrak{o}$ is a character of $Dist(\mathbb{T})$ such that $\lambda + \rho \in \mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$.*

(i) The functors $\mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}$ and $H^0(\mathfrak{X}, \bullet)$ are quasi-inverse equivalence of categories between the abelian categories of finitely generated (left) $\widehat{\mathrm{D}}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathfrak{o}} L$-modules and coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules.

(ii) The functor $\mathscr{L}oc^{(m)}_{\mathfrak{X},\lambda}$ is an exact functor.

Chapter 4

The sheaves $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$

In this chapter we will study the problem of passing to the inductive limit when m varies. Let us recall that $\xi : \widetilde{X} := \mathbb{G}/\mathbf{N} \to X := \mathbb{G}/\mathbb{B}$ is a locally trivial $\mathbb{T}$-torsor (subsection 2.4). For every couple of positive integers $m \leq m'$ there exists a canonical homomorphism of sheaves of filtered rings [6, (2.2.1.5)]

$$\rho_{m',m} : \mathcal{D}^{(m)}_{\widetilde{X}} \to \mathcal{D}^{(m')}_{\widetilde{X}}. \tag{4.1}$$

Let us fix a character $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$. As we have remarked for $m \leq m'$ we have a commutative diagram

$$\begin{array}{ccc} D^{(m)}(\mathbb{T}) & & \\ \downarrow & \overset{\lambda^{(m)}}{\searrow} \; \overset{\lambda^{(m')}}{\nearrow} & \mathfrak{o}. \\ D^{(m')}(\mathbb{T}) & & \end{array} \tag{4.2}$$

Moreover, by [6, (1.4.7.1)] we dispose of a canonical morphism $\mathcal{P}^n_{\tilde{X},(m')} \to \mathcal{P}^n_{\tilde{X},(m)}$.

In section 3.3 we have defined a $\mathbb{T}$-equivariant structure $\Phi^n_{(m)} : p_1^*\mathcal{P}^n_{\tilde{X},n} \to \sigma^*\mathcal{P}^n_{\tilde{X},n}$ on $\mathcal{P}^n_{\tilde{X},n}$ (w recall for the reader that σ denotes the right action of $\mathbb{T}$ on $\tilde{X}$ and p_1 is the first projection). By universal property of $\mathcal{P}^n_{\tilde{X},(m)}$ the preceding $\mathbb{T}$-equivariant structures fit into a commutative diagram

$$\begin{array}{ccc} p_1^*\mathcal{P}^n_{\tilde{X},(m')} & \xrightarrow{\Phi^n_{(m')}} & \sigma^*\mathcal{P}^n_{\tilde{X},(m')} \\ \downarrow & & \downarrow \\ p_1^*\mathcal{P}^n_{\tilde{X},(m)} & \xrightarrow{\Phi^n_{(m)}} & \sigma^*\mathcal{P}^{(m)}_{\tilde{X},(m)}. \end{array} \tag{4.3}$$

This implies that the morphisms $\mathcal{P}^n_{\tilde{X},(m')} \to \mathcal{P}^n_{\tilde{X},(m)}$ are $\mathbb{T}$-equivariant and therefore by lemma 2.2.1 and lemma 2.2.3, we can conclude that the canonical maps in (4.1) are $\mathbb{T}$-equivariant. In this way, we dispose of morphisms $\widetilde{\mathcal{D}^{(m)}} \to \widetilde{\mathcal{D}^{(m')}}$. The diagram (4.2) implies that we also have maps $\mathcal{D}^{(m)}_{X,\lambda} \to \mathcal{D}^{(m')}_{X,\lambda}$ and therefore an inductive system

$$\xi_*\widehat{(\rho_{m',m})}^{\mathbb{T}} : \widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda} \to \widehat{\mathscr{D}}^{(m')}_{\mathfrak{X},\lambda}. \tag{4.4}$$

Definition 4.0.1. *We will denote by* $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$ *the limit of the inductive system (4.4), tensored with* L

$$\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda} := \left(\varinjlim_{m} \widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda} \right) \otimes_{\mathfrak{o}} L.$$

4.1 The localization functor $\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}$

As in the subsection 4.3 let us denote by $\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda} := H^0(\mathfrak{X}, \mathscr{D}^{\dagger}_{\mathfrak{X},\lambda})$. In a completely analogous way as we have done in the subsection referred above, we define the localization functor $\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}$ from the category of finitely presented $\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules to the category of coherent $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules. This is, if E denotes a finitely presented $\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda}$-module, then $\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}(E)$ denotes the associated sheaf to the presheaf on $\mathfrak{X}$ defined by

$$\mathfrak{U} \subseteq \mathfrak{X} \mapsto \mathscr{D}^{\dagger}_{\mathfrak{X},\lambda} \otimes_{\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda}} E.$$

It is clear that $\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}$ is a functor from the category of finitely presented $\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules to the category of coherent $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules.

4.2 The arithmetic Beilinson-Bernstein theorem for the sheaves $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$

In this subsection we will concentrate our efforts to show the following Beilinson-Bernstein theorem for the sheaf of rings $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$. To do that, we will fix throughout this section a character $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ of the distribution algebra $\mathrm{Dist}(\mathbb{T})$ such that $\lambda + \rho \in \mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$. We want to show

Theorem 4.2.1. *Let us suppose that* $\lambda : Dist(\mathbb{T}) \to \mathfrak{o}$ *is a character of* $Dist(\mathbb{T})$ *such that* $\lambda + \rho \in \mathfrak{t}^*_L$ *is a dominant and regular character of* $\mathfrak{t}_L$.

(i) The functors $\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}$ *and* $H^0(\mathfrak{X}, \bullet)$ *are quasi-inverse equivalence of categories between the abelian categories of finitely presented (left)* $\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda}$*-modules and coherent* $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$*-modules.*

(ii) The functor $\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}$ *is an exact functor.*

To do this, we recall the following facts.

Remark 4.2.2. *(i) Let us recall that in remark 1.5.4 we have stated that* $D^{(m)}(\mathbb{T})$ *is isomorphic to the subspace of* $\mathbb{T}$*-invariants* $H^0(\mathbb{T}, \mathcal{D}^{(m)}_{\mathbb{T}})^{\mathbb{T}}$. *The isomorphism is in fact induced by the the action of* $\mathbb{T}$ *on itself by right translations [38, theorem 4.4.8.3] and is compatible with m variable. This means that if* Q_m *and* Q'_m *denotes those isomorphisms for* $m \leq m'$*, then we have a commutative diagram*

$$\begin{array}{ccc} D^{(m)}(\mathbb{T}) & \xrightarrow{Q_m} & H^0(\mathbb{T}, \mathcal{D}^{(m)}_{\mathbb{T}})^{\mathbb{T}} \\ \downarrow{\scriptstyle \phi_{m',m}} & & \downarrow{\scriptstyle (\gamma_{m',m})^{\mathbb{T}}} \\ D^{(m')}(\mathbb{T}) & \xrightarrow{Q_{m'}} & H^0(\mathbb{T}, \mathcal{D}^{(m')}_{\mathbb{T}})^{\mathbb{T}} \end{array}$$

where the morphisms $\phi_{m',m}$ *are obtained by dualizing the canonical morphisms* $\psi_{m',m}$ *in subsection 1.4 and the morphisms* $\gamma_{m',m}$ *are defined in (4.1).*

(ii) Again by remark 1.5.4 the isomorphism of proposition 2.5.8 are compatible for varying m.

Let us recall the following proposition.

Proposition 4.2.3. *[6, Proposition 3.6.1] Let Y be a topological space, and $\{\mathcal{D}_\pi\}_{i\in J}$ be a filtered inductive system of coherent sheaves of rings on Y, such that for any $i \leq j$ the morphisms $\mathcal{D}_i \to \mathcal{D}_j$ are flat. Then the sheaf $\mathcal{D}^\dagger := \varinjlim_{i\in J} \mathcal{D}_i$ is a coherent sheaf of rings.*

Proposition 4.2.4. *The sheaf of rings $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$ is coherent.*

Proof. The previous proposition tells us that we only need to show that the morphisms $\xi_*\widehat{(\rho_{m',m})}^{\mathbb{T}}{}_{\mathbb{Q}}$ are flats. As this is a local property we can take $U \in \mathcal{S}$ and to verify this property over the formal completion $\mathfrak{U}$. In this case, remark 4.2.2 and the argument used in the proof of the first part of proposition 2.5.12 give us, by functoriality, the following commutative diagram

$$\begin{array}{ccc}
\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}(\mathfrak{U}) & \xrightarrow{\xi_*\widehat{(\rho_{m',m})}^{\mathbb{T}}{}_{\mathbb{Q}}(\mathfrak{U})} & \widehat{\mathscr{D}}^{(m')}_{\mathfrak{X},\lambda,\mathbb{Q}}(\mathfrak{U}) \\
\downarrow \wr & & \downarrow \wr \\
\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X}\mathbb{Q}}(\mathfrak{U}) & \xrightarrow{\hat{\rho}_{m',m,\mathbb{Q}}(\mathfrak{U})} & \widehat{\mathscr{D}}^{(m')}_{\mathfrak{X},Q}(\mathfrak{U})
\end{array}$$

The flatness theorem [6, theorem 3.5.3] states that the lower morphism is flat and so is the morphism on the top. □

Notation: From now on we suppose that $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ induces a dominant and regular character of $\mathfrak{t}_L$, under the correspondence (2.26).

Lemma 4.2.5. *For every coherent $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-module $\mathscr{E}$ there exist $m \geq 0$, a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module $\mathscr{E}_m$ and an isomorphism of $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules*

$$\tau : \mathscr{D}^{\dagger}_{\mathfrak{X},\lambda} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathscr{E}_m \xrightarrow{\simeq} \mathscr{E}.$$

Moreover, if $(m', \mathscr{E}_{m'}, \tau')$ is another such triple, then there exist $l \geq max\{m, m'\}$ and an isomorphism of $\widehat{\mathscr{D}}^{(l)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules

$$\tau_l : \widehat{\mathscr{D}}^{(l)}_{\mathfrak{X},\lambda,\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathscr{E}_m \xrightarrow{\simeq} \widehat{\mathscr{D}}^{(l)}_{\mathfrak{X},\lambda,\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m')}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathscr{E}_{m'}$$

such that $\tau' \circ \left(id_{\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}} \otimes \tau_l \right) = \tau.$

Proof. This is [6, proposition 3.6.2 (ii)]. We remark that $\mathfrak{X}$ is quasi-compact and separated, and the sheaf $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ satisfies the conditions in [6, 3.4.1]. □

Proposition 4.2.6. *Let $\mathscr{E}$ be a coherent $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-module.*

(i) There exists an integer $r(\mathscr{E})$ such that, for all $r \geq r(\mathscr{E})$ there is $a \in \mathbb{N}$ and an epimorphism of $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules

$$\left(\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}(-r)\right)^{\oplus a} \to \mathscr{E} \to 0.$$

(ii) For all $i > 0$ one has $H^i(\mathfrak{X}\mathscr{E}) = 0$.

Proof. [1] Let $\mathscr{E}$ be a coherent $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-module. The preceding proposition tells us that there exist $m \in \mathbb{N}$, a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-

[1]This is exactly as in [36, theorem 4.2.8]

module $\mathscr{E}_m$ and an isomorphism of $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules

$$\tau : \mathscr{D}^{\dagger}_{\mathfrak{X},\lambda} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,Q}} \mathscr{E}_m \xrightarrow{\simeq} \mathscr{E}.$$

Now we use theorem 3.1.5 for $\mathscr{E}_m$ and we get the desired surjection in (i) after tensoring with $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$. To show (ii) we use the fact that, as $\mathfrak{X}$ is a noetherian topological space, cohomology commutes with direct limites and

$$\mathscr{E} = \mathscr{D}^{\dagger}_{\mathfrak{X},\lambda} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathscr{E}_m = \varinjlim_{l\geq m} \widehat{\mathscr{D}}^{(l)}_{\mathfrak{X},\lambda,\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathscr{E}_m.$$

Being $\mathscr{E}_m$ a coherent $\mathscr{D}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module we can conclude that $\widehat{\mathscr{D}}^{(l)}_{\mathfrak{X},\lambda,\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathscr{E}_m$ is a coherent $\mathscr{D}^{(l)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module for every $l \geq m$. Then for every $i > 0$

$$H^i(\mathfrak{X},\mathscr{E}) = \varinjlim_{l\geq m} H^i\left(\mathfrak{X}, \widehat{\mathscr{D}}^{(l)}_{\mathfrak{X},\lambda,\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathscr{E}_m\right) = 0,$$

by part (i) and theorem 3.1.5. □

Proposition 4.2.7. *Let $\mathscr{E}$ be a coherent $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-module. Then $\mathscr{E}$ is generated by its global sections as $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-module. Moreover, $\mathscr{E}$ has a resolution by finite free $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-modules and $H^0(\mathfrak{X},\mathscr{E})$ is a $\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda}$-module of finite presentation.*

Proof. [2] Theorem 3.1.5 gives us a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module $\mathscr{E}_m$ such that $\mathscr{E} \simeq \widehat{\mathscr{D}}^{\dagger}_{\mathfrak{X},\lambda} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}} \mathscr{E}_m$. Moreover, $\mathscr{E}_m$ has a resolution by finite free $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules (proposition 3.4.1). Both results clearly imply the first and the second part of the lemma. The final part of the lemma is therefore a consequence of the first part and the acyclicity of the the functor $H^0(\mathfrak{X},\bullet)$. □

Proof of theorem 4.2.1. All in all, we can follow the same arguments of [35, corollary 2.3.7]. We start by taking $(\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda})^{\oplus a} \to (\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda})^{\oplus b} \to E \to 0$ a finitely presented $\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda}$-module. By localizing and applying the global sections functor, we obtain a commutative diagram

$$\begin{array}{ccccccc}
(\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda})^{\oplus a} & \longrightarrow & (\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda})^{\oplus b} & \longrightarrow & E & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
(\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda})^{\oplus a} & \longrightarrow & (\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda})^{\oplus b} & \longrightarrow & H^0(\mathfrak{X},\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}(E)) & \longrightarrow & 0.
\end{array}$$

which tells us that $E \to H^0(\mathfrak{X},\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}(E))$ is an isomorphism. To show that if $\mathscr{E}$ is coherent $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}$-module then the canonical morphism $\mathscr{D}^{\dagger}_{\mathfrak{X},\lambda} \otimes_{\mathrm{D}^{\dagger}_{\mathfrak{X},\lambda}} H^0(\mathfrak{X},\mathscr{E}) \to \mathscr{E}$ is an isomorphism the reader can follow the same argument as before. As we have remarked, the second assertion follows because any equivalence between abelian categories is exact. □

Calculation of global sections

Let us recall that in the subsection 4.2 we have used the fact that associated to the linear form $\lambda \in \mathfrak{t}^*_L$ there exists a central character $\chi_\lambda : Z(\mathfrak{g}_L) \to L$, where $Z(\mathfrak{g}_L)$ denotes the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_L)$. In this case, if $\mathrm{Ker}(\chi_{\lambda+\rho})_{\mathfrak{o}} := D^{(m)}(\mathbb{G}) \cap \mathrm{Ker}(\chi_\lambda)$, we can consider the central redaction

$$D^{(m)}(\mathbb{G})_\lambda := D^{(m)}(\mathbb{G})/D^{(m)}(\mathbb{G})\mathrm{Ker}(\chi_{\lambda+\rho})_{\mathfrak{o}}$$

[2]This is exactly as in [34, theorem 5.1]

and its p-adic completion $\widehat{D}^{(m)}(\mathbb{G})_\lambda$. Following the notation introduced in subsection 3.1 we have $D^{(m)}(\mathbb{G})_\lambda \otimes_{\mathfrak{o}} L = \mathcal{U}(\mathfrak{g}_L)_\lambda$. We have shown that there exists a canonical isomorphism of $\mathfrak{o}$-algebras

$$\widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathfrak{o}} L \xrightarrow{\simeq} H^0\left(\mathfrak{X}, \widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\lambda,\mathbb{Q}}\right).$$

Taking inductive limits we can conclude that if

$$D^{\dagger}(\mathbb{G})_\lambda := \varinjlim_m \widehat{D}^{(m)}(\mathbb{G})_\lambda \otimes_{\mathfrak{o}} L,$$

then we also have a canonical isomorphism of $\mathfrak{o}$-algebras

$$D^{\dagger}(\mathbb{G})_\lambda \xrightarrow{\simeq} H^0(\mathfrak{X}, \mathscr{D}^{\dagger}_{\mathfrak{X},\lambda}).$$

Theorem 4.2.1 and the preceding calculation complete the Beilinson-Bernstein correspondence. We end this chapter with the following remark.

In [35] and [39] C. Huyghe and T. Schmidt studied the algebraic case. This means that λ is induced via derivation by a character $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$. In this setting, the consider arithmetic differential operators acting on the line bundle $\mathcal{L}(\lambda)$ induced by λ (cf. section 5.5). Let us denote those sheaves by $\widehat{\mathscr{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}(\lambda)$ and the inductive limit by $\mathscr{D}^{\dagger}_{\mathfrak{X}}(\lambda)$. They have showed analogous results to theorem 4.2.1, if $\lambda + \rho \in \mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$ ([39, Theorem 3.2.5] and [35, Theorem 3.1]). This in particular implies that if $\lambda' : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ is the character of the distribution algebra $\mathrm{Dist}(\mathbb{T})$ induced via the correspondence (2.26), then

$$H^0\left(\mathfrak{X}, \mathscr{D}^{\dagger}_{\mathfrak{X}}(\lambda)\right) = D^{\dagger}(\mathbb{G})_\lambda = H^0\left(\mathfrak{X}, \mathscr{D}^{\dagger}_{\mathfrak{X},\lambda'}\right).$$

For $\lambda' + \rho = \lambda + \rho \in \mathfrak{t}^*_L$ dominant and regular.[3] Therefore, we have the following equivalence of categories

$$\left\{\text{Coherent } \mathscr{D}^{\dagger}_{\mathfrak{X}}(\lambda) - \text{modules}\right\} \xrightarrow{H^0(\mathfrak{X},\bullet)} \left\{\text{Finitely presented } D^{\dagger}(\mathbb{G})_\lambda - \text{modules}\right\} \xrightarrow{\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda'}} \left\{\text{Coherent } \mathscr{D}^{\dagger}_{\mathfrak{X},\lambda'} - \text{modules}\right\}.$$

[3]By construction, if we tensor with L the characters λ and λ', then they induces the same character of $\mathfrak{t}_L$, cf. (2.26).

Chapter 5

Arithmetic differential operators with congruence levels

In this chapter we will introduce congruence levels to the constructions given in the preceding chapters. This means, deformations of our (integral) twisted differential operators. This notion will be a fundamental tool to define differential operators on an admissible blow-up of the flag $\mathfrak{o}$-scheme X. We also point out to the reader that we could have started this work by considering differential operators with congruences levels because, as we will explain later, the case "$k = 0$" naturally recovers all our preceding definitions. We have decided to treat this case apart because this is already a meaningful constructions which generalizes [35] and [39]. Many of the results in this chapter are analogues to the results obtained in our previous work and their proofs follow the same lines of reasoning. In all these cases we will refer to the respective analogue.

5.1 Congruence levels

In this section we retake the notations of sections 2.1 and 2.4. This means that $\tilde{X}$ and X will denote smooth separated schemes over $\mathfrak{o}$, such that $\tilde{X}$ is endowed with a right $\mathbb{T}$-action $\sigma : \tilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to \tilde{X}$. We will also denote by $\xi : \tilde{X} \to X$ a locally trivial $\mathbb{T}$-torsor for the Zariski topology and by $\mathcal{S}$ the set of open affine subschemes U of X that trivialises the torsor (Remark 2.1.1).

Finally, as usual, we will denote by $\mathcal{D}^{(m)}_{\widetilde{X}}$ (resp. by $\mathcal{D}^{(m)}_{X}$) the usual sheaf of level m differential operators on $\widetilde{X}$ (resp. on X). As we have remarked in the first chapter, those sheaves come equipped with a filtration

$$\mathcal{O}_{\widetilde{X}} \subseteq \mathcal{D}^{(m)}_{\widetilde{X},1} \subseteq ... \subseteq \mathcal{D}^{(m)}_{\widetilde{X},d} \subseteq ... \subseteq \mathcal{D}^{(m)}_{\widetilde{X}},$$

with $\mathcal{D}^{(m)}_{\widetilde{X},d}$ the sheaf of level m differential operators of order less or equal than d.

5.1.1 Associated Rees rings

[1] Let $\mathcal{A}$ be a sheaf of $\mathfrak{o}$-algebras endowed with a positive filtration $(F_d\mathcal{A})_{d\in\mathbb{N}}$ and such that $\mathfrak{o} \subset F_0\mathcal{A}$. The sheaf $\mathcal{A}$ gives rise to a subsheaf of graded rings $R(\mathcal{A})$ of the polynomial algebra $\mathcal{A}[t]$ over $\mathcal{A}$. This is defined by

$$R(\mathcal{A}) := \bigoplus_{i\in\mathbb{N}} F_i\mathcal{A} \cdot t^i,$$

[1] This digression can be found before the proof of [36, Proposition 3.3.7].

its associated Rees ring. This subsheaf comes equipped with a filtration by the sheaves of subgroups

$$R_d(\mathcal{A}) := \bigoplus_{i=0}^{d} F_i\mathcal{A}\cdot t^i \subseteq R(\mathcal{A}).$$

Specializing $R(\mathcal{A})$ in an element $\mu \in \mathfrak{o}$ we get a sheaf of filtered subrings $\mathcal{A}_\mu$ of $\mathcal{A}$. More exactly, $\mathcal{A}_\mu$ equals the image under the homomorphism of sheaves of rings $\phi_\mu : R(\mathcal{A}) \to \mathcal{A}$, sending $t \mapsto \mu$, and it is equipped with the filtration induced by $\mathcal{A}$. Moreover, if the sheaf of graded rings $\mathrm{gr}(\mathcal{A})$, associated to the filtration $(F_d\mathcal{A})_{d\in\mathbb{N}}$, is flat over $\mathfrak{o}$, then[2]

$$F_d\mathcal{A}_\mu = \sum_{i=0}^{d} \mu^i F_i\mathcal{A}. \tag{5.1}$$

If $\psi : \mathcal{A} \to \mathcal{B}$ is a morphism of positive filtered $\mathfrak{o}$-algebras (with $\mathfrak{o} \subseteq F_0\mathcal{A}$ and $\mathfrak{o} \subseteq F_0\mathcal{B}$), then the commutative diagram

$$\begin{array}{ccc} R(\mathcal{A}) & \xrightarrow{a_d t^d \mapsto \psi(a_d)t^d} & R(\mathcal{B}) \\ \downarrow{\scriptstyle \phi_\mu} & & \downarrow{\scriptstyle \phi_\mu} \\ \mathcal{A} & \xrightarrow{\quad \psi \quad} & \mathcal{B} \end{array}$$

gives us a filtered morphism of rings $\psi_\mu : \mathcal{A}_\mu \to \mathcal{B}_\mu$. This in particular implies that for $\mu \in \mathfrak{o}$ fixed, the preceding process is functorial.

Remark 5.1.1. *The previous digression is well-known for rings. In this setting we have results completely analogues to the ones presented so far ([43, Chapter 12, section 6]). We will use these results in the next sections.*

Finally, under the hypothesis (5.1), if we endow $Ker(\phi_\mu)$ with the filtration induced by $R(\mathcal{A})$, then for every $d \in \mathbb{Z}_{>0}$ we have $F_dKer(\phi_\mu) = (t-\mu)R_{d-1}(\mathcal{A})$. To see this, we take a local section $p(t) = \sum_{i=0}^{d} a_i t^i \in F_dKer(\phi_\mu)$ and a polynomial $q(t) = \sum_{j=0}^{d-1} b_j t^j$ such that $p(t) = (t-\mu)q(t) + c$, with $c \in \mathcal{A}$. As $0 = p(\mu) = c$, we conclude that

$$p(t) = (t-\mu)q(t) = \sum_{i=0}^{d-1} b_j t^{j+1} - \sum_{j=0}^{d-1} \mu b_j t^j.$$

The previous relation implies for example that $a_0 = -\mu b_0$ and therefore $b_0 \in F_0\mathcal{A}$. Furthermore, an inductive argument allows us to conclude that $b_j \in F_j\mathcal{A}$ for every $0 \le j \le d-1$. In other words $q(t) \in R_{d-1}(\mathcal{A})$.
The short exact sequence

$$0 \to Ker(\phi_\mu) \to R_d(\mathcal{A}) \xrightarrow{\phi_\mu} F_d\mathcal{A}_\mu \to 0$$

implies that $F_d(\mathcal{A}_\mu) = R_d(\mathcal{A})/(t-\mu)R_{d-1}(\mathcal{A})$ for every $d \in \mathbb{Z}_{>0}$. Of course, by (5.1), we have $F_0(\mathcal{A}_\mu) = F_0(\mathcal{A})$.

The following lemma is completely formal.

Lemma 5.1.2. *Let $f : Y \to Z$ be a morphism of schemes and let $\mathcal{A}$ be a sheaf of algebras on Z. Then there exists a canonical isomorphism of sheaf of algebras on Y*

$$f^*(\mathcal{A}[t]) \xrightarrow{\simeq} f^*(\mathcal{A})[t], \quad 1 \otimes \left(\sum a_i t^i\right) \mapsto \sum (1 \otimes a_i)t^i.$$

Let $p_1 : \tilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \to \tilde{X}$ denote the first projection. Let us retake the notations in 5.1.1 and let us also assume that $(\mathcal{A}, \Phi)$ is a $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\tilde{X}}$-module ((2.3) and (2.4)) via a filtered isomorphism Φ. By lemma 5.1.2 and

[2] This is [36, Claim 3.3.10.].

the $\mathbb{T}$-equivariant structure of $\mathcal{A}$ we have a canonical isomorphism

$$\Phi_0 : p_1^*(\mathcal{A}[t]) \xrightarrow{\simeq} \sigma^*(\mathcal{A}[t]); \ \ 1 \otimes \left(\sum a_i t^i \right) \mapsto 1 \otimes \left(\sum \Phi(a_i) t^i \right), \tag{5.2}$$

which defines a $\mathbb{T}$-equivariant structure on $\mathcal{A}[t]$. Moreover, the positive filtration $\left(F_d\mathcal{A}\right)_{d\in\mathbb{N}}$ of $\mathcal{A}$ induces two positive filtrations $\left(p_1^*(F_d\mathcal{A}[t])\right)_{d\in\mathbb{N}}$ and $\left(\sigma^*(F_d\mathcal{A}[t])\right)_{d\in\mathbb{N}}$, over $p_1^*(\mathcal{A}[t])$ and $\sigma^*(\mathcal{A}[t])$, respectively. Those filtrations make of Φ_0 a filtered isomorphism, and therefore it induces $\mathbb{T}$-quivariant structures over the Rees ring $R(A)$, the ideal $(t-\mu)R(\mathcal{A})$, and the subgroups $R_d(\mathcal{A})$ and $(t-\mu)R_{d-1}(\mathcal{A})$.

5.1.2 Congruence subgroups

Let us denote by $\mathbb{F}_q := \mathfrak{o}/(\varpi)$ the quotient field of $\mathfrak{o}$. We start this subsection by recalling the following notion.

5.1.3. ***Dilatations*** *In the following digression we will suppose that $Z = Spec(A)$ is an affine $\mathfrak{o}$-scheme of finite type. Let us recall the following definition [60, Definition 2.1].*

Definition 5.1.4. *Let $Z_L := Z \times_{Spec(\mathfrak{o})} Spec(L) = Spec(A \otimes_{\mathfrak{o}} L)$ be the generic fiber. We say that an $\mathfrak{o}$-scheme $Z_0 = Spec(B)$, such that $B \subseteq A \otimes_{\mathfrak{o}} L$ is an $\mathfrak{o}$-subalgebra of finite type, is a model of Z_L if $B[\varpi^{-1}] = A \otimes_{\mathfrak{o}} L$. A model Z_0 is smooth if $Z_0 \to Spec(\mathfrak{o})$ is smooth.*

Let us recall the construction of the *dilatation* of a closed subscheme Z_0 of the special fiber $Z_{\mathbb{F}_q} = Z \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathbb{F}_q)$ [15, Chapter 3, Section 3.2]. Let $\mathcal{J} \subsetneq A$ be the proper ideal of A defining Z_0. As A is noetherian, we can suppose that $\mathcal{J}$ is generated by finitely many elements $f_0 = \varpi, f_1, ..., f_n$ of A. We define the dilation of Z_0 as the affine A-scheme $Z_{(\varpi)} := \mathrm{Spec}(A_{(\varpi)})$, where

$$A_{(\varpi)} := A\left[\frac{f_1}{\varpi}, ..., \frac{f_n}{\varpi}\right] := \left(A\left[T_1, ..., T_n\right] / (f_1 - \varpi T_1, ..., f_n - \varpi T_n)\right) / (\varpi\text{-torsion}).$$

In particular, we see that $Z_{(\varpi)}$ is always flat over $\mathfrak{o}$, it is a model of the generic fiber Z_L [59, Proposition 1.1] and we have a canonical morphism

$$Z_{(\varpi)} \to Z. \tag{5.3}$$

As before, let $\mathbb{G}$ be a split, connected and reductive group scheme over $\mathfrak{o}$. We denote by $\mathbb{G}_L := \mathbb{G} \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(L)$ the *generic fiber* of $\mathbb{G}$ and by $\mathbb{G}_{\mathbb{F}_q} := \mathbb{G} \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathbb{F}_q)$ the *special fiber* of $\mathbb{G}$. For every $k \in \mathbb{N}$, there exists a smooth model $\mathbb{G}(k)$ such that $\mathrm{Lie}(\mathbb{G}(k)) = \varpi^k\mathfrak{g}$. In fact, we take $\mathbb{G}(0) := \mathbb{G}$ and we construct $\mathbb{G}(1)$ as the dilatation of the trivial subgroup of $\mathbb{G}_{\mathbb{F}_q}$ on $\mathbb{G}$. By the preceding construction, this is a smooth model of $\mathbb{G}_L$. Moreover, if we write $\mathfrak{o}[\mathbb{G}] = \mathfrak{o}[f_1, ..., f_n]$ with $f_i(e) = 0$ for every $1 \leq i \leq n$ (e being the identity element), then $(f_i \bmod \varpi)_{1\leq i\leq n}$ is the ideal of the trivial subgroup of $\mathbb{G}_{\mathbb{F}_q}$. By construction

$$\mathbb{G}(1) = \mathrm{Spec}\left(\mathfrak{o}\left[\frac{f_1}{\varpi}, ..., \frac{f_n}{\varpi}\right]\right).$$

This in particular implies that $\mathrm{Lie}(\mathbb{G}(1)) = \varpi\mathfrak{g}$.

We can now construct $\mathbb{G}(k)$ inductively as follows. As before, we put $\mathbb{G}(0) = \mathbb{G}$, and $\mathbb{G}(k+1)$ equals the dilatation of the trivial subgroup of $\mathbb{G}(k)_{\mathbb{F}_q}$. For every $k \in \mathbb{N}$ the $\mathfrak{o}$-group scheme $\mathbb{G}(k)$ is again smooth, its Lie algebra is $\varpi^k\mathfrak{g}$ and it is a smooth model of $\mathbb{G}_L$. We also point out to the reader that, by (5.3), we have for every $k \in \mathbb{N}$ a canonical morphism

$$\mathbb{G}(k+1) \to \mathbb{G}(k). \tag{5.4}$$

5.1.5. ***Arithmetic distribution algebras of finite level.***

Let us start this digression by recalling the following two facts from [38]. Let $N \in \mathbb{N}$. Let $\mathbb{G}_a = \mathrm{Spec}(\mathfrak{o}[t])$ be the additive group $\mathfrak{o}$-scheme and let us consider the group $\mathbb{G}_a^N$ endowed with coordinates $t_1, ..., t_N$. If $f_1, ..., f_N$ denotes a base of $\mathrm{Lie}(\mathbb{G}_a^N)$, then using the notations introduced in section 1.4 we have the following relation. For every multi-index $\underline{k} \in \mathbb{N}^N$ we have [38, Proposition 4.1.11]

$$\frac{\underline{k}!}{q_{\underline{k}}!}\underline{f}^{<\underline{k}>} = \underline{f}^{\underline{k}}. \tag{5.5}$$

On the other side, let us consider the multiplicative group $\mathfrak{o}$-scheme $\mathbb{G}_m = \mathrm{Spec}(\mathfrak{o}[t, t^{-1}])$ and let us take the group $\mathbb{G}_m^N$. If $h_1, ..., h_N$ denotes a base of $\mathrm{Lie}(\mathbb{G}_m^N)$, then using the preceding notation, we have the following relation [38, (50)]

$$\underline{h}^{<\underline{k}>} = q_{\underline{k}}!\binom{\underline{h}}{\underline{k}}. \tag{5.6}$$

Now, let us recall that in this work $\mathbb{B} \subseteq \mathbb{G}$ denotes a Borel subgroup of the split connected reductive group $\mathfrak{o}$-scheme $\mathbb{G}$, that $\mathbb{T} \subseteq \mathbb{B}$ denotes a split maximal torus of $\mathbb{G}$ and $\mathbf{N} \subseteq \mathbb{B}$ the unipotent radical of $\mathbb{B}$ (Section 2.4). Let $\overline{\mathbf{N}}$ be the opposite unipotent radical.[3] By [38, Proposition 4.1.11, (ii)] the open immersion $\mathbf{N} \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{T} \times_{\mathrm{Spec}(\mathfrak{o})} \overline{\mathbf{N}} \to \mathbb{G}$ induces an isomorphism of filtered $\mathfrak{o}$-modules

$$D^{(m)}(\mathbb{G}) \xrightarrow{\simeq} D^{(m)}(\mathbf{N}) \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{T}) \otimes_{\mathfrak{o}} D^{(m)}(\overline{\mathbf{N}}).$$

By construction (section 2.4), there exist $N_1, N_2 \in \mathbb{N}$ such that $\mathbf{N}$ and $\overline{\mathbf{N}}$ are isomorphic to $\mathbb{G}_a^{N_1}$ and by definition $\mathbb{T} \simeq \mathbb{G}_m^{N_2}$. Moreover, if we fix basis elements $(f_i)_{1\leq i\leq N_1}$, $(h_j)_{1\leq j\leq N_2}$ and $(e_l)_{1\leq l\leq N_1}$ of the $\mathfrak{o}$-Lie algebras $\mathfrak{n}$, $\mathfrak{t}$ and $\overline{\mathfrak{n}}$, respectively, then by (5.5) and (5.6) we can conclude that $D^{(m)}(\mathbb{G})$ equals the $\mathfrak{o}$-subalgebra of $\mathcal{U}(\mathfrak{g}_L)$ generated as an $\mathfrak{o}$-module by the elements

$$q_{\underline{v}}!\frac{\underline{f}^{\underline{v}}}{\underline{v}!}q_{\underline{v}'}!\binom{\underline{h}}{\underline{v}'}q_{\underline{v}''}!\frac{\underline{e}^{\underline{v}''}}{\underline{v}''!}. \tag{5.7}$$

This relation implies that $D^{(m)}(\mathbb{G}(k))$ equals the $\mathfrak{o}$-subalgebra of $\mathcal{U}(\mathfrak{g}_L)$ generated as an $\mathfrak{o}$-module by the elements

$$q_{\underline{v}}!\varpi^{k|\underline{v}|}\frac{\underline{f}^{\underline{v}}}{\underline{v}!}q_{\underline{v}'}!\varpi^{k|\underline{v}'|}\binom{\underline{h}}{\underline{v}'}q_{\underline{v}''}!\varpi^{k|\underline{v}''|}\frac{\underline{e}^{\underline{v}''}}{\underline{v}''!}. \tag{5.8}$$

An element of the preceding form has order $d = |v| + |v|' + |v|''$, and the $\mathfrak{o}$-span of elements of order less or equal that d forms an $\mathfrak{o}$-submodule $D_d^{(m)}(\mathbb{G}(k)) \subseteq D^{(m)}(\mathbb{G}(k))$. In this way $D^{(m)}(\mathbb{G}(k))$ becomes a filtered $\mathfrak{o}$-algebra. This construction also tells us that

$$D^{(m)}(\mathbb{G}(0))_{\varpi^k} = D^{(m)}(\mathbb{G}(k)). \tag{5.9}$$

5.1.3 Level m relative enveloping algebras of congruence level k

Now, let k be a non-negative integer called a *congruence level* [37, Subsection 2.1]. By using the order filtration $(\mathcal{D}_{\widetilde{X}}^{(m)})_{d\in\mathbb{N}}$ of the sheaf $\mathcal{D}_{\widetilde{X}}^{(m)}$, we can define the sheaf of *arithmetic differential operators of congruence level* k, $\mathcal{D}_{\widetilde{X}}^{(m,k)}$, as the subsheaf

[3] In the notation of section 2.4 $\overline{\mathbf{N}}$ is the closed subgroup of $\mathbb{G}$ generated by all U_α with $\alpha \in -\Lambda^+$.

of $\mathcal{D}^{(m)}_{\widetilde{X}}$ given by the specialization of $R(\mathcal{D}^{(m)}_{\widetilde{X}})$ in $\varpi^k \in \mathfrak{o}$. This means

$$\mathcal{D}^{(m,k)}_{\widetilde{X}} := \sum_{d\in\mathbb{N}} \varpi^{kd}\mathcal{D}^{(m)}_{\widetilde{X},d}.$$

By (1.3) and [34, Proposition 1.3.4.2] we can also conclude that, if $(\mathcal{D}^{(m,k)}_{\widetilde{X},d})_{d\in\mathbb{N}}$ denotes the order filtration induced by $\mathcal{D}^{(m)}_{\widetilde{X}}$, then

$$\mathcal{D}^{(m,k)}_{\widetilde{X},d} = \sum_{i=0}^{d} \varpi^{ki}\mathcal{D}^{(m)}_{\widetilde{X},i}.$$

In local coordinates we can describe the sheaf $\mathcal{D}^{(m,k)}_{\widetilde{X}}$ in the following way. Let $U \subseteq \widetilde{X}$ be an open affine subset endowed with coordinates $x_1, ..., x_N$. Let $dx_1, ..., dx_N$ be a basis of $\Omega_{\widetilde{X}}(U)$ and $\partial_{x_1}, ..., \partial_{x_N}$ the dual basis of $\mathcal{T}_{\widetilde{X}}(U)$. By using the notation in section 1.2, one has the following description [37, Subsection 2.1]

$$\mathcal{D}^{(m,k)}_{\widetilde{X}}(U) = \left\{ \sum_{\underline{\nu}}^{<\infty} \varpi^{k|\underline{\nu}|} a_{\underline{\nu}} \underline{\partial}^{<\underline{\nu}>} \mid a_{\underline{\nu}} \in \mathcal{O}_{\widetilde{X}}(U) \right\}.$$

Of course, we have analogue definitions on X.

On the other hand, using the short exact sequence

$$0 \to (t-\varpi^k)R(\mathcal{D}^{(m)}_{\widetilde{X}}) \to R(\mathcal{D}^{(m)}_{\widetilde{X}}) \to R(\mathcal{D}^{(m)}_{\widetilde{X}})/(t-\varpi^k)R(\mathcal{D}^{(m)}_{\widetilde{X}}) \overset{\phi_{\varpi^k}}{\cong} \mathcal{D}^{(m,k)}_{\widetilde{X}} \to 0, \tag{5.10}$$

and the fact that all the terms in the sequence are quasi-coherent $\mathcal{O}_{\widetilde{X}}$-modules, we can use proposition 2.2.2 and the final commentary of subsection 5.1.1 to get the following result.

Proposition 5.1.6. *For every non-negative integer k, the sheaf of arithmetic differential operators of congruence level k, $\mathcal{D}^{(m,k)}_{\widetilde{X}}$, is a $\mathbb{T}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module.*

Remark 5.1.7. *If $\widetilde{X}$ is also equipped with a right $\mathbb{G}$-action, then we can use the preceding reasoning to show that the sheaf $\mathcal{D}^{(m,k)}_{\widetilde{X}}$ is a $\mathbb{G}$-equivariant quasi-coherent $\mathcal{O}_{\widetilde{X}}$-module.*

Furthermore, for every $d \in \mathbb{Z}_{>0}$ the short exact sequence

$$0 \to (t-\varpi^k)R_{d-1}(\mathcal{D}^{(m)}_{\widetilde{X}}) \to R_d(\mathcal{D}^{(m)}_{\widetilde{X}}) \to R_d(\mathcal{D}^{(m)}_{\widetilde{X}})/(t-\varpi^k)R_{d-1}(\mathcal{D}^{(m)}_{\widetilde{X}}) \overset{\phi_{\varpi^k}}{\cong} \mathcal{D}^{(m,k)}_{\widetilde{X},d} \to 0$$

and again the final commentary in subsection 5.1.1 Imply that every term in the filtration $(\mathcal{D}^{(m,k)}_{\widetilde{X},d})_{d\in\mathbb{N}}$ (being $\mathcal{D}^{(m,k)}_{\widetilde{X},0} = \mathcal{O}_{\widetilde{X}}$) is a $\mathbb{T}$-equivariant coherent $\mathcal{O}_{\widetilde{X}}$-module.

Definition 5.1.8. *Let $\xi : \widetilde{X} \to X$ be a locally trivial $\mathbb{T}$-torsor. Following 2.3.5 we define the level m relative enveloping algebra of congruence level k of the torsor to be the sheaf of $\mathbb{T}$-invariants of $\xi_*\mathcal{D}^{(m,k)}_{\widetilde{X}}$:*

$$\widetilde{\mathcal{D}^{(m,k)}} := \left(\xi_*\mathcal{D}^{(m,k)}_{\widetilde{X}}\right)^{\mathbb{T}}.$$

The preceding sheaf is endowed with a canonical filtration

$$F_d\left(\widetilde{\mathcal{D}^{(m,k)}}\right) := \left(\xi_*\mathcal{D}^{(m,k)}_{\widetilde{X},d}\right)^{\mathbb{T}}, \qquad (d \in \mathbb{N}).$$

Proposition 5.1.9. *For any $U \in \mathcal{S}$ there exists an isomorphism of sheaves of filtered $\mathfrak{o}$-algebras*

$$\widetilde{\mathcal{D}^{(m,k)}}|_U \xrightarrow{\simeq} \mathcal{D}_X^{(m,k)}|_U \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{T}(k)).$$

Proof. Let $U \in \mathcal{S}$. We recall for the reader that this means that there is a $\mathbb{T}$-invariant isomorphism $\xi^{-1}(U) \xrightarrow{\alpha} U \times_{\mathfrak{o}} \mathbb{T}$ such that the diagram

$$\begin{array}{ccc} \xi^{-1}(U) & \xrightarrow{\quad\alpha\quad} & U \times_{\mathfrak{o}} \mathbb{T} \\ & {}_{\xi}\searrow \quad \swarrow_{p_1} & \\ & U & \end{array}$$

is commutative. We apply the same reasoning that in proposition 2.1.2 to get

$$\begin{aligned} \widetilde{\mathcal{D}^{(m,k)}}(U) &= \mathcal{D}_{\tilde{X}}^{(m,k)}(\xi^{-1}(U))^{\mathbb{T}} \\ &\simeq \left(\mathcal{D}_X^{(m,k)}(U) \otimes_{\mathfrak{o}} \mathcal{D}_{\mathbb{T}}^{(m,k)}(\mathbb{T})\right)^{\mathbb{T}} \\ &= \mathcal{D}_X^{(m,k)}(U) \otimes_{\mathfrak{o}} \left(\left(\mathcal{D}_{\mathbb{T}}^{(m)}(\mathbb{T})\right)_{\varpi^k}\right)^{\mathbb{T}} \\ &\simeq \mathcal{D}_X^{(m,k)}(U) \otimes_{\mathfrak{o}} \left(\left(\mathcal{D}_{\mathbb{T}}^{(m)}(\mathbb{T})\right)^{\mathbb{T}}\right)_{\varpi^k} \\ &\simeq \mathcal{D}_X^{(m,k)}(U) \otimes_{\mathfrak{o}} \left(D^{(m)}(\mathbb{T})\right)_{\varpi^k} \\ &\simeq \mathcal{D}_X^{(m,k)}(U) \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{T}(k)). \end{aligned}$$

The first equality is by definition. By (2.13) the isomorphisms in proposition 2.1.2 preserves the subsheaves $\mathcal{D}_X^{(m,k)}$ and $\mathcal{D}_{\mathbb{T}}^{(m,k)}$ which gives the first isomorphism. The second equality is again by definition. The second isomorphism is just the fact $\varpi^k \in \mathfrak{o}$ and the $\mathbb{T}$-action is $\mathfrak{o}$-linear. The last two isomorphisms are given by the first assertion in remark 1.5.4 and (5.9), respectively. □

Let us recall that the tangent sheaf $\mathcal{T}_{\tilde{X}}$ is a $\mathbb{T}$-equivariant coherent $\mathcal{O}_{\tilde{X}}$-module (remark 2.3.2) and therefore we can consider the subsheaf $\varpi^k \left(\xi_* \mathcal{T}_{\tilde{X}}\right)^{\mathbb{T}}$ of the sheaf of invariant sections $\left(\xi_* \mathcal{T}_{\tilde{X}}\right)^{\mathbb{T}}$. If $U \in \mathcal{S}$, then applying the same reasoning as in 2.3.3 we have

$$\varpi^k \left(\xi_* \mathcal{T}_{\tilde{X}}\right)^{\mathbb{T}}(U) = \varpi^k \mathcal{T}_{\tilde{X}} \left(\xi^{-1}(U)\right)^{\mathbb{T}} = \varpi^k \mathcal{T}_X(U) \oplus \left(\mathcal{O}_X(U) \otimes_{\mathfrak{o}} \varpi^k \mathfrak{t}\right).$$

Here $\mathfrak{t} = \mathrm{Lie}(\mathbb{T})$. As X is smooth, the preceding relation implies that $\varpi^k \left(\xi_* \mathcal{T}_{\tilde{X}}\right)^{\mathbb{T}}$ is a locally free $\mathcal{O}_X$-module of finite rank, and therefore we can consider the level m symmetric algebra $Sym^{(m)}\left(\varpi^k \left(\xi_* \mathcal{T}_{\tilde{X}}\right)^{\mathbb{T}}\right)$. Moreover, for $U \in \mathcal{S}$, we have

$$Sym^{(m)}\left(\varpi^k \left(\xi_* \mathcal{T}_{\tilde{X}}\right)^{\mathbb{T}}\right)(U) = Sym^{(m)}(\varpi^k \mathcal{T}_X(U)) \otimes_{\mathfrak{o}} Sym^{(m)}(\varpi^k \mathfrak{t}). \tag{5.11}$$

The proof of the following proposition follows word for word the arguments given in the proof of proposition 2.3.7.

Proposition 5.1.10. *If $\xi : \tilde{X} \to X$ is a locally trivial $\mathbb{T}$-torsor, then there exists a canonical and graded isomorphism*

$$Sym^{(m)}\left(\varpi^k \left(\xi_* \mathcal{T}_{\tilde{X}}\right)^{\mathbb{T}}\right) \xrightarrow{\simeq} gr_\bullet\left(\widetilde{\mathcal{D}^{(m,k)}}\right).$$

5.2 Relative enveloping algebras with congruence level on homogeneous spaces

We recall for the reader that in this work $\mathbb{G}$ denotes a split connected reductive group scheme over $\mathfrak{o}$, $\mathbb{B} \subseteq \mathbb{G}$ is a Borel subgroup, $\mathbb{T} \subseteq \mathbb{B}$ is a split maximal Torus of $\mathbb{G}$ and $\mathbf{N} \subseteq \mathbb{B}$ is the unipotent radical subgroup of $\mathbb{B}$. We have also denoted by $\widetilde{X} = \mathbb{G}/\mathbf{N}$, the basic affine space, and by $X := \mathbb{G}/\mathbb{B}$, the flag $\mathfrak{o}$-scheme. Those are smooth and separated schemes over $\mathfrak{o}$ (section 2.4). As we have remarked $\tilde{X}$ is endowed with right commutating $(\mathbb{G}, \mathbb{T})$-actions (resp. X is endowed with right commutating $(\mathbb{G}, \mathbb{T})$-actions, being trivial the right $\mathbb{T}$-action). We also recall that $\xi : \widetilde{X} \to X$ denotes the canonical projection, which is a locally trivial $\mathbb{T}$-torsor (Subsection 2.4).

Let us consider the morphism given by proposition 1.5.3

$$\Phi^{(m)} : D^{(m)}(\mathbb{G}(0)) \to H^0(\widetilde{X}, \mathcal{D}^{(m)}_{\widetilde{X}}).$$

We recall for the reader that this comes from functoriality from the right $\mathbb{G}$-action on $\widetilde{X}$. This map induces a morphism between the associated constant sheaves of filtered $\mathfrak{o}$-algebras

$$\underline{D^{(m)}(\mathbb{G}(0))} \to \underline{H^0(\tilde{X}, \mathcal{D}^{(m)}_{\widetilde{X}})}.$$

By composing with the canonical map of sheaves $\underline{H^0(\widetilde{X}, \mathcal{D}^{(m)}_{\widetilde{X}})} \to \mathcal{D}^{(m)}_{\widetilde{X}}$, we get a homomorphism of sheaves of filtered $\mathfrak{o}$-algebras $\underline{D^{(m)}(\mathbb{G}(0))} \to \mathcal{D}^{(m)}_{\widetilde{X}}$. Specialising in $\varpi^k \in \mathfrak{o}$ gives rise, by functoriality, to a filtered morphism of sheaves of filtered $\mathfrak{o}$-algebras

$$\underline{D^{(m)}(\mathbb{G}(0))}_{\varpi^k} \to \left(\mathcal{D}^{(m)}_{\widetilde{X}}\right)_{\varpi^k}.$$

By (5.9) we have that $D^{(m)}(\mathbb{G}(0))_{\varpi^k} = D^{(m)}(\mathbb{G}(k))$ as filtered subrings of $\mathcal{U}(\mathfrak{g}_L)$. We thus obtain a morphism

$$\underline{D^{(m)}(\mathbb{G}(k))} \to \mathcal{D}^{(m,k)}_{\widetilde{X}}, \tag{5.12}$$

which induces a homomorphism of filtered $\mathfrak{o}$-algebras

$$\Phi^{(m,k)} : D^{(m)}(\mathbb{G}(k)) \to H^0(\tilde{X}, \mathcal{D}^{(m,k)}_{\widetilde{X}})$$

Now, if $\widetilde{X}_L := \widetilde{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(L)$, then given that $\mathcal{D}^{(m,k)}_{\widetilde{X}}|_{\widetilde{X}_L} = \mathcal{D}_{\widetilde{X}_L}$ is the usual sheaf of differential operators on $\widetilde{X}_L$, we can apply the same reasoning that in section 2.5 to show that $\Phi^{(m,k)}$ factors through the homomorphism of filtered $\mathfrak{o}$-algebras

$$\Phi^{(m,k)} : D^{(m)}(\mathbb{G}(k)) \to H^0\left(\widetilde{X}, \mathcal{D}^{(m,k)}_{\widetilde{X}}\right)^{\mathbb{T}} = H^0\left(X, \xi_* \mathcal{D}^{(m,k)}_X\right)^{\mathbb{T}}. \tag{5.13}$$

Let us put $\mathcal{A}^{(m,k)}_X := \mathcal{O}_X \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}(k))$, and we equip this sheaf with the skew ring multiplication coming from the action of $D^{(m)}(\mathbb{G}(k))$ on $\mathcal{O}_X$ via $\Phi^{(m,k)}$ (we recall that, by lemma 2.2.9, we have an action of $\widetilde{\mathcal{D}^{(m,k)}}$ on $(\xi_* \mathcal{O}_{\widetilde{X}})^{\mathbb{T}} = \mathcal{O}_X$). The map $\Phi^{(m,k)}$ induces a unique $\mathcal{O}_X$-linear map

$$\Phi^{(m,k)}_X : \mathcal{A}^{(m,k)}_X \to \widetilde{\mathcal{D}^{(m,k)}}$$

which is also a morphism of sheaves of filtered $\mathfrak{o}$-algebras (the filtration on $\mathcal{A}^{(m,k)}_X$ is given by proposition 1.5.6).

Let $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ be a character of the distribution algebra (2.5.2) and for every level $m \in \mathbb{N}$, let us also denote by $\lambda : D^{(m)}(\mathbb{T}) \to \mathfrak{o}$ the induced character of $D^{(m)}(\mathbb{T})$. As before, we endow $\mathfrak{o}$ with the trivial filtration ($0 = \mathrm{F}_{-1}\mathfrak{o}$

and $F_i \mathfrak{o}_{\lambda^{(m)}} = \mathfrak{o}$ for all $i \geq 0$) and we consider it as a filtered $D^{(m)}(\mathbb{T})$-module via λ. By remark 5.1.1 and the fact that $D^{(m)}(\mathbb{T})_{\varpi^k} = D^{(m)}(\mathbb{T}(k))$, (5.9), we have a character $\lambda : D^{(m)}(\mathbb{T}(k)) \to \mathfrak{o}$. [4]

As in (45) we have

$$H^0\left(X_L, (\xi \times_{\mathfrak{o}} id_L)_* \mathcal{D}_{\tilde{X}_L}\right)^{\mathbb{T}_L} = H^0\left(\tilde{X}_L, \mathcal{D}_{\tilde{X}_L}\right)^{\mathbb{T}_L} = H^0\left(\widetilde{X}, \mathcal{D}^{(m,k)}_{\widetilde{X}}\right)^{\mathbb{T}} \otimes_{\mathfrak{o}} L.$$

By proposition 1.5.3 and the same reasoning that we have given at the beginning of this section, the right $\mathbb{T}$-action on $\widetilde{X}$ induces a canonical morphism of filtered $\mathfrak{o}$-algebras $\Phi^{(m,k)}_{\mathbb{T}} : D^{(m)}(\mathbb{T}(k)) \to H^0(\widetilde{X}, \mathcal{D}^{(m,k)}_{\widetilde{X}})$ and exactly as in page 45, we can show that $\Phi^{(m,k)}_{\mathbb{T}}$ factors through a morphism

$$\Phi^{(m,k)}_{\mathbb{T}} : D^{(m)}(\mathbb{T}) \to Z\left(H^0\left(X, \xi_* \mathcal{D}^{(m,k)}_{\tilde{X}}\right)^{\mathbb{T}}\right) = H^0\left(X, Z\left(\widetilde{\mathcal{D}^{(m,k)}}\right)\right).$$

Here $Z(\widetilde{\mathcal{D}^{(m,k)}})$ denotes the center of $\widetilde{\mathcal{D}^{(m,k)}}$. We have the following definition.

Definition 5.2.1. *Let $\lambda : D^{(m)}(\mathbb{T}(k)) \to \mathfrak{o}$ be a character of the distribution algebra $D^{(m)}(\mathbb{T}(k))$. We define the sheaf of level m integral twisted arithmetic differential operators with congruence level k, $\mathcal{D}^{(m,k)}_{X,\lambda}$, on the flag scheme X by*

$$\mathcal{D}^{(m,k)}_{X,\lambda} := \widetilde{\mathcal{D}^{(m,k)}} \otimes_{D^{(m)}(\mathbb{T}(k))} \mathfrak{o}.$$

By 5.1.9 we have the following result.

Proposition 5.2.2. *Let $U \in \mathcal{S}$. Then $\mathcal{D}^{(m,k)}_{X,\lambda}|_U$ is isomorphic to $\mathcal{D}^{(m,k)}_{X}|_U$ as a sheaf of filtered $\mathfrak{o}$-algebras.*

By using the preceding result, we can conclude as in proposition 2.5.8.

Proposition 5.2.3. *The sheaf $\mathcal{D}^{(m,k)}_{X,\lambda}$ is a sheaf of $\mathcal{O}_X$-rings with noetherian sections over all open affine subsets of X.*

Definition 5.2.4. *We will denote by*

$$\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda} := \varprojlim_j \mathcal{D}^{(m,k)}_{X,\lambda}/p^{j+1}\mathcal{D}^{(m,k)}_{X,\lambda}$$

the p-adic completion of $\mathcal{D}^{(m,k)}_{X,\lambda}$ and we consider it as a sheaf on $\mathfrak{X}$. Following the notation given at the beginning of this work, the sheaf $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ will denote our sheaf of level m twisted differential operators with congruence level k on the formal flag scheme $\mathfrak{X}$.

Using proposition 5.2.2, we can conclude, as in proposition 2.5.12, that

Proposition 5.2.5. *(i) There exists a basis $\mathscr{B}$ of the topology of $\mathfrak{X}$, consisting of open affine subsets, such that for every $\mathfrak{U} \in \mathscr{B}$ the ring $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda}(\mathfrak{U})$ is twosided notherian.*

(ii) The sheaf of rings $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ is coherent.

Using the morphism $\Phi^{(m,k)}_{X}$ and the canonical projection from $\widetilde{\mathcal{D}^{(m,k)}}$ onto $\mathcal{D}^{(m,k)}_{X,\lambda}$ we can define a canonical map

$$\Phi^{(m,k)}_{X,\lambda} : \mathcal{A}^{(m,k)}_{X} \to \mathcal{D}^{(m,k)}_{X,\lambda}. \tag{5.14}$$

The same reasoning given in proposition 2.5.13 shows the following result.

[4] We have abused of the notation and we have called all this maps λ. The reasons is that by 2.5.2 all these maps induce the same character of $\mathfrak{t}_L$.

Proposition 5.2.6. *(i) There exists a canonical isomorphism* $Sym^{(m)}(\varpi^k\mathcal{T}_X) \simeq gr_\bullet(\mathcal{D}^{(m,k)}_{X,\lambda})$.

(ii) The canonical morphism $\Phi^{(m,k)}_{X,\lambda}$ *is surjective.*

(iii) The sheaf $\mathcal{D}^{(m,k)}_{X,\lambda}$ *is a coherent* $\mathcal{A}^{(m,k)}_X$*-module.*

Notation: From now on, we will fix again a character $\lambda : \text{Dist}(\mathbb{T}) \to \mathfrak{o}$ of the distribution algebra Dist($\mathbb{T}$), such that $\lambda+\rho \in \mathfrak{t}^*_L$ is a dominant and regular character of the Lie algebra $\mathfrak{t}_L$ (cf. 2.5.3). By abuse of notation we will denote by $\lambda : D^{(m)}(\mathbb{T}(k)) \to \mathfrak{o}$ the character induced by specialising in the parameter $t = \varpi^k$.

By construction, we know that $\mathcal{D}^{(m,k)}_{X,\lambda}|_{X_L} = \mathcal{D}_\lambda$ is the usual sheaf of twisted differential operators on the flag variety X_L. The preceding proposition and the same arguments given in proposition 2.6.4 give us the following result

Proposition 5.2.7. *Let us suppose that* $\lambda : Dist(\mathbb{T}) \to \mathfrak{o}$ *is a character of the distribution algebra Dist*($\mathbb{T}$)*, such that* $\lambda+\rho \in \mathfrak{t}^*_L$ *is a dominant and regular character of* $\mathfrak{t}_L$.

(i) Let us fix $r \in \mathbb{Z}$*. For every positive integer* $l \in \mathbb{Z}_{>0}$*, the cohomology group* $H^l(X, \mathcal{D}^{(m,k)}_{X,\lambda}(r))$ *has bounded p-torsion.*

(ii) For every coherent $\mathcal{D}^{(m,k)}_{X,\lambda}$*-module* $\mathcal{E}$*, the cohomology group* $H^l(X,\mathcal{E})$ *has bounded p-torsion for all* $l > 0$.

5.2.1 Passing to formal completions and cohomological properties

Let $\mathscr{E}$ be a coherent $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda}$-module. By applying the same result as in proposition 3.1.2 we can find $r_1(\mathscr{E}) \in \mathbb{Z}$ such that, for all $r \geq r_1(\mathscr{E})$ there is $a \in \mathbb{Z}$ and an epimorphism of $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda}$-modules

$$\left(\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda}(-r)\right)^{\oplus a} \to \mathscr{E} \to 0. \tag{5.15}$$

Moreover, there exists $r_2(\mathscr{E}) \in \mathbb{Z}$ such that, for all $r \geq r_2(\mathscr{E})$ we have

$$H^l(\mathfrak{X}, \mathscr{E}(r)) = 0 \ \text{ for all } \ l > 0.$$

The same inductive argument exhibited in the second part of proposition 2.6.4 and (5.15) give us (cf. corollary 3.1.3)

Corollary 5.2.8. *Let* $\mathscr{E}$ *be a coherent* $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda}$*-module. There exists* $c = c(\mathscr{E}) \in \mathbb{N}$ *such that for all* $l > 0$ *the cohomology group* $H^l(\mathfrak{X},\mathscr{E})$ *is annihilated by* p^c.

Let us fix a coherent $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module $\mathscr{F}$. By definition, the sheaf $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$ satisfies [6, Conditions 3.4.1] and therefore by [6, Proposition 3.4.5] we can find a coherent $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda}$-module $\mathscr{E}$ such that $\mathscr{F} = \mathscr{E} \otimes_{\mathfrak{o}} L$. This relation and (5.15) allow us to find $r(\mathscr{F}) \in \mathbb{Z}$ such that, for every $r \geq r(\mathscr{F})$ there exist $a \in \mathbb{N}$ and an epimorphism of $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-modules

$$\left(\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}(-r)\right)^{\oplus a} \to \mathscr{F} \to 0. \tag{5.16}$$

Moreover, corollary 5.2.8 implies that

$$H^l(\mathfrak{X},\mathscr{F}) = H^l(\mathfrak{X},\mathscr{E}\otimes_{\mathfrak{o}} L) = H^l(\mathfrak{X},\mathscr{E})\otimes_{\mathfrak{o}} L = 0 \ \text{ for all } \ l > 0. \tag{5.17}$$

5.2.2 Calculation of global sections

Let $\chi_\lambda : Z(\mathfrak{g}_L) \to L$ be the central character induced by $\lambda \in \mathfrak{t}^*$ via base change and the classical Harish-Chandra isomorphism (section 3.2). As before, we denote by $\text{Ker}(\chi_{\lambda+\rho})_{\mathfrak{o}} := D^{(m)}(\mathbb{G}(k)) \cap \text{Ker}(\chi_{\lambda+\rho})$, and we consider the central redaction

$$D^{(m)}(\mathbb{G}(k))_\lambda := D^{(m)}(\mathbb{G}(k))/D^{(m)}(\mathbb{G}(k))\mathrm{Ker}(\chi_{\lambda+\rho})_{\mathfrak{o}},$$

which is clearly an integral model of $\mathcal{U}(\mathfrak{g}_L)_\lambda := \mathcal{U}(\mathfrak{g}_L)\otimes_{Z(\mathfrak{g}_L),\chi_{\lambda+\rho}} L$. Let $\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda$ be the p-adic completion.

Let us consider $\Phi_\lambda^{(m,k)} : D^{(m)}(\mathbb{G}(k)) \to H^0(X, \mathcal{D}_{X,\lambda}^{(m,k)})$ defined by taking global sections in (5.14). This morphism induces, by construction, the following commutative diagram

$$\begin{array}{ccc} D^{(m)}(\mathbb{G}(k)) & \xrightarrow{\Phi_\lambda^{(m,k)}} & H^0\left(X, \mathcal{D}_{X,\lambda}^{(m,k)}\right) \\ \big\downarrow & & \big\downarrow \\ \mathcal{U}(\mathfrak{g}_L) & \xrightarrow{\Phi_\lambda} & H^0\left(X_L, \mathcal{D}_\lambda\right). \end{array}$$

As before, by the classical Beilinson-Bernstein theorem [3] and the preceding commutative diagram, we have that $\Phi_\lambda^{(m,k)}$ factors through a morphism $\overline{\Phi}_\lambda^{(m,k)} : D^{(m)}(\mathbb{G}(k))_\lambda \to H^0(X, \mathcal{D}_{X,\lambda}^{(m,k)})$ which becomes an isomorphism after tensoring with L. Then lemma 3.2.1 implies that $\overline{\Phi}_\lambda^{(m,k)}$ gives rise to an isomorphism

$$\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L \xrightarrow{\simeq} H^0\left(\widehat{X, \mathcal{D}_{X,\lambda}^{(m,k)}}\right)\otimes_{\mathfrak{o}} L.$$

Proposition 3.1.1 together with the fact that $\mathfrak{X}$ is in particular a noetherian topological space gives us

$$\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L \xrightarrow{\simeq} H^0\left(\mathfrak{X}, \widehat{\mathscr{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m,k)}\right). \tag{5.18}$$

5.2.3 The arithmetic Beilinson-Bernstein theorem with congruence level

5.2.9. *Let E be a finitely generated $\widehat{D}^{(m)}(\mathbb{G}(k))_{\lambda,\mathbb{Q}} := \widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L$-module. As in section 4.1 we define $\mathscr{L}oc_{\mathfrak{X},\lambda}^{(m,k)}(E)$ as the associated sheaf to the presheaf on $\mathfrak{X}$ defined by*

$$\mathfrak{U} \subseteq \mathfrak{X} \mapsto \widehat{\mathscr{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m,k)}(\mathfrak{U}) \otimes_{\widehat{D}^{(m)}(\mathbb{G}(k))_{\lambda,\mathbb{Q}}} E.$$

It is clear that $\mathscr{L}oc_{\mathfrak{X},\lambda}^{(m,k)}$ is a functor from the category of finitely generated $\widehat{D}^{(m)}(\mathbb{G}(k))_{\lambda,\mathbb{Q}}$-modules to the category of coherent $\widehat{\mathscr{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m,k)}$-modules.

Following the same lines of reasoning in theorem 3.4.2 we have

Theorem 5.2.10. *Let us suppose that $\lambda : Dist(\mathbb{T}) \to \mathfrak{o}$ is a character of the distribution algebra $Dist(\mathbb{T})$, such that $\lambda + \rho \in \mathfrak{t}_L^*$ is a dominant and regular character of $\mathfrak{t}_L$.*

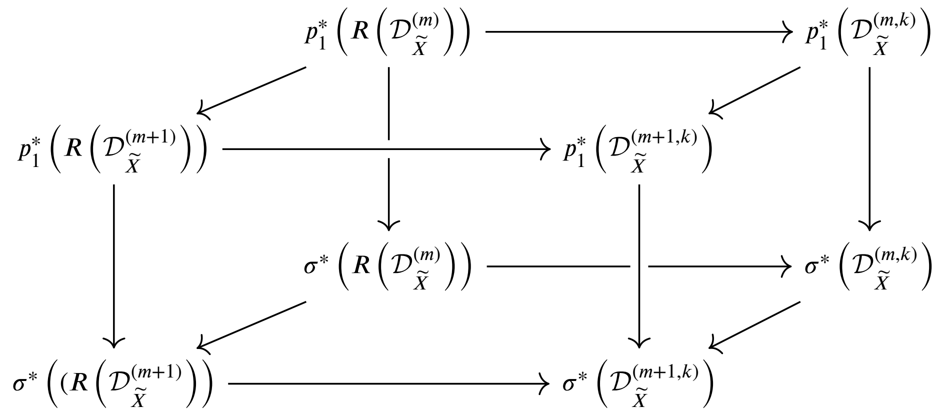
(i) The functors $\mathscr{L}oc_{\mathfrak{X},\lambda}^{(m,k)}$ and $H^0(\mathfrak{X}, \bullet)$ are quasi-inverse equivalences of categories between the abelian categories of finitely generated $\widehat{D}^{(m)}(\mathbb{G}(k))_{\lambda,\mathbb{Q}}$-modules and coherent $\widehat{\mathscr{D}}_{\mathfrak{X},\lambda,\mathbb{Q}}^{(m,k)}$-modules.

(ii) The functor $\mathscr{L}oc_{\mathfrak{X},\lambda}^{(m,k)}$ is an exact functor.

5.3 The sheaves $\mathscr{D}_{\mathfrak{X},k\lambda}^{\dagger}$

Thorough this section we will suppose that $\lambda : \mathrm{Dist}(\mathbb{T}) \to \mathfrak{o}$ induces, under the correspondence (2.26), a dominant and regular character of $\mathfrak{t}_L$. We recall for the reader that the induced character of $D^{(m)}(\mathbb{T}(k))$ will also be denoted by

$\mathfrak{o}$. We will study the problem of passing to the inductive limit when m varies. First of all, let us recall that the canonical morphism $\mathcal{D}^{(m)}_{\widetilde{X}} \to \mathcal{D}^{(m+1)}_{\widetilde{X}}$ is $\mathbb{T}$-equivariant (5.36) and by construction, the induced map between the Rees rings $R(\mathcal{D}^{(m)}_{\widetilde{X}}) \to R(\mathcal{D}^{(m+1)}_{\widetilde{X}})$ is $\mathbb{T}$-equivariant. We have the following diagram



Except for the right lateral face, the other faces of the cube form commutative diagrams either by $\mathbb{T}$-equivariance of the map $R(\mathcal{D}^{(m)}_{\widetilde{X}}) \to R(\mathcal{D}^{(m+1)}_{\widetilde{X}})$ or by functoriality on the commutative diagram (which comes from the exact sequence (5.10))

$$\begin{array}{ccc} R\left(\mathcal{D}^{(m)}_{\widetilde{X}}\right) & \longrightarrow & \mathcal{D}^{(m,k)}_{\widetilde{X}} \\ \downarrow & & \downarrow \\ R\left(\mathcal{D}^{(m+1)}_{\widetilde{X}}\right) & \longrightarrow & \mathcal{D}^{(m+1,k)}_{\widetilde{X}}, \end{array}$$

but, by construction (cf. proposition 5.1.6), this forces the commutativity on right lateral face, which means that the canonical map $\mathcal{D}^{(m,k)}_{\widetilde{X}} \to \mathcal{D}^{(m,k)}_{\widetilde{X}}$ is also $\mathbb{T}$-equivariant. We dispose therefore of a morphism $\widetilde{\mathcal{D}^{(m,k)}} \to \widetilde{\mathcal{D}^{(m+1,k)}}$. The commutativity of the diagram

$$\begin{array}{ccc} D^{(m)}(\mathbb{T}(k)) & \xrightarrow{\ \lambda\ } & \\ \big\downarrow & & \mathfrak{o} \\ D^{(m+1)}(\mathbb{T}(k)) & \xrightarrow{\ \lambda\ } & \end{array}$$

implies that we also have maps $\mathcal{D}^{(m,k)}_{X,\lambda} \to \mathcal{D}^{(m+1,k)}_{X,\lambda}$ and in consequence an inductive system

$$\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda} \to \widehat{\mathscr{D}}^{(m+1,k)}_{\mathfrak{X},\lambda}. \tag{5.19}$$

Definition 5.3.1. *We will denote by $\mathscr{D}^{\dagger}_{\mathfrak{X},k,\lambda}$ the limit of the inductive system (5.19), tensored with L*

$$\mathscr{D}^{\dagger}_{\mathfrak{X},k,\lambda} := \left(\varinjlim_{m} \widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda}\right) \otimes_{\mathfrak{o}} L. \tag{5.20}$$

Let $\mathscr{E}$ be a coherent $\mathscr{D}^{\dagger}_{\mathfrak{X},k\lambda}$-module. As in lemma 4.2.5 we can find $m \geq 0$, a coherent $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}$-module $\mathscr{E}_m$ and an isomorphism of $\mathscr{D}^{\dagger}_{\mathfrak{X},k,\lambda}$-modules

$$\mathscr{D}^{\dagger}_{\mathfrak{X},k,\lambda}\otimes_{\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}}\mathscr{E}_m\xrightarrow{\simeq}\mathscr{E}.$$

Using this isomorphism and (5.16) we can conclude that there exists $r(\mathscr{E})\in\mathbb{Z}$, such that for all $r\geq r(\mathscr{E})$ there is $a\in\mathbb{N}$ and an epimorphism of $\mathscr{D}^{\dagger}_{\mathfrak{X},k,\lambda}$-modules

$$\left(\mathscr{D}^{\dagger}_{\mathfrak{X},k,\lambda}(-r)\right)^{\oplus a}\rightarrow\mathscr{E}\rightarrow 0.$$

Moreover, by (5.17) we have for every $i>0$

$$H^i(\mathfrak{X},\mathscr{E})=\varinjlim_{l\geq m}H^i\left(\mathfrak{X},\widehat{\mathscr{D}}^{(l,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}\otimes_{\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\lambda,\mathbb{Q}}}\mathscr{E}_m\right)=0.$$

Let $D^{\dagger}(\mathbb{G}(k))_{\lambda}:=\varinjlim_{m\in\mathbb{N}}\widehat{D}^{(m)}(\mathbb{G}(k))_{\lambda}\otimes_{\mathfrak{o}}L$. By (5.18) and using the same reasoning that in theorem 4.2.1 we have

Theorem 5.3.2. *Let us suppose that $\lambda:Dist(\mathbb{T})\rightarrow\mathfrak{o}$ is a character of the distribution algebra $Dist(\mathbb{T})$, such that $\lambda+\rho\in\mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$.*

(i) The functors $\mathscr{L}oc^{\dagger}_{\mathfrak{X},k,\lambda}$ and $H^0(\mathfrak{X},\bullet)$ are quasi-inverse equivalence of categories between the abelian categories of coherent (left) $\mathscr{D}^{\dagger}_{\mathfrak{X},k,\lambda}$-modules and finitely presented $D^{\dagger}(\mathbb{G}(k))_{\lambda}$-modules.

(ii) The functor $\mathscr{L}oc^{\dagger}_{\mathfrak{X},k,\lambda}$ is an exact functor.

5.4 Linearization of group actions

Let us start with the following definition from [30, Chapter II, exercise 5.18] (cf. [16, Definition 3.1.1]).

Definition 5.4.1. *Let Y be an $\mathfrak{o}$-scheme. A (geometric) line bundle over Y is a scheme $\mathbf{L}$ together with a morphism $\pi:\mathbf{L}\rightarrow Y$ such that Y admits an open covering $(U_i)_{i\in I}$ satisfying the following two conditions:*

(i) For any $i\in I$ there exists an isomorphism $\psi_i:\pi^{-1}(U_i)\xrightarrow{\simeq}\mathbb{A}^1_{U_i}$.

(ii) For any $i,j\in I$ and for any open affine subset $V=Spec(A[x])\subseteq U_i\cap U_i$ the automorphism $\theta_{ij}:\psi_j\circ\psi_i^{-1}|_V:\mathbb{A}^1_V\rightarrow\mathbb{A}^1_V$ of $\mathbb{A}^1_V$ is given by a linear automorphism $\theta^{\natural}_{ij}$ of $A[x]$. This means, $\theta^{\natural}_{ij}(a)=a$ for any $a\in A$, and $\theta^{\natural}_{ij}(x)=a_{ij}x$ for a suitable $a_{ij}\in A$.

In the preceding definition, the scheme $\mathbf{L}$ is obtained by glueing the trivial line bundles $p_{1,i}:U_i\times\mathbb{A}^1_{\mathfrak{o}}\rightarrow U_i$ via the linear transition functions (a_{ij}). Thus, each fibre $\mathbf{L}_x$ is a line, in the sense that it has a canonical structure of a 1-dimensional affine space.

Definition 5.4.2. *Given a line bundle $\pi:\mathbf{L}\rightarrow Y$ and a morphism $\phi:Y'\rightarrow Y$, the pull-back $\phi^*(\mathbf{L})$ is the fiber product $\mathbf{L}\times_Y Y'$ equipped with its projection to Y'.*

Now, let $\pi:\mathbf{L}\rightarrow Y$ be a line bundle over Y, then a *section* of π over an open subset $U\subset Y$ is a morphism $s:U\rightarrow\mathbf{L}$ such that $\pi\circ s=id_U$. Moreover the presheaf $\mathcal{L}$ defined by

$$U\subseteq Y\quad\mapsto\quad\{s:U\rightarrow\mathbf{L}\mid\ s\ \text{ is a section over }\ U\}$$

is a sheaf called the *sheaf of sections* of the line bundle $\mathbf{L}$. This is an invertible sheaf (i.e., a locally free sheaf of rank 1).

On the other hand, if $\mathcal{E}$ is a locally free sheaf of rank 1 on Y and we let

$$\mathbf{V}(\mathcal{E}) := \underline{\mathrm{Spec}}_Y \left(\mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{E})\right)$$

be the line bundle over Y associated to $\mathcal{E}$ [27, 1.7.8], then we have a one-to-one correspondence between isomorphism classes of locally free sheaves of rank 1 on Y and isomorphic classes of (geometric) line bundles over Y [30, Chapter II, Exercises 5.1 (a) and 5.18 (d)]

$$\begin{array}{ccc} \{\text{Isomorphism classes of locally free sheaves of rank 1}\} & \leftrightarrow & \{\text{Isomorphic classes of line bundles}\} \\ \mathcal{E} & \mapsto & \mathbf{V}(\mathcal{E}^\vee) \\ \mathcal{L} & \leftarrow\!\shortmid & \mathbf{L} \end{array} \tag{5.21}$$

Let $\pi : \mathbf{L} \to Y$ be a line bundle over Y, let $\mathcal{L}$ be its sheaf of sections and $\phi : Y' \to Y$ a morphism of schemes. Let us calculate the sheaf of sections of the pull-back line bundle $\phi^*(\mathbf{L}) := \mathbf{L} \times_Y Y' \to Y'$. First of all, under the previous correspondence we have $\mathbf{L} = \mathbf{V}(\mathcal{L}^\vee)$. Therefore by [27, Proposition 1.7.11 (iv)] there exists canonical isomorphisms

$$\phi^*(\mathbf{L}) = \phi^*(\mathbf{V}(\mathcal{L}^\vee)) = \mathbf{V}(\phi^*(\mathcal{L}^\vee)) = \mathbf{V}((\phi^*(\mathcal{L}))^\vee),$$

where the third isomorphism is just the fact that $\mathcal{L}$ is free of finite rank. Again, by the preceding correspondence we can conclude that the sheaf of sections $\mathcal{L}_{\phi^*(\mathbf{L})}$ equals $((\phi^*(\mathcal{L}))^\vee)^\vee = \phi^*(\mathcal{L})$.

We end this digression about line bundles by pointing out to the reader that if $\pi : \mathbf{L} \to Y$ is a line bundle over Y and $\phi_1, \phi_2 : Y' \to Y$ are two morphisms from a scheme Y' to Y such that $\phi_1^*(\mathbf{L}) \simeq \phi_2^*(\mathbf{L})$, then $\phi_1^*(\mathcal{L}) \simeq \phi_2^*(\mathcal{L})$.

Let us suppose now that Y is endowed with a right $\mathbb{G}$-action, this means that we have a morphism $\alpha : Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G} \to Y$. In particular, for every $g \in \mathbb{G}(\mathfrak{o})$ we dispose of a translation morphism

$$\rho_g : Y = Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}) \xrightarrow{id_Y \times g} Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G} \xrightarrow{\alpha} Y$$

In the next lines we will study (geometric) line bundles which are endowed with a right $\mathbb{G}$-action.

Definition 5.4.3. *Let $\pi : \boldsymbol{L} \to Y$ be a line bundle. A $\mathbb{G}$-Linearization of $\boldsymbol{L}$ is a right $\mathbb{G}$-action $\beta : \boldsymbol{L} \times_{Spec(\mathfrak{o})} \mathbb{G} \to \boldsymbol{L}$ satisfying the following two conditions:*

(i) The diagram

$$\begin{array}{ccc} \boldsymbol{L} \times_{Spec(\mathfrak{o})} \mathbb{G} & \xrightarrow{\beta} & \boldsymbol{L} \\ \downarrow{\scriptstyle \pi \times id_{\mathbb{G}}} & & \downarrow{\scriptstyle \pi} \\ Y \times_{Spec(\mathfrak{o})} \mathbb{G} & \xrightarrow{\alpha} & Y \end{array}$$

(ii) The action on the fibers is linear. This means that for every $x \in Y$ and $g \in \mathbb{G}(o)$, the morphism on the fibers $\boldsymbol{L}_x \to \boldsymbol{L}_{x \cdot g}$, induced by the commutative diagram

$$\begin{array}{ccc} \boldsymbol{L} & \xrightarrow{\rho_g} & \boldsymbol{L} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\rho_g} & Y, \end{array}$$

is $\mathfrak{o}$-linear.

Let $g \in \mathbb{G}(\mathfrak{o})$ and let us suppose that $\Psi : \alpha^*(\mathbf{L}) \to p_1^*(\mathbf{L})$ is a morphism of line bundles over $Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G}$. Let us consider the translation morphism

$$\rho_g : Y = Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}) \xrightarrow{id_Y \times g} Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G} \xrightarrow{\alpha} Y.$$

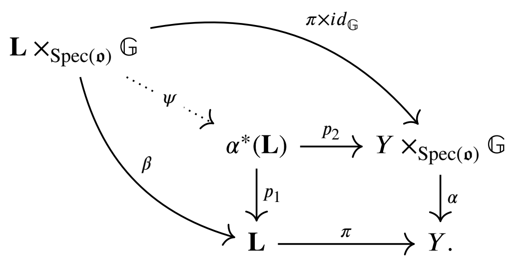
We have the relations $(id_Y \times g)^* \alpha^*(\mathbf{L}) = \rho_g^*(L)$ and $(id_Y \times g)^* p_1^*(\mathbf{L}) = \mathbf{L}$. So every morphism of line bundles $\Psi : \alpha^*(\mathbf{L}) \to p_1^*(\mathbf{L})$ induces morphisms $\Psi_g : \rho_g^*(L) \to L$ for all $g \in \mathbb{G}(\mathfrak{o})$. The following reasoning can be found in [23, Page 104] or [16, Lemma 3.2.4].

Proposition 5.4.4. *Let $\pi : \boldsymbol{L} \to Y$ be a line bundle over Y endowed with a $\mathbb{G}$-linearization $\beta : \boldsymbol{L} \times_{Spec(\mathfrak{o})} \mathbb{G} \to \boldsymbol{L}$. Then there exists an isomorphism*

$$\Psi : \alpha^*(\boldsymbol{L}) \to p_1^*(\boldsymbol{L})$$

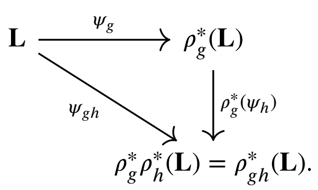
of line bundles over $\boldsymbol{L} \times_{Spec(\mathfrak{o})} \mathbb{G}$, such that $\Psi_{gh} = \Psi_g \circ \rho_g^*(\Psi_h)$ for all $g, h \in \mathbb{G}(\mathfrak{o})$.

Proof. By definition of linearization we have the following commutative diagram



By universal property there exists a unique morphism of line bundles $\psi : p_1^*(\mathbf{L}) \to \alpha^*(\mathbf{L})$, which is linear on the fibers since so is β. Let $g \in \mathbb{G}(\mathfrak{o})$. To see that ψ is an isomorphism we can use the correspondence (5.21). In this case, if $x \in Y$, $g \in \mathbb{G}$ and $\psi_{(x,g)} : \mathcal{L}_x \to \mathcal{L}_{xg}$ denotes the respective morphism between the stalks, then $\psi_{(x,g)}$ is an isomorphism being $\psi_{(xg,g^{-1})}$ the inverse.

Let $g, h \in \mathbb{G}(\mathfrak{o})$. Applying $(id_X \times g)^*$ to ψ we get the morphism $\psi_g : \mathbf{L} \to \rho_g^*(\mathbf{L})$ and given that β is a right action ($\rho_h \circ \rho_g = \rho_{gh}$), it fits into the following commutative diagram



Moreover, since $\psi_g : \mathbf{L} \to \rho_g^*(\mathbf{L})$ is an isomorphism for every $g \in \mathbb{G}(\mathfrak{o})$ (the fiber over $x \in Y$ coincides with $\psi_{(x,g)}$) then we can consider the morphism $\Psi_g := \psi_g^{-1} : \rho_g^*(\mathbf{L}) \to \mathbf{L}$ which coincides with the fibers of the morphism

$$\Psi := \psi^{-1} : \alpha^*(\mathbf{L}) \to p_1^*(\mathbf{L}).$$

By construction, these morphism satisfy the cocycle condition of the proposition. This means that for every $g, h \in \mathbb{G}(\mathfrak{o})$, we have

$$\Psi_{gh} = \Psi_g \circ \rho_g^*(\Psi_h).$$

□

Remark 5.4.5. *Let $\phi : \mathbf{L} \to Y$ be a line bundle endowed with a $\mathbb{G}$-linearization. Let $\mathcal{L}$ be the sheaf of section of $\mathbf{L}$. The morphism of the preceding proposition induces in a canonical way an isomorphism $\Phi : \alpha^*(\mathcal{L}) \xrightarrow{\simeq} p_1^*(\mathcal{L})$, and the cocycle condition for Ψ implies that Φ makes commutative the diagram (2.4) (to see the discussion at the beginning of [38, Subsection 3.2]).*

Let us suppose now that $X := \mathbb{G}/\mathbb{B}$ is again the smooth flag $\mathfrak{o}$-scheme. Let us recall that by (2.19) we have a canonical isomorphism $\mathbb{T} \simeq \mathbb{B}/\mathbf{N}$. This in particular implies that every algebraic character $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ induces a character of the Borel subgroup $\lambda : \mathbb{B} \to \mathbb{G}_m$. Let us consider the locally free action of $\mathbb{B}$ on the trivial fiber bundle $\mathbb{G}\times\mathfrak{o}$ over $\mathbb{G}$ given by

$$b.(g,u) := (gb^{-1}, \lambda(b)u); \quad (g \in \mathbb{G},\ b \in \mathbb{B},\ u \in \mathfrak{o}).$$

We denote by $\mathbf{L}(\lambda) := \mathbb{B}\diagdown(\mathbb{G}\times\mathfrak{o})$ the quotient space obtained by this action.
Let $\pi : \mathbb{G} \to X$ be the canonical projection. Since the map $\mathbb{G}\times\mathfrak{o} \to X$, $(g,u) \mapsto \pi(x)$ is constant on $\mathbb{B}$-orbits, it induces a morphism $\pi_\lambda : \mathbf{L}(\lambda) \to X$. Moreover, given that π is locally trivial [40, Part II, 1.10 (2)] $\pi_\lambda : \mathbf{L}(\lambda) \to X$ defines a line bundle over X [40, Part I, 5.16]. Furthermore, the right $\mathbb{G}$-action on $\mathbb{G}\times\mathfrak{o}$ given by

$$(g_0,u)\bullet g \mapsto (g^{-1}g_0, u) \quad (g \in \mathbb{G},\ (g_0,u) \in \mathbb{G}\times\mathfrak{o})$$

induces a right action on $\mathbf{L}(\lambda)$ for which $\mathbf{L}(\lambda)$ turns out to be a $\mathbb{G}$-linearized line bundle on X. By the preceding remark, the sheaf of sections $\mathcal{L}(\lambda)$ of the line bundle $\mathbf{L}(\lambda)$ is a $\mathbb{G}$-equivariant invertible sheaf. In fact, we can give a local description of the sheaf $\mathcal{L}(\lambda)$. If $U \subseteq X$ is an affine open subset, then [40, Part I, 5.8 (2) and 5.15 (1)]

$$\mathcal{L}(\lambda)(U) = \left(\mathfrak{o}_\lambda \otimes_{\mathfrak{o}} \mathfrak{o}[\pi^{-1}(U)]\right)^{\mathbb{B}}.$$

Here $\mathfrak{o}_\lambda = \mathfrak{o}$ is viewed as a $\mathbb{B}$-module and the $\mathbb{B}$-action is given by the action on $\mathfrak{o}$ via λ and the operation on $\mathfrak{o}[\pi^{-1}U]$ derived from the action on $\pi^{-1}(U) \subseteq \mathbb{G}$.

5.5 Arithmetic differential operators acting on a line bundle

We start this section by recalling to the reader that the sheaf $\mathcal{D}_X^{(m)}$ is endowed with a left and a right structure of $\mathcal{O}_X$-module. These structures come from the canonical morphisms of rings $d_1,\ d_2 : \mathcal{O}_X \to \mathcal{P}^n_{X,(m)}$, which are induced by universal property (proposition 1.1.3) and the projections $p_1,\ p_2 : X \times_{\mathrm{Spec}(\mathfrak{o})} X \to X$. We have the following definition.

Definition 5.5.1. *Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character. For every congruence level $k \in \mathbb{N}$, we define the sheaf of level m arithmetic differential operators acting on the line bundle $\mathcal{L}(\lambda)$, by*

$$\mathcal{D}_X^{(m,k)}(\lambda) := \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{D}_X^{(m,k)} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee.$$

The multiplicative structure of the sheaf $\mathcal{D}_X^{(m,k)}(\lambda)$ is defined as follows. If $\alpha^\vee, \beta^\vee \in \mathcal{L}(\lambda)^\vee$, $P, Q \in \mathcal{D}_X^{(m,k)}$ and $\alpha, \beta \in \mathcal{L}(\lambda)$ then

$$\alpha \otimes P \otimes \alpha^\vee \bullet \beta \otimes Q \otimes \beta^\vee = \alpha \otimes P \left\langle \alpha^\vee, \beta \right\rangle Q \otimes \beta^\vee. \tag{5.22}$$

Moreover, the action of $\mathcal{D}_X^{(m,k)}(\lambda)$ on $\mathcal{L}(\lambda)$ is given by

$$\left(t \otimes P \otimes t^\vee\right) \bullet s := \left(P \bullet < t^\vee, s >\right) t \qquad (s, t \in \mathcal{L}(\lambda) \text{ and } t^\vee \in \mathcal{L}(\lambda)^\vee).$$

Remark 5.5.2. *Given that the locally free $\mathcal{O}_X$-modules of rank one $\mathcal{L}(\lambda)^\vee$ and $\mathcal{L}(\lambda)$ are in particular flat, the sheaf*

$\mathcal{D}^{(m,k)}_X(\lambda)$ is filtered by the order of twisted differential operators. This is, the subsheaf $\mathcal{D}^{(m,k)}_{X,d}$ of $\mathcal{D}^{(m,k)}_X$, of differential operators of order less that d, induces a subsheaf of twisted differential operators of order less than d defined by

$$\mathcal{D}^{(m,k)}_{X,d}(\lambda) := \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{D}^{(m,k)}_{X,d} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee,$$

and given than the tensor product preserves inductive limits, we obtain

$$\mathcal{D}^{(m,k)}_X(\lambda) = \varinjlim_d \mathcal{D}^{(m,k)}_{X,d}(\lambda).$$

Moreover, the exact sequence

$$0 \to \mathcal{D}^{(m,k)}_{X,d-1} \to \mathcal{D}^{(m,k)}_{X,d} \to \mathcal{D}^{(m,k)}_{X,d}/\mathcal{D}^{(m,k)}_{X,d-1} \to 0$$

induces the exact sequence

$$0 \to \mathcal{D}^{(m,k)}_{X,d-1}(\lambda) \to \mathcal{D}^{(m,k)}_{X,d}(\lambda) \to \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{D}^{(m,k)}_{X,d}/\mathcal{D}^{(m,k)}_{X,d-1} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee \to 0$$

which tells us that

$$gr\left(\mathcal{D}^{(m,k)}_X(\lambda)\right) \simeq \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} gr\left(\mathcal{D}^{(m,k)}_X\right) \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee \simeq gr\left(\mathcal{D}^{(m,k)}_X\right).$$

The second isomorphism is defined by $\alpha \otimes P \otimes \alpha^\vee \mapsto \alpha^\vee(\alpha)P$. This is well defined because $gr\left(\mathcal{D}^{(m,k)}_X\right)$ is in particular a commutative ring.

Proposition 5.5.3. *There exists a canonical isomorphism of graded sheaves of algebras*

$$gr_\bullet\left(\mathcal{D}^{(m,k)}_X(\lambda)\right) \xrightarrow{\simeq} Sym^{(m)}(\varpi^k \mathcal{T}_X).$$

Proof. By (1.3), and the fact that $\mathcal{D}^{(m,k)}_X$ and $\varpi^k\mathcal{T}_X$ are locally free sheaves (and therefore free ϖ-torsion) we have the following short exact sequence

$$0 \to \mathcal{D}^{(m,k)}_{X,d-1} \to \mathcal{D}^{(m,k)}_{X,d} \to \mathrm{Sym}^{(m)}_d\left(\varpi^k \mathcal{T}_X\right) \to 0,$$

which gives us the isomorphisms

$$\mathrm{Sym}^{(m)}\left(\varpi^k \mathcal{T}_X\right) \simeq gr_\bullet\left(\mathcal{D}^{(m,k)}_X\right) \simeq gr_\bullet\left(\mathcal{D}^{(m,k)}_X(\lambda)\right).$$

□

Proposition 5.5.4. *Let $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character. For every congruence level $k \in \mathbb{N}$, the sheaf $\mathcal{D}^{(m,k)}_X(\lambda)$ is a $\mathbb{G}$-equivariant quasi-coherent $\mathcal{O}_X$-module.*

Proof. The proposition is an immediately consequence of remarks 5.1.7, 5.4.5 and proposition 2.2.4. □

5.5.5. *Sheaf of differential operators.*

Let us briefly recall the construction of the sheaf of differential operators over the smooth $\mathfrak{o}$-scheme X [29, 16.8.4].[5] If $\mathcal{I}$ is the ideal of the diagonal embedding $X \hookrightarrow X \times_{\mathfrak{o}} X$, we denote by $\mathcal{P}^n_X := \mathcal{O}_{X\times_{\mathfrak{o}}X}/\mathcal{I}^{n+1}$ the sheaf of principal parts. We

[5] This construction is in fact more general and is made for an arbitrary smooth $\mathfrak{o}$-scheme.

put

$$\mathcal{D}_X := \bigcup_{n\in\mathbb{N}} \mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{P}^n_X, \mathcal{O}_X\right).$$

The reader can take a look to [29, 16.8.10] to check that $\mathcal{D}_X$ is in fact a sheaf of rings. What will be important for us is the following local description. As before, let us suppose that $U \subset X$ is an affine open subset endowed with coordinates $x_1, ..., x_M$. Let $\partial_{x_1}, ..., \partial_{x_M}$ be the dual base of the sheaf of derivations, and for every $1 \leq i \leq M$ and $l \in \mathbb{N}$ we denote by $\partial^{[l]}_{x_i} \in \mathcal{D}_X(U)$ the differential operator defined by $l!\partial^{[l]}_{x_i} = \partial^l_{x_i}$. Finally, using multi-index notation, for $\underline{l} = (l_1, ...l_M) \in \mathbb{N}^M$, we put $\underline{\partial}^{[\underline{l}]} = \prod_{i=1}^M \partial^{[l_i]}_{x_i}$. One has the following description [29, 16.11.2.2]

$$\mathcal{D}_X(U) = \left\{ \sum_{\underline{l}}^{<\infty} a_{\underline{l}}\underline{\partial}^{[\underline{l}]} \mid a_{\underline{l}} \in \mathcal{O}_X(U) \right\}.$$

Furthermore, if $p_2 : X \times_{\mathfrak{o}} X \to X$ denotes the second projection, then we have a canonical map $d_2 : \mathcal{O}_X \to \mathcal{P}^n_X$ which induces a structure of (left) $\mathcal{D}_X$-module on $\mathcal{O}_X$. This is given by

$$\mathcal{O}_X \xrightarrow{d_2} \mathcal{P}^n_X \xrightarrow{P} \mathcal{O}_X \quad (P \in \mathcal{D}_X).$$

Finally, by [6, Proposition 1.4.5] we dispose of canonical morphisms $\mathcal{P}^n_X \to \mathcal{P}^n_{X,(m)}$, from the sheaf of principal parts to the sheaf of level m divided powers. Taking duals we get $\mathcal{O}_X$-linear homomorphisms $\mathcal{D}^{(m)}_{X,n} \to \mathcal{D}_X$ and passing to the inductive limit we get a canonical morphism of filtered rings [6, (2.2.1.5)]

$$\mathcal{D}^{(m)}_X \to \mathcal{D}_X. \tag{5.23}$$

In particular, by construction, we have a canonical homomorphism of filtered rings

$$\mathcal{D}^{(m,k)}_X \to \mathcal{D}_X. \tag{5.24}$$

Let $\mathscr{L}(\lambda)$ be the formal ϖ-adic completion of the sheaf of section $\mathcal{L}(\lambda)$ of the fiber bundle $\mathbf{L}(\lambda)$. We regard this sheaf as an invertible sheaf on $\mathcal{O}_{\mathfrak{X}}$. As before, we will consider the following sheaves of ϖ-complete algebras

$$\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X}} := \varprojlim_{j\in\mathbb{N}} \mathcal{D}^{(m,k)}_X/\varpi^{j+1}\mathcal{D}^{(m,k)}_X, \quad \mathscr{D}^{\dagger}_{\mathfrak{X},k} := \left(\varinjlim_{m\in\mathbb{N}} \mathscr{D}^{(m,k)}_{\mathfrak{X}}\right) \otimes_{\mathfrak{o}} L \quad \text{and} \quad \widehat{\mathscr{D}}_{\mathfrak{X}} := \varprojlim_{j\in\mathbb{N}} \mathcal{D}_X/\varpi^{j+1}\mathcal{D}_X, \tag{5.25}$$

Let $U \subset X$ be an affine open subset of X endowed with local coordinates $x_1, ..., x_M$ and let $\mathcal{U}$ be the formal completion of U along the special fiber $U_{\mathbb{F}_q}$. By using multi-index notation, every section $P \in \widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X}}(\mathcal{U})$ (resp. $P \in \widehat{\mathscr{D}}_{\mathfrak{X}}(\mathcal{U})$) can be written in a unique way [6, (2.4.1.2)]

$$P = \sum_{\underline{v}}^{\infty} \varpi^{k|\underline{v}|}a_{\underline{v}}\underline{\partial}^{<\underline{v}>} \quad (\text{resp. } P = \sum_{\underline{v}}^{\infty} a_{\underline{v}}\underline{\partial}^{[\underline{v}]}) \tag{5.26}$$

where the sequence $(a_{\underline{k}}) \in \Gamma(\mathcal{U}, \mathcal{O}_{\mathfrak{X}})$ tends to 0 for the ϖ-topology. Furthermore, if $|\bullet|$ is a Banach norm on the affinoid algebra $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ [6, 2.4.2] then by [37, Subsection 2.1] we have

$$\mathscr{D}^{\dagger}_{\mathfrak{X},k}(\mathcal{U}) = \left\{ \sum_{\underline{v}}^{\infty} \varpi^{k|\underline{v}|}a_{\underline{v}}\underline{\partial}^{[\underline{v}]} \mid a_{\underline{v}} \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(\mathcal{U}), \text{ and } \exists C > 0,\ \eta < 1 \mid |a_{\underline{v}}| < C\eta^{|\underline{v}|} \right\}.$$

The sheaves (5.25) define the twists

$$\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X}}(\lambda) := \mathscr{L}(\lambda)\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathscr{L}(\lambda)^{\vee}\ \text{ and }\ \widehat{\mathscr{D}}_{\mathfrak{X}} := \mathscr{L}(\lambda)\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathscr{D}}_{\mathfrak{X}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathscr{L}(\lambda)^{\vee},$$

and we can define

$$\mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda) := \mathscr{L}(\lambda)_{\mathbb{Q}}\otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}}\mathscr{D}^{\dagger}_{\mathfrak{X},k}\otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}}\mathscr{L}(\lambda)^{\vee}_{\mathbb{Q}}.$$

Remark 5.5.6. *The preceding sheaves can also be defined as the ϖ-completion of the sheaf of arithmetic differential operators acting on the line bundle $\mathcal{L}(\lambda)$. This means that*

$$\mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda) = \varinjlim_{m\in\mathbb{N}}\left(\varprojlim_{j\in\mathbb{N}}\mathcal{D}^{(m,k)}_{X}(\lambda)/\varpi^{j+1}\mathcal{D}^{(m,k)}_{X}(\lambda)\right)\otimes_{\mathfrak{o}} L.$$

We will use this isomorphism in the next subsection.

Remark 5.5.7. *If $U\subseteq X$ is an open affine subset of X, then proposition 5.5.3 tells us that the graded algebra $gr_{\bullet}\left(\mathcal{D}^{(m,k)}_{X}(\lambda)(U)\right)$ is isomorphic to $Sym^{(m)}\left(\varpi^{k}\mathcal{T}_{X}(U)\right)$ which is known to be noetherian (subsection 1.3). Therefore, the sheaf $\mathcal{D}^{(m,k)}_{X}(\lambda)$ has noetherian section over all open affine subset. By using the same reasoning that in lemma 2.5.12 we can conclude that the sheaf $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\mathbb{Q}}(\lambda)$ is coherent.*

5.5.1 Local description

We have the following local description for the sheaves $\mathcal{D}^{(m,k)}_{X}(\lambda)$.

Lemma 5.5.8. *There exists a covering $\mathscr{B}$ of X by affine open subsets such that, over every open subset $U\in\mathscr{B}$ the rings $\mathcal{D}^{(m,k)}_{U}(\lambda)$ and $\mathcal{D}^{(m,k)}_{U}$ are isomorphic.*

Proof. Let's first recall the following relations from [6]. First of all, for $v = v' + v''$, with $v', v''\in\mathbb{N}$, let $v = p^m q + r$, $v' = p^m q' + r'$ and $v'' = p^m q'' + r''$ be the euclidean division of v, v' and v'' by p^m. We define the modified binomial coefficients [6, 1.1.2.1]

$$\left\{\begin{matrix} v \\ v' \end{matrix}\right\} := \frac{q!}{q'!q''!}.$$

For multi-indexes $\underline{v},\underline{v}',\underline{v}''\in\mathbb{N}^M$ such that $\underline{v}=\underline{v}'+\underline{v}''$ we can define $\underline{v}! = \prod_{i=1}^{M} v_i!$ and

$$\left\{\begin{matrix} \underline{v} \\ \underline{v}' \end{matrix}\right\} := \frac{\underline{q}!}{\underline{q}'!\underline{q}''!}.$$

Finally, if $U\subset X$ is an affine open subset endowed with local coordinates $x_1,...,x_M$, for every $\underline{v}\in\mathbb{N}^M$ and $f\in\mathcal{O}_X(U)$ we have the following relation [6, proposition 2.2.4, iv]

$$\underline{\partial}^{<\underline{v}>}f = \sum_{\underline{v}'+\underline{v}''=\underline{v}}\left\{\begin{matrix} \underline{v} \\ \underline{v}' \end{matrix}\right\}\underline{\partial}^{<\underline{v}'>}(f)\underline{\partial}^{<\underline{v}''>}\in\mathcal{D}^{(m,0)}_{U} = \mathcal{D}^{(m)}_{U}.$$

Now, let's take an affine covering $\mathscr{B}$ of X such that for every $U\in\mathscr{B}$, U is endowed with local coordinates and there exists a local section $\alpha\in\mathcal{L}(\lambda)(U)$ such that $\mathcal{L}(\lambda)|_U = \alpha\mathcal{O}_U$ and $\mathscr{L}(\lambda)^{\vee}|_U = \alpha^{\vee}\mathcal{O}_U$, where $\alpha^{\vee}$ denotes the dual element associated to α. Let's show that

$$\mathcal{D}_U^{(m,k)}(\lambda) = \bigoplus_{\underline{v}} \varpi^{k|\underline{v}|}\mathcal{O}_U\alpha \otimes \underline{\partial}^{<\underline{v}>} \otimes \alpha^{\vee}. \tag{5.27}$$

It is enough to show that for every $\underline{v} \in \mathbb{N}^M$ and $f, g \in \mathcal{O}_U$ the section $\alpha \otimes \varpi^{k|\underline{v}|} f \underline{\partial}^{<\underline{v}>} \otimes g\alpha^{\vee}$ belongs to the right side of (5.27). In fact, from the first part of the proof

$$\begin{aligned}
\alpha\otimes\varpi^{k|\underline{v}|} f\underline{\partial}^{<\underline{v}>} \otimes g\alpha^{\vee}
&= \alpha \otimes \varpi^{k|\underline{v}|} f\underline{\partial}^{<\underline{v}>} g \otimes \alpha^{\vee}\\
&= \alpha \otimes \varpi^{k|\underline{v}|} f \sum_{\underline{v}'+\underline{v}''=\underline{v}} \left\{ \begin{matrix} \underline{v} \\ \underline{v}' \end{matrix} \right\} \underline{\partial}^{<\underline{v}'>}(g)\underline{\partial}^{<\underline{v}''>} \otimes \alpha^{\vee}\\
&= \sum_{\underline{v}'+\underline{v}''=\underline{v}} \varpi^{k|\underline{v}|} f \left\{ \begin{matrix} \underline{v} \\ \underline{v}' \end{matrix} \right\} \underline{\partial}^{<\underline{v}'>}(g)\alpha \otimes \underline{\partial}^{<\underline{v}''>} \otimes \alpha^{\vee}.
\end{aligned}$$

Let's define $\theta : \mathcal{D}_U^{(m,k)}(\lambda) \to \mathcal{D}_U^{(m,k)}$ by $\theta\left(\varpi^{k|\underline{v}|} f\alpha \otimes \underline{\partial}^{<\underline{v}>} \otimes \alpha^{\vee}\right) = \varpi^{k|\underline{v}|} f\underline{\partial}^{<\underline{v}>}$ and let's see that θ is a homomorphism of rings (the multiplication on the left is given by (5.22)). By (5.27), the elements in $\mathcal{D}_U^{(m,k)}(\lambda)$ are linear combinations of the terms $\varpi^{k|\underline{v}|} f\alpha \otimes \underline{\partial}^{<\underline{v}>} \otimes \alpha^{\vee}$ and therefore, it is enough to show that θ preserves the multiplicative structure over the elements of this form. So, let's take $\underline{v}, \underline{u} \in \mathbb{N}$ and $f, g \in \mathcal{O}_U$. On the one hand

$$\begin{aligned}
\theta(\varpi^{k|\underline{v}|} f\alpha \otimes \underline{\partial}^{<\underline{v}>} \otimes \alpha^{\vee} \bullet \varpi^{k|\underline{u}|} g\alpha \otimes \underline{\partial}^{<\underline{u}>} \otimes \alpha^{\vee})
&= \theta(\varpi^{k|\underline{v}|} f\alpha \otimes \underline{\partial}^{<\underline{v}>}\varpi^{k|\underline{u}|} g\underline{\partial}^{<\underline{u}>} \otimes \alpha^{\vee})\\
&= \theta\left(\sum_{\underline{v}'+\underline{v}''=\underline{v}} \varpi^{k|\underline{v}|} f \left\{ \begin{matrix} \underline{v} \\ \underline{v}' \end{matrix} \right\} \underline{\partial}^{<\underline{v}'>}(\varpi^{k|\underline{u}|} g)\alpha \otimes \underline{\partial}^{<\underline{v}''>}\underline{\partial}^{<\underline{u}>} \otimes \alpha^{\vee}\right)\\
&= \sum_{\underline{v}'+\underline{v}''=\underline{v}} \varpi^{k|\underline{v}|} f \left\{ \begin{matrix} \underline{v} \\ \underline{v}' \end{matrix} \right\} \underline{\partial}^{<\underline{v}'>}(\varpi^{k|\underline{u}|} g)\underline{\partial}^{<\underline{v}''>}\underline{\partial}^{<\underline{u}>},
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\theta(\varpi^{k|\underline{v}|} f\alpha \otimes \underline{\partial}^{<\underline{v}>} \otimes \alpha^{\vee}) \bullet \theta(\varpi^{k|\underline{u}|} g\alpha \otimes \underline{\partial}^{<\underline{u}>} \otimes \alpha^{\vee})
&= \varpi^{k|\underline{v}|} f\underline{\partial}^{<\underline{v}>} \bullet \varpi^{k|\underline{u}|} g\partial^{<\underline{u}>}\\
&= \sum_{\underline{v}'+\underline{v}''=\underline{v}} \varpi^{k|\underline{v}|} f \left\{ \begin{matrix} \underline{v} \\ \underline{v}' \end{matrix} \right\} \underline{\partial}^{<\underline{v}'>}(\varpi^{k|\underline{u}|} g)\underline{\partial}^{<\underline{v}''>}\underline{\partial}^{<\underline{u}>}.
\end{aligned}$$

Both equations show that θ is a ring homomorphism.
Finally, a reasoning completely analogous shows that the morphism $\theta^{-1} : \mathcal{D}_U^{(m,k)} \to \mathcal{D}_U^{(m,k)}(\lambda)$ defined by

$$\theta^{-1}(\varpi^{k|\underline{v}|} f\underline{\partial}^{<\underline{v}>}) = \varpi^{k|\underline{v}|} f\alpha \otimes \underline{\partial}^{<\underline{v}>} \otimes \alpha^{\vee}$$

is also a homomorphism of rings and $\theta\circ\theta^{-1} = \theta^{-1}\circ\theta = id$. This ends the proof of the lemma. □

By [37, Proposition 2.2.11] the morphisms of sheaves $\widehat{\mathscr{D}}_{\mathfrak{X},\mathbb{Q}}^{(m,k)}$ are left and right flat. Using this fact and the same arguments given in proposition 4.2.4 we can conclude the following result.

Proposition 5.5.9. *The sheaf of rings $\mathscr{D}_{\mathfrak{X},k}^{\dagger}(\lambda)$ is coherent.*

Remark 5.5.10. *The cohomological properties of the sheaves $\mathscr{D}_{\mathfrak{X},\mathbb{Q}}^{(m,0)}(\lambda)$ and $\mathscr{D}_{\mathfrak{X},0}^{\dagger}(\lambda)$ have been studied in [35] and [39].*

By definition $\mathcal{D}_X^{(m,k)}(\lambda)|_{X_L} = \mathcal{L}(\lambda_L) \otimes_{\mathcal{O}_{X_L}} \mathcal{D}_{X_L} \otimes_{\mathcal{O}_{X_L}} \mathcal{L}(\lambda_L)^\vee$, where $\lambda_L := \lambda \otimes_{\mathfrak{o}} 1_L$. Then in order to apply the arguments of Huyghe-Schmidt in [35] and [39], as we have done in section 5.1.3, we need to find an explicitly description of $gr_\bullet(\mathcal{D}_X^{(m,k)}(\lambda))$ and a canonical epimorphism of filtered $\mathfrak{o}$ algebras $\mathcal{A}_X^{(m,k)} \to \mathcal{D}_X^{(m,k)}(\lambda)$ as in (5.12).

Proposition 5.5.11. *There exists a canonical homomorphism of filtered $\mathfrak{o}$-algebras*

$$\Phi^{(m,k)} : D^{(m)}(\mathbb{G}(k)) \to H^0\left(X, \mathcal{D}_X^{(m,k)}(\lambda)\right).$$

Proof. By proposition 1.5.3, there exists a morphism of sheaves of filtered $\mathfrak{o}$-algebras

$$\mathcal{A}_X^{(m,0)} \to \mathcal{D}_X^{(m,0)}(\lambda). \tag{5.28}$$

Let's first show that after specialising in ϖ^k the Rees ring associated to the twisted order filtration of $\mathcal{D}_{X_0,\lambda}^{(m,0)}$ we get $\mathcal{D}_X^{(m,k)}(\lambda)$. To do that, we consider $\mathcal{D}_X^{(m,0)}$ filtered by the order of differential operators and we define the following homomorphisms of $\mathcal{O}_X$-modules

$$\mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} R\left(\mathcal{D}_X^{(m,0)}\right) \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee \underset{\theta^{-1}}{\overset{\theta}{\rightleftarrows}} R\left(\mathcal{D}_X^{(m,0)}(\lambda)\right), \tag{5.29}$$

by

$$\theta\left(\alpha \otimes \sum_i P_i t^i \otimes \alpha^\vee\right) = \sum_i (\alpha \otimes P_i \otimes \alpha^\vee) t^i$$

with $\operatorname{ord}(P_i) = i$ for every i in the sum, and

$$\theta^{-1}\left(\sum_j (\alpha_j \otimes P_j \otimes \alpha_j^\vee) t^j\right) = \sum_j \alpha_j \otimes P_j t^j \otimes \alpha_j^\vee$$

with $\operatorname{ord}(P_j) = j$ for every j. It's clear that $\theta \circ \theta^{-1} = \theta^{-1} \circ \theta = id$ and therefore (5.29) is an isomorphism of $\mathcal{O}_X$-modules. We remark that an easy calculation shows that (5.29) is in fact an isomorphism of rings.
Let's denote by $\sigma_1 : R\left(\mathcal{D}_X^{(m,0)}(\lambda)\right) \to \mathcal{D}_X^{(m,k)}(\lambda); t \mapsto \varpi^k$ and by $\sigma_2 : R\left(\mathcal{D}_X^{(m,0)}\right) \to \mathcal{D}_X^{(m,k)}; t \mapsto \varpi^k$, and let's consider the following diagrams

$$\begin{array}{ccc} \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} R\left(\mathcal{D}_X^{(m,0)}\right) \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee & \underset{\theta^{-1}}{\overset{\theta}{\rightleftarrows}} & R\left(\mathcal{D}_X^{(m,0)}(\lambda)\right) \\ & \searrow^{id_{\mathcal{L}(\lambda)} \otimes \sigma_2 \otimes id_{\mathcal{L}(\lambda)^\vee}} & \downarrow \sigma_1 \\ & & \mathcal{D}_X^{(m,k)}(\lambda) \end{array}$$

It is straightforward to check that both diagrams are commutative and given that θ and θ^{-1} are isomorphisms we can conclude that

$$\begin{aligned} \left(\mathcal{D}_{X_0}^{(m,0)}(\lambda)\right)_{\varpi^k} &= \operatorname{Im}(\sigma_1) = \operatorname{Im}(id_{\mathcal{L}(\lambda)} \otimes \sigma_2 \otimes id_{\mathcal{L}(\lambda)^\vee}) \\ &= \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \operatorname{Im}(\sigma_2) \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee \\ &= \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \left(\mathcal{D}_X^{(m,0)}\right)_{\varpi^k} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee \\ &= \mathcal{D}_X^{(m,k)}(\lambda). \end{aligned}$$

On the other hand, taking the natural filtration of $\mathcal{A}_X^{(m,0)}$ we have

$$R\left(\mathcal{A}_X^{(m,0)}\right) = \mathcal{O}_X \otimes_{\mathfrak{o}} R\left(D^{(m)}(\mathbb{G}(0))\right)$$

and therefore $(\mathcal{A}_X^{(m,0)})_{\varpi^k} = \mathcal{A}_X^{(m,k)}$. The above two calculations tell us that passing to the Rees rings and specialising in ϖ^k the map (5.28), we get a homomorphism of filtered sheaves of $\mathfrak{o}$-algebras

$$\Phi_X^{(m,k)} : \mathcal{A}_X^{(m,k)} \to \mathcal{D}_X^{(m,k)}(\lambda). \tag{5.30}$$

Taking global sections we obtain the morphism $\Phi^{(m,k)}(\lambda)$ of the proposition. □

We recall that the right $\mathbb{G}$-action on X (cf. remark 2.4.1) induces a canonical application

$$\mathcal{O}_X \otimes_{\mathfrak{o}} \mathfrak{g} \to \mathcal{T}_X \tag{5.31}$$

which is surjective by [35, Subsection 1.6]. By using proposition 5.5.3 and the fact that $gr(\mathcal{A}_X^{(m,k)}) = \mathcal{O}_X \otimes_{\mathfrak{o}} \mathrm{Sym}^{(m)}(\mathfrak{g})$ imply that $\Phi_X^{(m,k)}$ is also a surjective morphism.

By using the preceding proposition we can apply the arguments in [39, Subsection 3.2.4] (exactly as we have done in sections 3.2 and 5.2.2) to obtain

Proposition 5.5.12. *Let $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}_L^*$ is a dominant and regular character of $\mathfrak{t}_L$. Then*

$$H^0\left(\mathfrak{X}, \widehat{\mathscr{D}}_{\mathfrak{X},\mathbb{Q}}^{(m,k)}(\lambda)\right) = \widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L \quad \textit{and} \quad H^0\left(\mathfrak{X}, \mathscr{D}_{\mathfrak{X}}^{\dagger}(\lambda)\right) = D^{\dagger}(\mathbb{G}(k))_\lambda$$

Remark 5.5.13. *The preceding proposition implies that the operators introduced in this section and the ones introduced in the preceding section have the same global sections if $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*$ is a dominant and regular character of the Lie algebra $\mathfrak{t}_L^*$.*

By replying the same lines of reasoning given in section 5.3 we can define the localization functors $\mathcal{L}oc_{\mathfrak{X},k}^{\dagger}(\lambda)$ and $\mathcal{L}oc_{\mathfrak{X},k}^{(m,k)}(\lambda)$ in the setting of this section and we have the following central result.

Theorem 5.5.14. *Let $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character which induces, via derivation, a dominant and regular character of the L-Lie algebra $\mathfrak{t}_L$.*

(i) The functors $\mathscr{L}oc_{\mathfrak{X}}^{(m,k)}(\lambda)$ and $H^0(\mathfrak{X}, \bullet)$ (res. $\mathscr{L}oc_{\mathfrak{X},k}^{\dagger}(\lambda)$ and $H^0(\mathfrak{X}, \bullet)$) are quasi-inverse equivalence between the abelian categories of finitely generated $\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda \otimes_{\mathfrak{o}} L$ and coherent $\widehat{\mathscr{D}}_{\mathfrak{X},\mathbb{Q}}^{(m,k)}(\lambda)$-modules (resp. finitely presented $D^{\dagger}(\mathbb{G}(k))_\lambda$-modules and coherent $\mathscr{D}_{\mathfrak{X},k}^{\dagger}(\lambda)$).

(ii) The functor $\mathscr{L}oc_{\mathfrak{X}}^{(m,k)}(\lambda)$ (resp. $\mathscr{L}oc_{\mathfrak{X},k}^{\dagger}(\lambda)$) is an exact functor.

Remark 5.5.15. *Let $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character and let $\lambda' : Dist(\mathbb{T}) \to \mathfrak{o}$ be the character of the distribution algebra $Dist(\mathbb{T})$, induced by λ under the correspondence (2.26). For every $k \in \mathbb{N}$, we have built surjective canonical morphisms of filtered $\mathfrak{o}$-algebras ((5.14) and (5.30))*

$$\Phi_X^{(m,k)} : \mathcal{A}_X^{(m,k)} \to \mathcal{D}_X^{(m,k)}(\lambda) \quad \textit{and} \quad \Phi_{X,\lambda'}^{(m,k)} : \mathcal{A}_X^{(m,k)} \to \mathcal{D}_{X,\lambda'}^{(m,k)}.$$

As before, we can conclude that

$$\left\{\textit{Coh. } \mathscr{D}_{\mathfrak{X},k}^{\dagger}(\lambda) - \textit{modules}\right\} \xrightarrow{H^0(\mathfrak{X},\bullet)} \left\{\textit{Finitely presented } D^{\dagger}(\mathbb{G}(k))_\lambda - \textit{modules}\right\} \xrightarrow{\mathscr{L}oc_{\mathfrak{X},k,\lambda'}^{\dagger}} \left\{\textit{Coh. } \mathscr{D}_{\mathfrak{X},k,\lambda'}^{\dagger} - \textit{modules}\right\}.$$

5.6 Group actions

We start this section with the following notation (cf. 2.2.5).

We recall that $\mathbb{G}$ acts on the right on the flag $\mathfrak{o}$-scheme $X = \mathbb{G}/\mathbb{B}$. Let us denote by $\alpha : X \times_{\mathrm{Spec}(\mathfrak{o})} \mathbb{G} \to X$ this action. As in 2.2.5, let $X_i := X \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}/\varpi^{i+1})$. The scheme X_i is endowed with a right $\mathbb{G}_i$-action $\alpha_i : X_i \times_{\mathrm{Spec}(\mathfrak{o}/\varpi^{i+1})} \mathbb{G}_i \to X_i$. Let us denote by $\gamma_i : X_i \hookrightarrow X$ and $\theta_i : X_i \times_{\mathrm{Spec}(\mathfrak{o}/\varpi^{i+1})} \mathbb{G}_i \hookrightarrow X_{i+1} \times_{\mathrm{Spec}(\mathfrak{o}/\varpi^{i+2})} \mathbb{G}_{i+1}$ the closed embeddings. Let $\mathfrak{G}$ denote the formal completion of $\mathbb{G}$ along its special fiber $\mathbb{G}_{\mathbb{F}_q}$.

For every $i \in \mathbb{N}$ let $\mathcal{L}(\lambda)_i := \gamma_i^*(\mathcal{L}(\lambda))$. By remark 5.4.5 the sheaf $\mathcal{L}(\lambda)$ is endowed with a $\mathbb{G}$-equivariant structure $\Phi : p_1^*(\mathcal{L}(\lambda)) \xrightarrow{\simeq} \alpha^*(\mathcal{L}(\lambda))$ such that

$$\begin{array}{ccc} p_{1,i}^*(\mathcal{L}(\lambda)_i) & \xrightarrow{\quad \Phi_i \quad} & \sigma_i^*(\mathcal{L}(\lambda)_i) \\ \downarrow\wr & & \downarrow\wr \\ \theta_i^* p_{1,i+1}(\mathcal{L}(\lambda)_{i+1}) & \xrightarrow{\theta_i^*(\Phi_{i+1})} & \theta_i^*\alpha_{i+1}^*(\mathcal{L}(\lambda)_{i+1}). \end{array}$$

is a commutative diagram. This implies that $\mathscr{L}(\lambda) = \varprojlim_{i\in\mathbb{N}} \gamma_i^*(\mathcal{L}(\lambda))$ is a $\mathfrak{G}$-equivariant line bundle over $\mathfrak{X}$.

As we have done in proposition 5.4.4 (cf. [38, 3.3.2]), for every $g \in \mathfrak{G}(\mathfrak{o}) = \mathbb{G}(\mathfrak{o})$ there exists an isomorphism

$$\rho_g : \mathfrak{X} \xrightarrow{id_{\mathfrak{X}} \times g} \mathfrak{X} \times_{\mathrm{Spec}(\mathfrak{o})} \mathfrak{X} \xrightarrow{\alpha} \mathfrak{X}.$$

This morphism and the $\mathfrak{G}$-equivariant structure of $\mathscr{L}(\lambda)$ induces an $\mathcal{O}_{\mathfrak{X}}$-linear isomorphism $\Phi_g : \mathscr{L}(\lambda) \to (\rho_g)_*(\mathscr{L}(\lambda))$ verifying the cocycle condition

$$\Phi_{hg} = (\rho_g)_*(\Phi_h) \circ \Phi_g \quad \text{and} \quad (g, h \in \mathbb{G}(\mathfrak{o})). \tag{5.32}$$

Now, as $\mathfrak{X}$ is locally noetherian its ideal of ϖ-torsion is locally annihilated by a power of ϖ . By (5.26), for every $m \leq m'$ the morphisms $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\mathbb{Q}} \hookrightarrow \widehat{\mathscr{D}}^{(m',k)}_{\mathfrak{X},\mathbb{Q}}$, induced by functoriality on the canonical morphisms $\mathcal{D}^{(m)}_X \to \mathcal{D}^{(m')}_X$, are injective as well as the morphism $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\mathbb{Q}} \hookrightarrow \widehat{\mathscr{D}}_{\mathfrak{X}}$ induced by (5.24) and we get the injections ([6, 2.4.1.5])

$$\widehat{\mathscr{D}}^{(0,k)}_{\mathfrak{X},\mathbb{Q}} \hookrightarrow \widehat{\mathscr{D}}^{(1,k)}_{\mathfrak{X},\mathbb{Q}} \hookrightarrow \ldots \hookrightarrow \widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\mathbb{Q}} \hookrightarrow \ldots \hookrightarrow \mathscr{D}^{\dagger}_{\mathfrak{X},k} \hookrightarrow \widehat{\mathscr{D}}_{\mathfrak{X},\mathbb{Q}}. \tag{5.33}$$

On the other hand by passing to the projective limit, the sheaf $\widehat{\mathscr{D}}_{\mathfrak{X}}$ acts on $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ (resp. $\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$) and therefore $\widehat{\mathscr{D}}_{\mathfrak{X},\mathbb{Q}}(\lambda) := \mathscr{L}(\lambda)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \widehat{\mathscr{D}}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathscr{L}(\lambda)^{\vee}_{\mathbb{Q}}$ acts on the line bundle $\mathscr{L}(\lambda)_{\mathbb{Q}}$ (resp. the line bundle $\mathscr{L}(\lambda) \subset \mathscr{L}(\lambda)_{\mathbb{Q}}$) by

$$\left(s \otimes P \otimes s^{\vee}\right) \bullet t := P \bullet \left\langle s^{\vee}, t\right\rangle s \quad (P \in \widehat{\mathscr{D}}_{\mathfrak{X},\mathbb{Q}}, \; s, t \in \mathscr{L}(\lambda)_{\mathbb{Q}} \text{ and } s^{\vee} \in \mathscr{L}(\lambda)^{\vee}_{\mathbb{Q}}). \tag{5.34}$$

By lemma 5.5.8 and (5.33) we can conclude that $\mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda) \subseteq \widehat{\mathscr{D}}_{\mathfrak{X},\mathbb{Q}}(\lambda)$ and therefore the line bundle $\mathscr{L}(\lambda)_{\mathbb{Q}}$ is also a (left) $\mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$-module (resp. $\mathscr{L}(\lambda) \subseteq \mathscr{L}(\lambda)_{\mathbb{Q}}$). In particular, we have an induced $\mathbb{G}(\mathfrak{o})$-action on the sheaf $\mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda)$

$$T_g : \mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda) \to (\rho_g)_*\mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda), \quad P \mapsto \Phi_g \circ P \circ (\Phi_g)^{-1}. \tag{5.35}$$

Locally, if $\mathcal{U} \subseteq \mathfrak{X}$ and $P \in \mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda)(\mathcal{U})$ then $T_g(\mathcal{U})(P)$ is given by the following commutative diagram

$$\begin{array}{ccc}
\mathscr{L}(\lambda)(\mathcal{U}.g^{-1}) & \xrightarrow{T_{g,\mathcal{U}}(P)} & \mathscr{L}(\lambda)(\mathcal{U}.g^{-1}) \\
\Big\downarrow \Phi^{-1}_{g,\mathcal{U}} & & \Phi_{g,\mathcal{U}} \Big\uparrow \\
\mathscr{L}(\lambda)(\mathcal{U}) & \xrightarrow{P} & \mathscr{L}(\lambda)(\mathcal{U}).
\end{array}$$

The cocycle condition (5.32) tells that the diagram

$$\begin{array}{ccc}
\mathscr{L}(\lambda)(\mathcal{U}.(hg)^{-1}) = \mathscr{L}(\lambda)(\mathcal{U}.g^{-1}h^{-1}) & \xrightarrow{T_{gh,(\mathcal{U}}(P)} & \mathscr{L}(\lambda)(\mathcal{U}.g^{-1}h^{-1}) \\
\Big\downarrow \Phi^{-1}_{h,\mathcal{U}.g^{-1}} = (\rho_g)_* \Phi^{-1}_{h,\mathcal{U}} & & \Phi_{h,\mathcal{U}.g^{-1}} = (\rho_g)_* \Phi_{h,\mathcal{U}} \Big\uparrow \\
\mathscr{L}(\lambda)(\mathcal{U}.g^{-1}) & & \mathscr{L}(\mathcal{U}.g^{-1}) \\
\Big\downarrow \Phi^{-1}_{g,\mathcal{U}} & & \Phi_{g,\mathcal{U}} \Big\uparrow \\
\mathscr{L}(\lambda)(\mathcal{U}) & \xrightarrow{P} & \mathscr{L}(\lambda)(\mathcal{U})
\end{array} \tag{5.36}$$

is commutative and we get the relation

$$T_{hg} = \left(\rho_g\right)_* T_h \circ T_g \quad (g, h \in G_0). \tag{5.37}$$

Chapter 6

Twisted differential operators on formal model of flag varieties

Through out this chapter $X = \mathbb{G}/\mathbb{B}$ will denote the smooth flag $\mathfrak{o}$-scheme and $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ will always denote an algebraic character. As before, we will denote be $\mathcal{L}(\lambda)$ the (algebraic) line bundle on X induced by λ. In this chapter we will generalize the construction given in [36] by introducing sheaves of twisted differential operators on an admissible blow-up of the smooth formal flag $\mathfrak{o}$-scheme $\mathfrak{X}$. The reader will figure out that some reasoning are inspired in the results of Huyghe-Patel-Strauch-Schmidt in [36].

6.1 Differential operators on admissible blow-ups

We start with the following definition.

Definition 6.1.1. *Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf. We say that a blow-up $pr : Y \to X$ along the closed subset $V(\mathcal{I})$ is admissible if there is $k \in \mathbb{N}$ such that $\varpi^k \mathcal{O}_X \subseteq \mathcal{I}$.*

Let us fix $\mathcal{I} \subseteq \mathcal{O}_X$ an open ideal and $\mathrm{pr} : Y \to X$ an admissible blow-up along $V(\mathcal{I})$. We point out to the reader that $\mathcal{I}$ is not uniquely determined by the space Y. In the sequel we will denote by

$$k_Y := \min_{\mathcal{I}} \min\{k \in \mathbb{N} \mid \varpi^k \in \mathcal{I}\},$$

where the first minimum runs over all open ideal sheaves $\mathcal{I}$ such that the blow-up along $V(\mathcal{I})$ is isomorphic to Y.

Now, as $\mathcal{I}$ is an open ideal sheaf, the blow-up induces a canonical isomorphism $Y_L \simeq X_L$ between the generic fibers. Moreover, as ϖ is invertible on X_L, we have $\mathcal{D}_X^{(m,k)}|_{X_L} = \mathcal{D}_X|_{X_L} = \mathcal{D}_{X_L}$, the usual sheaf of (algebraic) differential operators on X_L. Therefore $\mathrm{pr}^{-1}\left(\mathcal{D}_X^{(m,k)}\right)|_{Y_L} = \mathcal{D}_{Y_L}$. In particular, $\mathcal{O}_{Y_L}$ has a natural structure of (left) $\mathrm{pr}^{-1}\left(\mathcal{D}_X^{(m,k)}\right)|_{Y_L}$-module. The idea is to find those congruence levels $k \in \mathbb{N}$ such that the preceding structure extends to a module structure on $\mathcal{O}_Y$ over $\mathrm{pr}^{-1}\left(\mathcal{D}_X^{(m,k)}\right)$. Let us denote by

$$\mathcal{D}_Y^{(m,k)} := \mathrm{pr}^*\left(\mathcal{D}_X^{(m,k)}\right) = \mathcal{O}_Y \otimes_{\mathrm{pr}^{-1}\mathcal{O}_X} \mathrm{pr}^{-1}\mathcal{D}_X^{(m,k)}. \tag{6.1}$$

The problem to find those congruence levels was studied in [36] and [37]. In fact, we have the following condition [36, Corollary 2.1.18].

Proposition 6.1.2. *Let $k \geq k_Y$. The sheaf $\mathcal{D}_Y^{(m,k)}$ is a sheaf of rings on Y. Moreover, it is locally free over $\mathcal{O}_Y$.*

Explicitly, if ∂_1, ∂_2 are both local sections of $\mathrm{pr}^{-1}\left(\mathcal{D}_X^{(m,k)}\right)$, and if f_1, f_2 are local sections of $\mathcal{O}_Y$, then

$$(f_1 \otimes \partial_1) \bullet (f_2 \otimes \partial_2) = f_1\partial_1(f_2) \otimes \partial_2 + f_1 f_2 \otimes \partial_1\partial_2.$$

We have all the ingredients that allow us to construct the desired sheaves over Y, this is, to extend the sheaves of rings defined in the preceding chapter to an admissible blow-up of X. Let $k \geq k_Y$ fix. Let us first recall that taking arbitrary sections $P, Q \in \mathcal{D}_X^{(m,k)}$, $s, t \in \mathcal{L}(\lambda)$ and $s^\vee, t^\vee \in \mathcal{L}(\lambda)^\vee$ (the last two not necessarily the duals of s and t) over an arbitrary open subset $U \subset X$, the multiplicative structure of the sheaf $\mathcal{D}_X^{(m,k)}(\lambda)$ is defined by (cf. (5.22))

$$s \otimes P \otimes s^\vee \bullet t \otimes Q \otimes t^\vee = s \otimes P \left\langle s^\vee, t\right\rangle Q \otimes t^\vee.$$

Now, if $\mathrm{pr}\colon Y \to X$ denotes the projection, we put

$$\mathcal{D}_Y^{(m,k)}(\lambda) := \mathrm{pr}^*\left(\mathcal{D}_X^{(m,k)}(\lambda)\right).$$

By the adjointness property of pr^* and pr_* we have a canonical isomorphism of $\mathcal{O}_X$-modules

$$\mathrm{pr}_*\mathscr{H}om_{\mathcal{O}_Y}(\mathrm{pr}^*\mathcal{E}, \mathcal{F}) \simeq \mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathrm{pr}_*\mathcal{F})$$

where $\mathcal{E}$ is an $\mathcal{O}_X$-module and $\mathcal{F}$ is an $\mathcal{O}_Y$-module. In particular, we have the following isomorphism

$$\mathrm{Hom}\left(\mathrm{pr}^*\mathcal{D}_X^{(m,k)} \otimes_{\mathcal{O}_Y} \mathrm{pr}^*\mathcal{L}(\lambda)^\vee, \mathcal{F}\right) = \mathrm{Hom}\left(\mathrm{pr}^*\left(\mathcal{D}_X^{(m,k)} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee\right), \mathcal{F}\right),$$

which tells us that $\mathrm{pr}^*\mathcal{D}_X^{(m,k)} \otimes_{\mathcal{O}_Y} \mathrm{pr}^*\mathcal{L}(\lambda)^\vee$ and $\mathrm{pr}^*\left(\mathcal{D}_X^{(m,k)} \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee\right)$ are canonically isomorphic. By applying once again the preceding reasoning we get

$$\mathcal{D}_Y^{(m,k)}(\lambda) = \mathrm{pr}^*\mathcal{L}(\lambda) \otimes_{\mathcal{O}_Y} \mathrm{pr}^*\mathcal{D}_X^{(m,k)} \otimes_{\mathcal{O}_Y} \mathrm{pr}^*\mathcal{L}(\lambda)^\vee.$$

In consequence, the preceding isomorphism and proposition 6.1.2 allow us to endow the sheaf of $\mathcal{O}_Y$-modules $\mathcal{D}_Y^{(m,k)}(\lambda)$ with a multiplicative structure for every $k \geq k_Y$. On local sections we have

$$s \otimes P \otimes s^\vee \bullet t \otimes Q \otimes t^\vee = s \otimes P \left\langle s^\vee, t\right\rangle Q \otimes t^\vee,$$

where $s, t \in \mathrm{pr}^*\mathcal{L}(\lambda)$, $s^\vee, t^\vee \in \mathrm{pr}^*\mathcal{L}(\lambda)^\vee$ and $P, Q \in \mathcal{D}_Y^{(m,k)}$.

Let $\mathfrak{Y}$ be the completion of Y along its special fiber $Y_{\mathbb{F}_q} = Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}/\varpi)$.

6.1.3. *In this work we will only consider formal blow-ups $\mathfrak{Y}$ arising from the formal completion along the special fiber of an admissible blow-up $Y \to X$ (cf. proposition 6.3.1 below). Under this assumption we will identify $k_y = k_{\mathfrak{Y}}$.*

Definition 6.1.4. *Let $pr : Y \to X$ be an admissible blow-up of the flag variety X and let $k \geq k_Y$. The sheaves*

$$\widehat{\mathscr{D}}_{\mathfrak{Y},\mathbb{Q}}^{(m,k)}(\lambda) := \left(\varprojlim_{i\in\mathbb{N}} \mathcal{D}_Y^{(m,k)}(\lambda)/\varpi^{i+1}\mathcal{D}_Y^{(m,k)}(\lambda)\right) \otimes_{\mathfrak{o}} L \quad \textit{and} \quad \mathscr{D}_{\mathfrak{Y},k}^{\dagger}(\lambda) := \varinjlim_{m\in\mathbb{N}} \widehat{\mathscr{D}}_{\mathfrak{Y},\mathbb{Q}}^{(m,k)}(\lambda).$$

are called sheaves of twisted arithmetic differential operators on $\mathfrak{Y}$.

Proposition 6.1.5. *(i) The sheaves $\mathcal{D}_Y^{(m,k)}(\lambda)$ are filtered by the order of twisted differential operators and there is a canonical isomorphism of graded sheaves of algebras*

$$gr\left(\mathcal{D}_Y^{(m,k)}(\lambda)\right) \simeq Sym^{(m)}\left(\varpi^k pr^*\mathcal{T}_X\right),$$

where $k \geq k_Y$.

(ii) There is a basis for the topology of Y*, consisting of affine open subsets, such that for any open subset* $U \in Y$ *in this basis, the ring* $\mathcal{D}_Y^{(m,k)}(\lambda)(U)$ *is noetherian. In particular, the sheaf of rings* $\mathcal{D}_Y^{(m,k)}(\lambda)$ *is coherent.*

(iii) The sheaf $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)$ *is coherent.*

Proof. We have an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{D}^{(m,k)}_{X,d-1} \to \mathcal{D}^{(m,k)}_{X,d} \to \mathrm{Sym}^{(m)}_d \left(\varpi^k \mathcal{T}_X\right) \to 0.$$

Taking the tensor product with $\mathcal{L}(\lambda)$ and $\mathcal{L}(\lambda)^\vee$ on the left and on the right, respectively, and applying pr^* we obtain the exact sequence (since $\mathrm{Sym}^{(m)}_d(\varpi^k\mathcal{T}_X)$ is a locally free $\mathcal{O}_X$-module of finite rank)

$$0 \to \mathcal{D}^{(m,k)}_{Y,d-1}(\lambda) \to \mathcal{D}^{(m,k)}_{Y,d}(\lambda) \to \mathrm{pr}^*\mathcal{L}(\lambda) \otimes_{\mathcal{O}_Y} \mathrm{Sym}^{(m)}_d \left(\varpi^k \mathrm{pr}^*\mathcal{T}_X\right) \otimes_{\mathcal{O}_Y} \mathrm{pr}^*\mathcal{L}(\lambda)^\vee \to 0,$$

which implies (i) because

$$\mathrm{pr}^*\mathcal{L}(\lambda) \otimes_{\mathcal{O}_Y} \mathrm{Sym}^{(m)} \left(\varpi^k \mathrm{pr}^*\mathcal{T}_X\right) \otimes_{\mathcal{O}_Y} \mathrm{pr}^*\mathcal{L}(\lambda)^\vee \simeq \mathrm{Sym}^{(m)} \left(\varpi^k \mathrm{pr}^*\mathcal{T}_X\right)$$

by commutativity of the symmetric algebra.

Let $U \subseteq X$ be an affine open subset endowed with local coordinates $x_1, ..., x_M$ and such that $\mathcal{L}(\lambda)|_U = s\mathcal{O}_U$ for some $s \in \mathcal{L}(\lambda)(U)$. Then, by lemma 5.5.8 we have the following local description for $\mathcal{D}_Y^{(m,k)}(\lambda)$ on $V = \mathrm{pr}^{-1}(U)$

$$\mathcal{D}^{(m,k)}_X(\lambda)(V) = \left\{ \sum_{\underline{v}}^{<\infty} \varpi^{k|\underline{v}|} a_{\underline{v}} \underline{\partial}^{\langle \underline{v} \rangle} \mid \underline{v} = (v_1, ..., v_M) \in \mathbb{N}^M \textit{ and } a_{\underline{v}} \in \mathcal{O}_Y(V) \right\}.$$

By (i), the graded algebra $gr_\bullet\left(\mathcal{D}^{(m,k)}_Y(\lambda)(V)\right)$ is isomorphic to $\mathrm{Sym}^{(m)}\left(\varpi^k \mathrm{pr}^*\mathcal{T}_X(V)\right)$ which is known to be noetherian [34, Proposition 1.3.6]. Therefore, taking as a basis the set of affine open subsets of Y that are contained in some $\mathrm{pr}^{-1}(U)$ we get (ii). We also remark that, as $\mathcal{D}^{(m,k)}_Y(\lambda)$ is $\mathcal{O}_Y$-quasi-coherent, and by (ii) in the actual proposition, it has noetherian sections over the affine open subsets of Y (cf. [37, Proposition 2.2.2 (iii)]), it is certainly a sheaf of coherent rings [6, proposition 3.1.3]. Finally, by definition, we see that $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y}}(\lambda)$ satisfies the conditions (a) and (b) of 3.3.3 in [6] and hence [6, Proposition 3.3.4] gives us (iii). □

Let us briefly study the problem of passing to the inductive limit when m varies.

Let $U \subset X$ such that $\mathcal{D}^{(m,k)}_X(\lambda)|_U \simeq \mathcal{D}^{(m,k)}_X|_U$ and let us take $V \subseteq Y$ an affine open subset such that $V \subseteq \mathrm{pr}^{-1}(U)$. We have the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{i_V} & Y \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr} \\ U & \xrightarrow{i_U} & X, \end{array}$$

which implies that $\mathcal{D}^{(m,k)}_Y(\lambda)|_V \simeq \mathcal{D}^{(m,k)}_Y|_V$, as sheaves of rings. In particular, if $\mathfrak{V}$ denotes the formal p-adic completion along the special fiber $V_{\mathbb{F}_q}$ we have the commutative diagram (cf. proposition 2.5.12)

$$\begin{array}{ccc} \widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)(\mathfrak{V}) & \longrightarrow & \widehat{\mathscr{D}}^{(m+1,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)(\mathfrak{V}) \\ \downarrow \wr & & \downarrow \wr \\ \widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\mathfrak{V}) & \longrightarrow & \widehat{\mathscr{D}}^{(m+1,k)}_{\mathfrak{Y},\mathbb{Q}}(\mathfrak{V}). \end{array} \tag{6.2}$$

Given that the morphism of sheaves $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}} \to \widehat{\mathscr{D}}^{(m+1,k)}_{\mathfrak{Y},\mathbb{Q}}$ is left and right flat [37, Proposition 2.2.11], the preceding diagram allows us to conclude that the morphism $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda) \to \widehat{\mathscr{D}}^{(m+1,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)$ is also left and right flat. By proposition 4.2.3 we have the following result.

Proposition 6.1.6. *The sheaf of rings $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$ is coherent.*

As we will explain later, there exists a canonical morphism of sheaves of filtered $\mathfrak{o}$-algebras[1]

$$\mathcal{A}^{(m,k)}_{Y} := \mathcal{O}_Y \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}(k)) \to \mathcal{D}^{(m,k)}_{Y}(\lambda)$$

which allows to conclude the following proposition exactly as we have done in the proof of proposition 3.4.1 (cf. [36, Proposition 4.3.1]).

Proposition 6.1.7. *Let $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$.*

(i) Let $\mathscr{E}$ be a coherent $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)$-module. Then $\mathscr{E}$ is generated by its global sections as $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)$-module. Furthermore, $\mathscr{E}$ has a resolution by finite free $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)$-modules.

(ii) Let $\mathscr{E}$ be a coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-module. Then $\mathscr{E}$ is generated by its global sections as $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-module. Furthermore, $\mathscr{E}$ has a resolution by finite free $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules.

6.2 An Invariance theorem for admissible blow-ups

Let $pr : \mathfrak{Y} \to \mathfrak{X}$ be an admissible blow-up along a closed subset $\mathbf{V}(\mathscr{I})$ defined by an open ideal sheaf $\mathscr{I} \subseteq \mathcal{O}_{\mathfrak{X}}$. Using (6.1.3), we can suppose that $\mathfrak{Y}$ is obtained as the formal completion of an admissible blow-up $Y \to X$ (we will abuse of the notation and we will denote again by $\mathrm{pr} : Y \to X$ the canonical morphism of this (algebraic) blow-up) along a closed subset $\mathbf{V}(\mathcal{I})$ defined by an open ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, such that $\mathscr{I}$ is the restriction of the formal p-adic completion of $\mathcal{I}$. Let us denote by $Y_i := Y \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}/\varpi^{i+1})$ the redaction module ϖ^{i+1} and by $\gamma_i : Y_i \to Y$ the canonical closed embedding. In [37] the authors have studied the cohomological properties of the sheaves

$$\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}} := \varprojlim_{i\in\mathbb{N}} \gamma_i^* \mathcal{D}^{(m,k)}_{Y} \otimes_{\mathfrak{o}} L \quad \text{and} \quad \mathscr{D}^{\dagger}_{\mathfrak{Y},k} := \varinjlim_{m\in\mathbb{N}} \widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}.$$

Let us consider the commutative diagram

$$\begin{array}{ccc} Y_i & \xrightarrow{pr_i} & X_i \\ \downarrow \gamma_i & & \downarrow \gamma_i \\ Y & \xrightarrow{pr} & X. \end{array}$$

[1] We construct this morphism in (6.39). The arguments given there are independent and we won't introduce a circular argument.

Here $\mathrm{pr}_i : Y_i \to X_i$ denotes the redaction of the morphism pr module ϖ^{i+1}. We put $\underline{\mathscr{L}(\lambda)^\vee} := \varprojlim_i \gamma_i^* \mathrm{pr}^* \mathcal{L}(\lambda)^\vee$ and $\underline{\mathscr{L}(\lambda)} := \varprojlim_i \gamma_i^* \mathrm{pr}^* \mathcal{L}(\lambda)$. By using the preceding commutative diagram we have

$$\begin{aligned}\gamma_i^* \mathcal{D}_Y^{(m,k)}(\lambda) &= \gamma_i^* \left(\mathrm{pr}^* \mathcal{L}(\lambda) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y^{(m,k)} \otimes_{\mathcal{O}_Y} \mathrm{pr}^* \mathcal{L}(\lambda)^\vee \right) \\ &= \gamma_i^* \left(\mathrm{pr}^* \mathcal{L}(\lambda) \right) \otimes_{\mathcal{O}_{Y_i}} \gamma_i^* \mathcal{D}_Y^{(m,k)} \otimes_{\mathcal{O}_{Y_i}} \gamma_i^* \left(\mathrm{pr}^* \mathcal{L}(\lambda)^\vee \right).\end{aligned}$$

Taking the projective limit we get the following description of the sheaves $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)$

$$\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda) = \underline{\mathscr{L}(\lambda)}_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}} \widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}} \underline{\mathscr{L}(\lambda)^\vee}_{\mathbb{Q}},$$

and by taking the inductive limit we get the characterization

$$\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) = \underline{\mathscr{L}(\lambda)}_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}} \mathscr{D}^\dagger_{\mathfrak{Y},k} \otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}} \underline{\mathscr{L}(\lambda)^\vee}_{\mathbb{Q}}. \tag{6.3}$$

As in the preceding chapter, the sheaf $\underline{\mathscr{L}(\lambda)}_{\mathbb{Q}}$ is endowed with the following (left) $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-action

$$\left(t \otimes P \otimes t^\vee\right) \bullet s \;:=\; \left(P \bullet < t^\vee, s >\right) t \quad (s, t \in \underline{\mathscr{L}(\lambda)} \text{ and } t^\vee \in \underline{\mathscr{L}(\lambda)^\vee}).$$

We end this first discussion by remarking that the relation $\mathrm{pr}_i^* \circ \gamma_i^* = \gamma_i^* \circ \mathrm{pr}^*$, which comes from the preceding commutative diagram, implies that

$$\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) = \mathrm{pr}^* \mathscr{D}^\dagger_{\mathfrak{X},k}(\lambda). \tag{6.4}$$

Let us suppose that $\pi : Y' \to Y$ is a morphism of admissible blow-ups (abusing of the notation, we will also denote by $\pi : \mathfrak{Y}' \to \mathfrak{Y}$ the respective morphism of formal admissible blow-ups in the sense of [13, Part II, chapter 8, section 8.2, definition 3]). This means that we have a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\pi} & Y \\ & \searrow^{pr'} & \downarrow{\scriptstyle pr} \\ & & X. \end{array} \qquad \text{resp.} \qquad \begin{array}{ccc} \mathfrak{Y}' & \xrightarrow{\pi} & \mathfrak{Y} \\ & \searrow^{pr'} & \downarrow{\scriptstyle pr} \\ & & \mathfrak{X}. \end{array}$$

Let $k \geq \{k_{Y'}, k_Y\}$. Let us denote by $\mathcal{D}^{(m,k)}_{X,i}(\lambda) := \mathcal{D}^{(m,k)}_X(\lambda)/p^{i+1}\mathcal{D}^{(m,k)}_Y(\lambda)$ (we will use the same notations over Y_i' and Y_i) and by $\pi_i : Y_i' \to Y_i$ the redaction module ϖ^{i+1}. The preceding commutative diagram implies that

$$\mathcal{D}^{(m,k)}_{Y',i}(\lambda) = (pr_i')^* \mathcal{D}^{(m,k)}_{X_i}(\lambda) = \pi_i^* \mathcal{D}^{(m,k)}_{Y_i}(\lambda). \tag{6.5}$$

In this way, she sheaf $\mathcal{D}^{(m,k)}_{Y',i}(\lambda)$ can be endowed with a structure of right $\pi_i^{-1}\mathcal{D}^{(m,k)}_{Y_i}(\lambda)$-module. Passing to the projective limit, the sheaf $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y}'}(\lambda)$ is a sheaf of right $\pi^{-1}\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y}}(\lambda)$-modules. So, passing to the inductive limit over m we can conclude that $\mathscr{D}^\dagger_{\mathfrak{Y}',k}(\lambda)$ is a right $\pi^{-1}\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-module. For a $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-module $\mathscr{E}$, we define

$$\pi^! \mathscr{E} := \mathscr{D}^\dagger_{\mathfrak{Y}',k}(\lambda) \otimes_{\pi^{-1}\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)} \pi^{-1}\mathscr{E},$$

with analogous definitions for $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X},\mathbb{Q}}(\lambda)$. We will need the following lemma whose proof can be found in [37, Lemma 2.3.5].

Lemma 6.2.1. *There exists $N \in \mathbb{N}$ such that*

(i) *For all* $i \geq 0$, *the kernel and the cokernel of the canonical map* $\mathcal{O}_{Y_i} \to \pi_{i*}\mathcal{O}_{Y_i'}$ *is killed by* ϖ^N.

(ii) *For all* $i \geq 0$, *for all* $j \geq 1$, $\varpi^N R^j \pi_{i*}\mathcal{O}_{Y_i'} = 0$.

We have the following theorem whose proof follows word for word the reasoning given in [37, Theorem 2.3.4].

Theorem 6.2.2. *Let* $\pi : Y' \to Y$ *be a morphism over* X *of admissible blow-ups. Let* $k \geq \max\{k_{Y'}, k_Y\}$.

(i) *If* $\mathscr{E}$ *is a coherent* $\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)$, *then* $R^j\pi_*\mathscr{E} = 0$ *for every* $j > 0$. *Moreover,* $\pi_*\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda) = \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$, *so* π_* *induces an exact functor between coherent modules over* $\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)$ *and* $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$, *respectively.*

(ii) *The formation* $\pi^!$ *is an exact functor from the category of coherent* $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$*-modules to the category of coherent* $\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)$*-modules.*

(iii) *The functors* π_* *and* $\pi^!$ *are quasi-inverse equivalences between the categories of coherent* $\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)$*-modules and coherent* $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$*-modules.*

We remark for the reader that this theorem has an equivalent version for the sheaves $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)$ and $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y}',\mathbb{Q}}(\lambda)$.

Proof. The question being local on $\mathfrak{Y}$ we can suppose that $\mathfrak{Y}$ is affine. Let us first assume that $\mathscr{E} = \mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)$. Since $R^j\pi_*$ commutes with inductive limits, we can even restrain our attention on the sheaves $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)$. By [58, Lemma 20.32.4] $R\varprojlim_i \mathcal{D}^{(m,k)}_{Y_i'}(\lambda) = \varprojlim_i \mathcal{D}^{(m,k)}_{Y_i'}(\lambda)$, and we have

$$R\pi_*\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y}'}(\lambda) \simeq R\pi_* R\varprojlim_{i\in\mathbb{N}} \mathcal{D}^{(m,k)}_{Y_i'}(\lambda) \simeq R\varprojlim_{i\in\mathbb{N}} R\pi_{i\,*}\mathcal{D}^{(m,k)}_{Y_i'}(\lambda),$$

the last isomorphism is [58, Lemma 20.32.2]. By the projection formula, there is a canonical isomorphism

$$R\pi_{i*}\mathcal{D}^{(m,k)}_{Y_i'}(\lambda) \simeq R\pi_{i*}\mathcal{O}_{Y_i'} \otimes_{\mathcal{O}_{Y_i}} \mathcal{D}^{(m,k)}_{Y_i}(\lambda)$$

and the canonical map $\mathcal{O}_{Y_i} \to R\pi_{i*}\mathcal{O}_{Y_i'}$ induces a canonical map of complexes

$$h : \mathcal{D}^{(m,k)}_{Y_i}(\lambda) \to \left(R\pi_{i*}\mathcal{D}^{(m,k)}_{Y_i'}(\lambda)\right).$$

By applying $R\varprojlim_{i\in\mathbb{N}}$ to h, we get also a canonical map $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y}}(\lambda) \to R\pi_*\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y}'}(\lambda)$. Moreover, for every $j \geq 0$ and for every $i \in \mathbb{N}$, we have

$$R^j\pi_*\mathcal{D}^{(m,k)}_{Y_i'}(\lambda) \simeq R^j\pi_{i\,*}\mathcal{O}_{Y_i'} \otimes_{\mathcal{O}_{Y_i}} \mathcal{D}^{(m,k)}_{Y_i}(\lambda),$$

which implies by the preceding lemma that the kernel and cokernel of h are annihilated by ϖ^N, as well as the projective systems $(R^j\pi_{i\,*}\mathcal{D}^{(m,k)}_{Y_i'}(\lambda))$ if $j \geq 1$. Let $\mathcal{C} := (\mathcal{C}_i)$ be the cone of h, then we have the exact sequence of projective systems of sheaves

$$0 \to (\mathcal{H}^{-1}(\mathcal{C}_i)) \to (\mathcal{D}^{(m,k)}_{Y_i}(\lambda)) \to (\pi_{i\,*}\mathcal{D}^{(m,k)}_{Y_i'}(\lambda)) \to (\mathcal{H}^0(\mathcal{C}_i)) \to 0,$$

and for all $j \geq 1$

$$(R^j\pi_{i\,*}\mathcal{D}^{(m,k)}_{Y_i}(\lambda)) \simeq (\mathcal{H}^j(\mathcal{C}_i)).$$

In particular the cohomology of $\mathcal{C}$ is annihilated by ϖ^N so by [37, Lemma 2.3.3] we obtain a quasi-isomorphism

$$\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda) \to R\pi_*\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y}',\mathbb{Q}}(\lambda).$$

By passing to the cohomology sheaves we get the second part of (i) for the sheaf $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y}',\mathbb{Q}}(\lambda)$ and hence for the sheaf $\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)$. To handle with the second part let us consider the following assertion for every $j \geq 1$. Let a_j: for any coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)$-module $\mathscr{E}$ and for all $l \geq j$, $R^l\pi_*\mathscr{E} = 0$. The assertion is true for $j = \dim(\mathfrak{Y}) + 1$. Let us suppose that a_{j+1} is true and let us take a coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)$-module $\mathscr{E}$. By proposition 6.1.7 there exists $b \in \mathbb{N}$ and a short exact sequence of coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)$-modules

$$0 \to \mathscr{F} \to \left(\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)\right)^{\oplus b} \to \mathscr{E} \to 0.$$

Since $R^j\pi_*\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda) = 0$, the long exact sequence for π_* gives us

$$R^j\pi_*\mathscr{E} \simeq R^{j+1}\pi_*\mathscr{F},$$

which is 0 by induction hypothesis. This ends the proof of (i).
Let us show (ii) for the sheaves $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$. The case for the sheaves $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)$ being equal. Given that $\pi^!\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) = \mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)$, and since the tensor product is right exact, we can conclude that $\pi^!$ preserves coherence.
Now, we have a morphism $\pi^{-1}\mathscr{E} \to \pi^!\mathscr{E}$ sending $m \mapsto 1 \otimes m$. This maps induces the morphism $\mathscr{E} \to \pi_*\pi^!\mathscr{E}$. To show that this is an isomorphism is a local question on $\mathfrak{Y}$. If $\mathfrak{V} \subseteq \mathfrak{Y}$ is the formal completion of an affine open subset $V \subseteq \mathrm{pr}^{-1}(U)$, and $U \subseteq X$ is an affine open subset such that $\mathcal{D}^{(m,k)}_X(\lambda)|_U \simeq \mathcal{D}^{(m,k)}_X|_U$ (lemma 5.5.8), then by (6.2) and [37, Corollary 2.2.15] we can conclude that the previous map is in fact an isomorphism over $\mathfrak{V}$. Finally, if $\mathscr{F}$ is a coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)$-module, then we have the map $\pi^!\pi_*\mathscr{F} \to \mathscr{F}$, sending $P \otimes m \mapsto Pm$. To see that this is an isomorphism we can use the preceding reasoning. □

Let us recall that if $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$, then by proposition 5.5.12 we have

$$H^0\left(\mathfrak{X}, \mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda)\right) = D^{\dagger}(\mathbb{G}(k))_{\lambda}.$$

The previous theorem implies

Corollary 6.2.3. *Let $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda+\rho \in \mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$. In the situation of the preceding theorem we have*

$$H^0\left(\mathfrak{Y}, \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)\right) = H^0\left(\mathfrak{X}, \mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda)\right) = D^{\dagger}(\mathbb{G}(k))_{\lambda} = H^0\left(\mathfrak{Y}', \mathscr{D}^{\dagger}_{\mathfrak{Y}',k}(\lambda)\right).$$

Theorem 6.2.4. *Let $pr\colon Y \to X$ be an admissible blow-up. Let us suppose that $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$.*

(i) For any coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-module $\mathscr{E}$ and for all $q > 0$ one has $H^q(\mathfrak{Y}, \mathscr{E}) = 0$.

(ii) The functor $H^0(\mathfrak{Y}, \bullet)$ is an equivalence between the category of coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules and the category of finitely presented $D^{\dagger}(\mathbb{G}(k))_{\lambda}$-modules.

The same statement holds for coherent modules over $\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{Y},\mathbb{Q}}(\lambda)$.

Proof. The first part of the theorem follows from the fact that $H^0(\mathfrak{Y}, \bullet) = H^0(\mathfrak{X}, \bullet) \circ \pi_*$. Now we only have to apply the preceding theorem and theorem 5.5.14.

Let us denote by $\mathcal{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)$ the exact functor defined by the composition

$$\text{Finitely presented } D^{\dagger}(\mathbb{G}(k))_{\lambda} - \text{modules} \xrightarrow{\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}} \text{Coherent } \mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda) - \text{modules} \xrightarrow{\pi!} \text{Coherent} \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) - \text{modules.}$$

Let us compute this functor. To do that, we may fix a finitely presented $D^{\dagger}(\mathbb{G}(k))_{\lambda}$-module E. Then

$$\pi^{!}\left(\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}(E)\right) = \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{\pi^{-1}\mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda)} \pi^{-1}\mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda) \otimes_{D^{\dagger}(\mathbb{G}(k))_{\lambda}} E = \mathscr{L}oc^{\dagger}_{\mathfrak{Y},\lambda}(E).$$

Now, to show that

$$H^{0}\left(\mathfrak{Y}, \pi^{!}\left(\mathscr{L}oc^{\dagger}_{\mathfrak{X},\lambda}(E)\right)\right) = H^{0}\left(\mathfrak{Y}, \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{D^{\dagger}(\mathbb{G}(k))_{\lambda}} E\right) = E,$$

we can take a resolution

$$\left(D^{\dagger}(\mathbb{G}(k))_{\lambda}\right)^{\oplus b} \to \left(D^{\dagger}(\mathbb{G}(k))_{\lambda}\right)^{\oplus a} \to E \to 0,$$

to get the following diagram

$$\begin{array}{ccccccc}
\left(D^{\dagger}(\mathbb{G}(k))_{\lambda}\right)^{\oplus b} & \longrightarrow & \left(D^{\dagger}(\mathbb{G}(k))_{\lambda}\right)^{\oplus a} & \longrightarrow & E & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\left(D^{\dagger}(\mathbb{G}(k))_{\lambda}\right)^{\oplus b} & \longrightarrow & \left(D^{\dagger}(\mathbb{G}(k))_{\lambda}\right)^{\oplus a} & \longrightarrow & H^{0}\left(\mathfrak{Y}, \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{D^{\dagger}(\mathbb{G}(k))_{\lambda}} E\right) & \longrightarrow & 0.
\end{array}$$

where the sequence on the top is clearly exact. By definition $\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(\bullet)$ is an exact functor and by (i) the global section functor $H^{0}(\mathfrak{Y}, \bullet)$ is also exact. This shows that the sequence at the bottom is also exact and we end the proof of the theorem. □

In the sequel we will denote by G_0 the compact locally L-analytic group $G_0 := \mathbb{G}(\mathfrak{o})$.

6.3 Group actions on Blow-ups

We start this section with the following proposition whose proof is given in [36, Proposition 2.2.9].

Proposition 6.3.1. *Let $\mathfrak{Y} \to \mathfrak{X}$ be an admissible blow-up, obtained by blowing up an open ideal sheaf $\mathscr{I} \subseteq \mathcal{O}_{\mathfrak{X}}$. Then there is an open ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{X}$ such that $\mathscr{I}$ is the restriction to $\mathfrak{X}$ of the ϖ-adic completion of $\mathcal{I}$, and $\mathfrak{Y}$ is therefore the completion of the blow-up Y of $\mathcal{I}$ along its special fiber.*

Let $\mathfrak{G}$ be the formal completion of the group $\mathfrak{o}$-scheme $\mathbb{G}$, along its special fiber $\mathbb{G}_{\mathbb{F}_p} := \mathbb{G} \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}/\varpi)$. Let us denote by $\alpha : \mathfrak{X} \times_{\mathrm{Spf}(\mathfrak{o})} \mathfrak{G} \to \mathfrak{X}$ the induced right $\mathfrak{G}$-action on the formal flag $\mathfrak{o}$-scheme $\mathfrak{X}$ (cf. section 5.6). For every $g \in \mathfrak{G}(\mathfrak{o}) = G_0$ we have an automorphism ρ_g of $\mathfrak{X}$ given by

$$\rho_g : \mathfrak{X} = \mathfrak{X} \times_{\mathrm{Spf}(\mathfrak{o})} \mathrm{Spf}(\mathfrak{o}) \xrightarrow{id_{\mathfrak{X}} \times g} \mathfrak{X} \times_{\mathrm{Spf}(\mathfrak{o})} \mathfrak{G} \xrightarrow{\alpha} \mathfrak{X}.$$

As $\mathfrak{G}$ acts on the right, we have the following relation

$$\left(\rho_g\right)_{*}\left(\rho^{\natural}_{h}\right) \circ \rho^{\natural}_{g} = \rho^{\natural}_{hg} \quad (g, h \in G_0). \tag{6.6}$$

Here $\rho^{\natural}_{g} : \mathcal{O}_{\mathfrak{X}} \to (\rho_g)_{*}\mathcal{O}_{\mathfrak{X}}$ denotes the comorphism of ρ_g.

Let $H \subseteq G_0$ be an open subgroup. We say that an open ideal sheaf $\mathscr{I} \subseteq \mathcal{O}_{\mathfrak{X}}$ is *H-stable* if for all $g \in H$ the comorphism $\rho_g^\natural$ maps $\mathscr{I} \subseteq \mathcal{O}_{\mathfrak{X}}$ into $(\rho_g)_*\mathscr{I} \subseteq (\rho_g)_*\mathcal{O}_{\mathfrak{X}}$. In this case $\rho_g^\natural$ induces a morphism of sheaves of graded rings

$$\bigoplus_{d\in\mathbb{N}} \mathscr{I}^d \to \left(\rho_g\right)_* \left(\bigoplus_{d\in\mathbb{N}} \mathscr{I}^d\right)$$

on $\mathfrak{X}$. This morphism of sheaves induces an automorphism of the blow-up $\mathfrak{Y} = \mathbf{Proj}\left(\oplus_{d\in\mathbb{N}}\mathscr{I}^d\right)$, let us say ρ_g by abuse of notation, and the action of H on $\mathfrak{X}$ lifts to a right action of H on $\mathfrak{Y}$, in the sense that for every $g, h \in G_0$ the relation (6.6) is verified and we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\rho_g} & \mathfrak{Y} \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathfrak{X} & \xrightarrow{\rho_g} & \mathfrak{X}. \end{array} \tag{6.7}$$

Definition 6.3.2. *Let $H \subseteq G_0$ be an open subgroup and pr $: \mathfrak{Y} \to \mathfrak{X}$ and admissible blow-up defined by an open ideal subsheaf $\mathscr{I} \subset \mathcal{O}_{\mathfrak{X}}$. We say that $\mathfrak{Y}$ is H-equivariant if $\mathscr{I}$ is H-stable.*

We will need the following result in the next sections. The reader can find its proof in [36, Lemma 5.2.3].

Lemma 6.3.3. *Let pr $: \mathfrak{Y} \to \mathfrak{X}$ be an admissible blow-up, and let us assume that $k \geq k_Y = k_{\mathfrak{Y}}$ (this notation is justified by proposition 6.3.1). Then $\mathfrak{Y}$ is $G_k = \mathbb{G}(k)(\mathfrak{o})$-equivariant and the induced action of every $g \in G_{k+1}$ on the special fiber of Y is the identity. Therefore, G_{k+1} acts trivially on the underlying topological space of $\mathfrak{Y}$.*

Let us recall that in the preceding chapter we have defined a G_0-action on $\mathscr{L}(\lambda)$. This means, for every $g \in G_0$ we have an isomorphism $\Phi_g : \mathscr{L}(\lambda) \to (\rho_g)_*\mathscr{L}(\lambda)$ satisfying the cocycle condition $\Phi_{hg} = (\rho_g)_*\Phi_h \circ \Phi_g$, for every $g, h \in G_0$. Let us consider the isomorphism $(\rho_g)^*\mathscr{L}(\lambda) \to \mathscr{L}(\lambda)$ given by adjonction on Φ_g. Let us suppose that $H \subseteq G_0$ is an open subgroup and that pr $: \mathfrak{Y} \to \mathfrak{X}$ is an H-equivariant admissible blow-up. Pulling back the isomorphism $(\rho_g)^*\mathscr{L}(\lambda) \to \mathscr{L}(\lambda)$, via $(\mathrm{pr})^*$, and using the preceding commutative diagram we get $\mathrm{pr}^*(\rho_g)^*\mathscr{L}(\lambda) = (\rho_g)^*\mathrm{pr}^*\mathscr{L}(\lambda) = (\rho_g)^*\underline{\mathscr{L}(\lambda)}$ (notation given at the beginning of the preceding section). By adjontion we get the map

$$\mathrm{R}_g : \underline{\mathscr{L}(\lambda)} \xrightarrow{\simeq} \left(\rho_g\right)_* \underline{\mathscr{L}(\lambda)}$$

which satisfies, by functoriality, the cocycle condition

$$R_{hg} = \left(\rho_g\right)_* R_h \circ R_g \quad (g, h \in H). \tag{6.8}$$

As in (5.35) we can define (from now on we will work on admissible blow-ups of $\mathfrak{Y}$ so we will use the same notation)

$$\begin{array}{cccc} T_g : & \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) & \to & \left(\rho_g\right)_* \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \\ & P & \to & R_g \circ P \circ R_g^{-1}. \end{array} \tag{6.9}$$

Locally, if $\mathcal{U} \subseteq \mathfrak{Y}$ is an open subset and $P \in \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)(\mathcal{U})$ then $T_{g\,\mathcal{U}}(P)$ is given by the following diagram

$$\begin{array}{ccc} \underline{\mathscr{L}(\lambda)}(\mathcal{U}.g^{-1}) & \xrightarrow{T_{g\,\mathcal{U}}(P)} & \underline{\mathscr{L}(\lambda)}(\mathcal{U}.g^{-1}) \\ \downarrow{\scriptstyle R^{-1}_{g\,\mathcal{U}}} & & \downarrow{\scriptstyle R_{g\,\mathcal{U}}} \\ \underline{\mathscr{L}(\lambda)}(\mathcal{U}) & \xrightarrow{P} & \underline{\mathscr{L}(\lambda)}(\mathcal{U}). \end{array},$$

and exactly as we have done in (5.36) we can conclude that

$$T_{hg} = \left(\rho_g\right)_* T_h \circ T_g,$$

for every $g, h \in H$.

6.4 Complete distribution algebras and locally analytic representations

6.4.1 Locally analytic representations and locally analytic distributions

Let us recall that in this work L denotes a finite extension of $\mathbb{Q}_p$. In particular, L is a nonarchimedian *spherically complete* field in the sense that for any decreasing sequence of closed balls $B_1 \supseteq B_2 \supseteq ...$ in L, the intersection $\cap_{n\in\mathbb{N}} B_n$ is nonempty. We thus assume throughout this section that the so-called coefficient-field, over which the topological vector will be defined, is equal to our base field L.

Let V be an L-vector space. Let us start by recalling that a *lattice* M in V is an $\mathfrak{o}$-submodule satisfying: for any vector $v \in V$ there exists a nonzero scalar $a \in L^\times$ such that $av \in M$. This means that the natural map $L \otimes_{\mathfrak{o}} M \to V$ sending $a \otimes v \mapsto av$, is a bijection.

In what follows, we will be always considering the category of Hausdorff locally convex topological L-vector spaces. This means, the category of L-vector spaces V, endowed with a family of (nonempty) lattices $(M_j)_{j\in J}$, which satisfies

(l_1) for any $j \in J$ and any $a \in L^\times$ there exists $k \in J$ such that $M_k \subset aM_j$.

(l_2) If $i, j \in J$, then there exists $k \in J$ such that $M_k \subseteq M_j \cap M_i$.

These conditions imply that the sets $v + M_j$, with $v \in V$ and $j \in J$, form a basis of a topology on V, which is called the *locally convex topology* on V (defined by the family $(M_j)_{j\in J}$).

Definition 6.4.1. *Let V be a Hausdorff locally convex topological L-vector space. We say that V is a **BH**-space if it admits a complete metric defined by a norm, such that this induces a locally convex topology finer that its given topology.*

In the preceding definition, if the metric topology equals the given locally convex topology, we say that V is a *Fréchet* space. Furthermore, if W is another locally convex L-vector space, then a continuous linear map between two Hausdorff locally convex topological L-vector spaces $f : V \to W$ is called a **BH**-map, if there exists an L-Banach space U such that f admits a factorization of the form $V \to U \to W$.
On the other hand, the continuous map $f : V \to W$ is called *compact* if there is an open lattice $M \subseteq V$ such that $\overline{f(M)}$ is compact.

Definition 6.4.2. *A locally convex vector space V is called of compact type if it is the locally convex inductive limit of a sequence*

$$V_1 \to ... \to V_k \xrightarrow{j_k} V_{k+1} \to ...$$

of L-Banach spaces with injective and compact transition maps (the topology on the inductive limit is the finest locally convex topology such that all the natural maps $V_k \to V$ are continuous [55, Chapter I, section 5 E.]).

We can define now an important class of topological L-algebras ([24, Definition 1.6] or [52, Section 3]).

Definition 6.4.3. *Let A be a topological L-algebra. We say that A is a Fréchet-Stein algebra if there exists a sequence $(A_k)_{k\in\mathbb{N}}$ of Notherian Banach L-algebras satisfying the following conditions.*

(i) *We can find an isomorphism*

$$A \xrightarrow{\simeq} \varprojlim_{k \in \mathbb{N}} A_k,$$

with flat transition maps $A_{k+1} \to A_k$.

(ii) *The maps* $A \to A_k$ *have dense image.*

If V is locally convex L-vector space, then we say that V is a nuclear Fréchet space over L if there exists a sequence $(V_k)_{k \in \mathbb{N}}$ of Banach spaces , such that

$$V \xrightarrow{\simeq} \varprojlim_{k \in \mathbb{N}} V_k$$

and the transition maps are compact (the right side is equipped with the projective limit topology [55, Chapter I, section 5 D]). We have [54, Theorem 1.3]

Proposition 6.4.4. *Passing to strong duals yields an anti-equivalence of categories between the category of spaces of compact type and the category of nuclear Fréchet spaces.*

We say that a locally convex L-vector space V is *hereditarily* complete if any quotient of V is complete. In this work we will use the following weaker version of the preceding definition.

Definition 6.4.5. *Let* A *be a locally convex topological* L*-algebra. We say that* A *is a weak Fréchet-Stein* L*-algebra if there exists a sequence* $(A_k)_{k \in \mathbb{N}}$ *of locally convex* L*-algebras satisfying the following properties.*

(i) *For every* $k \in \mathbb{N}$ *the* L*-algebra* A_k *is hereditarily complete.*

(ii) *For each* $k \in \mathbb{N}$ *there exists an* L*-algebra homomorphism* $A_{k+1} \to A_k$*, which is a* $\boldsymbol{BH}$*-map.*

(iii) *An isomorphism of locally convex topological* L*-algebras*

$$A \xrightarrow{\simeq} \varprojlim_{k \in \mathbb{N}} A_k.$$

It is possible to show that any two weak Fréchet-Stein structures on A are equivalent [25, Proposition 1.2.7].

Definition 6.4.6. *Let* A *be a weak Fréchet-Stein algebra, and let* $A \xrightarrow{\simeq} \varprojlim_{k \in \mathbb{N}} A_k$ *be a choice of a weak Fréchet-Stein structure on* A*. If* M *is a locally convex topological* A*-module, we say that* M *is coadmissible (with respect to the given weak Fréchet-Stein structure on* A*) if there exists a sequence* $(M_k)_{k \in \mathbb{N}}$ *satisfying the following conditions.*

(i) *For every* $k \in \mathbb{N}$*,* M_k *is a finitely generated locally convex topological* A_k*-module.*

(ii) *An isomorphism of topological* A_k*-modules* $A_k \widehat{\otimes}_{A_{k+1}} M_{k+1} \xrightarrow{\simeq} M_k$ *(the tensor product is understood in the sense of [25, Lemma 1.2.3]).*

(iii) *An isomorphism of topological* A*-modules* $M \xrightarrow{\simeq} \varprojlim_{k \in \mathbb{N}} M_k$.

- If $M \xrightarrow{\simeq} \varprojlim_{k \in \mathbb{N}} M_k$ is a coadmissible A-module, we will say that $(M_k)_{k \in \mathbb{N}}$ is an $(A_k)_{k \in \mathbb{N}}$-sequence.

Remark 6.4.7. *Let* A *be a locally convex topological* L*-algebra.*

(i) *A Fréchet-Stein structure on* A *is a weak Fréchet-Stein structure* $A \xrightarrow{\simeq} \varprojlim_{k \in \mathbb{N}} A_k$ *on* A*, such that for each* $n \geq 1$*, the algebra* A_k *is noetherian* L*-Banach algebra, and the transition maps* $A_{k+1} \to A_k$ *are flat.*

(*ii*) [2] *Let us suppose that* $A \xrightarrow{\simeq} \varprojlim_{k\in\mathbb{N}} A_k$ *is a Fréchet-Stein structure on A. If* $M \xrightarrow{\simeq} \varprojlim_{k\in\mathbb{N}} M_k$ *is a coadmissible A-module, where* $(M_k)_{k\in\mathbb{N}}$ *is an* $(A_k)_{k\in\mathbb{N}}$*-sequence, then every* M_k *is a Hausdorff quotient of* $A_k^{r_k}$*, for some* $r_k \geq 0$*, and hence an* A_k*-Banach module. By [14, Proposition 3.7.3 (6)] the natural map*

$$A_k \otimes_{A_{k+1}} M_{k+1} \to M_k \to A_k \widehat{\otimes}_{A_{k=1}} M_{k+1} \to M_k$$

is an isomorphism (this notation is introduced before the definition 6.4.11). Thus the notion of coadmissiblity for topological A-modules as defined before coincides with that defined in [52, Second definition of section 3].

Let $\mathbb{X}$ be an affinoid rigid analytic space over L. We will denote by $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, L) := \mathcal{O}(\mathbb{X})$ the Tate L-algebra of L-valued rigid analytic functions on $\mathbb{X}$. This is an Banach L-algebra [13, Part I, chapter 3, section 3.1, proposition 5].

Example 6.4.1. *Let* $r \in |L^\times|$ *and* $a \in L^d$ *be a fixed point. Let us consider the closed ball of radius r,*

$$B_r(a) := \left\{ x \in L^d \mid |x-a| \leq r \right\}.$$

This can be identified as the set of L-points of a rigid analytic ball $\mathbb{B}_r$*, and therefore the algebra of all L-valued rigid analytic functions on* $B_r(a)$ *is the Tate algebra*

$$\mathcal{O}(\mathbb{B}_r) := \left\{ f(x) = \sum_{\underline{i}} c_{\underline{i}}(x-a)^{\underline{i}} \mid c_{\underline{i}} \in L \ \ and \ \ \lim_{|\underline{i}|\to\infty} |c_{\underline{i}}| r^{|\underline{i}|} = 0 \right\}.$$

This is a Banach L-algebra if we endow it with the norm

$$|f| := \max_{\underline{i}} |c_{\underline{i}}| r^{|\underline{i}|}.$$

Here $\underline{i} \in \mathbb{N}^d$ *is a multi-index and* $(x-a)^{\underline{i}} := (x_1 - a_1)^{i_1} ... (x_d - a_d)^{i_d}$.

6.4.8. *Let* $\mathcal{M}$ *be a Hausforff space*

A chart for $\mathcal{M}$ *is an open subset* $U_i \subseteq \mathcal{M}$ *together with a homeomorphism*

$$\varphi_i : U_i \to B_{r_i} \subset L^{d_i} \tag{6.10}$$

where B_{r_i} *is a closed ball. We say that two charts* (U_i, φ_i) *and* (U_j, φ_j) *(with* $U_i \cap U_j \neq \emptyset$*) are compatible if both maps*

$$\varphi_j(U_i \cap U_j) \xrightarrow{\varphi_i \circ \varphi_j^{-1}} \varphi_j(U_i \cap U_j) \quad and \quad \varphi_i(U_i \cap U_j) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} \varphi_i(U_i \cap U_j)$$

are given by a collection of convergent power series (locally L-analytic functions). We have:

(*i*) *[56, II, section 7, lemma 7.1] Let* (U_i, φ_i) *and* (U_j, φ_j) *be two compatible charts, such that* $U_i \cap U_j \neq \emptyset$*. Then using the same notation given in (6.10), we have that* $d_i = d_j$*.*

We will say that a collection of compatible charts $\mathcal{A} = \left\{(U_j, \varphi_j)\right\}_{j\in J}$ *is an atlas for* $\mathcal{M}$ *if* $\mathcal{M} : \cup_{j\in J} U_j$*. Two atlas* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are called equivalent if* $\mathcal{A}_1 \cup \mathcal{A}_2$ *is again an atlas of M. The atlas* $\mathcal{A}_1$ *will be called maximal if any equivalent atlas* $\mathcal{B}$ *satisfies* $\mathcal{B} \subset \mathcal{A}_1$*.*

(*ii*) *[56, II, Section 7, remark 7.2] The equivalence of atlases is an equivalence relation. Every equivalence class contains exactly one maximal atlas.*

[2] This is as in [25, Page 26].

We will say that an atlas $\mathcal{A}$ is of dimension d, if the charts of $\mathcal{A}$ have dimension d.

Definition 6.4.9. *A locally L-analytic manifold $(\mathcal{M}, \mathcal{A})$ (or simply M) is a Hausdorff topological space $\mathcal{M}$ equipped with a maximal atlas $\mathcal{A}$. The manifold is called d-dimensional if the atlas $\mathcal{A}$ is d-dimensional.*

A function $f : \mathcal{M} \to L$ is called locally L-analytic if $f \circ \varphi_i^{-1} \in \mathcal{O}(\mathbb{B}_{r_i})$ for any chart $(U_i, \varphi_i, B_{r_i})$.

Lemma 6.4.10. *[57, 8.6] Any locally L-analytic manifold is strictly paracompact.*

Let V and W be Hausdorff locally convex topological L-vector spaces. If $M \subseteq V$ and $N \subseteq W$ are lattices of V and W, respectively, then $M \otimes_{\mathfrak{o}} N$ is a lattice of $V \otimes_L W$. The family of all lattices $M \otimes_{\mathfrak{o}} N$ satisfies the conditions (l_1) and (l_2) and defines a locally convex topology on the tensor project $V \otimes_L W$, which is called the projective tensor topology. We will denote by $V \widehat{\otimes}_L W$ its Hausdorff completion [55, Chapter I, section 7, proposition 7.5].

Definition 6.4.11. *If $\mathbb{X}$ is an affinoid rigid analytic space over L, and if W is an L-Banach space then we write $\mathcal{C}^{an}(\mathbb{X}, W) := \mathcal{C}^{an}(\mathbb{X}, L)\widehat{\otimes}_L W$, for the space of W-valued rigid analytic functions on $\mathbb{X}$.*

In fact, if $\mathbb{X}$ is a closed ball centered at zero and $|| \bullet ||$ denotes the norm of W then

$$\mathcal{C}^{an}(\mathbb{X}, W) = \left\{ \sum_{\underline{i}} a_{\underline{i}} x^{\underline{i}} \mid a_{\underline{i}} \in W \ \ and \ \ \lim_{|\underline{i}| \to \infty} ||a_{\underline{i}}|| r^{|\underline{i}|} = 0. \right\}.$$

Let M be a locally L-analytic manifold and V a locally convex L-vector space. We say that a function $f : M \to V$ is locally analytic if for each point $x \in M$ there is a chart $(M_i, \varphi_i, B_{r_i})$ containing x, a Banach space W_i equipped with a continuous L-linear map $\psi_i : W_i \to V$ and a rigid analytic function $f_i \in \mathcal{C}^{an}(B_{r_i}, W_i)$ such that $f|_{M_i} = \psi_i \circ f_i \circ \varphi_i$. We let $\mathcal{C}^{an}(M, V)$ denote the space of locally analytic L-valued functions on M. We have therefore an isomorphism

$$\mathcal{C}^{an}(M, V) \xrightarrow{\simeq} \varinjlim_{(M_i, \varphi_i, W_i)} \prod_{i \in I} \mathcal{C}^{an}(M_i, W_i).$$

We have the following result [53, Proposition 10.3].

Proposition 6.4.12. *if M is compact and V is of compact type, then $\mathcal{C}^{an}(M, V)$ is again of compact type.*

Let us suppose that V is a compact type convex L-vetor space and let G_0 be a compact locally L-analytic group (or any compact open subgroup of a locally L-analytic group). By the preceding proposition the space $\mathcal{C}^{\text{an}}(G_0, V)$ of V-valued locally L-analytic functions on G_0 is a compact type convex L-vector space, and hence its strong dual is a nuclear Fréchet space by proposition 6.4.4. The space of *V-valued locally analytic distributions* is denoted by [3]

$$D(G_0, V) := \left(\mathcal{C}^{\text{an}}(G_0, V)\right)'_b.$$

In particular, if $V = L$ then any element $g \in G_0$ gives rise to a Dirac delta function δ_g supported at g. This is defined by $\delta_g(f) := f(g)$. In this way, if $L[G_0]$ denotes the group ring of G_0 over L, we obtain an embedding $L[G_0] \hookrightarrow D(G_0, L)$ whose image is dense [54, Lemma 3.1]. This implies that the L-algebra structure on $L[G_0]$ extends, in a unique way, to a topological L-algebra structure on $D(G_0, L)$.

Definition 6.4.13. *A locally convex L-vector space V is called barrelled if every closed lattice in V is open.*

Examples of barrelled spaces are Fréchet spaces and vector spaces of compact type.
We can finally give one of the central definitions in this work.

[3] As in [53] the subscript indicates the dual endowed with its strong topology.

Definition 6.4.14. *Let V be a locally convex barrelled L-vector space, equipped with an action of G_0 by continuous L-linear automorphisms. We say that V is a locally analytic representation of G_0 if for any $v \in V$, the orbit map*

$$\begin{array}{rccc} o_v : & G_0 & \to & V \\ & g & \mapsto & g \cdot v, \end{array}$$

lies in $\mathcal{C}^{an}(G_0, L)$.

In order to relate locally analytic representation with modules over the ring of locally analytic distributions we need to introduce the exponential map. More exactly, the tangent space $\mathfrak{g} := T_e(G_0)$ to the identity of the locally analytic group G_0 has a structure of Lie algebra [56, Corollary 13.13]. The Campbell-Baker-Hausdorff formula converges p-adically in a neighborhood U of zero in $\mathfrak{g}$, defining an analytic map [56, Corollary 18.19]

$$\exp : U \to G_0.$$

In particular, to each $\eta \in \mathfrak{g}$ we can associate a linear continuous form

$$f \mapsto := \frac{d}{dt} f(\exp(t\eta))|_{t=0} \quad (f \in D(G_0, L)),$$

which induces a morphism of rings

$$\mathcal{U}(\mathfrak{g}) \to D(G_0, L). \tag{6.11}$$

Now, if V is a locally analytic representation of G_0, then V has a structure of $D(G_0, L)$-module [54, Section 3]. For example, if $g \in G_0$ and δ_g is the Dirac distribution supported at g, we have $\delta_g \bullet v = gv$. We have the following result [54, Proposition 3.2]

Proposition 6.4.15. *The map $(g, v) \mapsto g \bullet v$ is separately continuous. This structure extends the action of $\mathcal{U}(\mathfrak{g})$ on V and any continuous linear G_0-map between locally analytic G_0-representations gives rise to a $D(G_0, L)$-morphism.*

Let us denote by $\mathrm{Rep}_c^{\mathrm{an}}(G_0)$ the category of locally analytic G_0- representations on L-vector spaces of compact type with continuous linear G_0-maps and by $\mathcal{M}_L^{\mathrm{Fr}}(G_0)$ the category of continuous $D(G_0, L)$-modules on nuclear Fréchet spaces with continuous $D(G_0, L)$-module maps.

Definition 6.4.16. *An admissible G_0- representation over L is a locally analytic G_0-representation on an L-vector space of compact type V such that the strong dual V_b' is a coadmissible $D(G_0, L)$-module (cf. (6.13)) equipped with its canonical topology.*

Let us denote by $\mathrm{Rep}^{\mathrm{adm}}(G_0)$ the category of all admissible G_0-representations with continuous linear G_0-maps and by $\mathcal{C}_{G_0}$ the full subcategory of $\mathrm{Mod}(D(G_0, L))$ consisting of coadmissible modules. We have the following commutative diagram of functors ([54, Corollary 3.4] and [53, Theorem 20.1])

$$\begin{array}{ccc} \mathrm{Rep}_c^{\mathrm{an}}(G_0) & \xrightarrow{\simeq} & \mathcal{M}_L^{\mathrm{Fr}}(G_0) \\ \uparrow & & \uparrow \\ \mathrm{Rep}^{\mathrm{adm}}(G_0) & \xrightarrow{\simeq} & \mathcal{C}_{G_0}. \end{array} \tag{6.12}$$

The horizontal maps are the anti-equivalences of categories defined by $V \mapsto V_b'$ and the structure of $D(G_0, L)$-module on V_b' is given by

$$(\delta \bullet m)(v) := \delta\left(g \mapsto m(g^{-1} \bullet v)\right) \quad (\delta \in D(G_0, L),\ m \in V_b' \text{ and } v \in V). \tag{6.13}$$

Finally, let us explain the concept of locally analytic vectors in an admissible representation. Let us suppose that G is a locally L-analytic group and $H \subseteq G$ is a compact open subgroup which is a chart. As we have seen, this means that $H = \mathbb{H}(L)$ with $\mathbb{H}$ a rigid analytic closed ball. Let us take now an L-vector space V endowed with a continuous G-action. We put

$$V_{\mathbb{H}-\mathrm{an}} := \left\{ v \in V \mid o_v : H \to V \in \mathcal{C}^{\mathrm{an}}(\mathbb{H}, V) \right\}.$$

Let us suppose for the moment that $\mathbb{G}$ is a rigid analytic group defined over L, which admits an admissible cover

$$\mathbb{G} = \bigcup_{n \in \mathbb{Z}_{>0}} \mathbb{G}_n$$

where $(\mathbb{G}_n)_{n\in\mathbb{Z}_{>0}}$ is a decreasing sequence of admissible affiinoid open subgroups of $\mathbb{G}$. We write $G := \mathbb{G}(L)$ and $G_n := \mathbb{G}_n(L)$ for every $n \in \mathbb{Z}_{>0}$. We also assume that G_n is Zariski dense in $\mathbb{G}_n$ for each $n \in \mathbb{Z}_{>0}$, and thus G is Zariski dense in $\mathbb{G}$. In this case, we define [4]

$$V_{\mathbb{G}-\mathrm{an}} := \bigcup_{n \in \mathbb{Z}_{>0}} V_{\mathbb{G}_n-\mathrm{an}}.$$

6.4.2 Complete arithmetic distribution algebras and distribution algebra of an analytic group

Let $k \in \mathbb{N}$ be a natural number. Through this section $\mathfrak{G}(k)$ will denote the formal group $\mathfrak{o}$-scheme defined by the formal p-adic completion of the congruence group $\mathbb{G}(k)$ along its special fiber $\mathbb{G}_{\mathbb{F}_q}(k) = \mathbb{G}(k) \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}/\varpi)$ (being $\mathbb{G}(0) = \mathbb{G}$). As before, we will also denote by $\mathbb{G}_i(k) := \mathbb{G}(k) \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}/\varpi^{i+1})$ the redaction modulo ϖ^{i+1}. The morphisms $\mathbb{G}_{i+1}(k) \to \mathbb{G}_i(k)$ induce a morphism $D^{(m)}(\mathbb{G}_{i+1}(k)) \to D^{(m)}(\mathbb{G}_i(k))$ [39, Proposition 4.1.11]. By [39, Corollary 5.3.2] we have

Lemma 6.4.17. *Let e be the index of ramification of L over $\mathbb{Q}_p$. If $e < p - 1$, then the ring $D^{\dagger}(\mathbb{G}(k))$ is coherent.*

By construction, the group $\mathfrak{o}$-scheme $\mathfrak{G}(k)$ is topological of finite type. Let $\widehat{\mathfrak{G}}^{\circ}$ be the completion of $\mathfrak{G}(k)$ along its unit section $\mathrm{Spf}(\mathfrak{o}) \to \mathfrak{G}(k)$. This is a group scheme over $\mathfrak{o}$, but not a formal $\mathfrak{o}$-scheme because $\varpi\mathcal{O}_{\widehat{\mathfrak{G}}(k)^{\circ}}$ is not necessarily an ideal of definition. Let us denote by $\mathbb{G}(k)^{\circ}$ its associated rigid-analytic space in the sense of [5, (0.2.6)]. This is an analytic group over L with $\mathrm{Lie}(\mathbb{G}(k)^{\circ}) = \mathfrak{g}_L$, which is not affinoid in general. This can be built as follows. Let η be the closed point of $\mathfrak{o}$ and let $t_1, ..., t_N$ be a sequence of regular generators for the unit section of $\mathbb{G}(k)$ such that $\mathfrak{o}[\mathbb{G}(k)] = \mathfrak{o}[t_1, ..., t_N]$ and $\mathfrak{o}[\mathfrak{G}(k)] = \mathfrak{o}\left\{t_1, ..., t_N\right\}$. We have that $\mathfrak{o}[\widehat{\mathfrak{G}}^{\circ}] = \mathfrak{o}[[t_1, ..., t_N]]$ and the space $\mathbb{G}(k)^{\circ}$ is isomorphic to an open disk of dimension N. In fact, if

$$A_n := \mathfrak{o}[[t_1, ...t_N]]\left\{T_1, ..., T_N\right\}/(t_1^n - \varpi T_1, ..., t_N^n - \varpi T_N),$$

with $\mathfrak{o}[[t_1, ...t_N]]\left\{T_1, ..., T_N\right\}$ the p-adic completion of $\mathfrak{o}[[t_1, ...t_N]][T_1, ..., T_N]$, then A_n in an $\mathfrak{o}$-algebra topologically of finite type and therefore $B_n := A_n \otimes_{\mathfrak{o}} L$ is a Tate algebra [20, Lemma 7.1.2 (b)]. Moreover, for $n' \geq n$ we have a canonical isomorphism

$$\begin{array}{ccc} B_{n'}\left\{T_1', ..., T_N'\right\}/(t_1^n - \varpi T_1', ..., t_N^n - \varpi T_N') & \to & B_n \\ T_i' & \mapsto & t_i^n/\varpi \end{array}$$

identifying $\mathbb{G}(k)_n^{\circ} := \mathrm{Spm}(B_n)$ with the special domain of $\mathbb{G}(k)_{n'}^{\circ}$ defined by the equations $|t_i(x)| \leq |\varpi|^{1/n}$ (this means a closed ball of radius $r_n = |\varpi|^{1/n}$) [20, Lemma 7.1.2 (c)]. We define $\mathbb{G}(k)^{\circ}$ as the rigide analytic space defined by glueing the affinoid spaces $(\mathbb{G}(k)_n^{\circ})_{n\in\mathbb{Z}_{>0}}$. By construction, the space $\mathbb{G}(k)^{\circ}$ is the unit open ball. Furthermore, given that $\mathbb{G}(k)_{n'}^{\circ}$

[4] This is a consequence of the remark preceding the definition 2.1.18 in [25].

is already a closed ball of radius $|\varpi|^{1/n'}$ we can conclude that the inclusion $\mathbb{G}(k)^\circ_n \hookrightarrow \mathbb{G}(k)^\circ_{n+1}$ is relatively compact ([55, Final example of section 16]) and the algebra of analytic functions on $\mathbb{G}(k)^\circ$ is given by [25, definition 2.1.18]

$$\mathcal{C}^{\mathrm{an}}(\mathbb{G}(k)^\circ, L) := \mathcal{O}(\mathbb{G}(k)^\circ) = \varprojlim_{n\in\mathbb{Z}_{>0}} B_n.$$

By [25, Proposition 2.1.6] and [55, Page 107 and proposition 19.9] we have that $\mathcal{O}(\mathbb{G}(k)^\circ)$ is a nuclear Fréchet algebra over L (cf. [24, Example 1.7]).

Remark 6.4.18. *By construction, ror any $k' \geq k$ the canonical map $\mathbb{G}(k') \to \mathbb{G}(k)$ induces an open embedding of rigid analytic spaces $\mathbb{G}(k')^\circ \hookrightarrow \mathbb{G}(k)^\circ$. In particular, $\mathbb{G}(k)^\circ$ is a rigid analytic subgroup of $\mathbb{G}^\circ$ for every $k \in \mathbb{Z}_{>0}$.*

Definition 6.4.19. *The strong continuous dual*

$$\mathcal{D}^{an}(\mathbb{G}(k)^\circ) := \mathcal{O}(\mathbb{G}(k)^\circ)'_b$$

is called the analytic distribution algebra of the rigid analytic group $\mathbb{G}(k)^\circ$.

By [55, Proposition 16.5], if $(B_n)'_b$ (the strong dual of the Banach algebra B_n) is endowed with the Banach topology ([55, Remark 6.7]) we have a canonical topological isomorphism

$$\left(\varprojlim_{n\in\mathbb{Z}_{>0}} B_n\right)'_b \xrightarrow{\simeq} \varinjlim_{n\in\mathbb{Z}_{>0}} (B_n)'_b.$$

This means that $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)$ is a locally convex space which is in fact a topological L-algebra of compact type (proposition 6.4.4). This reference also gives us a canonical ring morphism $\delta : \mathbb{G}(k)^\circ \hookrightarrow \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)^\times$, where $\delta_g := \delta(g)$ is a "Dirac delta function" supported on g. Furthermore, to each $\eta \in \mathrm{Lie}(\mathbb{G}(k)^\circ)$ we can associate a linear continuous form

$$f \mapsto \frac{d}{dt} f(\exp(t\eta))|_{t=0}$$

which induces a morphism of rings

$$\mathcal{U}(\mathrm{Lie}(\mathbb{G}(k)^\circ)) \to \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ). \tag{6.14}$$

We have the following result from [39, Proposition 5.2.1].

Proposition 6.4.20. *The application $\mathcal{U}(Lie(\mathbb{G}(k)^\circ)) \to \mathcal{D}^{an}(\mathbb{G}(k)^\circ)$ induces a topological isomorphism of rings*

$$D^\dagger(\mathbb{G}(k)) \xrightarrow{\simeq} \mathcal{D}^{an}(\mathbb{G}(k)^\circ).$$

Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character and $\chi_\lambda : Z(\mathfrak{g}_L) \to L$ the central character induced by λ via the Harish-Chandra isomorphism. We put

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda := \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)\big/(\mathrm{Ker}(\chi_\lambda))\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ).$$

So, if $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ denotes an algebraic character we have an isomorphism

$$D^\dagger(\mathbb{G}(k))_\lambda \xrightarrow{\simeq} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda. \tag{6.15}$$

By proposition 5.5.12 we have

Corollary 6.4.21. *Now, let $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho + \mathfrak{t}_L^*$ is a dominant and regular character of $\mathfrak{t}_L$. Then*

$$H^0\left(\mathfrak{X}, \mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda)\right) = \mathcal{D}^{an}(\mathbb{G}(k)^{\circ})_{\lambda}.$$

6.4.3 Locally analytic representations and coadmissible modules

Let us recall that G_0 denotes the compact locally L-analytic group $G_0 = \mathbb{G}(\mathfrak{o})$. In subsection 6.4.1 we have introduced the following notations. Let $\mathcal{C}^{\text{an}}(G_0, L)$ be the space of L-valued locally L-analytic functions on G_0 and $D(G_0, L)$ the continuous strong dual of $\mathcal{C}^{\text{an}}(G_0, L)$. Given that G_0 is compact, this space carries a structure of nuclear Fréchet-Stein algebra.

The following digression is an adapted version of the proof of [25, Proposition 5.3.1] to our work. First of all, let us recall that G_0 acts on the space $\mathcal{C}^{\text{cts}}(G_0, L)$, of continuous L-valued functions, by the formula

$$(g \bullet f)(x) := f(g^{-1}x) \quad (g, x \in G_0,\ f \in \mathcal{C}^{\text{cts}}(G_0, L)).$$

Let us consider the sequence $(\mathbb{G}(k)^{\circ})_{k\in\mathbb{Z}_{>0}}$ of σ-affinoid rigid analytic open subgroups of $\mathbb{G}^{\circ}$[5] (remark 6.4.18). We recall for the reader that this notion was introduced in [25, Definition 2.1.17]. It states that if $\mathbb{X}$ is a rigid analytic space over L, then $\mathbb{X}$ is a σ-affinoid if there exists an increasing sequence $(\mathbb{X}_n)_{n\in\mathbb{Z}_{>0}}$ of affinoid open subsets of $\mathbb{X}$ such that $\mathbb{X} = \cup_{n\in\mathbb{Z}_{>0}}\mathbb{X}_n$ is an admissible covering.
By definition, for each $k \in \mathbb{Z}_{>0}$ there are continuous injections

$$\mathcal{C}^{\text{cts}}(G_0, L)_{\mathbb{G}(k)^{\circ}-\text{an}} \hookrightarrow \mathcal{C}^{\text{cts}}(G_0, L)_{\mathbb{G}(k+1)^{\circ}-\text{an}} \tag{6.16}$$

and

$$\mathcal{C}^{\text{cts}}(G_0, L)_{\mathbb{G}(k)^{\circ}-\text{an}} \hookrightarrow \mathcal{C}^{\text{an}}(G_0, L) \tag{6.17}$$

The latter being compatible with the former. Passing to the inductive limit we get an isomorphism

$$\varinjlim_{k\in\mathbb{Z}_{>0}} \mathcal{C}^{\text{cts}}(G_0, L)_{\mathbb{G}(k)^{\circ}-\text{an}} \xrightarrow{\simeq} \mathcal{C}^{\text{an}}(G_0, L). \tag{6.18}$$

As in [25, Proposition 5.3.1], for each $k \in \mathbb{Z}_{>0}$ we put

$$D(\mathbb{G}(k)^{\circ}, G_0) := \left(\mathcal{C}^{\text{cts}}(G_0, L)_{\mathbb{G}(k)^{\circ}-\text{an}}\right)'_b .$$

Given that G_k is compact, we can use [25, Proposition 3.4.11] to conclude that the restriction map induces

$$\mathcal{C}^{\text{cts}}(G_0, L)_{\mathbb{G}(k)^{\circ}-\text{an}} \to \mathcal{C}^{\text{cts}}(G_k, L)_{\mathbb{G}(k)^{\circ}-\text{an}} \xrightarrow{\simeq} \mathcal{C}^{\text{an}}(\mathbb{G}(k)^{\circ}, L) = \mathcal{O}(\mathbb{G}(k)^{\circ}).$$

This yields a closed embedding $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ}) \hookrightarrow D(\mathbb{G}(k)^{\circ}, G_0)$. The ring structure on $\mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})$ extends naturally to a ring structure on $D(\mathbb{G}(k)^{\circ}, G_0)$, such that

$$D(\mathbb{G}(k)^{\circ}, G_0) = \bigoplus_{g\in G_0/G_k} \mathcal{D}^{\text{an}}(\mathbb{G}(k)^{\circ})\delta_g. \tag{6.19}$$

[5] In fact, strictly σ-affinoid, meaning that the transition maps $\mathbb{G}(k)^{\circ}_n \hookrightarrow \mathbb{G}(k)^{\circ}_{n+1}$ are relative compact in the sense of [13, Part I, chapter 6 section 6.3, definition 6].

Dualizing the isomorphism (6.18) yields an isomorphism of topological L-algebras

$$D(G_0, L) \xrightarrow{\simeq} \varprojlim_{k\in\mathbb{Z}_{>0}} D(\mathbb{G}(k)^\circ, G_0). \tag{6.20}$$

Let us show that this defines a weak Fréchet-Stein structure on $D(G_0, L)$. Each of the algebras $D(\mathbb{G}(k)^\circ, G_0)$ are of compact type [25, Proposition 3.4.11] and the transition morphisms $D(\mathbb{G}(k+1)^\circ, G_0) \to D(\mathbb{G}(k)^\circ, G_0)$ are compact [25, Proposition 2.1.16]. Finally, for each $k \in \mathbb{Z}_{>0}$ the map (6.18) is a continuous injection of reflexive spaces, therefore the dual map $D(G, L) \to D(\mathbb{G}(k)^\circ, G_0)$ has dense image. This proves that $D(G_0, L)$ is a weak Fréchet-Stein algebra.

Let $V \in \mathrm{Rep}^{\mathrm{adm}}(G_0)$ and $M := V'_b$. By [25, Lemma 6.1.6] the subspace $V_{\mathbb{G}(k)^\circ-\mathrm{an}} \subseteq V$ is a nuclear Fréchet space and therefore its strong dual $M_k := \left(V_{\mathbb{G}(k)^\circ-\mathrm{an}}\right)'_b$ is a space of compact type and a finitely generated topological $D(\mathbb{G}(k)^\circ, G_0)$-module by [25, Lemma 6.1.13]. By [25, Theorem 6.1.20] and the diagram (6.12) the module M is a coadmissible $D(G_0, L)$-module relative to the weak Fréchet -Stein structure of $D(G_0, L)$ defined in the previous paragraph.

We have the following result from [36, Lemma 5.1.7].

Lemma 6.4.22. *(i) The $D(\mathbb{G}(k)^\circ, G_0)$-module M_k is finitely generated.*

(ii) There are natural isomorphisms

$$D(\mathbb{G}(k-1)^\circ, G_0) \otimes_{D(\mathbb{G}(k)^\circ, G_0)} M_k \xrightarrow{\simeq} M_{k-1}.$$

(iii) The natural map $D(\mathbb{G}(k-1)^\circ, G_0) \otimes_{D(G_0,L)} M \to M_k$ is bijective.

Now, let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho + \mathfrak{t}_L^*$ is a dominant and regular character of $\mathfrak{t}_L$. Let us recall that we have an isomorphism

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \xrightarrow{\simeq} D^\dagger(\mathbb{G}(k))_\lambda \xrightarrow{\simeq} \varinjlim_{m\in\mathbb{N}} \left(\widehat{D}^{(m)}(\mathbb{G}(k))_\lambda\right) \otimes_{\mathfrak{o}} L.$$

The preceding relation and the fact that the ring structure of $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)$ extends naturally to a ring structure on $D(\mathbb{G}(k)^\circ, G_0)$ allow us to consider the ring

$$D(\mathbb{G}(k)^\circ, G_0)_\lambda := D(\mathbb{G}(k)^\circ, G_0)/\mathrm{Ker}(\chi_\lambda) D(\mathbb{G}(k)^\circ, G_0).$$

From now on, we will denote $\mathcal{C}_{G_0}$ the full subcategory of $\mathrm{Mod}(D(\mathbb{G}_0, L))$ consisting of coadmissible modules, with respect to the preceding weak Fréchet-Stein structure on $D(G_0, L)$.

Definition 6.4.23. *We define the category $\mathcal{C}_{G_0,\lambda}$ of coadmissible $D(G_0, L)$-modules with central character $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ by*

$$\mathcal{C}_{G_0,\lambda} := Mod\left(D(G_0, L)\big/Ker(\chi_\lambda) D(G_0, L)\right) \cap \mathcal{C}_{G_0}.$$

We point out that the preceding definition is completely legal because the center $Z(\mathfrak{g}_L)$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_L)$ lies in the center of $D(G_0, L)$ [54, Proposition 3.7]. We also recall that the group $G_k := \mathbb{G}(k)(\mathfrak{o})$ is contained in $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)$ as a set of Dirac distributions. For each $g \in G_k$ we will write δ_g for the image of the Dirac distribution supported at g in

$$H^0\left(\mathfrak{Y}, \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)\right) = \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda$$

(cf. corollaries 6.2.3 and (6.15)). Inspired in [36, Definition 5.2.7] we have the following definition.

Definition 6.4.24. *Let $H \subset G_0$ be an open subset and $\mathfrak{Y}$ an H-equivariant admissible blow-up of $\mathfrak{X}$. Let us suppose that $k \geq k_{\mathfrak{Y}}$ (notation as in 6.1.3). A strongly H-equivariant $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-module is a $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-module $\mathscr{M}$ together with a family $(\varphi_g)_{g\in H}$ of isomorphisms*

$$\varphi_g : \mathscr{M} \to (\rho_g)_*\mathscr{M}$$

of sheaves of L-vector spaces, satisfying the following conditions:

(i) For all $g, h \in H$ we have $\left(\rho_g\right)_* \left(\varphi_h\right) \circ \varphi_g = \varphi_{hg}$.

(ii) For all open subset $\mathcal{U} \subset \mathfrak{Y}$, all $P \in \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)(\mathcal{U})$, and all $m \in \mathscr{M}(\mathcal{U})$ we have $\varphi_g(P \bullet m) = T_g(P) \bullet \varphi_g(m)$.

(iii) [6] For all $g \in H \cap G_{k+1}$ the map $\varphi_g : \mathscr{M} \to \left(\rho_g\right)_* \mathscr{M} = \mathscr{M}$ is equal to multiplication by $\delta_g \in H^0\left(\mathfrak{Y}, \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)\right)$.

A morphism between two strongly H-equivariant $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules $(\mathscr{M}, (\varphi^{\mathscr{M}}_g)_{g\in H})$ and $(\mathscr{N}, (\varphi^{\mathscr{N}}_g)_{g\in H})$ is a $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$ linear morphism $\psi : \mathscr{M} \to \mathscr{N}$ such that for all $g \in H$, the following diagram is commutative

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\psi} & \mathscr{N} \\ \downarrow{\scriptstyle \varphi^{\mathscr{M}}_g} & & \downarrow{\scriptstyle \varphi^{\mathscr{N}}_g} \\ \left(\rho_g\right)_* \mathscr{M} & \xrightarrow{(\rho_g)_*(\psi)} & \left(\rho_g\right)_* \mathscr{N} \end{array} \qquad \varphi^{\mathscr{N}}_g \circ \psi = (\rho_g)_*(\psi) \circ \varphi^{\mathscr{M}}_g .$$

Commentary 1. *Let $\mathscr{M} \in Coh\left(\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda), G_0\right)$. In what follows we will use the notation $gm := \varphi_{g,\mathcal{U}}(m) \in \mathscr{M}(\mathcal{U}.g^{-1})$, for $\mathcal{U} \subseteq \mathfrak{Y}$ an open subset, $g \in G_0$ and $m \in \mathscr{M}(\mathcal{U})$. This notation is inspired in property (ii) of the previous definition. In fact, if $g, h \in G_0$, then by (ii) we have $h(g\ m) = (hg)\ m$.*

We denote the category of strongly H-equivariant coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules by $\mathrm{Coh}\left(\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda), G_0\right)$.

Theorem 6.4.25. *Let $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character such that $\lambda + \rho \in \mathfrak{t}^*_L$ is a dominant and regular character of $\mathfrak{t}_L$. Let $pr : \mathfrak{Y} \to \mathfrak{X}$ be a G_0-equivariant admissible blow-up, and let $k \geq k_{\mathfrak{Y}}$. The functors $\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)$ and $H^0(\mathfrak{Y}, \bullet)$ induce quasi-inverse equivalences between the category of finitely presented $D(\mathbb{G}(k)^{\circ}, G_0)_{\lambda}$-modules and $Coh\left(\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda), G_0\right)$*

Before starting the proof, we recall for the reader that the functor $\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)$ has been defined in the proof of theorem 6.2.4. An explicitly expression is given in (6.21) below.

Proof. If $\mathscr{M} \in \mathrm{Coh}\left(\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda), G_0\right)$, then in particular $\mathscr{M}$ is a coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-module. Since by theorem 6.2.4, corollary 6.2.3 and proposition 6.4.21 we have that $H^0(\mathfrak{Y}, \mathscr{M})$ is a finitely presented $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}$-module, then by (6.19) we can conclude that $H^0(\mathfrak{Y}, \mathscr{M})$ is a finitely presented $D(\mathbb{G}(k)^{\circ}, \mathbb{G}_0)_{\lambda}$-module.

On the other hand, let us suppose that M is a finitely presented $D(\mathbb{G}(k)^{\circ}, \mathbb{G}_0)_{\lambda}$-module. By (6.19) we can consider

$$\mathscr{M} := \mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(M) = \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M. \tag{6.21}$$

For every $g \in G_0$ we want to define an isomorphism of sheaves of L-vector spaces

$$\varphi_g : \mathscr{M} \to \left(\rho_g\right)_* \mathscr{M}$$

[6] This conditions makes sense because the elements $g \in G_{k+1}$ acts trivially on the underlying topological space of $\mathfrak{Y}$, cf. Lemma 6.3.3.

satisfying the conditions (i), (ii) and (iii) in the preceding definition. As we have remarked, the Dirac distributions induce an injective morphism from G_0 to the group of units of $D(G_0, L)$, since by (6.20) M is in particular a G_0-module, we have an isomorphism

$$\mathscr{M} \to \left(\left(\rho_g\right)_* \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)\right) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M,$$

which on local sections is defined by $\varphi_{g,\,\mathcal{U}}(P \otimes m) := T_{g,\,\mathcal{U}}(P) \otimes gm$. Here $P \in \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)(\mathcal{U})$, $\mathcal{U} \subseteq \mathfrak{Y}$ is an open subset, $m \in M$ and T_g is the isomorphism defined in (6.9).

One has an isomorphism

$$\left(\rho_g\right)_* (\mathscr{M}) \xrightarrow{\simeq} \left(\left(\rho_g\right)_* \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)\right) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M.$$

Indeed, $(\rho_g)_*$ is exact and so choosing a finite presentation of M as $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}$-module reduces to the case $M = \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}$ which is trivially true. This implies that the preceding isomorphism extends to an isomorphism

$$\varphi_g : \mathscr{M} \to \left(\rho_g\right)_* \mathscr{M}.$$

Let $g, h \in G_0$, $\mathcal{U} \subseteq \mathfrak{Y}$ an open subset, $P, Q \in \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)(\mathcal{U})$ and $m \in M$. Then

$$\begin{aligned}
\varphi_{h,\,\mathcal{U}.g^{-1}}\left(\varphi_{g,\,\mathcal{U}}\right)(P \otimes m) &= \varphi_{h,\,\mathcal{U}.g^{-1}}(T_{g,\,\mathcal{U}}(P) \otimes g\, m)\\
&= T_{h,\,\mathcal{U}.g^{-1}}(T_{g,\,\mathcal{U}}(P)) \otimes hg\, m\\
&= T_{hg,\,\mathcal{U}}(P) \otimes (hg)\, m\\
&= \varphi_{hg,\,\mathcal{U}}(P \otimes m),
\end{aligned}$$

which verifies the first condition. Now, by definition $T_{g,\,\mathcal{U}}(PQ) = T_{g,\,\mathcal{U}}(P)T_{g,\,\mathcal{U}}(Q)$ and therefore $\varphi_{g,\,\mathcal{U}}(PQ \otimes m) = T_{g,\,\mathcal{U}}(P)\varphi_{g,\,\mathcal{U}}(Q \otimes m)$, which gives (ii). Finally, given that the delta distributions δ_g for g in the normal subgroup G_{k+1} of G_0 are contained in $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})$ we have $g.P := T_g(P) = \delta_g\ P\ \delta_{g^{-1}}$, and therefore

$$\begin{aligned}
\varphi_{g,\,\mathcal{U}}(P \otimes m) &= g.P \otimes\ g.m\\
&= \delta_g P \delta_{g^{-1}} \delta_g \otimes\ m\\
&= \delta_g P \otimes\ m.
\end{aligned}$$

and condition (iii) follows. □

Remark 6.4.26. *If $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ denotes the trivial character, then $\mathscr{D}^{\dagger}_{\mathfrak{X},k}(\lambda) = \mathscr{D}^{\dagger}_{\mathfrak{X},k}$ is the sheaf of arithmetic differential operators introduced in [36]. Moreover, by construction, if $pr : \mathfrak{Y} \to \mathfrak{X}$ denotes an H-equivariant admissible blow-up, then $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) = \mathscr{D}^{\dagger}_{\mathfrak{Y},k}$ and for every $g \in H$ the isomorphism T_g equals the isomorphism $Ad(g)$ defined in [36, (5.2.6)].*

Now, let us take $\pi : \mathfrak{Y}' \to \mathfrak{Y}$ a morphism of G_0-equivariant admissible blow-ups of $\mathfrak{X}$ (whose lifted actions we denote by $\rho^{\mathfrak{Y}'}$ and $\rho^{\mathfrak{Y}}$), and let us suppose that $k \geq k_{\mathfrak{Y}}$ and $k' \geq \max\{k'_{\mathfrak{Y}}, k\}$. By (6.5) and theorem 6.2.2 we have an injective morphism of sheaves

$$\Psi : \pi_* \mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda) = \mathscr{D}^{\dagger}_{\mathfrak{Y},k'}(\lambda) \hookrightarrow \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda). \tag{6.22}$$

Moreover, if $g \in G_0$ we have the commutative diagram

$$\begin{array}{ccc} \pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda) = \mathscr{D}^\dagger_{\mathfrak{Y},k'}(\lambda) & \xhookrightarrow{\Psi} & \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \\ \downarrow{\scriptstyle \pi_* T_g^{\mathfrak{Y}'}} & & \downarrow{\scriptstyle T_g^{\mathfrak{Y}}} \\ \pi_*\left(\rho_g^{\mathfrak{Y}'}\right)_* \mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda) = \left(\rho_g^{\mathfrak{Y}}\right)_* \mathscr{D}^\dagger_{\mathfrak{Y},k'}(\lambda) & \xhookrightarrow{(\rho_g^{\mathfrak{Y}})_*(\Psi)} & \left(\rho_g^{\mathfrak{Y}}\right)_* \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \end{array}$$

which implies that $\Psi : \pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda) = \mathscr{D}^\dagger_{\mathfrak{Y},k'}(\lambda) \hookrightarrow \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$ is G_0-equivariant, i.e. it satisfies

$$T_g^{\mathfrak{Y}} \circ \Psi = \left(\rho_g^{\mathfrak{Y}}\right)_*(\Psi) \circ \pi_*\left(T_g^{\mathfrak{Y}'}\right).$$

Now, let us suppose given two modules $\mathscr{M}_{\mathfrak{Y}'} \in \mathrm{Coh}\left(\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda), G_0\right)$ and $\mathscr{M}_{\mathfrak{Y}} \in \mathrm{Coh}\left(\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda), G_0\right)$ together with a morphism

$$\psi : \pi_*\mathscr{M}_{\mathfrak{Y}'} \to \mathscr{M}_{\mathfrak{Y}}$$

linear relative to $\Psi : \pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda) \hookrightarrow \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$ and which is G_0-equivariant, i.e. satisfying

$$\begin{array}{ccc} \pi_*\mathscr{M}_{\mathfrak{Y}'} & \xrightarrow{\psi} & \mathscr{M}_{\mathfrak{Y}} \\ \downarrow{\scriptstyle \pi_*(\varphi_g^{\mathscr{M}_{\mathfrak{Y}'}})} & & \downarrow{\scriptstyle \varphi_g^{\mathscr{M}_{\mathfrak{Y}}}} \\ \pi_*(\rho_g^{\mathfrak{Y}'})_*\mathscr{M}_{\mathfrak{Y}} = (\rho_g^{\mathfrak{Y}})_*\pi_*\mathscr{M}_{\mathfrak{Y}'} & \xrightarrow{(\rho_g^{\mathfrak{Y}})_*\psi} & (\rho_g^{\mathfrak{Y}})_*\mathscr{M}_{\mathfrak{Y}}. \end{array} \qquad \varphi_g^{\mathscr{M}_{\mathfrak{Y}}} \circ \psi = \left(\rho_g^{\mathfrak{Y}}\right)_* \psi \circ \pi_*\left(\varphi_g^{\mathscr{M}_{\mathfrak{Y}'}}\right).$$

for all $g \in G_0$. By using Ψ we obtain a morphism of $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$-modules

$$\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda)} \pi_*\mathscr{M}_{\mathfrak{Y}'} \to \mathscr{M}_{\mathfrak{Y}}.$$

Let us denote by $\mathscr{K}$ the submodule of $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda)} \pi_*\mathscr{M}_{\mathfrak{Y}'}$ locally generated by all the elements of the form $P\delta_h \otimes m - P \otimes (h \bullet m)$, where $h \in G_{k+1}$, m is a local section of $\pi_*\mathscr{M}_{\mathfrak{Y}'}$ and P is a local section of $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$. As in [36, Page 35] we will denote the quotient $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda)} \pi_*\mathscr{M}_{\mathfrak{Y}'} \big/ \mathscr{K}$ by

$$\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda),\, G_{k+1}} \pi_*\mathscr{M}_{\mathfrak{Y}'}. \tag{6.23}$$

Let us see that this module lies in $\mathrm{Coh}\big(\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda), G_0\big)$. To do that let us first show that

$$\left(\rho_g\right)_* \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{(\rho_g)_*\pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda)} \left(\rho_g\right)_* \pi_*\mathscr{M}_{\mathfrak{Y}'} = \left(\rho_g\right)_* \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda)} \pi_*\mathscr{M}_{\mathfrak{Y}'}. \tag{6.24}$$

As $\mathscr{M}_{\mathfrak{Y}'}$ is a coherent $\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda)$ we can find a finite presentation of $\mathscr{M}_{\mathfrak{Y}'}$

$$\left(\mathscr{D}_{\mathfrak{Y}',k'}(\lambda)\right)^{\oplus a} \to (\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda))^{\oplus b} \to \mathscr{M}_{\mathfrak{Y}'} \to 0$$

which induces, by exactness of $(\rho_g)_*$ and π_*, the exact sequence

$$\left(\left(\rho_g\right)_* \mathscr{D}^{\dagger}_{\mathfrak{Y},k'}(\lambda)\right)^{\oplus a} \to \left(\left(\rho_g\right)_* \mathscr{D}^{\dagger}_{\mathfrak{Y},k'}(\lambda)\right)^{\oplus b} \to \left(\rho_g\right)_* \pi_* \mathscr{M}_{\mathfrak{Y}'} \to 0.$$

By base change over the preceding exact sequence we obtain the following commutative diagram

$$\begin{array}{ccccccc} \left(\left(\rho_g\right)_* \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)\right)^{\oplus a} & \longrightarrow & \left(\left(\rho_g\right)_* \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)\right)^{\oplus b} & \longrightarrow & \left(\rho_g\right)_* \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_* \mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda)} \pi_* \mathscr{M}_{\mathfrak{Y}'} & \longrightarrow & 0 \\ \downarrow id & & \downarrow id & & \downarrow & & \\ \left(\left(\rho_g\right)_* \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)\right)^{\oplus a} & \longrightarrow & \left(\left(\rho_g\right)_* \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)\right)^{\oplus b} & \longrightarrow & \left(\rho_g\right)_* \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{(\rho_g)_* \pi_* \mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda)} \left(\rho_g\right)_* \pi_* \mathscr{M}_{\mathfrak{Y}'} & \longrightarrow & 0 \end{array}$$

(of course, here we have used theorem 6.2.2 to identify $\pi_* \mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda) = \mathscr{D}^{\dagger}_{\mathfrak{Y},k'}(\lambda)$). This shows (6.24) and therefore the diagonal action

$$\varphi_g : \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{*\pi_* \mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda)} \pi_* \mathscr{M}_{\mathfrak{Y}'} \to \left(\rho_g\right)_* \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{*\pi_* \mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda)} \pi_* \mathscr{M}_{\mathfrak{Y}'}$$

defined on simple tensor products by

$$g \bullet (P \otimes m) := g \bullet P \otimes g \bullet m, \tag{6.25}$$

for $g \in G_0$, and P and m are local sections of $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$ and $\pi_* \mathscr{M}_{\mathfrak{Y}'}$, respectively (in order to soft the notation we use the accord introduced in the commentary 1 after the definition 6.4.24). Now to see that (6.23) is a strongly G_0-equivariant $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-module, we only need to check that $\phi_g(\mathscr{K}) \subset \mathscr{K}$. This is, the diagonal actions fix the submodule $\mathscr{K}$. We have

$$\begin{aligned} g \bullet (P\delta_h \otimes m - P \otimes h \bullet m) &= g \bullet (P\delta_h) \otimes g \bullet m - g \bullet P \otimes g \bullet (h \bullet m) \\ &= (g \bullet P)(g \bullet \delta_h) \otimes g \bullet m - g \bullet P \otimes (ghg^{-1}) \bullet (g \bullet m) \\ &= (g \bullet P)\delta_{ghg^{-1}} \otimes g \bullet m - g \bullet P \otimes (ghg^{-1}) \bullet (g \bullet m) \end{aligned}$$

and G_{k+1} is a normal subgroup we can conclude that $ghg^{-1} \in G_{k+1}$ and G_0 fix $\mathscr{K}$. Moreover, since the target of the preceding morphism is strongly G_0-equivariant, this factors through the quotient and we thus obtain a morphism of $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules

$$\overline{\psi} : \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_* \mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda),\, G_{k+1}} \pi_* \mathscr{M}_{\mathfrak{Y}'} \to \mathscr{M}_{\mathfrak{Y}}. \tag{6.26}$$

By construction $\overline{\psi} \in \mathrm{Coh}\left(\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda), G_0\right)$.

6.5 Admissible blow-ups and formal models

The following discussion is given in [36, 3.1.1 and 5.2.13]. Let us start by considering the generic fiber of the flag variety $X_L := X \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(L)$ (the flag *variety*). For the rest of this work $\mathbb{X}^{\mathrm{rig}}$ will denote the rigid-analytic space associated via the GAGA functor to X_L [13, Part I, chapter 5, section 5.4, Definition and proposition 3]. Any admissible formal $\mathfrak{o}$-scheme $\mathfrak{Y}$ (in the sense of [13, Part II, chapter 7, section 7.4, Definitions 1 and 4]) whose associated rigid-analytic space is isomorphic to $\mathbb{X}^{\mathrm{rig}}$ will be called a formal model of $\mathbb{X}^{\mathrm{rig}}$. For any two formal models $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ there exists a third formal model $\mathfrak{Y}'$ and admissible formal blow-up morphisms $\mathfrak{Y}' \to \mathfrak{Y}_1$ and $\mathfrak{Y}' \to \mathfrak{Y}_2$ [13, Part II, chapter 8, section 8.2, remark 10].

Now, let us denote by $\mathcal{F}_{\mathfrak{X}}$ the set of admissible formal blow-ups $\mathfrak{Y} \to \mathfrak{X}$. This set is ordered by $\mathfrak{Y}' \geq \mathfrak{Y}$ if the blow-up morphism $\mathfrak{Y}' \to \mathfrak{X}$ factors as the composition of a morphism $\mathfrak{Y}' \to \mathfrak{Y}$ and the blow-up morphism $\mathfrak{Y} \to \mathfrak{X}$. In this case, the morphism $\mathfrak{Y}' \to \mathfrak{Y}$ is unique [13, Part II, chapter 8, section 8.2, proposition 9], and it is itself a blow-up morphism [41, Chapter 8, section 8.1.3, proposition 1.12 (d) and theorem 1.24]. By [13, Part II, chapter 8, section 8.2, remark 10] the set $\mathcal{F}_{\mathfrak{X}}$ is directed and it is cofinal in the set of all formal models. Furthermore, any formal model $\mathfrak{Y}$ of $\mathbb{X}^{\mathrm{rig}}$ is dominated by one which is a G_0-equivariant admissible blow-up of $\mathfrak{X}$ [36, Proposition 5.2.14]. In particular, if $\mathfrak{X}_\infty$ denotes the projective limit of all formal models of $\mathbb{X}^{\mathrm{rig}}$, then

$$\mathfrak{X}_\infty = \varprojlim_{\mathcal{F}_{\mathfrak{X}}} \mathfrak{Y}.$$

We will be interested in the following directed subset of $\mathcal{F}_{\mathfrak{X}}$.

Definition 6.5.1. *We denote by $\underline{\mathcal{F}_{\mathfrak{X}}}$ the set of pairs $(\mathfrak{Y}, k)$, where $\mathfrak{Y} \in \mathcal{F}_{\mathfrak{X}}$ and $k \in \mathbb{N}$ satisfies $k \geq k_{\mathfrak{Y}}$. This set is ordered by $(\mathfrak{Y}', k') \geq (\mathfrak{Y}, k)$ if and only if $\mathfrak{Y} \geq \mathfrak{Y}$ and $k' \geq k$.*

We will need the following auxiliary result. We will follow word for word the reasoning given in [36, Lemma 5.2.12] when $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is equal to the trivial character. Our case is completely analogous.

Lemma 6.5.2. *Let $\mathfrak{Y}'$, $\mathfrak{Y} \in \mathcal{F}_{\mathfrak{X}}$ be G_0-equivariant admissible blow-ups (definition 6.3.2). Suppose $(\mathfrak{Y}', k') \geq (\mathfrak{Y}, k)$ with canonical morphism $\pi : \mathfrak{Y}' \to \mathfrak{Y}$ over $\mathfrak{X}$ and let M be a coherent $D(\mathbb{G}(k')^\circ, G_0)_\lambda$-module with localization $\mathscr{M} = \mathscr{L}oc^\dagger_{\mathfrak{Y}',k'}(\lambda)(M) \in Coh\big(\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda), G_0\big)$. Then there exists a canonical isomorphism in $Coh\big(\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda), G_0\big)$ given by*

$$\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda),\, G_{k+1}} \pi_*\mathscr{M} \xrightarrow{\simeq} \mathscr{L}oc^\dagger_{\mathfrak{Y},k}(\lambda)\left(D(\mathbb{G}(k)^\circ, G_0) \otimes_{D(\mathbb{G}(k')^\circ, G_0)} M\right).$$

Proof. Let Σ be a system of representatives in G_{k+1} for the cosets in $G_{k+1}/G_{k'+1}$. By (6.19) we have a canonical map

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \to D(\mathbb{G}(k)^\circ, G_0)_\lambda \tag{6.27}$$

which is compatible with variation in k. Now, let us take M a $D(\mathbb{G}(k')^\circ, G_0)_\lambda$-module and let us consider the free $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda$-module

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma} := \bigoplus_{(m,h)\in M\times\Sigma} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda\, e_{m,h},$$

whose formation is functorial in M. In fact, if M' is another $D(\mathbb{G}(k')^\circ, G_0)_\lambda$and if $f : M \to M'$ is a $D(\mathbb{G}(k')^\circ, G_0)_\lambda$-linear map, then

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma} & \to & \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M'\times\Sigma} \\ e_{m,h} & \mapsto & e_{f(m),h} \end{array}$$

induces a linear map between the corresponding free modules. Me make the convention $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{\{0\}\times\Sigma} = \{0\}$ is the trivial submodule of $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma}$. Moreover, the free module $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma}$ comes with a linear map

$$\begin{array}{cccc} f_M : & \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma} & \to & \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda} M \\ & \bigoplus_{(m,h)} \lambda_{m,h} e_{m,h} & \mapsto & (\lambda_{m,h}\delta_h) \otimes m - \lambda_{m,h} \otimes (\delta_h.m). \end{array}$$

Here M is considered as a $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda$-module via the canonical map (6.27). Let us note that, since M is a $D(\mathbb{G}(k')^\circ, G_0)_\lambda$-module, and because G_{k+1} is contained in $D(\mathbb{G}(k')^\circ, G_0)_\lambda$, the expression $\delta_h.m$ is defined for any $h \in G_{k+1}$. If $f : M \to$

M' is a linear map of $D(\mathbb{G}(k')^\circ, G_0)_\lambda$-modules, then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma} & \xrightarrow{f_M} & \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda} M \\ \downarrow & & \downarrow \\ \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M'\times\Sigma} & \xrightarrow{f_{M'}} & \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda} M' \end{array}$$

and we have a sequence of linear maps

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma} \xrightarrow{f_M} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda} M \xrightarrow{\mathrm{can}_M} D(\mathbb{G}(k)^\circ, G_0)_\lambda \otimes_{D(\mathbb{G}(k')^\circ, G_0)_\lambda} M \to 0.$$

The final map being surjective by (6.27).

Claim 1. If M is a finitely presented $D(\mathbb{G}(k'), G_0)_\lambda$-module, then the above sequence is exact.

Proof. Let us start by remarking that if $\lambda_{m,h}\, e_{m,h} \in \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma}$ then

$$\begin{aligned} \mathrm{can}_M(f_M(\lambda_{m,h}\, e_{m,h})) &= \mathrm{can}_M(\lambda_{m,h}\delta_h \otimes m - \lambda_{m,h} \otimes \delta_h m) \\ &= \lambda_{m,h}\delta_h \otimes m - \lambda_{m,h} \otimes \delta_h m \\ &= 0. \end{aligned}$$

For the last equality we have used the fact that $\delta_h \in D(\mathbb{G}(k'), G_0)_\lambda$. Let us show now that $\ker(\mathrm{can}_M) \subseteq \mathrm{im}(f_M)$. Let us take a finite presentation of the $D(\mathbb{G}(k'), G_0)_\lambda$-module M

$$M_a := \left(D(\mathbb{G}(k'), G_0)_\lambda\right)^{\oplus a} \xrightarrow{\alpha} M_b := \left(D(\mathbb{G}(k'), G_0)_\lambda\right)^{\oplus b} \xrightarrow{\beta} M \to 0$$

which, by functoriality, induces the following commutative diagram

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma} & \longrightarrow & \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda} M & \longrightarrow & D(\mathbb{G}(k)^\circ, G_0)_\lambda \otimes_{D(\mathbb{G}(k')^\circ, G_0)_\lambda} M & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M_a\times\Sigma} & \longrightarrow & \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda} M_a & \longrightarrow & D(\mathbb{G}(k)^\circ, G_0)_\lambda \otimes_{D(\mathbb{G}(k')^\circ, G_0)_\lambda} M_a & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda^{M_b\times\Sigma} & \longrightarrow & \mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k')^\circ)_\lambda} M_b & \longrightarrow & D(\mathbb{G}(k)^\circ, G_0)_\lambda \otimes_{D(\mathbb{G}(k')^\circ, G_0)_\lambda} M_b & \longrightarrow & 0 \end{array}$$

The 3×3-lemma reduces us to the case of a finitely presented module of the form $M_a := D(\mathbb{G}(k')^\circ, G_0)_\lambda^{\oplus a}$, and since we need to show that $\ker(\mathrm{can}_{M_a})$ lies in the submodule generated by the images of the elements $e_{m_i,h}$ for generators $m_1, ..., m_a \in M_a$ and $h \in \Sigma$ we can even suppose that $a = 1$. In this case the claim follows from (6.19). □

Claim 2. If M is a finitely presented $D(\mathbb{G}(k')^\circ, G_0)_\lambda$-module and we let $\mathscr{M} := \mathcal{L}oc^\dagger_{\mathfrak{Y}',k'}(\lambda)(M)$, then the natural morphism

$$\mathcal{L}oc^\dagger_{\mathfrak{Y},k}(\lambda)(D(\mathbb{G}(k)^\circ, G_0)_\lambda \otimes_{D(\mathbb{G}(k')^\circ, G_0)_\lambda} M) \to \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_* \mathscr{D}_{\mathfrak{Y}',k'}(\lambda)} \pi_* \mathscr{M}$$

is bijective.

Proof. By theorem 6.2.2 the functor π_* is an exact functor on coherent $\mathscr{D}^\dagger_{\mathfrak{Y}',k'}$-modules. Taking a finite presentation of M reduces to the case $M = D(\mathbb{G}(k')^\circ, G_0)_\lambda$ which is clear. □

Now, let M be a finitely presented $D(\mathbb{G}(k')^\circ, G_0)_\lambda$-module. Let $m_1, ..., m_a$ be generators for M as a $\mathcal{D}^{\rm an}(\mathbb{G}(k')^\circ)_\lambda$-module. We have a sequence of $\mathcal{D}^{\rm an}(\mathbb{G}(k)^\circ)_\lambda$-modules

$$\bigoplus_{(i,h)} \mathcal{D}^{\rm an}(\mathbb{G}(k)^\circ)_\lambda\, e_{m_i,h} \xrightarrow{f_a} \mathcal{D}^{\rm an}(\mathbb{G}(k)^\circ)_\lambda \otimes_{\mathcal{D}^{\rm an}(\mathbb{G}(k')^\circ)_\lambda} M \xrightarrow{\mathrm{can}_M} D(\mathbb{G}(k)^\circ, G_0)_\lambda \otimes_{D(\mathbb{G}(k')^\circ, G_0)_\lambda} M \to 0$$

where f_a denotes the restriction of the map f_M to the free submodule of $\mathcal{D}^{\rm an}(\mathbb{G}(k)^\circ)_\lambda^{M\times\Sigma}$ generated by the finitely many vectors $e_{m_i,h}$, with $1 \le i \le a$ and $h \in \Sigma$. Since $\mathrm{im}(f_a) = \mathrm{im}(f_M)$ the sequence is exact by the first claim. Since it consists of finitely presented $\mathcal{D}^{\rm an}(\mathbb{G}(k)^\circ)_\lambda$-modules, we can apply the localisation functor $\mathcal{L}oc^\dagger_{\mathfrak{Y},k}(\lambda)$ to it. As

$$\mathcal{L}oc^\dagger_{\mathfrak{Y},k}(\lambda)\left(\bigoplus_{(i,h)} \mathcal{D}^{\rm an}(\mathbb{G}(k)^\circ)_\lambda\, e_{m_i,h}\right) = \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\mathcal{D}^{\rm an}(\mathbb{G}(k)^\circ)_\lambda} \bigoplus_{(i,h)} \mathcal{D}^{\rm an}(\mathbb{G}(k)^\circ)_\lambda\, e_{m_i,h} = \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)^{\oplus a|\Sigma|}$$

the second claims gives us the exact sequence

$$\begin{array}{ccccccc} \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)^{\oplus a|\Sigma|} & \to & \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_*\mathscr{D}_{\mathfrak{Y}',k'}(\lambda)} \pi_*\mathscr{M} & \to & \mathscr{L}oc^\dagger_{\mathfrak{Y},k}(\lambda)\left(D(\mathbb{G}(k)^\circ, G_0)_\lambda \otimes_{D(\mathbb{G}(k')^\circ, G_0)_\lambda} M\right) & \to & 0 \\ e_{m_i,h} \otimes P & \mapsto & (P\delta_h \otimes m_i - P \otimes \delta_h m) & & & & \end{array}$$

where $\mathscr{M} := \mathcal{L}oc^\dagger_{\mathfrak{Y}',k'}(\lambda)(M)$. The cokernel of the first map in this sequence equals by definition

$$\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda),\, G_{k+1}} \pi_*\mathscr{M},$$

whence an isomorphism

$$\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_*\mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda),\, G_{k+1}} \pi_*\mathscr{M} \simeq \mathscr{L}oc^\dagger_{\mathfrak{Y},k}(\lambda)\left(D(\mathbb{G}(k)^\circ, G_0) \otimes_{D(\mathbb{G}(k')^\circ, G_0)} M\right).$$

□

Now, let $\mathscr{I}$ be an open ideal sheaf on $\mathfrak{X}$, and let $g \in G_0$. Then $\mathscr{J} := (\rho_g^\natural)^{-1}((\rho_g)_*(\mathscr{I}))$ is again an open ideal sheaf on $\mathfrak{X}$. Let $\mathfrak{Y}$ be the blow-up of $\mathscr{I}$ and $\mathfrak{Y}.g$ the blow-up of $\mathscr{J}$, with canonical morphism $\mathrm{pr}_g : \mathfrak{Y}.g \to \mathfrak{X}$. We have the following result from [36, lemma 5.2.16].

Lemma 6.5.3. *There exists a morphism $\rho_g : \mathfrak{Y} \to \mathfrak{Y}.g$ such that the following diagram is commutative*

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\rho_g} & \mathfrak{Y}.g \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr_g} \\ \mathfrak{X} & \xrightarrow{\rho_g} & \mathfrak{X}. \end{array}$$

Moreover, we have $k_{\mathfrak{Y}.g} = k_{\mathfrak{Y}}$ and for any two elements $g, h \in G_0$, we have a canonical isomorphism $(\mathfrak{Y}.g).h \simeq \mathfrak{Y}.(gh)$, and the morphism $\mathfrak{Y} \to \mathfrak{Y}.g \to (\mathfrak{Y}.g).h \simeq \mathfrak{Y}.(gh)$ is equal to ρ_{gh}. This gives a right action of the group G_0 on the family $\mathcal{F}_{\mathfrak{X}}$.

Let $\mathrm{pr} : \mathfrak{Y} \to \mathfrak{X}$ be an admissible blow-up and let us denote by $\underline{\mathscr{L}(\lambda)}$ the invertible sheaf on $\mathfrak{Y}$ induced by pulling back the invertible sheaf on $\mathfrak{X}$ induced by the character λ. This is $\underline{\mathscr{L}(\lambda)} := pr^*\mathscr{L}(\lambda)$. Furthermore, for $g \in \mathbb{G}_0$ if $\rho_g : \mathfrak{Y} \to \mathfrak{Y}.g$

is the morphism given by the previous lemma and $pr._g : \mathfrak{Y}.g \to \mathfrak{X}$ is the blow-up morphism, then we will denote

$$\underline{\mathscr{L}_g(\lambda)} := pr_g^* \mathscr{L}(\lambda).$$

The notation being fixed, we prevent the reader that in order to simplify the notation, in the rest of this work we will avoid to underline these sheaves if the context is clear and there is not risk to any confusion.

Let us recall that in section 5.6 we have built for any $g \in G_0$ an $\mathcal{O}_{\mathfrak{X}}$-linear isomorphism $\Phi_g : \mathscr{L}(\lambda) \to (\rho_g)_* \mathscr{L}(\lambda)$, being $\rho_g := \alpha \circ (id_{\mathfrak{X}} \times g)$ the translation morphism (α the right $\mathfrak{G}$-action on $\mathfrak{X}$). By pulling back this morphism and using the commutative diagram in the previous lemma ($\rho_g^* \circ pr_g^* = pr^* \circ \rho_g^*$) we an $\mathcal{O}_{\mathfrak{Y}}$-linear isomorphism $(\rho_g)^* pr_g^* \mathscr{L}(\lambda) \to pr^* \mathscr{L}(\lambda)$. By adjointness and following the accord established in the previous paragraph, we get an $\mathcal{O}_{\mathfrak{Y}.g}$-liner morphism

$$R_g : \mathscr{L}_g(\lambda) \to (\rho_g)_* \mathscr{L}(\lambda).$$

By construction R_g satisfies the cocycle condition (6.8). This means that for every $g, h \in G_0$ we have

$$R_{hg} = \mathscr{L}_{hg}(\lambda) \xrightarrow{R_g} (\rho_g)_* \mathscr{L}_h(\lambda) \xrightarrow{(\rho_g)_* R_h} (\rho_{hg})_* \mathscr{L}(\lambda). \tag{6.28}$$

In particular R_g is an isomorphism for every $g \in G_0$.

Exactly as we have done in (6.9), and given that by construction $\mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda)$ acts on $\mathscr{L}(\lambda)$ (resp. $\mathscr{D}^\dagger_{\mathfrak{Y}.g,k}(\lambda)$ acts on $\mathscr{L}_g(\lambda)$), we can build an isomorphism

$$\begin{array}{cccc} T_g : & \mathscr{D}^\dagger_{\mathfrak{Y}.g,k}(\lambda) & \to & (\rho_g)_* \mathscr{D}^\dagger_{\mathfrak{Y},k}(\lambda) \\ & P & \mapsto & R_g \circ P \circ R_g^{-1}. \end{array}$$

Locally, if $\mathcal{U} \subset \mathfrak{Y}.g$ is an open subset and $P \in \mathscr{D}^\dagger_{\mathfrak{Y}.g,k}(\lambda)(\mathcal{U})$ then $T_{g\,\mathcal{U}}(P)$ is defined by the following diagram

$$\begin{array}{ccc} \mathscr{L}(\lambda)(\mathcal{U}.g^{-1}) & \overset{T_{g\,\mathcal{U}}(P)}{\dashrightarrow} & \mathscr{L}(\lambda)(\mathcal{U}.g^{-1}) \\ \downarrow R^{-1}_{g,\mathcal{U}} & & \uparrow R_{g,\mathcal{U}} \\ \mathscr{L}_g(\mathcal{U}) & \xrightarrow{\quad P \quad} & \mathscr{L}_g(\mathcal{U}) \end{array}$$

Exactly as we have done in (5.36) we get the following cocycle condition

$$T_{hg} = (\rho_g)_* T_h \circ T_g \qquad (g, h \in G_0). \tag{6.29}$$

From the previous lemma we get [36, Corollary 5.2.18]

Corollary 6.5.4. *Let us suppose that $(\mathfrak{Y}', k') \geq (\mathfrak{Y}, k)$ for $\mathfrak{Y}$, $\mathfrak{Y}' \in \mathcal{F}_{\mathfrak{X}}$ and let $\pi : \mathfrak{Y}' \to \mathfrak{Y}$ be the unique morphism over $\mathfrak{X}$. Let $g \in G_0$. Then $(\mathfrak{Y}'.g, k') \geq (\mathfrak{Y}.g, k)$ and if we denote by $\pi.g : \mathfrak{Y}'.g \to \mathfrak{Y}.g$ the unique morphism over $\mathfrak{X}$, we have a commutative diagram*

$$\begin{array}{ccc} \mathfrak{Y}' & \xrightarrow{\rho_g} & \mathfrak{Y}'.g \\ \downarrow \pi & & \downarrow \pi.g \\ \mathfrak{Y} & \xrightarrow{\rho_g} & \mathfrak{Y}.g. \end{array}$$

Based on [36, Definition 5.2.19] we have the following definition.

Definition 6.5.5. *A coadmissible G_0-equivariant $\mathscr{D}(\lambda)$-module on $\mathcal{F}_{\mathfrak{X}}$ consists of a family $\mathscr{M} := \left(\mathscr{M}_{\mathfrak{Y},k}\right)_{(\mathfrak{Y},k)}$ of coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules $\mathscr{M}_{\mathfrak{Y},k}$ for all $(\mathfrak{Y},k) \in \underline{\mathcal{F}_{\mathfrak{X}}}$, with the following properties:*

(a) For any $g \in G_0$ with morphism $\rho_g : \mathfrak{Y} \to \mathfrak{Y}.g$, there exists an isomorphism

$$\varphi_g : \mathscr{M}_{\mathfrak{Y}.g,k} \to \left(\rho_g\right)_* \mathscr{M}_{\mathfrak{Y},k}$$

of sheaves of L-vector spaces, satisfying the following properties:

(i) For all $g, h \in G_0$ we have $(\rho_g)_*(\varphi_h) \circ \varphi_g = \varphi_{hg}$.

(ii) For all open subset $\mathcal{U} \subseteq \mathfrak{Y}.g$, all $P \in \mathscr{D}^{\dagger}_{\mathfrak{Y}.g,k}(\lambda)(\mathcal{U})$, and all $m \in \mathscr{M}_{\mathfrak{Y}.g,k}(\mathcal{U})$ one has $\varphi_g(P \bullet m) = T_{g,\mathcal{U}}(P) \bullet \varphi_{g,\mathcal{U}}(m)$.

(iii) [7] For all $g \in G_{k+1}$ the map $\varphi_g : \mathscr{M}_{\mathfrak{Y}.g,k} = \mathscr{M}_{\mathfrak{Y},k} \to (\rho_g)_*\mathscr{M}_{\mathfrak{Y},k} = \mathscr{M}_{\mathfrak{Y},k}$ is equal to multiplication by $\delta_g \in H^0\big(\mathfrak{Y}, \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)\big)$.

(b) Suppose $\mathfrak{Y}, \mathfrak{Y}' \in \mathcal{F}_{\mathfrak{X}}$ are both G_0-equivariant, and assume further that $(\mathfrak{Y}',k') \geq (\mathfrak{Y},k)$, and that $\pi : \mathfrak{Y}' \to \mathfrak{Y}$ is the unique morphism over $\mathfrak{X}$. We require the existence of a transition morphism $\psi_{\mathfrak{Y}',\mathfrak{Y}} : \pi_*\mathscr{M}_{\mathfrak{Y}',k'} \to \mathscr{M}_{\mathfrak{Y},k}$, linear relative to the canonical morphism $\Psi : \pi_*\mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda) \to \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$. By using the commutative diagram in the preceding corollary, we required

$$\begin{array}{ccc} (\pi.g)_*(\rho_g)_*\mathscr{M}_{\mathfrak{Y}',k'} = (\rho_g)_*\pi_*\mathscr{M}_{\mathfrak{Y}',k'} & \xrightarrow{(\rho_g)_*(\psi_{\mathfrak{Y}',\mathfrak{Y}})} & (\rho_g)_*\mathscr{M}_{\mathfrak{Y},k} \\ {\scriptstyle (\pi.g)_*\varphi_g}\uparrow & & \uparrow{\scriptstyle \varphi_g} \\ (\pi.g)_*\mathscr{M}_{\mathfrak{Y}'.g,k'} & \xrightarrow{\psi_{\mathfrak{Y}'.g,\mathfrak{Y}.g}} & \mathscr{M}_{\mathfrak{Y}.g,k} \end{array} \qquad \varphi_g \circ \psi_{\mathfrak{Y}'.g,\mathfrak{Y}.g} = (\rho_g)_*(\psi_{\mathfrak{Y}',\mathfrak{Y}}) \circ (\pi.g)_*(\varphi_g).$$

The morphism induced by $\psi_{\mathfrak{Y}',\mathfrak{Y}}$

$$\overline{\psi}_{\mathfrak{Y}',\mathfrak{Y}} : \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{\pi_*\mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda),\, G_{k+1}} \pi_*\mathscr{M}_{\mathfrak{Y}'} \to \mathscr{M}_{\mathfrak{Y}} \tag{6.30}$$

is required to be an isomorphism of $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules. Additionally, the morphisms $\psi_{\mathfrak{Y}',\mathfrak{Y}}$ are required to satisfy the transitivity condition $\psi_{\mathfrak{Y}',\mathfrak{Y}} \circ \pi_*(\psi_{\mathfrak{Y}'',\mathfrak{Y}'}) = \psi_{\mathfrak{Y}'',\mathfrak{Y}}$ for $(\mathfrak{Y}'',k'') \geq (\mathfrak{Y}',k') \geq (\mathfrak{Y},k)$ in $\underline{\mathcal{F}_{\mathfrak{X}}}$. Moreover, $\psi_{\mathfrak{Y},\mathfrak{Y}} = id_{\mathscr{M}_{\mathfrak{Y},k}}$.

A morphism $\mathscr{M} \to \mathscr{N}$ between such modules consists of morphisms $\mathscr{M}_{\mathfrak{Y},k} \to \mathscr{N}_{\mathfrak{Y},k}$ of $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-modules, making commutative the following diagrams

$$\begin{array}{ccc} \mathscr{M}_{\mathfrak{Y}.g,k} & \longrightarrow & (\rho_g)_*\mathscr{M}_{\mathfrak{Y},k} \\ \downarrow & & \downarrow \\ \mathscr{N}_{\mathfrak{Y}.g,k} & \longrightarrow & (\rho_g)_*\mathscr{N}_{\mathfrak{Y},k} \end{array} \qquad \begin{array}{ccc} \pi_*\mathscr{M}_{\mathfrak{Y}',k'} & \longrightarrow & \mathscr{M}_{\mathfrak{Y},k} \\ \downarrow & & \downarrow \\ \pi_*\mathscr{N}_{\mathfrak{Y}',k'} & \longrightarrow & \mathscr{N}_{\mathfrak{Y},k}. \end{array}$$

We denote the resulting category by $\mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$.

Let us build now the bridge to the category $\mathcal{C}_{G_0,\lambda}$ of coadmissible $D(G_0,L)_\lambda$-modules. Given such a module M we have its associated admissible locally analytic G_0-representation $V := M'_b$ together with its subspace of $\mathbb{G}(k)^\circ$-analytic vectors $V_{\mathbb{G}(k)^\circ-\mathrm{an}} \subseteq V$. As we have remarked, this is stable under the G_0-action and its dual $M_k := \left(V_{\mathbb{G}(k)^\circ-\mathrm{an}}\right)'$ is a finitely presented $D(\mathbb{G}(k)^\circ, G_0)_\lambda$-module. In this situation we produce a coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$-module

$$\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(M_k) = \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_\lambda} M_k$$

[7] As is remarked in [36, Definition 5.2.19 (iii)], if $g \in G_{k+1}$, then $\mathfrak{Y}.g = \mathfrak{Y}$ and g acts trivially on the underlying topological space $|\mathfrak{Y}|$.

for any element $(\mathfrak{Y}, k) \in \underline{\mathcal{F}}_{\mathfrak{X}}$. We will denote the resulting family by

$$\mathscr{L}oc_{\lambda}^{G_0}(M) := \left(\mathscr{L}oc_{\mathfrak{Y},k}^{\dagger}(\lambda)(M_k)\right)_{(\mathfrak{Y},k)\in\underline{\mathcal{F}}_{\mathfrak{X}}}.$$

On the other hand, let $\mathscr{M}$ be an arbitrary coadmissible G_0-equivariant arithmetic $\mathscr{D}(\lambda)$-module on $\mathcal{F}_{\mathfrak{X}}$. The restriction morphisms $\psi_{\mathfrak{Y}',\mathfrak{Y}} : \pi_* \mathscr{M}_{\mathfrak{Y}',k'} \to \mathscr{M}_{\mathfrak{Y},k}$ induce maps $H^0\left(\mathfrak{Y}', \mathscr{M}_{\mathfrak{Y}',k'}\right) \to H^0\left(\mathfrak{Y}, \mathscr{M}_{\mathfrak{Y},k}\right)$ on global sections. We let

$$\Gamma(\mathscr{M}) := \varprojlim_{(\mathfrak{Y},k)\in\underline{\mathcal{F}}_{\mathfrak{X}}} H^0\left(\mathfrak{Y}, \mathscr{M}_{\mathfrak{Y},k}\right).$$

The projective limit is taken in the sense of abelian groups. We have the following theorem. Except for some technical details the proof follows word for word the reasoning given in [36, Theorem 5.2.23].

Theorem 6.5.6. *Let us suppose that $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}_L^*$ is a dominant and regular character of $\mathfrak{t}_L$. The functors $\mathscr{L}oc_{\lambda}^{G_0}$ and $\Gamma(\bullet)$ induce quasi-inverse equivalences between the categories $\mathcal{C}_{G_0,\lambda}$ (of coadmissible $D(G_0, L)_{\lambda}$-modules) and $\mathcal{C}_{\mathfrak{X},\lambda}^{G_0}$.*

Proof. Let us take $M \in \mathcal{C}_{G_0,\lambda}$ and $\mathscr{M} \in \mathcal{C}_{\mathfrak{X},\lambda}^{G_0}$. As in [36, Proof of theorem 5.2.23] we will organise the proof in four steps.

Step 1. We have $\mathscr{L}oc_{\lambda}^{G_0}(M) \in \mathcal{C}_{\mathfrak{X},\lambda}^{G_0}$ and $\mathscr{L}oc_{\lambda}^{G_0}(M)$ is functorial in M.

Proof. Let us start by defining

$$\varphi_g : \mathscr{L}oc_{\mathfrak{Y}.g,k}^{\dagger}(\lambda)(M_k) \to \left(\rho_g\right)_* \mathscr{L}oc_{\mathfrak{Y},k}^{\dagger}(\lambda)(M_k) \quad (g \in G_0)$$

satisfying (i), (ii) and (iii) in the preceding definition. We recall for the reader that

$$\mathscr{L}oc_{\mathfrak{Y}.g,k}^{\dagger}(\lambda)(M_k) = \mathscr{D}_{\mathfrak{Y}.g,k}^{\dagger}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M_k.$$

Let $\tilde{\varphi}_g : M_k \to M_k$ denote the map dual to the map $V_{\mathbb{G}(k)^{\circ}-\mathrm{an}} \to V_{\mathbb{G}(k)^{\circ}-\mathrm{an}}$ given by $w \mapsto g^{-1}w$. By definition $\tilde{\varphi}_h \circ \tilde{\varphi}_g = \tilde{\varphi}_{hg}$. Let $\mathcal{U} \subseteq \mathfrak{Y}.g$ be an open subset and $P \in \mathscr{D}_{\mathfrak{Y}.g,k}^{\dagger}(\lambda)(\mathcal{U})$, $m \in M_k$. We define

$$\varphi_{g,\mathcal{U}}(P \otimes m) := T_{g,\mathcal{U}}(P) \otimes \tilde{\varphi}_g(m).$$

Given that $(\rho_g)_*$ is exact we can choose a finite presentation of M_k as a $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}$-module to conclude that we have a canonical isomorphism

$$\left(\rho_g\right)_* \left(\mathscr{L}oc_{\mathfrak{Y},k}^{\dagger}(\lambda)(M_k)\right) \xrightarrow{\simeq} \left(\left(\rho_g\right)_* \mathscr{D}_{\mathfrak{Y},k}^{\dagger}(\lambda)\right) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M_k.$$

This means that the above definition extends to a map

$$\varphi_g : \mathscr{D}_{\mathfrak{Y}.g,k}^{\dagger}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M_k \to \left(\rho_g\right)_* \left(\mathscr{L}oc_{\mathfrak{Y},k,\lambda}^{\dagger}(M_k)\right).$$

For the first condition we need to show that the following diagram is commutative

$$\begin{array}{ccc} \mathscr{D}_{\mathfrak{Y}.(hg),k}^{\dagger}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M_k & \xrightarrow{\varphi_g} & (\rho_g)_* \left(\mathscr{D}_{\mathfrak{Y}.h,k}^{\dagger}(\lambda)\right) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M_k \\ & \searrow^{\varphi_{hg}} & \downarrow{\scriptstyle (\rho_g)_*\varphi_h} \\ & & (\rho_g)_*(\rho_h)_* \left(\mathscr{D}_{\mathfrak{Y},k}^{\dagger}(\lambda)\right) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M_k. \end{array}$$

Let $\mathcal{U} \subseteq \mathfrak{Y}.(hg)$ be an open subset, $P, Q \in \mathscr{D}^{\dagger}_{\mathfrak{Y}.(hg),k}(\lambda)(\mathcal{U})$ and $m \in M_k$. We have

$$\begin{aligned}\varphi_{h,\mathcal{U}.g^{-1}}(\varphi_{g,\mathcal{U}}(P\otimes m)) &= \varphi_{h,\mathcal{U}.g^{-1}}(T_{g,\mathcal{U}}(P)\otimes\tilde{\varphi}_g(m))\\ &= T_{h,\mathcal{U}.g^{-1}}(T_{g,\mathcal{U}}(P))\otimes\tilde{\varphi}_h(\tilde{\varphi}_g(m))\\ &= T_{hg,\mathcal{U}}(P)\otimes\tilde{\varphi}_{hg}(m)\\ &= \varphi_{hg,\mathcal{U}}(P\otimes m).\end{aligned}$$

Which implies that the diagram is commutative, and therefore the condition (i) is satisfied. Second condition follows from

$$\varphi_g(Q\bullet P\otimes m) = T_{g,\mathcal{U}}(QP)\otimes\tilde{\varphi}_g(m) = T_{g,\mathcal{U}}(Q)T_{g,\mathcal{U}}(P)\otimes\tilde{\varphi}_g(m) = T_{g,\mathcal{U}}(Q)\varphi_{g,\mathcal{U}}(P\otimes m).$$

Finally, condition (iii) follows from the fact that if $g \in G_{k+1}$ then $\tilde{\varphi}_g(m) = \delta_g m$. Let us verify condition (b). We suppose that $\mathfrak{Y}'$, $\mathfrak{Y}$ are G_0-equivariant and that $(\mathfrak{Y}', k) \succeq (\mathfrak{Y}, k)$ with canonical morphism $\pi : \mathfrak{Y}' \to \mathfrak{Y}$ over $\mathfrak{X}$. As π_* is exact we have an isomorphism

$$\pi_*\left(\mathscr{L}oc^{\dagger}_{\mathfrak{Y}',k'}(\lambda)(M_{k'})\right)\xrightarrow{\simeq}\pi_*\left(\mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda)\right)\otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})_{\lambda}} M_{k'}.$$

(This is an argument already given in the text for the functor $(\rho_g)_*$). On the other hand, we have remarked that $\mathbb{G}(k')^{\circ} \subseteq \mathbb{G}(k)^{\circ}$ and we have a map $\tilde{\psi}_{\mathfrak{Y}',\mathfrak{Y}} : M_{k'} \to M_k$ obtained as the dual map of the natural inclusion $V_{\mathbb{G}(k)^{\circ}-\mathrm{an}} \hookrightarrow V_{\mathbb{G}(k')^{\circ}-\mathrm{an}}$. Let $\mathcal{U} \subseteq \mathfrak{Y}$ be an open subset and $P \in \pi_*\mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda)(\mathcal{U})$, $m \in M_{k'}$. We define

$$\psi_{\mathfrak{Y}',\mathfrak{Y}}(P\otimes m) := \Psi_{\mathfrak{Y}',\mathfrak{Y}}(P)\otimes\tilde{\psi}_{\mathfrak{Y}',\mathfrak{Y}}(m),$$

where Ψ is the canonical injection $\pi_*\mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda) \hookrightarrow \mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$. By using the preceding isomorphism we can conclude that this morphisms extends naturally to a map

$$\psi_{\mathfrak{Y}',\mathfrak{Y}} : \pi_*\left(\mathscr{L}oc^{\dagger}_{\mathfrak{Y}',k'}(\lambda)(M_{k'})\right) \to \mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(M_k).$$

The cocycle condition translates into the diagram

$$\begin{array}{ccc}\left(\rho_g^{\mathfrak{Y}}\right)_*\pi_*\left(\mathscr{L}oc^{\dagger}_{\mathfrak{Y}',k'}(\lambda)(M_{k'})\right) = (\pi.g)_*\left(\rho_g^{\mathfrak{Y}}\right)_*\left(\mathscr{L}oc^{\dagger}_{\mathfrak{Y}',k'}(\lambda)(M_{k'})\right) & \xrightarrow{\left(\rho_g^{\mathfrak{Y}}\right)_*\psi_{\mathfrak{Y}',\mathfrak{Y}}} & \left(\rho_g^{\mathfrak{Y}}\right)_*\left(\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(M_k)\right)\\ \uparrow{\scriptstyle (\pi.g)_*\varphi_g} & & \uparrow{\scriptstyle \varphi_g}\\ (\pi.g)_*\left(\mathscr{L}oc^{\dagger}_{\mathfrak{Y}'.g,k'}(\lambda(M_{k'})\right) & \xrightarrow{\psi_{\mathfrak{Y}',\mathfrak{Y}}} & \mathscr{L}oc^{\dagger}_{\mathfrak{Y}.g,k}(\lambda)(M_k)\end{array} \tag{6.31}$$

By construction, the diagrams

$$\begin{array}{ccc}(\pi.g)_*\left(\rho_g^{\mathfrak{Y}'}\right)_*\mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda) = \left(\rho_g^{\mathfrak{Y}}\right)_*\pi_*\mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda) & \xrightarrow{(\rho_g^{\mathfrak{Y}})_*\Psi_{\mathfrak{Y}',\mathfrak{Y}}} & \left(\rho_g^{\mathfrak{Y}}\right)_*\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)\\ \uparrow{\scriptstyle (\pi.g)_*T_g} & & \uparrow{\scriptstyle T_g}\\ (\pi_g)_*\mathscr{D}^{\dagger}_{\mathfrak{Y}'.g,k'}(\lambda) & \xrightarrow{\Psi_{\mathfrak{Y}'.g,\mathfrak{Y}.g}} & \mathscr{D}^{\dagger}_{\mathfrak{Y}.g,k}(\lambda)\end{array} \qquad \begin{array}{ccc}M_{k'} & \xrightarrow{\tilde{\psi}_{\mathfrak{Y}',\mathfrak{Y}}} & M_k\\ \downarrow{\scriptstyle \tilde{\varphi}_g} & & \downarrow{\scriptstyle \tilde{\varphi}_g}\\ M_{k'} & \xrightarrow{\tilde{\psi}_{\mathfrak{Y}',\mathfrak{Y}}} & M_k\end{array} \tag{6.32}$$

are commutative and therefore (6.31) is also a commutative diagram. The transitivity properties are clear. Let us see that the induced morphism $\overline{\psi}_{\mathfrak{Y}',\mathfrak{Y}}$ is in fact an isomorphism. As in [36, Page 42] the morphism $\overline{\psi}_{\mathfrak{Y}',\mathfrak{Y}}$ corresponds under the isomorphism of lemma 6.5.2 to the linear extension

$$D(\mathbb{G}(k)^\circ, G_0) \otimes_{D(\mathbb{G}(k')^\circ, G_0)} M_{k'} \to M_k$$

of $\tilde{\psi}_{\mathfrak{Y}',\mathfrak{Y}}$ via functoriality of $\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k,\lambda}$. By lemma 6.4.22 this linear extension is an isomorphism and hence, so is $\overline{\psi}_{\mathfrak{Y}',\mathfrak{Y}}$. We conclude that $\mathscr{L}oc^{G_0}_{\lambda}(M) \in \mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$. Given a morphism $M \to N$ in $\mathcal{C}_{G_0,\lambda}$, we get, by definition, morphisms $M_k \to N_k$ for any $k \in \mathbb{Z}_{>0}$ compatible with $\tilde{\varphi}_g$ and $\tilde{\psi}_{\mathfrak{Y}',\mathfrak{Y}}$. By functoriality of $\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)$, they give rise to linear maps

$$\mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(M_k) \to \mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(N_k)$$

which are compatible with the maps φ_g and $\psi_{\mathfrak{Y}',\mathfrak{Y}}$.

Step 2. $\Gamma(\mathscr{M})$ is an object in $\mathcal{C}_{G_0,\lambda}$.

Proof. For $k \in \mathbb{N}$ we choose $(\mathfrak{Y}, k) \in \mathcal{F}_{\mathfrak{X}}$ and we put $N_k := H^0(\mathfrak{Y}, \mathscr{M}_{(\mathfrak{Y},k)})$. By (6.26), lemma 6.5.2 and the fact that $\mathscr{M} \in \mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$ we get linear isomorphisms

$$D(\mathbb{G}(k)^\circ, G_0) \otimes_{D(\mathbb{G}(k')^\circ, G_0)} N_{k'} \to N_k$$

for $k' \geq k$. This implies that the modules N_k form a $\left(D(\mathbb{G}(k)^\circ, G_0)\right)_{k\in\mathbb{N}}$-sequence and the projective limit is a coadmissible module.

Step 3. $\Gamma \circ \mathscr{L}oc^{G_0}_{\lambda}(M) \simeq M$.

Proof. If $V := M'_b$, then we have by definition compatible isomorphisms

$$H^0\left(\mathfrak{Y}, \mathscr{L}oc^{G_0}_{\lambda}(M)_{(\mathfrak{Y},k)}\right) = H^0\left(\mathfrak{Y}, \mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(M_k)\right) = \left(V_{\mathbb{G}(k)^\circ-\mathrm{an}}\right)'_b,$$

which imply that the coadmissible modules $\Gamma \circ \mathscr{L}oc^{G_0}_{\lambda}(M)$ and M have isomorphic $\left(D(\mathbb{G}(k)^\circ, G_0)\right)_{k\in\mathbb{N}}$-sequences.

Step 4. $\mathscr{L}oc^{G_0}_{\lambda} \circ \Gamma(\mathscr{M}) \simeq \mathscr{M}$.

Proof. Let $N := \Gamma(\mathscr{M})$ and $V := N'_b$ the corresponding admissible representation. Let $\mathscr{N} := \mathscr{L}oc^{G_0}_{\lambda}(N)$. According to the lemma 6.4.22

$$N_k := D(\mathbb{G}(k)^\circ, G_0) \otimes_{D(G_0,L)} N_{k'} \to N$$

produces a $\left(D(\mathbb{G}(k)^\circ, G_0)\right)_{k\in\mathbb{N}}$-sequence for the coadmissible module N which is isomorphic to its constituting sequence $\left(H^0(\mathfrak{Y}, \mathscr{M}_{\mathfrak{Y},k})\right)_{(\mathfrak{Y},k)\in\mathcal{F}_{\mathfrak{X}}}$ from step 2. Now let $(\mathfrak{Y}, k) \in \mathcal{F}_{\mathfrak{X}}$. We have the following isomorphisms

$$\mathscr{N}_{\mathfrak{Y},k} = \mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)(N_k) \simeq \mathscr{L}oc^{\dagger}_{\mathfrak{Y},k}(\lambda)\left(H^0(\mathfrak{Y}, \mathscr{M}_{\mathfrak{Y},k})\right) \simeq \mathscr{M}_{\mathfrak{Y},\lambda}.$$

By T_g-linearity the action maps $\varphi_g^{\mathscr{M}_{\mathfrak{Y},k}}$ and $\varphi_g^{\mathscr{N}_{\mathfrak{Y},k}}$, constructed in step 1, are the same. Similarly if $(\mathfrak{Y}', k') \succeq (\mathfrak{Y}, k)$ are G_0-equivariant then the transition maps $\psi^{\mathscr{M}_{\mathfrak{Y}',\mathfrak{Y}}}$ and $\psi^{\mathscr{N}_{\mathfrak{Y}',\mathfrak{Y}}}$ coincide, by $\Psi_{\mathfrak{Y}',\mathfrak{Y}}$-linearity. Hence $\mathscr{N} \simeq \mathscr{M}$ in $\mathcal{C}^{G_0}_{\mathfrak{X},\lambda}$. □

6.5.1 Coadmissible G_0-equivariant $\mathscr{D}(\lambda)$-modules on the Zariski-Riemann space

Let us recall that $\mathfrak{X}_\infty$ denotes the projective limit of all formal models of $\mathbb{X}^{rig}$ (the rigid-analytic space associated by the GAGA functor to the flag variety X_L). The set $\mathcal{F}_{\mathfrak{X}}$ of admissible formal blow-ups $\mathfrak{Y} \to \mathfrak{X}$ is ordered by setting $\mathfrak{Y}' \geq \mathfrak{Y}$ if the blow-up morphism $\mathfrak{Y}' \to \mathfrak{X}$ factors as $\mathfrak{Y}' \xrightarrow{\pi} \mathfrak{Y} \to \mathfrak{X}$, with π a blow-up morphism. The set $\mathcal{F}_{\mathfrak{X}}$ is directed in the sense that any two elements have a common upper bound, and it is cofinal in the set of all formal models. In particular, $\mathfrak{X}_\infty = \varprojlim_{\mathcal{F}_{\mathfrak{X}}} \mathfrak{Y}$. The space $\mathfrak{X}_\infty$ is also known as the Zariski-Riemann space [13, Part II, chapter 9, section 9.3][8]. In this subsection we indicate how to realize coadmissible G_0-equivariant $\mathscr{D}(\lambda)$-modules on $\mathcal{F}_{\mathfrak{X}}$ as sheaves on the Zariski-Riemann space $\mathfrak{X}_\infty$. We start with the following proposition whose proof can be found in [36, Proposition 5.2.14].

Proposition 6.5.7. *Any formal model $\mathfrak{Y}$ of $\mathbb{X}^{rig}$ s dominated by one which is a G_0-equivariant admissible blow-up of $\mathfrak{X}$.*

Remark 6.5.8. *As $\mathcal{F}_{\mathfrak{X}}$ is cofinal in the set of all formal models, the preceding proposition tells us that the set of all G_0-equivariant admissible blow-ups of $\mathfrak{X}$ is also cofinal in the set of all formal models of $\mathfrak{X}$. From now on, we will assume that if $\mathfrak{Y} \in \mathcal{F}_{\mathfrak{X}}$, then $\mathfrak{Y}$ also G_0-equivariant, and we will denoted by $\rho_g^{\mathfrak{Y}} : \mathfrak{Y} \to \mathfrak{Y}$ the morphism induced by every $g \in G_0$.*

For every $\mathfrak{Y} \in \mathcal{F}_{\mathfrak{X}}$ we denote by $\mathrm{sp}_{\mathfrak{Y}} : \mathfrak{X}_\infty \to \mathfrak{Y}$ the canonical projection map. Let $\mathfrak{Y}' \geq \mathfrak{Y}$ with blow-up morphism $\pi' : \mathfrak{Y}' \to \mathfrak{Y}$ and $g \in G_0$. Let us consider the following commutative diagram coming from the G_0-equivariance of the family $\mathcal{F}_{\mathfrak{X}}$

$$\begin{array}{ccccc} \mathfrak{X}_\infty & \xrightarrow{sp_{\mathfrak{Y}}} & \mathfrak{Y} & \xrightarrow{\rho_g^{\mathfrak{Y}}} & \mathfrak{Y} \\ \big\downarrow{\scriptstyle sp_{\mathfrak{Y}'}} & \nearrow{\scriptstyle \pi} & & \nearrow{\scriptstyle \pi'} & \\ \mathfrak{Y}' & \xrightarrow{\rho_g^{\mathfrak{Y}'}} & \mathfrak{Y}'. & & \end{array}$$

This diagram allows to define a continuous function

$$\begin{array}{cccc} \rho_g : & \mathfrak{X}_\infty & \to & \mathfrak{X}_\infty \\ & (a_{\mathfrak{Y}})_{\mathfrak{Y} \in \mathcal{F}_{\mathfrak{X}}} & \mapsto & (\rho_g^{\mathfrak{Y}}(a_{\mathfrak{Y}}))_{\mathfrak{Y} \in \mathcal{F}_{\mathfrak{X}}}. \end{array} \tag{6.33}$$

which defines a G_0-action on the space $\mathfrak{X}_\infty$.

Let $\mathcal{U} \subset \mathfrak{Y}$ be an open subset and let us take $V := \mathrm{sp}_{\mathfrak{Y}}^{-1}(\mathcal{U}) \subset \mathfrak{X}_\infty$. Using the commutative diagram

$$\begin{array}{ccc} \mathfrak{X}_\infty & \xrightarrow{\mathrm{sp}_{\mathfrak{Y}}} & \mathfrak{Y} \\ & \searrow{\scriptstyle \mathrm{sp}_{\mathfrak{Y}'}} & \big\uparrow{\scriptstyle \pi'} \\ & & \mathfrak{Y}' \end{array}$$

we see that

$$\begin{aligned} \mathrm{sp}_{\mathfrak{Y}'}(V) &= \mathrm{sp}_{\mathfrak{Y}'}(\mathrm{sp}_{\mathfrak{Y}}^{-1}(\mathcal{U})) \\ &= \mathrm{sp}_{\mathfrak{Y}'}(\mathrm{sp}_{\mathfrak{Y}'}^{-1}(\pi'^{-1}(\mathcal{U}))) \\ &= \pi'^{-1}(\mathcal{U}), \end{aligned}$$

which implies that $\mathrm{sp}_{\mathfrak{Y}'}(V)$ is an open subset of $\mathfrak{Y}'$. Now, let us suppose that $\mathfrak{Y}'' \xrightarrow{\pi''} \mathfrak{Y}' \xrightarrow{\pi'} \mathfrak{Y}$ are morphisms over $\mathfrak{Y}$.

[8] In this reference this space is denoted by $\langle \mathfrak{X} \rangle$, cf. [37, subsection 3.2].

The commutative diagram

$$\begin{array}{ccccc} & & \mathfrak{X}_\infty \supseteq V := \mathrm{sp}_{\mathfrak{Y}}^{-1}(\mathcal{U}) & & \\ & \swarrow^{\mathrm{sp}\,_{\mathfrak{Y}''}} & \downarrow^{\mathrm{sp}_{\mathfrak{Y}'}} & \searrow^{\mathrm{sp}\,_{\mathfrak{Y}}} & \\ & & \mathfrak{Y}' & & \\ & \nearrow_{\pi''} & & \nwarrow_{\pi'} & \\ \mathfrak{Y}'' & & \longrightarrow & & \mathfrak{Y} \supseteq \mathcal{U} \end{array}$$

implies that

$$\pi''^{-1}(\mathrm{sp}\,_{\mathfrak{Y}'}(V)) = \pi''^{-1}(\pi''(\mathrm{sp}\,_{\mathfrak{Y}}(V))) = \mathrm{sp}\,_{\mathfrak{Y}''}(V). \tag{6.34}$$

In this situation, the morphism (6.22)

$$\Psi_{\mathfrak{Y}'',\mathfrak{Y}'} : \pi''_* \mathscr{D}^\dagger_{\mathfrak{Y}'',k''}(\lambda) \to \mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda)$$

induces the ring homomorphism

$$\mathscr{D}^\dagger_{\mathfrak{Y}'',k''}(\lambda)(\mathrm{sp}\,_{\mathfrak{Y}''}(V)) = \pi''_* \mathscr{D}^\dagger_{\mathfrak{Y}'',k''}(\lambda)(\mathrm{sp}\,_{\mathfrak{Y}'}(V)) \xrightarrow{\Psi_{\mathfrak{Y}'',\mathfrak{Y}'}} \mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda)(\mathrm{sp}\,_{\mathfrak{Y}'}(V))$$

and we can form the projective limit as in [36, (5.2.25)]

$$\mathscr{D}(\lambda)(V) := \varprojlim_{\mathfrak{Y}'\to\mathfrak{Y}} \mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda)(\mathrm{sp}\,_{\mathfrak{Y}'}(V)).$$

By definition, the open subsets of the form $V := \mathrm{sp}_{\mathfrak{Y}}^{-1}(\mathcal{U})$ form a basis for the topology of $\mathfrak{X}_\infty$ and $\mathscr{D}(\lambda)$ is a presheaf on this basis. The associated sheaf on $\mathfrak{X}_\infty$ to this presheaf will also be denoted by $\mathscr{D}(\lambda)$.
Now, relation (6.34), and commutativity of the diagrams

$$\begin{array}{ccc} \mathfrak{Y}'' & \xrightarrow{\rho_g^{\mathfrak{Y}''}} & \mathfrak{Y}'' \\ \downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \pi''} \\ \mathfrak{Y}' & \xrightarrow{\rho_g^{\mathfrak{Y}'}} & \mathfrak{Y}' \end{array}$$

and the first one in (6.32) tells us that

$$\begin{array}{ccc} \mathscr{D}^\dagger_{\mathfrak{Y}'',k''}(\lambda)\left(\mathrm{sp}\,_{\mathfrak{Y}''}(V)\right) = \pi''_* \mathscr{D}^\dagger_{\mathfrak{Y}'',k''}(\lambda)\left(\mathrm{sp}\,_{\mathfrak{Y}'}(V)\right) & \xrightarrow{\Psi_{\mathrm{sp}\,_{\mathfrak{Y}'}(V)}} & \mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda)\left(\mathrm{sp}\,_{\mathfrak{Y}'}(V)\right) \\ \downarrow{\scriptstyle T^{\mathfrak{Y}''}_{g,\,\mathrm{sp}\,_{\mathfrak{Y}''}(V)}} & & \downarrow{\scriptstyle T^{\mathfrak{Y}'}_{g,\,\mathrm{sp}_{\mathfrak{Y}'}(V)}} \\ \mathscr{D}^\dagger_{\mathfrak{Y}'',k''}(\lambda)\left(\left(\rho_g^{\mathfrak{Y}''}\right)^{-1}\left(\mathrm{sp}\,_{\mathfrak{Y}''}(V)\right)\right) = \left(\rho_g^{\mathfrak{Y}'}\right)_* \pi''_* \mathscr{D}^\dagger_{\mathfrak{Y}'',k''}(\lambda)\left(\mathrm{sp}\,_{\mathfrak{Y}'}(V)\right) & \xrightarrow{\Psi_{\mathrm{sp}_{\mathfrak{Y}'}(\rho_g^{-1}(V))}} & \left(\rho_g^{\mathfrak{Y}'}\right)_* \mathscr{D}^\dagger_{\mathfrak{Y}',k'}(\lambda)\left(\mathrm{sp}\,_{\mathfrak{Y}'}(V)\right) \end{array}$$

is also a commutative diagram. Let us identify

$$\mathscr{D}(\lambda)(V) = \left\{ P := \left(P_{\mathfrak{Y}',k'}\right)_{(\mathfrak{Y}',k')\in\mathcal{F}_{\mathfrak{X}}} \in \prod_{(\mathfrak{Y}',k')\mathcal{F}_{\mathfrak{X}}} \mathscr{D}^{\dagger}_{\mathfrak{Y}',k'}(\lambda)\left(\mathrm{sp}_{\,\mathfrak{Y}'}(V)\right) \mid \Psi_{\mathfrak{Y}'',\mathfrak{Y}'}(P_{\mathfrak{Y}'',k''}) = P_{\mathfrak{Y}',k'} \right\}$$

and let us consider the sequence

$$g.P := \left(T^{\mathfrak{Y}''}_{g,\,\mathrm{sp}_{\,\mathfrak{Y}''}(V)}(P_{\mathfrak{Y}'',k''})\right)_{(\mathfrak{Y}'',k'')\in\mathcal{F}_{\mathfrak{X}}} \in \prod_{(\mathfrak{Y}'',k'')\in\mathcal{F}_{\mathfrak{X}}} \mathscr{D}^{\dagger}_{\mathfrak{Y}'',k''}(\lambda)\left(\left(\rho^{\mathfrak{Y}''}_g\right)^{-1}\mathrm{sp}_{\,\mathfrak{Y}''}(V)\right).$$

Using the commutativity of the preceding diagram we see that

$$\begin{aligned}\Psi_{\mathrm{sp}_{\mathfrak{Y}'}(\rho_g^{-1}(V))}\left(T^{\mathfrak{Y}''}_{g,\,(\pi''^{-1})(\mathrm{sp}_{\,\mathfrak{Y}'}(V))}(P_{\mathfrak{Y}'',k''})\right) &= T^{\mathfrak{Y}'}_{g,\,\mathrm{sp}_{\mathfrak{Y}'}(V)}\left(\Psi_{\mathrm{sp}_{\,\mathfrak{Y}'}(V)}(P_{\mathfrak{Y}'',k''})\right)\\ &= T^{\mathfrak{Y}'}_{g,\,\mathrm{sp}_{\mathfrak{Y}'}(V)}(P_{\mathfrak{Y}',k'})\end{aligned}$$

and therefore, for $g \in G_0$, the actions $T^{\mathfrak{Y}}_g$ assemble to an action

$$T_g : \mathscr{D}(\lambda) \xrightarrow{\simeq} (\rho_g)_*\mathscr{D}(\lambda).$$

This action is on the left, in the sense that if $g, h \in G_0$ then

$$(\rho_g)_* T_h \circ T_g = T_{hg}.$$

Let us suppose now that $\mathscr{M} = (\mathscr{M}_{\mathfrak{Y},k}) \in \mathscr{C}^{G_0}_{\mathfrak{X},\lambda}$. We have the transition maps $\psi_{\mathfrak{Y}'',\mathfrak{Y}'} : \pi''_*\mathscr{M}_{\mathfrak{Y}'',k''} \to \mathscr{M}_{\mathfrak{Y}',k'}$ which are linear relative to the morphism (6.22). As before, we have the map

$$\mathscr{M}_{\mathfrak{Y}'',k''}\left(\mathrm{sp}_{\,\mathfrak{Y}''}(V)\right) = \pi''_*\mathscr{M}_{\mathfrak{Y}'',k''}\left(\mathrm{sp}_{\,\mathfrak{Y}'}(V)\right) \xrightarrow{\psi_{\mathrm{sp}_{\,\mathfrak{Y}'}(V)}} \mathscr{M}_{\mathfrak{Y}',k'}\left(\mathrm{sp}_{\,\mathfrak{Y}'}(V)\right)$$

which allows us to define $\mathscr{M}_\infty$ as the sheaf on $\mathfrak{X}_\infty$ associated to the presheaf

$$\mathscr{M}_\infty(V) := \varprojlim_{\mathfrak{Y}'\to\mathfrak{Y}} \mathscr{M}_{\mathfrak{Y}',k'}\left(\mathrm{sp}_{\,\mathfrak{Y}'}(V)\right).$$

By definition, we have the following commutative diagram

$$\begin{array}{ccc}\mathscr{M}_{\mathfrak{Y}'',k''}\left(\mathrm{sp}_{\,\mathfrak{Y}''}(V)\right) = \pi''_*\mathscr{M}_{\mathfrak{Y}'',k''}(\left(\mathrm{sp}_{\,\mathfrak{Y}'}(V)\right) & \xrightarrow{\psi_{\mathrm{sp}_{\,\mathfrak{Y}'}(V)}} & \mathscr{M}_{\mathfrak{Y}',k'}\left(\mathrm{sp}_{\,\mathfrak{Y}'}(V)\right)\\ \downarrow \varphi^{\mathfrak{Y}''}_{g\,\mathrm{sp}_{\,\mathfrak{Y}''}(V)} & & \downarrow \varphi^{\mathfrak{Y}'}_{g,\,\mathrm{sp}_{\mathfrak{Y}'}(V)}\\ \mathscr{M}^{\dagger}_{\mathfrak{Y}'',k''}\left(\left(\rho^{\mathfrak{Y}''}_g\right)^{-1}\left(\mathrm{sp}_{\,\mathfrak{Y}''}(V)\right)\right) = \left(\rho^{\mathfrak{Y}'}_g\right)_*\pi''_*\mathscr{M}_{\mathfrak{Y}'',k''}\left(\mathrm{sp}_{\,\mathfrak{Y}'}(V)\right) & \xrightarrow{\psi_{\mathrm{sp}_{\mathfrak{Y}'}(\rho_g^{-1}(V))}} & \left(\rho^{\mathfrak{Y}'}_g\right)_*\mathscr{M}_{\mathfrak{Y}',k'}\left(\mathrm{sp}_{\,\mathfrak{Y}'}(V)\right).\end{array}$$

Identifying

$$\mathscr{M}_\infty(V) = \left\{ m := \left(m_{\mathfrak{Y}',k'}\right)_{(\mathfrak{Y}',k')\in\mathcal{F}_{\mathfrak{X}}} \in \prod_{(\mathfrak{Y}',k')\mathcal{F}_{\mathfrak{X}}} \mathscr{M}_{\mathfrak{Y}',k'}\left(\mathrm{sp}_{\,\mathfrak{Y}'}(V)\right) \mid \psi_{\mathfrak{Y}'',\mathfrak{Y}'}(m_{\mathfrak{Y}'',k''}) = m_{\mathfrak{Y}',k'} \right\}$$

we see as before that if

$$g.m := \left(\varphi^{\mathfrak{Y}''}_{g,\,\mathrm{sp}_{\,\mathfrak{Y}''}(V)}(m_{\mathfrak{Y}'',k''})\right)_{(\mathfrak{Y}'',k'')\in\mathcal{F}_{\mathfrak{X}}} \in \prod_{(\mathfrak{Y}'',k'')\in\mathcal{F}_{\mathfrak{X}}} \mathscr{M}_{\mathfrak{Y}'',k''}\left(\left(\rho_g^{\mathfrak{Y}''}\right)^{-1}\mathrm{sp}_{\,\mathfrak{Y}''}(V)\right),$$

then the preceding commutative diagram implies that

$$\begin{aligned}\psi_{\mathrm{sp}_{\mathfrak{Y}'}(\rho_g^{-1}(V))}\left(\varphi^{\mathfrak{Y}''}_{g,\,(\pi''^{-1})(\mathrm{sp}_{\,\mathfrak{Y}'}(V))}(m_{\mathfrak{Y}'',k''})\right) &= \varphi^{\mathfrak{Y}'}_{g,\,\mathrm{sp}_{\mathfrak{Y}'}(V)}\left(\psi_{\mathrm{sp}_{\,\mathfrak{Y}'}(V)}(m_{\mathfrak{Y}'',k''})\right)\\ &= \varphi^{\mathfrak{Y}'}_{g,\,\mathrm{sp}_{\mathfrak{Y}'}(V)}(m_{\mathfrak{Y}',k'}),\end{aligned}$$

and therefore we get a family $(\varphi_g)_{g\in G_0}$ of isomorphisms

$$\varphi_g : \mathscr{M}_\infty \to (\rho_g)_*\mathscr{M}_\infty. \tag{6.35}$$

of sheaves of L-vector spaces. By definition 6.5.5 we have that if $g,h\in G_0$ then $\varphi_{hg} = (\rho_g)_*\varphi_h \circ \varphi_g$. Furthermore, under the preceding identifications, if $P = (P_{\mathfrak{Y}',k'}) \in \mathscr{D}(\lambda)(V)$ and $m = (m_{\,\mathfrak{Y}',k'}) \in \mathscr{M}_\infty(V)$, then $P.m = (P_{\,\mathfrak{Y}',k'}.m_{\,\mathfrak{Y}',k'})_{(\mathfrak{Y}',k')\in\mathcal{F}_{\mathfrak{X}}}$ and therefore

$$\begin{aligned}\varphi_{g,\,V}(P.m) &= \left(\varphi^{\mathfrak{Y}'}_{g,\,\mathrm{sp}_{\mathfrak{Y}'}(V)}(P_{\,\mathfrak{Y}',k'}.m_{\,\mathfrak{Y}',k'})\right)_{(\mathfrak{Y}',k')\in\mathcal{F}_{\mathfrak{X}}}\\ &= \left(T^{\mathfrak{Y}'}_{g,\,\mathrm{sp}_{\mathfrak{Y}'}(V)}(P_{\mathfrak{Y}',k'}).\varphi^{\mathfrak{Y}'}_{g,\,\mathrm{sp}_{\mathfrak{Y}'}(V)}(m_{\mathfrak{Y}',k'})\right)_{(\mathfrak{Y}',k')\in\mathcal{F}_{\mathfrak{X}}}\\ &= T_{g,\,V}(P).\varphi_{g,\,V}(m).\end{aligned}$$

In particular, $\mathscr{M}_\infty$ is an equivariant $\mathscr{D}(\lambda)$-module on the topologial G_0-space $\mathfrak{X}_\infty$. Let us see that the formation of $\mathscr{M}_\infty$ is functorial. Let $\gamma : \mathscr{M} \to \mathscr{N}$ be a morphism in $\mathscr{C}^{G_0}_{\mathfrak{X},\lambda}$. Then, in particular we have the following commutative diagram

$$\begin{array}{ccc}\pi''_*\mathscr{M}_{\mathfrak{Y}'',k''} & \xrightarrow{\ \psi^{\mathscr{M}}_{\mathfrak{Y}'',\mathfrak{Y}'}\ } & \mathscr{M}_{\mathfrak{Y}',k'}\\ \Big\downarrow{\scriptstyle \pi''_*(\gamma_{\mathfrak{Y}'',k''})} & & \Big\downarrow{\scriptstyle \gamma_{\mathfrak{Y}',k'}}\\ \pi_*\mathscr{N}_{\mathfrak{Y},k} & \xrightarrow{\ \psi^{\mathscr{N}}_{\mathfrak{Y}'',\mathfrak{Y}'}\ } & \mathscr{N}_{\mathfrak{Y},k}.\end{array}$$

Let $m = (m_{\mathfrak{Y},k})_{(\mathfrak{Y},k)\in\mathcal{F}_{\mathfrak{X}}} \in \mathscr{M}_\infty(V)$ and

$$s := \left(\gamma_{\mathfrak{Y}'',k''}(m_{\mathfrak{Y}'',k''})\right)_{(\mathfrak{Y}'',k'')\in\mathcal{F}_{\mathfrak{X}}} \in \prod_{(\mathfrak{Y}'',k'')\in\mathcal{F}_{\mathfrak{X}}} \mathscr{N}_{\mathfrak{Y}'',k''}\left(\mathrm{sp}_{\,\mathfrak{Y}''}(V)\right).$$

Commutativity in the preceding diagram implies that

$$\begin{aligned}\psi^{\mathscr{N}}_{\mathrm{sp}_{\mathfrak{Y}''}(V)}\left(s_{\mathfrak{Y}'',k''}\right) &= \psi^{\mathscr{N}}_{\mathrm{sp}_{\,\mathfrak{Y}''}(V)}\left(\gamma_{\mathrm{sp}_{\mathfrak{Y}'}(V)}\left(m_{\mathfrak{Y}'',k''}\right)\right)\\ &= \gamma_{\mathrm{sp}_{\mathfrak{Y}'}(V)}\left(\psi^{\mathscr{M}}_{\mathrm{sp}_{\,\mathfrak{Y}'}(V)}(m_{\mathfrak{Y}'',k''})\right)\\ &= \gamma_{\mathrm{sp}_{\mathfrak{Y}'}(V)}\left(m_{\mathfrak{Y}',k'}\right)\\ &= s_{\mathfrak{Y}',k'},\end{aligned}$$

therefore $s \in \mathscr{N}_\infty(V)$ and γ induces a morphism $\gamma_\infty : \mathscr{M}_\infty \to \mathscr{N}_\infty$.This shows that the preceding construction is functorial. The next proposition is the twisted analogue of [36, Proposition 5.2.29].

Proposition 6.5.9. *Let $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ be an algebraic character which induces, via derivation, a dominant and regular character of $\mathfrak{t}_L^*$. The functor $\mathscr{M} \rightsquigarrow \mathscr{M}_\infty$ from the category $\mathscr{C}^{G_0}_{\mathfrak{X},\lambda}$ to G_0-equivariant $\mathscr{D}(\lambda)$-modules is a faithful functor.*

Proof. We start the proof by remarking that $\mathrm{sp}_{\mathfrak{Y}}(\mathfrak{X}_\infty) = \mathfrak{Y}$ for every $\mathfrak{Y} \in \mathcal{F}_{\mathfrak{X}}$. By remark 6.5.8, the global sections of $\mathscr{M}_\infty$ equal to

$$H^0(\mathfrak{X}_\infty, \mathscr{M}_\infty) = \varprojlim_{(\mathfrak{Y},k)\in \underline{\mathcal{F}}_{\mathfrak{X}}} H^0(\mathfrak{Y}, \mathscr{M}_{\mathfrak{Y},k}) = \Gamma(\mathscr{M}).$$

Now, let $f, h\mathscr{M} \to \mathscr{N}$ be two morphisms in $\mathscr{C}^{G_0}_{\mathfrak{X},\lambda}$ such that $f_\infty = h_\infty$. By theorem 6.5.6, it suffices to verify $\Gamma(f) = \Gamma(h)$ which is clear since $H^0(\mathfrak{X}_\infty, f_\infty) = H^0(\mathfrak{X}_\infty, h_\infty)$. □

If $(\bullet)_\infty$ denotes the previous functor, the we will denote by $\mathcal{L}oc^{G_0}_\infty(\lambda)$ the composition of the functor $\mathcal{L}oc^{G_0}_\lambda$ with $(\bullet)_\infty$, i.e.,

$$\{\text{Coadmissible } D(G_0, L)_\lambda - \text{modules}\} \xrightarrow{\mathcal{L}oc^{G_0}_\infty(\lambda)} \{G_0 - \text{equivariant } \mathscr{D}(\lambda) - \text{modules}\}.$$

Since $\mathcal{L}oc^{G_0}_\lambda$ is an equivalence of categories, the preceding proposition implies that $\mathcal{L}oc^{G_0}_\infty(\lambda)$ is a faithful functor.

6.6 G-equivariant modules

Thorough this section we will denote by $G = \mathbb{G}(L)$ and by $\mathcal{B}$ the semi-simple Bruhat-Tits building of the p-adic group G ([18] et [19]). This is a simplicial complex endowed with a natural right G-action.

The purpose of this section is to extend the above results from G_0-equivariant objects to objects equivariants for the whole group G.

We start by fixing some notation.[9] To each special vertex $v \in \mathcal{B}$ the Bruhat-Tits theory associates a connected reductive group $\mathfrak{o}$-scheme $\mathbb{G}_v$, whose generic fiber $(\mathbb{G}_v)_L := \mathbb{G}_v \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(L)$ is canonically isomorphic to $\mathbb{G}_L$. We denote by X_v the smooth flag scheme of $\mathbb{G}_v$ whose generic fiber $(X_v)_L$ is canonically isomorphic to the flag variety X_L. We will distinguish the next constructions by adding the corresponding vertex to them. For instance, we will write Y_v for an (algebraic) admissible blow-up of the smooth model X_v, $G_{v,0}$ for the group of points $\mathbb{G}_v(\mathfrak{o})$ and $G_{v,k}$ for the group of points $\mathbb{G}_v(k)(\mathfrak{o})$. We will use the same conventions if we deal with formal completions. This means that we will denote by $\mathfrak{X}_v$ the smooth formal flag $\mathfrak{o}$-scheme obtained by formal completion of X_v along its special fiber $(X_v)_{\mathbb{F}_q} := X_v \times_{\mathrm{Spec}(\mathfrak{o})} \mathrm{Spec}(\mathfrak{o}/\varpi)$. Moreover, $\mathfrak{Y}_v$ will always denote an admissible formal blow-up of $\mathfrak{X}_v$. We point out to the reader that the blow-up morphism $\mathfrak{Y}_v \to \mathfrak{X}_v$ will make part of the datum of $\mathfrak{Y}_v$, and that even if for another special vertex $v' \neq v$ the formal $\mathfrak{o}$-scheme $\mathfrak{Y}_v$ is also a blow-up of the smooth formal model $\mathfrak{X}_{v'}$, we will only consider it as a blow-up of $\mathfrak{X}_v$. We will denote by $\mathcal{F}_v := \mathcal{F}_{\mathfrak{X}_v}$, the set of all admissible formal blow-ups $\mathfrak{Y}_v \to \mathfrak{X}_v$ of $\mathfrak{X}_v$ and by $\underline{\mathcal{F}}_v : \underline{\mathcal{F}}_{\mathfrak{X}_v}$ the respective directed system of definition 6.5.1. By the preceding accord, the sets $\mathcal{F}_v$ and $\mathcal{F}_{v'}$ are disjoint if $v \neq v'$. Let

$$\mathcal{F} := \bigsqcup_v \mathcal{F}_v$$

where v runs over all special vertices of $\mathcal{B}$. We recall for the reader that $\mathfrak{X}_\infty$ is equal to the projective limit of all formal models of $\mathbb{X}^{\mathrm{rig}}$.

Remark 6.6.1. *The set $\mathcal{F}$ is partially ordered in the following way. We say that $\mathfrak{Y}_{v'} \geq \mathfrak{Y}_v$ if the projection $\mathrm{sp}_{\mathfrak{Y}_{v'}} : \mathfrak{X}_\infty \to$*

[9] This is exactly as in [36, 5.3.1].

$\mathfrak{Y}_{v'}$ *factors through the projection* $sp_{\mathfrak{Y}_v} : \mathfrak{X}_\infty \to \mathfrak{Y}_v$

$$\begin{array}{ccc} & \mathfrak{X}_\infty & \\ {}^{sp_{\mathfrak{Y}_{v'}}}\swarrow & & \searrow^{sp_{\mathfrak{Y}_v}} \\ \mathfrak{Y}_{v'} & \longrightarrow & \mathfrak{Y}_v. \end{array}$$

Definition 6.6.2. *We will denote by* $\underline{\mathcal{F}} := \bigsqcup_v \underline{\mathcal{F}}_v$, *where* v *runs over all the special vertices of* $\mathcal{B}$. *This set is partially ordered as follows. We say that* $(\mathfrak{Y}_{v'}, k') \geq (\mathfrak{Y}_v, k)$ *if* $\mathfrak{Y}_{v'} \geq \mathfrak{Y}_v$ *and* $Lie(\mathbb{G}_{v'}(k')) \subset Lie(\mathbb{G}_v(k))$ *(or equivalent* $\varpi^{k'} Lie(\mathbb{G}_{v'}) \subset \varpi^k Lie(\mathbb{G}_v)$, *cf. subsection 5.1.2) as lattices in* $\mathfrak{g}_L$.

For any special vertex $v \in \mathcal{B}$, any element $g \in G$ induces an isomorphism

$$\rho_g^v : X_v \to X_{v.g}.$$

The isomorphism induced by ρ_g^v on the generic fibers $(X_v)_L \simeq X_L \simeq (X_{v.g})_L$ coincides with right translation by g on X_L

$$\rho_g : X_L = X_L \times_{\mathrm{Spec}(L)} \mathrm{Spec}(L) \xrightarrow{id_{X_L} \times g} X_L \times_{\mathrm{Spec}(L)} \mathrm{Spec}(\mathbb{G}_L) \xrightarrow{\alpha_L} X_L.$$

where we have used $\mathbb{G}(L) = \mathbb{G}_L(L)$. Moreover, ρ_g^v induces a morphism $\mathfrak{X}_v \to \mathfrak{X}_{v.g}$, which we denote again by ρ_g^v, and which coincides with the right translation on $\mathfrak{X}_v$ if $g \in G_{v,0}$ (of course in this case $vg = v$). Let $(\rho_g^v)^\natural : \mathcal{O}_{\mathfrak{X}_{vg}} \to (\rho_g^v)_* \mathcal{O}_{\mathfrak{X}_v}$ be the comorphism of ρ_g^v. If $\pi : \mathfrak{Y}_v \to \mathfrak{X}_v$ is an admissible blow-up of an ideal $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}_v}$, then blowing-up $((\rho_g^v)^\natural)^{-1}((\rho_g^v)_* \mathcal{I})$ produced a formal scheme $\mathfrak{Y}_{vg}$ (cf. lemma 6.5.3), together with an isomorphism $\rho_g^v : \mathfrak{Y}_v \to \mathfrak{Y}_{vg}$. As in lemma 6.5.3 we have $k_{\mathfrak{Y}_v} = k_{\mathfrak{Y}_{vg}}$. For any $g, h \in G$ and any admissible formal blow-up $\mathfrak{Y}_v \to \mathfrak{X}_v$, we have $\rho_h^{vg} \circ \rho_g^v = \rho_{gh}^v : \mathfrak{Y}_v \to \mathfrak{Y}_{vgh}$. This gives a right G-action on the family $\mathcal{F}$ and on the projective limit $\mathfrak{X}_\infty$. Finally, if $\mathfrak{Y}_{v'} \geq \mathfrak{Y}_v$ with morphism $\pi : \mathfrak{Y}_{v'} \to \mathfrak{Y}_v$ and $g \in G$, then $\mathfrak{Y}_{v'g} \geq \mathfrak{Y}_{vg}$, and we have the following commutative diagram (cf. corollary 6.5.4)

$$\begin{array}{ccc} \mathfrak{Y}_{v'} & \xrightarrow{\pi} & \mathfrak{Y}_v \\ \downarrow{\scriptstyle \rho_g^{v'}} & & \downarrow{\scriptstyle \rho_g^v} \\ \mathfrak{Y}_{v'g} & \xrightarrow{\pi g} & \mathfrak{Y}_{vg}. \end{array} \tag{6.36}$$

Now, over every special vertex $v \in \mathcal{B}$ the algebraic character λ induces an invertible sheaf $\mathcal{L}_v(\lambda)$ on X_v, such that for every $g \in G$ there exists an isomorphism

$$R_g^v : \mathcal{L}_{vg}(\lambda) \to (\rho_g^v)_* \mathcal{L}_v(\lambda),$$

satisfying the cocycle condition

$$R_{hg}^{vhg} = \left(\rho_g^{vh}\right)_* R_h^v \circ R_g^{vh} \qquad (h, g \in G). \tag{6.37}$$

As usual, for every special vertex $v \in \mathcal{B}$, we will denote by $\mathscr{L}_v(\lambda)$ the p-adic completion of the sheaf $\mathcal{L}_v(\lambda)$, which is considered as an invertible sheaf on $\mathfrak{X}_v$. Let $(\mathfrak{Y}_v, k) \in \mathcal{F}$ with blow-up morphism $\mathrm{pr} : \mathfrak{Y}_v \to \mathfrak{X}_v$. At the level of differential operators, we will denote by $\mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda)$ the sheaf of arithmetic differential operators on $\mathfrak{Y}_v$ acting on the line bundle $\mathscr{L}_v(\lambda)$[10] we have the following important properties. Let $g \in G$. As in (6.9) the isomorphism (6.37) induces a left

[10]Here we abuse of the notation and we denote again by $\mathscr{L}_v(\lambda)$ the invertible sheaf $\mathrm{pr}^* \mathscr{L}_v(\lambda)$ on $\mathfrak{Y}_v$.

action

$$\begin{array}{cccc} T^v_g : & \mathscr{D}^\dagger_{\mathfrak{Y}_{vg},k}(\lambda) & \xrightarrow{\simeq} & \left(\rho^v_g\right)_* \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda) \\ & P & \mapsto & R^v_g\, P\, (R^v_g)^{-1}. \end{array}$$

Now, we identify the global sections $\Gamma(\mathfrak{Y}_v, \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda))$ with $\mathcal{D}^{\mathrm{an}}(\mathbb{G}_v(k)^\circ)_\lambda$ and obtain the group homomorphism

$$\begin{array}{ccc} G_{v,k+1} & \to & \Gamma\left(\mathfrak{Y}_v, \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda)\right)^\times \\ g & \mapsto & \delta_g \end{array}$$

Where $G_{v,k+1} = \mathbb{G}_v(k)^\circ(L)$ denotes the group of L-rational points (or $\mathfrak{o}$-points of $\mathbb{G}_v(k+1)$). We will follow the same lines of reasoning given in [36, Proposition 5.3.2] to prove the following proposition.

Proposition 6.6.3. *Suppose* $(\mathfrak{Y}_{v'}, k') \geq (\mathfrak{Y}_v, k)$ *for pairs* $(\mathfrak{Y}_{v'}, k')$, $(\mathfrak{Y}_v, k) \in \underline{\mathcal{F}}$ *with morphism* $\pi : \mathfrak{Y}_{v'} \to \mathfrak{Y}_v$*. There exists a canonical morphism of sheaves of rings*

$$\Psi : \pi_* \mathscr{D}^\dagger_{\mathfrak{Y}_{v'},k'}(\lambda) \to \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda)$$

which is G-equivariant in the sense that for every $g \in G$ *the following diagram is commutative*

$$\begin{array}{ccc} (\pi.g)_* \mathscr{D}^\dagger_{\mathfrak{Y}_{v'g},k'}(\lambda) & \xrightarrow{\quad\Psi\quad} & \mathscr{D}^\dagger_{\mathfrak{Y}_{vg},k}(\lambda) \\ \downarrow{\scriptstyle (\pi.g)_*(T_g)} & & \downarrow{\scriptstyle T_g} \\ (\pi.g)_*(\rho^{v'}_g)_* \mathscr{D}^\dagger_{\mathfrak{Y}_{v'},k'}(\lambda) = (\rho^v_g)_* \pi_* \mathscr{D}^\dagger_{\mathfrak{Y}_{v'},k'}(\lambda) & \xrightarrow{(\rho^v_g)_*(\Psi)} & (\rho^v_g)_* \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda). \end{array}$$

Proof. Let us denote by $\mathrm{pr}' : \mathfrak{Y}_{v'} \to \mathfrak{X}_{v'}$ and $\mathrm{pr} : \mathfrak{Y}_v \to \mathfrak{X}_v$ the blow-ups morphisms, and let us put $\widetilde{\mathrm{pr}} := \mathrm{pr} \circ \pi$. We have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{Y}_{v'} & \xrightarrow{\pi} & \mathfrak{Y}_v \\ \downarrow{\scriptstyle \mathrm{pr}'} & \searrow^{\widetilde{\mathrm{pr}}} & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathfrak{X}_{v'} & & \mathfrak{X}_v. \end{array}$$

Let us fix $m \in \mathbb{N}$. As in [36, Proposition 5.3.6] we show first the existence of a canonical morphism of sheaves of $\mathfrak{o}$-algebras

$$\mathcal{D}^{(m,k)}_{Y_{v'}}(\lambda) \to \widetilde{\mathrm{pr}}^* \mathcal{D}^{(m,k)}_{X_v}(\lambda). \tag{6.38}$$

Here $Y_{v'}$, Y_v, $X_{v'}$ and X_v denote the $\mathfrak{o}$-scheme of finite type whose completions are $\mathfrak{Y}_{v'}$, $\mathfrak{Y}_v$, $\mathfrak{X}_{v'}$ and $\mathfrak{X}_v$, respectively. The morphisms between these schemes will be denoted by the same letters, for instance $\mathrm{pr} : Y_v \to X_v$. We recall for the reader that the sheaf $\mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda)$ is filtered by locally free sheaves of finite rank

$$\begin{aligned} \mathcal{D}^{(m,k')}_{Y_{v'},d}(\lambda) &= \mathrm{pr}'^* \mathcal{L}_{v'}(\lambda) \otimes_{\mathcal{O}_{Y_{v'}}} \mathrm{pr}'^* \mathcal{D}^{(m,k')}_{X_{v'},d} \otimes_{\mathcal{O}_{Y_{v'}}} \mathrm{pr}'^* \mathcal{L}_{v'}(\lambda)^\vee \\ &= \mathrm{pr}'^* \left(\mathcal{D}^{(m,k')}_{X_{v'},d}(\lambda)\right), \end{aligned}$$

and therefore by the projection formula [30, Part II, Section 5, exercise 5.1 (d)] and given that $\mathrm{pr}'_* \mathcal{O}_{Y_{v'}} = \mathcal{O}_{X_{v'}}$ (cf. [36, Lemma 3.2.3]) we have for every $d \in \mathbb{N}$

$$\begin{aligned}\mathrm{pr}'_*\left(\mathcal{D}^{(m,k')}_{Y_{v'},d}(\lambda)\right) &= \mathrm{pr}'_*\left(\mathcal{O}_{Y_{v'}} \otimes_{\mathcal{O}_{Y_{v'}}} \mathrm{pr}'^*\mathcal{D}^{(m,k')}_{X_{v'},d}(\lambda)\right)\\ &= \mathrm{pr}'_*(\mathcal{O}_{Y_{v'}}) \otimes_{\mathcal{O}_{X_{v'}}} \mathcal{D}^{(m,k')}_{X_{v'},d}(\lambda)\\ &= \mathcal{D}^{(m,k')}_{X_{v'},d}(\lambda),\end{aligned}$$

which implies that

$$\mathrm{pr}'_*\left(\mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda)\right) = \mathcal{D}^{(m,k')}_{X_{v'}}(\lambda)$$

because the direct image commutes with inductive limits on a noetherian space. By proposition 5.5.11 and the preceding relation we have a canonical map of filtered $\mathfrak{o}$-algebras

$$D^{(m)}(\mathbb{G}_{v'}(k')) \to H^0\left(X_{v'}, \mathcal{D}^{(m,k')}_{X_{v'}}(\lambda)\right) = H^0\left(X_{v'}, \mathrm{pr}'_*\left(\mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda)\right)\right) = H^0\left(Y_{v'}, \mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda)\right),$$

in particular we get a morphism of sheaves of filtered $\mathfrak{o}$-algebras (this is exactly as we have done in (2.23))

$$\Phi^{(m,k')}_{Y_{v'}} : \mathcal{A}^{(m,k')}_{Y_{v'}} := \mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_{v'}(k')) \to \mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda). \tag{6.39}$$

Applying $\mathrm{Sym}^{(m)}(\bullet) \circ \varpi^{k'}\mathrm{pr}'^*(\bullet)$ to the surjection (5.31) we obtain a surjection

$$\mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} \mathrm{Sym}^{(m)}\left(\mathrm{Lie}(\mathbb{G}_{v'}(k'))\right) \to \mathrm{Sym}^{(m)}\left(\varpi^{k'}\mathrm{pr}'^*\,\mathcal{T}_{X_{v'}}\right)$$

which equals the associated graded morphism of (6.39) by proposition 6.1.5. Hence $\Phi^{(m,k')}_{Y_{v'}}$ is surjective. On the other hand, if we apply $\widetilde{\mathrm{pr}}^*$ to the surjection

$$\Phi^{(m,k)}_{X_v} : \mathcal{A}^{(m,k)}_{X_v} = \mathcal{O}_{X_v} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_v(k)) \to \mathcal{D}^{(m,k)}_{X_v}(\lambda)$$

we obtain the surjection $\mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_v(k)) \to \widetilde{\mathrm{pr}}^*\mathcal{D}^{(m,k)}_{X_v}(\lambda)$. Let us recall that $(\mathfrak{Y}_{v'}, k') \succeq (\mathfrak{Y}_v, k)$ implies, in particular, that $\mathrm{Lie}(\mathbb{G}_{v'}(k')) \subseteq \mathrm{Lie}(\mathbb{G}_v(k))$ and therefore $\varpi^{k'}\mathrm{Lie}(\mathbb{G}_{v'}) \subset \varpi^k\mathrm{Lie}(\mathbb{G}_v)$. By (5.8), the preceding inclusion gives rise to an injective ring homomorphism $D^{(m)}(\mathbb{G}_{v'}(k')) \hookrightarrow D^{(m)}(\mathbb{G}_v(k))$. Let us see that the composition

$$\mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_{v'}(k')) \hookrightarrow \mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_v(k)) \twoheadrightarrow \widetilde{\mathrm{pr}}^*\mathcal{D}^{(m,k)}_{X_v}(\lambda)$$

factors through $\mathcal{D}^{(m,k')}_{X_{v'}}(\lambda)$.

$$\begin{array}{ccc}\mathcal{O}_{Y_{v'}} \otimes_{\mathfrak{o}} D^{(m)}(\mathbb{G}_{v'}(k')) & \longrightarrow & \widetilde{\mathrm{pr}}^*\mathcal{D}^{(m,k)}_{X_v}(\lambda)\\ \downarrow & \nearrow & \\ \mathcal{D}^{(m,k')}_{Y_{v'}}(\lambda). & & \end{array}$$

Since by lemma 5.5.8 all those sheaves are ϖ-torsion free, this can be checked after tensoring with L in which case we have that $\mathcal{D}^{(m,k')}_{Y_{v'}} \otimes_{\mathfrak{o}} L \simeq \widetilde{\mathrm{pr}}^*\mathcal{D}^{(m,k)}_{X_v} \otimes_{\mathfrak{o}} L$ is the (push-forward of the) sheaf of algebraic differential operators on the generic fiber of $Y_{v'}$ (cf. discussion given at the beginning of section 6.1). We thus get the canonical morphism of sheaves (6.38). Passing to completions we get a canonical morphism $\widehat{\mathscr{D}}^{(m,k')}_{\mathfrak{Y}_{v'}}(\lambda) \to \widetilde{\mathrm{pr}}^*\widehat{\mathscr{D}}^{(m,k)}_{\mathfrak{X}_v}(\lambda)$. Taking inductive limit over all m and inverting ϖ gives a canonical morphism $\mathscr{D}^{\dagger}_{\mathfrak{Y}_{v'},k'}(\lambda) \to \widetilde{\mathrm{pr}}^*\mathscr{D}^{\dagger}_{\mathfrak{X}_v,k}(\lambda)$. Now, let us consider the formal scheme $\mathfrak{Y}_{v'}$

as a blow-up of $\mathfrak{X}_v$ via $\widetilde{\mathrm{pr}}$. Then π becomes a morphism of formal schemes over $\mathfrak{X}_v$ and we consider $\widetilde{\mathrm{pr}}^*\mathscr{D}^\dagger_{\mathfrak{X}_v,k}(\lambda)$ as the sheaf of arithmetic differential operators with congruence level k defined on $\mathfrak{Y}_{v'}$ via $\widetilde{\mathrm{pr}}^*$. Using the invariance theorem (theorem 6.2.2) we get $\pi_*\left(\widetilde{\mathrm{pr}}^*\mathscr{D}^\dagger_{\mathfrak{X}_v,k}(\lambda)\right) = \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}$. Then applying π_* to the morphism $\mathscr{D}^\dagger_{\mathfrak{Y}_{v'},x'}(\lambda) \to \widetilde{\mathrm{pr}}^*\mathscr{D}^\dagger_{\mathfrak{X}_v,k}(\lambda)$ gives the morphism

$$\Psi : \pi_*\mathscr{D}^\dagger_{\mathfrak{Y}_{v'},k'}(\lambda) \to \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}$$

of the statement. As in [36, Proposition 5.3.8], making use of the maps $\Phi^{(m,k)}_{Y_v}$, as above, the assertion about the G-equivariance is reduced to some obvious functorial properties of the rings $D^{(m)}(\mathbb{G}_v(k))$. □

Definition 6.6.4. *A coadmissible G-equivariant arithmetic $\mathscr{D}(\lambda)$-module on $\mathcal{F}$ consists of a family $\mathscr{M} := (\mathscr{M}_{\mathfrak{Y}_v,k})_{(\mathfrak{Y}_v,k)\in\mathcal{F}}$ of coherent $\mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda)$-modules with the following properties:*

(a) For any special vertex $v \in \mathcal{B}$ and $g \in G$ with isomorphism $\rho^v_g : \mathfrak{Y}_v \to \mathfrak{Y}_{vg}$, there exists an isomorphism

$$\phi^v_g : \mathscr{M}_{\mathfrak{Y}_{vg},k} \to \left(\rho^v_g\right)_* \mathscr{M}_{\mathfrak{Y},k}$$

of sheaves of L-vector spaces, satisfying the following conditions:

(i) For all $h, g \in G$ we have [11]

$$(\rho^{vh}_g)_*\phi^v_h \circ \phi^{vh}_g = \phi^v_{hg}.$$

(ii) For all open subsets $\mathcal{U} \subseteq \mathfrak{Y}_{vg}$, all $P \in \mathscr{D}^\dagger_{\mathfrak{Y}_{vg},k}(\lambda)(\mathcal{U})$, and all $m \in \mathscr{M}_{\mathfrak{Y}_{vg},k}(\mathcal{U})$ one has $\phi^v_{g,\mathcal{U}}(P.m) = T^v_{g,\mathcal{U}}(P).\phi^v_{g,\mathcal{U}}(m)$.

(iii) For all $g \in G_{k+1,v}$ the map $\phi^v_g : \mathscr{M}_{\mathfrak{Y},k} \to (\rho^v_g)_*\mathscr{M}_{\mathfrak{Y},k} = \mathscr{M}_{\mathfrak{Y},k}$ is equal to the multiplication by $\delta_g \in H^0(\mathfrak{Y}_v, \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda))$.

(b) For any two pairs $(\mathfrak{Y}_{v'}, k') \geq (\mathfrak{Y}_v, k)$ in $\mathcal{F}$ with morphism $\pi : \mathfrak{Y}_{v'} \to \mathfrak{Y}_v$ there exists a transition morphism $\psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v} : \pi_*\mathscr{M}_{\mathfrak{Y}_{v'}} \to \mathscr{M}_{\mathfrak{Y}_v}$, linear relative to the canonical morphism $\Psi : \pi_*\mathscr{D}^\dagger_{\mathfrak{Y}_{v'},k'}(\lambda) \to \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda)$ (in the preceding proposition) and making commutative the following diagram

$$\begin{array}{ccc} (\pi.g)_*\mathscr{M}_{\mathfrak{Y}_{v'g}} & \xrightarrow{\psi_{\mathfrak{Y}_{v'g},\mathfrak{Y}_{vg}}} & \mathscr{M}_{\mathfrak{Y}_{vg}} \\ \downarrow{\scriptstyle (\pi.g)_*\phi^{v'}_g} & & \downarrow{\scriptstyle \phi^v_g} \\ (\rho^v_g)_*\pi_*\mathscr{M}_{\mathfrak{Y}_{v'}} = (\pi.g)_*(\rho^{v'}_g)_*\mathscr{M}_{\mathfrak{Y}_{v'}} & \xrightarrow{(\rho^v_g)_*\psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v}} & (\rho^v_g)_*\mathscr{M}_{\mathfrak{Y}_v} \end{array} \qquad \phi^v_g \circ \psi_{\mathfrak{Y}_{v'g},\mathfrak{Y}_{vg}} = (\rho^v_g)_*\psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v} \circ (\pi.g)_*\phi^{v'}_g \quad (6.40)$$

for any $g \in G$ (where we have use the relation $(\rho^v_g)_* \circ \pi_* = (\pi.g)_* \circ (\rho^{v'}_g)_*$ coming from the commutative diagram (6.36)). If $v' = v$, and $(\mathfrak{Y}'_v, k') \geq (\mathfrak{Y}_v, k)$ in $\underline{\mathcal{F}}_v$, and if $\mathfrak{Y}'_v$, $\mathfrak{Y}_v$ are $G_{v,0}$-equivariant, then we require additionally that the morphism induced by $\psi_{\mathfrak{Y}'_v,\mathfrak{Y}_v}$ (cf. (6.26))

$$\overline{\psi}_{\mathfrak{Y}'_v,\mathfrak{Y}_v} : \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda) \otimes_{\pi_*\mathscr{D}^\dagger_{\mathfrak{Y}'_v,k'}(\lambda),G_{v,k+1}} \pi_*\mathscr{M}_{\mathfrak{Y}'_v,k'} \to \mathscr{M}_{\mathfrak{Y}_v,k} \qquad (6.41)$$

is an isomorphism of $\mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda)$-modules. As in theorem 6.5.6, the morphism $\psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v} : \pi_*\mathscr{M}_{\mathfrak{Y}_{v'},k'} \to \mathscr{M}_{\mathfrak{Y}_v,k}$ are

[11] Here we use the fact the action of G on $\mathcal{B}$ is on the right and therefore $(\rho^{vh}_g)_* \circ (\rho^v_h)_* = (\rho^v_{hg})_x$.

required to satisfy the transitive condition

$$\begin{array}{ccc} \mathfrak{Y}_{v''} & \xrightarrow{\pi'} & \mathfrak{Y}_{v'} \\ & \underset{\pi''}{\searrow} \quad \underset{\pi}{\swarrow} & \\ & \mathfrak{Y}_v & \end{array} \qquad \begin{array}{ccc} \pi_*\pi'_*\mathscr{M}_{\mathfrak{Y}_{v''},k''} = \pi''_*\mathscr{M}_{\mathfrak{Y}_{v''},k''} & \xrightarrow{\psi_{\mathfrak{Y}_{v''},\mathfrak{Y}_v}} & \mathscr{M}_{\mathfrak{Y}_v,k} \\ \quad \searrow^{\pi_*\psi_{\mathfrak{Y}_{v''},\mathfrak{Y}_{v'}}} & & \nearrow_{\psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v}} \\ & \pi_*\mathscr{M}_{\mathfrak{Y}'_{v'},k'} & \end{array}$$

$$\psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v} \circ \pi_*(\psi_{\mathfrak{Y}_{v''},\mathfrak{Y}_{v'}}) = \psi_{\mathfrak{Y}_{v''},\mathfrak{Y}_v},$$

whenever $(\mathfrak{Y}_{v''}, k'') \geq (\mathfrak{Y}_{v'}, k') \geq (\mathfrak{Y}_v, k)$ *in* $\underline{\mathcal{F}}$. *Moreover,* $\psi_{\mathfrak{Y}_v,\mathfrak{Y}_v} = id_{\mathscr{M}_{\mathfrak{Y}_v,k}}$.

A morphism $\mathscr{M} \to \mathscr{N}$ *between two coadmissible* G*-equivariant arithmetic* $\mathscr{D}(\lambda)$*-modules consists in a family of morphisms* $\mathscr{M}_{\mathfrak{Y},k} \to \mathscr{N}_{\mathfrak{Y},k}$ *of* $\mathscr{D}^{\dagger}_{\mathfrak{Y},k}(\lambda)$*-modules, such that for any couple of special vertex* $v, v' \in \mathcal{B}$ *we have the following commutative diagrams*

$$\begin{array}{ccc} \mathscr{M}_{\mathfrak{Y}_{vg},k} & \longrightarrow & (\rho^v_g)_*\mathscr{M}_{\mathfrak{Y},k} \\ \downarrow & & \downarrow \\ \mathscr{N}_{\mathfrak{Y}_{vg},k} & \longrightarrow & (\rho^v_g)_*\mathscr{N}_{\mathfrak{Y},k} \end{array} \qquad \begin{array}{ccc} \pi_*\mathscr{M}_{\mathfrak{Y}_{v'},k'} & \longrightarrow & \mathscr{M}_{\mathfrak{Y}_v,k} \\ \downarrow & & \downarrow \\ \pi_*\mathscr{N}_{\mathfrak{Y}_{v'},k'} & \longrightarrow & \mathscr{N}_{\mathfrak{Y}_v,k}. \end{array}$$

We denote the resulting category by $\mathscr{C}^{\mathcal{F}}_{G,\lambda}$.

We recall for the reader that $D(G_0, L)$ is a Fréchet-Stein algebra [52, Theorem 5.1]. Moreover, a $D(G, L)$-module is called coadmissible if it is coadmissible as a $D(H, L)$-module for every compact open subgroup $H \subseteq G$ (cf. remark 6.4.7 (*ii*)). Given that for any two compact open subgroups $H \subseteq H' \subseteq G$ the algebra $D(H', L)$ is finitely generated free and hence coadmissible as a $D(H, L)$-module, it follows from [52, Lemma 3.8] that the preceding condition needs to be tested only for a single compact open subgroup $H \subseteq G$. This motivates the following definition where we will consider the weak Freéchet-Stein structure of $D(G_0, L)$ defined in (6.20).

Definition 6.6.5. *We say that* M *is a coadmissble* $D(G, L)$*-module if* M *is coadmissisble as a* $D(G_0, L)$*-module.*

Let us construct now the bridge to the category of coadmisible $D(G, L)_\lambda$-modules. Let M be such a coadmissible $D(G, L)_\lambda$-module and let $V := M'_b$. We fix $v \in \mathcal{B}$ a special vertex. Let $V_{\mathbb{G}_v(k)^\circ-\mathrm{an}}$ [12] be the subspace of $\mathbb{G}_v(k)^\circ$-analytic vectors and let $M_{v,k}$ be its continuous dual. For any $(\mathfrak{Y}_v, k) \in \underline{\mathcal{F}}$ we have a coherent $\mathscr{D}^{\dagger}_{\mathfrak{Y}_v,k}(\lambda)$-module

$$\mathcal{L}oc^{\dagger}_{\mathfrak{Y}_v,k}(\lambda)(M_{v,k}) = \mathscr{D}^{\dagger}_{\mathfrak{Y}_v,k}(\lambda) \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}_v(k)^\circ)_\lambda} M_{v,k}$$

and we can consider the family

$$\mathcal{L}oc^G_\lambda(M) := \left(\mathcal{L}oc^{\dagger}_{\mathfrak{Y}_v,k}(\lambda)(M_{v,k})\right)_{(\mathfrak{Y}_v,k)\in\underline{\mathcal{F}}}.$$

On the other hand, given an object $\mathscr{M} \in \mathscr{C}^{\mathcal{F}}_{G,\lambda}$, we may consider the projective limit

$$\Gamma(\mathscr{M}) := \varprojlim_{(\mathfrak{Y},k)\in\underline{\mathcal{F}}} H^0(\mathfrak{Y}, \mathscr{M}_{\mathfrak{Y},k})$$

with respect to the transition maps $\psi_{\mathfrak{Y}',\mathfrak{Y}}$. Here the projective limit is taken in the sens of abelian groups and over the cofinal family of pairs $(\mathfrak{Y}_v, k) \in \underline{\mathcal{F}}$ with $G_{v,0}$-equivariant $\mathfrak{Y}_v$, cf. remark 6.5.8.

[12] Here we use the fact that $(\mathbb{G}_v)_L = \mathbb{G}_L$.

Theorem 6.6.6. *Let us suppose that $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}_L^*$ is a dominant and regular character of $\mathfrak{t}_L$. (and therefore, a dominant and regular character on every special vertex of $\mathcal{B}$). The functors $\mathcal{L}oc_\lambda^G(\bullet)$ and $\Gamma(\bullet)$ induce quasi-inverse equivalences between the category of coadmissible $D(G, L)_\lambda$-modules and $\mathscr{C}_{G,\lambda}^{\mathcal{F}}$.*

The proof follows the same lines of reasoning given in [36, Theorem 5.3.12].

Proof. The proof is an extension of the the proof of theorem 6.5.6, taking into account the additional G-action. Let M be a coadmissible $D(G, L)_\lambda$-module and let $\mathscr{M} \in \mathscr{C}_{\mathcal{F},\lambda}^G$. The proof of the theorem follows the following steps.

Claim 1. One has $\mathcal{L}oc_\lambda^G(M) \in \mathscr{C}_{G,\lambda}^{\mathcal{F}}$ and $\mathcal{L}oc_\lambda^G(\bullet)$ is functorial.

Proof. Let $g \in G$, $v \in \mathcal{B}$ a special vertex and $\rho_g^v : \mathfrak{Y}_v \to \mathfrak{Y}_{vg}$ the respective isomorphism. For conditions (a) for $\mathcal{L}oc_\lambda^G(M)$ we need the maps

$$\phi : \mathcal{L}oc_\lambda^G(M)_{\mathfrak{Y}_v,k} := \mathcal{L}oc^\dagger_{\mathfrak{Y}_v,k}(\lambda)(M_{v,k}) \to (\rho_g^v)_* \mathcal{L}oc_\lambda^G(M)_{\mathfrak{Y}_v,k}$$

satisfying the properties (i), (ii) and (iii). Let $\tilde{\phi}_g^v : M_{vg,k} \to M_{v,k}$ denote the dual map to [13]

$$\begin{array}{ccc} V_{\mathbb{G}_v(k)^\circ-\text{an}} & \to & V_{\mathbb{G}_{vg}(k)^\circ-\text{an}} \\ w & \mapsto & g^{-1}w. \end{array}$$

Let $\mathcal{U} \subseteq \mathfrak{Y}_{vg}$ be an open subset and $P \in \mathscr{D}^\dagger_{\mathfrak{Y}_{vg},k}(\lambda)(\mathcal{U})$, $m \in M_{vg,k}$. We define

$$\phi_{g,\mathcal{U}}^v(P \otimes m) := T_{g,\mathcal{U}}^v(P) \otimes \tilde{\phi}_g^v(m). \tag{6.42}$$

Exactly as we have done in theorem 6.5.6, the family (ϕ_g^v) satisfies the requirements (i), (ii) and (iii). Let us verify now condition (b). Given $(\mathfrak{Y}_{v'}, k') \geq (\mathfrak{Y}_v, k)$ in $\underline{\mathcal{F}}$, we have $\mathbb{G}_{v'}(k')^\circ \subseteq \mathbb{G}_v(k)^\circ$ in $\mathbb{G}^{\text{rig}}$ and we denote by $\tilde{\psi}_{\mathfrak{Y}_{v'},\mathfrak{Y}_v} : M_{v',k'} \to M_{v,k}$ the map dual to the natural inclusion $V_{\mathbb{G}_v(k)^\circ-\text{an}} \subseteq V_{\mathbb{G}_{v'}(k')^\circ-\text{an}}$. Let $\mathcal{U} \subseteq \mathfrak{Y}_{v'}$ be an open subset and $P \in \pi_* \mathscr{D}^\dagger_{\mathfrak{Y}_{v'},k'}(\lambda)(\mathcal{U})$, $m \in M_{v',k'}$. We then define [14]

$$\psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v}(P \otimes m) := \Psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v}(P) \otimes \tilde{\psi}_{\mathfrak{Y}_{v'},\mathfrak{Y}_v}(m)$$

where $\Psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v} : \pi_* \mathscr{D}^\dagger_{\mathfrak{Y}_{v'},k'}(\lambda) \to \mathscr{D}^\dagger_{\mathfrak{Y}_v,k}(\lambda)$ is the canonical morphism given by the preceding proposition. This definition extends to a map

$$\psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v} : \pi_* \mathcal{L}oc_\lambda^G(M)_{\mathfrak{Y}_{v'},k'} \to \mathcal{L}oc_\lambda^G(M)_{\mathfrak{Y}_v,k}$$

which satisfies all the required conditions. The functoriality of $\mathcal{L}oc_\lambda^G(\bullet)$ can be verified exactly as we have done for the functor $\mathcal{L}oc_\lambda^{G_0}(\bullet)$. □

Claim 2. $\Gamma(\mathscr{M})$ is a coadmissible $D(G, L)_\lambda$-module.

Proof. We already know that $\Gamma(\mathscr{M})$ is a coadmissible $D(G_{v,0}, L)_\lambda$-module for any v (theorem 6.5.6). So it suffices to exhibit a compatible G-action on $\Gamma(\mathscr{M})$. Let $g \in G$. The isomorphisms $\phi_g^v : \mathscr{M}_{\mathfrak{Y}_{vg},k} \to (\rho_g^v)_* \mathscr{M}_{\mathfrak{Y},k}$, which are compatibles with transitions maps, induce isomorphisms at the level of global sections (which we denote again by ϕ_g^v to soft the notation)

$$\phi_g^v : H^0(\mathfrak{Y}_{vg,k}, \mathscr{M}_{\mathfrak{Y}_{vg},k}) \to H^0(\mathfrak{Y}_v \mathscr{M}_{\mathfrak{Y},k}).$$

[13] Here we use $\mathbb{G}_{vg}(k)^\circ = g^{-1}\mathbb{G}_v(k)^\circ g$ in $\mathbb{G}^{\text{rig}}$.

[14] We avoid the subscript $\mathcal{U}$ in order to soft the notation.

Let us identify

$$\Gamma(\mathscr{M}) = \varprojlim_{(\mathfrak{Y}_{vg},k)\in\underline{\mathcal{F}}_{vg}} H^0(\mathfrak{Y}_{vg,k}, \mathscr{M}_{\mathfrak{Y}_{vg},k})$$
$$= \left\{ \left(m_{\mathfrak{Y}_{vg},k}\right)_{(\mathfrak{Y}_{vg},k)\in\underline{\mathcal{F}}_{vg}} \in \prod_{(\mathfrak{Y}_{vg},k)\in\underline{\mathcal{F}}_{vg}} H^0(\mathfrak{Y}_{vg,k}, \mathscr{M}_{\mathfrak{Y}_{vg},k}) \mid \psi_{\mathfrak{Y}'_{vg},\mathfrak{Y}_{vg}}(m_{\mathfrak{Y}'_{vg},k}) = m_{\mathfrak{Y}_{vg},k} \right\}$$

Where we have abused of the notation and we have denoted by $\psi_{\mathfrak{Y}'_{vg},\mathfrak{Y}_{vg}}$ the morphism obtained by taking global sections on the morphism $\psi_{\mathfrak{Y}'_{vg},\mathfrak{Y}_{vg}} : (\pi.g)_*\mathscr{D}^{\dagger}_{\mathfrak{Y}'_{vg},k'}(\lambda) \to \mathscr{D}^{\dagger}_{\mathfrak{Y}_{vg},k}(\lambda)$. For $g \in G$ and $m := (m_{\mathfrak{Y}_{vg},k})_{(\mathfrak{Y}_{vg},k)\in\underline{\mathcal{F}}_{vg}} \in \Gamma(\mathscr{M})$ we define

$$g.m := \left(\phi^v_g(m_{\mathfrak{Y}_{vg},k})\right)_{(\mathfrak{Y}_{vg},k)\in\underline{\mathcal{F}}_{vg}} \in \prod_{(\mathfrak{Y}_v,k)\in\underline{\mathcal{F}}_v} H^0(\mathfrak{Y}_v, \mathscr{M}_{\mathfrak{Y}_v,k}), \qquad g.m\,_{(\mathfrak{Y}_v,k)\in\underline{\mathcal{F}}_v} := \phi^v_g(m_{\mathfrak{Y}_{vg},k}) \tag{6.43}$$

We want to see that $g.m \in \Gamma(\mathscr{M}) = \varprojlim_{(\mathfrak{Y}_v,k)\in\underline{\mathcal{F}}_v} H^0(\mathfrak{Y}_v, \mathscr{M}_{\mathfrak{Y}_v,k})$ and that this assignment defines a left G-action on $\Gamma(\mathscr{M})$. Taking global sections on (6.40) we get the commutative diagram

$$\begin{array}{ccc} H^0(\mathfrak{Y}'_{vg}, \mathscr{M}_{\mathfrak{Y}'_{vg},k'}) & \xrightarrow{\psi_{\mathfrak{Y}'_{vg},\mathfrak{Y}_{vg}}} & H^0(\mathfrak{Y}_{vg}, \mathscr{M}_{\mathfrak{Y}_v,k}) \\ \downarrow \phi^v_g & & \downarrow \phi^v_g \\ H^0(\mathfrak{Y}'_v, \mathscr{M}_{\mathfrak{Y}'_v,k'}) & \xrightarrow{\psi_{\mathfrak{Y}'_v,\mathfrak{Y}_v}} & H^0(\mathfrak{Y}_v, \mathscr{M}_{\mathfrak{Y}_v,k}) \end{array}$$

which implies that

$$\begin{aligned} \psi_{\mathfrak{Y}'_v,\mathfrak{Y}_v}(g.m\,_{\mathfrak{Y}'_v,k'}) &= \psi_{\mathfrak{Y}'_v,\mathfrak{Y}_v}(\phi^v_g(m_{\mathfrak{Y}'_{vg},k'})) \\ &= \phi^v_g(\psi_{\mathfrak{Y}'_{vg},\mathfrak{Y}_{vg}}(m_{\mathfrak{Y}'_{vg},k'})) \\ &= \phi^v_g(m_{\mathfrak{Y}_{vg},k}) \\ &= g.m\,_{\mathfrak{Y}_v,k}. \end{aligned}$$

We obtain an isomorphism

$$\Gamma(\mathscr{M}) = \varprojlim_{\underline{\mathcal{F}}_{vg}} H^0(\mathfrak{Y}_{vg}, \mathscr{M}_{\mathfrak{Y}_{vg},k}) \xrightarrow{g} \varprojlim_{\underline{\mathcal{F}}_v} H^0(\mathfrak{Y}_v, \mathscr{M}_{\mathfrak{Y}_v,k}) = \Gamma(\mathscr{M}).$$

According to (i) in (a) we have the sequence

$$\phi^v_{hg} : H^0(\mathfrak{Y}_{vhg}, \mathscr{M}_{\mathfrak{Y}_{vhg},k}) \xrightarrow{\phi^{vh}_g} H^0(\mathfrak{Y}_{vh}, \mathscr{M}_{\mathfrak{Y}_{vh},k}) \xrightarrow{\phi^v_h} H^0(\mathfrak{Y}_v, \mathscr{M}_{\mathfrak{Y}_v,k})$$

which tells us that $h.(g.m) = (hg).m$, for $h, g \in G$ and $m \in \Gamma(\mathscr{M})$. This gives a G-action on $\Gamma(\mathscr{M})$ which, by construction, is compatible with its various $D(G_{v,0}, L)$-module structures. □

Claim 3. $\Gamma \circ \mathcal{L}oc^G_\lambda(M) \simeq M$.

Proof. By theorem 6.5.6 we know that this holds as a coadmissible $D(G_0, L)_\lambda$-module, so we need to identify the G-action on both sides. Let v be a special vertex. According to (6.42), the action

$$\Gamma \circ \mathcal{L}oc_{\lambda}^{G}(M) \simeq \varprojlim_{k} M_{vg,k} \to \varprojlim_{v} M_{v,k} \simeq \Gamma \circ \mathcal{L}oc_{\lambda}^{G}(M)$$

of an element $g \in G$ on $\Gamma \circ \mathcal{L}oc_{\lambda}^{G}(M)$ is induced by $\tilde{\phi}_{g}^{v} : M_{vg,k} \to M_{v,k}$. By dualizing

$$V = \bigcup_{k\in\mathbb{N}} V_{\mathbb{G}_{vg}(k)^{\circ}-\text{an}} = \bigcup_{k\in\mathbb{N}} V_{\mathbb{G}_{v}(k)^{\circ}-\text{an}}$$

we obtain the identification

$$M \simeq \varprojlim_{k} M_{vg,k} \simeq \varprojlim_{k} M_{v,k},$$

and therefore we get back the original action of g on M. □

Claim 4. $\mathcal{L}oc_{\lambda}^{G} \circ \Gamma(\mathscr{M}) \simeq \mathscr{M}$.

Proof. We know that $\mathcal{L}oc_{\lambda}^{G}(\Gamma(\mathscr{M}))_{\mathfrak{Y}_v,k} = \mathscr{M}_{\mathfrak{Y}_v,k}$ as $\mathscr{D}^{\dagger}_{\mathfrak{Y}_v,k}(\lambda)$-modules for any $(\mathfrak{Y}_v, k) \in \underline{\mathcal{F}}$, cf. theorem 6.5.6. It remains to verify that these isomorphisms are compatible with the maps ϕ_{g}^{v} and $\psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v}$ on both sides. To do that, let us see that the maps ϕ_{g}^{v} on the left-hand side are induced by the maps of the right-hand side. Given

$$\phi_{g}^{v} : \mathscr{M}_{\mathfrak{Y}_v,k} \to (\rho_{g}^{v})_{*}\mathscr{M}_{\mathfrak{Y}_v,k},$$

the corresponding map

$$\phi_{g}^{v} : \mathcal{L}oc_{\lambda}^{G}(\Gamma(\mathscr{M}))_{\mathfrak{Y}_{vg},k} \to (\rho_{g}^{v})_{*}(\mathcal{L}oc_{\lambda}^{G}(\Gamma(\mathscr{M}))_{\mathfrak{Y}_v,k})$$

equals the map

$$\mathscr{D}^{\dagger}_{\mathfrak{Y}_{vg},k}(\lambda) \otimes_{D^{\text{an}}(\mathbb{G}_{vg}(k)^{\circ})_{\lambda}} H^{0}\left(\mathfrak{Y}_{vg}, \mathscr{M}_{\mathfrak{Y}_{vg},k}\right) \to (\rho_{g}^{v})_{*}\left(\mathscr{D}^{\dagger}_{\mathfrak{Y}_v,k}(\lambda) \otimes_{D^{\text{an}}(\mathbb{G}_{v}(k)^{\circ})_{\lambda}} H^{0}\left(\mathfrak{Y}_v, \mathscr{M}_{\mathfrak{Y}_{v,k}}\right)\right)$$

given locally by $T^{v}_{g,\mathfrak{Y}_{gv}} \otimes H^{0}(\mathfrak{Y}_{vg}, \phi_{g}^{v})$, cf. (6.42). Let $\mathcal{U} \subseteq \mathfrak{Y}_v$ be an open subset and $P \in \mathscr{D}^{\dagger}_{\mathfrak{Y}_v,k}(\lambda)(\mathcal{U})$, $m \in M_{v,k} = H^{0}(\mathfrak{Y}_{vg}, \mathscr{M}_{\mathfrak{Y}_{vg},k})$. The isomorphism $\mathcal{L}oc_{\lambda}^{G}(\Gamma(\mathscr{M}))_{\mathfrak{Y}_v,k} \simeq \mathscr{M}_{\mathfrak{Y}_v,k}$ are induced (locally) by $P \otimes m \mapsto P.(m|_{\mathcal{U}})$. Condition (ii) tells us that these morphisms interchange the maps ϕ_{g}^{v}, as desired. The compatibility with transitions maps $\psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v}$ for two models $(\mathfrak{Y}_{v'}, k') \succeq (\mathfrak{Y}, k)$ in $\underline{\mathcal{F}}$ is deduced in a entirely similar manner as we have done in theorem 6.5.6 and the fact that $\psi_{\mathfrak{Y}_{v'},\mathfrak{Y}_v}$ is linear relative to the canonical morphism $\Psi : \pi_{*}\mathscr{D}^{\dagger}_{\mathfrak{Y}_{v'},k'}(\lambda) \to \mathscr{D}^{\dagger}_{\mathfrak{Y}_v,k}$. □

This ends the proof of the theorem. □

As in the case of the group G_0, we now indicate how objects from $\mathscr{C}^{\mathcal{F}}_{G,\lambda}$ can be realized as honest G-equivariant sheaves on the G-space $\mathfrak{X}_{\infty}$. The following discussion is an adaptation of the discussion given in [50, 5.4.3 and proposition 5.4.5] to our case.

Proposition 6.6.7. *The G_0-equivariant structure of the sheaf $\mathscr{D}(\lambda)$ extends to a G-equivariant structure.*

Proof. Let $g \in G$ and let $v, v' \in \mathcal{B}$ be special vertexes. Let us suppose that $(\mathfrak{Y}_{v'}, k') \succeq (\mathfrak{Y}_v, k)$ in $\underline{\mathcal{F}}$. The isomorphism $\rho_{g}^{v'} : \mathfrak{Y}_{v'} \to \mathfrak{Y}_{v'g}$ induces a ring isomorphism

$$T_{g}^{v'} : \mathscr{D}^{\dagger}_{\mathfrak{Y}_{v'g},k'}(\lambda) \to \left(\rho_{g}^{v'}\right)_{*}\mathscr{D}^{\dagger}_{\mathfrak{Y}_{v'},k'}(\lambda).$$

On the other hand, and exactly as we have done in (6.33), the commutative diagram

$$\begin{array}{ccccc} \mathfrak{X}_\infty & \xrightarrow{sp_{\mathfrak{Y}_v}} & \mathfrak{Y}_v & \xrightarrow{\rho_g^v} & \mathfrak{Y}_{vg} \\ \downarrow{\scriptstyle sp_{\mathfrak{Y}_{v'}}} & \nearrow{\scriptstyle \pi} & & \nearrow{\scriptstyle \pi.g} & \\ \mathfrak{Y}_{v'} & \xrightarrow{\rho_g^{v'}} & \mathfrak{Y}_{v'g}. & & \end{array}$$

defines a continuous function

$$\begin{array}{cccc} \rho_g : & \mathfrak{X}_\infty & \to & \mathfrak{X}_\infty \\ & (a_v) & \mapsto & (\rho_g^v(a_v)), \end{array}$$

which fits into a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}_\infty & \xrightarrow{\rho_g} & \mathfrak{X}_\infty \\ \downarrow{\scriptstyle \mathrm{pr}_{\mathfrak{Y}_{v'}}} & & \downarrow{\scriptstyle \mathrm{pr}_{\mathfrak{Y}_{v'g}}} \\ \mathfrak{Y}_{v'} & \xrightarrow{\rho_g^{v'}} & \mathfrak{Y}_{v'g}. \end{array}$$

In particular, if $V \subseteq \mathfrak{X}_\infty$ is an open subset of the $V := \mathrm{pr}_{\mathfrak{Y}_v}^{-1}(\mathcal{U})$ with $\mathcal{U} \subseteq \mathfrak{Y}_v$ an open subset. Then

$$\left(\rho_g^{v'}\right)^{-1}\left(\mathrm{pr}_{\mathfrak{Y}_{v'g}}(V)\right) = \mathrm{pr}_{\mathfrak{Y}_{v'}}\left(\rho_g^{-1}(V)\right)$$

and so the map $T_g^{v'}$ induces the morphism

$$\mathscr{D}^\dagger_{\mathfrak{Y}_{v'g},k'}(\lambda)\left(\mathrm{pr}_{\mathfrak{Y}_{v'g}}(V)\right) \to \mathscr{D}^\dagger_{\mathfrak{Y}_{v'},k'}(\lambda)\left(\mathrm{pr}_{\mathfrak{Y}_{v'}}\left(\rho_g^{-1}(V)\right)\right). \tag{6.44}$$

Moreover, if $(\mathfrak{Y}_{v''}, k'') \ge (\mathfrak{Y}_{v'}, k') \ge (\mathfrak{Y}_v, k)$ in $\underline{\mathcal{F}}$, and as before $V := \mathrm{pr}_{\mathfrak{Y}_v}^{-1}(\mathcal{U}) \subseteq \mathfrak{X}_\infty$ with $\mathcal{U} \subseteq \mathfrak{Y}_v$ an open subset, then the commutative diagram

$$\begin{array}{ccc} \mathscr{D}^\dagger_{\mathfrak{Y}_{v''g},k''}(\lambda)\left(\mathrm{pr}_{\mathfrak{Y}_{v''g}}(V)\right) & \longrightarrow & \mathscr{D}^\dagger_{\mathfrak{Y}_{v''},k''}(\lambda)\left(\mathrm{pr}_{\mathfrak{Y}_{v''}}\left(\rho_g^{-1}(V)\right)\right). \\ \downarrow & & \downarrow \\ \mathscr{D}^\dagger_{\mathfrak{Y}_{v'g},k'}(\lambda)\left(\mathrm{pr}_{\mathfrak{Y}_{v'g}}(V)\right) & \longrightarrow & \mathscr{D}^\dagger_{\mathfrak{Y}_{v'},k'}(\lambda)\left(\mathrm{pr}_{\mathfrak{Y}_{v'}}\left(\rho_g^{-1}(V)\right)\right). \end{array}$$

implies that if, by cofinality, we identify $\mathscr{D}(\lambda)(V) = \varprojlim_{(\mathfrak{Y}_{vg},k)\in\underline{\mathcal{F}}_{vg}} \mathscr{D}^\dagger_{\mathfrak{Y}_{vg},k}(\lambda)\left(\mathrm{pr}_{\mathfrak{Y}_{vg}}(V)\right)$ and we take projective limits in (6.44), then we get a ring homomorphism

$$T_{g,V} : \mathscr{D}(\lambda)(V) \to (\rho_g)_* \mathscr{D}(\lambda)(V)$$

which implies that the sheaf $\mathscr{D}(\lambda)$ is G-equivariant. Furthermore, from construction this G-quivariant structure extends the G_0-structure defined in (6.35). □

Finally, let us recall the faithful functor

$$\mathscr{M} \rightsquigarrow \mathscr{M}_\infty$$

from coadmissible G_0-equivariant arithmetic $\mathscr{D}(\lambda)$-modules on $\mathcal{F}_{\mathfrak{X}}$ to G_0-equivariant $\mathscr{D}(\lambda)$-modules on $\mathfrak{X}_\infty$. If $\mathscr{M}$ comes from a coadmissible G-equivariant $\mathscr{D}(\lambda)$-module on $\mathcal{F}$, then $\mathscr{M}_\infty$ is in fact G-equivariant (as in (6.35), this can be proved by using the family of L-linear isomorphisms $(\phi_g^v)_{g\in G}$. As in proposition 6.5.9, the preceding theorem gives us

Theorem 6.6.8. *Let us suppose that $\lambda \in Hom(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character such that $\lambda + \rho \in \mathfrak{t}_L^*$ is a dominant and regular character of $\mathfrak{t}_L$. The functor $\mathscr{M} \rightsquigarrow \mathscr{M}_\infty$ from the category $\mathscr{C}_{G,\lambda}^{\mathcal{F}}$ to G-equivariant $\mathscr{D}(\lambda)$-modules on $\mathfrak{X}_\infty$ is a faithful functor.*

We summarize the main results of this work with the following commutative diagrams of functors (cf. [50, Theorem 5.4.10])

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \text{Coadmissible} \\ D(G,L)_\lambda - \text{modules} \end{array} \right\} & \xrightarrow[\simeq]{\mathcal{L}oc_\lambda^G} & \left\{ \begin{array}{c} \text{Coadmissible } G - \text{equivariant} \\ \text{arithmetic } \mathscr{D}(\lambda) - \text{modules} \end{array} \right\} \\
\downarrow & & \downarrow \\
\left\{ \begin{array}{c} \text{Coadmissible} \\ D(G_0,L)_\lambda - \text{modules} \end{array} \right\} & \xrightarrow[\simeq]{\mathcal{L}oc_\lambda^{G_0}} & \left\{ \begin{array}{c} \text{Coadmissible } G_0 - \text{equivariant} \\ \text{arithmetic } \mathscr{D}(\lambda) - \text{modules} \end{array} \right\}
\end{array}$$

Here the left-hand vertical arrow is the restriction functor coming from the homomorphism

$$D(G_0, L)_\lambda \to D(G, L)_\lambda$$

and the right-hand vertical arrow is the forgetful functor.

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