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Représentations associées à des graduations d'algèbres de Lie et d'algèbres de Lie colorées

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Introduction

Cette thèse se place dans le contexte d'un principe de reconstruction d'un objet mathématique à partir de fragments de cet objet. Supposant que l'on brise en morceaux une structure possèdant certaines symétries, il se pose deux questions : comment les morceaux encodent-ils le fait qu'ils proviennent d'une structure mathématique supérieure ? comment effectuer la reconstruction à partir des morceaux ?

Dans la théorie des algèbres de Lie, ce principe s'applique notamment aux graduations d'algèbres de Lie. Une algèbre de Lie $\tilde{\mathfrak{g}}$ est graduée par un groupe abélien Γ, si elle se décompose sous la forme

$$\tilde{\mathfrak{g}} = \bigoplus_{a \in \Gamma} \mathfrak{g}_a$$

tel que

$$[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b} \qquad \forall a, b \in \Gamma.$$

En particulier, $\mathfrak{g}_0$ est une sous-algèbre de Lie de $\tilde{\mathfrak{g}}$ et pour tout a dans Γ, $\mathfrak{g}_a$ est une représentation de $\mathfrak{g}_0$. Cette idée de reconstruction peut ainsi s'exprimer de la manière suivante : comment reconstruire l'algèbre de Lie $\tilde{\mathfrak{g}}$ à partir de l'algèbre de Lie $\mathfrak{g}_0$ et de ses représentations $\mathfrak{g}_a$? Dans le cas où $\Gamma = \mathbb{Z}$ et la graduation est de longueur 3, ce problème a été beaucoup étudié par des mathématiciens strasbourgeois. Par exemple, dans [MRS86] les représentations d'algèbres de Lie $\rho : \mathfrak{g} \to End(V)$ associées aux espaces préhomogènes définissant une structure d'algèbre de Lie sur $\tilde{\mathfrak{g}} = V^* \oplus \mathfrak{g} \oplus V$ sont examinées.

B. Kostant a étudié un problème similaire dans le contexte des représentations orthogonales $\mathfrak{g} \to \mathfrak{so}(V, (\ ,\ )_V)$ d'algèbres de Lie quadratiques $(\mathfrak{g}, (\ ,\ )_{\mathfrak{g}})$ (voir [Kos56], [Kos99]). Il s'est posé la question : comment construire un crochet de Lie sur l'espace vectoriel $\mathfrak{g} \oplus V$ qui étend le crochet de $\mathfrak{g}$, l'action de $\mathfrak{g}$ sur V et telle que $(\ ,\ )_{\mathfrak{g}} \perp (\ ,\ )_V$ soit invariante ?

Il a d'abord montré que la composante dans $\mathfrak{g}$ d'un tel crochet est entièrement déterminée par l'application moment

$$\mu : V \times V \to \mathfrak{g}$$

associée à la représentation, et que la composante dans V de ce crochet définie une forme trilinéaire $\phi \in \Lambda^3(V)^*$ invariante par l'action de $\mathfrak{g}$. Puis, réciproquement, il a montré que le crochet sur $\mathfrak{g} \oplus V$ défini par le moment d'une représentation orthogonale V et par une forme trilinéaire $\mathfrak{g}$-invariante sur V est un crochet de Lie si et seulement si un certain invariant à valeurs dans l'algèbre de Clifford $C(V, (\ ,\ )_V)$ s'annule.

Dans le cas où ϕ est non-nulle et où $\mathfrak{g} \oplus V$ est une algèbre de Lie, Kostant ([Kos99]) a ensuite construit un opérateur cubique de Dirac, qui est un élément de $U(\mathfrak{g} \oplus V) \otimes C(V, (\ ,\ )_V)$, et il a montré une formule analogue à celle de Parthasarathy pour son carré (voir [Par72]).

Sur le corps des réels, le cas où $\mathfrak{g}$ est une somme directe d'une algèbre de Lie compacte $\mathfrak{h}$ et de l'algèbre de Lie $\mathfrak{su}(2)$ et où la forme quadratique sur V est un produit scalaire a été étudié en détails dans [MS13]. Ce type de graduation permet notamment de donner une construction des algèbres de Lie réelles exceptionnelles simples de type compact.

Sur un corps k de caractéristique différente de 2 et 3, le cas où $\mathfrak{g}$ est une somme directe d'une algèbre de Lie $\mathfrak{h}$ et de l'algèbre de Lie $\mathfrak{sl}(2, k)$ et où V est le produit tensoriel d'une représentation symplectique de $\mathfrak{h}$ et de la représentation naturelle symplectique k^2 de $\mathfrak{sl}(2, k)$ a été étudié en détails dans [SS15]. Ces décompositions permettent entre autres de donner une construction des algèbres de Lie exceptionnelles simples de type déployé.

B. Kostant a ensuite étendu ses résultats aux représentations symplectiques d'algèbres de Lie quadratiques provenant de la $\mathbb{Z}_2$-graduation canonique d'une superalgèbre de Lie superquadratique (voir [Kos01]). Dans ce cas, l'invariant pertinent est un élément d'une algèbre de Weyl plutôt que d'une algèbre de Clifford. Voir aussi [CK15] pour le cas des représentations superquadratiques des superalgèbres de Lie.

Bien que le but initial de la thèse ait été d'établir un lien direct entre les reconstructions considérées dans [MS13] et [SS15], son contenu se porte en grande partie (chapitres 3,4 et 5) sur une généralisation du problème de reconstruction de B. Kostant au cas des représentations d'algèbres de Lie colorées. Une algèbre de Lie colorée est une généralisation des algèbres de Lie et des superalgèbres de Lie introduite par M. Scheunert (voir [Sch79a]). Les autres chapitres de la thèse traitent respectivement d'algèbres de Lie simples de dimension 3 et de superalgèbres de Lie dont la partie paire est une algèbre de Lie simple de dimension 3. Ces derniers peuvent être lus indépendamment.

Nous allons maintenant donner une description plus détaillée du contenu des chapitres de cette thèse.

Dans la comparaison des graduations d'algèbres de Lie étudiées dans [MS13] et [SS15], on remarque qu'une algèbre de Lie simple de dimension 3 joue un rôle important. Ainsi, l'objet du premier chapitre est l'étude de la structure des algèbres de Lie simples de dimension 3 sur un corps k de caractéristique différente de 2.

Dans la première moitié de ce chapitre nous :

- rappelons la forme normale d'une algèbre de Lie simple de dimension 3 (voir [Jac58] ou [Mal92]) ;

- caractérisons les formes quadratiques ternaires qui sont des formes de Killing d'une algèbre de Lie simple de dimension 3 ;
- donnons la construction explicite d'une algèbre de Lie simple de dimension 3 à partir d'un trivecteur et d'une forme de Killing ;
- construisons l'algèbre de quaternions associée à une algèbre de Lie simple de dimension 3.

Dans la deuxième moitié du premier chapitre, on donne un procédé pour obtenir une algèbre de Lie simple non-déployée de dimension 3 à partir de $\mathfrak{sl}(2,k)$ en s'inspirant de la construction de $\mathfrak{su}(2)$ à partir d'une involution de Cartan de $\mathfrak{sl}(2,\mathbb{R})$. Pour cela, on part de $\mathfrak{sl}(2,k)$ muni d'une involution σ telle que $ad(X)$ n'est pas diagonalisable pour tout point fixe X de σ.

On démontre que l'algèbre de Lie obtenue par notre procédé est non-déployée si et seulement si $\frac{K(X,X)}{2}$ n'est pas une somme de deux carrés dans k pour tout point fixe X de σ (K désignant la forme de Killing de $\mathfrak{sl}(2,k)$). Les algèbres de Lie simples de dimension 3 non-déployées que l'on peut obtenir grâce à ce procédé sont caractérisées par le fait que leur forme de Killing est anisotrope et représente -2. Pour les corps locaux et globaux, on caractérise ces algèbres de Lie aussi en termes de symbole de Hilbert et de symbole de Legendre. On donne des exemples explicites d'algèbres de Lie simples de dimension 3 rationnelles non-déployées qui peuvent et ne peuvent pas être obtenues par ce procédé.

Dans la continuité du premier chapitre et du thème général de la thèse, celui de reconstruire une structure d'algèbre de Lie (colorée) via une représentation $\mathfrak{g} \to End(V)$, on se demande dans le deuxième chapitre : quelles sont les représentations $\mathfrak{s} \to End(V)$ des algèbres de Lie $\mathfrak{s}$ simples de dimension 3 permettant de définir une structure de superalgèbre de Lie sur l'espace vectoriel $\mathfrak{s} \oplus V$? Dans ce chapitre, contrairement à Kostant ([Kos01]), nous ne supposons pas que la représentation V est symplectique.

Sur le corps des complexes et avec l'hypothèse de simplicité de $\mathfrak{s} \oplus V$, la réponse à cette question peut se déduire de la classification des superalgèbres de Lie complexes simples de dimension finie de V. Kac (voir [Kac75]). Dans ce cas, la seule possibilité, à isomorphisme près, est la superalgèbre de Lie complexe $\mathfrak{osp}_{\mathbb{C}}(1|2)$. Sur le corps des nombres réels et de nouveau avec l'hypothèse de simplicité de $\mathfrak{s} \oplus V$, la réponse, qui peut se déduire de la même manière de la classification des superalgèbres de Lie réelles simples de dimension finie de V. Serganova (voir [Ser83]), est la même : la seule possibilité, à isomorphisme près, est la superalgèbre de Lie réelle $\mathfrak{osp}_{\mathbb{R}}(1|2)$. Cependant sur un corps général k, il n'y a actuellement pas de classification des superalgèbres de Lie simples de dimension finie que l'on pourrait utiliser de même pour répondre à notre question. Néanmoins, remarquons que, si k est algébriquement clos et si $car(k) > 5$, S. Bouarroudj et D. Leites ont classifié les superalgèbres de Lie de dimension finie sur k ayant une matrice de Cartan indécomposable sous des hypothèses techniques (voir [BL07]). Dans cette classification, la seule superalgèbre de Lie simple dont la partie paire est une algèbre de Lie simple de dimension 3 est $\mathfrak{osp}_k(1|2)$.

Nous démontrons le théorème de classification suivant :

Théorème 1. (voir [Mey17]) *Soit k un corps de caractéristique différente de 2 et 3. Soit $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ une superalgèbre de Lie sur k de dimension finie telle que $\mathfrak{g}_0$ est une algèbre de Lie simple de dimension 3 et soit*

$$\mathcal{Z}(\mathfrak{g}) := \{x \in \mathfrak{g} \mid \{x, y\} = 0 \qquad \forall y \in \mathfrak{g}\}.$$

Alors, il y a trois cas :

a) $\{\mathfrak{g}_1, \mathfrak{g}_1\} = \{0\}$ *;*

b) $\mathfrak{g} \cong \mathfrak{g}_0 \oplus (\mathfrak{g}_0 \oplus k) \oplus \mathcal{Z}(\mathfrak{g})$ *;*

c) $\mathfrak{g} \cong \mathfrak{osp}_k(1|2) \oplus \mathcal{Z}(\mathfrak{g})$*.*

Pour cette classification nous ne supposons pas que k est algébriquement clos et nous ne supposons pas que $\mathfrak{g}$ est simple. Remarquons que, dans cette liste, $\mathfrak{osp}_k(1|2)$ est la seule superalgèbre de Lie simple non-triviale et que les superalgèbres de Lie de la famille b) ne peuvent pas être obtenues par la construction de Kostant (voir [Kos01]).

Il découle aussi de ce théorème que si k est de caractéristique positive et si la restriction du crochet à $\mathfrak{g}_1$ est non nulle, alors $\mathfrak{g}$ est une superalgèbre de Lie restreinte au sens de V. Petrogradski (voir [Pet92]) et Y. Wang-Y. Zhang (voir [WZ00]).

Le principe de la preuve est d'abord de démontrer ce résultat quand k est algébriquement clos et quand $\mathfrak{g}_0 = \mathfrak{sl}(2, k)$. Un point essentiel ici est que pour de tels corps, la classification des représentations irréductibles de dimension finie de $\mathfrak{sl}(2, k)$ est connue (pour k de caractéristique positive et algébriquement clos voir par exemple [RS67] ou [SF88]). Ceci, avec l'étude de la restriction du crochet aux sous $\mathfrak{sl}(2, k)$-modules de $\mathfrak{g}_1$ est l'ingrédient principal de la preuve. Il se trouve que la difficulté principale a lieu quand k est de caractéristique positive et que les seuls sous-modules irréductibles de $\mathfrak{g}_1$ sont triviaux. Dans ce cas, il n'y a pas de complète réductibilité des représentations de dimension finie de $\mathfrak{sl}(2, k)$ mais, utilisant notamment une observation de H. Strade (voir [Str04]), on montre que le crochet restreint à $\mathfrak{g}_1$ est trivial comme attendu.

Une fois ce résultat prouvé sous les conditions restreintes ci-dessus, on utilise trois résultats d'ordre général pour passer à un corps non-algébriquement clos et à une algèbre de Lie $\mathfrak{g}_0$ simple de dimension 3 arbitraire.

Notre classification est valable sur un corps de caractéristique différente de 2 et 3. En caractéristique 3 un contre-exemple à cette classification est donné. Il s'agit d'une superalgèbre de Lie de la forme $\mathfrak{sl}(2,k) \oplus V$ où V est de dimension trois et ayant $\mathfrak{osp}_k(1|2)$ comme sous-superalgèbre de Lie. En caractéristique 2, on donne un exemple d'une superalgèbre de Lie dont la partie paire est $\mathfrak{sl}(2,k)$, dont la partie impaire correspond à la représentation adjointe de $\mathfrak{sl}(2,k)$ et qui n'est pas dans la liste du théorème. Cependant, $\mathfrak{sl}(2,k)$ n'étant plus simple, ce n'est pas à proprement parler un contre-exemple au théorème ci-dessus. En caractéristique 2 nous n'avons pas de contre-exemple à ce théorème.

Dans la suite de la thèse, on passe à la généralisation du problème de Kostant au cadre des représentations d'algèbres de Lie colorées.

Le troisième chapitre est consacré au développement des outils nécessaires à l'étude de ce problème :

- l'algèbre multilinéaire dans la catégorie des espaces vectoriels gradués par un groupe abélien Γ (voir [CK16]);
- l'application moment d'une représentation ϵ-orthogonale d'une algèbre de Lie colorée ϵ-quadratique.

Après les définitions et quelques exemples d'espaces vectoriels et d'algèbres gradués par un groupe abélien Γ, on rappelle la définition d'un facteur de commutation $\epsilon : \Gamma \times \Gamma \to k$ (voir [Bou70]). Un facteur de commutation permet de modifier l'action standard du groupe symétrique S_n sur les tenseurs d'ordre n d'un espace vectoriel Γ-gradué, ainsi que de définir pour les espaces vectoriels Γ-gradués une notion d'application multilinéaire ϵ-symétrique ou ϵ-antisymétrique qui prend compte de la Γ-graduation. Si $\Gamma = \mathbb{Z}_2$ et $\epsilon(n,m) = (-1)^{nm}$ pour tout n, m dans $\mathbb{Z}_2$, on retrouve les notions d'applications linéaires supersymétriques et antisupersymétrique pour les espaces vectoriels $\mathbb{Z}_2$-gradués. On définit ensuite la ϵ-algèbre extérieur $\Lambda_\epsilon(V)$ d'un espace vectoriel Γ-gradué V ainsi que la ϵ-algèbre de Clifford $C_\epsilon(V, (\ ,\ ))$ d'un espace vectoriel Γ-gradué muni d'une forme ϵ-symétrique $(\ ,\ )$ (voir [CK16]). L'algèbre $\Lambda_\epsilon(V)$ est $\Gamma \times \mathbb{Z}$-graduée et $\tilde{\epsilon}$-symétrique pour un certain $\tilde{\epsilon}$ de $\Gamma \times \mathbb{Z}$. L'algèbre extérieure usuelle $\Lambda(V)$ et l'algèbre symétrique usuelle $S(V)$ d'un espace vectoriel sont des exemples de ϵ-algèbres extérieures pour des choix de (Γ, ϵ) appropriés. De même, l'algèbre de Clifford usuelle et l'algèbre de Weyl usuelle sont des exemples de ϵ-algèbres de Clifford pour des choix de (Γ, ϵ) appropriés. Nous définissons un produit tensoriel $\widehat{\otimes}$ pour les algèbres Γ-graduées tel que le produit de deux algèbres ϵ-symétrique (resp. ϵ-antisymétrique) soit ϵ-symétrique. Ce produit permet notamment de montrer que si V et W sont des espaces vectoriels Γ-gradués, alors on a un isomorphisme naturel :

$$\Lambda_\epsilon(V \oplus W) \cong \Lambda_\epsilon(V) \widehat{\otimes} \Lambda_\epsilon(W).$$

De même il permet de montrer l'isomorphisme naturel des ϵ-algèbres de Clifford :

$$C_\epsilon((V, (\ ,\ )_V) \perp (W, (\ ,\ )_W)) \cong C_\epsilon(V, (\ ,\ )_V) \widehat{\otimes} C_\epsilon(W, (\ ,\ )_W)$$

où $(V, (\ ,\ )_V)$ et $(W, (\ ,\ )_W)$ sont des espaces vectoriels Γ-gradués munis de formes ϵ-quadratiques. Ces isomorphismes résument et généralisent tous les isomorphismes qui expriment l'algèbre extérieure (resp. symétrique, de Clifford, de Weyl) usuelle d'une somme en termes du produit des algèbres extérieures (resp. symétriques, de Clifford, de Weyl) usuelles des facteurs.

Ensuite, dans la deuxième partie du troisième chapitre, on s'intéresse aux algèbres de Lie colorées. Ces algèbres ont été initialement introduites par V. Rittenberg et D. Wyler (voir [RW78a] et [RW78b]) puis étudiées par M. Scheunert (voir [Sch79a] et [Sch83]). Les algèbres de Lie colorées ont notamment un intérêt en physique, par exemple dans [AKTT17] les auteurs montrent qu'une équation de Lévy-Leblond décrivant une onde de spin $\frac{1}{2}$ admet un ensemble de symétries ayant une structure d'algèbre de Lie colorée pour le groupe abélien $\mathbb{Z}_2 \times \mathbb{Z}_2$. Une algèbre de Lie (Γ, ϵ)-colorée (par la suite on dira simplement colorée) est un espace vectoriel Γ-gradué $\mathfrak{g}$ muni d'un crochet bilinéaire qui est ϵ-antisymétrique et qui satisfait à l'identité de Jacobi

$$\epsilon(z,x)\{x,\{y,z\}\} + \epsilon(x,y)\{y,\{z,x\}\} + \epsilon(y,z)\{z,\{x,y\}\} = 0 \qquad \forall x,y,z \in \mathfrak{g}.$$

Les algèbres de Lie et les superalgèbres de Lie sont des algèbres de Lie colorées. Une algèbre Γ-graduée associative A munie du crochet

$$\{a,b\} := ab - \epsilon(a,b)ba \qquad \forall a,b \in A$$

est l'exemple standard d'algèbre de Lie colorée et dans le cas où $A = End(V)$, pour un espace vectoriel Γ-gradué V, on note $\mathfrak{gl}_\epsilon(V)$ cette algèbre de Lie colorée. Pour une notion appropriée de trace, les matrices de trace nulle forment une sous-algèbre de Lie colorée $\mathfrak{sl}_\epsilon(V)$ de $\mathfrak{gl}_\epsilon(V)$. Les autres algèbres de Lie classiques ont aussi un analogue coloré et en particulier, si V est un espace vectoriel Γ-gradué muni d'une forme ϵ-symétrique $(\ ,\ )_V$, l'espace vectoriel

$$\mathfrak{so}_\epsilon(V, (\ ,\ )_V) = \{f \in \mathfrak{gl}_\epsilon(V) \mid (f(v),w)_V + \epsilon(f,v)(v,f(w))_V = 0 \qquad \forall v,w \in V\}$$

est une sous-algèbre de Lie colorée de $\mathfrak{gl}_\epsilon(V)$, l'algèbre de Lie colorée ϵ-orthogonale. Notons que si k est algébriquement clos et de caractéristique nulle, $\mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ est simple tandis que $\mathfrak{sl}_\epsilon(V)$ n'est pas toujours simple (voir [Moo99]).

Dans la dernière partie du troisième chapitre, nous introduisons l'application moment d'une représentation ϵ-orthogonale d'une algèbre de Lie colorée ϵ-quadratique. C'est cette application qui va jouer un rôle central dans les quatrième et cinquième chapitres.

Définition. *Soit $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ une représentation ϵ-orthogonale d'une algèbre de Lie colorée ϵ-quadratique $(\mathfrak{g}, (\ ,\ )_\mathfrak{g})$. L'application moment est l'unique application $\mu : \Lambda^2_\epsilon(V) \to \mathfrak{g}$ telle que*

$$(x, \mu(u,v))_\mathfrak{g} = (\rho(x)(u), v)_V \qquad \forall x \in \mathfrak{g},\ \forall u,v \in V.$$

L'exemple standard d'application moment est μ_{can}, associée à la représentation fondamentale de $\mathfrak{so}_\epsilon(V, (\ ,\ )_V)$. On montre qu'elle vérifie

$$\mu_{can}(u,v)(w) = \epsilon(v,w)(u,w)_V w - (v,w)_V u \qquad \forall u,v,w \in V \tag{1}$$

et qu'elle définit un isomorphisme $\mathfrak{g}$-équivariant entre $\Lambda^2_\epsilon(V)$ et $\mathfrak{so}_\epsilon(V, (\ ,\ )_V)$. Cet isomorphisme généralise le fait que si (V, ω) (resp. (V, g)) est un espace vectoriel symplectique (resp. orthogonal), alors $S^2(V)$ (resp. $\Lambda^2(V)$) est isomorphe à $\mathfrak{sp}(V, \omega)$ (resp. $\mathfrak{so}(V, g)$). Enfin, on note que le produit tensoriel de deux représentations ϵ-orthogonales est ϵ-orthogonale et pour conclure ce chapitre, nous montrons que l'application moment du produit est le produit des applications moments en un sens précis.

Dans le quatrième chapitre nous :

- rappelons les résultats de Kostant (voir [Kos99], [Kos01]) ;
- généralisons les résultats de B. Kostant ([Kos99], [Kos01]) et les résultats de Z. Chen et Y. Kang ([CK15]) aux représentations ϵ-orthogonales des algèbres de Lie colorées ϵ-quadratiques sur un corps de caractéristique différente de 2 et 3 ;
- introduisons la notion de représentation ϵ-orthogonale spéciale et établissons son lien avec la généralisation ci-dessus ;
- définissons les covariants d'une représentation ϵ-orthogonale spéciale et étudions leur géométrie.

Plus précisément, le quatrième chapitre commence par un rappel du travail originel de Kostant (voir [Kos99], [Kos01]) quant au problème de reconstruction d'une algèbre de Lie complexe $\mathbb{Z}_2$-graduée (resp. superalgèbre de Lie) à partir d'une représentation complexe orthogonale (resp. symplectique) V d'une algèbre de Lie complexe quadratique $\mathfrak{g}$.

Nous démontrons le théorème suivant :

Théorème 2. *Soit $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ une représentation ϵ-orthogonale d'une algèbre de Lie colorée ϵ-quadratique et soit $\mu : \Lambda^2_\epsilon(V) \to \mathfrak{g}$ l'application moment associée.*

a) S'il existe une structure d'algèbre de Lie colorée ϵ-quadratique sur $\mathfrak{g} \oplus V$ qui étend le crochet de $\mathfrak{g}$ et l'action de $\mathfrak{g}$ sur V, alors $\phi : \Lambda^2_\epsilon(V) \to \mathfrak{g} \oplus V$ définie par

$$\phi(v,w) = \{v,w\} - \mu(v,w) \qquad \forall v,w \in V$$

satisfait à :

- *$\phi(v,w) \in V$ pour tout $v,w \in V$;*
- *$(u,v,w) \mapsto (\phi(u,v),w)_V$ est une forme ϵ-alternée $\mathfrak{g}$-équivariante de degré 0 ;*

- $(\mu + \phi) \wedge (\mu + \phi) = 0$ *dans* $\Lambda^4_\epsilon(V)^*$.

b) Soit $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus V$*, soit* $\phi : \Lambda^2_\epsilon(V) \to V$ *telle que* $(u, v, w) \mapsto (\phi(u, v), w)_V$ *est une forme* ϵ*-alternée* $\mathfrak{g}$*-équivariante de degré* 0 *et soit* $\{\ ,\ \} : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ *l'unique application bilinéaire* ϵ*-antisymétrique qui étend le crochet de* $\mathfrak{g}$*, l'action de* $\mathfrak{g}$ *sur* V *et telle que*

$$\{v, w\} = \mu(v, w) + \phi(v, w) \qquad \forall v, w \in V.$$

Alors ce crochet définit une structure d'algèbre de Lie colorée ϵ*-quadratique sur* $\tilde{\mathfrak{g}}$ *si et seulement si* $(\mu + \phi) \wedge (\mu + \phi) = 0$ *dans* $\Lambda^4_\epsilon(V)^*$.

Par la suite, une représentation satisfaisant aux conditions (a) du théorème 2 sera appelée une représentation de type Lie ou de type $\mathbb{Z}_2$-Lie si en plus $\phi \equiv 0$.

L'invariant $(\mu + \phi) \wedge (\mu + \phi)$ du théorème précédent prend ses valeurs dans une ϵ-algèbre extérieure. Après identification de la ϵ-algèbre extérieure avec la ϵ-algèbre de Clifford, ce théorème implique les résultats de B. Kostant sur les représentations complexes orthogonales et symplectiques des algèbres de Lie complexes quadratiques (voir [Kos99], [Kos01]) ainsi que les résultats de Z. Chen et Y. Kang sur les représentations complexes orthosymplectiques des superalgèbres de Lie complexes superquadratiques (voir [CK15]).

On donne ensuite une autre interprétation des données (μ, ϕ) du théorème 2.(b) en termes de "tenseurs de courbure", à savoir, on leur associe le 4-tenseur

$$R_{\mu+\phi}(A, B, C, D) = (\{\{A, B\}, C\}, D)_V \qquad \forall A, B, C, D \in V.$$

On montre que la condition $(\mu + \phi) \wedge (\mu + \phi) = 0$ du Théorème 2.(b) est équivalente à une identité de "Bianchi" du type $\beta(R_{\mu+\phi}) = 0$ où $\beta(R_{\mu+\phi})$ est la "ϵ-somme cyclique" sur les arguments de $R_{\mu+\phi}$. Ce point de vue et le fait que $\beta^2 = 3\beta$ suggère une nouvelle condition algébrique naturelle sur les représentations ϵ-orthogonales et cela mène à la notion de représentation ϵ-orthogonale spéciale.

Définition. *Une représentation* ϵ*-orthogonale* $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ *d'une algèbre de Lie colorée* ϵ*-quadratique* $\mathfrak{g}$ *est dite spéciale si*

$$\Big(Id - \frac{1}{3}\beta\Big)(R_\mu) = R_{\mu_{can}}.$$

Cette comparaison entre les applications moment μ et μ_{can} est possible en tant que 4-tenseurs sur V. Remarquons que les représentations symplectiques spéciales d'algèbres de Lie considérées dans [SS15] sont des exemples de représentations ϵ-orthogonales spéciales. Rappelons aussi que parmi ces représentations, figurent

- une représentation demi-spinorielle de dimension 32 d'une algèbre de Lie de type $\mathfrak{so}(12)$ dont cette représentation est définie sur k (voir [SS15]) ;

- l'action de $\mathfrak{sl}(6,k)$ sur les formes trilinéaires alternées $\Lambda^3(k^6)$ ([Rei07], [Cap72]) ;
- l'action de $\mathfrak{sp}(6,\omega)$ sur les 3-formes alternées primitives (voir [SS15]) ;
- l'action de $\mathfrak{sl}(2,k)$ sur les cubiques binaires $S^3(k^2)$ (voir [SS12]) ;
- la représentation irréductible de dimension 56 de l'algèbre de Lie déployée $\mathfrak{e}_7$ (voir [SS15]).

Des définitions équivalentes de la notion de représentation ϵ-orthogonale spéciale s'énoncent de la manière suivante :

Proposition. *Soit $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V,(\ ,\ )_V)$ une représentation ϵ-orthogonale d'une algèbre de Lie colorée ϵ-quadratique $\mathfrak{g}$, soit $\mu : \Lambda^2_\epsilon(V) \to \mathfrak{so}_\epsilon(V,(\ ,\ )_V)$ l'application moment et soit μ_{can} l'application moment (1) ci-dessus. Les conditions suivantes sont équivalentes :*

a) V est une représentation ϵ-orthogonale spéciale ;

b) $\mu(A,B)(C)+\epsilon(B,C)\mu(A,C)(B) = (A,B)_V C + \epsilon(B,C)(A,C)_V B - 2(B,C)_V A$ pour tout $A,B,C \in V$;

c) $\mu(A,B)(C)+\epsilon(B,C)\mu(A,C)(B) = \mu_{can}(A,B)(C) + \epsilon(B,C)\mu_{can}(A,C)(B)$ pour tout $A,B,C \in V$.

Pour établir un lien concret entre les représentations ϵ-orthogonales spéciales et les représentations ϵ-orthogonales de type $\mathbb{Z}_2$-Lie du Théorème 2, on étudie les représentations $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V,(\ ,\ )_V)$ où $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ est une somme directe d'algèbres de Lie colorées ϵ-quadratiques et où $V = V_1 \otimes V_2$ est un produit tensoriel de représentations ϵ-orthogonales. On démontre :

Théorème 3. *Soit $\rho_1 : \mathfrak{g}_1 \to \mathfrak{so}_\epsilon(V_1,(\ ,\ )_{V_1})$ une représentation ϵ-orthogonale d'une algèbre de Lie colorée ϵ-quadratique et soit V_2 un espace vectoriel Γ-gradué muni d'une forme ϵ-symétrique $(\ ,\ )_{V_2}$. Si le produit tensoriel*

$$\mathfrak{g}_1 \oplus \mathfrak{so}_\epsilon(V_2,(\ ,\ )_{V_2}) \to \mathfrak{so}_\epsilon(V_1 \otimes V_2,(\ ,\ )_{V_1\otimes V_2})$$

est de type $\mathbb{Z}_2$-Lie, alors :

- *soit V_1 est la représentation fondamentale d'une algèbre de Lie colorée ϵ-orthogonale $\mathfrak{so}_\epsilon(V_1,(\ ,\ )_{V_1})$;*
- *soit V_2 est de dimension 1, $(\ ,\ )_{V_2}$ est symétrique et $\rho_1 : \mathfrak{g}_1 \to \mathfrak{so}_\epsilon(V_1,(\ ,\ )_{V_1})$ est de type $\mathbb{Z}_2$-Lie ;*
- *soit V_2 est de dimension 2, $(\ ,\ )_{V_2}$ est antisymétrique et $\rho_1 : \mathfrak{g}_1 \to \mathfrak{so}_\epsilon(V_1,(\ ,\ )_{V_1})$ est ϵ-orthogonale spéciale.*

Ainsi, une représentation $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ ϵ-orthogonale spéciale donne lieu à une algèbre de Lie colorée de la forme

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{sl}(2, k) \oplus (V \otimes k^2).$$

Par exemple, les représentations symplectiques spéciales d'algèbres de Lie donnent lieu à des algèbres de Lie (voir [SS15]) et les représentations orthogonales spéciales d'algèbres de Lie donnent lieu à des superalgèbres de Lie. On verra des exemples de ces dernières dans le cinquième chapitre.

Les algèbres de Lie colorées $\tilde{\mathfrak{g}}$ obtenues à partir des représentations ϵ-orthogonales spéciales ne sont pas arbitraires. Elles ont la particularité d'admettre une graduation de type Heisenberg ([Fau71] et [SS15]), c'est-à-dire qu'il existe un H dans $\tilde{\mathfrak{g}}$ de degré 0 tel que

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-2} \oplus \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$$

où $\tilde{\mathfrak{g}}_i = \{x \in \tilde{\mathfrak{g}} \mid \{H, x\} = ix\}$ et $dim(\tilde{\mathfrak{g}}_{-2}) = dim(\tilde{\mathfrak{g}}_2) = 1$. Il est vraisemblable que cette propriété caractérise les algèbres de Lie colorées que l'on peut obtenir à partir des représentations ϵ-orthogonales spéciales (pour une discussion de ce problème dans le cadre des algèbres de Lie, voir [Fau71], [Che87] et [SS15]).

Pour terminer le quatrième chapitre, nous étudions des propriétés géométriques des représentations ϵ-orthogonales spéciales. Il est connu depuis longtemps (voir [Eis44]) que les cubiques binaires, qui sont une représentation spéciale symplectique de l'algèbre de Lie $\mathfrak{sl}(2, k)$, admettent trois covariants et que ces covariants satisfont à des identités remarquables établies par G.B. Mathews (voir [Mat11]). Plus généralement, les représentations spéciales symplectiques d'algèbres de Lie admettent aussi trois covariants qui sont des fonctions polynômiales sur l'espace de la représentation et qui satisfont à des identités de Mathews généralisées (voir [SS15]). Par analogie nous définissons trois covariants pour les représentations ϵ-orthogonales spéciales d'algèbres de Lie colorées et établissons des identités de Mathews correspondantes. Une particularité de ces identités est que pour les formuler on doit étendre la notion de composition de fonctions polynômiales à une notion de composition de formes multilinéaires ϵ-antisymétriques.

Dans le cinquième et dernier chapitre, afin d'illustrer les résultats obtenus dans les chapitres précédents, on donne des exemples de représentations ϵ-orthogonales spéciales. Les représentations ϵ-orthogonales spéciales semblent être rares parmi toutes les représentations ϵ-orthogonales et dans la pratique il est difficile de montrer qu'une représentation ϵ-orthogonale donnée est spéciale.

On donne d'abord des exemples de représentations ϵ-orthogonales spéciales d'algèbres de Lie colorées classiques. En premier lieu, on remarque que la représentation fondamentale de l'algèbre de Lie colorée ϵ-quadratique $(\mathfrak{so}_\epsilon(V, (\ ,\ )), B)$ est une représentation ϵ-orthogonale spéciale. On montre ensuite que V est aussi une représentation ϵ-orthogonale spéciale du commutant $\mathfrak{m}$ d'une application structure appropriée $J \in \mathfrak{so}_\epsilon(V, (\ ,\ ))$ mais par rapport à une forme ϵ-quadratique $B_\mathfrak{m}$ différente de B. De cette manière, on montre que la représentation $U \oplus U^*$ de $\mathfrak{gl}_\epsilon(U)$ et la représentation fondamentale de $\mathfrak{u}_\epsilon(W, H)$ sont des représentations ϵ-orthogonales spéciales.

Ensuite, on donne trois exemples de représentations orthogonales spéciales d'algèbres de Lie quadratiques et on calcule les trois covariants associés.

On montre d'abord que, par rapport à une famille à un paramètre $\alpha \in k \setminus \{0, -1\}$ de formes quadratiques invariantes sur $\mathfrak{sl}(2, k) \times \mathfrak{sl}(2, k)$, le produit tensoriel des deux représentations fondamentales est une représentation spéciale orthogonale. Cela donne ainsi lieu à une famille, notée $D'(2, 1; \alpha)$, de superalgèbres de Lie de dimension 17 de la forme :

$$\mathfrak{sl}(2, k) \oplus \mathfrak{sl}(2, k) \oplus \mathfrak{sl}(2, k) \oplus k^2 \otimes k^2 \otimes k^2.$$

Le deuxième exemple concerne la représentation fondamentale de dimension 7 d'une algèbre de Lie $\mathfrak{g}$ de type G_2. Nous réalisons d'abord la représentation fondamentale de $\mathfrak{g}$ comme une sous-représentation Ξ de la représentation spinorielle Σ d'une algèbre de Lie $\mathfrak{h}$ de type $\mathfrak{so}(7)$. Cette réalisation nous permet d'exprimer l'application moment de Ξ en termes d'un produit vectoriel et de montrer que Ξ est une représentation spéciale orthogonale. On montre aussi que le covariant quadrilinéaire de Ξ, qui est une 4-forme sur Ξ, admet une décomposition en une somme de 7 formes décomposables qui correspondent naturellement aux 7 droites d'un plan de Fano. Le covariant quadrilinéaire ou son dual (voir [Bry06]) caractérise complètement l'action de G_2 sur Ξ. La superalgèbre de Lie correspondante, notée G'_3, est de dimension 31 et de la forme :

$$\mathfrak{g} \oplus \mathfrak{sl}(2, k) \oplus \Xi \otimes k^2.$$

Dans le troisième exemple, on montre que la représentation spinorielle Σ de l'algèbre de Lie $\mathfrak{h}$ (voir ci-dessus) est spéciale orthogonale en s'appuyant sur le deuxième exemple. On donne ensuite une décomposition du covariant quadrilinéaire de cette représentation en une somme de 14 formes décomposables qui correspondent naturellement aux 14 plans affines d'un espace affine de dimension 3 sur $\mathbb{Z}_2$. La superalgèbre de Lie correspondante, notée F'_4, est de dimension 40 et de la forme :

$$\mathfrak{h} \oplus \mathfrak{sl}(2, k) \oplus \Sigma \otimes k^2.$$

Sur le corps des complexes, il y a une classification des superalgèbres de Lie complexes simples de dimension finie due à V. Kac (voir [Kac75]). Il montre qu'il y a trois types de superalgèbres de Lie exceptionnelles : une famille à un paramètre $D(2, 1; \alpha)$, G_3 et F_4. Si $k = \mathbb{C}$, les superalgèbres de Lie obtenues dans les trois exemples précédents correspondent à ces superalgèbres de Lie. C'est-à-dire que pour tout $\alpha \in k \setminus \{0, -1\}$, $D'(2, 1; \alpha)$ est isomorphe à $D(2, 1; \alpha)$, G'_3 est isomorphe à G_3 et F'_4 est isomorphe à F_4.

Chapter 1

Simple three-dimensional Lie algebras

Throughout this chapter, the field k is always of characteristic not two.

1.1 Elementary definitions and properties

In this section we first recall the primary examples of simple three-dimensional Lie algebras over k : $\mathfrak{sl}(2,k)$ and k^3 together with a cross-product. We also recall a normal form for simple three-dimensional Lie algebras which depends on two non-zero parameters (see [Jac58]).

Definition 1.1.1. *A Lie algebra is called simple if it is not abelian and has no proper ideal.*

Remark 1.1.2. *a) A one-dimensional Lie algebra is necessarily abelian and therefore it is not simple.*

b) Let $\mathfrak{s}$ be a non-abelian two-dimensional Lie algebra. Let $\{x,y\}$ be a basis of $\mathfrak{s}$, then

$$[\mathfrak{s},\mathfrak{s}] = Vect < [x,y] >$$

is a one-dimensional ideal of $\mathfrak{s}$ and therefore $\mathfrak{s}$ is not simple. In fact, up to isomorphism, there is only one non-abelian two-dimensional Lie algebra.

Thus, the smallest simple Lie algebras over an arbitrary field are three-dimensional.

Example 1.1.3. *The two classical and most common examples of simple three-dimensional Lie algebras are the following :*

a) The vector space $\mathfrak{sl}(2,k) := \{M \in M_2(k) \mid Tr(M) = 0\}$ *with the commutator of matrices as Lie bracket. The standard basis of* $\mathfrak{sl}(2,k)$ *is*

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and the Lie bracket satisfies

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

We observe here that the linear map $ad(h)$ *is diagonalisable.*

There is another useful basis $\{x,y,z\}$ *of* $\mathfrak{sl}(2,k)$ *:*

$$x = \frac{h}{2}, \quad y = \frac{e-f}{2}, \quad z = \frac{e+f}{2}.$$

The Lie bracket satisfies

$$[x,y] = z, \quad [y,z] = x, \quad [z,x] = -y.$$

There is a different viewpoint on this example. If (V,ω) *is a symplectic vector space over* k*, we define the symplectic Lie algebra by :*

$$\mathfrak{sp}(V,\omega) := \{f \in End(V) \mid \forall v,v' \in V,\ \omega(f(v),v') + \omega(v,f(v')) = 0\}.$$

If V *is two-dimensional, we can choose a basis of* V *such that in this basis the matrix of* ω *is*

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and so an element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sp}(V,\omega)$ *verifies*

$$\Omega M + M^t \Omega = 0 \ \Leftrightarrow\ \begin{pmatrix} c & d \\ -a & -b \end{pmatrix} = \begin{pmatrix} c & -a \\ d & -b \end{pmatrix} \ \Leftrightarrow\ M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

Hence, we have an isomorphism

$$\mathfrak{sp}(V,\omega) \cong \mathfrak{sl}(2,k).$$

b) The vector space k^3 *with the cross product as Lie bracket. In the standard basis*

$$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

we have

$$i \times j = k, \quad j \times k = i, \quad k \times i = j.$$

Remark 1.1.4. *a) If* -1 *is a square in* k*, the two examples are isomorphic.*

b) It is well-known that over $\mathbb{C}$ *there is, up to isomorphism, just one simple three-dimensional Lie algebra :* $\mathfrak{sl}(2, \mathbb{C})$*. Over* $\mathbb{R}$ *the two previous examples are, up to isomorphism, the only simple three-dimensional Lie algebras.*

c) There are more possibilities over other fields. For example, over $\mathbb{Q}$*, there are, up to isomorphism, an infinite number of simple three-dimensional Lie algebras.*

Over an arbitrary field, there is a normal form for simple three-dimensional Lie algebras depending on two non-zero parameters.

Definition 1.1.5. *Let* α, β *be two elements of* k^* *and let* $\{x, y, z\}$ *be a basis of the vector space* k^3*. We define a bilinear skew-symmetric bracket* $[\ ,\ ] : k^3 \times k^3 \to k^3$ *by*

$$[x, y] = z, \quad [y, z] = \alpha x, \quad [z, x] = \beta y$$

and denote this algebra by $L(\alpha, \beta)$*.*

We can check that $L(\alpha, \beta)$ is a simple three-dimensional Lie algebra. Moreover, every simple three-dimensional Lie algebra is isomorphic to some $L(\alpha, \beta)$.

Proposition 1.1.6. *Every simple three-dimensional Lie algebra is isomorphic to some* $L(\alpha, \beta)$*, where* α *and* β *are non-zero elements in* k*.*

Proof. See [Jac58], [Mal92] or [SF88]. □

Remark 1.1.7. *a) The* α *and* β *in Proposition (1.1.6) are not unique.*

b) We have for the examples of (1.1.3) :

$$\mathfrak{sl}(2, k) \cong L(1, -1), \quad (k^3, \times) \cong L(1, 1).$$

c) In general, it is not easy to determine if $L(\alpha, \beta)$ *is isomorphic to* $\mathfrak{sl}(2, k)$*.*

1.2 The Killing form

In this section we collect some well-known and not so well-known properties of the Killing form of simple three-dimensional Lie algebras. In particular, we give a necessary and sufficient condition for a ternary quadratic form to be isometric to the Killing form of some simple three-dimensional Lie algebra.

Definition 1.2.1. *Let $\mathfrak{g}$ be a Lie algebra. The Killing form K of $\mathfrak{g}$ is the $ad_{\mathfrak{g}}$-invariant symmetric bilinear form on $\mathfrak{g}$ defined by*

$$K(x,y) := Tr(ad(x) \circ ad(y))$$

for x, y in $\mathfrak{g}$.

Remark 1.2.2. *a) The Killing form of $L(\alpha,\beta)$ is isometric to $< -2\beta, -2\alpha, -2\alpha\beta >$.*

b) Let $M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ be an arbitrary element of $\mathfrak{sl}(2,k)$. In the basis $\{x,y,z\}$ of $\mathfrak{sl}(2,k)$ (see Example (1.1.3) a)), we have :

$$M = 2a \cdot x + (b-c) \cdot y + (b+c) \cdot z.$$

This basis satisfies the relations of $L(1,-1)$, so by a), the Killing form K of $\mathfrak{sl}(2,k)$ is isometric to $< 2, -2, 2 >$ and we have

$$K(M,M) = 8a^2 - 2(b-c)^2 + 2(b+c)^2 = 8(a^2+bc).$$

In particular

$$K(M,M) = -8 \cdot det(M). \tag{1.1}$$

Let $N = \begin{pmatrix} \alpha & \beta \\ \delta & -\alpha \end{pmatrix}$ be another arbitrary element of $\mathfrak{sl}(2,k)$. In the basis $\{x,y,z\}$ we have

$$N = 2\alpha \cdot x + (\beta - \delta) \cdot y + (\beta + \delta) \cdot z$$

and so

$$K(M,N) = 8a\alpha - 2(b-c)(\beta-\delta) + 2(b+c)(\beta+\delta) = 4(2a\alpha + c\beta + b\delta).$$

Since

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \delta & -\alpha \end{pmatrix} = \begin{pmatrix} a\alpha + b\delta & * \\ * & c\beta + a\alpha \end{pmatrix}$$

we obtain the formula

$$K(M,N) = 4 \cdot Tr(MN). \tag{1.2}$$

Proposition 1.2.3. *a) Let $\mathfrak{s}$ be a three-dimensional Lie algebra. The Killing form of $\mathfrak{s}$ is non-degenerate if and only if $\mathfrak{s}$ is simple.*

b) Two simple three-dimensional Lie algebras are isomorphic if and only if their Killing forms are isometric.

Proof. a) For the fact that the Killing form of any Lie algebra is non-degenerate implies that it is a direct sum of simple Lie algebras, see [Die53]. Conversely, if $\mathfrak{s}$ is a simple three-dimensional Lie algebra, then $\mathfrak{s} \cong L(\alpha, \beta)$ for some α, $\beta \in k^*$ and so by Remark 1.2.2 a) the Killing form is non-degenerate.

b) See [Mal92].

□

Remark 1.2.4. *a) By Remark 1.2.2 a), the discriminant of the Killing form K of a simple three-dimensional Lie algebra is :*

$$disc(K) = [-8] = [-2] \in {}^{k^*}/_{k^{*2}}.$$

b) Conversely, if a ternary quadratic form q satisfies

$$disc(q) = [-2] \in {}^{k^*}/_{k^{*2}}$$

then q is isometric to a Killing form of a simple three-dimensional Lie algebra. To see this, we can check directly that $disc(q) = [-2]$ implies that q is isometric to $< -2\beta, -2\alpha, -2\alpha\beta >$ for some $\alpha, \beta \in k^*$ and this is to the Killing form of $L(\alpha, \beta)$. For a more constructive proof, we will give in Section 1.3 the explicit construction of a simple three-dimensional Lie algebra whose Killing form is q.

Corollary 1.2.5. *Let $\mathfrak{s}$ be a simple three-dimensional Lie algebra.*

a) All derivations of $\mathfrak{s}$ are inner.

b) The Killing form of $\mathfrak{s}$ is isotropic if and only if $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(2, k)$.

Proof. a) See Theorem 6 p.74 in [Jac62].

b) Let K be the Killing form of $\mathfrak{s}$. If K is isotropic, K is isometric to $H \perp < a >$ where H is a hyperbolic plane and $a \in k^*$. Since $disc(K) = [-2] \in {}^{k^*}/_{k^{*2}}$ and $disc(H) = [-1] \in {}^{k^*}/_{k^{*2}}$ we have $[a] = [2] \in {}^{k^*}/_{k^{*2}}$ and so K is isometric to the quadratic form $< 1, -1, 2 >$ which is isometric to the Killing form of $\mathfrak{sl}(2, k)$.

Conversely, if $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(2, k)$, the Killing form of $\mathfrak{s}$ is isotropic by (1.1) since $det \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$.

□

We now express the characteristic polynomial of an element of a simple three-dimensional Lie algebra in terms of the Killing form of the Lie algebra.

Proposition 1.2.6. *Let $\mathfrak{s}$ be a simple three-dimensional Lie algebra, let K be its Killing form and let $h \in \mathfrak{s}$. The characteristic polynomial $P_h(X)$ of $ad(h)$ is*

$$P_h(X) = -X(X^2 - \frac{K(h,h)}{2}).$$

Proof. If $\{h, E, F\}$ is a basis of $\mathfrak{s}$, then the matrix of $ad(h)$ in this basis is

$$\begin{pmatrix} 0 & a & d \\ 0 & b & e \\ 0 & c & f \end{pmatrix},$$

where $a, b, c, d, e, f \in k$ and so the characteristic polynomial of $ad(h)$ is

$$P_h(X) = -X(X^2 + X(-b - f) + (bf - ec)).$$

Since $\mathfrak{s}$ is simple, $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ and so $tr(ad(h)) = 0$. Thus $b = -f$ and

$$P_h(X) = -X(X^2 + (-b^2 - ec)).$$

We can check that

$$K(h,h) = Tr(ad(h) \circ ad(h)) = 2(b^2 + ec)$$

and then

$$P_h(X) = -X(X^2 - \frac{K(h,h)}{2}).$$

□

Corollary 1.2.7. *Let $\mathfrak{s}$ be a simple three-dimensional Lie algebra, let K be its Killing form and let $h \in \mathfrak{s}$.*

a) The linear map $ad(h)$ is diagonalisable if and only if h is anisotropic and $\frac{K(h,h)}{2}$ is a square in k.

b) If $ad(h)$ is diagonalisable, then $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(2, k)$.

c) We have

$$ad(h)^{2n+1} = \left(\frac{K(h,h)}{2}\right)^n ad(h) \qquad \forall n \in \mathbb{N}.$$

Remark 1.2.8. *Suppose that the field k is of positive characteristic $p > 2$ and let $\mathfrak{g}$ be a Lie algebra. It is well-known that for any $h \in \mathfrak{g}$ the linear map $ad(h)^p$ is a derivation.*

A simple Lie algebra $\mathfrak{g}$ is said to be restricted if for all $h \in \mathfrak{g}$ the derivation $ad(h)^p$ is interior and in that case, there is a map $^{[p]} : \mathfrak{g} \to \mathfrak{g}$ defined by

$$ad(h)^p = ad(h^{[p]}) \qquad \forall h \in \mathfrak{g}.$$

A Lie algebra with a non-degenerate Killing form is a restricted Lie algebra (see Corollary p.191 in [Jac62]) and hence a simple three-dimensional Lie algebra $\mathfrak{s}$ is restricted. By Corollary 1.2.7, the map $^{[p]} : \mathfrak{s} \to \mathfrak{s}$ is given by

$$h^{[p]} = \Big(\frac{K(h,h)}{2}\Big)^{\frac{p-1}{2}} h \qquad \forall h \in \mathfrak{s}.$$

Definition 1.2.9. *A simple three-dimensional Lie algebra $\mathfrak{s}$ is called split if $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(2,k)$.*

Beware : for an anisotropic element h in a split simple three-dimensional Lie algebra, the linear map $ad(h)$ is not necesseraly diagonalisable.

Example 1.2.10. *Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $\mathfrak{sl}(2,\mathbb{R})$. This element has Killing length*

$$K(J,J) = -8 \cdot det(J) = -8$$

which is not a square in $\mathbb{R}$ and so $ad(J)$ is not diagonalisable.

Remark 1.2.11. *Let $\mathfrak{s}$ be a simple three-dimensional Lie algebra.*

a) If $\mathfrak{s}$ is non-split, every element $H \in \mathfrak{s}$ satisfies

$$\frac{K(H,H)}{2} \notin (k^*)^2.$$

b) If $\mathfrak{s}$ is split, we have a partition of $\mathfrak{s}$ into three sets :

$$\begin{aligned} Q =& \{H \in \mathfrak{s} \mid \frac{K(H,H)}{2} \notin k^2\} \ ; \\ C =& \{H \in \mathfrak{s} \mid \frac{K(H,H)}{2} = 0\} \ ; \\ A =& \{H \in \mathfrak{s} \mid \frac{K(H,H)}{2} \in (k^*)^2\}. \end{aligned}$$

Let $\{x, y, z\}$ *be a basis of* $\mathfrak{s}$ *such that*

$$[x, y] = z, \quad [y, z] = x \quad [z, x] = -y.$$

Then :

$$\begin{aligned} Q =& \{Xx + Yy + Zz \in \mathfrak{s} \mid X^2 - Y^2 + Z^2 \notin k^2\} \ ; \\ C =& \{Xx + Yy + Zz \in \mathfrak{s} \mid X^2 - Y^2 + Z^2 = 0\} \ ; \\ A =& \{Xx + Yy + Zz \in \mathfrak{s} \mid X^2 - Y^2 + Z^2 \in (k^*)^2\}. \end{aligned}$$

Proposition 1.2.12. *Let* $\mathfrak{s}$ *be a simple split three-dimensional Lie algebra with Killing form* K*, let* $H \in \mathfrak{s}$ *be anisotropic and let* P *be the orthogonal plane to* $Vect < H >$*.*

a) If H *is split, then* P *is hyperbolic.*

b) If H *is non-split, then* P *is anisotropic.*

Proof. a) Since H is split, we have $[K(H, H)] = [2] \in {k^*}/{k^{*2}}$, and since

$$K \cong < 2, -2, 2 > \cong < K(H, H) > \perp K_{|P}$$

by Witt's cancellation theorem we obtain $K_{|P} \cong < -2, 2 >$.

b) For contradiction, suppose that P is hyperbolic. Since

$$K \cong < 2, -2, 2 > \cong < K(H, H) > \perp K_{|P}$$

by Witt's cancellation theorem we obtain

$$[K(H, H)] = [2] \in {k^*}/{k^{*2}}$$

which is not possible.

□

It seems well-known that, up to a constant, the Killing form of a simple three-dimensional Lie algebra $\mathfrak{s}$ is the only invariant symmetric bilinear form on $\mathfrak{s}$. However, we could not find a proof of this fact in the literature so we now give one.

Proposition 1.2.13. *Let* $\mathfrak{s}$ *be a simple three-dimensional Lie algebra, let* K *be its Killing form and let* $K_1 : \mathfrak{s} \times \mathfrak{s} \to k$ *be an* $ad_{\mathfrak{s}}$*-invariant non-degenerate symmetric bilinear form, then there exists* $\lambda \in k^*$ *such that*

$$K_1 = \lambda K.$$

Proof. To show the proposition we need the following lemma :

Lemma 1.2.14. *Let*

$$\mathcal{Z}_{\mathfrak{s}}(\mathfrak{s}) := \{f \in End(\mathfrak{s}) \mid \forall a \in \mathfrak{s} \quad ad(a) \circ f = f \circ ad(a)\}.$$

Then $\mathcal{Z}_{\mathfrak{s}}(\mathfrak{s})$ *is isomorphic to* k.

Proof. Let $\{x, y, z\}$ be a standard basis of $\mathfrak{s} \cong L(\alpha, \beta)$ for some $\alpha, \beta \in k^*$ by Proposition 1.1.6 and let $f \in \mathcal{Z}_{\mathfrak{s}}(\mathfrak{s})$. We have

$$f([a, b]) = [a, f(b)] \qquad \forall a, b \in \mathfrak{s}. \tag{1.3}$$

Thereby, with $a = b = x$ in Equation 1.3, we obtain that

$$[x, f(x)] = 0$$

which implies that there exists $\lambda_1 \in k$ such that $f(x) = \lambda_1 x$. In a similar way, using $a = b = y$ and $a = b = z$ in Equation 1.3, we obtain that there exist $\lambda_2, \lambda_3 \in k$ such that

$$f(y) = \lambda_2 y, \qquad f(z) = \lambda_3 z.$$

Next, with $a = x$ and $b = y$ in Equation 1.3 we obtain

$$f([x, y]) = [x, f(y)]$$

which means $\lambda_3 = \lambda_2$. Finally with $a = y$ and $b = z$ in Equation 1.3 we obtain $\lambda_1 = \lambda_3$ and then $f = \lambda_1 Id$. □

We now return to the proof of the proposition. Since K_1 is non-degenerate there exists $f \in End(\mathfrak{s})$ such that

$$K(x, y) = K_1(f(x), y) \qquad \forall x, y \in \mathfrak{s}.$$

For $x, y, z \in \mathfrak{s}$ we have

$$\begin{aligned} K_1(f \circ ad(x) \circ f^{-1}(y), z) &= K([x, f^{-1}(y)], z) \\ &= -K(f^{-1}(y), [x, z]) \\ &= -K_1(y, [x, z]) \\ &= K_1(ad(x)(y), z). \end{aligned}$$

Since K_1 is non-degenerate we have

$$f \circ ad(x) = ad(x) \circ f \qquad \forall x \in \mathfrak{s},$$

then $f \in \mathcal{Z}_{\mathfrak{s}}(\mathfrak{s})$ and by the previous lemma there exists $\lambda \in k^*$ such that

$$K_1 = \lambda K.$$

□

1.3 Canonical trivector and cross-product

In this section, we show that a simple three-dimensional Lie algebra $\mathfrak{s}$ determines a canonical element in $\Lambda^3(\mathfrak{s})$ called the canonical trivector of $\mathfrak{s}$. We see that the Lie bracket on $\mathfrak{s}$ can be written explicitly in terms of this trivector and the Killing form of $\mathfrak{s}$.

Let $\mathfrak{s}$ be a simple three-dimensional Lie algebra, let K be its Killing form and let $\{a, b, c\}$ be a basis of $\mathfrak{s}$. By the $\mathrm{ad}_{\mathfrak{s}}$-invariance of the Killing form we have

$$K([a,b],a) = -K(b,[a,a]) = 0,$$
$$K([a,b],b) = K(a,[b,b]) = 0$$

and hence

$$K([a,b],c) \neq 0$$

since K is non-degenerate.

Definition 1.3.1. *The trivector $\epsilon_{\{a,b,c\}} \in \Lambda^3(\mathfrak{s})$ associated to the basis $\{a,b,c\}$ of $\mathfrak{s}$ is defined by*

$$\epsilon_{\{a,b,c\}} := \frac{a \wedge b \wedge c}{K([a,b],c)}.$$

Proposition 1.3.2. *The trivector $\epsilon_{\{a,b,c\}}$ is non-zero and independent of the choice of basis $\{a,b,c\}$ of $\mathfrak{s}$. We denote $\epsilon_{\{a,b,c\}}$ by ϵ and call it the canonical trivector of $\mathfrak{s}$.*

Proof. Let $\{a', b', c'\}$ be another basis of $\mathfrak{s}$ and $M : \mathfrak{s} \to \mathfrak{s}$ be the linear map such that

$$M(a) = a', \quad M(b) = b', \quad M(c) = c'.$$

Then

$$a' \wedge b' \wedge c' = det(M)(a \wedge b \wedge c),$$
$$K([a',b'],c') = det(M)K([a,b],c)$$

and so

$$\frac{a' \wedge b' \wedge c'}{K([a',b'],c')} = \frac{a \wedge b \wedge c}{K([a,b],c)}.$$

□

Example 1.3.3. *In the standard basis $\{x,y,z\}$ of $L(\alpha,\beta)$*

$$\epsilon = -\frac{x \wedge y \wedge z}{2\alpha\beta}.$$

The quadratic form K induces a quadratic form K_{Λ^2} (resp. K_{Λ^3}) on $\Lambda^2(\mathfrak{s})$ (resp. $\Lambda^3(\mathfrak{s})$) given by

$$K_{\Lambda^2}(v_1 \wedge v_2, v_1 \wedge v_2) = det(K(v_i, v_j)) \qquad \forall v_1 \wedge v_2 \in \Lambda^2(\mathfrak{s})$$

and

$$K_{\Lambda^3}(v_1 \wedge v_2 \wedge v_3, v_1 \wedge v_2 \wedge v_3) = det(K(v_i, v_j)) \qquad \forall v_1 \wedge v_2 \wedge v_3 \in \Lambda^3(\mathfrak{s}).$$

Proposition 1.3.4. *With the notation above, we have*

$$K_{\Lambda^3}(\epsilon, \epsilon) = -2.$$

Proof. The Lie algebra $\mathfrak{s}$ isomorphic to $L(\alpha, \beta)$ for some $\alpha, \beta \in k^*$. By the previous example, in the standard basis $\{x, y, z\}$ of $L(\alpha, \beta)$ we have

$$K_{\Lambda^3}(\epsilon, \epsilon) = K_{\Lambda^3}(-\frac{x \wedge y \wedge z}{2\alpha\beta}, -\frac{x \wedge y \wedge z}{2\alpha\beta}) = \frac{1}{4\alpha^2\beta^2} \begin{vmatrix} K(x,x) & 0 & 0 \\ 0 & K(y,y) & 0 \\ 0 & 0 & K(z,z) \end{vmatrix} = -2.$$

□

Proposition 1.3.5. *The map $\beta : \Lambda^2(\mathfrak{s}) \to \mathfrak{s}$ given by $\beta(u \wedge v) = [u, v]$ satisfies*

$$u \wedge v \wedge w = K(\beta(u \wedge v), w)\epsilon$$

for u, v, w in $\mathfrak{s}$. This is the unique map satisfying this relation.

Proof. Since $\Lambda^3(\mathfrak{s})$ is one-dimensional, there exists a map $\alpha \in \Lambda^3(\mathfrak{s})^*$ such that

$$u \wedge v \wedge w = \alpha(u, v, w)\epsilon \qquad u, v, w \in \mathfrak{s}.$$

Since $\Lambda^3(\mathfrak{s})^*$ is one-dimensional there exists a constant $c \in k^*$ such that

$$\alpha(u, v, w) = cK([u, v], w) \qquad u, v, w \in \mathfrak{s}.$$

If $\{u, v, w\}$ is a basis of $\mathfrak{s}$, we have

$$\frac{u \wedge v \wedge w}{K([u, v], w)} = \epsilon$$

and so

$$\alpha(u, v, w) = K([u, v], w) \qquad u, v, w \in \mathfrak{s}.$$

□

In particular, we have a formula similar to the classical relation between the norm of the cross-product of two elements and the scalar product between these two elements (see [BG67]).

Proposition 1.3.6. *For v_1, v_2 in $\mathfrak{s}$, we have the formula*

$$K([v_1,v_2],[v_1,v_2]) = \frac{1}{2}\Big(K(v_1,v_2)^2 - K(v_1,v_1)K(v_2,v_2)\Big).$$

Proof. We have

$$v_1 \wedge v_2 \wedge \beta(v_1 \wedge v_2) = K(\beta(v_1 \wedge v_2), \beta(v_1 \wedge v_2))\epsilon. \tag{1.4}$$

Note that

$$K(v_1, \beta(v_1 \wedge v_2)) = K(v_2, \beta(v_1 \wedge v_2)) = 0$$

and so, applying K_{Λ^3} on each side of Equation (1.4), we obtain

$$K(v_1,v_1)K(v_2,v_2) - K(v_1,v_2)^2 = -2K(\beta(v_1 \wedge v_2), \beta(v_1 \wedge v_2)).$$

□

In fact, the Killing form and the canonical trivector completely determine the Lie bracket, as we see now.

Let :

- $\mathfrak{s}$ be a three-dimensional vector space ;
- $q \in S^2(\mathfrak{s}^*)$ be a quadratic form of discriminant equal to $[-2] \in {k^*}/{k^{*2}}$;
- $\epsilon \in \Lambda^3(\mathfrak{s})$ be a trivector such that $q_{\Lambda^3}(\epsilon) = -2$.

Proposition 1.3.7. *We denote by K_q the symmetric bilinear form associated by polarisation to q.*

a) There exists an unique map $\beta : \Lambda^2(\mathfrak{s}) \to \mathfrak{s}$ satisfying

$$u \wedge v \wedge w = K_q(\beta(u \wedge v), w)\epsilon \qquad \forall u, v, w \in \mathfrak{s}.$$

Furthermore, β verifies the Jacobi identity :

$$\beta(\beta(u \wedge v) \wedge w) + \beta(\beta(v \wedge w) \wedge u) + \beta(\beta(w \wedge u) \wedge v) = 0 \qquad \forall u, v, w \in \mathfrak{s}.$$

b) We can define a Lie bracket on $\mathfrak{s}$ by $[u,v] = \beta(u \wedge v)$ for $u, v \in \mathfrak{s}$ and so $(\mathfrak{s}, [\ ,\ ])$ is a simple three-dimensional Lie algebra.

c) *The Killing form K of $\mathfrak{s}$ is equal to K_q.*

Proof. a) Let $u \wedge v$ be in $\Lambda^2(\mathfrak{s})$. For w in $\mathfrak{s}$ there exists a constant C_w such that

$$u \wedge v \wedge w = C_w \cdot \epsilon$$

since $\Lambda^3(\mathfrak{s})$ is one-dimensional.

For u, v fixed, the trivector $u \wedge v \wedge w$ is linear in w, so C_w is a linear form on $\mathfrak{s}$. Since K_q is non-degenerate, we can define $\beta(u \wedge v)$ as the unique solution of

$$u \wedge v \wedge w = K_q(\beta(u \wedge v), w) \cdot \epsilon \qquad \forall w \in \mathfrak{s}.$$

Let $J : \Lambda^3(\mathfrak{s}) \to \mathfrak{s}$ be the Jacobi-map defined by

$$J(u, v, w) := \beta(\beta(u \wedge v) \wedge w) + \beta(\beta(v \wedge w) \wedge u) + \beta(\beta(w \wedge u) \wedge v).$$

Suppose that $\{e_1, e_2, e_3\}$ is an orthogonal basis relative to q. Clearly, $J(e_i, e_j, e_k)$ is zero if i, j, k are not distinct. For $J(e_1, e_2, e_3)$ we have :

$$\beta(e_i \wedge e_j) = ae_1 + be_2 + ce_3$$

for some $a, b, c \in k$. Since

$$K(\beta(e_1 \wedge e_2), e_1)\epsilon = e_1 \wedge e_2 \wedge e_1 = 0$$

and since $K(\beta(e_1 \wedge e_2), e_1) = aK(e_1, e_1)$, we obtain $a = 0$. By a similar argument we show that $b = 0$, so

$$\beta(e_1 \wedge e_2) = ce_3.$$

Since $e_1 \wedge e_2 \wedge e_3$ is non-zero, the element c is non-zero. We have,

$$\beta(e_1 \wedge e_2) \wedge e_3 = 0.$$

Similarly, we show that

$$\beta(e_2 \wedge e_3) = c'e_1, \qquad \beta(e_3 \wedge e_1) = c''e_2$$

for some $c', c'' \in k^*$. Hence, $J(e_1, e_2, e_3) = 0$.

b) The brackets between the elements of the orthogonal basis $\{e_1, e_2, e_3\}$ show that the Lie algebra $\mathfrak{s}$ is isomorphic to the Lie algebra $L(cc', cc'')$ and hence is simple.

c) Let $\{e_1, e_2, e_3\}$ be a basis of $\mathfrak{s}$ such that

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = ae_1, \quad [e_3, e_1] = be_2$$

for some $a, b \in k^*$. To show that K_q is equal to the Killing form of $\mathfrak{s}$ we have to show that

$$q(e_1) = -2b, \quad q(e_2) = -2a, \quad q(e_3) = -2ab.$$

Since K_q is ad-invariant, we have

$$q(e_3) = bq(e_2), \quad q(e_3) = aq(e_1), \quad aq(e_1) = bq(e_2).$$

Moreover, we have

$$e_1 \wedge e_2 \wedge e_3 = q(e_3)\epsilon$$

and so, applying q_{Λ^3} in both sides of this equation, we obtain

$$q(e_1)q(e_2) = -2q(e_3).$$

Since $q(e_3) = aq(e_1)$ we have $q(e_2) = -2a$. Similarly, we show $q(e_1) = -2b$ and $q(e_3) = -2ab$ and then K_q is equal to the Killing form of $\mathfrak{s}$.

□

This proposition provides a constructive proof of the following corollary as opposed to the proof given in Remark 1.2.4.

Corollary 1.3.8. *Let $\mathfrak{s}$ be a three-dimensional vector space together with a non-degenerate quadratic form q. Then, there exists a Lie bracket on $\mathfrak{s}$ whose Killing form is q if and only if $disc(q) = [-2] \in {k^*}/{k^{*2}}$.*

Remark 1.3.9. *If we apply the above process to $(\mathfrak{s}, q, -\epsilon)$, we obtain another Lie algebra (with Lie bracket equal to $-[\ ,\ ]$). However $(\mathfrak{s}, -[\ ,\ ])$ is isomorphic to $(\mathfrak{s}, [\ ,\ ])$.*

The conclusion of this section is that we have a correspondence which generalises the well-known link between the cross-product and the Lie algebra $\mathfrak{so}(3)$:

$$\left\{\begin{array}{c}\text{simple}\\ \text{three-dimensional}\\ \text{Lie algebras}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}\text{three-dimensional vector spaces}\\ \text{with a quadratic form q of discriminant } [-2] \in {k^*}/{k^{*2}}\\ \text{and a trivector } \epsilon \text{ such that } q_{\Lambda^3}(\epsilon) = -2\end{array}\right\}$$

1.4 Orientation preserving isometries and automorphisms

In this section, we study some properties of the group of isometries of the Killing form of a simple three-dimensional Lie algebra. In particular we see that the group of special isometries of the Killing form of a simple three-dimensional Lie algebra coincides with the group of automorphisms of this Lie algebra.

Definition 1.4.1. *Let $\mathfrak{s}$ be a vector space and K a quadratic form over $\mathfrak{s}$.*

Let $O(\mathfrak{s})$ (resp. $SO(\mathfrak{s})$) be the group of isometries (resp. special isometries) of $(\mathfrak{s}, K)$:

$$\begin{aligned} O(\mathfrak{s}) :=& \{f \in GL(\mathfrak{s}) \mid K(f(x), f(y)) = K(x,y) \quad \forall x, y \in \mathfrak{s}\} \\ SO(\mathfrak{s}) :=& \{g \in O(\mathfrak{s}) \mid det(g) = 1\}. \end{aligned}$$

Lemma 1.4.2. *For every anisotropic x in $\mathfrak{s}$, we denote by τ_x the reflection defined by*

$$\tau_x(v) := v - \frac{2K(x,v)}{K(x,x)}x \qquad \forall v \in \mathfrak{s}.$$

This map has the following properties :

a) $\tau_x \in O(\mathfrak{s})$ and τ_x is involutive.

b) $det(\tau_x) = -1$ and $\tau_x(x) = -x$.

Proof. See [Lam05]. □

Remark 1.4.3. *Suppose that $\mathfrak{s}$ is a simple three-dimensional Lie algebra together with its Killing form K. If $k = \mathbb{R}$, we have*

$$O(\mathfrak{sl}(2,\mathbb{R})) \cong O(1,2),$$

and $O(\mathfrak{sl}(2,\mathbb{R}))$ has four connected components whilst $SO(\mathfrak{sl}(2,\mathbb{R}))$ has two connected components. On the other hand,

$$O(\mathfrak{su}(2)) \cong O(3),$$

and $O(\mathfrak{su}(2))$ has two connected components whereas $SO(\mathfrak{su}(2))$ is connected.

Proposition 1.4.4. *(Witt) Suppose that $\mathfrak{s}$ is a simple three-dimensional Lie algebra together with its Killing form K. Let x and y be two non-zero elements in $\mathfrak{s}$ such that $K(x,x) = K(y,y)$. Then, there exists g in $SO(\mathfrak{s})$ such that $g(x) = y$.*

Proof. We first consider the anisotropic case :

$$K(x,x)=K(y,y)\neq 0.$$

We observe that

$$K(x+y,x+y)+K(x-y,x-y)=2(K(x,x)+K(y,y))=4K(x,x)\neq 0.$$

Thereby $x+y$ and $x-y$ cannot be both isotropic.

If $x-y$ is anisotropic, we can consider the reflection τ_{x-y}. Since

$$K(x-y,x-y)=K(x,x)+K(y,y)-2K(x,y)=2(K(x,x)-K(x,y))=2K(x,x-y)$$

we obtain

$$\tau_{x-y}(x)=x-2\frac{K(x,x-y)}{K(x-y,x-y)}(x-y)=x-(x-y)=y.$$

Now, let $v\in\mathfrak{s}$ be an anisotropic vector such that $K(x,v)=0$. Then

$$\tau_v(x)=x$$

so we have

$$(\tau_{x-y}\circ\tau_v)(x)=y \text{ and } \tau_{x-y}\circ\tau_v\in SO(\mathfrak{s}).$$

If $x-y$ is isotropic then $x+y$ is not and we can consider τ_{x+y}. Then

$$\tau_{x+y}(x)=-y,\quad \tau_y(y)=-y$$

and

$$(\tau_y\circ\tau_{x+y})(x)=y \text{ with } \tau_{x-y}\circ\tau_v\in SO(\mathfrak{s}).$$

Now we consider the isotropic case :

$$K(x,x)=K(y,y)=0.$$

Since x (resp. y) is isotropic, there exists a hyperbolic plane H_x (resp. H_y) such that $x\in H_x$ (resp. $y\in H_y$).

Furthermore the planes $(H_x,K|_{H_x})$ and $(H_y,K|_{H_y})$ are isometric, so there exists $u\in O(\mathfrak{s})$ such that

$$u(H_x)=H_y.$$

We choose a hyperbolic basis $\{e_1, e_2\}$ of H_y,

$$K(e_1, e_1) = K(e_2, e_2) = 0, \quad K(e_1, e_2) = 1.$$

We have

$$K(u(x), u(x)) = K(x, x) = 0,$$

so $u(x)$ and y are both on the isotropic lines $D_1 := Vect < e_1 >$ or $D_2 := Vect < e_2 >$.

If $u(x)$ and y are on the same line, we have

$$u(x) = ay$$

for an $a \in k^*$. Replacing the basis $\{e_1, e_2\}$ by $\{e_2, e_1\}$ if necessary we may assume that $u(x)$ and y are on $D_1 = Vect < e_1 >$. There exists $v \in O(\mathfrak{s})$ such that on $H_y = Vect < e_1, e_2 >$ its matrix is

$$v = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & a \end{pmatrix}$$

so $v(y) = \frac{1}{a}y$ and we have

$$v \circ u(x) = v(ay) = y.$$

If on the other hand $u(x)$ and y are not on the same line. Again, replacing the basis $\{e_1, e_2\}$ by $\{e_2, e_1\}$ if necessary, we may assume that $u(x) \in D_1$ and $y \in D_2$. There exist $a, b \in k^*$ such that

$$u(x) = ae_1, \quad y = be_2.$$

There exists $v \in O(\mathfrak{s})$ such that on $H_y = Vect < e_1, e_2 >$ its matrix is

$$v = \begin{pmatrix} 0 & \frac{a}{b} \\ \frac{b}{a} & 0 \end{pmatrix},$$

so

$$v \circ u(x) = v(ae_1) = be_2 = y.$$

We have shown the existence of $v \circ u \in O(\mathfrak{s})$ such that $v \circ u(x) = y$. We now correct this isometry to obtain another one in $SO(\mathfrak{s})$.

We have an orthogonal Witt decomposition

$$(\mathfrak{s}, K) = (H, K|_H) \perp (H^\perp, K|_{H^\perp})$$

where $(H, K|_H)$ is a hyperbolic plane and $(H^\perp, K|_{H^\perp})$ is one-dimensional and anistropic. Thus, in both cases above, if $det(v \circ u) = -1$, we can always correct by the isometry $\tau \in O(\mathfrak{s})$ with the property $det(\tau) = -1$ defined by

$$\tau|_A = -Id_A, \quad \tau|_H = Id_H.$$

We conclude that there exists an element $g \in SO(\mathfrak{s})$ such that $g(x) = y$. □

We now prove the key result which will allow us to relate automorphisms and isometries.

Proposition 1.4.5. *Let $\mathfrak{s}$ (resp. $\mathfrak{s}'$) be a simple three-dimensional Lie algebra with Killing form K (resp. K') and canonical trivector ϵ (resp. ϵ'). Let f be a linear map $: \mathfrak{s} \to \mathfrak{s}'$.*

Then f is an isomorphism of Lie algebras if and only if $K'(f(x), f(y)) = K(x, y)$ for all x, y in $\mathfrak{s}$ and $f(\epsilon) = \epsilon'$ (where by abuse of notation f also denote the induced map on $\Lambda^3(\mathfrak{s})$).

Proof. Let $f : \mathfrak{s} \to \mathfrak{s}'$ be an isomorphism and x, y be in $\mathfrak{s}$. Since

$$ad(f(x))(f(y)) = f \circ (ad(x)(y))$$

we have

$$ad(f(x)) = f \circ ad(x) \circ f^{-1}.$$

Hence

$$\begin{aligned} K'(f(x), f(y)) &= Tr(ad(f(x)) \circ ad(f(y))) \\ &= Tr(f \circ ad(x) \circ ad(y) \circ f^{-1})) \\ &= Tr(ad(x) \circ ad(y)) \\ &= K(x, y). \end{aligned}$$

Furthermore, if $\{a, b, c\}$ is a basis of $\mathfrak{s}$,

$$f(\epsilon) = f(\frac{a \wedge b \wedge c}{K([a,b],c)}) = \frac{1}{K([a,b],c)} f(a \wedge b \wedge c) = \frac{f(a) \wedge f(b) \wedge f(c)}{K'(f([a,b]), f(c))} = \frac{f(a) \wedge f(b) \wedge f(c)}{K'([f(a), f(b)], f(c))}$$

and since $\{f(a), f(b), f(c)\}$ is a basis of $\mathfrak{s}'$ we obtain

$$f(\epsilon) = \epsilon'.$$

Now, we prove the implication in the other direction. Let $\phi : \mathfrak{s} \to \mathfrak{s}'$ be such that $K'(\phi(x), \phi(y)) = K(x, y)$ for all x, y in $\mathfrak{s}$ and $\phi(\epsilon) = \epsilon'$. We want to prove that

$$\phi([a, b]) = [\phi(a), \phi(b)] \ \forall a, b \in \mathfrak{s}.$$

More precisely, since K' is non-degenerate and since $\mathfrak{s} \cong L(\alpha, \beta) = Vect < x, y, z \ / \ [x, y] = z, \ [y, z] = \alpha x, \ [z, x] = \beta y >$ for some $\alpha, \beta \in k^*$, we show that

$$K'(\phi([a, b]), X) = K'([\phi(a), \phi(b)], X)$$

for $a, b \in \{x, y, z\}$ and $X \in \{\phi(x), \phi(y), \phi(z)\}$ (which is a basis of $\mathfrak{s}'$).

Take $a = x$ and $b = y$. For $X = \phi(x)$ we have

$$K'(\phi([x, y]), \phi(x)) = K'(\phi(z), \phi(x)) = K(z, x) = 0$$
$$K'([\phi(x), \phi(y)], \phi(x)) = -K'(\phi(y), [\phi(x), \phi(x)]) = 0$$

and so

$$K'(\phi([x, y]), \phi(x)) = K'([\phi(x), \phi(y)], \phi(x)). \tag{1.5}$$

Similarly we have

$$K'(\phi([x, y]), \phi(y)) = K'([\phi(x), \phi(y)], \phi(y)). \tag{1.6}$$

For $X = \phi(z)$ we use the hypothesis $\phi(\epsilon) = \epsilon'$ which is equivalent to

$$\frac{\phi(x) \wedge \phi(y) \wedge \phi(z)}{K([x, y], z)} = \frac{\phi(x) \wedge \phi(y) \wedge \phi(z)}{K'([\phi(x), \phi(y)], \phi(z))}.$$

Since $K([x, y], z) = K'(\phi([x, y]), \phi(z))$ then

$$K'(\phi([x, y]), \phi(z)) = K'([\phi(x), \phi(y)], \phi(z)). \tag{1.7}$$

Thereby, from Equations (1.5), (1.6), (1.7) we deduce

$$\phi([x, y]) = [\phi(x), \phi(y)].$$

Simarly, we obtain

$$\phi([y, z]) = [\phi(y), \phi(z)], \quad \phi([z, x]) = [\phi(z), \phi(x)].$$

□

Corollary 1.4.6. *Let f be a linear endomorphism of $\mathfrak{s}$. Then f is an automorphism of $(\mathfrak{s}, [\ ,\ ])$ if and only if $f \in SO(\mathfrak{s})$.*

1.5 Correspondence with quaternion algebras

It is well-known that the imaginary part of a quaternion algebra $\mathcal{H}$ is a simple three-dimensional Lie algebra for the bracket defined by the commutator. In this section we show how to construct a quaternion algebra from a simple three-dimensional Lie algebra.

Recall from [Lam05] that we define the quaternion algebra $\left(\frac{\alpha,\beta}{k}\right)$ (with $\alpha, \beta \in k^*$) to be the k-algebra on two generators i, j with the defining relations :

$$i^2 = \alpha, \quad j^2 = \beta, \quad ij = -ji. \tag{1.8}$$

We note that the element ij verifies $(ij)^2 = -\alpha\beta$.

Furthermore, $\{1, i, j, ij\}$ form a k-basis for $\left(\frac{\alpha,\beta}{k}\right)$ which shows that $\left(\frac{\alpha,\beta}{k}\right)$ is a central simple four-dimensional associative, non-commutative composition algebra for the norm

$$N(a+bi+cj+dij) = (a+bi+cj+dij)\overline{(a+bi+cj+dij)} = a^2 - b^2\alpha - c^2\beta + d^2\alpha\beta,$$

where $\overline{(a+bi+cj+dij)} := a - (bi+cj+dij)$.

The Lie algebra $L(\alpha, \beta)$ is isomorphic to the imaginary part of the quaternion algebra $\left(\frac{-\alpha,-\beta}{k}\right)$ since $Im\left(\frac{-\alpha,-\beta}{k}\right) = Vect < i, j, ij >= Vect < \frac{i}{2}, \frac{j}{2}, \frac{ij}{2} >$ and :

$$\begin{aligned}
[\frac{i}{2}, \frac{j}{2}] &= \frac{1}{4}(ij - ji) = \frac{ij}{2}, \\
[\frac{j}{2}, \frac{ij}{2}] &= \frac{1}{4}(jij - ij^2) = -\frac{1}{2}ij^2 = \beta \cdot \frac{i}{2}, \\
[\frac{ij}{2}, \frac{i}{2}] &= \frac{1}{4}(iji - i^2 j) = -\frac{1}{2}i^2 j = \alpha \cdot \frac{j}{2}.
\end{aligned}$$

Hence, with the commutator as Lie bracket, $Im(\left(\frac{-\alpha,-\beta}{k}\right)) \cong L(\alpha, \beta)$.

Conversely, from a simple three-dimensional Lie algebra $\mathfrak{s}$ we can reconstruct a quaternion algebra.

Let $\mathcal{H}$ be the vector space defined by

$$\mathcal{H} := k \oplus \mathfrak{s}.$$

Definition 1.5.1. *Define the product $\cdot : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ by :*

a) for $a, b \in k$, $a \cdot b := ab$ (the field product on k) ;

b) for $a \in k$ and $v \in \mathfrak{s}$, $a \cdot v := v \cdot a := av$ (the scalar multiplication of k on $\mathfrak{s}$) ;

c) for $v, w \in \mathfrak{s}$,

$$v \cdot w := \frac{K(v,w)}{8} \cdot 1 + \frac{[v,w]}{2}.$$

Remark 1.5.2. *We define a conjugation $^- : \mathcal{H} \to \mathcal{H}$ by*

$$\overline{x} := a \cdot 1 - v \qquad \forall x = a \cdot 1 + v \in \mathcal{H}$$

and a norm map $N : \mathcal{H} \to k$ by

$$N(x) = x \cdot \overline{x} = (a^2 - \frac{K(v,v)}{8}) \cdot 1 \qquad \forall x = a \cdot 1 + v \in \mathcal{H}.$$

Proposition 1.5.3. *The vector space $\mathcal{H}$ with the product $\cdot$ above is a quaternion algebra. Furthermore, if $\mathfrak{s} \cong L(\alpha, \beta)$ then $\mathcal{H} \cong \left(\frac{-\alpha,-\beta}{k}\right)$.*

Proof. To show that $\mathcal{H}$ is a quaternion algebra, we will show that $\mathcal{H}$ together with the product $\cdot$ and the quadratic form N is a composition algebra.

Let $x = a \cdot 1 + v$ and $y = b \cdot 1 + w$ be two elements of $\mathcal{H}$. We first show that

$$N(xy) = N(x)N(y).$$

Since $N(x) = (a^2 - \frac{K(v,v)}{8}) \cdot 1$ and $N(y) = (b^2 - \frac{K(w,w)}{8}) \cdot 1$ we have

$$N(x)N(y) = (a^2b^2 - a^2\frac{K(w,w)}{8} - b^2\frac{K(v,v)}{8} + \frac{K(v,v)K(w,w)}{8^2}) \cdot 1.$$

On the other hand,

$$xy = (ab + \frac{K(v,w)}{8}) \cdot 1 + aw + bv + \frac{[v,w]}{2}.$$

Expanding and simplying, we obtain

$$N(xy) = (a^2b^2 - a^2\frac{K(w,w)}{8} - b^2\frac{K(v,v)}{8} + \frac{K(v,w)^2}{8^2} - \frac{K([v,w],[v,w])}{8 \times 4}) \cdot 1.$$

With the formula of Proposition 1.3.6 we obtain

$$-\frac{K([v,w],[v,w])}{8 \times 4} = \frac{K(v,v)K(w,w)}{8^2} - \frac{K(v,w)^2}{8^2}.$$

Hence

$$N(xy) = N(x)N(y),$$

and since the Killing form is non-degenerate, N is also non-degenerate. Thus, $\mathcal{H}$ is a composition algebra of dimension 4 and so a quaternion algebra.

If $\mathfrak{s} \cong L(\alpha, \beta)$, we can calculate by hand the norm on standard basis elements $\{x, y, z\}$ of $L(\alpha, \beta)$:

$$N(x) = -\frac{K(x,x)}{2} = \beta, \quad N(y) = -\frac{K(y,y)}{2} = \alpha, \quad N(z) = -\frac{K(z,z)}{2} = \alpha\beta$$

and so $\mathcal{H} \cong \left(\frac{-\alpha,-\beta}{k}\right)$. □

Example 1.5.4. *a) Over $\mathbb{R}$, the Lie algebra $\mathfrak{su}(2)$ is the imaginary part of the classical quaternion algebra $\mathbb{H}$.*

b) The Lie algebra $\mathfrak{sl}(2,k)$ is the imaginary part of the split quaternion algebra $M_2(k)$ (i.e. two by two matrices over k).

By combining Proposition 1.4.5 and results in [Lam05] we have the

Proposition 1.5.5. *For $\alpha, \beta, \alpha', \beta' \in k^*$ the following are equivalent :*

a) the Lie algebras $L(\alpha,\beta)$ and $L(\alpha',\beta')$ are isomorphic ;

b) the quaternion algebras $\left(\frac{-\alpha,-\beta}{k}\right)$ and $\left(\frac{-\alpha',-\beta'}{k}\right)$ are isomorphic ;

c) the quadratic forms $< \beta, \alpha, \alpha\beta >$ and $< \beta', \alpha', \alpha'\beta' >$ are equivalent.

There is a corresponding concept in dimension 2. For $\alpha \in k^*$, we define $\left(\frac{\alpha}{k}\right)$ to be the k-algebra on one generator i with the defining relation :

$$i^2 = \alpha.$$

This is a two-dimensional composition algebra for the norm map $N : \left(\frac{\alpha}{k}\right) \to k$ defined by

$$N(a + ib) = a^2 - \alpha b^2.$$

We observe that

- the imaginary part of $\left(\frac{\alpha}{k}\right)$ is trivially a Lie algebra. In this case this is a one-dimensional Lie algebra and recall that all the one-dimensional Lie algebras are isomorphic.
- the one-dimensional Lie algebra $Im\left(\frac{\alpha}{k}\right)$ inherits the one-dimensional quadratic form isometric to $< -\alpha >$.

Remark 1.5.6. *One can construct four-dimensional composition algebras from two-dimensional composition algebras by the doubling process (see Section 1.5 in [SV00]).*

Consider the direct sum of vector spaces :

$$\mathcal{H} := \left(\frac{\alpha}{k}\right) \oplus j\left(\frac{\alpha}{k}\right).$$

This is a four-dimensional vector space with basis $\{1, i, j, ji\}$; then for $\mathcal{H}$ with the product obtained by adding the two relations $j^2 = \beta \cdot 1$ (for some $\beta \in k^*$) and $ij = -ji$ we have an isomorphism of algebras

$$\mathcal{H} \cong \left(\frac{\alpha,\beta}{k}\right).$$

We have a result similar to Proposition 1.5.5 :

Proposition 1.5.7. *For $\alpha, \alpha' \in k^*$ the following are equivalent :*

a) the composition algebras $\left(\frac{\alpha}{k}\right)$ and $\left(\frac{\alpha'}{k}\right)$ are isomorphic ;

b) the quadratic forms $< \alpha >$ and $< \alpha' >$ are equivalent.

Remark 1.5.8. *There are also composition algebras of dimension 8 : the octonion algebras. However, since octonion algebras are not associative and not Lie-admissible, the commutator does not define a Lie algebra structure on the imaginary part.*

1.6 Construction of non-split simple three-dimensional Lie algebras from involutions of $\mathfrak{sl}(2,k)$

It is well-known that the only non-split simple three-dimensional real Lie algebra $\mathfrak{su}(2)$ can be constructed from $\mathfrak{sl}(2,\mathbb{R})$ together with a Cartan involution. In this section we give an analogous construction in the context of simple three-dimensional Lie algebras over a field of characteristic not two, and characterise the non-split simple three-dimensional Lie algebras which can be obtained in this way.

1.6.1 The construction

In this subsection, we define a notion of split and non-split pairs $(\mathfrak{s},\sigma)$ where $\mathfrak{s}$ is a simple three-dimensional Lie algebra and σ is an involutive automorphism of $\mathfrak{s}$. We give a construction which associates a split (resp. non-split) pair $(\mathfrak{s}',\sigma')$ to a non-split (resp. split) pair $(\mathfrak{s},\sigma)$.

We first introduce some notation. Let

$$k^*_{-1} := \{x^2 + y^2 \mid x, y \in k\} \setminus \{0\}.$$

This is a subgroup of k^* and if $-1 \in k^{*2}$, then ${k^*}/{k^*_{-1}} \cong \{1\}$ since

$$(\frac{1+\Delta}{2})^2 + (\sqrt{-1}\frac{1-\Delta}{2})^2 = \Delta$$

for all $\Delta \in k^*$.

Definition 1.6.1. *Let $\mathfrak{s}$ be a simple three-dimensional Lie algebra and $\sigma : \mathfrak{s} \to \mathfrak{s}$ be a non-trivial involutive automorphism.*

a) We say that $(\mathfrak{s},\sigma)$ is split if $\mathfrak{s}$ is split and if for $X \in \mathfrak{s}^\sigma$ we have

$$[K(X,X)] \neq 1 \in {k^*}/{k^*_{-1}}.$$

b) We say that $(\mathfrak{s},\sigma)$ is non-split if $\mathfrak{s}$ is non-split and if for $X \in \mathfrak{s}^\sigma$ we have

$$[K(X,X)] \neq 1 \in {k^*}/{k^*_{-1}}, \quad \exists H \perp X \text{ such that } [K(H,H)] = [K(X,X)] \in {k^*}/{k^{*2}}.$$

Note that an involutive automorphism of $\mathfrak{s}$ is either the identity or has a one-dimensional set of fixed-points.

Remark 1.6.2. *If $k = \mathbb{R}$, $(\mathfrak{sl}(2,\mathbb{R}),\sigma)$ is a split pair if and only if σ is a Cartan involution. The pair $(\mathfrak{su}(2),\sigma)$ is a non-split pair for any non-trivial involution σ.*

Let $(\mathfrak{s},\sigma)$ be a split or non-split pair. By definition $X \in \mathfrak{s}^\sigma$ verifies $\frac{K(X,X)}{2} \notin k^{*2}$. Let $\Lambda \in k^*$ be such that for $X \in \mathfrak{s}^\sigma$,

$$[\Lambda] = [\frac{K(X,X)}{2}] \in {k^*}/{k^{*2}},$$

and λ a square root of Λ in a quadratic extension of k. To the pair $(\mathfrak{s},\sigma)$ we are going to associate another pair $(\mathfrak{s}',\sigma')$.

Since σ is involutive we have

$$\mathfrak{s} \cong \mathfrak{l} \oplus \mathfrak{p},$$

where $\mathfrak{l}$ is the one-dimensional eigenspace for the eigenvalue 1 and $\mathfrak{p}$ is the two-dimensional eigenspace for the eigenvalue -1.

Let $\mathfrak{s}'$ be the simple three-dimensional k-Lie algebra

$$\mathfrak{s}' := \mathfrak{l} \oplus \lambda\mathfrak{p}$$

with the Lie bracket extended from $\mathfrak{s}$:

$$[a+\lambda b, c+\lambda d] = ([a,c] + \Lambda[b,d]) + \lambda([b,c]+[a,d]) \qquad \forall a+\lambda b, c+\lambda d \in \mathfrak{s}'.$$

Let $\sigma' : \mathfrak{s}' \to \mathfrak{s}'$ be the involutive automorphism defined by

$$\sigma'|_{\mathfrak{l}} = Id|_{\mathfrak{l}}, \quad \sigma'|_{\lambda\mathfrak{p}} = -Id|_{\lambda\mathfrak{p}}.$$

Remark 1.6.3. *Let X be a non-zero fixed-point of σ and X' be a non-zero fixed-point of σ'. We have*

$$[K_{\mathfrak{s}}(X,X)] = [K_{\mathfrak{s}'}(X',X')] \in {k^*}/{k^{*2}}$$

where $K_{\mathfrak{s}}$ (resp. $K_{\mathfrak{s}'}$) is the Killing form of $\mathfrak{s}$ (resp. $\mathfrak{s}'$).

Theorem 1.6.4. *Let $(\mathfrak{s},\sigma)$ be a split (respectively non-split) pair. The pair $(\mathfrak{s}',\sigma')$ associated to $(\mathfrak{s},\sigma)$ by the construction above is non-split (respectively split).*

Proof. We first prove the following lemma.

Lemma 1.6.5. *Let X be a non-zero element of a split simple three-dimensional Lie algebra $\mathfrak{s}$. Then, there exists H in $\mathfrak{s}$ such that $ad(H)$ is diagonalisable and which is orthogonal to X.*

Proof. Let $\{h,e,f\}$ be standard $\mathfrak{sl}(2,k)$-triple. Since $Vect < e,f >$ is a hyperbolic plane, there exists $X' \in Vect < e,f >$ such that $K(X,X) = K(X',X')$. This implies that there exists $g \in SO(\mathfrak{s})$ such that $g(X') = X$. Since h is orthogonal X', $g(h)$ is orthogonal to X, we have $K(g(h),g(h)) = K(h,h)$ and so the linear map $ad(g(h))$ is diagonalisable. □

Let $(\mathfrak{s},\sigma)$ be a non-split pair. Let $(\mathfrak{s}',\sigma')$ be the pair associated to $(\mathfrak{s},\sigma)$ by the construction above and let $K_{\mathfrak{s}}$ (resp. $K_{\mathfrak{s}'}$) be the Killing form of $\mathfrak{s}$ (resp. $\mathfrak{s}'$). Since there exists $H \in \mathfrak{s}$ orthogonal to X such that

$$[K_{\mathfrak{s}}(H,H)] = [K_{\mathfrak{s}}(X,X)] \in {}^{k^*}/_{k^{*2}}$$

we have

$$\sigma(H) = -H$$

and so $\lambda H \in \mathfrak{s}'$. The element λH verifies

$$[\frac{K_{\mathfrak{s}'}(\lambda H,\lambda H)}{2}] = [\Lambda \cdot \frac{K_{\mathfrak{s}}(H,H)}{2}] = [\Lambda \cdot \frac{K_{\mathfrak{s}}(X,X)}{2}] = [\Lambda^2] = [1] \in {}^{k^*}/_{k^{*2}},$$

and hence is a split grading operator. It follows that $\mathfrak{s}'$ is split and

$$[K_{\mathfrak{s}'}(X,X)] = [K_{\mathfrak{s}}(X,X)] \neq 1 \in {}^{k^*}/_{k^*_{-1}}.$$

Let $(\mathfrak{s},\sigma)$ be a split pair, let $(\mathfrak{s}',\sigma')$ be the pair associated to $(\mathfrak{s},\sigma)$ by the construction above and let $K_{\mathfrak{s}}$ (resp. $K_{\mathfrak{s}'}$) be the Killing form of $\mathfrak{s}$ (resp. $\mathfrak{s}'$). Let $X \in \mathfrak{s}^{\sigma}$. By Lemma 1.6.5, there exist $H,E,F \in \mathfrak{s}$ such that $\{H,E,F\}$ is a standard basis of $\mathfrak{s} \cong \mathfrak{sl}(2,k)$ and $X \in Vect < E,F >$. Since σ is involutive and $\sigma(H) = -H$ we obtain that σ is a reflection on the hyperbolic plane $Vect < E,F >$ and so there exists $a \in k^*$ such that $\sigma(E) = aF$ and $\sigma(F) = \frac{1}{a}E$.

The eigenspaces $\mathfrak{l}$ and $\mathfrak{p}$ are

$$\mathfrak{l} = Vect < E + aF >, \quad \mathfrak{p} = Vect < H, E - aF >,$$

and so

$$\mathfrak{s}' = \mathfrak{l} \oplus \lambda\mathfrak{p} = Vect < E + aF > \oplus Vect < \lambda H, \lambda(E - aF) > .$$

We now calculate the structure constants of $\mathfrak{s}'$:

$$[\frac{\lambda H}{2}, \frac{E+aF}{2}] = \frac{\lambda(E-aF)}{2},$$
$$[\frac{E+aF}{2}, \frac{\lambda(E-aF)}{2}] = -a\frac{\lambda H}{2},$$
$$[\frac{\lambda(E-aF)}{2}, \frac{\lambda H}{2}] = -\Lambda\frac{E+aF}{2}.$$

Since

$$[\Lambda] = [\frac{K_{\mathfrak{s}}(X,X)}{2}] = [\frac{K_{\mathfrak{s}'}(E+aF,E+aF)}{2}] = [a] \in {k^*}/{k^{*2}}$$

it follows that $\mathfrak{s}'$ is isomorphic to $L(-\Lambda,-\Lambda)$. Since the quadratic form $< 2\Lambda, 2\Lambda, -2\Lambda^2 >$ is isomorphic of $< -2, 2\Lambda, 2\Lambda >$, the Lie algebra $L(-\Lambda,-\Lambda)$ is isomorphic to $L(-\Lambda,1)$.

Lemma 1.6.6. *Let $\Delta \in k^*$. The Lie algebra $L(-\Delta,1)$ is split if and only if Δ is a sum of two squares.*

Proof. The Lie algebras $L(-\Delta,1)$ and $L(1,-1)$ are isomorphic if and only if their Killing form are isometric. Furthermore, by Witt's cancellation theorem, the quadratic form $< -2, 2\Delta, 2\Delta >$ is isometric to $< -2, 2, 2 >$ if and only if the binary quadratic form $< 2\Delta, 2\Delta >$ is isometric to $< 2, 2 >$. Since these binary quadratic forms have the same discriminant, they are equivalent if and only if they represent a common element (see Proposition 5.1 p.15 in [Lam05]), in other words if and only if $< 1, 1 >$ represents Δ. □

Since

$$[\Lambda] = [\frac{K_{\mathfrak{s}'}(E+aF,E+aF)}{2}] \neq 1 \in {k^*}/{k^*_{-1}}$$

the Lie algebra $\mathfrak{s}'$ is non-split by Lemma 1.6.6 and since

$$[\frac{K_{\mathfrak{s}'}(\lambda H,\lambda H)}{2}] = [\Lambda] = [\frac{K_{\mathfrak{s}'}(E+aF,E+aF)}{2}] \in {k^*}/{k^{*2}},$$

the pair $(\mathfrak{s}',\sigma')$ is non-split. □

Definition 1.6.7. *A non-split simple three-dimensional Lie algebra is said to be obtainable if there exists an involutive automorphism σ on $\mathfrak{s}$ such that $(\mathfrak{s},\sigma)$ is a non-split pair.*

1.6.2 Criteria for a non-split simple three-dimensional Lie algebra to be obtainable

Given a split pair $(\mathfrak{s},\sigma)$, the associated non-split Lie algebra $\mathfrak{s}'$ is isomorphic to $L(-\Delta,-\Delta)$ where $[\Delta]=[K(X,X)]$ is non-trivial in ${k^*}/{k^*_{-1}}$. This motivates the following question :

Question. Let $\mathfrak{s}$ be a non-split simple three-dimensional Lie algebra. Under what conditions on $\mathfrak{s}$ does there exist $\delta\in k^*$ such that $\mathfrak{s}$ is isomorphic to $L(\delta,\delta)$?

Proposition 1.6.8. *Let $\mathfrak{s}$ be a simple three-dimensional Lie algebra. Then $\mathfrak{s}$ is non-split and isomorphic to $L(\delta,\delta)$ for some $\delta\in k^*$ if and only if the Killing form of $\mathfrak{s}$ is anisotropic and represents -2.*

Proof. A Lie algebra is non-split if and only if its Killing form is anisotropic. If the Killing form K of $\mathfrak{s}$ represents -2, there exists δ and γ in k^* such that K is isometric to

$$<-2,\delta,\gamma>.$$

Since $disc(K)=[-2]\in {k^*}/{k^{*2}}$ and $disc(<-2,\delta,\gamma>)=[-2\delta\gamma]\in {k^*}/{k^{*2}}$ we have $[\gamma]=[\delta]\in {k^*}/{k^{*2}}$ and then K is isometric to the quadratic form $<-2,\delta,\delta>$ which is isometric to the Killing form of $L(\frac{-\delta}{2},\frac{-\delta}{2})$. Hence $\mathfrak{s}$ is isomorphic to $L(\frac{-\delta}{2},\frac{-\delta}{2})$.

Conversely, the Killing form of $L(\delta,\delta)$ is isometric to $<-2\delta,-2\delta,-2>$ and hence represents -2. □

We can summarise the various conditions we have for a Lie algebra to be obtainable in the following corollary.

Corollary 1.6.9. *Let $\mathfrak{s}$ be a non-split simple three-dimensional Lie algebra. The following are equivalent :*

a) The Lie algebra $\mathfrak{s}$ is obtainable,

b) There exist $X,H\in\mathfrak{s}$ such that

$$H\perp X,\quad [K(X,X)]\neq 1\in {k^*}/{k^*_{-1}} \text{ and } [K(H,H)]=[K(X,X)]\in {k^*}/{k^{*2}},$$

c) The Lie algebra $\mathfrak{s}$ is isomorphic to $L(-\Delta,-\Delta)$ for some $\Delta\in k^*$,

d) The Killing form represents -2.

Proof. Conditions a) and b) are equivalent by definition. As we have seen in the proof of Theorem 1.6.4, Conditions a) and b) are equivalent to the condition c). By Proposition 1.6.8, Conditions a), b) and c) are equivalent to Condition d). □

Remark 1.6.10. *Although the involution σ is not explicit in conditions b), c) and d), it is implicitly defined as the unique involution with fixed-point set $k\cdot X$ in case b) and $k\cdot x$ in cases c) and d), where $\{x,y,z\}$ is a standard basis of $L(-\Delta,-\Delta)$.*

1.6.3 The Hilbert symbol and the Legendre symbol

In this subsection, we recall the definition of the Hilbert symbol, which is an invariant of non-degenerate binary quadratic forms, and its expression in terms of the Legendre symbol when the field is a local field (for the definition of a local field see p.151 in [Lam05]).

Definition 1.6.11. *Let $\alpha, \beta \in k^*$. We define the Hilbert symbol $(\alpha, \beta) \in \{\pm 1\}$ as follows :*

$$(\alpha, \beta) := \begin{cases} 1 & \text{if the binary form } <\alpha, \beta> \text{ represents } 1, \\ -1 & \text{otherwise.} \end{cases}$$

Here are some properties of the Hilbert symbol.

Proposition 1.6.12. *Let $\alpha, \beta, \gamma \in k^*$.*

a) We have $(\alpha, \beta) = (\beta, \alpha)$.

b) If $\alpha \in k^{*2}$ then $(\alpha, \beta) = 1$.

c) If k is a local field, we have $(\alpha, \beta\gamma) = (\alpha, \beta)(\alpha, \gamma)$.

d) If k is a local field, $(\alpha, \beta) = 1$ for all $\beta \in k^*$ if and only if $\alpha \in k^{*2}$.

Proof. Properties 1 and 2 follow from the definition, and see Proposition 7 p.208 of [Ser79] for Properties 3 and 4. □

Remark 1.6.13. *Let $\alpha, \beta \in k^*$. By Proposition 1.5.5 and Theorem 2.7 p.58 of [Lam05], the Lie algebra $L(\alpha, \beta)$ is split if and only if $(-\alpha, -\beta) = 1$.*

We now introduce the Legendre symbol.

Definition 1.6.14. *For an odd prime p and $a \in \mathbb{Z}$, the Legendre symbol is defined by :*

$$\left(\frac{a}{p}\right) := \begin{cases} 0 & \text{if } p \text{ divides } a, \\ 1 & \text{if } a \text{ is a square modulo } p, \\ -1 & \text{otherwise.} \end{cases}$$

Remark 1.6.15. *There is a formula for the Legendre symbol :*

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}} \pmod{p}.$$

The Hilbert symbol over a local field can be re-written in terms of the Legendre symbol as follows. Suppose that $k_{\mathfrak{P}}$ is a non-dyadic local field, denote by $\overline{k_{\mathfrak{P}}}$ its residue class field and denote by $v_{\mathfrak{P}}$ its valuation. Let $\alpha, \beta \in k_{\mathfrak{P}}^*$. We note $a = v_{\mathfrak{P}}(\alpha)$ and $b = v_{\mathfrak{P}}(\beta)$. By Corollary p.211 of [Ser79] we have :

$$(\alpha, \beta) = \left((-1)^{ab}\frac{\alpha^b}{\beta^a}\right)^{\frac{|\overline{k_{\mathfrak{P}}}|-1}{2}}.$$

In particular if $|\overline{k_{\mathfrak{P}}}|$ is prime, we have

$$(\alpha, \beta) = \left(\frac{(-1)^{ab}\frac{\alpha^b}{\beta^a}}{|\overline{k_{\mathfrak{P}}}|}\right).$$

1.6.4 Criteria over a local field

Suppose that $k_{\mathfrak{P}}$ is a non-dyadic local field, denote by $\overline{k_{\mathfrak{P}}}$ its residue class field and denote by $v_{\mathfrak{P}}$ its valuation. Recall that for any prime p, the fields $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ are examples of local fields whose residue class field is isomorphic to $\mathbb{F}_p$.

By Proposition 1.5.5 and by Theorem 2.2 p.152 of [Lam05], there is (up to isomorphism) only one non-split simple three-dimensional Lie algebra over $k_{\mathfrak{P}}$. The standard model is $L(-\pi, -u)$ where $\pi, u \in k_{\mathfrak{P}}$, $v_{\mathfrak{P}}(u) = 0$, $\bar{u} \notin \overline{k_{\mathfrak{P}}}^{*2}$ and $v_{\mathfrak{P}}(\pi) = 1$.

Proposition 1.6.16. *Let $k_{\mathfrak{P}}$ be a non-dyadic local with residue class field $\overline{k_{\mathfrak{P}}}$. Then, a non-split simple three-dimensional Lie algebra over $k_{\mathfrak{P}}$ is obtainable if and only if $|\overline{k_{\mathfrak{P}}}| \equiv 3 \pmod 4$.*

Proof. We first prove the following lemma.

Lemma 1.6.17. *We have $-1 \in k_{\mathfrak{P}}^{*2}$ if and only if $|\overline{k_{\mathfrak{P}}}| \equiv 1 \pmod 4$ if and only if ${k_{\mathfrak{P}}^*}/{k_{\mathfrak{P}_{-1}}^*} = \{1\}$.*

Proof. We have $-1 \in k_{\mathfrak{P}}^{*2}$ if and only if $|\overline{k_{\mathfrak{P}}}| \equiv 1 \pmod 4$ by Corollary 2.6 p.154 of [Lam05]. We have ${k_{\mathfrak{P}}^*}/{k_{\mathfrak{P}_{-1}}^*} = \{1\}$ if and only if the quadratic form $< 1, 1, -\Delta >$ is isotropic for all $\Delta \in k_{\mathfrak{P}}^*$. The quadratic form $< 1, 1, -\Delta >$ is isotropic for all $\Delta \in k_{\mathfrak{P}}^*$ if and only if $(-1, \Delta) = 1$ for all $\Delta \in k_{\mathfrak{P}}^*$. However, since $k_{\mathfrak{P}}$ is a local field, $(-1, \Delta) = 1$ for all $\Delta \in k_{\mathfrak{P}}^*$ if and only if -1 is a square by Proposition 1.6.12. □

Hence, if $|\overline{k_{\mathfrak{P}}}| \equiv 1 \pmod 4$, the group ${k_{\mathfrak{P}}^*}/{k_{\mathfrak{P}_{-1}}^*}$ is trivial and so the construction of Subsection 1.6.1 cannot be applied.

If $|\overline{k_{\mathfrak{P}}}| \equiv 3 \pmod 4$, the group ${k^*_{\mathfrak{P}}}/{k^*_{\mathfrak{P}_{-1}}}$ is non-trivial and so the construction of Subsection 1.6.1 can be applied to a split pair $(\mathfrak{sl}(2, k_{\mathfrak{P}}), \sigma)$. The Lie algebra obtained from the construction is non-split and so isomorphic to $L(-\pi, -u)$. □

1.6.5 Criteria over a global field

Recall that the global fields are the number fields and the finite extensions of the function fields $\mathbb{F}_q(t)$. Using the Hasse-Minkowski theorem ([Lam05] p.170) and the results of the previous section we obtain the following characterisation of obtainable non-split simple three-dimensional Lie algebras over a global field.

Proposition 1.6.18. *Let k be a global field. Let $\mathfrak{s}$ be a non-split simple three-dimensional Lie algebra and let K be its Killing form. The Lie algebra $\mathfrak{s}$ is obtainable if and only if it satisfies the following conditions :*

a) over every non-archimedean completion $k_{\mathfrak{P}}$ of k such that $|\overline{k_{\mathfrak{P}}}| \equiv 1 \pmod 4$, the Killing form K is isotropic,

b) the Killing form K represents -2 over all dyadic completions.

Remark 1.6.19. *Condition b) is automatically satisfied if k is a function field since only number fields have dyadic completions.*

Proof. From Subsection 1.6.2 we know that $\mathfrak{s}$ is obtainable if and only if its Killing form represents -2. Moreover, the Killing form K represents -2 if and only if the quadratic form $K \perp < 2 >$ is isotropic. By the Hasse-Minkowski theorem ([Lam05] p.170) we know that $K \perp < 2 >$ is isotropic over k if and only if $K \perp < 2 >$ is isotropic over every completion $k_{\mathfrak{P}}$ of k (including the dyadic completions). We now show that this condition is automatically satisfied for archimedean completions and non-archimedean completions $k_{\mathfrak{P}}$ such that $|\overline{k_{\mathfrak{P}}}| \equiv 3 \pmod 4$.

If $|\cdot|_{\mathfrak{P}}$ is an archimedean absolute value on k then $k_{\mathfrak{P}}$ is either $\mathbb{R}$ or $\mathbb{C}$. If $k_{\mathfrak{P}} = \mathbb{C}$, the quadratic form $K \perp < 2 >$ is of course isotropic and if $k_{\mathfrak{P}} = \mathbb{R}$, the signature of $K \perp < 2 >$ is always indefinite and so the quadratic form is isotropic.

Furthermore, using Proposition 1.6.16 we have that over every non-archimedean completion $k_{\mathfrak{P}}$ of k such that $|\overline{k_{\mathfrak{P}}}| \equiv 3 \pmod 4$ the quadratic form $K \perp < 2 >$ is isotropic.

Finally, using again Proposition 1.6.16 we have that over every non-archimedean completion $k_{\mathfrak{P}}$ of k such that $|\overline{k_{\mathfrak{P}}}| \equiv 1 \pmod 4$ the quadratic form $K \perp < 2 >$ is isotropic if and only if K is isotropic. This complete the proof of the proposition. □

This result can be re-expressed as follows.

Corollary 1.6.20. *Let k be a global field. Let $L(\alpha,\beta)$ be a non-split simple three-dimensional Lie algebra, where $\alpha,\beta\in k^*$. The Lie algebra $L(\alpha,\beta)$ is obtainable if and only if it satisfies the following conditions :*

a) over every non-archimedean non-dyadic completion $k_{\mathfrak{P}}$ of k such that $|\overline{k_{\mathfrak{P}}}|\equiv 1 \pmod 4$ and such that $v_{\mathfrak{P}}(\alpha)$ or $v_{\mathfrak{P}}(\beta)$ is non-zero we have

$$(\alpha,\beta)_{k_{\mathfrak{P}}}=1 \tag{1.9}$$

where $v_{\mathfrak{P}}$ is the valuation associated to $k_{\mathfrak{P}}$.

b) the quadratic form $<\alpha,\beta,\alpha\beta,-1>$ is isotropic over all dyadic completions.

Proof. Let K be the Killing form of $L(\alpha,\beta)$. By Remark 1.6.13, the quadratic form K is isotropic over a non-archimedean completion $k_{\mathfrak{P}}$ of k such that $|\overline{k_{\mathfrak{P}}}|\equiv 1 \pmod 4$ if and only if $(-\alpha,-\beta)_{k_{\mathfrak{P}}}=1$. But since -1 is a square in $k_{\mathfrak{P}}$ by Lemma 1.6.17 this is equivalent to have $(\alpha,\beta)_{k_{\mathfrak{P}}}=1$. □

Over $\mathbb{Q}$ this implies the following result :

Proposition 1.6.21. *Let $L(\alpha,\beta)$ be a non-split simple three-dimensional Lie algebra over $\mathbb{Q}$, where $\alpha,\beta\in\mathbb{Q}^*$. The Lie algebra $L(\alpha,\beta)$ is obtainable if and only if for every prime $p\equiv 1 \pmod 4$ such that $v_p(\alpha)$ or $v_p(\beta)$ is non-zero we have*

$$\left(\frac{\frac{\alpha^{v_p(\beta)}}{\beta^{v_p(\alpha)}}}{p}\right)=1$$

where v_p is the p-adic valuation associated to the prime p.

Remark 1.6.22. *For fixed $\alpha,\beta\in\mathbb{Q}^*$, the number of primes p such that $v_p(\alpha)$ or $v_p(\beta)$ is non-zero is finite.*

Proof. The dyadic completion of $\mathbb{Q}$ is $\mathbb{Q}_2$ and

$$disc(<\alpha,\beta,\alpha\beta,-1>_{\mathbb{Q}_2})=-1\notin\mathbb{Q}_2^{2*}$$

by Corollary p.40 in [Cas78]. Hence, $<\alpha,\beta,\alpha\beta,-1>_{\mathbb{Q}_2}$ is isotropic by Lemma 2.6 p.59 of [Cas78] and so the Killing form of $L(\alpha,\beta)$ represents -2 over $\mathbb{Q}_2$.

We know that the non-archimedean completions of $\mathbb{Q}$ are the p-adic fields $\mathbb{Q}_p$ and the residue class field of $\mathbb{Q}_p$ is isomorphic to $\mathbb{F}_p$. Then, $|\overline{\mathbb{Q}_p}|\equiv 1 \pmod 4$ if and only if $p\equiv 1 \pmod 4$. Furthermore, if $v_p(\alpha)=0$ and $v_p(\beta)=0$, the condition (1.9) is automatically satisfied. □

Here are some examples of obtainable and not obtainable non-split simple three-dimensional Lie algebras over $\mathbb{Q}$.

Example 1.6.23. *Suppose that $k = \mathbb{Q}$.*

a) If $\alpha, \beta > 0$, then $(-\alpha, -\beta) = -1$ and so $L(\alpha, \beta)$ is non-split. In particular, the Lie algebras $L(2,3)$, $L(2,5)$ and $L(3,25)$ are non-split. The Lie algebra $L(2,3)$ is obtainable since there is no prime $p \equiv 1 \pmod 4$ such that $v_p(2)$ or $v_p(3)$ is non-zero. The Lie algebra $L(2,5)$ is not obtainable since for the prime $p = 5$, we have $v_5(2) = 0$, $v_5(5) = 1$ and

$$\left(\frac{\frac{2^{v_5(5)}}{5^{v_5(2)}}}{5}\right) = 2^2 = -1 \pmod 5.$$

The Lie algebra $L(3,25)$ is obtainable since for the prime $p = 5$, we have $v_5(3) = 0$, $v_5(25) = 2$ and

$$\left(\frac{\frac{3^{v_5(25)}}{25^{v_5(3)}}}{5}\right) = 9^2 = 1 \pmod 5.$$

b) We know from Example 2.17 p.63 of [Lam05] that the Lie algebra $L(3,-5)$ is non-split. Since for the prime $p = 5$, we have $v_5(3) = 0$, $v_5(-5) = 1$ and

$$\left(\frac{\frac{3^{v_5(-5)}}{(-5)^{v_5(3)}}}{5}\right) = 3^2 = -1 \pmod 5,$$

then $L(3,-5)$ is not obtainable.

c) Let p be an odd prime. We know from Example 2.14 p.62 of [Lam05] that $L(1,-p)$ is non-split if and only if $p \equiv 3 \pmod 4$. If $p \equiv 3 \pmod 4$, then the non-split Lie algebra $L(1,-p)$ is obtainable since there is no prime $p' \equiv 1 \pmod 4$ such that $v_{p'}(1)$ or $v_{p'}(-p)$ is non-zero.

d) Let p be an odd prime. We know from Example 2.15 p.62 of [Lam05] that $L(2,-p)$ is non-split if and only if $p \equiv 5 \pmod 8$ or $p \equiv 7 \pmod 8$. If $p \equiv 7 \pmod 8$, then the non-split Lie algebra $L(2,-p)$ is always obtainable since there is no prime $p' \equiv 1 \pmod 4$ such that $v_{p'}(2)$ or $v_{p'}(-p)$ is non-zero. If $p \equiv 5 \pmod 8$, the non-split Lie algebra $L(2,-p)$ is always not obtainable since for the prime p, we have $v_p(2) = 0$, $v_p(-p) = 1$ and

$$\left(\frac{2}{p}\right) = -1 \pmod p$$

by the Quadratic Reciprocity Law (see p.181 in [Lam05]).

Chapter 2

Lie superalgebras whose even part is a simple three-dimensional Lie algebra

2.1 Introduction

Since the work of E. Cartan, it has been well-known that the structure and representation theory of the smallest simple complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ are the keys to the classification of all finite-dimensional, simple complex Lie algebras. Lie superalgebras are generalisations of Lie algebras and from this point of view, it is natural to ask what are the Lie superalgebras whose even part is a three-dimensional simple Lie algebra.

Over the complex numbers and with the extra assumption that $\mathfrak{g}$ is simple, the answer to this question can be extracted from the classification of finite-dimensional, simple complex Lie superalgebras by V. Kac (see [Kac75]). In this case the only possibility, up to isomorphism, is the complex orthosymplectic Lie superalgebra $\mathfrak{osp}_{\mathbb{C}}(1|2)$. Over the real numbers and again with the extra assumption that $\mathfrak{g}$ is simple, the answer can similarly be extracted from the classification of finite-dimensional, real simple Lie superalgebras by V. Serganova (see [Ser83]) and is the same: the only possibility, up to isomorphism, is the real orthosymplectic Lie superalgebra $\mathfrak{osp}_{\mathbb{R}}(1|2)$. However if k is a general field, there is currently no classification of finite-dimensional, simple Lie superalgebras over k to which we can appeal to answer the question above. Nevertheless, let us point out, if k is algebraically closed and if $char(k) > 5$, S. Bouarroudj and D. Leites have conjectured a list of all the finite-dimensional, simple Lie superalgebras over k (see [BL07]) and S. Bouarroudj, P. Grozman and D. Leites have classified finite-dimensional Lie superalgebras over k with indecomposable Cartan matrices under the assumption that they have a Dynkin diagram with only one node (see [BGL09]). In both cases, the only Lie superalgebra whose even part is a three-dimensional simple Lie algebra which appears is $\mathfrak{osp}_k(1|2)$.

In this chapter we will give a classification of all finite-dimensional Lie superalgebras $\mathfrak{g}$ over a field k of characteristic not two or three whose even part is a three-dimensional simple Lie algebra. For our classification, we do not assume that k is algebraically closed and we do not assume that $\mathfrak{g}$ is simple. The main result (Theorem 2.6.7) is:

Theorem. *Let k be a field of characteristic not two or three. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional Lie superalgebra over k such that $\mathfrak{g}_0$ is a three-dimensional simple Lie algebra and let*

$$\mathcal{Z}(\mathfrak{g}) := \{x \in \mathfrak{g} \mid \{x, y\} = 0 \quad \forall y \in \mathfrak{g}\}.$$

Then, there are three cases:

a) $\{\mathfrak{g}_1, \mathfrak{g}_1\} = \{0\}$ *;*

b) $\mathfrak{g}_1 = (\mathfrak{g}_0 \oplus k) \oplus \mathcal{Z}(\mathfrak{g}),$ *(see Example 2.2.9) ;*

c) $\mathfrak{g} \cong \mathfrak{osp}_k(1|2) \oplus \mathcal{Z}(\mathfrak{g}),$ *(see Example 2.2.12).*

In *b*), the non-trivial brackets on $\mathfrak{g}_1 = (\mathfrak{g}_0 \oplus k) \oplus \mathcal{Z}(\mathfrak{g})$ are given by

$$\{v, \lambda\} = \lambda v \quad \forall v \in \mathfrak{g}_0,\ \forall \lambda \in k$$

where the right-hand side of this equation is to be understood as an even element of $\mathfrak{g}$.

It follows from this classification that $\mathfrak{g}$ is simple if and only if $\mathfrak{g} \cong \mathfrak{osp}_k(1|2)$ or $\mathfrak{g}_1 = \{0\}$. It also follows that, if k is of positive characteristic p and the restriction of the bracket to $\mathfrak{g}_1$ is non-zero, then $\mathfrak{g}$ is a restricted Lie superalgebra in the sense of V. Petrogradski ([Pet92]) and Y. Wang-Y. Zhang ([WZ00]). See Corollary 2.6.9 for explicit formulae for the $[p]|[2p]$-mapping.

We first prove the theorem when $\mathfrak{g}_0$ is $\mathfrak{sl}(2, k)$ and k is either of characteristic zero or of positive characteristic and algebraically closed. An essential point here is that for such fields, the classification of finite-dimensional, irreducible representations of $\mathfrak{sl}(2, k)$ is known (for k of positive characteristic and algebraically closed see for example [RS67] or [SF88]). This, together with a careful study of the restriction of the bracket to irreducible $\mathfrak{sl}(2, k)$-submodules of $\mathfrak{g}_1$, is the main ingredient of the proof. It turns out that the main difficulty occurs when k is of positive characteristic and the only irreducible submodules of $\mathfrak{g}_1$ are trivial. In this case we do not have complete reducibility of finite-dimensional representations of $\mathfrak{sl}(2, k)$ but nevertheless, using notably an observation of H. Strade (see [Str04]), we show that the bracket restricted to $\mathfrak{g}_1$ is trivial as expected.

Once we have proved our result under the restricted hypotheses above, we use three rather general results (see Propositions 2.6.5, 2.6.6 and 2.6.8) to extend it to the case when k is not algebraically closed and $\mathfrak{g}_0$ is not necessarily isomorphic to $\mathfrak{sl}(2, k)$. Recall that if k is not algebraically closed there are in general many three-dimensional simple Lie algebras over k, not just $\mathfrak{sl}(2, k)$ (see [Mal92]).

This chapter is organised as follows. In Section 2 we give a precise definition of the Lie superalgebras which appear in our classification. In Section 3 we give some general consequences of the Jacobi identities of a Lie superalgebra whose even part is simple. In Section 4 we recall what is known on the structure of finite-dimensional irreducible representations of $\mathfrak{sl}(2,k)$ and a criterion of complete reducibility in positive characteristic due to N. Jacobson (see [Jac58]). In Section 5, assuming that k is of characteristic zero or of positive characteristic and algebraically closed, we prove some vanishing properties of the bracket of a Lie superalgebra of the form $\mathfrak{g} = \mathfrak{sl}(2,k) \oplus \mathfrak{g}_1$. In the last section we prove the main results of this chapter (Theorems 2.6.1 and 2.6.7) and give some counter-examples in characteristic two and three.

Throughout this chapter, the field k is always of characteristic not two or three (except in the comments and examples given after Corollary 2.6.9).

2.2 Examples of Lie superalgebras

In a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the elements of $\mathfrak{g}_0$ are called even and those of $\mathfrak{g}_1$ are called odd. We denote by $|v| \in \mathbb{Z}_2$ the parity of a homogeneous element $v \in \mathfrak{g}$ and whenever this notation is used, it is understood that v is homogeneous.

Definition 2.2.1. *A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ together with a bilinear map $\{\ ,\ \} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that*

a) $\{\mathfrak{g}_\alpha, \mathfrak{g}_\beta\} \subseteq \mathfrak{g}_{\alpha+\beta}$ *for all* $\alpha, \beta \in \mathbb{Z}_2$,

b) $\{x, y\} = -(-1)^{|x||y|}\{y, x\}$ *for all* $x, y \in \mathfrak{g}$,

c) $(-1)^{|x||z|}\{x, \{y, z\}\} + (-1)^{|y||x|}\{y, \{z, x\}\} + (-1)^{|z||y|}\{z, \{x, y\}\} = 0$ *for all* $x, y, z \in \mathfrak{g}$.

Example 2.2.2. *Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space. The algebra* $\mathrm{End(V)}$ *is an associative superalgebra for the $\mathbb{Z}_2$-gradation defined by*

$$\mathrm{End(V)}_a := \{f \in \mathrm{End(V)} \mid f(V_b) \subseteq V_{a+b} \quad \forall b \in \mathbb{Z}_2\}$$

for all $a \in \mathbb{Z}_2$. Now, if we define for all $v, w \in$ $\mathrm{End(V)}$

$$\{v, w\} := vw - (-1)^{|v||w|}wv,$$

then $\mathrm{End(V)}$ *with the $\mathbb{Z}_2$-gradation defined above and the bracket $\{\ ,\ \}$ is a Lie superalgebra.*

A Lie superalgebra can also be thought as a Lie algebra and a representation carrying an extra structure.

Proposition 2.2.3. *Let* $\mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_1)$ *be a representation of a Lie algebra* $\mathfrak{g}_0$ *and let* $P : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ *be a symmetric bilinear map.*

The vector space $\mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *with the bracket* $\{\ ,\ \} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *defined by*

a) $\{x, y\} := [x, y]$ *for* $x, y \in \mathfrak{g}_0$,

b) $\{x, v\} := -\{v, x\} := x(v)$ *for* $x \in \mathfrak{g}_0$ *and* $v \in \mathfrak{g}_1$,

c) $\{v, w\} := P(v, w)$ *for* $v, w \in \mathfrak{g}_1$,

is a Lie superalgebra if and only if the map P *satisfies the two relations:*

$$[x, P(u, v)] = P(x(u), v) + P(u, x(v)) \qquad \forall x \in \mathfrak{g}_0,\ \forall u, v \in \mathfrak{g}_1, \tag{2.1}$$

$$P(u, v)(w) + P(v, w)(u) + P(w, u)(v) = 0 \qquad \forall u, v, w \in \mathfrak{g}_1. \tag{2.2}$$

Proof. Straightforward. □

Remark 2.2.4. *Given a Lie algebra* $\mathfrak{g}_0$*, one can ask which representations* $\mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_1)$ *arise as the odd part of a non-trivial Lie superalgebra* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$*. In [Kos01], this question is considered in the case of symplectic complex representations of quadratic complex Lie algebras (i.e. those admitting a non-degenerate symmetric ad-invariant bilinear form).*

Let $\rho : \mathfrak{g}_0 \to \mathfrak{sp}(\mathfrak{g}_1)$ *be a finite-dimensional, symplectic complex representation of a finite-dimensional, quadratic complex Lie algebra.*

B. Kostant shows that there exists a Lie superalgebra structure on $\mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *compatible with the natural super-symmetric bilinear form, if and only if the image of the invariant quadratic form on* $\mathfrak{g}_0$ *under the induced map from the envelopping algebra of* $\mathfrak{g}_0$ *to the Weyl algebra of* $\mathfrak{g}_1$ *satisfies a certain identity. In this case the map* P *is uniquely determined.*

Let $\mathfrak{g}$ be a Lie algebra. Using the adjoint representation of $\mathfrak{g}$ and a doubling process, we can construct a Lie superalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_0$ is isomorphic to $\mathfrak{g}$ and such that P is non-trivial.

Definition 2.2.5. *Let* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *be isomorphic Lie algebras and let* $\phi : \mathfrak{g} \to \mathfrak{g}'$ *be an isomorphism of Lie algebras. Let*

$$\tilde{\mathfrak{g}} := \mathfrak{g} \oplus (\mathfrak{g}' \oplus \mathcal{Z}_{\mathfrak{g}}(\mathfrak{g}')),$$

where

$$\mathcal{Z}_{\mathfrak{g}}(\mathfrak{g}') := \{f \in Hom(\mathfrak{g}, \mathfrak{g}') \mid f \circ ad(x) = ad(\phi(x)) \circ f \quad \forall x \in \mathfrak{g}\}.$$

We define a $\mathbb{Z}_2$*-gradation of* $\tilde{\mathfrak{g}}$ *by*

$$\tilde{\mathfrak{g}}_0 := \mathfrak{g}, \qquad \tilde{\mathfrak{g}}_1 := \mathfrak{g}' \oplus \mathcal{Z}_{\mathfrak{g}}(\mathfrak{g}')$$

and a $\mathbb{Z}_2$*-graded skew-symmetric bilinear bracket* $\{\ ,\ \}$ *on* $\tilde{\mathfrak{g}}$ *by:*

- $\{x, y\} := [x, y]$ *for* $x, y \in \mathfrak{g}$ *;*
- $\{x, v\} := [\phi(x), v]$ *for* $x \in \mathfrak{g},\ v \in \mathfrak{g}'$ *;*
- $\{x, f\} := 0$ *for* $x \in \mathfrak{g},\ f \in \mathcal{Z}_{\mathfrak{g}}(\mathfrak{g}')$ *;*
- $\{v, w\} := \{f, g\} := 0$ *for* $v, w \in \mathfrak{g}',\ f, g \in \mathcal{Z}_{\mathfrak{g}}(\mathfrak{g}')$ *;*
- $\{v, f\} := \phi^{-1}(f(\phi^{-1}(v)))$ *for* $v \in \mathfrak{g}',\ f \in \mathcal{Z}_{\mathfrak{g}}(\mathfrak{g}')$.

Remark 2.2.6. *The Lie algebra* $\mathfrak{g}'$ *is isomorphic to* $\mathfrak{g}$ *and so* $\mathcal{Z}_{\mathfrak{g}}(\mathfrak{g}') \cong \mathcal{Z}_{\mathfrak{g}}(\mathfrak{g})$.

Proposition 2.2.7. *The vector space* $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus (\mathfrak{g}' \oplus \mathcal{Z}_{\mathfrak{g}}(\mathfrak{g}'))$ *together with the* $\mathbb{Z}_2$*-gradation and bracket* $\{\ ,\ \}$ *above is a Lie superalgebra.*

Proof. We have to check the two relations (2.1) and (2.2). Let $x \in \mathfrak{g}$, $v \in \mathfrak{g}'$ and $f \in \mathcal{Z}_{\mathfrak{g}}(\mathfrak{g}')$, we have

$$\begin{aligned}[x, P(v, f)] &= [x, \phi^{-1}(f(\phi^{-1}(v)))] = \phi^{-1}([\phi(x), f(\phi^{-1}(v))]) = \phi^{-1}(f([x, \phi^{-1}(v)])) \\ &= \phi^{-1}(f(\phi^{-1}([\phi(x), v]))) = P([\phi(x), v], f) = P(\{x, v\}, f) + P(v, \{x, f\}),\end{aligned}$$

and so the relation (2.1) is satisfied. We only need to check the relation (2.2) for $f \in \mathcal{Z}_{\mathfrak{g}}(\mathfrak{g}')$ and $v, w \in \mathfrak{g}'$:

$$\begin{aligned}\{\{f, v\}, w\} + \{\{v, w\}, f\} + \{\{f, w\}, v\} &= \{\phi^{-1}(f(\phi^{-1}(v))), w\} + \{\phi^{-1}(f(\phi^{-1}(w))), v\} \\ &= [f(\phi^{-1}(v)), w] + [f(\phi^{-1}(w)), v] \\ &= f([\phi^{-1}(v), \phi^{-1}(w)]) + f([\phi^{-1}(w), \phi^{-1}(v)]) \\ &= 0.\end{aligned}$$

□

Remark 2.2.8. *a) This Lie superalgebra cannot be obtained by Kostant's construction since* $k \oplus \mathfrak{g}'$ *is not a symplectic representation of* $\mathfrak{g}$.

b) This Lie superalgebra is not simple.

From the point of view of this chapter, the most interesting case of this construction is when $\mathfrak{g}$ is a three-dimensional simple Lie algebra.

Example 2.2.9. *Let* $\mathfrak{s}$ *be a three-dimensional simple Lie algebra over* k *and let* $\bar{k}$ *be the algebraic closure of* k*. We have*

$$\mathcal{Z}_{\mathfrak{s}}(\mathfrak{s}) \otimes \bar{k} \cong \mathcal{Z}_{\mathfrak{s} \otimes \bar{k}}(\mathfrak{s} \otimes \bar{k}).$$

Since $\mathfrak{s} \otimes \bar{k} \cong \mathfrak{sl}(2, \bar{k})$ *is simple, by Schur's Lemma we obtain that* $\mathcal{Z}_{\mathfrak{s}\otimes\bar{k}}(\mathfrak{s} \otimes \bar{k}) \cong \bar{k}$ *and hence* $\mathcal{Z}_{\mathfrak{s}}(\mathfrak{s}) \cong k$.

In this case, the Lie superalgebra defined above is isomorphic to

$$\mathfrak{s} \oplus (\mathfrak{s} \oplus k).$$

When $\mathfrak{s}$ *is split, the Lie superalgebra* $\mathfrak{s} \oplus (\mathfrak{s} \oplus k)$ *is isomorphic to the "strange" Lie superalgebra* $\mathfrak{p}(1)$ *(see section 2.4 in [Mus12]).*

We now introduce the other type of Lie superalgebra which we will need later on in the chapter. These are the orthosymplectic Lie superalgebras whose definition and properties we now recall (for more details see [Sch79b]).

Definition 2.2.10. *Let* $V = V_0 \oplus V_1$ *be a finite-dimensional* $\mathbb{Z}_2$*-graded vector space together with a non-degenerate even supersymmetric bilinear form* B*, i.e.,* $(V_0, B|_{V_0})$ *is non-degenerate quadratic vector space,* $(V_1, B|_{V_1})$ *is a symplectic vector space and* V_0 *is* B*-orthogonal to* V_1*. We define the orthosymplectic Lie superalgebra to be the vector space* $\mathfrak{osp}_k(V, B) := \mathfrak{osp}_k(V, B)_0 \oplus \mathfrak{osp}_k(V, B)_1$ *where*

$$\mathfrak{osp}_k(V, B)_i := \{f \in End(V)_i \mid B(f(v), v') + (-1)^{|f||v|} B(v, f(v')) = 0 \quad \forall v, v' \in V\} \quad \forall i \in \mathbb{Z}_2.$$

We can check that $\mathfrak{osp}_k(V, B)$ is closed under the bracket defined in Example 2.2.2 and is in fact a simple Lie subsuperalgebra of End(V) with

$$\mathfrak{osp}_k(V, B)_0 \cong \mathfrak{so}(V_0, B|_{V_0}) \oplus \mathfrak{sp}(V_1, B|_{V_1}), \qquad \mathfrak{osp}_k(V, B)_1 \cong V_0 \otimes V_1.$$

Remark 2.2.11. *The Lie superalgebra* $\mathfrak{osp}_k(V, B)$ *can also be obtained from a symplectic representation of a quadratic Lie algebra as follows (cf. Remark 2.2.4).*

Since V_0 *is quadratic and* V_1 *is symplectic, there is a natural symplectic form* $B|_{V_0} \otimes B|_{V_1}$ *on* $V_0 \otimes V_1$ *and so the representation* $\mathfrak{so}(V_0, B|_{V_0}) \oplus \mathfrak{sp}(V_1, B|_{V_1}) \to \mathrm{End}(V_0 \otimes V_1)$ *is a symplectic representation of a quadratic Lie algebra.*

The orthosymplectic Lie superalgebra which is relevant in this chapter is the following.

Example 2.2.12. *Let*

$$\mathfrak{osp}_k(1|2) := \mathfrak{sl}(2, k) \oplus k^2$$

be the Lie superalgebra defined by the standard representation k^2 *of* $\mathfrak{sl}(2, k)$ *and by the moment map* $P : k^2 \times k^2 \to \mathfrak{sl}(2, k)$ *given by*

$$P\left((a, b), (c, d)\right) = \begin{pmatrix} -(ad + bc) & 2ac \\ -2bd & ad + bc \end{pmatrix}.$$

If $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space together with a non-degenerate even super-symmetric bilinear form B such that V_0 is one-dimensional and V_1 two-dimensional, it is easy to see that

$$\mathfrak{osp}_k(V, B) \cong \mathfrak{osp}_k(1|2).$$

2.3 Generalities on Lie superalgebras with simple even part

In this section we investigate some of the consequences of the identities (2.1) and (2.2) when the even part of the Lie superalgebra is simple.

Lemma 2.3.1. *Let V be a representation of a simple Lie algebra $\mathfrak{g}$, let $P : V \times V \to \mathfrak{g}$ be a symmetric bilinear map and let $W \subseteq V$ be a $\mathfrak{g}$-submodule.*

a) Suppose that the map P satisfies the relation (2.2) and $\mathfrak{g}$ acts non-trivially on W. If

$$P(V, W) = \{0\}$$

then we have $P \equiv 0$.

b) Suppose that P satisfies the relations (2.1) and (2.2) and that $P(W, W) \neq \{0\}$. Then

$$\mathfrak{g} \cdot V \subseteq W.$$

c) Suppose that the map P satisfies the relation (2.1) and $\mathfrak{g}$ acts trivially on W. Then we have $P(W, W) = \{0\}$.

Proof. a) Let $v, v' \in V$. By the relation (2.2), we obtain

$$P(v, v')(w) + P(v, w)(v') + P(v', w)(v) = 0 \quad \forall w \in W.$$

By hypothesis $P(v, w) = P(v', w) = 0$ and so $P(v, v')(w) = 0$ for all $w \in W$. The non-trivial representation W is faithful since $\mathfrak{g}$ is simple and so $P(v, v') = 0$.

b) By the identity (2.1), Span $< P(W, W) >$ is an ideal of $\mathfrak{g}$. Since it is non-trivial by assumption, we have Span $< P(W, W) >= \mathfrak{g}$. Hence, if $x \in \mathfrak{g}$ we have

$$x = \sum_i P(w_i, w'_i),$$

for some $w_1, w'_1, \ldots, w_n, w'_n$ in W. Let $v \in V$. Using the relation (2.2), this implies

$$x(v) = \sum_i P(w_i, w'_i)(v) = -\sum_i (P(w_i, v)(w'_i) + P(w'_i, v)(w_i))$$

and hence we observe that $x(v) \in W$.

c) Let $w, w' \in W$. Using the relation (2.1) we have

$$[x, P(w, w')] = P(x(w), w') + P(w, x(w')) = 0 \quad \forall x \in \mathfrak{g}$$

and so $P(w, w') = 0$ because $\mathfrak{g}$ is simple.

□

2.4 Reducibility of representations of $\mathfrak{sl}(2, k)$

Representation theory of Lie algebras in positive characteristic is quite different from representation theory of Lie algebras in characteristic zero. For example, in positive characteristic, a Lie algebra $\mathfrak{g}$ has no infinite-dimensional irreducible representations and the dimension of the irreducible representations of $\mathfrak{g}$ is bounded. For more details we refer to the survey [Jan98].

Notation. *a) An $\mathfrak{sl}(2, k)$-triple $\{E, H, F\}$ is a basis of $\mathfrak{sl}(2, k)$ satisfying:*

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

b) If k is of positive characteristic p, we denote by $Gk(p)$ (resp. $[a]$) the image of $\mathbb{Z}$ (resp. a) under the natural map $\mathbb{Z} \to k$. We will refer to elements of the image of this map as integers in k.

We first recall the structure of the irreducible representations of $\mathfrak{sl}(2, k)$ over a field of characteristic zero or an algebraically closed field of positive characteristic (see for example [SF88] (p. 207-208)). If the field k is of positive characteristic but not algebraically closed, there is no classification of the irreducible representations of $\mathfrak{sl}(2, k)$ to the best of the author's knowledge.

Theorem 2.4.1. *Let k be a field of characteristic zero or an algebraically closed field of positive characteristic. Let W be a finite-dimensional irreducible representation of $\mathfrak{sl}(2, k)$.*

a) There exist $\alpha, \beta \in k$, a basis $\{e_0, \ldots, e_m\}$ of W and an $\mathfrak{sl}(2, k)$-triple $\{E, H, F\}$ such that:

$$\begin{aligned}
&H(e_i) = (\alpha - 2[i])e_i \ ; \\
&E(e_0) = 0 \ ; && E(e_i) = [i](\alpha - ([i] - 1))e_{i-1}, \quad 1 \le i \le m \ ; \\
&F(e_m) = \beta e_0 \ ; && F(e_i) = e_{i+1}, \quad 0 \le i \le m - 1.
\end{aligned}$$

b) If $char(k) = 0$, then $\alpha = dim(W) - 1$ and $\beta = 0$.

c) If $char(k) = p > 0$, then $dim(W) \le p$. If $dim(W) = p$ and $\alpha \in Gk(p)$, then $\alpha = [dim(W) - 1]$. If $dim(W) < p$, then $\alpha = [dim(W) - 1]$, $\beta = 0$.

Remark 2.4.2. *In particular, we remark that*

$$Ann_{\mathfrak{sl}(2,k)}(e_i) := \{x \in \mathfrak{sl}(2,k) \mid x(e_i) = 0\} = \begin{cases} \{0\} & \textit{if } 1 \leq i \leq m-1,\ i \neq \frac{m}{2}\ ; \\ \text{Span} < H > & \textit{if } i = \frac{m}{2}\ ; \\ \text{Span} < E > & \textit{if } i = 0\ ; \\ \text{Span} < F > & \textit{if } i = m \textit{ and } \beta = 0\ ; \\ \{0\} & \textit{if } i = m \textit{ and } \beta \neq 0. \end{cases}$$

In other words, $Ann_{\mathfrak{sl}(2,k)}(e_i)$ can only be non-trivial for special values of i.

We now turn to the question of when a finite-dimensional representation of $\mathfrak{sl}(2,k)$ is completely reducible. The following theorem gives sufficient conditions for complete reducibility even if k is not algebraically closed.

Theorem 2.4.3. *Let k be an arbitrary field. Let $\rho : \mathfrak{sl}(2,k) \to \text{End(V)}$ be a finite-dimensional representation and let $\{E, H, F\}$ be an $\mathfrak{sl}(2,k)$-triple.*

a) If $char(k) = 0$, then V is completely reducible.

b) If $char(k) = p > 0$ and $\rho(E)^{p-1} = \rho(F)^{p-1} = 0$, then V is completely reducible.

Proof. The first part is well-known and follows from the Weyl's theorem on complete reducibility. For the second part see [Jac58]. □

Remark 2.4.4. *For other conditions implying complete reducibility of representations of $\mathfrak{sl}(2,k)$ over an algebraically closed field of positive characteristic, see [Str04] (p. 252-253).*

2.5 Vanishing properties of the bracket restricted to the odd part of a Lie superalgebra whose even part is $\mathfrak{sl}(2,k)$

In this section, we prove two preliminary results which are crucial to the proof of our main theorems. Let k be a field of characteristic zero or an algebraically closed field of positive characteristic. The first result shows that a Lie superalgebra whose even part is $\mathfrak{sl}(2,k)$ and whose odd part is an irreducible representation can only be non-trivial if the odd part is two-dimensional. The second result shows that if the restriction of the bracket to a non-trivial irreducible submodule W of the odd part vanishes, then the bracket vanishes identically unless W is three-dimensional.

Proposition 2.5.1. *Let k be a field of characteristic zero or an algebraically closed field of positive characteristic. Let W be a finite-dimensional irreducible representation of $\mathfrak{sl}(2,k)$ and let $P : W \times W \to \mathfrak{sl}(2,k)$ be a symmetric bilinear map which satisfies the relations* (2.1) *and* (2.2).

a) If $dim(W) \neq 2$, we have

$$P \equiv 0.$$

b) If $dim(W) = 2$, let $\{e_0, e_1\}$ be a basis of W and let $\{E, H, F\}$ be an $\mathfrak{sl}(2, k)$-triple as in Theorem 2.4.1 a). Then, there exists $\gamma \in k$ such that

$$P(e_0, e_0) = -2\gamma E, \quad P(e_0, e_1) = \gamma H, \quad P(e_1, e_1) = 2\gamma F.$$

Proof. If $dim(W) = 1$, it follows from Lemma 2.3.1 c) that $P \equiv 0$, so suppose $dim(W) \geq 2$ from now on. Let $m := dim(W) - 1$, let $\alpha, \beta \in k$, let $\{e_0, \ldots, e_m\}$ be a basis of W and let $\{E, H, F\}$ be an $\mathfrak{sl}(2, k)$-triple as in Theorem 2.4.1 a).

Suppose that α is not an integer in k. This is only possible if k is of positive characteristic, say p. We then know from Theorem 2.4.1 that

$$dim(W) = p \neq 2$$

and since

$$[H, P(e_i, e_j)] = 2(\alpha - [i] - [j])P(e_i, e_j) \quad \forall\ 0 \leq i, j \leq m$$

it follows that $P(e_i, e_j)$ is either zero or an eigenvector of H corresponding to the eigenvalue $2(\alpha - [i] - [j])$. The map $ad(H)$ is diagonalisable with eigenvalues which are integers. Since $(\alpha - [i] - [j])$ cannot be an integer by assumption, we have

$$P(e_i, e_j) = 0 \quad \forall\ 0 \leq i, j \leq m.$$

This proves a) if α is not an integer in k.

Suppose now that α is an integer in k. By Theorem 2.4.1 b) and c), we have $\alpha = [m]$ (with $1 \leq m \leq p - 1$ if $char(k) = p$), and so

$$[H, P(e_i, e_i)] = 2([m] - 2[i])P(e_i, e_i) \qquad \forall\ 0 \leq i \leq m.$$

The map $ad(H)$ is diagonalisable with eigenvalues $-2, 0, 2$ and so if $P(e_i, e_i) \neq 0$ we must have $[m] - 2[i]$ equals respectively $-1, 0$ or 1 in k and $P(e_i, e_i)$ is proportional to respectively F, H or E.

Suppose $[m] - 2[i] = 0$. If $char(k) = 0$ then m is even and $i = \frac{m}{2}$. If $char(k) = p$, since the conditions $1 \leq m \leq p - 1$ and $0 \leq i \leq m$ imply $-p < m - 2i < p$, it follows, again, that m is even and $i = \frac{m}{2}$.

Suppose $[m] - 2[i] = 1$. If $char(k) = 0$ then m is odd and $i = \frac{m-1}{2}$. If $char(k) = p$, then either m is odd and $i = \frac{m-1}{2}$ or $m = p - 1$ and $i = m$.

However, in the second case we would have $P(e_m, e_m) \in \text{Span} < E >$. Since $char(k) \neq 3$ we have (by Equation (2.2))

$$P(e_m, e_m)(e_m) = 0$$

and hence $P(e_m, e_m) = 0$ by Remark 2.4.2. Similarly, the relation $[m] - 2[i] = -1$ implies that m is odd and $i = \frac{m+1}{2}$.

In conclusion, if m is even we have:

$$\begin{aligned} P(e_{\frac{m}{2}}, e_{\frac{m}{2}}) &= aH, \\ P(e_i, e_i) &= 0 \quad \forall i \in [\![0, m]\!] \backslash \{\frac{m}{2}\} \end{aligned} \tag{2.3}$$

for some $a \in k$ and if m is odd we have:

$$\begin{aligned} P(e_{\frac{m-1}{2}}, e_{\frac{m-1}{2}}) &= bE, \\ P(e_{\frac{m+1}{2}}, e_{\frac{m+1}{2}}) &= cF, \\ P(e_i, e_i) &= 0 \quad \forall i \in [\![0, m]\!] \backslash \{\frac{m-1}{2}, \frac{m+1}{2}\} \end{aligned} \tag{2.4}$$

for some $b, c \in k$. We now show that $a = b = c = 0$.

If m is even, then $\frac{m}{2} \geq 1$ and by (2.3) we have

$$P(e_{\frac{m}{2}-1}, e_{\frac{m}{2}-1}) = 0$$

which by (2.1) means

$$[F, P(e_{\frac{m}{2}-1}, e_{\frac{m}{2}-1})] = 2P(e_{\frac{m}{2}}, e_{\frac{m}{2}-1}) = 0.$$

Furthermore, by (2.2) we have

$$P(e_{\frac{m}{2}}, e_{\frac{m}{2}})(e_{\frac{m}{2}-1}) + 2P(e_{\frac{m}{2}}, e_{\frac{m}{2}-1})(e_{\frac{m}{2}}) = 0$$

and from (2.3) it follows that

$$aH(e_{\frac{m}{2}-1}) = 0$$

and hence

$$2ae_{\frac{m}{2}-1} = 0.$$

We conclude that $a = 0$ and so

$$P(e_{\frac{m}{2}}, e_{\frac{m}{2}}) = 0.$$

If m is odd and $m \neq 1$, then $\frac{m-1}{2} \geq 1$, $\frac{m+1}{2} < m$ and since $char(k) \neq 3$, we have (by Equation (2.2))

$$P(e_{\frac{m-1}{2}}, e_{\frac{m-1}{2}})(e_{\frac{m-1}{2}}) = 0$$

and hence by (2.4)

$$bE(e_{\frac{m-1}{2}}) = 0$$

which means $b = 0$ since $Ker(E) = \text{Span} < e_0 >$ (cf. Theorem 2.4.1) and $\frac{m-1}{2} \neq 0$. Similarly, since $char(k) \neq 3$, we have (by Equation (2.2))

$$P(e_{\frac{m+1}{2}}, e_{\frac{m+1}{2}})(e_{\frac{m+1}{2}}) = 0$$

and so

$$cF(e_{\frac{m+1}{2}}) = 0$$

which means $c = 0$ since $Ker(F) \subseteq \text{Span} < e_m >$ (cf. Theorem 2.4.1) and $\frac{m+1}{2} \neq m$.

To summarise we have now shown that if $m \neq 1$, then

$$P(e_i, e_i) = 0 \quad \forall i \in [\![0, m]\!].$$

Now we suppose $m \neq 1$ and show by induction on n that, for all n in $[\![0, m]\!]$,

$$P(e_i, e_{i+n}) = 0 \quad \forall i \in [\![0, m-n]\!]. \tag{2.5}$$

Base case $(n = 0)$*:* We have already shown that

$$P(e_i, e_i) = 0 \quad \forall i \in [\![0, m]\!]$$

and so (2.5) is true if $n = 0$.

Induction: Suppose that the relation

$$P(e_i, e_{i+k}) = 0 \quad \forall i \in [\![0, m-k]\!]$$

is satisfied for all k in $[\![0, n-1]\!]$. We have

$$[F, P(e_i, e_{i+n-1})] = P(e_{i+1}, e_{i+n-1}) + P(e_i, e_{i+n}) \quad \forall i \in [\![0, m-n]\!]$$

but since equation (2.5) is satisfied for all k in $[\![0, n-1]\!]$ we obtain

$$P(e_i, e_{i+n-1}) = 0, \quad P(e_{i+1}, e_{i+n-1}) = 0 \quad \forall i \in [\![0, m-n]\!]$$

and hence

$$P(e_i, e_{i+n}) = 0 \quad \forall i \in [\![0, m-n]\!].$$

This completes the proof of (2.5) by induction and hence the proof of part a) of the Lemma.

Finally, to prove part b) of the Lemma, suppose that $m = 1$ and recall that by (2.4)

$$P(e_0, e_0) = bE, \qquad P(e_1, e_1) = cF.$$

By (2.1), we have

$$[H, P(e_0, e_1)] = P(H(e_0), e_1) + P(e_0, H(e_1)) = P(e_0, e_1) - P(e_0, e_1) = 0,$$

and hence there exists γ in k such that

$$P(e_0, e_1) = \gamma H.$$

Using the relation (2.2), we obtain

$$2P(e_0, e_1)(e_0) + P(e_0, e_0)(e_1) = 0$$

which means

$$2\gamma H(e_0) + bE(e_1) = 0$$

from which it follows that

$$2\gamma e_0 + be_0 = 0$$

and so $b = -2\gamma$. Similarly, using the relation (2.2) we also have

$$2P(e_0, e_1)(e_1) + P(e_1, e_1)(e_0) = 0$$

which means

$$2\gamma H(e_1) + cF(e_0) = 0$$

from which it follows that

$$-2\gamma e_1 + ce_1 = 0$$

and so $c = 2\gamma$. Thus, we have

$$P(e_0, e_0) = -2\gamma E, \quad P(e_0, e_1) = \gamma H, \quad P(e_1, e_1) = 2\gamma F,$$

and this proves b). □

For the next proposition we do not assume that the odd part of the Lie superalgebra is an irreducible representation of $\mathfrak{sl}(2,k)$. However, we show that if it contains a non-trivial irreducible submodule of dimension not three on which P vanishes, then P vanishes identically.

Proposition 2.5.2. *Let k be a field of characteristic zero or an algebraically closed field of positive characteristic. Let V be a finite-dimensional representation of $\mathfrak{sl}(2,k)$ and $W \subseteq V$ a non-trivial irreducible submodule. Let $P : V \times V \to \mathfrak{sl}(2,k)$ be a symmetric bilinear map which satisfies the relations (2.1), (2.2) and $P(W,W) = \{0\}$.*

a) If $dim(W) \neq 3$, then

$$P \equiv 0.$$

b) If $dim(W) = 3$, let $\{e_0, e_1, e_2\}$ be a basis of W and let $\{E, H, F\}$ be an $\mathfrak{sl}(2,k)$-triple as in Theorem 2.4.1 a). Then there exists $\gamma \in V^*$ such that for all v in V,

$$P(v, e_0) = -\gamma(v)E, \quad P(v, e_1) = \gamma(v)H, \quad P(v, e_2) = 2\gamma(v)F.$$

Proof. Let $v \in V$. Using the relation (2.2), we obtain

$$2P(v,w)(w) + P(w,w)(v) = 0 \quad \forall w \in W$$

which implies that

$$P(v,w)(w) = 0 \quad \forall w \in W.$$

Let $\{e_0, \dots, e_m\}$ be a basis of W as in Theorem 2.4.1 a). Since

$$P(v, e_i)(e_i) = 0 \quad \forall i \in [\![0, m]\!]$$

it follows from Remark 2.4.2 that:

- $\exists a, b \in k, \quad \text{s.t.} \quad P(v, e_0) = aE, \; P(v, e_m) = bF,$
- $P(v, e_i) = 0 \quad \forall i \in [\![1, m-1]\!], \; i \neq \frac{m}{2},$
- if m is even, $\exists c \in k, \quad \text{s.t.} \quad P(v, e_{\frac{m}{2}}) = cH.$

By (2.2), we have

$$P(v, e_0)(e_m) + P(v, e_m)(e_0) = 0$$

and hence

$$aE(e_m) + bF(e_0) = 0. \tag{2.6}$$

Suppose that the representation W is not three-dimensional so that $m - 1 \neq 1$. Since W is non-trivial, this implies that $a = b = 0$ and hence that

$$P(v, e_0) = P(v, e_m) = 0.$$

If m is even, again by (2.2), we have

$$P(v, e_{\frac{m}{2}})(e_0) + P(v, e_0)(e_{\frac{m}{2}}) + P(e_0, e_{\frac{m}{2}})(v) = 0$$

and hence $cH(e_0) = 0$. Since $H(e_0) \neq 0$, it follows that $c = 0$ and $P(v, e_{\frac{m}{2}}) = 0$. Therefore, we have

$$P(v, e_i) = 0 \quad \forall i \in [\![0, m]\!],$$

and hence by Lemma 2.3.1 a), we have $P \equiv 0$.

Suppose $dim(W) = 3$. By (2.6) and Theorem 2.4.1 a) we have

$$2ae_1 + be_1 = 0$$

and then $b = -2a$. By the relation (2.2) we also have

$$P(v, e_0)(e_1) + P(v, e_1)(e_0) = 0 \Rightarrow aE(e_1) + cH(e_0) = 0 \Rightarrow 2ae_0 + 2ce_0 = 0,$$

and so $a = -c$. Thus,

$$P(v, e_0) = aE, \quad P(v, e_1) = -aH, \quad P(v, e_2) = -2aF$$

and a clearly depends linearly on v. This proves b). □

2.6 Lie superalgebras with three-dimensional simple even part

In this section we prove the two main theorems of this chapter. The first is a classification of finite-dimensional Lie superalgebras whose even part is $\mathfrak{sl}(2, k)$ under the hypotheses that k is of characteristic zero or an algebraically closed field of positive characteristic. The second extends this classification to the case of finite-dimensional Lie superalgebras whose even part is any three-dimensional simple Lie algebra over an arbitrary field of characteristic not two or three.

Theorem 2.6.1. *Let k be a field of characteristic zero or an algebraically closed field of positive characteristic. Let $\mathfrak{g} = \mathfrak{sl}(2, k) \oplus \mathfrak{g}_1$ be a finite-dimensional Lie superalgebra over k and let*

$$\mathcal{Z}(\mathfrak{g}) := \{x \in \mathfrak{g} \mid \{x, y\} = 0 \quad \forall y \in \mathfrak{g}\}.$$

Then there are three cases:

a) $\{\mathfrak{g}_1, \mathfrak{g}_1\} = \{0\}$ *;*

b) $\mathfrak{g}_1 = (\mathfrak{sl}(2, k) \oplus k) \oplus \mathcal{Z}(\mathfrak{g})$, *(see Example 2.2.9) ;*

c) $\mathfrak{g} \cong \mathfrak{osp}_k(1|2) \oplus \mathcal{Z}(\mathfrak{g})$, *(see Example 2.2.12).*

Proof. Let $\{E, H, F\}$ be an $\mathfrak{sl}(2, k)$-triple. We denote by V the representation $\mathfrak{g}_1$ of $\mathfrak{sl}(2, k)$. Even if V is not completely reducible, it always has irreducible submodules. To prove the theorem we show the following four implications:

- V has an irreducible submodule of dimension strictly greater than 3 $\Rightarrow \mathfrak{g}$ as in a) ;
- V has an irreducible submodule of dimension 2 $\Rightarrow \mathfrak{g}$ as in a) or c) ;
- V has an irreducible submodule of dimension 3 $\Rightarrow \mathfrak{g}$ as in a) or b) ;
- V only has irreducible submodules of dimension 1 $\Rightarrow \mathfrak{g}$ as in a).

Case 1: If there is an irreducible representation $W \subseteq V$ such that $dim(W) > 3$, then by Proposition 2.5.1 we have $P|_{W \times W} \equiv 0$. Furthermore, by Proposition 2.5.2, we have

$$P \equiv 0,$$

and $\mathfrak{g}$ is as in a).

Case 2: Let $W \subseteq V$ be an irreducible representation of dimension 2. If $P|_{W \times W} \equiv 0$, then by Proposition 2.5.2 we have

$$P \equiv 0$$

and $\mathfrak{g}$ is as in a).

If $P|_{W \times W} \not\equiv 0$, by Lemma 2.3.1 b), we have

$$x(v) \in W \quad \forall x \in \mathfrak{sl}(2,k),\ \forall v \in V$$

and, since W is irreducible, this means V has no other non-trivial irreducible submodules. Since

$$E^2|_W = F^2|_W = 0,$$

we also have

$$E^3 = F^3 = 0.$$

Since $char(k) = 0$ or $char(k) > 3$ then V is completely reducible by Theorem 2.4.3 and so $V = W \oplus V_0$ where V_0 is a subspace of V on which $\mathfrak{sl}(2,k)$ acts trivially. Let v be in V_0. The vector space

$$I_v := \text{Span} < P(v,w) \quad \forall w \in W >$$

is at most of dimension 2 and is an ideal of $\mathfrak{sl}(2,k)$. Thus $I_v = \{0\}$ which implies

$$P(V_0, W) = \{0\}$$

since $v \in V_0$ was arbitrary. Furthermore, since $V = W \oplus V_0$ and

$$P(V_0, V_0) = \{0\}$$

by Lemma 2.3.1 c), we have

$$P(V_0, V) = \{0\}$$

and clearly $V_0 = \mathcal{Z}(\mathfrak{g})$. Let $\{e_0, e_1\}$ be a basis of W as in Theorem 2.4.1. By Proposition 2.5.1 there exists $\gamma \in k^*$ such that

$$P(e_0, e_0) = -2\gamma E, \quad P(e_0, e_1) = \gamma H, \quad P(e_1, e_1) = 2\gamma F$$

and the bracket defined on W is a moment map. Hence $\mathfrak{sl}(2,k) \oplus \mathfrak{g}_1 \cong \mathfrak{osp}_k(1|2) \oplus \mathcal{Z}(\mathfrak{g})$ (see Example 2.2.12) and $\mathfrak{g}$ is as in c).

Case 3: Let $W \subseteq V$ be an irreducible representation of dimension 3. Recall that by Proposition 2.5.1, $P|_{W \times W} \equiv 0$. Let $\{e_0, e_1, e_2\}$ be a basis of W as in Theorem 2.4.1. By Proposition 2.5.2, there exists a γ in V^* such that for all v in V, we have

$$P(v, e_0) = -\gamma(v)E, \quad P(v, e_1) = \gamma(v)H, \quad P(v, e_2) = 2\gamma(v)F.$$

If $\gamma(v) = 0$ for all v in V, then by Lemma 2.3.1 a) we have $P \equiv 0$ and $\mathfrak{g}$ is as in a). If there exists v in V such that $\gamma(v) \neq 0$ we proceed as follows.

Let v' in V be such that $\gamma(v') \neq 0$. Then by Relation (2.2) we have

$$2P(v', e_i)(v') = -P(v', v')(e_i) \in W \quad \forall i \in \{0, 1, 2\},$$

and since $\mathfrak{sl}(2,k) = \text{Span} < P(v', e_0), P(v', e_1), P(v', e_2) >$ this implies

$$x(v') \in W \quad \forall x \in \mathfrak{sl}(2,k). \tag{2.7}$$

Let v'' in V be such that $\gamma(v'') = 0$. Then

$$P(v, e_i)(v'') + P(v'', e_i)(v) + P(v, v'')(e_i) = 0 \quad \forall i \in \{0, 1, 2\}$$

implies

$$P(v, e_i)(v'') = -P(v, v'')(e_i) \in W \quad \forall i \in \{0, 1, 2\}$$

and since $\gamma(v) \neq 0$, we have

$$x(v'') \in W \quad \forall x \in \mathfrak{sl}(2,k). \tag{2.8}$$

From (2.7) and (2.8) it follows that

$$x \cdot V \subseteq W \quad \forall x \in \mathfrak{sl}(2,k)$$

and hence that

$$E^4 = F^4 = 0.$$

Since $char(k) = 0$ or $char(k) > 3$ then V is completely reducible by Theorem 2.4.3 and so, since W is irreducible, $V = W \oplus V_0$ where V_0 is a subspace of V on which $\mathfrak{sl}(2,k)$ acts trivially. By Lemma 2.3.1 c) $P|_{V_0 \times V_0}$ is trivial and hence

$$V_0 \cap Ker(\gamma) = \mathcal{Z}(\mathfrak{g})$$

since $Ker(\gamma)$ is the supercommutant of W in $\mathfrak{g}$. Furthermore, $\gamma : V \to k$ is a non-trivial linear form vanishing on W so it follows that $V_0 \cap Ker(\gamma) = \mathcal{Z}(\mathfrak{g})$ is of codimension one in V_0. Taking $v''' \in V_0$ such that $\gamma(v''') \neq 0$, we obtain

$$\mathfrak{g}_1 = (\mathfrak{sl}(2,k) \oplus \text{Span} < v''' >) \oplus \mathcal{Z}(\mathfrak{g})$$

and $\mathfrak{g}$ is as in b).

Case 4: Now, suppose that the only irreducible submodules of V are trivial. If $char(k) = 0$ then V is necessarily trivial and by Lemma 2.3.1 c) we have $P \equiv 0$.

We assume that k is algebraically closed and $char(k) > 0$. First, suppose that V is indecomposable and non-trivial. Since the only irreducible submodules of V are trivial, we can find a composition series

$$\{0\} \subset V_1 \subset \ldots \subset V_n = V$$

where for some $1 \leq l \leq n$, $W := V_l$ is a maximal trivial submodule of V.

Lemma 2.6.2. *Let $\mathfrak{g}$ be a simple Lie algebra. Let M be a representation of $\mathfrak{g}$ and let $N \subseteq M$ be a submodule of M such that N and M/N are trivial representations. Then M is a trivial representation.*

Proof. Let $y, y' \in \mathfrak{g}$ and $v \in M$. Since M/N is a trivial representation, we have $y'(v) \in N$ and so

$$y(y'(v)) = 0,$$

since N is trivial. Let $x \in \mathfrak{g}$. Since $\mathfrak{g}$ is simple, we have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ then $x = \Sigma[x_i, x_i']$ where $x_i, x_i' \in \mathfrak{g}$ and hence

$$x(v) = (\Sigma[x_i, x_i'])(v) = \Sigma(x_i(x_i'(v)) - x_i'(x_i(v))) = 0.$$

□

We recall:

Lemma 2.6.3. *(Remark 5.3.2 of [Str04]) A composition factor V_{i+1}/V_i is of dimension 1 or $p-1$.*

Proof. Since V is indecomposable, the Casimir element $\Omega = (H+Id)^2 + 4FE$ of $\mathfrak{sl}(2,k)$ has a unique eigenvalue. If we compute this eigenvalue on the first composition factor which is trivial, we obtain 1. Let $\alpha \in k^*$ and let $\{e_0, \ldots, e_m\}$ be a basis of the composition factor V_{i+1}/V_i as in Theorem 2.4.1. We have

$$\Omega(e_0) = (\alpha+1)^2 e_0$$

and so $(\alpha+1)^2 = 1$ which is equivalent to $\alpha(\alpha+2) = 0$. Hence $\alpha = [dim(V_{i+1}/V_i) - 1]$ by Theorem 2.4.1 and so $\alpha(\alpha+2) = 0$ is equivalent to $dim(V_{i+1}/V_i) = 1$ or $p-1$. □

Corollary 2.6.4. *The composition factor V_{l+1}/W is of dimension $p-1$.*

Proof. If $dim(V_{l+1}/W) = 1$, then V_{l+1} is trivial by Lemma 2.6.2 which is a contradiction to the maximality of V_l. Consequently V_{l+1}/W is of dimension $p-1$. □

Let $w \in W$ and consider

$$I_w := \text{Span} < P(v,w) \mid v \in V_{l+1} >,$$

which is an ideal of $\mathfrak{sl}(2,k)$. Suppose that $I_w \neq \{0\}$, or equivalently that $I_w = \mathfrak{sl}(2,k)$.

Since $\mathfrak{sl}(2,k)$ acts trivially on W by assumption and since $P(W,W) = \{0\}$ by Lemma 2.3.1 c), we have a well-defined equivariant linear map from V_{l+1}/W to $\mathfrak{sl}(2,k)$ given by

$$[v] \mapsto P(v,w).$$

This map is surjective by assumption, and injective since V_{l+1}/W is irreducible. Therefore the dimension of V_{l+1}/W is 3, which is impossible since we have seen above that V_{l+1}/W is of dimension $p-1 \neq 3$.

Consequently, for all w in W, $I_w = \{0\}$ and then the ideal

$$I := \text{Span} < P(v,w) \mid v \in V_{l+1},\ w \in W >,$$

is also trivial. Thus, there is a well-defined symmetric bilinear map $\dot{P} : V_{l+1}/W \times V_{l+1}/W \mapsto \mathfrak{sl}(2,k)$ given by

$$\dot{P}([v_1],[v_2]) = P(v_1,v_2)$$

and $\dot{P}$ satisfies the two relations (2.1) and (2.2). Since V_{l+1}/W is an irreducible representation of $\mathfrak{sl}(2,k)$ of dimension $p-1 \geq 4$ it follows from Proposition 2.5.1 that $\dot{P} \equiv 0$ and so

$$P|_{V_{l+1}\times V_{l+1}} \equiv 0.$$

Let $w \in W$ and $v \in V$. We have

$$P(v,v')(w) + P(v,w)(v') + P(v',w)(v) = 0 \quad \forall v' \in V_{l+1}$$

which implies that

$$P(v,w)(v') = 0 \quad \forall v' \in V_{l+1}.$$

Since V_{l+1} is a non-trivial representation of $\mathfrak{sl}(2,k)$, this means

$$P|_{V\times W} \equiv 0.$$

Again, there is a well-defined symmetric bilinear map $\ddot{P} : V/W \times V/W \mapsto \mathfrak{sl}(2,k)$ given by

$$\ddot{P}([v_1],[v_2]) = P(v_1,v_2)$$

and $\ddot{P}$ satisfies the two relations (2.1) and (2.2). However the representation V/W contains the irreducible representation V_{l+1}/W of dimension $p-1$ and hence, by Proposition 2.5.2, we have $\ddot{P} \equiv 0$, and so finally $P \equiv 0$.

Now, if the representation V is decomposable, we have

$$V \cong V_1 \oplus \ldots \oplus V_n \tag{2.9}$$

where V_i is indecomposable such that the only irreducible submodules of V_i are trivial. We have just seen that

$$P|_{V_i \times V_i} \equiv 0 \quad \forall i,$$

thus, it remains to prove that $P|_{V_i \times V_j} \equiv 0$ for two indecomposable summands V_i and V_j in the decomposition (2.9). Let

$$\{0\} \subset U_1 \subset \ldots \subset U_n = V_i, \quad \{0\} \subset U'_1 \subset \ldots \subset U'_m = V_j,$$

be two composition series where for some $1 \leq l \leq n$ (resp. $1 \leq k \leq m$), $W := U_l$ (resp. $W' := U'_k$) is a maximal trivial submodule of V_i (resp. V_j).

If V_i and V_j are trivial, $P|_{V_i \times V_j} \equiv 0$ By Lemma 2.3.1 c). If V_i is non-trivial, using the same reasoning as above, we will first show that

$$P|_{V_i \times W'} \equiv 0.$$

Let $w \in W'$ and consider the ideal

$$I_w = \text{Span} < P(v, w) \mid v \in U_{l+1} > .$$

By Corollary 2.6.4, we have $dim(U_{l+1}/W) = p-1$. Suppose that $I_w \neq \{0\}$. Since $P(W', W) = \{0\}$, we have a well-defined equivariant linear map from U_{l+1}/W to $\mathfrak{sl}(2, k)$ defined by

$$[v] \mapsto P(v, w).$$

This map is surjective by assumption and injective since U_{l+1}/W is irreducible. Therefore the dimension of U_{l+1}/W is 3, which is impossible since we have seen that the dimension of U_{l+1}/W is $p-1$. Thus, for all w' in W, $I_w = \{0\}$ and then $P(U_{l+1}, W') = \{0\}$.

Now, let $v \in V$, $w' \in W'$ and $u \in U_{l+1}$. We have

$$P(v, w')(u) + P(u, w')(v) + P(v, u)(w') = 0,$$

and since $P(u, w') = 0$ and $P(v, u) = 0$, this implies

$$P(v, w')(u) = 0 \quad \forall u \in U_{l+1}.$$

However U_{l+1} is non-trivial, so $P(v, w') = 0$ and hence

$$P|_{V_i \times W'} \equiv 0.$$

If V_j is trivial (which means that $V_j = W'$), this shows that $P|_{V_i \times V_j} \equiv 0$.

If V_j is non-trivial, as we have already seen,

$$P|_{V_i \times W'} \equiv 0, \quad P|_{V_j \times W} \equiv 0$$

and hence we have a well-defined symmetric bilinear map $\dot{P}$ from $(V_i/W \oplus V_j/W')^2$ to $\mathfrak{sl}(2, k)$ given by

$$\dot{P}([v_1], [v_2]) := P(v_1, v_2)$$

and $\dot{P}$ satisfies the two relations (2.1) and (2.2). By Corollary 2.6.4 there is an irreducible submodule of $V_i/W \oplus V_j/W'$ of dimension $p-1$ and then, by Propositions 2.5.1 and 2.5.2, we have $\dot{P} \equiv 0$. Then $\mathfrak{g}$ is as in a). □

In order to extend our result to fields of positive characteristic which are not necessarily algebraically closed and also to general three-dimensional simple Lie algebras, we need the two following observations.

Proposition 2.6.5. *Let $\mathfrak{s}$ be a non-split three-dimensional simple Lie algebra. The only two-dimensional representation of $\mathfrak{s}$ is the trivial representation.*

Proof. Let $\rho : \mathfrak{s} \to \mathrm{End(V)}$ be a two-dimensional representation of $\mathfrak{s}$. Since $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$, the representation ρ maps $\mathfrak{s}$ to $\mathfrak{sl}(V)$. Since $\mathfrak{s}$ is simple, the representation is either trivial or an isomorphism but since $\mathfrak{s} \not\cong \mathfrak{sl}(2, k)$, the representation is trivial. □

Lemma 2.6.6. *Let $\mathfrak{g}$ be a Lie superalgebra over k and let $\tilde{k}/k$ be an extension. We have*

$$\mathcal{Z}(\mathfrak{g} \otimes \tilde{k}) = \mathcal{Z}(\mathfrak{g}) \otimes \tilde{k}.$$

Proof. The inclusion $\mathcal{Z}(\mathfrak{g}) \otimes \tilde{k} \subseteq \mathcal{Z}(\mathfrak{g} \otimes \tilde{k})$ is clear.

Let $x \in \mathcal{Z}(\mathfrak{g} \otimes \tilde{k})$ and let $\{e_i\}_{i \in I}$ be a k-basis of $\tilde{k}$. Then there exist $\{v_i\}_{i \in I}$ such that $v_i \in \mathfrak{g}$ and $x = \sum_{i \in I} v_i \otimes e_i$. For all y in $\mathfrak{g}$ we have

$$\{x, y \otimes 1\} = 0$$

which implies

$$\sum_{i \in I} \{v_i, y\} \otimes e_i = 0$$

and hence we have

$$\{v_i, y\} = 0 \quad \forall i \in I, \ \forall y \in \mathfrak{g}.$$

Consequently $x \in \mathcal{Z}(\mathfrak{g}) \otimes \tilde{k}$. □

We can now prove the most important result of the chapter.

Theorem 2.6.7. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional Lie superalgebra such that $\mathfrak{g}_0$ is a three-dimensional simple Lie algebra and let*

$$\mathcal{Z}(\mathfrak{g}) := \{x \in \mathfrak{g} \mid \{x, y\} = 0, \ \forall y \in \mathfrak{g}\}.$$

Then, there are three cases:

a) $\{\mathfrak{g}_1, \mathfrak{g}_1\} = \{0\}$ *;*

b) $\mathfrak{g}_1 = (\mathfrak{g}_0 \oplus k) \oplus \mathcal{Z}(\mathfrak{g}),$ *(see Example 2.2.9) ;*

c) $\mathfrak{g} \cong \mathfrak{osp}_k(1|2) \oplus \mathcal{Z}(\mathfrak{g}),$ *(see Example 2.2.12).*

Proof. Let $\bar{k}$ be the algebraic closure of k and set $\bar{\mathfrak{g}} := \mathfrak{g} \otimes \bar{k}$, $\bar{\mathfrak{g}_0} := \mathfrak{g}_0 \otimes \bar{k}$ and $\bar{\mathfrak{g}_1} := \mathfrak{g}_1 \otimes \bar{k}$. Since $\bar{\mathfrak{g}_0} \cong \mathfrak{sl}(2, \bar{k})$ it follows from Theorem 2.6.1 that $\bar{\mathfrak{g}}$ satisfies one of the following:

a) $\{\bar{\mathfrak{g}_1}, \bar{\mathfrak{g}_1}\} = \{0\}$;

b) $\bar{\mathfrak{g}_1} \cong (\mathfrak{sl}(2, \bar{k}) \oplus \bar{k}) \oplus \mathcal{Z}(\bar{\mathfrak{g}})$;

c) $\bar{\mathfrak{g}} \cong \mathfrak{osp}_{\bar{k}}(1|2) \oplus \mathcal{Z}(\bar{\mathfrak{g}})$.

Case a): If $\{\bar{\mathfrak{g}_1}, \bar{\mathfrak{g}_1}\} = \{0\}$ then clearly $\{\mathfrak{g}_1, \mathfrak{g}_1\} = \{0\}$.

Case b): Suppose that $\bar{\mathfrak{g}_1} \cong \bar{V} \oplus \mathcal{Z}(\bar{\mathfrak{g}})$ where $\bar{V}$ is the direct sum of the adjoint representation and a one-dimensional trivial representation of $\bar{\mathfrak{g}_0} \cong \mathfrak{sl}(2, \bar{k})$. We recall Proposition 3.13 of [Bou60]:

Proposition 2.6.8. *Let $\mathfrak{h}$ be a Lie algebra, let V and W be two finite-dimensional representations of $\mathfrak{h}$ and let $\tilde{k}/k$ be an extension field. If $V \otimes \tilde{k}$ and $W \otimes \tilde{k}$ are isomorphic as representations of $\mathfrak{h} \otimes \tilde{k}$, then V and W are isomorphic as representations of $\mathfrak{h}$.*

From this, it follows that there is a direct sum decomposition

$$\mathfrak{g}_1 = V_1 \oplus V_0$$

where $\mathfrak{g}_0$ acts by the adjoint representation on V_1 and trivially on V_0. By Lemma 2.3.1 c), P restricted to V_0 vanishes identically. If $P|_{V_1 \times V_1} \not\equiv \{0\}$ then $\bar{\mathfrak{g}_0} \oplus V_1 \otimes \bar{k}$ would be a counter-example to Theorem 2.6.1 so we deduce that $P|_{V_1 \times V_1} \equiv \{0\}$.

Since $\mathcal{Z}(\mathfrak{g}) \subseteq V_0$ and since $V_0 \otimes \bar{k}$ is characterised as the subspace of $\bar{\mathfrak{g}_1}$ on which $\bar{\mathfrak{g}_0}$ acts trivially, it follows from Lemma 2.6.6 that $\mathcal{Z}(\mathfrak{g})$ is of codimension one in V_0. Hence there exists $v \in V_0$ such that

$$V_0 = \mathrm{Span} < v > \oplus \mathcal{Z}(\mathfrak{g})$$

and we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (V_1 \oplus \text{Span} < v > \oplus \mathcal{Z}(\mathfrak{g}))\,.$$

Since $P|_{V_1 \times V_1} \equiv \{0\}$ and $P(v,v) = 0$ we must have $P(v, V_1) \not\equiv \{0\}$. However the map $v_1 \mapsto P(v, v_1)$ is a $\mathfrak{g}_0$-equivariant isomorphism from V_1 to $\mathfrak{g}_0$ and hence is uniquely determined up to a constant (see Example 2.2.9). It is now easy to check that this implies b).

Case c): Now, suppose that $\mathfrak{g}_1 \otimes \bar{k} \cong \bar{V} \oplus \mathcal{Z}(\mathfrak{g} \otimes \bar{k})$ where $\bar{V}$ is the standard representation of $\mathfrak{g}_0 \otimes \bar{k} \cong \mathfrak{sl}(2, \bar{k})$. By Lemma 2.6.6, we have $\mathcal{Z}(\mathfrak{g} \otimes \bar{k}) = \mathcal{Z}(\mathfrak{g}) \otimes \bar{k}$ and so $\mathfrak{g}_1/\mathcal{Z}(\mathfrak{g})$ is an irreducible two-dimensional representation of $\mathfrak{g}_0$ which implies $\mathfrak{g}_0 \cong \mathfrak{sl}(2,k)$ by Proposition 2.6.5. By Proposition 2.6.8, there is a direct sum decomposition

$$\mathfrak{g}_1 = V_1 \oplus \mathcal{Z}(\mathfrak{g})$$

where $\mathfrak{g}_0 \cong \mathfrak{sl}(2,k)$ acts on V_1 by the standard representation. By Lemmas 2.3.1 and 2.5.1 b) the bracket on V_1 is a moment map and so $\mathfrak{g} \cong \mathfrak{osp}_k(1|2) \oplus \mathcal{Z}(\mathfrak{g})$. □

Over a field of positive characteristic, there are the important notions of *restricted* Lie algebra (see Remark 1.2.8) and *restricted* Lie superalgebra ([Pet92] or [WZ00]) and then we obtain the following result.

Corollary 2.6.9. *Let k be a field of positive characteristic p, let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional Lie superalgebra over k such that $\mathfrak{g}_0$ is a three-dimensional simple Lie algebra, such that $\{\mathfrak{g}_1, \mathfrak{g}_1\} \neq \{0\}$ and let K be the Killing form of $\mathfrak{g}_0$.*

a) The Lie superalgebra $\mathfrak{g}$ is restricted.

b) The $[p]$-map on $\mathfrak{g}_0$ satisfies

$$x^{[p]} = \Big(\frac{K(x,x)}{2}\Big)^{\frac{p-1}{2}} x \qquad \forall x \in \mathfrak{g}_0. \tag{2.10}$$

c) If $\mathfrak{g}_1 = (\mathfrak{g}_0 \oplus k) \oplus \mathcal{Z}(\mathfrak{g})$ the $[2p]$-map on $\mathfrak{g}_1$ satisfies

$$(x+\lambda)^{[2p]} = \lambda^p \Big(\frac{K(x,x)}{2}\Big)^{\frac{p-1}{2}} x \qquad \forall x + \lambda \in \mathfrak{g}_0 \oplus k$$

where the right-hand side of this equation is to be understood as an even element of $\mathfrak{g}$.

d) If $\mathfrak{g} \cong \mathfrak{osp}_k(1|2) \oplus \mathcal{Z}(\mathfrak{g})$ the $[2p]$-map on $\mathfrak{g}_1$ is trivial.

Proof. a) The Killing form of $\mathfrak{g}_0$ is non-degenerate (see [Mal92]) and hence $\mathfrak{g}_0$ is a restricted Lie algebra (see page 191 of [Jac62]). This means by definition that the adjoint representation of $\mathfrak{g}_0$ is a restricted representation. It is well-known that the trivial representation of $\mathfrak{g}_0$ and the standard representation k^2 of $\mathfrak{sl}(2,k)$ are also restricted representations. Hence, in the sense of V. Petrogradski ([Pet92]), the Lie superalgebras $\mathfrak{g}_0 \oplus (\mathfrak{g}_0 \oplus k) \oplus \mathcal{Z}(\mathfrak{g})$ and $\mathfrak{osp}_k(1|2) \oplus \mathcal{Z}(\mathfrak{g})$ appearing in Theorem 2.6.7 are restricted Lie superalgebras.

b) See Remark 1.2.8.

c) By [BLLS14] we have

$$x^{[2p]} = \Big(\frac{1}{2}P(x,x)\Big)^{[p]} \qquad \forall x \in \mathfrak{g}_1. \tag{2.11}$$

If $\mathfrak{g}_1 = (\mathfrak{g}_0 \oplus k) \oplus \mathcal{Z}(\mathfrak{g})$, since $(y+z)^{[2p]} = y^{[2p]}$ for all $y \in \mathfrak{g}_1$, for all $z \in \mathcal{Z}(\mathfrak{g})$, we will only consider $y \in \mathfrak{g}_1$ of the form $y = x + \lambda$ where $x \in \mathfrak{g}_0$ and $\lambda \in k$. Using Equation (2.11) and Definition 2.2.5 we have

$$\begin{aligned}(x+\lambda)^{[2p]} &= \Big(\frac{1}{2}P(x+\lambda, x+\lambda)\Big)^{[p]}\\ &= P(x,\lambda)^{[p]}\\ &= (\lambda x)^{[p]}\end{aligned}$$

and using Equation (2.10) we obtain

$$\begin{aligned}(x+\lambda)^{[2p]} &= \Big(\frac{K(\lambda x, \lambda x)}{2}\Big)^{\frac{p-1}{2}} \lambda x\\ &= \lambda^p \Big(\frac{K(x,x)}{2}\Big)^{\frac{p-1}{2}} x.\end{aligned}$$

d) Suppose that $\mathfrak{g} \cong \mathfrak{osp}_k(1|2) \oplus \mathcal{Z}(\mathfrak{g})$ and let $(a,b) \in k^2$. Using Equation (2.11) and Example 2.2.12 we have

$$\begin{aligned}(a,b)^{[2p]} &= \Big(\frac{1}{2}P((a,b),(a,b))\Big)^{[p]}\\ &= \begin{pmatrix} -ab & a^2 \\ -b^2 & ab \end{pmatrix}^{[p]}.\end{aligned}$$

Using Equation (2.10) and the well-known fact that

$$K(x,x) = -8 \cdot det(x) \qquad \forall x \in \mathfrak{sl}(2,k)$$

we obtain

$$(a,b)^{[2p]} = \Big(-4 \cdot det \begin{pmatrix} -ab & a^2 \\ -b^2 & ab \end{pmatrix} \Big)^{\frac{p-1}{2}} \begin{pmatrix} -ab & a^2 \\ -b^2 & ab \end{pmatrix}$$
$$= 0.$$

□

Throughout this chapter, we have always assumed the base field k to be of characteristic not two or three. Here are some comments on this assumption.

- If k is of characteristic three, the definition of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is usually modified by adding the property

$$\{x, \{x, x\}\} = 0 \qquad \forall x \in \mathfrak{g}_1 \tag{2.12}$$

  to those of Definition 2.2.1. We will give a counter-example to Theorem 2.6.1 with this definition of a Lie superalgebra in characteristic three.

- If k is of characteristic two, the definition of a Lie superalgebra is also usually modified, see [Leb10]. In this characteristic there are many reasons for which our proof totally fails. The most important of these, is that $\mathfrak{sl}(2,k)$ is nilpotent, not simple. On the other hand, an analogue of $\mathfrak{osp}_k(1|2)$ can be defined (see Remark 2.2.1 in [BLLS14]) and the Lie superalgebra of Definition 2.2.5 can also be defined. In characteristic two, there are counter-examples to Theorem 2.6.1 (see below). However, there are still simple three-dimensional Lie algebras and if k is algebraically closed, irreducible representations of the only simple three-dimensional Lie algebra over k are known (see [Dol78]). To the best of our knowledge, there is no counter-example to Theorem 2.6.7 (with $\mathfrak{osp}_k(1|2) \oplus \mathcal{Z}(\mathfrak{g})$ removed from the statement).

Example 2.6.10. *Suppose that $char(k) = 3$ and let $\{E, H, F\}$ be an $\mathfrak{sl}(2,k)$-triple.*

Let V be a three-dimensional vector space with a basis $\{e_0, e_1, v\}$. We define a representation $\rho : \mathfrak{sl}(2,k) \to \mathrm{End(V)}$ by:

$$\rho(H) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \rho(E) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \rho(F) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We define a symmetric bilinear map $P : V \times V \to \mathfrak{sl}(2,k)$ by

$$\begin{aligned} P(e_0,e_0) &= E, \quad P(e_0,e_1) = H, \quad P(e_1,e_1) = -F, \\ P(v,e_0) &= F, \quad P(v,v) \ \ = H, \quad P(v,e_1) \ = -E, \end{aligned}$$

which satisfies the identities (2.1), (2.2) and (2.12) and hence we obtain a structure of Lie superalgebra on the vector space $\mathfrak{sl}(2,k) \oplus V$.

In fact, the linear subspace $\mathfrak{sl}(2,k) \oplus \mathrm{Span} < e_0, e_1 >$ *is a Lie subsuperalgebra isomorphic to* $\mathfrak{osp}_k(1|2)$.

Example 2.6.11. *Suppose that char*$(k) = 2$.

Let $\mathfrak{g}_1$ *be a Lie algebra isomorphic to* $\mathfrak{sl}(2,k)$ *and let* $\phi : \mathfrak{sl}(2,k) \to \mathfrak{g}_1$ *be an isomorphism of Lie algebras. Let* $\{E, H, F\}$ *be an* $\mathfrak{sl}(2,k)$*-triple, i.e.,*

$$[E, F] = H, \quad [H, E] = 0, \quad [H, F] = 0.$$

The Lie algebra $\mathfrak{sl}(2,k)$ *acts on* $\mathfrak{g}_1$ *by the adjoint representation:*

$$y(x) := [\phi(y), x] \qquad \forall x \in \mathfrak{g}_1, \ \forall y \in \mathfrak{sl}(2,k).$$

If $x = a\phi(E) + b\phi(H) + c\phi(F) \in \mathfrak{g}_1$, *we define:*

$$x^2 := acH.$$

Let $y = a'\phi(E) + b'\phi(H) + c'\phi(F) \in \mathfrak{g}_1$ *and define*

$$P(x, y) := (x + y)^2 + x^2 + y^2 = (ac' + a'c)H.$$

We now show that the super vector space $\mathfrak{g} := \mathfrak{sl}(2,k) \oplus \mathfrak{g}_1$ *together with the bracket defined by the Lie bracket on* $\mathfrak{sl}(2,k)$*, the adjoint representation* $\mathfrak{sl}(2,k) \to \mathrm{End}(\mathfrak{g}_1)$ *and the bilinear symmetric map* $P(\ , \ )$ *is a Lie superalgebra. According to [Leb10] this is equivalent to*

$$\{x^2, y\} = \{x, \{x, y\}\} \qquad \forall x \in \mathfrak{g}_1, \ \forall y \in \mathfrak{g}$$

and it is easy to see that both sides of this equation vanish for all $x \in \mathfrak{g}_1, \ y \in \mathfrak{g}$.

Chapter 3

The colour category

Let k be a field of characteristic not two or three.

3.1 Vector spaces and algebras graded by an abelian group

In this section, we give definitions and examples of vector spaces and algebras graded by an abelian group. Let Γ be an abelian group.

Definition 3.1.1. *A vector space V with a decomposition $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ is said to be Γ-graded and an element $v \in V_\gamma$ is said to be homogeneous.*

Remark 3.1.2. *A Γ-gradation of a vector space realises it as the set of sections of a "vector bundle" over Γ.*

For an element $v \in V_\gamma$ we set $|v| := \gamma$ and we call $|v|$ the degree of v. For convenience, whenever the degree of an element is used in a formula, it is assumed that this element is homogeneous and that we extend by linearity the formula for non-homogeneous elements.

Definition 3.1.3. *An algebra A is said to be Γ-graded if it is Γ-graded as vector space and $|a \cdot b| = |a| + |b|$ for all homogeneous a, b in A.*

Example 3.1.4. *Let $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ and $W = \bigoplus_{\gamma\in\Gamma} W_\gamma$ be Γ-graded vector spaces.*

a) The base field k has a trivial Γ-gradation given by

$$|a| = 0 \qquad \forall a \in k.$$

b) The vector space $Hom(V,W)$ is Γ-graded by $Hom(V,W) = \bigoplus_{\gamma\in\Gamma} Hom(V,W)_\gamma$ where

$$Hom(V,W)_\gamma := \{f \in Hom(V,W) \mid f(V_a) \subseteq W_{a+\gamma}\ \forall a \in \Gamma\}.$$

In particular, the dual vector space $V^* = Hom(V, k)$ *of* V *is* Γ*-graded and*

$$\left(V^*\right)_\gamma \cong \left(V_{-\gamma}\right)^*.$$

c) *The vector space* $V \oplus W$ *is* Γ*-graded by* $V \oplus W = \bigoplus_{\gamma\in\Gamma} (V \oplus W)_\gamma$ *where*

$$(V \oplus W)_\gamma = V_\gamma \oplus W_\gamma.$$

d) *The vector space* $V \otimes W$ *is* Γ*-graded by* $V \otimes W = \bigoplus_{\gamma\in\Gamma} (V \otimes W)_\gamma$ *where*

$$(V \otimes W)_\gamma = \bigoplus_{a+b=\gamma} V_a \otimes W_b.$$

e) *As we have seen, the vector space* $End(V) = Hom(V, V)$ *is* Γ*-graded. In fact, the associative algebra* $End(V)$ *is also* Γ*-graded as an algebra.*

f) *Let* A *and* B *be* Γ*-graded algebras. As we have seen, the vector space* $A \oplus B$ *is* Γ*-graded. In fact, the algebra* $A \oplus B$ *is also* Γ*-graded as an algebra.*

3.2 Commutation factors and representations of the symmetric group

Let Γ be an abelian group. In this section we introduce the notion of a commutation factor of Γ. This allows us to define a notion of "commutative" and "anticommutative" for Γ-graded algebras which takes into account the Γ-gradation. It also allows us to modify the standard action of the symmetric group S_n on the n-fold tensor product of a Γ-graded vector space.

Definition 3.2.1. *(See III.116 in [Bou70]) Let* Γ *be an abelian group. A commutation factor* ϵ *of* Γ *is a map* $\epsilon : \Gamma \times \Gamma \to k^*$ *such that for all* $a, b, c \in \Gamma$

$$\begin{aligned} \epsilon(a,b)\epsilon(b,a) &= 1, \\ \epsilon(a+b,c) &= \epsilon(a,c)\epsilon(b,c), \\ \epsilon(a,b+c) &= \epsilon(a,b)\epsilon(a,c). \end{aligned}$$

The basic features of commutation factors are given in the following remark.

Remark 3.2.2. a) *The first two properties of this definition imply the third.*

b) *We have*

$$\epsilon(a,0) = \epsilon(0,a) = 1, \qquad \epsilon(a,-b) = \epsilon(b,a) \qquad \forall a,b \in \Gamma.$$

c) The map $-\epsilon : \Gamma \times \Gamma \to k^*$ is not a commutation factor.

d) For $a \in \Gamma$, we have $\epsilon(a, a) = \pm 1$ and hence a partition

$$\Gamma = \Gamma_0 \cup \Gamma_1$$

where

$$\Gamma_0 := \{a \in \Gamma \mid \epsilon(a, a) = 1\}, \qquad \Gamma_1 := \{a \in \Gamma \mid \epsilon(a, a) = -1\}.$$

The map $a \mapsto \epsilon(a, a) \in \mathbb{Z}_2$ is a group homomorphism so Γ_0 is a normal subgroup of index at most two.

e) If Γ is cyclic, then ϵ is abelian in the sense that $\epsilon(a, b) = \epsilon(b, a)$ for all $a, b \in \Gamma$.

f) The set of all the commutation factors of Γ is an abelian group where the identity is the trivial map and where the law is given by pointwise multiplication of maps.

Here are some examples of non-trivial commutation factors.

Example 3.2.3. *a) The most important non-trivial example of a commutation factor is obtained by taking $\Gamma = \mathbb{Z}_2$ and ϵ defined by*

$$\epsilon(a, b) := (-1)^{ab} \qquad \forall a, b \in \mathbb{Z}_2.$$

In this case Γ-graded vector spaces are "super" vector spaces and ϵ-symmetric objects correspond to "supersymmetric" objects in the usual sense.

b) Suppose that $k = \mathbb{C}$ and $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_n$. Let $q := exp\left(\frac{2i\pi}{n}\right) \in \mathbb{C}$ and let $\epsilon : (\mathbb{Z}_n \times \mathbb{Z}_n) \times (\mathbb{Z}_n \times \mathbb{Z}_n) \to$ be defined by

$$\epsilon((a, b), (c, d)) := q^{ad-bc} \qquad \forall (a, b), (c, d) \in \mathbb{Z}_n \times \mathbb{Z}_n.$$

Then ϵ is a commutation factor.

If the group Γ is cyclic the commutation factors are easy to determine.

Proposition 3.2.4. *Let ϵ be a commutation factor of Γ.*

a) If $\Gamma = \mathbb{Z}_n$ with n odd, then ϵ is trivial.

b) If $\Gamma = \mathbb{Z}$ or if $\Gamma = \mathbb{Z}_n$ with n even, then ϵ is either trivial or equal to

$$\epsilon(i, j) = (-1)^{ij} \qquad \forall i, j \in \Gamma.$$

Proof. If Γ is cyclic and generated by $g \in \Gamma$, then ϵ is determined by $\epsilon(g,g)$. By Remark 3.2.2 we have $\epsilon(g,g) = \pm 1$. However if Γ is of finite odd order n then we must have

$$1 = \epsilon(0,g) = \epsilon(ng,g) = \epsilon(g,g)^n = \epsilon(g,g).$$

□

In [Sch79a], Scheunert gives the general form of a commutation factor of a finitely generated abelian group in terms of a cyclic decomposition.

Example 3.2.5. *Let Γ be an abelian group together with commutation factor ϵ. Then, $\tilde{\epsilon} : (\mathbb{Z} \times \Gamma) \times (\mathbb{Z} \times \Gamma) \to k^*$ given by*

$$\tilde{\epsilon}((m,\gamma),(m',\gamma')) = (-1)^{mm'}\epsilon(\gamma,\gamma') \quad \forall m,m' \in \mathbb{Z},\ \forall \gamma,\gamma' \in \Gamma$$

is a commutation factor of $\mathbb{Z} \times \Gamma$ and the same formula defines a commutation factor of $\mathbb{Z}_2 \times \Gamma$.

Notation. *Let V be a Γ-graded vector space, let ϵ be a commutation factor of Γ and let $v, w \in V$. For brevity, we denote $\epsilon(|v|,|w|)$ by $\epsilon(v,w)$ and by $\mathcal{E}$ the canonical linear map $\mathcal{E} : V \to V$ given by*

$$\mathcal{E}(v) := \epsilon(v,v)v \qquad \forall v \in V.$$

We now show that if V is a Γ-graded vector space, then using a commutation factor one can modify the standard action of the symmetric group S_n on $V^{\otimes n}$ to take into account the Γ-grading of V.

Proposition 3.2.6. *Let V be a Γ-graded vector space and let ϵ be a commutation factor of Γ. There is a unique right group action $\rho : S_n \to GL(V^{\otimes n})$ of the permutation group S_n on $V^{\otimes n}$ such that the action of a transposition $\tau_{i,i+1} \in S_n$ is given by*

$$\rho(\tau_{i,i+1})(v_1 \otimes \ldots \otimes v_n) = \epsilon(v_i, v_{i+1}) v_1 \otimes \ldots \otimes v_{i+1} \otimes v_i \otimes \ldots \otimes v_n \tag{3.1}$$

for all $v_1, \ldots, v_n \in V$. For an arbitrary element $\sigma \in S_n$, this action is given by

$$\rho(\sigma)(v_1 \otimes \ldots \otimes v_n) = q(\sigma; v_1, \ldots, v_n) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$$

where

$$q(\sigma; v_1, \ldots, v_n) = \prod_{\substack{1 \le i < j \le n \\ \sigma^{-1}(i) > \sigma^{-1}(j)}} \epsilon(v_i, v_j).$$

Proof. For the existence, since S_n generated by the transpositions $\tau_{i,i+1}$ with the relations

$$\tau_{i-1,i}\tau_{i,i+1}\tau_{i-1,i} = \tau_{i,i+1}\tau_{i-1,i}\tau_{i,i+1}$$

we just need to check that

$$\rho(\tau_{i-1,i}) \circ \rho(\tau_{i,i+1}) \circ \rho(\tau_{i-1,i}) = \rho(\tau_{i,i+1}) \circ \rho(\tau_{i-1,i}) \circ \rho(\tau_{i,i+1}).$$

The left-hand side acting on $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$ gives

$$\begin{aligned}
&\rho(\tau_{i-1,i})(\rho(\tau_{i,i+1})(\rho(\tau_{i-1,i})(v_1 \otimes \ldots \otimes v_n))) \\
&= \epsilon(v_{i-1}, v_i)\rho(\tau_{i-1,i})(\rho(\tau_{i,i+1})(v_1 \otimes \ldots \otimes v_i \otimes v_{i-1} \otimes \ldots \otimes v_n)) \\
&= \epsilon(v_{i-1}, v_i)\epsilon(v_{i-1}, v_{i+1})\rho(\tau_{i-1,i})(v_1 \otimes \ldots \otimes v_i \otimes v_{i+1} \otimes v_{i-1} \otimes \ldots \otimes v_n) \\
&= -\epsilon(v_{i-1}, v_i)\epsilon(v_{i-1}, v_{i+1})\epsilon(v_i, v_{i+1})v_1 \otimes \ldots \otimes v_{i+1} \otimes v_i \otimes v_{i-1} \otimes \ldots \otimes v_n
\end{aligned}$$

and the right-hand side acting on $v_1 \otimes \ldots \otimes v_n \in V^{\otimes n}$ gives

$$\begin{aligned}
&\rho(\tau_{i,i+1})(\rho(\tau_{i-1,i})(\rho(\tau_{i,i+1})(v_1 \otimes \ldots \otimes v_n))) \\
&= \epsilon(v_i, v_{i+1})\rho(\tau_{i,i+1})(\rho(\tau_{i-1,i})(v_1 \otimes \ldots \otimes v_{i+1} \otimes v_i \otimes \ldots \otimes v_n)) \\
&= \epsilon(v_i, v_{i+1})\epsilon(v_{i-1}, v_{i+1})\rho(\tau_{i,i+1})(v_1 \otimes \ldots v_{i+1} \otimes v_{i-1} \otimes v_i \otimes \ldots \otimes v_n) \\
&= \epsilon(v_i, v_{i+1})\epsilon(v_{i-1}, v_{i+1})\epsilon(v_{i-1}, v_i)v_1 \otimes \ldots v_{i+1} \otimes v_i \otimes v_{i-1} \otimes \ldots \otimes v_n.
\end{aligned}$$

Hence the existence of a right representation $\pi : S_n \to GL(V^{\otimes n})$ satisfying (3.1) is proven and its unicity is clear. □

The "antisymmetric" analogue of this result is the following :

Proposition 3.2.7. *Let V be a Γ-graded vector space and let ϵ be a commutation factor of Γ. There is a unique right group action $\pi : S_n \to GL(V^{\otimes n})$ of the permutation group S_n on $V^{\otimes n}$ such that the action of a transposition $\tau_{i,i+1} \in S_n$ is given by*

$$\pi(\tau_{i,i+1})(v_1 \otimes \ldots \otimes v_n) = -\epsilon(v_i, v_{i+1})v_1 \otimes \ldots \otimes v_{i+1} \otimes v_i \otimes \ldots \otimes v_n \tag{3.2}$$

for all $v_1, \ldots, v_n \in V$. For an arbitrary element $\sigma \in S_n$, this action is given by

$$\pi(\sigma)(v_1 \otimes \ldots \otimes v_n) = p(\sigma; v_1, \ldots, v_n)v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$$

where

$$p(\sigma; v_1, \ldots, v_n) = sgn(\sigma) \prod_{\substack{1 \leq i < j \leq n \\ \sigma^{-1}(i) > \sigma^{-1}(j)}} \epsilon(v_i, v_j).$$

Remark 3.2.8. *There is also a unique left group action ρ' (resp. π') of S_n on $V^{\otimes n}$ such that transpositions act by (3.1) (resp. (3.2)) : $\rho'(\sigma) = \rho(\sigma^{-1})$ (resp. $\pi'(\sigma) = \pi(\sigma^{-1})$) for $\sigma \in S_n$.*

Remark 3.2.9. *a) For $v_1, \dots, v_n \in V$ and $\sigma, \sigma' \in S_n$, we have*

$$p(\sigma\sigma'; v_1, \dots, v_n) = p(\sigma'; v_{\sigma(1)}, \dots, v_{\sigma(n)})p(\sigma; v_1, \dots, v_n), \tag{3.3}$$
$$p(Id; v_1, \dots, v_n) = 1. \tag{3.4}$$

The map q also satisfies these relations. In general if $\nu : G \to End(W)$ is a right representation of a group G, a map $p : G \times W \to W$ which satisfies Equations (3.3) and (3.4) is called a multiplier. It is then the case that

$$\nu_p(g)(v) = p(v, g)\nu(g)(v) \qquad \forall g \in G,\ \forall v \in W$$

defines a representation of G.

b) There are group inclusions $S_n \subseteq S_{n+m}$ (S_n acts only on the first n coordinates) and $S_m \subseteq S_{n+m}$ (S_m acts only on the last m coordinates) and furthermore (with the obvious notations)

$$p_n(\sigma; v_1, \dots, v_n) = p_{n+m}(\sigma; v_1, \dots, v_{n+m}) \qquad \forall \sigma \in S_n,\ \forall v_1, \dots, v_{n+m} \in V,$$
$$p_m(\sigma; v_n, \dots, v_{n+m}) = p_{n+m}(\sigma; v_1, \dots, v_{n+m}) \qquad \forall \sigma \in S_m,\ \forall v_1, \dots, v_{n+m} \in V.$$

The map q also satisfies these relations.

3.3 ϵ-symmetric and ϵ-antisymmetric bilinear maps

Let Γ be an abelian group and let ϵ be a commutation factor of Γ. In this section we define ϵ-symmetric and ϵ-antisymmetric bilinear maps for Γ-graded vector spaces. With respect to a fixed quadratic extension field $\tilde{k}$ of k, we also give a notion of ϵ-hermitian and ϵ-antihermitian forms for Γ-graded $\tilde{k}$-vector spaces

Definition 3.3.1. *Let V and W be Γ-graded vector spaces and let $B : V \times V \to W$ be a bilinear map.*

a) We say that B is ϵ-symmetric if $B(v, w) = \epsilon(v, w)B(w, v)$ for all $v, w \in V$.

b) We say that B is ϵ-antisymmetric if $B(v, w) = -\epsilon(v, w)B(w, v)$ for all $v, w \in V$.

c) We say that B is of degree γ if $B(v, w) = 0$ when $|v| + |w| \neq \gamma$ for all $v, w \in V$.

d) If $W = V$, we say that V is an ϵ-commutative (resp. ϵ-anticommutative) algebra if B is ϵ-symmetric (resp. ϵ-antisymmetric).

Remark 3.3.2. *If $W = k$, unless otherwise stated, we always assume that B is of degree 0.*

Example 3.3.3. *a) If the commutation factor ϵ is trivial, an ϵ-symmetric bilinear form on a Γ-graded vector space V is a symmetric bilinear form in the usual sense.*

b) If $\epsilon(v, w) = -1$ for all v, w non-zero in a Γ-graded vector space V, an ϵ-symmetric bilinear form on V is an antisymmetric bilinear form in the usual sense.

c) Let V be a vector space and let $g : V \times V \to k$ be a symmetric bilinear form in the usual sense. With respect to certain choices of (Γ, ϵ) and Γ-gradations of V, the bilinear form g is ϵ-antisymmetric. For example, let $\Gamma = \mathbb{Z}_2$,

$$\epsilon(a, b) := (-1)^{ab} \qquad \forall a, b \in \mathbb{Z}_2$$

and $V_0 = \{0\}$, $V_1 = V$. Similarly an antisymmetric form on V will be ϵ-symmetric with respect to certain choices of (Γ, ϵ) and Γ-gradations of V.

In fact, modifying the Γ-gradation of a vector space V, an ϵ-symmetric (resp. ϵ-antisymmetric) bilinear form on V can become ϵ-antisymmetric (resp. ϵ-symmetric).

Remark 3.3.4. *Suppose that there exists $\delta \in \Gamma$ such that $\epsilon(\delta, \delta) = -1$ and $2\delta = 0$. Let $V = \bigoplus_{\gamma\in\Gamma} V_a$ be a Γ-graded vector space and let $(\ , \ ) : V \times V \to k$ be a bilinear form. We consider V with the grading $V = \bigoplus_{\gamma\in\Gamma} V'_\gamma$ where $V'_\gamma := V_{\gamma+\delta}$ for all $\gamma \in \Gamma$. We define $(\ , \ )' : V \times V \to k$ by $(v, w)' = (v, w)$ for all v, w in V.*

If $(\ , \ )$ is ϵ-symmetric (resp. ϵ-antisymmetric), then $(\ , \ )'$ is ϵ-antisymmetric (resp. ϵ-symmetric).

It turns out that an ϵ-symmetric bilinear form is the orthogonal sum of a symmetric bilinear form and an antisymmetric bilinear form in the following sense.

Proposition 3.3.5. *Let V be a Γ-graded vector space and let $(\ , \ ) : V \times V \to k$ be a ϵ-symmetric bilinear form. Let $V = V_0 \oplus V_1$ where*

$$V_0 = \{v \in V \mid \epsilon(v, v) = 1\}, \qquad V_1 = \{v \in V \mid \epsilon(v, v) = -1\}.$$

Then, the restriction of $(\ , \ )$ to V_0 is symmetric in the usual sense, the restriction of $(\ , \ )$ to V_1 is antisymmetric in the usual sense, and V_0 is orthogonal to V_1.

Proof. Let $v, w \in V_0$. We have $(v, w) = 0$ unless if $|v| = -|w|$. In that case we have

$$(v, w) = \epsilon(v, w)(w, v) = \epsilon(v, v)(w, v) = (w, v).$$

Similarly we have $(v, w) = -(w, v)$ if $v, w \in V_1$. Let $v \in V_0$, $w \in V_1$, then we have $|v| + |w| \neq 0$ and so $(v, w) = 0$. □

The analogue of the previous proposition for an ϵ-antisymmetric bilinear form is the following.

Proposition 3.3.6. *Let V be a Γ-graded vector space and let $(\ ,\ ) : V \times V \to k$ be a ϵ-antisymmetric bilinear form. Let $V = V_0 \oplus V_1$ where*

$$V_0 = \{v \in V \mid \epsilon(v,v) = 1\}, \qquad V_1 = \{v \in V \mid \epsilon(v,v) = -1\}.$$

Then, the restriction of $(\ ,\ )$ to V_0 is antisymmetric in the usual sense, the restriction of $(\ ,\ )$ to V_1 is symmetric in the usual sense, and V_0 is orthogonal to V_1.

We now define notions of ϵ-hermitian and ϵ-antihermitian forms with respect to a quadratic extension field of k.

Definition 3.3.7. *Let $\tilde{k}/k$ be a quadratic extension and let $\bar{\ } : \tilde{k} \to \tilde{k}$ be the associated involution. Let V be a Γ-graded $\tilde{k}$-vector space and let $H : V \times V \to \tilde{k}$ be a map such that*

$$H(au + bv, w) = aH(u,w) + bH(v,w) \qquad \forall u,v,w \in V\ \forall a,b \in \tilde{k},$$
$$H(u, av + bw) = \bar{a}H(u,v) + \bar{b}H(u,w) \qquad \forall u,v,w \in V\ \forall a,b \in \tilde{k}.$$

We say that H is an ϵ-hermitian form if

$$\overline{H(v,u)} = \epsilon(v,u)H(u,v) \qquad \forall u,v \in V$$

and we say that H is an ϵ-antihermitian form if

$$\overline{H(v,u)} = -\epsilon(v,u)H(u,v) \qquad \forall u,v \in V.$$

Here are the standard examples of ϵ-hermitian and ϵ-antihermitian forms.

Example 3.3.8. *Let $\tilde{k}/k$ be a quadratic extension and let $\bar{\ } : \tilde{k} \to \tilde{k}$ be the associated involution. Let V be a Γ-graded vector space. Since $\tilde{k}$ is Γ-graded by $|\tilde{k}| := 0$ and V is Γ-graded, their tensor product $V \otimes \tilde{k}$ is Γ-graded (see Examples 3.1.4). Let $(\ ,\ ) : V \times V \to k$ be a ϵ-(anti)symmetric bilinear form. Extend this form to $V \otimes \tilde{k}$ by*

$$(v_1 \otimes k_1, v_2 \otimes k_2) := k_1 k_2 (v_1, v_2) \qquad \forall v_1 \otimes k_1, v_2 \otimes k_2 \in V \otimes \tilde{k},$$

and define $H : (V \otimes \tilde{k}) \times (V \otimes \tilde{k}) \to \tilde{k}$ by

$$H(x,y) := (x, \bar{y}) \qquad \forall x,y \in V \otimes \tilde{k}.$$

It is easily checked that this is an ϵ-(anti)hermitian form.

3.4 ϵ-exterior and ϵ-Clifford algebras

Let Γ be an abelian group and let ϵ be a commutation factor of Γ. In this section we define and prove the basic properties of the ϵ-exterior algebra of a Γ-graded vector space and of the ϵ-Clifford algebra of a Γ-graded vector space with respect to an ϵ-symmetric bilinear form (see [CK16]).

Let V be a Γ-graded vector space and let $T(V) = \bigoplus_{i\in\mathbb{N}} T^i(V)$ be the tensor algebra over V. As in Examples 3.1.4, the Γ-gradation of V induces a Γ-gradation on $T(V) = \bigoplus_{i\in\mathbb{N}} T^i(V)$ by the rule

$$|1| := 0, \qquad |x_1 \otimes \ldots \otimes x_n| := |x_1| + \ldots + |x_n| \qquad \forall x_1, \ldots, x_n \in V.$$

Furthermore, $T(V)$ has a natural $\mathbb{Z}$-gradation given by

$$\alpha(1) := 0, \qquad \alpha(x_1 \otimes \ldots \otimes x_n) := n \qquad \forall x_1, \ldots, x_n \in V.$$

The tensor algebra $T(V)$ has the following universal property : let $f : V \to A$ be a linear map where A is a unital algebra and let $i : V \to T(V)$ be the canonical injective map. Then, there exists a unique algebra homomorphism $\tilde{f} : T(V) \to A$ such that $\tilde{f} \circ i = f$ and $\tilde{f}(1) = 1_A$. If A is a Γ-graded algebra and f is of degree 0, since i is of degree 0, then $\tilde{f}$ is a Γ-graded algebra homomorphism of degree 0.

Definition 3.4.1. *Let V be a Γ-graded vector space. The ϵ-exterior algebra $\Lambda_\epsilon(V)$ of V is defined by*

$$\Lambda_\epsilon(V) := T(V)/I(V),$$

where $T(V)$ is the tensor algebra over V and $I(V)$ is the two-sided ideal of $T(V)$ generated by elements of the form

$$x \otimes y + \epsilon(x, y) y \otimes x \qquad \forall x, y \in V.$$

Remark 3.4.2. *a) For all $x, y \in V$, since $|x \otimes y| = |y \otimes x|$, the Γ-gradation of $T(V)$ induces a Γ-gradation of $\Lambda_\epsilon(V)$ and since $\alpha(x \otimes y) = \alpha(y \otimes x)$, the $\mathbb{Z}$-gradation of $T(V)$ induces a $\mathbb{Z}$-gradation of $\Lambda_\epsilon(V)$. Hence, $\Lambda_\epsilon(V)$ is a $\mathbb{Z} \times \Gamma$-graded algebra and we have*

$$u \wedge v = (-1)^{\alpha(u)\alpha(v)} \epsilon(u, v) v \wedge u \qquad \forall u, v \in \Lambda_\epsilon(V).$$

In other words, we have

$$u \wedge v = \tilde{\epsilon}(u, v) v \wedge u \qquad \forall u, v \in \Lambda_\epsilon(V),$$

where $\tilde{\epsilon}$ is the commutation factor of $\mathbb{Z} \times \Gamma$ given in Example 3.2.5 and hence $\Lambda_\epsilon(V)$ is an $\tilde{\epsilon}$-commutative $\mathbb{Z} \times \Gamma$-algebra.

b) *For* $v_1, \ldots, v_n \in V$ *and* $\sigma \in S_n$*, we have*

$$v_1 \wedge \ldots \wedge v_n = p(\sigma; v_1, \ldots, v_n) v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(n)}$$

where p *is the map of Proposition 3.2.7 given by*

$$p(\sigma; v_1, \ldots, v_n) = sgn(\sigma) \prod_{\substack{1 \leq i < j \leq n \\ \sigma^{-1}(i) > \sigma^{-1}(j)}} \epsilon(v_i, v_j).$$

Similarly we define the ϵ-Clifford algebra of V with respect to an ϵ-symmetric bilinear form on V.

Definition 3.4.3. *Let* V *be a* Γ*-graded vector space and let* $(\ ,\ ) : V \times V \to k$ *be an* ϵ*-symmetric bilinear form. The* ϵ*-Clifford algebra* $C_\epsilon(V, (\ ,\ ))$ *of* V *is defined by*

$$C_\epsilon(V, (\ ,\ )) := T(V)/I(V),$$

where $T(V)$ *is the tensor algebra over* V *and* $I(V)$ *is the two-sided ideal of* $T(V)$ *generated by elements of the form*

$$x \otimes y + \epsilon(x, y) y \otimes x - 2(x, y) \cdot 1 \qquad \forall x, y \in V.$$

Remark 3.4.4. *For all* $x, y \in V$*, since* $|x \otimes y| = |y \otimes x|$*, the* Γ*-gradation of* $T(V)$ *induces a* Γ*-gradation of* $C_\epsilon(V, (\ ,\ ))$ *and since* $\alpha(x \otimes y) \equiv \alpha(y \otimes x) \equiv \alpha(1) \pmod 2$*, the* $\mathbb{Z}$*-gradation of* $T(V)$ *induces a* $\mathbb{Z}_2$*-gradation of* $C_\epsilon(V, (\ ,\ ))$*. Hence,* $C_\epsilon(V, (\ ,\ ))$ *is a* $\mathbb{Z}_2 \times \Gamma$*-graded algebra.*

Here are some basic properties of ϵ-exterior and ϵ-Clifford algebras.

Remark 3.4.5. a) *If the commutation factor* ϵ *is trivial, the* ϵ*-symmetric bilinear form* $(\ ,\ )$ *on* V *is a symmetric bilinear form in the usual sense. Hence,* $\Lambda_\epsilon(V)$ *is the usual exterior algebra* $\Lambda(V)$ *of* V *and* $C_\epsilon(V, (\ ,\ ))$ *is the usual Clifford algebra* $C(V, (\ ,\ ))$ *of* $(V, (\ ,\ ))$*.*

b) *If* $\epsilon(v, w) = -1$ *for all* v, w *non-zero in* V*, the* ϵ*-symmetric bilinear form* $(\ ,\ )$ *on* V *is an antisymmetric bilinear form in the usual sense. Hence,* $\Lambda_\epsilon(V)$ *is the usual symmetric algebra* $S(V)$ *of* V *and* $C_\epsilon(V, (\ ,\ ))$ *is the usual Weyl algebra* $W(V, (\ ,\ ))$ *of* $(V, (\ ,\ ))$*.*

c) *If* $(\ ,\ )$ *is totally degenerate then* $C_\epsilon(V, (\ ,\ )) = \Lambda_\epsilon(V)$*.*

As in usual cases, we have a Chevalley isomorphism between $C_\epsilon(V, (\ ,\ ))$ and $\Lambda_\epsilon(V)$.

Theorem 3.4.6. *Let k be a field of characteristic zero. Let V be a finite-dimensional Γ-graded vector space and let $(\ ,\ ) : V \times V \to k$ be a non-degenerate ϵ-symmetric bilinear form. As vector spaces, $C_\epsilon(V, (\ ,\ ))$ and $\Lambda_\epsilon(V)$ are isomorphic.*

Proof. See [CK16]. □

Remark 3.4.7. *It is almost certain that this result also holds in positive characteristic.*

Remark 3.4.8. *Let V be a finite-dimensional Γ-graded vector space. Following Remark 3.2.2 we define*

$$V_0 := \bigoplus_{a \in \Gamma_0} V_a, \qquad V_1 := \bigoplus_{a \in \Gamma_1} V_a$$

and obtain a $\mathbb{Z}_2$-gradation on V. Let $\{e_1, \ldots, e_n\}$ be a basis of V_0 and let $\{f_1, \ldots, f_m\}$ be a basis of V_1. The set

$$\{e_1^{\wedge k_1} \wedge \ldots \wedge e_n^{\wedge k_n} \wedge f_1^{\wedge l_1} \wedge \ldots \wedge f_m^{\wedge l_m} \mid i \in [\![0, n]\!], j \in [\![0, m]\!], k_i \in \{0, 1\}, l_j \in \mathbb{N}\}$$

is a basis of $\Lambda_\epsilon(V)$.

The ϵ-Clifford algebra has the following universal property.

Proposition 3.4.9. *(Universal property of $C_\epsilon(V, (\ ,\ ))$) Let V be a Γ-graded vector space and let $(\ ,\ ) : V \times V \to k$ be a ϵ-symmetric bilinear form. Let A be a unital associative algebra, let $i : V \to C_\epsilon(V, (\ ,\ ))$ be the canonical injective map and let $\phi : V \to A$ be a linear map such that*

$$\phi(x)\phi(y) + \epsilon(x, y)\phi(y)\phi(x) = 2(x, y)1_A \qquad \forall x, y \in V.$$

Then, there exists a unique algebra homomorphism $\tilde{\phi} : C_\epsilon(V, (\ ,\ )) \to A$ such that $\phi = \tilde{\phi} \circ i$ and $\tilde{\phi}(1) = 1_A$. If A is a Γ-graded algebra and ϕ is of degree 0, then $\tilde{\phi}$ is a Γ-graded algebra homomorphism of degree 0.

Proof. Abstract nonsense. □

3.5 Graded tensor product of graded algebras with respect to a commutation factor

Let Γ be an abelian group and let ϵ be a commutation factor of Γ.

In this section we define a product of Γ-graded algebras with respect to a commutation factor ϵ which has the property that the product of two "ϵ-commutative" (resp.

"ϵ-anticommutative") algebras is "ϵ-commutative". Furthermore, we will show that, with respect to this product, the ϵ-exterior algebra functor (resp. ϵ-Clifford algebra functor) is multiplicative for direct sums (resp. orthogonal sums). These results summarise and generalise the isomorphisms which express the usual exterior (resp. symmetric, Clifford, Weyl) algebra of a direct or orthogonal sum in terms of the product of the usual exterior (resp. symmetric, Clifford, Weyl) algebras of the summands.

Definition 3.5.1. *Let A and A' be Γ-graded algebras. We define the Γ-graded tensor product $A \otimes_\epsilon A'$ with respect to ϵ to be the Γ-graded vector space $A \otimes A'$ together with the unique product satisfying*

$$a \otimes a' \cdot b \otimes b' := \epsilon(a', b) ab \otimes a'b' \qquad \forall a \otimes a', b \otimes b' \in A \otimes A'.$$

This is a Γ-graded algebra.

This tensor product has the following universal property.

Proposition 3.5.2. *Let A, A', B be Γ-graded unital algebras. Let $f : A \to B$ and $g : A' \to B$ be homomorphisms of algebras such that*

$$f(a)g(a') = \epsilon(a, a')g(a')f(a) \qquad \forall a \in A,\ \forall a' \in A'.$$

Then there exists a unique homomorphism of algebras $\phi : A \otimes_\epsilon A' \to B$ such that

$$\phi(a \otimes 1) = f(a), \quad \phi(1 \otimes a') = g(a') \qquad \forall a \in A,\ \forall a' \in A'.$$

Moreover, if f and g are of degree 0, then ϕ is of degree 0.

Proposition 3.5.3. *The Γ-graded tensor product of two ϵ-(anti)commutative algebras is ϵ-commutative. The Γ-graded tensor product of an ϵ-commutative algebra with an ϵ-anticommutative algebra is ϵ-anticommutative.*

Proof. Let A and A' be ϵ-commutative Γ-graded algebras. Let $a \otimes a', b \otimes b' \in A \otimes_\epsilon A'$. We have

$$\begin{aligned}
\epsilon(a \otimes a', b \otimes b') b \otimes b' \cdot a \otimes a' &= \epsilon(a + a', b + b')\epsilon(b', a) ba \otimes b'a' \\
&= \epsilon(a + a', b + b')\epsilon(b', a)\epsilon(b, a)\epsilon(b', a') ab \otimes a'b' \\
&= \epsilon(a', b) ab \otimes a'b' \\
&= a \otimes a' \cdot b \otimes b'
\end{aligned}$$

and hence $A \otimes_\epsilon A'$ is ϵ-commutative. Similarly, we show that the Γ-graded tensor product of two ϵ-anticommutative algebras is ϵ-commutative and also that the Γ-graded tensor product of an ϵ-commutative algebra with an ϵ-anticommutative algebra is ϵ-anticommutative. □

Proposition 3.5.4. *Let V and W be Γ-graded vector spaces, let $(\ ,\ )_V : V \times V \to k$ and $(\ ,\ )_W : W \times W \to k$ be ϵ-symmetric bilinear forms. As $\mathbb{Z}_2 \times \Gamma$-graded algebras, there is a natural isomorphism*

$$C_\epsilon(V, (\ ,\ )_V) \otimes_{\tilde{\epsilon}} C_\epsilon(W, (\ ,\ )_W) \cong C_\epsilon(V \oplus W, (\ ,\ )_V \perp (\ ,\ )_W).$$

Proof. We have the canonical injective maps given by

$$i_V : V \to C_\epsilon(V \oplus W, (\ ,\ )_V \perp (\ ,\ )_W),$$
$$i_W : W \to C_\epsilon(V \oplus W, (\ ,\ )_V \perp (\ ,\ )_W).$$

By the universal property of the ϵ-Clifford algebra (see Proposition 3.4.9) there exists algebra homomorphisms

$$\tilde{i}_V : C_\epsilon(V, (\ ,\ )_V) \to C_\epsilon(V \oplus W, (\ ,\ )_V \perp (\ ,\ )_W)$$
$$\tilde{i}_W : C_\epsilon(W, (\ ,\ )_W) \to C_\epsilon(V \oplus W, (\ ,\ )_V \perp (\ ,\ )_W)$$

such that $\tilde{i}_V(v) = i_V(v)$ for all $v \in V$ and $\tilde{i}_W(w) = i_W(w)$ for all $w \in W$.

Moreover, V is orthogonal to W and so

$$i_V(v)i_W(w) = -\epsilon(v, w)i_W(w)i_V(v) \qquad \forall v \in V,\ \forall w \in W.$$

Since $\tilde{i}_V$ and $\tilde{i}_W$ are algebra homomorphisms we obtain

$$\tilde{i}_V(v)\tilde{i}_W(w) = \tilde{\epsilon}(v, w)\tilde{i}_W(w)\tilde{i}_V(v) \qquad \forall v \in C_\epsilon(V, (\ ,\ )_V),\ \forall w \in C_\epsilon(W, (\ ,\ )_W)$$

and so by the universal property of the $\tilde{\epsilon}$-tensor product (see Proposition 3.5.2) there exists an algebra homomorphism

$$I : C_\epsilon(V, (\ ,\ )_V) \otimes_{\tilde{\epsilon}} C_\epsilon(W, (\ ,\ )_W) \to C_\epsilon(V \oplus W, (\ ,\ )_V \perp (\ ,\ )_W)$$

such that $I(v \otimes 1) = \tilde{i}_V(v)$ for all $v \in C_\epsilon(V, (\ ,\ )_V)$ and $I(1 \otimes w) = \tilde{i}_W(w)$ for all $w \in C_\epsilon(W, (\ ,\ )_W)$.

We also have a canonical injective map

$$j : V \oplus W \to C_\epsilon(V, (\ ,\ )_V) \otimes_{\tilde{\epsilon}} C_\epsilon(W, (\ ,\ )_W)$$

and by the universal property of the ϵ-Clifford algebra (see Proposition 3.4.9) there exists an algebra homomorphism

$$J : C_\epsilon(V \oplus W, (\ ,\ )_V \perp (\ ,\ )_W) \to C_\epsilon(V, (\ ,\ )_V) \otimes_{\tilde{\epsilon}} C_\epsilon(W, (\ ,\ )_W)$$

such that $j = J \circ i_{V \oplus W}$, where

$$i_{V \oplus W} : V \oplus W \to C_\epsilon(V \oplus W, (\ ,\ )_V \perp (\ ,\ )_W)$$

is the canonical injective map. We have

$$I \circ J \circ i_{V \oplus W} = I \circ j = i_{V \oplus W}$$

and since $C_\epsilon(V \oplus W, (\ ,\ )_V \perp (\ ,\ )_W)$ is generated as algebra by $V \oplus W$ we obtain

$$I \circ J = Id.$$

We also have

$$J \circ I \circ j = J \circ i_{V \oplus W} = j$$

and since $C_\epsilon(V, (\ ,\ )_V) \otimes_{\tilde{\epsilon}} C_\epsilon(W, (\ ,\ )_W)$ is generated as algebra by $V \oplus W$ we obtain

$$J \circ I = Id$$

and then $C_\epsilon(V, (\ ,\ )_V) \otimes_{\tilde{\epsilon}} C_\epsilon(W, (\ ,\ )_W)$ and $C_\epsilon(V \oplus W, (\ ,\ )_V \perp (\ ,\ )_W)$ are isomorphic. □

In the same way, we can show the following.

Proposition 3.5.5. *Let V and W be Γ-graded vector spaces. As $\mathbb{Z} \times \Gamma$-graded algebras, there is a natural isomorphism*

$$\Lambda_\epsilon(V) \otimes_{\tilde{\epsilon}} \Lambda_\epsilon(W) \cong \Lambda_\epsilon(V \oplus W).$$

Let V be a finite-dimensional Γ-graded vector space. By Remarks 3.2.2 and 3.4.8, we have $\Gamma = \Gamma_0 \cup \Gamma_1$ and $V = V_0 \oplus V_1$. We define

$$Supp_V(\Gamma_0) := \{\gamma \in \Gamma_0 \mid V_\gamma \neq \{0\}\}, \quad Supp_V(\Gamma_1) := \{\gamma \in \Gamma_1 \mid V_\gamma \neq \{0\}\}$$

and hence since $dim(V) < \infty$ we can write

$$Supp_V(\Gamma_0) = \{\gamma_0, \ldots, \gamma_n\}, \quad Supp_V(\Gamma_1) = \{\tilde{\gamma}_0, \ldots, \tilde{\gamma}_m\}.$$

Proposition 3.5.6. *There is a natural isomorphism of $\mathbb{Z} \times \Gamma$-graded algebras*

$$\Lambda_\epsilon(V) \cong \Lambda(V_{\gamma_0}) \otimes_{\tilde{\epsilon}} (\ldots \otimes_{\tilde{\epsilon}} (\Lambda(V_{\gamma_n}) \otimes_{\tilde{\epsilon}} (S(V_{\tilde{\gamma}_0}) \otimes_{\tilde{\epsilon}} (\ldots \otimes_{\tilde{\epsilon}} (S(V_{\tilde{\gamma}_m})\ldots)))\ldots),$$

where on the right hand side, $\Lambda(V_{\gamma_i})$ (resp. $S(V_{\tilde{\gamma}_i})$) is the usual exterior algebra (resp. symmetric algebra).

Proof. Let $u, v \in V_{\gamma_i}$. Since $|u| = |v|$ and $\epsilon(u, u) = 1$, we have

$$u \wedge v = -v \wedge u.$$

Similarly, let $u, v \in V_{\tilde{\gamma}_i}$. Since $|u| = |v|$ and $\epsilon(u, u) = -1$, we have

$$u \wedge v = v \wedge u$$

and then the result is obtained applying Proposition 3.5.5 successively to

$$V_{\tilde{\gamma}_{m-1}} \oplus V_{\tilde{\gamma}_m},\ V_{\tilde{\gamma}_{m-2}} \oplus (V_{\tilde{\gamma}_{m-1}} \oplus V_{\tilde{\gamma}_m}),\ \ldots\ , V_{\gamma_0} \oplus (V_{\gamma_1} \oplus (\ldots(V_{\tilde{\gamma}_{m-1}} \oplus V_{\tilde{\gamma}_m})\ldots)).$$

□

3.6 ϵ-alternating multilinear maps

Let Γ be an abelian group and let ϵ be a commutation factor of Γ. In this section we define ϵ-alternating maps for Γ-graded vector spaces and then the product and composition of ϵ-alternating maps.

3.6.1 General definitions

Let V and W be Γ-graded vector spaces. Let $\tilde{\pi} : S_n \to GL(Hom(V^{\otimes n}, W))$ be the left group action given by

$$(\tilde{\pi}(\sigma)(f))(\bar{v}) := f(\pi(\sigma)(\bar{v})) \qquad \forall \sigma \in S_n,\ \forall f \in Hom(V^{\otimes n}, W),\ \forall \bar{v} \in V^{\otimes n}$$

where $\pi : S_n \to GL(V^{\otimes n})$ is the right group action given by Corollary 3.2.7.

Definition 3.6.1. *Let V and W be Γ-graded vector spaces. We define the vector space $Alt^i_\epsilon(V, W)$*

$$Alt^i_\epsilon(V, W) := \{f \in Hom(V^{\otimes n}, W) \mid \tilde{\pi}(\sigma)(f) = f \quad \forall \sigma \in S_n\}.$$

We also define the $\mathbb{Z} \times \Gamma$-graded vector space

$$Alt_\epsilon(V, W) := \bigoplus_{i \in \mathbb{N}} Alt^i_\epsilon(V, W)$$

and denote $Alt_\epsilon(V, k)$ by $Alt_\epsilon(V)$.

Remark 3.6.2. *Each $f \in Alt^i_\epsilon(V, W)$ is uniquely determined by the multilinear map $\tilde{f} : V^i \to W$ given by $\tilde{f}(v_1, \ldots, v_i) = f(v_1 \otimes \ldots \otimes v_i)$ where $v_1, \ldots, v_i \in V$. This map $\tilde{f}$ satisfies*

$$\tilde{f}(v_1, \ldots, v_l, v_{l+1}, \ldots, v_i) = -\epsilon(v_l, v_{l+1})\tilde{f}(v_1, \ldots, v_{l+1}, v_l, \ldots, v_i)$$

for all $v_1, \ldots, v_i \in V$ and $l \in [\![1, i-1]\!]$, and conversely given a multilinear map $g : V^i \to W$ satisfying this property there is a unique element $f \in Alt^i_\epsilon(V, W)$ such that $\tilde{f} = g$.

Proposition 3.6.3. *Let V be a finite-dimensional Γ-graded vector space. There is a natural $\mathbb{Z} \times \Gamma$-graded vector space isomorphism between $Alt_\epsilon(V)$ and $(\Lambda_\epsilon(V))^*$.*

Proof. For all $i \in \mathbb{N}$, we show that the vector space $Alt^i_\epsilon(V)$ is isomorphic to the vector space $(\Lambda^i_\epsilon(V))^*$. On the one hand we can define the linear map $\phi : (\Lambda^i_\epsilon(V))^* \to Alt^i_\epsilon(V)$ by

$$\phi(\alpha)(v_1 \otimes \ldots \otimes v_i) = \alpha(v_1 \wedge \ldots \wedge v_i) \quad \forall \alpha \in (\Lambda^i_\epsilon(V))^*,\ \forall v_1, \ldots, v_n \in V.$$

On the other, if $f \in Alt^i_\epsilon(V)$ then

$$f\left(v_1 \otimes \ldots \otimes v_j \otimes (v_{j+1} \otimes v_{j+2} + \epsilon(v_{j+1}, v_{j+2})v_{j+2} \otimes v_{j+1}) \otimes v_{j+3} \otimes \ldots \otimes v_i\right) = 0$$

for all $v_1, \ldots, v_i \in V$ and so f annihilates $I(V) \cap T^i(V)$ and factors to define $\bar{f} : \Lambda^i_\epsilon(V) \to k$. One checks that $\phi(\bar{f}) = f$ and $\overline{\phi(\alpha)} = \alpha$ which proves the proposition. ☐

3.6.2 Exterior product and composition of ϵ-alternating multilinear maps

Definition 3.6.4. *Let $I := [\![1,n]\!]$ and let $I_1,\ldots,I_m$ be disjoint subsets of I such that $\bigcup_{i\in[\![1,m]\!]} I_i = I$. We denote by $S(I_1,\ldots,I_m)$ the set of all permutations $\sigma \in S_n$ which satisfy*

$$\forall j \in [\![1,m]\!],\ a,b \in I_j \text{ and } a<b \quad \Rightarrow \quad \sigma(a)<\sigma(b).$$

Such a permutation is called a shuffle permutation.

Remark 3.6.5. *The cardinal of $S(I_1,\ldots,I_m)$ is given by*

$$|S(I_1,\ldots,I_m)| = \binom{n}{|I_1|}\binom{n-|I_1|}{|I_2|}\cdots\binom{n-|I_1|-\ldots-|I_{m-2}|}{|I_{m-1}|}.$$

Let T,U,V and W be Γ-graded vector spaces, let $f \in Alt^i_\epsilon(T,U)$, let $g \in Alt^j_\epsilon(T,V)$ and let $\phi : U\times V \to W$ be a bilinear map. Let $(fg)_\phi : T^{\otimes(i+j)} \to W$ be given by

$$(fg)_\phi(v_1\otimes\ldots\otimes v_{i+j}) := \phi(f(v_1\otimes\ldots\otimes v_i), g(v_{i+1}\otimes\ldots\otimes v_{i+j})) \quad \forall v_1,\ldots,v_{i+j}\in T.$$

The exterior product of f and g is now defined by "antisymmetrising" this product only with respect to shuffle permutations.

Definition 3.6.6. *With the notations above, the map $f\wedge_\phi g : T^{\otimes(i+j)} \to W$ is defined by*

$$f\wedge_\phi g := \sum_{\sigma\in S([\![1,i]\!],[\![i+1,i+j]\!])} \tilde{\pi}(\sigma)((fg)_\phi).$$

Remark 3.6.7. *The relation between this exterior product and "antisymmetrisation" over all permutations is :*

$$\sum_{\sigma\in S_{i+j}} \tilde{\pi}(\sigma)((fg)_\phi) = i!j!f\wedge_\phi g.$$

In characteristic zero the two possible definitions of an "exterior product" are equivalent.

Proposition 3.6.8. *The map $f\wedge_\phi g$ is in $Alt^{i+j}_\epsilon(T,W)$.*

Proof. This proof is based on the proof of the analogous result for classical exterior forms in [Car67]. Here, we use implicitly the formulae of Remark 3.2.9.

To prove the proposition it is sufficient to show that if $l\in[\![1,i+j-1]\!]$,

$$f\wedge_\phi g(v_1\otimes\ldots\otimes v_l\otimes v_{l+1}\otimes\ldots\otimes v_{i+j}) = -\epsilon(v_l,v_{l+1})f\wedge_\phi g(v_1\otimes\ldots\otimes v_{l+1}\otimes v_l\otimes\ldots\otimes v_{i+j})$$

for all $v_1, \ldots, v_{i+j}$ in T. Consider the right-hand side of this equation. We have

$$\begin{aligned}&-\epsilon(v_l, v_{l+1}) f \wedge_\phi g(v_1 \otimes \ldots \otimes v_{l+1} \otimes v_l \otimes \ldots \otimes v_{i+j})\\ &= -\epsilon(v_l, v_{l+1}) \sum_{\sigma \in S([\![1,i]\!],[\![i+1,i+j]\!])} (fg)_\phi \left(\pi(\sigma)(v_1 \otimes \ldots \otimes v_{l+1} \otimes v_l \otimes \ldots \otimes v_{i+j})\right).\end{aligned}$$

We divide the permutations $\sigma \in S([\![1,i]\!],[\![i+1,i+j]\!])$ in two categories :

1. Those σ for which $\sigma^{-1}(l)$ and $\sigma^{-1}(l+1)$ are both integers $\leq i$ or both $\geq i+1$. In the first case, v_l and v_{l+1} occur amongst the first i places in $f(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(i)})$ and hence, since f is ϵ-alternating, we have

$$-\epsilon(v_l, v_{l+1})(fg)_\phi \left(\pi(\sigma)(v_1 \otimes \ldots \otimes v_{l+1} \otimes v_l \otimes \ldots \otimes v_{i+j})\right) = (fg)_\phi \left(\pi(\sigma)(v_1 \otimes \ldots \otimes v_{i+j})\right).$$

In the second case, we have the same relation since g is ϵ-alternating.

2. The second category is itself divided into two sub-categories : those σ for which $\sigma^{-1}(l) \leq i$ and $\sigma^{-1}(l+1) \geq i+1$ and those σ for which $\sigma^{-1}(l) \geq i$ and $\sigma^{-1}(l+1) \leq i+1$. Let τ be the transposition which interchanges l and $l+1$. If σ is in the first sub-category, $\tau\sigma$ is in the second, and vice versa. We may therefore group in pairs the remaining terms as follows : for each σ such that $\sigma^{-1}(l) \leq i$ and $\sigma^{-1}(l+1) \geq i+1$ we have

$$\begin{aligned}&-\epsilon(v_l, v_{l+1}) \Big((fg)_\phi \left(\pi(\sigma)(v_1 \otimes \ldots \otimes v_{l+1} \otimes v_l \otimes \ldots \otimes v_{i+j})\right)\\ &+ (fg)_\phi \left(\pi(\tau\sigma)(v_1 \otimes \ldots \otimes v_{l+1} \otimes v_l \otimes \ldots \otimes v_{i+j})\right) \Big)\\ &= (fg)_\phi \left(\pi(\sigma)(v_1 \otimes \ldots \otimes v_{i+j})\right) + (fg)_\phi \left(\pi(\tau\sigma)(v_1 \otimes \ldots \otimes v_{i+j})\right).\end{aligned}$$

□

The most important example of the above construction is when $U = V = W = k$ and $\phi : k \times k \to k$ is the product. In this case we denote $\wedge_\phi$ by $\wedge_\epsilon$ and this defines a product on $Alt_\epsilon(V)$.

Proposition 3.6.9. *We respect to $\wedge_\epsilon$, the algebra $Alt_\epsilon(V)$ is $\mathbb{Z} \times \Gamma$-graded, $\tilde{\epsilon}$-commutative and associative.*

Later on, we will need the following definition.

Definition 3.6.10. *Let T, U, W be Γ-graded vector spaces and $\phi : U \times U \to W$ a bilinear map. The norm $N_\phi(f)$ of $f \in Alt_\epsilon(T, U)$ is defined by*

$$N_\phi(f) := f \wedge_\phi f.$$

This is an element of $Alt^{2i}_\epsilon(T, W)$ if $f \in Alt^i_\epsilon(T, W)$.

Finally to complete this subsection, we define a composition of ϵ-alternating multilinear maps. Let U, V and W be Γ-graded vector spaces, let $f \in Alt^i_\epsilon(U, V)$ and let $g \in Alt^j_\epsilon(W, U)$. Let $f * g : W^{\otimes(ij)} \to V$ be given by

$$f * g(v_1, \ldots, v_{ij}) := f(g(v_1 \otimes \ldots \otimes v_j) \otimes g(v_{j+1} \otimes \ldots \otimes v_{2j}) \otimes \ldots \otimes g(v_{(i-1)j+1} \otimes \ldots \otimes v_{ij}))$$

for all $v_1 \otimes \ldots \otimes v_{ij} \in W^{\otimes(ij)}$.

The exterior composition of f and g is now defined by "antisymmetrising" this composition only with respect to certain shuffle permutations.

Definition 3.6.11. *The map $f \circ g : W^{\otimes(ij)} \to V$ is defined by*

$$f \circ g := \sum_{\sigma \in S(\llbracket 1,j \rrbracket, \ldots, \llbracket (i-1)j+1, ij \rrbracket)} \tilde{\pi}(\sigma)(f * g).$$

Remark 3.6.12. *The relation between this and antisymmetrising over all permutations is*

$$\sum_{\sigma \in S_{ij}} \tilde{\pi}(\sigma)(f * g) = (j!)^i f \circ g.$$

In characteristic zero the possible definitions of an "exterior composition" are equivalent.

Proposition 3.6.13. *The map $f \circ g$ is in to $Alt^{ij}_\epsilon(W, V)$.*

Proof. This proof is similar to the proof of Proposition 3.6.8. □

3.7 Interior derivation

Let Γ be an abelian group and let ϵ be a commutation factor of Γ. In this section we define an analogue of classical interior derivation for ϵ-exterior algebras.

Remark 3.7.1. *Recall that if V is a Γ-graded vector space, its dual has a Γ-gradation given by :*

$$V^*_\gamma := \{f \in V^* \mid f(x) = 0 \text{ if } x \in V_\lambda \text{ with } \lambda + \gamma \neq 0\}.$$

Proposition 3.7.2. *Let V be a Γ-graded vector space and let $v \in V$. There exists a unique linear map $i_v : \Lambda_\epsilon(V^*) \to \Lambda_\epsilon(V^*)$ of degree $|v|$ such that :*

- $i_v(\Lambda^k_\epsilon(V^*)) \subseteq \Lambda^{k-1}_\epsilon(V^*)$;
- $i_v(\Lambda^0_\epsilon(V^*)) = 0$;
- $i_v(a \wedge b) = i_v(a) \wedge b + (-1)^{\alpha(v)\alpha(a)} \epsilon(v, a) a \wedge i_v(b)$ *for all* $a, b \in \Lambda_\epsilon(V^*)$;

- $i_v(a) = a(v)$ *for* $a \in V^*$.

Proof. If i_v exists, it is unique by the second, third and fourth properties since V^* generate $\Lambda_\epsilon(V^*)$. We now show the existence of i_v. Let $\tilde{i}_v : T(V^*) \to T(V^*)$ defined by

$$\tilde{i}_v(x_1 \otimes \ldots \otimes x_n) = \sum_{i=1}^{n} (-1)^{i-1} \epsilon(v, 0 + x_1 + \ldots + x_{i-1}) x_i(v) x_1 \otimes \ldots \otimes \widehat{x}_i \otimes \ldots \otimes x_n \quad (3.5)$$

for all $x_1, \ldots, x_n \in V^*$. Let $x_1, x_2 \in V^*$. We have

$$\begin{aligned}\tilde{i}_v(x_1 \otimes x_2 + \epsilon(x_1, x_2) x_2 \otimes x_1) &= x_1(v)x_2 - \epsilon(v, x_1)x_2(v)x_1 + \epsilon(x_1, x_2)x_2(v)x_1 - \epsilon(x_1, x_2)\epsilon(v, x_2)x_1(v)x_2 \\ &= x_1(v)x_2(1 - \epsilon(x_1, x_2)\epsilon(v, x_2)) + x_2(v)x_1(-\epsilon(v, x_1) + \epsilon(x_1, x_2)).\end{aligned}$$

We remark that if $|x_1| \neq -|v|$ then $x_1(v) = 0$, and that if $|x_1| = -|v|$ then

$$1 - \epsilon(x_1, x_2)\epsilon(v, x_2) = 0.$$

Similarly if $|x_2| \neq -|v|$ then $x_2(v) = 0$ and if $|x_2| = -|v|$ then

$$-\epsilon(v, x_1) + \epsilon(x_1, x_2) = 0.$$

Hence, we obtain

$$\tilde{i}_v(x_1 \otimes x_2 + \epsilon(x_1, x_2)x_2 \otimes x_1) = 0 \qquad \forall x_1, x_2 \in V^*. \quad (3.6)$$

Recall that we have $\Lambda_\epsilon(V^*) = T(V^*)/I(V^*)$ where $I(V^*)$ is the two-sided ideal of $T(V^*)$ generated by all elements of the form

$$x \otimes y + \epsilon(x, y) y \otimes x \qquad \forall x, y \in V^*.$$

Let $x \otimes y \otimes z$ be an arbitrary element of $I(V^*)$ such that $x, z \in T(V^*)$ and $y = x_1 \otimes x_2 + \epsilon(x_1, x_2)x_2 \otimes x_1$ for some $x_1, x_2 \in V^*$. It follows from (3.5) that

$$\tilde{i}_v(x \otimes y \otimes z) = \tilde{i}_v(x) \otimes y \otimes z + (-1)^{\alpha(x)} \epsilon(v, x) x \otimes \tilde{i}_v(y) \otimes z + (-1)^{\alpha(x)+\alpha(y)} \epsilon(v, x + y) x \otimes y \otimes \tilde{i}_v(z).$$

By Equation (3.6), we have $\tilde{i}_v(y) = 0$ and hence

$$\tilde{i}_v(x \otimes y \otimes z) = \tilde{i}_v(x) \otimes y \otimes z + (-1)^{\alpha(x)+\alpha(y)} \epsilon(v, x + y) x \otimes y \otimes \tilde{i}_v(z).$$

This equation implies $\tilde{i}_v(x \otimes y \otimes z) \in I(V^*)$ and so $\tilde{i}_v$ factors to define a map $i_v : \Lambda_\epsilon(V^*) \to \Lambda_\epsilon(V^*)$. One checks that i_v satisfies the four given properties and this proves the proposition. □

Proposition 3.7.3. *Let V be a finite-dimensional Γ-graded vector space. The linear map $\eta : \Lambda^l_\epsilon(V^*) \to (\Lambda^l_\epsilon(V))^*$ defined by*

$$\eta(f)(v_1 \wedge \ldots \wedge v_l) := i_{v_1} \circ \ldots \circ i_{v_l}(f) \qquad \forall v_1, \ldots, v_l \in V$$

is an isomorphism if $char(k) > l$ or if $char(k) = 0$.

Proof. By Remark 3.4.8 we have a $\mathbb{Z}_2$-gradation $V^* = V^*_0 \oplus V^*_1$. Let $\{e_1, \ldots, e_n\}$ be a basis of V^*_0 and let $\{f_1, \ldots, f_m\}$ be a basis of V^*_1. Hence

$$\mathcal{B} := \{e_1^{\wedge k_1} \wedge \ldots \wedge e_n^{\wedge k_n} \wedge f_1^{\wedge l_1} \wedge \ldots \wedge f_m^{\wedge l_m} \mid k_i \in \{0,1\},\ l_i \in \mathbb{N},\ \sum k_i + \sum l_i = l\}$$

is a basis of $\Lambda^l_\epsilon(V^*)$. Let $\{e^*_1, \ldots, e^*_n\}$ be the basis of V_0 dual to the basis $\{e_1, \ldots, e_n\}$ of V^*_0 and let $\{f^*_1, \ldots, f^*_m\}$ be a basis of V_1 dual to the basis $\{f_1, \ldots, f_m\}$ of V^*_1.

Let $e_{i_1} \wedge \ldots \wedge e_{i_{n'}} \wedge f_{j_1}^{\wedge l_{j_1}} \wedge \ldots \wedge f_{j_{m'}}^{\wedge l_{j_{m'}}}$ be an element of $\mathcal{B}$. Since

$$i_{e^*_i}(e_j) = \delta_{ij}, \quad i_{e^*_i}(f_j) = 0, \quad i_{f^*_i}(e_j) = 0, \quad i_{f_i^{*\wedge l_i}}(f_j^{\wedge l_j}) = l_i! \delta_{ij},$$

we have

$$i_{f_{j_{m'}}^{*\wedge l_{j_{m'}}} \wedge \ldots \wedge f_{j_1}^{*\wedge l_{j_1}} \wedge e^*_{i_{n'}} \wedge \ldots \wedge e^*_{i_1}} \left(e_{i_1} \wedge \ldots \wedge e_{i_{n'}} \wedge f_{j_1}^{\wedge l_{j_1}} \wedge \ldots \wedge f_{j_{m'}}^{\wedge l_{j_{m'}}} \right) \neq 0.$$

Hence the map $\eta : \Lambda^l_\epsilon(V^*) \to (\Lambda^l_\epsilon(V))^*$ given by

$$\eta(f)(v_1 \wedge \ldots \wedge v_l) := i_{v_1} \circ \ldots \circ i_{v_l}(f) \qquad \forall v_1, \ldots, v_l \in V$$

is injective and then an isomorphism since $dim(\Lambda^l_\epsilon(V^*)) = dim((\Lambda^l_\epsilon(V))^*)$. ☐

The Γ-graded vector space $V \oplus V^*$ has a canonical non-degenerate ϵ-symmetric bilinear form $(\ ,\ )$ given by

$$\begin{cases} (v, w) := 0 & \forall v, w \in V, \\ (\alpha, \beta) := 0 & \forall \alpha, \beta \in V^*, \\ (\alpha, v) = \epsilon(\alpha, v)(v, \alpha) := \alpha(v) & \forall \alpha \in V^*,\ \forall v \in V. \end{cases}$$

The map $\rho : V \oplus V^* \to End(\Lambda_\epsilon(V^*))$ defined by

$$\rho(v) := \begin{cases} i_v & \text{if } v \in V, \\ e_v & \text{if } v \in V^*, \end{cases}$$

where $e_v : \Lambda_\epsilon(V^*) \to \Lambda_\epsilon(V^*)$ is defined by $e_v(w) := v \wedge w$ for $v, w \in V^*$, satisfies

$$\rho(x)\rho(y) + \epsilon(x, y)\rho(y)\rho(x) = (x, y) Id_{\Lambda_\epsilon(V^*)} \qquad \forall x, y \in V \oplus V^*$$

as we now show.

Proposition 3.7.4. *Let V be a Γ-graded vector space. With the notations above we have the following properties.*

a) For $v, w \in V$, we have
$$i_v \circ i_w = -\epsilon(v,w) i_w \circ i_v.$$

b) For $v, w \in V^*$, we have
$$e_v \circ e_w = -\epsilon(v,w) e_w \circ e_v.$$

c) For $v \in V$ and $a \in V^*$, we have
$$i_v \circ e_a + \epsilon(v,a) e_a \circ i_v = a(v).$$

Proof. a) We show
$$i_v \circ i_w(a) = -\epsilon(v,w) i_w \circ i_v(a) \qquad \forall a \in \Lambda^i_\epsilon(V^*)$$
by induction on i.

Base case : We have $i_v(i_w(a)) = 0$ and $i_w(i_v(a))$ for all $a \in \Lambda^0_\epsilon(V^*) \oplus \Lambda^1_\epsilon(V^*)$.

Induction : Suppose that
$$i_v \circ i_w(a) = -\epsilon(v,w) i_w \circ i_v(a) \qquad \forall a \in \Lambda^i_\epsilon(V^*).$$
For $a \wedge b \in \Lambda^{i+1}_\epsilon(V^*)$ such that $a \in V^*$ we have
$$i_v(i_w(a \wedge b)) = i_v\Big(i_w(a) \wedge b + (-1)^{\alpha(a)} \epsilon(w,a) a \wedge i_w(b)\Big)$$
$$= (-1)^{\alpha(a)-1} \epsilon(v, w+a) i_w(a) \wedge i_v(b) + (-1)^{\alpha(a)} \epsilon(w,a) i_v(a) \wedge i_w(b) + \epsilon(w+v,a) a \wedge i_v i_w(b)$$
$$= -\epsilon(v,w)\Big((-1)^{\alpha(a)} \epsilon(v,a) i_w(a) \wedge i_v(b) + (-1)^{\alpha(a)-1} \epsilon(w, v+a) i_v(a) \wedge i_w(b)\Big) + \epsilon(w+v,a) a \wedge i_v i_w(b)$$
and since by assumption we have $i_v i_w(b) = -\epsilon(v,w) i_w i_v(b)$ then we obtain that
$$i_v(i_w(a \wedge b)) = -\epsilon(v,w) i_w(i_v(a \wedge b)).$$

b) Since $v \wedge w = -\epsilon(v,w) w \wedge v$ we have $e_v \circ e_w = -\epsilon(v,w) e_w \circ e_v$.

c) Let $b \in \Lambda_\epsilon(V^*)$. We have
$$\Big(i_v \circ e_a + \epsilon(v,a) e_a \circ i_v\Big)(b) = i_v(a \wedge b) + \epsilon(v,a) a \wedge i_v(b)$$
$$= i_v(a) \wedge b - \epsilon(v,a) a \wedge i_v(b) + \epsilon(v,a) a \wedge i_v(b) = a(v) b.$$
□

Corollary 3.7.5. *With the notations above, the map $\rho : V \oplus V^* \to End(\Lambda_\epsilon(V))$ extends to a Γ-graded injective algebra homomorphism $\tilde{\rho} : C_\epsilon(V \oplus V^*, \frac{1}{2}(\ ,\ )) \to End(\Lambda_\epsilon(V))$.*

3.8 Colour Lie algebras

Let Γ be an abelian group and let ϵ be a commutation factor of Γ. In this section we define colour Lie algebras which are generalisations of Lie algebras and Lie superalgebras originally introduced by V. Rittenberg and D. Wyler (see [RW78a] and [RW78b]) and then studied by M. Scheunert (see [Sch79a] and [Sch83]). We give examples of colour Lie algebras, the "classical" colour Lie algebras.

3.8.1 Definitions

Definition 3.8.1. *A colour Lie algebra is a Γ-graded vector space $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ together with a bilinear map $\{\ ,\ \} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that*

a) $\{\mathfrak{g}_\alpha, \mathfrak{g}_\beta\} \subseteq \mathfrak{g}_{\alpha+\beta}$ *for* $\alpha, \beta \in \Gamma$,

b) $\{x, y\} = -\epsilon(x, y)\{y, x\}$ *for* $x, y \in \mathfrak{g}$ *(ϵ-antisymmetry),*

c) $\epsilon(z, x)\{x, \{y, z\}\} + \epsilon(x, y)\{y, \{z, x\}\} + \epsilon(y, z)\{z, \{x, y\}\} = 0$ *for* $x, y, z \in \mathfrak{g}$ *(ϵ-Jacobi identity).*

Here are some examples of colour Lie algebras.

Example 3.8.2. *a) A Lie algebra $\mathfrak{g}$ is a colour Lie algebra where the Γ-gradation is given by $\mathfrak{g}_0 = \mathfrak{g}$.*

b) A Lie superalgebra $\mathfrak{g}$ is a colour Lie algebra for the group $\Gamma = \mathbb{Z}_2$ and the commutation factor

$$\epsilon(a, b) = (-1)^{ab} \qquad \forall a, b \in \mathbb{Z}_2.$$

c) An associative Γ-graded algebra $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ together with the bracket $\{a, b\} := ab - \epsilon(a, b)ba$ for all a, b in A is a colour Lie algebra.

d) Let V be a Γ-graded vector space. The associative Γ-graded algebra $End(V) = \bigoplus_{\gamma\in\Gamma} End(V)_\gamma$ is a colour Lie algebra for the bracket $\{a, b\} := ab - \epsilon(a, b)ba$ for all a, b in $End(V)$ and we denote this colour Lie algebra $\mathfrak{gl}_\epsilon(V)$.

e) Let $\mathfrak{g}$ be a colour Lie algebra. A linear map $f : \mathfrak{g} \to \mathfrak{g}$ is an ϵ-derivation if it satisfies the Leibniz rule

$$f(\{x, y\}) = \{f(x), y\} + \epsilon(f, x)\{x, f(y)\} \qquad \forall x, y \in \mathfrak{g}.$$

The set $Der(\mathfrak{g})$ of all ϵ-derivations of $\mathfrak{g}$ is a colour Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

Remark 3.8.3. *Let $\mathfrak{g}$ be a colour Lie algebra and let $x \in \mathfrak{g}$. The linear map $ad(x) \in \mathfrak{gl}(\mathfrak{g})$ given by $ad(x)(y) := \{x, y\}$ for $y \in \mathfrak{g}$ is a ϵ-derivation of degree 0.*

Proposition 3.8.4. *Let V be a Γ-graded vector space and let $(\ , \ )$ be an ϵ-symmetric bilinear form on V. Then the vector space $\Lambda^2_\epsilon(V)$ together with the bilinear bracket $\{ \ , \ \}$ defined, for all $v_1, v_2, v_3, v_4 \in V$, by*

$$\begin{aligned}\{v_1 \wedge v_2, v_3 \wedge v_4\} : &= (v_2, v_3)v_1 \wedge v_4 + \epsilon(v_1 + v_2, v_3)(v_2, v_4)v_3 \wedge v_1 \\ &+ \epsilon(v_2, v_3 + v_4)(v_1, v_3)v_4 \wedge v_2 + \epsilon(v_1, v_2 + v_3)(v_1, v_4)v_2 \wedge v_3 \end{aligned} \tag{3.7}$$

is a colour Lie algebra.

Proof. A long but straightforward calculation. □

We now define morphisms of colour Lie algebras and representations of colour Lie algebras.

Definition 3.8.5. *a) Let $\mathfrak{g}$ and $\mathfrak{g}'$ be colour Lie algebras. A linear map $f \in Hom(\mathfrak{g}, \mathfrak{g}')$ is a morphism of colour Lie algebras if $f(\{x, y\}) = \{f(x), f(y)\}$ for all $x, y \in \mathfrak{g}$. Furthermore, we say that $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic as colour Lie algebras if f is a linear isomorphism.*

b) Let $\mathfrak{g}$ be a colour Lie algebra. A representation V of $\mathfrak{g}$ is a Γ-graded vector space V together with a morphism of colour Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}_\epsilon(V)$.

Remark 3.8.6. *a) Unless otherwise stated it is always assumed that morphisms are of degree 0.*

b) For a representation $\rho : \mathfrak{g} \to \mathfrak{gl}_\epsilon(V)$ of a colour Lie algebra $\mathfrak{g}$, we sometimes write $x(v)$ instead of $\rho(x)(v)$ for $x \in \mathfrak{g}$ and $v \in V$.

Remark 3.8.7. *If Γ is a finitely generated abelian group and ϵ is a commutation factor of Γ, Scheunert shows in [Sch79a] that there is a natural bijection between (Γ, ϵ)-colour Lie algebras and Γ-graded Lie superalgebras. He also shows that there is a natural bijection between representations of a given (Γ, ϵ)-colour Lie algebra and Γ-graded representations of corresponding Γ-graded Lie superalgebra.*

3.8.2 Classical colour Lie algebras

In this subsection we define analogues of the classical Lie (super)algebras for colour Lie algebras. We begin with the analogue of special linear Lie (super)algebras.

Definition 3.8.8. *(see [Sch83]) Let V be a finite-dimensional Γ-graded vector space and let $f \in End(V)$. Recall that $\mathcal{E} : V \to V$ is defined by $\mathcal{E}(v) = \epsilon(v, v)v$ for all $v \in V$. The ϵ-trace of f is defined by :*

$$Tr_\epsilon(f) := Tr(\mathcal{E} \circ f).$$

Example 3.8.9. *Let V be a finite-dimensional Γ-graded vector space. Recall that $\mathcal{E} : V \to V$ is defined by $\mathcal{E}(v) = \epsilon(v, v)v$ for all $v \in V$. Define*

$$\mathfrak{sl}_\epsilon(V) := \{f \in \mathfrak{gl}_\epsilon(V) \mid Tr_\epsilon(f) = 0\}.$$

One can show that $\mathfrak{sl}_\epsilon(V)$ is stable under the bracket of $\mathfrak{gl}_\epsilon(V)$ and hence $\mathfrak{sl}_\epsilon(V)$ is a colour Lie subalgebra of $\mathfrak{gl}_\epsilon(V)$.

We now define the analogue of orthogonal and symplectic Lie algebras as well as orthosymplectic Lie superalgebras.

Example 3.8.10. *Let V be a finite-dimensional Γ-graded vector space and let $(\ ,\ ) : V \times V \to k$ be an ϵ-symmetric or ϵ-antisymmetric bilinear form. Define the vector space*

$$\mathfrak{so}_\epsilon(V, (\ ,\ )) := \bigoplus_{\gamma\in\Gamma} \mathfrak{so}_\epsilon(V, (\ ,\ ))_\gamma$$

where

$$\mathfrak{so}_\epsilon(V, (\ ,\ ))_\gamma := \{f \in End(V)_\gamma \mid (f(v), w) + \epsilon(f, v)(v, f(w)) = 0 \quad \forall v, w \in V\}.$$

One can show that $\mathfrak{so}_\epsilon(V, (\ ,\ ))$ is stable under the bracket of $\mathfrak{gl}_\epsilon(V)$ and hence $\mathfrak{so}_\epsilon(V, (\ ,\ ))$ is a colour Lie algebra. Furthermore, if $(\ ,\ )$ is non-degenerate and $dim(V) \geq 2$, $\mathfrak{so}_\epsilon(V, (\ ,\ ))$ is simple (see [Moo99]).

A given ϵ-symmetric bilinear form may be ϵ-antisymmetric with respect to a different Γ-gradation of the underlying vector space.

Remark 3.8.11. *Let $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ be a Γ-graded vector space and let $(\ ,\ ) : V \times V \to k$ be a ϵ-antisymmetric bilinear form. Suppose there exists $\delta \in \Gamma$ such that $\epsilon(\delta, \delta) = -1$ and $2\delta = 0$. Consider the vector space V with the Γ-gradation $V = \bigoplus_{\gamma\in\Gamma} V'_\gamma$ where $V'_\gamma := V_{\gamma+\delta}$ for all $\gamma \in \Gamma$. Then $(\ ,\ )$ is an ϵ-symmetric bilinear form of degree 0 with respect to this gradation.*

Since $\mathfrak{so}_\epsilon(V, (\ ,\ ))$ is the same for both gradations, the identity is an isomorphism of colour Lie algebras of degree δ between $\mathfrak{so}_\epsilon(\bigoplus_{\gamma\in\Gamma} V_\gamma, (\ ,\ ))$ and $\mathfrak{so}_\epsilon(\bigoplus_{\gamma\in\Gamma} V'_\gamma, (\ ,\ )')$.

Definition 3.8.12. *Let $\mathfrak{g}$ be a colour Lie algebra. An ϵ-orthogonal representation V of $\mathfrak{g}$ is a Γ-graded vector space V with a non-degenerate ϵ-symmetric bilinear form $(\ ,\ )$ on V together with a morphism of colour Lie algebras $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$.*

We now define the analogue of unitary and special unitary Lie algebras.

Example 3.8.13. *Let $\tilde{k}|k$ be a quadratic extension and let $^- : \tilde{k} \to \tilde{k}$ be the associated involution. Let V be a Γ-graded $\tilde{k}$-vector space and let $H : V \times V \to \tilde{k}$ be an ϵ-(anti)hermitian form. For $\gamma \in \Gamma$ we define*

$$\mathfrak{u}_\epsilon(V, H)_\gamma := \{f \in End(V)_\gamma \mid H(f(v), w) + \epsilon(f, v)H(v, f(w)) = 0 \quad \forall v, w \in V\}.$$

This is a k-vector space and we then define the k-vector space $\mathfrak{u}_\epsilon(V, H) := \bigoplus_{\gamma \in \Gamma} \mathfrak{u}_\epsilon(V, H)_\gamma$. One can show that $\mathfrak{u}_\epsilon(V, H)$ is stable under the bracket of $\mathfrak{gl}_\epsilon(V)$ and hence $\mathfrak{u}_\epsilon(V, H)$ is a colour k-Lie algebra.

We also define the colour k-Lie algebra $\mathfrak{su}_\epsilon(V, H) := \mathfrak{u}_\epsilon(V, H) \cap \mathfrak{sl}_\epsilon(V)$.

Proposition 3.8.14. *As $\tilde{k}$-colour Lie algebras, $\mathfrak{u}_\epsilon(V, H) \otimes \tilde{k}$ and $\mathfrak{gl}_\epsilon(V)$ are isomorphic, and $\mathfrak{su}_\epsilon(V, H) \otimes \tilde{k}$ and $\mathfrak{sl}_\epsilon(V)$ are isomorphic.*

Proof. Let $\Lambda \in k$ be such that $\tilde{k} = k(\sqrt{\Lambda})$ and let $f \in \mathfrak{gl}_\epsilon(V)$. We define $f^* \in \mathfrak{gl}_\epsilon(V)$ by

$$H(f^*(v), w) = \epsilon(f, v)H(v, f(w)) \qquad \forall v, w \in V.$$

We can check that $f - f^* \in \mathfrak{u}_\epsilon(V, H)$, $f + f^* \in \sqrt{\Lambda}\mathfrak{u}_\epsilon(V, H)$ and then

$$\mathfrak{gl}_\epsilon(V) = \mathfrak{u}_\epsilon(V, H) \oplus \sqrt{\Lambda}\mathfrak{u}_\epsilon(V, H)$$

since

$$f = \frac{f - f^*}{2} + \frac{f + f^*}{2}.$$

□

3.8.3 Colour Jacobi-Jordan algebra

Let Γ be an abelian group and let ϵ be a commutation factor of Γ. One might ask what kind of mathematical structure emerges if in the definition of a colour Lie algebra one requires the bracket to be ϵ-symmetric instead of ϵ-antisymmetric (without changing the ϵ-Jacobi identity). This question is considered for Lie algebras in [BF14] where it is shown that an algebra with a symmetric bilinear bracket which satisfies the usual Jacobi identity, that is to say, a Jacobi-Jordan algebra, is in particular a Jordan algebra. The Jacobi-Jordan algebras introduced in [BF14] are also called mock-Lie algebras or Lie-Jordan algebras and an example of an infinite-dimensional Jacobi-Jordan algebra already appeared in [Zev66]. Here, we show analogously that a colour Jacobi-Jordan algebra is in particular a colour Jordan algebra, in the sense of [BG01].

Definition 3.8.15. *A colour Jacobi-Jordan algebra is a* Γ*-graded vector space* $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ *together with a bilinear map* $\{\ ,\ \} : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ *such that*

a) $\{\mathfrak{g}_\alpha, \mathfrak{g}_\beta\} \subseteq \mathfrak{g}_{\alpha+\beta}$ *for* $\alpha, \beta \in \Gamma$*,*

b) $\{x, y\} = \epsilon(x, y)\{y, x\}$ *for* $x, y \in \mathfrak{g}$ *(*ϵ*-symmetry),*

c) $\epsilon(z, x)\{x, \{y, z\}\} + \epsilon(x, y)\{y, \{z, x\}\} + \epsilon(y, z)\{z, \{x, y\}\} = 0$ *for* $x, y, z \in \mathfrak{g}$ *(*ϵ*-Jacobi identity).*

An algebra A is called a nilalgebra of nilindex n if $x^n = 0$ for all $x \in A$. In particular, taking $x = y = z$ in the Jacobi identity we obtain $\{x, \{x, x\}\} = 0$ and hence a colour Jacobi-Jordan algebra is a nilalgebra of nilindex 3.

Here is the definition of a colour Jordan algebra (see [BG01]).

Definition 3.8.16. *A colour Jordan algebra is a* Γ*-graded vector space* $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ *together with a bilinear map* $\{\ ,\ \} : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ *such that*

a) $\{\mathfrak{g}_\alpha, \mathfrak{g}_\beta\} \subseteq \mathfrak{g}_{\alpha+\beta}$ *for* $\alpha, \beta \in \Gamma$*,*

b) $\{x, y\} = \epsilon(x, y)\{y, x\}$ *for* $x, y \in \mathfrak{g}$ *(*ϵ*-symmetry),*

c) for $x, y, z, r \in \mathfrak{g}$ *we have*

$$\begin{aligned}&\epsilon(z, x + r)\{\{x, y\}, \{r, z\}\} + \epsilon(y, z + r)\{\{z, x\}, \{r, y\}\} + \epsilon(x, y + r)\{\{y, z\}, \{r, x\}\}\\ &= \epsilon(z, x + r)\{\{\{x, y\}, r\}, z\} + \epsilon(y, z + r)\{\{\{z, x\}, r\}, y\} + \epsilon(x, y + r)\{\{\{y, z\}, r\}, x\}\end{aligned}$$

(ϵ*-Jordan identity).*

Example 3.8.17. *a) An associative* Γ*-graded algebra* $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ *together with the bracket* $\{a, b\} := ab + \epsilon(a, b)ba$ *for all* a, b *in* A*, is a colour Jordan algebra.*

b) Let $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ *be a* Γ*-graded vector space. The associative algebra* $End(V)$ *is* Γ*-graded and hence is a colour Jordan algebra for the bracket* $\{a, b\} := ab + \epsilon(a, b)ba$ *for all* a, b *in* A*. Suppose that* $(\ ,\ ) : V \times V \to k$ *is an* ϵ*-symmetric bilinear form and let*

$$\mathfrak{so}'_\epsilon(V, (\ ,\ )) := \bigoplus_{\gamma\in\Gamma} \mathfrak{so}'_\epsilon(V, (\ ,\ ))_\gamma$$

where

$$\mathfrak{so}'_\epsilon(V, (\ ,\ ))_\gamma := \{f \in End(V)_\gamma \mid (f(v), w) = \epsilon(f, v)(v, f(w))\ \forall v, w \in V\}.$$

We can show that $\mathfrak{so}'_\epsilon(V, (\ ,\ ))$ *is stable under the bracket of* $End(V)$ *and hence* $\mathfrak{so}'_\epsilon(V, (\ ,\ ))$ *is a colour Jordan algebra.*

Proposition 3.8.18. *A colour Jacobi-Jordan algebra is a colour Jordan algebra.*

Proof. Let $\mathfrak{g}$ be a colour Jacobi-Jordan algebra. We want to show that for all $x, y, z, r \in \mathfrak{g}$

$$\begin{aligned}&\epsilon(z, x+r)\{\{x,y\},\{r,z\}\} + \epsilon(y, z+r)\{\{z,x\},\{r,y\}\} + \epsilon(x, y+r)\{\{y,z\},\{r,x\}\} \\ &\quad = \epsilon(z, x+r)\{\{\{x,y\},r\},z\} + \epsilon(y, z+r)\{\{\{z,x\},r\},y\} + \epsilon(x, y+r)\{\{\{y,z\},r\},x\}.\end{aligned} \tag{3.8}$$

Using the Jacobi identity we can show that

$$\begin{aligned}\epsilon(z, x+r)\{\{x,y\},\{r,z\}\} &= \epsilon(x, y+r)\{\{\{y,z\},r\},x\} + \epsilon(y, z+r)\{\{\{z,x\},r\},y\} \\ &\quad + \epsilon(x+z, r)\epsilon(x,y)\{\{\{y,r\},z\},x\} + \epsilon(y, z+r)\{\{z,\{x,r\}\},y\}\ ; \\ \epsilon(y, z+r)\{\{z,x\},\{r,y\}\} &= \epsilon(z, x+r)\{\{\{x,y\},r\},z\} + \epsilon(z, x+r)\epsilon(x,y)\{\{y,\{x,r\}\},z\} \\ &\quad - \epsilon(x, y+r)\epsilon(z,r)\{\{\{y,r\},z\},x\}\ ; \\ \epsilon(x, y+r)\{\{y,z\},\{r,x\}\} &= -\epsilon(y, z+r)\{\{z,\{x,r\}\},y\} - \epsilon(z, x+r)\epsilon(x,y)\{\{y,\{x,r\}\},z\}.\end{aligned}$$

Thus, summing these three terms, we obtain Equation (3.8). □

Remark 3.8.19. *In [BF14], the authors give a classification of Jacobi-Jordan algebras of dimension strictly inferior to 7 over an algebraically closed field of characteristic not 2 or 3.*

3.8.4 ϵ-symmetric bilinear forms on colour Lie algebras

Let Γ be an abelian group and let ϵ be a commutation factor of Γ. In this subsection we define the notion of invariant ϵ-symmetric bilinear form and Killing form for colour Lie algebras. We show that the "ϵ-trace" in the fundamental representation defines an invariant, ϵ-symmetric and non-degenerate bilinear form on $\mathfrak{gl}_\epsilon$ and $\mathfrak{sl}_\epsilon$.

Definition 3.8.20. *a) Let $\mathfrak{g}$ be a colour Lie algebra. A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to k$ is ad-invariant if for all $x, y, z \in \mathfrak{g}$,*

$$B(\{x,y\},z) = -\epsilon(x,y)B(y,\{x,z\}).$$

b) An ϵ-quadratic colour Lie algebra is a colour Lie algebra together with a bilinear form which is ϵ-symmetric, ad-invariant and non-degenerate.

Definition 3.8.21. *Let $\mathfrak{g}$ be a finite-dimensional colour Lie algebra. We define the Killing form of $\mathfrak{g}$ to be the bilinear form $K : \mathfrak{g} \times \mathfrak{g} \to k$ given by*

$$K(x,y) := Tr_\epsilon(ad(x)ad(y)) \qquad \forall x, y \in \mathfrak{g}.$$

Proposition 3.8.22. *The Killing form of a finite-dimensional colour Lie algebra is an ad-invariant, ϵ-symmetric bilinear form.*

Proof. Let $\mathfrak{g}$ be a colour Lie algebra and let $x, y \in \mathfrak{g}$. In the Γ-graded vector $\mathfrak{gl}_\epsilon(V)$ since $|ad(x)| = |x|$ and $|ad(y)| = |y|$ then we have $Tr_\epsilon(ad(x)ad(y)) = 0$ if $|x| + |y| \neq 0$. Hence we suppose $|x| + |y| = 0$. Since $\mathcal{E} \circ ad(x) = \epsilon(x, x)ad(x) \circ \mathcal{E}$, we have

$$Tr(\mathcal{E}ad(x)ad(y)) = Tr(ad(y)\mathcal{E}ad(x)) = \epsilon(x, y)Tr(\mathcal{E}ad(y)ad(x))$$

and hence the bilinear form is ϵ-symmetric.

Let $x, y, z \in \mathfrak{g}$ such that $|z| = -|x| - |y|$, we have

$$\begin{aligned} Tr(\mathcal{E}\{ad(x), ad(y)\}ad(z)) &= Tr(\mathcal{E}ad(x)ad(y)ad(z)) - \epsilon(x, y)Tr(\mathcal{E}ad(y)ad(x)ad(z)) \\ &= \epsilon(x, x)Tr(ad(x)\mathcal{E}ad(y)ad(z)) - \epsilon(x, y)Tr(\mathcal{E}ad(y)ad(x)ad(z)) \\ &= \epsilon(x, x)Tr(\mathcal{E}ad(y)ad(z)ad(x)) - \epsilon(x, y)Tr(\mathcal{E}ad(y)ad(x)ad(z)) \\ &= \epsilon(x, y)\epsilon(x, z)Tr(\mathcal{E}ad(y)ad(z)ad(x)) - \epsilon(x, y)Tr(\mathcal{E}ad(y)ad(x)ad(z)) \\ &= -\epsilon(x, y)(-\epsilon(x, z)Tr(\mathcal{E}ad(y)ad(z)ad(x)) + Tr(\mathcal{E}ad(y)ad(x)ad(z))) \\ &= -\epsilon(x, y)Tr(\mathcal{E}ad(y)\{ad(x), ad(z)\}) \end{aligned}$$

and hence the form is ad-invariant. □

Proposition 3.8.23. *Let V be a finite-dimensional Γ-graded vector space. The colour Lie algebra $\mathfrak{gl}_\epsilon(V)$ together with the ad-invariant, ϵ-symmetric bilinear form defined by*

$$Tr_\epsilon(fg) \qquad \forall f, g \in \mathfrak{gl}_\epsilon(V) \tag{3.9}$$

is non-degenerate.

Proof. As in the proof of Proposition 3.8.22 we show that the bilinear form of Equation (3.9) is ad-invariant and ϵ-symmetric. The form is also non-degenerate since for in a homogeneous basis, an elementary matrix E_{ij} satisfies

$$Tr_\epsilon(E_{ij}E_{ji}) \neq 0.$$

□

Proposition 3.8.24. *Let V be a finite-dimensional Γ-graded vector space such that $dim(V) \geq 2$ together with a non-degenerate ϵ-symmetric bilinear form $(\ , \ )$ on V. The colour Lie algebra $\mathfrak{so}_\epsilon(V, (\ , \ ))$ together with the bilinear form*

$$Tr_\epsilon(fg) \qquad \forall f, g \in \mathfrak{so}_\epsilon(V, (\ , \ )) \tag{3.10}$$

is non-degenerate.

Proof. By Proposition 3.8.23 the bilinear form considered is ϵ-symmetric and ad-invariant. We need the following lemma.

Lemma 3.8.25. *Let V be a finite-dimensional Γ-graded vector space such that $dim(V) \geq 2$ together with a non-degenerate ϵ-symmetric bilinear form $(\ ,\ )$ on V. Let $u, v \in V$. The map $f(u,v) \in \mathfrak{so}_\epsilon(V, (\ ,\ ))$ defined by*

$$f(u,v)(w) := \epsilon(v,w)(u,w)v - (v,w)u \qquad \forall w \in V$$

satisfies

$$Tr_\epsilon(f \circ f(u,v)) = -2(f(u),v) \qquad \forall f \in \mathfrak{so}_\epsilon(V, (\ ,\ )).$$

Proof. Let $u, v \in V$. One can check that $f(u,v) \in \mathfrak{so}_\epsilon(V, (\ ,\ ))$. Let $\{e_i : 1 \leq i \leq dim(V)\}$ be a homogeneous basis of V. Let $e_i, e_j \in \{e_i : 1 \leq i \leq dim(V)\}$ and $f \in \mathfrak{so}_\epsilon(V, (\ ,\ ))$. Let $\{e^i : 1 \leq i \leq dim(V)\}$ be its dual basis in the sense that $(e_i, e^j) = \delta_{ij}$ for all $i, j \in [\![1, dim(V)]\!]$. We have $|e^i| = -|e_i|$ and

$$f(e_i, e_j)(e^k) = \epsilon(e_k, e_j)\delta_{ik}e_j - \delta_{jk}e_i.$$

After a straightforward calculation we have

$$\frac{(\mathcal{E}(f(\mu(e_i,e_j)(e^k))), e_k)}{\epsilon(e_k,e_k)} = -\delta_{ik}(f(e_k), e_j) - \delta_{jk}(f(e_i), e_k). \tag{3.11}$$

Since

$$\sum_{k=1}^{dim(V)} \frac{(f(e^k), e_k)}{\epsilon(e_k, e_k)} = Tr(f) \qquad \forall f \in \mathfrak{gl}_\epsilon(V)$$

by Equation (3.11) we obtain

$$Tr_\epsilon(f \circ f(e_i, e_j)) = -2(f(e_i), e_j). \tag{3.12}$$

□

Let $f \in \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be such that

$$Tr_\epsilon(f \circ g) = 0 \qquad \forall g \in \mathfrak{so}_\epsilon(V, (\ ,\ )).$$

In particular, we have

$$Tr_\epsilon(f \circ f(u,v)) = 0 \qquad \forall u, v \in V.$$

By Lemma 3.8.25, we have

$$(f(u), v) \qquad \forall u, v \in V$$

then, since $(\ ,\ )$ is non-degenerate, we obtain that $f \equiv 0$ and so the bilinear form (3.10) is non-degenerate. □

Remark 3.8.26. *If $\Gamma = \mathbb{Z}_2$, $\epsilon(m,n) = (-1)^{mn}$ for all $m, n \in \mathbb{Z}_2$ and K is the Killing form, then*

$$K(f,g) = 2(dim(V_0) - dim(V_1))Tr_\epsilon(fg) - 2Tr_\epsilon(f)Tr_\epsilon(g) \quad \forall f, g \in \mathfrak{gl}_\epsilon(V), \tag{3.13}$$
$$K(f,g) = (dim(V_0) - dim(V_1) - 2)Tr_\epsilon(fg) \quad \forall f, g \in \mathfrak{so}_\epsilon(V, (\ ,\ )), \tag{3.14}$$

see [Sch79b] pages 127 and 128.

3.9 ϵ-orthogonal representations of ϵ-quadratic colour Lie algebras

Let Γ be an abelian group and let ϵ be a commutation factor of Γ. In this section, we define the moment map of an ϵ-orthogonal representation $(V, (\ ,\ ))$ of a colour Lie algebra $\mathfrak{g}$. After giving some general properties, we study the moment map of the natural representation of $\mathfrak{so}_\epsilon(V, (\ ,\ ))$ which we will call the "canonical" moment map. Finally, we give a formula for the moment map of a tensor product of ϵ-orthogonal representations in terms of the moment maps of the factors.

Unless otherwise stated, we suppose all ϵ-orthogonal representations of dimension at least two.

3.9.1 The moment map of an ϵ-orthogonal representation

Definition 3.9.1. *Let $(\mathfrak{g}, B_\mathfrak{g})$ be a finite-dimensional ϵ-quadratic colour Lie algebra and let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of $\mathfrak{g}$. We define the moment map of the representation $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ to be the bilinear map $\mu : V \times V \to \mathfrak{g}$ given by*

$$B_\mathfrak{g}(x, \mu(v,w)) = (\rho(x)(v), w) \qquad \forall v, w \in V,\ \forall x \in \mathfrak{g}.$$

Remark 3.9.2. *More properly, the map μ defined here should be called the comoment map as what is usually called the moment map maps $V \times V$ to $\mathfrak{g}^*$. However since $\mathfrak{g}$ is ϵ-quadratic, one can identify $\mathfrak{g}$ and $\mathfrak{g}^*$ as vector spaces and then, as we now show, the comoment and moment are identified.*

The map $\tilde{\mu} : V \times V \to \mathfrak{g}^*$ is given by

$$\tilde{\mu}(v,w)(x) := \epsilon(v+w, x)(\rho(x)(v), w) \qquad \forall v, w \in V,\ \forall x \in \mathfrak{g}.$$

Define a linear isomorphism $\tau : \mathfrak{g} \to \mathfrak{g}^*$ by

$$\tau(x)(y) := B_\mathfrak{g}(x,y) \qquad \forall x, y \in \mathfrak{g}.$$

Let $v, w \in V$ and let $x \in \mathfrak{g}$. We have

$$\begin{aligned}\tau(\mu(v,w))(x) &= B_{\mathfrak{g}}(\mu(v,w),x)\\ &= \epsilon(v+w,x)B_{\mathfrak{g}}(x,\mu(v,w))\\ &= \epsilon(v+w,x)B_{\mathfrak{g}}(\rho(x)(v),w)\\ &= \tilde{\mu}(v,w)(x)\end{aligned}$$

and hence $\tau(\mu(v,w)) = \tilde{\mu}(v,w)$.

We now show that the moment map is ϵ-antisymmetric and equivariant. This generalises the fact that the moment map of a symplectic representation of a Lie algebra is symmetric and equivariant.

Proposition 3.9.3. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_{\mathfrak{g}})$ and let $\mu : V \times V \to \mathfrak{g}$ be its moment map.*

a) The moment map is of degree 0.

b) For v, w in V, we have $\mu(v,w) = -\epsilon(v,w)\mu(w,v)$.

c) For $x \in \mathfrak{g}$ and $v, w \in V$, we have

$$\{x, \mu(v,w)\} = \mu(x(v),w) + \epsilon(x,v)\mu(v,x(w)). \tag{3.15}$$

d) Let $\{e_i : 1 \le i \le dim(\mathfrak{g})\}$ be a basis of $\mathfrak{g}$ and let $\{e^i : 1 \le i \le dim(\mathfrak{g})\}$ be the dual basis of $\{e_i : 1 \le i \le dim(\mathfrak{g})\}$ in the sense that $B_{\mathfrak{g}}(e_i, e^j) = \delta_{ij}$. Then, for $v, w \in V$, we have

$$\mu(v,w) = \sum_{i=1}^{dim(\mathfrak{g})} (e_i(v), w)e^i. \tag{3.16}$$

Proof. a) The moment map is of degree 0 since the bilinear forms $B_{\mathfrak{g}}$ and $(\ ,\ )$ are of degree 0.

b) For $x \in \mathfrak{g}$, we have

$$\begin{aligned}B_{\mathfrak{g}}(x,\mu(v,w)) &= (\rho(x)(v),w)\\ &= -\epsilon(x,v)(v,\rho(x)(w))\\ &= -\epsilon(x,v)\epsilon(v,x+w)(\rho(x)(w),v)\\ &= -\epsilon(v,w)B_{\mathfrak{g}}(x,\mu(w,v)).\end{aligned}$$

c) For $y \in \mathfrak{g}$ we have

$$\begin{aligned} B_{\mathfrak{g}}(y, \{x, \mu(v,w)\}) &= -\epsilon(y,x) B_{\mathfrak{g}}(\{x,y\}, \mu(v,w)) \\ &= -\epsilon(y,x)(\{x,y\}(v), w) \\ &= -\epsilon(y,x)(xy(v), w) + (yx(v), w) \\ &= \epsilon(x,v)(y(v), x(w)) + B_{\mathfrak{g}}(y, \mu(x(v), w)) \\ &= \epsilon(x,v) B_{\mathfrak{g}}(y, \mu(v, x(w))) + B_{\mathfrak{g}}(y, \mu(x(v), w)). \end{aligned}$$

d) Since $\{e^i : 1 \leq i \leq dim(\mathfrak{g})\}$ is a basis of $\mathfrak{g}$, then we have $\mu(v,w) = \sum_{j=1}^{\dim(\mathfrak{g})} a_j e^j$ for some $a_j \in k$ and moreover

$$(e_i(v), w) = B(e_i, \mu(v,w)) = B(e_i, \sum_{j=1}^{dim(\mathfrak{g})} a_j e^j) = a_i.$$

□

Remark 3.9.4. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_{\mathfrak{g}})$ and let $\mu : V \times V \to \mathfrak{g}$ be its moment map. Let $\alpha, \beta \in k^*$. Then $\mathfrak{g}$ is also ϵ-quadratic for the bilinear form $\alpha \cdot K$ and $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, \beta \cdot (\ ,\ ))$ is also an ϵ-orthogonal representation of $\mathfrak{g}$. The corresponding moment map μ' satisfies*

$$\mu'(v,w) = \frac{\beta}{\alpha} \mu(v,w) \qquad \forall v, w \in V.$$

3.9.2 The canonical moment map of an ϵ-orthogonal representation

Let V be a Γ-graded vector space together with a non-degenerate ϵ-symmetric bilinear form $(\ ,\ )$. In this section we study the moment map $\mu_{can} : \Lambda^2_\epsilon(V) \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ and in particular show that it is an equivariant isomorphism. This generalises the fact that if (V, ω) is a symplectic vector space, then $S^2(V)$ is isomorphic to $\mathfrak{sp}(V, \omega)$. It also includes the fact that if $(V, (\ ,\ ))$ is an orthogonal vector space, then $\Lambda^2(V)$ is isomorphic to $\mathfrak{so}(V, (\ ,\ ))$.

Proposition 3.9.5. *Let V be a finite-dimensional Γ-graded vector space together with a non-degenerate ϵ-symmetric bilinear form $(\ ,\ )$. Consider the ϵ-orthogonal representation of the ϵ-quadratic colour Lie algebra $(\mathfrak{so}_\epsilon(V, (\ ,\ )), B)$ where $B(f,g) := -\frac{1}{2} Tr_\epsilon(fg)$ for all $f, g \in \mathfrak{so}_\epsilon(V, (\ ,\ ))$. Then, the corresponding moment map μ_{can} satisfies*

$$\mu_{can}(u,v)(w) = \epsilon(v,w)(u,w)v - (v,w)u \qquad \forall u, v, w \in V. \tag{3.17}$$

Proof. By Lemma 3.8.25 we have

$$\mu_{can}(u,v) = f(u,v) \qquad \forall u,v \in V$$

and so

$$\mu_{can}(u,v)(w) = \epsilon(v,w)(u,w)v - (v,w)u \qquad \forall u,v,w \in V.$$

□

Remark 3.9.6. *Let $u,v,w \in V$. The moment map*

$$\mu_{can}(u,v)(w) = \epsilon(v,w)(u,w)v - (v,w)u$$

can also be written

$$\epsilon(u+v,w)i_w(u^* \wedge v^*)$$

where $u^*, v^* \in V^*$ are defined by $u^*(w) := (w,u)$ and $v^*(w) := (w,v)$.

We now calculate μ_{can} for the standard symplectic plane.

Example 3.9.7. *Let $k^2 = Vect < p,q >$ be a two-dimensional vector space together with the symplectic form ω defined by $\omega(p,q) = 1$. This is an ϵ-orthogonal representation, where $\Gamma = \mathbb{Z}_2$ and $\epsilon(a,b) := (-1)^{ab}$ for all $a,b \in \mathbb{Z}_2$. The Lie algebra $\mathfrak{sp}(k^2,\omega)$ is isomorphic to the Lie algebra $\mathfrak{sl}(2,k)$ and the $\mathfrak{sl}(2,k)$-triple $\{E,F,H\}$ given by*

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

acts on k^2 by

$$E(p) = 0,\ E(q) = p,\ F(p) = q,\ F(q) = 0,\ H(p) = p,\ H(q) = -q.$$

The Lie algebra $\mathfrak{sl}(2,k)$ is quadratic with respect to the form $\frac{1}{2}Tr(XY)$ and the basis $\{E,F,H\}$ is dual to the basis $\{2F,2E,H\}$. Hence, the canonical moment map is given by

$$\mu_{can}(p,p) = -2E, \quad \mu_{can}(q,q) = 2F, \quad \mu_{can}(p,q) = H.$$

Proposition 3.9.8. *Let V be a finite-dimensional Γ-graded vector space together with a non-degenerate ϵ-symmetric bilinear form $(\ ,\ )$. If $\Lambda^2_\epsilon(V)$ is given the bracket of Example 3.8.4, then the map $\mu_{can} : \Lambda^2_\epsilon(V) \to \mathfrak{so}_\epsilon(V,(\ ,\ ))$ is an isomorphism of colour Lie algebras.*

Proof. Let $u, v \in V$ be linearly independent and such that $\mu_{can}(u, v) = 0$. By Proposition 3.9.5 we have

$$\epsilon(v, w)(u, w)v - (v, w)u = 0 \qquad \forall w \in V.$$

Since u and v are linearly independent, we obtain that $u = v = 0$ and so μ_{can} is injective.

Recall that $V = V_0 \oplus V_1$, where

$$V_0 := \{v \in V \mid \epsilon(v, v) = 1\}, \qquad V_1 := \{v \in V \mid \epsilon(v, v) = -1\}$$

and let $m := dim(V_0)$ and $n := dim(V_1)$. Using Remark 3.4.8 we have

$$dim(\Lambda^2_\epsilon(V)) = \frac{m(m-1)}{2} + mn + \frac{n(n+1)}{2}$$

and one can show that we also have

$$dim(\mathfrak{so}_\epsilon(V, (\ ,\ ))) = \frac{m(m-1)}{2} + mn + \frac{n(n+1)}{2}.$$

Hence μ_{can} is surjective and so is a linear isomorphism. By a straightforward calculation we have

$$\{\mu_{can}(v_1 \wedge v_2), \mu_{can}(v_3 \wedge v_4)\} = \mu_{can}(\{v_1 \wedge v_2, v_3 \wedge v_4\})$$

for all $v_1, v_2, v_3, v_4 \in V$. □

Proposition 3.9.9. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$ and let $\mu : V \times V \to \mathfrak{g}$ be its moment map. Suppose that ρ is injective and suppose that there exists $c \in k^*$ such that*

$$\mu(u, v)(w) = c \cdot \mu_{can}(u, v)(w) \quad \forall u, v, w \in V.$$

Then $\mathfrak{g}$ is isomorphic to $\mathfrak{so}_\epsilon(V, (\ ,\ ))$.

Proof. We want to show that $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ is an isomorphism of colour Lie algebras. Let $f \in \mathfrak{so}_\epsilon(V, (\ ,\ ))$. Since $\mu_{can} : \Lambda^2_\epsilon(V) \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ is an isomorphism of colour Lie algebras, there exists $u \in \Lambda^2_\epsilon(V)$ such that $\mu_{can}(u) = f$. But since

$$\mu(u)(w) = c \cdot \mu_{can}(u)(w) \quad \forall w \in V$$

it follows that $\rho(\mu(\frac{1}{c}u)) = f$ and hence ρ is surjective. It is also injective by assumption and then $\mathfrak{g}$ is isomorphic to $\mathfrak{so}_\epsilon(V, (\ ,\ ))$. □

Remark 3.9.10. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$ and let $\mu : V \times V \to \mathfrak{g}$ be its moment map. Consider the adjoint map $\rho^t : \mathfrak{so}_\epsilon(V, (\ ,\ )) \to \mathfrak{g}$ of $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ defined by*

$$B_\mathfrak{g}(x, \rho^t(f)) = B_{\mathfrak{so}_\epsilon(V,(\ ,\ ))}(\rho(x), f) \qquad \forall x \in \mathfrak{g},\ \forall f \in \mathfrak{so}_\epsilon(V, (\ ,\ )).$$

Let $x \in \mathfrak{g}$, $u, v \in V$. We have

$$\begin{aligned} B_\mathfrak{g}(x, \rho^t(\mu_{can}(u, v))) &= B_{\mathfrak{so}_\epsilon(V,(\ ,\ ))}(\rho(x), \mu_{can}(u, v)) \\ &= (\rho(x)(u), v) \\ &= B_\mathfrak{g}(x, \mu(u, v)) \end{aligned}$$

and hence $\mu = \rho^t \circ \mu_{can}$.

3.9.3 Multiplicativity of the moment map

Let Γ be an abelian group and let ϵ be a commutation factor of Γ. Let $\mathfrak{g}$ and $\mathfrak{h}$ be finite-dimensional ϵ-quadratic colour Lie algebras with respect to (Γ, ϵ) and let

$$\rho_\mathfrak{g} : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V), \qquad \rho_\mathfrak{h} : \mathfrak{h} \to \mathfrak{so}_\epsilon(W, (\ ,\ )_W)$$

be finite-dimensional ϵ-orthogonal representations of $\mathfrak{g}$ and $\mathfrak{h}$. On the vector space $V \otimes W$ we define the bilinear form

$$(v \otimes w, v' \otimes w')_{V\otimes W} := \epsilon(w, v')(v, v')_V (w, w')_W \qquad \forall v \otimes w, v' \otimes w' \in V \otimes W.$$

Proposition 3.9.11. *The bilinear form $(\ ,\ )_{V\otimes W}$ is non-degenerate and ϵ-symmetric for the natural Γ-grading of $V \otimes W$.*

Proof. Let $v \otimes w, v' \otimes w' \in V \otimes W$. We have

$$\begin{aligned} (v \otimes w, v' \otimes w')_{V\otimes W} &= \epsilon(w, v')(v, v')_V (w, w')_W \\ &= \epsilon(w, v')\epsilon(v, v')\epsilon(w, w')(v', v)_V (w', w)_W \\ &= \epsilon(v + w, v' + w')\epsilon(w', v)(v', v)_V (w', w)_W \\ &= \epsilon(v \otimes w, v' \otimes w')(v' \otimes w', v \otimes w)_{V\otimes W} \end{aligned}$$

and so $(\ ,\ )_{V\otimes W}$ is ϵ-symmetric.

Let $\{v_i : 1 \le i \le dim(V)\}$ be a basis of V, let $\{w_i : 1 \le i \le dim(W)\}$ be a basis of W and let $\{v^i : 1 \le i \le dim(V)\}$ and $\{w^i : 1 \le i \le dim(W)\}$ be the corresponding dual basis in the sense that $(v_i, v^j)_V = \delta_{ij}$ and $(w_i, w^j)_W = \delta_{ij}$. The set $\{v_i \otimes w_j : 1 \le i \le dim(V),\ 1 \le j \le dim(W)\}$ is a basis of $V \otimes W$ and for all $v_i \otimes w_j$ in $\{v_i \otimes w_j : 1 \le i \le dim(V),\ 1 \le j \le dim(W)\}$ we have

$$(v_i \otimes w_j, v^i \otimes w^j)_{V\otimes W} \neq 0$$

and so $(\ ,\ )_{V\otimes W}$ is non-degenerate. □

Hence, one can define an ϵ-orthogonal representation of the ϵ-quadratic colour Lie algebra $(\mathfrak{g} \oplus \mathfrak{h}, B_{\mathfrak{g}} \perp B_{\mathfrak{h}})$

$$\rho : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{so}_{\epsilon}(V \otimes W, (\ ,\ )_{V \otimes W})$$

by :

$$\rho(g+h)(v \otimes w) := \rho_{\mathfrak{g}}(g)(v) \otimes w + \epsilon(h, v) v \otimes \rho_{\mathfrak{h}}(w) \quad \forall g \in \mathfrak{g},\ \forall h \in \mathfrak{h},\ \forall v \otimes w \in V \otimes W. \tag{3.18}$$

The moment map of the tensor product is essentially the product of the moment maps of the factors as we now show.

Proposition 3.9.12. *The moment map $\mu_{V \otimes W} : \Lambda^2_{\epsilon}(V \otimes W) \to \mathfrak{g} \oplus \mathfrak{h}$ is given, for all $v \otimes w$, $v' \otimes w'$ in $V \otimes W$, by*

$$\mu_{V \otimes W}(v \otimes w, v' \otimes w') = \epsilon(w, v')\Big(\mu_V(v, v')(w, w')_W + (v, v')_V \mu_W(w, w')\Big), \tag{3.19}$$

where $\mu_V : \Lambda^2_{\epsilon}(V) \to \mathfrak{g}$ (resp. $\mu_W : \Lambda^2_{\epsilon}(W) \to \mathfrak{h}$) is the moment map of the ϵ-orthogonal representation V of $\mathfrak{g}$ (resp. W of $\mathfrak{h}$).

Proof. Let $g \in \mathfrak{g}$, $h \in \mathfrak{h}$ and $v \otimes w, v' \otimes w' \in V \otimes W$. We have

$$\begin{aligned} B_{\mathfrak{g} \oplus \mathfrak{h}}(g + h, \mu_{V \otimes W}(v \otimes w, v' \otimes w')) =& ((g + h)(v \otimes w), v' \otimes w')_{V \otimes W} \\ =& (g(v) \otimes w, v' \otimes w')_{V \otimes W} + \epsilon(h, v)(v \otimes h(w), v' \otimes w')_{V \otimes W} \\ =& \epsilon(w, v')(g(v), v')_V (w, w')_W \\ &+ \epsilon(h + w, v')\epsilon(h, v)(v, v')_V (h(w), w')_W \\ =& \epsilon(w, v')(B_{\mathfrak{g}}(g, \mu_V(v, v')(w, w')_W) \\ &+ B_{\mathfrak{h}}(h, \epsilon(h, v + v')\mu_W(w, w')(v, v')_V)) \end{aligned}$$

and since $(v, v')_V = 0$ if $|v| \neq -|v'|$, we obtain

$$B_{\mathfrak{g} \oplus \mathfrak{h}}(g+h, \mu_{V \otimes W}(v \otimes w, v' \otimes w')) = B_{\mathfrak{g} \oplus \mathfrak{h}}(g+h, \epsilon(w, v')(\mu_V(v, v')(w, w')_W + \mu_W(w, w')(v, v')_V)).$$

□

Chapter 4

Invariants associated to an ϵ-orthogonal representation

In this chapter we :

- recall the results of B. Kostant (see [Kos99], [Kos01]) ;
- generalise the results of B. Kostant ([Kos99], [Kos01]) and the results of Z. Chen et Y. Kang ([CK15]) to ϵ-orthogonal representations of ϵ-quadratic colour Lie algebras over a field of characteristic not two or three ;
- introduce the notion of special ϵ-orthogonal representation and state a link with the generalisation above ;
- define the covariants of a special ϵ-orthogonal representation and study their geometry.

Unless otherwise stated :

- k is a field of characteristic not two or three ;
- Γ is an abelian group ;
- ϵ is a commutation factor of Γ ;
- ϵ-orthogonal representations are of dimension at least two.

4.1 Review of Kostant's invariants

In this section we recall Kostant's original constructions of invariants for orthogonal (and resp. symplectic) representations of quadratic Lie algebras (see [Kos99] and [Kos01]). He showed that the vanishing of these invariants is equivalent to the existence of a quadratic Lie algebra (resp. Lie superalgebra) structure on the direct sum of the Lie algebra and the representation space.

4.1.1 An invariant associated to an orthogonal representation of a complex quadratic Lie algebra (see [Kos99])

Let $(\mathfrak{r}, B_\mathfrak{r})$ be a finite-dimensional, quadratic, complex Lie algebra, let $(\mathfrak{p}, B_\mathfrak{p})$ be a finite-dimensional, non-degenerate quadratic, complex vector space and let $\rho : \mathfrak{r} \to \mathfrak{so}(\mathfrak{p}, B_\mathfrak{p})$ be an orthogonal representation of the Lie algebra $\mathfrak{r}$.

Definition 4.1.1. *Let $\mathfrak{g} := \mathfrak{r} \oplus \mathfrak{p}$ be the complex vector space, quadratic with respect to the non-degenerate quadratic form $B_\mathfrak{g} := B_\mathfrak{r} \perp B_\mathfrak{p}$. We say that the representation $\rho : \mathfrak{r} \to \mathfrak{so}(\mathfrak{p}, B_\mathfrak{p})$ is of Lie type if there exists a Lie algebra structure $[\ ,\ ]$ on $\mathfrak{g}$ such that*

- $[x, y] = [x, y]_\mathfrak{r}$ *for x, y in $\mathfrak{r}$;*
- $[x, v] = -[v, x] = \rho(x)(v)$ *for x in $\mathfrak{r}$, for v in $\mathfrak{p}$;*
- *the quadratic form $B_\mathfrak{g}$ is $ad(\mathfrak{g})$-invariant.*

If we also have $\{\mathfrak{p}, \mathfrak{p}\} \subseteq \mathfrak{r}$ then the representation $\rho : \mathfrak{r} \to \mathfrak{so}(\mathfrak{p}, B_\mathfrak{p})$ is said to be of $\mathbb{Z}_2$-Lie type.

To state the necessary and sufficient criterion given by B. Kostant we need to recall some facts about the structure of the exterior algebra $\Lambda(\mathfrak{p})$ and about the structure of the Clifford algebra $C(\mathfrak{p})$ (with respect to $B_\mathfrak{p}$).

The quadratic form $B_\mathfrak{p}$ extends uniquely to a non-degenerate quadratic form on the exterior algebra $\Lambda(\mathfrak{p})$, also denoted by $B_\mathfrak{p}$, with property that $\Lambda^j(\mathfrak{p})$ is orthogonal to $\Lambda^k(\mathfrak{p})$ if $j \neq k$. More precisely, the quadratic form on $\Lambda(\mathfrak{p})$ is given by

$$B_\mathfrak{p}(x_1 \wedge \ldots \wedge x_j, y_1 \wedge \ldots \wedge y_j) = det(x_k, y_l) \quad \forall x_1 \wedge \ldots \wedge x_j, y_1 \wedge \ldots \wedge y_j \in \Lambda^j(\mathfrak{p}).$$

Let $v \in \Lambda(\mathfrak{p})$ and define $e_v \in End(\Lambda(\mathfrak{p}))$ by

$$e_v(w) := v \wedge w \qquad \forall w \in \Lambda(\mathfrak{p}).$$

Since $(\Lambda(\mathfrak{p}), B_\mathfrak{p})$ is a non-degenerate quadratic vector space, we also define $i_v \in End(\Lambda(\mathfrak{p}))$ by

$$i_v := e_v^t.$$

In particular, if $v, w \in \mathfrak{p}$, we have

$$i_v(w) = B_\mathfrak{p}(v, w).$$

C. Chevalley (see [Che97]) showed that the two algebras $\Lambda(\mathfrak{p})$ and $C(\mathfrak{p})$ are isomorphic as $\mathfrak{so}(\mathfrak{p}, B_{\mathfrak{p}})$-representations (not as algebras) as follows. Let $\gamma : \mathfrak{p} \to End(\Lambda(\mathfrak{p}))$ be the linear map defined by

$$\gamma(v) := e_v + i_v \qquad \forall v \in \mathfrak{p}.$$

By a straightforward calculation we have

$$\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = 2B_{\mathfrak{p}}(v, w) \quad \forall v, w \in \mathfrak{p}$$

and then by the universal property of the Clifford algebra $C(\mathfrak{p})$ we extend γ to an algebra homomorphism

$$\gamma : C(\mathfrak{p}) \to End(\Lambda(\mathfrak{p})).$$

Proposition 4.1.2. *(see [Che97]) The linear map $\phi : C(\mathfrak{p}) \to \Lambda(\mathfrak{p})$ given by*

$$\phi(v) = \gamma(v)(1) \qquad \forall v \in C(\mathfrak{p})$$

is a linear isomorphism of $\mathfrak{so}(\mathfrak{p}, B_{\mathfrak{p}})$-representations.

Thus, the vector space $\Lambda(\mathfrak{p})$ inherits a Clifford multiplication given by

$$vw := \phi\left(\phi^{-1}(v)\phi^{-1}(w)\right) \qquad \forall v, w \in \Lambda(\mathfrak{p}).$$

In particular, the Clifford commutator defines a Lie algebra structure on $\Lambda^2(\mathfrak{p})$ and by a straightforward calculation we have

$$ad(v)(w) := vw - wv = -2i_w(v) \quad \forall v \in \Lambda^2(\mathfrak{p}),\ \forall w \in \mathfrak{p},$$

and hence we obtain an isomorphism of Lie algebras

$$ad : \Lambda^2(\mathfrak{p}) \to \mathfrak{so}(\mathfrak{p}, B_{\mathfrak{p}}).$$

Consequently we have a morphism of Lie algebras

$$\rho_* : \mathfrak{r} \to \Lambda^2(\mathfrak{p})$$

given by

$$\rho_* = ad^{-1} \circ \rho.$$

Furthermore, since $(\mathfrak{r}, B_{\mathfrak{r}})$ and $(\Lambda^2(\mathfrak{p}), B_{\mathfrak{p}})$ are non-degenerate quadratic vector spaces, we obtain the adjoint map of ρ_* :

$$\rho_*^t : \Lambda^2(\mathfrak{p}) \to \mathfrak{r}$$

and since $C^{even}(\mathfrak{p}) = \bigoplus_{i \in \mathbb{N}} \Lambda^{2i}(\mathfrak{p})$ is a unital associative algebra, we can uniquely extend the map $\rho_* : \mathfrak{r} \to \Lambda^2(\mathfrak{p})$ to the universal enveloping algebra $U(\mathfrak{r})$ of the Lie algebra $\mathfrak{r}$:

$$\rho_* : U(\mathfrak{r}) \to C^{even}(\mathfrak{p}).$$

Recall that via the Poincaré–Birkhoff–Witt theorem the quadratic form $B_{\mathfrak{g}} \in S^2(\mathfrak{g}^*)$ defines a unique element of $U(\mathfrak{g})$.

Theorem 4.1.3. *The representation $\rho : \mathfrak{r} \to \mathfrak{so}(\mathfrak{p}, B_\mathfrak{p})$ is of Lie type if and only if there exists $u \in (\Lambda^3(\mathfrak{p}))^\mathfrak{r}$ such that*

$$\rho_*(B_\mathfrak{r}) + u^2 \in \mathbb{C} = \Lambda^0(\mathfrak{p}).$$

In this case, the Lie bracket on $\mathfrak{p}$ is given by :

$$[v, w] = -2\rho_*^t(v \wedge w) + 2i_v i_w(u) \quad \forall v, w \in \mathfrak{p}.$$

Remark 4.1.4. *In particular, the representation $\rho : \mathfrak{r} \to \mathfrak{so}(\mathfrak{p}, B_\mathfrak{p})$ is of $\mathbb{Z}_2$-Lie type if and only if*

$$\rho_*(B_\mathfrak{r}) \in \mathbb{C} = \Lambda^0(\mathfrak{p}).$$

Remark 4.1.5. *In fact, the proofs remain valid over a field of characteristic zero.*

Example 4.1.6. *a) Let $(\ , \ )$ be a negative definite symmetric bilinear form on $\mathbb{R}^3$. The fundamental representation of $\mathfrak{so}(\mathbb{R}^3, (\ , \ ))$ satisfies the condition of Remark 4.1.4 and the Lie algebra $\mathfrak{so}(\mathbb{R}^3, (\ , \ )) \oplus \mathbb{R}^3$ is isomorphic to the Lie algebra $\mathfrak{so}(4)$.*

b) Let $(\ , \ )$ be a positive definite symmetric bilinear form on $\mathbb{R}^3$. The fundamental representation of $\mathfrak{so}(\mathbb{R}^3, (\ , \ ))$ satisfies the condition of Remark 4.1.4 and the Lie algebra $\mathfrak{so}(\mathbb{R}^3, (\ , \ )) \oplus \mathbb{R}^3$ is isomorphic to the Lie algebra $\mathfrak{so}(3, 1)$.

c) In the chapter 5, we will study in detail the fundamental representation V of a simple exceptional Lie algebra $\mathfrak{g}$ of type G_2. We will see that, together with a non-trivial map $\phi : \Lambda^2(V) \to V$, the representation $\mathfrak{g} \to \mathfrak{so}(V, (\ , \ ))$ is of Lie type and the Lie algebra $\mathfrak{g} \oplus V$ is isomorphic to the Lie algebra $\mathfrak{spin}(V)$.

4.1.2 An invariant associated to a symplectic representation of a complex quadratic Lie algebra (see [Kos01])

Let $(\mathfrak{r}, B_\mathfrak{r})$ be a finite-dimensional, quadratic, complex Lie algebra, let $(\mathfrak{p}, B_\mathfrak{p})$ be a finite-dimensional, symplectic, complex vector space and let $\rho : \mathfrak{r} \to \mathfrak{sp}(\mathfrak{p}, B_\mathfrak{p})$ be a representation of the Lie algebra $\mathfrak{r}$.

Definition 4.1.7. *Let $\mathfrak{g} := \mathfrak{r} \oplus \mathfrak{p}$ be the complex vector space and let $B_\mathfrak{g}$ be the non-degenerate supersymmetric bilinear form $B_\mathfrak{g} := B_\mathfrak{r} \perp B_\mathfrak{p}$. We say that the representation $\rho : \mathfrak{r} \to \mathfrak{so}(\mathfrak{p}, B_\mathfrak{p})$ is of Lie type if there exists a Lie superalgebra structure $\{ \ , \ \}$ on $\mathfrak{g}$ such that*

- $\{x, y\} = [x, y]_\mathfrak{r}$ *for x, y in $\mathfrak{r}$;*
- $\{x, v\} = -\{v, x\} = \rho(x)(v)$ *for x in $\mathfrak{r}$, for v in $\mathfrak{p}$;*

- *the supersymmetric bilinear form $B_{\mathfrak{g}}$ is $ad(\mathfrak{g})$-invariant :*

$$B_{\mathfrak{g}}(\{x,y\},z) = -(-1)^{xy}B(x,\{y,z\}) \qquad \forall x,y,z \in \mathfrak{g}.$$

To state the necessary and sufficient criterion given by B. Kostant we need to recall some facts about the structure of the symmetric algebra $S(\mathfrak{p})$ and about the structure of the Weyl algebra $W(\mathfrak{p})$ (with respect to $B_{\mathfrak{p}}$).

The symplectic form $B_{\mathfrak{p}}$ extends uniquely to a non-degenerate bilinear form on the symmetric algebra $S(\mathfrak{p})$, also denoted by $B_{\mathfrak{p}}$, with the property that $S^j(\mathfrak{p})$ is orthogonal to $S^k(\mathfrak{p})$ if $j \neq k$. More precisely, the non-degenerate bilinear form on $S(\mathfrak{p})$ is given by

$$B_{\mathfrak{p}}(x_1\cdot\ldots\cdot x_j, y_1\cdot\ldots\cdot y_j) = \sum_{\sigma\in\mathcal{S}_n} B_{\mathfrak{p}}(x_1,y_{\sigma(1)})\cdot\ldots\cdot B_{\mathfrak{p}}(x_j,y_{\sigma(j)}) \quad \forall x_1\cdot\ldots\cdot x_j, y_1\cdot\ldots\cdot y_j \in S^j(\mathfrak{p}).$$

Note that $B_{\mathfrak{p}}$ restricted to $S^j(\mathfrak{p})$ is symmetric is j is even and antisymmetric if j is odd.

Let $v \in S(\mathfrak{p})$ and define the map $e_v \in End(S(\mathfrak{p}))$ by

$$e_v(w) := v\cdot w \qquad \forall w \in S(\mathfrak{p}).$$

For $v \in \mathfrak{p}$, we also define $i_v \in End(S(\mathfrak{p}))$ to be the unique derivation of $S(\mathfrak{p})$ of degree -1 such that

$$i_v(w) = B_{\mathfrak{p}}(v,w) \qquad \forall w \in \mathfrak{p}.$$

Now, if we define the map $\gamma : \mathfrak{p} \to End(S(\mathfrak{p}))$ by

$$\gamma(v) := e_v + i_v \qquad \forall v \in \mathfrak{p},$$

by a straightforward calculus we have

$$\gamma(v)\gamma(w) - \gamma(w)\gamma(v) = 2B_{\mathfrak{p}}(v,w) \quad \forall v,w \in \mathfrak{p}$$

and then by the universal property of the Weyl algebra $W(\mathfrak{p})$ we extend γ to an algebra homomorphism

$$\gamma : W(\mathfrak{p}) \to End(S(\mathfrak{p})).$$

Proposition 4.1.8. *The linear map $\phi : W(\mathfrak{p}) \to S(\mathfrak{p})$ given by*

$$\phi(v) = \gamma(v)(1) \qquad \forall v \in W(\mathfrak{p})$$

is a linear isomorphism of $\mathfrak{sp}(\mathfrak{p}, B_{\mathfrak{p}})$-representations.

Thus, the vector space $S(\mathfrak{p})$ inherits a Weyl multiplication given by

$$vw := \phi\left(\phi(v)^{-1}\phi^{-1}(w)\right) \qquad \forall v, w \in S(\mathfrak{p}).$$

Furthermore, the Weyl commutator defines a Lie algebra structure on $S^2(\mathfrak{p})$ and by a straightforward calculation we have

$$ad(v)(w) := vw - wv = -2i(w)(v) \quad \forall v \in S^2(\mathfrak{p}) \ \forall w \in \mathfrak{p},$$

and hence we obtain an isomorphism of Lie algebras

$$ad : S^2(\mathfrak{p}) \to \mathfrak{sp}(\mathfrak{p}, B_{\mathfrak{p}}).$$

Consequently we have a morphism of Lie algebras

$$\rho_* : \mathfrak{r} \to S^2(\mathfrak{p})$$

given by

$$\rho_* = ad^{-1} \circ \rho.$$

Furthermore, since $(\mathfrak{r}, B_{\mathfrak{r}})$ and $(S^2(\mathfrak{p}), B_{\mathfrak{p}})$ are non-degenerate quadratic vector spaces, we obtain the adjoint map of ρ_* :

$$\rho_*^t : S^2(\mathfrak{p}) \to \mathfrak{r}$$

and since $W^{even}(\mathfrak{p}) = \bigoplus_{i\in\mathbb{N}} S^{2i}(\mathfrak{p})$ is a unital associative algebra for Weyl multiplication, we can uniquely extend the map $\rho_* : \mathfrak{r} \to S^2(\mathfrak{p})$ to the universal enveloping algebra $U(\mathfrak{r})$ of the Lie algebra $\mathfrak{r}$:

$$\rho_* : U(\mathfrak{r}) \to W^{even}(\mathfrak{p}).$$

Recall that via the Poincaré–Birkhoff–Witt theorem the quadratic form $B_{\mathfrak{g}} \in S^2(\mathfrak{g}^*)$ defines a unique element of $U(\mathfrak{g})$.

Theorem 4.1.9. *The representation $\rho : \mathfrak{r} \to \mathfrak{sp}(\mathfrak{p}, B_{\mathfrak{p}})$ is of Lie type if and only if*

$$\rho_*(B_{\mathfrak{r}}) \in \mathbb{C} = W^0(\mathfrak{p}).$$

In this case, the Lie bracket on $\mathfrak{p}$ is given by :

$$\{v, w\} = -2\rho_*^t(v \cdot w).$$

4.2 Characterisation of ϵ-orthogonal representations of colour Lie type and the norm of the moment map

In this section, we state and prove a theorem on ϵ-orthogonal representations of ϵ-quadratic colour Lie algebras which contains and generalises the results of Kostant (see Section 4.1) on orthogonal and symplectic complex representations of quadratic Lie algebras.

Definition 4.2.1. *Let $(\mathfrak{g}, B_{\mathfrak{g}})$ be a finite-dimensional ϵ-quadratic colour Lie algebra and let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of the colour Lie algebra $\mathfrak{g}$. Let $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus V$ and let $B_{\tilde{\mathfrak{g}}}$ the non-degenerate ϵ-symmetric bilinear form on $\tilde{\mathfrak{g}}$ defined by $B_{\tilde{\mathfrak{g}}} := B_{\mathfrak{g}} \perp (\ ,\ )$. We say that the representation $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ is of colour Lie type if there exists a colour Lie algebra structure $\{\ ,\ \}$ on $\tilde{\mathfrak{g}}$ such that*

- *the ϵ-quadratic form $B_{\tilde{\mathfrak{g}}}$ is $ad(\tilde{\mathfrak{g}})$-invariant ;*
- *$\{x, y\} = \{x, y\}_{\mathfrak{g}}$ for x, y in $\mathfrak{g}$;*
- *$\{x, v\} = -\epsilon(x, v)\{v, x\} = \rho(x)(v)$ for x in $\mathfrak{g}$, for v in V.*

If we also have $\{V, V\} \subseteq \mathfrak{g}$ then the representation $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ is said to be of colour $\mathbb{Z}_2$-Lie type.

Example 4.2.2. *Let V be a finite-dimensional Γ-graded vector space and let $(\ ,\ )$ be a non-degenerate ϵ-symmetric bilinear form on V. Then, V is of colour $\mathbb{Z}_2$-Lie type as a representation of $(\mathfrak{so}_\epsilon(V, (\ ,\ )), B)$ where*

$$B(f, g) := -\frac{1}{2} Tr_\epsilon(fg) \qquad \forall f, g \in \mathfrak{so}_\epsilon(V, (\ ,\ ));$$
$$\{v, w\} := \mu_{can}(v, w) \qquad \forall v, w \in V.$$

The colour Lie algebra $\mathfrak{so}_\epsilon(V, (\ ,\ )) \oplus V$ is then isomorphic to the colour Lie algebra $\mathfrak{so}_\epsilon(V \oplus L, (\ ,\ ) \perp (\ ,\ )_L)$ where L is a one-dimensional trivially Γ-graded vector space with an appropriate non-degenerate ϵ-symmetric bilinear form $(\ ,\ )_L$ (see Example 4.1.6).

In order to determine which representations are of colour Lie type we look at the "extra" structure induced on V by the bracket $\{\ ,\ \}$.

Proposition 4.2.3. *If $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ is of colour Lie type, then*

$$\{v, w\} = \mu(v, w) + \phi(v, w) \qquad \forall v, w \in V,$$

where $\mu : \Lambda^2_\epsilon(V) \to \mathfrak{g}$ is the moment map of the ϵ-orthogonal representation $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ and where $\phi : \Lambda^2_\epsilon(V) \to V$ is a linear map such that $|\phi| = 0$ which satisfies the two following properties :

$$\rho(x)(\phi(v, w)) = \phi(\rho(x)(v), w) + \epsilon(x, v)\phi(v, \rho(x)(w)) \qquad \forall x \in \mathfrak{g},\ \forall v, w \in V, \tag{4.1}$$
$$(\phi(u, v), w) = -\epsilon(u, v)(v, \phi(u, w)) \qquad \forall u, v, w \in V. \tag{4.2}$$

Remark 4.2.4. *Property (4.1) is equivalent to the fact that ϕ is $\mathfrak{g}$-equivariant and Property (4.2) is equivalent to the fact that $(u, v, w) \mapsto (\phi(u, v), w)$ is an ϵ-alternating trilinear form.*

This proposition shows that the only structure induced on V by the bracket $\{\ ,\ \}$ which is not intrinsic to the representation $\mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ is the map $\phi : \Lambda^2_\epsilon(V) \to V$. Hence we reformulate the problem of determining which representations of $\mathfrak{g}$ are of colour Lie type as follows.

Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$, let $\mu : \Lambda^2_\epsilon(V) \to \mathfrak{g}$ be its moment map and let $\phi : \Lambda^2_\epsilon(V) \to V$ be a map of degree 0 which satisfies Properties (4.1) and (4.2). Define $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus V$ and $\{\ ,\ \} : \Lambda^2_\epsilon(V) \to \tilde{\mathfrak{g}}$ by $\{\ ,\ \} := \mu + \phi$ and $B_{\tilde{\mathfrak{g}}} := B_\mathfrak{g} \perp (\ ,\ )$. The problem now is to determine necessary and sufficient conditions on this data such that $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ is of colour Lie type.

Proposition 4.2.5. *With the notation above, the bilinear form $B_{\tilde{\mathfrak{g}}}$ is ad($\tilde{\mathfrak{g}}$)-invariant in the sense that*

$$B_{\tilde{\mathfrak{g}}}(\{x, y\}, z) = -\epsilon(x, y) B_{\tilde{\mathfrak{g}}}(y, \{x, z\}) \qquad \forall x, y, z \in \tilde{\mathfrak{g}}.$$

Proof. Let $x \in \mathfrak{g}$.

a) For $y, z \in \mathfrak{g}$, we have $B_{\tilde{\mathfrak{g}}}(\{x, y\}, z) = -\epsilon(x, y) B_{\tilde{\mathfrak{g}}}(y, \{x, z\})$ because $B_\mathfrak{g}$ is ad($\mathfrak{g}$)-invariant.

b) For $y \in \mathfrak{g}$ and for $v \in V$, we have $B_{\tilde{\mathfrak{g}}}(\{x, y\}, v) = 0$ and $B_{\tilde{\mathfrak{g}}}(y, \{x, v\}) = 0$ since $\mathfrak{g}$ is orthogonal to V.

c) For $v, w \in V$, we have

$$B_{\tilde{\mathfrak{g}}}(\{x, v\}, w) = (x(v), w) = -\epsilon(x, v)(v, x(w)) = -\epsilon(x, v) B_{\tilde{\mathfrak{g}}}(v, \{x, w\}).$$

Let $u \in V$.

a) For $v, w \in V$ using Equation (4.2) we have

$$B_{\tilde{\mathfrak{g}}}(\{u, v\}, w) = (\phi(u, v), w) = -\epsilon(u, v)(v, \phi(u, w)) = -\epsilon(u, v) B_{\tilde{\mathfrak{g}}}(v, \{u, w\}).$$

b) For $v \in V$ and $x \in \mathfrak{g}$ we have

$$\begin{aligned} B_{\tilde{\mathfrak{g}}}(\{u, v\}, x) &= B_\mathfrak{g}(\mu(u, v), x) = \epsilon(u + v, x) B_\mathfrak{g}(x, \mu(u, v)) = \epsilon(u + v, x)(x(u), v) \\ &= \epsilon(u + v, x)\epsilon(x + u, v) B_{\tilde{\mathfrak{g}}}(v, \{x, u\}) = -\epsilon(u, v) B_{\tilde{\mathfrak{g}}}(v, \{u, x\}). \end{aligned}$$

c) For $x, y \in \mathfrak{g}$, we have $B_{\tilde{\mathfrak{g}}}(\{u, x\}, y) = 0$ and $B_{\tilde{\mathfrak{g}}}(x, \{u, y\}) = 0$ since $\mathfrak{g}$ is orthogonal to V.

□

To see when $\{\ ,\ \}$ defines a colour Lie algebra structure on $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus V$ we just have to see when the ϵ-Jacobi identities are satisfied.

Definition 4.2.6. *The Jacobi tensor $J : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ is given by*

$$J(u,v,w) = \epsilon(w,u)\{\{u,v\},w\} + \epsilon(u,v)\{\{v,w\},u\} + \epsilon(v,w)\{\{w,u\},v\} \qquad \forall u,v,w \in \tilde{\mathfrak{g}}.$$

Remark 4.2.7. *The Jacobi tensor is not an ϵ-alternating multilinear map. However, the map $\tilde{J} : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ defined by*

$$\tilde{J}(u,v,w) := \epsilon(u,w)J(u,v,w) \qquad \forall u,v,w \in \tilde{\mathfrak{g}}$$

is an ϵ-alternating mutilinear map.

We now show that three elements satisfy the ϵ-Jacobi identity if any one of them is in $\mathfrak{g}$.

Proposition 4.2.8. *Let $u,v,w \in \tilde{\mathfrak{g}}$. Then $J(u,v,w) = 0$ if u,v or w is an element of $\mathfrak{g}$.*

Proof. a) Let $x,y,z \in \mathfrak{g}$. Then $J(x,y,z) = 0$ since $\mathfrak{g}$ is a colour Lie algebra.

b) Let $x,y \in \mathfrak{g}$ and $v \in V$. We have :

$$\begin{aligned} J(x,y,v) &= \epsilon(v,x)\{\{x,y\},v\} + \epsilon(x,y)\{\{y,v\},x\} + \epsilon(y,v)\{\{v,x\},y\} \\ &= \epsilon(v,x)\{x,y\}(v) + \epsilon(x,y)\{y(v),x\} - \epsilon(y,v)\epsilon(v,x)\{x(v),y\} \\ &= \epsilon(v,x)\{x,y\}(v) - \epsilon(x,y)\epsilon(y+v,x)xy(v) + \epsilon(y,v)\epsilon(v,x)\epsilon(x+v,y)yx(v) \\ &= \epsilon(v,x)\{x,y\}(v) - \epsilon(v,x)xy(v) + \epsilon(v,x)\epsilon(x,y)yx(v); \end{aligned}$$

however $\rho(\{x,y\}) = \rho(x)\rho(y) - \epsilon(x,y)\rho(y)\rho(x)$ and hence $J(x,y,v) = 0$.

c) Let $x \in \mathfrak{g}$ and $v,w \in V$. Since the moment map and ϕ are $\mathfrak{g}$-equivariant we have

$$\{x,\{v,w\}\} = \{x(v),w\} + \epsilon(x,v)\{v,x(w)\}$$

and then

$$\begin{aligned} J(x,v,w) &= \epsilon(w,x)\{\{x,v\},w\} + \epsilon(x,v)\{\{v,w\},x\} + \epsilon(v,w)\{\{w,x\},v\} \\ &= \epsilon(w,x)\{x(v),w\} - \epsilon(x,v)\epsilon(v+w,x)\{x,\{v,w\}\} - \epsilon(v,w)\epsilon(w,x)\{x(w),v\} \\ &= \epsilon(w,x)\{x(v),w\} - \epsilon(w,x)\left(\{x(v),w\} + \epsilon(x,v)\{v,x(w)\}\right) - \epsilon(v,w)\epsilon(w,x)\{x(w),v\} \\ &= 0. \end{aligned}$$

□

The final ϵ-Jacobi identity we need to consider is for three elements in V. A priori, the Jacobi tensor of three elements in V takes values in $\mathfrak{g} \oplus V$ but, using the $\mathfrak{g}$-equivariance of ϕ, the following calculation shows that its component in $\mathfrak{g}$ is zero.

Proposition 4.2.9. *Let $u, v, w \in V$. We have $B_{\mathfrak{g}}(J(u, v, w), x) = 0$ for all $x \in \mathfrak{g}$.*

Proof. Let $u, v, w \in V$ and $x \in \mathfrak{g}$. We have

$$\begin{aligned} B_{\mathfrak{g}}(\{u, \{v, w\}\}, x) &= \epsilon(u + v + w, x) B_{\mathfrak{g}}(x, \{u, \{v, w\}\}) \\ &= -\epsilon(u, v + w)\epsilon(u + v + w, x) B_{\mathfrak{g}}(x, \{\{v, w\}, u\}) \\ &= -\epsilon(u, v + w)\epsilon(u + v + w, x)(x(\{v, w\}), u) \\ &= -\epsilon(u, v + w)\epsilon(u + v + w, x)(x(\phi(v, w)), u) \end{aligned}$$

and since

$$\rho(x)(\phi(v, w)) = \phi(\rho(x)(v), w) + \epsilon(x, v)\phi(v, \rho(x)(w)),$$

then

$$\begin{aligned} B_{\mathfrak{g}}(\{u, \{v, w\}\}, x) &= -\epsilon(u, v + w)\epsilon(u + v + w, x)\Big((\phi(x(v), w), u) + \epsilon(x, v)(\phi(v, x(w)), u)\Big) \\ &= -\epsilon(u, v + w)\epsilon(u + v + w, x)\Big((\{x(v), w\}, u) + \epsilon(x, v)(\{v, x(w)\}, u)\Big) \\ &= -\epsilon(u, v + w)\epsilon(u + v + w, x)\Big(-\epsilon(x + v, w)(\{w, x(v)\}, u) + \epsilon(x, v)(\{v, x(w)\}, u)\Big) \\ &= -\epsilon(u, v + w)\epsilon(u + v + w, x)\Big((x(v), \{w, u\}) - \epsilon(x, v)\epsilon(v, x + w)(x(w), \{v, u\})\Big) \\ &= -\epsilon(u, v + w)\epsilon(u + v + w, x)\Big(B_{\mathfrak{g}}(x, \{v, \{w, u\}\}) - \epsilon(v, w) B_{\mathfrak{g}}(x, \{w, \{v, u\}\})\Big) \\ &= -\epsilon(u, v + w) B_{\mathfrak{g}}(\{v, \{w, u\}\}, x) + \epsilon(u, v + w)\epsilon(v, w) B_{\mathfrak{g}}(\{w, \{v, u\}\}, x). \end{aligned}$$

Hence, $B_{\mathfrak{g}}(J(u, v, w), x) = 0$. □

Recall that (see Definition 3.6.6) the bilinear form $B_{\tilde{\mathfrak{g}}}$ on $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus V$ allows us to define an exterior product

$$\wedge_{B_{\tilde{\mathfrak{g}}}} : Alt_\epsilon(V, \tilde{\mathfrak{g}}) \times Alt_\epsilon(V, \tilde{\mathfrak{g}}) \to Alt_\epsilon(V).$$

Theorem 4.2.10. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_{\mathfrak{g}})$, let $\mu : \Lambda^2_\epsilon(V) \to \mathfrak{g}$ be its moment map and let $\phi : \Lambda^2_\epsilon(V) \to V$ be a map of degree 0 which satisfies Properties (4.1) and (4.2). Let $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus V$, let $B_{\tilde{\mathfrak{g}}} := B_{\mathfrak{g}} \perp (\ ,\ )$ and let $\{\ ,\ \} : \Lambda^2_\epsilon(\tilde{\mathfrak{g}}) \to \tilde{\mathfrak{g}}$ be the unique ϵ-antisymmetric bilinear map which extends the bracket of $\mathfrak{g}$, the action of $\mathfrak{g}$ on V and such that*

$$\{v, w\} = \mu(v, w) + \phi(v, w) \qquad \forall v, w \in V.$$

Then the following are equivalent :

a) $(\tilde{\mathfrak{g}}, B_{\tilde{\mathfrak{g}}}, \{\ ,\ \})$ *is an ϵ-quadratic colour Lie algebra.*

b) $N_{B_{\tilde{\mathfrak{g}}}}(\mu + \phi) = 0$.

c) $N_{B_{\tilde{\mathfrak{g}}}}(\mu) = -N_{B_{\tilde{\mathfrak{g}}}}(\phi)$.

Proof. From Propositions 4.2.5, 4.2.8 and 4.2.9 it follows that $\tilde{\mathfrak{g}}$ is a colour Lie algebra if and only if $(J(v_1, v_2, v_3), v_4) = 0$ for all $v_1, v_2, v_3, v_4 \in V$. We set $\psi := \mu + \phi \in Alt^2_\epsilon(V, \tilde{\mathfrak{g}})$ and consider $N_{B_{\tilde{\mathfrak{g}}}}(\psi) = \psi \wedge_{B_{\tilde{\mathfrak{g}}}} \psi \in Alt^4_\epsilon(V) \cong \Lambda_\epsilon(V)^*$. Let $v_1, v_2, v_3, v_4 \in V$. We have

$$N_{B_{\tilde{\mathfrak{g}}}}(\psi)(v_1, v_2, v_3, v_4) = \sum_{\sigma \in S(\{1,2\},\{3,4\})} p(\sigma; v_1, v_2, v_3, v_4) B_{\tilde{\mathfrak{g}}}(\psi(v_{\sigma(1)}, v_{\sigma(2)}), \psi(v_{\sigma(3)}, v_{\sigma(4)})).$$

Since

$$S(\{1,2\},\{3,4\}) = \{id, (123), (1243), (23), (13)(24), (243)\}$$

and

$$B_{\tilde{\mathfrak{g}}}(x, y) = \epsilon(x, y) B_{\tilde{\mathfrak{g}}}(y, x) \qquad \forall x, y \in \tilde{\mathfrak{g}}$$

we obtain

$$\begin{aligned} N_{B_{\tilde{\mathfrak{g}}}}(\psi)(v_1, v_2, v_3, v_4) =& B_{\tilde{\mathfrak{g}}}(\psi(v_1, v_2), \psi(v_3, v_4)) + \epsilon(v_1, v_2 + v_3) B_{\tilde{\mathfrak{g}}}(\psi(v_2, v_3), \psi(v_1, v_4)) \\ & - \epsilon(v_3, v_4)\epsilon(v_1, v_2 + v_4) B_{\tilde{\mathfrak{g}}}(\psi(v_2, v_4), \psi(v_1, v_3)) \\ & - \epsilon(v_2, v_3) B_{\tilde{\mathfrak{g}}}(\psi(v_1, v_3), \psi(v_2, v_4)) \\ & + \epsilon(v_1 + v_2, v_3 + v_4) B_{\tilde{\mathfrak{g}}}(\psi(v_3, v_4), \psi(v_1, v_2)) \\ & + \epsilon(v_2 + v_3, v_4) B_{\tilde{\mathfrak{g}}}(\psi(v_1, v_4), \psi(v_2, v_3)) \\ =& 2\Big(B_{\tilde{\mathfrak{g}}}(\psi(v_1, v_2), \psi(v_3, v_4)) + \epsilon(v_1, v_2 + v_3) B_{\tilde{\mathfrak{g}}}(\psi(v_2, v_3), \psi(v_1, v_4)) \\ & - \epsilon(v_2, v_3) B_{\tilde{\mathfrak{g}}}(\psi(v_1, v_3), \psi(v_2, v_4)) \Big). \end{aligned}$$

Moreover, since

$$\begin{aligned} B_{\tilde{\mathfrak{g}}}(\psi(v_1, v_2), \psi(v_3, v_4)) &= B_{\mathfrak{g}}(\mu(v_1, v_2), \mu(v_3, v_4)) + (\phi(v_1, v_2), \phi(v_3, v_4)) \\ &= (\mu(v_1, v_2)(v_3), v_4) + \epsilon(v_1 + v_2, v_3 + v_4)(\phi(v_3, v_4), \phi(v_1, v_2)) \\ &= (\mu(v_1, v_2)(v_3), v_4) - \epsilon(v_3, v_4)\epsilon(v_1 + v_2, v_3 + v_4)(v_4, \phi(v_3, \phi(v_1, v_2))) \\ &= (\mu(v_1, v_2)(v_3), v_4) + (\phi(\phi(v_1, v_2), v_3), v_4) \\ &= (\{\{v_1, v_2\}, v_3\}, v_4) \end{aligned}$$

then we have

$$\begin{aligned} N_{B_{\tilde{\mathfrak{g}}}}(\psi)(v_1, v_2, v_3, v_4) =& 2\Big(\{\{v_1, v_2\}, v_3\} + \epsilon(v_1, v_2 + v_3)\{\{v_2, v_3\}, v_1\} - \epsilon(v_2, v_3)\{\{v_1, v_3\}, v_2\}, v_4 \Big) \\ =& 2\epsilon(v_1, v_3)\Big(\epsilon(v_3, v_1)\{\{v_1, v_2\}, v_3\} + \epsilon(v_1, v_2)\{\{v_2, v_3\}, v_1\} + \epsilon(v_2, v_3)\{\{v_3, v_1\}, v_2\}, v_4 \Big) \\ =& 2\epsilon(v_1, v_3)\Big(J(v_1, v_2, v_3), v_4 \Big). \end{aligned}$$

As pointed out above $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus V$ is a colour Lie algebra if and only if $(J(v_1, v_2, v_3), v_4) = 0$ for all v_1, v_2, v_3, v_4 in V and so this is equivalent to

$$N_{B_{\tilde{\mathfrak{g}}}}(\mu + \phi) = 0.$$

This proves that $a)$ is equivalent to $b)$. Note that since V is orthogonal to $\mathfrak{g}$, we have

$$\mu \wedge_{B_{\tilde{\mathfrak{g}}}} \phi = 0,$$

hence

$$N_{B_{\tilde{\mathfrak{g}}}}(\mu + \phi) = N_{B_{\tilde{\mathfrak{g}}}}(\mu) + N_{B_{\tilde{\mathfrak{g}}}}(\phi)$$

and so $b)$ is equivalent to $c)$. □

Corollary 4.2.11. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$ and let $\mu : \Lambda^2_\epsilon(V) \to \mathfrak{g}$ be its moment map. Let $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus V$, let $B_{\tilde{\mathfrak{g}}} := B_\mathfrak{g} \perp (\ ,\ )$ and let $\{\ ,\ \} : \Lambda^2_\epsilon(\tilde{\mathfrak{g}}) \to \tilde{\mathfrak{g}}$ be the unique ϵ-antisymmetric bilinear map which extends the bracket of $\mathfrak{g}$, the action of $\mathfrak{g}$ on V and such that*

$$\{v, w\} = \mu(v, w) \qquad \forall v, w \in V.$$

Then the following are equivalent :

a) $(\tilde{\mathfrak{g}}, B_{\tilde{\mathfrak{g}}}, \{\ ,\ \})$ is an ϵ-quadratic colour Lie algebra.

b) $N_{B_{\tilde{\mathfrak{g}}}}(\mu) = 0$.

c) V is of colour $\mathbb{Z}_2$-Lie type.

Remark 4.2.12. *The invariants of the previous theorem and corollary take values in the ϵ-exterior algebra. After identifying the ϵ-exterior algebra with the ϵ-Clifford algebra (see Theorem 3.4.6) one can show that these results contain and generalise the results of Kostant (see Section 4.1) on orthogonal and symplectic complex representations of quadratic complex Lie algebras. They also generalise the results of Z. Chen and Y. Kang (see [CK15]) on orthosymplectic complex representations of supersymmetric complex Lie superalgebras.*

4.3 The Bianchi map and special ϵ-orthogonal representations

In the previous section we saw (Theorem 4.2.10) that one can associate a colour Lie algebra $\tilde{\mathfrak{g}}$ to certain data on an ϵ-orthogonal representation V of an ϵ-quadratic colour Lie algebra if a particular invariant of this data vanishes. In this section we will give a different interpretation of the data as an element of a space of "curvature tensors" on V. We show that the vanishing of the above invariant is equivalent to an algebraic "Bianchi identity" for the corresponding curvature tensor. Further analysis of this identity makes it clear that there are other natural conditions we can impose on ϵ-orthogonal representations and this leads to the notion of special ϵ-orthogonal representations.

Definition 4.3.1. *Let V be a Γ-graded vector space. We define $\mathcal{R}(V)$ to be the vector space of all multilinear maps $R : V \times V \times V \times V \to k$ which satisfy*

$$R(A, B, C, D) = -\epsilon(A, B)R(B, A, C, D) \qquad \forall A, B, C, D \in V, \tag{4.3}$$
$$R(A, B, C, D) = \epsilon(A + B, C + D)R(C, D, A, B) \qquad \forall A, B, C, D \in V. \tag{4.4}$$

Remark 4.3.2. *A map $R \in \mathcal{R}(V)$ satisfies*

$$R(A, B, C, D) = -\epsilon(C, D)R(A, B, D, C) \qquad \forall A, B, C, D \in V. \tag{4.5}$$

In general, for a Γ-graded vector space V, a map $R \in \mathcal{R}(V)$ is not ϵ-alternating but we can define $\beta : \mathcal{R}(V) \to \mathcal{R}(V)$, called the Bianchi map, with the property that $\beta(R) \in Alt^4_\epsilon(V)$ for all $R \in \mathcal{R}(V)$.

Definition 4.3.3. *Let V be a Γ-graded vector space.*

a) The Bianchi map $\beta : \mathcal{R}(V) \to \mathcal{R}(V)$ is defined by

$$\beta(R)(A, B, C, D) := R(A, B, C, D)+\epsilon(A, B+C)R(B, C, A, D)+\epsilon(A+B, C)R(C, A, B, D)$$

for all $A, B, C, D \in V$.

b) The vector space $Ker(\beta)$ is called the space of formal curvature tensors of V.

Proposition 4.3.4. *Let V be a Γ-graded vector space and let $R \in \mathcal{R}(V)$. Then :*

a) $\beta(R) \in Alt^4_\epsilon(V)$.

b) $R \in Alt^4_\epsilon(V)$ if and only if $\beta(R) = 3R$.

c) $\mathcal{R}(V) = Alt^4_\epsilon(V) \oplus Ker(\beta)$.

Proof. a) By Remark 3.6.2 the map $\beta(R)$ is an alternating multilinear map if and only if

$$\beta(R)(A, B, C, D) = -\epsilon(A, B)\beta(R)(B, A, C, D),$$
$$\beta(R)(A, B, C, D) = -\epsilon(B, C)\beta(R)(A, C, B, D),$$
$$\beta(R)(A, B, C, D) = -\epsilon(C, D)\beta(R)(A, B, D, C)$$

for all $A, B, C, D \in V$. These three identities can be shown by straightforward calculation.

b) If $\beta(R) = 3R$, we have $R \in Alt^4_\epsilon(V)$ by $a)$. Conversely, if $R \in Alt^4_\epsilon(V)$, then for all $A, B, C, D \in V$ we have

$$\epsilon(A, B + C)R(B, C, A, D) = R(A, B, C, D),$$
$$\epsilon(A + B, C)R(C, A, B, D) = R(A, B, C, D),$$

and so

$$\begin{aligned}\beta(R)(A, B, C, D) =& R(A, B, C, D) + \epsilon(A, B + C)R(B, C, A, D) + \epsilon(A + B, C)R(C, A, B, D)\\ =& 3R(A, B, C, D).\end{aligned}$$

c) Follows from $b)$.

□

Remark 4.3.5. *Let V be a Γ-graded vector space and let $R \in \mathcal{R}(V)$. With respect to the decomposition $\mathcal{R}(V) = Alt^4_\epsilon(V) \oplus Ker(\beta)$ of the previous proposition, the component of R in $Alt^4_\epsilon(V)$ is $\frac{1}{3}\beta(R)$ and its component in $Ker(\beta)$ is $R - \frac{1}{3}\beta(R)$.*

We now show that, to the data of Theorem 4.2.10 one can associate an element of $\mathcal{R}(V)$ and interpret the vanishing condition in terms of the Bianchi map β introduced above.

Definition 4.3.6. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$, let $\mu : \Lambda^2_\epsilon(V) \to \mathfrak{g}$ be its moment map and let $\phi : \Lambda^2_\epsilon(V) \to V$ be a map of degree 0 which satisfies Properties (4.1) and (4.2). Let $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus V$, let $B_{\tilde{\mathfrak{g}}} := B_\mathfrak{g} \perp (\ ,\ )$ and let $\{\ ,\ \} : \Lambda^2_\epsilon(\tilde{\mathfrak{g}}) \to \tilde{\mathfrak{g}}$ be the unique ϵ-antisymmetric bilinear map which extends the bracket of $\mathfrak{g}$, the action of $\mathfrak{g}$ on V and such that*

$$\{v, w\} = \mu(v, w) + \phi(v, w) \qquad \forall v, w \in V.$$

Define the multilinear map $R_\psi : V \times V \times V \times V \to k$ by

$$R_\psi(A, B, C, D) := B_{\tilde{\mathfrak{g}}}(\{\{A, B\}, C\}, D) \qquad \forall A, B, C, D \in V.$$

If $\phi = 0$ we denote R_ψ by R_μ.

Proposition 4.3.7. *With the notation above, we have :*

a) The map R_ψ is an element of $\mathcal{R}(V)$.

b) We have $N_{B_{\tilde{\mathfrak{g}}}}(\psi) = 2\beta(R_\psi)$.

Proof. a) Since the bracket is ϵ-antisymmetric, Equation (4.3) is satisfied. For $A, B, C, D \in V$, we also have

$$\begin{aligned} R_\psi(A,B,C,D) &= (\mu(A,B)(C),D) + (\phi(\phi(A,B),C),D) \\ &= B_{\mathfrak{g}}(\mu(A,B),\mu(C,D)) + (\phi(A,B),\phi(C,D)) \\ &= \epsilon(A+B,C+D)\Big(B_{\mathfrak{g}}(\mu(C,D),\mu(A,B)) + (\phi(C,D),\phi(A,B))\Big) \\ &= \epsilon(A+B,C+D)\Big((\mu(C,D)(A),B) + (\phi(\phi(C,D),A),B)\Big) \\ &= \epsilon(A+B,C+D)R_\psi(C,D,A,B) \end{aligned}$$

and then Equation (4.4) is satisfied.

b) For $A, B, C, D \in V$, we have

$$\begin{aligned} \beta(R_\psi)(A,B,C,D) &= R_\psi(A,B,C,D) + \epsilon(A,B+C)R_\psi(B,C,A,D) + \epsilon(A+B,C)R_\psi(C,A,B,D) \\ &= \Big(\{\{A,B\},C\} + \epsilon(A,B+C)\{\{B,C\},A\} + \epsilon(A+B,C)\{\{C,A\},B\},D\Big) \\ &= \epsilon(A,C)\Big(\epsilon(C,A)\{\{A,B\},C\} + \epsilon(A,B)\{\{B,C\},A\} + \epsilon(B,C)\{\{C,A\},B\},D\Big) \\ &= \epsilon(A,C)(J(A,B,C),D). \end{aligned}$$

As we have seen in the proof of the theorem 4.2.10 we have

$$N_{B_{\tilde{\mathfrak{g}}}}(\psi)(A,B,C,D) = 2\epsilon(A,C)(J(A,B,C),D)$$

and so $N_{B_{\tilde{\mathfrak{g}}}}(\psi) = 2\beta(R_\psi)$.

□

Corollary 4.3.8. *With the notation above the following are equivalent :*

a) $(\tilde{\mathfrak{g}}, B_{\tilde{\mathfrak{g}}}, \{\ ,\ \})$ *is an ϵ-quadratic colour Lie algebra.*

b) $\beta(R_\psi) = 0$.

Apart from the zero map, the space of formal curvature tensors $Ker(\beta)$ has another canonical element : $R_{\mu_{can}}$. Since for $R \in \mathcal{R}(V)$, its projection in $Ker(\beta)$ is $R - \frac{1}{3}\beta(R)$ (see Remark 4.3.5), it is natural to ask which ϵ-quadratic representations have curvature R_μ such that $R_\mu - \frac{1}{3}\beta(R_\mu)$ is equal to $R_{\mu_{can}}$.

Definition 4.3.9. *A special ϵ-orthogonal representation is a finite-dimensional ϵ-orthogonal representation $\mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ of a finite-dimensional ϵ-quadratic colour Lie algebra $\mathfrak{g}$ satisfying*

$$R_\mu - \frac{1}{3}\beta(R_\mu) = R_{\mu_{can}}.$$

We now characterise special ϵ-orthogonal representations in terms of their moment map.

Proposition 4.3.10. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be an ϵ-orthogonal representation. The following four properties are equivalent :*

a) $(V, (\ ,\ ))$ *is a special ϵ-orthogonal representation.*

b) $R_\mu - \frac{1}{3}\beta(R_\mu) = R_{\mu_{can}}$.

c) $\mu(A, B)(C) + \epsilon(B, C)\mu(A, C)(B) = (A, B)C + \epsilon(B, C)(A, C)B - 2(B, C)A$ $\forall A, B, C \in V$.

d) $\mu(A, B)(C) + \epsilon(B, C)\mu(A, C)(B) = \mu_{can}(A, B)(C) + \epsilon(B, C)\mu_{can}(A, C)(B)$ $\forall A, B, C \in V$.

Proof. Property c) is equivalent to d) by Equation (3.17).

Let $A, B, C, D \in V$. We now show that c) implies b). Expanding the left-hand side of b), we have :

$$
\begin{aligned}
&R_\mu(A, B, C, D) - \frac{1}{3}\beta(R_\mu)(A, B, C, D) \\
&= \frac{1}{3}\Big(2\mu(A, B)(C) - \epsilon(A, B + C)\mu(B, C)(A) + \epsilon(B, C)\mu(A, C)(B), D\Big) \\
&= \frac{1}{3}\Big(\mu(A, B)(C) + \epsilon(B, C)\mu(A, C)(B) - \epsilon(A, B)(\mu(B, A)(C) + \epsilon(A, C)\mu(B, C)(A)), D\Big).
\end{aligned} \tag{4.6}
$$

Now, using c) twice, we have

$$
\begin{aligned}
\mu(A, B)(C) + \epsilon(B, C)\mu(A, C)(B) &= (A, B)C + \epsilon(B, C)(A, C)B - 2(B, C)A, \\
\mu(B, A)(C) + \epsilon(A, C)\mu(B, C)(A) &= (B, A)C + \epsilon(A, C)(B, C)A - 2(A, C)B
\end{aligned}
$$

and hence, substituting in Equation (4.6)

$$
\begin{aligned}
&R_\mu(A, B, C, D) - \frac{1}{3}\beta(R_\mu)(A, B, C, D) \\
&= \frac{1}{3}\Big((A, B)C + \epsilon(B, C)(A, C)B - 2(B, C)A - \epsilon(A, B)((B, A)C + \epsilon(A, C)(B, C)A - 2(A, C)B), D\Big) \\
&= \Big(\epsilon(B, C)(A, C)B - (B, C)A, D\Big) \\
&= \Big(\mu_{can}(A, B)(C), D\Big) \\
&= R_{\mu_{can}}(A, B, C, D).
\end{aligned}
$$

This proves b). Conversely, if $R_\mu - \frac{1}{3}\beta(R_\mu) = R_{\mu_{can}}$, we have

$$\begin{aligned}
&\Big((A,B)C + \epsilon(B,C)(A,C)B - 2(B,C)A, D\Big) \\
&= \Big(\mu_{can}(A,B)(C) + \epsilon(B,C)\mu_{can}(A,C)(B), D\Big) \\
&= R_{\mu_{can}}(A,B,C,D) + \epsilon(B,C)R_{\mu_{can}}(A,C,B,D) \\
&= \Big(R_\mu - \frac{1}{3}\beta(R_\mu)\Big)(A,B,C,D) + \epsilon(B,C)\Big(R_\mu - \frac{1}{3}\beta(R_\mu)\Big)(A,C,B,D) \\
&= R_\mu(A,B,C,D) + \epsilon(B,C)R_\mu(A,C,B,D) \\
&= \Big(\mu(A,B)(C) + \epsilon(B,C)\mu(A,C)(B), D\Big),
\end{aligned}$$

and since $(\ , \ )$ is non-degenerate, we obtain

$$\mu(A,B)(C) + \epsilon(B,C)\mu(A,C)(B) = (A,B)C + \epsilon(B,C)(A,C)B - 2(B,C)A.$$

□

Example 4.3.11. *Let $\mathfrak{g}$ be a Lie algebra and let $(V, (\ , \ ))$ be a symplectic representation of $\mathfrak{g}$. Then it follows from Proposition 4.3.10 that $\mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ , \ )) = \mathfrak{sp}(V, (\ , \ ))$ is a special ϵ-orthogonal representation if and only if it is a special symplectic representation in the sense of [SS15].*

4.4 When are tensor products of ϵ-orthogonal representations of colour Lie type ?

In this section we investigate the question of when the tensor product of two ϵ-orthogonal representations is of colour Lie type if one of them, W, is the fundamental representation of an ϵ-orthogonal colour Lie algebra. It turns out that a necessary and sufficient condition for this to be the case is that the other representation, V, is also the fundamental representation of an ϵ-orthogonal colour Lie algebra unless $dim(W) = 1$ or $dim(W) = 2$. If $dim(W) = 1$, the necessary and sufficient condition is that V is of colour Lie type and if $dim(W) = 2$, that V is a special ϵ-orthogonal representation as definied in Section 4.3.

Let $(\mathfrak{g}, B_\mathfrak{g})$ and $(\mathfrak{h}, B_\mathfrak{h})$ be finite-dimensional ϵ-quadratic colour Lie algebras with respect to (Γ, ϵ) and let

$$\rho_\mathfrak{g} : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ , \ )_V), \qquad \rho_\mathfrak{h} : \mathfrak{h} \to \mathfrak{so}_\epsilon(W, (\ , \ )_W)$$

be finite-dimensional ϵ-orthogonal representations of $\mathfrak{g}$ and $\mathfrak{h}$. As we have seen in Section 3.9.3 we have an ϵ-orthogonal representation

$$\rho : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{so}_\epsilon(V \otimes W, (\ , \ )_{V\otimes W})$$

of the ϵ-quadratic colour Lie algebra $(\mathfrak{g} \oplus \mathfrak{h}, B_{\mathfrak{g}} \perp B_{\mathfrak{h}})$. Its moment map $\mu_{V\otimes W} : \Lambda^2_\epsilon(V \otimes W) \to \mathfrak{g} \oplus \mathfrak{h}$ satisfies

$$\mu_{V\otimes W}(v \otimes w, v' \otimes w') = \epsilon(w, v')\Big(\mu_V(v, v')(w, w')_W + \mu_W(w, w')(v, v')_V\Big)$$

for all $v \otimes w, v' \otimes w' \in V \otimes W$, where $\mu_V : \Lambda^2_\epsilon(V) \to \mathfrak{g}$ (resp. $\mu_W : \Lambda^2_\epsilon(W) \to \mathfrak{h}$) is the moment map of the representation $\rho_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ (resp. $\rho_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{so}_\epsilon(W, (\ ,\ )_W)$). Hence we define two maps $\tilde{\mu}_V : \Lambda^2_\epsilon(V \otimes W) \to \mathfrak{g}$ and $\tilde{\mu}_W : \Lambda^2_\epsilon(V \otimes W) \to \mathfrak{h}$ by

$$\begin{aligned} \tilde{\mu}_V(v \otimes w, v' \otimes w') &:= \epsilon(w, v')\mu_V(v, v')(w, w')_W \qquad \forall v \otimes w, v' \otimes w' \in V \otimes W, \\ \tilde{\mu}_W(v \otimes w, v' \otimes w') &:= \epsilon(w, v')\mu_W(w, w')(v, v')_V \qquad \forall v \otimes w, v' \otimes w' \in V \otimes W. \end{aligned}$$

Proposition 4.4.1. *Let $(\mathfrak{g}, B_{\mathfrak{g}})$ and $(\mathfrak{h}, B_{\mathfrak{h}})$ be finite-dimensional ϵ-quadratic colour Lie algebras with respect to (Γ, ϵ) and let*

$$\rho_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V), \qquad \rho_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{so}_\epsilon(W, (\ ,\ )_W)$$

be finite-dimensional ϵ-orthogonal representations of $\mathfrak{g}$ and $\mathfrak{h}$. The following are equivalent

a) $\rho : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{so}_\epsilon(V \otimes W, (\ ,\ )_{V\otimes W})$ *is of colour $\mathbb{Z}_2$-Lie type.*

b) $N_{B_{\mathfrak{g}}}(\tilde{\mu}_V) = -N_{B_{\mathfrak{h}}}(\tilde{\mu}_W)$.

Proof. Since $\mathfrak{g}$ is orthogonal to $\mathfrak{h}$ we obtain

$$N_{B_{\mathfrak{g}} \perp B_{\mathfrak{h}}}(\mu_{V\otimes W}) = N_{B_{\mathfrak{g}} \perp B_{\mathfrak{h}}}(\tilde{\mu}_V + \tilde{\mu}_W) = N_{B_{\mathfrak{g}}}(\tilde{\mu}_V) + N_{B_{\mathfrak{h}}}(\tilde{\mu}_W)$$

and by Theorem 4.2.10, the representation $\rho : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{so}_\epsilon(V \otimes W, (\ ,\ )_{V\otimes W})$ is of colour $\mathbb{Z}_2$-Lie type if and only if $N_{B_{\mathfrak{g}} \perp B_{\mathfrak{h}}}(\mu_{V\otimes W}) = 0$. □

We now study the representations $\rho : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{so}_\epsilon(V \otimes W, (\ ,\ )_{V\otimes W})$ of colour $\mathbb{Z}_2$-Lie type when $\mathfrak{h} = \mathfrak{so}_\epsilon(W, (\ ,\ )_W)$ and $\rho_{\mathfrak{h}}$ is the fundamental representation of $\mathfrak{so}_\epsilon(W, (\ ,\ )_W)$.

Proposition 4.4.2. *Suppose that there exist three homogeneous non-zero elements w, w', w'' in W such that*

$$(w, w')_W \neq 0, \qquad (w, w'')_W = 0, \qquad (w', w'')_W = 0.$$

Suppose that the representation $\rho : \mathfrak{g} \oplus \mathfrak{so}_\epsilon(W, (\ ,\ )_W) \to \mathfrak{so}_\epsilon(V \otimes W, (\ ,\ )_{V\otimes W})$ of the ϵ-quadratic colour Lie algebra $(\mathfrak{g} \oplus \mathfrak{so}_\epsilon(W, (\ ,\ )_W), B_{\mathfrak{g}} \perp B_{\mathfrak{h}})$ where $B_{\mathfrak{h}}(f, g) = -\frac{1}{2}Tr_\epsilon(fg)$ for all $f, g \in \mathfrak{so}_\epsilon(W, (\ ,\ )_W)$ is of colour $\mathbb{Z}_2$-Lie type, then

$$\mu_V(v, v')(v'') = \mu_{can}(v, v')(v'') \qquad \forall v, v', v'' \in V.$$

Proof. Let $v, v', v'' \in V$. We now compute the ϵ-Jacobi identity for the three elements $v \otimes w, v' \otimes w', v'' \otimes w''$ using Properties (3.18), (3.19) and the moment map of W given by formula (3.17).

The first term of this identity is

$$\begin{aligned}
&\epsilon(v''+w'', v+w)\mu_{V\otimes W}(v\otimes w, v'\otimes w')(v''\otimes w'') \\
&= \epsilon(v''+w'', v+w)\epsilon(w, v')\mu_V(v, v')(v'')\otimes(w, w')_W w'' \\
&= \epsilon(v'', v)\epsilon(v'', w)\epsilon(w'', v)\epsilon(w'', w)\epsilon(w, v')\mu_V(v, v')(v'')\otimes(w, w')_W w'',
\end{aligned} \tag{4.7}$$

the second term is

$$\begin{aligned}
&\epsilon(v+w, v'+w')\mu_{V\otimes W}(v'\otimes w', v''\otimes w'')(v\otimes w) \\
&= \epsilon(v+w, v'-w)\epsilon(-w, v'')\epsilon(-w+w'', v)(v', v'')_V v'\otimes \epsilon(w'', w)\epsilon(w, w)(w, w')_W w'' \\
&= \epsilon(v, v')\epsilon(w, v')\epsilon(v'', w)\epsilon(w'', v)\epsilon(w'', w)(v', v'')_V v'\otimes(w, w')_W w''
\end{aligned} \tag{4.8}$$

and the last term is

$$\begin{aligned}
&\epsilon(v'+w', v''+w'')\mu_{V\otimes W}(v''\otimes w'', v\otimes w)(v'\otimes w') \\
&= \epsilon(v'-w, v''+w'')\epsilon(w'', v)\epsilon(w''+w, v')(v'', v)_V v'\otimes -(w, w')_W w'' \\
&= -\epsilon(v', v'')\epsilon(v'', w)\epsilon(w'', w)\epsilon(w'', v)\epsilon(w, v')(v'', v)_V v'\otimes(w, w')_W w''.
\end{aligned} \tag{4.9}$$

Taking the sum of Equations (4.7), (4.8) and (4.9) we see that the ϵ-Jacobi identity for the three elements $v \otimes w, v' \otimes w', v'' \otimes w''$ is satisfied if and only if

$$\epsilon(v'', v)\mu_V(v, v')(v'') + \epsilon(v, v')(v', v'')_V v' - \epsilon(v', v'')(v'', v)_V v' = 0,$$

and by (3.17) this is equivalent to

$$\mu_V(v, v')(v'') = \mu_{can}(v, v')(v'').$$

□

Corollary 4.4.3. *With the hypotheses of Proposition 4.4.2, if $\rho_{\mathfrak{g}}$ is injective, then $\rho : \mathfrak{g} \oplus \mathfrak{so}_\epsilon(W, (\ ,\ )_W) \to \mathfrak{so}_\epsilon(V \otimes W, (\ ,\ )_{V\otimes W})$ is of colour $\mathbb{Z}_2$-Lie type if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{so}_\epsilon(V, (\ ,\ )_V)$.*

Proof. See Proposition 3.9.9. □

We now show that there are very few ϵ-orthogonal spaces which do not satisfy the hypotheses of Proposition 4.4.2.

Proposition 4.4.4. *Let W be a finite-dimensional Γ-graded vector space and let $(\ ,\ )_W$ be a non-degenerate ϵ-symmetric bilinear form on W. Recall that we have a decomposition $W = W_0 \oplus W_1$ where*

$$W_0 = \{w \in W \mid \epsilon(w,w) = 1\}, \qquad W_1 = \{w \in W \mid \epsilon(w,w) = -1\}.$$

Suppose that $dim(W_0) + \frac{dim(W_1)}{2} \geq 2$. Then there exists $w, w', w'' \in W$ such that

$$(w,w')_W \neq 0, \qquad (w,w'')_W = 0, \qquad (w',w'')_W = 0.$$

Thus, the only possibilities to have an ϵ-orthogonal representation

$$\rho : \mathfrak{g} \oplus \mathfrak{so}_\epsilon(W, (\ ,\ )_W) \to \mathfrak{so}_\epsilon(V \otimes W, (\ ,\ )_{V\otimes W})$$

of colour $\mathbb{Z}_2$-Lie type such that $\mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ is not the fundamental representation of $\mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ are $W = W_0$ and $dim(W) = 1$ or $W = W_1$ and $dim(W) = 2$.

Remark 4.4.5. *Let W be a Γ-graded vector space such that $W = W_0$ and $dim(W) = 1$ and let $(\ ,\ )_W$ be a non-degenerate ϵ-symmetric bilinear form on W. Then we have $\mathfrak{so}_\epsilon(W, (\ ,\ )_W) = \{0\}$. Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$. Then, the representation*

$$\mathfrak{g} \oplus \mathfrak{so}_\epsilon(W, (\ ,\ )_W) \to \mathfrak{so}_\epsilon(V \otimes W, (\ ,\ )_{V\otimes W})$$

is of colour $\mathbb{Z}_2$-Lie type if and only if the representation $\mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ is of colour $\mathbb{Z}_2$-Lie type.

Remark 4.4.6. *Let W be a Γ-graded vector space such that $W = W_1$ and $dim(W) = 2$ and let $(\ ,\ )_W$ be a non-degenerate ϵ-symmetric bilinear form on W. Let $\{p, q\}$ be a homogeneous basis of W such that $(p,q)_W = 1$. Then $\mathfrak{so}_\epsilon(W, (\ ,\ )_W)$ is isomorphic to $\mathfrak{sl}(2,k)$. Let $\{E, H, F\}$ be the standard $\mathfrak{sl}(2,k)$-triple, we have*

$$|E| = 2|p|, \qquad |H| = 0, \qquad |F| = -2|p|.$$

Theorem 4.4.7. *Let $\mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$ with respect to (Γ, ϵ) and suppose that $\mathfrak{sl}(2,k) \to \mathfrak{sp}(k^2, \omega)$ is an ϵ-orthogonal representation with respect to (Γ, ϵ). Then the representation*

$$\mathfrak{g} \oplus \mathfrak{sl}(2,k) \to \mathfrak{so}_\epsilon(V \otimes k^2, (\ ,\ )_V \otimes \omega)$$

is of colour $\mathbb{Z}_2$-Lie type if and only the representation $\mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ )_V)$ is special ϵ-orthogonal.

Proof. Let $\{p, q\}$ be a homogeneous symplectic basis of (k^2, ω). From Remark 4.4.6 there exists $\gamma \in \Gamma$ such that $\epsilon(\gamma, \gamma) = -1$, $|p| = \gamma$ and $|q| = -\gamma$.

Recall from Example 3.9.7 that

$$\mu_{k^2}(p, p) = -2E, \quad \mu_{k^2}(q, q) = 2F, \quad \mu_{k^2}(p, q) = H.$$

Let $v, v', v'' \in V$. We first see that the ϵ-Jacobi identity of $v \otimes p, v' \otimes p, v'' \otimes p$ and the ϵ-Jacobi identity of $v \otimes q, v' \otimes q, v'' \otimes q$ are satisfied. Furthermore, after calculation, the ϵ-Jacobi identity of $v \otimes p, v' \otimes q, v'' \otimes q$ is satisfied if and only if the ϵ-Jacobi identity of $v \otimes p, v' \otimes p, v'' \otimes q$ is satisfied.

We now compute the Jacobi tensor $J(v \otimes p, v' \otimes p, v'' \otimes q)$. The first term of $J(v \otimes p, v' \otimes p, v'' \otimes q)$ is

$$\epsilon(v'' + \gamma, v + \gamma)\mu_{V \otimes k^2}(v \otimes p, v' \otimes p)(v'' \otimes q) = 2\epsilon(v'', v)\epsilon(\gamma, v'')\epsilon(v, \gamma)^2(v, v')_V v'' \otimes p.$$

The second term of $J(v \otimes p, v' \otimes p, v'' \otimes q)$ is

$$\epsilon(v+\gamma, v'+\gamma)\mu_{V\otimes k^2}(v'\otimes p, v''\otimes q)(v\otimes p) = \epsilon(v+\gamma, v'+\gamma)\epsilon(\gamma, v'')\Big(\mu_V(v', v'')(v)\otimes p+(v', v'')_V v\otimes p\Big)$$

and the last term of $J(v \otimes p, v' \otimes p, v'' \otimes q)$ is

$$\epsilon(v'+\gamma, v''+\gamma)\mu_{V\otimes k^2}(v''\otimes q, v\otimes p)(v'\otimes p) = \epsilon(v'+\gamma, v''-\gamma)\epsilon(v, \gamma)\Big(-\mu_V(v'', v)(v')\otimes p+(v'', v)_V v'\otimes p\Big).$$

Hence, summing these three terms, $J(v \otimes p, v' \otimes p, v'' \otimes q) = 0$ if and only if we have

$$2\epsilon(v'', v)\epsilon(\gamma, v'')\epsilon(v, \gamma)^2(v, v')_V v''+\epsilon(v+\gamma, v'+\gamma)(v', v'')_V v+\epsilon(v'+\gamma, v''-\gamma)\epsilon(v, \gamma)(v'', v)_V v'$$

$$= -\epsilon(v + \gamma, v' + \gamma)\epsilon(\gamma, v'')\mu_V(v', v'')(v) + \epsilon(v' + \gamma, v'' - \gamma)\epsilon(v, \gamma)\mu_V(v'', v)(v)$$

and now if we multiply both sides by $\epsilon(v'' - \gamma, v' + \gamma)\epsilon(\gamma, v)$, this is equivalent to

$$-2(v, v')_V v'' + \epsilon(v, v')(v'', v')_V v + (v'', v)_V v' = -\epsilon(v + v'', v')\mu_V(v', v'')(v) + \mu_V(v'', v)(v')$$

and this is equivalent to

$$\mu_V(v, v')(v'') + \epsilon(v', v'')\mu_V(v, v'')(v') = (v, v')_V v'' + \epsilon(v', v'')(v, v'')_V v' - 2(v', v'')_V v. \quad (4.10)$$

□

4.5 Mathews identities for the covariants of a special ϵ-orthogonal representation

In this section we introduce and study a bilinear, a trilinear and a quadrilinear ϵ-alternating multilinear map associated to a special ϵ-orthogonal representation. These maps generalise the three classical covariants of the space of binary cubics (see [Eis44]) which is a special ϵ-orthogonal representation of the Lie algebra $\mathfrak{sl}(2,k)$. We prove a set of identities satisfied by them which generalise the Mathews identities for binary cubics (see [Mat11]) and their analogues for special symplectic representations of Lie algebras (see [SS15]).

Definition 4.5.1. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$ and let $\mu : \Lambda^2_\epsilon(V) \to \mathfrak{g}$ be its moment map. We define the multilinear maps $\psi : V \times V \times V \to V$, $B_\psi : V \times V \times V \to V$ and $B_Q : V \times V \times V \times V \to k$ as follows. For all $v_1, v_2, v_3, v_4 \in V$:*

$$\begin{aligned}
\psi(v_1, v_2, v_3) &= \mu(v_1, v_2)(v_3),\\
B_\psi(v_1, v_2, v_3) &= \psi(v_1, v_2, v_3) + \epsilon(v_1 + v_2, v_3)\psi(v_3, v_1, v_2) + \epsilon(v_1, v_2 + v_3)\psi(v_2, v_3, v_1),\\
B_Q(v_1, v_2, v_3, v_4) &= (v_1, B_\psi(v_2, v_3, v_4)) - \epsilon(v_1 + v_2 + v_3, v_4)(v_4, B_\psi(v_1, v_2, v_3))\\
&\quad + \epsilon(v_1 + v_2, v_3 + v_4)(v_3, B_\psi(v_4, v_1, v_2)) - \epsilon(v_1, v_2 + v_3 + v_4)(v_2, B_\psi(v_3, v_4, v_1)).
\end{aligned}$$

The maps μ, B_ψ and B_Q are called the covariants of the ϵ-orthogonal representation ρ.

In fact, the maps μ, B_ψ and B_Q are ϵ-alternating mutilinear maps.

Proposition 4.5.2. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$ and let $\mu : \Lambda^2_\epsilon(V) \to \mathfrak{g}$ be its moment map. We have $\mu \in Alt^2_\epsilon(V, \mathfrak{g})$, $B_\psi \in Alt^3_\epsilon(V, V)$ and $B_Q \in Alt^4_\epsilon(V)$.*

Proof. Let $v_1, v_2, v_3, v_4 \in V$. We have

$$B_\psi(v_1, v_2, v_3) = \frac{1}{2} \sum_{\sigma \in S_3} p(\sigma; v_1, v_2, v_3)\psi(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}), \tag{4.11}$$

$$B_Q(v_1, v_2, v_3, v_4) = \frac{1}{2} \sum_{\sigma \in S_4} p(\sigma; v_1, v_2, v_3, v_4)(v_{\sigma(1)}, \psi(v_{\sigma(2)}, v_{\sigma(3)}, v_{\sigma(4)})) \tag{4.12}$$

and so B_ψ and B_Q are ϵ-alternating multilinear maps. □

Proposition 4.5.3. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional special ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$. For all $v_1, v_2, v_3, v_4 \in V$, we have*

$$\begin{aligned}
B_\psi(v_1, v_2, v_3) &= 3(\psi(v_1, v_2, v_3) - \mu_{can}(v_1, v_2)(v_3)),\\
B_Q(v_1, v_2, v_3, v_4) &= 4(v_1, B_\psi(v_2, v_3, v_4)).
\end{aligned}$$

Proof. We prove the first identity. Using Equation (c)), we have

$$\begin{aligned}\epsilon(v_1+v_2,v_3)\psi(v_3,v_1,v_2) &= -\epsilon(v_2,v_3)\psi(v_1,v_3,v_2)\\ &= \psi(v_1,v_2,v_3)-\epsilon(v_2,v_3)(v_1,v_3)v_2-(v_1,v_2)v_3+2\epsilon(v_2,v_3)(v_3,v_2)v_1\end{aligned}$$

and

$$\begin{aligned}\epsilon(v_1,v_2+v_3)\psi(v_2,v_3,v_1) =& -\epsilon(v_1,v_2)\psi(v_2,v_1,v_3)+(v_2,v_3)v_1\\ &+\epsilon(v_1,v_2)(v_2,v_1)v_3-2\epsilon(v_1,v_2+v_3)(v_3,v_1)v_2.\end{aligned}$$

Hence,

$$\begin{aligned}B_\psi(v_1,v_2,v_3) =&3\psi(v_1,v_2,v_3)-\epsilon(v_2,v_3)(v_1,v_3)v_2-(v_1,v_2)v_3+2\epsilon(v_2,v_3)(v_3,v_2)v_1+(v_2,v_3)v_1\\ &+\epsilon(v_1,v_2)(v_2,v_1)v_3-2\epsilon(v_1,v_2+v_3)(v_3,v_1)v_2\\ =&3\psi(v_1,v_2,v_3)-3\epsilon(v_2,v_3)(v_1,v_3)v_2+3(v_2,v_3)v_1\\ =&3(\psi(v_1,v_2,v_3)-\mu_{can}(v_1,v_2)(v_3)).\end{aligned}$$

We prove the second identity. Using the first one, we have

$$\begin{aligned}-\epsilon(v_1+v_2+v_3,v_4)(v_4,B_\psi(v_1,v_2,v_3)) =& -\epsilon(v_1+v_2+v_3,v_4)\epsilon(v_1,v_2+v_3)(v_4,B_\psi(v_2,v_3,v_1))\\ =& -3\epsilon(v_1+v_2+v_3,v_4)\epsilon(v_1,v_2+v_3)(v_4,\psi(v_2,v_3,v_1)\\ &-\mu_{can}(v_2,v_3)(v_1))\\ =&3\epsilon(v_1,v_2+v_3+v_4)(\psi(v_2,v_3,v_4)-\mu_{can}(v_2,v_3)(v_4),v_1)\\ =&3(v_1,\psi(v_2,v_3,v_4)-\mu_{can}(v_2,v_3)(v_4))\\ =&(v_1,B_\psi(v_2,v_3,v_4)).\end{aligned}$$

Similarly,

$$\begin{aligned}\epsilon(v_1+v_2,v_3+v_4)(v_3,B_\psi(v_4,v_1,v_2)) &= (v_1,B_\psi(v_2,v_3,v_4)),\\ -\epsilon(v_1,v_2+v_3+v_4)(v_2,B_\psi(v_3,v_4,v_1)) &= (v_1,B_\psi(v_2,v_3,v_4))\end{aligned}$$

and hence it follows that

$$B_Q(v_1,v_2,v_3,v_4)=4(v_1,B_\psi(v_2,v_3,v_4)).$$

□

Example 4.5.4. *If $\mu=\mu_{can}$ then the covariants B_ψ and B_Q are trivial.*

In fact, at least in the case of special ϵ-orthogonal representations, the quadrilinear covariant is essentially the norm of the moment map.

Proposition 4.5.5. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional special ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$. We have*

$$B_Q = -2N_{B_\mathfrak{g}}(\mu) = -4\beta(R_\mu).$$

Proof. Let $v_1, v_2, v_3, v_4 \in V$. We know from the proof of Theorem 4.2.10 that we have

$$N_{B_\mathfrak{g}}(\mu)(v_1, v_2, v_3, v_4) = -2\epsilon(v_2, v_4)(v_1, \epsilon(v_2, v_3)\psi(v_3, v_4, v_2) + \epsilon(v_3, v_4)\psi(v_4, v_2, v_3) + \epsilon(v_4, v_2)\psi(v_2, v_3, v_4)),$$

by Proposition 4.5.3 we have also

$$B_Q(v_1, v_2, v_3, v_4) = 4(v_1, \psi(v_2, v_3, v_4) + \epsilon(v_2 + v_3, v_4)\psi(v_4, v_2, v_3) + \epsilon(v_2, v_3 + v_4)\psi(v_3, v_4, v_2))$$

and hence $B_Q = -2N_{B_\mathfrak{g}}(\mu)$. Furthermore, by Proposition 4.3.7 we also have $B_Q = -4\beta(R_\mu)$. □

Recall that (see Definition 3.6.6) the representation ρ allows us to define an exterior product

$$\wedge_\rho : Alt_\epsilon(V, \mathfrak{g}) \times Alt_\epsilon(V, V) \to Alt_\epsilon(V, V)$$

and the scalar multiplication allows us to define exterior products

$$\wedge_\epsilon : Alt_\epsilon(V, k) \times Alt_\epsilon(V, k) \to Alt_\epsilon(V, k),$$
$$\wedge_\times : Alt_\epsilon(V, k) \times Alt_\epsilon(V, V) \to Alt_\epsilon(V, V),$$
$$\wedge_\times : Alt_\epsilon(V, k) \times Alt_\epsilon(V, \mathfrak{g}) \to Alt_\epsilon(V, \mathfrak{g}).$$

Recall also that (see Definition 3.6.11) we have composition maps

$$\circ : Alt_\epsilon(V, \mathfrak{g}) \times Alt_\epsilon(V, V) \to Alt_\epsilon(V, \mathfrak{g}),$$
$$\circ : Alt_\epsilon(V, V) \times Alt_\epsilon(V, V) \to Alt_\epsilon(V, V),$$
$$\circ : Alt_\epsilon(V, k) \times Alt_\epsilon(V, V) \to Alt_\epsilon(V, k).$$

With this notation we now prove some identities satisfied by the covariants of a special ϵ-orthogonal representation which generalise the classical Mathews identities (see [Mat11]).

Theorem 4.5.6. *Let $\rho : \mathfrak{g} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be a finite-dimensional special ϵ-orthogonal representation of a finite-dimensional ϵ-quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$. We have the following identities :*

$$\mu \wedge_\rho B_\psi = -\frac{3}{2} B_Q \wedge_\times Id_V \quad \in Alt^5_\epsilon(V, V), \tag{4.13}$$
$$\mu \circ B_\psi = 3B_Q \wedge_\times \mu \quad \in Alt^6_\epsilon(V, \mathfrak{g}), \tag{4.14}$$
$$B_\psi \circ B_\psi = -\frac{27}{2} B_Q \wedge_\epsilon B_Q \wedge_\times Id_V \quad \in Alt^9_\epsilon(V, V), \tag{4.15}$$
$$B_Q \circ B_\psi = -54 B_Q \wedge_\epsilon B_Q \wedge_\epsilon B_Q \quad \in Alt^{12}_\epsilon(V, k). \tag{4.16}$$

Proof. In this proof, for $v_1, \ldots, v_n \in V$ and $\sigma \in S_n$ we denote $p(\sigma; v_1, \ldots, v_n)$ by $p(\sigma; v)$. Let $v_1, \ldots, v_{12} \in V$.

1. Using Equation (c), we have

$$\begin{aligned}
&\mu(v_{\sigma(1)}, v_{\sigma(2)})(\mu(v_{\sigma(3)}, v_{\sigma(4)})(v_{\sigma(5)})) = \epsilon(v_{\sigma(1)}, v_{\sigma(2)})(v_{\sigma(1)}, \mu(v_{\sigma(3)}, v_{\sigma(4)})(v_{\sigma(5)}))v_{\sigma(2)} \\
&- 2(v_{\sigma(2)}, \mu(v_{\sigma(3)}, v_{\sigma(4)})(v_{\sigma(5)}))v_{\sigma(1)} + (v_{\sigma(1)}, v_{\sigma(2)})\mu(v_{\sigma(3)}, v_{\sigma(4)})(v_{\sigma(5)}) \\
&- \epsilon(v_{\sigma(2)}, v_{\sigma(3)} + v_{\sigma(4)} + v_{\sigma(5)})\mu(v_{\sigma(1)}, \mu(v_{\sigma(3)}, v_{\sigma(4)})(v_{\sigma(5)}))(v_{\sigma(2)}).
\end{aligned}$$

Let $H := \{id, (12)\}$ be a subgroup of S_5 which acts by right on S_5. Hence, we have a partition $S_5 = \cup \mathcal{O}_\sigma$ where $\mathcal{O}_\sigma$ is the orbit of the element σ under the action of H. Futhermore, for all $\sigma \in S_5$ we have $|\mathcal{O}_\sigma| = 2$ and the number of orbits is 60. Since

$$p(\sigma; v)(v_{\sigma(1)}, v_{\sigma(2)})\mu(v_{\sigma(3)}, v_{\sigma(4)})(v_{\sigma(5)}) + p(\sigma(12); v)(v_{\sigma(2)}, v_{\sigma(1)})\mu(v_{\sigma(3)}, v_{\sigma(4)})(v_{\sigma(5)}) = 0$$

then we have

$$\sum_{\sigma \in S_5} p(\sigma; v)(v_{\sigma(1)}, v_{\sigma(2)})\mu(v_{\sigma(3)}, v_{\sigma(4)})(v_{\sigma(5)}) = 0.$$

We now show that

$$-\sum_{\sigma \in S_5} p(\sigma; v)\epsilon(v_{\sigma(2)}, v_{\sigma(3)} + v_{\sigma(4)} + v_{\sigma(5)})\mu(v_{\sigma(1)}, \mu(v_{\sigma(3)}, v_{\sigma(4)})(v_{\sigma(5)}))(v_{\sigma(2)}) = 0.$$

First of all, by setting $\sigma' := (2345) \in S_5$ we have

$$\begin{aligned}
&-\sum_{\sigma \in S_5} p(\sigma; v)\epsilon(v_{\sigma(2)}, v_{\sigma(3)} + v_{\sigma(4)} + v_{\sigma(5)})\mu(v_{\sigma(1)}, \mu(v_{\sigma(3)}, v_{\sigma(4)})(v_{\sigma(5)}))(v_{\sigma(2)}) \\
&= -\sum_{\sigma \in S_5} p(\sigma; v)\epsilon(v_{\sigma\sigma'(5)}, v_{\sigma\sigma'(2)} + v_{\sigma\sigma'(3)} + v_{\sigma\sigma'(4)})\mu(v_{\sigma\sigma'(1)}, \mu(v_{\sigma\sigma'(2)}, v_{\sigma\sigma'(3)})(v_{\sigma\sigma'(4)}))(v_{\sigma\sigma'(5)}) \\
&= \sum_{\sigma \in S_5} p(\sigma; v)\mu(v_{\sigma(1)}, \mu(v_{\sigma(2)}, v_{\sigma(3)})(v_{\sigma(4)}))(v_{\sigma(5)}).
\end{aligned}$$

Let H be the subgroup of S_5 defined by

$$H := \{id, (14), (12)(34), (1342), (23), (14)(23), (1243), (13)(24)\}$$

which acts by right on S_5. Hence, we have a partition $S_5 = \cup \mathcal{O}_\sigma$ where $\mathcal{O}_\sigma$ is the orbit of the element σ under the action of H. Futhermore, for all $\sigma \in S_5$ we have $|\mathcal{O}_\sigma| = 8$ and the number of orbits is 15. Using the ϵ-antisymmetry

$$\mu(v_{\sigma(2)}, v_{\sigma(3)}) = -\epsilon(v_{\sigma(2)}, v_{\sigma(3)})\mu(v_{\sigma(3)}, v_{\sigma(2)})$$

we obtain

$$\sum_{\sigma'\in\mathcal{O}_\sigma} p(\sigma';v)\mu(v_{\sigma'(1)},\mu(v_{\sigma'(2)},v_{\sigma'(3)})(v_{\sigma'(4)}))(v_{\sigma'(5)})$$
$$=2\sum_{\sigma'\in\tilde{\mathcal{O}}_\sigma} p(\sigma';v)\mu(v_{\sigma'(1)},\mu(v_{\sigma'(2)},v_{\sigma'(3)})(v_{\sigma'(4)}))(v_{\sigma'(5)})$$

where

$$\tilde{\mathcal{O}}_\sigma=\{\sigma,\sigma(14),\sigma(12)(34),\sigma(1342)\}.$$

We have

$$\sum_{\sigma'\in\tilde{\mathcal{O}}_\sigma} p(\sigma';v)\mu(v_{\sigma'(1)},\mu(v_{\sigma'(2)},v_{\sigma'(3)})(v_{\sigma'(4)}))(v_{\sigma'(5)})$$
$$=p(\sigma;v)\mu(v_{\sigma(1)},\mu(v_{\sigma(2)},v_{\sigma(3)})(v_{\sigma(4)}))(v_{\sigma(5)})+p(\sigma(14);v)\mu(v_{\sigma(4)},\mu(v_{\sigma(2)},v_{\sigma(3)})(v_{\sigma(1)}))(v_{\sigma(5)})$$
$$+p(\sigma(12)(34);v)\mu(v_{\sigma(2)},\mu(v_{\sigma(1)},v_{\sigma(4)})(v_{\sigma(3)}))(v_{\sigma(5)})+p(\sigma(1342);v)\mu(v_{\sigma(3)},\mu(v_{\sigma(1)},v_{\sigma(4)})(v_{\sigma(2)}))(v_{\sigma(5)})$$
$$=p(\sigma;v)\epsilon(v_{\sigma(1)},v_{\sigma(2)}+v_{\sigma(3)})[\mu(v_{\sigma(2)},v_{\sigma(3)}),\mu(v_{\sigma(1)},v_{\sigma(4)})](v_{\sigma(5)})$$
$$+p(\sigma;v)\epsilon(v_{\sigma(2)}+v_{\sigma(3)},v_{\sigma(4)})[\mu(v_{\sigma(1)},v_{\sigma(4)}),\mu(v_{\sigma(2)},v_{\sigma(3)})](v_{\sigma(5)})$$
$$=0.$$

Hence

$$\sum_{\sigma\in S_5} p(\sigma;v)\mu(v_{\sigma(1)},v_{\sigma(2)})(\mu(v_{\sigma(3)},v_{\sigma(4)})(v_{\sigma(5)}))$$
$$=\sum_{\sigma\in S_5} p(\sigma;v)\Big(\epsilon(v_{\sigma(1)},v_{\sigma(2)})(v_{\sigma(1)},\mu(v_{\sigma(3)},v_{\sigma(4)})(v_{\sigma(5)}))v_{\sigma(2)}-2(v_{\sigma(2)},\mu(v_{\sigma(3)},v_{\sigma(4)})(v_{\sigma(5)}))v_{\sigma(1)}\Big).$$

But, if we set $\sigma'=(2345)$, we have

$$\sum_{\sigma\in S_5} p(\sigma;v)\epsilon(v_{\sigma(1)},v_{\sigma(2)})(v_{\sigma(1)},\mu(v_{\sigma(3)},v_{\sigma(4)})(v_{\sigma(5)}))v_{\sigma(2)}$$
$$=-\sum_{\sigma\in S_5} p(\sigma\sigma';v)(v_{\sigma\sigma'(1)},\mu(v_{\sigma\sigma'(2)},v_{\sigma\sigma'(3)})(v_{\sigma\sigma'(4)}))v_{\sigma\sigma'(5)}$$
$$=-\sum_{\sigma\in S_5} p(\sigma;v)(v_{\sigma(1)},\mu(v_{\sigma(2)},v_{\sigma(3)})(v_{\sigma(4)}))v_{\sigma(5)}.$$

Again, if we set $\sigma'=(12345)$, we have

$$-2\sum_{\sigma\in S_5} p(\sigma;v)(v_{\sigma(2)},\mu(v_{\sigma(3)},v_{\sigma(4)})(v_{\sigma(5)}))v_{\sigma(1)}$$
$$=-2\sum_{\sigma\in S_5} p(\sigma\sigma';v)(v_{\sigma\sigma'(1)},\mu(v_{\sigma\sigma'(2)},v_{\sigma\sigma'(3)})(v_{\sigma\sigma'(4)}))v_{\sigma\sigma'(5)}$$
$$=-2\sum_{\sigma\in S_5} p(\sigma;v)(v_{\sigma(1)},\mu(v_{\sigma(2)},v_{\sigma(3)})(v_{\sigma(4)}))v_{\sigma(5)}.$$

Therefore, we obtain

$$\sum_{\sigma\in S_5} p(\sigma;v)\mu(v_{\sigma(1)},v_{\sigma(2)})(\mu(v_{\sigma(3)},v_{\sigma(4)})(v_{\sigma(5)})) = -3\sum_{\sigma\in S_5} p(\sigma;v)(v_{\sigma(1)},\mu(v_{\sigma(2)},v_{\sigma(3)})(v_{\sigma(4)}))v_{\sigma(5)}. \tag{4.17}$$

Furthermore, since

$$\mu(v_{\sigma(1)},v_{\sigma(2)}) = \frac{1}{2}\Big(\mu(v_{\sigma(1)},v_{\sigma(2)}) - \epsilon(v_{\sigma(1)},v_{\sigma(2)})\mu(v_{\sigma(2)},v_{\sigma(1)})\Big)$$

and using Equations (4.11) and (4.12) we obtain

$$\mu\wedge_\rho B_\psi(v_1,v_2,v_3,v_4,v_5) = \frac{1}{4}\sum_{\sigma\in S_5} p(\sigma;v)\mu(v_{\sigma(1)},v_{\sigma(2)})(\mu(v_{\sigma(3)},v_{\sigma(4)})(v_{\sigma(5)})),$$
$$B_Q\wedge_\times Id_V(v_1,v_2,v_3,v_4,v_5) = \frac{1}{2}\sum_{\sigma\in S_5} p(\sigma;v)(v_{\sigma(1)},\mu(v_{\sigma(2)},v_{\sigma(3)})(v_{\sigma(4)}))v_{\sigma(5)},$$

and then we have

$$\mu\wedge_\rho B_\psi = -\frac{3}{2}B_Q\wedge_\times Id_V.$$

2. By equivariance, we have

$$\sum_{\sigma\in S_6} p(\sigma;v)\mu\left(\mu(v_{\sigma(1)},v_{\sigma(2)})(v_{\sigma(3)}),\mu(v_{\sigma(4)},v_{\sigma(5)})(v_{\sigma(6)})\right)$$
$$= \sum_{\sigma\in S_6} p(\sigma;v)\Big(\Big[\mu(v_{\sigma(1)},v_{\sigma(2)}),\mu(v_{\sigma(3)},\mu(v_{\sigma(4)},v_{\sigma(5)})(v_{\sigma(6)}))\Big]$$
$$- \epsilon(v_{\sigma(1)}+v_{\sigma(2)},v_{\sigma(3)})\mu(v_{\sigma(3)},\mu(v_{\sigma(1)},v_{\sigma(2)})(\mu(v_{\sigma(4)},v_{\sigma(5)})(v_{\sigma(6)})))\Big).$$

We will show that

$$\sum_{\sigma\in S_6} p(\sigma;v)\left[\mu(v_{\sigma(1)},v_{\sigma(2)}),\mu(v_{\sigma(3)},\mu(v_{\sigma(4)},v_{\sigma(5)})(v_{\sigma(6)}))\right] = 0.$$

Let H be the subgroup of S_6 defined by

$$H := \{id,(36),(34)(56),(3564),(45),(36)(45),(3465),(35)(64)\}$$

which acts by right on S_6. Hence, we have a partition $S_6 = \cup\mathcal{O}_\sigma$ where $\mathcal{O}_\sigma$ is the orbit of the element σ under the action of H. Futhermore, for all $\sigma\in S_6$ we have $|\mathcal{O}_\sigma| = 8$ and the number of orbits is 90. We now show that

$$\sum_{\sigma'\in\mathcal{O}_\sigma} p(\sigma';v)\left[\mu(v_{\sigma'(1)},v_{\sigma'(2)}),\mu(v_{\sigma'(3)},\mu(v_{\sigma'(4)},v_{\sigma'(5)})(v_{\sigma'(6)}))\right] = 0.$$

Using the symmetry $\mu(v_{\sigma'(4)}, v_{\sigma'(5)}) = -\epsilon(v_{\sigma'(4)}, v_{\sigma'(5)})\mu(v_{\sigma'(5)}, v_{\sigma'(4)})$ we have

$$\begin{aligned}&\sum_{\sigma'\in\mathcal{O}_\sigma} p(\sigma';v)\left[\mu(v_{\sigma'(1)}, v_{\sigma'(2)}), \mu(v_{\sigma'(3)}, \mu(v_{\sigma'(4)}, v_{\sigma'(5)})(v_{\sigma'(6)}))\right]\\ &= 2\sum_{\sigma'\in\tilde{\mathcal{O}}_\sigma} p(\sigma';v)\left[\mu(v_{\sigma'(1)}, v_{\sigma'(2)}), \mu(v_{\sigma'(3)}, \mu(v_{\sigma'(4)}, v_{\sigma'(5)})(v_{\sigma'(6)}))\right]\end{aligned}$$

where

$$\tilde{\mathcal{O}}_\sigma := \{\sigma, \sigma(36), \sigma(34)(56), \sigma(3564)\}.$$

Then, we have

$$\begin{aligned}&\sum_{\sigma'\in\tilde{\mathcal{O}}_\sigma} p(\sigma';v)\left[\mu(v_{\sigma'(1)}, v_{\sigma'(2)}), \mu(v_{\sigma'(3)}, \mu(v_{\sigma'(4)}, v_{\sigma'(5)})(v_{\sigma'(6)}))\right]\\
=&p(\sigma;v)\left[\mu(v_{\sigma(1)}, v_{\sigma(2)}), \mu(v_{\sigma(3)}, \mu(v_{\sigma(4)}, v_{\sigma(5)})(v_{\sigma(6)}))\right]\\
&+ p(\sigma(36);v)\left[\mu(v_{\sigma(1)}, v_{\sigma(2)}), \mu(v_{\sigma(6)}, \mu(v_{\sigma(4)}, v_{\sigma(5)})(v_{\sigma(3)}))\right]\\
&+ p(\sigma(34)(56);v)\left[\mu(v_{\sigma(1)}, v_{\sigma(2)}), \mu(v_{\sigma(4)}, \mu(v_{\sigma(3)}, v_{\sigma(6)})(v_{\sigma(5)}))\right]\\
&+ p(\sigma(3564);v)\left[\mu(v_{\sigma(1)}, v_{\sigma(2)}), \mu(v_{\sigma(5)}, \mu(v_{\sigma(3)}, v_{\sigma(6)})(v_{\sigma(4)}))\right]\\
=&p(\sigma;v)\epsilon(v_{\sigma(3)}, v_{\sigma(4)} + v_{\sigma(5)})\left[\mu(v_{\sigma(1)}, v_{\sigma(2)}), [\mu(v_{\sigma(4)}, v_{\sigma(5)}), \mu(v_{\sigma(3)}, v_{\sigma(6)})]\right]\\
&+ p(\sigma;v)\epsilon(v_{\sigma(4)} + v_{\sigma(5)}, v_{\sigma(6)})\left[\mu(v_{\sigma(1)}, v_{\sigma(2)}), [\mu(v_{\sigma(3)}, v_{\sigma(6)}), \mu(v_{\sigma(4)}, v_{\sigma(5)})]\right]\\
=&0.\end{aligned}$$

Hence,

$$\sum_{\sigma\in S_6} p(\sigma;v)\left[\mu(v_{\sigma(1)}, v_{\sigma(2)}), \mu(v_{\sigma(3)}, \mu(v_{\sigma(4)}, v_{\sigma(5)})(v_{\sigma(6)}))\right] = 0.$$

Thus

$$\sum_{\sigma\in S_6} p(\sigma;v)\mu\left(\mu(v_{\sigma(1)}, v_{\sigma(2)})(v_{\sigma(3)}), \mu(v_{\sigma(4)}, v_{\sigma(5)})(v_{\sigma(6)})\right)$$

is equal to

$$-\sum_{\sigma\in S_6} p(\sigma;v)\epsilon(v_{\sigma(1)} + v_{\sigma(2)}, v_{\sigma(3)})\mu(v_{\sigma(3)}, \mu(v_{\sigma(1)}, v_{\sigma(2)})(\mu(v_{\sigma(4)}, v_{\sigma(5)})(v_{\sigma(6)})))$$

and with a change of index using $\sigma' := (132)$ this is equal to

$$-\sum_{\sigma\in S_6} p(\sigma;v)\mu(v_{\sigma(1)}, \mu(v_{\sigma(2)}, v_{\sigma(3)})(\mu(v_{\sigma(4)}, v_{\sigma(5)})(v_{\sigma(6)})))$$

and this is equal to

$$= - \sum_{\sigma' \in S(\llbracket 1 \rrbracket, \llbracket 2,6 \rrbracket)} \Big(p(\sigma'; v) \mu(v_{\sigma'(1)}, \sum_{\sigma \in S_5} p(\sigma; v_{\sigma'}) \mu(v_{\sigma\sigma'(2)}, v_{\sigma\sigma'(3)})(\mu(v_{\sigma\sigma'(4)}, v_{\sigma\sigma'(5)})(v_{\sigma\sigma'(6)}))) \Big).$$

Using Equation (4.17), we have

$$\begin{aligned}
&\sum_{\sigma \in S_6} p(\sigma; v) \mu \Big(\mu(v_{\sigma(1)}, v_{\sigma(2)})(v_{\sigma(3)}), \mu(v_{\sigma(4)}, v_{\sigma(5)})(v_{\sigma(6)}) \Big) \\
&= - \sum_{\sigma' \in S(\llbracket 1 \rrbracket, \llbracket 2,6 \rrbracket)} \Big(p(\sigma'; v) \mu(v_{\sigma'(1)}, \sum_{\sigma \in S_5} p(\sigma; v_{\sigma'}) \mu(v_{\sigma\sigma'(2)}, v_{\sigma\sigma'(3)})(\mu(v_{\sigma\sigma'(4)}, v_{\sigma\sigma'(5)})(v_{\sigma\sigma'(6)}))) \Big) \\
&= 3 \sum_{\sigma' \in S(\llbracket 1 \rrbracket, \llbracket 2,6 \rrbracket)} \Big(p(\sigma'; v) \mu(v_{\sigma'(1)}, \sum_{\sigma \in S_5} p(\sigma; v_{\sigma'})(v_{\sigma\sigma'(2)}, \mu(v_{\sigma\sigma'(3)}, v_{\sigma\sigma'(4)})(v_{\sigma\sigma'(5)}))v_{\sigma\sigma'(6)} \Big) \\
&= 3 \sum_{\sigma \in S_6} \Big(p(\sigma; v) \mu(v_{\sigma(1)}, (v_{\sigma(2)}, \mu(v_{\sigma(3)}, v_{\sigma(4)})(v_{\sigma(5)}))v_{\sigma(6)}) \Big) \\
&= 3 \sum_{\sigma \in S_6} \Big(p(\sigma; v)(v_{\sigma(1)}, \mu(v_{\sigma(2)}, v_{\sigma(3)})(v_{\sigma(4)})) \mu(v_{\sigma(5)}, v_{\sigma(6)}) \Big)
\end{aligned}$$

Since

$$\begin{aligned}
\mu \circ B_\psi(v_1, v_2, v_3, v_4, v_5, v_6) &= \frac{1}{4} \sum_{\sigma \in S_6} p(\sigma; v) \mu \Big(\mu(v_{\sigma(1)}, v_{\sigma(2)})(v_{\sigma(3)}), \mu(v_{\sigma(4)}, v_{\sigma(5)})(v_{\sigma(6)}) \Big), \\
B_Q \wedge_\times \mu(v_1, v_2, v_3, v_4, v_5, v_6) &= \frac{1}{4} \sum_{\sigma \in S_6} p(\sigma; v)(v_{\sigma(1)}, \mu(v_{\sigma(2)}, v_{\sigma(3)})(v_{\sigma(4)})) \mu(v_{\sigma(5)}, v_{\sigma(6)}),
\end{aligned}$$

and then

$$\mu \circ B_\psi = 3 B_Q \wedge_\times \mu.$$

3. We have

$$\begin{aligned}
&\sum_{\sigma \in S_9} p(\sigma; v) \mu\Big(\mu(v_{\sigma(1)}, v_{\sigma(2)})(v_{\sigma(3)}), \mu(v_{\sigma(4)}, v_{\sigma(5)})(v_{\sigma(6)}) \Big)(\mu(v_{\sigma(7)}, v_{\sigma(8)})(v_{\sigma(9)})) \\
&= 8 \sum_{\sigma \in S(\llbracket 1,6 \rrbracket, \llbracket 7,9 \rrbracket)} p(\sigma; v) \Big(\mu \circ B_\psi(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, v_{\sigma(4)}, v_{\sigma(5)}, v_{\sigma(6)}) \Big)(B_\psi(v_{\sigma(7)}, v_{\sigma(8)}, v_{\sigma(9)}))
\end{aligned}$$

and using Equation (4.14) this is equal to

$$24 \sum_{\sigma\in S([\![1,6]\!],[\![7,9]\!])} p(\sigma;v)\Big(B_Q \wedge_\times \mu(v_{\sigma(1)},v_{\sigma(2)},v_{\sigma(3)},v_{\sigma(4)},v_{\sigma(5)},v_{\sigma(6)})\Big)(B_\psi(v_{\sigma(7)},v_{\sigma(8)},v_{\sigma(9)}))$$
$$= 3\sum_{\sigma\in S_9} p(\sigma;v)(v_{\sigma(1)},\mu(v_{\sigma(2)},v_{\sigma(3)})(v_{\sigma(4)}))\mu(v_{\sigma(5)},v_{\sigma(6)})(\mu(v_{\sigma(7)},v_{\sigma(8)})(v_{\sigma(9)}))$$
$$= 3 \sum_{\sigma'\in S([\![1,4]\!],[\![5,9]\!])} p(\sigma';v)\sum_{\sigma\in S_4} p(\sigma;v_{\sigma'})(v_{\sigma\sigma'(1)},\mu(v_{\sigma\sigma'(2)},v_{\sigma\sigma'(3)})(v_{\sigma\sigma'(4)}))$$
$$\sum_{\sigma\in S_5} p(\sigma;v_{\sigma'})\mu(v_{\sigma\sigma'(5)},v_{\sigma\sigma'(6)})(\mu(v_{\sigma\sigma'(7)},v_{\sigma\sigma'(8)})(v_{\sigma\sigma'(9)}))$$

and using Equation (4.17), this is equal to

$$-9 \sum_{\sigma'\in S([\![1,4]\!],[\![5,9]\!])} p(\sigma';v)\sum_{\sigma\in S_4} p(\sigma;v_{\sigma'})(v_{\sigma\sigma'(1)},\mu(v_{\sigma\sigma'(2)},v_{\sigma\sigma'(3)})(v_{\sigma\sigma'(4)}))$$
$$\sum_{\sigma\in S_5} p(\sigma;v_{\sigma'})(v_{\sigma\sigma'(5)},\mu(v_{\sigma\sigma'(6)},v_{\sigma\sigma'(7)})(v_{\sigma\sigma'(8)}))v_{\sigma\sigma'(9)}$$
$$= -9\sum_{\sigma\in S_9} p(\sigma;v)(v_{\sigma(1)},\mu(v_{\sigma(2)},v_{\sigma(3)})(v_{\sigma(4)}))(v_{\sigma(5)},\mu(v_{\sigma(6)},v_{\sigma(7)})(v_{\sigma(8)}))v_{\sigma(9)}$$
$$= -36 B_Q \wedge_\epsilon B_Q \wedge_\times Id_V(v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9).$$

Since

$$B_\psi \circ B_\psi(v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9)$$

$$= \frac{3}{8}\sum_{\sigma\in S_9} p(\sigma;v)\mu\Big(\mu(v_{\sigma(1)},v_{\sigma(2)})(v_{\sigma(3)}),\mu(v_{\sigma(4)},v_{\sigma(5)})(v_{\sigma(6)})\Big)(\mu(v_{\sigma(7)},v_{\sigma(8)})(v_{\sigma(9)}))$$

then

$$B_\psi \circ B_\psi = -\frac{27}{2} B_Q \wedge_\epsilon B_Q \wedge_\times Id_V.$$

4. We have

$$\sum_{\sigma\in S_{12}} p(\sigma;v)\Big(\mu(v_{\sigma(1)},v_{\sigma(2)})(v_{\sigma(3)}),\mu(\mu(v_{\sigma(4)},v_{\sigma(5)})(v_{\sigma(6)}),\mu(v_{\sigma(7)},v_{\sigma(8)})(v_{\sigma(9)}))(\mu(v_{\sigma(10)},v_{\sigma(11)})(v_{\sigma(12)}))\Big)$$
$$=\frac{16}{3}\sum_{\sigma\in S(\llbracket 1,3\rrbracket,\llbracket 4,12\rrbracket)} p(\sigma;v)\Big(B_\psi(v_{\sigma(1)},v_{\sigma(2)},v_{\sigma(3)}),$$
$$B_\psi\circ B_\psi(v_{\sigma(4)},v_{\sigma(5)},v_{\sigma(6)},v_{\sigma(7)},v_{\sigma(8)},v_{\sigma(9)},v_{\sigma(10)},v_{\sigma(11)},v_{\sigma(12)})\Big)$$
$$=-72\sum_{\sigma\in S(\llbracket 1,3\rrbracket,\llbracket 4,12\rrbracket)} p(\sigma;v)\Big(B_\psi(v_{\sigma(1)},v_{\sigma(2)},v_{\sigma(3)}),$$
$$B_Q\wedge_\epsilon B_Q\wedge_\times Id_V(v_{\sigma(4)},v_{\sigma(5)},v_{\sigma(6)},v_{\sigma(7)},v_{\sigma(8)},v_{\sigma(9)},v_{\sigma(10)},v_{\sigma(11)},v_{\sigma(12)})\Big)$$
$$=-9\sum_{\sigma\in S_{12}} p(\sigma;v)(\mu(v_{\sigma(1)},v_{\sigma(2)})(v_{\sigma(3)}),v_{\sigma(12)})(v_{\sigma(4)},\mu(v_{\sigma(5)},v_{\sigma(6)})(v_{\sigma(7)}))(v_{\sigma(8)},\mu(v_{\sigma(9)},v_{\sigma(10)})(v_{\sigma(11)}))$$
$$=-9\sum_{\sigma\in S_{12}} p(\sigma;v)\epsilon(v_{\sigma(1)}+v_{\sigma(2)}+v_{\sigma(3)},v_{\sigma(12)})$$
$$(v_{\sigma(12)},\mu(v_{\sigma(1)},v_{\sigma(2)})(v_{\sigma(3)}))(v_{\sigma(4)},\mu(v_{\sigma(5)},v_{\sigma(6)})(v_{\sigma(7)}))(v_{\sigma(8)},\mu(v_{\sigma(9)},v_{\sigma(10)})(v_{\sigma(11)}))$$
$$=-9\sum_{\sigma\in S_{12}} p(\sigma;v)(v_{\sigma(1)},\mu(v_{\sigma(2)},v_{\sigma(3)})(v_{\sigma(4)}))(v_{\sigma(5)},\mu(v_{\sigma(6)},v_{\sigma(7)})(v_{\sigma(8)}))(v_{\sigma(9)},\mu(v_{\sigma(10)},v_{\sigma(11)})(v_{\sigma(12)}))$$
$$=-72B_Q\wedge_\epsilon B_Q\wedge_\epsilon B_Q(v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9,v_{10},v_{11},v_{12}).$$

Since

$$B_Q\circ B_\psi(v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9,v_{10},v_{11},v_{12})$$
$$=\frac{3}{4}\sum_{\sigma\in S_{12}} p(\sigma;v)\Big(\mu(v_{\sigma(1)},v_{\sigma(2)})(v_{\sigma(3)}),\mu(\mu(v_{\sigma(4)},v_{\sigma(5)})(v_{\sigma(6)}),\mu(v_{\sigma(7)},v_{\sigma(8)})(v_{\sigma(9)}))(\mu(v_{\sigma(10)},v_{\sigma(11)})(v_{\sigma(12)}))\Big)$$

then

$$B_Q\circ B_\psi=-54B_Q\wedge_\epsilon B_Q\wedge_\epsilon B_Q.$$

□

Chapter 5

Examples of special ϵ-orthogonal representations

5.1 Special ϵ-orthogonal representations of classical colour Lie algebras

Let Γ be an abelian group and let ϵ be a commutation factor of Γ. In this section, we first observe that the fundamental representation of the ϵ-quadratic colour Lie algebra $(\mathfrak{so}_\epsilon(V, (\ ,\ )), B)$ is a special ϵ-orthogonal representation. We then show that V is also a special ϵ-orthogonal representation of the commutant $\mathfrak{m}$ of an appropriate structure map $J \in \mathfrak{so}_\epsilon(V, (\ ,\ ))$ *but with respect to an ϵ-quadratic bilinear form $B_\mathfrak{m}$ different from* B. In this way, we show that the representation $U \oplus U^*$ of $\mathfrak{gl}_\epsilon(U)$ and the fundamental representation of $\mathfrak{u}_\epsilon(W, H)$ are special ϵ-orthogonal representations.

Throughout this section we suppose that the fundamental representation k^2 of $\mathfrak{sl}(2, k)$ is an ϵ-orthogonal representation with respect to (Γ, ϵ).

5.1.1 The fundamental representation of $\mathfrak{so}_\epsilon(V, (\ ,\ ))$

Let V be a finite-dimensional Γ-graded vector space and let $(\ ,\ )$ be a non-degenerate ϵ-symmetric bilinear form on V. The colour Lie algebra $\mathfrak{so}_\epsilon(V, (\ ,\ ))$ is ϵ-quadratic for the bilinear form

$$B(f, g) = -\frac{1}{2} Tr_\epsilon(f \circ g) \qquad \forall f, g \in \mathfrak{so}_\epsilon(V, (\ ,\ )),$$

and the fundamental representation of $\mathfrak{so}_\epsilon(V, (\ ,\ ))$ has moment map μ_{can} which satisfies (d) of Proposition 4.3.10. Hence the fundamental representation of $\mathfrak{so}_\epsilon(V, (\ ,\ ))$ is a special ϵ-orthogonal representation and by Theorem 4.4.7 there is a colour Lie algebra of the form

$$\mathfrak{so}_\epsilon(V, (\ ,\ )) \oplus \mathfrak{sl}(2, k) \oplus V \otimes k^2$$

and one can check that it is isomorphic to the colour Lie algebra $\mathfrak{so}_\epsilon(V \oplus k^2, (\ ,\ ) \perp \omega)$.

Proposition 5.1.1. *The covariants $B_\psi \in Alt^3_\epsilon(V, V)$ and $B_Q \in Alt^4_\epsilon(V)$ of the fundamental representation of $\mathfrak{so}_\epsilon(V, (\ ,\ ))$ are trivial.*

Proof. Let $v_1, v_2, v_3 \in V$. We have

$$\psi(v_1, v_2, v_3) = \mu_{can}(v_1, v_2)(v_3),$$

then $B_\psi \equiv 0$ and $B_Q \equiv 0$ by Proposition 4.5.3. □

5.1.2 Restriction to $\mathfrak{gl}_\epsilon(V)$ and $\mathfrak{u}_\epsilon(V, H)$

Let V be a finite-dimensional Γ-graded vector space together with a non-degenerate ϵ-symmetric bilinear form $(\ ,\ ) : V \times V \to k$. Let $J \in \mathfrak{so}_\epsilon(V, (\ ,\ ))$ be such that $|J| = 0$ and $J^2 = \lambda Id$, where $\lambda \in k^*$. Let

$$\mathfrak{m} := \{f \in \mathfrak{so}_\epsilon(V, (\ ,\ )) \mid f \circ J = J \circ f\}.$$

This is a colour Lie subalgebra of $\mathfrak{so}_\epsilon(V, (\ ,\ ))$.

Proposition 5.1.2. *a) If $\lambda \in k^{*2}$, we have $V = W \oplus W'$ where W (resp. W') is the eigenspace of J for the eigenvalue $\sqrt{\lambda}$ (resp. $-\sqrt{\lambda}$) and the colour Lie algebra $\mathfrak{m}$ is isomorphic to $\mathfrak{gl}_\epsilon(W)$.*

b) If $\lambda \notin k^{*2}$, let $\tilde{k} = k(\sqrt{\lambda})$. We have $V \otimes \tilde{k} = W \oplus W'$ where W (resp. W') is the eigenspace of J for the eigenvalue $\sqrt{\lambda}$ (resp. $-\sqrt{\lambda}$). The map $H : W \times W \to \tilde{k}$ defined by

$$H(v, w) := (J(v), \bar{w}) \qquad \forall v, w \in W$$

is an ϵ-antihermitian form and the colour Lie algebra $\mathfrak{m}$ is isomorphic to $\mathfrak{u}_\epsilon(W, H)$.

Proof. a) Let $f \in \mathfrak{m}$. Since $f \circ J = J \circ f$ we have $f(v) \in W$ (resp. W') if $v \in W$ (resp. W'). Let $\phi : \mathfrak{m} \to \mathfrak{gl}_\epsilon(W)$ be the morphism of colour Lie algebras defined by

$$\phi(f)(v) := f(v) \qquad \forall v \in W.$$

We show that ϕ is injective. Let $f, g \in \mathfrak{m}$ be such that $\phi(f) = \phi(g)$. Then, we have

$$f(v) = g(v) \qquad \forall v \in W.$$

Moreover, for $v \in W$ and $v' \in W'$ we have

$$(f(v'), v) = -\epsilon(v', v + v')(v', f(v)),$$
$$(g(v'), v) = -\epsilon(v', v + v')(v', g(v)),$$

since $f(v) = g(v)$ we obtain

$$(f(v'), v) = (g(v'), v) \qquad \forall v \in W,\ \forall v' \in W'.$$

The ϵ-symmetric bilinear form $(\ ,\ )$ is non-degenerate and $(v, v') = 0$ for all $v, v' \in W$ (resp. W') so we have

$$f(v') = g(v') \qquad \forall v' \in W'.$$

Thus, ϕ is injective and since $(\ ,\ )$ restricted to W is totally degenerate, ϕ is surjective and then an isomorphism of colour Lie algebras.

b) One can check that $\mathfrak{m} \subseteq \mathfrak{u}_\epsilon(W, H)$. Moreover, by a), we have $\mathfrak{m} \otimes \tilde{k} \cong \mathfrak{gl}_\epsilon(W)$ and since $\mathfrak{u}_\epsilon(W, H) \otimes \tilde{k} \cong \mathfrak{gl}_\epsilon(W)$ by Proposition 3.8.14, the colour Lie algebra $\mathfrak{m}$ is isomorphic to $\mathfrak{u}_\epsilon(W, H)$.

□

Suppose from now on that $Tr_\epsilon(J|_W) \neq 0$.

Proposition 5.1.3. *We have a decomposition of colour Lie algebras :*

$$\mathfrak{m} = \{\mathfrak{m}, \mathfrak{m}\} \oplus Vect < J > .$$

Proof. Since $Tr_\epsilon(J|_W) \neq 0$, we have $\{\mathfrak{m}, \mathfrak{m}\} \cap Vect < J >= \{0\}$ and so $\{\mathfrak{m}, \mathfrak{m}\} \oplus Vect < J >$ is a colour Lie subalgebra of $\mathfrak{m}$.

Suppose that $\lambda \in k^{*2}$. We have $\mathfrak{m} \cong \mathfrak{gl}_\epsilon(W)$. One can check that $dim(\{\mathfrak{m}, \mathfrak{m}\}) = dim(W)^2 - 1$ and so $dim(\{\mathfrak{m}, \mathfrak{m}\}) = dim(W)^2 - 1$. Thus,

$$\{\mathfrak{m}, \mathfrak{m}\} \oplus Vect < J >= \mathfrak{m}.$$

Suppose that $\lambda \notin k^{*2}$ and let $\tilde{k} = k(\sqrt{\lambda})$. We have $\mathfrak{m} \cong \mathfrak{u}_\epsilon(W, H)$ and $\mathfrak{m} \otimes \tilde{k} \cong \mathfrak{gl}_\epsilon(W)$. Furthermore, since

$$\{\mathfrak{m} \otimes \tilde{k}, \mathfrak{m} \otimes \tilde{k}\} = \{\mathfrak{m}, \mathfrak{m}\} \otimes \tilde{k}$$

we obtain $dim(\{\mathfrak{m}, \mathfrak{m}\}) = dim(W)^2 - 1$ and so

$$\{\mathfrak{m}, \mathfrak{m}\} \oplus Vect < J >= \mathfrak{m}.$$

□

We can now define an ϵ-symmetric bilinear form on $\mathfrak{m}$ using the decomposition of the previous proposition.

Definition 5.1.4. *Let $B : \mathfrak{m} \times \mathfrak{m} \to k$ be the unique bilinear form satisfying :*

$$\begin{aligned}
&B(f,g) := -\frac{1}{4}Tr_\epsilon(f \circ g) \qquad \forall f,g \in \{\mathfrak{m},\mathfrak{m}\},\\
&B(f,J) := 0 \qquad \forall f \in \{\mathfrak{m},\mathfrak{m}\},\\
&B(J,J) := \lambda.
\end{aligned}$$

One can check that this bilinear form is ϵ-symmetric, ad-invariant and non-degenerate and so $(\mathfrak{m}, B)$ is an ϵ-quadratic colour Lie algebra. Hence, we can consider the moment map of the ϵ-quadratic representation $\rho : \mathfrak{m} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$.

Lemma 5.1.5. *The moment map $\mu : \Lambda^2_\epsilon(V) \to \mathfrak{m}$ satisfies*

$$\mu(v,w) = \mu_{can}(v,w) - \frac{1}{\lambda}\mu_{can}(J(v),J(w)) + \frac{1}{\lambda}(J(v),w)J \qquad \forall v,w \in V.$$

Proof. Consider $f : \Lambda^2_\epsilon(V) \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ defined by

$$f(v,w) = \mu_{can}(v,w) - \frac{1}{\lambda}\mu_{can}(J(v),J(w)) + \frac{1}{\lambda}(J(v),w)J \qquad \forall v,w \in V.$$

One can check that $f(v,w) \in \mathfrak{m}$ for all $v,w \in V$. We want to show that

$$B(f, f(v,w)) = (f(v),w) \qquad \forall f \in \mathfrak{m},\ \forall v,w \in V.$$

Let $v,w \in V$ and $f \in \{\mathfrak{m},\mathfrak{m}\}$. We have

$$\begin{aligned}
B(f,f(v,w)) &= -\frac{1}{4}Tr_\epsilon(f \circ f(v,w))\\
&= -\frac{1}{4}Tr_\epsilon(f \circ \mu_{can}(v,w)) + \frac{1}{4\lambda}Tr_\epsilon(f \circ \mu_{can}(J(v),J(w))).
\end{aligned}$$

By Proposition 3.9.5 we obtain

$$\begin{aligned}
Tr_\epsilon(f \circ \mu_{can}(v,w)) &= -2(f(v),w)\\
Tr_\epsilon(f \circ \mu_{can}(J(v),J(w))) &= -2(f(J(v)),J(w)) = 2\lambda(f(v),w)
\end{aligned}$$

and then

$$B(f,f(v,w)) = (f(v),w).$$

Furthermore we have

$$B(J,f(v,w)) = \frac{1}{\lambda}(J(v),w)B(J,J) = (J(v),w)$$

and hence $\mu(v,w) = f(v,w)$ for all $v,w \in V$. □

The main result of this section is :

Proposition 5.1.6. *The ϵ-orthogonal representation $\mathfrak{m} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ of the ϵ-quadratic colour Lie algebra $(\mathfrak{m}, B)$ is a special ϵ-orthogonal representation.*

Proof. We show that the map μ satisfies Equation (c)). Let $u, v, w \in V$. We have

$$\begin{aligned}
&-\mu_{can}(J(v), J(w))(u) - \epsilon(w,u)\mu_{can}(J(v), J(u))(w) + (J(v), w)J(u) + \epsilon(w,u)(J(v), u)J(w) \\
=&-\epsilon(w,u)(J(v), u)J(w) + (J(w), u)J(v) - (J(v), w)J(u) + \epsilon(w,u)(J(u), w)J(v) \\
&+ (J(v), w)J(u) + \epsilon(w,u)(J(v), u)J(w) \\
=&(J(w), u)J(v) + \epsilon(w,u)(J(u), w)J(v) \\
=&0.
\end{aligned} \tag{5.1}$$

Since $\mu(v, w)(u) + \epsilon(w, u)\mu(v, u)(w)$ is equal to

$$\begin{aligned}
=&\mu_{can}(v, w)(u) + \epsilon(w,u)\mu_{can}(v,u)(w) + \frac{1}{\lambda}\Big(- \mu_{can}(J(v), J(w))(u) - \epsilon(w,u)\mu_{can}(J(v), J(u))(w) \\
&+ (J(v), w)J(u) + \epsilon(w,u)(J(v), u)J(w)\Big).
\end{aligned}$$

Using Equation (5.1) we then obtain

$$\begin{aligned}
\mu(v, w)(u) + \epsilon(w,u)\mu(v,u)(w) =&\mu_{can}(v, w)(u) + \epsilon(w,u)\mu_{can}(v,u)(w) \\
=&(v, w)u + \epsilon(w,u)(v,u)w - 2(w,u)v
\end{aligned}$$

and so the ϵ-orthogonal representation $\mathfrak{m} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ is special by Proposition 4.3.10. ☐

By Theorem 4.4.7, this proposition shows that the representation $\mathfrak{m} \to \mathfrak{so}_\epsilon(V, (\ ,\ ))$ gives rise to a colour Lie algebra of the form

$$\mathfrak{g} := \mathfrak{m} \oplus \mathfrak{sl}(2, k) \oplus V \otimes k^2.$$

- If $\lambda \in k^{*2}$, one can check that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_\epsilon(W \oplus k^2)$.
- If $\lambda \notin k^{*2}$, let $\tilde{k} = k(\sqrt{\lambda})$. If we consider the Γ-graded vector space $\tilde{k}^2 = k^2 \underset{k}{\otimes} \tilde{k}$ together with the ϵ-antihermitian form $\Omega : \tilde{k}^2 \times \tilde{k}^2 \to \tilde{k}$ defined by

  $$\Omega\Big(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \Big) := a\bar{c} - b\bar{d} \qquad \forall a, b, c, d \in \tilde{k},$$

  then we have $\mathfrak{su}_\epsilon(\tilde{k}^2, \Omega) \cong \mathfrak{sl}(2, k)$. One can check that $\mathfrak{g}$ is isomorphic to $\mathfrak{su}_\epsilon(W \oplus k^2, H \perp \Omega)$.

Proposition 5.1.7. *The trilinear covariant $B_\psi \in Alt^3_\epsilon(V,V)$ of the representation $\mathfrak{m} \to \mathfrak{so}_\epsilon(V,(\ ,\ ))$ satisfies: for all $v_1, v_2, v_3 \in V$,*

$$B_\psi(v_1,v_2,v_3) = \frac{3}{\lambda}\Big((J(v_1),v_2)J(v_3) + \epsilon(v_1+v_2,v_3)(J(v_3),v_1)J(v_2) + (J(v_2),v_3)J(v_1)\Big).$$

Proof. Let $v_1, v_2, v_3 \in V$. By Proposition 4.5.3, we have

$$\begin{aligned} B_\psi(v_1,v_2,v_3) &= 3\Big(\psi(v_1,v_2,v_3) - \mu_{can}(v_1,v_2)(v_3)\Big) \\ &= \frac{3}{\lambda}\Big(-\mu_{can}(J(v_1),J(v_2))(v_3) + (J(v_1),v_2)J(v_3)\Big) \\ &= \frac{3}{\lambda}\Big(-\epsilon(v_2,v_3)(J(v_1),v_3)J(v_2) + (J(v_2),v_3)J(v_1) + (J(v_1),v_2)J(v_3)\Big) \\ &= \frac{3}{\lambda}\Big(\epsilon(v_1+v_2,v_3)(J(v_3),v_1)J(v_2) + (J(v_2),v_3)J(v_1) + (J(v_1),v_2)J(v_3)\Big). \end{aligned}$$

□

Remark 5.1.8. *It is almost certain that the restriction of V to $\mathfrak{sl}_\epsilon(W)$ (resp. $\mathfrak{su}_\epsilon(W,H)$) is not a special ϵ-orthogonal representation for any non-degenerate ϵ-symmetric bilinear form on $\mathfrak{sl}_\epsilon(W)$ (resp. $\mathfrak{su}_\epsilon(W,H)$). In any case this is a known to be true if (Γ,ϵ) are trivial and $k=\mathbb{R}$.*

5.2 A one-parameter family of special orthogonal representations of $\mathfrak{sl}(2,k)\times\mathfrak{sl}(2,k)$

In this section we will show that with respect to a one parameter family of invariant quadratic forms on the Lie algebra $\mathfrak{sl}(2,k)\times\mathfrak{sl}(2,k)$, the tensor product of the two fundamental representations is a special orthogonal representation.

Let (V_1,ω_1) and (V_2,ω_2) be two-dimensional symplectic vector spaces. We have canonical moment maps

$$\mu_1 : S^2(V_1) \to \mathfrak{sp}(V_1,\omega_1), \qquad \mu_2 : S^2(V_2) \to \mathfrak{sp}(V_2,\omega_2)$$

given by

$$\mu_i(v,v')(v'') = -\omega_i(v,v'')v' - \omega_i(v',v'')v \qquad \forall v,v',v'' \in V_i$$

where the quadratic bilinear forms K_i on $\mathfrak{sp}(V_i,\omega_i)$ are

$$K_i(f,g) = \frac{1}{2}Tr(f\circ g) \qquad \forall f,g \in \mathfrak{sp}(V_i,\omega_i).$$

Let $\alpha,\beta \in k^*$. The Lie algebras $\mathfrak{sp}(V_1,\omega_1)$ and $\mathfrak{sp}(V_2,\omega_2)$ are quadratic for the bilinear forms $\tilde{K}_1 := \frac{1}{\alpha}K_1$ and $\tilde{K}_2 := \frac{1}{\beta}K_2$ and the corresponding moment maps satisfy

$$\tilde{\mu}_1 = \alpha\mu_1, \qquad \tilde{\mu}_2 = \beta\mu_2.$$

We consider now the orthogonal representation

$$\mathfrak{sp}(V_1,\omega_1) \times \mathfrak{sp}(V_2,\omega_2) \to \mathfrak{so}(V_1 \otimes V_2, \omega_1 \otimes \omega_2)$$

of the quadratic Lie algebra $(\mathfrak{sp}(V_1,\omega_1) \times \mathfrak{sp}(V_2,\omega_2), \frac{1}{\alpha}K_1 \perp \frac{1}{\beta}K_2)$. The moment map satisfies

$$\mu_{\alpha,\beta}(v \otimes w, v' \otimes w') = -\Big(\alpha\mu_1(v,v')\omega_2(w,w') + \beta\mu_2(w,w')\omega_1(v,v')\Big)$$

for all $v \otimes w, v' \otimes w' \in V_1 \otimes V_2$.

Proposition 5.2.1. *Let $\alpha, \beta \in k^*$. The orthogonal representation*

$$\mathfrak{sp}(V_1,\omega_1) \times \mathfrak{sp}(V_2,\omega_2) \to \mathfrak{so}(V_1 \otimes V_2, \omega_1 \otimes \omega_2)$$

of the quadratic Lie algebra $(\mathfrak{sp}(V_1,\omega_1) \times \mathfrak{sp}(V_2,\omega_2), \frac{1}{\alpha}K_1 \perp \frac{1}{\beta}K_2)$ is a special orthogonal representation if and only if $\alpha + \beta = -1$.

Proof. Let $v \otimes w, v' \otimes w', v'' \otimes w'' \in V_1 \otimes V_2$. We want to know under what conditions on α and β do we have

$$\mu_{\alpha,\beta}(v \otimes w, v' \otimes w')(v'' \otimes w'') + \mu_{\alpha,\beta}(v \otimes w, v'' \otimes w'')(v' \otimes w') = \omega_1 \otimes \omega_2(v \otimes w, v' \otimes w')v'' \otimes w''$$
$$+ \omega_1 \otimes \omega_2(v \otimes w, v'' \otimes w'')v' \otimes w' - 2\omega_1 \otimes \omega_2(v' \otimes w', v'' \otimes w'')v \otimes w. \tag{5.2}$$

Since V_1 and V_2 are two-dimensional (and after a permutation of v, v', v'' or w, w', w'' if necessary) we have $v'' = av + bv'$ and $w'' = cw + dw'$ where $a, b, c, d \in k$. Hence we have

$$\omega_1 \otimes \omega_2(v \otimes w, v' \otimes w')v'' \otimes w'' + \omega_1 \otimes \omega_2(v \otimes w, v'' \otimes w'')v' \otimes w' - 2\omega_1 \otimes \omega_2(v' \otimes w', v'' \otimes w'')v \otimes w$$
$$= \omega_1(v,v')\omega_2(w,w')\Big(-av \otimes dw' - bv' \otimes cw - 2bv' \otimes dw' + av \otimes cw\Big)$$

On the other hand we have

$$\mu_{\alpha,\beta}(v \otimes w, v' \otimes w')(v'' \otimes w'') + \mu_{\alpha,\beta}(v \otimes w, v'' \otimes w'')(v' \otimes w')$$
$$= -\Big(\alpha\mu_1(v,v')(v'') \otimes \omega_2(w,w')w'' + \beta\omega_1(v,v')v'' \otimes \mu_2(w,w')(w'') + \alpha\mu_1(v,v'')(v') \otimes \omega_2(w,w'')w'$$
$$+ \beta\omega_1(v,v'')v' \otimes \mu_2(w,w'')(w')\Big)$$
$$= (\alpha+\beta)\omega_1(v,v'')v' \otimes \omega_2(w,w')w'' + (\alpha+\beta)\omega_1(v,v')v'' \otimes \omega_2(w,w'')w' + \alpha\omega_1(v',v'')v \otimes \omega_2(w,w')w''$$
$$+ \alpha\omega_1(v'',v')v \otimes \omega_2(w,w'')w' + \beta\omega_1(v,v')v'' \otimes \omega_2(w',w'')w + \beta\omega_1(v,v'')v' \otimes \omega_2(w'',w')w$$
$$= (\alpha+\beta)\omega_1(v,v')bv' \otimes \omega_2(w,w')cw + (\alpha+\beta)\omega_1(v,v')bv' \otimes \omega_2(w,w')dw'$$
$$+ (\alpha+\beta)\omega_1(v,v')av \otimes \omega_2(w,w')dw' + (\alpha+\beta)\omega_1(v,v')bv' \otimes \omega_2(w,w')dw'$$
$$+ \alpha\omega_1(v',v)av \otimes \omega_2(w,w')cw + \alpha\omega_1(v',v)av \otimes \omega_2(w,w')dw' + \alpha\omega_1(v,v')av \otimes \omega_2(w,w')dw'$$
$$+ \beta\omega_1(v,v')av \otimes \omega_2(w',w)cw + \beta\omega_1(v,v')bv' \otimes \omega_2(w',w)cw + \beta\omega_1(v,v')bv' \otimes \omega_2(w,w')cw$$
$$= (\alpha+\beta)\omega_1(v,v')\omega_2(w,w')\Big(bv' \otimes cw + 2bv' \otimes dw' + av \otimes dw' - av \otimes cw\Big).$$

Hence, Equation (5.2) is satisfied if and only if $\alpha + \beta = -1$ and so by Proposition 4.3.10 the representation $\mathfrak{sp}(V_1, \omega_1) \times \mathfrak{sp}(V_2, \omega_2) \to \mathfrak{so}(V_1 \otimes V_2, \omega_1 \otimes \omega_2)$ is special orthogonal if and only if $\alpha + \beta = -1$. $\square$

Proposition 5.2.2. *The trilinear covariant $B_\psi \in Alt^3(V_1 \otimes V_2, V_1 \otimes V_2)$ of the representation $\mathfrak{sp}(V_1, \omega_1) \times \mathfrak{sp}(V_2, \omega_2) \to \mathfrak{so}(V_1 \otimes V_2, \omega_1 \otimes \omega_2)$ satisfies: for all $v \otimes w$, $v' \otimes w'$ and $v'' \otimes w''$ in $V_1 \otimes V_2$,*

$$B_\psi(v\otimes w, v'\otimes w', v''\otimes w'') = 3(\alpha-\beta)\Big(\omega_1(v, v'')v'\otimes\omega_2(w'', w')w+\omega_1(v', v'')v\otimes\omega_2(w, w'')w'\Big).$$

In particular, if $\alpha + \beta = -1$, the covariant B_ψ satisfies: for all $v \otimes w$, $v' \otimes w'$ and $v'' \otimes w''$ in $V_1 \otimes V_2$,

$$B_\psi(v\otimes w, v'\otimes w', v''\otimes w'') = 3(2\alpha+1)\Big(\omega_1(v, v'')v'\otimes\omega_2(w'', w')w+\omega_1(v', v'')v\otimes\omega_2(w, w'')w'\Big).$$

Proof. Let $v\otimes w, v'\otimes w', v''\otimes w'' \in V_1\otimes V_2$. Since V_1 and V_2 are two-dimensional (and after a permutation of v, v', v'' or w, w', w'' if necessary) we have $v'' = av+bv'$ and $w'' = cw+dw'$ where $a, b, c, d \in k$. We have

$$\begin{aligned}\mu_{\alpha,\beta}(v \otimes w, v' \otimes w')(v'' \otimes w'') = {} & (\alpha + \beta)\omega_1(v', v'')v \otimes \omega_2(w'', w')w + (\alpha + \beta)\omega_1(v, v'')v' \otimes \omega_2(w, w'')w' \\ & + (\alpha - \beta)\omega_1(v', v'')v \otimes \omega_2(w, w'')w' + (\alpha - \beta)\omega_1(v, v'')v' \otimes \omega_2(w'', w')w,\end{aligned} \tag{5.3}$$

$$\begin{aligned}\mu_{\alpha,\beta}(v' \otimes w', v'' \otimes w'')(v \otimes w) = {} & (\alpha + \beta)\omega_1(v', v'')v \otimes \omega_2(w', w'')w + 2\alpha\omega_1(v'', v)v' \otimes \omega_2(w', w'')w \\ & + 2\beta\omega_1(v', v'')v \otimes \omega_2(w'', w)w',\end{aligned} \tag{5.4}$$

$$\begin{aligned}\mu_{\alpha,\beta}(v'' \otimes w'', v \otimes w)(v' \otimes w') = {} & (\alpha + \beta)\omega_1(v, v'')v' \otimes \omega_2(w'', w)w' + 2\alpha\omega_1(v'', v')v \otimes \omega_2(w'', w)w' \\ & + 2\beta\omega_1(v'', v)v' \otimes \omega_2(w'', w')w.\end{aligned} \tag{5.5}$$

Hence, summing Equations (5.3), (5.4) and (5.5), we obtain

$$B_\psi(v\otimes w, v'\otimes w', v''\otimes w'') = 3(\alpha-\beta)\Big(\omega_1(v, v'')v'\otimes\omega_2(w'', w')w+\omega_1(v', v'')v\otimes\omega_2(w, w'')w'\Big).$$

$\square$

If $\alpha + \beta = -1$, this proposition shows that the representation

$$\mathfrak{sp}(V_1, \omega_1) \times \mathfrak{sp}(V_2, \omega_2) \to \mathfrak{so}(V_1 \otimes V_2, \omega_1 \otimes \omega_2)$$

of the quadratic Lie algebra $(\mathfrak{sp}(V_1, \omega_1) \times \mathfrak{sp}(V_2, \omega_2), \frac{1}{\alpha}K_1 \perp \frac{1}{\beta}K_2)$ is a special orthogonal representation and so by Theorem 4.4.7 we have a Lie superalgebra of the form

$$\mathfrak{sp}(V_1, \omega_1) \oplus \mathfrak{sp}(V_2, \omega_2) \oplus \mathfrak{sl}(2, k) \oplus V_1 \otimes V_2 \otimes k^2.$$

We denote this Lie superalgebra by $D'(2, 1; \alpha)$. It is natural to ask when $D'(2, 1; \alpha)$ and $D'(2, 1; \alpha')$ are isomorphic.

Remark 5.2.3. *Let $\alpha \in k^* \setminus \{0,-1\}$. The representation*

$$\mathfrak{sp}(V_1,\omega_1)\times\mathfrak{sp}(V_2,\omega_2)\to\mathfrak{so}(V_1\otimes V_2,\omega_1\otimes\omega_2)$$

of the quadratic Lie algebras $(\mathfrak{sp}(V_1,\omega_1)\times\mathfrak{sp}(V_2,\omega_2),\frac{1}{\alpha}K_1\perp\frac{1}{-1-\alpha}K_2)$ *and* $(\mathfrak{sp}(V_1,\omega_1)\times\mathfrak{sp}(V_2,\omega_2),\frac{1}{-1-\alpha}K_1\perp\frac{1}{\alpha}K_2)$ *give rise to isomorphic Lie superalgebras and so we have*

$$D'(2,1;\alpha)\cong D'(2,1;-1-\alpha).$$

In some cases it is possible to identify $D'(2,1;\alpha)$.

Proposition 5.2.4. *The Lie superalgebra $D'(2,1;-\frac{1}{2})$ is isomorphic to $\mathfrak{osp}(W_0\oplus W_1,(\ ,\ )\perp\omega)$ where $(W_0,(\ ,\ ))$ is a four-dimensional hyperbolic vector space and (W_1,ω) is a two-dimensional symplectic vector space.*

Proof. Let $(W_0,(\ ,\ ))$ be a four-dimensional hyperbolic vector space. As we have seen in Subsection 5.1.1, the fundamental representation of the quadratic Lie algebra $(\mathfrak{so}(W_0,(\ ,\ )),B)$ where $B(f,g):=-\frac{1}{2}Tr(fg)$ for all $f,g\in\mathfrak{so}(W_0,(\ ,\ ))$ is special orthogonal and so give rise to a Lie superalgebra isomorphic to

$$\mathfrak{osp}(W_0\oplus W_1,(\ ,\ )\perp\omega)$$

where (W_1,ω) is a two-dimensional symplectic vector space. One can show that $\mathfrak{so}(W_0,(\ ,\ ))$ is isomorphic to $\mathfrak{sp}(V_1,\omega_1)\times\mathfrak{sp}(V_2,\omega_2)$ where (V_1,ω_1) and (V_2,ω_2) are two-dimensional symplectic vector spaces. Under this isomorphism, the quadratic form B is isometric to the quadratic form

$$-Tr(f_1g_1)-Tr(f_2g_2)\qquad\forall f_1,g_1\in\mathfrak{sp}(V_1,\omega_1),\ \forall f_2,g_2\in\mathfrak{sp}(V_2,\omega_2)$$

which is equal to the quadratic form $\frac{1}{\alpha}K_1+\frac{1}{\alpha}K_2$, where $\alpha=-\frac{1}{2}$, described above on $\mathfrak{sp}(V_1,\omega_1)\times\mathfrak{sp}(V_2,\omega_2)$ and so $D'(2,1;\alpha)$ is isomorphic to $\mathfrak{osp}(W_0\oplus W_1,(\ ,\ )\perp\omega)$. $\square$

In [Kac75], Kac gives a one-parameter family of simple complex Lie superalgebras denoted $D(2,1;\alpha)$. He shows that if α is not equal to $-\frac{1}{2},-2$ or 1, then $D(2,1;\alpha)$ is an exceptional simple Lie superalgebra and if α is equal to $-\frac{1}{2},-2$ or 1, then $D(2,1;\alpha)$ is isomorphic to $\mathfrak{osp}(2,1)$.

Proposition 5.2.5. *If $k=\mathbb{C}$ then $D'(2,1;\alpha)$ is isomorphic to $D(2,1;\alpha)$ for all $\alpha\in k^*\setminus\{0,-1\}$.*

Proof. Let $\alpha\in k^*\setminus\{0,-1\}$. In Section 4.2 of [Mus12], there is a construction of the Lie superalgebra $D(2,1;\alpha)$ of the form

$$\mathfrak{sp}(W_1,\psi_1)\times\mathfrak{sp}(W_2,\psi_2)\times\mathfrak{sp}(W_3,\psi_3)\oplus W_1\otimes W_2\otimes W_3$$

where (W_1, ψ_1), (W_2, ψ_2) and (W_3, ψ_3) are two-dimensional symplectic vector spaces together the bracket defined for all $x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3, z_1 \otimes z_2 \otimes z_3 \in W_1 \otimes W_2 \otimes W_3$ by

$$\begin{aligned}
&\{\{x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3\}, z_1 \otimes z_2 \otimes z_3\} \\
&= (-1-\alpha)\psi_1(x_1, y_1)\psi_2(x_2, y_2) z_1 \otimes z_2 \otimes \Big(\psi_3(x_3, z_3)y_3 + \psi_3(y_3, z_3)x_3\Big) \\
&+ \psi_1(x_1, y_1)\psi_3(x_3, y_3) z_1 \otimes \Big(\psi_2(x_2, z_2)y_2 + \psi_2(y_2, z_2)x_2\Big) \otimes z_3 \\
&+ \alpha\psi_2(x_2, y_2)\psi_3(x_3, y_3)\Big(\psi_1(x_1, z_1)y_1 + \psi_1(y_1, z_1)x_1\Big) \otimes z_2 \otimes z_3.
\end{aligned} \tag{5.6}$$

On the other hand, $D'(2, 1; \alpha)$ is of the form

$$\mathfrak{sp}(V_1, \omega_1) \times \mathfrak{sp}(V_2, \omega_2) \times \mathfrak{sl}(2, k) \oplus V_1 \otimes V_2 \otimes k^2$$

together the bracket defined for all $x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3 \in V_1 \otimes V_2 \otimes k^2$ by

$$\begin{aligned}
&\{x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3\} \\
&= \mu_{\alpha,-1-\alpha}(x_1 \otimes x_2, y_1 \otimes y_2)\omega(x_3, y_3) + w_1 \otimes w_2(x_1 \otimes x_2, y_1 \otimes y_2)\mu_{can}(x_3, y_3) \\
&= -\alpha\mu_1(x_1, y_1)\omega_2(x_2, y_2)\omega(x_3, y_3) + (1+\alpha)\mu_2(x_2, y_2)\omega_1(x_1, y_1)\omega(x_3, y_3) \\
&\quad - \omega_1(x_1, y_1)\omega_2(x_2, y_2)\mu_{can}(x_3, y_3)
\end{aligned}$$

where ω is the usual symplectic form on k^2. Then, for all $x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3, z_1 \otimes z_2 \otimes z_3 \in V_1 \otimes V_2 \otimes k^2$ we have

$$\begin{aligned}
&\{\{x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3\}, z_1 \otimes z_2 \otimes z_3\} \\
&= \alpha\omega_2(x_2, y_2)\omega(x_3, y_3)\Big(\omega_1(x_1, z_1)y_1 + \omega_1(y_1, z_1)x_1\Big) \otimes z_2 \otimes z_3 \\
&+ (-1-\alpha)\omega_1(x_1, y_1)\omega(x_3, y_3) z_1 \otimes \Big(\omega_2(x_2, z_2)y_2 + \omega_2(y_2, z_2)x_2\Big) \otimes z_3 \\
&+ \omega_1(x_1, y_1)\omega_2(x_2, y_2) z_1 \otimes z_2 \otimes \Big(\omega(x_3, z_3)y_3 + \omega(y_3, z_3)x_3\Big).
\end{aligned} \tag{5.7}$$

Hence, by Equations (5.6) and (5.7), we obtain

$$D'(2, 1; \alpha) \cong D(2, 1; \alpha).$$

□

Remark 5.2.6. *In [Ser83], Serganova shows that there are three families of simple real Lie superalgebras which are real forms of Kac's $D(2, 1; \alpha)$ (see also [Par80] for a discussion about the real forms of $D(2, 1; \alpha)$). If $k = \mathbb{R}$, by Proposition 5.2.5, the family $D'(2, 1; \alpha)$ defined above corresponds to one these families.*

5.3 The fundamental representation of G_2 is special orthogonal

In this section, we show that the irreducible fundamental 7-dimensional representation of an exceptional Lie algebra $\mathfrak{g}$ of type G_2 is special orthogonal. To do this we first realise $\mathfrak{g}$ as a subalgebra of a Lie algebra of type $\mathfrak{so}(7)$ acting in an 8-dimensional spinor representation Σ. As a representation of $\mathfrak{g}$, Σ is the direct sum of a trivial representation and an irreducible 7-dimensional representation which, using Clifford calculus, we show is special orthogonal.

5.3.1 Spinorial construction of the exceptional Lie algebra G_2

In this subsection, we construct an exceptional Lie algebra of type G_2 starting from a Clifford representation of a quadratic vector space of dimension 7.

Let (V, B_V) be a 7-dimensional non-degenerate quadratic vector space and suppose there exist an 8-dimensional non-degenerate quadratic vector space (Σ, B_Σ) and a linear map $\rho : V \to \mathfrak{so}(\Sigma, B_\Sigma)$ such that

$$\rho^2(v) = B_V(v,v)Id \qquad \forall v \in V.$$

For general (V, B_V) such a map does not exist. For example, if $k = \mathbb{R}$, such a map ρ exists if and only if (V, B_V) is of signature $(7,0)$ or $(3,4)$.

For future reference, we denote by q_V (resp. q_Σ) the quadratic form associated to B_V (resp. B_Σ).

Let $C := C(V, B_V)$ be the Clifford algebra of V with respect to B_V. By the universal property of the Clifford algebra $C(V, B_V)$ the linear map $\rho : V \to \mathfrak{so}(\Sigma, B_\Sigma)$ extends to a morphism of unital algebras $C(V, B_V) \to End(\Sigma)$ also denoted by ρ. Sometimes for $c \in C$ and $\alpha \in \Sigma$, we write $c \cdot \alpha$ or $c(\alpha)$ instead of $\rho(c)(\alpha)$.

It is well-known that $C(V, B_V)$ has an $O(V, B_V)$-equivariant decomposition $C = \bigoplus_{k\in\mathbb{N}} C^k$ and that C^2 and $C^1 \oplus C^2$ are Lie algebras for the bracket defined by the commutator. We identify V with C^1.

Proposition 5.3.1. *a) The bracket of C^2 with C^1 defines an isomorphism of the Lie algebra C^2 with $\mathfrak{so}(V, B_V)$.*

b) The morphism $\rho : C^1 \oplus C^2 \to End(\Sigma)$ is a Lie algebra isomorphism onto $\mathfrak{so}(\Sigma, B_\Sigma)$.

Proof. See [Che97]. □

Definition 5.3.2. *Let ψ_0 be an anisotropic element of Σ. We set*

$$\mathfrak{g} := \{c \in C^2 \mid c \cdot \psi_0 = 0\}.$$

Proposition 5.3.3. *The vector space $\mathfrak{g}$ is a simple exceptional Lie subalgebra of C^2 of dimension 14 of type G_2.*

Proof. Reference ? □

Example 5.3.4. *Let $\mathbb{O}$ be an octonion (or Cayley) algebra over k.*

Recall from [SV00] that $\mathbb{O}$ has a natural involution $^- : \mathbb{O} \to \mathbb{O}$ and hence $\mathbb{O} = k \oplus Im(\mathbb{O})$ where $Im(\mathbb{O}) := \{x \in \mathbb{O} \mid \bar{x} = -x\}$. Furthermore, the quadratic form $q_{\mathbb{O}} : \mathbb{O} \to k$ given by $q_{\mathbb{O}}(x) = x\bar{x}$ for all $x \in \mathbb{O}$ is non-degenerate and also non-degenerate when restricted to $Im(\mathbb{O})$. We denote by $B_{\mathbb{O}} : \mathbb{O} \times \mathbb{O} \to k$ the symmetric bilinear form associated by polarisation to $q_{\mathbb{O}}$.

Hence, if we set $V := Im(\mathbb{O})$, $(\Sigma, B_\Sigma) := (\mathbb{O}, B_{\mathbb{O}})$ and $\rho(v)(x) := vx$ for $v \in V$ and $x \in \Sigma$, we have

$$\rho(v)^2 = q_V(v)Id \qquad \forall v \in Im(\mathbb{O})$$

where $q_V : Im(\mathbb{O}) \to k$ is defined by $q_V(v) := -q_{\mathbb{O}}(v)$. In this context, if we take $\psi_0 = 1$ one can show that $\mathfrak{g}$ of Definition 5.3.2 is the set of derivations of $\mathbb{O}$.

Recall from [SV00] (Section 2.4) that the set of all derivations $Der(\mathbb{O})$ of $\mathbb{O}$ is a simple Lie algebra of type G_2.

5.3.2 The fundamental representation of G_2

With the notation of the previous subsection, we show in this subsection that the 7-dimensional subspace $\psi_0^\perp$ of Σ is a special orthogonal representation of $\mathfrak{g}$. To prove this it is convenient to express the moment map in terms of a "cross-product" and use spinorial calculations.

Our first step is to show that $\psi_0^\perp$ is isomorphic to V as a representation of $\mathfrak{g}$.

Proposition 5.3.5. *Let $\Xi := \psi_0^\perp$. The linear map $\eta : V \to \Xi$ defined by $v \mapsto \rho(v)(\psi_0)$ is an isomorphism of $\mathfrak{g}$-representations.*

Proof. We first need the following lemma.

Lemma 5.3.6. *We have*

$$B_\Sigma(v \cdot \psi, w \cdot \phi) + B_\Sigma(v \cdot \phi, w \cdot \psi) = -2B_V(v,w)B_\Sigma(\psi,\phi) \quad \forall v,w \in V,\ \forall \psi,\phi \in \Sigma. \quad (5.8)$$

Proof. Let $v, w \in V$ and $\psi, \phi \in \Sigma$. By substituting v by $v + w$ in the equation

$$q_\Sigma(v \cdot \psi) = -q_V(v)q_\Sigma(\psi)$$

we obtain

$$B_\Sigma(v \cdot \psi, w \cdot \psi) = -B_V(v,w)q_\Sigma(\psi) \qquad \forall v,w \in V. \quad (5.9)$$

and by substituting ψ by $\psi + \phi$ in Equation (5.9) we obtain Equation (5.8). □

We now prove the proposition. Let $v \in V$ such that $\eta(v) = 0$. We have

$$B_\Sigma(v \cdot \psi_0, w \cdot \psi_0) = -B_V(v, w) q_\Sigma(\psi_0) = 0 \qquad \forall w \in V.$$

Hence $v = 0$ and so η is injective. Since $dim(V) = dim(\Xi)$, it follows that η is an isomorphism. □

Using the previous isomorphism, we define a cross-product on V as follows. First note that if $v, w \in V$ then

$$B_\Sigma((vw - wv) \cdot \psi_0, \psi_0) = 0$$

since $vw - wv \in C^2$ acts antisymmetrically on Σ.

Definition 5.3.7. *Let $v, w \in V$. We define $v \times w \in V$ to be the unique element in V which satisfies*

$$\eta(v \times w) = \frac{1}{2}(vw - wv) \cdot \psi_0$$

and call it the cross-product of v and w.

Proposition 5.3.8. *For $u, v, w \in V$ we have the following formulas:*

a)

$$q_V(v \times w) = B_V(v, w)^2 - q_V(v) q_V(w); \tag{5.10}$$

b)

$$B_V(u, v \times w) = -B_V(v, u \times w); \tag{5.11}$$

c)

$$u \times (v \times w) + v \times (u \times w) = 2B_V(u, v)w - B_V(u, w)v - B_V(v, w)u. \tag{5.12}$$

Proof. We have

$$\begin{aligned}
-q_V(v \times w) q_\Sigma(\psi_0) &= q_\Sigma((v \times w) \cdot \psi_0) = \frac{1}{4} q_\Sigma((vw - wv) \cdot \psi_0) \\
&= \frac{1}{4}(q_\Sigma((vw) \cdot \psi_0) + q_\Sigma((wv) \cdot \psi_0) - 2B_\Sigma((vw) \cdot \psi_0, \cdot (wv)\psi_0) \\
&= \frac{1}{4}(2q_V(v) q_V(w) q_\Sigma(\psi_0) + 2B_\Sigma(w \cdot \psi_0, (vwv) \cdot \psi_0) \\
&= \frac{1}{4}(2q_V(v) q_V(w) q_\Sigma(\psi_0) + 2B_\Sigma(w \cdot \psi_0, ((2B_V(v, w) - wv)v) \cdot \psi_0) \\
&= \frac{1}{4}(2q_V(v) q_V(w) q_\Sigma(\psi_0) + 4B_V(v, w) B_\Sigma(w \cdot \psi_0, v \cdot \psi_0) + 2q_V(v) q_V(w) q_\Sigma(\psi_0) \\
&= q_V(v) q_V(w) q_\Sigma(\psi_0) - B_V(v, w)^2 q_\Sigma(\psi_0)
\end{aligned}$$

which proves a). Similarly,

$$\begin{aligned} -B_V(u, v\times w)q(\psi_0) &= B_\Sigma(u\cdot\psi_0, (v\times w\cdot\psi_0) = -B_\Sigma(u\cdot\psi_0, (wv)\cdot\psi_0) \\ &= B_\Sigma((wu)\cdot\psi_0, v\cdot\psi_0) = B_V(v, u\times w)q_\Sigma(\psi_0) \end{aligned}$$

which proves b). Finally, since

$$(u\times(v\times w))\cdot\psi_0 = (uvw)\cdot\psi_0 - B_V(v,w)u\cdot\psi_0 - B_V(u, v\times w)\psi_0,$$

it follows that $(u\times(v\times w)+v\times(u\times w))\cdot\psi_0$ is equal to

$$(uvw)\cdot\psi_0 - B_V(v,w)u\cdot\psi_0 - B_V(u,v\times w)\psi_0 + (vuw)\cdot\psi_0 - B_V(u,w)v\cdot\psi_0 - B_V(v,u\times w)\psi_0$$

and by b), this is equal to

$$2B_V(u,v)w\cdot\psi_0 - B_V(v,w)u\cdot\psi_0 - B_V(u,w)v\cdot\psi_0.$$

Part c) now follows from Proposition 5.3.5. □

We now introduce a "product" $\times : \Lambda^2(\Sigma)\to\Xi$ on the space of spinors which allows us to define bases of Σ particularly adapted to calculations involving the moment map.

Definition 5.3.9. *a) Let $\{e_i\ ;\ 1\le i\le 7\}$ be an orthogonal and anisotropic basis of V. We define $\delta : \Lambda^2(\Sigma)\to V$ by*

$$\delta(\alpha,\beta) := \sum_{1\le i\le 7}\frac{1}{q_V(e_i)}B_\Sigma(e_i\cdot\alpha,\beta)e_i \qquad \forall\alpha,\beta\in\Sigma.$$

This map is independent of the choice of orthogonal and anistropic basis and $\mathfrak{so}(V, B_V)$-equivariant.

b) The bilinear antisymmetric product $\times : \Sigma\times\Sigma\to\Xi$ is given by

$$\alpha\times\beta := -\frac{1}{q_\Sigma(\psi_0)}\delta(\alpha,\beta)(\psi_0) \qquad \forall\alpha,\beta\in\Sigma.$$

This map is $\mathfrak{g}$-equivariant.

The product $\times$ satisfies identities similar to (5.10), (5.11) and (5.12) as follows.

Proposition 5.3.10. *For $\alpha,\beta,\gamma\in\Xi$ we have the following formulas:*

a)

$$q_\Sigma(\alpha\times\beta) = \frac{-1}{q_\Sigma(\psi_0)}(B_\Sigma(\alpha,\beta)^2 - q_\Sigma(\alpha)q_\Sigma(\beta)); \tag{5.13}$$

b)

$$B_\Sigma(\alpha, \beta \times \gamma) = -B_\Sigma(\beta, \alpha \times \gamma); \tag{5.14}$$

c)

$$\alpha \times (\beta \times \gamma) + \beta \times (\alpha \times \gamma) = \frac{-1}{q_\Sigma(\psi_0)}(2B_\Sigma(\alpha, \beta)\gamma - B_\Sigma(\alpha, \gamma)\beta - B_\Sigma(\beta, \gamma)\alpha). \tag{5.15}$$

Proof. One can check that

$$\delta(u \cdot \psi_0, v \cdot \psi_0)(\psi_0) = -q_\Sigma(\psi_0) u \times v \cdot \psi_0 \qquad \forall u, v \in V$$

and so *a*), *b*) and *c*) follow from Proposition 5.3.5 and Equations (5.10), (5.11) and (5.12). □

Corollary 5.3.11. *Let $\alpha, \beta \in \Xi$ be anisotropic and such that $B_\Sigma(\alpha, \beta) = 0$. If $\gamma \in \Xi$ is anisotropic and orthogonal to α, β and $\alpha \times \beta$, then*

$$\mathcal{B} = \{\alpha, \beta, \alpha \times \beta, \gamma, \alpha \times \gamma, \beta \times \gamma, (\alpha \times \beta) \times \gamma\}$$

is an orthogonal basis of Ξ of anisotropic spinors.

Proof. The elements of $\mathcal{B}$ are anisotropic by (5.13) and orthogonal by (5.14). □

Define the quadratic form $B_\mathfrak{g} : \mathfrak{g} \times \mathfrak{g} \to k$ by

$$B_\mathfrak{g}(f, g) := -\frac{1}{3} Tr(\rho(f) \circ \rho(g)) \qquad \forall f, g \in \mathfrak{g}. \tag{5.16}$$

It is clear that $B_\mathfrak{g}$ is non-degenerate and *ad*-invariant and now we consider the moment map μ_Ξ of the representation $\mathfrak{g} \to \mathfrak{so}(\Xi, B_\Sigma)$ of the quadratic Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$. Using bases as in Corollary 5.3.11, we express μ_Ξ in terms of the product $\times$ on Ξ.

Proposition 5.3.12. *The moment map $\mu_\Xi : \Xi \times \Xi \to \mathfrak{g}$ of the representation $\mathfrak{g} \to \mathfrak{so}(\Xi, B_\Sigma)$ satisfies*

$$\mu_\Xi(\alpha, \beta)(\gamma) = -\frac{q_\Sigma(\psi_0)}{2}\Big(\gamma \times (\alpha \times \beta) - \alpha \times (\beta \times \gamma) - \beta \times (\gamma \times \alpha)\Big) \qquad \forall \alpha, \beta, \gamma \in \Xi.$$

Proof. Let $\alpha, \beta \in \Xi$. We define $D_{\alpha,\beta} \in \mathfrak{so}(\Sigma, B_\Sigma)$ by

$$\begin{aligned} D_{\alpha,\beta}(\gamma) &:= \gamma \times (\alpha \times \beta) - \alpha \times (\beta \times \gamma) - \beta \times (\gamma \times \alpha) \qquad \forall \gamma \in \Xi, \\ D_{\alpha,\beta}(\psi_0) &:= 0. \end{aligned} \tag{5.17}$$

Since $C^1 \oplus C^2 \cong \mathfrak{so}(\Sigma, B_\Sigma)$, we consider $D_{\alpha,\beta}$ as an element of $C^1 \oplus C^2$ and show that this element is in $\mathfrak{g}$.

Lemma 5.3.13. *a) Let $\alpha, \beta, \gamma \in \Xi$ such that $B_\Sigma(\alpha, \beta) = 0$, $B_\Sigma(\alpha, \gamma) = 0$ and $B_\Sigma(\beta, \gamma) = 0$. We have*

$$D_{\alpha,\beta}(\alpha) = -2\frac{q_\Sigma(\alpha)}{q_\Sigma(\psi_0)}\beta, \qquad D_{\alpha,\beta}(\gamma) = (\alpha \times \beta) \times \gamma, \qquad D_{\alpha,\beta}(\alpha \times \beta) = 0.$$

b) Let $\alpha, \beta \in \Xi$. We have $D_{\alpha,\beta} \in \mathfrak{g}$.

Proof. $a)$ We have

$$D_{\alpha,\beta}(\alpha) = 2\alpha \times (\alpha \times \beta)$$

and using (5.15) we obtain

$$2\alpha \times (\alpha \times \beta) = -2\frac{q_\Sigma(\alpha)}{q_\Sigma(\psi_0)}\beta.$$

The other two identities follow similarly from the definition of $D_{\alpha,\beta}$ and (5.15).

$b)$ Since $C^1 \oplus C^2$ is isomorphic to $\mathfrak{so}(\Sigma, B_\Sigma)$, the quadratic form $B : C^1 \oplus C^2 \times C^1 \oplus C^2 \to k$ defined by

$$Tr(\rho(c) \circ \rho(c')) \qquad \forall c, c' \in C^1 \oplus C^2$$

is non-degenerate by Proposition 3.8.24 and one can check that C^1 is orthogonal to C^2. Let $\alpha, \beta \in \Xi$ be anisotropic and such that $B_\Sigma(\alpha, \beta) = 0$. To show that $D_{\alpha,\beta} \in C^2$ we now prove that $Tr(\rho(v) \circ \rho(D_{\alpha,\beta})) = 0$ for all $v \in C^1$.

Let $v \in C^1$. By Corollary 5.3.11, there exists an anisotropic $\gamma \in \Xi$ such that

$$\mathcal{B} = \{\psi_0, \alpha, \beta, \alpha \times \beta, \gamma, \alpha \times \gamma, \beta \times \gamma, (\alpha \times \beta) \times \gamma\}$$

is an orthogonal basis of anisotropic spinors and then

$$Tr(\rho(v) \circ \rho(D_{\alpha,\beta})) = -\sum_{c \in \mathcal{B}} \frac{B_\Sigma(\rho(v)(c), D_{\alpha,\beta}(c))}{q_\Sigma(c)}.$$

Using $a)$ of Lemma 5.3.13, we have the following identities:

$$\frac{B_\Sigma(\rho(v)(\psi_0), D_{\alpha,\beta}(\psi_0))}{q_\Sigma(\psi_0)} = 0; \tag{5.18}$$

$$\frac{B_\Sigma(\rho(v)(\alpha), D_{\alpha,\beta}(\alpha))}{q_\Sigma(\alpha)} = \frac{-2}{q_\Sigma(\psi_0)} B_\Sigma(\rho(v)(\alpha), \beta); \tag{5.19}$$

$$\frac{B_\Sigma(\rho(v)(\beta), D_{\alpha,\beta}(\beta))}{q_\Sigma(\beta)} = \frac{-2}{q_\Sigma(\psi_0)} B_\Sigma(\rho(v)(\alpha), \beta); \tag{5.20}$$

$$\frac{B_\Sigma(\rho(v)(\alpha \times \beta), D_{\alpha,\beta}(\alpha \times \beta))}{q_\Sigma(\alpha \times \beta)} = 0. \tag{5.21}$$

A straightforward calculation using a) of Lemma 5.3.13 shows:

$$\frac{B_\Sigma(\rho(v)(\gamma), D_{\alpha,\beta}(\gamma))}{q_\Sigma(\gamma)} = \frac{B_\Sigma(\rho(v)(\alpha), \beta)}{q_\Sigma(\psi_0)}; \tag{5.22}$$

$$\frac{B_\Sigma(\rho(v)(\alpha\times\gamma), D_{\alpha,\beta}(\alpha\times\gamma))}{q_\Sigma(\alpha\times\gamma)} = \frac{B_\Sigma(\rho(v)(\alpha), \beta)}{q_\Sigma(\psi_0)}; \tag{5.23}$$

$$\frac{B_\Sigma(\rho(v)(\beta\times\gamma), D_{\alpha,\beta}(\beta\times\gamma))}{q_\Sigma(\beta\times\gamma)} = \frac{B_\Sigma(\rho(v)(\alpha), \beta)}{q_\Sigma(\psi_0)}; \tag{5.24}$$

$$\frac{B_\Sigma(\rho(v)((\alpha\times\beta)\times\gamma), D_{\alpha,\beta}((\alpha\times\beta)\times\gamma))}{q_\Sigma((\alpha\times\beta)\times\gamma)} = \frac{B_\Sigma(\rho(v)(\alpha), \beta)}{q_\Sigma(\psi_0)}. \tag{5.25}$$

Summing Equations (5.18),(5.19),(5.20),(5.21),(5.22),(5.23),(5.24) and (5.25) we obtain

$$Tr(\rho(v)\circ\rho(D_{\alpha,\beta})) = 0.$$

Hence $D_{\alpha,\beta} \in C^2$ and, since $D_{\alpha,\beta}(\psi_0) = 0$, this implies $D_{\alpha,\beta} \in \mathfrak{g}$. □

Lemma 5.3.14. *We have*

$$Tr(\rho(D)\circ\rho(D_{\alpha,\beta})) = \frac{6}{q_\Sigma(\psi_0)} B_\Sigma(D(\alpha), \beta) \qquad \forall D \in \mathfrak{g},\ \forall \alpha, \beta \in \Xi. \tag{5.26}$$

Proof. It is sufficient to show (5.26) for all $\alpha, \beta \in \Xi$ such that $B_\Sigma(\alpha, \beta) = 0$ and $q_\Sigma(\alpha)q_\Sigma(\beta) \neq 0$. Let $D \in \mathfrak{g}$ and let $\alpha, \beta \in \Xi$ be such that $B_\Sigma(\alpha, \beta) = 0$ and $q_\Sigma(\alpha)q_\Sigma(\beta) \neq 0$. By Corollary 5.3.11, there exists an anisotropic $\gamma \in \Xi$ such that

$$\mathcal{B} = \{\alpha, \beta, \alpha\times\beta, \gamma, \alpha\times\gamma, \beta\times\gamma, (\alpha\times\beta)\times\gamma\}$$

is an orthogonal basis of anisotropic spinors and

$$Tr(\rho(D)\circ\rho(D_{\alpha,\beta})) = -\sum_{c\in\mathcal{B}} \frac{B_\Sigma(D(c), D_{\alpha,\beta}(c))}{q_\Sigma(c)}.$$

Using a) of Lemma 5.3.13 we obtain

$$\frac{B_\Sigma(D(\alpha), D_{\alpha,\beta}(\alpha))}{q_\Sigma(\alpha)} = \frac{-2B_\Sigma(D(\alpha), \beta)}{q_\Sigma(\psi_0)}; \tag{5.27}$$

$$\frac{B_\Sigma(D(\beta), D_{\alpha,\beta}(\beta))}{q_\Sigma(\beta)} = \frac{-2B_\Sigma(D(\alpha), \beta)}{q_\Sigma(\psi_0)}; \tag{5.28}$$

$$\frac{B_\Sigma(D(\alpha\times\beta), D_{\alpha,\beta}(\alpha\times\beta))}{q_\Sigma(\alpha\times\beta)} = 0. \tag{5.29}$$

Similarly, we have

$$\frac{B_\Sigma(D(\alpha\times\gamma), D_{\alpha,\beta}(\alpha\times\gamma))}{q_\Sigma(\alpha\times\gamma)} = \frac{-B_\Sigma(D(\alpha), \beta)}{q_\Sigma(\psi_0)} + \frac{B_\Sigma(D(\gamma), (\gamma\times\beta)\times\alpha)}{q_\Sigma(\gamma)}$$

and hence

$$\frac{B_\Sigma(D(\alpha\times\gamma),D_{\alpha,\beta}(\alpha\times\gamma))}{q_\Sigma(\alpha\times\gamma)}+\frac{B_\Sigma(D(\gamma),D_{\alpha,\beta}(\gamma))}{q_\Sigma(\gamma)}=\frac{-B_\Sigma(D(\alpha),\beta)}{q_\Sigma(\psi_0)}. \tag{5.30}$$

We also have

$$\frac{B_\Sigma(D(\beta\times\gamma),D_{\alpha,\beta}(\beta\times\gamma))}{q_\Sigma(\beta\times\gamma)}=\frac{-B_\Sigma(D(\alpha),\beta)}{q_\Sigma(\psi_0)}-\frac{B_\Sigma(D(\gamma),\gamma\times(\beta\times\alpha))}{q_\Sigma(\gamma)},$$
$$\frac{B_\Sigma(D((\alpha\times\beta)\times\gamma),D_{\alpha,\beta}((\alpha\times\beta)\times\gamma))}{q_\Sigma((\alpha\times\beta)\times\gamma)}=-\frac{B_\Sigma(D(\gamma),\gamma\times(\alpha\times\beta))}{q_\Sigma(\gamma)}$$

and so

$$\frac{B_\Sigma(D(\beta\times\gamma),D_{\alpha,\beta}(\beta\times\gamma))}{q_\Sigma(\beta\times\gamma)}+\frac{B_\Sigma(D((\alpha\times\beta)\times\gamma),D_{\alpha,\beta}((\alpha\times\beta)\times\gamma))}{q_\Sigma((\alpha\times\beta)\times\gamma)}=\frac{-B_\Sigma(D(\alpha),\beta)}{q_\Sigma(\psi_0)}. \tag{5.31}$$

Thus, summing Equations (5.27), (5.28), (5.29), (5.30) and (5.31) we obtain

$$Tr(\rho(D)\circ\rho(D_{\alpha,\beta}))=\frac{6}{q_\Sigma(\psi_0)}B_\Sigma(D(\alpha),\beta).$$

□

These two lemmas imply the proposition as follows. Let $\alpha,\beta\in\Xi$. Since $\mu_\Xi(\alpha,\beta)$ is the unique element of $\mathfrak{g}$ satisfying

$$B_{\mathfrak{g}}(D,\mu_\Xi(\alpha,\beta))=B_\Sigma(D(\alpha),\beta)$$

it follows from 5.16, Lemmas 5.3.13 and 5.3.14 that

$$\mu_\Xi(\alpha,\beta)=-\frac{q_\Sigma(\psi_0)}{2}D_{\alpha,\beta}$$

which proves the proposition by (5.17). □

Corollary 5.3.15. *For all $\alpha,\beta,\gamma\in\Xi$,*

$$\mu_\Xi(\alpha,\beta)(\gamma)=\frac{1}{2}\Big(3\mu_{can}(\alpha,\beta)(\gamma)-q_\Sigma(\psi_0)(\alpha\times\beta)\times\gamma\Big). \tag{5.32}$$

Proof. By proposition 5.3.12, we have

$$\mu_\Xi(\alpha,\beta)(\gamma)+\frac{q_\Sigma(\psi_0)}{2}(\alpha\times\beta)\times\gamma=\frac{q_\Sigma(\psi_0)}{2}\Big(2(\alpha\times\beta)\times\gamma+\alpha\times(\beta\times\gamma)+\beta\times(\gamma\times\alpha)\Big).$$

Using Equation (5.15) twice we obtain

$$\begin{aligned}
&2(\alpha \times \beta) \times \gamma + \alpha \times (\beta \times \gamma) + \beta \times (\gamma \times \alpha) \\
&= -\gamma \times (\alpha \times \beta) - \alpha \times (\gamma \times \beta) + \gamma \times (\beta \times \alpha) + \beta \times (\gamma \times \alpha) \\
&= \frac{3}{q_\Sigma(\psi_0)} \mu_{can}(\alpha, \beta)(\gamma),
\end{aligned}$$

which proves the corollary. □

We now give the main result of this section.

Proposition 5.3.16. *The representation $\rho : \mathfrak{g} \to \mathfrak{so}(\Xi, B_\Sigma)$ of the quadratic Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$ is a special orthogonal representation.*

Proof. Let $\alpha, \beta, \gamma \in \Xi$. We have

$$\mu_\Xi(\alpha, \beta)(\gamma) + \mu_\Xi(\alpha, \gamma)(\beta) = q_\Sigma(\psi_0)\Big(\gamma \times (\beta \times \alpha) + \beta \times (\gamma + \alpha)\Big).$$

Using Equation (5.15) we obtain

$$\mu_\Xi(\alpha, \beta)(\gamma) + \mu_\Xi(\alpha, \gamma)(\beta) = B_\Sigma(\alpha, \beta)\gamma + B_\Sigma(\alpha, \gamma)\beta - 2B_\Sigma(\beta, \gamma)\alpha$$

and so by Proposition 4.3.10, the representation $\rho : \mathfrak{g} \to \mathfrak{so}(\Xi, B_\Sigma)$ is a special orthogonal representation. □

By the previous proposition and Theorem 4.4.7, there is a Lie superalgebra of the form

$$\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathfrak{sl}(2, k) \oplus \Xi \otimes k^2.$$

If $k = \mathbb{C}$, Kac (see [Kac75]) showed that there is a simple exceptional Lie superalgebra G_3 of dimension 31 whose even part is isomorphic to $G_2 \times \mathfrak{sl}(2, \mathbb{C})$ and whose odd part is isomorphic to the tensor product of the fundamental representations.

Proposition 5.3.17. *If $k = \mathbb{C}$ then $\tilde{\mathfrak{g}}$ is isomorphic to G_3.*

Proof. Since the representation $\Xi \otimes k^2$ is a faithful and irreducible representation of $\mathfrak{g} \oplus \mathfrak{sl}(2, k)$ and since

$$\{\Xi \otimes k^2, \Xi \otimes k^2\} = \mathfrak{g} \oplus \mathfrak{sl}(2, k)$$

then $\tilde{\mathfrak{g}}$ is a simple Lie superalgebra (see Lemma 3 p.95 in [Sch79b]). One can check in the classification of Kac (see [Kac75]) that the only simple Lie superalgebra whose even part is isomorphic to $G_2 \times \mathfrak{sl}(2, \mathbb{C})$ is G_3 and then $\tilde{\mathfrak{g}}$ is isomorphic to G_3. □

Remark 5.3.18. *If $k = \mathbb{R}$, Serganova (see [Ser83]) showed that there are two real forms of G_3 whose even parts are isomorphic to the compact (resp. split) exceptional simple real Lie algebra of type G_2 in direct sum with $\mathfrak{sl}(2,\mathbb{R})$ and whose odd parts are isomorphic to the tensor product of the fundamental representations. In our construction, if (V, B_V) is of signature $(7,0)$ (resp. $(4,3)$), the Lie algebra $\mathfrak{g}$ is the compact (resp. split) exceptional simple real Lie algebra of type G_2. Hence, by Proposition (5.3.17), both real forms of G_3 are obtained by our construction.*

Since the representation $\mathfrak{g} \to \mathfrak{so}(\Xi, B_\Sigma)$ is special, we can explicitly calculate its covariants B^Ξ_ψ, B^Ξ_Q and the identities they satisfy. Recall that these are (see Theorem 4.5.6)

$$\mu_\Xi \circ B^\Xi_\psi = 3B^\Xi_Q \wedge_\times \mu_\Xi \quad \in Alt^6_\epsilon(\Xi, \mathfrak{g}), \tag{5.33}$$

$$B^\Xi_\psi \circ B^\Xi_\psi = -\frac{27}{2} B^\Xi_Q \wedge_\epsilon B^\Xi_Q \wedge_\times Id_\Xi \quad \in Alt^9_\epsilon(\Xi, \Xi), \tag{5.34}$$

$$B^\Xi_Q \circ B^\Xi_\psi = -54 B^\Xi_Q \wedge_\epsilon B^\Xi_Q \wedge_\epsilon B^\Xi_Q \quad \in Alt^{12}_\epsilon(\Xi, k). \tag{5.35}$$

As a vector space, Ξ is of dimension 7 and so

$$B^\Xi_\psi \circ B^\Xi_\psi = B^\Xi_Q \wedge_\epsilon B^\Xi_Q \wedge_\times Id = B^\Xi_Q \circ B^\Xi_\psi = B^\Xi_Q \wedge_\epsilon B^\Xi_Q \wedge_\epsilon B^\Xi_Q = 0$$

since as alternating maps, these are of degree $9, 9, 12$ and 12 respectively. It turns out, as we now show, that both-sides of Equation (5.33) also vanish identically.

Proposition 5.3.19. *Let B^Ξ_ψ and B^Ξ_Q be the covariants of the representation $\rho : \mathfrak{g} \to \mathfrak{so}(\Xi, B_\Sigma)$. We have*

a) $B^\Xi_\psi(\alpha,\beta,\gamma) = \frac{q_\Sigma(\psi_0)}{2}\Big(\alpha \times (\beta \times \gamma) + \beta \times (\gamma \times \alpha) + \gamma \times (\alpha \times \beta)\Big) \qquad \forall \alpha,\beta,\gamma \in \Xi.$

b) $\mu_\Xi \circ B^\Xi_\psi = 0$ *and* $B^\Xi_Q \wedge_\times \mu_\Xi = 0$.

Proof. a) This can be proved by a straightforward calculation from the definition of B^Ξ_ψ and Proposition 5.3.12.

b) We will need the following lemma.

Lemma 5.3.20. *For all $\alpha, \beta, \gamma \in \Xi$, we have*

$$\mu_\Xi(\alpha \times \beta, \gamma) + \mu_\Xi(\beta \times \gamma, \alpha) + \mu_\Xi(\gamma \times \alpha, \beta) = 0. \tag{5.36}$$

Proof. It is sufficient to prove Equation (5.36) for $\alpha, \beta, \gamma \in \Sigma$ anisotropic. Moreover, if two of them are equal or if $\gamma = \alpha \times \beta$, Equation (5.36) is easily checked.

Hence, consider $\alpha, \beta, \gamma \in \Sigma$ anisotropic and such that $B_\Sigma(\alpha,\beta) = 0$, $B_\Sigma(\alpha,\gamma) = 0$, $B_\Sigma(\beta,\gamma) = 0$ and $B_\Sigma(\alpha \times \beta, \gamma) = 0$. Let $\delta \in \Sigma$. By Equation (5.15) we have

$$(\beta \times \gamma) \times \alpha = (\gamma \times \alpha) \times \beta = (\alpha \times \beta) \times \gamma$$

and then using Corollary 5.3.15 we have

$$\mu_\Xi(\alpha\times\beta,\gamma)(\delta)+\mu_\Xi(\beta\times\gamma,\alpha)(\delta)+\mu_\Xi(\gamma\times\alpha,\beta)(\delta)$$
$$=\frac{3}{2}\Big(\mu_{can}(\alpha\times\beta,\gamma)(\delta)+\mu_{can}(\beta\times\gamma,\alpha)(\delta)+\mu_{can}(\gamma\times\alpha,\beta)(\delta)-q_\Sigma(\psi_0)(((\alpha\times\beta)\times\gamma)\times\delta)\Big).$$

By Corollary 5.3.11,

$$\mathcal{B}:=\{\alpha,\beta,\alpha\times\beta,\gamma,\alpha\times\gamma,\beta\times\gamma,(\alpha\times\beta)\times\gamma\}$$

is an orthogonal basis of Ξ and by a straightforward calculation we have

$$\mu_{can}(\alpha\times\beta,\gamma)(\delta)+\mu_{can}(\beta\times\gamma,\alpha)(\delta)+\mu_{can}(\gamma\times\alpha,\beta)(\delta)-q_\Sigma(\psi_0)(((\alpha\times\beta)\times\gamma)\times\delta)=0\quad\forall\delta\in\mathcal{B}.$$

Hence we obtain

$$\mu_\Xi(\alpha\times\beta,\gamma)(\delta)+\mu_\Xi(\beta\times\gamma,\alpha)(\delta)+\mu_\Xi(\gamma\times\alpha,\beta)(\delta)=0.$$

□

Let $v_1,\dots,v_6\in\Xi$. We have

$$\mu_\Xi\circ B^\Xi_\psi(v_1,\dots,v_6)=\sum_{\sigma\in S([\![1,3]\!],[\![4,6]\!])}\epsilon(\sigma)\mu_\Xi(B^\Xi_\psi(v_{\sigma(1)},v_{\sigma(2)},v_{\sigma(3)}),B^\Xi_\psi(v_{\sigma(4)},v_{\sigma(5)},v_{\sigma(6)}))$$
$$=2\sum_{\sigma\in S'}\epsilon(\sigma)\mu_\Xi(B^\Xi_\psi(v_{\sigma(1)},v_{\sigma(2)},v_{\sigma(3)}),B^\Xi_\psi(v_{\sigma(4)},v_{\sigma(5)},v_{\sigma(6)}))$$

where $S':=\{Id,(14),(15),(16),(24),(25),(26),(34),(35),(36)\}$.

Without loss of generality, we can assume that the vectors v_i are anisotropic and that $B_\Sigma(v_i,v_j)=0$ if $i\neq j$. Hence we have $B^\Xi_\psi(v_i,v_j,v_k)=\frac{3}{2}q_\Sigma(\psi_0)v_i\times(v_j\times v_k)$ for $i\neq j$, $i\neq k$ and $j\neq k$ and so

$$\mu_\Xi\circ B^\Xi_\psi(v_1,\dots,v_6)=\frac{9}{4}q_\Sigma(\psi_0)^2\sum_{\sigma\in S'}\epsilon(\sigma)\mu_\Xi(v_{\sigma(1)}\times(v_{\sigma(2)}\times v_{\sigma(3)}),v_{\sigma(4)}\times(v_{\sigma(5)}\times v_{\sigma(6)})).$$

Furthermore, we can also assume that $v_3=v_1\times v_2$, $v_5=v_1\times v_4$ and $v_6=v_2\times v_4$. We have

$$\mu_\Xi\circ B^\Xi_\psi(v_1,\dots,v_6)=-\frac{9q_\Sigma(v_1)q_\Sigma(v_2)q_\Sigma(v_4)}{2q_\Sigma(\psi_0)}\Big(\mu_\Xi(v_2\times v_4,v_1)+\mu_\Xi(v_4\times v_1,v_2)+\mu_\Xi(v_1\times v_2,v_4)\Big).$$

Using Lemma (5.3.20) we obtain $\mu_\Xi\circ B^\Xi_\psi=0$ and by Theorem 4.5.6 we have $B^\Xi_Q\wedge_\times\mu_\Xi=0$. □

Proposition 5.3.21. *Let $B_Q^\Xi \in \Lambda^4(\Xi)^*$ be the quadrilinear covariant of the representation $\mathfrak{g} \to \mathfrak{so}(\Xi, B_\Sigma)$. Let*

$$\mathcal{B} := \{\alpha, \beta, \alpha \times \beta, \gamma, \alpha \times \gamma, \beta \times \gamma, (\alpha \times \beta) \times \gamma\}$$

be a basis of Ξ as in Corollary 5.3.11 and set:

$$v_1 := \alpha, v_2 := \beta, v_3 := \alpha \times \beta, v_4 := \gamma, v_5 := \alpha \times \gamma, v_6 := \beta \times \gamma, v_7 := (\alpha \times \beta) \times \gamma.$$

Let $\Phi : \Lambda^4(\Xi) \to \Lambda^4(\Xi)^*$ be the isomorphism defined by

$$\Phi(u_1 \wedge \ldots \wedge u_4)(w_1 \wedge \ldots \wedge w_4) := det(B_\Sigma(u_i, w_j)_{i,j}) \quad \forall u_1 \wedge \ldots \wedge u_4, w_1 \wedge \ldots \wedge w_4 \in \Lambda^4(\Xi).$$

Then:

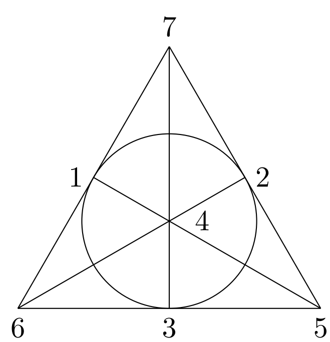
$$\begin{aligned}\Phi^{-1}(B_Q^\Xi) =& \frac{6q_\Sigma(\psi_0)}{q_\Sigma(v_1)q_\Sigma(v_2)q_\Sigma(v_4)} v_1 \wedge v_2 \wedge v_4 \wedge v_7 - \frac{6q_\Sigma(\psi_0)}{q_\Sigma(v_1)q_\Sigma(v_2)q_\Sigma(v_4)} v_1 \wedge v_2 \wedge v_5 \wedge v_6 \\ &- \frac{6q_\Sigma(\psi_0)}{q_\Sigma(v_1)q_\Sigma(v_2)q_\Sigma(v_4)} v_1 \wedge v_3 \wedge v_4 \wedge v_6 - \frac{6q_\Sigma(\psi_0)^2}{q_\Sigma(v_1)^2 q_\Sigma(v_2)q_\Sigma(v_4)} v_1 \wedge v_3 \wedge v_5 \wedge v_7 \\ &+ \frac{6q_\Sigma(\psi_0)}{q_\Sigma(v_1)q_\Sigma(v_2)q_\Sigma(v_4)} v_2 \wedge v_3 \wedge v_4 \wedge v_5 - \frac{6q_\Sigma(\psi_0)^2}{q_\Sigma(v_1)q_\Sigma(v_2)^2 q_\Sigma(v_4)} v_2 \wedge v_3 \wedge v_6 \wedge v_7 \\ &- \frac{6q_\Sigma(\psi_0)^2}{q_\Sigma(v_1)q_\Sigma(v_2)q_\Sigma(v_4)^2} v_4 \wedge v_5 \wedge v_6 \wedge v_7. \end{aligned} \tag{5.37}$$

Proof. Let $u_1, \ldots, u_4 \in \Xi$ be independent and orthogonal. From Propositions 4.5.3, 5.3.19 and Equation (5.15) we have

$$B_Q^\Xi(u_1, u_2, u_3, u_4) = 6q_\Sigma(\psi_0)B_\Sigma(u_1, u_2 \times (u_3 \times u_4))$$

and then the decomposition (5.37) follows from a long but straightforward calculation. □

Remark 5.3.22. *In the decomposition (5.37), there are seven 4-vectors of the form $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4}$. The seven quadruples of indices $\{i_1, i_2, i_3, i_4\}$ appearing are not arbitrary. There is a numbering of the seven points of the Fano plane such that these quadruples are exactly the complements of the seven lines.*



Example 5.3.23. *If we identify* Ξ *with* $Im(\mathbb{O})$ *as in Example 5.3.4, the product* $\times$ *and the covariant* B_ψ^Ξ *are essentially the octonion bracket and associator. More precisely:*

$$\alpha \times \beta = \frac{1}{2}(\alpha\beta - \beta\alpha) \qquad \forall \alpha, \beta \in \Xi,$$
$$B_\psi^\Xi(\alpha,\beta,\gamma) = \frac{3}{4}((\alpha\beta)\gamma - \alpha(\beta\gamma)) \qquad \forall \alpha,\beta,\gamma \in \Xi.$$

5.4 The spinor representation of a Lie algebra of type $\mathfrak{so}(7)$ is special orthogonal

Suppose as in the previous section that (V, B_V) is a 7-dimensional non-degenerate quadratic vector space and $\rho : V \to \mathfrak{so}(\Sigma, B_\Sigma)$ is a Clifford representation of dimension 8. In this section, we will show that Σ is a special ϵ-orthogonal representation of $\mathfrak{so}(V, B_V)$. Recall that if $\psi_0 \in \Sigma$ is an anisotropic spinor and if $\mathfrak{g}$ is the isotropy in $\mathfrak{so}(V, B_V)$, we show that $\psi_0^\perp$ is a special ϵ-orthogonal representation of $\mathfrak{g}$.

5.4.1 The embedding of G_2 in C^2

We now study the orthogonal complement of $\mathfrak{g}$ in C^2.

Definition 5.4.1. *We define* $\Omega' \in \Lambda^3(V)^*$ *and* $\Omega \in \Lambda^3(V)$ *by*

$$\Omega'(u,v,w) := -B_V(u \times v, w) \qquad \forall u,v,w \in V,$$
$$\Omega := \Phi^{-1}(\Omega')$$

where $\Phi : \Lambda^3(V) \to \Lambda^3(V)^*$ *is the isomorphism given by*

$$\Phi(u_1 \wedge u_2 \wedge u_3)(w_1 \wedge w_2 \wedge w_3) := det(B_V(u_i, w_j)_{i,j}) \quad \forall u_1 \wedge u_2 \wedge u_3, w_1 \wedge w_2 \wedge w_3 \in \Lambda^3(V).$$

It is well-known that there is a unique $O(V, B_V)$-equivariant isomorphism $Q : \Lambda^3(V) \to C^3$ such that

$$Q(e_1 \wedge e_2 \wedge e_3) = e_1 \cdot e_2 \cdot e_3$$

if $e_1, e_2, e_3 \in V$ are orthogonal and independent. For brevity, in what follows we will denote $Q(\Omega)$ by Ω and determine its action on Σ.

Proposition 5.4.2. *Let* α *be an anisotropic spinor of* Ξ*. We have*

$$\rho(\Omega)(\psi_0) = 7\psi_0, \qquad \rho(\Omega)(\alpha) = -\alpha.$$

Proof. Suppose that α is anisotropic and let $v := \eta^{-1}(\alpha)$. There exist $v, w \in V$ such that

$$\mathcal{B} := \{u, v, w, u \times v, u \times w, v \times w, (u \times v) \times w\}$$

be an orthogonal basis of V of anisotropic vectors. A tedious calculation shows that

$$\begin{aligned}\Omega = &-\frac{u \cdot v \cdot u \times v}{q_V(u)q_V(v)} - \frac{u \cdot w \cdot u \times w}{q_V(u)q_V(w)} - \frac{v \cdot w \cdot v \times w}{q_V(v)q_V(w)} - \frac{u \cdot v \times w \cdot (u \times v) \times w}{q_V(u)q_V(v)q_V(w)} \\ &+ \frac{v \cdot u \times w \cdot (u \times v) \times w}{q_V(u)q_V(v)q_V(w)} - \frac{w \cdot u \times v \cdot (u \times v) \times w}{q_V(u)q_V(v)q_V(w)} - \frac{u \times v \cdot u \times w \cdot v \times w}{q_V(u)q_V(v)q_V(w)}.\end{aligned}$$

and another tedious calculation that

$$\rho(\Omega)(\psi_0) = 7\psi_0, \qquad \rho(\Omega)(\alpha) = -\alpha.$$

□

Since the Clifford algebra $C = C^+ \oplus C^-$ is $\mathbb{Z}_2$-graded, it is a Lie superalgebra for the bracket $\{\ ,\ \} : C \times C \to C$ given by

$$\{c, d\} := cd + (-1)^{|c||d|}dc \qquad \forall c, d \in C$$

where $|c|$ and $|d|$ denotes the parity of the two elements c and d.

Definition 5.4.3. *Let $v \in V$. We define $c_v \in C^2$ by*

$$c_v := \{v, \Omega\}$$

and we define the 7-dimensional subspace $W \subset C^2$ by

$$W := Vect < \{c_v \mid v \in V\} > .$$

The subspace W acts on Σ as follows.

Proposition 5.4.4. *Let $v, w \in V$ such that v is anisotropic and such that $B_V(v, w) = 0$. We have*

$$\rho(c_v)(\psi_0) = 6v \cdot \psi_0, \quad \rho(c_v)(v \cdot \psi_0) = 6q_V(v)\psi_0, \quad \rho(c_v)(w \cdot \psi_0) = -2v \times w \cdot \psi_0.$$

Proof. We have

$$\begin{aligned}\rho(c_v)(\psi_0) &= v \cdot \Omega \cdot \psi_0 + \Omega \cdot v \cdot \psi_0 = 7v \cdot \psi_0 - v \cdot \psi_0 = 6v \cdot \psi_0, \\ \rho(c_v)(v \cdot \psi_0) &= v \cdot \Omega \cdot v \cdot \psi_0 + \Omega \cdot v^2 \cdot \psi_0 = 6v^2\psi_0 = 6q(v)\psi_0.\end{aligned}$$

Since $B_V(v, w) = 0$ we have $v \times w \cdot \psi_0 = v \cdot w \cdot \psi_0$ and hence

$$\rho(c_v)(w \cdot \psi_0) = v \cdot \Omega \cdot w \cdot \psi_0 + \Omega \cdot v \cdot w \cdot \psi_0 = -2v \cdot w \cdot \psi_0 = -2v \times w \cdot \psi_0.$$

□

The Lie algebra C^2 is quadratic for the bilinear form $B_{\mathfrak{s}} : C^2 \times C^2 \to k$ defined by

$$B_{\mathfrak{s}}(f,g) := -\frac{3}{8} Tr(\rho(f) \circ \rho(g)) \qquad \forall f, g \in C^2.$$

We can now prove that W is orthogonal to $\mathfrak{g}$ in C^2.

Proposition 5.4.5. *Let v, w be anisotropic vectors in V such that $B_V(v,w) = 0$ and let D in $\mathfrak{g}$. We have*

$$B_{\mathfrak{s}}(D, c_v) = 0, \qquad B_{\mathfrak{s}}(c_v, c_w) = 0, \qquad B_{\mathfrak{s}}(c_v, c_v) = -36 q_V(v).$$

Proof. By Corollary 5.3.11, there exists u in V anisotropic such that

$$\mathcal{B} := \{\psi_0, v\cdot\psi_0, w\cdot\psi_0, v\times w\cdot\psi_0, u\cdot\psi_0, v\times u\cdot\psi_0, w\times u\cdot\psi_0, (v\times w)\times u\cdot\psi_0\}$$

is an orthogonal basis of Σ of anisotropic spinors. We have

$$Tr(\rho(D)\circ\rho(c_v)) = \sum_{c\in\mathcal{B}} \frac{B_\Sigma(D(c_v(c)), c)}{q_\Sigma(c)}.$$

Furthermore, using Proposition 5.4.4

$$\frac{B_\Sigma(D(c_v(\psi_0)), \psi_0)}{q_\Sigma(\psi_0)} = 0, \tag{5.38}$$

$$\frac{B_\Sigma(D(c_v(v\cdot\psi_0)), v\cdot\psi_0)}{q_\Sigma(v\cdot\psi_0)} = 0, \tag{5.39}$$

$$\frac{B_\Sigma(D(c_v(w\cdot\psi_0)), w\cdot\psi_0)}{q_\Sigma(w\cdot\psi_0)} = 2\frac{B_\Sigma(v\cdot\psi_0 \times D(w\cdot\psi_0), w\cdot\psi_0)}{q_V(w)q_\Sigma(\psi_0)}, \tag{5.40}$$

$$\frac{B_\Sigma(D(c_v(v\times w\cdot\psi_0)), v\times w\cdot\psi_0)}{q_\Sigma(v\times w\cdot\psi_0)} = -2\frac{B_\Sigma(v\cdot\psi_0 \times D(w\cdot\psi_0), w\cdot\psi_0)}{q_V(w)q_\Sigma(\psi_0)}, \tag{5.41}$$

$$\frac{B_\Sigma(D(c_v(u\cdot\psi_0)), u\cdot\psi_0)}{q_\Sigma(u\cdot\psi_0)} = 2\frac{B_\Sigma(v\cdot\psi_0 \times D(u\cdot\psi_0), u\cdot\psi_0)}{q_V(u)q_\Sigma(\psi_0)}, \tag{5.42}$$

$$\frac{B_\Sigma(D(c_v(v\times u\cdot\psi_0)), v\times u\cdot\psi_0)}{q_\Sigma(v\times u\cdot\psi_0)} = 2\frac{B_\Sigma(D(u\cdot\psi_0), v\times u\cdot\psi_0)}{q_V(u)q_\Sigma(\psi_0)}, \tag{5.43}$$

$$\begin{aligned}\frac{B_\Sigma(D(c_v(w\times u\cdot\psi_0)), w\times u\cdot\psi_0)}{q_\Sigma(v\times u\cdot\psi_0)} &= 2(\frac{B_\Sigma(v\cdot\psi_0 \times D(w\cdot\psi_0), w\cdot\psi_0)}{q_V(w)q_\Sigma(\psi_0)} \\ &+ \frac{B_\Sigma(v\cdot\psi_0 \times D(u\cdot\psi_0), u\cdot\psi_0)}{q_V(u)q_\Sigma(\psi_0)}),\end{aligned} \tag{5.44}$$

$$\begin{aligned}\frac{B_\Sigma(D(c_v((v\times w)\times u\cdot\psi_0)), (v\times w)\times u\cdot\psi_0)}{q_\Sigma(v\times u\cdot\psi_0)} &= 2(\frac{B_\Sigma(v\times u\cdot\psi_0, D(u\cdot\psi_0))}{q_V(u)q_\Sigma(\psi_0)} \\ &+ \frac{B_\Sigma(v\times w, D(w\cdot\psi_0))}{q_V(w)q_\Sigma(\psi_0)}).\end{aligned} \tag{5.45}$$

Hence, summing Equations (5.38),(5.39),(5.40),(5.41),(5.42),(5.43),(5.44) and (5.45), we obtain that $Tr(\rho(D) \circ \rho(c_v)) = 0$ and so $B_{\mathfrak{s}}(D, c_v) = 0$. We have

$$Tr(\rho(c_v) \circ \rho(c_w)) = -\sum_{c \in \mathcal{B}} \frac{B_\Sigma(c_v(c), c_w(c))}{q_\Sigma(c)}$$

but since $B_V(v, w) = 0$ we have $B_\Sigma(c_v(c), c_w(c)) = 0$ for all $c \in \mathcal{B}$ and so $B_{\mathfrak{s}}(c_v, c_w) = 0$. We have

$$Tr(\rho(c_v) \circ \rho(c_v)) = -\sum_{c \in \mathcal{B}} \frac{q_\Sigma(c_v(c))}{q_\Sigma(c)}.$$

Furthermore

$$\frac{q_\Sigma(c_v(\psi_0))}{q_\Sigma(\psi_0)} = -36 q_V(v), \qquad \frac{q_\Sigma(c_v(v \cdot \psi_0))}{q_\Sigma(\psi_0)} = -36 q_V(v)$$

and for $c \in Vect < \psi_0 >^\perp$ such that $B_\Sigma(c, v \cdot \psi_0) = 0$ we have

$$\frac{q_\Sigma(c_v(c))}{q_\Sigma(c)} = -4 q_V(v).$$

Hence $Tr(\rho(c_v) \circ \rho(c_v)) = 96 q_V(v)$ and so $B_{\mathfrak{s}}(c_v, c_v) = -36 q_V(v)$. □

Corollary 5.4.6. *We have the orthogonal decomposition of vector spaces*

$$C^2 = \mathfrak{g} \oplus W.$$

The subset $\mathfrak{g}$ is a Lie subalgebra of C^2 but the decomposition $C^2 = \mathfrak{g} \oplus W$ is not $\mathbb{Z}_2$-graded as we now show.

Proposition 5.4.7. *We have*

$$[c_v, c_w] = -4 c_{v \times w} + \frac{32}{q_\Sigma(\psi_0)} \mu_\Xi(v \cdot \psi_0, w \cdot \psi_0) \qquad \forall v, w \in V.$$

Proof. Let $v, w \in V$ be anisotropic vectors and such that $B_V(v, w) = 0$. We have

$$\Sigma = Vect(\{\psi_0, v \cdot \psi_0, w \cdot \psi_0\}) \oplus Vect(\{\psi_0, v \cdot \psi_0, w \cdot \psi_0\})^\perp.$$

By a straightforward calculation one can check

$$\rho([c_v, c_w])(\alpha) = -4 \rho(c_{v \times w})(\alpha) + \frac{32}{q_\Sigma(\psi_0)} \mu_\Xi(v \cdot \psi_0, w \cdot \psi_0)(\alpha)$$

for $\alpha \in \{\psi_0, v \cdot \psi_0, w \cdot \psi_0\}$ and $\alpha \in Vect(\{\psi_0, v \cdot \psi_0, w \cdot \psi_0\})^\perp$. □

Remark 5.4.8. *By the previous proposition, the representation $\mathfrak{g} \to \mathfrak{so}(W, B_{\mathfrak{s}}|_W)$ of the quadratic Lie algebra $(\mathfrak{g}, B_{\mathfrak{s}}|_{\mathfrak{g}})$ together with a non-trivial map $\phi : \Lambda^2(W) \to W$ is of Lie type in the sense of Kostant (see Theorem 4.1.3 if $k = \mathbb{R}, \mathbb{C}$ or Theorem 4.2.10).*

5.4.2 The spinor representation of C^2

In this subsection, we give formulae for the moment map of the representation $\rho : C^2 \to \mathfrak{so}(\Sigma, B_\Sigma)$ of the quadratic Lie algebra $(C^2, B_\mathfrak{s})$ in order to show that Σ is a special orthogonal representation of C^2.

Proposition 5.4.9. *The moment map $\mu_\Sigma : \Lambda^2(\Sigma) \to C^2$ of the representation $\rho : C^2 \to \mathfrak{so}(\Sigma, B_\Sigma)$ of the quadratic Lie algebra $(C^2, B_\mathfrak{s})$ satisfies:*

$$\mu_\Sigma(\alpha,\beta) = -\frac{q_\Sigma(\psi_0)}{18} c_{\eta^{-1}(\alpha\times\beta)} + \frac{8}{9}\mu_\Xi(\alpha,\beta) \qquad \forall \alpha,\beta \in \Xi,$$
$$\mu_\Sigma(\psi_0,\alpha) = \frac{q_\Sigma(\psi_0)}{6} c_{\eta^{-1}(\alpha)} \qquad \forall \alpha \in \Xi.$$

Proof. Let $\alpha, \beta \in \Xi$ and $D \in \mathfrak{g}$. By Proposition 5.3.12, we have

$$B_\mathfrak{s}(D, \frac{8}{9}\mu_\Xi(\alpha,\beta)) = B_\mathfrak{g}(D, \mu_\Xi(\alpha,\beta)) = B_\Sigma(D(\alpha),\beta).$$

From Propositions 5.4.5 and 5.4.4, we obtain

$$B_\mathfrak{s}(c_{\eta^{-1}(\alpha\times\beta)}, -\frac{q_\Sigma(\psi_0)}{18} c_{\eta^{-1}(\alpha\times\beta)}) = B_\Sigma(c_{\eta^{-1}(\alpha\times\beta)}(\alpha),\beta)$$

and so

$$\mu_\Sigma(\alpha,\beta) = -\frac{q_\Sigma(\psi_0)}{18} c_{\eta^{-1}(\alpha\times\beta)} + \frac{8}{9}\mu_\Xi(\alpha,\beta).$$

We have

$$B_\mathfrak{s}(D, \mu_\Sigma(\psi_0,\alpha)) = B_\Sigma(D(\psi_0),\alpha) = 0$$

and from Propositions 5.4.5 and 5.4.4

$$B_\mathfrak{s}(c_{\eta^{-1}(\alpha)}, \frac{q_\Sigma(\psi_0)}{6} c_{\eta^{-1}(\alpha)}) = B_\Sigma(c_{\eta^{-1}(\alpha)}(\psi_0),\alpha)$$

and so

$$\mu_\Sigma(\psi_0,\alpha) = \frac{q_\Sigma(\psi_0)}{6} c_{\eta^{-1}(\alpha)}.$$

□

We now give the main result of this section.

Proposition 5.4.10. *The representation $\rho : C^2 \to \mathfrak{so}(\Sigma, B_\Sigma)$ of the quadratic Lie algebra $(C^2, B_\mathfrak{s})$ is a special orthogonal representation.*

Proof. By Theorem 4.3.10 we have to show that:

$$\mu_\Sigma(\alpha,\beta)(\gamma)+\mu_\Sigma(\alpha,\gamma)(\beta)=B_\Sigma(\alpha,\beta)\gamma+B_\Sigma(\alpha,\gamma)\beta-2B_\Sigma(\beta,\gamma)\alpha \quad \forall\alpha,\beta,\gamma\in\Sigma. \tag{5.46}$$

To prove this, it is enough to show that:

$$\mu_\Sigma(\alpha,\beta)(\alpha)=q_\Sigma(\alpha)\beta, \tag{5.47}$$
$$\mu_\Sigma(\alpha,\beta)(\gamma)+\mu(\alpha,\gamma)(\beta)=0 \tag{5.48}$$

for $\alpha,\beta,\gamma\in\Sigma$ anisotropic such that $B_\Sigma(\alpha,\beta)=0$, $B_\Sigma(\alpha,\gamma)=0$ and $B_\Sigma(\beta,\gamma)=0$.

For $\alpha=\psi_0$ we have

$$\mu_\Sigma(\psi_0,\beta)(\psi_0)=\frac{q_\Sigma(\psi_0)}{6}c_{\eta^{-1}(\beta)}(\psi_0)=q_\Sigma(\psi_0)\beta.$$

For $\beta=\psi_0$ we have

$$\mu_\Sigma(\alpha,\psi_0)(\alpha)=-\frac{q_\Sigma(\psi_0)}{6}c_{\eta^{-1}(\alpha)}(\alpha)=q_\Sigma(\alpha)\psi_0.$$

For α,β orthogonal to ψ_0 we have

$$\mu_\Sigma(\alpha,\beta)(\alpha)=-\frac{q_\Sigma(\psi_0)}{18}c_{\eta^{-1}(\alpha)\times\eta^{-1}(\beta)}(\alpha)+\frac{8}{9}\mu_\Xi(\alpha,\beta)(\alpha)=q_\Sigma(\alpha)\beta$$

and hence equation (5.47) is proven.

To prove (5.48), let $\alpha=\psi_0$. Then

$$\mu_\Sigma(\psi_0,\beta)(\gamma)+\mu_\Sigma(\psi_0,\gamma)(\beta)=\frac{q_\Sigma(\psi_0)}{6}c_{\eta^{-1}(\beta)}(\gamma)+\frac{q_\Sigma(\psi_0)}{6}c_{\eta^{-1}(\gamma)}(\beta)=0.$$

Similarly, for $\beta=\psi_0$ we have

$$\begin{aligned}\mu_\Sigma(\alpha,\psi_0)(\gamma)+\mu_\Sigma(\alpha,\gamma)(\psi_0)&=-\frac{q_\Sigma(\psi_0)}{6}c_{\eta^{-1}(\alpha)}(\gamma)-\frac{q_\Sigma(\psi_0)}{18}c_{\eta^{-1}(\alpha)\times\eta^{-1}(\gamma)}(\psi_0)\\&=\frac{q_\Sigma(\psi_0)}{3}\alpha\times\gamma-\frac{q_\Sigma(\psi_0)}{3}\alpha\times\gamma\\&=0.\end{aligned}$$

This proves (5.48) for $\alpha=\psi_0$ or $\beta=\psi_0$ and also for $\gamma=\psi_0$ by the symmetry of the equation.

Now suppose that α,β and γ are orthogonal to ψ_0. Then by Proposition 5.4.9 we have

$$\begin{aligned}\mu_\Sigma(\alpha,\beta)(\gamma)+\mu_\Sigma(\alpha,\gamma)(\beta)=&-\frac{q_\Sigma(\psi_0)}{18}c_{\eta^{-1}(\alpha)\times\eta^{-1}(\beta)}(\gamma)+\frac{8}{9}\mu_\Xi(\alpha,\beta)(\gamma)\\&-\frac{q_\Sigma(\psi_0)}{18}c_{\eta^{-1}(\alpha)\times\eta^{-1}(\gamma)}(\beta)+\frac{8}{9}\mu_\Xi(\alpha,\gamma)(\beta).\end{aligned}$$

Since

$$c_{\eta^{-1}(\alpha)\times\eta^{-1}(\beta)}(\gamma) + c_{\eta^{-1}(\alpha)\times\eta^{-1}(\gamma)}(\beta) = -2((\alpha\times\beta)\times\gamma + (\alpha\times\gamma)\times\beta) = 0, \tag{5.49}$$

by Equation (5.15) and since

$$\mu_\Xi(\alpha,\beta)(\gamma) + \mu_\Xi(\alpha,\gamma)(\beta) = 0 \tag{5.50}$$

by Proposition 5.3.16, it follows that

$$\mu_\Sigma(\alpha,\beta)(\gamma) + \mu_\Sigma(\alpha,\gamma)(\beta) = 0.$$

This proves equation (5.48) and the representation $\rho : C^2 \to \mathfrak{so}(\Sigma, B_\Sigma)$ is special orthogonal. □

Since the orthogonal representation $C^2 \to \mathfrak{so}(\Sigma, B_\Sigma)$ is special, by Theorem 4.4.7, there is a Lie superalgebra of the form

$$\tilde{\mathfrak{f}} := C^2 \oplus \mathfrak{sl}(2,k) \oplus \Sigma \otimes k^2.$$

If $k = \mathbb{C}$, Kac (see [Kac75]) showed that there is a simple exceptional Lie superalgebra F_4 of dimension 40 whose even part is isomorphic to $\mathfrak{so}(7)\times\mathfrak{sl}(2,\mathbb{C})$ and whose odd part is isomorphic to the tensor product of the spinor representation of $\mathfrak{so}(7)$ and the fundamental representation of $\mathfrak{sl}(2,\mathbb{C})$.

Proposition 5.4.11. *If $k = \mathbb{C}$ then $\tilde{\mathfrak{f}}$ is isomorphic to F_4.*

Proof. Since the representation $\Sigma \otimes k^2$ is a faithful and irreducible representation of $C^2 \oplus \mathfrak{sl}(2,k)$ and since

$$\{\Sigma\otimes k^2, \Sigma\otimes k^2\} = C^2 \oplus \mathfrak{sl}(2,k)$$

then $\tilde{\mathfrak{f}}$ is a simple Lie superalgebra (see Lemma 3 p.95 in [Sch79b]). One can check in the classification of Kac (see [Kac75]) that the only simple Lie superalgebra of dimension 40 whose even part is isomorphic to $\mathfrak{so}(7)\times\mathfrak{sl}(2,\mathbb{C})$ is F_4 and then $\tilde{\mathfrak{f}}$ is isomorphic to F_4. □

Remark 5.4.12. *If $k = \mathbb{R}$, Serganova (see [Ser83]) showed that there are four real forms of F_4. In particular, two of them have an even part isomorphic to $\mathfrak{so}(7)\oplus\mathfrak{sl}(2,\mathbb{R})$ (resp. $\mathfrak{so}(4,3)\oplus\mathfrak{sl}(2,\mathbb{R})$) and an odd part isomorphic to the tensor product of the spinor representation of $\mathfrak{so}(7)$ (resp. $\mathfrak{so}(4,3)$) and $\mathbb{R}^2$. By our construction, if (V, B_V) is of signature $(7,0)$ or $(4,3)$, we obtain these two real forms of F_4 by Proposition 5.4.11.*

Since the representation $C^2 \to \mathfrak{so}(\Sigma, B_\Sigma)$ is special, we can explicitly calculate its covariants B_ψ^Σ and B_Q^Σ. Recall that from Theorem 4.5.6 we have the identities

$$\mu_\Sigma \circ B_\psi^\Sigma = 3B_Q^\Sigma \wedge_\times \mu_\Sigma \quad \in Alt_\epsilon^6(\Sigma, C^2), \tag{5.51}$$

$$B_\psi^\Sigma \circ B_\psi^\Sigma = -\frac{27}{2} B_Q^\Sigma \wedge_\epsilon B_Q^\Sigma \wedge_\times Id_\Sigma \quad \in Alt_\epsilon^9(\Sigma, \Sigma), \tag{5.52}$$

$$B_Q^\Sigma \circ B_\psi^\Sigma = -54 B_Q^\Sigma \wedge_\epsilon B_Q^\Sigma \wedge_\epsilon B_Q^\Sigma \quad \in Alt_\epsilon^{12}(\Sigma, k). \tag{5.53}$$

As a vector space Σ is of dimension 8 so

$$B_\psi^\Sigma \circ B_\psi^\Sigma = B_Q^\Sigma \wedge_\epsilon B_Q^\Sigma \wedge_\times Id = B_Q^\Sigma \circ B_\psi^\Sigma = B_Q^\Sigma \wedge_\epsilon B_Q^\Sigma \wedge_\epsilon B_Q^\Sigma = 0$$

since as alternating maps, these are of degree $9, 9, 12$ and 12 respectively.

Proposition 5.4.13. *Let $B_\psi^\Sigma \in Alt^3(\Sigma, \Sigma)$ be the trilinear covariant of the representation $C^2 \to \mathfrak{so}(\Sigma, B_\Sigma)$. Let α, β, γ be anisotropic spinors such that $B_\Sigma(\alpha, \beta) = 0$, $B_\Sigma(\alpha, \gamma) = 0$ and $B_\Sigma(\beta, \gamma) = 0$.*

a) If $\alpha, \beta, \gamma \in \Xi$ and $B_\Sigma(\alpha \times \beta, \gamma) = 0$, then:

$$B_\psi^\Sigma(\alpha, \beta, \gamma) = q_\Sigma(\psi_0)\alpha \times (\beta \times \gamma).$$

b) If $\alpha, \beta \in \Xi$, then:

$$B_\psi^\Sigma(\alpha, \beta, \alpha \times \beta) = \frac{q_\Sigma(\alpha) q_\Sigma(\beta)}{q_\Sigma(\psi_0)} \psi_0.$$

c) If $\gamma = \psi_0$, then:

$$B_\psi^\Sigma(\alpha, \beta, \gamma) = -q_\Sigma(\psi_0)\alpha \times \beta.$$

Proof. A straightforward calculation using Proposition 5.4.9. □

Proposition 5.4.14. *Let $B_Q^\Sigma \in \Lambda^4(\Sigma)^*$ be the quadrilinear covariant of the representation $C^2 \to \mathfrak{so}(\Sigma, B_\Sigma)$. Let*

$$\mathcal{B} := \{\psi_0, \alpha, \beta, \alpha \times \beta, \gamma, \alpha \times \gamma, \beta \times \gamma, (\alpha \times \beta) \times \gamma\}$$

be a basis of Σ as in Corollary 5.3.11 and set:

$v_1 := \psi_0, v_2 := \alpha, v_3 := \beta, v_4 := \alpha \times \beta, v_5 := \gamma, v_6 := \alpha \times \gamma, v_7 := \beta \times \gamma, v_8 := (\alpha \times \beta) \times \gamma.$

Let $\Phi : \Lambda^4(\Sigma) \to \Lambda^4(\Sigma)^*$ be the isomorphism defined by

$$\Phi(u_1 \wedge \ldots \wedge u_4)(w_1 \wedge \ldots \wedge w_4) := det(B_\Sigma(u_i, w_j)_{i,j}) \quad \forall u_1 \wedge \ldots \wedge u_4, w_1 \wedge \ldots \wedge w_4 \in \Lambda^4(\Sigma).$$

Then:

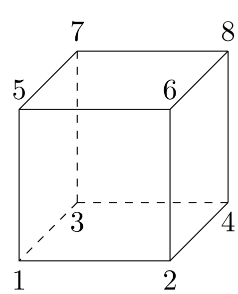
$$\begin{aligned}\Phi^{-1}(B_Q^\Sigma) = &\frac{4}{q_\Sigma(v_2)q_\Sigma(v_3)} v_1\wedge v_2\wedge v_3\wedge v_4 + \frac{4}{q_\Sigma(v_2)q_\Sigma(v_5)} v_1\wedge v_2\wedge v_5\wedge v_6\\ &-\frac{4q_\Sigma(v_1)}{q_\Sigma(v_2)q_\Sigma(v_3)q_\Sigma(v_5)} v_1\wedge v_2\wedge v_7\wedge v_8 + \frac{4}{q_\Sigma(v_3)q_\Sigma(v_5)} v_1\wedge v_3\wedge v_5\wedge v_7\\ &+\frac{4q_\Sigma(v_1)}{q_\Sigma(v_2)q_\Sigma(v_3)q_\Sigma(v_5)} v_1\wedge v_3\wedge v_6\wedge v_8 + \frac{4q_\Sigma(v_1)}{q_\Sigma(v_2)q_\Sigma(v_3)q_\Sigma(v_5)} v_1\wedge v_4\wedge v_5\wedge v_8\\ &-\frac{4q_\Sigma(v_1)}{q_\Sigma(v_2)q_\Sigma(v_3)q_\Sigma(v_5)} v_1\wedge v_4\wedge v_6\wedge v_7 + \frac{4q_\Sigma(v_1)}{q_\Sigma(v_2)q_\Sigma(v_3)q_\Sigma(v_5)} v_2\wedge v_3\wedge v_5\wedge v_8\\ &-\frac{4q_\Sigma(v_1)}{q_\Sigma(v_2)q_\Sigma(v_3)q_\Sigma(v_5)} v_2\wedge v_3\wedge v_6\wedge v_7 - \frac{4q_\Sigma(v_1)}{q_\Sigma(v_2)q_\Sigma(v_3)q_\Sigma(v_5)} v_2\wedge v_4\wedge v_5\wedge v_7\\ &-\frac{4q_\Sigma(v_1)^2}{q_\Sigma(v_2)^2q_\Sigma(v_3)q_\Sigma(v_5)} v_2\wedge v_4\wedge v_6\wedge v_8 + \frac{4q_\Sigma(v_1)}{q_\Sigma(v_2)q_\Sigma(v_3)q_\Sigma(v_5)} v_3\wedge v_4\wedge v_5\wedge v_6\\ &-\frac{4q_\Sigma(v_1)^2}{q_\Sigma(v_2)q_\Sigma(v_3)^2q_\Sigma(v_5)} v_3\wedge v_4\wedge v_7\wedge v_8 - \frac{4q_\Sigma(v_1)^2}{q_\Sigma(v_2)q_\Sigma(v_3)q_\Sigma(v_5)^2} v_5\wedge v_6\wedge v_7\wedge v_8.\end{aligned} \tag{5.54}$$

Proof. This can be proved by a long but straightforward calculation using Propositions 4.5.3 and 5.4.13. □

Corollary 5.4.15. *We have*

$$\Phi^{-1}(B_Q^\Sigma)\wedge_\epsilon \Phi^{-1}(B_Q^\Sigma) = \frac{-224q_\Sigma(v_1)^2}{q_\Sigma(v_2)^2q_\Sigma(v_3)^2q_\Sigma(v_5)^2} v_1\wedge\ldots\wedge v_8.$$

Remark 5.4.16. *a) In the decomposition (5.54), there are fourteen 4-vectors of the form $v_{i_1}\wedge v_{i_2}\wedge v_{i_3}\wedge v_{i_4}$. The fourteen quadruples of indices $\{i_1,i_2,i_3,i_4\}$ appearing are not arbitrary. There is a numbering of the eight points of the affine space $(\mathbb{Z}_2)^3$ such that each quadruple corresponds to one of the fourteen affine planes.*



b) If $k=\mathbb{R}$, B_Q^Σ is an $\mathfrak{so}(V,B_V)$-invariant 4-form and if fact this characterises $\mathfrak{so}(V,B_V)$ as a Lie subalgebra of $\mathfrak{gl}(V)$. Recall that the definition of a Spin(7)-structure on a 8-dimensional vector space is as the isotropy group of certain type of 4-form.

Remark 5.4.17. *If char(k)$\neq 7$, $B_Q^\Sigma \wedge_\epsilon B_Q^\Sigma$ is non-zero by Corollary 5.4.15 and then is a basis of $\Lambda^8(\Sigma)^*$. In contrast with the special orthogonal representation $\mathfrak{g}\to\mathfrak{so}(\Xi,B_\Sigma)$ (see Proposition 5.3.19), both-sides of (5.51) do not vanish identically. We have $B_Q^\Sigma \wedge_\times \mu_\Sigma \neq 0$ since*

$$B_Q^\Sigma \wedge_\times \mu_\Sigma \wedge_{B_\mathfrak{s}} \mu_\Sigma = B_Q^\Sigma \wedge_\epsilon N_{B_\mathfrak{s}}(\mu_\Sigma) = -\frac{1}{2} B_Q^\Sigma \wedge_\epsilon B_Q^\Sigma.$$

By Theorem 4.5.6 we also have $B_Q^\Sigma \wedge_\times \mu_\Sigma \neq 0$.

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Philippe MEYER

Représentations associées à des graduations d'algèbres de Lie et d'algèbres de Lie colorées

Soit k un corps de caractéristique différente de 2 et de 3. Les algèbres de Lie colorées généralisent à la fois les algèbres de Lie et les superalgèbres de Lie. Dans cette thèse on étudie des représentations V d'algèbres de Lie colorées g provenant de structures d'algèbres de Lie colorées sur l'espace vectoriel g⊕V. En premier lieu, on s'intéresse à la structure générale des algèbres de Lie simples de dimension 3 sur k. Puis, on classifie à isomorphisme près les superalgèbres de Lie de dimension finie dont la partie paire est une algèbre de Lie simple de dimension 3. Ensuite, pour un groupe abélien Γ et un facteur de commutation ε de Γ, on développe l'algèbre multilinéaire associée aux espaces vectoriels Γ-gradués. Dans ce contexte, les algèbres de Lie colorées jouent le rôle des algèbres de Lie. Ce langage nous permet d'énoncer et prouver un théorème de reconstruction d'une algèbre de Lie colorée ε-quadratique g⊕V à partir d'une représentation ε-orthogonale V d'une algèbre de Lie colorée ε-quadratique g. Ce théorème fait intervenir un invariant qui prend ses valeurs dans la ε-algèbre extérieure de V et généralise des résultats de Kostant et Chen-Kang. Puis, on introduit la notion de représentation ε-orthogonale spéciale V d'une algèbre de Lie colorée ε-quadratique g et on montre qu'elle permet de définir une structure d'algèbre de Lie colorée ε-quadratique sur l'espace vectoriel g⊕sl(2,k)⊕V⊗k². Enfin on donne des exemples de représentations ε-orthogonales spéciales, notamment des représentations orthogonales spéciales d'algèbres de Lie dont : une famille à un paramètre de représentations de sl(2,k)⊕sl(2,k) ; la représentation fondamentale de dimension 7 d'une algèbre de Lie de type G_2 ; la représentation spinorielle de dimension 8 d'une algèbre de Lie de type so(7).

Mots-Clés : extensions d'algèbres de Lie colorée - invariant de Kostant - représentations spéciales.

Let k be a field of characteristic not 2 or 3. Colour Lie algebras generalise both Lie algebras and Lie superalgebras. In this thesis we study representations V of colour Lie algebras g arising from colour Lie algebras structures on the vector space g⊕V. Firstly, we study the general structure of simple three-dimensional Lie algebras over k. Then, we classify up to isomorphism finite-dimensional Lie superalgebras whose even part is a simple three-dimensional Lie algebra. Next, to an abelian group Γ and a commutation factor ε of Γ, we develop the multilinear algebra associated to Γ-graded vector spaces. In this context, colour Lie algebras play the role of Lie algebras. This language allows us to state and prove a theorem reconstructing an ε-quadratic colour Lie algebra g⊕V from an ε-orthogonal representation V of an ε-quadratic colour Lie algebra g. This theorem involves an invariant taking its values in the ε-exterior algebra of V and generalises results of Kostant and Chen-Kang. We then introduce the notion of a special ε-orthogonal representation V of an ε-quadratic colour Lie algebra g and show that it allows us to define an ε-quadratic colour Lie algebra structure on the vector space g⊕sl(2,k)⊕V⊗k². Finally we give examples of special ε-orthogonal representations and in particular examples of special orthogonal representations of Lie algebras amongst which are: a one-parameter family of representations of sl(2,k)⊕sl(2,k) ; the 7-dimensional fundamental representation of a Lie algebra of type G_2 ; the 8-dimensional spinor representation of a Lie algebra of type so(7).

Keywords: extensions of colour Lie algebras - Kostant invariant - special representations.