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Points entiers généralisés sur les variétés abéliennes

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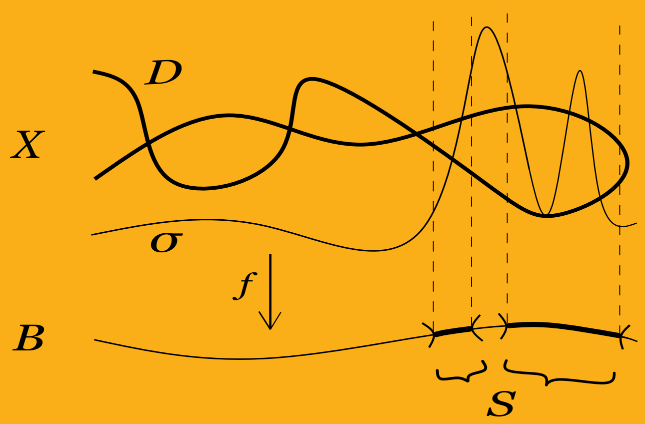

Résumé

Soit A une variété abélienne définie sur le corps de fonctions K d'une surface de Riemann compacte B. Fixons un modèle $f\colon \mathcal{A} \to B$ de A/K et un certain divisor effectif horizontal $\mathcal{D} \subset \mathcal{A}$. Dans cette thèse, les propriétés concernant *la finitude, l'ordre de croissance*, *la nonexistence générique* et *l'uniformité* de l'ensemble des sections $(S, \mathcal{D})$-entières de $\mathcal{A}$ sont étudiées. Le sous ensemble $S \subset B$ n'est pas nécessairement fini ou dénombrable. Ces sections entières σ correspondent aux points rationnels de $A(K)$ qui vérifient la condition géométrique $f(\sigma(B) \cap \mathcal{D}) \subset S$. Cette notion de section entière généralise la notion usuelle de solution entière d'un système des *équations diophantiennes*. Dans ce context, nous introduisons une certaine machinerie appelée *hauteur hyperbolique-homotopique* qui joue le rôle de la théorie d'intersection. Nous démontrons plusieurs nouveaux résultats sur la finitude de certaines unions larges de sections $(S, \mathcal{D})$-entières ainsi que leur croissance polynomiale en fonction de $\#S \cap U$, où nous demandons que les sous ensembles $S \subset B$ ne sont finis que dans un certain petit ouvert U de B. Ces résultats sont hors de portée des méthodes purement algébriques, notamment de la théorie des hauteurs standards. Nos travaux mettent en particulier en évidence certains phénomènes nouveaux qui sont en faveur de la version géométrique de la *conjecture de Lang-Vojta*. Si A est une courbe elliptique, nous obtenons les mêmes résultats pour certaines unions de sections $(S, \mathcal{D})$-entières, où non seulement $S \subset B$ mais $\mathcal{D} \subset \mathcal{A}$ peuvent aussi varier en familles. Nous en déduisons la nonexistence générique des sections entières. Si A est constante, une borne uniforme qui ne dépend que des invariants numériques est établie sur l'ensemble des sections entières. Nous démontrons également un résultat négatif concernant le *théorème de Parshin-Arakelov*. Ainsi, nos résultats motivent de nouvelles questions intéressantes sur la finitude des sections entières que ce soit sur les corps de nombres ou sur les corps de fonctions.

Abstract

Let A be an abelian variety over the function field K of a compact Riemann surface B. Fix a model $f\colon \mathcal{A} \to B$ of A/K and a certain effective horizontal divisor $\mathcal{D} \subset \mathcal{A}$. We study various properties concerning the *finiteness, growth order, generic emptiness, and uniformity* of the set of $(S, \mathcal{D})$-integral sections in $\mathcal{A}$. Here, $S \subset B$ is an arbitrary subset and thus not necessarily finite nor countable. These integral sections σ correspond to rational points in $A(K)$ which satisfy the extra geometric condition $f(\sigma(B) \cap \mathcal{D}) \subset S$. This notion of integral points generalizes the usual notion of integral solutions of a system of *Diophantine equations*. We introduce in this context the so-called *hyperbolic-homotopic height* as a substitute for the classical intersection theory. We then establish several new results concerning the finiteness of various large unions of $(S, \mathcal{D})$-integral points and their polynomial growth in terms of $\#S \cap U$, where the sets $S \subset B$ is required to be finite only in a certain small open subset U of B. Such results are out of reach of a purely algebraic method. Thereby, we give some new evidence and phenomena to the *Geometric Lang-Vojta conjecture*. When A is an elliptic curve, we obtain the same results for certain unions of $(S, \mathcal{D})$-integral points, where both $S \subset B$ and $\mathcal{D} \subset \mathcal{A}$ are allowed to vary in certain families. As an application, we obtain certain generic emptiness properties of integral points on abelian varieties over function fields. When A is constant, we prove a uniform finiteness result involving only numerical invariants on the number of integral points for $S \subset B$ finite. A negative finiteness result concerning the *Parshin-Arakelov theorem* is also given. Our work provides evidence for some new interesting questions on the finiteness of integral points in both the number field and the function field cases.

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Introduction

1. Contexte

Notations. Dans cette thèse, sauf contre-indication, la lettre B désigne toujours une courbe projective lisse de genre g définie sur un corps k de caractéristique 0 (e.g., $k = \mathbb{C}$) et $K = k(B)$ désigne le corps des fonctions rationnelles sur B. Les symboles comme X_K, Y_K sont souvent utilisés pour noter un schéma défini sur K. Si X est un B-schéma, le symbole X_K désigne également la fibre générique de X et si $U \subset B$, nous notons X_U la restriction de X au-dessus de U.

La notation d_Y est réservée pour la pseudométrique hyperbolique de Kobayashi sur un espace complexe Y. Les disques plongés dans une surface de Riemann sont supposés d'avoir un bord suffisamment lisse par morceaux. Le cardinal d'un ensemble E est noté par $\#E$.

L'étude des problèmes diophantiens, c'est-à-dire l'étude de l'ensemble des solutions entières d'un système des équations polynomiales en plusieurs variables, connaît une longue histoire qui remonte jusqu'au premier travail de Diophante d'Alexandrie au troisième siècle. Les deux points culminants de cette théorie sont le fameux dernier théorème de Fermat et la conjecture de Mordell démontrés respectivement par Andrew Wiles en 1995 (cf. [**115**]) et par Gerd Faltings en 1991 (cf. [**36**]). Alors que le premier résultat concerne la *nonexistence* des solutions non-triviales dans $\mathbb{Z}^3$ de l'équation $x^n + y^n = z^n$ quand $n \geq 3$, le second affirme la finitude de l'ensemble des points rationnels sur toute courbe algébrique de genre au moins 2.

L'idée centrale est de donner une interprétation géométrique pour les équations polynomiales ainsi qu'aux ensembles des solutions rationnelles ou entières. Par exemple, supposons que nous voulons étudier l'ensemble $I_{\mathbb{Q}}$ des solutions $(x, y) \in \mathbb{Q}^2$ et l'ensemble $I_{\mathbb{Z}}$ des solutions $(x, y) \in \mathbb{Z}^2$ de l'équation de Weierstrass :

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}. \tag{1.1}$$

L'équation (1.1) induit une courbe plane $E/\mathbb{Q}$ dans $\mathbb{P}^2_{\mathbb{Q}}$ et aussi une surface fermée $\mathcal{E}/\operatorname{Spec}(\mathbb{Z})$ dans $\mathbb{P}^2_{\mathbb{Z}}$ qui sont toutes définies par l'équation homogène $y^2 z = x^3 +$

$axz^2 + bz^3$ et qui vérifient le diagramme cartésien suivant :

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathrm{Spec}(\mathbb{Q}) \arrow[r, hook] & \mathrm{Spec}(\mathbb{Z}). \end{tikzcd}$$

Comme $\mathcal{E}$ est propre sur $\mathrm{Spec}(\mathbb{Z})$, l'ensemble des *points rationnels* $E(\mathbb{Q})$ est naturellement identifié avec l'ensemble des *sections* $\mathcal{E}(\mathrm{Spec}(\mathbb{Z}))$ d'après le critère valuatif de propreté. Soit $\mathcal{D} = \{z = 0\} \cap \mathcal{E} \subset \mathbb{P}^2_{\mathbb{Z}}$ un diviseur hyperplan. Le tire-en-arrière de $\mathcal{D}$ à E est $O = (1\colon 0\colon 0) \in E(\mathbb{Q})$. Soit $(O) \in \mathcal{E}(\mathrm{Spec}(\mathbb{Z}))$ la section associée au point O. Nous avons alors :

$$I_{\mathbb{Q}} = E(\mathbb{Q}) \setminus \{O\} = \mathcal{E}(\mathrm{Spec}(\mathbb{Z})) \setminus \{(O)\}$$

et

$$I_{\mathbb{Z}} = (\mathcal{E} \setminus \mathcal{D})(\mathrm{Spec}(\mathbb{Z})).$$

En particulier, l'ensemble $I_{\mathbb{Z}}$ des solutions entières $(x, y) \in \mathbb{Z}^2$ de (1.1) admet une interprétation géométrique comme l'ensemble des sections $\sigma\colon \mathrm{Spec}(\mathbb{Z}) \to \mathcal{E}$ qui évitent le diviseur $\mathcal{D}$, c'est-à-dire :

$$\sigma(\mathrm{Spec}(\mathbb{Z})) \cap \mathcal{D} = \varnothing.$$

L'exemple ci-dessus peut être généralisé, par exemple, quand la base $\mathrm{Spec}(\mathbb{Z})$ est remplacée par un schéma de Dedekind de dimension au plus 1, e.g., $\mathrm{Spec}(\mathcal{O})$ où $\mathcal{O}$ est l'anneau des entiers d'un corps de nombres. Par conséquent, l'étude des points rationnels ou entiers d'un problème diophantien est équivalente à l'étude des sections dans certains objets géométriques.

Dans cette présente thèse, notre objectif principal est d'analyser l'ensemble des points $(S, \mathcal{D})$-entiers d'une variété algébrique définie sur un corps de fonctions en caractéristique 0.

Définition 0.1 ($(S, \mathcal{D})$-points entiers et sections entières)**.** Soit B/k une courbe projective lisse définie sur un corps k. Soit $K = k(B)$ le corps de fonctions de B. Soit $f\colon X \to B$ un morphisme plat et propre des variétés intègres. Soient $S \subset B$ un sous-ensemble (pas nécessairement fini) et $\mathcal{D} \subset X$ un sous-ensemble (pas nécessairement un diviseur effectif). Une section $\sigma\colon B \to X$ est dite $(S, \mathcal{D})$-*entière* si elle vérifie la condition suivante (voir Figure 0.1) :

$$f(\sigma(B) \cap \mathcal{D}) \subset S. \tag{1.2}$$

Pour chaque $P \in X_K(K)$, soit $\sigma_P\colon B \to X$ la section induite (voir Remarque 0.2). Alors P est dit $(S, \mathcal{D})$-*entier* si σ_P est $(S, \mathcal{D})$-entière.

Nous remarquons d'abord quelques propriétés suivantes.

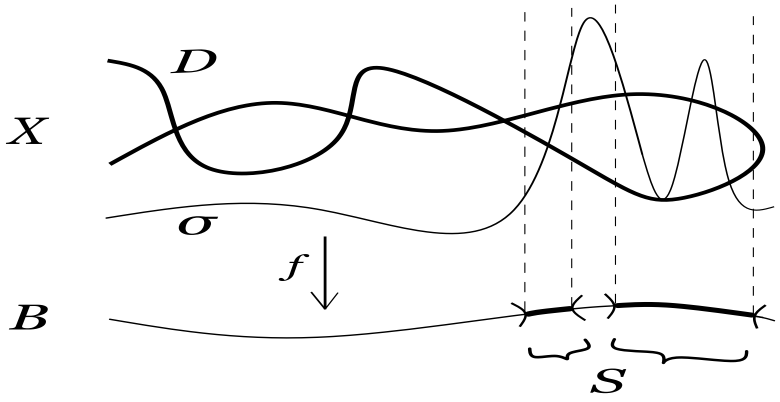

FIGURE 0.1. An $(S,\mathcal{D})$-integral section σ

Remarque 0.2. Avec les notations comme dans Définition 0.1, il y a une identification naturelle

$$X(B) = X_K(K)$$

par le critère valuatif de propreté. En effet, chaque section $\sigma\colon B \to X$ induit par le changement de base $\mathrm{Spec}(K) \to B$ un point rationnel $\sigma_K\colon \mathrm{Spec}\, K \to X_K$. Réciproquement, la fermeture de Zariski de chaque point rationnel $P \in X(K)$ induit une section dans X. L'ensemble des sections $(S,\mathcal{D})$-entières de $X \to B$ est alors identifié avec un certain sous-ensemble de $X_K(K)$.

Remarque 0.3. Si $S = B$ ou $\mathcal{D} = \varnothing$, Condition (1.2) est vide. L'ensemble des sections $(B,\mathcal{D})$-entières et l'ensemble des sections $(S,\varnothing)$-entières sont donc identifiés avec l'ensemble $X_K(K)$ des points rationnels sur la fibre générique X_K. Une autre situation extrême est quand $S = \varnothing$. Dans ce cas, Condition (1.2) implique que l'ensemble des sections $(S,\mathcal{D})$-entières est vide à chaque fois que $\mathcal{D}$ contient une fibre verticale ou $\mathcal{D}$ est un diviseur ample.

Remarque 0.4. Dans Définition 0.1, nous insistons sur le fait que l'ensemble des sections $(S,\mathcal{D})$-entières est défini par rapport à un modèle fixé préalable

$$X \to B$$

et il est important de faire la différence avec la notion usuelle dans la littérature des *ensembles quasi-entiers* des points rationnels de X_K (cf. [**99**, §7.1]). Si $S \subset B$ est un sous-ensemble fini, nous notons $\mathcal{O}_{K,S} \subset K$ l'ensemble des S-entiers. Nous fixons aussi un diviseur effectif $D \subset X$ qui est réduit et ample.

Définition 0.5 (Ensemble quasi-entier)**.** Un sous-ensemble $A \subset X_K(K) \setminus D$ est dit (S,D)-*quasi-entier* s'il existe un modèle $\mathcal{X} \to B$ de X_K, i.e., $\mathcal{X}_K = X_K$, et un diviseur effectif $\mathcal{D} \subset \mathcal{X}$ qui est un modèle de D, i.e., $\mathcal{D}_K = D$, tel que pour chaque $P \in A$, la section associée $\sigma_P : B \to \mathcal{X}$ est une section $(S, \mathcal{D})$-entière.

La notion des ensembles quasi-entiers des points rationnels est relative aux choix du modèle. Notamment, tout sous-ensemble fini $A \subset X_K(K) \setminus D$ est (S, D)-quasi-entier. Par conséquent, la notion des points (S, D)-*quasi-entiers* n'est pas intéressante. En effet, soit $\mathcal{X} \to B$ un modèle propre et plat de X_K. Soit $\mathcal{D} \subset \mathcal{X}$ la fermeture de Zariski D dans $\mathcal{X}$. Comme $A \subset X_K(K) \setminus D$ est un ensemble fini, l'ensemble des points d'intersection

$$I_A = \cup_{P \in A} \sigma_P(B) \cap \mathcal{D} \subset \mathcal{X}$$

est fini et la multiplicité de l'intersection en des points de I_A entre des sections $\sigma_P(B)$ (où $P \in A$) et le diviseur $\mathcal{D}$ est aussi finie. Donc, après un nombre fini d'éclatements de $\mathcal{X}$ le long des points de I_A, nous obtiendrons un autre modèle propre et plat $\mathcal{X}' \to B$ tel que l'ensemble A devient $(S, \mathcal{D}')$-quasi-entier où $\mathcal{D}'$ est la transformée stricte de $\mathcal{D}$ dans $\mathcal{X}'$.

L'intérêt principal de la notion des ensembles quasi-entiers est le suivant : nous pouvons souvent démontrer la finitude de ces ensembles indépendamment du choix d'un modèle. Ainsi, la finitude des ensembles quasi-entiers devient une propriété *intrinsèque*. Néanmoins, afin d'obtenir des résultats *quantitatifs* dans cette thèse, nous devons fixer un modèle comme expliqué ci-dessus. De ce point de vue, nous travaillons toujours dans la présente thèse avec la notion des points $(S, \mathcal{D})$-entiers par rapport à un modèle fixé comme dans Définition 0.1.

Si X_K/K est une courbe, nous avons le résultat fameux suivant sur la conjecture de Mordell, qui est valable sur des corps de fonctions ainsi que sur des corps de nombres (voir [**78**], [**66**], [**41**], [**86**], [**36**], [**37**]) :

Théorème 0.6 (Conjecture de Mordell)**.** *Supposons que X est une famille non isotriviale des courbes de genre au moins 2. Alors $X_K(K)$ est fini.*

L'une des conjectures les plus importantes en géométrie arithmétique est la conjecture suivante qui généralise la conjecture de Mordell (voir [**112**, Conjecture 4.4], [**63**]) :

Conjecture 0.7 (Lang)**.** *Soit X une variété projective lisse de type général définie sur un corps de nombres K. L'ensemble des points rationnels $X(K)$ n'est pas de Zariski dense dans X.*

Plus généralement, pour les variétés quasi-projectives et lisses (voir par exemple [**53**, F.3.5] et aussi Conjecture 1.28 pour la version *géometrique*) :

Conjecture 0.8 (Lang-Vojta)**.** *Soit X une variété lisse de type log-general définie sur un corps de nombres K. Soit S un ensemble fini de places de K et soit $\mathcal{O}_{K,S} \subset K$ l'anneau des S-entiers. Soit $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_{K,S})$ un modèle de X, i.e., $\mathcal{X}_K = X$. Alors l'ensemble des points entiers $\mathcal{X}(\mathcal{O}_{K,S})$ n'est pas de Zariski dense dans $\mathcal{X}$.*

Une variété projective lisse X est de type général si le diviseur canonique K_X est gros, c'est-à-dire $h^0(\mathcal{O}(nK_X)) \gg n^{\dim X}$. En particulier, une courbe projective lisse est de type général si et seulement si elle est de genre au moins 2. Une variété quasi-projective lisse Y est de type log-général si Y admet une compactification $\bar{Y}$ dont le diviseur de bord D est à accroissements normaux et tel que $K_{\bar{Y}} + D$ est gros.

Ces conjectures restent encore ouvertes de nos jours. Toutefois, dans [**17**], les auteurs ont établi une conséquence très intéressante suivante de la conjecture de Lang-Vojta, qui est connue parfois sous le nom de la *Conjecture de la borne uniforme* pour des courbes hyperboliques définies sur un corps de nombres (voir aussi [**84**] pour un énoncé plus fort) :

Conjecture 0.9 (Uniformité)**.** *Soit $g \geq 2$ un entier. Alors il existe un nombre $N(g)$ tel que pour tout corps de nombres F, il existe au plus un nombre fini de courbes de genre g définies sur F et qui ont plus de $N(g)$ F-points rationnels.*

L'ingrédient clé dans l'article [**17**] pour démontrer Conjecture 0.9 est le théorème de corrélation suivante. Soit $X \to B$ une famille des courbes de genre $g \geq 2$. Alors $X^n_B = X \times_B \cdots \times_B X$ (n-fois) admet un morphisme rationnel dominant à une variété projective lisse de type général V_n si n est suffisamment grand. Les F-points rationnels sur les fibres X_b définies sur un corps de nombres F se projettent sur les F-points rationnels de V_n. En prenant $X \to B$ une famille qui paramétrise toutes les courbes projectives lisses de genre $g \geq 2$, nous pouvons ensuite appliquer la conjecture de Lang à V_n et effectuer une récurrence noethérienne pour obtenir une borne uniforme (c'est-à-dire Conjecture 0.9) sur le nombre de points rationnels sur une courbe.

La conjecture 0.9 de la borne uniforme n'est pas connue même dans le cas des corps de fonctions. Néanmoins, en utilisant sa version uniforme du théorème de Parshin-Arakelov 1.35, Caporaso a obtenu une certaine version uniforme de la conjecture de Mordell sur des corps de fonctions (voir [**18**]). Rappelons que g désigne le genre de la courbe B.

Théorème 0.10 (Caporaso)**.** *Il existe un nombre $M(q, g, s) \in \mathbb{N}$ tel que pour chaque surface minimale non isotriviale X fibrée sur B dont les fibres sont des courbes projectives lisses de genre $q \geq 2$ en dehors d'un ensemble fini $S \subset B$ de cardinal s, alors $\#X_K(K) \leq M(q, g, s)$.*

Dans le cas des courbes elliptiques, nous devons nous restreindre aux points entiers afin d'obtenir des résultats de finitude car l'ensemble des points rationnels est infini sauf si le rang de Mordell-Weil de cette courbe elliptique est nul. Soit S un ensemble fini de places contenant 2 and 3 d'un corps de nombres F. Soit $f\colon \mathcal{E} \to \mathrm{Spec}(\mathcal{O}_F)$ le modèle de Néron sur l'anneau des entiers $\mathcal{O}_F$ d'une courbe elliptique E/F. Soit $(O) \subset \mathcal{E}$ la section zéro et soit $V_a \subset \mathcal{E}$ l'union de tous les composants zéros des fibres additives de $\mathcal{E}$. Soit S_a le sous-ensemble fini des places additives. Notons $\mathcal{D} = (O) \cup V_a$ qui est un diviseur effectif connexe de $\mathcal{E}$. Abramovich a introduit la notion des points S-*entiers stables* de E qui sont par définition les points rationnels $P \in E(K)$ dont la section correspondante $\sigma_P\colon \mathrm{Spec}(\mathcal{O}_F) \to \mathcal{E}$ est $(S \setminus S_a, \mathcal{D})$-entière dans $\mathcal{E}$ dans le sens de Définition 0.1, c'est-à-dire,

$$f(\sigma_P(\mathrm{Spec}(\mathcal{O}_F)) \cap \mathcal{D}) \subset S \setminus S_a.$$

Autrement dit, un point rationnel $P \in E(K)$ est S-entier stable si et seulement si la section σ_P est $(S, (O))$-entière dans $\mathcal{E}$ et $\sigma_P(\mathrm{Spec}(\mathcal{O}_F)) \cap V_a = \varnothing$. En travaillant avec le sous-ensemble $E_{stab}(F, S)$ des points S-entiers stables, Abramovich a démontré le résultat suivant (voir [**1**, Theorem 1]) :

Théorème 0.11 (Abramovich)**.** *Supposons que la conjecture de Lang-Vojta est vraie. Il existe un nombre $N(F, S)$ tel que pour chaque courbe elliptique E/F, $\#E_{stab}(F, S) \leq N(F, S)$.*

Quelques généralisations dans le même style de Théorème 0.11 sont établies (voir [**2**]).

Dans le contexte expliqué ci-dessus, il est naturel d'espérer des résultats de finitude similaires sur l'ensemble des points entiers d'une courbe elliptique définie sur un corps de fonctions et plus généralement d'une variété abélienne définie sur un corps de fonctions.

Nous remarquons également l'observation suivante dans le cas des corps de nombres qui justifie le grand intérêt de l'étude générale de diverses réunions des points entiers ou des sections entières de dénominateurs bornés. Soit $\mathcal{P}$ l'ensemble des nombres premiers. Si $S \subset \mathcal{P}$ est un sous-ensemble fini, nous notons $\mathbb{Z}_S$ la localisation de l'anneau des entiers $\mathbb{Z}$ en S. Ainsi, $\mathbb{Z}_S^*$ est l'ensemble des S-unités de $\mathbb{Z}$. Pour chaque $s \in \mathbb{N}$, nous considérons l'ensemble

$$R_{\mathbb{Q},s} := \{x, y \in \mathbb{Z}_S^*\colon x + y = 1\}$$

qui est la réunion des solutions des équations en S-unités avec S borné en cardinal par s. Observons d'abord que $R_{\mathbb{Q},s} \subset R_{\mathbb{Q},s+1}$ pour tout $s \in \mathbb{N}$ et que $R_{\mathbb{Q},1} = \{(2,-1), (-1,2), (\frac{1}{2}, \frac{1}{2})\}$. Nous démontrons (voir Proposition 1.45) que la finitude de $\mathbb{R}_{\mathbb{Q},2}$ est déjà un problème très difficile :

Proposition 0.12. *L'ensemble $R_{\mathbb{Q},2}$ est infini si et seulement si la réunion de l'ensemble des nombres premiers de Fermat et l'ensemble des nombres premiers de Mersenne est infinie.*

Nous allons donner un résumé des résultats principaux obtenus dans cette thèse dans la section suivante.

2. Survol des résultats principaux

Dans cette thèse, nous allons étudier en détail divers résultats de finitude des sections entières d'une famille des variétés abéliennes $f\colon X \to B$ de dimension relative n paramétrisée par une surface de Riemann compacte B de genre g. Quelques applications aux équations en unités en deux inconnues (voir (4.1), Chaptire 1) sur les corps de fonctions sont également données dans Chapitre 8 et Chapitre 11. Soit $T \subset B$ le sous-ensemble fini de B qui supporte des fibres singulières de $X \to B$. Soient $K = \mathbb{C}(B)$ et $\mathcal{D} \subset X$ un diviseur effectif dont la partie générique $\mathcal{D}_K$ ne contient pas de translations d'une sous variété abélienne non nulle de X_K. Cette condition sur $\mathcal{D}$ est automatique si X est une variété abélienne simple, e.g., une courbe elliptique. D'après le théorème de Lang-Néron (voir Théorème 2.20), $X_K(K)/\operatorname{Tr}_{K/\mathbb{C}}(X)(\mathbb{C})$ est un groupe commutatif de rang fini où $\operatorname{Tr}_{K/\mathbb{C}}(X)$ est la trace de X_K (voir Définition 2.17). Si X est simple et non isotriviale, sa trace est nulle.

Nos résultats (numérotés par des lettres majuscules) présentés dans la section "Plan of the thesis, statement of main results and discussion" au chapitre 1 peuvent être regroupés en trois thèmes : *finitude/croissance, nonexistence générique* et *uniformité*. Les résultats principaux de cette thèse sont contenus dans les articles suivants :

1) *Large unions of generalized integral sections on elliptic surfaces*, Preprint, 2019 ([**88**])
2) *Generalized integral points on abelian varieties and the Geometric Lang-Vojta conjecture*, Preprint 2019 ([**89**])
3) *On Parshin-Arakelov theorem and uniformity of S-integral sections in elliptic surfaces*, Preprint, 2019 ([**90**])
4) *Finiteness criteria and uniformity of integral sections in some families of abelian varieties*, Preprint, 2019 ([**91**])

2.1. Résultas sur la finitude et la croissance. Quant aux problèmes liés à la finitude, il est naturel de demander les questions suivantes. Nous commençons par la question la plus basique :

Question 0.13. Pour quels sous-ensembles $S \subset B$ (pas nécessairement finis) et pour quels diviseurs effectifs $\mathcal{D} \subset X$, l'ensemble des points $(S, \mathcal{D})$-entiers est-il fini ?

Remarquons que tout point $(S, \mathcal{D})$-entier appartient à un seul ensemble $X(B)$. Par ailleurs, nous pouvons regarder chaque couple $(S, \mathcal{D})$ comme un couple de paramètre qui définit une équation diophantienne. Il est donc possible de considérer la réunion de toutes solutions de plusieurs équations diophantiennes induites par des couples $(S, \mathcal{D})$ différents. Ainsi, nous posons la question suivante :

Question 0.14. Pour quelle réunion de sous-ensembles $S \subset B$ et de diviseurs effectifs $\mathcal{D} \subset X$, la réunion des points $(S, \mathcal{D})$-entiers est-elle encore finie ?

D'une manière plus quantitative, nous demandons la taille de l'ensemble des points $(S, \mathcal{D})$-entiers en fonction du cardinal de l'ensemble S. Plus concrètement, quel est le taux de croissance asymptotique de l'ensemble des points $(S, \mathcal{D})$-entiers quand $\#S \to \infty$? Encore plus généralement, nous nous intéressons particulièrement à la question suivante :

Question 0.15. Soit $s \in \mathbb{N}$. Est-ce que la réunion des points $(S, \mathcal{D})$-entiers sur tous les sous-ensembles $S \subset B$ vérifiant $\#S \leq s$ est finie ? Dans ce cas, quel est l'ordre de croissance de telle réunion quand $s \to \infty$?

Si nous combinons Question 0.13 et Question 0.15, nous avons une question naturelle :

Question 0.16. Soit $s \in \mathbb{N}$. Pour quel sous-ensemble infini $Z \subset B$, la réunion des points $(Z \cup S, \mathcal{D})$-entiers, sur tous les sous-ensembles $S \subset B$ de cardinal au plus s, est-elle encore finie ? Dans ce cas, quel est l'ordre de croissance de cette réunion en fonction de s quand $s \to \infty$?

L'état de l'art de la recherche actuelle concernant ces questions est le suivant. Grâce à un théorème de Parshin [**87**] (voir Théorème 1.18), il est connu que Question 0.13 admet une réponse positive quand S est *fini* et nous pouvons affirmer la finitude de l'ensemble des points $(S, \mathcal{D})$-entiers modulo la trace de X_K. Si X/K est une courbe elliptique, ce théorème de Parshin et équivalent au théorème de Siegel sur des corps de fonctions. Dans ce cas, nous savons de plus une croissance polynomiale comme réponse à Question 0.15 (voir [**52**], Corollaire 1.37).

Quant à Questions 0.15 - 0.16, quelques réponses ne sont connues dans la littérature (et qui donnent une croissance polynomiale) que dans le cas où ($Z = \varnothing$ et) X, si elle n'est pas une surface elliptique, est soit une famille constante (voir [**82**]) ou soit la trace de X_K est nulle (voir [**14**]). Tous ces résultats connus établissent également une borne uniforme sur la multiplicité de l'intersection entre le diviseur $\mathcal{D}$ et une section de la famille. Comme l'intersection ne peut avoir lieu qu'au-dessus de $S \subset B$ pour les sections $(S, \mathcal{D})$-entières, la hauteur des points entiers correspondants, qui n'est rien d'autre que la somme de toutes les multiplicités de l'intersection, est donc bornée si S est fini. Un argument de comptage (voir Lemme 2.24, Théorème 2.21) implique immédiatement la croissance polynomiale en fonction de s.

Quant à Question 0.13, notre Théorème A fournit un critère simple sur $\mathcal{D}$ qui demande $\mathcal{D}(K) = \varnothing$ pour pouvoir conclure la finitude sans passer au modulo la trace de l'ensemble des points $(S, \mathcal{D})$-entiers pour tout sous-ensemble fini $S \subset B$. Quand X est une famille constante, notre Théorème I fournit également un critère pour la finitude (qui sera alors uniforme) de l'ensemble des sections $(S, \mathcal{D})$-entières non constantes. De plus, notre Théorème principal F réponds à Questions 0.13 - 0.14 et affirme la finitude (modulo la trace de X_K) de la réunion des points $(S, \mathcal{D})$-entiers où S parcourt certains sous-ensembles infinis indénombrables de B. Soit d une métrique riemannienne arbitraire sur B. À tire d'exemple, Théorème F implique le résultat de finitude suivant :

Théorème. *Pour chaque réunion finie disjointe W des disques fermés dans B telle que des points distincts de T sont contenus dans des disques distincts de W, alors l'ensemble des points $(W, \mathcal{D})$-entiers est fini (modulo la trace de X_K).*

À la limite de notre connaissance, Théorème F établit pour la première fois un tel résultat dans la littérature. Ce résultat est hors de portée des méthodes purement algébriques comme la théorie des hauteurs standards sous la forme de la théorie de l'intersection (voir [**39**]) qui demande inévitablement que l'ensemble S soit fini afin d'établir une borne. Comme W peut être choisi de sorte que sa mesure soit arbitrairement proche à celle de B, notre résultat fournit aussi quelques informations de nature topologique sur l'intersection entre le diviseur $\mathcal{D}$ et l'ensemble des sections. Nous allons aborder plus en détail ce point dans Théorème G. Théorème F donne non seulement une réponse à Question 0.16 sur la finitude mais aussi une borne sur la croissance polynomiale dans tous les cas. Soit d une métrique riemannienne sur B, nous déduisons de Théorème F l'estimation uniforme suivante sur la croissance du nombre des sections entières généralisées :

Théorème. *Pour chaque $s \in \mathbb{N}$ et pour chaque réunion finie disjointe W des disques fermés dans B qui contiennent les points de T dans les disques distincts, la réunion I_s des points $(W \cup S, \mathcal{D})$-entiers, où $S \subset B$ parcourt tous les sous-ensembles de cardinal au plus s, est finie (modulo la trace de X_K). La croissance* *de cette réunion est au plus polynomiale en fonction de s et de degré inférieur ou égal à*

$$2 \dim X_K \operatorname{rank} \pi_1(B_0).$$

Dans l'énoncé ci-dessus, $\operatorname{rank} \pi_1(B_0)$ est par définition le nombre minimal de générateurs nécessaires pour engendrer le groupe fondamental $\pi_1(B_0, b_0)$ où $b_0 \in B_0$ est un point base arbitraire. Typiquement, nous avons :

$$\operatorname{rank} \pi_1(B_0) = 2g - 1 + \#T.$$

Ainsi, le degré de la croissance polynomiale de la réunion I_s des points entiers est bornée par une constante uniforme de nature topologique $2 \dim X_K \operatorname{rank} \pi_1(B_0)$.

À un facteur de $\frac{1}{2}$ près, nous remarquons en utilisant la formule de Ogg-Shafarevich que la borne $2 \dim X_K \operatorname{rank} \pi_1(B_0)$ sur la croissance polynomiale en fonction de s dans le théorème généralise les résultats connus dans la littérature dans le cas spécial de nature purement algébrique où B_0 est le complémentaire dans B d'un ensemble fini de points.

En particulier, Théorème F fournit des phénomènes nouveaux en faveur de la version géométrique de la conjecture de Lang-Vojta (voir Conjecture 1.28 et la discussion qui la suit) dans le cas des sous-variétés d'une variété abélienne (voir aussi Corollaire 1.30) :

Corollaire. *Soit W une réunion finie disjointe des disques fermés dans B qui contiennent les points de T dans les disques distincts. Soit $B_0 = B \setminus W$. Pour chaque $s \in \mathbb{N}$, il existe $M = M(\mathcal{A}, \mathcal{D}, B_0, s) \in \mathbb{R}_+$ tel que pour toute section $\sigma \colon B \to \mathcal{A}$ vérifiant la propriété $\#f(\sigma(B_0) \cap \mathcal{D}) \leq s$, la borne uniforme suivante est satisfaite :*

$$\deg_B \sigma^* \mathcal{D} < M.$$

La version géométrique de la conjecture de Lang-Vojta, couplée avec la pseudo-hyperbolicité algébrique dans le sens de Demailly, prédit dans la situation algébrique où $B_0 = B$ que la constante M est bornée supérieurement par une fonction linéaire en s. Cela revient à affirmer que nous disposons une borne uniforme sur la multiplicité de l'intersection entre les sections σ et le diviseur $\mathcal{D}$. Comme expliqué au début de la section, nous rappelons que cette conjecture reste ouverte en toute généralité dans notre contexte des fibrations abéliennes.

Dans le cas où X est une surface elliptique (que ce soit isotriviale ou non isotriviale), notre Théorème H raffine de manière assez remarquable Théorème F et confirme non seulement la finitude comme réponse à Question 0.14 mais aussi une croissance polynomiale en fonction de s comme réponse à Question 0.16 où le diviseur $\mathcal{D}$, et donc non seulement l'ensemble S, est autorisé à varier en une famille compacte T des diviseurs horizontaux. Nous soulignons le fait que la croissance de la double réunion

$$J_s = \cup_{\mathcal{D} \in T} \cup_{\# S \leq s} \{\text{points } (S, \mathcal{D})\text{-entiers de } X_K\},$$

qui est à priori “deux fois” plus large que la réunion des points entiers I_s, est également bornée par une croissance polynomiale en s de degré $2 \operatorname{rank} \pi_1(B_0)$ (remarquons que $\dim X_K = 1$).

L'un des ingrédients originaux pour démontrer nos résultats sur la finitude et la croissance des points entiers est Théorème D qui affirme une borne linéaire sur la longueur hyperbolique de certains lacets dans divers complémentaires d'une surface de Riemann. Soit $B_0 = B \setminus U$ le complémentaire dans B d'une réunion finie disjointe U des disques fermés. Si nous fixons une classe d'homotopie libre α des courbes fermées dans B_0, alors Théorème D implique le résultat suivant :

Théorème. *Il existe une constante finie $L > 0$ vérifiant la propriété suivante. Pour chaque sous-ensemble fini $S \subset B_0$, il existe un lacet lisse par morceaux $\gamma \subset B_0 \setminus S$ qui représente la classe de conjugasion de α et dont la longueur hyperbolique dans $B_0 \setminus S$ satisfait l'inégalité suivante :*

$$\text{length}_{d_{B_0 \setminus S}}(\gamma) \leq L(\# S + 1). \tag{2.1}$$

En combinant avec les propriétés de croissance de la géométrie des groupes fondamentaux, Théorème D dans notre contexte joue le rôle de la partie hyperbolique d'une certaine *hauteur hyperbolique-homotopique* qui remplace la théorie des hauteurs standards comme la théorie de l'intersection. Notre hauteur hyperbolique-homotopique est composée en trois parties indépendantes. La première partie, Théorème D, est un énoncé de nature purement hyperbolique sur les surfaces de Riemann. La deuxième partie est la géométrie des groupes fondamentaux de $\mathcal{A}_{B_0}$ (voir Chapitre 6, Appendice 7). Une réduction *homotopique* à la Parshin fournit la dernière composante importante de cette machinerie.

En fait, en utilisant les propriétés fondamentales de la métrique hyperbolique au sens de Kobayashi, nous pouvons donner une borne inférieure sur la croissance polynomiale en fonction de s dans Théorème D comme suivant. Pour chaque classe d'homotopie de conjugaison des lacets $\alpha \in \pi_1(B_0) \setminus \{0\}$ et pour chaque entier positif $s \in \mathbb{N}$, nous définissons :

$$L(\alpha, s) \coloneqq \sup_{\# S \leq s} \inf_{[\gamma] = \alpha} \text{length}_{d_{B_0 \setminus S}}(\gamma) \in \mathbb{R}_+,$$

où S parcourt tous les sous-ensembles finis de B_0 de cardinal au plus s et où γ parcourt tous les lacets dans $B_0 \setminus S$ qui représentent la classe α. Si α est fixée, Théorème D affirme que la fonction $L(\alpha, s)$ est majorée par une fonction linéaire en

s. Notre Théorème E fournit également une borne inférieure de $L(\alpha, s)$ en fonction de s. Plus précisément, nous démontrons :

Théorème. *Pour chaque $\alpha \in \pi_1(B_0) \setminus \{0\}$, il existe une constante finie $c > 0$ telle que la borne uniforme suivante est satisfaite pour tout $s \in \mathbb{N}$:*

$$L(\alpha, s) \geq \frac{cs^{1/2}}{\log(s+2)}.$$

Considérons les bornes supérieures et inférieures sur la croissance polynomiale en fonction de s des longueurs hyperbolique des lacets dans la classe α :

$$\deg^-(\alpha) := \liminf_{s\to\infty} \frac{\log L(\alpha, s)}{\log s}, \quad \deg^+(\alpha) := \limsup_{s\to\infty} \frac{\log L(\alpha, s)}{\log s}.$$

Nous obtenons ainsi l'encadrement général suivant :

$$\frac{1}{2} \leq \deg^-(\alpha) \leq \deg^+(\alpha) \leq 1.$$

2.2. Résultats sur la nonexistence générique. Comme l'a montré la preuve du dernier théorème de Fermat, établir un résultat de nonexistence d'une équation diophantienne est un problème particulièrement difficile en général. En fait, le théorème de MRDP (parfois connu comme Théorème de Matiyasevich), qui n'est pas moins spectaculaire, affirme une réponse négative au dixième problème de Hilbert et nous dit qu'il n'existe pas un algorithme capable de décider si une équation diophantienne donnée admet une solution entière (voir [**68**]). Par conséquent, nous nous contentons d'étudier un problème plus faisable, la *nonexistence générique* de l'ensemble des solutions entières d'une équation diophantienne. Par exemple, nous pouvons naturellement demander dans notre contexte la question suivante :

Question 0.17. Est-ce que l'ensemble des points $(S, \mathcal{D})$-entiers de X est vide pour un choix général du sous-ensemble fini S ?

En suivant la convention usuelle dans la littérature, un choix général d'un sous-ensemble fini $S \subset B$ signifie que pour chaque $s \geq 1$, nous considérons des sous-ensembles S de cardinal s et dont les images dans l'espace de paramétrisation B^s appartiennent à un sous-ensemble ouvert et dense $U_s \subset B^s$ pour la topologie de Zariski. Notre Corollaire B affirme Question 0.17 quand la trace de X_K est nulle et si le diviseur $\mathcal{D}$ est suffisamment positif. En fait, Corollaire B fournit une description encore plus forte sur la nonexistence générique. Soit d une métrique riemannienne sur B, nous prouvons que :

Théorème. *Pour chaque $\varepsilon > 0$, il existe un sous-ensemble complexe fermé $U \subset B$ avec $\mathrm{vol}_d(B \setminus U) < \varepsilon$ et tel que l'ensemble des points $(U \cup S, \mathcal{D})$-entiers est vide pour un choix général d'un sous-ensemble fini $S \subset B \setminus U$.*

Un choix général d'un sous-ensemble fini S de $B_0 = B \setminus U$ dans le théorème précédent signifie que pour chaque $s \in \mathbb{N}$, il existe un sous-ensemble fini $E_s \subset B_0$ (qui dépend de s) tel que la conclusion du théorème est vraie pour tout sous-ensemble fini $S \subset B_0$ de cardinal s et tel que $S \cap E_s = \varnothing$.

Par symétrie, en échangeant le rôle de S et de $\mathcal{D}$, nous demandons :

Question 0.18. Fixons un sous-ensemble fini $S \subset B$ et soit T un espace qui paramétrise une famille des diviseurs effectifs $\mathcal{D}$ sur X. Est-ce que l'ensemble des points $(S, \mathcal{D})$-entiers de X est vide pour un choix général du diviseur $\mathcal{D}$ dans la famille T ?

Dans la question précédente, un choix général d'un diviseur $\mathcal{D}$ signifie que nous considérons les diviseurs $\mathcal{D}$ paramétrisés par un sous-ensemble ouvert dense U de T pour la topologie de Zariski. Encore une fois, Question 0.18 admet aussi une réponse positive donnée par Corollaire C quand X_K est une courbe elliptique non isotriviale et quand les diviseurs de la famille T sont suffisamment positifs.

2.3. Résultats sur l'uniformité. Motivé par Conjecture de l'uniformité et Théorème 1.11 ou Théorème 1.13, nous étudions quelques propriétés *uniformes* dans le contexte des points entiers de certaines variétés abéliennes.

Supposons d'abord que X est une famille constante, c'est-à-dire que $X = A \times B$ où A est une variété abélienne complexe. Ainsi, nous avons $X(B) = A(K)$. Soit $S \subset B$ un sous-ensemble de cardinal $s \in \mathbb{N}$. Soit D un diviseur effectif ample sur A de degré $D^n = d$ et notons $\mathcal{D} = D \times B$. Soit $I \subset A(K) \setminus A(\mathbb{C})$ l'ensemble des points $(S, \mathcal{D})$-entiers non constants (qui ne sont pas définis sur le corps complexe). En utilisant des résultats de Noguchi-Wikelmann dans l'article [**82**], notre théorème principal I implique que :

Théorème. *Il existe un nombre fini positif $N(g, s, n, d) \geq 0$ tel que :*

(i) *Si I n'est pas infini, alors $\#I \leq N(g, s, n, d)$;*

(ii) *Si $n = 2$, $d > 2g - 2$ et D est un diviseur intègre, alors $\#I \leq N(g, s, n, d)$.*

Soit $S \subset B$ un sous-ensemble fini. Le fameux théorème de Parshin-Arakelov (voir [**86**], [**5**]) affirme la finitude de l'ensemble $F_q(B, S)$ des surfaces non isotriviales minimales fibrées au-dessus de B ayant bonne réduction en dehors de S et dont les fibres générales sont de genre $q \geq 2$. En développant la méthode de revêtement de Parshin, Théorème M implique le résultat négatif uniforme suivant :

Théorème. *Pour des entiers q et s suffisamment grands qui ne dépendent que du genre g de B, la réunion $\cup_{S \subset B, \#S \leq s} F_q(B, S)$ est indénombrable. Plus précisement, il existe un nombre fini $N = N(g, q, s) \in \mathbb{N}^*$ et un ouvert de Zariski dense $I \subset \mathbb{P}^N$*

tels que nous avons une application dont les fibres sont finies et uniformément bornées :

$$I \to \cup_{S\subset B, \#S\leq s} F_q(B,S).$$

De plus, nous pouvons choisir $N \to \infty$ quand $q, s \to \infty$.

Soit $\Delta \subset \mathcal{M}_{q,n}$ le lieu des courbes singulières dans l'espace modulaire compact $\mathcal{M}_{q,n}$ des courbes stables de genre $q \geq 2$ et de structure de niveau n (avec $n \geq 3$). Nous obtenons la conséquence suivante du théorème précédent :

Corollaire. *Pour des entiers positifs q, s suffisamment grands, il existe un nombre indénombrable des morphismes non constants $h\colon B \to \mathcal{M}_{q,n}$ deux-à-deux non-équivalents, tels que l'intersection ensembliste $h(B) \cap \Delta$ contient au plus s points.*

Ce résultat se résulte de notre tentation non-réussie d'utiliser la version uniforme (démontrée par Caporaso, voir [**18**]) du théorème de Parshin-Arakelov afin de donner une nouvelle preuve de certain résultat sur la finitude uniforme de la réunion des sections (S, Δ)-entières d'une surface elliptique non isotriviale $X \to B$ où S parcourt tous les sous-ensembles de B de cardinal au plus s. Néanmoins, si nous nous restreignons à l'ensemble des points $(S, \mathcal{D})$-entiers par rapport à un seul sous-ensemble fini fixé S au lieu de considérer la réunion sur tous les sous-ensembles S de cardinal bornés, nous obtenons effectivment une nouvelle preuve de la borne uniforme connue sur l'ensemble des points $(S, \mathcal{D})$-entiers (voir Théorème L pour plus de détails).

3. Plan de la thèse

Interdépendance des chapitres. Nous remarquons que la plupart des chapitres dans cette présente thèse peuvent être traités de manière indépendante. Les résultats principaux de Chapitre 5 seront utilisés dans Chapitres 6, 7 et 8. Par ailleurs, la méthode développée dans Chapitre 7 sera appliquée de manière essentielle dans Chaptires 7 et 8. Notamment, nous avons collecté et regroupé les prérequis et outils de base nécessaires dans Chapitre 2.

Organisation de la thèse. Le plan de ce mémoire de thèse est le suivant. Chapitre 2 présente les définitions, les outils et quelques théorèmes préliminaires qui seront utilisés par les autres chapitres. Quelques preuves y sont données, en particulier quand il est difficile de trouver une bonne référence dans la littérature. Chapitre 3 est élémentaire mais sera utile pour illustrer quelques idées générales et exemples de certains résultats obtenus dans les chapitres ultérieurs. Nous y nous restreignons au cas le plus simple mais déjà nontrivial des surfaces elliptiques constantes. Ainsi, les experts sont invités à passer directement aux chapitres suivants. Dans Chapitre 4, nous démontrons un critère de finitude de l'ensemble des points entiers classiques (c'est-à-dire S est fini et fixé) des variétés abéliennes définies sur un corps de fonctions. Nous étudions dans Chapitre 5 la croissance des longueurs hyperboliques dans le sens de Kobayashi des lacets dans divers complémentaires des surfaces de Riemann (pas nécessairement compactes) d'un sous-ensemble fini de points. Ensuite, Chapitres 6, 7 et 8 développent la méthode hyperbolique de Parshin afin d'obtenir les résultats quantitatifs sur certaines unions larges des sections entières d'une famille des variétés abéliennes, des courbes elliptiques ou d'une surface réglée respectivement. Dans Chapitre 9, nous obtenons une borne uniforme sur le nombre des sections entières d'une famille constante des variétés abéliennes. Nous donnerons une nouvelle preuve purement géométrique dans Chapitre 10 des bornes uniformes connues sur la hauteur canonique des sections entières d'une surface elliptique en utilisant seulement l'inégalité tautologique. Dans Chapitre 11, nous appliquons la version uniforme de Caporaso du fameux théorème de Parshin-Arakelov pour établir une nouvelle preuve d'une certaine uniformité de l'ensemble des points entiers classiques d'une courbe elliptique et pour obtenir également des informations sur les espaces modulaires des courbes stables concernant l'intersection d'une courbe donnée avec les diviseurs de singularité.

CHAPTER 1

Introduction

1. Context

Notations. Throughout this thesis, unless stated otherwise, the letter B denotes a smooth projective curve of genus g over a field k of characteristic 0 (e.g., $k = \mathbb{C}$) and K denotes its field of functions $K = k(B)$. We usually use the symbols such as X_K, Y_K to denote a scheme over K. On the other hand, for any B-scheme X, the symbol X_K also denotes the generic fibre of X and if $U \subset B$, we denote by X_U the restriction of X above U. The notation d_Y means the Kobayashi hyperbolic pseudo-metric on a complex space Y. Discs in Riemannian surfaces are assumed to have sufficiently piecewise smooth boundary. The symbol # stands for cardinality.

The study of *Diophantine problems*, i.e., the study of the set of integral solutions of a system of polynomial equations in several unknowns, has a long history dated back to the first work of Diophantus of Alexandria in the third century. The two most important achievements in the theory are the famous Fermat's Last Theorem and the Mordell conjecture proved respectively by Andrew Wiles in 1995 (cf. [**115**]) and by Gerd Faltings in 1991 (cf. [**36**]). While the first result concerns the *emptiness* of the set of nontrivial solutions in $\mathbb{Z}^3$ of the equation $x^n + y^n = z^n$ when $n \geq 3$, the second confirms the *finiteness* of the set of rational points on every algebraic curve of genus at least 2.

The proofs of both results are based on the sophisticated machinery of algebraic geometry developed in the whole twentieth century. The central idea is to give geometric meanings for polynomial equations as well as their sets of rational or integral solutions. For example, suppose that we want to study the set $I_{\mathbb{Q}}$ of solutions $(x, y) \in \mathbb{Q}^2$ and the set $I_{\mathbb{Z}}$ of solutions $(x, y) \in \mathbb{Z}^2$ of the Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}. \tag{1.1}$$

Equation (1.1) induces a plane curve $E/\mathbb{Q}$ in $\mathbb{P}^2_{\mathbb{Q}}$ as well as a closed surface $\mathcal{E}/\operatorname{Spec}(\mathbb{Z})$ in $\mathbb{P}^2_{\mathbb{Z}}$ both defined by the homogenous equation $y^2z = x^3 + axz^2 + bz^3$

and which fit in a cartesian diagram:

$$\begin{array}{ccc} E & \hookrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathbb{Q}) & \hookrightarrow & \operatorname{Spec}(\mathbb{Z}). \end{array}$$

Since $\mathcal{E}$ is proper over $\operatorname{Spec}(\mathbb{Z})$, the set of *rational points* $E(\mathbb{Q})$ is naturally identified with the set of *sections* $\mathcal{E}(\operatorname{Spec}(\mathbb{Z}))$ by the valuative criterion of properness. Consider the hyperplan divisor $\mathcal{D} = \{z = 0\} \cap \mathcal{E} \subset \mathbb{P}^2_{\mathbb{Z}}$. The pullback of $\mathcal{D}$ to E is $O = (1\colon 0\colon 0) \in E(\mathbb{Q})$. Denote by $(O) \in \mathcal{E}(\operatorname{Spec}(\mathbb{Z}))$ the corresponding section of O. It turns out that:

$$I_{\mathbb{Q}} = E(\mathbb{Q}) \setminus \{O\} = \mathcal{E}(\operatorname{Spec}(\mathbb{Z})) \setminus \{(O)\}$$

and

$$I_{\mathbb{Z}} = (\mathcal{E} \setminus \mathcal{D})(\operatorname{Spec}(\mathbb{Z})).$$

In particular, the set $I_{\mathbb{Z}}$ of integral solutions $(x, y) \in \mathbb{Z}^2$ of (1.1) admits a geometric interpretation as the set of sections $\sigma\colon \operatorname{Spec}(\mathbb{Z}) \to \mathcal{E}$ which avoid the divisor $\mathcal{D}$, i.e.,

$$\sigma(\operatorname{Spec}(\mathbb{Z})) \cap \mathcal{D} = \varnothing.$$

The above example can be generalized to more general situations, for example, when $\operatorname{Spec}(\mathbb{Z})$ is replaced by any one dimensional Dedekind scheme, e.g., $\operatorname{Spec}(\mathcal{O})$ for $\mathcal{O}$ the ring of integers of a number field. Therefore, the study of rational and integral solutions of Diophantine problems is equivalent to the study of sections in certain geometric objects.

In the present thesis, our main object of interest is the set of $(S, \mathcal{D})$-integral points of algebraic varieties over one dimensional function fields in characteristic zero:

Definition 1.1 ($(S, \mathcal{D})$-integral point and section)**.** Let B/k be a smooth projective connected curve over a field k. Let $K = k(B)$ be the function field of B. Let $f\colon X \to B$ be a proper flat morphism of integral varieties. Let $S \subset B$ be a subset (not necessarily finite) and let $\mathcal{D} \subset X$ be a subset (not necessarily an effective divisor). We say that a section $\sigma\colon B \to X$ is $(S, \mathcal{D})$-*integral* if it satisfies the condition (cf. Figure 1.1):

$$f(\sigma(B) \cap \mathcal{D}) \subset S. \tag{1.2}$$

For every $P \in X_K(K)$, denote by $\sigma_P\colon B \to X$ the corresponding section (cf. Remark 1.2). Then P is said to be $(S, \mathcal{D})$-*integral* if σ_P is $(S, \mathcal{D})$-integral.

Several important remarks are in order.

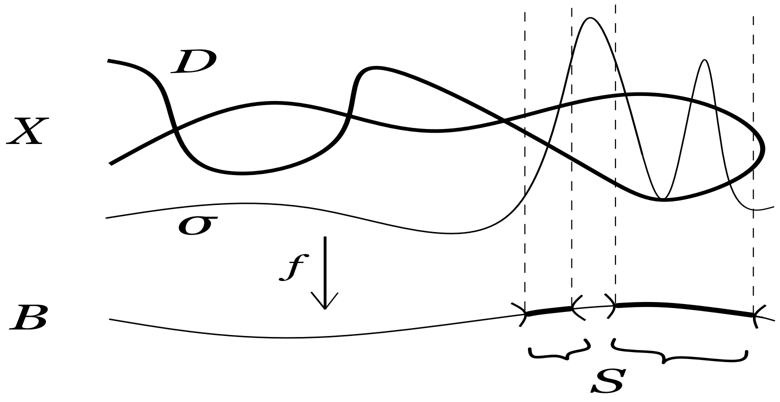

FIGURE 1.1. An $(S, \mathcal{D})$-integral section σ

Remark 1.2. With the notations be as in Definition 1.1, we have a natural identification

$$X(B) = X_K(K)$$

by the valuative criterion for properness. Indeed, every section $\sigma \colon B \to X$ induces by the base change $\mathrm{Spec}(K) \to B$ a rational point $\sigma_K \colon \mathrm{Spec}\, K \to X_K$. Conversely, every rational point $P \in X(K)$ gives rise to a section by taking the Zariski closure in X. The set of $(S, \mathcal{D})$-integral sections of $X \to B$ is then identified with a certain subset of $X_K(K)$.

Remark 1.3. For $S = B$ or $\mathcal{D} = \varnothing$, Condition (1.2) is empty. The set of $(B, \mathcal{D})$-integral sections and the set of $(S, \varnothing)$-integral sections are thus identified with the set $X_K(K)$ of rational points of the generic fibre X_K. Another extreme situation is when $S = \varnothing$. In this case, Condition (1.2) implies that the set of $(S, \mathcal{D})$-integral sections is empty whenever $\mathcal{D}$ contains a vertical fibre or $\mathcal{D}$ is an ample divisor.

Remark 1.4. In Definition 1.1, we emphasize that the set of $(S, \mathcal{D})$-integral sections are considered with respect to a given fixed $X \to B$ and should be distinguished from the following usual notion in the literature of *quasi-integral sets* of rational points on X_K (cf. [**99**, §7.1]). For every finite subset $S \subset B$, denote $\mathcal{O}_{K,S} \subset K$ the ring of S-integers. Fix an ample reduced effective divisor $D \subset X_K$.

Definition 1.5 (Quasi-integral set)**.** A subset $A \subset X_K(K) \setminus D$ is said to be (S, D)-*quasi integral* if there exists a model $\mathcal{X} \to B$ of X_K, i.e., $\mathcal{X}_K = X_K$, and an effective divisor $\mathcal{D} \subset \mathcal{X}$ model of D, i.e., $\mathcal{D}_K = D$, such that for every point $P \in A$, the corresponding section $\sigma_P \colon B \to \mathcal{X}$ is an $(S, \mathcal{D})$-integral section.

The notion of quasi-integral sets of rational points is relative to a choice of models. It turns out that every finite subset $A \subset X_K(K) \setminus D$ is (S, D) quasi-integral. Therefore, it is meaningless to talk about (S, D)-*quasi integral points.* Indeed, consider any proper flat model $\mathcal{X} \to B$ of X_K. Let $\mathcal{D} \subset \mathcal{X}$ be the Zariski closure of D in $\mathcal{X}$. Since $A \subset X_K(K) \setminus D$ is finite, the set of intersection points

$$I_A = \cup_{P \in A} \sigma_P(B) \cap \mathcal{D} \subset \mathcal{X}$$

is finite and the intersection multiplicities at the points in I_A of the sections $\sigma_P(B)$ (where $P \in A$) with $\mathcal{D}$ are also finite. Hence, by blowing up $\mathcal{X}$ sufficiently many finitely times along the points in I_A, we obtain another proper flat model $\mathcal{X}' \to B$ such that the set A becomes $(S, \mathcal{D}')$-quasi integral where $\mathcal{D}'$ is the strict transform of $\mathcal{D}$ in $\mathcal{X}'$.

The main interest of the notion of quasi-integral sets is that we can usually prove the finiteness of them in an arbitrary model. Hence, the finiteness property of quasi-integral sets becomes an *intrinsic* property. However, to obtain *quantitative* results (as in Theorem F or Theorem H), we have to fix the model as explained above. In this point of view, we shall always prefer in this thesis the notion of $(S, \mathcal{D})$-integral points with respect to fixed model as in Definition 1.1.

If X_K/K is a curve, we have the following famous result on the Mordell conjecture over function fields, which holds also for number fields (cf. [**78**], [**66**], [**41**], [**86**], [**36**], [**37**]):

Theorem 1.6 (Mordell conjecture)**.** *Suppose that X is a nonisotrivial family of curves of genus at least 2, then $X_K(K)$ is finite.*

One of the most important open conjectures in Arithmetic geometry is the following conjecture which implies the Mordell conjecture (cf. [**112**, Conjecture 4.4], [**63**]):

Conjecture 1.7 (Lang)**.** *Let X be a smooth projective variety of general type over a number field K. The set of rational points $X(K)$ is not Zariski dense in X.*

More generally, we have for quasi-projective smooth varieties (cf. for example [**53**, F.3.5], see also Conjecture 1.28 for the *geometric* version):

Conjecture 1.8 (Lang-Vojta)**.** *Let X be smooth variety of log-general type over a number field K. Let S be a finite set of places in K and let $\mathcal{O}_{K,S} \subset K$ be the ring of S-integers. Assume $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_{K,S})$ is a model of X, i.e., $\mathcal{X}_K = X$. Then the set of integral points $\mathcal{X}(\mathcal{O}_{K,S})$ is not Zariski dense in $\mathcal{X}$.*

Here, a smooth projective variety X is of general type if the canonical divisor K_X is big, i.e., $h^0(\mathcal{O}(nK_X)) \gg n^{\dim X}$ (cf. Definition 2.27). In particular, a smooth projective curve is of general type if and only if it has genus at least 2. A smooth quasi-projective variety Y is of log-general type if Y admits a compactification $\bar{Y}$ with normal crossing boundary divisor D with $K_{\bar{Y}} + D$ big.

These conjectures are still wide open nowadays. However, in [**17**], the authors established the following very interesting consequence of the Lang-Vojta conjecture usually known as the *Uniform boundedness conjecture* for hyperbolic curves over number fields (see also [**84**] for a stronger statement):

Conjecture 1.9 (Uniformity)**.** *Fix an integer $g \geq 2$. Then there exists a number $N(g)$ such that for any number field F there exist at most finitely many curves of genus g defined over F and which have more than $N(g)$ F-rational points.*

The key ingredient in [**17**] to obtain Conjecture 1.9 is the following correlation theorem. Let $X \to B$ be a family of curves of genus $g \geq 2$. Then $X_B^n = X \times_B \cdots \times_B X$ (n-times) admits a dominant rational map to a smooth projective variety of general type V_n for n large enough. F-rational points on fibres X_b defined over a number field F project to F-rational points of V_n. By taking $X \to B$ a parameter family of all smooth curves of genus $g \geq 2$, one applies the Lang conjecture to V_n and use the Noetherian induction to obtain a uniform bound (i.e., Conjecture 1.9) on the number of rational points on curves.

The Uniformity conjecture 1.9 is also unknown in the function field case. Nevertheless, by establishing a uniform version of Parshin-Arakelov theorem 1.35, Caporaso obtained a certain uniform version of the Mordell conjecture over function fields (cf. [**18**]). Recall that g denotes the genus of the curve B.

Theorem 1.10 (Caporaso)**.** *There exists a number $M(q, g, s) \in \mathbb{N}$ such that for every nonisotrivial minimal surface X fibered over B with smooth fibres of genus $q \geq 2$ outside a finite subset $S \subset B$ of cardinality s, we have $\#X_K(K) \leq M(q, g, s)$.*

In the case of elliptic curves, we must restrict ourselves to integral points to obtain finiteness statements since the set of rational points is infinite unless the Mordell-Weil rank of the elliptic curve is 0. Let S be a finite set of places containing 2 and 3 of a number field F. Let $f \colon \mathcal{E} \to \mathrm{Spec}(\mathcal{O}_F)$ be the Néron model over the ring of integers $\mathcal{O}_F$ of an elliptic curve E/F. Let $(O) \subset \mathcal{E}$ be the zero section and let $V_a \subset \mathcal{E}$ be the union of all zero components of additive fibres in $\mathcal{E}$. Let S_a be the finite subset of additive places. Denote $\mathcal{D} = (O) \cup V_a$ which is an effective connected divisor in $\mathcal{E}$. Abramovich introduced the notion of *stably S-integral points* of E, namely, rational points $P \in E(K)$ whose corresponding section $\sigma_P \colon \mathrm{Spec}(\mathcal{O}_F) \to \mathcal{E}$ is $(S \setminus S_a, \mathcal{D})$-integral in $\mathcal{E}$ in the sense of Definition 1.1, i.e.,

$$f(\sigma_P(\mathrm{Spec}(\mathcal{O}_F)) \cap \mathcal{D}) \subset S \setminus S_a.$$

In other words, a rational point $P \in E(K)$ is stably S-integral if and only if the section σ_P is $(S, (O))$-integral in $\mathcal{E}$ and $\sigma_P(\mathrm{Spec}(\mathcal{O}_F)) \cap V_a = \varnothing$. Restricting to the set $E_{stab}(F, S)$ of stably S-integral points, Abramovich proved that (cf. [**1**, Theorem 1]):

Theorem 1.11 (Abramovich). *Assume the Lang - Vojta conjecture. There exists a number* $N(F,S)$ *such that for every elliptic curve* E/F*, we have* $\#E_{stab}(F,S) \leq N(F,S)$*.*

Subsequent generalizations in the same style of Theorem 1.11 are established (cf. [**2**]).

Given the above contexts, it is then very natural to ask for similar uniform finiteness results for the set of integral points on elliptic curves over function fields and more generally on abelian varieties over function fields.

2. Overview of main results

In this thesis, we concentrate on families $f\colon X \to B$ of abelian varieties of dimension n over a compact Riemann surface B of genus g. Applications to the unit equations (cf. (4.1)) over function fields are also given in Chapter 8, Chapter 11. Let $T \subset B$ be the finite subset supporting singular fibres of $X \to B$. Let $K = \mathbb{C}(B)$ and let $\mathcal{D} \subset X$ be an effective divisor whose generic part does not contain a translation of a nonzero abelian subvariety in X_K. This condition on $\mathcal{D}$ is empty when X is a simple abelian variety, e.g., an elliptic curve. By the Lang-Néron theorem (cf. Theorem 2.20), $X_K(K)/\operatorname{Tr}_{K/\mathbb{C}}(X)(\mathbb{C})$ is an abelian group of finite rank where $\operatorname{Tr}_{K/\mathbb{C}}(X)$ is the trace of X_K (cf. Definition 2.17). When X is simple and nonisotrivial, its trace is zero.

Our results (numbered with capital letters) presented in the next section can be regrouped into the following three themes: *finiteness/growth, generic emptiness* and *uniformity*. The main results of this thesis are contained in the papers [**88**], [**89**], [**90**] and [**91**].

2.1. Finiteness/growth results.

Concerning finiteness and quantitative related problems, it is very natural to ask the following. We begin with the most basic question:

Question 1.12. For which subsets $S \subset B$ (not necessarily finite) and for which effective divisors $\mathcal{D} \subset X$, is the set of $(S,\mathcal{D})$-integral points finite?

All $(S,\mathcal{D})$-integral points belong to a single set $X(B)$. Besides, we can see each couple $(S,\mathcal{D})$ as a Diophantine equation. We can thus go further to consider the union of solutions of many Diophantine equations induced by different $(S,\mathcal{D})$. Hence, we ask:

Question 1.13. For which union of subsets $S \subset B$ and of effective divisors $\mathcal{D} \subset X$, is the union of $(S,\mathcal{D})$-integral points still finite?

More quantitatively, one may ask how large can the set of $(S, \mathcal{D})$-integral points be in terms of the cardinality of S. Concretely, what is the growth order of the set of $(S, \mathcal{D})$-integral points when $\#S \to \infty$? Even more generally:

Question 1.14. Let $s \in \mathbb{N}$. Is the union of $(S, \mathcal{D})$-integral points, over all $S \subset B$ such that $\#S \leq s$, finite? In this case, what is the growth order when $s \to \infty$?

Combining Question 1.12 and Question 1.14, arises a natural question:

Question 1.15. Let $s \in \mathbb{N}$. For which infinite subsets $Z \subset B$, is the union of $(Z \cup S, \mathcal{D})$-integral points, over all $S \subset B$ of cardinality at most s, still finite? In this case, what is the growth order in terms of s when $s \to \infty$?

The state-of-the-art concerning these questions is as follows. By a theorem of Parshin [**87**] (cf. Theorem 1.18), it is known that Question 1.12 has a positive answer when S is *finite* for the finiteness of the set of $(S, \mathcal{D})$-integral points modulo the trace of X_K. When X/K is an elliptic curve, this theorem of Parshin recovers the Siegel theorem over function fields. It is known in this case that we have a polynomial growth as an answer to Question 1.14 (cf. [**52**], Corollary 1.37).

For Questions 1.14 - 1.15, answers are known in the literature (which give a polynomial growth) only when ($Z = \varnothing$ and) X, if not an elliptic surface, is either a constant family (cf. [**82**]) or the trace of X_K is zero (cf. [**14**]). These known results all establish a uniform bound on the intersection multiplicities between a section with the divisor $\mathcal{D}$. Since intersection can only happen above $S \subset B$ for $(S, \mathcal{D})$-integral points, the height of these points, as sum over all intersection multiplicities, is thus bounded, provided S is finite. A counting argument (cf. Lemma 2.24, Theorem 2.21) then implies a polynomial growth in s.

Concerning Question 1.12, our Theorem A gives a simple criterion on $\mathcal{D}$, that is $\mathcal{D}(K) = \varnothing$, which assures the finiteness, without taking modulo the trace, of the set $(S, \mathcal{D})$-integral points for every finite $S \subset B$. When X is a constant family, our Theorem I also gives a criterion on the finiteness (which is then uniform) of the set of nonconstant $(S, \mathcal{D})$-integral sections. Moreover, our main Theorem F addreses Questions 1.12 - 1.13 and confirms the finiteness (modulo the trace of X_K) of the union of $(S, \mathcal{D})$-integral points where S runs over certain very large uncountably infinite subsets of B. Let d be any Riemannian metric on B. As an example, Theorem F implies:

Theorem. *For every finite disjoint union of closed discs W in B such that distinct points of T are contained in different discs of W, the set of $(W, \mathcal{D})$-integral points is finite (modulo the trace of X_K).*

To our knowledge, Theorem F establishes the first of such results in the literature. The result is out of reach of a purely algebraic method such as intersection height theory (cf. [**39**]) which requires inevitably that S is finite to establish a bound. As W can be chosen to be of measure arbitrarily close to the measure of B, our result gives some topological information on the intersection between $\mathcal{D}$ and the sections. We discuss this more in Theorem G. Answering Question 1.15 , Theorem F also assures that we have at most a polynomial growth in all cases. Let d be any Riemannian metric on B, we show, for example, in Theorem F that:

Theorem. *For every $s \in \mathbb{N}$ and for every finite disjoint union of closed discs W in B containing T in different discs, the union of $(W \cup S, \mathcal{D})$-integral points, where $S \subset B$ runs over all subsets of cardinality at most s, is finite (modulo the trace of X_K). The growth order is at most polynomial in s of explicit effective degree.*

In particular, Theorem F gives some new evidence and phenomena to the Geometric Lang-Vojta conjecture (cf. Conjecture 1.28 and the discussion that follows) in the case of subvarieties of abelian varieties (cf. Corollary 1.30).

One of the key new ingredients for our finiteness/growth results is Theorem D which gives a certain linear bound on the hyperbolic length of certain loops in various complements of a Riemann surface. Let $B_0 = B \setminus U$ be the complement in B of a finite disjoint union U of closed discs. If we fix a free homotopy (conjugation) class α of closed curves in B_0 then Theorem D implies that:

Theorem. *There exists $L > 0$ with the following property. For any finite subset $S \subset B_0$, there exists a piecewise smooth loop $\gamma \subset B_0 \setminus S$ which represents α up to conjugation and whose hyperbolic length in $B_0 \setminus S$ satisfies:*

$$\text{length}_{d_{B_0 \setminus S}}(\gamma) \leq L(\#S + 1). \tag{2.1}$$

Combining with the geometry of fundamental groups, Theorem D will serve in our context as a the hyperbolic part of a certain *hyperbolic-homotopic height* replacing the classical geometric height theory (i.e., intersection theory). As such, our hyperbolic-homotopic height is divided into two independent parts. The first part, Theorem D, is a purely *hyperbolic* statement on Riemann surfaces. The second part is the *geometry* of the fundamental group of $\mathcal{A}_{B_0}$ (cf. Chapter 6, Appendix 7). A *homotopy* reduction step due to Parshin completes the picture.

Remarkably, if X is a nonisotrivial elliptic curve, Theorem H refines Theorem F to confirm not only the finiteness for Question 1.13 but also a polynomial growth in s for Question 1.15 where $\mathcal{D}$, thus not just S, is now even allowed to vary in a compact family of divisors.

2.2. Emptiness results. Establishing an emptiness statement for a given Diophantine equation is extremely difficult, as shown by Fermat's Last Theorem. In fact, the equally spectacular MRDP theorem (Matiyasevich's theorem), which gives a negative answer to the Hilbert's Tenth Problem, tells us that there exists no algorithm which can decide whether a given Diophantine equation admits or not solutions in rational integers (cf. [**68**]). We thus restrict ourselves to consider a more feasible problem that is the *generic emptiness* of integral points of a Diophantine equation. For example, we ask in our context:

Question 1.16. Is it true that the set of $(S, \mathcal{D})$-integral points on X is empty for a general choice of a finite subset S?

As a convention in the literature, a general choice of a finite subset $S \subset B$ means for every $s \geq 1$, we consider subsets S of cardinality at most s whose images in the parameter space B^s belong to a Zariski dense open subset U_s. Our Corollary B confirms Question 1.16 whenever the trace of X_K is zero and $\mathcal{D}$ is positive enough. In fact, Corollary B shows even more. Let d be a Riemannian metric on B, we prove that:

Theorem. *For every $\varepsilon > 0$, there exists a closed subset $U \subset B$ with $\mathrm{vol}_d(B \backslash U) < \varepsilon$ such that the set of $(U \cup S, \mathcal{D})$-integral points is empty for a general choice of a finite subset $S \in B \setminus U$.*

A general choice of a finite subset S in $B_0 = B \setminus U$ in the above theorem means the following: for every $s \in \mathbb{N}$, there exists a finite subset $E_s \subset B_0$ (which depends on s) and the conclusion of the theorem holds for every finite subset $S \subset B_0$ of cardinality s with $S \cap E_s = \varnothing$.

Similarly, exchanging the role of S and $\mathcal{D}$, we ask:

Question 1.17. Fix a finite subset $S \subset B$ and consider a family T of effective divisors $\mathcal{D}$ in X. Is it true that the set of $(S, \mathcal{D})$-integral points on X is empty for a general choice of the divisor $\mathcal{D}$ in the family T?

Here, a general choice of $\mathcal{D}$ means we consider $\mathcal{D}$ parametrized by a Zariski dense open subset U of T. Again, Question 1.17 admits a positive answer given by Corollary C when X_K is a nonisotrivial elliptic curve and the family of divisors T is positive enough.

2.3. Uniform results. In the spirit of the Uniformity conjecture and Theorem 1.11 or Theorem 11.3, we study some *uniform* properties in the context of integral points in certain abelian varieties.

Assume first that X is a constant family, i.e., $X = A \times B$ for some complex abelian variety A. Thus $X(B) = A(K)$. Let $S \subset B$ be a subset of cardinality $s \in \mathbb{N}$. Let D be an effective ample divisor on A of degree $D^n = d$ and assume that $\mathcal{D} = D \times B$. Denote by $I \subset A(K) \setminus A(\mathbb{C})$ the set of nonconstant $(S, \mathcal{D})$-integral points. Using results of Noguchi-Wikelmann in [**82**], our main Theorem I implies that:

Theorem. *There exists a number $N(g, s, n, d) \geq 0$ such that:*

(i) *Either I is infinite or $\#I \leq N(g, s, n, d)$;*
(ii) *If $n = 2$, $d > 2g - 2$ and D is integral then $\#I \leq N(g, s, n, d)$.*

Let $S \subset B$ be a finite subset. The famous Parshin-Arakelov theorem ([**86**], [**5**]) confirms the finiteness of the set $F_q(B, S)$ consisting of non-isotrivial minimal surfaces over B with good reduction outside of S and whose general fibres have genus $q \geq 2$. Developing the covering trick of Parshin, Theorem M implies following uniform negative result.

Theorem. *For all large enough q and s depending only on the genus g of B, the union $\cup_{S \subset B, \#S \leq s} F_q(B, S)$ is uncountably infinite.*

Let $\Delta \subset \mathcal{M}_{q,n}$ be the locus of singular curves in the compact fine moduli space $\mathcal{M}_{q,n}$ of stable curves of genus $q \geq 2$ with level n-structure ($n \geq 3$). We obtain:

Corollary. *For large enough q, s, there exists uncountably many nonconstant morphisms $h \colon B \to \mathcal{M}_{q,n}$, up to automorphisms of B, whose set-theoretic intersection $h(B) \cap \Delta$ has no more than s points.*

The above results are a byproduct of our non successful attempt to use the uniform version due to Caporaso ([**18**]) of the Parshin-Arakelov theorem to reprove certain uniform finiteness of the union of (S, Δ)-integral points in a nonisotrivial elliptic surface $X \to B$ where S runs over all subset of B such that $\#S \leq s$. Nevertheless, restricting to the set of $(S, \mathcal{D})$-integral points with respect to a fixed S finite instead of taking the union over all such S, we obtain a new proof for the known uniform bound on $(S, \mathcal{D})$-integral points available in the literature (cf. Theorem L).

3. Plan of the thesis, statement of main results and discussion

The chapters in this thesis can be read independently, except for Chapter 6, Chapter 7, and Chapter 8 which are closely related and require the main results in Chapter 5.

The plan and the main results of the thesis are as follows. Chapter 2 collects basic definitions and recalls briefly known results and tools necessary for the thesis. Some proofs are given when appropriate. Chapter 3 illustrates general ideas and examples for some results obtained in Chapters 4 - 11 by restricting to the simplest but nontrivial case of constant elliptic surfaces. As such, experts are welcome to skip Chapter 3. Chapter 4 presents a criterion for the finiteness of integral points on abelian varieties. In Chapter 5, we study the growth of the hyperbolic lengths of loops in various complements of punctured Riemann surfaces. Chapters 6 - 8 develop the hyperbolic method of Parshin to obtain quantitative results on large unions of integral points on abelian varieties, on elliptic curves and on ruled surfaces respectively. Chapter 9 represents a uniform bound on the number of integral sections in a constant abelian family. We present a new proof in Chapter 10 of known uniform bounds on the canonical heights and of integral sections on elliptic surfaces using only the tautological inequality. In Chapter 11, we use Caporaso's uniform result on the Parshin-Arakelov theorem to give a new proof for certain uniformity of integral points on elliptic curves and to obtain certain set-theoretic information of the moduli space of curves.

We fix throughout a compact Riemann surface B of genus g. Denote $K = \mathbb{C}(B)$ its function field. In [**87**, Theorem 3.2], Parshin proved the following general finiteness theorem with a beautiful hyperbolic method:

Theorem 1.18 (Parshin)**.** *Let $S \subset B$ be a finite subset. Let A/K be an abelian variety and let D be an effective reduced ample divisor on A. Let $\mathcal{D}$ be the Zariski closure of D in the Néron model $f\colon \mathcal{A} \to B$ of A. Assume that D does not contain any translate of nonzero abelian subvarieties of A. Then the number of $(S, \mathcal{D})$-integral points is finite modulo $\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$.*

In the paper *loc.cit*, Parshin also proved the following result of Raynaud, which is a consequence of Raynaud's theorem on Manin-Mumford conjecture (cf. [**95**]).

Theorem 1.19 (Raynaud)**.** *Let A/K be an abelian variety and let $X \subset A$ be a closed subvariety not containing any translate of nonzero abelian subvarieties of A. Then $X(K)$ is finite modulo $\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$.*

Using the above theorems and properties of the Néron models with some further works, we prove in Chapter 4 a simple criterion for the finiteness of the set of integral points even without taking modulo the trace:

Theorem A. *Let $S \subset B$ be a finite subset. Let A/K be an abelian variety with a model $f\colon \mathcal{A} \to B$. Assume that $X, D \subset A$ are respectively a closed subvariety and an effective ample divisor on A. Let $\mathcal{D}$ be the Zariski closure of D in $\mathcal{A}$. Suppose that X and D do not contain any translate of nonzero abelian subvarieties of A.*

(i) *If $D(K) = \varnothing$ then the set of $(S, \mathcal{D})$-integral points is finite;*

(ii) *If $\dim \operatorname{Tr}_{K/\mathbb{C}}(A) \leq 1$ then $X(K)$ is finite.*

Theorem A generalizes the finiteness statement in Theorem I below for integral points in a constant abelian variety $\mathcal{A} = A \times B$ with a constant divisor $D \times B$ where $D \subset A$ is an effective ample divisor in a complex abelian variety A.

The condition $D(K) = \varnothing$ should be interpreted as the divisor D being in a very general position. In fact, we can prove that (cf. Chapter 4):

Theorem B. *Let A/K be an abelian variety with $D \subset A$ an effective ample divisor. Let $P \in A(K)$ be a rational point. Assume that for every $a \in \operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$, $a + P \notin D$. Then for every finite subset $S \subset B$, the set $P + \operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ contains only finitely many $(S, \mathcal{D})$-integral points of A with respect to any model $\mathcal{A} \to B$ of A with $\mathcal{D}$ being the Zariski closure of D in $\mathcal{A}$.*

For each $P \in A(K)$, the notation σ_P will stand for the section of $\mathcal{A} \to B$ induced by $P \in A(K)$. We can moreover prove that (cf. Chapter 4):

Theorem C. *Let A/K be an abelian variety with a Néron model $f\colon \mathcal{A} \to B$. Assume that $D \subset A$ is an effective ample divisor on A. Let $\mathcal{D}$ be the Zariski closure of D in $\mathcal{A}$. Let $P \in A(K)$ and let $\sigma_P \in \mathcal{A}(B)$ be the induced section. Assume $D \cap (P + \operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})) = \varnothing$. Then for $r \coloneqq \deg_B \sigma_P^* \mathcal{D}$, we have a finite morphism of complex schemes*

$$\pi\colon \operatorname{Tr}_{K/\mathbb{C}}(A) \to B^{(r)}, \quad a \mapsto (\sigma_{a+P})^* \mathcal{D}. \tag{3.1}$$

The condition $D \cap (P + \operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})) = \varnothing$ for some $P \in A(K)$ is a reasonably mild condition on D. For example, it is always satisfied if D does not contain any rational point in $A(K)$, i.e., if $D(K) = \varnothing$. Moreover, let $\Gamma = A(K)/\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ be the Mordell-Weil group of A. Then we can see $A(K) = \coprod_{[P_i] \in \Gamma} P_i + \operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ as a countable disjoint union of copies of $\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$. Then it suffices for D to avoid one of these copies, says, $P + \operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$, so that $P + a \notin D$ for every $a \in \operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$.

In Chapter 6, we shall focus on the hyperbolic-homotopy method developing the idea of Parshin in his proof of Theorem 1.18. One of the new ingredients, whose proof is given in Chapter 5, is the following linear bound on the hyperbolic length of loops in various complements of a Riemann surface.

Recall that for every complex space X, the pseudo Kobayashi hyperbolic metric on X is denoted by d_X (cf. Definition 2.38). Let U be a finite union of disjoint closed discs in B and denote $B_0 \coloneqq B \setminus U$. We show that:

Theorem D. *Let $\alpha \in \pi_1(B_0)\setminus\{0\}$. There exists $L > 0$ with the following property. For any finite subset $S \subset B_0$, there exists a piecewise smooth loop $\gamma \subset B_0 \setminus S$ which represents the free homotopy conjugation class α in B_0 and satisfies:*

$$\text{length}_{d_{B_0 \setminus S}}(\gamma) \leq L(\#S + 1). \tag{3.2}$$

Remark 1.20. A stronger statement is given Theorem 5.22 where we provide some more geometric information on the loop γ. Moreover, Theorem 5.2 proven in Chapter 5 shows that we can even require the loop γ to avoid furthermore certain bounded moving discs besides the finite subset S.

We sketch briefly below the interest of Theorem D to the study of integral points.

Consider an abelian fibration $f\colon \mathcal{A} \to B$ with an effective divisor $\mathcal{D} \subset \mathcal{A}$ whose generic part $\mathcal{D}_K \subset \mathcal{A}_K$ does not contain any translate of nonzero abelian subvarieties. Let $B_0 \coloneqq B \setminus U$ where U is certain finite union of disjoint closed discs in B. For every finite subset $S \subset B_0$, the image of every holomorphic section

$$\sigma\colon B_0 \setminus S \to (\mathcal{A} \setminus \mathcal{D})|_{B_0 \setminus S} \tag{3.3}$$

is a totally geodesic subspace with respect to the Kobayashi hyperbolic metrics $d_{B_0 \setminus S}$ and $d_{(\mathcal{A}\setminus\mathcal{D})|_{B_0\setminus S}}$. In particular, $\text{length}_{d_{(\mathcal{A}\setminus\mathcal{D})|_{B_0\setminus S}}} \sigma(\gamma) = \text{length}_{d_{B_0\setminus S}}(\gamma)$ for every loop $\gamma \subset B_0 \setminus S$. By Ehresmann's theorem, $\mathcal{A}_{B_0} \to B_0$ is a fibre bundle in the differential category. Let $w_0 \in \mathcal{A}_{B_0}$ and $b_0 = f(w_0) \in B_0$. Every algebraic section $\sigma\colon B_0 \to \mathcal{A}_{B_0}$ induces a homotopy section i_σ (more precisely a conjugacy class of sections) of the short exact sequence:

$$0 \to \pi_1(\mathcal{A}_{b_0}, w_0) \to \pi_1(\mathcal{A}_{B_0}, w_0) \xrightarrow{f_*} \pi_1(B_0, b_0) \to 0.$$

A reduction step of Parshin says that it is enough to bound the number of homotopy sections i_σ in order to bound the number of algebraic sections σ. Now fix a system of generators $\alpha_1, \dots, \alpha_k$ of $\pi_1(B_0, b_0)$. A theorem of Green says that $(\mathcal{A} \setminus \mathcal{D})_{B_0}$ is hyperbolically embedded in $\mathcal{A}$. Therefore, the number of homotopy sections will be controlled if we can bound $\text{length}_{d_{(\mathcal{A}\setminus\mathcal{D})|_{B_0}}} \sigma(\gamma_i) = \text{length}_{d_{B_0}}(\gamma_i)$ for some representative loop γ_i of α_i for every $i = 1, \dots, k$.

Let $S \subset B_0$ be a finite subset. An $(S \cup U, \mathcal{D})$-integral section $\sigma\colon B \to \mathcal{A}$ does not induce a section $B_0 \to \mathcal{A}_{B_0}$ in general but only induces a section $B_0 \setminus S \to (\mathcal{A} \setminus \mathcal{D})|_{B_0\setminus S}$. However, as $d_{B_0\setminus S} \geq d_{B_0}|_{B_0\setminus S}$, it suffices to bound $\text{length}_{d_{B_0\setminus S}}(\gamma_i)$ for *some* loop $\gamma_i \subset B_0 \setminus S$ representing α_i for $i = 1, \dots, k$. Therefore, Theorem D will play a crucial role for the quantitative growth estimation in terms of $s \in \mathbb{N}$ (cf.

Theorem F, Theorem H) of the union of $(S \cup U, \mathcal{D})$-integral sections of $\mathcal{A}$ over all finite subsets $S \subset B_0$ with $\#S \leq s$.

Now, for each free homotopy class $\alpha \in \pi_1(B_0)$ and each $s \in \mathbb{N}$, we can associate a constant

$$L(\alpha, s) \coloneqq \sup_{\#S \leq s} \inf_{[\gamma]=\alpha} \mathrm{length}_{d_{B_0 \setminus S}}(\gamma) \in \mathbb{R}_+, \tag{3.4}$$

where S runs over all subsets of B_0 of cardinality at most s and γ runs over all loops in $B_0 \setminus S$ representing the homotopy class α. Theorem D then asserts that the function $L(\alpha, s)$ grows at most linearly in terms of s. Moreover, we prove the following lower bound on the polynomial growth (cf. Chapter 5, Remark 5.6 for the optimality):

Theorem E. *Given $\alpha \in \pi_1(B_0) \setminus \{0\}$, there exists $c > 0$ such that for every $s \in \mathbb{N}$:*

$$L(\alpha, s) \geq \frac{cs^{1/2}}{\ln(s+2)}. \tag{3.5}$$

It would be interesting to understand the asymptotic behavior of $L(\alpha, s)$ in terms of s. For example, we may ask:

Question 1.21. What are the limits:

$$\deg^-(\alpha) \coloneqq \liminf_{s \to \infty} \frac{\ln L(\alpha, s)}{\ln s}, \quad \deg^+(\alpha) \coloneqq \limsup_{s \to \infty} \frac{\ln L(\alpha, s)}{\ln s}, \tag{3.6}$$

which correspond respectively to the lower and upper polynomial growth degrees of $L(\alpha, s)$ in terms of s?

Theorem D and Theorem E imply that for every $\alpha \in \pi_1(B_0) \setminus \{0\}$, we have:

$$1/2 \leq \deg^-(\alpha) \leq \deg^+(\alpha) \leq 1. \tag{3.7}$$

If we require α to belong to a certain base of $\pi_1(B_0)$, the constant $L > 0$ in Theorem D depends only on U and the Riemann surface B (cf. Theorem 5.22, Lemma 5.10). Our proof does not provide an explicit estimate of L. But with more care, an upper bound of L only in terms of some invariants of B and U could be established in principle. We expect also that the bound in Theorem D can be improved (as closed as possible) to be linear in $(\#S)^{1/2}$, which is best possible by Theorem E in the sense of Question 1.21. They are our ongoing projects.

Remark 1.22. Results in Theorem D and Theorem E are in a sense *orthogonal* to various remarkable results on the growth of the counting functions $c_X(L), s_X(L)$ for the number of certain type of closed geodesics of length $\leq L$ on a complete hyperbolic bordered Riemann surface X of finite area (cf., for example, [**67**], [**8**], [**56**], [**77**]).

Remark 1.23. It may be helpful to illustrate the difference of our results to the closely related Bers' theorem (cf. [**16**, Theorem 5.2.6]). Let X be a complete hyperbolic Riemann surface of genus g with n cusps. A *partition* on X is a set of $3g-3+n$ pairwise disjoint simple closed geodesics. However, it is well-known that these curves *do not* generate the free homotopy group $\pi_1(X)$. A partition is in fact the same as a decomposition into pants. Bers' theorem asserts that X has a partition with geodesics of hyperbolic length bounded by $13(3g-3+n)$.

Remark that Bers' theorem applies to complete hyperbolic Riemann surfaces with cusps that have *finite area*. On the contrary, punctured Riemann surfaces $B_0 \setminus S$ (as in Theorem D and Theorem E) have *infinite area* (whenever U is not finite) and look like the punctured Poincaré disc locally around the punctured points.

Therefore, Bers' theorems do not apply to the punctured Riemann surfaces $B_0 \setminus S$ which are equipped with the intrinsic hyperbolic metric $d_{B_0\setminus S}$ (cf. Definition 2.38). Moreover, our proofs of Theorem D and of Theorem E work with the very definition of the pseudo Kobayashi hyperbolic metric and do not require any tools from the hyperbolic trigonometry.

Before giving some more thorough discussion on Theorem D in Remarks 1.29 - 1.33 below, we continue with our main results obtained on integral points.

Setting (P). Assume that B is equipped with a Riemannian metric d. Let A/K be an abelian variety. Let $D \subset A$ be an effective ample divisor of A not containing any translate of nonzero abelian subvarieties. Let $\mathcal{D}$ be its Zariski closure in a model $f\colon \mathcal{A} \to B$ of A. Let $T \subset B$ be the finite subset containing $b \in B$ such that $\mathcal{A}_b$ is not smooth.

Using Theorem D and the geometry of fundamental groups, we establish the following finiteness and growth of certain large unions of $(S, \mathcal{D})$-integral points (cf. Chapter 6). In fact, we can obtain in Theorem 6.1 an even stronger statement allowing furthermore the intersection of integral sections with $\mathcal{D}$ over some bounded moving discs in B.

Theorem F. *In Setting (P), let $W \supset T$ be any finite union of disjoint closed discs in B such that distinct points of T are contained in distinct discs. Let $B_0 = B\setminus W$. There exists $m > 0$ such that:*

(∗) *For I_s ($s \in \mathbb{N}$) the union of $(S, \mathcal{D})$-integral points over all subsets $S \subset B$ such that $\#(S \cap B_0) \leq s$, we have:*

$$\#\left(I_s \text{ mod } \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\right) < m(s+1)^{2\dim A.\,\mathrm{rank}\,\pi_1(B_0)}, \quad \textit{for all } s \in \mathbb{N}. \tag{3.8}$$

Remark 1.24. Here, $\mathrm{rank}\,\pi_1(B_0)$ denotes the minimal number of generators of the finitely generated group $\pi_1(B_0)$. Typically, we have $\mathrm{rank}\,\pi_1(B_0) = 2g-1+\#T$.

Since I_s clearly contains the union of $(S, \mathcal{D})$-integral points over all subsets $S \subset B$ such that $\#S \leq s$, Theorem F implies Theorem 1.18 and recovers the polynomial bound in $\#S$ proved in [**52**] for elliptic curves or in [**14**] when $\operatorname{Tr}_{K/\mathbb{C}} A = 0$ and in [**82**] when A is defined over $\mathbb{C}$ (cf. Remark 1.26).

Remark 1.25. The subsets S are not required to be finite or even countably finite in the union I_s in Theorem F. In fact, for every $\varepsilon > 0$, S can be taken to be of measure ε-close to $\operatorname{vol}_d(B)$ by either adding more discs to W or by enlarging W in the first place. To the limit of our knowledge, even finiteness results of $(S, \mathcal{D})$-integral points for certain $S \subset B$ countably infinite is not stated elsewhere before in the literature. To obtain such results, establishing a height bound as in traditional approaches, which depends on the cardinality of S, is clearly not sufficient. In the case of number fields, the closest related finiteness result of $(S, \mathcal{D})$-integral points where S may be an infinite set of places seems to be a result of Silverman [**103**] (see Theorem 1.39) for elliptic curves but it requires some strong restrictions on the set of solutions.

Remark 1.26. Continue with the notations in Theorem F. By a standard counting argument on the lattice $\Gamma = A(K)/\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ (cf. Lemma 2.24, Theorem 2.21), a bound on the canonical height of $(S, \mathcal{D})$-integral points, if it is a polynomial p in $\#S$, would imply also a polynomial bound in $\#S$ (of the form $(c\sqrt{p(\#S)}+1)^{\operatorname{rank}\Gamma}$) on the number of such integral points (modulo the trace). Conversely, with height theory, it is necessary to establish a uniform bound on the intersection multiplicities (as predicted by the Geometric Lang-Vojta conjecture 1.28 below) in order to obtain a polynomial bound in $\#S$ on $(S, \mathcal{D})$-integral points, where $S \subset B$ is finite.

Such uniform bounds on the intersection multiplicities, and thus a linear bound on the canonical height $\widehat{h}_{A,L}$, where L is a symmetric ample line bundle on A, are only available in the case $\operatorname{Tr}_{K/\mathbb{C}}(A) = 0$ (established in [**14**] using Kolchin differentials, or in [**52**] for elliptic curves) or when the family $\mathcal{A} \to B$ is trivial (cf. [**82**] for $\mathcal{D}$ constant, where the main tool is jet-differentials, see Theorem 9.10 for general $\mathcal{D}$). However, no similar results were known for a general abelian variety A/K.

To this point, we could, motivated by Theorem F, possibly expect the following (with $N = 1$ in the above two special cases and $N \leq 2$ in general as suggested by the bound (4.1) in Theorem F):

Conjecture A. *Let A/K be an abelian variety with a model $f\colon \mathcal{A} \to B$, i.e., $\mathcal{A}_K = A$. Let $D \subset A$ be an effective ample divisor and let $\mathcal{D}$ be its Zariski closure in $\mathcal{A}$. There exists $m, N > 0$ depending only on $\mathcal{A}, B, \mathcal{D}$ such that for every $(S, \mathcal{D})$-integral point $P \in A(K)$ with $S \subset B$, we have:*

$$\widehat{h}_{A,L}(P) \leq m(\#S + 1)^N. \tag{3.9}$$

Remark 1.27. Let the notations be as in Theorem F. We argue below that the exponent $2\dim A \operatorname{rank} \pi_1(B_0)$ in the bound (4.1) is as reasonable, up to a factor $\frac{1}{2}$, as we can expect. Remark again that no quantitative estimations as in the bound (4.1) were known in the literature except when $\mathrm{Tr}_{K/\mathbb{C}}(A) = 0$ or when $\mathcal{A}$ is a constant family (and with $B_0 = B$ in both cases).

Let $T \subset B$ be the finite subset above which the fibres of $f\colon \mathcal{A} \to B$ is not smooth and let $t = \#T$. We shall consider an arbitrary disjoint union W of t closed discs centered at the points of T so that $\operatorname{rank} \pi_1(B_0) = 2g - 1 + t$. Let $r = \operatorname{rank} A(K)/\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ be the Mordell-Weil rank of A. For $s \in \mathbb{N}$, let J_s denote the union of $(S, \mathcal{D})$-integral points over all subsets $S \subset B$ such that $\#S \leq s$. It is clear that $J_s \subset I_s$ for every $s \in \mathbb{N}$.

Assume first that A is a nonisotrivial elliptic curve so that the trace of A is zero. Note that $\frac{1}{2} \deg \mathfrak{f}_{A/K} \leq t \leq \deg \mathfrak{f}_{A/K}$ where $\mathfrak{f}_{A/K}$ is the conductor divisor of the elliptic curve A/K (cf. [**104**, Ex. III.3.36]). Shioda's result in [**101**, Theorem 2.5] provides a very general bound on the Mordell-Weil rank:

$$r \leq 2(2g - 2 + t). \tag{3.10}$$

Best known results using height theory due to Hindry-Silverman [**52**, Corollary 8.5] (see Corollary 1.37) tell us that $\#J_s$ is bounded by a polynomial in s of degree

$$\frac{r}{2} \leq (2g - 1 + t) = \frac{1}{2}(2 \operatorname{rank} \pi_1(B_0)).$$

Therefore, Theorem F implies, up to a factor $\frac{1}{2}$, the known polynomial growth for the set $J_s \subset I_s$.

More generally, when A is an abelian variety with $\mathrm{Tr}_{K/\mathbb{C}}(A) = 0$, the Ogg-Shafarevich formula (cf. [**100**], [**83**], see also [**49**]) implies that:

$$r \leq 2 \dim A.(2g - 2 + t). \tag{3.11}$$

In this case, best results using height theory (due to Buium [**14**], see Remark 1.26) tell us that $\#J_s$ is bounded by a polynomial in s of degree $r/2 \leq \dim A.(2g-2+t)$. As $\mathrm{vol}_d(W)$ can be arbitrarily closed to $\mathrm{vol}_d(B)$, our union I_s of integral points is *a priori* much larger than J_s. However, Theorem F assures that the polynomial growth degree of $\#I_s$ is still at most $\sim \dim A. \operatorname{rank} \pi_1(B_0)$ just as J_s. This feature cannot be detected by Hindry-Silverman's and Buium's results.

If $\mathcal{A}$ is a constant family of simple abelian varieties then $T = \varnothing$. In this case, Noguchi-Winkelmann's results for a constant ample divisor $\mathcal{D} = D \times B$ (cf. [**82**], see Remark 1.26) or Theorem J below for general $\mathcal{D}$ imply that modulo the trace $\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$, the cardinality of J_s is bounded by a polynomial in s of degree $\frac{r}{2} \leq 2g \dim A$. Hence, Theorem F also improves known upper bounds on the growth degree to the much larger set $I_s \supset J_s$.

Let's return to some more important remarks on Theorem D (and Theorem F).

Suppose that we are given a smooth projective curve $\bar{C}$ defined over an algebraically closed field k of characteristic zero. Let S be a finite set of points of $\bar{C}$ and denote $C = \bar{C} \setminus S$. Consider a smooth affine variety $X/k(\bar{C})$ of log-general type with a model $\mathcal{X} \to C$. Let $\mathcal{D}$ be the hyperplane at infinity in a compactification $\bar{\mathcal{X}}$ of $\mathcal{X}$. The *Geometric Lang-Vojta conjecture*, which implies Conjecture A (cf. Remark 1.29 below), says that:

Conjecture 1.28 (Lang-Vojta)**.** *There exists a proper closed subset $Z \subset X$ and a finite number $m > 0$ with the following property. Let $\mathcal{Z}$ be the Zariski closure of Z in $\mathcal{X}$. For every section $\sigma \colon C \to \mathcal{X}$ with $\sigma(C) \not\subset \mathcal{Z}$,*

$$\deg_C \bar{\sigma}^* \mathcal{D} \leq m \cdot \max\{1, 2g(\bar{C}) - 2 + \#S\}, \tag{3.12}$$

where $\bar{\sigma} \colon \bar{C} \to \bar{\mathcal{X}}$ is the extension of σ.

In what follows, it may be helpful to regard $\#S$ as a bound for the number of points in the set-theoretic intersection $\bar{\sigma}(\bar{C}) \cap \mathcal{D}$.

Remark 1.29. The bound (3.12) is well-known when X is a curve. Moreover, when X is an elliptic curve, the bound is in fact effective (cf. [**52**, Corollary 8.5]). Besides that, several results are known when $\bar{X} = \mathbb{P}^2$ in all arithmetic, analytic and algebraic settings (cf. for example, [**24**], [**25**], [**110**] and the references therein). When X is the complement of an effective ample divisor in an abelian variety $\bar{X}$, the exceptional set Z is taken to be empty in Conjecture 1.28 (by an immediate induction using [**113**, Lemma 4.2.1]). It is also known, for S finite, that Conjecture 1.28 holds when X is the complement of an effective ample divisor in an abelian variety $A/k(\bar{C})$ such that $\mathrm{Tr}_{\overline{k(\bar{C})}/k}(A) = 0$ (cf. [**14**]) or when A is defined over $k = \mathbb{C}$ (cf. [**82**]). No similar results are known in the literature for a general abelian variety.

From Theorem F, we obtain a generalization of Conjecture 1.28 for certain polarized abelian varieties (A, D) as in Setting (P) defined above Theorem F. We remark that even if the condition $\#f(\sigma(B_0) \cap \mathcal{D}) \leq s$ in the statement below is replaced by the much weaker condition $\#f(\sigma(B) \cap \mathcal{D}) \leq s$, the conclusion is only known in the literature when the trace of A is zero or when $\mathcal{A}$ is isotrivial.

Corollary 1.30. *In Setting* (P)*, let $W \supset T$ be any finite union of disjoint closed discs in B such that distinct points of T are contained in distinct discs. Let $B_0 = B \setminus W$. Then for each $s \in \mathbb{N}$, there exists $M = M(\mathcal{A}, \mathcal{D}, B_0, s) \in \mathbb{R}_+$ such that for every section $\sigma \colon B \to \mathcal{A}$ with $\#f(\sigma(B_0) \cap \mathcal{D}) \leq s$, we have:*

$$\deg_B \sigma^* \mathcal{D} < M. \tag{3.13}$$

In fact, using Theorem 6.1 instead of Theorem F, we can obtain an even stronger conclusion than Corollary 1.30 (cf. Corollary 6.2).

Proof of Corollary 1.30 . We equip A/K with a symmetric ample line bundle L and consider the corresponding canonical Néron-Tate height function $\widehat{h}_L \colon A(K)/\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}) \to \mathbb{R}_+$. Since L is ample, there exists $n \in \mathbb{N}^*$ such that the line bundle $L^{\otimes n} \otimes \mathcal{O}(-D)$ is very ample on A. By standard positivity properties of height theory (cf. [**53**], [**23**]), there exists a finite number $c > 0$ such that for every $P \in A(K)$ with $\sigma_P \in \mathcal{A}(B)$ the corresponding section, we have:

$$n\widehat{h}_L(P) + c > \deg_B \sigma_P^* \mathcal{O}(\mathcal{D}). \tag{3.14}$$

Let $s \in \mathbb{N}$ and let $\Sigma_s \subset \mathcal{A}(B)$ be the union of $(S, \mathcal{D})$-integral sections over all subsets $S \subset B$ with $\#(S \cap B_0) \leq s$. Under the identification $A(B) = \mathcal{A}(B)$ (cf. Remark 1.2) and by Definition 1.1, we have $\Sigma_s = I_s$ where $I_s \subset A(K)$ is defined by $(*)$ in Theorem F.

As I_s mod $\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ is finite by Theorem F and as the canonical height $\widehat{h}_L$ is invariant under translations by the trace, we can define a finite number $H = \max_{P \in I_s} \widehat{h}_L(P)$. Since every section $\sigma \colon B \to \mathcal{A}$ with $\# f(\sigma(B_0) \cap \mathcal{D}) \leq s$ belongs to Σ_s, (3.14) implies that:

$$\deg_B \sigma^* \mathcal{D} \leq \sup_{\tau \in \Sigma_s} \deg_B \tau^* \mathcal{D} \leq n \max_{P \in I_s} \widehat{h}_L(P) + c \leq nH + c.$$

The conclusion follows by setting $M = nH + c \in \mathbb{R}_+$. ☐

In particular, the Lefschetz Principle and Corollary 1.30 imply immediately that Conjecture 1.28 is true for $X = \bar{X} \setminus D$ with moreover an empty exceptional set Z, where $\bar{X}/k(\bar{C})$ is an abelian variety and $D \subset \bar{X}$ is any effective ample divisor not containing any translates of nonzero abelian subvarieties of $\bar{X}$. We continue with several remarks.

Remark 1.31. It is worth notice the similarity between the bound (1.2) in Theorem D and the bound (3.12) in their linearity in terms of $\#S$. An important implication of the linear bound (3.12) when $\bar{X} = A$ is an abelian variety is the polynomial bound in $\#S$ on the number of $(S, \mathcal{D})$-integral sections of A modulo the trace (cf. Remark 1.26).

In this aspect, the bound (1.2) in Theorem D will serve, together with the geometry of fundamental groups, as a certain *hyperbolic-homotopic height* to establish a polynomial bound in $\#S$ of the number of $(U \cup S, \mathcal{D})$-integral sections of $\bar{X}$ modulo the trace when $k = \mathbb{C}$ and $U \subset \bar{C}$ is certain union of disjoint discs (cf. Theorem

F, Theorem H). Notably, we only need to care about the cardinality of the set-theoretic intersection $\sigma(\bar{C}) \cap \mathcal{D}$ above the complement $\bar{C} \setminus U$ which can be taken to be as small as we want (e.g., in terms of volume).

Remark 1.32. Another interesting aspect of the linear bound (1.2) in Theorem D is that it is a purely geometric statement on the base B. The linearity in terms of S is thus an intrinsic property seen already on the base B, independently of the abelian variety $\bar{X}$ and the divisor $\mathcal{D}$. This feature is predicted by the Geometric Lang-Vojta conjecture where $2g(\bar{C}) - 2 + \#S =: \chi(C)$ is nothing but the *Euler characteristic* of the affine curve C.

Remark 1.33. Theorem D, Theorem F and Corollary 1.30 thus provide some new evidence and phenomena to the Geometric Lang-Vojta conjecture. They suggest, for example, that in many situations, the constant m in Conjecture 3.12 should depend only on $\bar{C}$, the model $\mathcal{X}$, the divisor $\mathcal{D}$ and thus completely *independent* of the finite set S, as for the constants L, m in Theorem D, Theorem F and Conjecture A. As another evidence, [**24**, Theorem 1.1] (see also [**25**, Theorem 1]) confirms that we can take $m = 2^{17}.35$ when $\chi(C) > 0$ and $\mathcal{X} = \bar{C} \times (\mathbb{P}^2 \setminus D)$, where $D \subset \mathbb{P}^2$ is a quartic consisting of the union of a smooth conic and two lines in general position.

Concerning the topology of the intersection of sections of an abelian scheme with a horizontal divisor, we prove the following result (cf. Chapter 6).

Let A/K be an abelian variety. Let $D \subset A$ be an effective ample divisor not containing any translate of nonzero abelian subvarieties. Let $\mathcal{D}$ be its Zariski closure in a model $f \colon \mathcal{A} \to B$ of A. Recall that for each $P \in A(K)$, $\sigma_P \in \mathcal{A}(B)$ stands for the section induced by P. For every subset $R \subset A(K) \setminus D$, we define the intersection locus:

$$I(R, \mathcal{D}) \coloneqq \cup_{P \in R} f(\sigma_P(B) \cap \mathcal{D}) \subset B. \tag{3.15}$$

Theorem G. *Assume that* $\mathrm{Tr}_{K/\mathbb{C}}(A) = 0$ *and that* R *is infinite. We have:*

(a) $I(R, \mathcal{D})$ *is countably infinite but it is not analytically closed in* B*;*

(b) $I(R, \mathcal{D})$ *has uncountably many limit points* $I(R, \mathcal{D})_\infty$ *in* B*;*

(c) $I(R, \mathcal{D})_\infty$ *is not contained in any union* $W \supset T$ *of disjoint closed discs in* B *such that distinct points of* T *are contained in distinct discs.*

On the other hand, when the trace $\mathrm{Tr}_{K/\mathbb{C}}(A)$ is not zero and under some mild condition on the divisor D, a consequence of Theorem C says that:

Corollary A. *Let* A/K *be an abelian variety with a Néron model* $f \colon \mathcal{A} \to B$*. Assume that* $D \subset A$ *is an effective ample divisor on* A*. Let* $\mathcal{D}$ *be the Zariski closure of* D *in* $\mathcal{A}$*. Let* $P \in A(K)$ *be such that* $D \cap (P + \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})) = \varnothing$*. Let* $R \subset \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ *and* $I(R) \coloneqq \cup_{a \in R} f(\sigma_{a+P}(B) \cap \mathcal{D}) \subset B$*. Then:*

(i) *if R is infinite, $I(R)$ is Zariski dense, i.e., infinite, in B;*
(ii) *if R is analytically dense in a complex algebraic curve $C \subset \mathrm{Tr}_{K/\mathbb{C}}(A)$, $I(R)$ is analytically dense in B.*

Here are some motivations for Theorem G and Corollary A. Assume that B is a smooth projective curve defined over a number field k. Let $K = k(B)$ and let $f\colon \mathcal{A} \to B$ be a nonisotrivial elliptic surface. Assume that $\mathcal{D}$ is the zero section of f and $R = \{nQ\colon n \in \mathbb{N}^*\} \subset \mathcal{A}_K(K)$ for some non torsion point $Q \in \mathcal{A}_K(K)$. Let $I(R, \mathcal{D}) \coloneqq \cup_{P \in R} f(\sigma_P(B(\bar{k}) \cap \mathcal{D})) \subset B(\bar{k})$ where $\sigma_P \in \mathcal{A}(B)$ denotes the induced section of $P \in \mathcal{A}_K(K)$. It is known that the intersection locus $I(R, \mathcal{D}) \subset B(\bar{k})$ is analytically dense in $B(\mathbb{C})$ (cf. [**117**, Notes to chapter 3]). Theorem G and Corollary A thus provide some evidence that analogous density results on the intersection locus could be true in higher dimensional abelian varieties over function fields. In fact, recent results in [**27**] imply that $I(R, \mathcal{D})$ is even equidistributed in $B(\mathbb{C})$ with respected to a certain Galois-invariant measure.

As the parameter space $B^{(s)}$ of subsets $S \subset B$ of cardinality at most $s \geq 1$ is a variety of dimension s, the finiteness of the union I_s (cf. Theorem F) of $(S, \mathcal{D})$-integral points should imply that for a general choice of such S, there is very few or no $(S, \mathcal{D})$-integral points at all. In fact, using Theorem F we can show an even stronger property (cf. Chapter 6, Section 5):

Corollary B. *Let the notations be as in Theorem F. Assume $\mathrm{Tr}_{K/\mathbb{C}}(A) = 0$ and $\mathcal{D}$ horizontally strictly nef, i.e., $\mathcal{D} \cdot C > 0$ for all curves $C \subset \mathcal{A}$ not contained in a fibre. We have:*

(i) *for each $s \in \mathbb{N}$, there exists a finite subset $E \subset B$ such that for any $S \subset B \setminus E$ with $\#(S \cap B_0) \leq s$, the set of $(S, \mathcal{D})$-integral points is empty. Moreover, we can choose E such that $\#E \cap B_0 \leq ms(s+1)^{2 \dim A . \operatorname{rank} \pi_1(B_0)}$;*
(ii) *for each $s \in \mathbb{N}$, there exists a Zariski proper closed subset $\Delta \subset B^{(s)}$ such that for any $S \subset B$ of cardinality s whose image $[S] \in B^{(s)} \setminus \Delta$, there is no $(S, \mathcal{D})$-integral points.*

It is clear that Corollary B.(i) cannot not be obtained by means of classical algebraic methods (e.g., height theory). The subsets $S \subset B$ satisfying Corollary B.(i) can be taken as $S = (B \setminus (B_0 \cup E)) \cup N = (U \setminus E) \cup N$ where $N \subset B_0 \setminus E$ is any finite subset of cardinality at most s. Since B_0 can be taken arbitrarily small (cf. Theorem F), such transcendental subsets S are very large. In this sense, we find that Corollary B.(i) is quite surprising.

In Chapter 7, we shall further develop methods in Chapter 6 to the case of elliptic curves. One of the advantages in this case is that the condition requiring the divisor D to not contain any translates of nonzero abelian subvarieties is always satisfied. We obtain therein (Theorem H) a strong property on the finiteness and the growth of a very large union of $(S, \mathcal{D})$-integral points in which both S and $\mathcal{D}$ are allowed to vary in families.

An effective divisor D on a fibered variety $f \colon X \to B$ is called *horizontal* if the induced map $D \to B$ is dominant. Otherwise, D is said to be *vertical.*

Now let $f \colon X \to B$ be a nonisotrivial elliptic surface. Let $T \subset B$ be the finite subset above which the fibres of f are not smooth. Let $\mathcal{D} \subset X \times \tilde{Z}$ be an algebraic family of relative horizontal effective Cartier divisors on X. Assume that $\mathcal{D} \to B \times \tilde{Z}$ is flat. Let $Z \subset \tilde{Z}$ be a relatively compact subset with respect to the complex topology. The following finiteness and growth rate of the double union over S and $\mathcal{D}$ of $(S, \mathcal{D})$-integral points is proved (cf. Chapter 7):

Theorem H. *Let the notations and hypotheses be as above. Consider any finite union of disjoint closed discs $V \subset B$ containing T such that distinct points of T are contained in distinct discs. For each $s \in \mathbb{N}$, the union*

$$J_s \coloneqq \cup_{z \in Z} \cup_{S \subset B, \#(S \setminus V) \leq s} \{(S, \mathcal{D}_z)\text{-}\textit{integral points of } X_K\} \subset X_K(K)$$

is finite. Moreover, there exists $m > 0$ such that for every $s \in \mathbb{N}$, we have:

$$\#J_s \leq m(s+1)^{2 \operatorname{rank} \pi_1(B \setminus V)}. \tag{3.16}$$

In Theorem H, if $\mathcal{D}_z$ is a strictly nef divisor for some $z \in Z$ then Corollary B.(ii) implies the generic emptiness of $(S, \mathcal{D}_z)$-integral points for any general choice of the finite subset S. On the other hand, if we fix a finite subset $S \in B$ then we also have the emptiness of the set of $(S, \mathcal{D}_z)$-integral points for a general choice of $z \in Z$ whenever the family of divisors $\mathcal{D}$ is positive enough. This is the content of the following corollary (cf. Chapter 7, Section 2 for the proof).

We say that a family of effective Cartier divisors $R \subset Y \times T/T$, where Y, T are algebraic varieties, is *base-point-free* if

$$\cap_{t \in T} R_t = \varnothing.$$

Corollary C. *Let the notations be as in Theorem H. Assume that Z is integral and that $\mathcal{D}$ is base-point-free and that $\mathcal{D}_{z_0}$ is ample for some $z_0 \in Z$. Fix a finite subset $S \subset B$. There exists a Zariski dense open subset V of Z such that there is no $(S, \mathcal{D}_z)$-integral points for every $z \in V$.*

Therefore, along with Corollary B, we see that certain general Diophantine equations over function fields admit no integral points.

In Chapter 8, we shall apply the methods of Chapter 7 and Chapter 6 to study certain generalized unit equations over function fields.

In Chapter 9, inspired by the results and the jet-differential method of Noguchi-Winkelman ([**82**]), we shall establish the following uniform finiteness of integral points in a constant abelian variety with respect to a constant effective ample divisor. Consider a finite subset $S \subset B$ of cardinality at most s. Consider an abelian variety A of dimension n and an effective ample divisor D on A of degree $D^n = d$. Denote by I the set of nonconstant algebraic morphisms $f\colon (B \setminus S) \to (A \setminus D)$. Recall that g denotes the genus of B.

Theorem I. *There exists a number $N(g,s,n,d) \in \mathbb{N}$ satisfying:*

(i) *either I is infinite or $\#I \leq N(g,s,n,d)$;*
(ii) $\#\{f \in I \colon a + \operatorname{Im} f \not\subseteq D, \forall a \in A\} \leq N(g,s,n,d)$;
(iii) *if $n = 2$, $d > 2g-2$ and D is integral then $\#I \leq N(g,s,n,d)$.*

Remark 1.34. Let the notations be as in Theorem I. Let $\mathcal{A} = A \times B$. Then the set I of nonconstant algebraic morphisms $f\colon (B \setminus S) \to (A \setminus D)$ is exactly the set nonconstant $(S,\mathcal{D})$-integral points of $\mathcal{A} \to B$ where $\mathcal{D} = D \times B$.

The case of an arbitrary effective divisor $\mathcal{D} \subset A \times B$ is also treated with the tools of jet-differentials as in [**82**] as follows (cf. Theorem 9.10).

Theorem J. *Let A be a complex abelian variety. Let $\mathcal{D}$ be an integral divisor in $A \times B$. There exists a number $M > 0$ satisfying the following property. For every morphism $f\colon B \to A$ such that $(f \times \mathrm{Id}_B)(B) \not\subseteq \mathcal{D}$, we have an estimation*

$$\operatorname{mult}_x (f \times \mathrm{Id})^*\mathcal{D} \leq M \quad \textit{for all } x \in B. \tag{3.17}$$

As an application, we establish in Chapter 9 the following semi-effective bound on the number of integral points of *bounded denominators*. Note that in constrast to Theorem F, the ample divisor $\mathcal{D}_K$ in the statement below may a priori contain a translation of a nonzero abelian subvariety.

Corollary D. *Let A be a complex abelian variety of dimension n. Let $\mathcal{D} \subset A \times B$ be an effective divisor such that $\mathcal{D}_K$ is ample. For each integer $s \geq 1$, let $W(s,\mathcal{D})$ be the set of morphisms $f\colon B \to A$ such that $\#(f \times \mathrm{Id}_B)(B) \cap \mathcal{D} \leq s$. Then there exists a number $H > 0$ such that for any $s \geq 1$ we have a semi-effective bound*

$$\#W(s,\mathcal{D}) \textit{ modulo } A(\mathbb{C}) \leq (2\sqrt{sH}+1)^{4gn}.$$

Using only the tautological inequality (cf. Definition 2.42), we shall establish in Chapter 10 the following uniform bound on the canonical height of integral points in every *relative maximal variation* family of semisimple elliptic surfaces (cf. Definition 10.1) and with respect to an *adaptive* family of ample effective divisors (cf. Definition 10.3):

Theorem K. *Let $\mathcal{X} \xrightarrow{f} \mathcal{C} \to Z$ be a relative maximal variation family of semistable elliptic surfaces with a zero section $O\colon \mathcal{C} \to \mathcal{X}$. Let $\mathcal{D} \subset \mathcal{X}$ be an adaptive family of ample effective divisors. There exists numbers $c_1, c_2 > 0$ such that for every closed point $z \in Z$ and every $P \in \mathcal{X}_z(k(\mathcal{C}_z)) \setminus \mathcal{D}_z$, we have:*

$$\widehat{h}_{O_z}(P) \leq c_1 s + c_2, \qquad \text{where } s = \#\sigma_P(\mathcal{C}_z) \cap \mathcal{D}_z. \tag{3.18}$$

Here, $\widehat{h}_{O_z}$ is the Néron-Tate height on $\mathcal{X}_z(k(\mathcal{C}_z))$ associated to the origin O_z and $\sigma_P \in \mathcal{X}_z(\mathcal{C}_z)$ is the corresponding section of P.

When $\mathcal{D} = (O)$ is the zero section, the same method with the tautological inequality shows that the constants c_1, c_2 depend only on topological invariants $\chi = \chi(\mathcal{X}_z)$ and $q = g(\mathcal{C}_z)$ independent of $z \in Z$ (cf. Theorem 10.4). Thereby, we establish a new proof of a uniform consequence of a well-known result of Hindry-Silverman (cf. [**52**, Corollary 8.5]). However, our proof is more geometric and does not make use of the Weierstrass equation.

In Chapter 11, we obtain a new proof for the known uniform finiteness of integral points on elliptic curves (cf. [**52**]). The proof develops Parshin's idea in his proof of the Mordell conjecture over function fields. In particular, no height bound is established in our approach. For the statement, let $f\colon X \to B$ be a nonisotrivial elliptic surface and $T \subset B$ the finite set of cardinality t above which f is not smooth. Let $\mathcal{D}$ be an effective reduced horizontal divisor on X and let $S \subset B$ be a finite subset of cardinality s.

Theorem L. *The set of $(S, \mathcal{D})$-integral points is finite and uniformly bounded by a function depending only on g, s, t, $\deg \mathcal{D}_K$, the number of ramified points in the cover $\mathcal{D} \to B$, and the number of singular points on $\mathcal{D}$.*

Let's recall the celebrated Parshin-Arakelov theorem. Fix $S \subset B$ a finite subset. Let $F_q(B, S)$ be the set of non-isotrivial minimal surfaces over B with good reductions outside of S and with general fibres of genus q.

Theorem 1.35 (Parshin-Arakelov, [**86**], [**5**])**.** *$F_q(B, S)$ is finite for every $q \geq 2$.*

With the method proving Theorem L, it turns out that a finiteness result on the union $\cup_{S \subset B, \#S \leq s} F_q(B, S)$ will imply the finiteness of integral point of bounded denominators on elliptic surfaces such as Theorem H, Corollary 1.37, or Corollary 10.5. Unfortunately, we shall see in Chapter 11, Section 4 the following strong negative result:

Theorem M. *For large enough q, s depending only on the genus g of B, the union*

$$\cup_{S \subset B, \#S \leq s} F_q(B, S)$$

is uncountably infinite. Moreover, there exists $N(g,s,q) \in \mathbb{N}^*$, a Zariski dense open subset $I \subset \mathbb{P}^N$, and a map $I \to \cup_{S \subset B, \#S \leq s} F_q(B,S)$ with uniformly bounded finite fibres. We can take $N \to \infty$ when $q, s \to \infty$.

Consider the compact fine moduli space $\mathcal{M}_{q,n}$ of stable curves of genus $q \geq 2$ with suitable level n-structure ($n \geq 3$). Let $\Delta \subset \mathcal{M}_{q,n}$ be the divisor locus of singular curves. Then the Parshin-Arakelov theorem states the finiteness of the set of nonequivalent nonconstant morphisms from the affine curve $B \setminus S$ to $\mathcal{M}_{q,n} \setminus \Delta$. In a certain "union of (S, Δ)-integral section" style, we deduce from Theorem M the following geometric information on the moduli space of stable curves:

Corollary E. *For large enough $q, s \in \mathbb{N}$, there exists uncountably many non-constant morphisms $h \colon B \to \mathcal{M}_{q,n}$, up to automorphisms of B, such that the set-theoretic intersection $h(B) \cap \Delta$ has no more than s points.*

We mention here that by a well-established principle of Harris-Mumford, compactified moduli spaces of stable curves of sufficiently large genus $g \geq 24$ are varieties of general type (cf. [**51**]).

4. Some further motivations and digressions in number fields

To further motivate our studies on the finiteness of certain unions of integral points as in Theorem F and Theorem H, we present several observations in function fields as well as some simple but interesting digressions in number fields. From our geometric results, we shall settle a reasonable question on the finiteness of certain unions of integral points in an elliptic curve defined over a number field (cf. Question 1.38).

In this aspect, we mention that the main *difference* between number fields and function fields is the contributions of different places. In the case of number fields, different primes contribute differently in the height formula via the normalization factor $\sim \log p$ at the prime p. In the case of function fields, each place contributes with equal weights to the intersection product since the later is just the sum of intersection multiplicities at different places. This difference is another avatar of the following fundamental reason. While all places of a one dimensional complex function field are just points of a compact curve, places of a number field K are points in $\mathrm{Spec}(\mathcal{O}_K)$, which is not compact, plus the archimedean places. Compactify $\mathrm{Spec}(\mathcal{O}_K)$ using the archimedian places is a nontrivial task tackled by the theory of Arakelov geometry which we shall not explain here.

The main motivation of this section is Proposition 1.51 which stems from a conversation with Pietro Corvaja during the workshop *Diophantine Approximation and Value Distribution Theory* held at UQAM, Canada in May 2019.

4.1. Elliptic curves.

4.1.1. *The function field case.* Our starting point is a well-known effective bound of the canonical height $\widehat{h}$ of integral points on elliptic surfaces (cf. [**52**, Corollary 8.5]):

Theorem 1.36 (Hindry-Silverman)**.** *Let B be a smooth projective curve over a field k of characteristic 0. Let $K = k(B)$ be the function field of B and let S be a finite set of places of K. Suppose that $X \to B$ is a minimal elliptic surface with a section (O). Then for any $(S,(O))$-integral point $P \in X_K(K)$, we have*

$$\widehat{h}(P) \leq 25\chi(X) + 6g + 2s$$

where $\chi(X)$ is the Euler-Poincaré characteristic of X and $s = \#S$.

From Theorem 1.36, we can easily obtain the following effective finiteness result on integral points with *bounded denominators* which is our main motivation for Theorem F and Theorem H. The finiteness is already remarked without proof for example in [**96**, 2.9]. Let $f \colon X \to B$ be a minimal non-isotrivial elliptic surface with a zero section (O). Let $T \subset B$ be the type of X, i.e. the finite subset above which f is not smooth.

Corollary 1.37. *There exists $\alpha, \beta, \gamma > 0$ depending only on g and $t = \#T$ such that for every $s \in \mathbb{N}$, the union:*

$$I_s := \cup_{S \subset B, \#S \leq s} \{(S,(O))\text{-integral points of } X_K(K)\} \subset X_K(K)$$

is finite and $\#I_s \leq (\alpha s + \beta)^\gamma$. Moreover, the same result holds when (O) is replaced by any horizontal integral divisor $\mathcal{D} \subset X$ but with α, β, γ depending also on the genus of $\mathcal{D}$.

Proof. Since the canonical height bound in Theorem 1.36 depends only on $\#S$, the exact same proof of [**52**, Theorem 8.1] can be applied without change to give the result. Notice that α, β can be given explicitly as a function of the genus g of the base curve B and that γ is half of the Mordell-Weil rank of $X_K(K)$, which is bounded by $2(2g - 2 + t)$ by the Shioda-Tate formula (cf. [**101**, Theorem 2.5]). Now let $\mathcal{D} \subset X$ be an integral curve which is finite over B. Let $C \to \mathcal{D}$ be the normalization morphism and let $h \colon C \to B$ be the induced finite morphism of degree d. For each finite subset $S \subset B$ of cardinality $\#S \leq s$, let $S' = f'^{-1}(S) \subset C$ then $\#S' \leq s' := ds$. Consider the elliptic surface $f' \colon X' = X \times_B C \to C$ which is also nonisotrivial (cf. Theorem 2.5). Let T' be the type of X' then $\#T' \leq dt$. It is clear that $\mathcal{D} \times_B C$ splits into sections of f'. Let R be one of the these sections and let $K' = \mathbb{C}(C)$. It follows that we have

$$I_s \subset I'_{s'} := \cup_{S' \subset B, \#S' \leq s'} \{(S', R)\text{-integral points of } X'(K')\} \subset X'(K').$$

The desired properties of I_s are then obtained from those of $I'_{s'}$. Remark however that the constants α, β, γ now depend also on the genus of the divisor $\mathcal{D}$. $\square$

4.1.2. *The number field case.* Now let $s \geq 1$ be an integer. Motivated by Corollary 1.37 and notably by Theorem H, it is natural to ask if analogous finiteness results still hold in the case of number fields:

Question 1.38. Let K be a number field. Consider an elliptic curve E/K given by a Weierstrass equation

$$E\colon \quad y^2 = x^3 + Ax + B, \quad A, B \in K$$

Is the following union of integral points

$$E(s) := \{(x,y) \in K^2 \colon y^2 = x^3 + Ax + B,\, x \in \mathcal{O}_{K,S} \text{ for some } S \text{ with } \#S \leq s\}$$

finite for all $s \in \mathbb{N}$?

Any answer to the above question would be very interesting. While a positive answer will certainly encourage more research in this direction, a negative answer will help us better understand the *difference* between number fields and function fields.

It is worth mentioning the following related result of Silverman (cf. [**103**]). Let:

(1) K be a number field and $S \subset M_K$ be a finite subset of places of K;

(2) $E\colon y^2 = x^3 + Ax + B$ be an elliptic curve with $A, B \in \mathcal{O}_{K,S}$;

(3) $\mathcal{O}_{K,S}(\varepsilon) \coloneqq \{x \in K\colon \sum_{\nu \in S} \max(-\nu(x), 0) \geq \varepsilon h(x)\}$ for every $1 \geq \varepsilon > 0$;

It is clear that $\mathcal{O}_{K,S}(1) = \mathcal{O}_{K,S} \subset \mathcal{O}_{K,S}(\varepsilon) \subset \mathcal{O}_{K,S}(\varepsilon')$ for all $0 < \varepsilon' < \varepsilon < 1$. When $K = \mathbb{Q}$ and $S = \varnothing$, we have $\mathcal{O}_{\mathbb{Q},\varnothing}(\varepsilon) = \{\frac{p}{q} \in \mathbb{Q}\colon |q| \leq |p|^{1-\varepsilon}\}$. Remark that unless $\varepsilon = 1$, the set $\mathcal{O}_{K,S}(\varepsilon)$ is not contained in any integral ring $\mathcal{O}_{K,T}$ where T is a finite set of places.

Theorem 1.39 (Silverman)**.** *The following set of integral solutions:*

$$\{(x, y) \in K^2\colon y^2 = x^3 + Ax + B, x \in \mathcal{O}_{K,S}(\varepsilon)\}$$

is finite for every $0 < \varepsilon < 1$.

4.2. Unit equations. Let $\mathcal{O}_S^*$ be the group of S-units of a field K where S is a finite set of places of K. Consider the following equation

$$x + y = 1, \quad x, y \in \mathcal{O}_S^* \tag{4.1}$$

called the *S-unit equation.* In the case when K is a number filed, the set of solutions of (4.1) is finite (cf. Theorem 2.31). Analogous results also hold in the case of function field (cf. Theorem 2.32). For more details, there exists a vast literature on the finiteness of solutions of S-unit equations and generalized S-unit equations, cf. [**33**], [**34**], [**35**], etc.

In parallel to the studies of integral points of elliptic curves, we now give some observations in the case of S-unit equations where S is allowed to vary with bounded cardinality. Most of them are negative results on the finiteness but will serve as motivations for the finiteness result of Theorem 8.5 on some generalized unit equations in Section 8.

4.2.1. *The function field case.* We begin by showing that the analogue of Theorem H does not holds in the case of rational ruled surfaces.

Proposition 1.40. *Let $s \geq 0$ be an integer and let $K = \mathbb{C}(t)$. Then the union*

$$R_s \coloneqq \cup_{S \subset \mathbb{P}^1(\mathbb{C}), \#S \leq s} \{(x, y) \in (\mathcal{O}_{K,S}^*)^2\colon x + y = 1\}$$

of S-unit solutions with $\#S \leq s$ is infinite modulo $\mathbb{C}^*$ if and only if $s \geq 3$. The same conclusion holds when $\mathbb{C}$ is replaced by any algebraically closed field of characteristic 0.

Remark 1.41. By being finite (resp. infinite) modulo $\mathbb{C}^*$, we mean that the classes of x modulo $\mathbb{C}^*$ in K^* is finite (resp. infinite).

PROOF. Observe first that $R_0 \subset R_1 \subset \cdots \subset R_n \subset \cdots$. Hence, it suffices to show that R_3 is infinite modulo $\mathbb{C}^*$ while R_0, R_1, R_2 are finite modulo $\mathbb{C}^*$. Since any nonvanishing rational function on $\mathbb{P}^1$ must be a nonzero constant, we have $R_0 \simeq \mathbb{C}\backslash\{0,1\}$ and thus R_0 modulo $\mathbb{C}^*$ is finite. As $\deg \operatorname{div}(f) = \sum_{t\in\mathbb{P}^1} \operatorname{ord}_t(f) = 0$ for every $f \in \mathbb{C}(t) \setminus \{0\}$, the set of poles and zeros of a rational function cannot consist of a single element. Therefore, $R_1 \setminus R_0 = \varnothing$ and thus $R_1 = R_0$ is also finite modulo $\mathbb{C}^*$. Similarly, let $(x, y) \in R_2$ such that $x, y \notin \mathbb{C}$. By a change of variable, we can suppose that x and thus y have a pole at $\infty = (1\colon 0)$. But since $(x, y) \in R_2$, x and y must have a common zero. This is impossible as $x + y = 1$. Hence, $R_2 = R_1 = R_0 \simeq \mathbb{C} \setminus \{0, 1\}$ is finite modulo $\mathbb{C}^*$.

For R_3, we clearly have for every $(a, b) \in \mathbb{C}^2$ with $a \neq 0$ that

$$(x_{a,b}, y_{a,b}) = (at + b, 1 - at - b) \in (\mathcal{O}_{K,S}^*)^2$$

and $x_{a,b} + y_{a,b} = 1$ where $S = \{(-\frac{b}{a}\colon 1), (\frac{1-b}{a}\colon 1), (1\colon 0)\}$. It follows that $R_2 \supset \{(x_{a,b}, y_{a,b})\colon a \in \mathbb{C}^*, b \in \mathbb{C}\}$ and thus $R_2/\mathbb{C}^* \supset (\mathbb{C}^* \times \mathbb{C})/\mathbb{C}^*$ is clearly uncountably infinite.

In fact, for every integer $m \geq 1$, the fundamental theorem of algebra implies that:

$$R_{K,2m+1} \supset A_m \coloneqq \{(P(t), 1 - P(t))\colon P \in \mathbb{C}[t], \deg P = m\}$$

and thus

$$R_{K,2m+1}/\mathbb{C}^* \supset A_m/\mathbb{C}^* \simeq (\mathbb{C}^* \times \mathbb{C}^m)/\mathbb{C}^* \simeq \mathbb{C}^m.$$

$\square$

Corollary 1.42. *Let $s \geq 0$ be an integer and let $K = \mathbb{C}(B)$ be the function field of a connected smooth projective complex curve B. Then the union of unit solutions*

$$R_{K,s} \coloneqq \cup_{S\subset B, \#S\leq s}\{(x, y) \in (\mathcal{O}_{K,S}^*)^2\colon x + y = 1\}$$

is infinite modulo $\mathbb{C}^*$ for all large enough s.

PROOF. Any nonconstant rational function $f \in \mathbb{C}(B)$ induces a finite surjective morphism $\varphi\colon B \to \mathbb{P}^1$ and thus an inclusion of fields $\mathbb{C}(t) \to K = \mathbb{C}(B)$. Note that for every finite subset $S \subset \mathbb{P}^1$, we have $\varphi^*\mathcal{O}_{\mathbb{C}(t),S} \subset \mathcal{O}_{K,\varphi^*S}$. It is then easy to see that $\varphi^*R_2 \subset R_{K,2\deg(f)}$. Since R_2 is infinite modulo $\mathbb{C}^*$ by Proposition 1.40, the conclusion follows since $R_{K,2\deg(f)} \subset R_{K,s}$ for every $s \geq 2\deg(f)$. $\square$

Hence, in order to obtain a finiteness result, we may need to add more obstructions to the unit solutions and ask, for example:

Question 1.43. Consider the split ruled surface $\pi\colon X = \mathbb{P}^1 \times B \to B$ and a horizontal reduced divisor $\mathcal{D} \subset X$ of relative degree ≥ 2 with respect to B. Consider the following union of integral points of X

$$I(\mathcal{D}, s) := \cup_{S \subset B, \# S \leq s} \{P \in X(B)\colon P \text{ is } (S, \mathcal{D} + (0\colon 1) + (1\colon 0))\text{-integral}\}.$$

Is the set $I(\mathcal{D}, s)$ finite for every $s \geq 0$?

Remark 1.44. Let $K = \mathbb{C}(B)$ be the function field of a connected smooth projective complex curve B. Let $U \subset B$ be a nonempty analytic open subset. Motivated by Theorem F and Theorem H, it is natural to ask whether the following union of unit solutions is finite:

$$R_{K,U} := \cup_{S \subset U} \{(x, y) \in (\mathcal{O}_{K,S}^*)^2\colon x + y = 1\}.$$

However, we claim that the x-coordinate set $R'_{K,U} := \{x \in \mathcal{O}_{K,S}^*\colon 1 - x \in \mathcal{O}_{K,S}^*\}$ is even infinite modulo $\mathbb{C}^*$ in general. Indeed, we suppose first that $B = \mathbb{P}^1$ is the projective line and thus $K = \mathbb{C}(t)$. Hence, for every $a, b \in U$ such that $a \neq b$, we have $\left(\frac{a-b}{t-b}, \frac{t-a}{t-b}\right) \in R_{K,U}$. It follows immediately that $R'_{K,U}$ is infinite modulo $\mathbb{C}^*$. For general B, it suffices to choose a ramified covering map $f\colon B \to \mathbb{P}^1$ so that $R'_{K,f^{-1}(V)} \supset f^* R'_{\mathbb{C}(t),V}$ is infinite modulo $\mathbb{C}^*$ where $V \subset \mathbb{P}^1$ is a small open disc.

Therefore, to obtain a finiteness result in this context, it is necessary to restrict the set $R_{K,U}$ furthermore. Some positive answers are given in Theorem 8.5 and Corollary 8.7. The key point is to require $x \in \mathcal{O}_S^*$ for some fixed finite subset $S \subset B$.

4.2.2. *The number field case.* Let $\mathcal{P}$ denotes the set of all prime numbers. For every $s \geq 0$, consider the following union of unit solutions:

$$R_{\mathbb{Q},s} := \cup_{S \subset \mathcal{P}, \# S \leq s} \{(x, y) \in (\mathbb{Z}_S^*)^2\colon x + y = 1\}.$$

Unlike the situation in the function field case, the finiteness of the sets $R_{\mathbb{Q},s}$ is much more complicated as we shall see. In fact, it turns out that the set $R_{\mathbb{Q},s}$ is related to several important conjectures and theorems in number theory (cf. Proposition 1.45, Proposition 1.47, Remark 1.50) as well as some interesting classes of whole numbers such as Fermat and Mersenne numbers (cf. Proposition 1.51).

The first observation is that we can reduce the finiteness question to a first few sets $R_{\mathbb{Q},s}$. For example, we have the following non trivial result.

Proposition 1.45. *For every $s \geq 6$, the set $R_{\mathbb{Q},s}$ is infinite.*

Proof. By the recent famous theorem on gaps between prime numbers of Yitang Zhang [**114**] and by latter improvements of the same result by the group PolyMath (organized by Tarence Tao), we know that the set

$$Z = \{(p, q) \in \mathcal{P}^2\colon 0 < p - q \leq 246\} \tag{4.2}$$

is infinite. It is clear that the number of distinct prime divisors of an integer $n \in \{2, 4, \cdots, 246\}$ is at most 4 since otherwise, we would have $n \geq 2.3.5.7.11 > 246$. It follows that for every $(p,q) \in Z$, the total number of distinct prime divisors of $p, q, p-q$ is at most $1+1+4=6$. Hence, we have for every $s \geq 6$ that

$$R_{\mathbb{Q},s} \supset R_{\mathbb{Q},6} \supset \left\{ \left(\frac{p}{p-q}, \frac{-q}{p-q} \right) : (p,q) \in Z \right\}$$

which is clearly infinite since Z is infinite and $p-q$ is bounded. $\square$

We ask naturally the following question:

Question 1.46. Is there a more elementary explanation of Proposition 1.45 ?

It turns out that another explanation is available, but again nontrivial. A prime number $p \in \mathcal{P}$ is a *Chen prime* if $p+2$ is a prime or is it a product of two primes p_1, p_2. By Chen's theorem (cf. [**19**]), the set $\mathcal{C} \subset \mathcal{P}$ of Chen primes is infinite. In particular, for every $p \in \mathcal{C}$, the couple $(-p/2, (p+2)/2)$ belongs to $R_{\mathbb{Q},4}$ since the number of possible prime factors involved are at most 4, namely, 2, p, p_1, p_2. Thus, Proposition 1.45 can be in fact improved to:

Proposition 1.47. *For every $s \geq 4$, the set $R_{\mathbb{Q},s}$ is infinite.*

Similar to the function field case, we deduce that:

Corollary 1.48. *Let K be a number field. Let $\mathcal{P}_K$ denotes the set of all prime ideals of the ring of integers $\mathcal{O}_K$. For all integer $s \geq 4[K\colon \mathbb{Q}]$, the set*

$$R_{K,s} \coloneqq \cup_{S \subset \mathcal{P}_K, \#S \leq s} \{(x,y) \in (\mathcal{O}_{K,S}^*)^2 \colon x + y = 1\}$$

is infinite.

Proof. It suffices to observe that the number of prime ideals in $\mathcal{O}_K$ lying above each rational prime number p is at most $[K\colon \mathbb{Q}]$. Therefore, we have $R_{K,s} \supset R_{\mathbb{Q}, \lfloor s/[K\colon \mathbb{Q}] \rfloor} \supset R_{\mathbb{Q},4}$ and the conclusion follows. $\square$

It is natural to ask the following as in the function field case (Proposition 1.40):

Question 1.49. What is the minimum $s \in \mathbb{N}^*$ such that the set $R_{\mathbb{Q},s}$ is infinite?

By Proposition 1.47, such number s exists and ≤ 4 . Since it can be easily checked that $R_{\mathbb{Q},1} = \{(2,-1), (-1,2), (\frac{1}{2}, \frac{1}{2})\}$, the next step is to verify if the set $R_{\mathbb{Q},2}$ is infinite. Before this, we have the following observation for $R_{\mathbb{Q},3}$.

Remark 1.50. By definition, $R_{\mathbb{Q},3}$ is infinite if and only if the equation

$$p^{a_1} q^{b_1} r^{c_1} = p^{a_2} q^{b_2} r^{c_2} + p^{a_3} q^{b_3} r^{c_3}, \quad p, q, r \in \mathcal{P},\ a_i, b_i, c_i \in \mathbb{N} \tag{4.3}$$

has infinitely many solutions with $a_1a_2a_3 = b_1b_2b_3 = c_1c_2c_3 = 0$. A direct case by case study using the last conditions shows that $R_{\mathbb{Q},3}$ is infinite if and only at least one of the following equations has infinite solutions in $p, q, r \in \mathcal{P}$, $a, b, c \in \mathbb{N}$:

(1) $p^a q^b = r^c \pm 1$;

(2) $p^a = q^b + r^c$.

In particular, the Golbach's strong conjecture, which states that every even integer greater than 2 can be written as the sum of two prime numbers, implies that $R_{\mathbb{Q},3}$ is infinite. Similarly, the Twin prime conjecture also implies that $R_{\mathbb{Q},3}$ is infinite.

In the first equation $p^a q^b = r^c \pm 1$, at least one of p, q, r must equal to 2 by the parity reason. In particular, Fermat and Mersenne prime numbers $p = 2^n \pm 1$ induce elements of $R_{\mathbb{Q},3}$. In fact, we shall see below that these numbers induce all the set $R_{\mathbb{Q},2}$ up to a finite number of elements.

Return to $R_{\mathbb{Q},2}$, we now consider $(x, y) \in R_{\mathbb{Q},2} \setminus R_{\mathbb{Q},1}$ with $x > 0$. Then there exists $p, q \in \mathcal{P}$ and $m, n, a, b \in \mathbb{Z}$ such that $x = p^m q^n$, $y = \pm p^a q^b$ and $x + y = 1$. By working out mod 2, we see that p, q cannot be both odd or both even. Hence, we can suppose that $q = 2$ and p is an odd prime. The equation $x + y = 1$ becomes $p^m 2^n \pm p^a 2^b = 1$. We distinguish several cases:

(1) if $mn > 0$ (resp. $ab > 0$), there is no solutions in this case. For example, assume $m, n > 0$. Then $\pm p^a 2^b = p^m 2^n - 1 \in \mathbb{Z}$ implies that $a = b = 0$ by working out mod 2 and mod p. Thus $p^m 2^n = 0$ (impossible) or $p^m 2^n = 2$. Since p is odd, we find $m = 0$ which is a contradiction;

(2) similarly, if $m < 0 < n$ (resp. $a < 0 < b$) then $b = 0$, $a = m$ and we find $2^n \pm 1 = p^{-m}$, (resp. $n = 0$, $m = a$ and $1 + 2^b = p^{-m}$);

(3) if $n > 0$ and $m = 0$ then we find $b = 0$ since p is odd and thus $2^n = 1 + p^a$; similarly, if $b < 0$ and $a = 0$ then $n = 0$ and we have $p^m = 1 + 2^b$;

(4) if $n < 0$ then $b = n$ by working out mod 2. It follows that $m, a \geq 0$ by (1). Thus, $2^{-n} = p^m \pm p^a$, which is divisible by p if $m, a > 0$. Since p is an odd prime, we find $ma = 0$. If $m = 0$ then $2^{-n} = 1 + p^a$. If $m > 0$ then $a = 0$ and $2^{-n} \pm 1 = p^m$.

(5) if $n = 0$ then $p^m \pm p^a 2^b = 1$; we have $b \geq 0$ by working out mod 2; since $p \geq 3$, we find $b > 0$ and $m, a \leq 0$; thus $p^{m-a} \pm 2^b = p^{-a}$ and hence $m = a$; so $1 + 2^b = p^{-a}$;

(6) if $n = 1$ then $m \leq 0$ by (1). By (2) and (3), $b = 0$. If $m = 0$, we find $a = 0$ so that $(x, y) = (2, -1)$. If $m < 0$ then $a = m$ by (2). Thus $2 \pm 1 = p^{-m}$ and we find $m = 0$ (impossible) or $p = 3$, $m = -1$. Hence $(x, y) = (\frac{2}{3}, \frac{1}{3})$.

It follows from the above analysis that besides the following solutions ($x \geq y$):

$$(x, y) = \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{2}{3}, \frac{1}{3}\right), (2, -1),$$

we reduce to find $n \geq 2$ such that $2^n \pm 1$ is a power of an odd prime.

Now recall that the Catalan's conjecture (proved in 2002 by Mihailescu, cf. [**70**]) asserts that the only integer solutions of the equation

$$x^a - y^b = 1$$

where $x, y \geq 1$ and $a, b \geq 2$ is $(x, y, a, b) = (3, 2, 2, 3)$. Hence, if $2^n - 1 = p^c$ for some integer $c \geq 2$ then $2^n - p^c = 1$ and there is no solutions in this case by Catalan's conjecture. Similarly, if $2^n + 1 = p^c$ for some integer $c \geq 2$ then $p^c - 2^n = 1$ and Catalan's conjecture implies that $(p, n, c) = (3, 3, 2)$. The corresponding solutions $(x, y) \in R_{\mathbb{Q},2}$ with $x \geq y$ in this case are $(9, -8)$, $\left(\frac{9}{8}, \frac{-1}{8}\right)$, $\left(\frac{8}{9}, \frac{1}{9}\right)$.

Our problem thus reduces to find $n \geq 2$ such that $2^n \pm 1$ is a prime number.

To summarize, we have shown the following explicit description of the set $R_{\mathbb{Q},2}$:

Proposition 1.51. *We have* $R_{\mathbb{Q},2} = R^+_{\mathbb{Q},2} \cup \{(y, x) \colon (x, y) \in R^+_{\mathbb{Q},2}\}$ *with*

$$\begin{aligned} R^+_{\mathbb{Q},2} &\coloneqq \cup_{S \subset \mathcal{P}, \# S \leq 2} \left\{(x, y) \in (\mathbb{Z}_S^*)^2 \colon x + y = 1, x \geq y\right\} \\ &= \left\{\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{8}{9}, \frac{1}{9}\right), \left(\frac{9}{8}, \frac{-1}{8}\right), (2, -1), (9, -8)\right\} \cup F \cup M \end{aligned}$$

where

$$F = \cup_{2^n + 1 = p \text{ is prime}} \left\{(p, -2^n), (p2^{-n}, -2^{-n}), \left(\frac{2^n}{p}, \frac{1}{p}\right)\right\},$$

$$M = \cup_{2^n - 1 = p \text{ is prime}} \left\{(2^n, -p), (p2^{-n}, 2^{-n}), \left(\frac{2^n}{p}, \frac{-1}{p}\right)\right\}.$$

In particular, the set $R_{\mathbb{Q},2}$ *is infinite if and only if there are infinitely many Fermat prime numbers or infinitely many Mersenne prime numbers.*

We just remark here that the problem whether there are infinitely many Fermat or Mersenne prime numbers is still wide open. It is a straight forward verification that the above problem is not even a consequence of the important abc conjecture of Oesterlé-Masser, which we mention here for ease of reading:

Conjecture (Oesterlé-Masser)**.** *Given* $\varepsilon > 0$*, there are only finitely many pairwise coprime integers* $a, b, c > 0$ *with* $a + b = c$ *and* $c \geq (\mathrm{rad}(abc))^{1+\varepsilon}$*, where* $\mathrm{rad}(abc)$ *denotes the product of prime divisors of abc.*

CHAPTER 2

Preliminaries

1. Basic definitions

Let $K = k(B)$ be the field of functions of a smooth projective connected curve B over a field k of characteristic 0.

Definition 2.1. Given an elliptic curve E/K defined by a Weierstrass equation

$$y^2 = x^3 + Ax + B$$

with $A, B \in K$ and the discriminant $\Delta = 4A^3 + 27B^2 \neq 0$, we say that:

(1) E is *constant* if there is an elliptic curve E_0 over k such that $E \simeq E_0 \otimes_k K$. Alternatively, E is constant if E can be defined by a Weierstrass equation $y^2 = x^3 + ax + b$ with $a, b \in k$;

(2) E is *isotrivial* if $E \otimes_K L$ is constant for some finite extension L/K. Equivalently, E is isotrivial if and only if $j(E) \in k$ where $j(E) = 1728\frac{4A^3}{4A^3+27B^2}$ is the j-invariant of E;

(3) E is *non-isotrivial* if it is not isotrivial. Likewise, E is non-constant if it is not constant.

Example 2.2. The Legendre elliptic curve E_1 defined over the function field $k(t)$ given by the equation $y^2 = x(x-1)(x-t)$ is nonisotrivial while the elliptic curve $E_2/k(t)$ defined by $y^2 = x^3 + t$ is isotrivivial but nonconstant. We have $j(E_2) = 0$ and in fact E_2 becomes constant after the finite base change $k(t^{1/6})/k(t)$ where $t' = t^{1/6}$ is any sixth root of t. Indeed, under the change of variables $y' = y/t'^3$ and $x' = x/t'^2$, the curve E_2 is given by $y'^2 = x'^3 + 1$ which is clearly defined over k and thus is constant.

Definition 2.3. Let $f\colon X \to \operatorname{Spec}(K)$ be a smooth projective variety. The *Kodaira-Spencer class* $KS(X)$ is defined as the extension class of the following exact sequence:

$$0 \to f^*\Omega^1_{K/k} \to \Omega^1_{X/k} \to \Omega^1_{X/K} \to 0.$$

The importance of the Kodaira-Spencer is that it characterizes the isotriviality of varieties over function fields.

Theorem 2.4. *Let B be a curve over a field k of characteristic 0 and let $K = k(B)$. Let $X \to \mathrm{Spec}(K)$ be a smooth projective variety over K. Then X is isotrivial if and only if the Kodaira-Spencer class $KS(X)$ of X vanishes.*

Proof. See [**40**, Theorem 3.34]. □

Theorem 2.5. *Let $X' \to \mathrm{Spec}(K)$ be a smooth projective isotrivial variety. Suppose that $X' \to X$ is a dominant K-morphism of smooth projective varieties. Then X is also isotrivial over K.*

Proof. See [**40**, Lemma 3.30]. □

We do not know whether the following proof in the case of curves exists already in the literature.

Another proof of Theorem 2.5 in the case of curves. Without loss of generality, we can suppose that k is algebraically closed (or even $k = \mathbb{C}$ by Lefschetz principal). Consider any models $\mathcal{X}' \to B$ and $\mathcal{X} \to B$ of X' and X respectively. By the resolution of indeterminacy of surfaces, we can find a dominant B-morphism $\mathcal{X}' \to \mathcal{X}$ commuting with $\mathcal{X} \to B$ and $\mathcal{X}' \to B$.

Suppose on the contrary that X is nonisotrivial but X' is isotrivial. Let C denote a general fibre of $\mathcal{X}$. Since $\mathcal{X}'$ is isotrivial, its general curves over B are all isomorphic to each other which we can then denote by a single curve C'. As X is nonisotrivial, the dominating B-morphism $\mathcal{X}' \to \mathcal{X}$ induces infinitely many pairwise non isomorphic curves which are fibres of $\mathcal{X}$ and which are dominated by C'. Note that these dominating maps are all of the same degree $d = [k(X') \colon k(X)]$. Let g', g be respectively the genus of X' and X. If $g \geq 2$ then $g' \geq 2$ by the Riemann-Hurwitz formula. However, the de Franchis theorem (cf. Theorem 2.33.(i)) says that up to isomorphisms, there are only finitely many curves D of genus g' such that there is a dominant morphism $C' \to D$. We thus obtain a contradiction in this case. Similarly, Theorem 2.33.(iii) also implies a contradiction in the case $g = 1$. The case $g = 0$ cannot occur since otherwise X would be the projective line and hence trivial. We can thus conclude Theorem 2.5 in the case of curves. □

Corollary 2.6. *Let B be a curve over a field k of characteristic 0 and let $K = k(B)$. Let $X \to B$ be a nonisotrivial elliptic surface. Suppose that $X' \to X$ is a finite covers of X. Then $X' \to B$ is also nonisotrivial.*

1.1. Geometry of elliptic surfaces.

The base field k is assumed to be perfect. Let B/k be a smooth projective curve.

Theorem 2.7. *Let $f \colon X \to B$ be a minimal elliptic surface with a section (O) and let E/K be the associated elliptic curve. Then we have the followings:*

(1) *For each $P \in E(K) = X(B)$, consider the rational B-morphism $\tau_P \colon X \to X$ induced by the translation by P on each regular fibre. Then τ_P extends, by minimality of X, to an B-automorphism.*

(2) *Let $Aut_B(X) = \{automorphisms\ p \colon X \to X \mid f \circ p = f\}$ then we have a homomorphism of groups*

$$E(K) \to Aut_B(X), \quad P \mapsto \tau_P$$

In particular, τ_{-P} is the inverse of τ_P for any $P \in E(K)$.

(3) *We have a canonical isomorphism of groups $Isom_K(E) \simeq \mathrm{Aut}_B(X)$.*

(4) *The isomorphism group of E over K is given by*

$$Isom_K(E) \simeq E(K) \rtimes \mathrm{Aut}_K(E)$$

where $\# \mathrm{Aut}_{\overline{K}}(E) = 2, 4, 6$ according to $j_E \neq 0, 1728, j_E = 1728$ or $j_E = 0$.

Proof. See for example [**104**, Theorem III.9.1] for (1), (2), (3), [**86**, Theorem 0] for (3), and [**105**, AEC III.10.1] for (4). □

1.2. Néron models. In this section, we denote by B be a Dedekind scheme with field of functions K.

Definition 2.8. Let X_K be a separated scheme of finite type over K. *A model of X_K over B* is a locally finite type, separated and flat scheme over B endowed with an isomorphism from its generic fibre to X_K.

In what follows, we always implicitly fix an isomorphism between X_K and the generic fibre of a model of X_K.

We shall frequently use the following property stating that flatness is automatic for a very large class of reduced schemes over a Dedekind scheme.

Proposition 2.9. *Let Y be a Dedekind scheme. Let $f \colon X \to Y$ be a morphism of schemes where X is reduced. Then f is flat if and only if every irreducible component of X dominates Y.*

Proof. See [**65**, Proposition 3.9]. □

It follows from Proposition 2.9 that every separated integral scheme of finite type over B is a model of its generic fibre.

Definition 2.10. ([**9**, 10.1, 1.2]) Let X_K be a separated smooth algebraic variety over K. *A Néron model of X_K over B* is a finite type smooth model X of X_K over B satisfying the following universal property, called *the Néron mapping property*:

for every smooth scheme morphism $Y \to B$, the following canonical map is a bijection

$$\mathrm{Mor}_B(Y, X) \to \mathrm{Mor}_K(Y_K, X_K). \tag{1.1}$$

Remark 2.11. Up to a unique isomorphism, the Néron mapping property implies the uniqueness of the Néron model whenever it exists. Moreover, suppose that X is a model of X_K. Since X is separated by definition, it is easy to see that the map (1.1) is injective by a standard argument with the equalizer. Hence, it is enough to check the surjectivity for the Néron mapping property to prove that X is a Néron model.

Definition 2.12. *A group scheme X over a scheme S* is a representable functor $X\colon \mathrm{Sch}_S \to \mathrm{Grp}$, i.e., a scheme X such that $X(T)$ has the structure of a group for every S-schemes T.

A group scheme X over a scheme S is called *an abelian scheme* if X is smooth and proper over S with connected fibres.

Remark 2.13. By the Néron mapping property, the group structure on an abelian variety A_K extends to a group structure on the Néron model of A_K over B. However, the Néron model may not be an abelian scheme as it may not be proper.

Abelian schemes provide an important class of Néron models as shown in the following proposition:

Proposition 2.14. *Suppose that X/B be an abelian scheme. Then X is the Néron model of its generic fibre.*

Proof. See [**9**, Proposition 1.2.8]. □

Conversely, we have the following fundamental existence result:

Theorem 2.15. *Every abelian variety A_K over K admits a Néron model over B.*

Proof. See [**9**, Theorem 1.4.3]. □

For a proper smooth connected curve of positive genus X_K over K, a canonical smooth model of X_K is the smooth locus X_{sm} of the minimal proper regular model of X_K over B. In the case when X_K is an elliptic curve, X_{sm} turns out to be also the Néron model of X_K over B:

Theorem 2.16. *Let E/K be an elliptic curve over K. Then the Néron model of E over B is obtained as the open subscheme of smooth points X_{sm} of the minimal elliptic surface $X \to B$ associated to E.*

Proof. See [**65**, Theorem 10.2.14]. □

1.3. Intersection theory. Since the thesis will concentrate on the case of varieties over function fields, we shall freely use the language and standard notations of Intersection theory to talk about heights of rational points in a model. For a detailed and formal treatment of Intersection theory, see the canonical reference [**39**] for the general theory, or [**104**, Chapter II] for the case of elliptic surfaces.

1.4. Chow's traces of abelian varieties and the Lang-Néron theorem. In this section, let K/k be a finitely generated regular extension of fields, i.e., K/k is separable and k is algebraically closed in K. This turns out to be equivalent to require that K arises as the function field of a smooth geometrically connected scheme over k.

We recall the notion of traces of abelian varieties defined over K introduced by Chow (cf. [**20**], [**21**]).

Definition 2.17 (Chow)**.** Let A be an abelian variety over K. A *K/k-trace* of A is a final object $(\operatorname{Tr}_{K/k}(A), \lambda)$ in the category of pairs (B, f) where B/k is an abelian variety and $f\colon B_K \to A$ is a homomorphism of abelian varieties.

Theorem 2.18. *Let K/k be a finitely generated regular extension of fields. Let A be an abelian variety over K. The K/k-trace $\lambda\colon \operatorname{Tr}_{K/k}(A) \otimes_k K \to A$ exists.*

Morally speaking, the K/k-trace $\operatorname{Tr}_{K/k}(A)$ is the largest abelian subvariety of A that can be defined over k. If A/K is an elliptic curve then A is a *nonconstant* elliptic curve if and only if the K/k-trace $\operatorname{Tr}_{K/k}(A)$ of A vanishes. This is justified by the following theorem.

Theorem 2.19. *Let A be an abelian variety over K with K/k-trace $(\operatorname{Tr}_{K/k}(A), \lambda)$. Then the map $\lambda\colon \operatorname{Tr}_{K/k}(A) \otimes_k K \to A$ is injective on K-points. In particular, $\operatorname{Tr}_{K/k}(A)(k)$ is naturally a subgroup of $A(K)$.*

PROOF. See [**23**, Theorem 6.12]. ☐

In fact, when k is of characteristic zero, the map $\lambda\colon \operatorname{Tr}_{K/k}(A) \otimes_k K \to A$ is actually a closed immersion (see the discussion after Theorem 6.2 in [**23**]).

Therefore, it is meaningful to write $A(K)/\operatorname{Tr}_{K/k}(A)(k)$ and we can now state the fundamental Lang-Néron theorem which generalizes the famous Mordell-Weil theorem for elliptic curves.

Theorem 2.20 (Lang-Néron)**.** *Let A/K be an abelian variety. The abelian group $A(K)/\operatorname{Tr}_{K/k}(A)(k)$ is finitely generated.*

PROOF. See [**61**]. See also [**23**]. ☐

1.5. Canonical height on abelian varieties. We restrict to the case K is the function field of of a smooth projective curve B/k. Let A/K be an abelian variety with a *symmetric* line bundle L on A, i.e., $L \simeq [-1]^*L$. Let $\mathcal{A} \to B$ be a model of A and let $\mathcal{L}$ be a line bundle on $\mathcal{A}$ such that $\mathcal{L}_K \simeq L$. For $P \in A(K)$, denote $\sigma_P \in \mathcal{A}(B)$ the corresponding section and $(P) = \sigma_P(B)$. Consider the height function $h_{\mathcal{L}} \colon P \mapsto \deg \sigma_P^* \mathcal{L}$. Let $\Gamma \coloneqq A(K)/\operatorname{Tr}_{K/k}(A)(k)$.

The Lang-Néron theorem can be strengthen with properties of the canonical height. The following theorem is critical to quantitative results on rational points.

Theorem 2.21 (Néron-Tate)**.** *Assume that L is ample. There exists a constant $c > 0$ and a canonical quadratic function $\widehat{h}_L \colon \Gamma \to \mathbb{R}_+$ such that:*

(a) $|\widehat{h}_L(\cdot) - h_{\mathcal{L}}(\cdot)| \leq c$;

(b) $\widehat{h}_L \otimes \mathbb{R} \colon \Gamma \otimes \mathbb{R} \to \mathbb{R}$ *is positive-definite.*

Proof. See [**23**, Theorem 9.15]. □

Here are some remarks in the case of elliptic curves.

Lemma 2.22. *Let $X \to B$ be a minimal elliptic surface over a curve B with zero section (O). Let $K = k(B)$. Then every rational point $P \in X(K)$ satisfies*

$$|\widehat{h}_X(P) - (P) \cdot (O)| \leq -(O)^2. \tag{1.2}$$

Proof. See for example [**32**, Lemma 3]. Remark that we choose a normalization of the canonical height $\widehat{h}$, which may differ by a factor of 2 to some definitions in the literature, so that (1.2) holds. □

Recall known effective results on nontorsion rational points:

Proposition 2.23. *Let $X \to B$ be an elliptic surface over a curve B of genus g. Let $\chi(X)$ denote the Eular-Poincaré characteristic of X. Suppose that $P \in X(B)$ is non torsion. Then the following hold:*

(i) *If X is nonisotrivial then $\#X(B)_{tors} < 144(g+1)^{2/3}$.*
(ii) *If $\chi(X) \geq 2(g-1)$ then $\widehat{h}(P) \geq 10^{-11.5}\chi(X)$.*
(iii) *If $\chi(X) < 2(g-1)$ then $\widehat{h}(P) \geq 10^{-11-23g}\chi(X)$.*

Proof. See [**52**]. □

To obtain effective bound on rational points of bounded canonical heights, we remark the elementary counting lemma:

Lemma 2.24 (Lenstra)**.** *Let Γ be a finitely generated abelian group of rank r. Let q be a positive definite quadratic form on Γ/Γ_{tors}. Define*

$$\mu := \min\{q(x) \colon x \in \Gamma/\Gamma_{tors}\};$$

Then for each $\alpha > 0$, we have the following estimation:

$$\#\{x \in \Gamma \colon q(x) \leq \alpha\} \leq \#\Gamma_{tors} \left(2\sqrt{\frac{\alpha}{\mu}} + 1\right)^r.$$

PROOF. See for example [**106**, Lemma 6]. □

From the Lang-Néron theorem, Proposition 2.23 and Lemma 2.24, we obtain immediately the following consequence.

Lemma 2.25. *Let $X \to B$ be a nonisotrivial elliptic surface with zero section (O) over a curve B of genus g. For every $H > 0$, we have*

$$\#\{P \in X(B), \widehat{h}_O(P) \leq H\} \leq 144(g+1)^{2/3} \left(2\sqrt{\frac{H}{10^{-11-23g}\chi}} + 1\right)^r$$

where $r = \operatorname{rank} X(B)$ and χ is the Euler-Poincaré characteristic of X.

1.6. Local systems. A sheaf L on a (connected) topological space X is called a *locally constant sheaf* or a *local system* if every point $x \in X$ admits a neighborhood $U \subset X$ such that for all $y \in U$, the canonical map $L(U) \to L_y$ is an isomorphism.

Let $x_0 \in X$ and consider the fundamental group $\pi_1(X, x_0)$. By the definition of the local system, we obtain a monodromy action of $\pi_1(X, x_0)$ on the stalk L_{x_0}. Thus, the monodromy representation induces a functor from the full subcategory of locally constant sheaves of $\mathrm{Sh}(X)$ to the category of $\pi_1(X, x_0)$-sets.

Proposition 2.26. *The monodromy functor defines an equivalence of categories whenever X admits a universal cover, e.g., when X is path connected, locally path connected and locally simply connected.*

1.7. Uniformity conjecture.

Definition 2.27. An smooth projective algebraic variety X is said to be *of general type* if the canonical divisor K_X is big, i.e.,

$$\lim_{n\to\infty} \frac{\dim \log H^0(X, nK_X)}{\log n} = \dim X.$$

For example, curves of general type are exactly those are of genus at least 2 which are also exactly hyperbolic curves. One of the most important of Arithmetic geometry is the following conjecture (cf. [**63**] for more).

Conjecture 2.28 (Lang-Vojta)**.** *Let X be a smooth projective variety of general type over a number field K. Then the set of rational points $X(K)$ is not Zariski dense in X.*

The conjecture is still wide open nowadays. However, in [**17**], the authors established the following very interesting consequence of Lang-Vojta conjecture which is usually known as the uniform boundedness conjecture.

Conjecture 2.29 (Uniformity)**.** *Fix an integer $g \geq 2$. Then there exists a number $N(g)$ such that for any number field F there exist at most finitely many curves of genus g, defined over F, and having more than $N(g)$ F-rational points.*

2. Classical finiteness theorems

Theorem 2.30 (Siegel)**.** *Let $E/\mathbb{Q}$ be an elliptic curve together with an affine model $\mathcal{E}$ over $\mathbb{Q}$. Let $E(\mathbb{Z})$ denote the set of sections $\mathcal{E}(\operatorname{Spec}\mathbb{Z})$. Then $\#E(\mathbb{Z}) < \infty$.*

The same result also holds for elliptic surfaces under mild conditions. Moreover, the finiteness property still holds in this case even when we allow bounded denominators as already remarked in [**96**, 2.9] for example. One of the goals of the presenting thesis is to generalize the above remark to integral points on abelian varieties over function fields.

We recall here the famous finiteness theorem on S-unit equations which is first proved by S. Lang in 1960.

Theorem 2.31 (Lang)**.** *Let K be a number field and let S be a finite number of places of K. Then the number of solutions of the S-unit equation $x + y = 1$, $x, y \in \mathcal{O}^*_{K,S}$ is finite.*

Proof. See [**62**]. □

Similarly, a similar effective result holds in the case of function fields. It is remarkable that the bound is independent of the genus of the function fields.

Theorem 2.32 (Evertse)**.** *Let B be a smooth projective connected curve over a field k of characteristic 0. Let $K = k(B)$ be the function field of B and $S \subset B$ a finite subset. Then the set of solutions $(x, y) \in (\mathcal{O}^*_{K,S})^2$ with $x/y \notin k$ of the S-unit equation $x + y = 1$ has at most $2 \times 7^{2\#S}$ elements.*

Proof. See [**33**]. □

2.1. de Franchis theorems. We collect in this section several finiteness results concerning coverings of curves.

Theorem 2.33 (de Franchis). *Let $g \geq 1, g' \geq 2, d \geq 1$ be integers and C be a smooth projective complex curve of genus g, then:*

(i) *Up to isomorphism, there are only finitely many curves X of genus g' such that there is a non constant morphism $C \to X$.*

(ii) *For any smooth curve X of genus g', there are only finitely many non constant morphisms $C \to X$.*

(iii) *Up to isomorphism, there are only finitely many curves X of genus 1 such that there is a degree-d morphism $C \to X$.*

(iv) *For any smooth curve X of genus 1, up to composition with a translation of X, there are only finitely many degree-d morphisms $C \to X$.*

Proof. See [**6**, Theorem XXI.8.27]. □

Remark the following finiteness result on the number of subfields of bounded index of a function field.

Theorem 2.34 (Tamme-Kani). *Let K be a field and let C/K be a smooth, geometrically connected, projective curve of genus q. Let $F = K(C)$ and let $m \in \mathbb{N}$. The number $N_F(m)$ of subfields of F/K of genus 1 and of index at most m is bounded by*

$$N_F(m) \leq 2^{3r/2-1} m^{r-2} s_r(m),$$

where $r \leq r(q) = 4q^2$ and $s_r(m) < \frac{\zeta(2)}{2} m^2 + \frac{m}{2}(\log(m) + 1)$.

Proof. See [**57**, Theorem 4] and the corollary that follows. □

With the notations of Theorem 2.34, each subfield $K \subsetneq F' \subset F$ corresponds 1-to-1 with an K-isomorphism class of a smooth, geometrically connected, projective curve C' such that the corresponding non constant map $f\colon C \to C'$ verifies $f^*(K(C')) = F'$.

An immediate consequence of Theorem 2.34 that will be used latter is the following effective version of the de Franchis Theorem 2.33.(iii).

Corollary 2.35. *Let K be a field and let $m \in \mathbb{N}$. Let E/K be an elliptic curve and let C/K be a smooth projective geometrically connected curve of genus q. Then, up to composition with an element of $\mathrm{Aut}_K(E)$, the number of degree-m K-covers $h\colon C \to E$ is uniformly bounded by an effective function $M(q, m)$ depending only on q and m.*

PROOF. It suffices to apply Theorem 2.34 and define

$$M(q,m) = 2^{6q^2-1}m^{4q^2-2}\left(\frac{\zeta(2)}{2}m^2 + \frac{m}{2}(\log(m)+1)\right) \geq N_F(m).$$

□

For two covers $f_1 \colon X \to Y_1$ and $f_2 \colon X \to Y_2$ of compact Riemann surfaces, we say that f_1 and f_2 are *equivalent* if there is a biholomorphic map $p \colon Y_1 \to Y_2$ such that $f_2 = p \circ f_1$.

Theorem 2.36 (Bujalance-Gromadzki)**.** *Let X be a compact Riemann surface of genus $q \geq 2$. Then we have:*

(i) *The set of non equivalent ramified double covers $X \to Y$ is bounded by a function $D(q)$;*
(ii) *Suppose that g is even. Let $\pi_i \colon X \to Y_i$, $i = 1, \dots, k$ be all nonequivalent ramified double coverings over compact Riemann surfaces Y_i. Then $k = 1$ or $k = 3$.*

PROOF. See [**15**]. □

2.2. Shafarevich problems. The goal of the section is to give an effective version of the Shafarevich problem which can be regarded as a version of the Parshin-Arakelov theorem in the case of genus 1. We recall briefly the modular curves $X_1(N)$ and $Y_1(N)$ (see [**29**, Chapter 1.5] for more). Let $N \in \mathbb{N}$, we have $Y_1(N) = \Gamma_1(N)\backslash H$ where

$$\Gamma_1(N) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \right\}$$

is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ which acts on the upper half-plan of Poincaré H. It turns out that $Y_1(N)$ is a Riemann surface. Moreover, it is the moduli space classifying equivalence classes of pairs (E,Q) where E is a complex elliptic curve and Q is a point of E of order N. The surface $Y_1(N)$ can be compactified into a compact Riemann surface denoted $X_1(N)$. The genus of $X_1(N)$ can be given as an explicit function of N (cf. [**29**, Ex.3.1.6]). In particular, the genus of $X_1(n_0)$ is greater than 2 with $n_0 = 1728$, for example.

Let B be a smooth projective complex curve of genus g and let $S \subset B$ be a finite subset of cardinality $s \geq 1$. Two elliptic surfaces $\mathcal{E} \to B$ and $\mathcal{E}' \to B$ are said to be *equivalent* if there exists a B-isomorphism $\mu \colon \mathcal{E} \to \mathcal{E}'$.

Theorem 2.37 (Uniform Shafarevich's theorem for elliptic surfaces)**.** *There exists a function $A \colon \mathbb{N}^2 \to \mathbb{N}$ such that the set of equivalence classes of nonisotrivial* *minimal elliptic surfaces $\mathcal{E} \to B$ with good reduction away from S has no more than $A(g, s)$ elements.*

Proof. Let $U = B \backslash S$ then U is an open affine curve. We claim that for every elliptic scheme $\mathcal{E} \to U$, there is a finite étale cover $U' \to U$ of uniformly bounded degree (in terms of g, s) over which $\mathcal{E}$ has a section of order n_0. Indeed, since $\pi_1(U)$ has only finitely many quotients of a given size (which corresponds to the degree of $U' \to U$), there are only finitely many possibilities for U' up to U-isomorphism. The number of such U' is bounded in terms of the free group $\pi_1(U) \simeq F_{2g-1+s}$ and thus uniformly bounded in terms of g, s. Each connected component U'_i of U' has a moduli map to $Y_1(n_0)$ which is dominant if the j-invariant $j(E)$ is not constant. Moreover, since $X_1(n_0)$ has genus at least 2, the de Franchis theorem (Theorem 2.33.(i)) tells us that there are only finitely many such moduli maps. Therefore, there exists a finite étale extension $V \to U$ of uniformly bounded degree over which every nonisotrivial elliptic scheme $\mathcal{E} \to U$ has a section of order n_0 as claimed. Again, an effective version of the de Franchis theorem (cf. [**4**]) implies that the number of nonconstant maps $V \to Y_1(n_0)$ is uniformly bounded. The proof is completed. $\square$

3. Some standard tools

3.1. The pseudo Kobayashi hyperbolic metric. Let X be a complex manifold. The *pseudo Kobayashi hyperbolic metric* $d_X \colon X \times X \to X$ is defined as follows. Let ρ be the Poincaré metric on the unit disc $\Delta = \{z \in \mathbb{C} \colon |z| = 1\}$.

Let $x, y \in X$. Consider the data L consisting of a finite sequence of points $x_0 = x, x_1, \ldots, x_n = y$ in X, a sequence of holomorphic maps $f_i \colon \Delta \to X$ and of pairs $(a_i, b_i) \in \Delta^2$ for $i = 0, \ldots, n$ such that $f_i(a_i) = x_i$ and $f(b_i) = x_{i+1}$. Let $H(x, y; L) = \sum_{i=0}^{n} \rho(a_i, b_i)$.

Definition 2.38 (cf. [**59**])**.** For $x, y \in X$, we define $d_X(x, y) \coloneqq \inf_L H(x, y; L)$.

When $d_X(x, y) > 0$ for all distinct $x, y \in X$, i.e., when d_X is a metric, we say that X is a *hyperbolic* manifold. The most fundamental property of d_X is the distance-decreasing property:

Lemma 2.39. *Let $f \colon X \to Y$ be a holomorphic map of complex manifolds. Then for all $x, y \in X$, $d_Y(f(x), f(y)) \leq d_X(x, y)$. In particular, if $X \subset Y$, we have $d_Y|_X \leq d_X$.*

Proof. For every data $L = \{x_i, f_i, a_i, b_i\}$ associated to the points x, y, we have a data $f(L) = \{f(x_i), f \circ f_i, a_i, b_i\}$ associated to the points $f(x), f(y)$ and $H(x, y; L) = H(f(x), f(y); f(L))$. The lemma now follows from the definition. $\square$

When $X = \Delta$, the pseudo metric d_X is a metric and coincides with Poincaré hyperbolic metric ρ on Δ which can be given by a closed formula:

$$\rho(z_1, z_2) = 2\tanh^{-1}\left|\frac{z_1 - z_2}{1 - z_1\bar{z}_2}\right|, \quad z_1, z_2 \in \Delta. \tag{3.1}$$

3.2. Jet bundles. Let V be a complex algebraic variety. Following Noguchi's notations, for each integer $k \geq 1$, we regard the k-jets over a point $x \in V$ as morphisms from $\operatorname{Spec}\mathbb{C}\{t\}/(t^{k+1})$ to V sending the geometric point of $\operatorname{Spec}\mathbb{C}\{t\}/(t^{k+1})$ to x. The k-jet space is defined as $J^k(V) = \operatorname{Mor}(\operatorname{Spec}\mathbb{C}\{t\}/(t^{k+1}), V)$. We denote by $\mathfrak{m}(k)$ the maximal ideal (t) of $\mathbb{C}\{t\}/(t^{k+1})$. When V is a complex manifold, the space of k-jets is defined similarly using local charts and holomorphic germs of maps $\mathbb{C} \to V$.

If Y is a closed complex subspace of a complex manifold X, then $J^k(Y)$ is a naturally defined closed complex subspace of $J^k(X)$ for every $k \in \mathbb{N}$. See **[82]** for more details.

Now suppose that $V = A$ is an abelian variety. We have $\operatorname{Lie}(A) = T_0A$ as a vector space and the exponential map $\exp\colon T_0A \to A$ is a holomorphic universal cover.

Lemma 2.40. *For every $k \geq 0$, we have a holomorphic trivialization*

$$J^k(A) \simeq A \times (\mathfrak{m}(k) \otimes \operatorname{Lie}(A)) \tag{3.2}$$

given as follows. For each $x \in A$ and $\alpha = \sum_i \alpha_i v_i \in \mathfrak{m}(k) \otimes \operatorname{Lie}(A)$, we associate the following degree-k-germ of holomorphic map from $\operatorname{Spec}\mathbb{C}\{t\}/(t^{k+1})$ to A:

$$f_{x,\alpha}\colon t \mapsto \exp\left(\sum_i \alpha_i(t) v_i\right) \cdot x. \tag{3.3}$$

Proof. See **[82]**. $\square$

3.3. The tautological inequality. Let X be a smooth projective variety over a field k. Let $D \subset X$ be a *simple normal crossing* divisor, i.e., in some local coordinates $z_1, \cdots, z_n$ at any point $x \in X$, D is given by an equation of the form $z_1 \cdots z_k = 0$ with $k \leq n$.

Definition 2.41. The sheaf of differentials $V_1 = \Omega_{X/k}(\log D)$ with logarithmic poles along D is well-defined vector bundle. It is given locally at $x \in X$ by $\frac{dz_1}{z_1}, \cdots, \frac{dz_k}{z_k}, dz_{k+1}, \cdots, dz_n$. $X_1(D) \coloneqq \mathbb{P}V_1$ is defined as $\operatorname{Proj}\bigoplus_{d\geq 0} S^d V_1$.

Thus, points of $X_1(D)$ lying over a point $x \in X$ correspond to hyperplanes in the fibre $V_{1,x}/\mathfrak{m}_x V_{1,x}$ of V_1 at x. We have a canonical morphism $\pi_1\colon X_1(D) \to X$. Denote by $\mathcal{O}(1)$ the tautological line bundle on $X_1(D)$.

Consider a k-morphism $f\colon Y \to X$ where Y/k is a smooth projective curve. Assume that $f(Y) \not\subset D$. Then f can be lifted to a morphism (sometimes called the *derivative*) $f'\colon Y \to X_1(D)$ such that $\pi_1 \circ f' = f$ as follows. Sine the pullback of every logarithmic differential with poles along D is a logarithmic differential with poles along the support $(f^*D)_{red}$, we have a natural map $f^*(V_1) \xrightarrow{f^*} \Omega_{Y/k}((f^*D)_{red})$ which gives rise to a quotient map $f^*(V_1) \to f'^*\mathcal{O}(1)$ and thus the induced map f' (by the universal property of $\mathbb{P}V_1$). Moreover, we have an inclusion of sheaves

$$f'^*\mathcal{O}(1) \hookrightarrow \Omega_{Y/k}((f^*D)_{red}).$$

It follows that

$$(3.4)\quad c_1(\mathcal{O}(1))\cdot Y = \deg f'^*\mathcal{O}(1) \leq \deg \Omega_{Y/k}((f^*D)_{red}) = 2g(Y)-2+\deg(f^*D)_{red},$$

where $g(Y)$ denotes the genus of Y.

Definition 2.42. We call (3.4) the *tautological inequality* associated to $f\colon Y \to X$.

3.4. Simple cyclic covers. In this section, k is an algebraically closed field of characteristic 0. Hence a smooth k-algebraic variety is the same as a regular algebraic variety. We will now briefly recall the construction and basic properties of simple cyclic covers.

Lemma 2.43. *Let $X = \operatorname{Spec}(A)$ be a nonsingular affine variety over k, i.e., A is an integral regular k-algebra of finite type. Let $n \geq 1$ and let $P(T) = c_0T^n + \cdots + c_n \in A[T]$ be a polynomial of degree n. Hence we have an inclusion $A \to A[T]/(P(T))$ and a dominant morphism of affine k-algebraic varieties*

$$f\colon Y = \operatorname{Spec}(A[T]/(P(T)) \to X = \operatorname{Spec}(A)$$

Consider Y as the subvariety of $X \times \mathbb{A}^1$ given by the family of equations $P(x,T)$ for $x \in X$.

- (i) *f is finite if and only if $c_0 \in A^*$.*
- (ii) *f is étale at the point (x,t) where $x \in X(\overline{k})$ if and only if t is a simple root of $P(x,T) = c_0(x)T^n + \cdots + c_n(x) \in k[T]$.*

Proof. See for example [**74**, Example 2.5]. □

We recall the following standard proposition/definition:

Proposition 2.44 (Standard étale morphism)**.** *Let R be a ring. Let $g, f \in R[x]$. Suppose that f is monic and f' is invertible in the localization $R[x]_g/(f)$. Then the ring map $R \to R[x]_g/(f)$ is étale and we call it a standard étale morphism.*

Now let X be a smooth variety. Let $(D, s_D) \in \operatorname{Div}(X)$ be an effective divisor such that $\operatorname{div}(s_D) = D$. Suppose that L is a line bundle on X such that $D \sim L^m$ for some $m \in \mathbb{N}^*$. Then we have an $\mathcal{O}_X$-algebras $\mathcal{A} = \mathcal{O}_X \oplus L \cdots \oplus L^{m-1}$ defined as follows. For $0 \leq a, b \leq m-1$ and $s_q \in L^a$, $s_b \in L^b$, let $0 \leq r \leq a-1$ be such that $a+b = nm+r$ for some $n \in \mathbb{N}$. We then have $s_a s_b / s_D^n = s_r \in L^r$ and we define the multiplication $s_a.s_b \coloneqq s_r$. This give the $\mathcal{O}_X$-algebra structure on $\mathcal{O}_X \oplus L \cdots \oplus L^{m-1}$. The cyclic covering of degree m over X with branched locus D can be described as the morphism

$$f \colon X' = \mathrm{Spec}_X(\mathcal{A}) \to X$$

Indeed, suppose that the line bundle L is given by $\{U_i; g_{ij}\}$ where (U_i) is an affine covering of X and $g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$ are transition functions. Since $D \sim L^m$, $\mathcal{O}(s_D)$ can be given as $\{f_i \in \Gamma(U_i, \mathcal{O}_X) \mid f_i = g_{ij}^m f_j\}$. Let $U_i = \mathrm{Spec}(A_i)$ then $f_i \in A_i$. Consider the morphism

$$h_i \colon V_i = \mathrm{Spec}(A_i[Z_i]/(Z_i^m - f_i)) \to U_i = \mathrm{Spec}(A_i).$$

We claim that h_i is an étale morphism above $\mathrm{Spec}(A_i)_{f_i}$. Indeed, it suffices to apply Proposition 2.44 with $R = A_i$, $x = Z_i$, $f = Z_i^m - f_i$ and $g = Z_i$. Then $f' = mZ_i^{m-1}$. The branch locus is exactly where $Z_i = f_i = 0$ and the ramification index is exactly m. Note that this is true for rings A_i of characteristic 0 (e.g., X an arithmetic surface, etc.).

We define the relations $Z_i = g_{ij} Z_j$ on the variables Z_i. Observe that $Z_i^m - f_i = g_{ij}^m (Z_j^m - f_j)$. We deduce that the affine varieties V_i glue together to exactly $\mathrm{Spec}_X(\mathcal{A})$. Hence, Lemma 2.43 implies that f is a finite morphism of degree m which is totally ramified exactly above D. In particular, X' is also an projective variety over k if X is projective and that X' is irreducible if X is irreducible.

By Lemma 2.43, we see that $X' \to X$ is smooth above $X \backslash D$. Consider a regular point $x \in D$. Note that a closed point $x \in X$ is regular if and only if we have an isomorphism (by Cohen structure theorem, cf. [**65**, Theorem 4.2.27]) $\mathcal{O}_{X,x} \simeq \kappa(x)[[t_1, \ldots, t_q]]$ where $q = \dim(\mathcal{O}_{X,x})$. Suppose that $\kappa(x) = \overline{k} = k$. Since D is regular at x, we can suppose D is given by the equation $t_1 = 0$ around x. We have a commuting square

$$\begin{array}{ccc} \hat{\mathbb{A}}^q \times \hat{\mathbb{A}}^1 \supset (k[[t_1, \ldots, t_q]][Z]/(Z^m - t_1)) & \longrightarrow & \mathrm{Spec}(A[Z]/(Z^m - t_1)) \\ \big\downarrow & & \big\downarrow \\ \hat{\mathbb{A}}^q & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Observe that the closed sub-formal-scheme $V \coloneqq \{Z^m - t_1 = 0\} \subset \hat{\mathbb{A}}^{q+1}$ is smooth by Jacobson's criterion. Therefore, X' is also regular at all points above x. We have just proven the following result:

Proposition 2.45 (Simple cyclic covers)**.** *Let $m \in \mathbb{N}$. Let X be an irreducible projective smooth variety over an algebraically closed field k of characteristic* 0.

Let D be an effective divisor of X such that $D \sim L^{\otimes m}$ for some line bundle L of X. There exists a unique finite cyclic cover of degree m of irreducible projective k-varieties

$$f_{D,L,m} \colon X' \to X$$

such that f is totally ramified and ramified only above D and every point of $f_{D,L,m}^{-1}(X \setminus D_{sing})$ is a regular point of X'.

PROOF. See also for example [**85**], in particular [**85**, Proposition 3.1]. □

Moreover, it is clear that construction of simple cyclic covers is functorial:

Proposition 2.46. *Let X, Y be smooth irreducible projective varieties. Let $g \colon Y \to X$ be a morphism such that g^*D is well defined where D is an effective divisor on X. Let $m \in \mathbb{N}$ and L a line bundle such that $L^{\otimes m} \sim \mathcal{O}_X(D)$. We have a cartesian square:*

$$\begin{tikzcd} Y' \arrow[r, "f_{g^*D,g^*L,m}"] \arrow[d] & Y \arrow[d, "g"] \\ X' \arrow[r, "f_{D,L,m}"] & X. \end{tikzcd}$$

3.5. The generalized Riemann-Hurwitz formula.

Lemma 2.47 (Riemann-Hurwitz for singular curves)**.** *For any covering of possibly singular projective integral curves $f : X \to Y$ of degree k, the following generalized Riemann-Hurwitz's formula holds:*

$$2p_a(X) - 2 = k(2p_a(Y) - 2) + deg(R) + 2\sum_{P \in X} \delta_P - 2k \sum_{Q \in Y} \delta_Q$$

where :

- *R is the ramification divisor of the normalisation morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$;*
- *$\delta_x := dim_k \tilde{\mathcal{O}}_x / \mathcal{O}_x$ denotes the singularity degree of a point $x \in X$ or $x \in Y$ with $\tilde{\mathcal{O}}_x$ the normalisation of the local ring $\mathcal{O}_x$.*

If Z is a reduced projective complex curve with irreducible components $Z_1, \cdots, Z_m$,

$$p_a(Z) - 1 = \sum_{i=1}^{m} (p_a(\tilde{Z}_i) - 1) + \sum_{z \in Z} \delta_z.$$

PROOF. For the proof of the second formula, see [**65**, Proposition 7.5.4]. The first formula follows from the classical Riemann-Hurwitz's formula ([**65**, Proposition 7.4.16])

$$p_a(\tilde{X}) - 2 = k(2p_a(\tilde{Y}) - 2) + \deg(R)$$

and the relation $p_a(X) = p_a(\tilde{X}) + \sum_{P\in X} \delta_P$ which is a particular case of the second formula. $\square$

3.6. Hilbert schemes of algebraic groups. By the general theory of Hilbert schemes of subvarieties developed by Grothendieck, Altmann-Kleiman, we have the following important properties needed in the proof of Theorem 1.18 and Theorem F.

Lemma 2.48. *Let $\pi\colon G \to S$ be a (quasi-)projective group scheme over a scheme S. Let $\mathcal{D} \subset G$ be a closed subscheme and let $\mathcal{V} = G\backslash\mathcal{D}$. Consider the contravariant functors $\mathcal{F}_{G/S}\colon \mathrm{Sch}_S \to \mathrm{Ens}$ defined for $T \to S$ an S-scheme by*

$$\mathcal{F}_{G/S}(T) \coloneqq \left\{ \begin{array}{l} \textit{(connected) group subschemes of } G_T, \textit{ that are} \\ \textit{flat, proper, and of finite presentation over } T \end{array} \right\}.$$

The following holds:

(a) *$\mathcal{F}_{G/S}$ is representable by a locally of finite type S-scheme denoted also by $\mathcal{F}_{G/S}$;*

(b) *There exist natural immersions of S-schemes*

$$\mathcal{F}_{G/S}, \mathrm{Mor}_S(S,G), \mathrm{Hilb}^{\dim>0}_{\mathcal{D}/S}, \mathrm{Hilb}^{\dim>0}_{\mathcal{V}/S} \subset \mathrm{Hilb}_{G/S};$$

where $\mathrm{Hilb}^{\dim>0}_{X/S}$ denotes the complement in $\mathrm{Hilb}_{X/S}$ of the S-relative Hilbert schemes of points, i.e., of zero dimensional closed subschemes, of an S-scheme X.

(c) *The S-scheme $\mathcal{F}_{G/S,\mathcal{D}} \coloneqq \mathrm{Hilb}^{\dim>0}_{\mathcal{D}/S} \times_{\mathrm{Hilb}_{G/S}} \left(\mathcal{F}_{G/S} \times_S \mathrm{Mor}_S(S,G)\right)$ represents the contravariant functor $\mathrm{Sch}_S \to \mathrm{Ens}$ given by*

$$T \mapsto \left\{ \begin{array}{l} \textit{translates of } (\varphi\colon H \to G_T) \in \mathcal{F}_{G/S}(T) \textit{ by a } T\textit{-section } \sigma\colon T \to G_T \\ \textit{such that } \dim H > 0 \textit{ and } \mathrm{Im}(\sigma.\varphi) \subseteq \mathcal{D}_T \end{array} \right\},$$

where $\sigma.\varphi(h) \coloneqq \sigma(\pi_T \circ \varphi(h))\varphi(h)$ for every $h \in H$. That is, $\mathcal{F}_{G/S,\mathcal{D}}$ is the moduli space of translates of positive dimensional group subschemes of G_s contained in $\mathcal{D}_s$ for $s \in S$;

(d) *Similarly, the S-scheme $\mathcal{F}_{G/S,\mathcal{V}} \coloneqq \mathrm{Hilb}^{\dim>0}_{\mathcal{V}/S} \times_{\mathrm{Hilb}_{G/S}} \left(\mathcal{F}_{G/S} \times_S \mathrm{Mor}_S(S,G)\right)$ is the moduli space of translates of positive dimensional group subschemes of G_s that have empty intersection with $\mathcal{D}_s$ for $s \in S$;*

(e) *The schemes $\mathcal{F}_{G/S,\mathcal{D}}$ and $\mathcal{F}_{G/S,\mathcal{V}}$ have only countably many irreducible components.*

Proof. For assertion (a), see [**28**, Exposé XI, Remarque 3.13]. The existence of other schemes in (b) is standard since G/S is quasi-projective and so are $\mathcal{D}/S$, $\mathcal{V}/S$ as $\mathcal{D} \subset G$ is assumed to be closed. Observe that $\mathrm{Hilb}_{G/S}$ is the disjoint union of quasi-projective S-schemes $\mathrm{Hilb}_{G/S}^{q(x)}$ with $q(x) \in \mathbb{Q}[x]$ runs over all numerical polynomials of degree $\leq \dim G_s$ where G_s is a general fibre. Moreover, the S-schemes $\mathcal{F}_{G/S}, \mathrm{Mor}_S(S, G)$, $\mathrm{Hilb}_{\mathcal{D}/S}^{\dim>0}$ and $\mathrm{Hilb}_{\mathcal{V}/S}^{\dim>0}$ are also stratified by the same Hilbert polynomials and that $\mathrm{Hilb}_{\mathcal{D}/S}^{\dim>0}$ and $\mathrm{Hilb}_{\mathcal{V}/S}^{\dim>0}$ do not take into account zero degree polynomials.

Since each S-scheme of finite type has finitely many irreducible components, it follows from the stratification by Hilbert polynomials that all Hilbert schemes in the lemma have only countably many irreducible components. In particular, this proves (e). The assertions (c) and (d) are also clear. $\square$

Lemma 2.49. *Let B be a Dedekind scheme with fraction field K. Suppose that $\pi\colon G \to B$ is a quasi-projective B-group scheme. Let $\mathcal{D} \subset G$ be a closed subset with generic fibre $D = \mathcal{D}_K \subset G_K$. Then:*

(i) *if D does not contain any translates of positive dimensional $\overline{K}$-algebraic subgroups of G_K, then the exceptional set $Z(G, \mathcal{D})$ given by*

$$\{t \in B \text{ closed point}\colon \exists x \in G_t, \exists\, H \subset G_t \text{ a subgroup}, \dim H > 0,\, xH \subset \mathcal{D}_t\}$$

is countable;

(ii) *if $G_K \setminus D$ does not contain any translates of positive dimensional $\overline{K}$-algebraic subgroups of G_K, then the exceptional set $U(G, \mathcal{D})$ given by*

$$\{t \in B \text{ closed point}\colon \exists x \in G_t,\, \exists H \subset G_t \text{ a subgroup}, \dim H > 0,\, xH \subset G_t \setminus \mathcal{D}_t\}$$

is countable.

Proof. For (i), let X be an irreducible component of the Hilbert B-scheme $\mathcal{F}_{(G/B),\mathcal{D}}$ defined in Lemma 2.48. Then X is a scheme of finite type over B. We claim that the induced morphism $f_X\colon X \to B$ is not dominant. The point (i) will then be proved since the scheme $\mathcal{F}_{(G/B),\mathcal{D}}$ does not dominate B. Indeed, as $\mathcal{F}_{(G/B),\mathcal{D}}$ contains only countably many irreducible components each of which being of finite type over B (cf. Lemma 2.48.(c)), the image of $\mathcal{F}_{(G/B),\mathcal{D}}$ in B is thus a countable subset of closed points of B by Chevalley's theorem and the conclusion follows.

Suppose on the contrary that X dominates B. Then we can find a 1-dimensional irreducible closed subscheme $C \subset X$ such that C dominates B. Up to replacing X by C, we can thus assume that $\dim X = 1$. Let η be the generic point of X and let $V \hookrightarrow \mathcal{F}_{(G/B),\mathcal{D}} \times_B G$ be the universal group scheme. Then $L = \kappa(\eta) \subset \overline{K}$ is a finite extension of K. On the other hand, it follows from the functorial property of Hilbert schemes that V_L is the universal group scheme of G_L/L avoiding $\mathcal{D}_L$ but $\mathcal{F}_{(G_L/L),\mathcal{D}_L}$ is empty by the hypothesis of (i). This contradiction shows that X cannot dominate B and (i) is proved as explained in the above paragraph.

The proof of (ii) is similar. Therefore, the proof of the corollary is completed. $\square$

CHAPTER 3

Integral points on constant elliptic surfaces

1. Motivation and preliminary

The goal of the presenting chapter is to present general ideas and examples of some results obtained in this thesis by restricting ourselves to the simplest but nontrivial case of constant elliptic surfaces. We try to keep the presentation as elementary as possible. At the end of the chapter, we formulate a reasonable question on the emptiness of $(S, \mathcal{D})$-integral sections in nonisotrivial elliptic surfaces (cf. Question 3.16).

We fix throughout the following notations. Let B/k be a smooth projective genus-g curve over a field k of characteristic 0. Let $K = k(B)$ be the function field of B. Let E_0/k be an elliptic curve. Consider the trivial elliptic surface $\pi \colon X = E_0 \times B \to B$ where π is the second projection. Then we have a canonical identification $E_0(K) = \mathrm{Mor}_k(B, E_0) = X(B)$. Denote $\Gamma \coloneqq E_0(K)/E_0(k)$.

Note that $E_0(k) = \mathrm{Mor}_{B,cte}(B, X)$ is also the set of constant B-sections of X. We can equipped $X(B)$ with an abelian group structure by fiberwise addition induced by the group law of the elliptic curve E_0. Recall the content of the effective split Mordell-Weil Theorem.

Theorem 3.1. *Γ is a finitely generated abelian group of rank $r \leq 4g$.*

Proof. Let $J(B)/k$ be the Jacobian of B, we have:

$$X(B)/\mathrm{Mor}_{B,cte}(B, X) = \mathrm{Mor}_k(B, E_0)/\mathrm{Mor}_{k,cte}(B, E_0) \simeq \mathrm{Hom}_k(J(B), E_0).$$

Notice that by Abel-Jacobi Theorem, the Albanese variety $\mathrm{Alb}(B)$ of B is isomorphic to $J(B)$ (in general, $\mathrm{Alb}(V) = \mathrm{Pic}^0(V)^\vee$ need not be isomorphic to $\mathrm{Pic}^0(V)$). The last isomorphism follows from the property that B generates $J(B)$ and the universal property of Albanese varieties.

For every prime number $l \neq \mathrm{char}(k) = 0$, we have an injective Tate homomorphism

$$\mathrm{Hom}_k(J(B), E_0) \otimes \mathbb{Z}_l \to \mathrm{Hom}(V_l(J(B)), V_l(E_0))$$

where $V_l(J(B))$ and $V_l((E_0))$ are respectively the Tate modules of $J(B)$ and E_0 (cf. [**71**, Theorem 10.15]). Since we have $\dim_{\mathbb{Q}_l} V_l(J(B)) = 2 \dim J(B) = 2g$ and $\dim_{\mathbb{Q}_l} V_l(E_0) = 2 \dim E_0 = 2$, the homomorphism group $\mathrm{Hom}_k(J(B), E_0)$ is finitely generated of rank at most

$$\dim_{\mathbb{Q}_l} \operatorname{Hom}(V_l(J(C), V_l(E_0)) = 4g.$$

It follows that $\Gamma = X(B)/\operatorname{Mor}_{B,cte}(B, X)$ is a finitely generated of rank $r \leq 4g$. $\square$

We summarize below the properties of the height function on split elliptic curves.

Proposition 3.2. *The group $\Gamma = E_0(K)/E_0(k)$ is finitely generated and all torsion points of $E_0(K)$ belong to $E_0(k)$, i.e., $E_0(K)_{tors} \in E_0(k)$. The function*

$$h\colon E_0(K) \to \mathbb{Z}^+, \quad \phi \mapsto \deg(\phi) = (O) \cdot (\phi(B))$$

coincides with the canonical Néron-Tate height function $\hat{h}\colon E_0(K) \to \mathbb{Z}^+$. Moreover, $\hat{h}$ descends to a positive definite quadratic form $\hat{h}\colon \Gamma \to \mathbb{Z}^+$.

Proof. Recall the identification $E_0(K) \simeq \operatorname{Mor}_k(B, E_0) = X(B)$. By the Mordell-Weil theorem, Γ is a finitely generated abelian group.

Now suppose that $\phi \in E_0(K)$. Then it is a k-morphism $\phi\colon C \to E_0$ of smooth projective curves. ϕ is thus finite and the function $h(\phi) = \deg(\phi)$ is well-defined. Suppose that E_0/k is given by the reduced Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a, b \in k. \tag{1.1}$$

Working with the Weierstrass coordinates x, y on E_0, the morphism $\phi\colon B \to E_0$ is given by $B \ni t \mapsto (X(t), Y(t))$ where $(X, Y) \in K^2$ are rational functions on B verifying (1.1). Then we have

$$\deg(\phi) = \frac{1}{2}\deg(X)$$

where $\deg(X) = \deg(X\colon B \to \mathbb{P}^1)$ is the usual Weil height on E_0 used to defined the canonical height $\hat{h}$. Indeed, let $q = (x_0, y_0) \in E_0(\overline{k})$ be any general point,

$$\begin{aligned}\deg(\phi) &= \#\phi^{-1}(q) = \{t \in B\colon (X(t), Y(t)) = (x_0, y_0)\}\\ &= \frac{1}{2}\{t \in B\colon X(t) = x_0\} = \frac{1}{2}\deg(X).\end{aligned}$$

For any $n \in \mathbb{N}$, $h([n]\phi) = \deg([n])\deg(\phi) = n^2 h(\phi)$: take a general point $Q \in E_0(\bar{k})$ then there are n^2 points in the preimage of the multiplication $[n]\colon E_0 \to E_0$ and for each of these points, there are $\deg(\phi)$ points in the preimage of ϕ. Therefore, the definition of the Néron-Tate canonical height $\hat{h}\colon E_0(K) \to \mathbb{R}$ implies that

$$\hat{h}(\phi) := \lim_{n\to\infty} \frac{1}{n^2} h([n]\phi) = \lim_{n\to\infty} \frac{1}{n^2} n^2 h(\phi) = h(\phi) \in \mathbb{Z}^+.$$

Now let $\phi_0 \in E_0(k)$ be a constant morphism. Then $\deg(\phi + \phi_0) = \deg(\phi)$ since the degree of ϕ does not change under a translation of E_0. Hence, $h(\phi + \phi_0) = h(\phi)$. It follows that $\hat{h}$ descends to the quotient Γ to define a function $\hat{h}\colon E_0(K)/E_0(k) \to \mathbb{Z}^+$. It is well-known that $\hat{h}$ is a quadratic form (cf. [**104**, III.4.3]). Observe that $h(\phi) = 0$ if and only if ϕ is a constant morphism. In other words, $h(\phi) = 0$ if and only if $\phi \in E_0(k)$. Therefore, $E_0(K)/E_0(k)$ is a lattice equipped with positive definite quadratic form $\hat{h}$. ☐

Remark 3.3. Let the notations be as in Proposition 3.2. Let B' be a smooth projective curve and $h\colon B' \to B$ a ramified cover. Let $K' = k(B')$. Let $P \in E_0(K')$ and $(P) \subset E_0 \times B'$ the corresponding section. Then for any constant section (R) of $E_0 \times B' \to B'$ with $R \in E_0(k)$, we have $\hat{h}(P) = (R) \cdot (P)$.

Corollary 3.4. *Let $N \geq 0$ be a real number. Then we have an estimation:*

$$\#\{[P] \in \Gamma\colon \hat{h}(P) \leq N\} \leq (2\sqrt{N} + 1)^r$$

where $r \coloneqq \operatorname{rank} \Gamma \leq 4g$.

Proof. This is a direct application of the following estimation (cf. Lemma 2.24) on a finite n-dimensional lattice L equipped with a norm $\sigma\colon L \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$:

$$\#\{x \in L\colon \sigma(x) \leq N\} \leq \left(\frac{2N}{\mu_{L,\sigma}} + 1\right)^n$$

where $\mu_{L,\sigma} \coloneqq \inf\{\sigma(x)\colon x \in L \setminus \{0\}\}$. To conclude, it suffices to apply the above inequality for $L = E_0(K)/E_0(k)$ equipped with the norm

$$\sigma = \sqrt{\hat{h}} \otimes \mathbb{R}\colon E_0(K)/E_0(k) \otimes \mathbb{R} \to \mathbb{R}^+.$$

Remark that since $\hat{h}\colon E_0(K)/E_0(k) \to \mathbb{Z}^+$ is a definite positive quadratic form by Proposition 3.2 and verifies the Northcott property by the de Franchis Theorem (Theorem 2.33), σ is indeed a norm. Since $\hat{h}$ has integral values, $1 \leq \mu_{\Gamma,\sigma}$. ☐

Remark 3.5. A positive quadratic form over $\mathbb{Z}$ can be not positive after a $\mathbb{R}$-linear extension (as pointed out by Cassels (cf. [**53**, B.5.3]) by considering for exemple $L = \mathbb{Z} + \sqrt{5}\mathbb{Z}$ and $q\colon L \to \mathbb{R}$, $q(x) = |x|^2$). This phenomenon can happen if the value set (over $\mathbb{Z}$) is not discrete in $\mathbb{R}$.

As an easy form of the abc inequality for elliptic curves, the following height bound refines Theorem 1.36 in our context:

Theorem 3.6. *For every $\phi \in E_0(K) \setminus E_0(k)$, we have $\hat{h}(\phi) \leq 2g - 2 + \#(\phi^{-1}(O))$.*

Proof. Let R_ϕ be the ramification divisor of the finite morphism $\phi\colon B \to E_0$. The Riemann-Hurwitz formula tells us that $2g - 2 = \deg R_\phi \geq \sum_{x \in \phi^{-1}(O)} (e_x - 1)$

where e_x denotes the ramification index at x. We deduce that

$$\hat{h}(\phi) = \deg \phi = \sum_{x \in \phi^{-1}(O)} (e_x - 1) + \#(\phi^{-1}(O)) \leq 2g - 2 + \#(\phi^{-1}(O))$$

and the conclusion follows. $\square$

We introduce the notion of *parallel sections* in a constant elliptic surface.

Definition 3.7. Let $Q \in E_0(K)$. We say that $P \in E_0(K)$ is *Q-trivial*, or P is *parallel* to Q, if $P - Q \in E_0(k)$, i.e., whenever the classes of P and Q in Γ coincide. In this case, we denote also $P \parallel Q$. Since $E_0(k)$ is an abelian group, $\parallel$ is clearly an equivalence relation on $E_0(K)$.

The following easy observation will be frequently used in the proofs in this chapter. Let $f\colon X \to B$ be a minimal elliptic surface with generic fibre E/K. Let $D \subset X$ be an effective reduced divisor. For every rational point $a \in E(K)$, we denote by $\tau_a\colon X \to X$ the B-automorphism induced by the fiberwise translation by a, i.e., $\tau_a(x, b) = (x + a(b), b)$ for every $x \in X_b$ where $b \in B$ is such that X_b/k is an elliptic curve (cf. [**104**, Proposition III.9.1]). Then we have the following certain duality of integral points:

Lemma 3.8. *Consider rational points $P, Q \in E(K)$. We have:*

(i) *P is (S, D)-integral if and only if $P + R$ is $(S, \tau_R(D))$-integral for every rational point $R \in E(K)$.*

(ii) *P is $(S, (Q))$-integral if and only Q is $(S, (P))$-integral.*

Proof. The assertion (ii) follows immediately from the definition of integral points (cf. Definition 1.1). Similarly, since τ_R is a B-automorphism of X and $\tau_R((P)) = (P + R)$, we have an equality of the intersection locus

$$f((P) \cap D) = f((P + R) \cap \tau_R(D)) \subset B, \tag{1.2}$$

and the assertion (i) is proved. $\square$

2. Effective bounds of integral points

We are now in position to establish the following uniform bound on integral points on trivial elliptic surfaces.

Proposition 3.9. *Let $s \geq 0$ an integer. Let $X = E_0 \times B$ and let $D \in X(B)$ be any section of X. Then we have an estimation*

$$\begin{aligned} &\# \cup_{\# S \leq s} \{[P] \in \Gamma \colon [P] \textit{ contains an } (S, D)\textit{-integral, } D\textit{-nontrivial point}\} \\ &\leq (2\sqrt{2g - 2 + s} + 1)^{4g} \end{aligned}$$

where the union runs over all finite subsets S of B of cardinality at most s. Moreover, if S is any finite subset of B of cardinality s then we have

$$\#\{P \in E_0(K) \colon P \text{ is } (S,D)\text{-integral, } D\text{-nontrivial}\} \leq s(2\sqrt{2g-2+s}+1)^{4g}.$$

Remark 3.10. We give an interpretation of the above "arithmetic" statement in terms of the geometry of sections in an isotrivial elliptic surface. If we fix a section P and an integer $s \geq 0$, then there exist a finite (uniformly bounded) number

$$n \leq N(g,s) := (2\sqrt{2g-2+s}+1)^{4g}$$

of sections $P_1, \cdots, P_n$ such that every section Q must either intersect P at more than s pairwise distinct points (counted without multiplicities) in X or Q is parallel to one of the P_i's. Since $E_0(k)$ is "large" (e.g., when $k = \mathbb{C}$) and $E_0(K)$ is a disjoint union of countably infinite copies of $E_0(k)$ (when $r > 0$), we can say that for any integer s and almost all sections $Q \in E_0(K)$, the set theoretic intersection $Q \cap P$ contains at least s points. Intuitively, if we choose randomly two sections in $E_0(K)$, the cardinality q of their set theoretic intersection must be as large as a randomly chosen positive integer.

Similarly, the algebraic meaning of the proposition is the following. Consider a reduced Weierstrass equation over a constant field k:

$$y^2 = x^3 + ax + b, \quad a, b \in k, \quad \Delta = 4a^3 - 27b^2 \neq 0. \tag{2.1}$$

Let K be a finite field extension of $k(t)$ (e.g., $K = k(t)$). Then if the rational functions $(x_1, y_1), (x_2, y_2) \in K^*$ are distinct and verify the relation (2.1) then the equation $x_1 - x_2 = 0$ must have a lot of pairwise distinct solutions in general.

Proof of Proposition 3.9. Since X is isotrivial, we have $\chi(X) = 0$ (cf. [**76**]). Consider the zero section section $O \in E_0(k)$. so that E_0 becomes an elliptic curve with a fixed group law and with O the zero section. For each finite subset $S \subset B$ and each section Q of X, let $A(S,Q)$ be the set of (S,Q)-integral and Q-nontrivial sections. Then we must show that

$$\# \cup_{\#S \leq s} A(S,D) \bmod E_0(k) \leq (2\sqrt{2g-2+s}+1)^{4g}.$$

Observe that we have the following canonical bijection

$$\begin{aligned} \sigma(D,O) \colon \cup_{\#S \leq s} A(S,D) &\longrightarrow \cup_{\#S \leq s} A(S,O) \\ P &\longmapsto P - D \end{aligned}$$

and the inverse map is given by $\sigma(O,D)$ which sends Q to $Q + D$. This follows immediately from the very definition of S-integral points. Notice that the translation by $-D$ also gives a bijection between $A(S,D)$ and $A(S,O)$ for any subset $S \subset B$. Since the bijection $\sigma(D,O)$ is a single translation, it induces also a bijection when we take modulo $E_0(k)$: $[P_1], [P_2] \in \Gamma$ are two distinct classes if and only if $[P_1 - D], [P_2 - D] \in \Gamma$ are distinct. Therefore, we reduce the problem to show that

$$\# \cup_{\#S \leq s} A(S, O) \bmod E_0(k) \leq (2\sqrt{2g-2+s}+1)^{4g}.$$

By Theorem 3.6, for any $P \in \cup_{\#S \leq s} A(S, O)$, we have $\hat{h}_{E_0}(P) \leq 2g - 2 + s$. Therefore, if we define

$$H(m) := \{[P] \in \Gamma \colon \hat{h}(P) \leq m\} \subset \Gamma$$

then $\# \cup_{\#S \leq s} A(S, O) \bmod E_0(k) \subset H(2g-2+s)$. On the other hand, we deduce from Corollary 3.4 that

$$\#H(2g-2+s) \leq (2\sqrt{2g-2+s}+1)^r \leq (2\sqrt{2g-2+s}+1)^{4g} \tag{2.2}$$

where the last inequality results form the bound $r \leq 4g$ in Néron-Lang theorem 3.1. This establishes the first statement of the proposition.

Let $P_1, P_2, \cdots, P_n \in E_0(K)$ where $n \leq (2\sqrt{2g-2+s}+1)^{4g}$ be a set of representatives of $H(2g-2+s)$. Suppose that $P \in A(S, O)$ for some $S \subset B$ with $\#S \leq s$. We claim that in the class $[P] = P + E_0(k)$, there are at most s sections which belongs to $A(S, O)$. Indeed, assume that $Q = P + R \in A(S, O)$ where $R \in E_0(k)$. Then clearly we must have $R \notin -P(B \setminus S)$. As P is not O-trivial, $-P$ induces a surjective k-morphism also denoted by $-P \colon B \to E_0$. This implies that

$$\begin{aligned}
\#E_0(k) \setminus -P(B \setminus S) &\leq \#E_0(k) \setminus (-P(B) \setminus -P(S)) \\
&\leq \#E_0(k) \setminus (E_0(k) \setminus -P(S)) \\
&\leq \#P(S) \leq s.
\end{aligned}$$

Hence, there are at most s possibilities for $R \in E_0(k)$ and the claim follows. It suffices now to combine the above claim and the estimate (2.2) to conclude. □

3. Emptiness of integral points

Proposition 3.9 allows us to prove the following strong statement of generic emptiness of integral points on isotrivial elliptic surfaces. The result can be seen as an easy elliptic version of Theorem 8.8 for parametrized S-unit equations $\alpha x + \beta y = 1$.

Corollary 3.11. *Let $O \in E_0(k)$ be a rational point. Then there exist a finite number of sections $Q_1, \cdots, Q_n \in E_0(K)$ where*

$$n \leq (s(2\sqrt{2g-2+s}+1)^{4g}+1)^2$$

such that for $D \in E_0(K)$, the set of integral sections

$$Z(S, (O)+(D)) := \{P \in E_0(K) \colon P \text{ is } (S, (O)+(D))\text{-integral}\}$$

is empty whenever $D \notin \{Q_1, \cdots, Q_n\} + E_0(k)$. *In general, if* $D \in E_0(K) \setminus E_0(k)$,

$$\#Z(S, (O) + (D)) \leq 2s(2\sqrt{2g - 2 + s} + 1)^{4g}.$$

Proof. By Proposition 3.9, there exists an integer $m \geq 0$ satisfying

$$m \leq s(\sqrt{2g - 2 + s} + 1)^{4g} \tag{3.1}$$

and sections $P_1, \cdots, P_m \in E_0(K) \setminus E_0(k)$ such that

$Z(S, (O)) = \{P \in E_0(K) \colon P \text{ is } (S, (O))\text{-integral}\} = (E_0(k) \setminus \{O\}) \cup \{P_1, \cdots, P_m\}$.

Let $D \in E_0(K)$ and suppose that $P \in E_0(K)$ is $(S, (O)+D)$-integral. In particular, P and $D - P$ are $(S, (O))$-integral. Hence, P and $D - P$ belong to $Z(S, (O))$. Let $P_0 = O$, we deduce that

$$D = P + (D - P) \in Z(S, (O)) + Z(S, (O)) \subset \{P_i + P_j \colon 0 \leq i, j \leq m\} + E_0(k).$$

Since (3.1) implies that

$$n \coloneqq \#\{P_i + P_j \colon 0 \leq i, j \leq m\} \leq (m + 1)^2 \leq (s(2\sqrt{2g - 2 + s} + 1)^{4g} + 1)^2,$$

the first statement follows by taking $\{Q_1, \cdots, Q_n\} = \{P_i + P_j \colon 0 \leq i, j \leq m\}$. For the last statement, we have for any $D \in E_0(K)$ the inclusion:

$$\begin{aligned} Z(S, (O) + (D)) &= Z(S, (O)) \cap Z(S, (D)) \\ &\subset (E_0(k) \cup \{P_1, \cdots, P_m\}) \cap ((D + E_0(k)) \cup \{P_1 + D, \cdots, P_m + D\}). \end{aligned} \tag{3.2}$$

Assume that $D \notin E_0(k)$ then $(D + E_0(k)) \cap E_0(k) = \varnothing$. Hence,

$$\begin{aligned} (E_0(k) \cup \{P_1, \cdots, P_m\}) \cap (D + E_0(k)) &= \{P_1, \cdots, P_m\} \cap ((D + E_0(k)) \\ &\subset \{P_1, \cdots, P_m\}. \end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that $Z(S, (O) + (D)) \subset \{P_1, \cdots, P_m\} \cup \{P_1 + D, \cdots, P_m + D\}$. Therefore, we can conclude from (3.1) that

$$\#Z(S, (O) + (D)) \leq 2m \leq 2s(2\sqrt{2g - 2 + s} + 1)^{4g}.$$

□

Corollary 3.12. *Let* $S \subset B$ *be a finite subset of cardinality* s. *Let* $m \geq 2 + (s(2\sqrt{2g - 2 + s} + 1)^{4g} + 1)^2$ *and let* $R_1, \cdots, R_m \in E_0(K)$ *be pairwise nonparallel sections. Then the following set of integral points is empty:*

$$Z(S, (R_1) + \cdots + (R_m)) = \{P \in E_0(K) \colon P \textit{ is } (S, (R_1) + \cdots + (R_m))\textit{-integral}\}.$$

Proof. Let $Q_1, \cdots, Q_n \in E_0(K)$ be points satisfying the conclusion of Corollary 3.11. Since the classes $R_i + E_0(k)$'s are pairwise distinct and

$$m - 1 \geq 1 + (s(2\sqrt{2g - 2 + s} + 1)^{4g} + 1)^2 \geq n + 1,$$

there exists by pigeonhole principle a section R_i, $2 \leq i \leq m$, such that $R_i - R_1 \notin \{Q_1, \cdots, Q_n\} + E_0(k)$. The same corollary *loc.cit* implies that there is no $(S, (O) + (R_i - R_1))$-integral points. Thus, there is no $(S, (R_1) + (R_i))$-integral points either since $(R_1) + (R_i)$ is the translation of $(O) + (R_i - R_1)$ by the section (R_1). In particular, the set $Z(S, (R_1) + \cdots + (R_m))$ is empty since it is a subset of $Z(S, (R_1) + (R_i))$. $\square$

Remark 3.13. Let $F(s) \coloneqq 2+((2\sqrt{2g-2+s}+1)^{4g}+1)^2$ then Corollary 3.12 says that among arbitrary $n \geq F(0)+1$ pairwise nonparallel sections of $X = E_0 \times B \to B$, there always exist 2 sections whose set theoretic intersection contains at least $F^{-1}(n)+1$ distinct points. This geometric property is certainly false in the usual Euclidean plane with usual straight lines.

Even in the case of trivial elliptic surfaces, we have the following finiteness theorem of (S, D)-integral points whenever the divisor D is not reduced to a union of sections. This observation will be generalized to higher dimensional (and not necessarily constant) abelian varieties in Theorem A.

Let $q \colon X = E_0 \times B \to E_0$ and $\pi \colon X \to B$ be respectively the first and second projections.

Proposition 3.14. *With the above notations, assume that $\pi|_D \colon D \to B$ is a ramified cover of degree $d \geq 2$. Then there exists $c > 0$ (depending on D) such that for every finite subset $S \subset B$, the set of (S, D)-integral points of X has at most $c(\#S)^{4g+1}$ elements.*

Proof. We can clearly suppose that $k = \overline{k}$. Let $f \colon C \to D$ be the normalization morphism. Then $h = \pi \circ f \colon C \to B$ is a ramified cover of degree d. Let $K' = k(C)$. Fix a finite subset $S \subset B$. Denote $S' = h^{-1}(S) \subset C$ then $\#S' \leq d\#S$.

Consider the second projection $\pi' \colon X' = X \times_B C \to C$. It is clear that $D_C = D \times_B C$ splits into d sections of π'. Let R be any of the these sections. Remark that each (S, D)-integral section (P) of X with $P \in E_0(K)$ induces an (S', R)-integral section $(P') = (P) \times_B C$ of X' via the base change $h \colon C \to B$. On the other hand, since D_C splits into d sections, Theorem 1.36 (or Theorem 3.6) tells us that $D_C \cdot (P')$ is bounded by a constant $c_1 > 0$ depending linearly on $\#S'$ but not on the subset $S' \subset C$. As $\#S' \leq d\#S$, we deduce from Corollary 3.4 that modulo $E_0(k)$, the set of (S, D)-integral points of X has at most $c_2(\#S)^{4g}$ elements for some constant $c_2 > 0$ independent of $\#S$.

To complete the proof, we claim that for any section $P \in E_0(K)$, there is at most $d\#S$ points $Q \in P + E_0(k)$ such that Q is (S, D)-integral. This will complete the proof by setting $c = c_2 d$. Since $E_0(K)$ acts on X/B by translations, $D' \coloneqq D - P$ is also an integral horizontal divisor of X which is also a ramified cover of degree

d over B. Consider $R \in E_0(k)$ such that $P+R$ is (S,D)-integral. Then for every $b \in B$, we have by definition of integral points that

$$R \notin \cup_{b\in B\setminus S} q(D_b - P_b) = \cup_{b \in B\setminus S} q(D'_b) \tag{3.4}$$

where D_b, D'_b and P_b denote respectively the intersection of D, D' and the section (P) with the fibre $\pi^{-1}(b)$. Since the divisor D' is integral and horizontal, we have

$$\# D' \cap \pi^{-1}(S) \leq D' \cdot \pi^* \left(\sum_{b \in S} b \right) = d \# S.$$

On the other hand, D' is not constant, i.e., D' is not a section associated to a point in $E_0(k)$, since it is integral and has degree $d > 1$ over B. It follows that $q(D') = E_0$ and thus

$$\begin{aligned} \# E_0 \setminus \left(\cup_{b \in B \setminus S} q(D'_b) \right) &\leq \# E_0 \setminus \left(q(D') \setminus q(D' \cap \pi^{-1}(S)) \right) \\ &= \# E_0 \setminus \left(E_0 \setminus q(D' \cap \pi^{-1}(S)) \right) \\ &= \# q(D' \cap \pi^{-1}(S)) \leq d \# S. \end{aligned} \tag{3.5}$$

The relations (3.4) and (3.5) imply that there are at most $d\#S$ points $Q \in P + E_0(k)$ such that Q is (S,D)-integral as claimed. ☐

4. A similar emptiness result for nonisotrivial elliptic surfaces

The same type of emptiness result in the above section holds for nonisotivial elliptic surface. Corollary 3.11 and Corollary 3.12 suggest the generic emptiness of integral points, e.g., for divisors of large enough degree as follows, at least for totally splitting divisors.

Proposition 3.15. *Let $X \to B$ be a nonisotrivial minimal elliptic surface. Let $S \subset B$ be a finite subset of cardinality $s \geq 1$. Let $r = \operatorname{rank} X_K(K)$ and*

$$N = N(g,s,r) \coloneqq 144(g+1)^{2/3}(10^{7+12g}s)^r$$

Assume that $R_1, \cdots, R_m \in X_K(K)$ are pairwise distinct sections with $m \geq 1 + N^2$. Then the following set of integral points is empty:

$$Z(S, (R_1) + \cdots + (R_m)) = \{P \in X_K(K) \colon P \text{ is } (S, (R_1) + \cdots + (R_m))\text{-integral in } X\}.$$

Proof. The proof is similar to the proof of Corollary 3.11. Theorem [**52**, Theorem 0.6] implies that X_K contains no more than N rational points $Q_1, \cdots, Q_n \in X_K(K)$ which are $(S,(O))$-integral, where $n \leq N$. Since $m \geq 1 + N^2$, there exists by the pigeonhole principle some $i \in \{2, \cdots, m\}$ such that

$$R_i - R_1 \notin \{Q_u + Q_v \colon 1 \leq u, v \leq n\}$$

where $Q_u - Q_v \in X_K(K)$ is obtained by subtracting Q_v from Q_u using the elliptic group law. We claim that there is no $(S, (O) + (R_i - R_1))$-integral points. Indeed, assume that $Q \in X_K(K)$ is $(S, (O) + (R_i - R_1))$-integral then Q is in particular $(S, (O))$-integral. Hence, $Q = Q_v$ for some $v \in \{1, \dots, n\}$. On the other hand, since Q is also $(S, (R_i - R_1))$-integral, $R_i - R_1$ is $(S, (Q))$-integral by Lemma 3.8. This in turns implies that $R_i - R_1 - Q_v$ is $(S, (O))$-integral. Thus, $R_i - R_1 - Q_v = Q_u$ for some $u \in \{1, \cdots, n\}$ so that $R_i - R_1 = Q_u + Q_v$ which is a contradiction to the choice of R_i. The claim is prove.

Therefore, again by Lemma 3.8, there is no $(S, (R_1) + (R_i))$-integral points as well. Since every $(S, (R_1) + \cdots + (R_m))$-integral point is clearly $(S, (R_1) + (R_i))$-integral, the proof of the proposition is completed. $\square$

Consider now the following heuristic argument. Let $X \to B$ be a nonisotrivial minimal elliptic surface. Let S be a finite subset of B of cardinality $s \geq 1$. Let $\mathcal{D} \subset X$ be an effective integral horizontal divisor. Denote $m = \deg \mathcal{D}_K$ which is also the degree of the cover $\mathcal{D} \to B$. "Hence", we can regard $\mathcal{D}$ as a union of m disjoint sections of X, at least when making the base change $\tilde{\mathcal{D}} \to B$ to the normalization of $\mathcal{D}$. Proposition 3.15 thus suggests the following question:

Question 3.16. Let the notations be as above. Assume $r = \operatorname{rank} X(B) > 0$. Is it true that whenever $m = \deg \mathcal{D}_K$ is large enough (depending only on X, B, S), there exists no $(S, \mathcal{D})$-integral points?

Remark 3.17. A direct use of Proposition 3.15 by making the base change $\pi \colon \tilde{\mathcal{D}} \to B$ is not enough to give a positive answer. Indeed, let $X' = X \times_B \tilde{\mathcal{D}}$, $\mathcal{D}' = \mathcal{D} \times_B \tilde{\mathcal{D}} \subset X'$ and $S' = \pi^{-1} S$. Denote $B' = \tilde{\mathcal{D}}$ then $r' = \operatorname{rank} X'(B') \geq r$. By construction, $\mathcal{D}'$ is a union of m disjoint sections of $X' \to B'$. For S general so that π is not ramified over S, we have $s' = \# S' = ms$. It is clear that $(S, \mathcal{D})$-integral points lift to $(S', \mathcal{D}')$-integral points. However, we have (assume $s \geq 1$).

$$N(g', s', r') = 144(g' + 1)^{2/3} (10^{7+12g'} s')^{r'} > (10^7 ms) > m.$$

Hence, Proposition 3.15 cannot be applied.

Remark 3.18. Let $X \to B$ be an elliptic surface with section (O). One can naturally expect that higher degree horizontal integral sections $\mathcal{D}$ in X would present more obstruction for a rational point $P \in X_K(K)$ to be $(S, \mathcal{D})$-integral than to be $(S, (O))$-integral. However, it is equally possible that the divisors $\mathcal{D}$ would have more freedom to, in a dual way, avoid the set of sections of X above the finite subset S! In this sense, numerical invariants could not capture all geometric properties.

CHAPTER 4

On the finiteness of the set of integral points in abelian varieties

1. Statement of the finiteness criteria

Fix a complex smooth projective curve B of function field $K = \mathbb{C}(B)$, and let $S \subset B$ be a finite subset. Let A/K be an abelian variety with a model $f\colon \mathcal{A} \to B$. Assume that $D \subset A$ is an effective ample divisor on A. Let $\mathcal{D}$ be the Zariski closure of D in $\mathcal{A}$. We have seen in Theorem 1.18 that when taking modulo $\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$, the set of $(S, \mathcal{D})$-integral points on $\mathcal{A}$ is finite under the assumption that D does not contain any translate of nonzero abelian subvarieties. The main purpose of this chapter is to show that if moreover the divisor D admits no rational points, then even without taking modulo $\mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$, the set of $(S, \mathcal{D})$-integral points on $\mathcal{A}$ is actually finite. More specifically, we shall prove that:

Theorem A. *Suppose that D does not contain any translate of nonzero abelian subvarieties of A (e.g., when A is simple). The following hold:*

(i) *If $D(K) = \varnothing$ then the set of $(S, \mathcal{D})$-integral points is finite;*

(ii) *If $\dim \mathrm{Tr}_{K/\mathbb{C}}(A) \leq 1$ then $X(K)$ is finite.*

The condition $D(K) = \varnothing$ should be interpreted as the divisor D being in a very general position so that it does not contain any rational points. In fact, we can obtain the following stronger useful statement which will be used later in the proof of Theorem I in Chapter 9:

Theorem B. *Let A/K be an abelian variety with $D \subset A$ an effective ample divisor. Let $P \in A(K)$ be a rational point. Assume that for every $a \in \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$, $a + P \notin D$. Then for every finite subset $S \subset B$, the set $P + \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ contains only finitely many $(S, \mathcal{D})$-integral points of A with respect to any model $\mathcal{A} \to B$ of A with $\mathcal{D}$ being the Zariski closure of D in $\mathcal{A}$.*

For each $a \in A(K)$ and $\sigma \in \mathcal{A}(B)$, the notation $a + \sigma_P$ will stand for the section of $\mathcal{A} \to B$ induced by $a + P \in A(K)$. We can moreover prove that:

Theorem C. *Let A/K be an abelian variety with a Néron model $f\colon \mathcal{A} \to B$. Assume that $D \subset A$ is an effective ample divisor on A. Let $\mathcal{D}$ be the Zariski* *closure of D in $\mathcal{A}$. Let $P \in A(K)$ and let $\sigma_P \in \mathcal{A}(B)$ be the induced section. Assume $D \cap (P + \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})) = \varnothing$. Then for $r := \deg_B \sigma_P^* \mathcal{D}$, we have a finite morphism of complex schemes*

$$\pi\colon \mathrm{Tr}_{K/\mathbb{C}}(A) \to B^{(r)}, \quad a \mapsto (a + \sigma_P)^* \mathcal{D}. \tag{1.1}$$

If A and D are defined over $\mathbb{C}$, the finiteness statement in Theorem B can be made uniform in terms of certain numerical invariants (cf. Theorem I.(ii)).

As an application, we obtain the following result on the topology of the intersection locus of sections with a divisor in a family of abelian varieties.

Corollary A. *Let the hypotheses be as in Theorem C. Let $R \subset \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}) \subset A(K)$ be a subset and let $I(R) := \cup_{a \in R} f((a + \sigma_P)(B) \cap \mathcal{D})) \subset B$. We have:*

- (i) *if R is infinite, then $I(R)$ is Zariski dense, i.e., infinite, in B;*
- (ii) *if R is analytically dense in a complex algebraic curve $C \subset \mathrm{Tr}_{K/\mathbb{C}}(A)$, then $I(R)$ is analytically dense in B.*

2. Preliminary lemmata on the Néron models

We begin with the following lemma:

Lemma 4.1. *Let B be a compact smooth complex curve of function field $K = \mathbb{C}(B)$. Let A/K be an abelian variety with trace $\mathrm{Tr}_{K/\mathbb{C}}(A)$. Let $f\colon \mathcal{A} \to B$ be proper flat model of A/K. Then there exists a nonempty Zariski open subset $U \subset B$ such that for every $b \in U$, the evaluation map*

$$\mathrm{val}_b\colon\ \mathrm{Tr}_{K/\mathbb{C}} A(\mathbb{C}) \to \mathcal{A}_b, \quad P \mapsto \sigma_P(b),$$

where $\sigma_P\colon B \to \mathcal{A}$, is an injective morphism of varieties, where $\sigma_P\colon B \to \mathcal{A}$ is the corresponding section of P.

Proof. Let $f_{\textit{Néron}}\colon \mathcal{A}^{\textit{Néron}} \to B$ be the minimal Néron model of A/K. By definition of the trace, the canonical map $\iota_K\colon \mathrm{Tr}_{K/\mathbb{C}}(A) \otimes_{\mathbb{C}} K \to A$ is a closed immersion homomorphism of K-abelian varieties. Consider the smooth B-abelian scheme $\mathrm{Tr}_{K/\mathbb{C}}(A) \times B$. Then by the Néron mapping property (cf. Definition 2.10), the map ι_K extends to a unique B-morphism $\iota\colon \mathrm{Tr}_{K/\mathbb{C}}(A) \times B \to \mathcal{A}^{\textit{Néron}}$. By [**46**, Proposition 9.6.1.(ix)], there exist a nonempty Zariski open subset U_1 such that for every $b \in U_1$, the induced base change map

$$\iota_b\colon (\mathrm{Tr}_{K/\mathbb{C}}(A) \times B) \otimes \kappa(b) \to \mathcal{A}_b^{\textit{Néron}}$$

is also a closed immersion and in particular it is injective.

Similarly, the identity K-map $A \to A$ extends to a unique B-morphism $h\colon \mathcal{A} \to \mathcal{A}^{\textit{Néron}}$ by the Néron mapping property. It is clear that there exists a nonempty

Zariski open subset $U_2 \subset B$ above which h is an isomorphism fiberwise (again by [**46**, Proposition 9.6.1.(ix)]). Let $U = U_1 \cap U_2$. We claim that the map $h \circ \mathrm{val}_b$ is exactly the reduction map ι_b for every $b \in U$. Indeed, let $P \in \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ and let $\tau_P \colon B \to \mathcal{A}^{Néron}$ be the corresponding section (exists by the valuative criterion for properness or by the Néron mapping property). Then we must have $h \circ \sigma_P = \tau_P$ by the Néron mapping property and this proves $h \circ \mathrm{val}_b = \iota_b$. Therefore, the conclusion follows since val_b is a closed immersion and the fact that $h_b \colon \mathcal{A}_b \to \mathcal{A}_b^{Néron}$ is an isomorphism for every $b \in U$. ☐

Recall the following universal property of the symmetric powers of a curve which is needed in the proof of Lemma 4.4.

Proposition 4.2. *Let B be a complete smooth curve over a field k and T a k-scheme. Let $\mathrm{Div}^r_B(T)$ be the set of relative effective Cartier divisors on $C \times T \to T$ of degree r. Then $\mathrm{Div}^r_B \colon \mathrm{Sch}_k \to \mathrm{Set}$ is a functor represented by $B^{(r)}$. Moreover, for any relative effective divisor D on $B \times T/T$ of degree r, there exists a unique morphism $\varphi \colon T \to B^{(r)}$ such that $D = (\mathrm{Id} \times \varphi)^* D_{can}$ where $D_{can} = S_r \backslash (\sum_{i=1}^r \Delta_i)$ with $\Delta_i \subset B \times B^r$ the image of the section $B^r \to B \times B^r$ given by $(b_1, \cdots, b_r) \mapsto (b_i, b_1, \cdots, b_r)$.*

PROOF. See [**73**, Theorem 3.13]. ☐

Remark 4.3. Let $f \colon X \to Y$ be a morphism of schemes. Let D be an effective Cartier divisor on Y. Then the pullback divisor f^*D is defined if and only if $f(\mathrm{Ass}(X)) \cap \mathrm{supp}(D) = \varnothing$, if and only if $f^{-1}(D)$ is an effective divisor. In this case, we have $\mathrm{supp}\, f^*D = f^{-1}(D)$. Here, $\mathrm{Ass}(X)$ denotes the set of associated points of X (cf. [**65**, Definition 7.1.1]).

Lemma 4.4. *Let the notations be as in Theorem A.(i) and assume furthermore that $\mathcal{A}$ is a Néron model of A over B. Let $\sigma \colon B \to \mathcal{A}$ be a section and let $r := \deg_B \sigma^* \mathcal{D}$. Then we have $\deg_B (a + \sigma)^* \mathcal{D} = r$ for every $a \in \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$. Moreover, we have a well-defined morphism of complex schemes:*

$$\pi \colon \mathrm{Tr}_{K/\mathbb{C}}(A) \to B^{(r)}, \quad a \mapsto (a + \sigma)^* \mathcal{D}.$$

In the above lemma, the notation $a + \sigma$ means simply the section of $\mathcal{A} \to B$ associated to the rational point $a + P_\sigma \in A(K)$ where $P_\sigma \in A(K)$ is the rational point corresponding to $\sigma \in \mathcal{A}(B)$.

PROOF OF LEMMA 4.4. As in Lemma 4.1, we have a B-morphism $\lambda \colon \mathrm{Tr}_{K/\mathbb{C}}(A) \times B \to \mathcal{A}$ which extends the closed immersion $\tau_\sigma \circ \iota_K \colon \mathrm{Tr}_{K/\mathbb{C}}(A) \otimes_{\mathbb{C}} K \to A$. Here, $\tau_\sigma \colon A \to A$ denotes the translation by $\sigma \in A(K)$. Hence, we obtain a section

$$\Sigma = (\lambda, \pi_2) \colon B \times \mathrm{Tr}_{K/\mathbb{C}}(A) \to \mathcal{A} \times \mathrm{Tr}_{K/\mathbb{C}}(A)$$

of the morphism $\pi = f \times \mathrm{Id}\colon \mathcal{A} \times \mathrm{Tr}_{K/\mathbb{C}}(A) \to B \times \mathrm{Tr}_{K/\mathbb{C}}(A)$. Here $\pi_2\colon B \times \mathrm{Tr}_{K/\mathbb{C}}(A) \to \mathrm{Tr}_{K/\mathbb{C}}(A)$ denotes the second projection.

By the assumption $D(K) = \varnothing$ and by the dimensional reason, the divisor $\mathcal{D} \times \mathrm{Tr}_{K/\mathbb{C}}(A)$ of $\mathcal{A} \times \mathrm{Tr}_{K/\mathbb{C}}(A)$ is not contained in the image of Σ. We can thus define the effective Cartier divisor $R \coloneqq \Sigma^*(\mathcal{D} \times \mathrm{Tr}_{K/\mathbb{C}}(A))$ on $B \times \mathrm{Tr}_{K/\mathbb{C}}(A)$ (cf. Remark 4.3). We claim that R is a relative Cartier divisor of $B \times \mathrm{Tr}_{K/\mathbb{C}}(A) \to \mathrm{Tr}_{K/\mathbb{C}}(A)$ with $R_a = (a+\sigma)^*\mathcal{D}$ for every $a \in \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$. Indeed, fix $a \in \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ and consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{A} \times \{a\} \arrow[r, hook, "i_a"] \arrow[d, "f_a"'] & \mathcal{A} \times \mathrm{Tr}_{K/\mathbb{C}}(A) \arrow[d, "\pi"'] & \mathcal{D} \times \mathrm{Tr}_{K/\mathbb{C}}(A) \arrow[l, hook'] \\
B \times \{a\} \arrow[r, hook, "j_a"] \arrow[u, bend right, "\Sigma_a"'] & B \times \mathrm{Tr}_{K/\mathbb{C}}(A) \arrow[u, bend right, "\Sigma"'] &
\end{tikzcd}$$

Since $D(K) = \varnothing$, we have $\mathrm{Im}(a+\sigma) \not\subset \mathcal{D}$ and $\mathrm{Im}(j_a) \not\subset R$. As $\Sigma_a = a + \sigma$ and $\Sigma_a \circ i_a = j_a \circ \Sigma$, we deduce the following equality of effective Cartier divisors:

$$R_a = j_a^*\left(\Sigma^*\left(\mathcal{D} \times \mathrm{Tr}_{K/\mathbb{C}}(A)\right)\right) = (a+\sigma)^*\left(i_a^*\left(\mathcal{D} \times \mathrm{Tr}_{K/\mathbb{C}}(A)\right)\right) = (a+\sigma)^*\mathcal{D}.$$

Therefore, every fibre R_a is an effective Cartier divisors. R is clearly a locally principal closed subscheme of $B \times \mathrm{Tr}_{K/\mathbb{C}}(A)$ as it is an effective Cartier divisor. We deduce from [**108**, Lemma 062Y] that R is a relative effective Cartier divisor of $B \times \mathrm{Tr}_{K/\mathbb{C}}(A)/\mathrm{Tr}_{K/\mathbb{C}}(A)$, i.e., R as a closed subscheme is flat over $\mathrm{Tr}_{K/\mathbb{C}}(A)$. In particular, the curves $(a+\sigma)(B) \subset \mathcal{A}$ are algebraically equivalent. It follows that $\deg_B(a+\sigma)^*\mathcal{D} = r$ for every $a \in \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$. By the universal property of the symmetric power $B^{(r)} = \mathrm{Hilb}_B^r$ (cf. Proposition 4.2), we conclude that

$$\pi\colon\ \mathrm{Tr}_{K/\mathbb{C}}(A) \to B^{(r)}, \quad a \mapsto R_a = (a+\sigma)^*\mathcal{D}$$

is indeed a morphism of complex schemes. □

3. Proof of the mains results

We can now return to the proof of Theorem A].

Proof of Theorem A. By Theorem 1.19 and 1.18, the set $X(K)$ and the set of $(S, \mathcal{D})$-integral points are finite modulo $\mathrm{Tr}_{K/\mathbb{C}}(A)$. If $\mathrm{Tr}_{K/\mathbb{C}}(A) = 0$, then the corollary is an obvious consequence. Suppose now that $\dim \mathrm{Tr}_{K/\mathbb{C}}(A) = 1$ then the trace $E = \mathrm{Tr}_{K/\mathbb{C}}(A)$ is a complex elliptic curve. Since we are in characteristic 0, the canonical map $E_K \to A$ is a closed immersion homomorphism (cf. page 20 in [**23**]). Assume that $\sigma\colon B \to \mathcal{X}$ is a section corresponding to a point $P \in X(K)$. We have to show that $(P + E(\mathbb{C})) \cap X(K)$ is finite.

Suppose on the contrary that there exists an infinite sequence of pairwise distinct points $(a_i)_{i\in I} \subset E(\mathbb{C})$ such that $\{P+a_i\}_{i\in I} \in X(K)$. By taking the Zariski closure and remark that any infinite subset of an integral curve is Zariski dense, we deduce that $P + E_K \subset X$ which is clearly a contradiction. Therefore, we conclude that the set $(P + E(\mathbb{C})) \cap X(K)$ is finite and the proof of (ii) is completed.

Suppose now that $D(K) = \varnothing$, i.e., $\mathcal{D}$ does not contain any section of $f\colon \mathcal{A} \to B$. We first consider the case when $\mathcal{A}$ is a Néron model of A_K over B. Since the divisor $\mathcal{D}$ is generically ample, it is ample over a dense open affine subset $U \subset B \setminus S$. In particular, $\mathcal{D}|_U$ is an ample divisor of $\mathcal{A}|_U$ so that $(\mathcal{A}\setminus\mathcal{D})|_U$ is affine as $\mathcal{A}|_U$ is proper over U. Let $\sigma\colon B \to \mathcal{A}$ be a section and let $r \coloneqq \deg_B \sigma^*\mathcal{O}(\mathcal{D})$. Let $P \in A(K)$ be the corresponding rational points of σ. Then the condition $D(K) = \varnothing$ implies (cf. Lemma 4.4) that we have a well-defined morphism of complex schemes:

$$\pi\colon \operatorname{Tr}_{K/\mathbb{C}}(A) \to B^{(r)}, \quad a \mapsto (a+\sigma)^*\mathcal{D}.$$

Let S_i be an effective divisor of B of degree r such that $\operatorname{supp} S_i \subset S$. There are clearly only finitely many such S_i. Let $[S_i] \in B^{(r)}$ be the image of S in $B^{(r)}$. Let $W_i = \pi^{-1}([S_i])$ then W_i is a closed subset of $\operatorname{Tr}_{K/\mathbb{C}}(A)$. Remark that the union of the finitely many sets $P + W_i$ is exactly the set of *all* $(S,\mathcal{D})$-integral points of A in the translation class modulo $\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ of P with respect to the model $\mathcal{A} \to B$ and the divisor $\mathcal{D}$.

Since $\operatorname{Tr}_{K/\mathbb{C}}(A)$ is an abelian variety, it is proper thus so is W_i. Let $x_0 \subset U$ be a general point. Since $U \subset B \setminus S$, we can consider the morphism (cf. Lemma 4.1)

$$\Sigma_i\colon W_i \to \mathcal{A}_{x_0} \setminus \mathcal{D}_{x_0} \subset (\mathcal{A}\setminus\mathcal{D})|_U, \quad a \mapsto a(x_0) + \sigma(x_0).$$

Since W_i is proper, $\operatorname{Im}\Sigma_i$ is also proper. Moreover, since $\mathcal{A}_{x_0} \setminus \mathcal{D}_{x_0}$ is affine, we deduce that $\operatorname{Im}\Sigma_i$ is a proper affine scheme. Hence $\operatorname{Im}\Sigma_i$ is finite. It follows immediately that W_i is also finite since Σ_i is an injective morphism for a general choice of x_0 (cf. Lemma 4.1).

Thus, the number of $(S,\mathcal{D})$-integral points in each class modulo $\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ is finite. On the other hand, as mentioned at the beginning of the proof, the set of $(S,\mathcal{D})$-integral points is finite modulo $\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$. Therefore, we conclude that the number of $(S,\mathcal{D})$-integral points is finite.

Now consider the general case when $\mathcal{A}$ is not necessarily a Néron model of A over B. Let $f'\colon \mathcal{A}' \to B$ be a Néron model of A over B. By the Néron mapping property, there exists a B-morphism $h\colon \mathcal{A} \to \mathcal{A}'$ extending the identity K-map $\mathrm{Id}\colon A_K \to A_K$. Let $T \subset B$ be the finite subset above which the fibres of $f\colon \mathcal{A} \to B$ are not smooth. It is clear that $\mathcal{A}_{B\setminus T}$ and $\mathcal{A}'_{B\setminus T}$ are Néron models of A_K over $B\setminus T$ respectively by Proposition 2.14 and by the Néron mapping property. Again by the Néron mapping property (over $B \setminus T$), we find that $h|_{(B\setminus T)}$ must be a $(B \setminus T)$-isomorphism. On the other hand, for every section $\sigma\colon B \to \mathcal{A}$ associated to a rational point $P \in A(K)$, $h \circ \sigma\colon B \to \mathcal{A}'$ is exactly the corresponding section of P in $\mathcal{A}'$. Hence, it follows from Definition 1.1 that every (S, D)-integral point of $\mathcal{A}$ is an $(S \cup T, \mathcal{D}')$-integral point of $\mathcal{A}'$ where $\mathcal{D}'$ is the Zariski closure of D in $\mathcal{A}'$. Since $S \cup T$ is finite, Theorem A.(i) now follows from the corresponding results for $(S \cup T, \mathcal{D}')$-integral points in $\mathcal{A}'$. $\square$

In fact, the proof of Theorem B is already contained in the above arguments.

Proof of Theorem B. Let $\sigma_P\colon B \to \mathcal{A}$ be a section. It is not hard to see that in the course of the proofs of Lemma 4.4 and of Theorem A, we only need to use the condition $(a + \sigma_P)(B) \not\subset \mathcal{D}$ for every $a \in \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ to show that the number of $(S, \mathcal{D})$-integral sections in the translation class of σ_P by the trace is finite.

On the other hand, the condition $(a + \sigma_P)(B) \not\subset \mathcal{D}$ for every $a \in \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ is equivalent to $a + P \notin D$ for every $a \in \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$. Hence, the choice of the models $\mathcal{A} \to B$ and $\mathcal{D}$ of A, D does not matter. Therefore, Theorem B is proved. $\square$

Similarly, we can deduce immediately Theorem C.

Proof of Theorem C. Lemma 4.4 and the first part in the proof of Theorem A actually show that the morphism

$$\pi\colon\ \mathrm{Tr}_{K/\mathbb{C}}(A) \to B^{(r)}, \quad a \mapsto (a + \sigma_P)^*\mathcal{D} \tag{3.1}$$

is well-defined and is quasi-finite where $r = \deg_B \sigma_P^* \mathcal{O}(\mathcal{D}) \in \mathbb{N}$. Since $\mathrm{Tr}_{K/\mathbb{C}}(A)$ and $B^{(r)}$ are proper varieties, π is in fact a proper morphism. Therefore, π is a quasi-finite proper morphism of varieties. By [**46**, Théorème 8.11.1], it follows that π is indeed a finite morphism as desired. $\square$

The proof of Corollary A of Theorem C can also be given now as follows.

Proof of Corollary A. Let $r = \deg_B \sigma_P^* \mathcal{O}(\mathcal{D}) \in \mathbb{N}$, then the map

$$\pi\colon\ \mathrm{Tr}_{K/\mathbb{C}}(A) \to B^{(r)}, \quad a \mapsto (a + \sigma_P)^*\mathcal{D} \tag{3.2}$$

is a finite morphism by Theorem C. Suppose first that the set $R \subset \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ is infinite. It follows that the image $\pi(R) \subset B^{(r)}$ is also infinite. Consider the canonical finite morphism $q_r\colon B^r \to B^{(r)}$. Then the preimage $E \coloneqq q_r^{-1}(\pi(R)) \subset B^r$ is also infinite. Let $p_i\colon B^r \to B$ be the i-th projection for $i = 1, \dots, r$. It is a straightforward verification from the definition of the map π in (3.2) that

$$I(R) \coloneqq \cup_{a \in R} f((a + \sigma_P)(B) \cap \mathcal{D})) = \cup_{i=1}^r p_i(E).$$

But since E is infinite, it is clear that $\cup_{i=1}^r p_i(E)$ is also infinite and thus Zariski dense in B. The point (i) is thus proved.

Now suppose that R is analytically dense in a complex algebraic curve $C \subset \mathrm{Tr}_{K/\mathbb{C}}(A)$. The argument for the point (ii) is similar as above. The image $\pi(R) \subset B^{(r)}$ is now analytically dense in the image complex curve $\pi(C) \subset B^{(r)}$. The preimage $E = q_r^{-1}(\pi(R))$ is also analytically dense in the curve $q_r^{-1}(\pi(C))$. It follows that $\cup_{i=1}^r p_i(E)$ is analytically dense in B and the proof of (ii) and of Corollary A is therefore completed. $\square$

CHAPTER 5

Linear growth of the hyperbolic length of loops in certain complements of compact Riemann surfaces

"Dans certaines situations [...], il est bien plus élégant, voire indispensable [...] de travailler avec des groupoïdes fondamentaux par rapport à un paquet de points base convenable"
– Alexander Grothendieck, Esquisse d'un Programme

1. Introduction

Notations. Let B be a compact Riemann surface. Let U be finite union of disjoint closed discs in B and $b_0 \in B_0 := B \setminus U$. Fix a base of generators $\alpha_1, \dots, \alpha_k$ of $\pi_1(B_0, b_0)$. Equip B and thus B_0 with a fixed Riemannian metric d. We also fix an arbitrary collection $\{c_{b_0 b}\}$ consisting of bounded d-length and smooth directed paths contained in B_0 such that $c_{b_0 b}$ goes from b_0 to $b \in B_0$ for each $b \in B_0$.

Recall that for every complex space X, the Kobayashi pseudo hyperbolic metric on X is denoted by d_X. The goal of this chapter is to prove the following linear bound on the hyperbolic length of loops in various complements of B, which is crucial for Theorem F and Theorem H as well as their consequences.

Theorem D. *There exists a constant $L > 0$ with the following property. For any finite subset $S \subset B_0$, there exists $b \in B_0 \setminus S$ and piecewise smooth loops $\gamma_1, \dots, \gamma_k \subset B_0 \setminus S$ based at b representing the classes $\alpha_1, \dots, \alpha_k$ in $\pi_1(B_0, b_0)$ up to a single conjugation and such that:*

$$\text{length}_{d_{B_0 \setminus S}}(\gamma_i) \leq L(\#S + 1). \tag{1.1}$$

Definition 5.1. In this thesis, by *a single conjugation* (with respect to the collection of paths $\{c_{b_0 b}\}$ which is fixed throughout), we mean that each loop γ_i represents the conjugation of the class $\alpha_i \in \pi_1(B_0, b_0)$ by the change of base points from b to b_0 using the specific chosen path $c_{b_0 b}$, i.e., $[c_{b_0 b}^{-1} \circ \gamma_i \circ c_{b_0 b}] = \alpha_i \in \pi_0(B_0, b_0)$ for *every* i. This extra condition on the loops γ_i's will be useful for the next chapters.

The constant $L > 0$ depends only on U and the Riemann surface B. Our proof does not provide an explicite estimate of L. But with more care, an estimation of L only in terms of some invariants of B and U could be established in principle.

In fact, we shall prove a slightly stronger result (cf. Theorem 5.22) which asserts moreover the existence, for some fixed Riemannian metric on B, of neighborhoods of width $O((\#S)^{-1})$ contained entirely in $B \setminus (U \cup S)$ around the desired loops. Moreover, the proof of Theorem 5.22 actually shows the following generalization of Theorem D by even allowing a certain bounded moving discs besides the finite subset S:

Theorem 5.2. *For each $p \in \mathbb{N}$, there exists constants $L, R > 0$ with the following property. For each finite subset $S \subset B_0$ and every union Z of p discs in B each of d-radius R, there exists $b \in B_0 \backslash (S \cup Z)$ and piecewise smooth loops $\gamma_1, \ldots, \gamma_k \subset B_0 \backslash S$ based at b representing $\alpha_1, \ldots, \alpha_k$ in $\pi_1(B_0, b_0)$ up to a single conjugation and:*

$$\text{length}_{d_{B_0 \backslash (S \cup Z)}}(\gamma_i) \leq L(\#S + 1). \tag{1.2}$$

Now let $s \in \mathbb{N}$ and consider the parameter space B_0^s of subsets S of at most s points in B_0. Each free homotopy class $\alpha \in \pi_1(B_0)$ gives rise to a constant

$$L(\alpha, s) \coloneqq \sup_{\#S \leq s} \inf_{[\gamma] = \alpha} \text{length}_{d_{B \backslash S}}(\gamma), \tag{1.3}$$

where S runs over all subsets of B of cardinality at most s and γ runs over all loops representing the free homotopy class α. Theorem D thus gives an upper bound on the growth of $L(\alpha, s)$. The following optimal lower bound on the polynomial growth is shown (see also Remark 5.6):

Theorem E. *Given $\alpha \in \pi_1(B_0) \backslash \{0\}$, there exists $c > 0$ such that for every $s \in \mathbb{N}$:*

$$L(\alpha, s) \geq \frac{c s^{1/2}}{\ln(s + 2)}. \tag{1.4}$$

It would be interesting to understand the behavior of the function $L(\alpha, s)$ in terms of s. For example, we may ask:

Question 5.3. What are the limits:

$$\deg^-(\alpha) \coloneqq \liminf_{s \to \infty} \frac{\ln L(\alpha, s)}{\ln s}, \quad \deg^+(\alpha) \coloneqq \limsup_{s \to \infty} \frac{\ln L(\alpha, s)}{\ln s}, \tag{1.5}$$

which correspond respectively to the lower and upper polynomial growth degrees of $L(\alpha, s)$ in terms of s?

For this question, Theorem D and Theorem E tell us that for every $\alpha \in \pi_1(B_0) \backslash \{0\}$, we have:

$$1/2 \leq \deg^-(\alpha) \leq \deg^+(\alpha) \leq 1. \tag{1.6}$$

The chapter is organized as follows. We begin with a quick proof of Theorem E in Section 2. A weak form of Theorem D is then presented in Section 3 to better understand the Proof of Theorem D (cf. Remark 5.8). Section 4 recalls the notion of simple loops together with elementary properties needed for latter use. We prepare some lemmata in Section 5 before describing the main constructions in Section 6 and Section 7. The proof of Theorem D is then given in Section 8. Finally, we present in Section 9 another proof of the weak form of Theorem D that may be generalized to higher dimension manifolds.

2. Proof of Theorem E

Let X be a hyperbolic surface. Let $x \in X$ and $v \in T_X$. We denote the infinitesimal hyperbolic metric by $\lambda_X(x, v)$. Then $\lambda_X(x, v) = \inf 2/R$ where the minimum is taken over all $R > 0$ for which there exists a holomorphic map $f\colon \Delta(0, R) \to X$ such that $f'(0) = v$. Here, $\Delta(0, R) := \{z \in \mathbb{C}\colon |z| < R\} \subset \mathbb{C}$.

Let $\Omega = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ and consider the Fubini-Study metric d_{FS} on $\mathbb{CP}^1$ given by $d_{FS} z = |dz|/(1 + |z|^2)$ where z is the standard affine coordinate chart on $\mathbb{CP}^1$. We denote by $T_1\mathbb{CP}^1$ the unit tangent space with respect to the metric d_{FS}. The following fundamental estimation says that the hyperbolic metric on Ω near the cusp 0 behaves exactly as the hyperbolic metric of the punctured unit disc:

Theorem 5.4 (Ahlfors)**.** *There exists $\delta > 0$ and $C > 0$ such that for every $(z, v) \in T_1\mathbb{CP}^1$ such that $z \in \Omega$ and $d_{FS}(z, 0) < \delta$, we have*

$$|\log \lambda_\Omega(z, v) + \log d_{FS}(z, 0) + \log\log((d_{FS}(z, 0))^{-1})| < C \tag{2.1}$$

Proof. See [**3**, Theorem 1-12], in particular (1-24). It suffices to remark there that the Fubini-Study metric and the Euclidean metric are locally bilipschitz. □

Lemma 5.5. *There exists $r_0 > 0$ such that for every $s \geq 2$, we can cover the Riemann sphere $\mathbb{CP}^1$ by s closed discs of d_{FS}-radius $r_0 s^{-1/2}$.*

Proof. It is clear that we only need to show the existence of r_0 for s large enough. Let $\Delta_2 = \{z \in \mathbb{C}\colon |z| \leq 2\} \subset \mathbb{C}$. Denote d the Euclidean metric on $\mathbb{C}$. For each $\varepsilon > 0$, let $N(\varepsilon)$ denotes the minimum number of closed discs of d-radius ε in $\mathbb{C}$ which can cover Δ_2. Then Kershner's theorem (cf. [**58**]) tells us that $\lim_{s\to\infty} s^{-1}N(s^{-1/2}) = 8\pi 3^{1/2}/9$.

Remark that $N(s^{-1/2})$ is an increasing function in s. It follows that there exists a real number $c > 1$ such that $N(s^{-1/2}) < cs$ for all $s \geq 1$. Replacing s by s/c, we deduce that for every $s \geq c$, there exists a covering of Δ_2 by $\leq s$ discs of d-radius $(c/s)^{1/2}$. In particular, since $(c/s)^{1/2} \leq 1$ for $s \geq c$, we can find a subset of $k \leq s$ discs $D_1, \dots, D_k$ which cover Δ_1 and whose centres $z_1, \dots, z_k$ belong to

Δ_2. Consider the stereographic projection p_N from the north pole of $\mathbb{CP}^1$ onto $\mathbb{C}$. We obtain a cover of the Southern hemisphere of $\mathbb{CP}^1$ by $p_N^{-1}D_1, \ldots, p_N^{-1}D_k$. Since $d_{FS}z = |dz|/(1+|z|^2) \geq |dz|/5$ for every $z \in \Delta_2$, it follows that every $p_N^{-1}D_i$, where $i = 1, \ldots, k$, is contained in the disc centered at $p_N^{-1}(z_i)$ of d_{FS}-radius $5(c/s)^{1/2}$. By symmetry, we obtain a cover of $\mathbb{CP}^1$ by $2s$ discs of d_{FS}-radius $\leq 5(c/s)^{1/2}$ for every $s \geq c$ and the conclusion follows. $\square$

We can now return to the proof of Theorem E.

Proof of Theorem E. Consider an arbitrary ramified cover $\pi\colon B \to \mathbb{CP}^1$ of B to the Riemann sphere. Let d_{FS} be the Fubini-Study metric on $\mathbb{CP}^1$. We denote $\tilde{d}$ the induced metric on $B \setminus R_\pi$ where $R_\pi \subset B$ is the branch locus of π which is finite.

Since $\alpha \in \pi_1(B_0)\setminus\{0\}$, it is well-known that there exists $c_0 > 0$ such that every loop $\gamma \subset B_0$ representing α has bounded $\tilde{d}$-length from below by c_0, i.e., $\mathrm{length}_{\tilde{d}}(\gamma) > c_0$ (cf., for example, [**16**, Theorem 1.6.11]). In particular, it follows that

$$\mathrm{length}_{d_{FS}}(\pi(\gamma)) = \deg(\pi)^{-1}\,\mathrm{length}_{\tilde{d}}(\gamma) > c_0 \deg(\pi)^{-1}. \tag{2.2}$$

By Lemma 5.5, there exists $r_0 > 0$ such that for every $s \geq 2$, we can cover the Riemann sphere $\mathbb{CP}^1$ in a certain regular manner by s discs of d_{FS}-radius $r_0 s^{-1/2}$. Let $Z \subset \mathbb{CP}^1$ be the set containing the centres of these discs. For each point $z \in Z$, let $z' \in \mathbb{CP}^1$ be any point on the equator relative to z as a pole. Consider the stereographic projection P_w from the opposite pole $w \in \mathbb{CP}^1$ of z to the complex plane such that $P_w(z) = 0$ and $P_w(z') = 1$. Then P_w induces a biholomorphic isometry with respect to the induced Fubini-Study metric d_{FS}:

$$\tilde{P}_w\colon \mathbb{CP}^1 \setminus \{w, z, z'\} \to \Omega = \mathbb{CP}^1 \setminus \{0, 1, \infty\}. \tag{2.3}$$

Define $T = \{w, z, z'\colon z \in Z\}$ and $S = \pi^{-1}Z \subset B$. Then $\pi(B_0 \setminus S) \subset \mathbb{CP}^1 \setminus T$ and:

$$\#S \sim 3\deg(\pi)s. \tag{2.4}$$

For every $x \in \mathbb{CP}^1 \setminus T$, we can find by construction some $z \in Z$ such that $d_{FS}(x, z) < r_0 s^{-1/2}$. By Theorem 5.4 applied to $x \in \mathbb{CP}^1 \setminus \{w, z, z'\} \simeq \Omega$, we deduce that for all s large enough so that $r_0 s^{-1/2} < \delta$ and for every unit vector $v \in (T_1\mathbb{CP}^1)_x$, we have:

$$\begin{aligned} \lambda_{\mathbb{CP}^1\setminus T}(x, v) &\geq \lambda_{\mathbb{CP}^1\setminus\{w,z,z'\}}(x, v) && \text{(by Lemma 2.39)} \\ &\gtrsim d_{FS}(x, z)^{-1}(\log d_{FS}(x, z))^{-1} && \text{(by (2.3) and Theorem 5.4)} \\ &\gtrsim s^{1/2}(\log(s))^{-1}. && \text{(as } d_{FS}(x, z) < r_0 s^{-1/2}) \end{aligned} \tag{2.5}$$

Now let $\gamma \subset B_0 \backslash S$ be any piecewise smooth loop representing α and is parametrized by a map $f\colon [0,1] \to B_0 \setminus S$. We find that:

$$
\begin{aligned}
\operatorname{length}_{B_0\backslash S}(\gamma) &= \int_0^1 \lambda_{B\backslash S}\left(f(t), f'(t)\right) dt \\
&\geq \int_0^1 \lambda_{\mathbb{CP}^1\backslash T}\left(\pi \circ f(t), (\pi \circ f)'(t)\right) dt && \text{(by Lemma 2.39)} \\
&\gtrsim \int_0^1 s^{1/2}(\log(s))^{-1} |(\pi \circ f)'(t)|_{d_{FS}} dt && \text{(by (2.5))} \\
&= s^{1/2}(\log(s))^{-1} \operatorname{length}_{d_{FS}}(\pi(\gamma)) \\
&\gtrsim \#S^{1/2}(\log(\#S+2))^{-1}. && \text{(by (2.2) and (2.4))}
\end{aligned}
$$

□

Remark 5.6. The lower asymptotic polynomial growth $s^{1/2}$ in Theorem E is in fact optimal. Indeed, assume that $B = \mathbb{CP}^1$. Then in the above proof, we can use directly the bi-liptschitz metric equivalence (2.1) in Theorem 5.4 instead of the inequality (2.5). Therefore, for the choice given in the above proof of certain uniform distribution of the s points on $\mathbb{CP}^1$, the asymptotic polynomial growth $s^{1/2}$ of the function $L(\alpha, s)$ is attained.

3. A weak form of Theorem D and a discussion

In this section, we shall give a short proof of the following weak form of Theorem D to present some of the ideas as well as the difficulties that we have to overcome in the proof of Theorem D which will occupy the rest of the chapter.

Lemma 5.7. *Let $s \in \mathbb{N}$. Let B be a compact Riemann surface. Let $U \subset B$ be a finite disjoint union of closed discs. Let $b_0 \in B_0 = B \setminus U$ be contained in a small open disc $U_0 \subset B_0$. Let $\alpha_1, \dots, \alpha_m \in \pi_1(B_0, b_0)$ be a set of generators. There exists $\delta > 0$ such that for any subset $S \subset B_0 \setminus U_0$ of cardinality at most s, there exists loops $\gamma_1, \dots, \gamma_m \subset B_0 \setminus S$ representing $\alpha_1, \dots, \alpha_m$ respectively with $\operatorname{length}_{d_{B_0\backslash S}}(\gamma_i) < \delta$ for every $i = 1, \dots, m$.*

We notice that this weak form of Theorem D is already sufficient for a certain weaker finiteness result not concerning the growth of integral points (cf. Remark 6.10) as shown in Theorem F, Corollary B, and Theorem H except for the quantitative bounds given in (4.1) and (3.16).

Another proof of Lemma 5.7 is also presented in Section 9 which might be generalized to higher dimension varieties B.

First proof of Lemma 5.7. Let $B_* = (B_0 \cup \partial U) \setminus U_0$ be a compact subspace of B. It is clear that the product B_*^s parametrizes all subsets $S \subset B_*$ such that $\#S \leq s$ by assigning to each point $x = (b_1, \dots, b_s) \in B_*^s$ the support $S_x = \{b_1, \dots, b_s\}$. We claim that for each $x = (b_1, \dots, b_s) \in B_*^s$, there exists

(i) small analytic open discs $U_{x,1}, \cdots, U_{x,s} \subset B_*$ containing respectively $b_1, \cdots, b_s$ such that $U_{x,i} \cap U_{x,j} = \varnothing$ if $b_i \neq b_j$ and $U_{x,i} = U_{x,j}$ if $b_i = b_j$;

(ii) a set of loops $\gamma_{x,1}, \cdots, \gamma_{x,m} \subset B_0 \setminus \cup_{i=1}^s U_{x,i}$ based at b_0 which represent respectively $\alpha_1, \dots, \alpha_m$;

This will imply Lemma 5.7 as follows. Let $U_x = \prod_{i=1}^s U_{x,i} \subset B^s$ then U_x is an analytic open neighborhood of x in B_*^s. Since B_*^s is compact, we can find a finite cover $B_*^s = \cup_{x \in I} U_x$ where $I \subset B_*$ is a finite subset. Hence, we can define:

$$\delta := \max_{x \in I} \max_{1 \leq i \leq s} \operatorname{length}_{d_{B_0 \setminus \cup_{i=1}^s U_{x,i}}}(\gamma_{x,i}) > 0. \tag{3.1}$$

Then for every finite subset S of cardinality at most s, we can find $y \in B_*^s$ and $x \in I$ such that $S_y = S$ and $y \in U_x$. Since $S \subset \cup_{i=1}^s U_{x,i}$, we find $B_0 \setminus \cup_{i=1}^s U_{x,i} \subset B_0 \setminus S$. We can thus conclude from (3.1) and the distance-decreasing property of the pseudo hyperbolic metric:

$$\operatorname{length}_{d_{B_0 \setminus S}}(\gamma_{x,i}) \leq \operatorname{length}_{d_{B_0 \setminus \cup_{i=1}^s U_{x,i}}}(\gamma_{x,i}) \leq \delta.$$

To prove the claim, let $x = (b_1, \cdots, b_s) \in B_*^s$ and choose arbitrary loops $\gamma_1, \cdots, \gamma_m \subset B_0$ based at b_0 representing respectively the classes $\alpha_1, \cdots, \alpha_m$. Since $b_0 \notin S_x = \{b_1, \cdots, b_s\}$, we can obviously deform slightly these loops so that $S_x \cap \gamma_j = \varnothing$ for $j = 1, \cdots, m$. The claim follows immediately by taking small enough open discs around each point of S_x. ☐

As it may be helpful to better understand the proof of Theorem D, we mention some technical difficulties that we must tackle:

Remark 5.8. Given Lemma 5.7 and its proof above, we must carry out a careful analysis to overcome the following problems:

(1) to better appreciate the steps in our proof and the result of Theorem D, it would be helpful to keep in mind the configurations where points in the set S are "evenly distributed" in B_0 and/or they can accumulate at any point in B_0;

(2) when S *varies*, even when of bounded cardinality, the hyperbolic metric on the spaces $B_0 \setminus S$ are not at all the same nor induced by a *single* given metric on B_0; we notice that even the analysis of the hyperbolic metric on the punctured complex plan $\mathbb{C} \setminus \{a_1, \dots, a_s\}$ ($s \geq 2$) is already a nontrivial research area in the literature (cf., for instance, **[75]**, **[12]**, **[109]** and the references therein);

(3) even for the finiteness part of Theorem D, we have to get rid of the open set U_0 containing b_0 in the statement of Lemma 5.7. A compactness argument

with a finite open cover of B_0 by the open sets (U_0, b_0) is not enough with the above proof, as easily seen when S is "evenly distributed" and becomes denser in B_0;

(4) the base point b_0 should not be fixed nor belong to a finite subset since otherwise, an accumulation of many points of the set S near b_0 would increase to infinity the hyperbolic length of loops based at b_0;

(5) a certain construction on loops need to be carried out to obtain a bound which is *linear* in $\#S$ but independent of the choice of S in B_0.

(6) to obtain a construction as above, one should consider only certain classes of "nice" loops and avoid pathological loops such as Peano curves;

To deal with the points (5) and (6), we shall describe a detailed *algorithm* on *simple* loops called Procedure $(*)$ given in Lemma 5.18 (see also Lemma 5.21 for the global case independent of the base points).

4. Simple base of loops

Definition 5.9. Let $\gamma\colon [0,1] \to X$ be a non-nullhomotopic closed loop on an orientable surface X. The loop γ is called *simple* if γ is injective on $[0,1[$. The loop γ is called *primitive* if its homotopy class $[\gamma] \in \pi_1(X, \gamma(0))$ is *primitive*, i.e., $[\gamma] \neq 0$ and there does not exist $n \geq 2$ and $\alpha \in \pi_1(X, \gamma(0))$ such that $[\gamma] = \alpha^n$.

It is well-known that every non-nullhomotopic simple closed curve in an orientable surface is primitive (cf. [**38**, Proposition 1.4]). The converse is false in general. Nevertheless, we remark the following elementary property:

Lemma 5.10. *Let X be an orientable connected compact surface possibly with boundary. Then for every $x_0 \in X$, $\pi_1(X, x_0)$ admits a canonical system of primitive generators $\alpha_1, \cdots, \alpha_k$ such that each α_i admits a representative simple piecewise smooth closed loop $\gamma_i\colon [0,1] \to X \setminus \partial X$ based at x_0. The classes $\alpha_1, \cdots, \alpha_k$ can be chosen to be unique up to conjugation and independently of x_0.*

Proof. By classification of surfaces, X is a compact orientable surface of genus g and with k boundary components, i.e., homeomorphic to $\Sigma_{g,k}$. It is clear that there exists a primitive canonical basis of $\pi_1(\Sigma_{g,k}, x_0)$ with the desired property. Using local charts and the compactness, the loops can be obviously made to be piecewise-smooth (or even smooth). $\square$

Definition 5.11 (Simple base)**.** Let (X, x_0) be an orientable connected compact surface possibly with boundary. A homotopy class $\alpha \in \pi_1(X, x_0)$ is called *simple* if it contains a simple piecewise smooth loop. Every system of simple generators

$\alpha_1, \cdots, \alpha_k$ of $\pi_1(X, x_0)$ as in Lemma 5.10 is called a *simple base* or a *simple system of generators* of $\pi_1(X, x_0)$.

The notion of simple homotopy classes and the existence of a simple base (Lemma 5.10) shall be necessary in Procedure $(*)$ described in Section 6 below.

By the remark below, we can suppose in the rest of the chapter that the classes $\alpha_1, \ldots, \alpha_k$ in the statement of Theorem D form a simple system of generators.

Remark 5.12. It suffices to prove Theorem D for a simple base $\beta_1, \ldots, \beta_m$ of $\pi_1(B_0, b_0) = \pi_1(B_0 \cup \partial B_0, b_0)$. Indeed, let $\alpha_1, \ldots, \alpha_k \in \pi_1(B_0, b_0)$ then each α_i can be written as a finite product of the β_j's. Hence, by taking a suitable finite multiple of the constant L given by Theorem D applied for the simple base $\beta_1, \ldots, \beta_m$, we obtain a constant that satisfies Theorem D for the classes $\alpha_1, \ldots, \alpha_k$.

Remark 5.13. We mention here a useful elementary property of a simple piecewise smooth loop γ in a compact Riemannian surface (X, d) with $\gamma \cap \partial X = \varnothing$. Remark that γ only has a finite number of singular points (which are thus isolated). Hence, for every $x \in \gamma$ and $\varepsilon > 0$ smaller than the injectivity radius of X at x, we can find an arbitrarily small contractible closed region $\Delta \subset V(x, \varepsilon)$ where $V(x, \varepsilon)$ is the closed disc of d-radius ε, such that $x \in \Delta$ and

(a) $\partial\Delta$ is a non self-intersecting smooth loop homeomorphic to a circle in $V(x, \varepsilon)$;

(b) $\gamma \cap \Delta$ contains only one connected branch of γ.

5. Preliminaries

To recall the notations, we denote $\Delta(x, r) \subset \mathbb{C}$ the open complex disc centered at a point $x \in \mathbb{C}$ and of radius $r > 0$. For a complex space X, the infinitesimal Kobayashi-Royden pseudo metric λ_X on X corresponding to the Kobayashi pseudo hyperbolic metric d_X can be defined as follows. For $x \in X$ and every vector v in the tangent cone of X at x,

$$\lambda_X(x, v) = \inf 2/R$$

where the minimum is taken over all $R > 0$ for which there exists a holomorphic map $f\colon \Delta(0, R) \to X$ such that $f'(0) = v$. Remark that when $x \in X$ is regular, the tangent cone of X at x is the same as the tangent space T_xX.

We begin with the following simple estimation: fix a Riemannian metric d on a compact Riemann surface B. Define the *d-unit tangent space* of B by:

$$T_1B \coloneqq \{(z, v) \in TB\colon |v|_d = 1\}.$$

For every $z \in B$ and $r > 0$, let $D(z, r) = \{b \in B\colon d(b, z) < r\}$.

Lemma 5.14. *There exists a constant* $c(B,d) > 0$ *and a constant* $r(B,d) > 0$ *such that for every* $(z,v) \in T_1B$ *and* $0 < r < r(B,d)$*, we have*

$$\lambda_{D(z,r)}(z,v) \leq c(B,d)/r.$$

Proof. As B is a compact Riemann surface, it is clear that there exists a finite subset $I \subset B$ and $r_0 > 0$ such that $B = \cup_{t\in I} f_t(\Delta(0,r_0))$ and every $t \in I$ admits an open neighborhood $U_t \subset B$ and a biholomorphism $f_t\colon \Delta(0,3r_0) \to U_t$ satisfying $f_t(0) = t$, and

$$1 \leq |f_t'(x)|_d \quad \text{for all } x \in \Delta(0,2r_0). \tag{5.1}$$

Since I is finite, there exists $a > 1$ such that for every $t \in I$, the pullback metric $d_t = (f_t^* d)|_{\Delta(0,2r_0)}$ and the Euclidean metric λ_{euc} on $\Delta(0,2r_0) \subset \mathbb{C}$ are bi-Lipschitz:

$$a^{-1} d_t \leq \lambda_{euc} \leq a d_t. \tag{5.2}$$

Now, let $(z,v) \in T_1B$ and $0 < r < r_0$. Then there exists $t \in I$ such that $z \in f_t(\Delta(0,r_0))$, says, $z = f_t(x)$ for some $x \in \Delta(0,r_0)$. As $r < r_0$ and $a > 1$, we have $\Delta(x,a^{-1}r) \subset \Delta(0,2r_0)$. Then (5.2) implies that

$$f_t(\Delta(x,a^{-1}r)) \subset D(f_t(x),r) \cap f_t(\Delta(0,2r_0)) \subset D(z,r).$$

Since $|v|_d = 1$ and $|f_t'(x)|_d \geq 1$ by (5.1), $|v/f_t'(x)|_{euc} \leq a$ by (5.2). Thus, we have a holomorphic map $h\colon \Delta(0,a^{-2}r) \to D(z,r)$ given by $h(y) = f_t\left(\frac{v}{f_t'(x)} y + x\right)$.

As $h(0) = z$ and $h'(0) = v$, we deduce that $\lambda_{D(z,r)}(z,v) \leq 2a^2/r$. The conclusion follows by setting $c(B,d) = 2a^2$ and $r(B,d) = r_0$. $\square$

Let the notations and constants $c(B,d)$, $r(B,d) > 0$ be as in Lemma 5.14 above. For every $r > 0$ and every subset $\Omega \subset B$, we define

$$D(\Omega,r) = \{b \in B \colon d(b,\Omega) < r\} \subset B.$$

Using the distance-decreasing property of the Kobayashi pseudo hyperbolic metric, we obtain the following consequence:

Corollary 5.15. *Let* $\gamma \subset B$ *be a piecewise smooth closed curve. For every* $0 < r < r(B,d)$*, we have:*

$$\mathrm{length}_{d_{D(\gamma,r)}}(\gamma) \leq \frac{c(B,d)}{r} \mathrm{length}_d(\gamma).$$

Proof. Let γ be parametrized by a map $f\colon [0,1] \to B$. For every $0 < r < r_0(B,d)$, we find that:

$$
\begin{aligned}
\mathrm{length}_{d_{D(\gamma,r)}}(\gamma) &\le \int_0^1 \lambda_{D(\gamma,r)}\left(f(t), f'(t)\right) dt \\
&\le \int_0^1 \lambda_{D(f(t),r)}\left(f(t), f'(t)\right) dt && (\text{as } D(f(t),r) \subset D(\gamma,r)) \\
&= \int_0^1 \lambda_{D(f(t),r)}\left(f(t), \frac{f'(t)}{|f'(t)|_d}\right) |f'(t)|_d dt \\
&\le \frac{c(B,d)}{r} \int_0^1 |f'(t)|_d dt && (\text{by Lemma 5.14}) \\
&= \frac{c(B,d)}{r}\, \mathrm{length}_d(\gamma).
\end{aligned}
$$

□

Lemma 5.16. *Let (M,d) be a compact Riemannian surface possibly with boundary. Then there exist constants $r_0 = r_0(M,d) > 0$, $c_0 = c_0(M,d) > 0$ such that for every disc $D(x,r)$ of d-radius $r \le r_0$ with $x \in M$, we have:*

$$\mathrm{length}_d\, \partial D(x,r) \le c_0 r, \qquad \mathrm{vol}_d(D(x,r)) \le c_0 r^2.$$

Proof. By the compactness of M, we see that the Gaussian curvature K with respect to the metric d is bounded on M. Therefore, we can define

$$0 < \mu := \max_{x\in M} |K(x)| + 1 \in \mathbb{R}.$$

On the other hand, the Bertrand-Diguet-Puiseux theorem (cf. [**107**]) tells us that for every $x \in M$:

$$
\begin{aligned}
-\mu + 1 \le K(x) &= \lim_{r\to 0^+} \frac{3}{\pi r^3}(2\pi r - \mathrm{length}_d\, \partial D(x,r)) \\
&= \lim_{r\to 0^+} \frac{12}{\pi r^4}(\pi r^2 - \mathrm{vol}_d\, D(x,r)) \le \mu - 1.
\end{aligned}
$$

Hence, for each $x \in M$, there exists $r_x > 0$ such that for every $0 < r \le r_x$, we have:

$$
\begin{aligned}
2\pi r - \frac{\mu\pi r^3}{3} &\le \mathrm{length}_d\, \partial D(x,r) \le 2\pi r + \frac{\mu\pi r^3}{3}, \text{ and} \\
\pi r^2 - \frac{\mu\pi r^4}{12} &\le \mathrm{vol}_d(D(x,r)) \le \pi r^2 + \frac{\mu\pi r^4}{12}.
\end{aligned}
$$

Consider the open covering $M = \cup_{x\in M} D(x,r_x)$, we obtain a finite set $I \subset M$ and a finite subcovering $M = \cup_{x\in I} D(x,r_x)$ by the compactness of M. Let $r_0 =$

$\min_{x\in I} r_x > 0$ then for all $x \in M$ and for all $0 < r \le r_0$, we find that

$$\mathrm{length}_d\, \partial D(x,r) \le 2\pi r + \frac{\mu\pi r^3}{3}, \qquad \mathrm{vol}_d(D(x,r)) \le \pi r^2 + \frac{\mu\pi r^4}{12}$$

and the conclusion follows immediately by setting $c_0 = \pi(2+\mu r_0^2)/3$. □

6. Procedure $(*)$ for simple loops

Given a subset Ω of a metric space (X,d) and let $r \ge 0$, we recall the notations:

$$V(\Omega,r) = \{x \in X \colon d(x,\Omega) \le r\}, \quad D(\Omega,r) = \{x \in X \colon d(x,\Omega) < r\}.$$

Now let (B,d) be a compact Riemannian surface with boundary. Let $\gamma \subset B \setminus \partial B$ be *simple* piecewise smooth loop with base point $b_0 \in B\setminus\partial B$. Denote by $\mathrm{rad}(\gamma,d)$ the infimum of the injectivity radii in $(B\setminus\partial B,d)$ at all points $x\in\gamma$. Since γ is compact, it is clear that $\mathrm{rad}(\gamma,d)>0$. We define

$$L = L(B,d,\gamma) \coloneqq \min(d(\gamma,\partial B), \mathrm{rad}(\gamma,d)) > 0.$$

We shall consider $a>0$ small enough (depending only on γ, B, d) such that:

(P1) $4a < L(B,d,\gamma)$.

We require furthermore that a satisfies:

(P2) every point $x\in\gamma$ admits a simply connected open neighborhood U_x in $B\setminus\partial B$ such that $V(x,2a)\subset U_x$ and that the restriction $\gamma\cap U_x$ contains exactly one connected branch of γ.

Lemma 5.17. *There exists $a>0$ depending only on γ, B, d which satisfies* (P1) *and* (P2).

Proof. Suppose on the contrary that for every $n\ge 5$, there exists $x_n\in\gamma$ and for all simply connected open neighborhood U_n of x_n in B with $V(x,2L/n)\subset U_n\subset V(x_n,L/2)$, the restriction $\gamma\cap U_n$ has more than two connected components. By the compactness of γ, we can suppose, up to passing to a subsequence, that for some $x\in\gamma$, we have $x_n\to x$ as $n\to\infty$. Remark that since γ is piecewise smooth, it has only a finite number of singular points. Therefore, x is either a smooth point of γ or an isolated singular point of $\gamma.As\gamma$ is also compact, it can be seen easily in either case, for example using a local chart at x, that there exists a simply connected open neighborhood $U\subset V(x,L/3)$ of x such that $\gamma\cap U$ has exactly one connected component (cf. Remark 5.13). We can choose $r>0$ small enough such that $V(x,r)\subset U$. Then since $x_n\to x$ and $2L/n\to 0$, we have by the triangle inequality that for all $n \gg 1$:

$$V(x_n, 2L/n) \subset V(x, r) \subset U \subset V(x, L/3) \subset V(x_n, L/2).$$

Thus, we obtain a contradiction to the hypothesis on the x_n's for all $n \gg 1$ and the conclusion follows. □

Let $c_0(B, d) > 0$ be given by Lemma 5.16. We show that:

Lemma 5.18 (Procedure $(*)$)**.** *Suppose that $a > 0$ satisfies* (P1) *and* (P2). *For every finite subset $S \subset B$ of cardinality $s > 0$ such that $d(b_0, S) = \min_{x \in S} d(b, x) > a/s$, there exists a piecewise smooth loop $\gamma' \subset B \setminus D(S \cup \partial B, a/s)$ based at b_0 of the same homotopy class in $\pi_1(B, b_0)$ as γ and satisfies*

$$\text{length}_d(\gamma') \leq \text{length}_d(\gamma) + c_0(B, d)a.$$

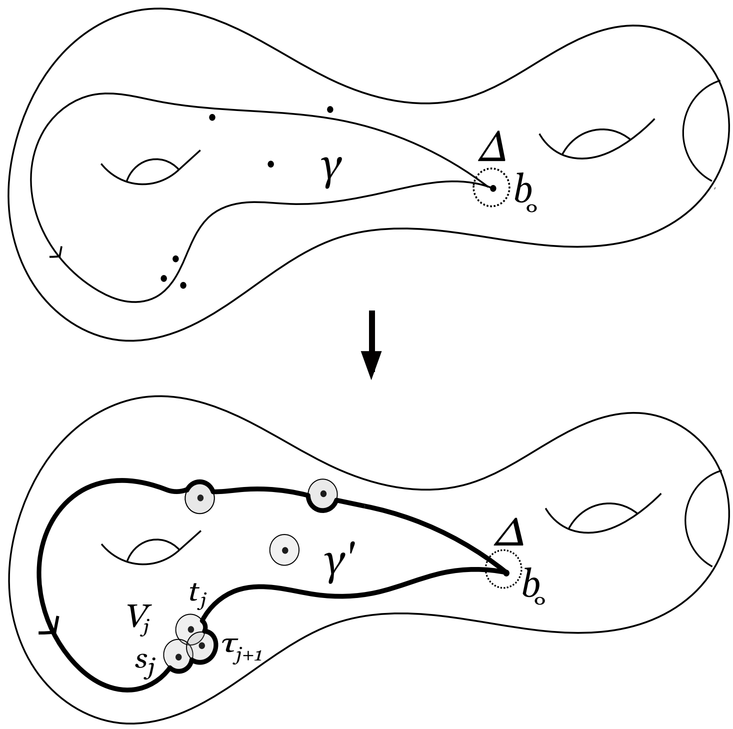

FIGURE 5.1. Procedure $(*)$ applied to γ (with $\Delta = V(b_0, a/s)$)

PROOF. We can write the union of discs $V(S, a/s) = \cup_{x \in S} V(x, a/s) \subset B$ as a disjoint union of connected components $V_1, \cdots, V_m$. Remark that the V_j's are also path connected. For every $j \in \{1, \ldots, m\}$, let n_j be the number of elements of S

in V_j then

$$\sum_j n_j \leq \#S = s. \tag{6.1}$$

By the triangle inequality, we also find that $\mathrm{diam}_d(V_j) \leq n_j 2a/s \leq s \times 2a/s = 2a$.

Consider the following m-step algorithm. Define $\gamma_0 := \gamma$. Suppose that we are given the curve $\gamma_j \subset B$ where $j \in \{0, \ldots, m-1\}$. Denote by $s_j, t_j \in \gamma_j$ the first and the last points, if they exist, on the intersection between the directed curve γ_j and V_{j+1}. If there is no such points, we set $\gamma_{j+1} = \gamma_j$ and continue the algorithm. Otherwise, we replace the directed part of γ_j between s_j and t_j by any directed simple curve τ_{j+1} lying on the boundary of V_j which connects s_j and t_j (cf. Figure 5.1). This is possible since V_j is path connected. Define γ_{j+1} the resulting curve and continue until we reach γ_m.

As $\min_{x \in S} d(b, x) > a/s$ by hypothesis, $b \notin V_j$ for every j and thus the base point $b_0 \in \gamma$ is not modified at any step. The boundary of each V_j is piecewise smooth since each $\partial V(x, a/s)$, $x \in S$, is smooth by the property of the injectivity radius as $a/s \leq L/4$ by (P1). It follows that $\gamma_1, \ldots, \gamma_m$ are piecewise smooth loops in B based at b_0. Moreover, it is evident that for every $0 \leq j \leq m-1$, we have:

$$\mathrm{length}_d(\gamma_{j+1}) \leq \mathrm{length}_d(\gamma_j) + \mathrm{length}_d(\tau_{j+1}). \tag{6.2}$$

Since $V_1, \ldots, V_m$ are disjoint, a direct induction shows that $s_j, t_j \in \gamma \cap \gamma_j$ and each of the curves $\tau_1, \ldots, \tau_j$ (whenever they are defined) is either non modified or does not appear in γ_{j+1} at the j-th step for every $0 \leq j \leq m-1$. Hence, γ_m contains some pairwise disjoint curves $\tau_{i_1}, \ldots, \tau_{i_k}$ and γ_m coincides with γ outside the directed paths $\sigma_{i_p} \subset \gamma$ between s_{i_p-1} and t_{i_p-1} where $p = 1, \ldots, k$.

We claim that there exists a homotopy in B with base point b_0 between γ_m and γ. Let $p \in \{1, \ldots, k\}$. Since $\mathrm{diam}_d(V_{i_p}) \leq 2a$, we have $\tau_{i_p} \subset V_{i_p} \subset V(s_{i_p-1}, 2a)$. By Property (P2) applied to $s_{i_p-1} \in \gamma$, there exists a simply connected open neighborhood U_p of s_{i_p-1} in B such that $V(s_{i_p-1}, 2a) \subset U_p$ and that $\gamma \cap U_p$ contains exactly one connected branch of γ. Remark that the curve τ_{i_p} is entirely contained in $V(s_{i_p-1}, 2a) \subset U_p$ with end points $s_{i_p-1}, t_{i_p-1} \in \gamma \cap U$. As $\gamma \cap U_p$ contains exactly one connected branch of γ and U_p is simply connected, replacing σ_{i_p} (see the above paragraph) by τ_{i_p} for every $p = 1, \ldots, k$ will not change the homotopy class of γ. As the resulting loop is γ_m, the claim is proved.

Since $\mathrm{diam} V(S, a/s) \leq s \times 2a/s = 2a$ and $4a < d(\gamma, \partial B)$ by Property (P1), it follows that $d(\gamma_m, \partial B) > 2a$. Thus by construction, $d(\gamma_m, S \cup \partial B) \geq a/s$ since $s \geq 1$. To summarize, $\gamma' = \gamma_m$ is a piecewise smooth loop based at b_0 homotopic to γ and $\gamma' \subset B \setminus D(S \cup \partial B, a/s)$.

Let $c_0 = c_0(B, d) > 0$ be given by Lemma 5.16 then we have by Lemma *loc. cit.* that

$$\text{length}_d(\tau_j) \leq \sum_{x \in S \cap V_j} \text{length}_d \, \partial V(x, a/s) \leq n_j \times c_0 a/s. \tag{6.3}$$

It follows that we have:

$$\begin{aligned} \text{length}_d(\gamma') &\leq \text{length}_d(\gamma) + \sum_j \text{length}_d(\tau_j) && \text{(by (6.2))} \\ &\leq \text{length}_d(\gamma) + \sum_j n_j \times c_0 a/s && \text{(by (6.3))} \\ &\leq \text{length}_d(\gamma) + s \times c_0 a/s && \text{(by (6.1))} \\ &= \text{length}_d(\gamma) + c_0 a. \end{aligned} \tag{6.4}$$

□

Remark 5.19. The hypothesis saying that γ is simple is necessary in Lemma 5.18. For example, when we replace γ by the loop $n\gamma$ ($n \geq 1$) then in Inequality 6.4, we can only obtain $\text{length}_d(\gamma') \leq \text{length}_d(\gamma) + n \sum_j \text{length}_d(\tau_j)$ and the conclusion must read

$$\text{length}_d(\gamma') \leq \text{length}_d(\gamma) + n c_0(B, d) a.$$

Similarly, Condition (P2) on a is necessary for the proof of Lemma 5.18. For example, when the connected component V_j intersects $n \geq 2$ different connected branches of γ, then Inequality 6.2 must also read $\text{length}_d(\gamma_j) \leq \text{length}_d(\gamma_{j-1}) + n \sum_j \text{length}_d(\tau_j)$.

Now, let $U \subset B \setminus \partial B$ be a simply connected open neighborhood of the base point b_0 such that the restriction $\gamma \cap U$ contains exactly one connected branch of γ. Let $\Delta \subset B \setminus \partial B$ be closed subset such that:

(a) $b_0 \subset \Delta \subset U$;

(b) for every $x \in \Delta$, there exists a simple loop $\gamma_x \subset B$ based at x such that γ_x and γ coincides as paths over $B \setminus \Delta$.

Lemma 5.20. *There exists $A > 0$ such that for every $x \in \Delta$, the constant A satisfies* (P1) *and* (P2) *for the loop γ_x in the Riemannian surface (B, d).*

Proof. Let $\Gamma = \gamma \cup \Delta$ then it is a compact subset contained in $B \setminus \partial B$. Thus, $\text{rad}(\Gamma, d)$, the infimum of the injectivity radii in $(B \setminus \partial B, d)$ at all points $x \in \Gamma$, is strictly positive. Moreover, $d(\Gamma, \partial) > 0$. Hence, $L = L(B, d, \Gamma) := \min(d(\Gamma, \partial B), \text{rad}(\Gamma, d)) > 0$.

For every $x \in \Delta$, we have $\gamma_x \subset \Gamma$ by Condition (b) above and thus $L(B, d, \gamma_x) \geq L$.

Suppose on the contrary that for every $n \geq 5$, there exists $x_n \in \Delta$, $z_n \in \gamma_x$ such that for all simply connected open neighborhood U_n of x_n in B with $V(z_n, 2L/n) \subset U_n$, the restriction $\gamma_x \cap U_n$ has more than two connected components.

By the compactness of Γ, we can suppose, up to passing to a subsequence, that $z_n \to z$ as $n \to \infty$ for some $z \in \Gamma$. We distinguish two cases. First, assume that $z \in U$. Then for $n \gg 1$, we have $z_n \in U$ and $V(z_n, 2L/n) \subset U$. This is a contradiction since $U \subset B \setminus \partial B$ is simply connected by hypothesis and $\gamma_x \cap U$ has only one connected branch of γ_x by Condition (b). Assume now that $z \in B \setminus U$. Then as in Lemma 5.17, we also find a contradiction since γ_x and γ coincides as paths over $B \setminus \Delta$ and Δ is closed and contained in U. The conclusion thus follows. □

7. Bound of the Riemannian lengths of certain loops with varying base points

We continue with the following global version of Procedure $(*)$ described in Lemma 5.18 for simple loops with arbitrary base points. In other words, we show that the constant a verifying Conditions (P1) and (P2) necessary to trigger Procedure $(*)$ can be chosen independently of the base point.

Let (B, d) be a compact Riemannianian surface without boundary. Let V be finite union of disjoint closed discs in B (with sufficiently smooth boundary ∂V). Let $U = V \setminus \partial V$. We regard (B_0, d), where $B_0 = B \setminus U$, as an Riemannian surface with boundary. Let $b_0 \in B_0$ and fix a base of *simple* generators $\alpha_1, \cdots, \alpha_k$ of $\pi_1(B_0, b_0)$ as in Lemma 5.10. The main result of the section is the following:

Lemma 5.21. *Let $\varepsilon > 0$. There exists constants $a, H > 0$ with the following property. For every $x \in B_0$ such that $d(x, \partial B_0) \geq \varepsilon$, there exists simple piecewise smooth loops $\gamma_1, \dots, \gamma_k \subset B_0$ based at x representing $\alpha_1, \dots, \alpha_k$ respectively up to a single conjugation (cf. Definition 5.1) with:*

$$\mathrm{length}_d(\gamma_i) \leq H, \quad \textit{for every } i \in \{1, \dots, k\}.$$

Moreover, for every $i \in \{1, \dots, k\}$, a verifies (P1) *and* (P2) *for the data (γ_i, B_0, d).*

Proof. For each $b \in B_0$ with $d(b, \partial B_0) \geq \varepsilon$, we can choose simple piecewise smooth loops $\gamma_{b,1}, \cdots, \gamma_{b,k} \subset B_0 \setminus \partial B_0$ based at b representing the classes $\alpha_1, \dots, \alpha_k$ respectively up to a single conjugation. We denote

$$H_b := \max_{1 \leq i \leq k} (\mathrm{length}_d \, \gamma_{b,i}) > 0.$$

By Lemma 5.17, there exists $a_b > 0$ satisfying (P1) and (P2) for the data ($\gamma_{b,i}$, B_0, d) for all $i = 1, \dots, k$. In particular, for each $i \in \{1, \dots, k\}$, b admits a simply connected neighborhood $U_{b,i} \subset B_0 \setminus \partial B_0$ such that $V(b, 2a_b) \subset U_{b,i}$ and that the restriction $\gamma \cap U_{b,i}$ contains exactly one connected branch of $\gamma_{b,i}$ (cf. Remark 5.13).

Consider a small enough closed region $\Delta_b \subset V(b, a_b) \subset B_0 \setminus \partial B_0$ containing b such that $\partial\Delta_b$ is a non self-intersecting smooth loop homeomorphic to a circle in $V(b, a_b)$ and $\gamma_{b,i} \cap \Delta_b$ contains only one connected branch of γ for every $i = 1, \ldots, k$. Let $l_b := \text{length}_d(\partial\Delta_b) > 0$.

Since the set $B' = \{b \in B_0 \colon d(x, \partial B_0) \geq \varepsilon\}$ is compact, there exists a finite subset $I \subset B'$ such that $B' \subset \cup_{b \in I} \Delta_b$.

Consider the following construction for every $x \in B'$. We can choose $b \in I$ such that $x \in \Delta_b$. For each $i \in \{1, \ldots, k\}$, let $s_{b,i}, t_{b,i} \in \gamma_{b,i} \cap \Delta_b$ be respectively the first and the last intersections of the directed curve $\gamma_{b,i}$ with the boundary $\partial\Delta_b$. Notice that $s_{b,i} \neq t_{b,i}$. Let σ be any maximal geodesic segment passing through x and contained in Δ_b. Let $s, t \in \sigma \cap \partial\Delta_b$ be the two extremal points of σ such that s and $s_{b,i}$ do not lie on distinct arcs delimited by t_{b_i} and t on $\partial\Delta_b$ (cf. Figure 5.2). The two directed geodesic segments δ_s from x to s and δ_t from t to x do not intersect except at x. We replace the restriction $\gamma_{b,i} \cap \Delta_b$ by the union of the directed arc $T_i \subset \partial\Delta_b$ from $t_{b,i}$ to t not containing $s_{b,i}$, the paths δ_t, δ_s and the directed arc $S_i \subset \partial\Delta_b$ from s to $s_{b,i}$ not containing $t_{b,i}$. Denote the resulting loop by $\gamma_{x,i}$ with base point x.

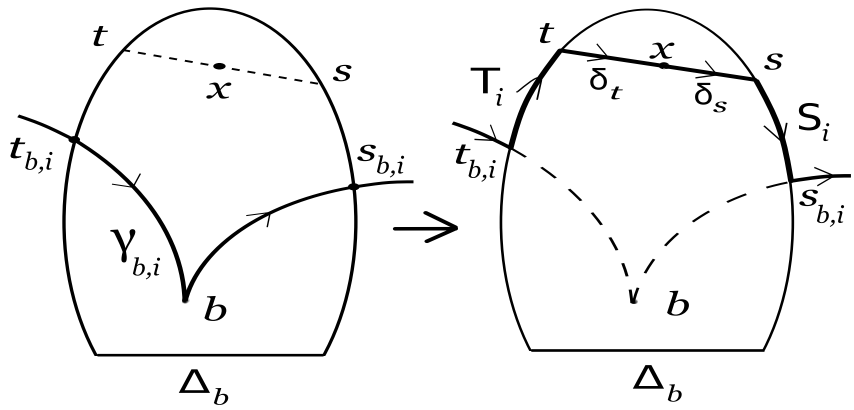

FIGURE 5.2. Local modification of $\gamma_{b,i} \cap \Delta_b$

It is clear by construction that the loop $\gamma_{x,i}$ is also simple and piecewise smooth. Moreover, the restrictions of $\gamma_{b,i}$ and $\gamma_{x,i}$ to $B_0 \setminus \Delta_b$ coincide. As $\Delta_b \subset U_{b,i}$ and $U_{b,i}$ is simply connected and contains only one branch of $\gamma_{b,i}$, the loops $\gamma_{b,i}$ and $\gamma_{x,i}$ are homotopic up to a conjugation induced by the change of base points. Setting

$$H := \max_{b \in I}(H_b + l_b + 2a_b) > 0,$$

we find that:

$$
\begin{aligned}
\operatorname{length}_d(\gamma_{x,i}) &= \operatorname{length}_d(\gamma_{x,i}|_{B\setminus\Delta_b}) \\
&\quad + \operatorname{length}_d(T_i) + \operatorname{length}_d(S_i) + \operatorname{length}_d(\delta_t) + \operatorname{length}_d(\delta_s) \\
&\leq \operatorname{length}_d(\gamma_{b,i}) + \operatorname{length}_d(\partial\Delta_b) + \operatorname{length}_d(\sigma) \\
&\leq H_b + l_b + 2a_b \qquad (\text{As } \sigma \subset \Delta_b \subset V(b, a_b)) \\
&\leq H. \qquad (7.1)
\end{aligned}
$$

By construction, $\Delta_b \subset V(b, a_b) \subset U_{b,i}$ for every $b \in I$ and $i \in \{1, \ldots, k\}$. Therefore, Lemma 5.20 applied to $\gamma_{b,i}$, b, U_b, and Δ_b implies that there exists $A_{b,i} > 0$ such that for every $x \in \Delta_b$, the constant $A_{b,i}$ satisfies Properties (P1) and (P2) for the loop $\gamma_{x,i}$ in the Riemannian surface (B_0, d). As I is finite, we can define:

$$a = \min_{b\in I} \min_{1\leq i\leq k} A_{b,i} > 0.$$

It is clear that $H, a > 0$ are independent of $x \in B'$ and verify the desired properties. □

8. Proof of the main result

We are now in position to prove the main Theorem D on the linear bound of hyperbolic length of geodesics representing fixed simple homotopy classes.

Let U be finite union of disjoint closed discs in a compact Riemann surface B. Let $b_1 \in B_1 := B \setminus U$. Fix a base of simple generators $\alpha_1, \ldots, \alpha_k$ of $\pi_1(B_1, b_1)$ as in Lemma 5.10 (see also Remark 5.12). Let d be a Riemannian metric on B. For every subset $\Omega \subset B$ and $R > 0$, recall the following notations:

$$V(\Omega, R) = \{x \in B \colon d(x, \Omega) \leq R\}, \quad D(\Omega, R) = \{x \in B \colon d(x, \Omega) < R\}.$$

We regard $B_0 = B_1 \cup \partial U$ as a compact Riemannian surface with boundary ∂U equipped with the induced Riemannian metric also denoted d. A slightly stronger result than Theorem D will be proved. In fact, we shall establish moreover the existence of certain "rings" in $B_1 \setminus S$ of width proportional to $(\#S)^{-1}$ around the desired loops.

Theorem 5.22. *Let the notations be as above. There exists $A > 0$ and $L > 0$ satisfying the following properties. For any finite subset $S \subset B_1$, there exists $b \in B_1 \setminus S$ and piecewise smooth loops $\gamma_i \subset B_1 \setminus S$ based at b representing α_i up to a single conjugation such that $V(\gamma_i, A(\#S)^{-1}) \subset B_1 \setminus S$ for $i = 1, \ldots, k$. Moreover,*

$$\operatorname{length}_{d_{B_1\setminus S}}(\gamma_i) \leq L(\#S + 1). \qquad (8.1)$$

PROOF OF THEOREM 5.22. Fix a number $\varepsilon > 0$ small enough so that we have $\mathrm{vol}_d(B_\varepsilon) > 0$ where $B_\varepsilon = \{b \in B_0 \colon d(x, \partial B_0) \geq \varepsilon\} \subset B_1$.

Let $a, H > 0$ be the constants given by Lemma 5.21 applied to the Riemannian surface (B_0, d), to $\varepsilon > 0$ and to the base $\alpha_1, \ldots, \alpha_k$ of $\pi_1(B_0, b_1) = \pi_1(B_1, b_1)$.

Let $c_0 = c_0(B_0, d) > 0$ and $r_0 = r_0(B_0, d)$ be the constant given in Lemma 5.16 applied to (B_0, d). Let $c = c(B, d) > 0$ and $r = r(B, d) > 0$ be given in Lemma 5.14 applied to (B, d). Define:

$$A := \frac{1}{2} \min \left(a, r, r_0, \sqrt{\frac{\mathrm{vol}_d(B_\varepsilon)}{4c_0}} \right) > 0, \qquad L := c \left(\frac{H}{A} + c_0 \right) > 0. \tag{8.2}$$

We claim that L satisfies the conclusion of Theorem D. Indeed, let $S \subset B_0$ be a finite subset of cardinality $s \geq 1$. We find that:

$$\begin{aligned}
\mathrm{vol}_d V(S, 2A/s) &\leq \sum_{x \in S} \mathrm{vol}_d(V(x, 2A/s)) && (\text{as } V(S, 2A/s) = \cup_{x \in S} V(x, 2A/s)) \\
&\leq s \times c_0 \times (2A/s)^2 = 4c_0 A^2/s && (\text{by Lemma 5.16 and } 2A/s \leq r_0 \text{ by (8.2)}) \\
&\leq 4c_0 A^2 \leq \frac{\mathrm{vol}_d(B_\varepsilon)}{4} < \mathrm{vol}_d(B_\varepsilon). && (\text{since } s \geq 1 \text{ and by (8.2)})
\end{aligned}$$

It follows that there exists $b \in B_\varepsilon \subset B_1$ such that $b \cap V(S, 2a/s) = \varnothing$, i.e., $d(b, S) \geq 2a/s$. Therefore, Lemma 5.21 implies that there exists simple piecewise smooth loops $\sigma_1, \ldots, \sigma_k \subset B_1$ based at b representing $\alpha_1, \ldots, \alpha_k$ respectively up to a single conjugation with:

$$\mathrm{length}_d(\sigma_i) \leq H, \quad \text{for every } i \in \{1, \ldots, k\}. \tag{8.3}$$

Moreover, as $0 < A < a$ by (8.2), we also have by Lemma 5.21 that A verifies (P1) and (P2) for the data (σ_i, B_0, d) for every $i \in \{1, \ldots, k\}$.

Since $d(b, S) = \min_{x \in S} d(b, x) \geq 2a/s > A/s$, we can thus apply Lemma 5.18 to the loops $\sigma_1, \ldots, \sigma_k$ to obtain piecewise smooth loops $\gamma_1, \ldots, \gamma_k \subset B_0 \setminus V(S \cup \partial B_0, A/s)$ based at b of the same homotopy classes as $\sigma_1, \ldots, \sigma_k$ in $\pi_1(B_0, b_1)$ and satisfy:

$$\mathrm{length}_d(\gamma_i) \leq \mathrm{length}_d(\sigma_i) + c_0 A, \quad i = 1, \ldots, k. \tag{8.4}$$

In particular, for every $i \in \{1, \ldots, k\}$, we have $V(\gamma_i, A/s) \subset B_1 \setminus S$. It follows from the distance-decreasing property of the hyperbolic metric that:

$$
\begin{aligned}
\operatorname{length}_{d_{B_1\setminus S}}(\gamma_i) &\leq \operatorname{length}_{d_{V(\gamma_i, A/s)}}(\gamma_i) && (\text{as } D(\gamma_i, A/s) \subset B_1 \setminus S)\\
&\leq \frac{c(B,d)}{A/s} \operatorname{length}_d(\gamma_i) && (\text{by Corollary 5.15 and } A/s \leq A < r \text{ by (8.2)})\\
&\leq \frac{cs}{A}(\operatorname{length}_d(\sigma_i) + c_0 A) && (\text{by (8.4)})\\
&\leq \frac{cs}{A}(H + c_0 A) && (\text{by (8.3)})\\
&= Ls \leq L(\#S+1). && (\text{by definition of } L \text{ in (8.2)})
\end{aligned}
$$

The conclusion follows since by construction, the loops $\gamma_1, \dots, \gamma_k \subset B_1 \setminus S$ represent the homotopy classes $\alpha_1, \dots, \alpha_k$ respectively by a (single) conjugation. $\square$

A slight modification of the above proof also allows us to conclude Theorem 5.2] as follows.

Proof of Theorem 5.2. Let the notations be as above. For a given $p \in \mathbb{N}^*$, we set $R = A/(4p)$ and consider the constant $A' = A/4 < A < a$ which which still satisfies the conclusion of Lemma 5.21. Then R and the constant $4L > 0$ defined in the above proof (cf. (8.2)) will verify the conclusion of Theorem 5.2. Indeed, for every p discs Z in B each of d-radius R, the only two modifications needed in the above proof are the following. First, by the same volume argument, we can find $b \in B_\varepsilon \subset B_1$ such that $b \cap (V(S, 2a/s) \cup Z) = \varnothing$. The second change lies in the use of Lemma 5.18 in the last step: we apply the procedure $(*)$ described in the proof of Lemma 5.18 for the decomposition into connected components of the closed set $V(S, A'/s) \cup Z$ instead of the set $V(S, A'/s)$. $\square$

The following remark suggests that in our machinery of *hyperbolic-homotopic height*, the hyperbolic part given in Theorem D cannot be better than a linear function in $\#S$, at least when using strictly the presented method in this chapter. Therefore, modifications are needed to obtain a better bound. This is an ongoing project where our goal is to obtain in Theorem D an optimal bound which as close as possible to a linear function in $(\#S)^{1/2}$.

Remark 5.23. Let the notions be as in the proof of Theorem D. Suppose that we want to construct a loop $\gamma' \subset B_1 \setminus V(S, a/s^p)$ for some real number $p > 0$ and some $a > 0$ small enough such that γ' is homotopic to γ and then control its hyperbolic length in $B_1 \setminus S$. The total volume of $V(S, a/s^p)$ is of order $s/s^{2p} = s^{1-2p}$ in s. Then we must require that $p \geq 1/2$ since otherwise, Lemma 5.16 implies that $V(S, a/s^p)$ will eventually cover all the surface B for s large enough and for S in general position and thus no such loop γ' can exist. Then the the procedure $(*)$

in Lemma 5.18, if applicable, tells us that we can deform γ in the region near S into another loop $\gamma' \subset B_1 \setminus V(S, a/s^p)$ satisfying:

$$\mathrm{length}_d(\gamma') \leq H + s \times ca/s^p = H + cas^{1-p}.$$

Therefore,

$$\begin{aligned}\mathrm{length}_{d_{B_1\setminus S}}(\gamma') &\leq \mathrm{length}_{d_{B_1\setminus V(S,a/s^p)}}(\gamma') \leq \mathrm{length}_{d_{V(\gamma,a/s^p)}}(\gamma') \\ &\leq c(a/s^p)^{-1}(H + c_0 as^{1-p}) = (c/a)(Hs^p + c_0 as)\end{aligned}$$

which is at least linear in $s = \#S$ for every $p \geq 1/2$ as claimed. 5.14 and Lemma 5.18.

9. Another proof for the weak form of the main result

In this section, we shall give another proof of the following weak form of Theorem D:

Lemma 5.24. *Let $s \in \mathbb{N}$. Let B be a compact Riemann surface. Let $U \subset B$ be a finite disjoint union of closed discs. Let $b_0 \in B_0 = B \setminus U$ be contained in a small open disc $U_0 \subset B_0$. Let $\alpha_1, \ldots, \alpha_m \in \pi_1(B_0, b_0)$ be a set of generators. There exists $\delta > 0$ such that for any subset $S \subset B_0 \setminus U_0$ of cardinality at most s, there exists loops $\gamma_1, \ldots, \gamma_m \subset B_0 \setminus S$ representing $\alpha_1, \ldots, \alpha_m$ respectively with* $\mathrm{length}_{d_{B_0\setminus S}}(\gamma_i) < \delta$ *for every $i = 1, \ldots, m$.*

This second proof is longer but might be applied to more general situations, e.g., to higher dimensional spaces. The main tool is the following semi-upper continuity property of hyperbolic metrics in families.

Theorem 5.25 (Wright). *Let X be a complete hyperbolic manifold with infinitesimal Kobayashi metric F_X. Let $S \subset \mathbb{C}^N$ be an analytic variety containing 0. Suppose that $\Phi = \{\varphi(s) \colon s \in S\} \subset \mathrm{End}(T|X|)$ is a smooth family of integrable almost complex structures on the underlying differentiable manifold $|X|$. Assume that $\varphi(0)$ coincides with the complex structure of X. Let X_s denote the induced complex manifold structure on X given by $\varphi(s)$. Let $x \in X$ and $\varepsilon > 0$. Then there exist a neighborhood $U \subset X$ of x and a number $\delta > 0$ such that for all $s \in S$ with $|s| < \delta$ and $(y, \xi) \in TX|_U$, we have*

$$F_{X_s}(y, \overline{\varphi(s)}(\xi)) \leq F_X(y, \xi) + \varepsilon ||\xi||,$$

where $|| \cdot ||$ is defined with respect to a Hermitian metric on X or can be taken to be the norm induced by an coordinate chart of U.

Proof. See [**116**, Proposition 2]. Remark that in the *loc.cit* paper, the Cartan-Maurer equation $\overline{\partial}\phi - [\phi, \phi]/2 = 0$ is used as an equivalent definition of the integrability of a deformation of an almost complex structure $\phi = \mathrm{Id} - \varphi$ (cf. Lemma [**55**, 6.1.2]). $\square$

Here, an *almost complex structure* J on a $2n$-dimensional smooth differential manifold M is a real vector bundle endomorphism $J\colon TM \to TM$ satisfying $J^2 = -\mathrm{Id}$. We have:

Lemma 5.26. *Let M, N be differentiable manifolds. Let J be an almost complex structures on N. Assume that $f\colon M \to N$ is a $\mathcal{C}^\infty$-diffeomorphism of real manifolds with tangent maps $f_*\colon TM \to TN$. Then the almost complex structure $f_*^{-1} J f_*$ on M is integrable if and only if J is integrable.*

PROOF. This is a standard application of Newlander-Nirenberg Theorem. $\square$

We recall two more standard tools in differential geometry:

Theorem 5.27 (Global smooth Urysohn lemma)**.** *Let $V \subset M$ be an open subset of a smooth manifold and let $Z \subset V$ be a subset that is closed in M. There is a smooth $\mathcal{C}^\infty$-function $f\colon M \to \mathbb{R}$ such that $f|_Z \equiv 1$, $\mathrm{supp}(f) \subset V$ and $0 \leq f \leq 1$.*

PROOF. See [**22**, Corollary 3.5.5, Theorem 2.6.1]. $\square$

Theorem 5.28 (Global flows)**.** *Let M be a smooth differential manifold. Let $V \in \mathcal{X}(M)$ be a smooth vector field with compact support, i.e., $\mathrm{supp}\, V \coloneqq \{x \in M\colon V(x) \neq 0\}$ is contained in a compact subset of M. Then there exists a unique smooth map $\Psi\colon M \times \mathbb{R} \to M$ satisfying for all $(x,t) \in M \times \mathbb{R}$:*

$$\partial_t \Psi = V \circ \Psi, \quad \Psi(x, 0) = x.$$

Moreover, the maps $(\Psi_t)_{t\in\mathbb{R}}$ form a one-parameter group of diffeomorphisms of M.

PROOF. This is a standard result, see any reference on differential manifolds. $\square$

We can now begin with an auxiliary construction.

Lemma 5.29. *Let X be a complex manifold and let $z_1, \ldots, z_k \in X$ be distinct points. For every $i = 1, \ldots, k$, let $U_i \subsetneq V_i$ be small open balls in X containing z_i such that the V_i's are disjoint. Then over $S \coloneqq U_1 \times \cdots \times U_k$, there exists a smooth family Ψ of $\mathcal{C}^\infty$- diffeomorphisms $\Psi_s\colon X \to X$ verifying for all $s = (s_1, \ldots, s_k) \in S$:*

(i) $\Psi_s|_{X \setminus V_1 \cup \cdots \cup V_k} = \mathrm{Id}$;

(ii) $\Psi_s(s_i) = z_i$ *for every* $i = 1, \ldots, k$;

(iii) $\Psi_{(z_1,\ldots,z_k)} = \mathrm{Id}_X$.

Moreover, Ψ induces a smooth family $\{\varphi(s)\colon s \in S\}$ of complex structures on the manifold $X_0 \coloneqq X \setminus \{z_1, \ldots, z_k\}$ given by:

$$\varphi(s) = ((\Psi_s)_*|_{TX_s}) J_s ((\Psi_s)_*^{-1}|_{TX_0}),$$

where $J_s \in \mathrm{End}(TX_s)$ is the natural complex structure on the complex manifold $X_s \subset X$ and $(\Psi_s)_*\colon TX \to TX$ denotes the induced tangent map.

Proof. Let $n = \dim_{\mathbb{C}} X$ and denote $Z \coloneqq \coprod_{i=1}^k \overline{U_i} \subset V \coloneqq \coprod_{i=1}^k V_i \subset X$, where the closures $\overline{U}_i$'s are taken in the analytic topology. By Urysohn's lemma (Theorem 5.27), there exists a smooth $\mathcal{C}^\infty$-function $\rho\colon X \to \mathbb{R}$ such that $\rho|_Z \equiv 1$ and $\operatorname{supp} \rho \subset V$. Let $W_i \subset X$ be a slightly larger open balls centered at z_i containing V_i for every $i = 1, \ldots, k$. For each $s = (s_1, \ldots, s_k) \in S$, let $\mathcal{W}_s$ be the smooth constant vector field of the differentiable manifold $W \coloneqq \coprod_{i=1}^k W_i \subset X$ with

$$(\mathcal{W}_s)_x = (z_1 - s_1, \ldots, z_k - s_k) \in \mathbb{C}^{nk} \simeq T\mathbb{R}^{2nk}$$

for every $x \in W$. Here and after, by fixing the diffeomorphisms, we regard $S \subset V \subset W$ as relatively compact submanifolds of $\mathbb{R}^{2nk}$ endowed with the usual coordinates. Consider the following vector fields $\mathcal{V} \in \mathcal{X}(S \times X)$ defined by

$$S \times X \ni (s, x) \mapsto \mathcal{V}_{(s,x)} = (0, \rho(x)\widehat{\mathcal{W}_s}) \in T_s S \times T_x X = T_{(s,x)}(S \times X),$$

where $\widehat{\mathcal{W}_s}$ is the extension by 0 of $\mathcal{W}_s$ in X. Since ρ is smooth on X supported in $V \subset W$ and $\mathcal{W}$ is a smooth vector field on W, it follows immediately that $\mathcal{V}$ is a smooth vector field on $S \times X$. Moreover, $\mathcal{V}$ is supported in $S \times V$ which is relatively compact. Therefore, Theorem 5.28 implies that there exists a unique $\mathcal{C}^\infty$-smooth map $\Psi\colon S \times X \times \mathbb{R} \to S \times X$ satisfying

$$\partial_t \Psi = \mathcal{V} \circ \Psi, \quad \Psi(s, x, 0) = (s, x),$$

for every $(s, x, t) \in S \times X \times \mathbb{R}$. Denote $z_0 = (z_1, \ldots, z_k)$. As $\rho|_Z \equiv 1$, $S \subset Z$ and $\mathcal{V} = (0, z_0 - s)$ on $S \times Z$, we deduce that

$$\Psi(s, z, t) = (s, z + t(z_0 - s)),$$

for all $(s, z, t) \in S \times Z \times \mathbb{R}$ such that $z + t(z_0 - s) \in Z$ by the (local) uniqueness of Ψ. In particular, $\Psi(s, s, 1) = (s, z_0)$ for every $s \in S$. Hence by Theorem 5.28, $\{\Psi_s = \Psi(s, \cdot, 1)\colon X \to X,\ s \in S\}$ is a smooth family of $\mathcal{C}^\infty$-diffeomorphisms of X verifying Properties (i), (ii) and (iii). For each $s \in S$, the integrable almost complex structure J_s is nothing but the restriction $\tilde{J}|_{TX_s}$ where $\tilde{J} \in \mathrm{End}(TX)$ is the natural complex structure of the complex manifold X. Therefore,

$$\varphi(s) = ((\Psi_s)_*|_{TX_s}) J_s ((\Psi_s)_*^{-1}|_{TX_0}) = \Big((\Psi_s)_* \tilde{J} (\Psi_s)_*^{-1}\Big)\Big|_{TX_0}.$$

The last statement then follows immediately from Lemma 5.26 and the smoothness of the families $\{\Psi_s\}_{s\in S}$, $\{\Psi_s^{-1}\}_{s\in S}$ induced by Ψ. For the smoothness of $\{\Psi_s^{-1}\}_{s\in S}$, remark that $\Psi_s^{-1}(\cdot) = \Psi(s, \cdot, -1)$ for $s \in S$ as $\{\Psi(\cdot, \cdot, t)\}_{t\in\mathbb{R}}$ forms a one-parameter group of diffeomorphisms of $S \times X$ by Theorem 5.28. $\square$

Lemma 5.30. *Let $k \geq 1$ be an integer. Let X be a complete hyperbolic complex manifold and let $\gamma \subset X$ be a piecewise smooth compact connected curve. Let $z_1, \dots, z_k \in X \setminus \gamma$ and let $(x_{i,n})_{n\geq 1} \to z_i$ be converging sequences for $i = 1, \dots, k$ (with respect to a Riemannian metric on X). Then we have:*

$$\limsup_{n\to\infty} \operatorname{length}_{d_{X\setminus\{x_{1,n},\dots,x_{k,n}\}}}(\gamma) \leq \operatorname{length}_{d_{X\setminus\{z_1,\dots,z_k\}}}(\gamma).$$

Proof. Since $z_1, \dots, z_k \in X \setminus \gamma$, we can choose disjoint small open discs V_i's centered at z_i's respectively and all disjoint from γ. Let $U_i \subset V_i$ be smaller nonempty open balls for each i which are also centered at z_i for all i's. Let $S \coloneqq U_1 \times \cdots \times U_k$ be regarded as a pyolyballs submanifold of $\mathbb{C}^{k \dim_{\mathbb{C}} X}$ (centered at the origin). By Lemma 5.29, we can choose a smooth family Ψ of $\mathcal{C}^\infty$- diffeomorphisms $\Psi_s \colon X \to X$ where $s = (s_1, \dots, s_k) \in S$ such that Ψ_s are identities outside of $V = V_1 \cup \dots \cup V_k$ and Ψ_s sends s_i to z_i for all i's. From these maps Ψ_s, we obtain also by Lemma 5.29 a smooth family $\{\varphi(s) \colon s \in S\}$ of complex structures on $X_0 \coloneqq X \setminus \{z_1, \dots, z_k\}$ induced by the natural complex structures of the manifolds $X_s \coloneqq X \setminus \{s_1, \dots, s_k\}$. Remark that since X is complete hyperbolic, $X_0 \subset X$ is also a complete hyperbolic manifold. Let $\sigma \colon [0, 1] \to X$ be a piecewise smooth parametrization of γ. Since σ is piecewise smooth and $[0, 1]$ is compact, $C \coloneqq \sup_{t\in[0,1]} ||\dot\sigma(t)||$ is finite where $|| \cdot ||$ denotes a fixed Hermitian metric on X.

Let $\varepsilon > 0$. Since the Ψ_s's are identities on $X \setminus V_1 \cup \dots \cup V_k$ which contains γ, Theorem 5.25 implies that for each $t \in [0, 1]$, there exist a neighborhood $W_t \subset [0, 1]$ of t and a number $\delta_t > 0$ such that

$$F_{X_s}(\sigma(w), \dot\sigma_s(w)) \leq F_{X_0}(\sigma(w), \dot\sigma(w)) + \varepsilon||\dot\sigma(w)|| \leq F_{X_0}(\sigma(w), \dot\sigma(w)) + C\varepsilon,$$

for all $w \in W_t$ and all $s \in S$ with $|s| \leq \delta_t$. The last inequality follows from the definition of the constant C. Since $\cup_{t\in[0,1]} W_t$ forms an open covering of $[0, 1]$, there exists a finite subcorvering $W_{t_1} \cup \dots \cup W_{t_m} = [0, 1]$ for some $0 \leq t_1 \leq \dots \leq t_m \leq 1$. We define $\delta = \min(\delta_{t_1}, \dots, \delta_{t_m}) > 0$ then it follows that for every $w \in [0, 1]$ and $s \in S$ with $|s| \leq \delta$, we have:

$$F_{X_s}(\sigma(w), \dot\sigma(w)) \leq F_{X_0}(\sigma(w), \dot\sigma(w)) + C\varepsilon.$$

Since $(x_{1,n}, \dots, x_{k,n})_{n\geq 1}$ converges to $(z_1, \dots, z_k)$, we deduce that:

$$\begin{aligned}\limsup_{n\to\infty} \mathrm{length}_{d_{X\setminus\{x_{1,n},\ldots,x_{k,n}\}}}(\gamma) &= \limsup_{n\to\infty} \int_{[0,1]} F_{X\setminus\{x_{1,n},\ldots,x_{k,n}\}}(\sigma(t),\dot\sigma(t))dt \\ &\leq \limsup_{n\to\infty} \int_{[0,1]} \left(F_{X_0}(\sigma(t),\dot\sigma(t)) + C\varepsilon\right) dt \\ &= \mathrm{length}_{d_{X_0}}(\gamma) + C\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can conclude the proof of the lemma. $\square$

Second proof of Lemma 5.7/5.24. Suppose the contrary, then for each integer $n > 0$, there exists a finite subset $S_n \subset B_0 \setminus U_0$ of cardinality at most s such that for any loops $\gamma_1, \ldots, \gamma_k \subset B_0 \setminus S_n$ representing the classes $\alpha_1, \ldots, \alpha_m \in \pi_1(B_0, b_0)$, we have

$$\mathrm{length}_{d_{B_0\setminus S_n}}(\gamma_i) \geq n$$

for some $i \in \{1, \ldots, k\}$. Since we can regard $(S_n)_{n\geq 1}$ as a sequence of elements of the compact complete metric space $(B\setminus U_0)^s$, there exists a convergent subsequence which we will denote also (by abuse of notation) by $(S_n)_{n\geq 1}$. The metric on $(B\setminus U_0)^s$ is defined to be induced from a Riemannian metric. Let $S^* = \{x_1, \ldots, x_t\} \subset B\setminus U_0$ be the set of limit points of $(S_n)_n$ where $t \leq s$. Since $b \in U_0$, we can choose a set of piecewise smooth loops $\gamma_1^*, \ldots, \gamma_m^* \subset B_0 \setminus S^*$ representing $\alpha_1, \ldots, \alpha_m$. Clearly, as S^* is finite, this can be done by slightly deform any given set of loops based at b_0 in B_0 representing the α_i's to avoid S^* if necessary. We set

$$\delta = \max_{1\leq i\leq m} \mathrm{length}_{d_{B_0\setminus S^*}}(\gamma_i^*) + 1.$$

Remark that since $(S_n)_n$ converges to the points of S^*, we have $\gamma_1, \ldots, \gamma_m \subset B_0\setminus S_n$ for all $n \geq N_1$ for some $N_1 \in \mathbb{N}$. By Lemma 5.30, we deduce that there exists an integer $N_2 \geq N_1$ such that for all $n \geq N_2$, we have

$$\max_{1\leq i\leq m} \left(\mathrm{length}_{d_{B_0\setminus S_n}}(\gamma_i^*) - \mathrm{length}_{d_{B_0\setminus S^*}}(\gamma_i^*)\right) < 1/2.$$

In particular, it follows from our constructions that $n \leq \max_{1\leq i\leq m} \mathrm{length}_{d_{B_0\setminus S_n}}(\gamma_i^*) < \delta$ for all $n \geq N_2$, which is clearly a contradiction. $\square$

CHAPTER 6

Hyperbolic and homotopy method for integral points

1. Introduction

Notations. Let B be a compact Riemann surface of function field $K = \mathbb{C}(B)$ of genus g equipped with a Riemannian metric d. Let A/K be an abelian variety. Let $D \subset A$ be an effective ample divisor of A not containing any translate of nonzero abelian subvarieties. Let $\mathcal{D}$ be its Zariski closure in a model $f \colon \mathcal{A} \to B$ of A. Let $T \subset B$ be the finite subset containing the points $b \in B$ such that $\mathcal{A}_b$ is not smooth.

The main results of this chapter are the following:

Theorem F. *Let $W \supset T$ be any finite union of disjoint closed discs in B such that distinct points of T are contained in distinct discs. Let $B_0 = B \setminus W$. There exists $m > 0$ such that:*

($*$) *For I_s ($s \in \mathbb{N}$) the union of $(S, \mathcal{D})$-integral points over all subsets $S \subset B$ such that $\#(S \cap B_0) \leq s$, we have:*

$$\#\big(I_s \text{ mod } \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\big) < m(s+1)^{2\dim A.\,\mathrm{rank}\,\pi_1(B_0)}, \quad \text{for all } s \in \mathbb{N}. \tag{1.1}$$

As a consequence, we obtain a generic emptiness of integral points on abelian varieties over function fields. More precisely, let the notations be as in Theorem F, we have:

Corollary B. *Assume $\mathrm{Tr}_{K/\mathbb{C}}(A) = 0$ and $\mathcal{D}$ horizontally strictly nef, i.e., $\mathcal{D}\cdot C > 0$ for all curves $C \subset \mathcal{A}$ not contained in a fibre.*

(i) *for each $s \in \mathbb{N}$, there exists a finite subset $E \subset B$ such that for any $S \subset B \setminus E$ with $\#(S \cap B_0) \leq s$, the set of $(S, \mathcal{D})$-integral points is empty. Moreover, we can choose E such that*

$$\#E \cap B_0 \leq sm(s+1)^{2\dim A.\,\mathrm{rank}\,\pi_1(B_0)};$$

(ii) *for each $s \in \mathbb{N}$, there exists a Zariski proper closed subset $\Delta \subset B^{(s)}$ such that for any $S \subset B$ of cardinality s whose image $[S] \in B^{(s)} \setminus \Delta$, there is no $(S, \mathcal{D})$-integral points.*

For every subset $R \subset A(K) \setminus D$, we define the intersection locus:

$$I(R, \mathcal{D}) \coloneqq \cup_{P \in R} f(\sigma_P(B) \cap \mathcal{D}) \subset B. \tag{1.2}$$

We obtain the following interesting information on the intersection locus $I(R, \mathcal{D})$:

Theorem G. *Assume that* $\mathrm{Tr}_{K/\mathbb{C}}(A) = 0$ *and that* R *is infinite. We have:*

(a) $I(R, \mathcal{D})$ *is countably infinite but it is not analytically closed in* B*;*

(b) $I(R, \mathcal{D})$ *has uncountably many limit points* $I(R, \mathcal{D})_\infty$ *in* B*;*

(c) $I(R, \mathcal{D})_\infty$ *is not contained in any union* $W \supset T$ *of disjoint closed discs in* B *such that distinct points of* T *are contained in distinct discs.*

The main idea for the proof our results is a combination of a *homotopy* reduction step due to Parshin to the study of morphisms between certain fundamental groups (cf. Proposition 6.4), and the following *hyperbolic* result of Chapter 5 which is of independent interest (cf. Definition 5.1 for the meaning of "a single conjugation"):

Theorem D. *Let* $U \subset B$ *be finite union of disjoint closed discs and let* $b_0 \in B_0 \coloneqq B \setminus U$. *Fix a base of simple generators* $\alpha_1, \dots, \alpha_k$ *of* $\pi_1(B_0, b_0)$. *There exists* $L > 0$ *such that for any finite subset* $S \subset B_0$, *there exists* $b \in B_0 \setminus S$ *and piecewise smooth loops* $\gamma_1, \dots, \gamma_k \subset B_0 \setminus S$ *based at* b *representing respectively* $\alpha_1, \dots, \alpha_k$ *up to a single conjugation and:*

$$\mathrm{length}_{d_{B_0 \setminus S}}(\gamma_i) \leq L(\#S + 1).$$

The bridge connecting the above two blocks to prove Theorem F is given by the *Fundamental Lemma* of the geometry of groups, which we formulated in Proposition 6.20 and in Lemma 6.21 (cf. Appendix 7). The latter can be regarded in our approach as an analogous counting tool of the Counting Lemma of Lenstra (cf. Lemma 2.24) frequently used in the classical height theory.

We remark here that using Theorem 5.2, the proof of Theorem F can be easily modified to obtain the following stronger statement allowing furthermore the intersection of integral sections with $\mathcal{D}$ over some bounded moving discs in B_0:

Theorem 6.1. *Let* $W \supset T$ *be any finite union of disjoint closed discs in* B *such that distinct points of* T *are contained in distinct discs. Let* $B_0 = B \setminus W$. *Let* $p \in \mathbb{N}$. *There exists* $m > 0$ *and* $r > 0$ *such that:*

($*$) *For* I_s^p ($s \in \mathbb{N}$) *the union of* $(S, \mathcal{D})$*-integral points over all subsets* $S \subset B$ *such that* $\#\left(S \cap (B_0 \setminus Z)\right) \leq s$ *where* Z *is any union of* p *discs of* d*-radius* r *in* B*:*

$$\#\left(I_s^p \text{ mod } \mathrm{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\right) < m(s+1)^{2 \dim A . \operatorname{rank} \pi_1(B_0)}, \quad \text{for all } s \in \mathbb{N}. \tag{1.3}$$

As an application, we obtain the following generalization of Corollary 1.30 presented in Chapter 1. The proof is similar to that of Corollary 1.30 but we use Theorem 6.1 instead of Theorem F.

Corollary 6.2. *Let $W \supset T$ be any finite union of disjoint closed discs in B such that distinct points of T are contained in distinct discs. Let $B_0 = B \setminus W$. Let $p, s \in \mathbb{N}$. There exists $r > 0$ and $M = M(\mathcal{A}, \mathcal{D}, B_0, p, s) \in \mathbb{R}_+$ such that for every section $\sigma\colon B \to \mathcal{A}$ with $\#f(\sigma(B_0 \setminus Z) \cap \mathcal{D}) \leq s$ where Z is any union of p discs of d-radius r in B, we have:*

$$\deg_B \sigma^* \mathcal{D} < M. \tag{1.4}$$

2. The homotopy reduction step of Parshin

The key step in the approach of Parshin in his proof of Theorem 1.18 is the following homotopy reduction Proposition 6.3 which is stated without proof in [**87**]. From this, we obtain Proposition 6.4 to reduce the problem of finiteness of integral points to a finiteness problem of certain bounded morphisms between certain fundamental groups.

Proposition 6.3 (Parshin)**.** *Let $W \supset T$ be any finite union of disjoint closed discs in B such that distinct points of T are contained in distinct discs. Let $B_0 = B \setminus W$. Let $b_0 \in B_0 := B \setminus W$ and denote $\Gamma = H_1(A_{b_0}, \mathbb{Z})$, $G = \pi_1(B \setminus U, b_0)$. Assume $A[n] \subset A(K)$ for some integer $n \geq 2$. Then we have a natural commutative diagram of homomorphisms:*

$$\begin{tikzcd} A(K)/nA(K) \arrow[r, hook, "\delta"] & H^1(\widehat{G}, A[n]) \arrow[drr, "\simeq"] & & \\ A(K) \arrow[u] \arrow[r, "\alpha"] & H^1(G, \Gamma) \arrow[rr, "\beta"] & & H^1(G, A[n]). \end{tikzcd} \tag{2.1}$$

It is worth giving a detailed proof of Proposition 6.3 in Section 8 which contains explicit descriptions of the maps α, β and δ. A concise proof is also given below in Section 2.1. Before that, we first indicate below how Proposition 6.3 allows us to reduce the problem to the finiteness of certain morphisms between certain fundamental groups. Since $\mathcal{A}_{B_0} \to B_0$ is a proper submersion, it is a fibre bundle by Ehresmann's fibration theorem (cf. [**30**]). Thus, we have an exact sequence of fundamental groups induced by the fibre bundle $A_{b_0} \to \mathcal{A}_{B_0} \to B_0$ of $K(\pi, 1)$-spaces, where we denote $A_{b_0} = \mathcal{A}_{b_0}$:

$$0 \to \pi_1(A_{b_0}, w_0) = H_1(A_{b_0}, \mathbb{Z}) \to \pi_1(\mathcal{A}_{B_0}, w_0) \xrightarrow{\rho = f_*} \pi_1(B_0, b_0) \to 0. \tag{2.2}$$

To fix the notations, w_0 is chosen here and in the rest of the thesis to be the zero point of A_{b_0}, which also lies on the zero section of $\mathcal{A}_{B_0}$. Moreover, suppose that we fix a collection $\{l_{w_0w}\}$ consisting of bounded length and smooth directed paths contained in A_{b_0} going from w_0 to $w \in A_{b_0}$. Then there is a natural way to define a homotopy section i_σ of (2.2) in its conjugacy class induced by a section σ of $\mathcal{A}|_{B_0} \to B_0$ by using the paths l_{w_0w} (cf. (8.3) in Appendix 8).

Proposition 6.4. *Let the notations be as in Proposition 6.3. Then there exists a constant $N_0 > 0$ such that every section class i of the exact sequence* (2.2) *is induced by at most N_0 rational points $P \in A(K)$ modulo the trace* $\operatorname{Tr}_{K/\mathbb{C}} A(\mathbb{C})$.

Proof. (cf. [**87**, Proposition 2.1]) Let $P, Q \in A(K)$ and assume that the conjugacy classes of i_P and i_Q of the exact sequence (2.2) are equal. Up to making a finite base change $B' \to B$ étale outside of T, we can suppose without loss of generality that $A[n] \subset A(K)$ for some integer $n \geq 2$ (this asumption is only necessary in Proposition 6.3). It follows that $\alpha(P) = \alpha(Q)$. Since α is a homomorphism, we deduce that $\alpha(P - Q) = 0$. Proposition 6.3 then implies that $\delta(P - Q) = 0$ and that $P - Q \in nA(K)$. Thus, we have $P - Q = nR$ for some $R \in A(K)$. Observe that $n\alpha(R) = \alpha(nR) = \alpha(P - Q) = 0$ in the torsion free abelian group $H^1(G, H_1(A_{b_0}, \mathbb{Z}))$. We deduce that $\alpha(R) = 0$ since $n \neq 0$. Therefore, by induction, the same argument shows that $P - Q \in n^k A(K)$ for every $k \in \mathbb{N}$. Since $\Omega := A(K)/\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})$ is a finitely generated abelian group by the Lang-Néron theorem (cf. [**61**]), we must have $P - Q \mod \operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}) \in \Omega_{tors}$ because $n \geq 2$. As the latter set is finite, the conclusion follows by setting $N_0 = \#\Omega_{tors}$. □

Remark 6.5. It may be useful to recall the strategy for Theorem F here. Using Proposition 6.4, we only need to give a quantitative bound on the number of possible conjugacy classes of homotopy sections induced by integral sections in the set I_s defined in Theorem F. For this, it suffices to show that each such conjugacy class admits a representative $\pi_1(B_0, b_0) \to \pi_1(\mathcal{A}_{B_0}, w_0)$ whose images of the generators $\alpha_1, \dots, \alpha_k$ belong to certain finite subset of $\pi_1(\mathcal{A}_{B_0}, w_0)$ of controlled cardinality.

2.1. A conceptual proof of Proposition 6.3. Let $\mathcal{A}_0$ be the restriction of $\mathcal{A}$ over B_0. Let $\mathcal{L}$ $(= \mathcal{N}_{\sigma_O(B_0)/\mathcal{A}_0})$ be the complex Lie algebra of $\mathcal{A}_0$, viewed as the vector bundle over B_0. Identifying the kernel Γ of the "relative" exponential map $\mathcal{L} \to \mathcal{A}_0$ with $(R^1 f_* \mathbb{Z})^\vee$, we obtain a canonical short exact sequence of locally constant *analytic* sheaves over B_0:

$$0 \to (R^1 f_* \mathbb{Z})^\vee \to \mathcal{L} \to \mathcal{A}_0^{an} \to 0. \tag{2.3}$$

When the trace of $\mathcal{A}$ is zero, we obtain a map $\mathcal{A}_0^{an}(B_0) \to H^1(B_0, (R^1 f_* \mathbb{Z})^\vee)$ whose restriction to the set of rational sections $A(K) \subset \mathcal{A}_0(B_0)$ is injective and this is already good enough for the proof of Proposition 6.4.

In general, consider the multiplication-by-n B_0-morphism $[n]\colon \mathcal{A}_0 \to \mathcal{A}_0$. The induced map $[n]\colon \mathcal{L} \to \mathcal{L}$ on $\mathcal{L}$ is an isomorphism with inverse $[n^{-1}]\colon \mathcal{L} \to \mathcal{L}$ given by the multiplication by n^{-1}. Notice that we assume $\mathcal{A}_0[n] \subset \mathcal{A}(K) \subset \mathcal{A}(B_0)$. The map $[n]$ and the sequence (2.3) induce the following commutative diagram in the analytic category:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathcal{A}_0[n] \arrow[d] & \\
0 \arrow[r] & (R^1 f_*\mathbb{Z})^\vee \arrow[r] \arrow[d, "n"] & \mathcal{L} \arrow[r] \arrow[d, "\simeq"] & \mathcal{A}_0 \arrow[r] \arrow[d, "n"] & 0 \\
0 \arrow[r] & (R^1 f_*\mathbb{Z})^\vee \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{A}_0 \arrow[r] \arrow[d] & 0 \\
 & Q \arrow[r] \arrow[d] & 0 \arrow[r] & 0 & \\
 & 0 & & &
\end{tikzcd} \tag{2.4}$$

By the snake lemma, $\mathcal{A}_0[n] \simeq Q$. Let $S \supset T$ be the centres of the discs in W and let K_S be the maximal Galois extension of K unramified outside of S. Then $\mathrm{Gal}(K_S/K) \simeq \widehat{G}$ where $G = \pi_1(B_0, b_0)$. Notice also the natural isomorphism $H^1(\widehat{G}, A[n]) \simeq H^1(G, A[n])$ induced by the injection $G \to \widehat{G}$ (cf. [**98**, I.2.6.b]). The Kummer exact sequence $0 \to A[n] \to A(\bar{K}) \xrightarrow{n} A(\bar{K}) \to 0$ gives rise to an exact sequence of Galois cohomology (thus of *algebraic* nature)

$$A(K) \xrightarrow{n} A(K) \to H^1(\widehat{G}, A[n]) \tag{2.5}$$

It is well-known that the category of local systems on a Eilenberg-MacLane $K(\pi,1)$-space X (i.e., $\pi_i(X) = 0$ for all $i > 0$) is equivalent to the category of π-modules. As $\mathcal{A}_0$ and B_0 are $K(\pi,1)$-spaces, there are canonical isomorphisms

$$H^1(B_0, (R^1 f_*\mathbb{Z})^\vee) \simeq H^1(G, \Gamma), \quad H^1(B_0, \mathcal{A}_0[n]) \simeq H^1(G, A[n]), \tag{2.6}$$

where $\Gamma = H_1(A_{b_0}, \mathbb{Z}) \simeq (R^1 f_*\mathbb{Z})^\vee_{b_0}$. Here, the group G acts naturally on Γ by monodromy (cf. *The description of* α in Appendix 8) and acts trivially on $A[n]$.

Combining (2.5) with the cohomology long exact sequences induced by Diagram (2.4), we obtain a natural commutative diagram:

$$
\begin{tikzcd}
A(K) \arrow[r] \arrow[d, "n"] & \mathcal{A}_0(B_0) \arrow[r] \arrow[d, "n"] & H^1(B_0, (R^1 f_* \mathbb{Z})^\vee) \arrow[d, "n"] \\
A(K) \arrow[r] \arrow[d] & \mathcal{A}_0(B_0) \arrow[r] \arrow[d] & H^1(B_0, (R^1 f_* \mathbb{Z})^\vee) \arrow[d] \\
H^1(\widehat{G}, A[n]) \arrow[r, "\simeq"] & H^1(B_0, \mathcal{A}_0[n]) \arrow[r, "\simeq"] & H^1(B_0, Q)
\end{tikzcd}
\tag{2.7}
$$

By decomposing the first column into $A(K)/nA(K) \hookrightarrow H^1(\widehat{G}, A[n])$ and using the isomorphisms in (2.6), we obtain as claimed the commutative diagram (2.1).

The detailed descriptions of the involved homomorphisms α, β, δ and the monodromy action of G are given in Appendix 8.

3. Some geometry of surfaces and hyperbolic manifolds

In the proof of Theorem F, we shall need the following general auxiliary lemma to control the geometry of a countable closed subset.

Lemma 6.6. *Let R be a closed countable subset of a compact Riemann surface B with boundary equipped with a Riemannian metric ρ. Let $T \subset B$ be a finite subset. Then for every $\varepsilon > 0$, $R \cup T$ is contained in a finite union Z of disjoint closed discs each of radius at most ε such that $\mathrm{vol}_\rho Z \leq \varepsilon$ and such that any two distinct points in T are contained in distinct discs.*

Proof. Write $T = \{t_1, \ldots, t_p\}$. Since the set $R \cup T$ is countable, it can be written as a sequence $R \cup T = (x_n)_{n \geq 1}$ such that $x_1 = t_1, \ldots, x_p = t_p$. Let $\delta = (\varepsilon/c)^{1/2} > 0$ where $c \gg 1$ is some large constant to be chosen latter.

We define by recurrence a (possibly finite) sequence $(y_n)_{n \geq 1} \subset R$ such that each y_n is contained in a small closed disc V_n of B of radius $r_n < \delta/2^n$. For $n = 1$, let $y_1 = x_1 = t_1$ and let $D \subset B$ be the closed disc of radius $\delta/2$ centered at y_1. Since R is countable and $]0, \delta/2[$ is uncountable, there exists clearly a smaller closed disc $V_1 \subset D$ of radius $r_1 \in]0, \delta/2[$ also centered at x_1 such that $\partial V_1 \cap R = \varnothing$ and that $V_1 \cap T = \varnothing$. Similarly, we can find successively for $i = 2, \ldots, p$ a closed disc V_i of radius $r_i \in]0, \delta/2^i[$ such that $t_i \in V_i$, $\partial V_i \cap R = V_i \cap T = \varnothing$ and that $V_i \cap (V_1 \cup \cdots \cup V_{i-1}) = \varnothing$.

Now for $n = k+1 > p$, let $m \geq 1$ be the smallest integer such that $x_m \notin V_1 \cup \cdots \cup V_k$. Define $y_{k+1} = x_m$ and let $V_{k+1} \subset B$ be a closed disc of radius $r_{k+1} \in]0, \delta/2^{k+1}[$ centered at y_{k+1} such that $\partial V_{k+1} \cap R = \varnothing$ and $V_{k+1} \cap (V_1 \cup \cdots \cup V_k) = \varnothing$. Observe again that such V_{k+1} exists since R is countable. By construction, it is clear that $R \cup T \subset \cup_{n\geq 1}(V_n \setminus \partial V_n)$. As $R \cup T \subset B$ is closed and B is compact, $R \cup T$ is compact. Hence, $R \cup T$ is contained in some finite union Z of disjoint closed discs $Z = V_{n_1} \cup \cdots \cup V_{n_q}$. It is immediate that $\{1, \dots, m\} \subset \{n_1, \dots, n_q\}$ since $t_i \in V_i$ but $t_i \notin V_j$ for all $i \in \{1, \dots, p\}$ and $j \neq i$. In other words, any two distinct points in T are contained in different discs of Z.

By compactness of B, there exists a constant $c_0 > 0$ such that for every small enough $r > 0$, we have for every point $b \in B$ (cf. Lemma 5.16):

$$\operatorname{vol}_\rho D(b, r) \leq c_0 r^2,$$

where $D(b, r) \subset B$ be the closed disc centered at b and of radius r. Therefore, we find for all $c \gg c_0$ that :

$$\operatorname{vol} Z \leq \sum_{n\geq 1} \operatorname{vol} V_n < c_0 \sum_{n\geq 1} (\delta/2^n)^2 < c_0 \frac{\varepsilon}{c} \sum_{n\geq 1} 4^{-n} < \varepsilon.$$

□

We recall below some fundamental properties of the pseudo Kobayashi hyperbolic metric (cf. Definition 2.38) and of hyperbolic manifolds due to Green. A complex space X is said to be *Brody hyperbolic* if there are no nonconstant holomorphic maps $\mathbb{C} \to X$.

Theorem 6.7 (Green)**.** *Let X be a relatively compact open subset of a complex manifold M. Let $D \subset X$ be a closed complex subspace. Denote by $\bar{X}, \bar{D}$ the closures of X and D in M. Assume that $\bar{D}$ and $\bar{X} \setminus \bar{D}$ are Brody hyperbolic. Then $X \setminus D$ is hyperbolic and we have an estimation*

$$d_{X\setminus D} \geq \rho|_{X\setminus D}$$

for some Hermitian metric ρ on M. In particular, if M is compact and λ is any Riemannian metric on $|M|$ then there exists $c > 0$ such that

$$d_{X\setminus D} \geq c\lambda|_{X\setminus D}.$$

Proof. See [**43**, Theorem 3] for the proof. □

Theorem 6.8 (Green)**.** *Let $X \subset A$ be a complex subspace and let D be a hypersurface of a complex torus A. The following hold:*

- (i) *X is hyperbolic if and only if X does not contain any translate of a nonzero complex subtorus of A;*
- (ii) *if D does not contain any translate of nonzero subtori of T then $A \setminus D$ is complete hyperbolic.*

Proof. See [**43**, Theorems 1-2]. □

Thanks to the distance-decreasing property of the pseudo-Kobayashi hyperbolic metric, we have the following important property of sections.

Lemma 6.9. *Let $f\colon X \to Y$ be a holomorphic map between complex spaces. Suppose that $\sigma\colon Y \to X$ is a holomorphic section. Then $\sigma(Y)$ is a totally geodesic subspace of X, i.e., for all $x, y \in Y$, we have:*

$$d_Y(x,y) = d_X(\sigma(x), \sigma(y)).$$

PROOF. The lemma is a direct consequence of distance-decreasing property of the pseudo-Kobayashi hyperbolic metric (cf. [**60**, Proposition 3.1.6]):

$$d_Y(x,y) = d_Y(f(\sigma(x)), f(\sigma(y))) \leq d_X(\sigma(x), \sigma(y)) \leq d_Y(x,y).$$

□

4. Proof of Theorem F

Recall the statement of Theorem F. Let B be a compact Riemann surface equipped with a Riemannian metric d. Let A/K be an abelian variety of dimension n. Let $D \subset A$ be an effective ample divisor of A not containing any translate of nonzero abelian subvarieties. Let $\mathcal{D}$ be its Zariski closure in a model $f : \mathcal{A} \to B$ of A.

Theorem F. *Let $W \supset T$ be any finite union of disjoint closed discs in B such that distinct points of T are contained in distinct discs. Let $B_0 = B \setminus W$. There exists $m > 0$ such that:*

(*) *For I_s ($s \in \mathbb{N}$) the union of $(S, \mathcal{D})$-integral points of $\mathcal{A}$ over all subsets $S \subset B$ such that $\#(S \cap B_0) \leq s$, we have:*

$$\#\left(I_s \ mod \ \mathrm{Tr}_{K/\mathbb{C}} A(\mathbb{C})\right) < m(s+1)^{2n.\,\mathrm{rank}(\pi_1(B_0))}, \quad \textit{for all } s \in \mathbb{N}. \tag{4.1}$$

PROOF OF THEOREM F. We can clearly suppose that W contains exactly $t = \#T$ disjoint closed discs $(W_t)_{t \in T}$ centered at the points of T. Define

$$\varepsilon = \frac{1}{3} \min\left(\mathrm{rad}(B,d), \min\{d(W_u, W_t) \colon u, t \in T, u \neq t\}\right) \tag{4.2}$$

where $d(W_u, W_t)$ denotes the d-distance between W_u and W_t and $\mathrm{rad}(B,d) > 0$ denotes the injectivity radius of B. Since the discs W_t's are disjoint and closed, $\varepsilon > 0$.

Consider the non-hyperbolic locus (which is simply a subset of T when $\mathcal{A} \to B$ is a family of elliptic curves)

$$V \coloneqq \{b \in B \colon \mathcal{D}_b \text{ is not hyperbolic}\}. \tag{4.3}$$

Then V is an analytic closed subset of B since hyperbolicity is an analytic open property on the base in a proper holomorphic family (cf. [**11**]). Observe that

$V \subset Z(\mathcal{A}, \mathcal{D})$ by Theorem 6.8 where $Z(\mathcal{A}, \mathcal{D})$ is defined in Lemma 2.49. Since $\mathcal{D}_K = D$ does not contain any translate of nonzero abelian subvarieties of A, it follows that $Z(\mathcal{A}, \mathcal{D})$ and thus V are at most countable by Lemma 2.49. We can thus apply Lemma 6.6 to obtain a finite union Z_ε of disjoint closed discs of d-radius at most ε and such that $V \subset Z_\varepsilon$. Then it follows from the definition of ε in (4.2) that we can enlarge the discs W_t's into larger disjoint closed discs W_t''s such that $B_0' := B \setminus (\cup_{t\in T} W_t') \subset B_0$ is a deformation retract of B_0 and $Z_\varepsilon \subset \cup_{t \in T} W_t'$. In particular, $\pi_1(B_0') = \pi_1(B_0)$. Clearly, it suffices to prove the theorem for $B_0' \subset B_0$. Hence, up to replacing B_0 by B_0', we can suppose that $V \subset W$.

B_0 is an unbordered hyperbolic Riemann surface. Let $\alpha_1, \dots, \alpha_k$ be a fixed system of simple generators of the fundamental group $\pi_1(B_0, b_0)$ for some $b_0 \in B_0$ (cf. Lemma 5.10). Notice that $k = \operatorname{rank}(\pi_1(B_0))$.

Fix a Hermitian metric ρ on $\mathcal{A}$. Let $P \in I_s$, i.e., $P \in X(K)$ such that P is an $(S, \mathcal{D})$-integral point for some $S \subset B$ with $\#(S \cap B_0) \leq s$.

By Theorem D (cf. Chapter 5), there exists $b \in B_0$ and a system of simple loops $\gamma_1, \dots, \gamma_k$ based at b representing respectively the classes $\alpha_1, \dots, \alpha_k$ up to a single conjugation by using a fixed collection of chosen paths $(c_{b_0,b})_{b\in B_0}$ in B_0 (cf. Definition 5.1) such that $\gamma_j \subset B_0 \setminus S$ and that

$$\text{length}_{d_{B_0\setminus S}}(\gamma_j) \leq L(s+1) \tag{4.4}$$

for some constant $L > 0$ independent of s, S, b and P.

By Theorem 6.8 and by our reduction to the case $V \subset W$, the varieties $\mathcal{D}_b$ and $\mathcal{A}_b \setminus \mathcal{D}_b$ are hyperbolic for every $b \in B_0$. Since we can always suppose that there is at least one closed disc (of strictly positive radius) in the union W, B_0 is hyperbolic as well. Let $h\colon \mathbb{C} \to (\mathcal{A} \setminus \mathcal{D})|_{B_0}$ be a holomorphic map. The holomorphic map $f \circ h\colon \mathbb{C} \to B_0$ must be constant since B_0 is hyperbolic. Thus, h factors through $\mathcal{A}_b \setminus \mathcal{D}_b$ for some $b \in B_0$. Since $b \in B_0$, $b \notin V$ by the definition of B_0. Thus, by the definition of V (cf. (4.3)), Theorem 6.8 implies that $\mathcal{A}_b \setminus \mathcal{D}_b$ is hyperbolic. It follows that h is constant. Therefore, up to enlarging slightly furthermore W if necessary, we see that the analytic closure of $(\mathcal{A} \setminus \mathcal{D})|_{B_0}$ in $\mathcal{A}$ is Brody hyperbolic. Similarly, the analytic closure $\mathcal{D}|_{B_0}$ is also Brody hyperbolic. Hence, Theorem 6.7 implies that there exists $c > 0$ such that $d_{(\mathcal{A}\setminus\mathcal{D})|_{B_0}} \geq c\rho|_{(\mathcal{A}\setminus\mathcal{D})|_{B_0}}$.

Now, let $\sigma_P\colon B \to \mathcal{A}$ be the corresponding section of the rational point P. Notice that for every $j \in \{1, \dots, k\}$, we have by the definition of $(S, \mathcal{D})$-integral points that:

$$\sigma_P(\gamma_j) \subset \sigma_P(B_0 \setminus S) \subset (\mathcal{A} \setminus \mathcal{D})|_{B_0 \setminus S}.$$

It follows that for every $j \in \{1, \dots, k\}$:

$$
\begin{aligned}
\operatorname{length}_\rho(\sigma_P(\gamma_j)) &\leq c^{-1} \operatorname{length}_{d_{(\mathcal{A}\setminus\mathcal{D})|_{B_0}}}(\sigma_P(\gamma_j)) \\
&\leq c^{-1} \operatorname{length}_{d_{(\mathcal{A}\setminus\mathcal{D})|_{B_0\setminus S}}}(\sigma_P(\gamma_j)) \quad (\text{as } (\mathcal{A}\setminus\mathcal{D})|_{B_0\setminus S} \subset (\mathcal{A}\setminus\mathcal{D})|_{B_0}) \\
&= c^{-1} \operatorname{length}_{d_{B_0\setminus S}}(\gamma_j) \quad (\text{by Lemma 6.9 }) \\
&\leq c^{-1} L(s+1) \quad (\text{by (4.4)})
\end{aligned}
\tag{4.5}
$$

Let σ_O be the zero section of $\mathcal{A} \to B$. Denote $w_0 = \sigma_O(b_0) \in A_{b_0} := \mathcal{A}_{b_0} \subset \mathcal{A}$.

Recall the following short exact sequence associated to the locally trivial fibration $\mathcal{A}_{B_0} \to B_0$ (cf. (2.2) after Proposition 6.3):

$$0 \to \pi_1(A_{b_0}, w_0) \to \pi_1(\mathcal{A}_{B_0}, w_0) \to \pi_1(B_0, b_0) \to 0 \tag{4.6}$$

The zero section of $\mathcal{A}$ induces a section $i_O\colon \pi_1(B_0, b_0) \to \pi_1(\mathcal{A}_{B_0}, w_0)$ of (4.6) which in turn induces a semi-direct product $\pi_1(\mathcal{A}_{B_0}, w_0) = \pi_1(A_{b_0}, w_0) \rtimes_\varphi \pi_1(B_0, b_0)$. Here, $\pi_1(B_0, b_0)$ acts on $\pi_1(A_{b_0}, w_0)$ by conjugation (see *The description of* α in Appendix 8), which is also the monodromy action and is denoted by:

$$\varphi\colon \pi_1(B_0, b_0) \to \operatorname{Aut}(\pi_1(A_{b_0}, w_0)), \quad \alpha \mapsto \varphi_\alpha. \tag{4.7}$$

Let δ_0 be the diameter of the analytic closure of $\mathcal{A}_{B_0}$ in $\mathcal{A}$ with respect to the metric ρ. Let $\{(c_{b_0 b})_{b \in B_0}\}$ be the collection of smooth directed paths of bounded d-length going from b_0 to every point $b \in B_0$. Then by the compactness of the closure of $\mathcal{A}_{B_0}$, the ρ-lengths of the induced collection of directed smooth paths $\{\sigma_O(c_{b_0 b})\}$ are also uniformly bounded, let's say by a constant δ_0'.

We deduce from (4.5) that the conjugacy class of the section i_P of (4.6) associated to σ_P admits a representative (also denoted i_p) which sends the basis $(\alpha_j)_{1 \leq j \leq k}$ to some homotopy classes in $\pi_1(\mathcal{A}_{B_0}, w_0)$ which admit representative loops of ρ-lengths bounded by

$$H(s) := c^{-1} L(s+1) + 2(\delta_0 + \delta_0'). \tag{4.8}$$

The term $2(\delta_0 + \delta_0')$ corresponds to the upper bound for the conjugation induced by the change of base points of the loops $\sigma_P(\gamma_i)$'s from $\sigma_P(b)$ to $w_0 = \sigma_O(b_0)$ using a short path living in the fibre A_b of ρ-length bounded by δ_0 which goes from $\sigma_P(b)$ to $\sigma_O(b)$ and the other path $\sigma_O(c_{b_0 b})$ whose ρ-length is bounded by δ_0' by construction. Notice that one cannot simply use the path $\sigma_P(c_{b_0 b})$ (and another path from $\sigma_P(b_0)$ to w_0) since we have no control on its ρ-length.

Via the semi-direct product $\pi_1(\mathcal{A}_{B_0}, w_0) = \pi_1(A_{b_0}, w_0) \rtimes_\varphi \pi_1(B_0, b_0)$, we can write $i_P(\alpha_j) = (\beta_j, \alpha_j)$ where $\beta_j \in \pi_1(A_{b_0}, w_0)$ for every $j \in \{1, \dots, k\}$.

As already remarked above, we can replace B_0 by $B_0 \cup \partial B_0$ without loss of generality. Thus, $(\mathcal{A}_{B_0}, \rho)$ can be regarded as a compact Riemannian manifold with boundary. Let $\pi\colon \tilde{\mathcal{A}}_0 \to \mathcal{A}_{B_0}$, $u\colon \mathbb{R}^{2n} \to A_{b_0}$, and $v\colon \Delta \to B_0$ be the universal covering maps. We have $\tilde{\mathcal{A}}_0 \simeq \mathbb{R}^{2n} \times \Delta$ and a commutative diagram where the composition of the top row is π:

$$\begin{tikzcd} \tilde{\mathcal{A}}_0 \simeq \mathbb{R}^{2n} \times \Delta \arrow[r, "u\times \mathrm{Id}"] & A_{b_0} \times \Delta \arrow[r] \arrow[d, "\mathrm{pr}_2"'] & \mathcal{A}_{B_0} \arrow[d, "f_{B_0}"] \\ & \Delta \arrow[r, "v"] & B_0. \end{tikzcd}$$

The right-most square is the pullback of the fibre bundle $\mathcal{A}_{B_0} \to B_0$ over the contractible open unit disc Δ (in the category of differential manifolds).

Fix a point $\tilde{w} = (\tilde{x}_0, \tilde{y}_0) \in \mathbb{R}^{2n} \times \Delta$ in the fibre $\pi^{-1}(w_0)$ above $w_0 \in \mathcal{A}_{B_0}$. Denote $x_0 = u(\tilde{x}_0) = w_0 \in A_{b_0}$ and $y_0 = v(\tilde{y}_0) = b_0$.

Let $j \in \{1, \dots, k\}$. We claim that the deck transformation $i_P(\alpha_j)w_0 = (\beta_j, \alpha_j)w_0 \in \mathbb{R}^{2n} \times \Delta$ is then simply given by the couple of deck transformations

$$(\varphi_{\alpha_j}(\beta_j)x_0, \alpha_j y_0) \in \mathbb{R}^{2n} \times \Delta$$

(cf. the monodromy action φ in (4.7)) with respect to the chosen points $\tilde{w} \in \pi^{-1}(w_0)$, $\tilde{x}_0 \in u^{-1}(x_0)$ and $\tilde{y}_0 \in v^{-1}(y_0)$ in the universal covers. Indeed, let $\gamma\colon [0,1] \to \mathcal{A}_{B_0}$ the path representing (β_j, α_j). Let $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^{2n} \times \Delta$ be the lifting of γ such that $\tilde{\gamma}(0) = (\tilde{x}_0, \tilde{y}_0) = \tilde{w}$. Define $\gamma' = (u \times \mathrm{Id}) \circ \tilde{\gamma}\colon [0,1] \to A_{b_0} \times \Delta$. Then γ' is the lifting of γ such that $\gamma'(0) = (w_0, \tilde{y}_0)$. Note that α_j is represented by $f_{B_0} \circ \gamma\colon [0,1] \to B_0$. By the definition of the monodromy action φ (via the Homotopy Lifting Property [**111**, page 45]), the homotopy class of γ' in $\pi_1(A_{b_0}, w_0)$ under the projection $A_{b_0} \times \Delta \to A_{b_0}$ is exactly $\varphi_{\alpha_j}(\beta_j)$. The claim thus follows.

The metric ρ on $\mathcal{A}_{B_0}$ pullbacks to a geodesic Riemannian metric $\tilde{\rho}$ on $\mathbb{R}^{2n} \times \Delta \simeq \tilde{\mathcal{A}}_0$. The metric $\tilde{\rho}$ induces in turn a geodesic Riemannian metric $d_j = \tilde{\rho}|_{\mathbb{R}^{2n} \times \{\alpha_j y_0\}}$ on $\mathbb{R}^{2n} \times \{\alpha_j y_0\}$ for every $j = 1, \dots, k$. Then $u\colon \mathbb{R}^{2n} \to A_{b_0}$ makes A_{b_0} into a compact geodesic Riemann manifold with the induced metric d_j for every $j = 1, \dots, k$. By construction and by (4.8), we find that:

$$d_j(\varphi_{\alpha_j}(\beta_j)x_0, \tilde{x}_0) \leq \tilde{\rho}((\beta_j, \alpha_j)w_0, \tilde{w}) \leq H(s).$$

Therefore, as $\pi_1(A_{b_0}, w_0) \simeq \mathbb{Z}^{2n}$ is an abelian group of finite rank, Lemma 6.21.(i) and Proposition 6.20 imply that for every $j = 1, \dots, k$, there exists a constant $m_j > 1$ independent of s such that there are at most $m_j(H(s)+1)^{2n}$ possibilities for $\varphi_{\alpha_j}(\beta_j)$ and thus for $i_P(\alpha_j) = (\beta_j, \alpha_j)$ as well.

Let $m_0 = (\max_{1 \leq j \leq k} m_j)^k > 1$. Then the number of (conjugacy classes of) sections i_P of (4.6), where P is an $(S, \mathcal{D})$-integral point, is at most $m_0(H(s)+1)^{2nk}$. We can therefore conclude from Proposition 6.4 that:

$$\#\left(I_s \text{ mod } \operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C})\right) \leq m(s+1)^{2nk}, \qquad \text{for all } s \geq 0$$

where $m = t_A m_0(c^{-1}L + c^{-1} + 2\delta_0)^{2nk}$ and $t_A = \#(A(K)/\operatorname{Tr}_{K/\mathbb{C}}(A)(\mathbb{C}))_{tors}$. □

Remark 6.10. Let the notations be as in the above proof. Using Lemma 5.24, we can give another proof for certain weaker finiteness parts of Theorem F under the additional assumption that the sets S are taken outside a certain open subset $U_0 \subset B$ containing the point $b_0 \subset B_0$. In fact, instead of obtaining a bound $\operatorname{length}_\rho(\sigma_P(\gamma_j)) \leq c^{-1}(Ls+1)$ which is linear in s as in the above proof, we obtain a weaker bound but sufficient for the finiteness:

$$\begin{aligned}
\operatorname{length}_\rho(\sigma_P(\gamma_j)) &\leq c^{-1} \operatorname{length}_{d_{(\mathcal{A}\setminus\mathcal{D})|_{B_0}}}(\sigma_P(\gamma_j)) \\
&\leq c^{-1} \operatorname{length}_{d_{(\mathcal{A}\setminus\mathcal{D})|_{B_0\setminus S}}}(\sigma_P(\gamma_j)) \quad (\text{as } (\mathcal{A}\setminus\mathcal{D})|_{B_0\setminus S} \subset (\mathcal{A}\setminus\mathcal{D})|_{B_0}) \\
&= c^{-1} \operatorname{length}_{d_{B_0\setminus S}}(\gamma_j) && \text{(by Lemma 6.9)} \\
&\leq c^{-1}\delta && \text{(by Lemma 5.24)}
\end{aligned}$$

where the constant $\delta > 0$ is given in Lemma 5.24. The rest of the proof goes verbatim as in the above proof and we obtain again the finiteness of the set I_s modulo the trace of $\mathcal{A}_K$.

5. Proof of Corollary B

We can now give the proof of the following consequence of Theorem F whose statement is slightly stronger than the one mentioned in Introduction.

Corollary B. *Keep the notations in Theorem F. Assume that* $\operatorname{Tr}_{K/\mathbb{C}}(A) = 0$ *and* $\mathcal{D}$ *is horizontally strictly nef, i.e.,* $\mathcal{D}\cdot C > 0$ *for all curves* $C \subset \mathcal{A}$ *not contained in a fibre. The following hold:*

(i) *for each* $s \in \mathbb{N}$*, there exists a finite subset* $E \subset B$ *such that for any* $S \subset B\setminus E$ *with* $\#(S \cap B_0) \leq s$*, the set of* $(S,\mathcal{D})$*-integral points is empty. Moreover, we can choose* E *such that*

$$\#E \cap B_0 \leq ms(s+1)^{2\dim A.\operatorname{rank}\pi_1(B_0)}.$$

(ii) *for each* $s \in \mathbb{N}$*, there exists a Zariski proper closed subset* $\Delta \subset B^{(s)}$ *such that for any* $S \subset B$ *of cardinality* s *and whose image* $[S] \in B^{(s)} \setminus \Delta$*, there is no* $(S,\mathcal{D})$*-integral points of* $\mathcal{A} \to B$*.*

Proof of Corollary B.(i). The hypothesis $\operatorname{Tr}_{K/\mathbb{C}}(A) = 0$ and Theorem F imply that the union of all $(S,\mathcal{D})$-integral points of A, where $S \subset B$ with $\#S \cap B_0 \leq s$, is finite. Let $P_1, \ldots, P_q \in A(K)$ be all such integral points where $q \leq m(s+1)^r$

for some $m = m(\mathcal{A}, B_0) > 0$ given in Theorem F and $r = 2 \dim A. \operatorname{rank} \pi_1(B_0)$. By the definition of $(S, \mathcal{D})$-integral points, for each $i = 1, \dots, q$, we have $f(\sigma_{P_i}(B) \cap \mathcal{D}) \cap B_0 = S_i$ for some finite subset $S_i \subset B_0$ of cardinality at most s. In particular, for every $i = 1, \dots, q$, we have $\sigma_{P_i}(B) \not\subset \mathcal{D}$ so that $\sigma_{P_i} \cap \mathcal{D}$ is finite as σ_{P_i} is algebraic. Hence, we can define the finite intersection loci in B by

$$E \coloneqq \cup_{i=1}^{q} f(\sigma_{P_i}(B) \cap \mathcal{D}) \subsetneq B.$$

Moreover, we have:

$$\# E \cap B_0 = \# \cup_{i=1}^{q} S_i \leq qs \leq m(s+1)^r s.$$

We claim that E verifies the point (i). Indeed, let $S \subset B \setminus E$ be any subset such that $\# S \cap B_0 \leq s$ and suppose on the contrary that $P \in A(K)$ is an $(S, \mathcal{D})$-integral point. Then $P = P_j$ for some $1 \leq j \leq q$ by the definition of the P_i's. Hence, $f(\sigma_P(B) \cap \mathcal{D}) \subset E$. But P is $(S, \mathcal{D})$-integral so that $f(\sigma_P(B) \cap \mathcal{D}) \subset S$. As $S \cap E = \varnothing$, we deduce that $f(\sigma_P(B) \cap \mathcal{D}) = \varnothing$ and thus $\deg_B \sigma_P^* \mathcal{D} = 0$. This is a contradiction since $\mathcal{D}$ is strictly nef by hypothesis. We conclude the proof of Corollary B.(i). ☐

Proof of Corollary B.(ii). Let E and r be given in Corollary B.(i). Let $\Delta \subset B^{(s)}$, where $B^{(s)} = B^s / \mathfrak{S}_s$ is the s-th symmetric product, be the image of the $(s-1)$-dimensional closed subset $E \times B^{s-1} \subset B^s$ under the quotient map $p \colon B^s \to B^{(s)}$. Since p is a finite morphism of algebraic schemes, Δ is an $(s-1)$-dimensional algebraic closed subset of $B_0^{(s)}$. Now let $[S] \in B^{(s)} \setminus \Delta$ and let $S = \operatorname{supp}[S] \subset B$. Hence $\# S \leq s$ and it follows from the construction of Δ that $S \cap E = \varnothing$. We claim that non of the P_i's is an $(S, \mathcal{D})$-integral point. Indeed, if P_i is $(S, \mathcal{D})$-integral then $\sigma_{P_i}(B) \cap \mathcal{D}$ is non empty since $\mathcal{D}$ is strictly nef by hypothesis. On the other hand, $f(\sigma_{P_i}(B) \cap \mathcal{D}) \subset S \cap E = \varnothing$ by the definition of E. We arrive at a contradiction and the claim is proved. The proof of Corollary B is thus completed. ☐

Remark 6.11. In fact, we also have a quantitative statement for Corollary B.(ii). Let $B_0^{(s)} = B_0^s / \mathfrak{S}_s$ be the s-th symmetric product of B_0. Then $B_0^{(s)} \subset B^{(s)}$. Let $E_0 = E \cap B_0 \subset B$. Let $\Delta_0 = \Delta \cap B_0^{(s)}$ then Δ_0 the union of $\# E_0 \leq ms(s+1)^r$ closed subspaces $E_0 \times B_0^{s-1} \subset B_0^s$. Δ_0 is an $(s-1)$-dimensional algebraic closed subspace of $B_0^{(s)}$. Now let $[S_0] \in B_0^{(s)} \setminus \Delta_0$ and let $S_0 = \operatorname{supp}[S_0] \subset B_0$. Then $\# S_0 \leq s$ and it follows from the construction of Δ_0 that $S_0 \cap E_0 = \varnothing$. For every $P \in A(K)$, $\sigma_P(B) \cap \mathcal{D}$ is non empty since $\mathcal{D}$ horizontally strictly nef. We deduce as above that non of the P_i's is an $(S_0, \mathcal{D})$-integral point by the definition of E. By the definition of the P_i's, it follows that there is no $(S_0, \mathcal{D})$-integral points whenever $[S_0] \in B_0^{(s)} \setminus \Delta_0$.

6. Proof of Theorem G

PROOF OF THEOREM G. By the Lang-Néron theorem, $A(K)$ is a finitely generated abelian group. In particular, the subset $R \subset A(K)$ is at most countable. On the other hand, for every $P \in R$, the set $\sigma_P(B) \cap \mathcal{D}$ is finite since $P \notin D$. It follows that the set $I(R, \mathcal{D})$ is at most countable.

Let $V := \{b \in B \colon \mathcal{D}_b \text{ is not hyperbolic}\} \subset B$. Then V is an analytically closed subset of B (cf. [**11**]) which is at most countable by Lemma 2.49 since $V \subset Z(\mathcal{A}, \mathcal{D})$ by Theorem 6.8 where $Z(\mathcal{A}, \mathcal{D})$ is defined in Lemma 2.49.

For (a), we suppose on the contrary that $I(R, \mathcal{D})$ is analytically closed in B. It follows that $V \cup I(R, \mathcal{D})$ is analytically closed and countable in B. We fix an arbitrary finite disjoint union W of closed discs centered at the points of T (points of bad reduction of the family). Then the same argument at the beginning of the proof of Theorem F implies that we can enlarge the discs in W to contain $V \cup I(R, \mathcal{D})$ so that they are still closed, disjoint and contain points of T separately. Theorem F applied for $B_0 = B \backslash W$ tells us that $A(K)$ and thus $R \subset A(K)$ contains only finitely many $(W, \mathcal{D})$-integral points, since $\mathrm{Tr}_{K/\mathbb{C}}(A) = 0$. On the other hand, since $I(R, \mathcal{D}) \subset W$, every $P \in R$ is an $(U, \mathcal{D})$-integral point. It follows that R must be a finite subset, which is a contradiction to the assumption that R is infinite. Hence, $I(R, \mathcal{D})$ is not analytically closed in B and in particular, it must be infinite. The proof of (a) is thus completed. Observe that the same argument proves also the property (c).

Since the set of limit points of any subset is closed, it is clear that $I(R, \mathcal{D})_\infty$ is closed in the analytic topology. For (b), suppose also on the contrary that $I(R, \mathcal{D})_\infty$ is countable. Then (a) implies that the closure

$$\overline{I(R, \mathcal{D})} = I(R, \mathcal{D}) \cup I(R, \mathcal{D})_\infty$$

is a countable and analytically closed subset of B. Therefore, the same argument as above shows that R is a finite subset of $A(K)$ which is again a contradiction. This proves (b) and the proof is completed. $\square$

7. Appendix: Geometry of the fundamental groups

In this section, we collect standard results on the geometry of the fundamental groups which are necessary for the proof of Theorem F. For the convenience of future works, we decided to give the main statements Proposition 6.20 and Lemma 6.21 which are slightly more general than what we shall actually need.

We first recall the following notion of quasi-isometry introduced by Gromov:

Definition 6.12 (Gromov)**.** Let X, Y be metric spaces. Let $L, A > 0$. We say that a map $f\colon X \to Y$ (not necessarily continuous) is (L, A)-quasi-isometry if:

(1) (equivalence) $\frac{1}{L}d(x,y) - A \le d(f(x), f(y)) \le Ld(x,y) + A$, for all $x, y \in X$;
(2) (quasi-surjective) there exists $R > 0$ such that for every point $y \in Y$, we have $f(X) \cap B(y, R) \neq \varnothing$.

Proposition 6.13. *Let X, Y be metric spaces. Suppose that X is quasi-isometric to Y. Then Y is also quasi isometric to X. Moreover, being "quasi-isometric to" is an equivalence relation.*

Proof. By hypothesis, there exists an (L, A)-quasi-isometry $f\colon X \to Y$ for some $L, A > 0$. Hence, by definition, there exists $R > 0$ such that $f(X) \cap B(y, R) \neq \varnothing$ for every $y \in Y$. We can clearly assume also that

$$R > \max\{A(1-L)/2L, A(L-1)/2\}.$$

We will construct a quasi-isometry $h\colon Y \to X$ as follows. For every $y \in Y$, choose any $x \in X$ such that $f(x) \in B(y, R)$, and define $h(y) = x$. Such x exists by the above remark so that h is well-defined. We claim that h is an $(L, A + 2R)$-quasi-isometry. Indeed, let $y_1, y_2 \in Y$ and $x_1 = h(y_1)$, $x_2 = h(y_2)$ then we have $d(f(x_1), y_1) < R$ and $d(f(x_2), y_2) < R$. Thus, by the (L, A)-equivalence property of f and the triangle inequality on (Y, d), we find:

$$\begin{aligned} d(h(y_1), h(y_2)) = d(x_1, x_2) &\le L(d(f(x_1), f(x_2)) + A) \\ &\le L(d(f(x_1), y_1) + d(y_1, y_2) + d(y_2, f(x_2)) + A) \\ &\le Ld(y_1, y_2) + (A + 2R), \end{aligned}$$

and similarly,

$$\begin{aligned} d(y_1, y_2) = d(f(x_1), f(x_2)) &\le Ld(x_1, x_2) + A \\ &\le Ld(h(y_1), h(y_2)) + A \\ &< L(d(h(y_1), h(y_2)) + A + 2R) \end{aligned}$$

so that we obtain the $(L, A + 2R)$-equivalence of h:

$$\frac{1}{L}d(y_1, y_2) - (A + 2R) \le d(h(y_1), h(y_2)) \le Ld(y_1, y_2) + A + 2R.$$

For the quasi-surjectivity, observe that for every $x \in X$ and $y = f(x) \in Y$, we have by the the definition of h that $d(f(h(y)), y) < R$. It follows that

$$d(x, h(y)) \le Ld(f(x), f(h(y))) + LA = Ld(y, f(h(y))) + LA \le LR + LA$$

and therefore $h(Y) \cap B(x, LR + LA) \neq \varnothing$ for every $x \in X$. Other properties of an equivalence relation can be easily checked. The proof is completed. $\square$

We recall also the notion of the word metric of a finitely generated group.

Definition 6.14. Let G be a finitely generated group and let S be a finite generating subset. The function $d_S\colon G \times G \to \mathbb{N}$ given by

$$(g,h) \mapsto \begin{cases} 0 & \text{if } g = h \\ \min\{n\colon g^{-1}h = s_1 \cdots s_n \text{ for some } s_1, \dots s_n \in S \cup S^{-1}\} & \text{otherwise} \end{cases}$$

defines a metric on G called the *word metric* associated to the generating set S.

The following sufficient conditions of quasi-isometry is due to Milnor and Svarc. It is sometimes called the Fundamental Lemma of Geometric Group Theory.

Theorem 6.15 (Milnor-Svarc)**.** *Let (X,d) be a geodesic metric space and G a finitely generated group acting on X by isometries, i.e., $d(gx,gy) = d(x,y)$. Let $x_0 \in X$ and suppose also that:*

(1) *The action is cobounded: there exists $R > 0$ such that the translates of the ball $B(x_0,R)$ cover X, i.e.,*
$$X = \cup_{g\in G}(gB(x_0,R)) = \cup_{g\in G}B(gx_0,R);$$
(2) *The action is metrically proper: for any $r > 0$, the following set is finite:*
$$\{g \in G\colon B(x_0,r) \cap gB(x_0,r) \neq \varnothing\}.$$

Then G is finitely generated and the map $p\colon G \to X$ defined by $p(g) = gx_0$ is a quasi-isometry where G is equipped with an arbitrary word metric associated to a finite system of generators.

Proof. See [**10**, Proposition I.8.19]. □

Lemma 6.16. *Every finitely generated group G has at most exponential growth, i.e., for every finite generating subset $S \subset G$, there exists a constant $N > 1$ such that for every $R > 0$, we have*

$$\#B((G,d_S),R) \leq N^R. \tag{7.1}$$

Proof. We will show that every $N \geq 2s+1$ satisfies the inequality (7.1), where $s = \#S$ is the cardinality of S. Moreover, the equality holds only if G is a free group of finite rank. Indeed, observe that we have by definition of the word metric d_S that for every $R > 0$,

$$B((G,d_S),R) \subset \left\{ \prod_{j=1}^{\lfloor R \rfloor} s_j \colon s_j \in S \cup S^{-1} \cup \{1_G\} \right\},$$

where $\lfloor R \rfloor$ denotes the largest integer smaller than or equal to R. It follows that

$$\begin{aligned}\#B((G,d_S),R) &\leq \# \left\{ \prod_{j=1}^{\lfloor R \rfloor} s_j \colon s_j \in S \cup S^{-1} \cup \{1_G\} \right\} \\ &\leq (\#S + \#S^{-1} + 1)^{\lfloor R \rfloor} \\ &\leq (2s+1)^R\end{aligned}$$

and the lemma is proved by taking $N = 2s+1$. Remark finally that if the equality $\#B((G,d_S),R) \leq (2s+1)^R$ is to hold for every $R > 0$ then we must have

$$B((G,d_S),R) = \left\{ \prod_{j=1}^{\lfloor R \rfloor} s_j \colon s_j \in S \cup S^{-1} \cup \{1_G\} \right\}.$$

In particular, there does not exist any relation between the elements in the generating set S. Therefore, G must be a free group of finite rank. □

We mention here the celebrated theorem of Gromov classifying all groups of polynomial growth.

Theorem 6.17 (Gromov)**.** *Let $(G, 1_G)$ be a finitely generated group. Fix a finite generating system $S \subset G$ and consider the corresponding word metric d_S. Let $n(L)$ be the number of elements $g \in G$ such that $d_S(g, 1_G) \leq L$. Then there exist $a, r > 0$ such that:*

$$n(L) \leq a(L+1)^r \quad \text{for all } L > 0$$

if and only if G is virtually nilpotent, i.e., G admits a finite index subgroup H which is nilpotent, i.e., $H_m = 0$ for some $m \geq 0$ where $H_{k+1} := [H_k, H]$, $H_0 = H$.

PROOF. See [**44**, Main Theorem]. □

We recall without proof the following standard theorem.

Theorem 6.18. *Let M be a compact (Hausdorff) space which admits a finite cover by open simply connected sets, and which is locally path connected (i.e., there is a base for the topology consisting of path connected sets). Then $\pi_1(M)$ is finitely generated. In particular, this holds for all compact semi-locally simply connected spaces and thus for all compact Riemann manifold with or without boundary.*

Definition 6.19 (Equivalence of growth functions)**.** Two increasing functions $f, g \colon \mathbb{R}_+ \to \mathbb{R}_+$ are said to have the *same order of growth* if there exist constants $a, c > 0$ such that for all $r > 0$, we have:

$$f(r) \leq cg(ar), \quad \text{and } g(r) \leq cf(ar).$$

In particular, if f, g have the same order of growth, then f is bounded from above (resp. from below) by a polynomial of degree m if and only if so is g.

The main statement of the section is the following.

Proposition 6.20. *Let (M, d) be a connected compact Riemann manifold with or without boundary. Let $\pi\colon \tilde{M} \to M$ be the universal cover of M. Then for any point $x_0 \in M$ not lying on the boundary, the map $p\colon \pi_1(M, x_0) \to \pi^{-1}(x_0)$ given by $g \mapsto gx_0$ is a bijective quasi-isometry. In particular, the growths of $\pi_1(M, x_0)$ are the same whether calculated with respect to the induced geometric norm by d on $\pi^{-1}(x_0)$ or with respect to the algebraic word norm associated to an arbitrary finite system of generators of $\pi_1(M, x_0)$.*

Proof. First observe that $\pi_1(M)$ is a finitely generated group by Theorem 6.18. Since M is a connected compact Riemann manifold with boundary, it is also a geodesic metric space. Fix a point $\tilde{x}_0 \in \pi^{-1}(x_0)$. Theorem 6.15 implies immediately that the action (the deck transformation) of the fundamental group $\pi_1(M, x_0)$ on the universal cover $\tilde{M}$ induces a quasi-isometry between $(\pi_1(M, x_0), d_{word})$ and $(\tilde{M}, \tilde{d})$, say an (L, A)-quasi-isometry. Here d_{word} denotes the word metric of the group $\pi_1(M, x_0)$ associated to some finite generating set and $\tilde{d}$ denotes the induced metric of d on $\tilde{M}$. Since the action of $\pi_1(M, x_0)$ commutes with the projection $\pi\colon \tilde{M} \to M$, we deduce from the above paragraph that the same quasi-isometry $g \mapsto gx_0$ gives us a bijective (L, A)-quasi-isometry between $(\pi_1(M, x_0), d_{word})$ and the fibre $\pi^{-1}(x_0)$ equipped with the induced metric $\tilde{d}|_{\pi^{-1}(x_0)}$:

$$L^{-1} d_{word}(g, h) - A \leq \tilde{d}|_{\pi^{-1}(x_0)}(gx_0, hx_0) \leq L d_{word}(g, h) + A$$

for every $g, h \in \pi_1(M, x_0)$. Let $D > 0$, it follows from the above estimations and the bijection $g \mapsto gx_0$ between $\pi_1(M, x_0)$ and $\pi^{-1}(x_0)$ that:

$$\#B(\pi_1(M, x_0), L^{-1}D + A) \leq \#B((\pi^{-1}(x_0), \tilde{x}_0), D) \leq \#B(\pi_1(M, x_0), LD + LA).$$

Here, $B(\pi_1(M, x_0), r)$ denotes the ball of d_{word}-radius r in $\pi_1(M, x_0)$ centered at 0. Similarly, $B((\pi^{-1}(x_0), \tilde{x}_0), r)$ denotes the ball of $\tilde{d}|_{\pi^{-1}(x_0)}$-radius r in $\pi^{-1}(x_0)$ centered at $\tilde{x}_0$. Since LD+A and $LD + LA$ are fixed linear functions in D, the growths of $\pi^{-1}(x_0)$ and of $\pi_1(M, x_0)$ are clearly of the same order. $\square$

An important application of Proposition 6.20 is the following lemma which can be seen as the hyperbolic-homotopy analogue of Lenstra's Counting Lemma 2.24:

Lemma 6.21. *Let (M, d) be a connected compact Riemannian manifold with (possibly empty) boundary. Let $x_0 \in M$. For every $L > 0$, define $n(L)$ to be the number of homotopy classes in $\pi_1(M, x_0)$ which admit some representative loops of length at most L with respect to the metric d. The following hold:*

(i) *if $\pi_1(M)$ is virtually nilpotent, then there exists $a, r > 0$ such that:*

$$n(L) \leq a(L+1)^r \quad \textit{for all } L > 0.$$

If $\pi_1(M)$ is abelian, r can be chosen to be the rank of $\pi_1(M)$;

(ii) *In general, there exists $p > 0$ such that:*

$$n(L) \leq \exp(p(L+1)) \quad \textit{for all } L > 0.$$

Proof. Let $\pi\colon \tilde{M} \to M$ be the universal cover of M and fix a point $\tilde{x}_0 \in \pi^{-1}(x_0)$. Then $\tilde{M}$ is naturally a connected geodesic Riemannian manifold with metric $\tilde{d}$ which makes π into a local isometry. The length of loops in (M, x_0) is the same as the length of their lifts in $\tilde{M}$. The metric on $\pi^{-1}(x_0)$ is of geometric nature and is induced by $\tilde{d}$. Hence, a loop γ in M based at x_0 of length at most L is uniquely determined by a point $[\gamma]x_0 \in \pi^{-1}(x_0)$ such that $\tilde{d}([\gamma]x_0, \tilde{x}_0) \leq L$.

By Theorem 6.15, $\pi_1(M, x_0)$ is quasi-isometric to the fibre $\pi^{-1}(x_0)$. Hence, they have the same order of growth (Proposition 6.20) where the metric on $\pi^{-1}(x_0)$ is induced by $\tilde{d}$ while the one on $\pi_1(M, x_0)$ is the word metric with respect to a finite system of generators (which exists since $\pi_1(M, x_0)$ is finitely generated by Theorem 6.18). Since the growth of $\pi_1(M, x_0)$ is at most exponential (cf. Lemma 6.16), the point (ii) is proved. The above discussion and Theorem 6.17 of Gromov altogether imply the point (i) except for the second statement.

If $\Gamma = \pi_1(M, x_0)$ is an abelian, it is admits a finite rank $r \geq 0$ (since it is finitely generated by Theorem 6.18). Hence, $\Gamma = \mathbb{Z}g_1 \oplus \cdots \oplus \mathbb{Z}g_r \oplus \Gamma_{tors}$ for some $g_1, \ldots, g_r \in \Gamma$. Recall the word metric d_S on Γ where $S = \{g_1, \ldots, g_r\}$. It is easy to see that:

$$\begin{aligned} \#\{g \in \Gamma\colon d_S(g, 1_\Gamma) \leq L\} &\leq \#\Gamma_{tors}.\#\left\{g = \sum_{j=1}^{r} n_j g_j\colon -L \leq n_j \leq L\right\} \\ &\leq \#\Gamma_{tors}(2L+1)^r \end{aligned}$$

The conclusion follows by Proposition 6.20. □

8. Appendix: An explicit proof of Proposition 6.3

Recall the diagram in Proposition 6.3 that we are going to describe in details:

(8.1)
$$\begin{tikzcd} A(K)/nA(K) \arrow[r, hook, "\delta"] & H^1(\widehat{G}, A[n]) \arrow[drr, "\simeq"] & & \\ A(K) \arrow[u] \arrow[rr, "\alpha"] & & H^1(G, \Gamma) \arrow[r, "\beta"] & H^1(G, A[n]). \end{tikzcd}$$

Experts are welcome to skip this section. Let's begin with some observations:

Preliminary remarks: Let $S \subset B$ be the centres of the discs in W. Note that $T \subset S$ by construction. First, remark that we have

$$\widehat{G} = G_{K,S} \coloneqq \mathrm{Gal}(K_S/K)$$

where $\widehat{G}$ is the profinite completion of G and K_S/K is the maximal Galois extension of K which is unramified outside of S. The equality is well-known and follows from the theory of étale fundamental group.

Second, observe that $f_{B_0}\colon \mathcal{A}_{B_0} \to B_0$ is an abelian scheme since $T \subset S$. For every integer $m \geq 1$ and $P \in A(K)$, consider the following cartesian diagram

$$\begin{tikzcd} C_{P,m} \arrow[r, hook] \arrow[d, "\times m"] & \mathcal{A}_{B_0} \arrow[d, "\times m"] & \\ \sigma_P(B_0) \arrow[r, hook] & \mathcal{A}_{B_0} \arrow[r, "f"] & B_0. \end{tikzcd}$$

The multiplication-by-m map $[m]\colon \mathcal{A} \to B$ is finite flat of degree $m^{2\dim A}$. Since we are in characteristic 0, it follows that each fibre of $\mathcal{A}[m]$ above point in B_0 is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^{2\dim A}$. Therefore, $[m]$ is finite étale over B_0 of degree $m^{2\dim A}$. The induced morphism $f_{C_{P,m}}\colon C_{P,m} \to B_0$ is also finite étale over B_0 since $\sigma_P(B_0) \to B_0$ is an isomorphism.

Third, since $A[n]$ is finite and G is discrete, it follows from [**98**, I.2.6.b] that the natural map $H^1(\widehat{G}, A[n]) \to H^1(G, A[n])$ induced by the injection $G \to \widehat{G}$ (note that G is residually finite) is an isomorphism.

Proof of Proposition 6.3. We begin with:

The description of δ: Since $A[n] \subset A(K)$,$\mathrm{Gal}(\bar{K}/K)$ and thus $G_{K,S} \subset \mathrm{Gal}(\bar{K}/K)$ act trivially on $A[n]$. Hence, $H^1(G_{K,S}, A(K)[n]) = \mathrm{Hom}(G_{K,S}, A(K)[n])$ by the definition of the first cohomology group. The Kummer exact sequence

$$0 \to A[n] \to A(K_S) \xrightarrow{\times n} A(K_S) \to 0$$

induces a long exact sequence

$$0 \to A[n]^{G_{K,S}} \to A(K_S)^{G_{K,S}} \to A(K_S)^{G_{K,S}} \to H^1(G_{K,S}, A[n]).$$

Since $A[n]^{G_{K,S}} = A[n]$ and $A(K_S)^{G_{K,S}} = A(K)$, we obtain in particular from the above exact sequence the injective homomorphism δ described as follows:

$$\begin{aligned} \delta\colon A(K)/nA(K) &\to \mathrm{Hom}(G_{K,S}, A[n]) \\ P &\mapsto (\phi_P\colon G_{K,S} \to A[n],\ \sigma \mapsto Q^\sigma - Q) \end{aligned}$$

for any $Q \in A(K_S)$ such that $nQ = P$. Then ϕ_P is well-defined and independent of the choice of Q. Indeed, since A is defined over K, the map $[n]$ is defined over K and G_K acts as identity on $A(K)$. Note that $n(Q^\sigma - Q) = (nQ)^\sigma - nQ = P^\sigma - P = O$ so $Q^\sigma - Q \in A[n]$. If $P = nQ = nQ'$ then $n(Q - Q') = O$ and $Q - Q' \in A[n] \subset A(K)$ so that $(Q - Q')^\sigma = Q - Q'$ and thus $Q^\sigma - Q = Q'^\sigma - Q'$ as desired. To check that ϕ_P is a homomorphism, let $\sigma, \tau \in G_{K,S}$ and $Q \in A(\bar{K})$ such that $nQ = P$, then

$$\begin{aligned}\phi_P(\sigma\tau) = Q^{\sigma\tau} - Q &= (Q^\sigma)^\tau - Q^\sigma + Q^\sigma - Q \\ &= Q^\tau - Q + Q^\sigma - Q \\ &= \phi_P(\tau) + \phi_P(\sigma)\end{aligned}$$

where the second equality follows from the fact that $[n]Q^\sigma = ([n]Q)^\sigma = P^\sigma = P \in A(K)$ and the well-defined property of ϕ_P. Similarly, we verify easily that δ is a homomorphism.

The description of α: recall that from the fibre bundle $A_{b_0} \to \mathcal{A}_{B_0} \to B_0$ (recall that $b_0 \in B_0$ is fixed in the statement of Proposition 6.3), we have an exact sequence of fundamental groups

$$0 \to \pi_1(A_{b_0}, w_0) = H_1(A_{b_0}, \mathbb{Z}) \to \pi_1(\mathcal{A}_{B_0}, w_0) \xrightarrow{\rho = f_*} \pi_1(B_0, b_0) \to 0. \tag{8.2}$$

The right exactness is due to the existence of the zero section while the left exactness is because $\pi_2(B_0) = 0$ as B_0 admits a simply connected universal cover (the complex disc). Since A_{b_0} is path connected, we can fix a collection of smooth (geodesic) paths $l_{w_0,w} \colon [0,1] \to A_{b_0}$ such that $l_{w_0,w}(0) = w_0$ and $l_{w_0,w}(1) = w$. Now, each section (analytic or algebraic) $\sigma_P \colon B_0 \to \mathcal{A}_{B_0}$ induces "naturally" a section $i_P \colon \pi_1(B_0, b_0) \to \pi_1(\mathcal{A}_{B_0}, w_0)$ of (8.2) as follows. Take any loop γ of B_0 based at b_0, we define the section i_P by the formula:

$$i_P([\gamma]) = [l^{-1}_{w_0,\sigma_P(b_0)} \circ \sigma_P(\gamma) \circ l_{w_0,\sigma_P(b_0)}] \in \pi_1(\mathcal{A}_{B_0}, w_0). \tag{8.3}$$

As a convention, we shall hereafter concatenate oriented paths as above, as oppose to the usual composition of homotopy classes, so the multiplication order reverses. As i_P, i_O are sections of ρ, the difference $i_P - i_O$ satisfies

$$\rho(i_P - i_O) = \rho(i_P) - \rho(i_O) = 0.$$

Therefore, we have $\mathrm{Im}(i_P - i_O) \subset \operatorname{Ker} \rho = H_1(A_{b_0}, \mathbb{Z})$. Thus, we have just defined a map $i_P - i_O \colon \pi_1(B_0, b_0) \to H_1(A_{b_0}, \mathbb{Z})$. This map is well-defined modulo a principal crossed homomorphism induced by different choices of the paths $l_{w_0,w}$ (these choices of paths also give rise to the conjugation class of i_P).

From the exact sequence (8.2), we claim that $i_P - i_O$ is a 1-cocycle of the group $G = \pi_1(B_0, b_0)$ with coefficients in $\Gamma = H_1(A_{b_0}, \mathbb{Z}) \simeq \mathbb{Z}^{2 \dim A}$ where the monodromy

G-action is given by conjugation. Remark that in general, this map is not an element of $\mathrm{Hom}(\pi_1(B_0, b_0), H_1(A_{b_0}, \mathbb{Z}))$ since the action may be not trivial.

Let $F := A_{b_0}$, the action of G on Γ is described as follows. Let $\lambda\colon I \to F$ be a loop where $I = [0,1]$ such that $\lambda(0) = \lambda(1) = w_0$. Let $\gamma\colon I \to B_0$ be another loop in B_0 with $\gamma(0) = \gamma(1) = b_0$. Let $\gamma' = \sigma_O \circ \gamma$. By the exact sequence (8.2), $\gamma' \circ \lambda \circ \gamma'^{-1}$ defines an element in $\pi_1(A_{b_0}, w_0)$ denoted $[\gamma] \cdot [\lambda]$. It is clear that $[\gamma] \cdot [\lambda]$ depends only on the homotopy classes $[\lambda]$ and $[\gamma]$ (with base points). Thus we obtain the action of G on Γ by conjugation.

Writing the group laws of fundamental groups multiplicatively, we have:

$$\begin{aligned}(i_P - i_O)([\gamma_1][\gamma_2]) &= i_P[\gamma_1] i_P[\gamma_2] i_O[\gamma_2]^{-1} i_O[\gamma_1]^{-1} \\ &= (i_P[\gamma_1])([\gamma_1']^{-1})[\gamma_1']))(i_P[\gamma_2])[\gamma_2']^{-1})[\gamma_1']^{-1} \\ &= (i_P[\gamma_1][\gamma_1']^{-1})[\gamma_1'](i_P[\gamma_2][\gamma_2']^{-1})[\gamma_1']^{-1} \\ &= ((i_P - i_O)[\gamma_1])([\gamma_1] \cdot (i_P - i_O)[\gamma_2]).\end{aligned}$$

Therefore, $i_P - i_O$ is a 1-cocycle of G as claimed. Notice that $H^1(G, \Gamma)$ is an abelian group since Γ is abelian. We claim that the induced natural map

$$\alpha\colon A(K) \to H^1(G, \Gamma), \quad P \mapsto i_P - i_O, \tag{8.4}$$

is a homomorphism, i.e., we must show that $\alpha(P+Q) = \alpha(P) + \alpha(Q)$ for all $P, Q \in A(K)$. This is due to the fact that $H_1(F)$ is compatible with the group law of F. More precisely, since $f\colon \mathcal{A}_{B_0} \to B_0$ is an abelian scheme over B_0, the translation by σ_P induces a B_0-automorphism of $\mathcal{A}_{B_0}$ which is trivial on $H_1(F)$: let $\gamma\colon I \to F$ and $a \in F$ then $[a + \gamma] = [\gamma] \in H_1(F, \mathbb{Z})$ as they are homologous under the map

$$I \times I \to F, \quad (x, y) \mapsto \gamma(x) + ya.$$

In particular, for all $[\gamma] \in \pi_1(B_0, b_0)$, we have the following equalities in $H_1(F, \mathbb{Z})$:

$$(i_{P+Q} - i_P)[\gamma] = \sigma_P((i_Q - i_O)[\gamma]) = (i_Q - i_O)[\gamma].$$

It follows that $i_{P+Q} - i_P = i_Q - i_O$ and hence $\alpha(P+Q) = \alpha(P) + \alpha(Q)$.

The description of β: Since F is a complex abelian variety, we can write $F = V/\Gamma$ where $V \simeq \mathbb{C}^{\dim A}$ and $\Gamma = H_1(F, \mathbb{Z}) = \pi_1(F) \subset V$. Let $\mathbb{Z}(1) = \mathrm{Ker}(\exp\colon \mathbb{C} \to \mathbb{C}^*) = 2i\pi\mathbb{Z}$. The map $\mathbb{C} \to \mathbb{C}^*$ given by $x \mapsto \exp(x/n)$ induces an isomorphism $\mathbb{Z}(1)/n\mathbb{Z}(1) \simeq \mu_n$. The dual abelian variety of F is given by $F^\vee = V'/\Gamma'$ where $V' = \bar{V}^\vee$ is the space of $\mathbb{C}$-conjugate linear maps on V and

$$\Gamma' = \{h \in V' \colon h(\bar{\lambda}) \in \mathbb{Z}(1),\ \forall \lambda \in \Gamma\}.$$

We have natural maps

$$\begin{aligned} H_1(F,\mathbb{Z}) = \Gamma &\longrightarrow \mathrm{Hom}(\Gamma', \mu_n) = H^1(F^\vee, \mu_n) \\ &= \mathrm{Hom}(\Gamma'/n\Gamma', \mu_n) = \mathrm{Hom}(\frac{1}{n}\Gamma'/\Gamma', \mu_n) \\ &= \frac{1}{n}\Gamma/\Gamma = F[n] = A[n]. \end{aligned}$$

described as follows. The map $\Gamma \to \mathrm{Hom}(\Gamma', \mu_n) = \mathrm{Hom}(\Gamma'/n\Gamma', \mu_n)$ is given by $x \mapsto (h \mapsto \exp(h(x)/n))$. Since $H_0(F,\mathbb{Z}) = 0$, the Universal coefficient gives us the natural isomorphism $H^1(F^\vee, \mu_n) \simeq \mathrm{Hom}(\Gamma', \mu_n)$. The division by n gives us the identification $\Gamma'/n\Gamma' = \frac{1}{n}\Gamma'/\Gamma'$. Finally, the canonical isomorphism $\frac{1}{n}\Gamma/\Gamma \simeq \mathrm{Hom}(\frac{1}{n}\Gamma'/\Gamma', \mu_n)$ is given by

$$\frac{1}{n}x + \Gamma \mapsto \left(\frac{1}{n}h + \Gamma' \mapsto \exp(h(\bar{x})/n)\right).$$

In other words, the homomorphism $\beta\colon H^1(G,\Gamma) \to H^1(G, A[n])$ is simply induced by the homomorphism of G-modules $\Gamma \to A[n]$ given by $x \mapsto \frac{1}{n}x + \Gamma$. The geometric meaning of the latter map is the following. We regard x as an element of $\Gamma = \pi_1(A, O)$. Then by the isomorphism $T\colon \pi_1(A,O) \to \Gamma$ defined in the proof of Lemma 6.23, we have $T(x) = x \cdot 0$ as an element in the lattice $\Gamma \subset \mathbb{C}^d$. Then $\frac{1}{n}x + \Gamma$ is viewed as $\frac{1}{n}T(x) + \Gamma \in \frac{1}{n}\Gamma/\Gamma = A[n]$.

Commutativity of the diagram: To finish the proof of Proposition 6.3, we need to check that the diagram commutes. This is the content of Lemma 6.22 below. ☐

Lemma 6.22. *For every $P \in A(K)$ and $Q \in A(K_S)$ such that $nQ = P$, and every $\tau \in G$, we have*

$$i_P(\tau) - i_O(\tau) = n(Q^\tau - Q) \mod nH_1(F,\mathbb{Z}).$$

Proof. Recall that $f_{C_{P,n}}\colon C_{P,n} \to B_0$ is a finite étale cover of B_0. Choose a point $R \in C_{P,n} \cap A_{b_0}$ and a representative loop $L\colon [0,1] \to B_0$ of τ. Denote by R^τ the image of R under the deck transformation action of τ on $C_{P,n}$. Then $Q^\tau - Q \in A[n]$ can be identified with $R^\tau - R \in A_{b_0}[n]$. Let $P_0 = \sigma_P(b_0) \in A_{b_0}$ then remark that $nR^\tau = nR = P_0$. Observe also that the loop L lifts to a unique path $\tilde{L}\colon [0,1] \to C_{P,n}$ with $\tilde{L}(0) = R$ and $\tilde{L}(1) = R^\tau$.

Let $\tilde{R}, \tilde{R}^\tau \in \mathbb{C}^{\dim A}$ be any points lying above R and R^τ under the universal cover map $\mathbb{C}^{\dim A} \to A_{b_0} = \mathbb{C}^{\dim A}/\Gamma$. Let l_{OR}, l_{RR^τ} and l_{OR^τ} be paths in A_{b_0} going from O_{b_0} to R and R^τ respectively which are images of the linear paths $l_{\overline{0\tilde{R}}}, l_{\overline{\tilde{R}\tilde{R}^\tau}}$ and $l_{\overline{0\tilde{R}^\tau}}$ in $\mathbb{C}^{\dim A}$.

Recall the multiplication-by-n map $[n]\colon \mathcal{A}_{B_0} \to \mathcal{A}_{B_0}$. By the exact sequence (8.2) and the properties $[n]R = [n]R^\tau = P_0$, $[n]\tilde{L} = \sigma_P(L)$, the homotopy class of the loop

$$[n]\left(l_{OR}^{-1}\circ\tilde{L}\circ l_{OR}-\sigma_O(L)\right)=[n]l_{OR}^{-1}\circ\tilde{L}\circ l_{OR}-\sigma_O(L)$$

is clearly equal to $i_P(\tau)-i_O(\tau)$ and belongs to $\pi_1(A_{b_0},w_0)=\Gamma$. On the other hand, since

$$f(l_{OR}^{-1}\circ l_{RR^\tau}^{-1}\circ\tilde{L}\circ l_{OR}-\sigma_O(L))=L-L=0,$$

the same exact sequence (8.2) implies that $l_{OR}^{-1}\circ l_{RR^\tau}^{-1}\circ\tilde{L}\circ l_{OR}-\sigma_O(L)\in\Gamma$. By the commutativity of Γ, we have an equality of homotopy classes modulo the group $n\Gamma$ by applying the map $[n]$ (cf. Lemma 6.23 below):

$$[n]l_{RR^\tau}=[n]l_{OR}^{-1}\circ\tilde{L}\circ l_{OR}-\sigma_O(L)\mod n\Gamma.$$

By the invariance of homotopy by translations in the abelian variety A_{b_0}, it follows that

$$n(\tilde{R}^\tau-\tilde{R})=[n]l_{\overline{RR^\tau}}=[n]\left(l_{OR}^{-1}\circ\tilde{L}\circ l_{OR}-\sigma_O(L)\right)=i_P(\tau)-i_O(\tau)\mod n\Gamma.$$

We can thus conclude that

$$n(Q^\tau-Q)=i_P(\tau)-i_O(\tau)\mod n\Gamma.$$

□

Lemma 6.23. *Let $A=\mathbb{C}^d/\Gamma$ be a complex abelian variety. Let $O\in A$ be the origin and consider the deck transformation action $\pi_1(A,O)\times\Gamma\to\Gamma$, $(g,h)\mapsto g\cdot h\in\mathbb{C}^d$. Then the action is compatible with the group law of A and for every integer $n\geq 1$ and $\gamma\in\Gamma$:*

$$([n](\gamma))\cdot 0=(n\gamma)\cdot 0=n(\gamma\cdot 0)\in\Gamma\subset\mathbb{C}^d,$$

where $[n]\colon A\to A$ is the multiplication-by-n map and $0\in\mathbb{C}^d$ is the origin.

PROOF. The map $T\colon\pi_1(A,O)\to\Gamma$ given by $g\mapsto g.0$ is clearly an isomorphism of of abelian groups where Γ is considered as a lattice inside $\mathbb{C}^d$. The inverse map $T^{-1}\colon\Gamma\to\pi_1(A,O)$ is defined as follows. Let $x\in\Gamma\subset\mathbb{C}^d$. Consider the linear path l_{0x} in $\mathbb{C}^d$ going from 0 to x. Then the image $\pi(l_{0x})$ is a loop based at $O\in A$. We then associate to x the class of $\pi(l_{0x})$ in $\pi_1(A,O)$.

Therefore, the second equality in the last statement follows. By using the above description of the isomorphism T^{-1} and the commutative diagram:

$$\begin{tikzcd}
\mathbb{C}^d \arrow[r, "\times n"] \arrow[d, "\pi"] & \mathbb{C}^d \arrow[d, "\pi"] \\
\mathbb{C}^d/\Gamma \arrow[r, "\times n"] & \mathbb{C}^d/\Gamma,
\end{tikzcd}$$

we deduce easily that $[n](\gamma)=n\gamma\in\pi_1(A,O)$ and the first equality is proved. □

CHAPTER 7

Hyperbolic and homotopy method in the case of elliptic curves

1. Statement of the main result

In this chapter, we shall discuss the hyperbolic and homotopy method studied in Chapter 6 in the case of elliptic surfaces over a compact Riemann surface B. It turns out that we can obtain in this situation a very strong property on the finiteness of certain unions J_s "twice larger" than the union I_s (cf. (4.1)) consisting of $(S, \mathcal{D})$-integral points in which both the set S and the divisor $\mathcal{D}$ are allowed to vary in families. Moreover, it is shown that the growth of J_s in terms of s is still at most polynomial of degree $2 \operatorname{rank} \pi_1(B_0)$ as in Theorem F.

Recall that an effective divisor D on a fibered variety X over B is called *horizontal* if the induced map $D \to B$ is dominant. Otherwise, D is said to be *vertical*.

We equip B with a Riemannian metric d. Our setting is as follows.

Setting (E). Fix a nonisotrivial elliptic surface $f\colon X \to B$. Denote by $T \subset B$ the *type* of X, i.e., the finite subset above which the fibres of f are not smooth. Let $\tilde{Z}$ be a smooth complex algebraic variety. Let $\mathcal{D} \subset X \times \tilde{Z}$ be an algebraic family of relative horizontal effective Cartier divisors on X. Assume that $\mathcal{D} \to B \times \tilde{Z}$ is flat. Let $Z \subset \tilde{Z}$ be a compact subset with respect to the complex topology.

The main result of the chapter is the following (cf. Section 4 for the proof).

Theorem H. *In Setting (E), consider any finite union of disjoint closed discs $V \subset B$ containing T such that distinct points in T are contained in different discs. For each $s \in \mathbb{N}$, the following union*

$$J_s \coloneqq \cup_{z \in Z} \cup_{S \subset B, \#(S \setminus V) \leq s} \{(S, \mathcal{D}_z)\text{-integral points of } X_K\} \subset X_K(K)$$

is finite. Moreover, there exists $m > 0$ such that for every $s \in \mathbb{N}$, we have:

$$\# J_s \leq m(s+1)^{2 \operatorname{rank} \pi_1(B \setminus V)}.$$

Remark 7.1. In fact, the exact same proof of Theorem H presented in this chapter shows that the conclusion of Theorem H also holds when $X \to B$ is an isotrivial elliptic surface, up to replacing $\# J_s$ by $\#(J_s \bmod \operatorname{Tr}_{K/\mathbb{C}}(X_K)(\mathbb{C}))$.

The proof of Theorem H is a combination of the hyperbolic-homotopic method developed in Chapter 6 with the technical Lemma 7.5 which controls locally the hyperbolic metric on certain smaller subsets of $X \setminus \mathcal{D}_z$ when z varies. In particular, our proof does not use the usual height theory and thus no height bound is established.

Remark 7.2. By miracle flatness theorem (cf. [**108**, Lemma 00R4], [**69**, Theorem 23.1]), the condition requiring $\mathcal{D} \to B \times \tilde{Z}$ flat is equivalent to the condition saying that for every $z \in \tilde{Z}$, the divisor $\mathcal{D}_z$ contains no vertical components (or also by Proposition 2.9 since B is an algebraic curve).

Theorem H can be seen as a certain generalization of the following theorem of Hindry-Silverman (cf. [**52**, Theorem 0.6])) which is obtained using the Néron-Tate height theory.

Theorem (Hindry-Silverman)**.** *In Setting (E), let (O) be the zero section in X and $r = \operatorname{rank} X(K)$, there exists an explicit function $c(g) > 0$ such that:*

$$\#\{(S,(O))\text{-}\textit{integral points of } X\} \leq c(g)((\#S)^{1/2}+1)^r.$$

The above result of Hindry-Silverman holds when B is defined over any algebraically closed field k of characteristic 0. But in the case $k = \mathbb{C}$, our Theorem H gives a much stronger finiteness result without losing the polynomial bound of reasonable degree order (cf. Remark 1.27).

2. Generic emptiness of integral points with respect to a general choice of divisors

In this section we shall give an application of Theorem H on generic emptiness of integral points with respect to a general choice of divisors in an elliptic surface. Recall that Corollary B implies the generic emptiness of $(S,\mathcal{D})$-integral points on X for a general choice of a finite subset $S \in B$. For $S \subset B$ finite and fixed, we shall present in this section the following corollary of Theorem H which confirms the generic emptiness of $(S,\mathcal{D})$-integral points when we vary the divisor $\mathcal{D}$. We say that a family of effective Cartier divisors $R \subset Y \times T/T$, where Y, T are algebraic varieties, is *base-point-free* if

$$\cap_{t \in T} R_t = \varnothing.$$

Corollary C. *Let the notations be as in Theorem H. Let $S \subset B$ be a finite subset. Assume that $\tilde{Z}$ is integral, $\tilde{\mathcal{D}}$ is base-point-free and that $\mathcal{D}_{z_0}$ is ample for some $z_0 \in Z$. Then there exists a Zariski dense open subset $V \subset \tilde{Z}$ such that there is no $(S,\mathcal{D}_z)$-integral points for a every $z \in V \cap Z$.*

We begin with a simple observation:

Lemma 7.3. *Let $X, \tilde{Z}$ be algebraic integral varieties and let $\tilde{\mathcal{D}} \to X \times \tilde{Z}/\tilde{Z}$ be a family of effective divisors in X. Consider the following conditions:*

(a) *$B(\tilde{\mathcal{D}}) = \varnothing$ where $B(\tilde{\mathcal{D}}) \coloneqq \cap_{z \in \tilde{Z}} \tilde{\mathcal{D}}_z$;*

(b) *$H_x \coloneqq \{z \in \tilde{Z} \colon x \in \tilde{\mathcal{D}}_z\}$ is a proper closed subset of $\tilde{Z}$ for every $x \in X$;*

Then we have $(a) \iff (b)$. Hence, a complete linear system on X is base-point-free, i.e., satisfying Condition (a), if and only if it satisfies Condition (b).

Proof. The implication $(b) \implies (a)$ is obvious since $H_x = \tilde{Z}$ if and only if $x \in B(\tilde{\mathcal{D}})$. For the converse implication, consider the projection $\pi_2 \colon X \times \tilde{Z} \to \tilde{Z}$. Observe that for every $x \in X$, we have

$$H_x = \pi_2((\{x\} \times \tilde{Z}) \cap \tilde{\mathcal{D}}) = \pi_2(\tilde{\mathcal{D}}_x).$$

Since $B(\tilde{\mathcal{D}}) = \varnothing$, it follows that $H_x \subsetneq \tilde{Z}$ as we have remarked above. It suffices to show that H_x is Zariski closed in Z. But this is true since $\tilde{\mathcal{D}}_x \to \tilde{Z}$ is a closed immersion as $\tilde{\mathcal{D}} \to X \times \tilde{Z}$ is. We conclude that $(a) \iff (b)$. The last statement follows by considering the universal divisor parametrized by $\mathbb{P}(|L|)$ for any linear system $|L|$ on X. ☐

Proof of Corollary C. By Theorem H, the following union of integral points:

$$J_S = \cup_{z \in Z} \{(S, \mathcal{D}_z)\text{-integral points of } X(K)\} \subset I_{\#S}$$

is finite. Define the intersection locus

$$E = \cup_{P \in J_S} f(\sigma_P(B)) \cap f^{-1}(S) \subset X.$$

Then E is a finite subset since each intersection $f(\sigma_P(B)) \cap f^{-1}(S)$ is finite and since J_S is finite. As $\tilde{\mathcal{D}}$ is base-point-free, Lemma 7.3 implies that

$$H = \cup_{x \in E} \{z \in \tilde{Z} \colon x \in \tilde{\mathcal{D}}_z\} = \cup_{x \in E} H_x \subsetneq \tilde{Z} \quad \text{(see Lemma 7.3.(b))}$$

is a proper Zariski closed subset of $\tilde{Z}$. Since $\tilde{\mathcal{D}}_{z_0} = \mathcal{D}_{z_0}$ is ample by hypothesis, there exists a Zariski dense open subset $U \subset \tilde{Z}$ containing z_0 such that $\tilde{\mathcal{D}}_z$ is ample for every $z \in U$. Thus, for every $z \in U \cap Z$, the effective divisor $\mathcal{D}_z$ intersects every section σ_P for $P \in E$ in at least one point. Therefore, if there exists an $(S, \mathcal{D}_z)$-integral point for some $z \in U \cap Z$, $\mathcal{D}_z$ must intersect one of the sections $\sigma_P(B)$ where $P \in J_S$ at some point $x \in X$ lying above S. Then by construction of H, $z \in H_x \subset H$. We deduce that for every $z \in (U \setminus H) \cap Z$, there is no $(S, \mathcal{D})$-integral points. The conclusion follows since $V = U \setminus H$ is Zariski dense open in $\tilde{Z}$. ☐

3. A key technical lemma

For the proof of Theorem H, we fix a Hermitian metric ρ on the smooth complex surface X. It is clear that we can assume $\tilde{Z}$ integral. Define $B_0 \coloneqq B \setminus V$. For a complex space Y, the symbol d_Y always denotes the Kobayashi hyperbolic pseudometric on Y (cf. Definition 2.38).

Remark 7.4. Let M be a smooth complex manifold. Let $\Delta(0,R) = \{z \in \mathbb{C} \colon |z| < R\}$ for every $R > 0$. Recall that the infinitesimal Kobayashi-Royden pseudo metric λ_M on M corresponding to the Kobayashi pseudo hyperbolic metric d_M can be defined as follows. For $x \in M$ and every vector $v \in T_x M$, $\lambda_M(x,v) \coloneqq \inf 2/R$, where the minimum is taken over all $R > 0$ for which there exists a holomorphic map $f \colon \Delta(0,R) \to M$ such that $f'(0) = v$.

The proof of Theorem H is based on the following key technical lemma of local nature. The main idea is the following. For each $z \in Z$, the hyperbolic metric on $(X \setminus \mathcal{D}_z)|_{B_0}$ dominates the Hermitian metric ρ by Green's theorem 6.7 up to a certain strictly positive factor. When z varies in $\tilde{Z}$, these factors vary as well and may *a priori* tend to 0. The point of Lemma 7.5 below is that for z in a small neighborhood of Z, we can, up to restricting further $(X \setminus \mathcal{D}_z)|_{B_0}$ over some fixed nice complement of B_0 (cf. Property (b) below), these factors are in fact bounded below by a strictly positive constant (cf. Property (P) below).

Lemma 7.5. *Let the notations be as in Theorem H. Let $\varepsilon > 0$. Then there exists $M > 0$ such that for each $z_i \in \tilde{Z}$, we have the following data:*

(a) *an analytic open neighborhood U_i of z_i in $\tilde{Z}$;*

(b) *a disjoint union $V_i \subset B_0$ consisting of $\leq M$ closed discs each of radius $\leq \varepsilon$;*

(c) *a constant $c_i > 0$;*

with the following property:

(P) *for each $z \in U_i$, we have $d_{(X\setminus\mathcal{D}_z)|_{(B_0\setminus V_i)}} \geq c_i \rho|_{(X\setminus\mathcal{D}_z)|_{(B_0\setminus V_i)}}$.*

We remark first a standard lemma.

Lemma 7.6. *Let $f \colon X \to Y$ be a proper flat morphism of integral complex algebraic varieties of the same dimension. Assume that Y is smooth and for some $y \in Y$, the fibre X_y is reduced and finite. Then there exists an analytic open neighborhood $U \subset Y$ of y such that $f^{-1}(U) \to U$ is a finite étale cover.*

Proof. Since X_y is finite and reduced and f is flat, every point $x \in X_y$ is a smooth point of f since we are in characteristic 0 (cf. [**65**, Lemma 3.20]). Since Y is regular, we deduce from the open property of smooth morphisms (cf. [**42**, Definition 6.14]) that every point of X_y is a regular point of X (cf. [**65**, Theorem 4.3.36]). Since $\dim X = \dim Y$ and Y is a smooth manifold, [**42**, Definition 6.14] implies that the map f is submersive at each point of the fibre X_y. Let $S \subset X$ be the set of singular points of X (i.e., where X is not locally a manifold) and let $S' \subset X$ be the set where f is not submersive. Since f is proper, $A = f(S) \cup f(S')$ is a closed subset of Y. Then by the proof of [**50**, Theorem 21, Chapter III], $X \setminus f^{-1}(A) \to Y \setminus A$ is an étale cover. We have seen that $X_y \cap (S \cup S') = \varnothing$. Therefore, $y \notin A$ and there exists an analytic open neighborhood $U \subset Y \setminus A$ of y such that $f^{-1}(U) \to U$ is a finite étale cover. □

For ease of reading, we mention here the following key theorem of Brody:

Theorem 7.7 (Brody's reparametrization lemma). *Let M be a complex manifold with (possibly empty) boundary. Let H be a Hermitian metric on M. Suppose that $f\colon \Delta_R \to M$ be a holomorphic map with $|df(0)|_H > c$ for some $c > 0$. Then there exists a holomorphic map $g\colon \Delta_R \to M$ satisfying the following conditions:*

(a) $|dg(0)|_H = c$;

(b) $|dg(z)|_H \leq \frac{cR^2}{R^2 - |z|^2}$ *for all* $z \in \Delta_R$;

(c) $\mathrm{Im}(g) \subset \mathrm{Im}(f)$.

Proof. See Brody's reparametrization lemma, page 616 in [**43**]. □

For the proof of Lemma 7.5, we claim first that there exists $N > 0$ such that the total number of irreducible components (counted with multiplicities) of each effective divisor $\mathcal{D}_z$ is at most N for every $z \in \tilde{Z}$. Indeed, let H be any ample divisor on X then $C \cdot H \geq 1$ for every irreducible curve $C \subset X$. Remark that the divisors $\mathcal{D}_z$ are all algebraic equivalent (since $\tilde{Z}$ is integral hence connected by curves) thus numerically equivalent. Therefore, $N := \mathcal{D}_z \cdot H$ is a constant independent of $z \in \tilde{Z}$. By the linearity of the intersection pairing, the above two remarks clearly show that the total number of irreducible components (counted with multiplicities) of $\mathcal{D}_z$ is at most N as claimed.

For each $z \in \tilde{Z}$, notice that the effective divisor $\mathcal{D}_z$ contains only horizontal components with respect to the fibration $f\colon X \to B$ by Remark 7.2. We denote by $(\mathcal{D}_z)_{red}$ the induced reduced scheme structure of $\mathcal{D}_z$. By the adjunction formula, we find that

$$p_1 := p_a(\mathcal{D}_z) = \frac{\mathcal{D}_z(\mathcal{D}_z + K_X)}{2} + 1 \geq 0$$

is a constant independent of $z \in \tilde{Z}$. Now, since the arithmetic genus of $\mathcal{D}_z$ are uniformrly bounded, a version of the Riemann-Hurwitz theorem (cf. [**65**, Propositions 7.4.16, 7.5.4]) for the ramified cover of algebraic curves $\pi_z\colon (\mathcal{D}_z)_{red} \to B$

implies that for all $z \in \tilde{Z}$:

$$2(p_1 - 2) + 2N \geq \#\{\text{ramification points of } \pi_z\} + \#\{\text{singular points of } (\mathcal{D}_z)_{red}\}.$$

Let $T_z \subset B$ be the image in B of the union of the ramification points of π_z and of the singular points of $(\mathcal{D}_z)_{red}$. It follows that T_z is finite and we have:

$$\#T_z \leq M := 2(p_1 - 2) + 2N, \quad \text{for all } z \in \tilde{Z}. \tag{3.1}$$

We can now return to the proof of Lemma 7.5.

Proof of Lemma 7.5. Let us fix $\varepsilon > 0$ sufficiently small and $z_i \in \tilde{Z}$. Clearly, we can choose a finite disjoint union V_i of at most M (defined in (3.1)) nonempty open discs in B_0 of ρ-radius $\leq \varepsilon$ to cover the points of $T_{z_i} \setminus V$. Recall that $T_{z_i} \subset B$ is the image in B of the union of the ramification points of $\pi_{z_i} \colon (\mathcal{D}_{z_i})_{red} \to B$ and of the singular points of $(\mathcal{D}_{z_i})_{red}$. This gives the data V_i for (b).

Denote $Y_z := (X \setminus \mathcal{D}_z)|_{B_0 \setminus V_i}$ for each $z \in \tilde{Z}$. To show the existence of U_i and c_i so that (i) is satisfied, we suppose the contrary. Hence, by the continuity of ρ on the compact unit tangent space of X, there would exist a sequence $(z_n)_{n \geq 1} \subset \tilde{Z}$ such that $z_n \to z_i$ in the analytic topology and a sequence of holomorphic maps

$$h_n \colon \Delta_{R_n} \to Y_{z_n}$$

such that $R_n \to \infty$ and $|dh_n(0)|_\rho > 1$. Here, $\Delta_R \subset \mathbb{C}$ denotes the open disc of radius R in the complex plane. By Brody's reparametrization lemma (Theorem 7.7), we obtain a sequence of holomorphic maps

$$g_n \colon \Delta_{R_n} \to Y_{z_n}$$

such that $|dg_n(0)|_\rho = 1$ and $|dg_n(z)|_\rho \leq R_n^2/(R_n^2 - |z|^2)$ for all $z \in \Delta_{R_n}$. In particular, $|dg_n(z)|_\rho \leq 4/3$ for all $z \in \Delta_{R_n/2}$. It follows that the family $(g_n)_{n \geq 1}$ is equicontinuous with image inside the compact space $X|_{B_0 \setminus V_i}$. By the Arzela-Ascoli theorem, we deduce, up to passing to a subsequence, that $(g_n)_{n \geq 1}$ converges uniformly on compact subsets of $\mathbb{C}$ to a map $g \colon \mathbb{C} \to X|_{B_0 \setminus V_i}$. A standard argument using Cauchy's theorem and Morera's theorem shows that g is a holomorphic map and we also have $|dg(0)|_\rho = 1$.

Consider the holomorphic composition map $\pi \circ g \colon \mathbb{C} \to B_0 \setminus V_i$. Since $B_0 \setminus V_i$ is hyperbolic, it is Brody hyperbolic and thus the map $\pi \circ g$ is a constant $b_* \in B_0 \setminus V_i$. Remark that $B \times \tilde{Z}$ is smooth and $\mathcal{D} \to B \times \tilde{Z}$ is a proper flat morphism of relative dimension 0 by hypotheses. Since moreover the fibre of $\mathcal{D}_{red}$ over (b_*, z_i) is reduced and finite, we can apply Lemma 7.6. It follows that there exists a constant $\lambda \in \mathbb{N}$, a small analytic open disc $\Delta \subset B_0 \setminus V_i$ containing b_* and a small analytic connected open neighborhood U_i of z_i in $\tilde{Z}$ such that $\mathcal{D}|_{\Delta \times U_i} \to \Delta \times U_i$ is an étale cover of degree λ. Up to shrinking Δ and U_i, we can suppose that $\mathcal{D}|_{\Delta \times U_i}$ consists of λ disjoint connected components. In particular, $\mathcal{D}_z|_\Delta \to \Delta$ is an étale cover consisting of disjoint λ-sheets for every $z \in U_i$.

Fix a connected component $\mathcal{D}^0$ of $\mathcal{D}|_{\Delta\times U_i}$. We thus obtain a family of 1-sheeted covers $D^0_z \to \Delta$ with $z \in U_i$. Each D^0_z is a complex submanifold of X_Δ via the inclusion $\iota_z \colon D^0_z \to X_\Delta$. The composition $\pi \circ \iota_z \colon D^0_z \to \Delta$ is thus holomorphic and bijective for every $z \in U_i$. Since every injective holomorphic map is biholomorphic to its image (cf. [**92**, Theorem 2.14, Chapter I]), we deduce that the map $s_z \colon \Delta \to D^0_z \to X_\Delta$ given by $t \mapsto D^0_z(t) = \pi^{-1}(t) \cap D^0_z$ is holomorphic for every $z \in U_i$.

We can write $X_\Delta \subset \mathbb{P}^2 \times \Delta$ as defined by the Weierstrass equation:

$$y^2 = x^3 + A(t)x + B(t)$$

where $A(t), B(t)$ are holomorphic functions on Δ such that the discriminant $4A^3 + 27B^2$ does not vanish on Δ (since $\Delta \cap T = \varnothing$ where we recall that $T \subset B$ is the finite subset above which the fibres of f are not smooth).

We can thus write $s_z(t) = (u(t,z), v(t,z))$ where $u, v \colon \Delta \times U_i \to \mathbb{C}$ are holomorphic functions. Since the translation maps on the elliptic fibration X_Δ are algebraic thus holomorphic, the maps $\Psi_z \colon X_\Delta \to X_\Delta$ given by the translations by $s_z - s_{z_i}$:

$$\Psi_z(x) = x + s_z(f(x)) - s_{z_i}(f(x)), \quad x \in X_\Delta,$$

form a smooth family of biholomorphisms commuting with the map $f \colon X_\Delta \to \Delta$.

Since $g_n \to g$ uniformly on compact subsets of $\mathbb{C}$ and $\mathrm{Im}(g) \subset f^{-1}(b_*)$, we can, up to passing to a subsequence with suitable restrictions of domains of definitions, suppose that $\mathrm{Im}(g_n) \subset X_\Delta$ and that the holomorphic maps $g_n \colon \Delta_{R_n} \to X_\Delta \setminus \mathcal{D}_{z_n}$ still satisfy the properties:

$$R_n \to \infty, \quad |dg_n(0)|_\rho = 1.$$

Consider now the sequence of holomorphic maps

$$f_n \coloneqq \psi_{z_n} \circ g_n \colon \Delta_{R_n} \to X_\Delta \setminus \mathcal{D}_{z_i}$$

into the fixed space $X_\Delta \setminus \mathcal{D}_{z_i}$. Then by the smoothness of the family of biholomorphisms $(\psi_z)_{z\in U_i}$ and the compactness of the ρ-unit tangent bundle of X, there exists a constant $c > 1$ such that

$$c^{-1} \leq |df_n(0)|_\rho \leq c, \quad \text{for all } n \geq 1.$$

Since $R_n \to \infty$, Remark 7.4 then implies immediately a contradiction to the fact that $X_\Delta \setminus \mathcal{D}_{z_i}$ is hyperbolically embedded in $X_\Delta = f^{-1}(\Delta)$ (for the Hermitian metric $\rho|_{f^{-1}(\Delta)}$) by Green's theorem 6.7. To check the latter fact, it suffices to remark that $\bar{\Delta}$, thus $\mathcal{D}_{z_i}|_{\bar{\Delta}}$ are Brody hyperbolic and moreover, the fibres of $X_{\bar{\Delta}} \setminus (\mathcal{D}_{z_i}|_{\bar{\Delta}})$ are also Brody hyperbolic.

Hence, the existence of the data in (a), (b) and (c) such that (P) is satisfied. □

4. Proof of the main result

We can now return to Theorem H. Given Lemma 7.5, the proof of Theorem H follows the same strategy as in the proof of Theorem F.

PROOF OF THEOREM H. We have defined $B_0 = B \setminus V$ at the beginning of S ection 3 and we equip B thus B_0 with a Riemannian metric d.

Fix $\varepsilon > 0$ sufficiently small. We can enlarge slightly the discs in V if necessary. Then we obtain a constant $M > 0$ and a set of data (U_i, V_i, c_i) for each element $z_i \in Z$ as in Lemma 7.5. Consider the open covering $\cup_{z_i \in Z} U_i$ of Z in $\tilde{Z}$. Since Z is compact in the complex topology by hypothesis, there exists a finite subset $Z_* \subset Z$ such that $Z \subset \cup_{z_i \in Z_*} U_i$. Since the set Z_* is finite and $Z \subset \cup_{z_i \in Z_*} Z_i$, it suffices to consider one $z_i \in Z_*$ and to prove Theorem H for $P \in J_s$ such that P is an $(S, \mathcal{D}_z)$-integral point for some $z \in U_i$ and for some $S \subset B$ such that $\#(S \cap B_0) \leq s$.

Denote $B_i \coloneqq B_0 \setminus V_i$. Fix a base point $b_0 \in B_i \subset B_0$. To make the proof more concrete, we fix a system of smooth paths $\{(c_{b_0 b})_{b \in B_0}\}$ of bounded d-length which go from b_0 to b for every $b \in B_0$. We then fix a system consisting of simple generators $\alpha_1, \dots, \alpha_k$ of $\pi_1(B_0, b_0)$ where $k = \operatorname{rank} \pi_1(B_0, b_0)$. Note that V_i is a union of disjoint closed discs which are contractible in B_0. Hence, we can choose a system of simple homotopy classes $\alpha_1^i, \dots, \alpha_k^i \in \pi_1(B_i, b_0)$ whose images in $\pi_1(B_0, b_0)$ are respectively $\alpha_1, \dots, \alpha_k$.

Denote by $L_i > 0$ the constant given by Theorem D (as stated in Chapter 5) applied to the the compact Riemann surface B and the disjoint union of closed discs W_i and to the homotopy classes $\alpha_1^i, \dots, \alpha_k^i \in \pi_1(B_i, b_0)$.

Now consider an $(S, \mathcal{D}_z)$-integral point $P \in X_K(K)$ for some $z \in U_i$ and for some $S \subset B$ such that $\#(S \cap B_0) \leq s$.

By Theorem D applied to B_i, there exists $b_i \in B_i$ and a system of loops $\gamma_1, \dots, \gamma_k$ based at b_i representing the homotopy classes $\alpha_1, \dots, \alpha_k$ up to a single conjugation using the path $c_{b_0 b_i}$ (cf. Definition 5.1) such that $\gamma_j \subset B_i \setminus S$ and that

$$\operatorname{length}_{d_{B_i \setminus S}}(\gamma_j) \leq L_i(s+1). \tag{4.1}$$

Now, let $\sigma_P \colon B \to X$ be the corresponding section of P. For every $j \in \{1, \dots, k\}$, we find that

$$\sigma_P(\gamma_j) \subset (X \setminus \mathcal{D}_z)|_{B_i \setminus S} \subset (X \setminus \mathcal{D}_z)|_{B_i}.$$

This is true because P is $(S, \mathcal{D}_z)$-integral so that $\sigma_P(\gamma_j)$ cannot intersect $\mathcal{D}_z$ outside of $f^{-1}(S)$ and because $\gamma_j \subset B_i \setminus S$. It follows that:

$$
\begin{aligned}
\operatorname{length}_\rho(\sigma_P(\gamma_j)) &\leq c_i^{-1} \operatorname{length}_{d_{(X \setminus \mathcal{D}_z)|_{B_0 \setminus V_i}}}(\sigma_P(\gamma_j)) && \text{(by Lemma 7.5)}\\
&\leq c_i^{-1} \operatorname{length}_{d_{(X \setminus \mathcal{D}_z)|_{B_i \setminus S}}}(\sigma_P(\gamma_j)) && (\text{as } (X \setminus \mathcal{D}_z)|_{B_i \setminus S} \subset (X \setminus \mathcal{D}_z)|_{B_i})\\
&\leq c_i^{-1} \operatorname{length}_{d_{B_i \setminus S}}(\gamma_j) && \text{(by Lemma 6.9)}\\
&\leq c_i^{-1} L_i(s+1) && \text{(by (4.1))}
\end{aligned}
$$

The second inequality in the above follows from Lemma 2.39 and from the definition $B_i = B_0 \setminus V_i \supset B_i \setminus S$.

Remark that by hypothesis, distinct points of the non smooth locus $T \subset B$ of f are contained in distinct discs of V. It follows that there exists a uniform constant $\delta > 0$ (independent of P) such that the homotopy section class i_P (cf. (3.2)) of the short exact sequence (2.2)

$$0 \to \pi_1(X_{b_0}, w_0) \to \pi_1(X_{B_0}, w_0) \to \pi_1(B_0, b_0) \to 0$$

admits a representative which sends the basis $(\alpha_j)_{1 \leq j \leq k}$ of $\pi_1(B_0, b_0)$ to the homotopy classes in $\pi_1(X_{B_0}, w_0)$ which admit representative loops of ρ-lengths bounded by the function

$$H(s) \coloneqq c_i^{-1} L_i(s+1) + \delta.$$

The uniform constant δ bounds the total length of the extra paths induced by the change of base points from $\sigma_P(b_i)$ to $w_0 = \sigma_O(b_0)$ using appropriate short paths from $\sigma_P(b_i)$ to $\sigma_O(b_i)$ and from $\sigma_O(b_i)$ to w_0 (cf. the proof of Theorem F for details).

From the above bound $H(s)$ on the length of the image loops by $(S, \mathcal{D}_z)$-integral sections for every $z \in U_i$ and every $S \subset B$ such that $\#(S \cap B_0) \leq s$, the rest of the proof follows exactly the same lines as in the proof of Theorem F in Chapter 6 which uses the homotopy reduction Proposition 6.4 and the geometry of the fundamental groups $\pi_1(X_b, w_b)$ for $b \in B_0$ and $w_b \in X_b$ as a counting lemma (Lemma 6.21). Therefore, we obtain at the end a constant $m > 0$ (which is independent of s) such that:

$$\# J_s \leq m(s+1)^{2 \operatorname{rank} \pi_1(B_0)}, \quad \text{for every } s \in \mathbb{N}.$$

$\square$

CHAPTER 8

Application of hyperbolic and homotopy method to unit equations

1. Statement of the main result

The goal of the present section is to apply the method used to prove results in Chapter 7 to obtain similar results in the case of ruled surfaces. Throughout this section, we fix a compact connected Riemann surface B of function field $K = \mathbb{C}(B)$, and a finite subset $S \subset B$.

Moreover, the following definitions and notations are used:

(1) $B_S := B \setminus U$ where U is a finite disjoint union of closed discs centered at points of S;

(2) $B_0 := B_S \setminus V$ where $V \subset B_S$ is a finite disjoint union of closed discs;

(3) $X = \mathbb{P}^1_{\mathbb{C}} \times B$ and $f\colon X \to B$ is the second projection;

(4) ρ is a fixed Hermitian metric on X;

(5) $\sigma_x\colon B \to X$ denotes the section induced by $x \in K$ and let $(x) := \sigma_x(B)$;

(6) $F = \mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$ is the fibre of f and $(0), (\infty) \subset X$ are the constant sections associated to the points $0, \infty \in \mathbb{P}^1(K)$;

(7) $Y = F \times B_0 = f^{-1}(B_0) \setminus ((0) \cup (\infty)) \subset X_0 = \mathbb{P}^1 \times B_0$.

Remark 8.1. The set of sections of $F \times (B \setminus S) \to B \setminus S$ is canonically identified with the set $\mathcal{O}_S^*$ of S-units of K. Indeed, sections of the surface $F \times (B \setminus S)$ are exactly sections of X which do not intersect (0) and (∞) at points lying above $B \setminus S$. On the other hand, each element $x \in K^*$ corresponds canonically to a non zero section denoted $\sigma_x \subset X$ and vice versa by the valuative critera for properness. The condition $x \in \mathcal{O}_S$ (resp. $x^{-1} \in \mathcal{O}_S$) means exactly that $\sigma_x(B)$ does not intersect (∞) (resp. (0)) at points lying above $B \setminus S$. Therefore, $x \in \mathcal{O}_S^*$ if and only if σ_x is a section of $F \times (B \setminus S) \to B \setminus S$ as claimed.

Definition 8.2. Let $D \subset X$ be an effective divisor and let $R \subset B$ be a subset. We say that a point $x \in K^*$ is (R, D)-*integral* if it satisfies

$$f(\sigma_x(B) \cap D) \subset R.$$

Definition 8.3. For each subset $R \subset B$, the *generalized ring of R-integers* of K is denoted by $\mathcal{O}_R = \{x \in K \colon \operatorname{val}_\nu(x) \geq 0, \forall \nu \in B \setminus R\} \subset K$.

Definition 8.4. For each $\varepsilon \geq 0$, we define the (B_0, ε)*-interior subset* of $\mathcal{O}_S^*$ by

$$\mathcal{O}_S^*(B_0, \varepsilon) \coloneqq \left\{ x \in \mathcal{O}_S^* \colon |x(t)|, |x(t)^{-1}| > \varepsilon, \forall t \in B_0 \right\}.$$

Equivalently, the image of every meromorphic function $x \in \mathcal{O}_S^*(B_0, \varepsilon)$ on B_0 does not meet the ε-neighborhoods of 0 and ∞ in $\mathbb{P}^1$. Remark that $\mathcal{O}_S^*(B_0, 0) = \mathcal{O}_S^*$ since $0, \infty \notin B_0$. Moreover, we have $\mathcal{O}_S^* = \cup_{\varepsilon > 0} \mathcal{O}_S^*(B_0, \varepsilon)$.

The main result of this chapter is the following quantitative finiteness result for large unions of integral points on rational curves over function fields.

Theorem 8.5. *Let $\tilde{Z}$ be a smooth complex algebraic variety and $Z \subset \tilde{Z}$ a compact subset with respect to the complex topology. Suppose that $\mathcal{D} \subset X \times \tilde{Z}$ is a family of effective divisors such that $\mathcal{D} \to B \times \tilde{Z}$ is flat and $\mathcal{D}_z$ is not contained in $(0) \cup (\infty)$ for every $z \in \tilde{Z}$. For every $r \in \mathbb{N}$ and $\varepsilon > 0$, the following union of integral points:*

$$J_{r,\varepsilon} \coloneqq \cup_{z \in Z} \cup_{R \subset B, \# R \cap B_0 \leq r} \{ x \in \mathcal{O}_S^*(B_0, \varepsilon) \colon x \text{ is } (R, \mathcal{D}_z)\text{-integral in } X \} \subset \mathcal{O}_S^*,$$

is finite modulo $\mathbb{C}^*$. Moreover, there exists a constant $m > 0$ such that:

$$\#(J_{r,\varepsilon} \bmod \mathbb{C}^*) \leq m(r+1)^{2 \operatorname{rank} \pi_1(B_0)} \text{ for every } r \in \mathbb{N}.$$

The proof of Theorem 8.5 given in Section 3 will follow closely the steps in the proof of Theorem H. It might be helpful to first consider the following counter-example explaining why we cannot take the whole set of S-units $\mathcal{O}_S^*$, i.e., $\varepsilon = 0$, in the union $J_{r,\varepsilon}$.

Example 8.6. Let the notations and hypotheses be as in Theorem 8.5. Assume moreover that $B = \mathbb{CP}^1$ is the Riemann sphere and that $S = \{0, \infty\}$. Let t be the inhomogenous coordinate on $\mathbb{P}^1$ then $\mathcal{O}_S^* = \mathbb{C}^*.\{t^n \colon n \in \mathbb{Z}\} = \{ct^n \colon c \in \mathbb{C}^*, n \in \mathbb{Z}\}$. Suppose also that $Z = \tilde{Z} = \{\cdot\}$ and $\mathcal{D} = (1) \subset X$ is the section induced by $1 \in K$.

For every $n \in \mathbb{Z}$, we define $c_n = \left(2 \sup_{t \in B_0} |t^n|\right)^{-1}$. Then $c_n > 0$ and is a finite number since $B_0 \subset \mathbb{CP}^1$ is a complement of a finite union $U \cup V$ of closed discs with nonempty interior containing 0 and ∞. It follows that $0 < |c_n t^n| < 1$ for all $t \in B_0$. Therefore, for each $n \in \mathbb{Z}$, $x_n \coloneqq c_n t^n \in \mathcal{O}_S^*$ is $(R, \mathcal{D})$-integral in X in the sense of Definition 8.3 where $R = U \cup V$. In particular, $\{x_n \colon n \in \mathbb{Z}\} \subset J_{r,0}$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{N}$. However, $\{x_n \colon n \in \mathbb{Z}\}$ modulo $\mathbb{C}^*$ is $\mathbb{Z}$ and thus $J_{r,0}$ is infinite. Remark that there exists $t_0 \in B_0 \subset \mathbb{C}$ such that $|t_0| \neq 0, 1$. Hence, if $|t_0| > 1$, we see that $c_n \to 0$ when $n \to -\infty$. Otherwise, if $0 < |t_0| < 1$ then $c_n \to 0$ as $n \to +\infty$.

Therefore, it is necessary to restrict to the union of integral points $J_{r,\varepsilon}$ where $\varepsilon > 0$ to obtain a finiteness result as in Theorem 8.5.

2. Some applications to generalized unit equations

We illustrate in this section several applications of Theorem 8.5 in a context generalizing the classical S-unit equations (cf. [**33**], [**34**], [**35**]).

Let $D_S := \sum_{b \in S}[b]$ be the effective divisor of B associated to the finite subset $S \subset B$. Let $r \in \mathbb{N}$, consider the following subset of K:

$$\begin{aligned} \mathcal{O}_{B_0,r} &:= \cup_{R \subset B_0, \#R \leq r}\{x \in K\colon \ \mathrm{val}_\nu(x) \geq 0, \ \text{for every } \nu \notin R \cup B_0\} \\ &= \cup_{R \subset B_0, \#R \leq r}\mathcal{O}_R. \end{aligned} \tag{2.1}$$

Remark that $\mathcal{O}_{B_0,r} \subset \mathcal{O}_{B_0,r+1}$ for every $r \geq 0$ and $K = \cup_{r \geq 0}\mathcal{O}_{B_0,r}$. Moreover, $\mathcal{O}_S \subset \mathcal{O}_{B_0,0}$ and $\mathcal{O}_{B_0,r}$ is not a ring unless $r = 0$.

For integers $n \geq 1$, $r \geq 0$ and a real number $\varepsilon > 0$, we consider the Diophantine equation

$$x + y = z \tag{2.2}$$

with $(x, y, z) \in K^3$ satisfying the following conditions:

(i) $x \in \mathcal{O}_S^*(B_0, \varepsilon)$;

(ii) $y^{-1} \in \mathcal{O}_{B_0,r}$;

(iii) $z \in L(nD_S) \setminus \{0\} = \{h \in K^*\colon \ \mathrm{div}(h) + nD_S \geq 0\} \subset K^*$.

In other words, we consider the union of solutions $(x, y) \in \mathcal{O}_S^*(B_0, \varepsilon) \times (\mathcal{O}_{B_0,r} \setminus \{0\})^{-1}$ of the parametrized equations $x + y = z$ with z varying in the space $\in L(nD_S) \setminus \{0\}$.

In the case $r = \varepsilon = 0$ and $B_0 = B \setminus S$, we recover the usual S-unit equation $x + y = 1$ with $x, y \in \mathcal{O}_S^*$ by setting $z = 1 \in L(nD_S) \setminus \{0\}$. Indeed, $r = 0$ and $B_0 = B \setminus S$ imply $\mathcal{O}_{B_0,r} = \mathcal{O}_S$. On the other hand, if $x \in \mathcal{O}_S^*$ and $y^{-1} \in \mathcal{O}_S$ such that $x + y = 1$, then $y = 1 - x \in \mathcal{O}_S$ since $\mathcal{O}_S$ is a ring and thus $y \in \mathcal{O}_S^*$. Therefore, (2.2) generalizes the usual S-unit equation over function fields.

The finiteness of the numbers of solutions $x, y \in \mathcal{O}_S^*$ with $x/y \notin \mathbb{C}^*$ of the unit equation $x + y = 1$ is well-known. It turns out that Theorem 8.5 actually implies that a similar property still holds for the generalized equation (2.2).

Corollary 8.7. *There are only finitely many $x \in \mathcal{O}_S^*(B_0, \varepsilon)$ modulo $\mathbb{C}^*$ such that the equation* (2.2) *admits a solution.*

Proof of Corollary 8.7. Denote $d = \dim L(nD_S) = \dim H^0(B, \mathcal{O}(nD_S))$. Fixing a basis $(z_1, \ldots, z_d)$ of the complex vector space $L(nD_S)$, we consider the compact unit sphere

$$\mathbf{S}^d = \left\{ \sum_{k=1}^{d} a_k z_k \in L(nD_S) \colon ||(a_1, \dots, a_k)|| = 1 \right\} \subset L(nD_S)$$

where $||(a_1, \dots, a_k)|| = \left(\sum_{k=1}^{d} a_k \right)^{1/2}$. Recall that $D_S \coloneqq \sum_{b \in S}[b]$.

Define $\tilde{Z} \coloneqq L(nD_S) \setminus \{0\} \simeq \mathbb{C}^d \setminus \{0\}$ then $\tilde{Z}$ is a integral smooth algebraic variety. We have a canonical valuation morphism

$$\mathrm{val} \colon B \times \tilde{Z} \to \mathbb{P}^1, \quad (b, z) \mapsto z(b).$$

Consider the flat family of divisors $\mathcal{D} \subset X \times \tilde{Z}$ given by the image of the algebraic section

$$\Sigma \colon B \times \tilde{Z} \to \mathbb{P}^1 \times B \times \tilde{Z}, \quad (b, z) \mapsto (z(b), b, z).$$

For each $z \in \tilde{Z} \subset K^*$, let $(z) \subset X$ be the induced section of the projection $f \colon X \to B$. It is clear that $(z) = \mathcal{D}_z \not\subset (0) \cup (\infty)$ for every $z \in \tilde{Z}$ by the construction of $\mathcal{D}$. Now let $z \in \tilde{Z} \subset K^*$, $x \in \mathcal{O}_S^*(B_0, \varepsilon)$ and $y = z - x$. It is not hard to see that the condition $y^{-1} \in \mathcal{O}_R$ for a certain subset $R \subset B$ verifying $\#(R \cap B_0) \leq r$ means exactly that

$$x \in \cup_{R \subset B, \# R \cap B_0 \leq r} \{x' \in \mathcal{O}_S^*(B_0, \varepsilon) \colon x' \text{ is } (R, \mathcal{D}_z)\text{-integral in } X\}. \tag{2.3}$$

We cannot apply directly Theorem 8.5 since $\tilde{Z} = L(nD_S) \setminus \{0\}$ is not compact. However, it suffices to restrict ourselves to the case $z \in Z \coloneqq \mathbf{S}^d$ since the compact subspace $\mathbf{S}^d$ contains all classes modulo $\mathbb{C}^*$ of $\tilde{Z}$. Therefore, Theorem 8.5 says that the set

$$J_{r,\varepsilon} = \cup_{z \in \mathbf{S}^d} \cup_{R \subset B, \# R \cap B_0 \leq r} \{x' \in \mathcal{O}_S^*(B_0, \varepsilon) \colon x' \text{ is } (R, \mathcal{D}_z)\text{-integral in } X\}$$

is finite modulo $\mathbb{C}^*$. Combining with (2.3), the proof of Corollary 8.7 is completed. □

Following the general idea that parametrized Diophantine equations have no or very few integral solutions under a general choice of parameters, we mention below a remarkable theorem on unit equations in the case of number fields.

Theorem 8.8 (Evertse-Győry-Stewart-Tijdeman). *Let K be a number field and S a finite number of places. There exists a finite set of triples $A \subset (K^*)^3$ with the following property. For every $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (K^*)^3$ whose class $[\alpha] \in (K^*)^3/(K^*(\mathcal{O}_S^*)^3)$ does not belong to $[A] \subset (K^*)^3/(K^*(\mathcal{O}_S^*)^3)$, the S-unit equations*

$$\alpha_1 x + \alpha_2 y = \alpha_3 \tag{2.4}$$

has at most 2 solutions.

Proof. See [**35**, Theorem 1]. Note that the natural action of $K^*(\mathcal{O}_S^*)^3$ on $(K^*)^3$ is given by: $(c,(u,v,w))\cdot(\alpha_1,\alpha_2,\alpha_3) = (cu\alpha_1, cv\alpha_2, cw\alpha_3)$. □

Theorem 8.8 implies that almost all equations of the form (2.4) have no more than 2 unit solutions. As an analogous result for certain Diophantine equations in function fields, Corollary 8.7 can be directly reformulated as follows.

Corollary 8.9. *Given $\varepsilon > 0$, let $\omega \in \mathcal{O}_S^*(B_0,\varepsilon)$ and $n \geq 1$. Consider the equation*

$$x + y = \omega \tag{2.5}$$

with unknowns $x, y \in K^*$ satisfying $x \in L(nD_S)\setminus\{0\}$ and $y^{-1} \in \mathcal{O}_{B_0,r}$ (cf. (2.1)). There exists a finite subset $A \subset \mathcal{O}_S^*(B_0,\varepsilon)$ such that whenever $\omega \notin \mathbb{C}^*A$, the equation (2.5) has no solutions. Moreover, there exists $m > 0$ such that we can take A having no more than $m(r+1)^{2\operatorname{rank}\pi_1(B_0)}$ elements for every $r \in \mathbb{N}$.

Note that the above last statement follows immediately from Theorem 8.5.

To finish this section, we remark the following generic emptiness of the set of solutions of parametrized S-unit equation:

$$x + y = z, \quad x, y \in \mathcal{O}_S^*, \quad z \in \mathcal{O}_S. \tag{2.6}$$

Corollary 8.10. *The equation (2.6) has no solutions $(x,y) \in (\mathcal{O}_S^*)^2$ for a general $z \in \mathcal{O}_S$. More precisely, for any finite dimensional complex vector subspace V of $\mathcal{O}_S$, there exists a finite union L of linear subsurfaces contained in V such that (2.6) has no solutions $(x,y) \in (\mathcal{O}_S^*)^2$ for every $z \in V \setminus L$.*

Proof. Let V be a finite dimensional complex vector subspace of $\mathcal{O}_S$. Then:

$$\begin{aligned} \mathcal{O}_S &= \{x \in K\colon \operatorname{val}_\nu(x) \geq 0 \text{ for all } \nu \notin S\} \\ &= \cup_{n\geq 0}\{x \in \mathcal{O}_S\colon \operatorname{val}_\nu(x) \geq -n \text{ for all } \nu \in S\} \\ &= \cup_{n\geq 0} L(nD_S) \end{aligned} \tag{2.7}$$

where $D_S = \sum_{b\in S} b$ is the reduced effective divisor on B associated to S. Therefore, there exists $n \in \mathbb{N}$ such that $V \subset L(nD_S)$ since V is finite dimensional. Hence, it suffices to prove the result for $z \in L(nD_S)$.

Every $z \in L(nD_S)\setminus\{0\}$ is an S'-unit where S' is the set of all zeros and poles of z which has no more than $ns + ns = 2ns$ elements. Applying Theorem 8.11 below to $m = 3$ and to the equation $x + y - z = 0$ with $x, y \in \mathcal{O}_S^*$ and $z \in L(nD_S)\setminus\{0\}$, we deduce a bound on the heights:

$$\max(H(x), H(y)) \leq s + s + ns + 2g - 2 = 2g + (2n+3)s - 2.$$

It follows that such x, y then belong to at most $s^{2g+(2n+3)s-2}$ classes modulo $\mathbb{C}^*$ by Proposition 8.12. From this, we obtain easily a union L_n of no more than $s^{2(2g+(2n+3)s-2)}$ linear subsurfaces in $L(nD_S)$ such that (2.6) has no solutions $(x,y) \in (\mathcal{O}_S^*)^2$ for every $z \in L(nD_S) \setminus L_n$. The conclusion follows. □

Recall the general abc theorem over function fields for linear equations.

Theorem 8.11 (Brownawell-Masser)**.** *Let $K = \mathbb{C}(B)$. Suppose that $u_1, \dots, u_m \in K^*$ form a solution of*

$$u_1 + \cdots + u_m = 0$$

such that no proper subsum is zero. Assume that u_i is an S_i-unit $(1 \leq i \leq m)$ *for some finite subsets $S_1, \dots, S_m$ of B. Then*

$$\begin{aligned} \max(H(u_1), \dots, H(u_m)) &\leq \sum_{\nu \in M_K} \max(\operatorname{ord}_\nu u_1, \dots, \operatorname{ord}_\nu u_m) \\ &\leq (m-2)(|S_1| + \cdots + |S_m|) + \frac{1}{2}(n-1)(n-2)(2g-2). \end{aligned}$$

Proof. See [**13**]. □

3. Proof of the main result

3.1. Preliminaries. We begin with the following easy analogue of the Lang-Néron theorem for the multiplicative group $\mathbb{G}_m$.

Proposition 8.12. *$\mathcal{O}_S^*/\mathbb{C}^*$ is a torsion-free abelian group of rank $\leq \#S$.*

Proof. Consider the following homomorphism of groups

$$\rho \colon \mathcal{O}_S^* \to \oplus_{\nu \in S} \mathbb{Z}, \quad f \mapsto (\operatorname{mult}_\nu(\operatorname{div} f))_{\nu \in S}.$$

We claim that $\operatorname{Ker} \rho = \mathbb{C}^*$. Indeed, suppose that $f \in \mathcal{O}_S^*$ satisfies $\rho(f) = 0$. Since $f \in \mathcal{O}_S^*$, all poles and zeros of f belongs to S. However, $\rho(f) = 0$ implies that these poles and zeros are all of order 0. It follows that the corresponding morphism $f \colon B \to \mathbb{P}^1$ must be constant and thus $f \in \mathbb{C}^*$ as claimed. Therefore, $\mathcal{O}_S^*/\mathbb{C}^* \to \oplus_{\nu \in S} \mathbb{Z}$ is injective and $\mathcal{O}_S^*/\mathbb{C}^*$ is torsion-free of rank $\leq \#S$. □

Remark 8.13. In general, let K be a field whose all places are non-archimedean. Then the constant field $k \coloneqq \{x \in K \colon |x|_\nu \leq 1, \forall \nu \in M_K\}$ is algebraically closed in K. Let S be a finite subset of places of K. Then $\mathcal{O}_S^*/k^*$ is a finitely generated abelian group whenever K satisfies the product formula (cf. [**33**] for more details).

Now each $z \in \mathcal{O}_S^* = \mathbb{G}_m(\mathcal{O}_S)$ induces a section σ_z of the projection $Y \to B_0$ and thus a section i_z of the following exact sequence of fundamental groups:

$$0 \to \pi_1(F, w_0) \to \pi_1(Y, w_0) \xrightarrow{\eta} \pi_1(B_0, b_0) \to 0, \tag{3.1}$$

where we fix $w_0 = 1 \in Y_{b_0} = \mathbb{C}^*$ above a fixed point $b_0 \in B_0$. Fix a collection of geodesics $l_{w_0,w} \colon [0,1] \to Y_{b_0}$ on Y_{b_0} such that $l_{w_0,w}(0) = w_0$ and $l_{w_0,w}(1) = w \in Y_{b_0}$. Every $x \in \mathcal{O}_S^*$ induces a section $\sigma_x \colon B_0 \to Y_{B_0}$ which in turn gives rise to a section $i_x \colon \pi_1(B_0, b_0) \to \pi_1(Y_{B_0}, w_0)$ of the exact sequence (2.2) as follows. For every loop γ of B_0 based at b_0, we define:

$$i_x([\gamma]) = [l^{-1}_{w_0,\sigma_x(b_0)} \circ \sigma_x(\gamma) \circ l_{w_0,\sigma_x(b_0)}] \in \pi_1(Y_{B_0}, w_0). \tag{3.2}$$

Denote $G = \pi_1(B_0, b_0)$ and let $\widehat{G}$ be the profinite completion of G.

As in the case of abelian varieties, we have the following reduction result.

Theorem 8.14. *Let $n \geq 2$ be an integer. We have the following commutative diagram of morphisms of groups:*

$$\begin{tikzcd}
\mathcal{O}_S^*/(\mathcal{O}_S^*)^n \arrow[r, hook, "\delta"] & H^1(\widehat{G}, \mu_n) \arrow[dr, "\simeq"] & \\
\mathcal{O}_S^* \arrow[u] \arrow[r, "\alpha"] & H^1(G, \mathbb{Z}) \arrow[r, "\beta"] & H^1(G, \mu_n).
\end{tikzcd}$$

Moreover, two elements of $\mathcal{O}_S^*$ induces the same class of sections of the exact sequence of fundamental groups (3.1) *if and only if they differs by a factor in $\mathbb{C}^*$.*

The proof of Theorem 8.14 will be given in Appendix 4.

3.2. Key lemma. Recall that d_M means the Kobayashi hyperbolic pseudometric on a complex space M and ρ is a fixed Hermitian metric on the smooth surface X. As in the case of elliptic fibrations, the main additional ingredient in the proof of Theorem 8.5 for ruled surfaces is the following analogous technical lemma of Lemma 7.5:

Lemma 8.15. *Let $\varepsilon > 0$. Then there exists $M > 0$ such that for each $z_i \in \tilde{Z}$, we have the following data:*

(a) *an analytic open neighborhood U_i of z_i in $\tilde{Z}$;*

(b) *a disjoint union V_i consisting of $\leq M$ discs each of ρ-radius $\leq \varepsilon$ in B_0;*

(c) *a constant $c_i > 0$;*

with the following property:

(Q) *for each $z \in U_i$, we have $d_{(Y\setminus\mathcal{D}_z)|_{(B_0\setminus V_i)}} \geq c_i\rho|_{(Y\setminus\mathcal{D}_z)|_{(B_0\setminus V_i)}}$.*

The proof of Lemma 8.15 applies, *mutatis mutandis*, the proof of Lemma 7.5 with some minor modifications. The same remark at the beginning of the proof of Lemma 7.5 shows that there exists $N' > 0$ such that the total number of irreducible components (counted with multiplicities) of each effective divisor $\mathcal{D}_z$ is at most N' for every $z \in Z$.

Moreover, since $\mathcal{D} \to B \times \tilde{Z}$ is flat, every divisor $\mathcal{D}_z$, $z \in \tilde{Z}$, contains no vertical components with respect to the projection $f \colon X \to B$. The divisors $\mathcal{D}_z$ are numerically equivalent and are not contained in the curve $(0) \cup (\infty) \subset X$. In particular, it follows that for some constant $N'' > 0$, we have

$$\#\mathcal{D} \cap ((0) \cup (\infty)) \leq N'', \quad \text{for all } z \in Z.$$

By the adjunction formula,

$$p_1 \coloneqq p_a(\mathcal{D}_z) = \frac{\mathcal{D}_z(\mathcal{D}_z + K_X)}{2} + 1 \geq 0$$

is a constant independent of $z \in \tilde{Z}$. Let $T'_z \subset B$ be the image in B of the union of the ramification points of the ramified cover of algebraic curves $\pi_z \colon (\mathcal{D}_z)_{red} \to B$, and of the singular points of $(\mathcal{D}_z)_{red}$. We have as in the relation (3.1) that:

$$\#T'_z \leq M' \coloneqq 2N' + 2(p_1 - 2), \quad \text{for all } z \in \tilde{Z}.$$

Define $T_z = T'_z \cup f(\mathcal{D} \cap ((0) \cup (\infty))) \subset B$ then it follows from the above discussion that

$$\#T_z \leq M \coloneqq M' + N'', \quad \text{for all } z \in \tilde{Z}. \tag{3.3}$$

We return to the proof of Lemma 8.15. Again, the idea of the proof is the same as in Lemma 7.5 but we indicate in details the needed modifications.

Proof of Lemma 8.15. Fix $\varepsilon > 0$ and $z_i \in \tilde{Z}$. Recall that $B_0 = B \setminus (U \cup V)$. Let M be the constant defined in (3.3). We can clearly choose a finite disjoint union V_i of at most M nonempty closed discs in B_0 of ρ-radius $\leq \varepsilon$ to cover the points of $T_{z_i} \setminus V$. Thus, we obtain the data V_i for Lemma 8.15.(b).

Define $\tilde{\mathcal{D}} = \mathcal{D} \cup (((0) \cup (\infty)) \times \tilde{Z}) \subset X \times \tilde{Z}$ and for each $z \in \tilde{Z}$, let

$$Y_z \coloneqq (Y \setminus \mathcal{D}_z)|_{B_0 \setminus V_i} = (X \setminus \tilde{\mathcal{D}}_z)|_{B_0 \setminus V_i} \subset X.$$

To show the existence of U_i, c_i satisfying (i), we suppose the contrary. By the continuity of ρ on the compact unit tangent space of X, there would exist a sequence $(z_n)_{n \geq 1} \subset \tilde{Z}$ such that $z_n \to z_i$ in the analytic topology and a sequence of holomorphic maps

$$h_n \colon \Delta_{R_n} \to Y_{z_n}$$

such that $R_n \to \infty$ and $|dh_n(0)|_\rho > 1$. Here, $\Delta_R \subset \mathbb{C}$ denotes the open disc of radius R in the complex plane. By Brody's reparametrization lemma (Theorem 7.7), we obtain a sequence of holomorphic maps

$$g_n \colon \Delta_{R_n} \to Y_{z_n}$$

such that $|dg_n(0)|_\rho = 1$ and $|dg_n(z)|_\rho \leq R_n^2/(R_n^2 - |z|^2)$ for all $z \in \Delta_{R_n}$. In particular, $|dg_n(z)|_\rho \leq 4/3$ for all $z \in \Delta_{R_n/2}$. It follows that the family $(g_n)_{n\geq 1}$ is equicontinuous with image inside the compact space $X|_{B_0\backslash V_i}$. Up to passing to a subsequence, $(g_n)_{n\geq 1}$ converges uniformly on compact subsets of $\mathbb{C}$ to a holomorphic map $g\colon \mathbb{C} \to X|_{B_0\backslash V_i}$ with $|dg(0)|_\rho = 1$.

Since $B_0\backslash V_i$ is hyperbolic, $f\circ g\colon \mathbb{C} \to B_0\backslash V_i$ is a constant $b_* \in B_0\backslash V_i$. As in Lemma 7.5, we can apply Lemma 7.6 to find a constant $\lambda \in \mathbb{N}$, a small analytic open disc $\Delta \subset B_0 \setminus V_i$ containing b_* and a small analytic connected open neighborhood U_i of z_i in $\tilde{Z}$ with:

$$\tilde{\mathcal{D}}|_{\Delta\times U_i} \to \Delta \times U_i$$

an étale cover of degree λ. Shrinking Δ and U_i if necessary, $\tilde{\mathcal{D}}|_{\Delta\times U_i}$ consists of λ disjoint connected components including $(0)|_\Delta \times U_i$ and $(0)|_\Delta \times U_i$.

Fixing a connected component $\mathcal{D}^0$ of $\mathcal{D}|_{\Delta\times U_i}$, we obtain a family of 1-sheeted covers $D_z^0 \to \Delta$ with $z \in U_i$ satisfying $D_z^0 \cap ((0) \cup (\infty)) = \varnothing$. Hence, each D_z^0 for $z \in U_i$ is a complex submanifold of Y_Δ via the inclusion $\iota_z\colon D_z^0 \to Y_\Delta$. The composition $f \circ \iota_z\colon D_z^0 \to \Delta$ is thus holomorphic and bijective for every $z \in U_i$. As injective holomorphic maps are biholomorphic to their images ([**92**, Theorem 2.14, Chapter I]), the map $s_z\colon \Delta \to D_z^0 \to Y_\Delta$ given by $t \mapsto D_z^0(t) = f^{-1}(t) \cap D_z^0$ is holomorphic for every $z \in U_i$. Moreover, the map

$$\Delta \times U_i \to Y_\Delta \times U_i, \quad (t, z) \mapsto (D_z^0(t), z) = (f^{-1}(t) \cap D_z^0, z)$$

is holomorphic. As $Y_\Delta = F\times\Delta$, we can write $s_z(t) = (u(t,z), t)$ where $u\colon \Delta\times U_i \to \mathbb{C}^*$ is a holomorphic function. The maps $\Psi_z\colon Y_\Delta \to Y_\Delta$, $z \in U_i$, given by the fibrewise multiplication by $s_z s_{z_i}^{-1}$, i.e.,

$$\Psi_z(x,t) = (xu(t,z)u(t,z_i)^{-1}, t), \quad (x,t) \in Y_\Delta = \mathbb{C}^* \times \Delta,$$

form a smooth family of biholomorphisms commuting with the projection $Y_\Delta \to \Delta$.

By passing to a subsequence with suitable restrictions of domains of definitions, we can assume that $\mathrm{Im}(g_n) \subset X_\Delta$ and that the holomorphic maps $g_n\colon \Delta_{R_n} \to X_\Delta\backslash\tilde{\mathcal{D}}_{z_n}$ still satisfy

$$R_n \to \infty, \quad |dg_n(0)|_\rho = 1.$$

Since $D_z^0 \subset \mathcal{D}_z$, we can consider the sequence of holomorphic maps

$$f_n \coloneqq \psi_{z_n} \circ g_n\colon \Delta_{R_n} \to Y_\Delta \setminus D_{z_i}^0 = X_\Delta \setminus (D_{z_i}^0 \cup (0) \cup (\infty))$$

into the fixed space $Y_\Delta \setminus D_{z_i}^0$. By the smoothness of the family $(\psi_z)_{z\in U_i}$ and the compactness of the ρ-unit tangent bundle of X, there exists $c > 1$ such that

$$c^{-1} \leq |df_n(0)|_\rho \leq c, \quad \text{for all } n \geq 1. \tag{3.4}$$

Now, consider any holomorphic map $h\colon \mathbb{C} \to Y_\Delta \setminus D^0_{z_i}$. The composition $f \circ h$ must be a constant since Δ is hyperbolic. Thus, h factors through a fibre of $X_\Delta \setminus (D^0_{z_i} \cup (0) \cup (\infty))$. However, as each such fibre is the complement of at least 3 points in $\mathbb{P}^1$ and thus is hyperbolic, h must be constant. Similarly, it is clear that each holomorphic map $\mathbb{C} \to (D^0_z \cup (0) \cup (\infty))|_{\bar{\Delta}}$ is constant. Green's theorem 6.7 implies that $Y_\Delta \setminus \mathcal{D}_{z_i}$ is hyperbolically embedded in $X_\Delta = f^{-1}(\Delta)$ (with respect to the metric $\rho|_{X_\Delta}$). But since $R_n \to \infty$, we clearly obtain a contradiction using (3.4) and Remark 7.4. We have therefore proved the existence of the data in (a), (b) and (c) such that (Q) is satisfied. $\square$

3.3. Proof of Theorem 8.5. We can now return to the main result which will be very similar to the proof of Theorem H. Let $(\alpha_1, \dots, \alpha_k)$ be a fixed system of generators of the fundamental group $\pi_1(B_S, b_0)$ with a fixed based point $b_0 \in B_0$. Let $w_0 = 1 \in Y_b = \mathbb{C}^*$.

Proof of Theorem 8.5. Fix $\varepsilon > 0$. We can enlarge slightly the discs in V if necessary. Then we obtain a constants $M > 0$ and a set of data (U_i, V_i, c_i) for each element $z_i \in Z$ as in Lemma 8.15. Consider the open covering $Z \subset \cup_{z_i \in Z} U_i$. Since Z is compact, there exists $Z_* \subset Z$ finite such that $Z \subset \cup_{z_i \in Z_*} U_i$. As $c_i > 0$ for every $z_i \in Z$ by Lemma 8.15, we have

$$c_* \coloneqq \min_{z_i \in Z_*} c_i > 0. \tag{3.5}$$

For each $z_i \in Z_*$, denote by $L_i > 0$ the maximum of the constants given by Theorem D applied to $B_i \coloneqq B \setminus (V \cup V_i) = B_0 \setminus V_i$ and to each free homotopy classes $\alpha_1, \dots, \alpha_k$ regarded as elements of $\pi_1(B_i) \supset \pi_1(B_0)$. Let

$$L = \max_{z_i \in \mathbb{Z}_*} L_i > 0. \tag{3.6}$$

Now let $x \in J_{r,\varepsilon}$, that is, $x \in \mathcal{O}^*_S(B_0, \varepsilon)$ is $(R, \mathcal{D}_z)$-integral in X for some $z \in Z$ and $R \subset B$ such that $\#R \cap B_0 \leq r$. As $Z =\subset \cup_{z_i \in Z_*} U_i$, there exists $z_i \in Z_*$ such that $z \in U_i$.

By Theorem D applied to B_i, there exists $b_i \in B_i$ and a system of loops $\gamma_1, \dots, \gamma_k$ based at b representing respectively the homotopy classes $\alpha_1, \dots, \alpha_k$ up to a single conjugation such that $\gamma_j \subset B_i \setminus R$ for every $j = 1, \dots, k$ and that

$$\mathrm{length}_{d_{B_i \setminus R}}(\gamma_j) \leq L_i(\#R \cap B_i + 1) \leq L_i(r + 1). \tag{3.7}$$

The second inequality follows from $\#R \cap B_i \leq \#R \cap B_0 \leq r$.

Let $\sigma_x\colon B \to X$ be the section induced by x. For every $j \in \{1, \dots, k\}$, we find that $\sigma_x(\gamma_j) \subset (Y \setminus \mathcal{D}_z)|_{B_i \setminus R} \subset (Y \setminus \mathcal{D}_z)|_{B_i}$ by the definition of $(R, \mathcal{D})$-integral points and because $x \in \mathcal{O}^*_S$. It follows that:

$$
\begin{aligned}
\operatorname{length}_\rho(\sigma_x(\gamma_j)) &\le c_i^{-1} \operatorname{length}_{d_{(Y\setminus\mathcal{D}_z)|_{B_0\setminus V_i}}}(\sigma_x(\gamma_j)) && \text{(by Lemma 8.15)}\\
&= c_i^{-1} \operatorname{length}_{d_{(Y\setminus\mathcal{D}_z)|_{B_i}}}(\sigma_x(\gamma_j)) && (\text{as } B_i := B_0 \setminus V_i \subset B_0)\\
&\le c_i^{-1} \operatorname{length}_{d_{(Y\setminus\mathcal{D}_z)|_{B_i\setminus R}}}(\sigma_x(\gamma_j)) && (\text{as } (Y\setminus\mathcal{D}_z)|_{B_i\setminus R} \subset (Y\setminus\mathcal{D}_z)|_{B_i})\\
&\le c_*^{-1} \operatorname{length}_{d_{B_i\setminus R}}(\gamma_j) && \text{(by (3.5) and Lemma 6.9)}\\
&\le c_*^{-1} L(r+1). && \text{(by (3.6) and (3.7))}
\end{aligned}
$$

Then homotopy section i_x associated to x of the short exact sequence (cf. the sequence (3.1) in Theorem 8.14)

$$0 \to \pi_1(Y_{b_0}, w_0) \to \pi_1(Y_{B_0}, w_0) \to \pi_1(B_0, b_0) \to 0$$

sends the basis $(\alpha_j)_{1\le j\le k}$ of $\pi_1(B_0, b_0)$ to the classes in $\pi_1(Y_{B_0}, w_0)$ which admit representative loops of ρ-lengths bounded by $H(r) := c_*^{-1}L(r+1) + \delta$ for some uniform constant δ (cf. the proof of Theorem F).

The rest of the proof follows tautologically the same lines of the proof of Theorem F in Chapter 6. We thus obtain a finite number $m > 0$ such that:

$$\#(J_{r,\varepsilon} \bmod \mathbb{C}^*) \le m(r+1)^{2\operatorname{rank}\pi_1(B_0)}, \quad \text{for every } r \in \mathbb{N}.$$

The only needed modification is the following. We can clearly assume that $0 < \varepsilon < 1$. Consider the compact bordered manifold $E_\varepsilon := \{z \in \mathbb{C} \colon \varepsilon \le |z| \le \varepsilon^{-1}\}$. Remark that $\pi_1(E_\varepsilon, w_0) = \mathbb{Z}$ (recall that $w_0 = 1 \in Y_{b_0}$). Since $x \in \mathcal{O}_S^*(B_0, \varepsilon)$, it actually induces a homotopy section $i_{x,\varepsilon}$ of the short exact sequence

$$0 \to \pi_1(E_\varepsilon, w_0) \to \pi_1(E_\varepsilon \times B_0, w_0) \to \pi_1(B_0, b_0) \to 0.$$

Since $\pi_1(E_\varepsilon \times B_0, w_0) = \pi_1(E_\varepsilon, w_0) \times \pi_1(B_0, b_0)$, the homotopy section $i_{x,\varepsilon}$ is determined by the $\pi_1(E_\varepsilon, w_0)$-component of $i_{x,\varepsilon}(\alpha_j)$ for every $j = 1, \dots, k$. But we have shown above that the induced ρ-length of certain representative loops of these components are bounded by $H(r) := c_*^{-1}L(r+1) + 2\delta$. The representative loops are in fact $\operatorname{pr}_1(\sigma_x(\gamma_j))$ where $\operatorname{pr}_1 \colon E_\varepsilon \times B_0 \to E_\varepsilon$ is the first projection.

As E_ε is compact and $\pi_1(E_\varepsilon, w_0) = \mathbb{Z}$, we can thus conclude as in the proof of Theorem F by applying Lemma 6.21. The proof is thus completed. $\square$

4. Appendix: Proof of Theorem 8.14

4.1. A conceptual proof. Recall that $Y = \mathbb{C}^* \times B_0$. We regard $f \colon Y \to B_0$ as a constant sheaf of (multiplicative) abelian groups over B_0. There is a canonical short exact sequence of sheaves over B_0 induced by the exponential map:

$$0 \to (R^1 f_* \mathbb{Z})^\vee \to T_Y \to Y \to 0. \tag{4.1}$$

Consider the n-th power B_0-morphism $\wedge n \colon Y \to Y$. The induced differential map $d(\wedge n) \colon T_Y \to T_Y$, is an isomorphism since we are in characteristic 0 so that n is invertible. The group of global sections $Y(B_0)$ is naturally identified with a subgroup of $Y_K(K) = K^*$. Clearly, $Y[n] := \mathrm{Ker}(\wedge n)$ is the constant sheaf μ_n on B_0. The map $\wedge n$ and the sequence (4.1) induce a commutative digram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Y[n] \arrow[d] & \\
0 \arrow[r] & (R^1 f_* \mathbb{Z})^\vee \arrow[r] \arrow[d, "n"] & T_Y \arrow[r] \arrow[d, "d(\wedge n)"] & Y \arrow[r] \arrow[d, "\wedge n"] & 0 \\
0 \arrow[r] & (R^1 f_* \mathbb{Z})^\vee \arrow[r] \arrow[d] & T_Y \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
 & Q \arrow[r] \arrow[d] & 0 \arrow[r] & 0 & \\
 & 0 & & &
\end{tikzcd} \tag{4.2}$$

We have a natural isomorphism $Y[n] \simeq Q$ (by the snake lemma). The cohomology long exact sequences induced by Diagram (4.2) give a commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_S^* \arrow[r] \arrow[d, "n"] & Y(B_0) \arrow[r] \arrow[d, "n"] & H^1(B_0, (R^1 f_* \mathbb{Z})^\vee) \arrow[d, "n"] \\
\mathcal{O}_S^* \arrow[r] \arrow[d] & Y(B_0) \arrow[r] \arrow[d] & H^1(B_0, (R^1 f_* \mathbb{Z})^\vee) \arrow[d] \\
H^1(\widehat{G}, \mu_n) \arrow[r] & H^1(B_0, Y[n]) \arrow[r, "\simeq"] & H^1(B_0, Q)
\end{tikzcd} \tag{4.3}$$

Since Y and B_0 are $K(\pi, 1)$-spaces, we have canonical isomorphisms:

$$H^1(B_0, \mu_n) \simeq H^1(G, \mu_n), \quad H^1(B_0, (R^1 f_* \mathbb{Z})^\vee) \simeq H^1(G, \Gamma),$$

where $G = \pi_1(B_0, b_0)$ and $\Gamma = H_1(Y_{b_0}, \mathbb{Z}) \simeq (R^1 f_* \mathbb{Z})^\vee_{b_0} \simeq \mathbb{Z}$. The actions of the group G on Γ and on μ_n are trivial (the monodromy is trivial in a trivial fibration).

By [**98**, I.2.6.b], there is a natural isomorphism $H^1(\widehat{G},\Gamma) \simeq H^1(G,\Gamma)$ induced by the injection $G \to \widehat{G}$. Hence, one obtains easily the diagram in Theorem 8.14.

4.2. An explicit proof. Experts are welcome to skip the section. We will need the following property.

Remark 8.16. For an integer $n \geq 1$, denote $(\mathcal{O}_S^*)^{1/n^\infty} \subset \bar{\mathcal{O}}_S \subset \bar{K}$ the set consisting of all elements $x \in \bar{K}$ such that $x^{n^m} \in \mathcal{O}_S^*$ for some integer $m \geq 0$. Then $(\mathcal{O}_S^*)^{1/n^\infty}$ is an abelian group closed under the action of taking n-th roots in $\bar{K}$.

Let U/k be a curve. Let $q \geq 1$ be an integer and consider the multiplication-by-n endomorphism $[q]\colon H \to H$ of the U-group scheme $H = \mathbb{G}_m \times U$ where $\mathbb{G}_m$ denotes the usual multiplicative group. Then $[q]$ is a finite étale map of degree q.

Let K_S/K is the maximal Galois extension of K which is unramified outside of S. Denote $G_S \coloneqq \mathrm{Gal}(K_S/K)$ the group of K_S over K. We apply the above remark to the case $U = B \setminus S$. Every $x \in \mathcal{O}_S^*$ corresponds exactly to a section Σ_x of the second projection $H = \mathbb{G}_m \times U \to U$ (cf. Remark 8.1). It follows that the preimage $[n^m]^{-1}(\Sigma_x)$ is étale over U. Thus by definition of G_S, we find that G_S acts transitively on the set of n^m-th roots of every element of $\mathcal{O}_S^*$. Hence, we obtain the following equality necessary for the proof of Theorem 8.14.

$$\left((\mathcal{O}_S^*)^{1/n^\infty}\right)^{G_S} = \mathcal{O}_S^*. \tag{4.4}$$

Proof of Theorem 8.14. Suppose first that the diagram

$$\begin{tikzcd}
\mathcal{O}_S^*/(\mathcal{O}_S^*)^n \arrow[r, hook, "\delta"] & H^1(\widehat{G},\mu_n) \arrow[dr, "\simeq"] & \\
\mathcal{O}_S^* \arrow[u] \arrow[r, "\alpha"] & H^1(G,\mathbb{Z}) \arrow[r, "\beta"] & H^1(G,\mu_n).
\end{tikzcd}$$

is commutative. If two elements of $\mathcal{O}_S^*$ induces the same section of (3.1), their quotient must be an element of $(\mathcal{O}_S^*)^n$ for every $n \geq 2$. Since $\mathcal{O}_S^*/\mathbb{C}^*$ is finitely generated (cf. Proposition 8.12), the quotient belongs to $\mathbb{C}^*$ and the last statement of Theorem 8.14 follows.

<u>Preliminary remarks</u>: Let $S \subset B$ be the centres of the discs in U. First, observe that we have

$$\widehat{G} = G_S \coloneqq \mathrm{Gal}(K_S/K)$$

where $\widehat{G}$ is the profinite completion of G and K_S/K is the maximal Galois extension of K which is unramified outside of S.

Second, $f\colon Y \to B_0$ is obviously a commutative group scheme. For every integer $m \geq 1$ and $x \in \mathcal{O}_S^*$, consider the following cartesian diagram

$$\begin{tikzcd} C_{x,m} \arrow[r, hook] \arrow[d, "\wedge m"] & Y \arrow[d, "\wedge m"] & \\ \sigma_x(B_0) \arrow[r, hook] & Y \arrow[r, "f"] & B_0. \end{tikzcd} \tag{4.5}$$

where $\sigma_x \colon B \to X$ is the corresponding section of x. Remark that $x \in \mathcal{O}_S^*$ means σ_x restricts to a section $B \setminus S \to F \times B \setminus S$ and in particular to a section $B_0 \to Y$. The m-th power B_0-map $\wedge m \colon Y \to Y$ is finite étale of degree m. As $\sigma_x(B_0) \to B_0$ is an isomorphism, $f|_{C_{x,m}} \colon C_{x,m} \to B_0$ is also finite étale.

<u>*The description of* δ</u>: Since $\mu_n \subset \mathbb{C} \subset K$, the Galois group $\mathrm{Gal}(\bar{K}/K)$ and thus $G_{K,S} \subset \mathrm{Gal}(\bar{K}/K)$ act trivially on μ_n. Hence, $H^1(G_S, \mu_n) = \mathrm{Hom}(G_S, \mu_n)$ by the definition of the first cohomology group.

Denote $(\mathcal{O}_S^*)^{1/n^\infty} \subset \bar{\mathcal{O}}_S \subset \bar{K}$ the set consisting of all elements $x \in \bar{K}$ such that $x^{n^m} \in \mathcal{O}_S^*$ for some integer $m \geq 0$. The Kummer exact sequence

$$1 \to \mu_n \to (\mathcal{O}_S^*)^{1/n^\infty} \xrightarrow{\wedge n} (\mathcal{O}_S^*)^{1/n^\infty} \to 1$$

where $\wedge n \colon z \to z^n$ is the n-th power map, induces a long exact sequence

$$0 \to \mu_n^{G_S} \to \left((\mathcal{O}_S^*)^{1/n^\infty}\right)^{G_S} \to \left((\mathcal{O}_S^*)^{1/n^\infty}\right)^{G_S} \to H^1(G_S, \mu_n).$$

We have $\mu_n^{G_S} = \mu_n$ and by Remark 8.16,

$$\left((\mathcal{O}_S^*)^{1/n^\infty}\right)^{G_S} := \{z \in (\mathcal{O}_S^*)^{1/n^\infty} : \sigma(z) = z, \text{ for all } z \in G_S\} = \mathcal{O}_S^*.$$

Therefore, we obtain an exact sequence:

$$0 \to \mu_n \to \mathcal{O}_S^* \to \mathcal{O}_S^* \to H^1(\widehat{G}, \mu_n) \tag{4.6}$$

and an injective homomorphism $\delta \colon \mathcal{O}_S^*/\mathcal{O}_S^{*n} \hookrightarrow H^1(\widehat{G}, \mu_n)$. Since $H^1(\widehat{G}, \mu_n) = \mathrm{Hom}(G_S, \mu_n)$, the map δ is given as follows:

$$\begin{aligned} \delta \colon \mathcal{O}_S^*/\mathcal{O}_S^{*n} &\to \mathrm{Hom}(G_S, \mu_n) \\ x &\mapsto (\phi_x \colon G_S \to \mu_n,\ \sigma \mapsto \sigma(t)/t) \end{aligned}$$

for any $t \in (\mathcal{O}_S^*)^{1/n^\infty}$ such that $t^n = x$. As in the proof of Proposition 6.3, the map ϕ_x is a well-defined homomorphism and is independent of the choice of t. Moreover, if $b_0 \in B \setminus S$ then $x_0 = x(b_0) \in \mathbb{C}^*$ and we can write $\delta(x) \colon \sigma \mapsto \sigma(t_0)/t_0$ for any t_0 such that $t_0^n = x_0$.

<u>*The description of* α</u>: Recall that the fibre bundle $Y_{b_0} \to Y \to B_0$ induces an exact sequence of fundamental groups:

$$0 \to \pi_1(Y_{b_0}, w_0) \to \pi_1(Y, w_0) \xrightarrow{\eta} \pi_1(B_0, b_0) \to 0. \tag{4.7}$$

We fix a collection of paths $l_{w_0,w}\colon [0,1] \to Y_{b_0}$ such that $l_{w_0,w}(0) = w_0$ and $l_{w_0,w}(1) = w \in Y_{b_0}$. Every section $\sigma_x\colon B_0 \to Y$ induces a section $i_x\colon \pi_1(B_0, b_0) \to \pi_1(Y, w_0)$ of (4.7) as follows. Take any loop γ of B_0 based at b_0, we define $i_x([\gamma])$ by:

$$i_x([\gamma]) = [l^{-1}_{w_0,\sigma_x(b_0)} \circ \sigma_x(\gamma) \circ l_{w_0,\sigma_x(b_0)}] \in \pi_1(Y, w_0).$$

As i_x, i_1 are sections of $\eta\colon \pi_1(Y) \to \pi_1(B_0)$, the difference $i_x - i_1$ satisfies

$$\eta(i_x - i_1) = \eta(i_x) - \eta(i_1) = 0.$$

Therefore, $\mathrm{Im}(i_x - i_1) \subset \operatorname{Ker}\eta = H_1(Y_{b_0}, \mathbb{Z})$. We have a map $i_x - i_1\colon \pi_1(B_0, b_0) \to H_1(Y_{b_0}, \mathbb{Z})$ which is well defined up to a principal cross homomorphism induced by different choices of the paths $l_{w_0,w}$. By (4.7), $i_x - i_1$ is a 1-cocycle of the group $G = \pi_1(B_0, b_0)$ with coefficients in $\Gamma = H_1(Y_{b_0}, \mathbb{Z}) \simeq \mathbb{Z}^2$ with the G-monodromy action given by conjugation as follows. Let $\alpha\colon I \to Y_{b_0}$ and $\gamma\colon I \to B_0$ be loops base at w_0 and b_0. Let $\gamma' = \sigma_1 \circ \gamma$. Then by (4.7), $\gamma'\alpha\gamma'^{-1}$ defines an element in $\pi_1(F, w_0)$, denoted $[\gamma]\cdot[\alpha]$ which does not depend on the choices of α and γ.

Writing the group laws of fundamental groups multiplicatively, we have exactly as in the proof of Proposition 6.3 that

$$(i_x - i_1)([\gamma_1][\gamma_2]) = ((i_x - i_1)[\gamma_1])([\gamma_1]\cdot(i_x - i_1)[\gamma_2]).$$

It follows that $i_x - i_1$ is a 1-cocycle of G. We claim that the induced natural map

$$\alpha\colon \mathcal{O}_S^* \to H^1(G, \Gamma), \quad x \mapsto i_x - i_1,$$

is a homomorphism. Indeed, the fiberwise multiplication by σ_x induces a B_0-automorphism of Y which is trivial on $\pi_1(Y)$ (up to conjugation) and on $H_1(F, \mathbb{Z})$. Therefore, we have the following equalities in $H_1(F, \mathbb{Z})$ for all $[\gamma] \in \pi_1(B_0, b_0)$:

$$(i_{xu} - i_x)[\gamma] = \sigma_x((i_u - i_1)[\gamma]) = (i_u - i_1)[\gamma].$$

It follows that $i_{xu} - i_x = i_u - i_1$ hence $i_{xu} - i_1 = i_u - i_1 + i_x - i_1$ and $\alpha(xu) = \alpha(x) + \alpha(u)$ as claimed.

The description of β: From the exponential exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp(2i\pi\cdot)} \mathbb{C}^* \to 1 \tag{4.8}$$

we have a canonical isomorphism $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ given by $z \mapsto \exp(2i\pi z)$. The projection $\pi\colon (\mathbb{C}, 0) \to (\mathbb{C}^*, 1)$ is the universal cover of $\mathbb{C}^*$ and we have a canonical isomorphism given by the deck transformation:

$$T\colon \Gamma = \pi_1(\mathbb{C}^*, 1) \to \mathbb{Z}, \quad \gamma \mapsto \gamma \cdot 0. \tag{4.9}$$

The homomorphism $\beta\colon H^1(G, \Gamma) \to H^1(G, \mu_n)$ is simply induced by the homomorphism of G-modules

$$e_n\colon \Gamma \to \mu_n, \quad g \mapsto \exp\left(\frac{2i\pi T(g)}{n}\right). \tag{4.10}$$

Commutativity of the diagram: To finish the proof of Proposition 6.3, we need to check that the diagram commutes, i.e., for every $\tau \in \Gamma$, $e_n(i_x(\tau) - i_1(\tau)) = \tau(u)/u \in \mu_n$. This follows from Lemma 8.17 below. □

Lemma 8.17. *Let $x \in \mathcal{O}_S^*$ and $u \in \bar{\mathcal{O}}_S^*$ such that $u^n = x$. For every $\tau \in G$,*

$$i_x(\tau) - i_1(\tau) = nT^{-1}(\tilde{u}' - \tilde{u}) \mod n\Gamma,$$

where $\tilde{u}, \tilde{u}' \in \mathbb{C}$ are arbitrary points lying respectively above $u, \tau(u)$ under the universal covering $\pi\colon \mathbb{C} \to \mathbb{C}^*$.

Proof. Recall that $f_{C_{x,n}}\colon C_{x,n} \to B_0$ is a finite étale cover of B_0 (cf. (4.5)). Choose a point $u \in C_{x,n} \cap Y_{b_0} \subset \mathbb{C}^*$ and a loop $L\colon [0,1] \to B_0$ representing τ. Let $\tau(u) \in C_{x,n} \cap Y_{b_0}$ be the image of u under the deck transformation action of τ on the cover $f_{C_{x,,n}}$. Let $x_0 = \sigma_x(b_0) \in Y_{b_0}$ then $u^n = x_0$ and $\tau(u)^n = \tau(u^n) = \tau(x_0) = x_0$ by the definition of $C_{x,n}$. Hence $\tau(u)/u \in \mu_n$. By the lifting property, the loop L lifts to a unique path $\tilde{L}\colon [0,1] \to C_{x,n}$ with $\tilde{L}(0) = u$ and $\tilde{L}(1) = \tau(u)$.

For every $z, t \in \mathbb{C}$, denote by $l_{zt}\colon [0,1] \to \mathbb{C}$ the linear path joining z and t.

Recall the n-th power B_0-map $\wedge n\colon Y \to Y$. Using the exact sequence (4.7), the properties $(\wedge n)(\tilde{L}) = \sigma_x(L)$ and $\wedge n(\sigma_1(L)) = \sigma_1(L)$, we find that the class of

$$\begin{aligned}(\wedge n)\left(\pi(l_{0u}^{-1}) \circ \tilde{L} \circ \pi(l_{0u}) - \sigma_1(L)\right) &= (\wedge n)\left(\pi(l_{0u}^{-1}) \circ \tilde{L} \circ \pi(l_{0u})\right) - \sigma_1(L) \\ &= \pi(l_{0u^n}^{-1}) \circ (\wedge n)(\tilde{L}) \circ \pi(l_{0u^n}) - \sigma_1(L) \\ &= \pi(l_{0x}^{-1}) \circ \sigma_x(L) \circ \pi(l_{0x}) - \sigma_1(L)\end{aligned} \tag{4.11}$$

is equal to $i_x(\tau) - i_1(\tau) \in \Gamma$. On the other hand, since

$$f(\pi(l_{0u}^{-1}) \circ \pi(l_{u\tau(u)}^{-1}) \circ \tilde{L} \circ \pi(l_{0u}) - \sigma_1(L)) = L - L = 0,$$

the sequence (4.7) implies that $\pi(l_{0u}^{-1}) \circ \pi(l_{u\tau(u)}^{-1}) \circ \tilde{L} \circ \pi(l_{0u}^{-1}) - \sigma_1(L) \in \Gamma$. Therefore,

$$(\wedge n)\pi(l_{u\tau(u)}) = (\wedge n)\left(\pi(l_{0u})^{-1} \circ \tilde{L} \circ \pi(l_{0u}) - \sigma_1(L)\right) \mod n\Gamma \tag{4.12}$$

and thus modulo $n\Gamma$, we conclude that:

$$\begin{aligned}nT^{-1}(\tilde{u}' - \tilde{u}) &= (\wedge n)\pi(l_{u\tau(u)}) && \text{(See Lemma 8.18)} \\ &= (\wedge n)\left(\pi(l_{0u})^{-1} \circ \tilde{L} \circ \pi(l_{0u}) - \sigma_1(L)\right) && \text{(By (4.12))} \\ &= i_x(\tau) - i_1(\tau). && \text{(By (4.11))}\end{aligned}$$

□

Therefore, for every $\tau \in G$, we have:

$$\begin{aligned} e_n(i_x(\tau) - i_1(\tau)) &= \exp\left(\frac{2i\pi T(i_x(\tau) - i_1(\tau))}{n}\right) && \text{(By definition of } e_n \text{ in (4.10))} \\ &= \exp\left(\frac{2i\pi T(nT^{-1}(\tilde{u}' - \tilde{u}))}{n}\right) && \text{(By Lemma 8.17)} \\ &= \exp\left(\frac{2i\pi(\tilde{u}' - \tilde{u})}{n}\right) = \tau(u)/u && \text{(By definition of } \tilde{u}', \tilde{u}\text{),} \end{aligned}$$

and the commutativity of the diagram in Theorem 8.14 follows. To complete the proof of Lemma 8.17, we prove:

Lemma 8.18. *Fix $0 \in \mathbb{C}$ as the point lying above $1 \in \mathbb{C}^*$ under the universal map $\pi\colon \mathbb{C} \to \mathbb{C}^*$, $x \mapsto \exp(2i\pi x)$. The corresponding deck transformation action*

$$\pi_1(\mathbb{C}^*, 1) \times \mathbb{Z} \to \mathbb{Z}, \quad (g, x) \mapsto g \cdot x \in \mathbb{C}.$$

is compatible with the group law of $\mathbb{C}^*$. For $n \geq 1$, $\gamma \in \Gamma = \pi_1(\mathbb{C}^*, 1)$, we have:

$$(\wedge n)(\gamma) = n\gamma \in \Gamma, \quad (n\gamma) \cdot 0 = n(\gamma \cdot 0) \in \mathbb{Z} \subset \mathbb{C}.$$

Proof. Recall from (4.9) the isomorphism of groups $T\colon \pi_1(\mathbb{C}^*, 1) \to \mathbb{Z}$ given by the deck transformation $g \mapsto g.0$. The inverse map $T^{-1}\colon \mathbb{Z} \to \pi_1(\mathbb{C}^*, 1)$ is defined as follows. Let $x \in \mathbb{Z} \subset \mathbb{C}$ and consider the linear path $l_{0x}\colon [0,1] \to \mathbb{C}$ going from 0 to x. The image $\pi(l_{0x})$ is a loop based at $1 \in \mathbb{C}^*$. We set $T^{-1}(x)$ to be the class of $\pi(l_{0x})$ in $\pi_1(\mathbb{C}^*, 1)$.

Now let $\gamma \in \pi_1(\mathbb{C}^*, 1)$. The loop $\pi(l_{0x})\colon [0,1] \to \mathbb{C}^*$ represents γ where $x = T(\gamma) \in \mathbb{Z} \subset \mathbb{C}$. Similarly, the loop $\pi(l_{0(nx)})\colon [0,1] \to \mathbb{C}^*$ represents $n\gamma$ where $nx = T(n\gamma)$. The commutative diagram:

$$\begin{tikzcd} {[0,1]} \arrow[r, "l_{0x}"] \arrow[dr, "\pi(l_{0x})"'] & \mathbb{C} \arrow[r, "\times n"] \arrow[d, "\pi"] & \mathbb{C} \arrow[d, "\pi"] \\ & \mathbb{C}^* \arrow[r, "\wedge n"] & \mathbb{C}^*, \end{tikzcd}$$

imply that $(\wedge n)(\gamma) = n\gamma \in \pi_1(\mathbb{C}^*, 1)$ and the first equality is proved. Since $\gamma \cdot 0 = T(\gamma) = x$, it follows that $n(\gamma \cdot 0) = nx = T(n\gamma) = (n\gamma) \cdot 0$ and the second equality follows. $\square$

CHAPTER 9

Uniformity of integral sections on constant abelian varieties

1. Introduction

Most of the results in this chapter are motivated by the following works of Noguchi-Winkelmann (cf. [**82**]) on the intersection multiplicities of curves with an ample divisor in an abelian variety. For the notations, we fix throughout an algebraically closed field k of characteristic 0 unless stated otherwise.

Theorem 9.1 (Noguchi-Winkelmann)**.** *There is a function $M_{NW}\colon \mathbb{N}^3 \to \mathbb{N}$ satisfying the following property. Let C be a smooth compact curve of genus g, let A be an abelian variety of dimension n, let D be an ample effective divisor on A with intersection number $D^n = d$. Let $f\colon C \to A$ be a morphism. Then either $f(C) \subset D$ or $\mathrm{mult}_x f^*D \leq M_{NW}(g, n, d)$ for all $x \in C$.*

In the other extreme situation where the abelian variety is traceless, we also have the following uniform bound of intersection multiplicities of Buium (cf. [**14**]).

Theorem 9.2 (Buium)**.** *Let $K = k(B)$ be the function field of an integral curve B/k. Let A/K be an abelian variety with $\mathrm{Tr}_{\overline{K}/k}(A_{\overline{K}}) = 0$. Then for each rational function $f \in K(A)$, there exists a positive constant $m(A, K, f)$ such that for every $P \in A(K)$ where f is defined and does not vanish, the zeroes and poles of $f(P) \in K^*$ have orders at most $m(A, K, f)$.*

The goal of the presenting chapter is to obtain the following uniform bound on the number of integral points in a constant abelian variety with respect to a constant effective divisor.

Let $n, g, d, s \geq 0$ be integers. Consider a smooth projective C/k of genus g with a finite subset $S \subset C$ of cardinality at most s. Consider an abelian variety A/k of dimension n and an effective ample divisor D on A of degree $D^n = d$. Denote by W the set of nonconstant algebraic morphisms $f\colon (C \setminus S) \to (A \setminus D)$.

Theorem I. *There exists a number $N(g, s, n, d) \geq 0$ such that:*

(i) *either W is infinite or $\#W \leq N(g, s, n, d)$;*

(ii) *$\#\{f \in W\colon a + \mathrm{Im}\, f \nsubseteq D, \forall a \in A\} \leq N(g, s, n, d)$;*

(iii) *if* $n=2$, $d>2g-2$ *and* D *is integral then* $\#W \leq N(g,s,n,d)$.

The case of an arbitrary effective divisor $\mathcal{D} \subset A \times C$ is also treated with the tools of jet-differentials as in the proof of Theorem 9.1 as follows (cf. Theorem 9.10).

Theorem J. *Let* A/k *be an abelian variety and let* C/k *be a smooth projective curve. Let* $\mathcal{D}$ *be an integral divisor in* $A \times C$. *There exists a number* $M>0$ *satisfying the following property. For every morphism* $\phi \colon C \to A$ *such that* $(\phi \times \mathrm{Id}_C)(C) \nsubseteq \mathcal{D}$, *we have an estimation*

$$\mathrm{mult}_x(\phi \times \mathrm{Id})^* \mathcal{D} \leq M \quad \text{for all } x \in C.$$

As an application, we establish the following semi-effective bound on the number of integral points of *bounded denominators* (cf. Theorem 9.13):

Corollary D. *Let* C/k *be a curve of genus* g *and let* A/k *be an abelian variety of dimension* n. *Let* $\mathcal{D} \subset A \times C$ *be an effective divisor such that* $\mathcal{D}_K$ *is ample. For each integer* $s \geq 1$, *let* $W(s,\mathcal{D})$ *be the set of morphisms* $f \colon C \to A$ *such that* $\#(f \times \mathrm{Id}_C)(C) \cap \mathcal{D} \leq s$. *Then there exists a number* $H>0$ *such that for any* $s \geq 1$ *we have a semi-effective bound*

$$\#W(s,\mathcal{D}) \text{ modulo } A(k) \leq (2\sqrt{sH}+1)^{4gn}.$$

An application on the generic emptiness of $(S,\mathcal{D})$-integral points is also given in Corollary 9.15.

2. Preliminaries

The following lemmata are well-known.

Lemma 9.3. *Let* $f \colon X \to Y$ *be a morphism of finite presentation of quasi-compact and separated schemes. Assume that* f *is quasi-finite. Then there exists a number* $n \geq 0$ *such that for every* $y \in Y$, *the fibre* $X_y = f^{-1}(y)$ *has at most* n *points.*

Proof. By Zariski's Main Theorem ([**65**, Chapter 4.4]), there exists a finite morphism $h \colon Z \to Y$ and an open immersion $\iota \colon X \to Z$ such that $f = h \circ \iota$. Hence, it suffices to prove the statement for h. We can clearly suppose that Z and Y are integral. Then the degree $n_0 = \deg h = [\kappa(Z) \colon \kappa(Y)] \in \mathbb{N}$ is well-defined. By generic flatness, there exists a dense open subset $U \subset Y$ such that $h_U \colon h^{-1}(U) \to U$ is flat and thus (cf. [**65**, Exercise 5.1.25]) every fibre of h_U has at most n_0 points. We continue the process with the finite morphism $h_1 \colon h^{-1}(Y_1) \to Y_1$. Similarly, we obtain another bound n_1 on the cardinality of the fibres of h_1 over a dense open subset U_1 of the proper closed subscheme $Y_1 = Y \setminus U$ of $Y_0 = Y$. Since Y is a Noetherian topological space, the procedure terminates after finitely many steps (i.e., the closed subset $Y_i \subset Y_{i-1}$ is empty for some $i \geq 1$). We can take n to be the maximum of the n_i's to conclude. $\square$

Lemma 9.4. *Let S be a Noetherian scheme. Let X and Y be respectively projective and quasi-projective schemes over S. Suppose that X is flat over S then the functor $\mathcal{M}or_S(X,Y)\colon\ U \mapsto \mathrm{Mor}_U(X \times_S U, Y \times_S U)$ is representable by an open S-subscheme $\mathrm{Mor}_S(X,Y)$ of $\mathrm{Hilb}_{X\times Y/S}$. Moreover, the natural paring*

$$\sigma\colon X \times_S \mathrm{Mor}_S(X,Y) \to Y, \quad (x,f) \mapsto f(x) \tag{2.1}$$

is an S-scheme morphism.

Proof. The first statement is a well-known consequence of Grothendieck's theorem on Quot-schemes (cf. [**48**, Les schémas de Hilbert]). The second statement is an application of Yoneda's lemma. Let $Z = X \times_S \mathrm{Mor}_S(X,Y)$ and consider the covariant transformation of point functors $\Sigma\colon h_Z \to h_Y$ given by composition:

$$\begin{aligned} \Sigma(U)\colon h_Z(U) = X(U) \times \mathrm{Mor}_S(X,Y)(U) &\longrightarrow h_Y(U) = Y(U) \\ (x,f) &\longmapsto f \circ x, \end{aligned}$$

for every S-scheme U. Here we identify naturally $X(U) \simeq X_U(U)$ and $Y(U) \simeq Y_U(U)$ by the universal property of fibre products. Observe that Σ is exactly the pairing (2.1). It is a direct verification that Σ is a natural transformation, i.e., for every S-scheme morphism $q\colon U \to V$, we have $\Sigma(V) \circ h_Z(q) = h_Y(q) \circ \Sigma(U)$. Therefore, Yoneda's lemma implies that Σ is representable by an S-scheme morphism $\sigma\colon Z \to Y$.

$\square$

Example 9.5. Let $S = \mathrm{Spec}(k)$ where k is a field.

(1) Let $X = \mathbb{P}^1_k$ and $Y = \mathbb{A}^1_k$. Then we have $\mathrm{Mor}_k(X,Y) = \mathbb{A}^1_k$ and the pairing $X \times_k \mathrm{Mor}_k(X,Y) \to Y$ is simply the second projection morphism $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$.

(2) Let $X = Y = \mathbb{P}^1_k$ then we can write $\mathrm{Mor}_k(X,Y) = \cup_{0 \leq d} \mathrm{Mor}^d_k(\mathbb{P}^1,\mathbb{P}^1)$, where $\mathrm{Mor}^d_k(\mathbb{P}^1,\mathbb{P}^1) = \mathbb{P}^{2d+1}_k$ parametrizes morphisms from $\mathbb{P}^1$ to $\mathbb{P}^1$ of degree at most d. Indeed, each morphism of degree at most d is given by a rational function $\sum_{i=0}^d a_i t^i / \sum_{i=0}^d b_i t^i \in k(t)$ where the coefficients a_i's, b_i's are not all zeros. Hence, such morphisms are in bijection with points $(a_0\colon \dots \colon a_d\colon b_0\colon \dots \colon b_d) \in \mathbb{P}^{2d+1}$. The pairing $X \times_k \mathrm{Mor}_k(X,Y) \to Y$ is then induced by the canonical scheme morphisms

$$\mathbb{P}^1 \times \mathbb{P}^{2d+1} \to \mathbb{P}^1, \quad ((t\colon 1),(a_0\colon \dots \colon a_d\colon b_0\colon \dots \colon b_d)) \mapsto \left(\sum_{i=0}^d a_i t^i\colon \sum_{i=0}^d b_i t^i\right).$$

3. Two uniform lemmata on abelian varieties

A *polarization* on an abelian variety is the algebraic equivalence class of an ample line bundle, or equivalently, an ample line bundle defined up to translation. Let L be an ample invertible sheaf on an abelian variety A of dimension n. Associated to L there is a canonical homomorphism $\phi_L\colon A \to A^\vee$ defined by $a \mapsto \tau_a^* L \otimes L^{-1}$. We have by the Riemann-Roch theorem that $h^0(A, L) = (L^n)/n!$, and $h^i(A, L) = 0$ for all $i > 0$. Moreover, $\deg \phi_L = (L^n/n!)^2$ and we call this number the *degree* of the polarization induced by L. In particular, $n!$ divides (L^n) and L admits global sections. L is a *principal polarization* if $(L^n) = n!$, i.e., $h^0(A, L) = 1$, or equivalently, ϕ_L is an isomorphism.

We first recall the following well-known result of Mumford on the moduli space of abelian varieties. Let $n, d, N \geq 1$ be integers. For each $\mathbb{Z}[1/N]$-scheme T, let $\mathcal{M}_{g,d^2,N}(T)$ be the set of isomorphism classes of triples (A, ϕ, j), where A is an abelian scheme over T of relative dimension n, ϕ is a degree-d^2 polarization, and $j\colon (\mathbb{Z}/N\mathbb{Z})^{2n} \to A[N]$ is an isomorphism of T-groups. We thus obtain a contravariant functor $\mathcal{M}_{g,d^2,N}\colon \mathrm{Sch}_{\mathbb{Z}[1/N]} \to \mathrm{Set}$.

Theorem 9.6 (Mumford). *The functor $\mathcal{M}_{n,d^2,N}$ is representable by a quasi-projective $\mathbb{Z}$-scheme $M_{n,d^2,N}$ whenever $N \geq 6^n d\sqrt{n!}$.*

PROOF. See [**79**, Theorem 7.9] □

Now return to the notations of Theorem I. Denote $\mathcal{A} = A \times C$ the constant abelian family over C induced by A. Let $K = k(C)$ and $A_K = A \otimes K$ the generic fibre of $\mathcal{A}$. We can thus view W as a subset of rational points of $A(K)$. The strategy is to establish first a uniform bound on the canonical height associated to the symmetric ample divisor $D + [-1]^*D$. We shall need the following lemma.

Lemma 9.7. *For each integers $n, d \geq 1$, there exists a number $m(n, d)$ with the following property. For every ample divisor D of degree d on an abelian variety A of dimension n,*

$$mD - (D + [-1]^*D)$$

is ample whenever $m \geq m(n, d)$.

Notice that if $n = 1$, i.e., A is an elliptic curve, then we can simply take $m(n, d) = 3$. Indeed, since $\deg D = \deg[-1]^*D$,

$$\deg(3D - (D + [-1]^*D)) = \deg(2D - [-1]^*D) = \deg D > 0.$$

It follows that $3D - (D + [-1]^*D)$ is ample.

Proof of Lemma 9.7. We consider a parameter space of n-dimensional abelian varieties $\pi\colon \mathcal{A} \to T$ which is also an *abelian scheme* over T, i.e., it is equipped with a zero section $e\colon T \to \mathcal{A}$, an inverse morphism $[-1]\colon \mathcal{A} \times_T \mathcal{A} \to \mathcal{A}$ and an associative abelian sum morphism $\sigma\colon \mathcal{A} \times_T \mathcal{A} \to \mathcal{A}$. Let $\mathcal{D}$ be the universal effective ample divisor of degree d on $\mathcal{A}$. For example, we can take $\mathcal{A}$, $\mathcal{D}$ to be respectively the universal abelian scheme with large enough level structure N and the universal ample divisor of degree d^2 over the Mumford moduli space $T = M_{n,d^2,N}$ in Theorem 9.6.

It suffices to find $m = m(n,d)$ such that $m\mathcal{D} - (\mathcal{D} + [-1]^*\mathcal{D})$ is ample. This can be done by Noetherian induction as follows. Up to replacing T by each of its irreducible components, we can suppose that T is integral. Let η be the generic point of T. Then $\mathcal{D}_\eta$ is ample on the generic abelian variety $\mathcal{A}_\eta$. Hence, there exists an integer $m_0 \geq 1$ such that $m_0\mathcal{D}_\eta - [-1]^*\mathcal{D}_\eta$ is ample. As ampleness is an open property on the base, we deduce that there exists a dense open subset $U_1 \subset T$ over which $m_0\mathcal{D} - [-1]^*\mathcal{D}$ is ample. Let $T_1 = T \setminus U_1$. As T_1 contains only a finite number of irreducible components, we can suppose that T_1 is irreducible to simplify the notations without affecting the uniform conclusion on $m(n,d)$. We continue the above procedure for T_1 and for the abelian scheme $\pi_1\colon \mathcal{A}_1 \to T_1$ and for and $\mathcal{D}_1 = \mathcal{D}|_{T_1}$, where $\pi_1 = \pi|_{\pi^{-1}T_1}$ and $\mathcal{A}_1 = \pi^{-1}T_1$. We then obtain a closed subset $T_2 \subset T_1$ (which we can again assume irreducible without loss of generality as above) and so on. In this ways, we obtain a sequence of integers $m_i \geq 1$ and a sequence of decreasing closed subsets $T_0 = T \supset T_1 \supset \cdots$ such that the divisor

$$\left(\max_{0 \leq i \leq p} m_i\right)\mathcal{D} - [-1]^*\mathcal{D}$$

is relatively ample over $T \setminus T_{p+1}$. Notice that $\dim T_{i+1} < \dim T_i$ for every i. It follows that the sequence $(T_i)_i$ is finite, i.e., $T_{q+1} = \varnothing$ for some integer $q \geq 0$. It suffices to set $m = m(n,d) = 1 + \max_{0 \leq i \leq q} m_i$ to see that

$$m\mathcal{D} - (\mathcal{D} + [-1]^*\mathcal{D}) = \left(\max_{0 \leq i \leq q} m_i\right)\mathcal{D} - [-1]^*\mathcal{D}$$

is relatively ample over $T \setminus T_{q+1} = T$. Therefore, $m\mathcal{D}_t - (\mathcal{D}_t + [-1]^*\mathcal{D}_t)$ is ample for every $t \in T$ and the conclusion follows. □

Now assume that C/k is a smooth projective connected curve of genus g and let A/k be an abelian variety of dimension n. Let L be an effective ample *numerical class* of divisors on A with top degree $d = L^n/n!$. For each finite subset $S \subset C$ and every effective divisor $D \in L$, let $Z(S,D)$ be the set of nonconstant morphisms $f\colon C \to A$ such that $f^{-1}(D) \subset S$. In the theorem below, the notation $D \geq 0$ means that D is an effective divisor.

Lemma 9.8. *For every $s \in \mathbb{N}$, we have*

$$\# \cup_{0 \leq D \in L} \cup_{\# S \leq s} Z(S, D) \text{ modulo } A(k) \leq (2\sqrt{m(n,d) s M_{NW}(g,n,d)} + 1)^{4ng}$$

where $M_{NW}(g,s,n,d)$ and $m(n,d)$ are defined in Theorem 9.1 and Lemma 9.7 respectively.

PROOF. For any ample divisor $D \in L$, we known that $m(n,d)D - (D + [-1]^* D)$ is ample by Lemma 9.7. It follows from Theorem 9.1 that the degree with respect to D of any nonconstant (S,D)-integral point $P \in A(K)$ is bounded by $sM_{NW(g,n,d)}$. Therefore, the canonical height of such points with respect to the symmetric ample divisor $D + [-1]^* D$ is bounded uniformly above by $m(n,d)sM_{NW}(g,n,d)$. We can thus conclude by using Lemma 2.24 applied to the lattice $A(K)/A(k)$ and the canonical positive definite quadratic form associated to $D + [-1]^* D$. $\square$

We are now in position to prove the first main result of the chapter.

PROOF OF THEOREM I. Let $f\colon (C \backslash S) \to (A \backslash D)$ be a nonconstant morphism. By Theorem 9.1, we have $\deg_C f^* D \leq H \coloneqq sM_{NW}(g,n,d)$. Since D is ample, it follows that

$$f \in \coprod_{1 \leq N \leq H} \mathrm{Mor}^{Nt+1-g}(C, A), \tag{3.1}$$

where the Hilbert polynomial is calculated with respect to the line bundle $\mathcal{O}(D)$.

By Theorem 9.6, there exists a universal abelian variety $A(n, d^2, N) \to M_{n,d^2,N}$ of n-dimensional degree-d^2 polarized abelian varieties of large level N-structure. Let $\mathcal{L}$ be the universal degree-d^2 ample line bundle on $A(n,d^2,N)$. Then $\mathcal{D} \to Y$, where $Y \coloneqq A(n,d^2,N) \times (\mathrm{Proj}\mathcal{L})^*$, is the universal effective ample divisor of degree d^2 on $A(n,d^2,N)$. In what follows, we denote $\mathcal{A} = A(n,d^2,N) \times Y$ and $T_1 = M_{n,d^2,N} \times Y$. Let T_2 be a coarse moduli space of curves of genus g with the universal curve $\mathcal{C} \to T_2$.

Let $T = T_1 \times T_2$ and let p_1, p_2 be respectively the first and second projection. Up to making the base change $p_1\colon T \to T_1$ for $\mathcal{A}, \mathcal{D}, T_1$ and $p_2\colon T \to T_2$ for $\mathcal{C}, \sigma, T_2$, we can suppose that $T_1 = T_2 = T$. Define

$$\mathcal{M} = \coprod_{1 \leq N \leq H} \mathrm{Mor}_T^{Nt+1-g}(\mathcal{C}, \mathcal{A}),$$

then $\mathcal{M}$ is a quasi-projective T-scheme by the standard Hilbert scheme theory. The Hilbert polynomial is calculated using the line bundle $\mathcal{O}(\mathcal{D})$. Now for each $i = 1, \ldots, s$, consider the image Δ_i of the following closed immersion of T-schemes

(diagonal morphisms)

$$h_i\colon \mathcal{C}\times_T(\mathcal{C}\times_T\cdots\times_T\hat{\mathcal{C}}_i\times_T\cdots\times_T\mathcal{C})\longrightarrow\mathcal{C}\times_T\mathcal{C}_T^s$$
$$(x,(x_1,\dots,x_{i-1},x_{i+1},\dots,x_s))\longmapsto(x,(x_1,\dots,x_{i-1},x,x_{i+1},\dots,x_s)),$$

where $\hat{\mathcal{C}}_i$ denotes the omitted i-th copy of $\mathcal{C}$ in the fibre product $\mathcal{C}_T^s$. Let $\Delta=\cup_{1\leq i\leq s}\Delta_i$ be a closed subset of $\mathcal{C}\times_T\mathcal{C}_T^s$. Consider the following map

$$\Psi\colon \mathcal{D}\times_T(\mathcal{C}\times_T\mathcal{C}_T^s\setminus\Delta)\times_T\mathcal{M}\longrightarrow\mathcal{A}\times_T\mathcal{C}_T^s\times_T\mathcal{M}$$
$$(x,y,z,f)\longmapsto(x-f(y),z,f).$$

We claim that Ψ is a morphism of quasi-projective T-schemes. Indeed, Ψ is a composition of the following T-morphisms:

$\iota\colon \mathcal{D}\times_T(\mathcal{C}\times_T\mathcal{C}_T^s\setminus\Delta)\times_T\mathcal{M}\to\mathcal{D}\times_T\mathcal{C}\times_T\mathcal{C}_T^s\times_T\mathcal{M}\simeq\mathcal{D}\times_T\mathcal{C}\times_T\mathcal{M}\times_T\mathcal{C}_T^s$ (immersion),

$\mathrm{Id}\times(\sigma,\mathrm{pr}_2)\times\mathrm{Id}\colon \mathcal{D}\times_T\mathcal{C}\times_T\mathcal{M}\times\mathcal{C}_T^s\to\mathcal{D}\times_T\mathcal{A}\times_T\mathcal{M}\times\mathcal{C}_T^s$, (cf. Lemma 9.4)

and the subtraction morphism $\mathcal{A}\times_T\mathcal{A}\to\mathcal{A}$ restricted to the first two factors $\mathcal{D}\times_T\mathcal{A}$.

Now define $\Sigma\coloneqq\mathcal{A}\times_T\mathcal{C}_T^s\times_T\mathcal{M}\setminus\operatorname{Im}\Psi$ to be the complement of the image of Ψ. By the Chevalley theorem, $\operatorname{Im}\Psi$ is a quasi-projective T-scheme thus so is Σ. Let $\pi\colon\Sigma\to\mathcal{C}_T^s\times_T\mathcal{M}\times_T T$ be the induced projection morphism. By [**46**, Proposition 9.2.6.(iv)], the subset $E\subset\mathcal{C}_T^s\times_T\mathcal{M}\times_T T$ above which π has finite fibres is constructible. We equip E with the induced reduced scheme structure. Then (cf. Lemma 9.3 below) there exists a number $N_1(g,s,n,d)$ such that for every $q\in E$, we have

$$\#\Sigma_q=\#\pi^{-1}(q)\leq N_1(g,s,n,d). \tag{3.2}$$

On the other hand, Lemma 9.8 implies that for each data set (C,S,A,D) with prescribed invariants (g,s,n,d), the number of translation classes of nonconstant morphisms $f\colon(C\setminus S)\to(A\setminus D)$ is bounded above by $(2\sqrt{m(n,d)sM_{NW}(g,n,d)}+1)^{4ng}$. Here, $M_{NW}(g,s,n,d)$ and $m(n,d)$ are defined respectively in Theorem 9.1 and Lemma 9.7.

From (3.1) and the definitions of T_1 and T_2, Theorem 9.1 implies that there exists $t\in T$ such that some translate of f belongs to $\mathcal{M}_t$. Moreover, the constructions of Ψ, Σ and of E which satisfies (3.2) imply that either there are at most $N_1(g,s,n,d)$ translates of f that belong to W or either there are an infinite number of them. By Theorem B, the latter case is excluded if no translates of $\operatorname{Im}f$ are contained in D. This allows us to conclude the first two statements (i) and (ii) by setting

$$N(g,s,n,d)=N_1(g,s,n,d)(2\sqrt{m(n,d)sM_{NW}(g,n,d)}+1)^{4ng}.$$

For the last statement (iii), suppose that $n = 2$, $d > 2g - 2$ and that D is an integral curve. If $g = 0$ then $C \simeq \mathbb{P}^1$. Since abelian varieties do not contain rational curves, every morphism $\mathbb{P}^1 \to A$ is constant and thus $\#W = 0$. Suppose now that $g \geq 1$. Since the canonical divisor K_A of A is trivial, we deduce from the adjunction formula that

$$p_a(D) = \frac{D^2 + D \cdot K_A}{2} + 1 = \frac{d}{2} + 1 > g \geq 1.$$

Therefore, the Riemann-Roch theorem for curves implies that any algebraic morphism $f\colon C \to D$ must be constant. It follows that for any non constant morphism $f \in W$, no translates of $\operatorname{Im} f$ can be contained in D. Hence, W must be finite by (ii) and we can conclude from (i) that $\#W \leq N(g, s, n, d)$. □

4. A generalization

4.1. Intersection multiplicities for varying divisors. We mention here without proof the following well-known property.

Lemma 9.9 (Identity principle)**.** *Let $f\colon C \to X$ be a holomorphic map from a connected complex curve C to a complex manifold X. Let $p \in C$ and let Y be an effective divisor of X such that $\mathit{mult}_p f^* Y \geq n$ for all $n \in \mathbb{N}$. Then $f(C) \subset Y$.*

By adopting the proof of Theorem 9.1 of Noguchi-Winkelmann, we obtain the following result on local intersection multiplicities in the setting of constant abelian varieties but with varying divisors.

Theorem 9.10. *Let A/k be an abelian variety and let C/k be a smooth projective curve. Let $\mathcal{D}$ be an integral divisor in $A \times C$. There exists a number $M > 0$ satisfying the following property. For every morphism $\phi\colon C \to A$ such that $(\phi \times \mathrm{Id}_C)(C) \not\subseteq \mathcal{D}$, we have an estimation*

$$\operatorname{mult}_x(\phi \times \mathrm{Id})^* \mathcal{D} \leq M \quad \textit{for all } x \in C.$$

Proof. We can suppose that $k = \mathbb{C}$ by the Lefschetz principle. Let $\pi\colon A \times C \to C$ be the canonical projection. Let $B = J(C)$ be the Jacobian variety of C and fix an embedding $C \to B$. For all analytic varieties X, Y and for every integer $n \geq 1$, we have a canonical identification $J^n(X \times Y) = J^n(X) \times J^n(Y)$. Let $\mathfrak{m}(n)$ be the maximal ideal of $\mathbb{C}\{t\}/(t^{n+1})$. Then by the exponential map, we have a natural *holomorphic* trivialization of the n-th jet bundle of A (cf. [**82**, Proposition 3.11]):

$$J^n(A) \simeq A \times (\mathfrak{m}(n) \otimes \operatorname{Lie} A).$$

Similarly, we have $J^n(B) \simeq B \times (\mathfrak{m}(n) \otimes \operatorname{Lie} B)$. This makes $J^n(A)$ and $J^n(B)$ become algebraic varieties since for example $\mathfrak{m}(n) \otimes \operatorname{Lie} A \simeq \mathbb{C}^{n \dim A}$ is algebraic.

Let $\phi\colon C \to A$ be an algebraic morphism. Then ϕ induces canonically a homomorphism $\tilde{\phi}\colon B \to A$ and a morphism $d^n\tilde{\phi}\colon J^n(B) \to J^n(A)$. Moreover, since morphisms from B to A *lift* canonically to homomorphisms of the associated Lie algebras, there exists a canonical linear map $\varphi_\phi\colon \operatorname{Lie} B \to \operatorname{Lie} A$ (as $\operatorname{Lie} A, \operatorname{Lie} B$ are commutative) which is independent of n and such that (cf. [**82**, Lemma 4.1]):

$$\Phi = \tilde{\phi} \times (\mathrm{Id}_{\mathfrak{m}(n)} \otimes \varphi_\phi)\colon B \times (\mathfrak{m}(n) \otimes \operatorname{Lie} B) \to A \times (\mathfrak{m}(n) \otimes \operatorname{Lie} A)$$

is compatible with $d^n\tilde{\phi}$, i.e., such that the diagram:

$$\begin{tikzcd}
J^n(B) \arrow[rr, "d^n\tilde{\phi}"] \arrow[d, "\simeq"] & & J^n(A) \arrow[d, "\simeq"] \\
B \times (\mathfrak{m}(n) \otimes \operatorname{Lie} B) \arrow[rr, "\Phi"] & & A \times (\mathfrak{m}(n) \otimes \operatorname{Lie} A)
\end{tikzcd}$$

is commutative. Notice that the linear map φ_ϕ determines locally, hence globally (since C is a curve and A is abelian) the morphism ϕ up to a translation of A. By abuse of notations, we will denote simply $d_\phi\colon \operatorname{Lie} B \to \operatorname{Lie} A$ the linear map φ_ϕ for every $\phi\colon C \to A$.

Denote $V = \operatorname{Hom}_{lin}(\operatorname{Lie} B, \operatorname{Lie} A) \simeq \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^g, \mathbb{C}^{\dim A}) \simeq \mathbb{A}^{g \dim A}$. Consider the algebraic variety $T \coloneqq C \times_C \mathcal{D} \times_k V$ where the morphism $\mathcal{D} \to C$ is induced by the projection $\pi\colon A \times C \to C$. We have natural inclusion of closed complex spaces:

$$J^n(\mathcal{D}) \subset J^n(A \times C) \subset J^n(A \times B) = J^n(A) \times J^n(B),$$

and similarly

$$J^n(C) \subset J^n(B).$$

Using the trivializations of $J^n(A)$, $J^n(B)$, the sets $J^n(\mathcal{D})$ and $J^n(C)$ can be regarded respectively as closed algebraic subvarieties of $\mathcal{D}\times (\mathfrak{m}(n) \otimes (\operatorname{Lie} A \times \operatorname{Lie} B))$ and $C \times (\mathfrak{m}(n) \otimes \operatorname{Lie} B)$. For each integer $n \geq 1$, define

$$W_n \coloneqq \{(c, d, \varphi) \in T\colon \left((\mathrm{Id}_{\mathfrak{m}(n)} \otimes \varphi) \times \mathrm{Id}_{J^n(B)}\right)(J^n_c(C)) \subset J^n_d(\mathcal{D})\} \subset T.$$

We claim that W_n is a closed algebraic subset of T for every $n \geq 1$. Indeed, we have the valuation morphism $\operatorname{Lie} B \times V \to \operatorname{Lie} A$, $(v, \varphi) \mapsto \varphi(v)$ which is clearly algebraic. Consider the following T-morphism of algebraic varieties

$$\Psi\colon J^n(C) \times_C T \to (\mathfrak{m}(n) \otimes (\operatorname{Lie} A \times \operatorname{Lie} B)) \times_k T$$
$$\left(\sum_i \alpha_i \otimes v_i, c, d, \varphi\right) \mapsto \left(\sum_i \alpha_i \otimes (\varphi(v_i), v_i), c, d, \varphi\right).$$

Denote $\Sigma \coloneqq \operatorname{Im}\Psi \setminus J^n(\mathcal{D}) \times_{\mathcal{D}} T$. Let $p_T\colon (\mathfrak{m}(n) \otimes (\operatorname{Lie} A \times \operatorname{Lie} B)) \times_k T \to T$ be the second projection then $W_n = T \setminus p_T(\Sigma)$. By Chevalley's theorem applied to Ψ and p_T, W_n is an algebraic subset of T. We shall show that the induced map

$p\colon \operatorname{Im}\Psi \to T$ is flat. This will prove the claim since $p_T(\Sigma)$ is then a Zariski open subset of T by the open property of flat morphisms and as Σ is Zariski open in $\operatorname{Im}\Psi$.

Clearly, it suffices to prove that the image of the map

$$\Phi\colon J^n(C) \times_C C \times V \to (\mathfrak{m}(n) \otimes (\operatorname{Lie} A \times \operatorname{Lie} B)) \times_k C \times V$$
$$\left(\sum_i \alpha_i \otimes v_i, c, \varphi\right) \mapsto \left(\sum_i \alpha_i \otimes (\varphi(v_i), v_i), c, \varphi\right)$$

is smooth (and thus flat) over $C \times V$. This is true because the fibre of $\operatorname{Im}\Phi$ over $(c, \varphi) \in C \times V$ is smooth as the image under the linear map $\operatorname{Id}_{\mathfrak{m}(n)} \otimes (\varphi, \operatorname{Id}_{\operatorname{Lie} B})$ of the smooth linear subspace $J_c^n(C) \subset \mathfrak{m}(n) \otimes \operatorname{Lie} B$. Hence, W_n is indeed a closed algebraic subset of T for every $n \geq 1$ as claimed.

Now we verify that $(W_n)_{n\geq 1}$ is a descending sequence of algebraic varieties. Indeed, suppose that $(c, d, \varphi) \in W_{n+1}$, this means by the definition that

$$d^{n+1}(f \times \operatorname{Id}_C)(J_c^{n+1}(C)) \subset J_d^{n+1}(\mathcal{D})$$

for any holomorphic germ map $f\colon (C, c) \to (A, d)$ such that $df_c(v) = \varphi(v)$ for all $v \in \operatorname{Lie} B$. Clearly, this is just a reformulation of the local condition:

$$\operatorname{mult}_c(f \times \operatorname{Id}_C)^*\mathcal{D} \geq n + 1.$$

It follows that $(c, d, \varphi) \in W_n$ and thus $W_{n+1} \subset W_n$ as desired.

Therefore, since the Zariski topology is Noetherian, the sequence $(W_n)_{n\geq 1}$ is stationary for $n \geq M$ for some integer $M \geq 1$. If a morphism $f\colon C \to A$ verifies $\operatorname{mult}_c(f \times \operatorname{Id}_C)^*\mathcal{D} \geq M + 1$ for some $c \in C$ then $(c, (c, f(c)), df_c) \in W_n$ for all $n \geq M$. This implies that $\operatorname{mult}_c(f \times \operatorname{Id}_C)^*\mathcal{D} \geq n$ for all $n \geq M$ and we must have $(f \times \operatorname{Id}_C)(C) \subset \mathcal{D}$ by Lemma 9.9. The conclusion thus follows. $\square$

4.2. Semi-effective bound for integral points with varying divisors. We continue with the following standard lemma.

Lemma 9.11. *Let X/k be a proper algebraic variety and let D be an effective integral divisor on X. Let $f\colon C \to X$ be a morphism from a curve C/k to X. We have for $\tilde{f} = f \times \operatorname{Id}_C$ that:*

$$\deg_C f^*\mathcal{O}(D) = (f_*C) \cdot D = (\tilde{f}_*C) \cdot (D \times C).$$

PROOF. Case $f(C) \not\subseteq D$ and D is smooth: Then there is an obvious bijection between the set theoretic intersections $f(C) \cap D$ and $(f \times \operatorname{Id}_C)(C) \cap D \times C$ given by $x \mapsto (x, f(x))$. Hence, it suffices to check that for every $x \in C$, we have

$$\operatorname{mult}_x f^*D = \operatorname{mult}_{(f(x),x)}(f \times \operatorname{Id}_C)^*(D \times C).$$

Let n_1 be the largest integer $n \geq 0$ such that $\operatorname{Im} d_x^n f \subset J_{f(x)}^n(D)$ where $d_x^n f \colon J_x^n(C) \to J_{f(x)}^n(X)$ is induced by f. Define n_2 to be the largest integer $n \geq 0$ such that for $d_x^n(f \times \mathrm{Id}_C) \colon J_x^n(C) \to J_{(f(x),x)}^n(X \times C)$, we have $\operatorname{Im} d_x^n(f \times \mathrm{Id}_C) \subset J_{(f(x),x)}^n(D \times C)$. Clearly $\operatorname{mult}_x f^*D = n_1 + 1$ and $\operatorname{mult}_{(f(x),x)}(f \times \mathrm{Id}_C)^*(D \times C) = n_2 + 1$. Hence, we must show that $n_1 = n_2$. Remark that $J_{(f(x),x)}^n(X \times C) = J_{f(x)}^n(X) \times J_x^n(C)$ and $J_{(f(x),x)}^n(D \times C) = J_{f(x)}^n(D) \times J_x^n(C)$. Under these natural identifications, $d_x^n(f \times \mathrm{Id}_C)$ is nothing but the morphism

$$d_x^n f \times d_x^n \mathrm{Id} \colon J_x^n(C) \to J_{f(x)}^n(X) \times J^n(C).$$

Since $d_x^n \mathrm{Id} = 0$ for all $n \geq 1$, we deduce that $n_1 = n_2$ as desired.

General case: If $\mathcal{O}(D)$ is a very ample line bundle then by Bertini's theorem and the invariance of intersection products by linear equivalence, we can suppose that D is a smooth integral divisor and that $f(C) \nsubseteq D$. We can then conclude by the above particular case. Note that we can replace D by any multiple nD where $n \geq 1$. Hence we can also conclude in the case when D is an ample divisor.

Any divisor can be written as a difference of two ample divisors. Therefore, $D = D_1 - D_2$ for some ample divisors D_1, D_2 on X. By the linearity of intersection products, the above paragraph concludes the proof of the lemma. $\square$

We remark the following comparison of the positivity of divisors.

Lemma 9.12. *Let C/k be a curve of function field $K = k(C)$ and let A/k be an abelian variety. Let $\mathcal{D} \subset A \times C$ be an effective integral divisor such that $\mathcal{D}_K$ is ample. Let $t_0 \in C$. Then $D = \mathcal{D}_{t_0}$ is an ample divisor on A and there exists $N, c > 0$ such that for every morphism $f \colon C \to A$ and $\tilde{f} = f \times \mathrm{Id}_C$, we have:*

$$-c + N^{-1}(\tilde{f}_*C) \cdot \mathcal{D} \leq (f_*C) \cdot D \leq N(\tilde{f}_*C) \cdot \mathcal{D} + c. \tag{4.1}$$

Proof. $\mathcal{D} \subset A \times C$ is a flat family of effective Cartier divisors parametrized by C (cf. Proposition 2.9). For $t \in C$, the divisors $\mathcal{D}_t$ are all algebraically equivalent on A. Moreover, they are ample since $\mathcal{D}_K$ is ample. Hence, for $N = 2$, the divisors $N\mathcal{D}_t - D$ and $ND - \mathcal{D}_t$ are ample on $A_t = A$ for every $t \in C$. It follows that $N\mathcal{D} - D \times C$ and $ND \times C - \mathcal{D}$ are relatively ample over C. Therefore, there exists vertical divisors V_1 and V_2 in $A \times C$ with finite images on C such that $\mathcal{O}(N\mathcal{D} - D \times C + V_1)$ and $\mathcal{O}(ND \times C - \mathcal{D} + V_2)$ are ample (cf. [**64**, Theorem 1.7.8, Proposition 1.7.10]).

Let $c = \max(|\deg_C \pi_* V_1|, |\deg_C \pi_* V_2|) \in \mathbb{N}$. For every morphism $f \colon C \to A$ and $\tilde{f} = f \times \mathrm{Id}_C$, we deduce that:

$$0 \leq (\tilde{f}_*C) \cdot (N\mathcal{D} - D \times C + V_1) \leq N(\tilde{f}_*C) \cdot \mathcal{D} - N(\tilde{f}_*C) \cdot (D \times C) + c.$$

The second inequality follows from the fact that $\tilde{f}_*C$ is reduced and transverse to every vertical integral divisor V of $A \times C \to C$. Since

$$(\tilde{f}_*C) \cdot (D \times C) = f_*(C) \cdot D$$

by Lemma 9.11, we deduce that

$$(f_*C) \cdot D \leq N(\tilde{f}_*C) \cdot \mathcal{D} + c.$$

The left inequality of (4.1) is obtained in a similar way by considering the ample line bundle $\mathcal{O}(ND \times C - \mathcal{D} + V_2)$. □

Using Theorem 9.10, we obtain the following semi-effective bound on integral points in contant abelian varieties with respect to a general effective divisor.

Theorem 9.13. *Let C/k be a curve of genus g and let A/k be an abelian variety of dimension n. Let $\mathcal{D} \subset A \times C$ be an effective divisor such that $\mathcal{D}_K$ is ample. For each integer $s \geq 1$, let $W(s, \mathcal{D})$ be the set of morphisms $f\colon C \to A$ such that $\#(f \times \mathrm{Id}_C)(C) \cap \mathcal{D} \leq s$. Then there exists a number $H > 0$ such that for any $s \geq 1$ we have a semi-effective bound*

$$\#W(s, \mathcal{D}) \text{ modulo } A(k) \leq (2\sqrt{sH} + 1)^{4gn}.$$

Remark 9.14. The set $W(s, \mathcal{D})$ is in fact a union of integral points:

$$W(s, \mathcal{D}) = \cup_{\#S \leq s}\{(S, \mathcal{D})\text{-integral sections of } A \times C \to C\}$$

where S describes all finite subsets of C of cardinality at most s.

Proof of Theorem 9.13. For each morphism $f\colon C \to A$, we will denote $\tilde{f} = f \times \mathrm{Id}_C\colon C \to A \times C$.

Since we work modulo translations by A, we can always and do assume that $\tilde{f}(C) \not\subseteq \mathcal{D}$ for each morphism $f\colon C \to A$ after a suitable translation. Indeed, suppose on the contrary that $\tilde{f}_a(C) \subset \mathcal{D}$ for all $a \in A$ where $f_a = f + t_a$ denotes the compposition of f with the translation-by-a map $t_a\colon A \to A$, $x \mapsto x + a$. Then

$$A \times C \subset \cup_{a \in A} \tilde{f}_a(C) \subset \mathcal{D},$$

which is a contradiction since $\dim \mathcal{D} = \dim A \times C - 1$. Therefore, by Theorem 9.10 there exists a number $M > 0$ such that

$$\mathrm{mult}_x \tilde{f}^*\mathcal{D} \leq M \quad \text{ for all } x \in C.$$

Theorem 9.10 implies that there exists a constant $M > 0$ such that $\mathrm{mult}_b \tilde{f}^*\mathcal{D} \leq M$ for every $b \in B$ and for every $f \in W(s, \mathcal{D})$ with $\tilde{f}(C) \not\subseteq \mathcal{D}$. It follows that:

$$\deg_C \tilde{f}^*\mathcal{D} = \sum_{c \in C} \mathrm{mult}_c \tilde{f}^*\mathcal{D} \leq \left(\#\tilde{f}(C) \cap \mathcal{D}\right).M \leq sM. \tag{4.2}$$

By Lemma 9.12, there exists $N, c > 0$ such that for every $f \in W(s, \mathcal{D})$ with $\tilde{f}(C) \nsubseteq \mathcal{D}$, we have:

$$\deg_C f^*\mathcal{O}(D) \leq N \deg \tilde{f}^*\mathcal{D} + c$$

where $D = \mathcal{D}_{t_0} \subset A_{t_0} = A$ for some $t_0 \in C$. By the same lemma *loc. cit*, D is ample. Combining with (4.2), we find that for every $f \in W(s, \mathcal{D})$ such that $\tilde{f}(C) \nsubseteq \mathcal{D}$, we have:

$$\deg_C f^*\mathcal{O}(D) \leq sNM + c \leq s(NM + c). \tag{4.3}$$

Since D is ample, mD dominates a symmetric ample divisor D_0 on A for some integer $m \geq 1$. We identify each $f \in W(s, \mathcal{D})$ with the associated rational point $P_f \in A(K)$. Let $H = m(NM + c)$. Then the canonical height of P_f in $A(K)$ with respect to D_0, which is given by $\deg_C f^*\mathcal{O}(D_0)$ and is invariant to the translation class of f, is bounded from the above by sH by (4.3). A standard counting argument on the lattice $A(K)/A(k)$ (cf. Lemma 2.24, Theorem 2.21) completes the proof. Remark that all torsion points of $A(K)$ belong to $A(\mathbb{C})$. □

5. Application on the generic emptiness of integral sections

Let A/k be an abelian variety and let C/k be a smooth projective curve of function field $K = k(C)$. Let $\mathcal{A} = A \times C$ and let $\mathcal{D} \subset \mathcal{A}$ be an effective divisor such that $\mathcal{D}_K$ is ample. The following result says that the set of $(S, \mathcal{D})$-integral points on $\mathcal{A}$ is empty for a general choice of the finite subset $S \subset C$. Moreover, there are much less integral points as the size $s = \#S$ of S goes to infinity. More precisely, the result says that for each $s \in \mathbb{N}$, the space of subsets $S \subset C$ of cardinality s such that there exists $(S, \mathcal{D})$-integral points lies in a closed subspace of dimension at most $\dim A$ in the s-dimensional parameter space $C^{(s)}$.

Corollary 9.15. *Assume $\mathcal{D}_K(K) = \varnothing$. For every $s \in \mathbb{N}$, there exists a closed algebraic subset $V_s \subset C^{(s)}$ such that the following hold:*

(i) $\dim V_s \leq \dim A$;

(ii) *for $S \subset C$ a finite subset of cardinality s and $U = C \setminus S$, if the projection $(\mathcal{A} \setminus \mathcal{D})|_U \to U$ admits an algebraic section then the image of S in $C^{(s)}$ belongs to V_s.*

Proof of Corollary 9.15. Fix $s \geq 1$. By Theorem 9.13, the projection $\mathcal{A} \to C$ admits only a finite number, modulo $A(\mathbb{C})$, of sections $\sigma \in \mathcal{A}(C)$ which are also sections of $(\mathcal{A} \setminus \mathcal{D})|_U \to U$ for $U = C \setminus S$ where $S \in C$ is some finite subset of cardinality s. Let I_s be the finite subset of such sections σ with $\deg \sigma^*\mathcal{D} = s$. Since $\mathcal{D}(K) = \varnothing$, Lemma 4.4 implies that we have a well-defined morphism $f_\sigma \colon A \to C^{(s)}$, $a \mapsto (a + \sigma)^*\mathcal{D}$. Here, $a + \sigma$ denotes simply the composition of

σ by the translation-by-a map. Define $V_s = \cup_{\sigma \in I_s} f_\sigma(A)$. By the definition of the morphisms f_σ, the image in $C^{(s)}$ of every finite subset $S \subset C$ verifying the condition (ii) must be contained in V_s. On the other hand, since the morphisms f_σ's are proper, V_s is a closed algebraic subset of $C^{(s)}$. Clearly, $\dim V_s \leq \dim A$ and the conclusion follows. $\square$

CHAPTER 10

Uniform results using the tautological inequality

1. Introduction

We fix throughout an integral quasi-projective variety Z/k over an algebraically closed field k of characteristic 0.

Definition 10.1. A family of k-elliptic surfaces $\mathcal{X} \to \mathcal{C} \to Z$ is called *a relative maximal variation family* if

(a) $\mathcal{C} \to Z$ is a family of smooth projective curves;

(b) $\mathcal{X}_z \to \mathcal{C}_z$ is an elliptic surface for every $z \in Z$;

(c) for every $z \in Z$ (not necessarily closed) and for ξ the generic point of $\mathcal{C}_z$, the Kodaira-Spencer class (cf. Definition 2.3) of the curve $X_\xi/\kappa(\xi)$ is nonzero.

Remark 10.2. Let $z \in Z$. Let g be the genus of $\mathcal{C}_z$ and let χ be the Euler-Poincaré characteristic $\chi(\mathcal{O}_{\mathcal{X}_z})$ of $\mathcal{X}_z$. Since Z is integral, g, χ are independent of $z \in Z$.

In particular, Definition 10.1 implies that every constant family of elliptic surfaces $X \times Z \to B \times Z \to Z$ is of relative maximal variation whenever the elliptic surface $X \to B$ is nonisotrivial (cf. Theorem 2.4).

Let $\chi, g \in \mathbb{N}$. The existence of relative maximal variation families of (resp. semistable) elliptic surfaces with section which parametrize all elliptic surfaces $X \to C$ such that $\chi(\mathcal{O}_X) = \chi$ and $g(C) = g$ is proved in [**97**, Theorem 7].

Definition 10.3. Consider a family of elliptic surfaces $\mathcal{X} \xrightarrow{f} \mathcal{C} \to Z$ with a section. Let $T \subset \mathcal{X}$ be the reduced divisor of singular fibres of $f\colon \mathcal{X} \to \mathcal{C}$. An integral divisor $\mathcal{D} \subset \mathcal{X}$ is called an *adaptive family of ample divisors* if $\mathcal{D} + T$ is simple normal crossing and for every $z \in Z$:

(a) $\mathcal{D}_z \coloneqq \mathcal{D} \cap \mathcal{X}_z$ is an effective ample divisor on $\mathcal{X}_z$;

(b) $\mathcal{D}_z + T_z$ is a simple normal crossing divisor.

Without searching for the optimal result, we restrict to families of semistable elliptic surfaces. The goal of the chapter is to prove the following uniform boundedness results on the canonical heights of rational points by means of the tautological inequality (cf. Section 3.3).

Theorem K. *Let $\mathcal{X} \xrightarrow{f} \mathcal{C} \to Z$ be a relative maximal variation family of semistable elliptic surfaces with a zero section $O\colon \mathcal{C} \to \mathcal{X}$. Let $\mathcal{D} \subset \mathcal{X}$ be an adaptive family of ample effective divisors. There exists numbers $c_1, c_2 > 0$ such that for every closed point $z \in Z$ and every $P \in \mathcal{X}_z(k(\mathcal{C}_z)) \setminus \mathcal{D}_z$, we have:*

$$\widehat{h}_{O_z}(P) \leq c_1 s + c_2, \qquad \text{where } s = \#\sigma_P(\mathcal{C}_z) \cap \mathcal{D}_z. \tag{1.1}$$

Here, $\widehat{h}_{O_z}$ is the Néron-Tate height on $\mathcal{X}_z(k(\mathcal{C}_z))$ associated to the origin O_z and $\sigma_P \in \mathcal{X}_z(\mathcal{C}_z)$ is the corresponding section of P.

In fact, we can show with the method of tautological inequality that:

Theorem 10.4. *If $\mathcal{D} = (O)$ is the zero section, the conclusion of Theorem K still holds. Moreover, the constants c_1, c_2 depend only g, χ (cf. Remark 10.2).*

Therefore, we obtain a new proof of the uniform consequence of Hindry-Silverman result [**52**, Corollary 8.5] (see Theorem 1.36 for the statement).

The uniform bound of height of integral points in Theorem K allows us to establish the following uniform bound which is polynomial in $s \in \mathbb{N}$ on the number of $(S, \mathcal{D})$-integral points of *bounded denominators* $\#S \leq s$.

Corollary 10.5. *Let the notations be as in Theorem K.*

(i) *There exists $\alpha, \beta, \gamma > 0$ depending only on $\mathcal{X}, \mathcal{C}, \mathcal{D}$ such that for every integer $s \geq 0$ and for every $z \in Z$, we have:*

$$\# \cup_{S \subset \mathcal{C}_z, \#S \leq s} \{(S, \mathcal{D}_z)\text{-integral points of } \mathcal{X}_z\} \leq (\alpha s + \beta)^\gamma; \tag{1.2}$$

(ii) *For each $z \in Z$ and each $n \in \mathbb{N}$, there exists a Zariski dense open subset $U_n \subset \mathcal{C}_z^n$ such that for every $x = (x_1, \dots, x_n) \in U_n$, the set of $(S_x, \mathcal{D}_z)$-integral points of the elliptic surface $\mathcal{X}_z \to \mathcal{C}_z$ is empty where $S_x = \{x_1, \dots, x_n\}$.*

Proof. (i) is a direct consequence of the uniform bound 3.12, Lemma 2.25 and Lemma 10.7 below. For (ii), see the proof of Corollary B.(ii). □

When the family of elliptic surfaces $\mathcal{X}$ is induced by a single nonisotrivial elliptic surface, Corollary 10.5.(i) can be strengthen by taking the union of integral points over arbitrary divisors $\mathcal{D}_z$ ($z \in Z$) as follows:

Corollary 10.6. *Let $X \to B$ be a nonisotrivial elliptic surface. Let Z be a variety and let $\mathcal{D} \subset X$ be an adaptive family of ample effective divisors with respect to the family $X \times Z \to B \times Z \to Z$. There exists $\alpha, \beta, \gamma > 0$ such that for every $s \in \mathbb{N}$,*

$$\# \cup_{z \in Z} \cup_{S \subset B, \#S \leq s} \{(S, \mathcal{D}_z)\text{-integral points of } X\} \leq (\alpha s + \beta)^\gamma. \tag{1.3}$$

PROOF. Up to a finite base change $B' \to B$, $X \to B$ becomes a nonisotrivial semistable elliptic surface. Thus, (i) is a direct consequence of the uniform bound 3.12, Lemma 2.25 and Lemma 10.7 below. □

2. Uniform boundedness of the Mordell-Weil rank in families

The following lemma is well-known to experts.

Lemma 10.7. *Let $f\colon X \to B$ be a nonisotrivial elliptic surface over a smooth projective curve B. Then we have*

$$r = \operatorname{rank} X(B) \le \rho(X) \le 12\chi(\mathcal{O}_X) + 4g(B) - 2,$$

where $\rho(X)$ is the Picard number of X, $g(B)$ is the genus of B and $\chi(\mathcal{O}_X)$ is the Euler-Poincaré characteristic of X.

PROOF. From the Shioda-Tate formula (cf. [**102**, Theorem 1.3, Corollary 5.3] or [**96**]), we find that $r = \operatorname{rank} X(B) \le \rho(X)$. For the second inequality, let $e(X)$ denote the topological Euler characteristic of X, then

$$b_2(X) = e(X) - 2 + 2b_1(X) = e(X) - 2 + 4g(B). \tag{2.1}$$

The first equality follows from the Poincaré duality. The second equality is a consequence of the following equalities:

$$b_1(X) = b_1(B) = 2g(B).$$

The first equality uses the properties $f_*\mathcal{O}_X = \mathcal{O}_B$, $R^1f_*\mathcal{O}_X = \mathbb{L}^{-1}$ with $\mathbb{L}$ the fundamental line bundle of X satisfying $\deg \mathbb{L} = \chi(\mathcal{O}_X) > 0$ (since X is nonisotrivial, cf. [**76**]), and the Leray Spectral sequence:

$$0 \to H^1(B, \mathcal{O}_B) \to H^1(X, \mathcal{O}_X) \to H^0(B, \mathbb{L}^{-1}) = 0.$$

The Dolbeault isomorphism says that $H^1(X, \mathcal{O}_X) \simeq H^0(X, \Omega^1_X)$ and $H^1(B, \mathcal{O}_B) \simeq H^0(B, \Omega^1_B)$. Hence, by Hodge decomposition, we see finally that

$$b_1(X) = h^{1,0} + h^{0,1} = 2h^{0,1} = 2h^1(X, \mathcal{O}_X)$$
$$b_1(B) = h^{1,0} + h^{0,1} = 2h^{0,1} = 2h^1(B, \mathcal{O}_B)$$

and thus $b_1(X) = b_1(B)$. From the injectivity of the cycle class map $\mathrm{NS}(X) \to H^2(X, \mathbb{Q})$, we deduce that $\rho(X) \le b_2(X)$.

On the other hand, we find from Noether's formula $\chi(\mathcal{O}_X) = (K_X^2 + e(X))/12$ that $12\chi(\mathcal{O}_X) = e(X)$ since $K_X^2 = 0$. Therefore, we deduce from (2.1) that:

$$\rho(X) \le b_2(X) = 12\chi(\mathcal{O}_X) - 2 + 4g(B).$$

□

3. Proof of the main results

We begin by stating here a useful criterion of ampleness on ruled surfaces.

Lemma 10.8. *Let V be a vector bundle of rank 2 on a smooth projective curve X over a field L of characteristic 0. Assume that V satisfies a non splitting short exact sequence:*

$$0 \to \mathcal{O}_X \to V \to \mathcal{O}(D) \to 0 \tag{3.1}$$

where D is an effective divisor on X such that $\deg D > 0$. Then the tautological bundle $\mathcal{O}(1)$ on the ruled surface $\mathbb{P}_X(V)$ is ample.

Proof. The class of the exact sequence (3.1) in $H^1(X, \mathcal{O}(D))$ is nonzero if and only if its class in $H^1(X_{\bar{L}}, \mathcal{O}(D_{\bar{L}}))$ is non zero. The Cohomological criterion for ampleness as in [**46**, Proposition III.2.6.1] and the invariant of cohomology under flat base change allows us to suppose that L is algebraically closed. We can now apply the Nakai-Moishezon criterion to prove that $\mathcal{O}(1)$ is ample. The proof goes as in [**40**, Lemma 6.27]. □

Return now to the notations of Theorem K.

3.1. Preliminary reductions. Since k is a field of characteristic zero, Z admits an integral resolution of singularities Z^{sm} (cf. [**54**]). Up to making a base change of $\mathcal{X} \to \mathcal{C} \to Z$ and of O to Z^{sm}, we can assume without loss of generality that Z is smooth. It follows that $\mathcal{C}$ and thus $\mathcal{X}$ are smooth varieties. In particular, if we denote $(O) \subset \mathcal{X}$ the image of the section O equipped with the reduced scheme structure then (O) is a smooth divisor of $\mathcal{X}$. We can clearly assume moreover that $\mathcal{X}$ and $\mathcal{C}$ are integral.

Let $F \subset \mathcal{C}$ be the effective reduced divisor of singular locus of the morphism $f\colon \mathcal{X} \to \mathcal{C}$. Then $T = f^*F$ is the divisor of singular fibres of $f\colon \mathcal{X} \to \mathcal{C}$.

Consider the embedded resolution of singularities $\mu\colon \mathcal{X}' \to \mathcal{X}$ of the effective Cartier divisor $(O) + T$ in $\mathcal{X}$ (cf. [**54**], see also [**64**, Theorem 4.1.3]). We remark here that μ can be obtained as a finite sequence of blowups along smooth centers supported in the singular loci of $(O) + T$ (thus contained in T).

Up to replacing $\mathcal{X}$ by $\mathcal{X}'$ and $(O) + T$ by (the support of) $\mu^*((O) + T) + E$ where E is the exceptional divisor of μ, we can suppose that the divisor

$$D := (O) + T$$

is simple normal crossing. As the family $\mathcal{X} \to \mathcal{C}$ is semistable and admits a section by hypotheses, the fibres D_z are also simple normal crossing for all $z \in Z$.

Therefore, the logarithmic cotangent bundles $\Omega_{\mathcal{X}}(\log D)$ and $\Omega_{\mathcal{X}_z}(\log D_z)$ are well-defined for every $z \in Z$ (cf. Definition 2.41).

For the notations, we denote for every $z \in Z$,

$$\mathcal{X}(D) \coloneqq \mathbb{P}_{\mathcal{X}}(\Omega_{\mathcal{X}/Z}(\log D)) \xrightarrow{\pi} \mathcal{X}, \qquad \mathcal{X}_z(D_z) \coloneqq \mathbb{P}(\Omega_{\mathcal{X}_z}(\log D_z)) \xrightarrow{\pi_z} \mathcal{X}_z. \tag{3.2}$$

Here, $\Omega_{\mathcal{X}/Z}(\log D)$ denotes the relative logarithmic cotangent bundle which by definition fits in the following short exact sequence:

$$0 \longrightarrow f^*\Omega_Z \longrightarrow \Omega_{\mathcal{X}}(\log D) \longrightarrow \Omega_{\mathcal{X}/Z}(\log D) \longrightarrow 0. \tag{3.3}$$

3.2. Main induction step. Let $\eta \in Z$ be the generic point of Z. Consider the universal elliptic surface $\mathcal{X}_\eta \to \mathcal{C}_\eta$ over the universal curve $\mathcal{C}_\eta$. Let $\mathbb{L} = (R^1 f_{\eta *} O_{\mathcal{X}_\eta})^{-1}$ be the fundamental line bundle of $\mathcal{X}_\eta \to \mathcal{C}_\eta$. Then $\deg(\mathbb{L}) = \chi(\mathcal{O}_X) > 0$ since $\mathcal{X}_\eta \to \mathcal{C}_\eta$ is nonisotrivial and $\omega_{\mathcal{X}_\eta} = f_\eta^*(\mathbb{L} \otimes \omega_{\mathcal{C}_\eta})$ (cf. **[76]**).

Lemma 10.9. *We have an exact sequence of vector bundles:*

$$0 \longrightarrow f_\eta^*\Omega_{\mathcal{C}_\eta/\kappa(\eta)}(F_\eta) \longrightarrow \Omega_{\mathcal{X}_\eta/\kappa(\eta)}(\log D_\eta) \longrightarrow \mathcal{O}_{\mathcal{X}_\eta}((O)_\eta) \otimes f_\eta^*(\mathbb{L}) \longrightarrow 0. \tag{3.4}$$

Proof. Let W be the quotient of $\Omega_{\mathcal{X}_\eta/\kappa(\eta)}(\log D_\eta)$ by $f_\eta^*\Omega_{\mathcal{C}_\eta/\kappa(\eta)}(F_\eta)$. The local freeness of W follows by a local calculation at points P lying on the divisor T of singular fibres. Moreover, the exact sequence (3.4) implies that:

$$\begin{aligned} W &= \det(\Omega_{\mathcal{X}_\eta/\kappa(\eta)}(\log D_\eta)) \otimes \left(f_\eta^*\Omega_{\mathcal{C}_\eta/\kappa(\eta)}(F_\eta)\right)^{-1} \\ &= f_\eta^*(\mathbb{L} \otimes \omega_{\mathcal{C}_\eta}) \otimes \mathcal{O}(D_\eta) \otimes f_\eta^*(\omega_{\mathcal{C}_\eta}^{\otimes(-1)}) \otimes f_\eta^*\mathcal{O}(-F_\eta) \\ &= \mathcal{O}_{\mathcal{X}_\eta}((O)_\eta) \otimes f_\eta^*(\mathbb{L}). \end{aligned}$$

□

Let $\xi \in \mathcal{C}_\eta$ be the generic point of $\mathcal{C}_\eta$ and $\kappa(\xi) = \kappa(\eta)(\mathcal{C}_\eta)$ the function field of $\mathcal{C}_\eta$. Since $\mathcal{X} \xrightarrow{f} \mathcal{C} \to Z$ is of relative maximal variation, the Kodaira-Spencer class of $\mathcal{X}_\eta/\kappa(\eta)$ is nonzero. In other words, the following exact sequence of vector bundles is non splitting:

$$0 \longrightarrow f_\xi^*\Omega_{\kappa(\xi)/\kappa(\eta)} \longrightarrow (\Omega_{\mathcal{X}_\eta/\kappa(\eta)}(\log D_\eta))_\xi \longrightarrow \Omega_{\mathcal{X}_\xi/\kappa(\xi)}(\log D_\xi) \longrightarrow 0. \tag{3.5}$$

As $\kappa(\xi)/\kappa(\eta)$ is a separable 1-dimensional transcendental field extension, $f_\xi^*\Omega_{\kappa(\xi)/\kappa(\eta)} \simeq \mathcal{O}_{\mathcal{X}_\xi}$. Since $\mathcal{X}_\xi/\kappa(\xi)$ is an elliptic curve, $\Omega_{\mathcal{X}_\xi/\kappa(\xi)}(\log D_\xi) = \mathcal{O}_{\mathcal{X}_\xi}(D_\xi)$. Hence, it follows from Lemma 10.8 that the tautological line bundle $\mathcal{O}_\xi(1)$ is ample on the elliptic ruled surface $\mathbb{P}(\mathcal{E}_\xi) \to \mathcal{X}_\xi$ where we define

$$\mathcal{E} \coloneqq \Omega_{\mathcal{X}_\eta/\kappa(\eta)}(\log D_\eta).$$

Lemma 10.10. *There exists an integer $N \geq 1$ and an effective divisor $\mathcal{V}_\eta$ of $\mathcal{C}_\eta$ such that the following line bundle on $\mathbb{P}(\mathcal{E}) = \mathcal{X}_\eta(D_\eta)$ is globally generated:*

$$\mathcal{L}_\eta := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(N) \otimes \pi_\eta^* \mathcal{O}(-D_\eta + f_\eta^* \mathcal{V}_\eta).$$

PROOF. As $\mathcal{O}_\xi(1)$ is ample, the line bundle $\mathcal{O}_\xi(N_1)$ is very ample on $\mathbb{P}(\mathcal{E}_\xi)$ for some integer $N_1 \geq 1$. Take any basis $s_1, \dots, s_k$ of the linear system $|\mathcal{O}_\xi(N_1)|$ on $\mathbb{P}(\mathcal{E}_\xi)$. As $\mathcal{C}_\eta/\kappa(\eta)$ is a curve, there exists an effective divisor $\mathcal{V}_1$ on $\mathcal{C}_\eta$ such that $s_1, \dots, s_k$ extend to global sections of $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(N_1) \otimes \pi_\eta^* \mathcal{O}(f_\eta^* \mathcal{V}_1))$.

Since $\mathcal{O}_\xi(N_1)$ is very ample on the generic fibre $\mathbb{P}(\mathcal{E}_\xi)$ of $\mathbb{P}(\mathcal{E})$, the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(N_1) \otimes \pi_\eta^* \mathcal{O}(\mathcal{V}_1)$ is a big line bundle on $\mathbb{P}(\mathcal{E})$ by the very definition of bigness (its the augmented base locus cannot not be all of $\mathcal{X}_\eta(D_\eta)$, cf. [**7**]).

Since $\deg(\mathbb{L}) = \chi(\mathcal{O}_{\mathcal{X}_\eta}) = -(O)_\eta^2 > 0$, the line bundle $\mathcal{O}_{\mathcal{X}_\eta}((O)_\eta) \otimes f_\eta^*(\mathbb{L})$ is nef on $\mathcal{X}_\eta$. On the other hand, $\deg(\Omega_{\mathcal{C}_\eta/\kappa(\eta)}(F_\eta)) \geq 0$. Indeed, it is enough to consider the case $g = 0$. But in this case, the Shioda-Tate formula [**96**] implies that $2a + m \geq 2\chi(\mathcal{X}_\eta) + 2$ where a, m are respectively the number of additive and multiplicative singular fibres of $\mathcal{X}_\eta \to \mathcal{C}_\eta$. Since $\mathcal{X}_\eta$ is nonisotrivial, $\chi(\mathcal{O}_{\mathcal{X}_\eta}) > 0$ and it follows that $\deg F = a + m \geq 2$. Hence $f_\eta^* \Omega_{\mathcal{C}_\eta/\kappa(\eta)}(F_\eta)$ is nef on $\mathcal{X}_\eta$.

Therefore, $\Omega_{\mathcal{X}_\eta/\kappa(\eta)}(\log D_\eta)$ is an extension of nef line bundles (cf. (3.4)) thus it is a nef vector bundle on $\mathcal{X}_\eta$. Thus, the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(N_1) \otimes \pi_\eta^* \mathcal{O}(\mathcal{V}_1)$ is actually nef and big.

Since the pullback of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(N_1) \otimes \pi_\eta^* \mathcal{O}(\mathcal{V}_1)$ to the generic fibre over $\mathcal{C}_\eta$ is $\mathcal{O}_\xi(N_1)$ which is very ample, Nakamaye's theorem [**80**] (which is valid over any field, cf. [**7**]) implies immediately that the augmented base locus $B_+(\mathcal{O}_{\mathcal{X}_\eta(D_\eta)}(N_1) \otimes \pi_\eta^* \mathcal{O}(\mathcal{V}_1)) \subset \mathbb{P}(\mathcal{E})$ is vertical over $\mathcal{C}_\eta$, i.e., it does not dominate $\mathcal{C}_\eta$. By the definition of the augmented base locus (cf. [**31**], [**7**]), there exists $N \geq 1$ and an effective vertical divisor $\mathcal{V}_\eta$ such that $\mathcal{L}_\eta$ is globally generated as desired. □

Let $\mathcal{V}$ be any effective divisor on $\mathcal{X}$ extending $\mathcal{V}_\eta$. Since $\mathcal{V}_\eta + T_\eta$ is vertical with respect to the projection $f_\eta \colon \mathcal{X}_\eta \to \mathcal{C}_\eta$, The pushforward $(f_\eta)_*(\mathcal{V}_\eta + T_\eta)$ is an effective divisor on $\mathcal{C}_\eta$ and we can set $M = \deg(f_\eta)_*(\mathcal{V}_\eta + T_\eta) \in \mathbb{N}$. It follows that for every z in some Zariski dense open subset $U_0 \subset Z$, we have

$$\deg(f_z)_*(\mathcal{V}_z + T_z) = M. \tag{3.6}$$

Since $D \subset \mathcal{X}$ and $D_z \subset \mathcal{X}_z$ are simple normal crossing, the formation of the relative logarithmic differential sheaves $\Omega_{\mathcal{X}/Z}(\log D)$ commutes with the localizations to fibres above points $z \in Z$. This means that for every point $z \in Z$,

$(\Omega_{\mathcal{X}/Z}(\log D))_z = \Omega_{\mathcal{X}_z}(\log D_z)$. Moreover, we have a commutative cartesian diagram

$$\begin{tikzcd} \mathcal{X}_z(D_z) \arrow[r, "\pi_z"] \arrow[d, hook] & \mathcal{X}_z \arrow[r, "f_z"] \arrow[d, hook] & \mathcal{C}_z \arrow[r, "h_z"] \arrow[d, hook] & z \arrow[d, hook] \\ \mathcal{X}(D) \arrow[r, "\pi"] & \mathcal{X} \arrow[r, "f"] & \mathcal{C} \arrow[r, "h"] & Z. \end{tikzcd} \tag{3.7}$$

Let $\mathcal{O}(1)$ be the tautological line bundle on $\mathcal{X}(D)$ and consider the following line bundle:

$$\mathcal{L} := \mathcal{O}(N) \otimes \pi^*\mathcal{O}(-D + \mathcal{V}).$$

By Lemma 10.10, $\mathcal{L}_\eta$ is globally generated where η is the generic point of Z. Denote $\varphi := h \circ f \circ \pi \colon \mathcal{X}(D) \to Z$ and $\varphi_z = h_z \circ f_z \circ \pi_z \colon \mathcal{X}_z(D_z) \to z$ the induced structure morphism of $\mathcal{X}_z(D_z)$ for every $z \in Z$. By the local constructibility of the set of $z \in Z$ satisfying the surjectivity of the map

$$(\varphi^*\varphi_*\mathcal{L})_z = (\varphi_z)^*(\varphi_z)_*\mathcal{L}_z \to \mathcal{L}_z, \tag{3.8}$$

there exists a nonempty Zariski open subset $U \subset U_0 \subset Z$ such that the map (3.8) is surjective, i.e., $\mathcal{L}_z$ is globally generated, for every $z \in U$.

Therefore, for every section $\sigma_P \in \mathcal{X}_z(\mathcal{C}_z)$ corresponding to a rational point $P \in \mathcal{X}_z(k(\mathcal{C}_z))$ for some $z \in U$, we have $\deg_B(\sigma'_P)^*\mathcal{L}_z \geq 0$, where $\sigma'_P \colon B \to \mathcal{X}_z(D_z)$ is the derivative map which lifts σ_P (cf. Defintion 2.42).

Since $\sigma_P = \pi_z \circ \sigma'_P$ and $\mathcal{L}_z = \mathcal{O}_{\mathcal{X}_z(D_z)}(N) \otimes \pi_z^*\mathcal{O}(-D_z + \mathcal{V}_z)$, it follows immediately for every $z \in U$ that:

$$D_z \cdot \sigma_P(B) \leq \deg(\sigma'_P)^*\mathcal{O}_{\mathcal{X}_z(D_z)}(N) + \mathcal{V}_z \cdot \sigma_P(B). \tag{3.9}$$

Remark that $\sigma_P(B) \cdot F \leq 1$ for every integral fibre F of $\mathcal{X}_z \to \mathcal{C}_z$. Combining with the tautological inequality applied to P (cf. Definition 2.42), we deduce that

$$D_z \cdot \sigma_P(B) \leq N(2g - 2 + \#D_z \cap \sigma_P(B)) + \#\mathcal{V}_z \cap \sigma_P(B). \tag{3.10}$$

Hence, by the definition of M (cf. (3.6)) and of U, we find for every $z \in U$ that:

$$\begin{aligned} (O)_z \cdot \sigma_P(B) &\leq N(2g - 2 + \#(O)_z \cap \sigma_P(B)) + \#\mathcal{V}_z \cap \sigma_P(B) + (N-1)\#T_z \cap \sigma_P(B). \\ &\leq N(2g - 2 + \#(O)_z \cap \sigma_P(B)) + NM. \end{aligned}$$

On the other hand, we have $|(O)_z \cdot \sigma_P(B) - \widehat{h}_{O_z}| \leq -(O)_z^2 = \chi$ by Lemma 2.22. Thus, we have proven the following main induction step:

Lemma 10.11. *There exists a Zariski dense open subset $U \subset Z$ and $M, N \in \mathbb{N}$ such that for every rational point $P \in \mathcal{X}_z(k(\mathcal{C}_z))$ with $z \in U$, we have:*

$$\widehat{h}_{O_z}(P) \leq N(2g - 2 + \#(O)_z \cap \sigma_P(B)) + NM + \chi. \tag{3.11}$$

3.3. Proof of Theorem 10.4. Now we apply the above procedure (3.11) for each integral component of $Z \setminus U$ and so on. Since $\dim Z \setminus U < \dim Z$, the whole process has only finitely steps. It is then clear that there exists $c_1, c_2 > 0$ such that for every $z \in Z$ and every section $\sigma_P \in \mathcal{X}_z(\mathcal{C}_z)$ associated to a rational point $P \in \mathcal{X}_z(k(\mathcal{C}_z))$, we have:

$$\widehat{h}_{O_z}(P) \leq c_1 s + c_2, \qquad \text{where } s = \#\sigma_P(B) \cap (O)_z. \tag{3.12}$$

By a result of Seiler ([**97**, Theorem 7]), the family $\mathcal{X} \to \mathcal{C} \to Z$ can be taken to be a parameter space of all semistable elliptic surfaces with section of parameters χ, g (cf. Remark 10.2). The constants c_1, c_2 thus depends only on χ, g and the proof of Theorem K is completed.

3.4. Proof of Theorem K. By hypotheses, $\mathcal{D} \subset \mathcal{X}$ is an adaptive family of ample effective divisor. Thus, we can apply, *mutatis mutandis*, the above *Main induction step*, by replacing D by $\mathcal{D} + T$, to obtain a Zariski dense open subset $U \subset Z$ and $M, N \in \mathbb{N}$ such that for every rational point $P \in \mathcal{X}_z(k(\mathcal{C}_z))$ with $z \in U$, we have (cf. (3.10)):

$$\mathcal{D}_z \cdot \sigma_P(B) \leq N(2g - 2 + \#\mathcal{D}_z \cap \sigma_P(B)) + M. \tag{3.13}$$

To estimate the canonical heights, we need the following auxiliary result.

Lemma 10.12. *There exists $A \in \mathbb{N}$ and a Zariski dense open subset $W \subset Z$ such that for every $z \in W$, the linear system $|A\mathcal{D}_z - (O)_z|$ is base-point-free where $(O)_z$ is the zero section of the elliptic surface $f_z \colon \mathcal{X}_z \to \mathcal{C}_z$.*

Proof. Consider the divisors $\mathcal{D}_\eta$ and $(O)_\eta$ on $\mathcal{X}_\eta$ where η is the generic point of Z. Since $\mathcal{D}$ is a family of ample divisors by hypothesis, $\mathcal{D}_\eta$ is ample on $\mathcal{X}_\eta$. Define $\mathcal{L}(A) \coloneqq \mathcal{O}(A\mathcal{D} - (O))$ then there exists $A \in \mathbb{N}$ such that the line bundle

$$\mathcal{L}(A)_\eta = \mathcal{O}(A\mathcal{D}_\eta - (O)_\eta)$$

is globally generated on $\mathcal{X}_\eta$. Let $\phi \coloneqq h \circ f \colon \mathcal{X} \to Z$ (cf. (3.7)). It follows that the canonical morphism

$$(\phi^* \phi_* \mathcal{L})_\eta = (\phi_\eta)^* (\phi_\eta)_* \mathcal{L}(A)_\eta \to \mathcal{L}(A)_\eta$$

is surjective. By the local constructibility property over the base of the surjectivity of morphism of coherent sheaves, there exists a Zariski dense open subset $W \subset Z$

such that for every $z \in W$, the following natural map is surjective:

$$(\phi_z)^*(\phi_z)_*\mathcal{L}(A)_z \to \mathcal{L}(A)_z.$$

It follows that the linear system $|A\mathcal{D}_z - (O)_z|$ is base-point-free for every $z \in W$. $\square$

Denote $U' = W \cap U$. Then U' is a Zariski dense open subset of Z. For every $z \in U'$, we find that:

$$\begin{aligned}
\widehat{h}_{O_z}(P) &\leq (O)_z \cdot \sigma_P(B) + \chi && \text{(by Lemma 2.22)}\\
&\leq A\mathcal{D}_z \cdot \sigma_P(B) + \chi && \text{(by Lemma 10.12)}\\
&\leq AN(2g - 2 + \#\mathcal{D}_z \cap \sigma_P(B)) + AM + \chi. && \text{(by (3.13))}
\end{aligned}$$

We continue the above procedure for each integral component of $Z \setminus U'$ and so on. As $\dim Z \setminus U' < \dim Z$, the process terminates after a finite number of steps. Hence, there exists $c_1, c_2 > 0$ such that for every $z \in Z$ and every $P \in \mathcal{X}_z(k(\mathcal{C}_z))$,

$$\widehat{h}_{O_z}(P) \leq c_1 s + c_2, \qquad \text{where } s = \#\sigma_P(B) \cap \mathcal{D}_z.$$

The proof of Theorem K is thus completed.

CHAPTER 11

Parshin-Arakelov theorem and integral points

1. Introduction

It is well-known that in the case of function fields, the Parshin-Arakelov theorem (cf. Theorem 11.1) implies the Mordell conjecture (cf. [**86**]). Moreover, by establishing a uniform version of the Parshin-Arakelov theorem (cf. Theorem 11.2), Caporaso obtained a certain uniform version of the Mordell conjecture over function fields. We will explain this in more details.

Notations. In this chapter, we fix throughout an irreducible smooth projective complex curve B of genus g and let $S \subset B$ be a subset of cardinality $s \in \mathbb{N}$. For a nonisotrivial minimal surface $X \to B$, we call the *type* of X the finite subset of B above which the fibres of X are not smooth. We also say that X is *of type* S if it has good reduction outside of S. Let $F_q(B,S)$ be the set of non-isotrivial minimal surfaces over B of type contained in S and whose general fibres have genus q.

When we restrict to the case $q \geq 2$, i.e., to families of hyperbolic curves, we have the following important result.

Theorem 11.1 (Parshin-Arakelov)**.** *The set $F_q(B,S)$ is finite for every $q \geq 2$.*

Proof. See [**86**] in the case $S = \varnothing$ and [**5**] for the general case. □

The above beautiful theorem was latter reinforced by Caporaso as follows.

Theorem 11.2 (Caporaso)**.** *There exists a function $C \colon \mathbb{N}^3 \to \mathbb{N}$ such that for all $q \geq 2$, we have*

$$\#F_q(B,S) \leq C(q,g,s).$$

Proof. See [**18**, Theorem 3.1]. □

Let $(X \to B) \in F_q(B,S)$ where $q \geq 2$. In [**86**], Parshin constructed a map α_X which associates to each section $\sigma \in X(B)$ a nonisotrivial families of curves of bounded genus $Y_\sigma \to B$ which factors through $X \to B$ and such that $\sigma(B)$ is exactly the branch locus of the cover $Y_\sigma \to X$. The obtained nonisotrivial families of curves Y_σ are all defined over a finite number of curves B' and the possible types $S' \subset B'$ are also finite. The key property of the map α_X is that all its fibres are finite. This is direct implication of the property that σ is uniquely determined as the branch locus of the cover $Y_\sigma \to X$ and that the number of possible covers $Y_\sigma \to X$ is finite by the de Franchis theorem (over function fields) since $q \geq 2$. Therefore, the Parshin-Arakelov theorem implies that the set $X(B) = X_K(K)$ must be finite, which is the content of the Mordell conjecture over function fields. Together with some additional arguments proving B', S' can be taken in a uniformly bounded family and the cardinalities of the fibres of α_X are also uniformly bounded, Caporaso's uniform version of the Parshin-Arakelov theorem 11.2 implies the following uniform version of the Mordell conjecture.

Theorem 11.3 (Caporaso)**.** *There exists a function $M\colon \mathbb{N}^3 \to \mathbb{N}$ with the following property. For every $q \geq 2$ and $X \in F_q(B,S)$, the set $X_K(K) = X(B)$ contains at most $M(q,g,s)$ rational points.*

Proof. See [**18**, Theorem 4.2]. □

The first goal of this chapter is to construct a map analogous to Parshin's map α_X to obtain a new proof for known uniform finiteness results on integral points on a nonisotrivial elliptic surface (cf. Section 3). Let $f\colon X \to B$ be a nonisotrivial elliptic surface of type T of cardinality t. Let $\mathcal{D}$ be an effective reduced horizontal divisor in X and $S \subset B$ of cardinality $s \in \mathbb{N}$.

Theorem L. *The set of $(S,\mathcal{D})$-integral points is finite and uniformly bounded by a function depending only on g, s, t, $\deg \mathcal{D}_K$, the number of ramified points in the cover $\mathcal{D} \to B$, and the number of singular points on $\mathcal{D}$.*

When $\mathcal{D}$ is a section, $\deg \mathcal{D}_K = 1$ and $\#\mathcal{D}_{ram} \cup \mathcal{D}_{sing} = \varnothing$. We recover in particular the uniform result in [**52**] whose proof uses height theory:

Corollary 11.4. *Let (O) be the zero section of X. The set of $(S,\mathcal{D})$-integral points is uniformly bounded by a function depending only on g, s, t.*

The main idea of our construction is similar to the method of Parshin in the following way. For each nonisotrivial elliptic surface $X \to B$ with a finite subset $S \subset B$ and a horizontal effective divisor $\mathcal{D}$, we define a map $\beta_{X,\mathcal{D},S}$ which associates to each $(S,\mathcal{D})$-integral section $\sigma \in X(B)$ a cover $Y_\sigma \to X$ whose (horizontal part of the) branch locus is $\sigma(B) \cup \mathcal{D}$. The induced maps $Y_\sigma \to B$ are nonisotrivial families of curves of uniformly bounded genus ≥ 2 and of uniformly bounded type.

The new key point is that the extra presence of $\mathcal{D}$, and not just the section $\sigma(B)$, in the branch locus turns out to be exactly what we need, altogether with the de

Franchis theorem on elliptic subfields (cf. Corollary 2.35), to show that the fibres of the map $\beta_{X,\mathcal{D},S}$ are uniformly bounded. Thereby, we obtain from Theorem 11.3 a uniform result of Siegel theorem in the case of function fields with a new method other than the classical methods using heights (e.g., Corollary 1.37 or Corollary 10.5) or using the hyperbolic method as in Theorem H.

It is then natural to expect that the method can be applied to obtain the known finiteness result on integral points of bounded denominators on elliptic curves over function fields. For this to be done using the construction of the map $\beta_{X,\mathcal{D},S}$, it turns out that we need the property saying that the union $\cup_{S\subset B, \# S \leq s} F_q(B,S)$ is finite for every $s \in \mathbb{N}$. Unfortunately, the second goal of this chapter is to establish a very negative result about the finiteness of the union $\cup_{S\subset B, \# S \leq s} F_q(B,S)$ (cf. Section 4):

Theorem M. *For all large enough q and s depending only on the genus g of B, the union*

$$\cup_{S\subset B, \# S \leq s} F_q(B,S)$$

is uncountably infinite. Moreover, there exists $N(q,s,g) \in \mathbb{N}$, a Zariski dense open subset $I \subset \mathbb{P}^N$, and a map $I \to \cup_{S\subset B, \# S \leq s} F_q(B,S)$ with uniformly bounded finite fibres.

The above negative result shows that we cannot extend our method, at least in a straightforward manner, to recover the conclusions of Corollary 1.37, Corollary 10.5) or Theorem H for integral points of bounded denominators.

Reformulated in a certain 'union of $(S,\mathcal{D})$-integral section" style, we obtain the following geometric information on the compact fine moduli spaces $\mathcal{M}_{q,n}$ of stable curves of genus q with level $n \geq 3$-structure.

Corollary. *Let $\Delta \subset \mathcal{M}_{q,n}$ be the divisor locus of singular curves. For large enough $q, s \in \mathbb{N}$, there exists uncountably many nonconstant morphisms $h \colon B \to \mathcal{M}_{q,n}$, up to automorphisms of B, such that the set-theoretic intersection $h(B) \cap \Delta$ has no more than s points.*

Finally, we shall apply our strategy to prove a certain uniform finiteness result on unit equations over function fields in Section 5 (cf. Theorem 11.11). The result obtained is nontrivial but far from being optimal (cf. Remark 11.13). But again, as in the case of integral points on nonisotrival elliptic surfaces, our proof is new in the sense that it does not reduce to establish any height bound on the set of solutions as in traditional approaches in the literature.

2. Preliminaries

Let X be a connected scheme. We fix a geometric point $\bar{x}$ to obtain an étale fundamental group $\pi_1^{et}(X) = \pi_1^{et}(X, \bar{x})$. An étale degre d cover of X is equivalent to a $\pi_1^{et}(X)$-set of cardinality d, i.e., a continuous morphism of topological groups

$$\pi_1^{et}(X) \to S_d,$$

where S_d denotes the discret group of permutations of a set of d elements. Hence, the number of degree d étale covers of X is given by $\#\mathrm{Hom}(\pi_1^{et}(X), S_d)$. If $\pi_1^{et}(X)$ is finitely generated with m generators then $\#\mathrm{Hom}(\pi_1^{et}(X), S_d) \leq (d!)^m$. Moreover, for a regular connected variety X over $\mathbb{C}$, we have $\mathrm{Hom}(\pi_1^{et}(X), S_d) = \mathrm{Hom}(\pi_1(X(\mathbb{C})), S_d)$. If $X = B\backslash T$, where T is a finite subset of cardinality t, then the topological fundamental group $\pi_1(X_{\mathbb{C}}(\mathbb{C}))$ is a free group F_{2g+t-1} of rank $2g-1+t$. Therefore, we see that

$$\pi_1^{et}(X) = \hat{F}_{2g-1+t}$$

is the completion of the free group of $2g-1+t$ generators. In this case, the number of degree d étale covers of $B\backslash T$ is bounded above by

$$N(d, g, t) = (d!)^{2g+t-1}.$$

Now return to the situation when $f\colon X \to B$ is a minimal elliptic surface with a zero section O and $T \subset B$ the type of X. Let $Y = X\backslash f^{-1}(T)$ then Y is an abelian scheme over $B \setminus T$. The multiplication-by-2 morphism [2] on Y is finite étale of degree 4. Moreover, its kernel $\mathrm{Ker}[2] \subset Y$, obtained by the base change associated to the zero section, is also finite étale of degree 4 over $B \setminus T$.

$$\begin{array}{ccc} \mathrm{Ker}[2] & \longrightarrow & B \setminus T \\ \big\downarrow & & \big\downarrow \sigma_O \\ Y & \xrightarrow{[2]} & Y. \end{array}$$

Similarly, by composing with the translation $\tau_P \in \mathrm{Aut}_B(X)$ for $P \in E(K)$, we see that $\mathrm{Ker}(\tau_P \circ [2])$ is also finite étale of degree 4 over $B\backslash T$. Every point $Q \in E(\bar{K})$ such that $[2]Q = P$ is an element of Ker[2]. Therefore, by composing all the finite étale covers of $B \setminus T$ of degree at most 4, we obtain a finite cover $B' \to B$ of degree at most $4^{N(4,g,t)}$ which is ramified only over T and that $2E(K') \supset E(K)$ where $K' = k(B')$. In particular, by the Riemann-Hurwitz formula, the genus g' of B' is bounded by an explicit function in g, t given by

$$g'(B) \leq 4^{N(4,g,t)}(g + t - 1) + 1.$$

It is clear that the same properties hold if we replace [2] by $[d]$ where $d \in \mathbb{N}$. We reformulate the above discussion as follows:

Lemma 11.5. *Let $d \in \mathbb{N}$ and let $f\colon X \to B$ be an elliptic surface with section of type $T \subset B$. Let E/K be the associated elliptic curve. Then there exists a finite cover $h\colon B' \to B$ of projective smooth curves ramified only over T such that $dE(K') \supset E(K)$ where $K' = \mathbb{C}(B')$. Moreover, h is of degree at most $(d^2)^{(d!)^{2g+t-1}}$ where $t = \#T$ and the genus g' of B' is bounded by:*

$$g' \leq (d^2)^{(d!)^{2g+t-1}}(g+t-1)+1.$$

Lemma 11.6. *Let $L \sim 0$ be a line bundle on the elliptic surface $f\colon X \to B$ with a section (O). Then for each vertical divisor V on X, the linear system $|(O)+V|$ consists of effective divisors of the form $(O)+F$ where $F \sim V$.*

Proof. Let $\mathcal{D} \in H^0(X,(O)+V)$ then $\mathcal{D} \sim (O)+V$. We write $\mathcal{D} = H+F$ where H is an effective horizontal divisor and F is vertical. Then the degree of H on the generic fibre is the same as (O) which is 1. This means that H is a section of $X \to B$ corresponding to a rational point $P \in X_K(K)$. Hence, the image of $H-(O)$ in $NS(X)/T(X)$ is zero. Let $T(X) \subset NS(X)$ be the subgroup generated by the zero section and all vertical divisors. We deduce by the Shioda-Tate isomorphism (cf. [**102**, Theorem 1.3, Corollary 5.3] or [**96**]) $E(K) \simeq NS(X)/T(X)$ that $H = (O)$. This implies in particular that $F \sim V$. We have $\mathcal{D} = (O)+F$ as claimed. $\square$

Lemma 11.7. *Let E/K be an elliptic curve over a field K. Let D be a divisor of degree $d \geq 1$ on E. Let $\iota = -\mathrm{Id}$ be the involution. Suppose that $j(E) \neq 0, 1728$ then we have:*

(i) $\mathrm{Isom}_K(E) = E(K) \rtimes \{\pm \mathrm{Id}\}$. *For all $P \in E(K)$, we have $\tau_P \circ \iota = \iota \circ \tau_{-P}$ where τ_P is the translation-by-P map.*

(ii) *There exists at most one point $R \in E(K)$ modulo the d-torsion group $E[d]$ such that all isomorphisms $u \in Isom_K(E)$ verifying $u(D) \sim D$ are of the form τ_P or $\iota \circ \tau_{R+P}$ where P is a rational d-torsion point of $E(K)$.*

Proof. For (i) see Theorem 2.7.(4) and remark that $\mathrm{Aut}(E) = \{\pm\tau\}$ since $j(E) \neq 0, 1728$. The second statement is easily checked. For (ii), let $u \in \mathrm{Isom}_K(E)$ be such that $u(D) \sim D$. From (i), we know that u is of the form $\pm\,\mathrm{Id} \circ \tau_P$ for some $P \in E(K)$. Consider the K-divisor $D - d[O]$ on E. We have $\deg(D - d[O]) = 0$. Hence the isomorphism of groups

$$E(K) \to \mathbf{Pic}^0_E(K), \quad Q \mapsto \mathcal{L}([Q]-[O])$$

implies that $D - d[O] \sim [Q]-[O]$ for some rational point $Q \in E(K)$. Thus, we find that $D \sim (d-1)[O]+[Q]$. Suppose first that $u = \tau_P$ for some $P \in E(K)$. Then $D \sim u(D) \sim (d-1)[P]+[P+Q] \sim (d-1)[P]+[P]+[Q]-[O]$. We deduce that $d[O] \sim d[P]$, i.e., $d([P]-[O]) \sim 0$. This means exactly that $[d]P = O$, i.e.,

P is a rational d-torsion point of E. Suppose now that $u = (-\operatorname{Id}) \circ \tau_P$ for some $P \in E(K)$. Similarly, we find that

$$u(D) \sim (d-1)[-P]+[-P-Q] \sim (d-1)(2[O]-[P])+(2[O]-[P])+(2[O]-[Q])-[O].$$

As $D \sim u(D)$, we deduce that $(d+2)[O] \sim d[P] + 2[Q]$. Hence $d([P]-[O]) \sim 2([Q]-[O])$ and thus $[d]P = [2]Q \in E(K)$. It suffices to let R a d-division point of $[2]Q$ to conclude. $\square$

3. Main result on uniform finiteness of integral points

We are now in position to construct the map $\beta_{X,\mathcal{D},S}$ defining on the set of $(S, \mathcal{D})$-integral points of a nonisotrvial elliptic surface $X \to B$ to prove the uniform result in Theorem L without establishing any height bound for integral points.

Proof of Theorem L. The symbol $\sim$ will be used to denote the linear equivalence and we usually denote by $(P) = \sigma_P(B) \subset X$ the section associated to a rational point $P \in X_K(K)$.

Let $D = \mathcal{D}_K$ be the pullback of $\mathcal{D}$ to the generic fibre. Let $d = \deg D$ and let $O \in E(K)$ be the zero element. We can clearly assume that $\mathcal{D}$ is integral thus contains no vertical components. By Lemma 11.5, every rational point in $E(K)$ has a $(d+1)$-th root in a base extension K'/K of degree at most 16^{2g+t-1} which is ramified only above T and with genus $g(K')$ bounded in terms of g, t. Therefore, up to making the corresponding base change K'/K, all $(S, \mathcal{D})$-integral points belong to $(d+1)E(K)$. Consider the zero-degree divisor $D - d[O]$. Since $E(K) \neq \emptyset$, we have

$$\boldsymbol{P}^0_E(K) \simeq E(K) \quad \text{(cf. [\textbf{73}, Chapter 1])}$$

where $\boldsymbol{P}^0_E(-)$ is the functor which associates to each K-scheme V the group of families of invertible sheaves on E of degree 0 parametrized by V, modulo the trivial family. In particular, $\boldsymbol{P}^0_E(K)$ is the group of equivalence classes of degree 0 invertible sheaves on E. We deduce that $D - d[O] \sim [Q] - [O]$ for some $Q \in E(K)$. Now let $P \in E(K)$ be an $(S, \mathcal{D})$-integral point. We can suppose $P \in (d+1)E(K)$ by the first paragraph. Denote by $R_P, R_Q \in E(K)$ certain $(d+1)$-th roots of P and Q respectively. In particular,

$$[P]+D \sim (-d[O]+(d+1)[R_P])+(d-1)[O]-d[O]+(d+1)[R_Q] \sim (d+1)([R_P]+[R_Q]-[O])$$

Since $\mathbb{C}(X) = K(E)$, we can extend linear equivalence relations on E to linear equivalence relations on X modulo vertical divisors. Thus, for some vertical divisor F on X, we find that

$$(P) + \mathcal{D} \sim (d+1)((R_P) + (R_Q) - (O)) + F$$

Remark that $f^*(\operatorname{Pic} B) \subset \operatorname{Pic} X$ and every fibre X_b is integral whenever $b \in B \setminus T$. Since $\operatorname{Pic}^0(B)$ is a divisible group, we can write $F \sim (d+1)W - W_T - \varepsilon f^*[b_0]$ for some vertical divisor W, W_T and some fixed point $b_0 \in T$ with $0 \leq \varepsilon \leq d$ such that W_S is effective and $f(W_T) \subset T$. Therefore, for $L_P = (R_P) + (R_Q) - (O) + W$, we find that

$$Z = (P) + \mathcal{D} + W_T + \varepsilon f^*[b_0] \sim (d+1)L_P.$$

We now determine a minimal nonisotrivial fibration of curves associated to P using the tool of cyclic covers. By Proposition 2.45, we obtain a degree $d+1$ simple cyclic cover $X' \to X$ of surfaces associated to the data $Z \sim (d+1)L_P$. Let $Y \to X'$ be the strict resolution of singularity of X'. Consider the composition $f_P \colon Y \to X' \to X \to B$. Then by the Riemann-Hurwitz formula calculated on general fibres $Y_b \to X_b$, we find that f_P is a family of curves of genus q satisfying $2q - 2 = d(d+1)$ thus

$$q = (d^2 + d + 2)/2. \tag{3.1}$$

Let Z_{sm} be the set of regular points of Z and $\mathcal{D}_{ram} \subset \mathcal{D}$ be the set of ramified points of the finite cover $\mathcal{D} \to B$. Then $W = f(Z_m \cup \mathcal{D}_{ram}) \subset B$ is finite. Consider $b \in B \setminus (W \cup T)$. Then X_b is a smooth curve and $Z_b \subset f^{-1}(b)$ is a smooth divisor since $b \notin W \cup T$. By Proposition 2.46, $Y_b = f_P^{-1}(b) \to X_b$ is also the degree $d+1$ simple cyclic cover associated to the data $Z_b \sim (d+1)(L_P)_b$. Hence, Proposition 2.45 implies that the fibre $Y_b = f_P^{-1}(b)$ is smooth whenever $b \in B \setminus (W \cup T)$. By Theorem 2.5, $Y \to B$ is nonisotrivial since X is nonisotrivial by hypothesis. Contract any rational curves on Y if necessary, we obtain a minimal nonisotrivial family over B of type $W \cup T$ of curves of genus q and thus $Y \in F_q(B, W \cup T)$.

Denote $\mathcal{D}_{sm}$ the set of regular points of $\mathcal{D}$. Then $\mathcal{D}_{sm} \subset Z_{sm} \cup f^{-1}(S \cup T)$ since the section (P) is smooth and does not intersect $\mathcal{D}$ at fibres lying over $B \setminus S$ by definition of $(S, \mathcal{D})$-integral points. Hence, $Y \in F_q(B, S')$ where $S' = W \cup S \cup T$ is independent of the choice of the $(S, \mathcal{D})$-integral point P. Let $\mathcal{D}_{sing} = \mathcal{D} \setminus \mathcal{D}_{sm}$. Note that

$$\#S' \leq s + t + \#\mathcal{D}_{ram} + \#f(\mathcal{D}_{sing}). \tag{3.2}$$

If $\mathcal{D}$ is the zero section then $S' = S \cup T$ so that $\#S' \leq s + t$. Therefore, we have constructed a Parshin-type map denoted β and given by

$$\begin{aligned} \{(S, \mathcal{D})\text{-integral points}\} &\xrightarrow{\beta} F_q(B, S') \\ P &\longmapsto (f_P \colon Y \to X \to B) \end{aligned}$$

with the property that the horizontal part of the branch locus of $Y \to X$ is $(P) \cup \mathcal{D}$.

We claim that the fibres of the map β is uniformly bounded. Indeed, let $Y \to B$ be an element of $F_q(B, S)$ that belongs to the image of β. Corollary 2.35 implies that there is at most $M(q,d+1)$ possible $(d+1)$-covers $Y \to X$ up to composition with an element of $\mathrm{Aut}_B(X) = \mathrm{Aut}_K(E)$. Here, the function $M(q,d+1)$ is given in Corollary 2.35. Hence, it suffices to prove that each such class of $(d+1)$-covers $Y \to X$ (modulo $\mathrm{Aut}_B(X)$) is the image of at most $4d^2$ possible $(S,\mathcal{D})$-integral points. Indeed, let $P, P' \in E(K)$ be $(S,\mathcal{D})$-integral points with $\beta(P) = \beta(P') \in F_q(B,S)$ and suppose that there exists $u \in \mathrm{Aut}_B(X) = \mathrm{Aut}_K(E) = E(K) \rtimes \{\pm \mathrm{Id}\}$ (cf. Theorem 2.7 and Lemma 11.7) such that

$$h_P = u \circ h_{P'}$$

where $f_P \colon Y \xrightarrow{h_P} X \to B$ and $f_{P'} \colon Y' \xrightarrow{h_{P'}} X \to B$. Since the horizontal parts of the branch loci of $Y \to X$ and of $Y' \to X$ are respectively $(P) \cup \mathcal{D}$ and $(P') \cup \mathcal{D}$, we must have $u_E\{P', D\} = \{P, D\}$. We consider two cases:

(1) Suppose that $d \geq 2$. We deduce that $u_E(P') = P$ and $u_E(D) = D$. Hence, Lemma 11.7 implies that there is at most one rational point $R \in E(K)$ modulo $E[d]$ which depends only on $\mathcal{L}(D)$ such that u_E is of the form τ_U or $(-\mathrm{Id}) \circ \tau_{R+U}$ where $U \in E[d]$. As $u_E(P') = P$, there exists at most $2\#E[d] = 2d^2$ possibilities for the point P'.

(2) Suppose that $d = 1$. Then Lemma 11.6 implies that $D \in E(K)$ is a fixed rational point and $\mathcal{D}$ is a section. Since $\mathrm{Aut}_K(E) = E(K) \rtimes \{\pm \mathrm{Id}\}$ and $u_E\{P', D\} = \{P, D\}$, we need to find $R \in E(K)$ verifying one of the following conditions:

   (a) $R + D = D, R + P' = P$;

   (b) $R + D = P, R + P' = D$;

   (c) $-(R + D) = D, -(R + P') = P$;

   (d) $-(R + D) = P, -(R + P') = D$.

   Each case gives at most one choice for R by the first equation and hence at most one choice for P' by the second one. Thus, there are at most 4 possibilities for P' in total.

In conclusion, we have shown that the fibres of β are uniformly bounded by $4d^2 M(q,d{+}1)$. Since $F_q(B,S')$ is also uniformly bounded by a function $C(q,g,\#S')$ (cf. Theorem 11.2), the number of $(S,\mathcal{D})$-integral points is bounded uniformly by

$$4d^2 M(q,d+1) C(q,g,\#S'),$$

which a function depending only on g, s, t, d, and $\#\mathcal{D}_{ram} \cup \mathcal{D}_{sing}$ by the relations (3.2) and (3.1). The proof is thus completed. $\square$

4. A negative result on the Parshin-Arakelov theorem

Keep the notations as in Theorem L, it is natural to attempt to generalize the above proof of Theorem L to show, for example, that for every $n \in \mathbb{N}$, the following union of integral points of bounded denominators

$$I_n = \cup_{S \subset B, \#S \leq n} \{(S, \mathcal{D})\text{-integral points}\} \tag{4.1}$$

is finite (which is true by Theorem H or Corollary 1.37). If we follow the same steps as in the proof of Theorem L, we will then obtain a map with uniformly bounded finite fibres

$$\begin{aligned} \cup_{S \subset B, \#S \leq n} \{(S, \mathcal{D})\text{-integral points}\} &\xrightarrow{\beta'} \cup_{S' \subset B, \#S' \leq n'} F_q(B, S') \\ P &\longmapsto (f_P \colon Y \to X \to B) \end{aligned}$$

for some $n' \in \mathbb{N}$. Thus we reduce to prove the finiteness of $\cup_{S' \subset B, \#S' \leq n'} F_q(B, S')$.

Similarly, let L be a very ample line bundle on X and suppose that want to prove the finiteness of the following union of integral points with respect to a varying divisor

$$J_L = \cup_{\mathcal{D} \in |L|_{sm}} \{(S, \mathcal{D})\text{-integral points}\}, \tag{4.2}$$

where $|L|_{sm}$ denotes the open dense algebraic subset of $|L|$ consisting of smooth effective divisors (Bertini's theorem). Remark that J_L is known to be finite since the height of points in J_L is bounded by, for example, Theorem K, or by Theorem 1.36 using the base change to curves in $|L|_{sm}$ whose genus is constant.

Proceeding as in the proof of Theorem L, we can also obtain a map

$$\begin{aligned} \cup_{\mathcal{D} \in |L|_{sm}} \{(S, \mathcal{D})\text{-integral points}\} &\xrightarrow{\beta''} \cup_{S'' \subset B, \#S'' \leq n''} F_q(B, S'') \\ P &\longmapsto (f_P \colon Y \to X \to B) \end{aligned}$$

for some $n'' \in \mathbb{N}$ since the divisors $\mathcal{D} \in |L|_{sm}$ are linearly equivalent, $\mathcal{D}_{sing} = \varnothing$, and $\#\mathcal{D}_{rm}$ is uniformly bounded by the Riemann-Hurwitz formula (Lemma 2.47). It can be shown that the map β'' has uniformly bounded finite fibres (cf. the last part in the proofs of Theorem L and Theorem M). Therefore, we reduce again to show the finiteness of the set $\cup_{S'' \subset B, \#S'' \leq n''} F_q(B, S'')$.

Unfortunately, it turns out that the set $\cup_{S \subset B, \#S \leq s} F_q(B, S)$ is even very far from being finite because of the following strong negative finiteness result:

Theorem M. *For all large enough q and s depending only on the genus g of B,*

$$\cup_{S \subset B, \#S \leq s} F_q(B, S)$$

is uncountably infinite. There exists $N(q, s, g) \in \mathbb{N}^*$, a Zariski dense open subset $I \subset \mathbb{P}^N$, and a map $I \to \cup_{S \subset B, \#S \leq s} F_q(B, S)$ with uniformly bounded finite fibres.

Therefore, additional arguments will certainly be needed with the above method to obtain known finiteness results for the above sets I_n, J_L (cf. (4.1) and (4.2)) and the corresponding consequences on generic emptiness results as in Corollary B.(ii) or Corollary C.

Before giving the proof of Theorem M, we need to begin with several lemmata.

Lemma 11.8. *Let B/k be a smooth projective integral curve of genus g over an algebraically closed field k. Then there exists a finite k-morphism $f\colon B \to \mathbb{P}^1$ of degree at most $2g+1$.*

Proof. Let $b \in B$ and consider the effective divisor $D = (2g+1)[b]$. Since $\deg D = 2g+1$, the line bundle $L = \mathcal{O}(D)$ is very ample on B. By the Riemann-Roch theorem, we have $\dim H^0(B, L) = g+2$. Fixing a basis of $H^0(B, L)$, we obtain an immersion embedding $j\colon B \to \mathbb{P}^{g+1} = \mathbb{P}|L|$. Let $z_0, \dots, z_{g+1}$ be the coordinate in $\mathbb{P}^{g+1}$. It is clear from the construction that for some i, the rational function $j^*z_i\colon B \to \mathbb{P}^1$ is non constant and of degree $\deg(j^*z_i) = 2g+1$. □

We are now in position to prove Theorem M.

Proof of Theorem M. The idea of the proof is very similar to the proof of Theorem L. Consider a non-isotrivial elliptic surface $f\colon X \to B$. Such a surface always exists since we can obtain one after a finite base change $B \to \mathbb{P}^1$ of the Legendre family $\mathcal{E} \to \mathbb{P}^1$ of elliptic curves defined in the affine plane by the equation

$$y^2 = x(x-1)(x-\lambda),$$

where λ denotes the inhomogeneous coordinate on $\mathbb{P}^1$. By Theorem 2.5, the obtained surface X is non-isotrivial since $\mathcal{E}$ is non-isotrivial and we are in characteristic 0. Here the model $\mathcal{E}$ can be taken to be for example the minimal elliptic surface associated to the generic elliptic curve $\mathcal{E}_{\mathbb{C}(\lambda)}$ over the function field $\mathbb{C}(\lambda)$. Remark that the finite morphism $B \to \mathbb{P}^1$ can be taken to have degree $m \leq 2g+1$ by Lemma 11.8. As $\mathcal{E}$ has only three singular fibres lying above $\lambda = 0, 1, \infty$, It follows that the set $T \subset B$ supporting singular fibres of X has no more than $3(2g+1)$ points.

Let H be a very ample line bundle on $\mathcal{E}$ and let L be its pull back to X, which is also very ample. Let $d \in \mathbb{N}^*$ be any fixed integer. By Bertini's theorem, we obtain a Zariski open dense subset $I \subset \mathbb{P}|2dL|$ parametrizing an uncountable family of smooth and irreducible horizontal curves $(C_i)_{i \in I} \subset X$ belonging to the complete linear system $|2dL|$. By the adjunction formula, the (arithmetic) genus g_i of the curves C_i's is a constant

$$g_i = \frac{C_i(C_i + K_X)}{2} + 1 = \frac{2dL(2dL + K_X)}{2} + 1 = m\frac{2dH(2dH + K_{\mathcal{E}})}{2} + 1 \tag{4.3}$$

which is bounded only in terms of g since $m \leq 2g+1$. From (4.3) and the Riemann-Hurwitz Formula 2.47, it is clear that for every $i \in I$, the degree of the ramification divisor $C_{i,ram}$ of the induced cover $C_i^{hor} \to B$ is bounded in function of g. Define $W_i = f(C_{i,ram}) \subset B$. It follows that $\# W_i$ is finite and bounded only in terms of g.

We can now obtain an uncountable number of pairwise non-equivalent non-isotrivial minimal families $Y_i \to B$ of curves of genus $q \geq 2$ which factor through $X \to B$ as follows. Let X_b ($b \in B$) and F be general fibres of X and $\mathcal{E}$ respectively. We denote

$$n = L \cdot X_b = m(H \cdot F) \leq (2g+1)(H \cdot F).$$

By Proposition 2.45, we obtain a cyclic double cover $X_i' \to X$ of surfaces associated to the data $\mathcal{O}(C_i) \sim 2ndL$. Let $Y_i \to X_i'$ be the strict resolution of singularity of X_i' and we blow down in Y_i any (-1)-curves if they exist. Denote $f_i \colon Y_i \to X_i' \to X \to B$ the induced composition. Then by the Riemann-Hurwitz formula applied to the general fibres $Y_{i,b} \to X_b$, we find that f_i is a family of curves of genus q satisfying $2q-2 = 2nd$ thus $q = nd+1 \geq 2$.

Let $b \in B \setminus (W_i \cup T)$ where we recall that $W_i = f(C_{i,ram})$. Then X_b is a smooth curve and $C_{i,b} \subset f^{-1}(b)$ is a smooth divisor by construction. By Proposition 2.46, $Y_{i,b} = f_i^{-1}(b) \to X_b$ is also the degree $d+1$ simple cyclic cover associated to the data $C_{i,b} \sim (2dL)_b$. Hence, Proposition 2.45 implies that the fibre $Y_{i,b} = f_i^{-1}(b)$ is smooth whenever $b \in B \setminus (W_i \cup T)$. By Theorem 2.5, $Y_i \to B$ is nonisotrivial since X is nonisotrivial by hypothesis. Therefore, Y_i is a minimal nonisotrivial family over B of type $W_i \cup T$ of curves of genus q and

$$Y_i \in F_q(B, W_i \cup T) \subset \cup_{S \subset B, \# S \leq s} F_q(B,S)$$

where the number s depends only on g. To summarize, for the Zariski dense open locus of integral smooth curves $I \subset \mathbb{P}|2dL|$, we have defined a map

$$I \to \cup_{S \subset B, \# S \leq s} F_q(B,S), \quad i \mapsto (Y_i \to B).$$

To finish, we shall show that for any fixed $i \in I$, there are at most a uniformly bounded number (in terms of g) of $j \in I$ such that $Y_j \simeq Y_i$ over B.

Corollary 2.35 tells us that there is no more than $M(q,2)$ possible double covers $Y_i \to X$ up to composition with an element of $\mathrm{Aut}_B(X) = \mathrm{Aut}_K(E)$. Therefore, it is enough to prove that each such class of double covers has at most $8d^2n^2$ covers of the form $Y_j \xrightarrow{h_j} X$ where $f_i \colon Y_i \xrightarrow{h_i} X \to B$ and $f_j \colon Y_j \xrightarrow{h_j} X \to B$ are constructed as above. For such j, there exists $u \in \mathrm{Aut}_B(X) = \mathrm{Aut}_K(E) = E(K) \rtimes \{\pm \mathrm{Id}\}$ (cf. Theorem 2.7 and Lemma 11.7) such that $h_i = u \circ h_j$. Since the horizontal parts of the branch loci of $Y_i \xrightarrow{h_i} X$ and of $Y_j \xrightarrow{h_j} X$ are respectively C_i and C_j, we must have $u(C_j) = C_i$. Since $C_i \sim C_j$, Lemma 11.7 implies that there is at most one rational point $R \in E(K)$ modulo $E[2nd]$ with u of the form τ_U or $(-\mathrm{Id}) \circ \tau_{R+U}$

where $U \in E[2nd]$. It follows that there are at most $2\#E[2nd] = 2(2nd)^2 = 8d^2n^2$ choices for such u and thus at most $8d^2n^2$ possibilities for such j as claimed since $C_j = u^{-1}(C_i)$. Therefore, the theorem is proved. One also remarks that when $d \to \infty$, we also have $N \to \infty$ and $q, s \to \infty$. $\square$

5. Application to unit equations

We shall apply the method presented in the present chapter to give a new proof for the uniform finiteness of unit equation over function fields as follows.

Let's begin with a general construction of covers associated to integral sections in a ruled surface. Let k be an algebraically closed field of characteristic 0. Let B/k be a curve of genus g and $K = k(B)$. Let $S \subset B$ be a finite subset of cardinality s. Let $f\colon X \to B$ be the trivial ruled surface over B and let $\pi\colon X \to B$ be the second projection. Let D be a reduced divisor of odd degree $d \geq 3$ on $\mathbb{P}^1_K$. Let $\mathcal{D}$ be the closure of D in X. Let $P \in X_K(K) \setminus D$. Let $(P) \subset X$ be the corresponding section $B \to X$ of P and denote

$$Z = (P) + \mathcal{D}.$$

Consider the divisor $Z_K = D + [P]$ on $\mathbb{P}^1_K$. Let $b \in S$ be a fixed point. Since $\deg(Z_K) = \deg(D)+1 = d+1$ and since the group $\mathrm{Pic}^0_K(\mathbb{P}^1_K)$ of equivalent classes of degree 0 line bundles of $\mathbb{P}^1_K$ is trivial, $Z_K \sim_K (d+1)[P]$. As $k(X) = K(t) = K(\mathbb{P}^1_K)$, we can extend the linear equivalence on the surface X up to a vertical divisor. Hence, we deduce that there exists a vertical divisor F on X such that

$$Z \sim_k (d+1)(P) + F.$$

We can write $F = \sum_i n_i \pi^*[b_i]$ for $b_i \in B$ and $n_i \in \mathbb{Z}$ where $[b_i]$ denotes the effective divisor associated to b_i. We have $\sum_i n_i = 2m - r$ for some $m, r \in \mathbb{N}$ and $0 \leq r \leq 1$. Thus $\deg(\sum_i n_i[b_i] - (2m-r)[b]) = 0$. By properties of the Jacobian $J(B)$, we find that $\sum_i n_i[b_i] \sim \mathcal{O}(2m[b] + r[b])$. It follows that $F = \sum_i n_i \pi^*[b_i] \sim 2\mathcal{O}(m\pi^*[b]) + r[b]$. Thus for $L = \mathcal{O}(\frac{d+1}{2}(P) + m\pi^*[b])$, we have

$$Z + r\pi^*[b] \sim_k L^{\otimes 2}.$$

By Proposition 2.45, we obtain a double cyclic cover $X' \to X$ associated to the data $(Z + r\pi^*[b], L^{\otimes 2})$. The cover $X' \to X$ is totally ramified above $Z + r\pi^*[b]$. Moreover, it is smooth above $B \setminus S$ whenever (P) is $(S, \mathcal{D})$-integral. By a strong resolution of singularity and by contracting all possible (-1)-curves, we obtain a minimal family of curves

$$f_P\colon Y_P \to X \to B.$$

Definition 11.9. We say that P is $\mathcal{D}$-*nontrivial* (or simply *nontrival* if no there is no possible confusion) if the induced family $f_P\colon Y_P \to B$ is nonisotrivial.

Example 11.10. When $\mathcal{D}$ is the sum of the three sections (0), (1), (∞), a rational point $x \in X(K)$ is $\mathcal{D}$-nontrivial if and only if $x \notin k^*$.

The main result of the section is the following.

Theorem 11.11. *Let the notations be as above. There is a function $U\colon \mathbb{N}^5 \to \mathbb{N}$ such that the following set of integral points*

$$\{\textit{nontrivial } (S, \mathcal{D})\textit{-integral points of } X \to B\}$$

has at most $U(g, s, d, d_{sing}, d_{ram})$ elements where d_{sing}, d_{ram} denote the number of singular points on $\mathcal{D}$ and the number of ramified points of the induced degree-d cover $\mathcal{D} \to B$.

From Theorem 11.11, we can recover the following known uniform finiteness result on unit equation over function fields.

Corollary 11.12. *The number of the solutions (x, y) with $x/y \notin k$ of the S-unit equation $x + y = 1$, $x, y \in \mathcal{O}_S^*$ is uniformly bounded in terms of g, s.*

Proof. It suffices to show that the number of $x \in \mathcal{O}_S^*$ with $1 - x \in \mathcal{O}_S^*$ and $x \notin k$ is uniformly bounded. Such a point x is exactly a nontrivial $(S, \mathcal{D})$-integral points in $X = \mathbb{P}^1 \times B$ with $\mathcal{D} = (0) + (1) + (\infty)$. Since $\mathcal{D}$ has no singular points and the cover $\mathcal{D} \to B$ is étale, the corollary follows from Theorem 11.11. $\square$

Remark 11.13. Theorem 11.11 and Corollary 11.12 are in fact consequences of the following remarkable result of Evertse (cf. **[33]**) whose proof uses height theory combined with the so-called gap principle:

Theorem (Evertse)**.** *Let B be a smooth projective curve over a field k of characteristic 0. Let $K = k(B)$ be the function field of B and $S \subset B$ a finite subset. The set of $(x, y) \in (\mathcal{O}_{K,S}^*)^2$ with $x/y \notin k$ and $x + y = 1$ has at most $2 \times 7^{2\#S}$ elements.*

Evertse's effective bound is totally independent of the function field K. To obtain Theorem 11.11, it suffices to make the base change $h\colon \tilde{D} \to B$ of degree $\deg(h) = d = \deg D_K$ where $\tilde{D}$ is the normalization of $\mathcal{D}$. The set of nontrivial $(S, \mathcal{D})$-integral points of $X \to B$ then becomes a subset of the set of solutions of the S'-unit equation $x + y = 1$ where $S' = h^{-1}(S)$ so that $\#S' \leq \deg(h)\#S = ds$.

However, our proof of Theorem 11.11 below does not use height theory and thus we give a new proof of a weak (but nontrivial) version of Evertse's theorem.

Remark first that since an isomorphism of $\mathbb{P}^1$, i.e., a Möbius transformation, is completely determined by the images of three distinct points. We can thus easily obtain the following elementary lemma.

Lemma 11.14. *Let k be a field and let $P, Q \in \mathbb{P}^1_k \backslash \{0, 1, \infty\}$ be two distinct points. Then there is at most $4! = 24$ isomorphisms of $\mathbb{P}^1$ sending the set $\{0, 1, \infty, P\}$ to the set $\{0, 1, \infty, Q\}$.*

Proof of Theorem 11.11. Let $P \in X(K)$ be a nontrivial $(S, \mathcal{D})$-integral point of X. Then $Z = (P) + \mathcal{D}$ has degree $d+1$ on general fibres of $Y_P \to B$. By the Riemann-Hurwitz formula applied to the double cover $Y_{P,t} \to X_t$ of general fibres ($t \in B$), $Y_P \to B$ is a family of curves of genus q such that $2q - 2 = 2(-2) + d + 1$ so that $q = (d-1)/2$. By Proposition 2.45 and Proposition 2.46, $Y_P \to B$ is of type $S' = S \cup \pi(\mathcal{D}_{sing} \cup \mathcal{D}_{ram})$ where $\mathcal{D}_{sing}$ is the set of singular points of $\mathcal{D}$ and $\mathcal{D}_{ram} \subset \mathcal{D}$ is the set of ramified points of the finite cover $\mathcal{D} \to B$.

Since P is a nontrivial $(S, \mathcal{D})$-integral point, the family f_P is nonisotrivial and has good reductions outside of S. To summarize, we have constructed a map

$$\{\text{nontrivial } (S, \mathcal{D})\text{-integral points of } X \to B\} \xrightarrow{\gamma} F_q(B, S')$$

$$P \longmapsto (f_P \colon Y_P \xrightarrow{h_P} X \to B)$$

where $F_q(B, S')$ denotes the set of non equivalent classes of minimal families of curves of genus q of type S'. Since $q \geq 1$ as $d \geq 3$, Shafarevich theorem (Theorem 2.37) and the Parshin-Arakelov theorem (Theorem 11.3) imply that $\#F_q(B, S')$ is uniformly bounded in terms of $q, g, \#S'$. To finish, we need to show that the map γ above has uniformly bounded fibres. So let $f_P \colon Y_P \xrightarrow{h_P} X \to B$ be in the image of γ for a some integral point P. We distinguish two cases.

Suppose first that $d \geq 5$ so that $q \geq 2$. Then Theorem 2.36 implies that up to an isomorphism of $\mathbb{P}^1$, there exists only a uniformly bounded finite number of double covers $Y_P \to X$. Thus, it suffices to show that if Q a nontrivial $(S, \mathcal{D})$-integral point such that $h_P = \mu \circ h_Q$ for some $\mu \in \mathrm{Aut}_B(X) = \mathrm{PGL}_2(k)$, then there are at most 24 choices for Q. But since μ must send the branch points of h_Q to branch points of h_P, we have $\mu\{Q, D\} = \{P, D\}$ and thus there are indeed no more than 24 possibilities for Q by Lemma 11.14.

Suppose now that $d = 3$ so that $q = 1$ and $Y_P \to B$ is an elliptic surface. It is well-known that up to an isomorphism of $\mathbb{P}^1$ and an automorphism of $(Y_P)_K$, there is only one double cover from $(Y_P)_K$ to $\mathbb{P}^1$ given by $(x, y) \mapsto y$ where Y_P is given by $y^2 = x^3 + Ax + B$ for some coordinates x, y. Hence, as in the case $d \geq 5$, at most 24 integral points Q give rise to the same family of curves $Y_Q \simeq Y_P$. The conclusion thus follows. □

CHAPTER 12

Concluding remarks and perspectives

The theme of the present thesis is the geometry and the arithmetic of sections in one dimensional families of varieties in the context of complex function fields as in the spirit of the Geometric Lang-Vojta conjecture. More specifically, the main objects studied are generalized integral sections of families $\mathcal{X} \to B$ where B is a Riemann surface and $\mathcal{X}$ is a family of abelian varieties. We have seen that one of the major advantages of the notion of generalized integral points is that it allows us to keep track also of some of the topological properties of sections. For such objects, we have developed the beautiful hyperbolic-homotopical method of Parshin to obtain some quantitative results on the finiteness of the set of generalized integral sections. The main tool is a new general estimation concerning the growth of the hyperbolic length of certain loops in complements of a non compact Riemann surface with a finite number of moving points removed. While the present thesis contributes some new phenomena and results with respect to the state-of-the-art of the fascinating known arithmetic and geometric properties of such families in the literature, many interesting questions remain open.

It would be interesting to continue the hyperbolic-homotopical method in the relation with the theory of the Geometry of the fundamental groups. To conclude, we mention below some of the project plans that we would like to carry out in the near future:

(i) topological properties of the intersection locus $I(R, \mathcal{D})$ (cf. (1.2)) of sections with horizontal effective divisors in fibrations of abelian varieties;

(ii) towards effective and uniform results (in the spirit of the Uniformity conjecture) concerning the finiteness of the generalized integral sections in abelian and Jacobian fibrations;

(iii) generalizations to higher dimensional bases B, to semi-abelian families and to the context of the p-adic world;

(iv) towards an effective bound of Theorem H when $\mathcal{D}$ is the zero section.

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Points entiers généralisés sur les variétés abéliennes

Résumé : L'objectif de cette thèse est l'étude des propriétés concernant la finitude, la croissance, la nonexistence générique et l'uniformité de l'ensemble des sections (S,D)-entières d'une famille des variétés abéliennes A fibrée au-dessus d'une surface de Riemann compacte B. Le sous-ensemble S de B est arbitraire et n'est pas nécessairement fini. Ces sections entières correspondent aux points rationnels de la fibre générique de A et qui ne peuvent intersecter le diviseur D de A qu'au-dessus de S. Dans ce contexte, une machinerie appelée hauteur hyperbolique-homotopique est introduite pour jouer le rôle de la théorie d'intersection. Nous démontrons plusieurs nouveaux résultats sur la finitude de certaines unions larges de sections (S, D)-entières ainsi que leur croissance polynomiale en fonction du cardinal de la restriction de S à un certain ouvert complexe petit U de B. Ces résultats sont hors de portée des méthodes purement algébriques. Ainsi, nos travaux mettent en évidence certains phénomènes nouveaux en faveur de la version géométrique de la conjecture de Lang-Vojta. Si A est une surface elliptique, les mêmes conclusions restent vraies où non seulement S mais D peuvent aussi varier en familles. Nous démontrons également un résultat négatif concernant le théorème de Parshin-Arakelov.

Mots-clés : points entiers généralisés, Conjecture de Lang-Vojta géométrique, variétés abéliennes, surfaces elliptiques, surfaces de Riemann

Summary : We study the finiteness, growth order, generic emptyness, and uniformity of the set of (S,D)-integral sections in an abelian fibration A over a compact Riemann surface B. Here, S is an arbitrary subset of B and not necessarily finite. These integral sections correspond to rational points of the generic fibre of A and which intersect the divisor D only possibly above S. We introduce in this context the so-called hyperbolic-homotopic height as a substitute for the classical intersection theory. We then establish several new results concerning the finiteness of various large unions of (S,D)-integral points and their polynomial growth in terms of the caradinality of the restriction of S in U, where the sets S is required to be finite only in a certain small open subset U of B. Such results are out of reach of a purely algebraic method. Thereby, we give some new evidence and phenomena to the Geometric Lang-Vojta conjecture. When A is an elliptic surface, we obtain the same results for certain unions of (S,D)-integral points, where both S and D are allowed to vary in certain families. A negative finiteness result concerning the Parshin-Arakelov theorem is also given.

Keywords : generalized integral points, geometric Lang-Vojta conjecture, abelian varieties, elliptic surfaces , Riemann surfaces