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Finitely generated normal subgroups of Kähler groups

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Contents

Introduction en français

Une *variété kählérienne* est une variété complexe munie d'une métrique hermitienne dont la partie imaginaire est une 2-forme fermée. Un groupe est dit *groupe kählérien* s'il peut être réalisé comme le groupe fondamental d'une variété kählérienne compacte. Une référence classique sur ce sujet est [2] et une référence plus récente est [18].

Il existe de nombreuses restrictions topologiques sur les variétés kählériennes compactes. Par exemple, la théorie de Hodge permet de munir le premier groupe de cohomologie à coefficients dans $\mathbb{R}$ d'une variété kählérienne compacte d'une structure complexe (voir chapitre 6 de [75]). Ceci implique que le rang de l'abélianisation d'un groupe kählérien doit être pair. Puisque tout sous-groupe d'indice fini d'un groupe kählérien est à nouveau un groupe kählérien, cette dernière affirmation doit également être valable pour tout sous-groupe d'indice fini d'un groupe kählérien. De cette manière, on peut voir qu'un groupe libre n'est jamais kählérien car pour un tel groupe, on peut toujours trouver un sous-groupe d'indice fini dont l'abélianisation a un rang impair. Voici quelques exemples de groupes kählériens.

• Le groupe fondamental d'une surface de Riemann fermée est un groupe kählérien car toute 2-forme sur un tel espace est fermée.

• Pour tout entier positif n, le groupe $\mathbb{Z}^{2n}$ est kählérien car il peut-être réalisé comme le groupe fondamental du tore complexe $\mathbb{C}^n/\mathbb{Z}^{2n}$. Dans ce cas, toute forme hermitienne définie positive sur $\mathbb{C}^n$ induit une métrique kählérienne sur $\mathbb{C}^n/\mathbb{Z}^{2n}$.

• L'espace projectif complexe $\mathbb{CP}^n$ muni de la métrique de Fubini-Study est un variété kählérienne. Ainsi, toute sous-variété complexe de $\mathbb{CP}^n$ est munie d'une structure kählérienne. Les groupes fondamentaux de telles variétés sont appelés *groupes projectifs* et en particulier ils sont des groupes kählériens[1].

• Dans [67], Serre a prouvé que tous les groupes finis sont des groupes projectifs et donc des groupes kählériens.

• Le produit direct de groupes kählériens et les sous-groupes d'indice fini de groupes kählériens, sont à nouveau des groupes kählériens. Ceci découle du fait que la métrique donnée par le produit des métriques kählériennes et la métrique du tiré en arrière sous un difféomorphisme local holomorphe d'une métrique kählérienne, sont à nouveau des métriques kählériennes.

• Les espaces symétriques Hermitiens et leurs quotients par des sous-groupes discrets sans torsion de leur groupe d'isométries holomorphes, sont des variétés kählériennes. Alors, les réseaux uniformes sans torsion du groupe d'isométries holomorphes d'espaces symétriques Hermitiens sont des groupes kählériens. Une astuce de Kollár, permet d'étendre ce résultat aux réseaux uniformes avec de la torsion (voir [3] pour une preuve de cette dernière affirmation). Un travail de Toledo [72] montre que de nombreux réseaux non-uniformes d'isométries holomorphes d'espaces symétriques Hermitiens sont également des groupes kählériens.

[1] Une question importante est de savoir si ces deux classes de groupes coïncident (voir [27]).

Dans l'étude des groupes kählériens il y a deux types de résultats : il y a des résultats négatifs qui disent que certaines familles de groupes ne contiennent pas de groupes kählériens. Mentionnons (parmi les nombreux résultats de ce type) les suivants : Carlson et Toledo ont prouvé dans [22] qu'un réseau dans le groupe d'isométries de l'espace hyperbolique réel de dimension au moins 3, n'est jamais kählérien. Delzant (en suivant certains résultats d'Arapura-Nori [5], de Brudnyi [17] et de Campana [19]) a prouvé dans [29] qu'un groupe kählérien résoluble doit être virtuellement nilpotent. En étudiant la cohomologie L_2 des variétés et des groupes kählériens, Gromov a prouvé dans [38] qu'un groupe kählérien infini est un groupe à un bout. Une preuve plus détaillée de ce dernier résultat peut être trouvée dans le travail de Arapura, Bressler et Ramachandran [4]. D'autre part, il existe des résultats positifs, c'est-à-dire des constructions de groupes kählériens avec des propriétés intéressantes. Mentionnons l'existence de groupes kählériens non résiduellement finis [73] ou de groupes kählériens nilpotents qui ne sont pas virtuellement Abéliens [20]. Dans cette thèse, nous contribuerons à ces deux lignes de recherche.

1 Contexte général de ce travail

Supposons que nous avons une suite exacte courte

$$1 \longrightarrow G \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1, \tag{1}$$

où Γ est un groupe kählérien et G est un groupe de type fini. Étant donné une telle suite exacte courte, nous pouvons demander :

Question 1. *Que peut-on dire de l'action par conjugaison de Γ sur G ?*

Nous allons restreindre notre étude aux suites exactes courtes comme (1), où G et Q sont des groupes infinis. L'action par conjugaison de Γ sur G induit un homomorphisme de groupes de Γ dans le groupe d'automorphismes de G.

$$\begin{aligned} \Gamma &\longrightarrow \mathrm{Aut}(G) \\ \gamma &\mapsto (x \mapsto \gamma x \gamma^{-1}). \end{aligned}$$

Rappelons que le *groupe d'automorphismes intérieurs de G*, noté par $\mathrm{Inn}(G)$, est composé des automorphismes de G induits par la conjugaison d'un élément de G, c'est-à-dire d'automorphismes de la forme $x \mapsto gxg^{-1}$ pour un certain g dans G. Le groupe d'automorphismes intérieurs est normal dans $\mathrm{Aut}(G)$ et le groupe quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$, noté $\mathrm{Out}(G)$, est appelé le *groupe d'automorphismes extérieurs de G*. Une suite exacte courte comme (1), induit un homomorphisme de groupes $\Gamma \to \mathrm{Out}(G)$ qui se factorise par Q, c'est-à-dire, le diagramme suivant

$$\begin{array}{ccc} \Gamma & \longrightarrow & \mathrm{Out}(G) \\ & \searrow \quad \nearrow & \\ & Q & \end{array}$$

commute. L'homomorphisme $Q \to \mathrm{Out}(G)$ obtenu de cette manière est appelé la *monodromie* de la suite exacte courte (1).

La géométrie complexe fournit des exemples de suites exactes courtes comme (1). Par exemple, une fonction holomorphe surjective à fibres connexes $f : X \to S$ entre un variété complexe compacte X et une surface de Riemann fermée S de genre positif, induit une suite exacte courte

$$1 \longrightarrow G \longrightarrow \pi_1(X) \xrightarrow{f_*} \pi_1^{orb}(S) \longrightarrow 1,$$

où G est l'image du groupe fondamental de la fibre générique dans $\pi_1(X)$ et S est munie d'une structure d'orbifold qui prend en compte les multiplicités des fibres singulières (voir Section 1.4.1). Dans [31], Dimca, Papadima et Suciu ont étudié certaines propriétés de finitude du groupe fondamental d'une fibre générique d'une fonction holomorphe comme avant, dans le cas particulier où X est un produit direct de surfaces de Riemann de genre supérieur ou égal à deux, et S est de genre un. Cette étude sur les propriétés de finitude des groupes fondamentaux des fibres génériques, a été poursuivie par Llosa Isenrich [49, 50] et par Bridson et Llosa Isenrich [14]. Ils construisent de nouveaux groupes kählériens avec des propriétés de finitude intéressantes. On se pose alors la question suivante.

Question 2. *Pour une fonction holomorphe surjective à fibres connexes $f : X \to S$ entre une variété complexe compacte X et une surface de Riemann fermée S de genre positif, que peut-on dire des propriétés de finitude du groupe fondamental de sa fibre générique ?*

Nous nous intéresserons au cas où X est une variété kählérienne compacte. Ce travail donnera des réponses partielles aux Questions 1 et 2.

2 Restrictions sur la monodromie

Un problème classique est l'étude de la monodromie : étant donné une submersion holomorphe propre $f : X \to Y$ (qui n'est pas un fibré holomorphe localement trivial) à fibres de dimension 1, on peut étudier la monodromie $\pi_1(Y) \to \mathrm{MCG}(S)$, où S est la fibre générique de f qui est une surface topologique et $\mathrm{MCG}(S)$ est le groupe modulaire de S. Dans [68], Shiga a prouvé le résultat suivant.

Théorème (Shiga)**.** *Soit $f : X \to B$ une submersion holomorphe qui n'est pas un fibré holomorphe localement trivial d'une surface complexe compacte sur une surface hyperbolique avec fibre générique une surface hyperbolique fermée S. Alors, l'image de la monodromie $\pi_1(B) \to \mathrm{MCG}(S)$ est infinie et elle ne peut préserver aucune classe d'isotopie d'une courbe fermée simple dans S.*

L'un des résultats qui a motivé le travail présenté dans le Chapitre 1, est le résultat suivant de Bregman et Zhang (voir [15]).

Théorème (Bregman-Zhang)**.** *Soit S une surface fermée de genre $g \geq 2$, Γ un groupe kählérien et k un entier positif tel qu'il existe une suite exacte courte*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \xrightarrow{P} \mathbb{Z}^k \longrightarrow 1.$$

Alors, il existe un sous-groupe d'indice fini Γ_1 de Γ contenant $\pi_1(S)$, tel que la suite exacte courte restreinte

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma_1 \xrightarrow{P} P(\Gamma_1) \simeq \mathbb{Z}^k \longrightarrow 1$$

est un produit direct.

Nous étudierons ici les suites exactes courtes comme (1), où G est le groupe fondamental d'une surface fermée orientée S de genre $g \geq 2$, sans supposer que S est la fibre d'une submersion holomorphe. Une des clés de cette étude est le fait que tout élément de $\mathrm{Out}(\pi_1(S))$ peut être représenté par un difféomorphisme de la surface S (voir Dehn [28] pour la preuve originale et Nielsen [57] pour la première preuve publiée). Nous donnerons également quelques restrictions sur les suites exactes courtes comme (1), où G se décompose comme produit amalgamé ou extension HNN.

Présentation des résultats sur les groupes kählériens et les groupes de type fini agissant sur des arbres

Nous présentons ici les résultats du préprint [56]. Notre hypothèse principale est qu'un groupe kählérien Γ, admet comme sous-groupe normal un groupe de type fini G qui agit sur un arbre T. Dans de nombreux cas, nous verrons que l'action de G sur T s'étend à Γ, ce qui nous permettra d'appliquer le résultat suivant de Gromov et Schoen (voir[37]).

Théorème (Gromov-Schoen)**.** *Soit X une variété kählérienne compacte dont le groupe fondamental Γ agit sur un arbre qui n'est pas isomorphe à une droite ni à un point. Supposons que l'action est minimale et sans point fixe au bord. Alors, il existe une fonction surjective holomorphe à fibres connexes de X sur un orbifold hyperbolique fermé Σ, induisant la suite exacte courte*

$$1 \longrightarrow N \longrightarrow \Gamma \xrightarrow{\Pi} \pi_1^{orb}(\Sigma) \longrightarrow 1,$$

de sorte que la restriction de l'action à N est triviale.

Voir la Section 1.4.1 pour la définition du groupe fondamental orbifold $\pi_1^{orb}(\Sigma)$. Le problème d'étendre l'action de G sur un arbre au groupe $\mathrm{Aut}(G)$ a été étudié dans la littérature. Dans plusieurs cas, on peut prouver des résultats d'extension. Ceci est le cas si le groupe G est l'un des exemples suivant :

1. Un produit libre $A*B$ avec A et B des groupes indécomposables, non cycliques infinis, agissant sur son arbre de Bass-Serre.

2. Le groupe de Baumslag-Solitar $\langle x, t | tx^p t^{-1} = x^q \rangle$, agissant sur son arbre de Bass-Serre, où p, q sont des entiers avec $p, q > 1$, et tels qu'aucun n'est un multiple de l'autre (voir Gilbert, Howie, Metaftsis et Raptis [35] et Pettet [59]).

3. Un groupe hyperbolique à un bout sans torsion, avec un groupe d'automorphismes extérieurs infini et qui n'est pas virtuellement un groupe de surface, agissant sur son arbre JSJ (Sela [63], voir aussi Bowditch [11]).

Afin d'appliquer ces résultats à l'étude des groupes kählériens, supposons que le groupe G n'a pas de centre, que l'arbre T n'est pas une droite ni un point et que l'action de G sur T est minimale, fidèle et sans points fixes au bord de T. Soit $\overline{\Gamma}$ l'image du morphisme $\Gamma \to \mathrm{Aut}(G)$ induit par l'action par conjugaison de Γ sur G. Sous ces conditions nous prouvons le résultat suivant dans le Chapitre 1.

Théorème A. *Supposons qu'il existe un sous-groupe d'indice fini* $\overline{\Gamma}_0$ *de* $\overline{\Gamma}$ *contenant* $G \simeq \mathrm{Inn}(G)$*, de sorte que l'action de* G *sur* T *puisse être étendue en une action* G*-compatible de* $\overline{\Gamma}$ *sur* T*. Alors* G *est virtuellement un groupe de surface. De plus, il existe un sous-groupe d'indice fini* Γ_1 *de* Γ *contenant* G *tel que la suite exacte courte restreinte*

$$1 \longrightarrow G \longrightarrow \Gamma_1 \xrightarrow{P} P(\Gamma_1) \longrightarrow 1$$

est un produit direct.

On renvoie à la Section 1.3.1.b pour la définition d'action G-compatible. La première application de ce résultat est la suivante.

Théorème B. *Soient* S *une surface fermée de genre* $g \geq 2$ *et* Γ *un groupe kählérien tel qu'il existe une suite exacte courte*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1.$$

Supposons que l'action par conjugaison de Γ *sur* $\pi_1(S)$ *préserve la classe de conjugaison d'une courbe fermée simple dans* S*. Alors, il existe un sous-groupe d'indice fini* Γ_1 *de* Γ *contenant* $\pi_1(S)$*, tel que la suite exacte courte restreinte*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma_1 \longrightarrow Q_1 \longrightarrow 1$$

est un produit direct (où Q_1 *est l'image de* Γ_1 *dans* Q*). En particulier, la monodromie* $Q \to \mathrm{Out}(\pi_1(S))$ *est finie.*

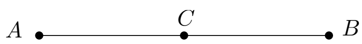
Ce résultat peut être vu comme une version topologique du résultat de Shiga. Nous verrons également que nous pouvons retrouver le résultat de Bregman et Zhang, en utilisant des résultats classiques sur le groupe de monodromie d'une surface hyperbolique et l'hyperbolisation des variétés de dimension 3. D'autres applications sont obtenues.

Théorème C. *Soit* G *un groupe qui se décompose comme un produit libre non trivial* $A * B$ *avec* A *et* B *indécomposables et non cycliques infinis. Alors* G *ne se plonge pas comme sous-groupe normal dans un groupe kählérien.*

Remarque. *Pour l'arbre de Bass-Serre d'un produit libre* $G = A * B * C$*, il existe des automorphismes de* G *qui ne peuvent pas être étendus. Par exemple si* β *est un élément de* B*, l'automorphisme défini par*

$$a \mapsto a, \quad b \mapsto b, \quad \text{and} \quad c \mapsto \beta c \beta^{-1}$$

pour tout $a \in A$*,* $b \in B$ *et* $c \in C$ *(appelé automorphisme de Fouxe-Rabinovitch), ne s'étend pas à l'arbre de Bass-Serre dont la graphe de groupes correspondant est :*



Cela implique que notre preuve du Théorème C ne s'étend pas à un produit libre d'au moins trois facteurs.

Théorème D. *Soit* G *le groupe de Baumslag-Solitar* $\langle x, t | t x^p t^{-1} = x^q \rangle$*, où* p, q *sont des entiers avec* $p, q > 1$*, et tels qu'aucun n'est un multiple de l'autre. Alors* G *ne se plonge pas comme sous-groupe normal dans un groupe kählérien.*

Théorème E. *Supposons que* G *est un groupe hyperbolique sans torsion à un bout, qui se plonge comme sous-groupe normal dans un groupe kählérien* Γ*. Si* $\mathrm{Out}(G)$ *est infini, alors* G *est virtuellement un groupe de surface.*

Nous observons que dans ce dernier théorème il n'y a pas d'hypothèse sur la fonction de monodromie. Nous avons seulement besoin de savoir que $\mathrm{Out}(G)$ est infini.

3 Groupes kählériens et propriétés de finitude

Nous présentons ici quelques résultats sur les propriétés de finitude des groupes kählériens. Une introduction détaillée aux propriétés de finitude sera donnée dans la Section 2.2. En particulier, les propriétés de finitude FP_n et $\mathscr{F}_n$ y seront définies.

Soit X un variété complexe compacte connexe de dimension complexe $n \geq 2$ et soit S une surface de Riemann fermée de genre positif. Une fonction holomorphe surjective à fibres connexes $f : X \to S$ est appelée *un pinceau irrationnel*. Dans [43], Kapovich a prouvé le résultat suivant

Théorème (Kapovich)**.** *Soient X une surface kählérienne compacte asphérique, S une surface de Riemann fermée de genre positif et $f : X \to S$ un pinceau irrationnel dont les fibres singulières sont de multiplicité un. Alors f est une submersion ou le noyau de la fonction induite au niveau des groupes fondamentaux $f_* : \pi_1(X) \to \pi_1(S)$, n'est pas finiment présenté.*

Ceci était le premier résultat qui faisait un lien entre l'existence de points critiques d'un pinceau irrationnel et les propriétés de finitude du noyau de la fonction induite au niveau des groupes fondamentaux. Comme conséquence de ce résultat, Kapovich a prouvé l'existence de surfaces hyperboliques complexes compactes dont les groupes fondamentaux admettent un sous-groupe normal de type fini qui n'est pas finiment présentable. Autrement dit, Kapovich a donné les premiers exemples de réseaux uniformes non cohérents dans $\mathrm{PU}(2,1)$. Rappelons qu'un groupe est dit *cohérent* si chaque sous-groupe de type fini est aussi finiment présentable. D'autres exemples de groupes kählériens non cohérents ont été donnés par Kapovich [42] et Py [60]. L'un des principaux ingrédients de ces exemples est le résultat suivant, essentiellement dû à Kapovich (voir [42] et [60]), qui découle de la combinaison du Théorème 4 dans [60] (dû à Kapovich) avec le résultat de Bregman et Zhang cité précédemment.

Théorème. *Soit Γ un groupe kählérien qui peut être réalisé comme le groupe fondamental d'une surface kählérienne compacte asphérique X avec premier nombre de Betti positif. Supposons que Γ ne possède aucun sous-groupe abélien de type fini dont le normalisateur a un indice fini dans Γ. Alors, au moins un des cas suivants se produit :*

1. *Γ est non cohérent.*
2. *Il existe un revêtement fini de X, qui admet une submersion holomorphe sur une surface hyperbolique compacte à fibres hyperboliques connexes.*

Friedl et Vidussi donnent un raffinement de ces résultats sur la cohérence des groupes kählériens dans [33]. Outre l'étude de la cohérence de certains groupes, on peut également étudier l'existence de sous-groupes normaux qui ne sont pas de type fini. Par exemple, rappelons qu'un sous-groupe normal d'un groupe libre non abélien ou d'un groupe de surface, est finiment engendré si et seulement s'il est d'indice fini (voir [62] pour la preuve originale dans le cas des groupes libres et les lemmes 3.3 et 3.4 dans [25]). En général, les sous-groupes coabéliens normaux des groupes kählériens qui ne sont pas de type fini, sont liés aux fibrations sur des surfaces de Riemann. En étudiant l'invariant de Bieri-Neumann-Strebel des groupes kählériens, Delzant a prouvé dans [29] le résultat suivant.

Théorème (Delzant). *Considérons la suite exacte courte*

$$1 \longrightarrow G \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1,$$

où Γ *est un groupe kählérien,* Q *est un groupe abélien et* G *n'est pas de type fini. Alors, si* X *est une variété kählérienne compacte qui réalise* Γ *comme son groupe fondamental, il existe une fonction holomorphe surjective à fibres connexes de* X *sur un orbifold hyperbolique de dimension* 2*.*

Nous renvoyons le lecteur à [29] pour les définitions d'un orbifold hyperbolique de dimension 2 et d'une fonction holomorphe d'une variété complexe compacte sur un tel espace. Lorsque $Q = \mathbb{Z}$, le théorème ci-dessus a été établi précédemment par Napier et Ramachandran dans [54] en utilisant des techniques différentes.

Nous allons introduire quelques résultats récents sur la construction de groupes kählériens ayant des propriétés de finitude exotiques et le contexte qui a motivé ces constructions. Dans [45, §0.3.1], Kollár a posé la question de savoir si un groupe projectif est toujours commensurable (à noyaux finis près) à un groupe admettant un espace classifiant qui soit une variété quasi-projective. Puisque toute variété quasi-projective a le type d'homotopie d'un complexe fini [30, p. 27], une réponse positive à cette question impliquerait que tout groupe projectif est commensurable à un groupe ayant un espace classifiant fini (voir Section 2.2.1 pour la définition de la commensurabilité). Cependant, une réponse négative à la question de Kollár a été donnée par Dimca, Papadima et Suciu dans [31]. Dans [31], les auteurs ont prouvé les deux résultats suivants (voir Théorème C et §2 dans [31]).

Théorème (Dimca, Papadima and Suciu). *Si* $n \geq 3$ *et si* $f : X \to S$ *est un pinceau irrationnel avec des points critiques isolés, alors le groupe fondamental d'une fibre lisse de* f *se plonge dans celui de* X *et coïncide avec le noyau de l'homomorphisme induit au niveau des groupes fondamentaux* $f_* : \pi_1(X) \to \pi_1(S)$*.*

Remarquez que sous l'hypothèse de ce dernier résultat, si on suppose que X est une variété kählérienne, on obtient un sous-groupe normal de type fini du groupe fondamental de X, qui est aussi un groupe kählérien.

Théorème (Dimca, Papadima and Suciu). *Soit* $X = \Sigma_1 \times \cdots \times \Sigma_n$ *un produit direct de* n *surfaces de Riemann de genre supérieur à 1 et soit* S *de genre 1. Si* $f : X \to S$ *est un pinceau irrationnel avec des points critiques isolés, alors* $H_n(\ker(f_*), \mathbb{Q})$ *est de dimension infinie.*

En combinant ces deux derniers résultats, Dimca, Papadima et Suciu ont donnée une réponse négative à la question de Kollár. En effet, dans la situation de ce dernier résultat, le noyau de l'homomorphisme $f_* : \pi_1(X) \to \pi_1(S)$, qui est le groupe fondamental d'une fibre lisse de f si $n \geq 3$, ne peut pas être de type FP_n car le groupe $H_n(\ker(f_*), \mathbb{Z})$ n'est pas de type fini (voir Proposition 87). La propriété d'être de type FP_n est invariante par la relation de commensurabilité [7, 31]. Par conséquent, aucun groupe commensurable au noyau de f_* ne peut avoir un espace classifiant fini.

Suite à l'article [31], d'autres exemples de groupes projectifs ayant des propriétés de finitude exotiques ont été construits et étudiés par Llosa Isenrichh [49, 50] et par Bridson et Llosa Isenrich [14]. Tous les exemples étudiés dans [14, 49, 50] sont soit des sous-groupes de produits directs de groupes de surface ou des extensions de tels sous-groupes, comme dans [14].

Présentation de résultats sur les groupes kählériens ayant des propriétés de finitude exotiques

Nous présentons ici nos résultats sur la construction de nouveaux groupes kählériens ayant des propriétés de finitude exotiques. Voir [55] pour une prépublication issue de ce travail, à paraître dans les *Annales de la Faculté des Sciences de Toulouse*. Il s'agit d'un travail en collaboration avec Pierre Py.

Soit $f : X \to S$ un pinceau irrationnel avec $\dim_{\mathbb{C}} X = n \geq 2$. Nous supposons que les points critiques de f sont isolés et que f n'est pas une submersion; son ensemble de points critiques est alors non vide. Soit $\widehat{X} \to X$ le revêtement de X tel que $\pi_1(\widehat{X}) \simeq \ker(f_*)$. Les résultats principaux sont les suivants :

Théorème F. *Le groupe d'homologie* $H_n(\widehat{X}, \mathbb{Q})$ *est de dimension infinie.*

Théorème G. *Si* X *est asphérique, le groupe* $H_n(\ker(f_*), \mathbb{Q})$ *est de dimension infinie. En particulier* $\ker(f_*)$ *n'est pas de type* FP_n*.*

Dans le cas particulier où X est un produit direct de surfaces de Riemann et que S est de genre 1, nous retrouvons à partir du Théorème G le second résultat de Dimca, Papadima et Suciu cité précédemment. Il est intéressant de chercher d'autres exemples de variétés projectives (ou kählériennes fermées) munies d'une fonction holomorphe à laquelle on peut appliquer les Théorèmes F et G. Une façon de construire de nouveaux exemples de groupes kählériens ayant des propriétés de finitude exotiques est d'utiliser *la surface de Cartwright-Steger* (voir [23, 24]) pour une définition et des propriétés de cette surface kählérienne compacte) que nous désignerons par Y. La fonction d'Albanese de cette surface complexe $h : Y \to E$ est une fonction holomorphe dont le but est une courbe elliptique. Il a été prouvé que ses singularités sont isolées (voir [23]) et non dégénérées (voir [46] et [61]). On peut donc considérer le produit Y^b de Y avec lui-même b fois et la fonction

$$h + \cdots + h : Y^b \to E. \tag{2}$$

Cela fournit des exemples naturels auxquels on peut appliquer les Théorèmes F et G. En désignant par

$$\Gamma < \mathrm{PU}(2,1)$$

le groupe fondamental de la surface de Cartwright-Steger, la dernière construction avec le Théorème G implique immédiatement :

Théorème H. *Le produit direct de* b *copies de* Γ *contient un sous-groupe normal coabélien* N *qui est de type* FP_{2b-1} *mais qui satisfait que* $H_{2b}(N, \mathbb{Q})$ *est de dimension infinie.*

Le groupe N apparaissant ci-dessus, est le noyau du morphisme au niveau des groupes fondamentaux induit par la fonction (2). Le fait que N soit de type FP_{2b-1} découle des résultats de [31]; nous expliquerons à nouveau ce fait dans ce texte. En particulier, le Théorème H implique que N n'est pas de type FP_{2b}. Dans le Chapitre 2, nous prouvons qu'aucun sous-groupe d'indice fini de N ne se plonge dans un produit direct de groupes de surface. Ceci implique que les groupes construits de cette manière sont d'une nature différente de celle des exemples construits dans [31, 49, 50].

Introduction

A *Kähler manifold* is a complex manifold endowed with a Hermitian metric whose imaginary part is a closed 2-form. A group is called a *Kähler group* if it can be realized as the fundamental group of a compact Kähler manifold. We refer the reader to [2] for a classical reference on this subject and to [18] for a more recent survey.

There are many topological restrictions on compact Kähler manifolds. For instance, Hodge Theory enables to endow the first cohomology group of a compact Kähler manifold with coefficients in $\mathbb{R}$ with a complex structure (see Chapter 6 of [75]). This implies that the rank of the Abelianization of a Kähler group must be even. Since any finite index subgroup of a Kähler group is again Kähler, the latter assertion must hold as well for any finite index subgroup of a Kähler group. One can see in this way that a free group is never Kähler since for such a group one can always find a finite index subgroup whose Abelianization has odd rank. Let us see some examples of Kähler groups.

• The fundamental group of a closed Riemann surface is a Kähler group since any 2-form on such a space is closed.

• For any positive integer n, the group $\mathbb{Z}^{2n}$ is Kähler since it is the fundamental group of the complex torus $\mathbb{C}^n/\mathbb{Z}^{2n}$. In this case, any positive-definite Hermitian form on $\mathbb{C}^n$ induces a Kähler metric on $\mathbb{C}^n/\mathbb{Z}^{2n}$.

• The complex projective space $\mathbb{CP}^n$ endowed with the Fubini-Study metric is a Kähler manifold. Therefore, any complex submanifold of $\mathbb{CP}^n$ is endowed with a Kähler structure. The fundamental groups of such manifolds are called *projective groups*; they are in particular Kähler groups[1].

• In [67], Serre proved that all finite groups are projective groups and therefore Kähler groups.

• The direct product of Kähler groups and finite index subgroups of Kähler groups are again Kähler groups. This follows from the fact that the product metric of Kähler metrics and the pullback metric under a holomorphic local diffeomorphism of a Kähler metric are again Kähler metrics.

• Hermitian symmetric spaces and their quotients by torsion-free discrete subgroups of their group of holomorphic isometries, are Kähler manifolds. Then, torsion-free uniform lattices of holomorphic isometries of Hermitian symmetric spaces are Kähler groups. A trick due to Kollár allows to extend this result to uniform lattices with non-trivial torsion (see [3] for a proof of the latter assertion). A work of Toledo [72] shows that many non-uniform lattices of holomorphic isometries of Hermitian symmetric spaces are also Kähler groups.

There are two types of results in the study of Kähler groups: there are some negative results which say that certain families of groups do not contain Kähler groups. Let us mention (among many other such results) the following: Carlson and Toledo proved in [22] that a lattice in the isometry group of the real hyperbolic space of dimension at least 3

[1] An important question is whether these two classes of groups coincide (see [27]).

is never Kähler. Delzant (following certain results of Arapura-Nori [5], Brudnyi [17] and Campana [19]) proved in [29] that a solvable Kähler group must be virtually nilpotent. By studying the L_2 cohomology of Kähler manifolds and Kähler groups, Gromov proved in [38] that an infinite Kähler group has one end. A more detailed proof of the latter result can be found in the work of Arapura, Bressler and Ramachandran [4]. On the other hand there are positive results, *i.e.* some constructions of Kähler groups with interesting properties. Let us mention the existence of non-residually finite Kähler groups [73] or of non-virtually Abelian nilpotent Kähler groups [20]. In this thesis we will contribute to these two lines of research.

1 General context of this work

We assume that we have a short exact sequence

$$1 \longrightarrow G \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1, \tag{1}$$

where Γ is a Kähler group and G is a finitely generated group. Given such a short exact sequence we can ask:

Question 1. *What can be said about the conjugation action of Γ on G ?*

We will restrict our study to short exact sequences as in (1) where G and Q are infinite groups. The conjugation action of Γ on G induces a group homomorphism from Γ to the group of automorphisms of G

$$\begin{aligned} \Gamma &\longrightarrow \mathrm{Aut}(G) \\ \gamma &\mapsto (x \mapsto \gamma x \gamma^{-1}). \end{aligned}$$

Recall that the *group of inner automorphisms of* G, denoted by $\mathrm{Inn}(G)$, is composed of the automorphisms of G induced by the conjugation by an element of G, *i.e.* automorphisms of the form $x \mapsto gxg^{-1}$ for some g in G. The group of inner automorphisms is normal in $\mathrm{Aut}(G)$ and the quotient group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ denoted by $\mathrm{Out}(G)$ is called the *outer automorphism group of* G. A short exact sequence as in (1) induces a group homomorphism $\Gamma \to \mathrm{Out}(G)$ that factors through Q, *i.e.* the following diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & \mathrm{Out}(G) \\ & \searrow \quad \nearrow & \\ & Q & \end{array}$$

commutes. The homomorphism $Q \to \mathrm{Out}(G)$ obtained in this way is called the *monodromy* of the short exact sequence (1).

The main examples of short exact sequences as in (1) come from complex geometry. For instance, a surjective holomorphic map with connected fibers $f : X \to S$ between a compact complex manifold X and a closed Riemann surface S of positive genus, induces a short exact sequence

$$1 \longrightarrow G \longrightarrow \pi_1(X) \xrightarrow{f_*} \pi_1^{orb}(S) \longrightarrow 1,$$

where G is the image of the fundamental group of the generic fiber in $\pi_1(X)$ and S is endowed with an orbifold structure that considers the multiplicities of singular fibers (see

Section 1.4.1). In [31], Dimca, Papadima and Suciu studied some finiteness properties of the fundamental group of a generic fiber of a holomorphic map as before, in the particular case when X is a direct product of Riemann surfaces of genus greater or equal than two and S is of genus 1. This study on finiteness properties of fundamental groups of generic fibers was continued by Llosa Isenrich [49, 50] and by Bridson and Llosa Isenrich [14]. They construct new Kähler groups with interesting finiteness properties. Then one has the following question.

Question 2. *For a surjective holomorphic map with connected fibers $f : X \to S$ between a compact complex manifold X and a closed Riemann surface S of positive genus, what can be said of the finiteness properties of the fundamental group of its generic fiber ?*

We will be interested in the case when X is a compact Kähler manifold. This work will give partial answers to Questions 1 and 2.

2 Restrictions on monodromy

A classical problem is the study of the monodromy: given a proper holomorphic submersion $f : X \to Y$ (which is not a locally trivial holomorphic fiber bundle) with fibers of dimension 1, one can study the monodromy $\pi_1(Y) \to \mathrm{MCG}(S)$, where S is the generic fiber of f, which is a topological surface and $\mathrm{MCG}(S)$ is the mapping class group of S. In [68], Shiga proved the following result.

Theorem (Shiga). *Let $f : X \to B$ be a holomorphic submersion which is not a locally trivial holomorphic fiber bundle from a compact complex surface onto a hyperbolic surface with generic fiber a closed hyperbolic surface S. Then the monodromy $\pi_1(B) \to \mathrm{MCG}(S)$ has infinite image and it cannot preserve any isotopy class of a simple closed curve in S.*

One of the results that motivated the work presented in Chapter 1, is the following result of Bregman and Zhang (see [15]).

Theorem (Bregman-Zhang). *Let S be a closed surface of genus $g \geq 2$, Γ be a Kähler group and k be a positive integer such that there is a short exact sequence*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \xrightarrow{P} \mathbb{Z}^k \longrightarrow 1.$$

Then there is a finite index subgroup Γ_1 of Γ containing $\pi_1(S)$ such that the restricted short exact sequence

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma_1 \xrightarrow{P} P(\Gamma_1) \simeq \mathbb{Z}^k \longrightarrow 1$$

splits as a direct product.

Here we will study short exact sequences as in (1), where G is the fundamental group of a closed oriented surface S of genus $g \geq 2$ and without assuming that S is the fiber of a holomorphic submersion. One of the keys of this study is the fact any element of $\mathrm{Out}(\pi_1(S))$ can be represented by a diffeomorphism of the surface S (see Dehn [28] for the original proof and Nielsen [57] for the first published proof). We will also give some restrictions on short exact sequences as in (1) where G splits as an amalgamated product or an HNN extension.

Presentation of results on Kähler groups and finitely generated groups acting on trees

Let us present the results of the preprint [56]. Our main assumption is that a Kähler group Γ admits as a normal subgroup a finitely generated group G acting on a tree T. In many cases we will see that the action of G on T extends to Γ, which will allow us to apply the following result of Gromov and Schoen (see [37]).

Theorem (Gromov-Schoen)**.** *Let X be a compact Kähler manifold whose fundamental group Γ acts on a tree which is not isomorphic to a line nor a point. Suppose that the action is minimal with no fixed points on the boundary. Then there is a surjective holomorphic map with connected fibers from X to a closed hyperbolic orbifold Σ inducing the short exact sequence*

$$1 \longrightarrow N \longrightarrow \Gamma \xrightarrow{\Pi} \pi_1^{orb}(\Sigma) \longrightarrow 1,$$

such that the restriction of the action to N is trivial.

See Section 1.4.1 for the definition of the orbifold fundamental group $\pi_1^{orb}(\Sigma)$. The problem of extending the action of G on a tree to the group $\mathrm{Aut}(G)$ has been studied in the literature. In several cases, one can prove extension results. This is the case if the group G is of one of the following forms:

1. A free products $A * B$ with A and B indecomposable groups, not infinite cyclic, acting on their Bass-Serre tree.

2. The Baumslag-Solitar group $\langle x, t | t x^p t^{-1} = x^q \rangle$, acting on its Bass-Serre tree, where p, q are integers with $p, q > 1$ and such that neither is a multiple of the other (see Gilbert, Howie, Metaftsis and Raptis [35] and Pettet [59]).

3. A one-ended hyperbolic groups without torsion, with infinite outer automorphism group and which is not virtually a surface group, acting on their JSJ tree. (Sela [63], see also Bowditch [11]).

In order to apply these results to the study of Kähler groups, let us assume that the group G is centerless, the tree T is not a line nor a point and that the action of G on T is minimal, faithful and without fixed points on the boundary of T. Let $\overline{\Gamma}$ be the image of the morphism $\Gamma \to \mathrm{Aut}(G)$ induced by the conjugation action of Γ on G. Under these conditions we prove the following result in Chapter 1.

Theorem A. *Suppose that there is a finite index subgroup $\overline{\Gamma}_0$ of $\overline{\Gamma}$ containing $G \simeq \mathrm{Inn}(G)$ such that the action of G on T can be extended to a G-compatible action of $\overline{\Gamma}_0$ on T. Then G is virtually a surface group. Moreover, there is a finite index subgroup Γ_1 of Γ containing G such that the restricted short exact sequence*

$$1 \longrightarrow G \longrightarrow \Gamma_1 \xrightarrow{P} P(\Gamma_1) \longrightarrow 1$$

splits as a direct product.

See Section 1.3.1.b for the definition of G-compatible action. The first application of this result is the following.

Theorem B. *Let S be a closed surface of genus $g \geq 2$ and Γ be a Kähler group such that there is a short exact sequence*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1.$$

Suppose that the conjugation action of Γ on $\pi_1(S)$ preserves the conjugacy class of a simple closed curve in S. Then there is a finite index subgroup Γ_1 of Γ containing $\pi_1(S)$ such that the restricted short exact sequence

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma_1 \longrightarrow Q_1 \longrightarrow 1$$

splits as a direct product (where Q_1 is the image of Γ_1 in Q). In particular the monodromy $Q \to \mathrm{Out}(\pi_1(S))$ is finite.

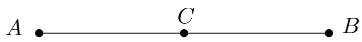
This result can be seen as a topological version of Shiga's result. We will also see that we can recover the result by Bregman and Zhang by using classical results on the mapping class group and the hyperbolization of three manifolds. More applications are obtained.

Theorem C. *Let G be a group that splits as a non-trivial free product $A * B$ with A and B indecomposable, not infinite cyclic. Then G does not embed as a normal subgroup in a Kähler group.*

Remark. *For the Bass-Serre tree of a free product $G = A * B * C$, there are automorphisms of G which cannot be extended. For instance if β is a fixed element of B, the automorphism such that*

$$a \mapsto a, \quad b \mapsto b, \quad \text{and} \quad c \mapsto \beta c \beta^{-1}$$

for all $a \in A$, $b \in B$ and $c \in C$ (called automorphism of Fouxe-Rabinovitch), does not extend to the Bass-Serre tree whose corresponding graph of groups is:



This implies that our proof of Theorem C does not extend to a free product of at least three factors.

Theorem D. *Let G be the Baumslag-Solitar group*

$$\langle x, t | t x^p t^{-1} = x^q \rangle,$$

where p, q are integers with $p, q > 1$ and such that neither one is a multiple of the other. Then G does not embed as a normal subgroup in a Kähler group.

Theorem E. *Suppose that G is a one-ended torsion-free hyperbolic group that embeds as a normal subgroup in a Kähler group Γ. If $\mathrm{Out}(G)$ is infinite, then G is virtually a surface group.*

We observe that in this last theorem there is no hypothesis on the monodromy map. We only need to know that $\mathrm{Out}(G)$ is infinite.

3 Kähler groups and finiteness properties

Let us now present some results on finiteness properties of Kähler groups. A detailed introduction to finiteness properties will be given in Section 2.2. In particular, the finiteness properties FP_n and $\mathscr{F}_n$ will be defined there.

Let X be a connected compact complex manifold of complex dimension $n \geq 2$ and let S be a closed Riemann surface of positive genus. A surjective holomorphic map with connected fibers $f : X \to S$ is called an *irrational pencil*. In [43], Kapovich proved the following result.

Theorem (Kapovich). *Let X be an aspherical compact Kähler surface, S be a closed Riemann surface of positive genus and $f : X \to S$ an irrational pencil whose singular fibers have multiplicity one. Then f is a submersion or the kernel of the induced map on fundamental groups $f_* : \pi_1(X) \to \pi_1(S)$, is not finitely presented.*

This was the first result relating the existence of critical points of an irrational pencil to the finiteness properties of the kernel of the induced map on fundamental groups. As a consequence of this result, Kapovich proved the existence of compact complex hyperbolic surfaces whose fundamental groups admit a finitely generated normal subgroup which is not finitely presentable. In other words, Kapovich gave the first examples of noncoherent uniform lattices in $PU(2, 1)$. Recall that a group is called *coherent* if every finitely generated subgroup is also finitely presentable. Further examples of noncoherent Kähler groups were given by Kapovich [42] and Py [60]. One of the main ingredients for these examples is the following result essentially due to Kapovich (see [42] and [60]), which follows by combining Theorem 4 in [60] (due to Kapovich) with the result of Bregman and Zhang quoted earlier.

Theorem. *Let Γ be a Kähler group that can be realized as the fundamental group of an aspherical compact Kähler surface X with positive first Betti number. Assume that Γ has no finitely generated Abelian subgroup whose normalizer has finite index in Γ. Then, at least one of the following cases occurs:*

1. *Γ is noncoherent.*
2. *There is a finite cover of X that admits a holomorphic submersion onto a compact hyperbolic surface with connected hyperbolic fibers.*

Friedl and Vidussi give a refinement of these results about coherence of Kähler groups in [33]. Besides studying the coherence of certain groups, one can also study the existence of non-finitely generated normal subgroups. For instance, recall that a normal subgroup of a non-Abelian free group or a surface group is finitely generated only if it has finite index (see [62] for the original proof in the case of free groups and Lemmas 3.3 and 3.4 in [25]). In general, non-finitely generated normal coabelian subgroups of Kähler groups are related to fibrations on Riemann surfaces. By studying the Bieri-Neumann-Strebel invariant of Kähler groups, Delzant proved in [29] the following result.

Theorem (Delzant). *Let us consider the short exact sequence*

$$1 \longrightarrow G \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1,$$

where Γ is a Kähler group, Q an Abelian group and G is not finitely generated. Then, if X is a compact Kähler manifold that realizes Γ as its fundamental group, there exists a surjective holomorphic map with connected fibers from X to a 2-dimensional hyperbolic orbifold.

We refer the reader to [29] for the definitions of a 2-dimensional hyperbolic orbifold and a holomorphic map from a compact complex manifold to such a space. When $Q = \mathbb{Z}$, the above theorem was previously established by Napier and Ramachandran in [54] using different techniques.

Let us introduce some recent results on the construction of Kähler groups with exotic finiteness properties and the context that motivated these constructions. In [45, §0.3.1], Kollár asked whether a projective group is always commensurable (up to finite kernels) to a group admitting a classifying space which is a quasi-projective variety. Since any quasi-projective variety has the homotopy type of a finite complex [30, p. 27], a positive answer to this question would imply that any projective group is commensurable to a group having a finite classifying space (see Section 2.2.1 for the definition of commensurability). However, a negative answer to Kollár's question was given by Dimca, Papadima and Suciu in [31]. In [31] the authors proved the following two results (see Theorem C and §2 in [31]).

Theorem (Dimca, Papadima and Suciu)**.** *If $n \geq 3$ and if $f : X \to S$ is an irrational pencil with isolated critical points, then the fundamental group of a smooth fiber of f embeds into that of X and coincides with the kernel of the induced homomorphism $f_* : \pi_1(X) \to \pi_1(S)$.*

Notice that under the hypothesis of the latter result, if we assume that X is Kähler, one obtains a finitely generated normal subgroup of the fundamental group of X, which is again a Kähler group.

Theorem (Dimca, Papadima and Suciu)**.** *Let $X = \Sigma_1 \times \cdots \times \Sigma_n$ be a direct product of n Riemann surfaces of genus greater than* 1 *and let S have genus* 1*. If $f : X \to S$ is an irrational pencil with isolated critical points then $H_n(\ker(f_*), \mathbb{Q})$ has infinite dimension.*

Combining the latter two results, Dimca, Papadima and Suciu answered negatively Kollár's question. Indeed in the situation of the latter result, the kernel of the induced homomorphism $f_* : \pi_1(X) \to \pi_1(S)$, which is the fundamental group of a smooth fiber of f if $n \geq 3$, cannot be of type FP_n as the group $H_n(\ker(f_*), \mathbb{Z})$ is not finitely generated (see Proposition 87). The property of being of type FP_n is invariant by the commensurability relation [7, 31]. Therefore, no group commensurable to the kernel of f_* can have a finite classifying space.

Building on the work [31], further examples of projective groups with exotic finiteness properties were constructed and studied by Llosa Isenrich [49, 50] and by Bridson and Llosa Isenrich [14]. All the examples studied in [14, 49, 50] are either subgroups of direct products of surface groups or extensions of such subgroups as in [14].

Presentation of results on Kähler groups with exotic finiteness properties

Here we present our results on the construction of new Kähler groups with exotic finiteness properties. See [55] for a preprint of this work which will appear in the *Annales de la Faculté des Sciences de Toulouse*. This is a joint work with Pierre Py.

Let $f : X \to S$ be an irrational pencil with $\dim_{\mathbb{C}} X = n \geq 2$. We assume that the critical points of f are isolated and that f is not a submersion; its critical set is then a nonempty finite set. Let $\widehat{X} \to X$ be the covering space of X such that $\pi_1(\widehat{X}) \simeq \ker(f_*)$. The main results are the following:

Theorem F. *The homology group $H_n(\widehat{X}, \mathbb{Q})$ has infinite dimension.*

Theorem G. *If X is aspherical, the group $H_n(\ker(f_*), \mathbb{Q})$ has infinite dimension. In particular $\ker(f_*)$ is not of type FP_n.*

In the special case when X is a product of Riemann surfaces and S has genus 1, we recover from Theorem G the second result of Dimca, Papadima and Suciu quoted before. It is of course interesting to look for more examples of projective (or closed Kähler) manifolds endowed with a holomorphic map to which one can apply Theorems F and G. One way to build new examples of Kähler groups satisfying an exotic finiteness property is to use the *Cartwright-Steger surface* (we refer to [23, 24] for a definition and properties of this compact Kähler surface) that we will denote by Y. The Albanese map of this complex surface $h : Y \to E$ is a holomorphic map whose target is an elliptic curve and it has been proven that its singularities are isolated (see [23]) and nondegenerate (see [46] and [61]). We can thus consider the product Y^b of Y with itself b times and the map

$$h + \cdots + h : Y^b \to E. \tag{2}$$

This provides natural examples to which one can apply Theorems F and G. Denoting by

$$\Gamma < \mathrm{PU}(2,1)$$

the fundamental group of the Cartwright-Steger surface, the latter construction together with Theorem G immediately implies:

Theorem H. *The direct product of b copies of Γ contains a coabelian normal subgroup N which is of type FP_{2b-1} but satisfies that $H_{2b}(N,\mathbb{Q})$ has infinite dimension.*

The group N appearing above, is the kernel of the morphism on fundamental groups induced by the map (2). The fact that N is of type FP_{2b-1} follows from the results in [31]; we will explain it again in this text. In particular, Theorem H implies that N is not of type FP_{2b}. In Chapter 2, we prove that no finite index subgroup of N, embeds in a direct product of surface groups. This implies that the groups constructed in this way are of a different nature compared to the examples from [31, 49, 50].

Chapter 1

Kähler groups acting on trees

In this chapter we prove that if a surface group embeds as a normal subgroup in a Kähler group and the conjugation action of the Kähler group on the surface group preserves the conjugacy class of a non-trivial element, then the Kähler group is virtually given by a direct product, where one factor is a surface group. As explained in the introduction, this can be seen as a generalization of a result due to Shiga [68]. Moreover we prove that if a one-ended hyperbolic group with infinite outer automorphism group embeds as a normal subgroup in a Kähler group then it is virtually a surface group. More generally we give restrictions on normal subgroups of Kähler groups which are amalgamated products or HNN extensions.

1.1 Introduction

Let G be a finitely generated group. Bass-Serre Theory (see [65] and [66]) establishes a dictionary between decompositions of G as an amalgamated product or an HNN extension and actions of G on a simplicial tree without inversions which are transitive on the set of edges. Of course, the theory also deals with more complicated graphs of groups but we will mainly deal with amalgamated products and HNN extensions.

Question. *Given a short exact sequence of finitely generated groups*

$$1 \longrightarrow G \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1, \tag{1.1.1}$$

where G acts on a tree T, under which conditions can we (virtually) extend this action to Γ ?

If the center of G acts trivially on T, we get an induced action of the group of inner automorphisms of G on T, and then this question can be approached by looking at the inclusion $\mathrm{Inn}(G) \hookrightarrow \mathrm{Aut}(G)$ and by trying to extend the action on T to a larger subgroup of $\mathrm{Aut}(G)$. Several authors have studied the latter question. Karrass, Pietrowski and Solitar [44] studied the case of an amalgamated product $A *_C B$, where C is maximal among all its conjugates in A and B. Pettet [59] studied the general situation of a graph of groups with a more restrictive condition on the edge stabilizers of its Bass-Serre tree (*"edge group incomparability hypothesis"*) which is equivalent to the *conjugate maximal condition* of Karrass, Pietrowski and Solitar when there is one orbit of edges. A particular case of this situation was studied by Gilbert, Howie, Metaftsis and Raptis [35] where they proved that the action of the Baumslag-Solitar group on its Bass-Serre tree can be extended to the whole group of automorphisms. In the context of a one-ended torsion-free hyperbolic group

G, Sela [63] proved the existence of a "canonical" tree T on which G acts named the JSJ tree. This means that $\mathrm{Aut}(G)$ contains a subgroup of finite index such that the action of G on T extends to such subgroup. By studying the action of $\mathrm{Aut}(G)$ on ∂G, Bowditch [11] constructed a finer tree T_B such that the action of G on T_B can be extended to $\mathrm{Aut}(G)$.

We will apply these results to study finitely generated normal subgroups of Kähler groups. The main ingredient to apply these results about actions on trees that extend to a group of automorphisms is a classical result of Gromov and Schoen [37] about Kähler groups acting on trees (see Theorem 47).

For the reader's convenience, we will state again four theorems from the introduction that will be proved in this chapter. We will study short exact sequences as in (1.1.1), where Γ is a Kähler group and G is a surface group, *i.e.*, when G can be realized as the fundamental group of a closed surface S of genus $g \geq 2$.

Theorem B. *Suppose that the conjugation action of Γ on $\pi_1(S)$ preserves the conjugacy class of a simple closed curve in S. Then there is a finite index subgroup Γ_1 of Γ containing $\pi_1(S)$ such that the restricted short exact sequence*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma_1 \longrightarrow Q_1 \longrightarrow 1$$

splits as a direct product (where Q_1 is the image of Γ_1 in Q). In particular the monodromy $Q \to \mathrm{Out}(\pi_1(S))$ is finite.

In Section 1.5 we will see that using Theorem B one can recover Bregman and Zhang's result on Kähler extensions of Abelian groups by surface groups and that this result can be extended to Kähler extensions by surface groups whose monodromy maps have Abelian image (see Corollaries 54 and 59).

Let us go back to the study of extensions as in (1.1.1) where G need not be a surface group. The folllowing three results are an application of Gromov and Schoen's theorem and the works [11, 35, 44, 59, 63]. Recall that a group H is indecomposable if for any decomposition $H = H_1 * H_2$ as a free product, H_1 or H_2 is trivial.

Theorem C. *Let G be a group that splits as a non-trivial free product $A * B$ with A and B indecomposable, not infinite cyclic. Then G does not embed as a normal subgroup in a Kähler group.*

Theorem D. *Let G be the Baumslag-Solitar group*

$$\langle x, t | t x^p t^{-1} = x^q \rangle,$$

where p, q are integers with $p, q > 1$ and such that neither one is a multiple of the other. Then G does not embed as a normal subgroup in a Kähler group.

Finally, our last result of this chapter deals with the case where the normal subgroup G is a one-ended hyperbolic group.

Theorem E. *Suppose that G is a one-ended torsion-free hyperbolic group that embeds as a normal subgroup in a Kähler group Γ. If $\mathrm{Out}(G)$ is infinite, then G is virtually a surface group.*

We observe that in this last theorem there is no hypothesis on the monodromy map $\Gamma \to \mathrm{Out}(G)$. We only need to know that $\mathrm{Out}(G)$ is infinite.

The structure of this chapter is the following. In Section 1.2, we give the definitions of an amalgamated product and HNN extensions and we recall some basic aspects of Bass-Serre Theory. We study as well the particular case of decompositions of surface groups over maximal cyclic subgroups and the Bass-Serre trees associated to these decompositions. In Section 1.3, we explain how to extend the action of a group on a tree to a subgroup of its group of automorphisms. This section is mainly expository. It gives a more geometric approach to the results of Karras, Pietrowski, Solitar and Pettet. It is only Section 1.3.4 that deals with surface groups, that we will use to prove Theorem B (in Section 1.4). Hence the reader only interested in the proof of Theorems A B, C, D and E can skip Sections 1.3.1, 1.3.2 and 1.3.3. In Section 1.4, we apply these results to the study of Kähler groups admitting as a normal subgroup a group acting on a tree. Theorem A is proved in Section 1.4.1 and Theorems B, C, D and E are proved in Section 1.4.2. Let us summarize which results about extensions of group actions are used in each proof.
• To prove Theorem B, we use the results we establish in Section 1.3.4.
• To prove Theorem C we can use either Karras, Pietrowski and Solitar's work (Theorem 40) or a special case of it that we reprove here (Theorem 28 in Section 1.3.2).
• To prove Theorem D we apply Gilbert, Howie, Metaftsis and Raptis's work (Theorem 43).
• Finally, to prove Theorem E we apply directly the work of Bowditch and Sela (Theorem 45).
In Section 1.5 we study the monodromy map of a Kähler extension by a surface group and establish variations on Theorem B.

1.2 Groups acting on trees

In this section we introduce the definitions of an amalgamated product and an HNN extension and we state the principal results of Bass-Serre Theory for such groups. We refer the reader to [65] and [66] for an introduction to this subject. We will also describe the decompositions of a surface group as an amalgamated product or as an HNN extension over a cyclic group (corresponding to a simple closed curve) and we will give a geometric construction of the Bass-Serre tree for such groups as a dual tree embedded in the hyperbolic plane.

1.2.1 Amalgamated products and HNN extensions

1.2.1.a Definitions

The *free product* of two groups A and B, denoted by $A * B$ can be described as the set of words

$$g_1 g_2 \cdots g_n,$$

where $g_i \in A \setminus \{1\}$ or $g_i \in B \setminus \{1\}$ for all $i = 1, \ldots, n$ and such that if $g_i \in A$, then $g_{i+1} \in B$ and if $g_i \in B$, then $g_{i+1} \in A$. The operation of this group is given by the juxtaposition of words (up to simplification) and the empty word represents the identity element. Given presentations of A and B

$$A = \langle X \mid R \rangle \quad \text{and} \quad B = \langle Y \mid S \rangle, \tag{1.2.1}$$

we obtain that the free product $A * B$ is the group given by the presentation

$$\langle X \sqcup Y \mid R \sqcup S \rangle.$$

Now, suppose that there exists a common subgroup C of A and B and let $i_A : C \hookrightarrow A$ $i_B : C \hookrightarrow B$ be the respective inclusions. Let N denote the normal closure in $A * B$ of the set of elements

$$\{i_A(c)i_B^{-1}(c) \mid c \in C\}.$$

Then, the *amalgamated product* of A and B along C denoted by $A *_C B$ is defined as the quotient $A * B/N$. Given presentations of A and B as in (1.2.1), we get that the amalgamated product $A *_C B$ is given by

$$\langle X \sqcup Y \mid R \sqcup S,\, i_A(c)i_B^{-1}(c)\ \forall c \in C\rangle.$$

Notice that this construction is interesting only if C is properly contained in A and B, otherwise the amalgamated product is isomorphic to A or B.

Using the general construction of amalgams given by Serre in [66], one can prove the non-triviality of groups defined by certain presentations. An instance of this is the following result of Higman, B.H. Neumann and H. Neumann (see [66, p. 8]).

Proposition 3. *Let C be a subgroup of a group A and let $\theta : C \to A$ be an injective homomorphism. Then, there exists a group G containing A and an element t of $G \setminus A$ such that $tct^{-1} = \theta(c)$ for all c in C. Furthermore, if A is countable (or finitely generated, or torsion-free) one can choose G to be a group with the same property.*

In the latter proposition, the subgroup generated by A and t is called *HNN extension* of A relative to θ and it is denoted by $A*_{C,\theta}$. Given a presentation of A as in (1.2.1) and an injective homomorphism $\theta : C \to A$ as in Proposition 3, we get that the HNN extension $A*_{C,\theta}$ is given by

$$\langle X, t \mid R,\, tct^{-1}\theta(c^{-1})\ \forall c \in C\rangle.$$

A priori, a group given by the latter presentation could be trivial, but Proposition 3 guarantees the non-triviality of such a group.

Definition 4. We say that a group G *splits over a group* C if it decomposes as an amalgamated product $A *_C B$ or as an HNN extension $A*_{C,\theta}$.

1.2.1.b Examples

• If A and B are the free groups of rank n and m respectively, then the free product $A * B$ is the free group of rank $n + m$.

• More interesting examples coming from topology are given by Van Kampen's Theorem. If X is the union of two open sets U and V whose intersection is path connected, then the fundamental group of X is given by the amalgamated product of the fundamental groups of U and V along the fundamental group of $U \cap V$ (where all the fundamental groups are based at a point in $U \cap V$), if the inclusions $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$ induce injections on fundamental group.

• Given a group A and an automorphism $\theta : A \to A$, the HNN extension $A*_{A,\theta}$ coincides with the semidirect product $A \rtimes \mathbb{Z}$: the morphism $A*_{A,\theta} \to \mathbb{Z}$ that sends t to the generator of $\mathbb{Z}$ and all the elements of A to zero, has the group A as its kernel. Therefore we obtain a short exact sequence

$$1 \to A \to A*_{A,\theta} \to \mathbb{Z} \to 1.$$

Since the map $\mathbb{Z} \to A*_{A,\theta}$ that sends the generator of $\mathbb{Z}$ to t is a section of the latter short exact sequence, we obtain that $A*_{A,\theta}$ splits as the semidirect product $A \rtimes \mathbb{Z}$.

• Let p, q be two non-zero integers. The Baumslag-Solitar group $G(p,q)$ is defined as the HNN extension given by the presentation

$$\langle x, t \mid tx^pt^{-1} = x^q \rangle.$$

Notice that if $p = 1 = q$ we obtain the fundamental group of a torus and if $p = 1 = -q$, we obtain the fundamental group of the Klein bottle.

Proposition 3 guarantees the non-triviality of such groups: let $A = \mathbb{Z}$ and $C = p\mathbb{Z}$. If we denote by x the generator of $\mathbb{Z}$, then in multiplicative notation, x^p is the generator of $p\mathbb{Z}$. Now, let $\theta : p\mathbb{Z} \to \mathbb{Z}$ be the injective homomorphism that sends x^p to x^q (in multiplicative notation). Hence, the Baumslag-Solitar group is precisely the HNN extension $\mathbb{Z}*_{p\mathbb{Z},\theta}$, which by Proposition 3 is non-trivial.

• In Section 1.2.3, we will see that any simple closed curve in a closed surface of genus $g \geq 2$, induces a decomposition of the fundamental group of such a surface as an amalgamated product or as an HNN extension over a cyclic group.

1.2.2 Bass-Serre dictionary

Recall that a (*simplicial*) *tree* is a connected graph $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges with the additional property that for any edge e in $\mathcal{E}$, the graph $(\mathcal{V}, \mathcal{E} \setminus \{e\})$ obtained by removing the edge e from the graph $(\mathcal{V}, \mathcal{E})$, is not connected.

An *isomorphism between two trees* $T = (\mathcal{V}, \mathcal{E})$ and $T' = (\mathcal{V}', \mathcal{E}')$ is a pair of bijections

$$\varphi_V : \mathcal{V} \to \mathcal{V}' \quad \text{and} \quad \varphi_E : \mathcal{E} \to \mathcal{E}'$$

such that if $e = \{v, w\}$ is an edge of T, then

$$\{\varphi_V(v), \varphi_V(w)\} = \varphi_E(\{v, w\}).$$

The latter condition means that the bijection between the sets of vertices of the trees T and T' respect the adjacency relation.

An *automorphism of a tree* T is an isomorphism between T and itself. Given a tree T and a group G, we abbreviate the expression G acts on T by automorphisms by G acts on T.

Notations. Suppose that G acts on a tree T and let e and v be an edge and a vertex of T respectively. We will denote by G_e and G_v the subgroups of G given by the respective stabilizers of e and v.

The dictionary established by Bass-Serre theory is given by the following results (see [65, 66]).

Theorem 5 (Serre)**.** *Let G be a group acting on a tree T without inversions in such a way that the action is transitive on the set of edges. Let $e = (v_1, v_2)$ be an edge of T.*

1. *If v_1 and v_2 are in different orbits, then G splits as the amalgamated product $G_{v_1} *_{G_e} G_{v_2}$.*

2. *If v_1 and v_2 are in the same orbit, then G splits as the HNN extension $G_{v_1}*_{G_e,\theta}$ where $\theta : G_e \to G_{v_1}$ is the monomorphism given by the conjugation by an element t in $G \setminus G_{v_1}$ that sends v_1 to v_2.*

Theorem 6 (Serre)**.** *Let G be a group that splits as an amalgamated product or as an HNN extension. Then, G acts on a tree T without inversions and this action is transitive on the set of edges.*

1. *If G splits as the amalgamated product $A *_C B$, then the set of vertices of T is the disjoint union of left cosets $G/A \sqcup G/B$ and the set of edges of T is the set of left cosets G/C. The adjacency is given by the maps $G/C \to G/A$ and $G/C \to G/B$.*

2. *If G splits as the HNN extension $A*_{C,\theta}$, where $\theta : C \to A$ is a monomorphism given by the conjugation by an element t in $G \setminus A$, then the set of vertices of T is the set of left cosets G/A and the set of edges of T is the set of left cosets G/C. The adjacency is given by the maps $\iota_1 : G/C \to G/A$ and $\iota_2 : G/C \to G/A$ that send xC to xA and xC to $xt^{-1}A$ respectively.*

The tree T associated to a decomposition of a group G as in Theorem 6 is called *the Bass-Serre tree of* G. Notice that the action of G on T is induced by the natural action of G on the sets of left cosets defining the vertices and edges of T. If G decomposes as an amalgamated product, then by Theorem 6 we have that for any element x of G one can associate the edge xC of T to its pair of vertices $\{xA, xB\}$. Therefore, the edges of T sharing the vertex xA is given by

$$\{xaC \mid a \in A\}.$$

Similarly, we obtain that the edges of T joined to the vertex xB is given by

$$\{xbC \mid b \in B\},$$

(see Figure 1.2.1). If G decomposes as an HNN extension $A*_{C,\theta}$, then by Theorem 6 we

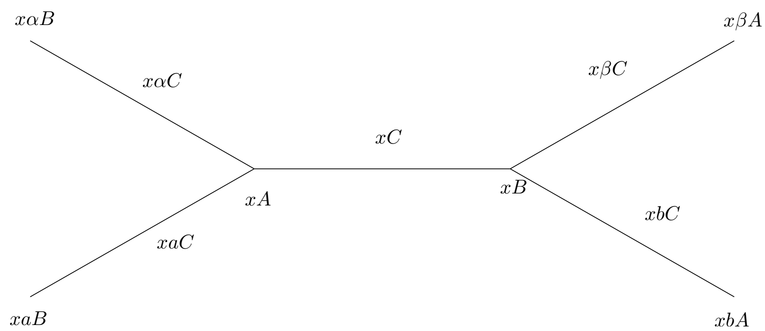


Figure 1.2.1: Bass-Serre tree of an amalgamated product $A *_C B$

have that for any element x of G one can associate the edge xC of T to its pair of vertices $\{xt^{-1}A, xA\}$. Therefore, the edges of its Bass-Serre tree T joined to the vertex xA is given by

$$\{xaC \mid a \in A\} \cup \{xatC \mid a \in A\},$$

(see Figure 1.2.2). The action of G on its Bass-Serre tree T has the following properties:

• It is minimal, *i.e.* there is no G-invariant proper subtree of T. To see this, let T' be a G-invariant subtree of T. If G decomposes as an amalgamated product $A *_C B$ and xA is a vertex of T', then by the transitivity of the action of G on G/A, we obtain that G/A is contained in T'. Then for all b in B, the path between A and bA is contained in T'. In particular, the vertex B and the edge C are contained in T'. By the transitivity of the

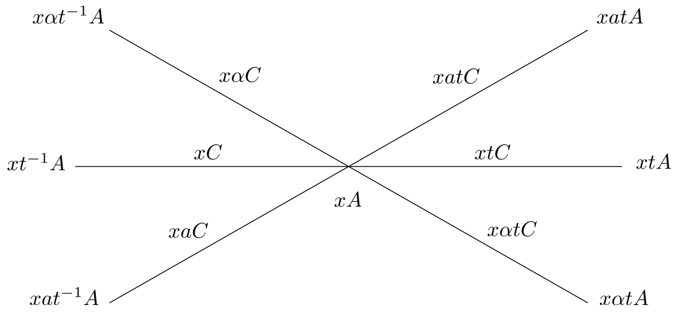


Figure 1.2.2: Bass-Serre tree of an amalgamated product $A*_{C,\theta}$

action of G on G/B and G/C we conclude that $T = T'$. The same arguments hold if G splits as an HNN extension.

• If G splits as an amalgamated product $A *_C B$ and $[A : C] > 2$ or $[B : C] > 2$, then the Bass-Serre tree of G is not a line. This follows from Theorem 6, since the number of vertices joined to A (respectively to B) is equal to $[A : C]$ (respectively to $[B : C]$).

• If G splits as an HNN extension $A*_{C,\theta}$ and $[A : C] > 1$ or $[A : \theta(C)] > 1$, then the Bass-Serre tree of G is not a line. Once again this follows from Theorem 6, since the number of vertices joined to A is equal to $[A : C] + [A : \theta(C)]$.

Recall that the boundary of a tree is given by the set of infinite paths without backtracking starting at a fixed point of the tree (this is one definition among many others).

• If G splits as an amalgamated product $A *_C B$ and C is properly contained in A and B, then the action of G on the boundary of T has no fixed points. All the infinite paths without backtracking starting at A, have as their first edge an element aC in the set of left cosets A/C. Since the action of A on A/C is transitive and C is properly contained in A, none of these paths is fixed under the action of A and thus neither under the action of G.

• If G splits as an HNN extension $A*_{C,\theta}$, and C and $\theta(C)$ are properly contained in A, then the action of G on the boundary of T has no fixed points. The argument is very similar to the case of an amalgamated product. In this case the infinite paths starting at A have as their first edge an element of the form aC or atC for some a in A, and the action of A is transitive on each of the following subsets of edges

$$\{aC \mid a \in A\} \quad \text{and} \quad \{atC \mid a \in A\}.$$

See Proposition 4.13 of [53] for a characterization of a group that splits as the fundamental group of a graph of groups whose action on the corresponding Bass-Serre tree fixes a point on the boundary

1.2.3 Decomposition of a surface group

Definition 7. A group G is called a surface group if it can be realized as the fundamental group of a closed oriented surface of genus $g \geq 2$.

Let γ be a simple closed curve in a closed oriented surface S of genus $g \geq 2$. The curve is called *separating*, if we obtain 2 connected components when we cut S along γ, and *nonseparating* if we obtain 1 connected component. Van Kampen's Theorem allows us to express the fundamental group of S in terms of the fundamental groups of the surfaces obtained after cutting S along γ.

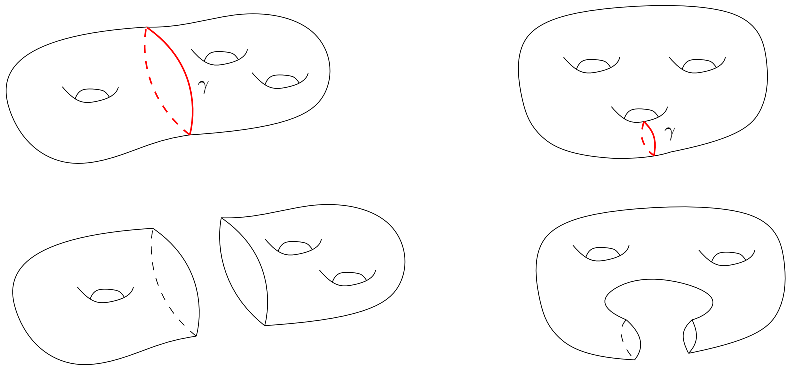


Figure 1.2.3: Cutting along a simple closed curve

1.2.3.a Cutting along a separating curve

Let γ be a separating curve in a closed surface S of genus $g \geq 2$ and let S_1 and S_2 be small neighborhoods of the surfaces obtained after cutting S along γ, such that $S_1 \cap S_2$ is a small neighborhood of γ that deformation retracts onto γ. We fix a base point x_0 on γ. Then, if we denote by A, B and C the fundamental groups based at x_0 of S_1, S_2 and $S_1 \cap S_2$ respectively, by Van Kampen's Theorem we obtain that

$$\pi_1(S, x_0) \simeq A *_C B.$$

Notice that A and B are free groups and since $S_1 \cap S_2$ deformation retracts onto γ, we get that C is a cyclic group generated by the homotopy class of γ. Hence, for each non-nullhomotopic separating curve of S we obtain a decomposition of the fundamental group of S as an amalgamated product of two free groups along a cyclic group.

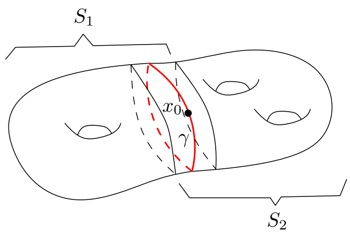


Figure 1.2.4: Surfaces S_1 and S_2

1.2.3.b Cutting along a nonseparating curve

Let γ be a nonseparating curve in a closed surface S of genus $g \geq 2$. As before, we fix a base point x_0 on γ and let τ be a loop as in Figure 1.2.5. Let S_1 be the surface obtained after cutting S along γ and let S_2 be a small neighborhood of γ and τ. Now, one can apply Van Kampen's Theorem to S_1 and S_2. For this, one must choose a "copy" of γ (containing a "copy" of x_0) in the surface S_1. Let A, B and C be the fundamental groups based at x_0 of S_1, S_2 and $S_1 \cap S_2$ respectively. Then, we obtain that

$$\pi_1(S, x_0) \simeq A *_C B.$$

We will simplify the latter expression by understanding the maps on fundamental groups

$$\pi_1(S_1 \cap S_2) \to \pi_1(S_1) \quad \text{and} \quad \pi_1(S_1 \cap S_2) \to \pi_1(S_2)$$

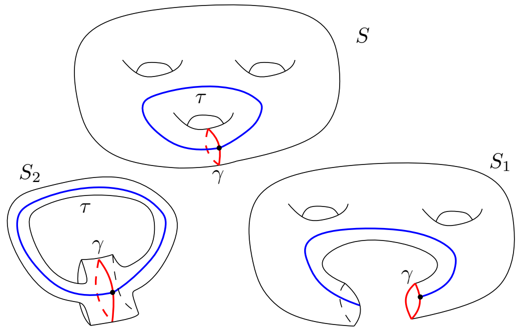


Figure 1.2.5: Surfaces S, S_1 and S_2

induced by the inclusions $S_1 \cap S_2 \hookrightarrow S_1$ and $S_1 \cap S_2 \hookrightarrow S_2$. Let $\tau_1 : [0,1] \to S_1$ be the path in S_1 obtained from τ after cutting S along γ and let σ be the loop based at $\tau_1(1)$ defined by the boundary component of S_1 which is not homotopic to γ. If S is a closed oriented surface of genus g, we obtain that S_1 is a surface of genus $g-1$ without two disks removed. Hence, one can obtain S_1 from a $4g-4$ polygon without two disks removed. Up to changing the orientation of σ, we may assume that the (oriented) boundary of the $4g-4$ polygon deformation retracts onto the concatenation $\bar{\gamma}\tau_1\bar{\sigma}\bar{\tau}_1$, where $\bar{\gamma}, \bar{\sigma}$ and $\bar{\tau}_1$ denote the inverse curves of γ, σ and τ_1 respectively (see Figure 1.2.6). Hence, if we denote by c_1 the homotopy class of γ and by c_2 the homotopy class of $\tau_1\sigma\bar{\tau}_1$, we obtain that $\pi_1(S, x_0)$ has the following presentation

$$\langle a_1, b_1, \ldots, a_{g-1}b_{g-1}, c_1, c_2 \,|\, [a_1, b_1] \cdots [a_{g-1}, b_{g-1}]c_1c_2 \rangle.$$

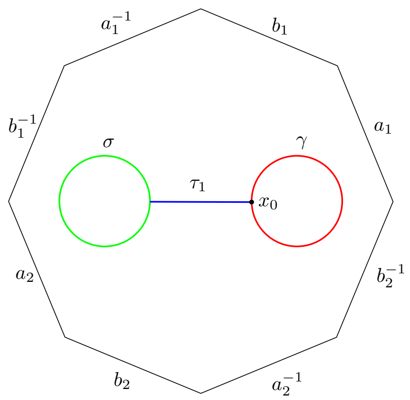


Figure 1.2.6: $4g-1$-polygon without two disks

If we denote by c and t the homotopy classes of γ and τ in S_2, we get that $\pi_1(S_2, x_0)$ is a free group of rank 2 generated by c and t, since S_2 deformation retracts onto the wedge-sum of the loops defined by γ and τ.

Finally, notice that the intersection between S_1 and S_2 is given by the union of a neighborhood of the boundary components of S_1 with a neighborhood of the path τ_1 (see Figure 1.2.7). Hence $\pi_1(S_1 \cap S_2, x_0)$ is a free group of rank 2, since $S_1 \cap S_2$ deformation retracts onto two loops joined by a path, where one of these loops is precisely γ.

From this, we deduce that the inclusions $S_1 \cap S_2 \hookrightarrow S_1$ and $S_1 \cap S_2 \hookrightarrow S_2$ identify c

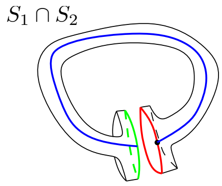


Figure 1.2.7: intersection of S_1 and S_2

with c_1 and tc_1t^{-1} with c_2^{-1}. Thus, we obtain the following presentation of $\pi_1(S, x_0)$

$$\langle a_1, b_1, \dots, a_{g-1}b_{g-1}, c_1, c_2, t \,|\, [a_1, b_1] \cdots [a_{g-1}, b_{g-1}]c_1c_2 \,,\, tc_1t^{-1}c_2\rangle.$$

If we denote by C the cyclic subgroup of $\pi_1(S_1, x_0)$ generated by c_1, we obtain that $\pi_1(S, x_0)$ is given by the HNN extension

$$\pi_1(S_1, x_0)*_{C,\theta},$$

where

$$\begin{aligned} \theta : C &\to \pi_1(S_1, x_0) \\ c_1 &\mapsto tc_1t^{-1}. \end{aligned}$$

1.2.4 Dual tree of a simple closed curve in a surface group

As we saw in Section 1.2.3, the fundamental group of a closed surface S of genus $g \geq 2$ splits over the cyclic subgroup generated by a simple closed curve. By Bass-Serre Theory, associated to such a splitting there exists a tree endowed with an action of the fundamental group of such a surface. Here we will give the construction of a dual tree associated to a simple closed curve in a closed surface of genus $g \geq 2$ and we will see that this tree coincides with the Bass-Serre tree associated to the decompositions studied in Section 1.2.3. This allows us to have a better understanding of the Bass-Serre tree for such decompositions of a surface group. This material is very classical, here we simply give a self-contained exposition of it.

Let S be a Riemann surface of genus greater than one. By the Uniformization Theorem we have that S is biholomorphic to $\mathbb{H}^2/\Gamma$, where Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. The application $\mathbb{H}^2 \to \mathbb{H}^2/\Gamma$ is its universal covering space and it induces a hyperbolic structure on S. Let us recall the classical identification of the fundamental group of S with the group $\Gamma < \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$. If we denote by $p : (\mathbb{H}^2, z_0) \to (S, x_0)$ the universal covering based map of S, then we have a map

$$\begin{aligned} \pi_1(S, x_0) &\to \mathrm{Aut}(\mathbb{H}^2) \\ [\sigma] &\mapsto \varphi_{[\sigma]}, \end{aligned} \tag{1.2.2}$$

where $\varphi_{[\sigma]}$ is the unique automorphism of $\mathbb{H}^2$ such that $\varphi_{[\sigma]}(z_0) = \widehat{\sigma}(1)$, with $\widehat{\sigma}$ a lift of σ starting at the point z_0. By the uniqueness of path liftings we have that for loops $\sigma, \tau : [0, 1] \to S$

$$\widehat{\sigma\tau}(1) = \varphi_{[\sigma]}(\widehat{\tau}(1)) = \varphi_{[\sigma]} \circ \varphi_{[\tau]}(z_0),$$

which implies that (1.2.2) is a group homomorphism, its image is the discrete group Γ and it is an isomorphism from $\pi_1(S, x_0)$ onto Γ. From now on we identify these two groups.

As in Section 1.2.3, let γ be a simple closed curve in S based at x_0. Recall that every non-nullhomotopic simple closed curve in S is homotopic to a unique closed geodesic (see

[32] p. 24). Hence, we may assume that γ is a closed geodesic for the hyperbolic structure on S.

Let $\mathcal{E}$ be the set of bi-infinite geodesics in $\mathbb{H}^2$, whose images under p are equal to γ and let $\mathcal{V}$ be the set of connected components of $\mathbb{H}^2 \setminus \cup\mathcal{E}$. We say that two elements of $\mathcal{V}$ are related if they are separated by an element of $\mathcal{E}$. In other words, V, W in $\mathcal{V}$ are related if there exists l in $\mathcal{E}$ such that $V \cup l \cup W$ is connected. This defines a symmetric binary relation on $\mathcal{V}$. Hence, $\mathcal{V}$ endowed with this relation defines a graph, where the set of edges is precisely given by $\mathcal{E}$ (see Figure 1.2.8).

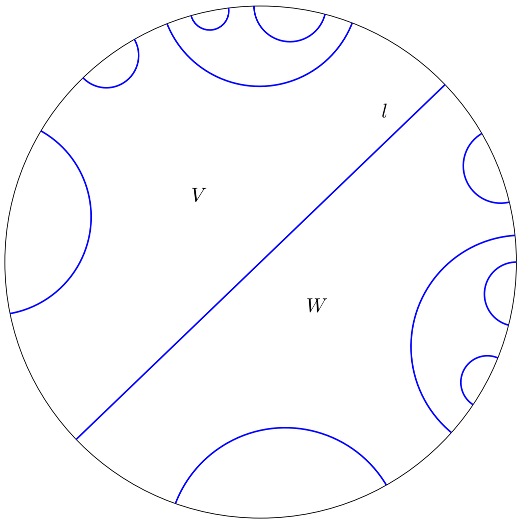


Figure 1.2.8: Connected components of $\mathbb{H}^2 \setminus \cup\mathcal{E}$ separated by an element of $\mathcal{E}$

For all x in $\pi_1(S, x_0)$, let l_x be the element of $\mathcal{E}$ passing through $\varphi_x(z_0)$. Then we get that $l_x = \varphi_x(l_1)$. This follows from the fact that $p \circ \varphi_x$ and thereby

$$p \circ \varphi_x(l_1) = p(l_1) = \gamma,$$

i.e. $\varphi_x(l_1)$ is a bi-infinite geodesic in $\mathbb{H}^2$ passing through $\varphi_x(z_0)$, whose image under p is equal to γ. Hence, the action of $\pi_1(S, x_0)$ preserves the set $\mathcal{E}$, and by continuity it preserves the set $\mathcal{V}$ with its symmetric binary relation.

A combinatorial path in the graph $(\mathcal{V}, \mathcal{E})$ between two vertices V, W of $(\mathcal{V}, \mathcal{E})$ is given by finitely many connected components $V_1, \ldots, V_{k+1}$ of $\mathbb{H}^2 \setminus \cup\mathcal{E}$, such that

- $V_1 = V$, $V_{k+1} = W$.
- For all $i = 1, \ldots, k$, $V_i \neq V_{i+1}$.
- For all $i = 1, \ldots, k$, there is a geodesic l_i such that $V_i \cup l_i \cup V_{i+1}$ is connected.

As a consequence, we get that $V_1 \cup l_1 \cup V_2 \cup \cdots \cup V_k \cup l_k \cup V_{k+1}$ is connected. This observation allows us to prove the following result

Proposition 8. *The graph* $(\mathcal{V}, \mathcal{E})$ *is a tree.*

Proof. Let l be an element of $\mathcal{E}$ defining an edge between the connected components V and W of $\mathbb{H}^2 \setminus \cup\mathcal{E}$. Then V and W are in different connected components of $\mathbb{H}^2 \setminus l$ and by the latter observation there is no combinatorial path from V to W in the graph $(\mathcal{V}, \mathcal{E} \setminus \{l\})$, which implies the result. □

The tree $(\mathcal{V}, \mathcal{E})$ is called the *dual tree of* S *associated to* γ. Now, we will see that this tree coincides with the Bass-Serre tree associated to the splitting of $\pi_1(S, x_0)$ over the cyclic subgroup generated by the homotopy class of γ, in such a way that the action of $\pi_1(S,x_0)$ on the dual tree is consistent with the action of $\pi_1(S,x_0)$ on the Bass-Serre tree.

Lemma 9. *The stabilizer of the bi-infinite geodesic l_1 under the action of $\pi_1(S,x_0)$ on $\mathbb{H}^2$ is equal to the cyclic group C generated by the homotopy class of γ.*

Proof. The cyclic group C stabilizes l_1 since the concatenation of γ (or $\bar{\gamma}$) with itself n times, lifts to a geodesic arc contained in l_1 for all positive integer n. Now, let x be an element of $\pi_1(S)$ such that

$$l_x = \varphi_x(l_1) = l_1.$$

Recall that l_x is the bi-infinite geodesic in $\mathbb{H}^2$ passing through $\varphi_x(z_0)$ whose image under p is equal to γ. If σ is a closed curve in S based at x_0 whose homotopy class is equal to x, we get that

$$\varphi_x(z_0) = \varphi_{[\sigma]}(z_0) = \widehat{\sigma}(1) \in l_1.$$

Hence, the geodesic arc contained in l_1 joining z_0 and $\varphi_{[\sigma]}(z_0)$ is homotopic to $\widehat{\sigma}$ relative to $\{0,1\}$, which implies that $[\sigma] = x$ is contained in C, and the result follows. □

As a consequence of the latter result, we obtain a bijection between the set of edges of the dual tree $(\mathcal{V},\mathcal{E})$ and the set of edges of the Bass-Serre tree T:

Corollary 10. *The map*

$$\begin{aligned} \mathcal{E} &\to \pi_1(S,x_0)/C \\ \varphi_x(l_1) = l_x &\mapsto \quad xC \end{aligned}$$

is well-defined and it is a bijection.

Proof. By Lemma 9, we get that the map $\mathcal{E} \to \pi_1(S,x_0)/C$ is well-defined and it is injective. The surjectivity of such a map follows directly by definition. □

We recall a topological result that will be useful to define the bijection between the set of vertices of the dual tree $(\mathcal{V},\mathcal{E})$ and the set of vertices of the Bass-Serre tree T.

Proposition 11. *Let Y be a connected smooth manifold and X a connected smooth submanifold of Y. Let us fix a base point x_0 in X. If we denote by $i : (X,x_0) \to (Y,x_0)$ the inclusion map and by $p : (\widehat{Y},\hat{x}_0) \to (Y,x_0)$ the universal covering space of Y, then $i_* : \pi_1(X,x_0) \to \pi_1(Y,x_0)$ is injective if and only if, the connected component of $p^{-1}(X)$ containing $\hat{x}_0$ is simply connected.*

Remark 12. *Under the hypothesis of the latter result, if we denote by $\widehat{X}$ the connected component of $p^{-1}(X)$ containing $\hat{x}_0$, we get that the lifts based at $\hat{x}_0$ of $i_*(\pi_1(X,x_0))$ are contained in $\widehat{X}$, i.e. $i_*(\pi_1(X,x_0))$ stabilizes $\widehat{X}$ under the action of $\pi_1(Y,x_0)$ on $\widehat{Y}$. Hence, if $\widehat{X}$ is simply connected, Proposition 11 allows us to consider $\pi_1(X,x_0)$ as a subgroup of $\pi_1(Y,x_0)$ that stabilizes $\widehat{X}$.*

Corollary 13. *If X is a connected component of $S \setminus \gamma$, then $\pi_1(X,x_0) \to \pi_1(S,x_0)$ is injective.*

Proof. Every connected component of $\mathbb{H}^2 \setminus \cup\mathcal{E}$ is convex (for the hyperbolic metric) since it is the intersection of a countable set of half-planes which are convex. Therefore, every connected component of $\mathbb{H}^2 \setminus \cup\mathcal{E}$ is simply connected and the result follows from Proposition 11. □

Let us fix X a connected component of $S \setminus \gamma$. By Corollary 13, we can consider $\pi_1(X, x_0)$ as a subgroup of $\pi_1(S, x_0)$. Let V_1 be the connected component of $\mathbb{H}^2 \setminus \cup\mathcal{E}$ containing all the lifts based at z_0 of the elements of $\pi_1(X, x_0)$. We denote by $\mathcal{V}_X$ the set of connected components of $\mathbb{H}^2 \setminus \cup\mathcal{E}$ whose image under p is equal to X. Observe that the action of $\pi_1(S, x_0)$ on $\mathbb{H}^2$ preserves $\mathcal{V}_X$. For all x in $\pi_1(S, x_0)$, we write

$$V_x = \varphi_x(V_1).$$

Lemma 14. *The stabilizer of V_1 under the action of $\pi_1(S, x_0)$ on $\mathbb{H}^2$ is equal to $\pi_1(X, x_0)$.*

Proof. By the proof of Corollary 13, we get that $p\restriction_{V_1} : V_1 \to X$ is the universal covering space of X and by Remark 12 we get that $\pi_1(X, x_0)$ stabilizes V_1. Now, let x be an element of $\pi_1(S, x_0)$ such that

$$V_x = \varphi_x(V_1) = V_1.$$

Then, we have that $l_x = \varphi_x(l_1)$ is a boundary component of V_1 containing $\varphi_x(z_0)$. If we denote by $\widehat{\sigma}$ the geodesic arc in $\mathbb{H}^2$ joining z_0 with $\varphi_x(z_0)$, we get that $\widehat{\sigma}$ is contained in V_1 (by the convexity of V_1.) Hence $\sigma = p \circ \widehat{\sigma}$ defines a closed curve in X such that $\varphi_x(z_0) = \varphi_{[\sigma]}(z_0)$. Therefore x is an element of $\pi_1(X, x_0)$ and the result follows. □

As a consequence of this result we obtain the following corollary that will allow us to give the bijection between the vertices of the dual tree $(\mathcal{V}, \mathcal{E})$ and the Bass-Serre tree T. We omit the proof since it is the same as the proof of Corollary 10.

Corollary 15. *The map*

$$\begin{aligned} \mathcal{V}_X &\to \pi_1(S, x_0)/\pi_1(X, x_0) \\ V_x &\mapsto x\pi_1(X, x_0) \end{aligned}$$

is well-defined and it is a bijection.

We conclude this section by studying separately the cases when γ is a separating curve and when it is a nonseparating curve.

1.2.4.a Case of a separating curve

As before, we denote by S_1 and S_2 the surfaces obtained after cutting S along γ and by A and B their respective fundamental groups based at x_0. Let V_1 and W_1 be the connected components of $\mathbb{H}^2 \setminus \cup\mathcal{E}$ that contain the lifts based at z_0 of A and B respectively. Then V_1 and W_1 are separated by l_1. We write

$$V_x = \varphi_x(V_1) \quad \text{and} \quad W_x = \varphi_x(W_1).$$

Then we have that

$$\mathcal{V} = \{V_x \,|\, x \in \pi_1(S, x_0)\} \sqcup \{W_x \,|\, x \in \pi_1(S, x_0)\},$$

and by Corollary 15, we have a bijection

$$\begin{aligned} \mathcal{V} &\to \pi_1(S, x_0)/A \sqcup \pi_1(S, x_0)/B \\ V_x &\mapsto xA \\ W_x &\mapsto xB. \end{aligned} \tag{1.2.3}$$

Notice that since l_1 separates V_1 and W_1, then by continuity, l_x separates V_x and W_x for all x in $\pi_1(S, x_0)$. Hence V_x and W_x are vertices in the dual tree $(\mathcal{V}, \mathcal{E})$ joined by the edge l_x. Finally, by Corollary 10, the map

$$\begin{aligned} \mathcal{E} &\to \pi_1(S)/C \\ l_x &\mapsto xC \end{aligned} \tag{1.2.4}$$

sends the edge l_x in the dual tree $(\mathcal{V}, \mathcal{E})$ to the edge xC in the Bass-Serre tree T which by Theorem 6 joins the vertices xA and xB. Therefore, the bijections (1.2.3) and (1.2.4) define an isomorphism between such trees.

1.2.4.b Case of a nonseparating curve

As before, we denote by S_1 the surface obtained after cutting S along γ and by A the fundamental group of S_1 based at x_0. Now, let V_1 be the connected component of $\mathbb{H}^2 \setminus \cup\mathcal{E}$ that contains the lifts based at z_0 of A. We write

$$V_x = \varphi_x(V_1).$$

Then we have that

$$\mathcal{V} = \{V_x \,|\, x \in \pi_1(S, x_0)\},$$

and by Corollary 15, we have a bijection

$$\begin{aligned} \mathcal{V} &\to \pi_1(S, x_0)/A \\ V_x &\mapsto xA \end{aligned} \tag{1.2.5}$$

Recall that $p \restriction_{V_1}: V_1 \to S_1$ is the universal covering space of S_1. Since S_1 has two boundary components, we get that V_1 has two types of boundary components, corresponding to the preimages of the boundary components of S_1. Recall that t is given by the homotopy class of a simple closed curve τ in S that becomes a path in S_1 joining its two boundary components. Notice that the lift of this path based at z_0 is a curve contained in V_1 joining z_0 with $\varphi_t(z_0)$. Hence l_1 (which contains z_0) and l_t (which contains $\varphi_t(z_0)$) are boundary components of V_1 of different type. Furthermore, the two types of boundary components of V_1 are given by

$$\{l_a \,|\, a \in A\} \quad \text{and} \quad \{l_{at} \,|\, a \in A\}.$$

Notice that l_t separates the connected components V_1 and V_t and l_1 separates de connected components $V_{t^{-1}}$ and V_1. More generally, by continuity, we get that l_x separates the connected components $V_{xt^{-1}}$ and V_x, *i.e.* $V_{xt^{-1}}$ and V_x are vertices in the dual tree $(\mathcal{V}, \mathcal{E})$ joined by the edge l_x. Finally, by Corollary 10, the map

$$\begin{aligned} \mathcal{E} &\to \pi_1(S, x_0)/C \\ l_x &\mapsto xC \end{aligned} \tag{1.2.6}$$

sends the edge l_x in the dual tree $(\mathcal{V}, \mathcal{E})$ to the edge xC in the Bass-Serre tree T which by Theorem 6 joins the vertices $xt^{-1}A$ and xA. We conclude that the bijections (1.2.5) and (1.2.6) define an isomorphism between such trees.

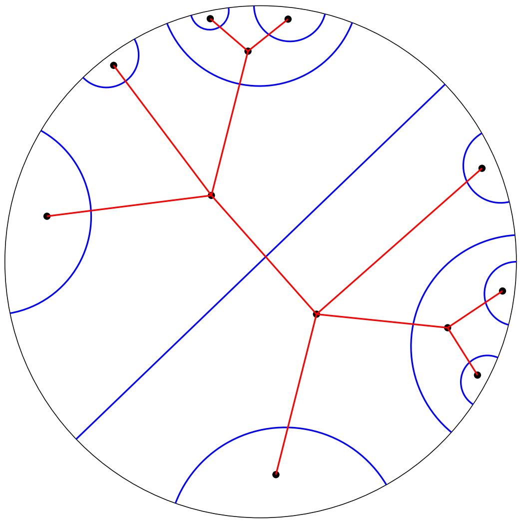


Figure 1.2.9: Dual tree embedded in $\mathbb{H}^2$

1.3 Extending actions on the Bass-Serre tree

As explained at the beginning of this chapter, this section is mainly expository and serves as a (geometric) introduction to [35, 44, 59].

1.3.1 Extending an action to a group of automorphisms

Let G be a finitely generated group that splits as an amalgamated product or an HNN extension and let T be its Bass-Serre tree. If the center $Z(G)$ of G acts trivially on T, the action of G on T factors through the quotient $G/Z(G)$. Since this quotient is isomorphic to the group $\mathrm{Inn}(G)$ of inner automorphisms of G, we obtain an induced action of $\mathrm{Inn}(G)$ on T. The aim of this section is to extend (under suitable hypothesis) this induced action of $\mathrm{Inn}(G)$ on T to a larger group of automorphisms of G.

1.3.1.a Conditions to obtain a trivial action of the center of the group

The following two results exhibit sufficient conditions to guarantee that the center of G acts trivially on T.

Lemma 16. *If there is a vertex v of T whose stabilizer is its own normalizer in G, then $Z(G)$ acts trivially on T.*

Proof. Since for a subgroup of G the property of being its own normalizer is preserved by conjugation we get that the stabilizer of any vertex in the orbit of v is its own normalizer. By definition $Z(G)$ commutes with all the elements of G. Hence, it normalizes the stabilizer of any vertex of T. In particular $Z(G)$ normalizes the stabilizer of any vertex in the orbit of v. Therefore, using the hypothesis of the Lemma, we get that $Z(G)$ is contained in the stabilizer of any vertex in the orbit of v, *i.e.* the action of G on T restricted to $Z(G)$ fixes each vertex in the orbit of v. Hence $Z(G)$ fixes every path between any two vertices in the orbit of v and as a consequence, it fixes the whole tree T. □

Remark 17. *In the latter proof we used the fact that if a group G acts on a tree T, and there is an element x of G that fixes two vertices v and w of T, then x fixes the path between*

v and w. This follows from the fact that the action of G on T is by automorphisms. Then, the action of G on T sends paths to paths, and the assertion follows from observing that the geodesic paths in a tree are unique.

For any two vertices v and v' of T we will denote by $[v, v']$ the combinatorial geodesic joining v with v'.

Lemma 18. *Suppose that there is an edge of T whose stabilizer is properly contained in the stabilizers of its vertices. Then the stabilizer of every vertex of T is its own normalizer.*

Proof. Since the action of G on T is transitive on the set of edges, the stabilizer of any edge is properly contained in the stabilizers of its vertices. Given a vertex v of T and an element x of G, we know that the stabilizer of $x \cdot v$ is xG_vx^{-1}. Now, let v be a vertex of T and x an element of G that normalizes G_v. We need to prove that $x \cdot v = v$. Let us assume by contradiction that $x \cdot v \neq v$ and let e be the first edge of the path $[v, x \cdot v]$. Then, the stabilizer $G_{[v,x \cdot v]}$ of the path $[v, x \cdot v]$ is contained in G_e. Since x normalizes G_v, the stabilizer of v and $x \cdot v$ are equal. From this, we obtain that G_v fixes v and $x \cdot v$, and thereby the path $[v, x \cdot v]$. Finally, we obtain that

$$G_v \subset G_{[v,x \cdot v]} \subset G_e \subsetneq G_v,$$

which is a contradiction. □

1.3.1.b Induced isometries of the Bass-Serre tree

Definition 19. We define the subgroup $\mathrm{Aut}_T(G)$ of $\mathrm{Aut}(G)$ by the following property: an automorphism $\varphi : G \to G$ is an element of $\mathrm{Aut}_T(G)$ if for every edge $e = (v_1, v_2)$ of T there is an element x in G such that

$$\varphi(G_{v_1}) = xG_{v_1}x^{-1}, \quad \varphi(G_{v_2}) = xG_{v_2}x^{-1}, \quad \text{and} \quad \varphi(G_e) = xG_ex^{-1}.$$

Lemma 20. *Suppose that the stabilizer of any vertex of T is its own normalizer and let $\varphi : G \to G$ be an element of $\mathrm{Aut}_T(G)$. Then, φ induces an isometry $\overline{\varphi} : T \to T$ such that for all g in G*

$$\overline{\varphi}(g \cdot \_) = \varphi(g) \cdot \overline{\varphi}(\_). \tag{1.3.1}$$

Moreover, if φ is the inner automorphism given by the conjugation by x, then $\overline{\varphi}$ is given by the action of x on T.

Remark 21. *By Lemma 16, we get that the hypothesis of Lemma 20 on the vertex stabilizers implies that the center of G acts trivially on T.*

Proof of Lemma 20. Let us fix an edge $e = (v_1, v_2)$. We warn the reader that the following construction of $\overline{\varphi}$ depends on the choice of e. We write

$$\varphi(G_{v_1}) = xG_{v_1}x^{-1}, \quad \varphi(G_{v_2}) = xG_{v_2}x^{-1}, \quad \text{and} \quad \varphi(G_e) = xG_ex^{-1}.$$

We define an isometry $\overline{\varphi} : T \to T$ as follows. If v is a vertex of T we write $v = g \cdot v_i$ for $i = 1, 2$ and define

$$\overline{\varphi}(v) = \varphi(g)x \cdot v_i.$$

If f is an edge of T we write $f = g \cdot e$ and define

$$\overline{\varphi}(f) = \varphi(g)x \cdot e.$$

These definitions are independent of the choice of the element g. In the case of an HNN extension, we must check that the action on a vertex v is independent of whether we represent v as the image of v_1 or v_2 (as v_1 and v_2 are in the same orbit). But if $v_2 = t \cdot v_1$ one checks that $t^{-1}x^{-1}\varphi(t)x$ normalizes G_{v_1}, and thus by the hypothesis of the lemma we get that $t^{-1}x^{-1}\varphi(t)x$ lies in G_{v_1}. Using this observation one proves that $\overline{\varphi}(v)$ is well defined. In this way we obtain two bijections, one of the set of vertices of T and one of the set of edges of T, which define an isometry of T. If there is another element y in G such that

$$\varphi(G_{v_1}) = yG_{v_1}y^{-1}, \quad \varphi(G_{v_2}) = yG_{v_2}y^{-1}, \quad \text{and} \quad \varphi(G_e) = yG_ey^{-1}$$

then $x^{-1}y$ is in the normalizer of G_{v_1} and G_{v_2}. Since G_{v_1} and G_{v_2} are their own normalizers we get that $x^{-1}y$ is in the intersection of G_{v_1} and G_{v_2}, which is G_e. This implies that $\overline{\varphi}$ is well defined and it has the desired property

$$\overline{\varphi}(g \cdot \_) = \varphi(g) \cdot \overline{\varphi}(\_).$$

Finally if φ is the inner automorphism given by the conjugation by x we get that

$$\overline{\varphi}(g \cdot e) = \varphi(g)x \cdot e = (xgx^{-1})x \cdot e = x \cdot (g \cdot e).$$

Similarly, for $i = 1, 2$ we obtain that $\overline{\varphi}(g \cdot v_i) = x \cdot (g \cdot v_i)$, which implies that $\overline{\varphi}$ is given by the action of x on T. □

Definition 22. An isometry $\overline{\varphi} : T \to T$ induced by an automorphism φ of G that satisfies (1.3.1) is called *G-compatible*. An extension of the induced action of $\mathrm{Inn}(G)$ on T to a subgroup Λ of $\mathrm{Aut}(G)$ is called *G-compatible* if every automorphism of Λ defines a G-compatible isometry of T.

Lemma 23. *Suppose that the stabilizer of any vertex of T is its own normalizer. Then the induced action of $\mathrm{Inn}(G)$ on T extends to a G-compatible action of $\mathrm{Aut}_T(G)$ on T.*

Proof. Let us fix an edge $e = (v_1, v_2)$. By Lemma 20, it suffices to show that for all φ, ψ in $\mathrm{Aut}_T(G)$ the isometries $\overline{\varphi}$ and $\overline{\psi}$ defined as before satisfy:

$$\overline{\varphi^{-1}} = \overline{\varphi}^{-1} \quad \text{and} \quad \overline{\varphi \circ \psi} = \overline{\varphi} \circ \overline{\psi}.$$

Let φ be an element in $\mathrm{Aut}_T(G)$ such that

$$\varphi(G_{v_1}) = xG_{v_1}x^{-1}, \quad \varphi(G_{v_2}) = xG_{v_2}x^{-1}, \quad \text{and} \quad \varphi(G_e) = xG_ex^{-1}.$$

Then, if $y = \varphi^{-1}(x^{-1})$ we get that

$$\varphi^{-1}(G_{v_1}) = yG_{v_1}y^{-1}, \quad \varphi^{-1}(G_{v_2}) = yG_{v_2}y^{-1}, \quad \text{and} \quad \varphi^{-1}(G_e) = yG_ey^{-1}.$$

Therefore the isometry $\overline{\varphi^{-1}}$ is given by

$$\begin{aligned} \overline{\varphi^{-1}}(g \cdot v_i) &= \varphi^{-1}(g)y \cdot v_i \quad \text{for } i = 1, 2; \\ \overline{\varphi^{-1}}(g \cdot e) &= \varphi^{-1}(g)y \cdot e. \end{aligned}$$

A direct computation shows that $\overline{\varphi^{-1}} = \overline{\varphi}^{-1}$.

Now, let φ and ψ be two elements of $\mathrm{Aut}(G)$ such that

$$\varphi(G_{v_1}) = xG_{v_1}x^{-1}, \quad \varphi(G_{v_2}) = xG_{v_2}x^{-1}, \quad \text{and} \quad \varphi(G_e) = xG_ex^{-1},$$

and

$$\psi(G_{v_1}) = yG_{v_1}y^{-1}, \quad \psi(G_{v_2}) = yG_{v_2}y^{-1}, \quad \text{and} \quad \psi(G_e) = yG_ey^{-1}.$$

Then, if $z = \psi(x)y$ we get that

$$\psi \circ \varphi(G_{v_1}) = zG_{v_1}z^{-1}, \quad \psi \circ \varphi(G_{v_2}) = zG_{v_2}z^{-1}, \quad \text{and} \quad \psi \circ \varphi(G_e) = zG_ez^{-1}.$$

Hence the isometry $\overline{\psi \circ \varphi}$ is given by

$$\begin{aligned} \overline{\psi \circ \varphi}(g \cdot v_i) &= \psi \circ \varphi(g)z \cdot v_i \quad \text{for } i = 1, 2; \\ \overline{\psi \circ \varphi}(g \cdot e) &= \psi \circ \varphi(g)z \cdot e. \end{aligned}$$

Once again, a direct computation shows that $\overline{\psi \circ \varphi} = \overline{\psi} \circ \overline{\varphi}$. □

Note that the isometry $\overline{\varphi} : T \to T$ defined in Lemma 20 depends on the selected edge $e = (v_1, v_2)$ when φ is not an inner automorphism of G. Therefore, the action of $\mathrm{Aut}_T(G)$ on T also depends on this edge.

Definition 24. We denote by Λ_T the subgroup of $\mathrm{Aut}(G)$ that preserves the conjugacy class of each vertex stabilizer and each edge stabilizer of the Bass-Serre tree T of G, *i.e.* an automorphism φ of G is an element of Λ_T if for any edge $e = (v_1, v_2)$ there are elements x, y, z in G such that

$$\varphi(G_{v_1}) = xG_{v_1}x^{-1}, \quad \varphi(G_{v_2}) = yG_{v_2}y^{-1}, \quad \text{and} \quad \varphi(G_e) = zG_ez^{-1}.$$

It is clear that $\mathrm{Aut}_T(G)$ is always contained in Λ_T. In the following, we will see some situations where both groups almost coincide.

1.3.2 First examples of extended actions

1.3.2.a Malnormal condition

A subgroup H of a group G is called malnormal if $xHx^{-1} \cap H$ is trivial for all x in $G \setminus H$.

Theorem 25. *Let G be a group that splits as an amalgamated product $A *_C B$ and let T be its Bass-Serre tree. If A and B are malnormal subgroups of G and C is a non-trivial subgroup properly contained in A and in B, then the induced action of* $\mathrm{Inn}(G)$ *extends to a G-compatible action of Λ_T on T.*

Proof. By Lemma 20 and Lemma 23 it suffices to show that Λ_T is contained in $\mathrm{Aut}_T(G)$: Let φ be an automorphism of G contained in Λ_T and let x, y, z be elements in G such that

$$\varphi(A) = xAx^{-1}, \quad \varphi(B) = yBy^{-1}, \quad \text{and} \quad \varphi(C) = zCz^{-1}.$$

Then C is contained in $z^{-1}xAx^{-1}z$ and $z^{-1}yBy^{-1}z$. By the malnormality of A and B in G, we get that $z^{-1}x$ and $z^{-1}y$ are elements of A and B respectively. Hence

$$\varphi(A) = zAz^{-1}, \quad \varphi(B) = zBz^{-1}, \quad \text{and} \quad \varphi(C) = zCz^{-1}.$$

Therefore Λ_T is contained in $\mathrm{Aut}_T(G)$. □

1.3.2.b Free product of two groups

Let G be a group that splits as a free product $A * B$ and let T be its Bass-Serre tree. In this case the group Λ_T is the subgroup of $\mathrm{Aut}(G)$ such that for all φ in Λ_T there are elements x, y in G such that

$$\varphi(A) = xAx^{-1} \quad \text{and} \quad \varphi(B) = yBy^{-1}.$$

Lemma 26. *Let x be an element of G such that A and xBx^{-1} generate G. Then there is an element a of A such that $xBx^{-1} = aBa^{-1}$.*

Proof. The result is clear if x is an element of A or B. We claim that x cannot be of the form

$$x = b_1 a_1 b_2 \cdots b_n a_n,$$

with a_i and b_i non-trivial elements of A and B respectively for all i. Otherwise, since A and xBx^{-1} generate G, there would be elements $\alpha_1, \ldots, \alpha_{k+1}$ of A and elements $\beta_1, \ldots, \beta_k$ of B such that

$$b_1 = \alpha_1(x\beta_1 x^{-1})\alpha_2 \cdots \alpha_k(x\beta_k x^{-1})\alpha_{k+1},$$

with $\alpha_i \neq 1$ for $i = 2, \ldots, k$ and $\beta_i \neq 1$ for $i = 1, \ldots, k$, which is impossible since the element on the right side of the latter equation cannot be reduced to a word of length one.

Finally, if $x = \alpha y \beta$ for some $\alpha \in A$, $\beta \in B$ and $y \in G$, we obtain that A and yBy^{-1} generate G. We deduce from this and the previous argument that x must be of the form

$$x = ab,$$

for some $a \in A$ and some $b \in B$, which implies the result. □

Proposition 27. *The induced action of* $\mathrm{Inn}(G)$ *on T extends to Λ_T.*

Proof. By Lemma 20 and Lemma 23 it suffices to show that Λ_T is contained in $\mathrm{Aut}_T(G)$. Let φ be an automorphism of G contained in Λ_T and let x, y be elements in G such that

$$\varphi(A) = xAx^{-1} \quad \text{and} \quad \varphi(B) = yBy^{-1}.$$

If we denote by ψ the inner automorphism of G given by the conjugation by x^{-1} we get that

$$\psi \circ \varphi(A) = A \quad \text{and} \quad \psi \circ \varphi(B) = x^{-1}yBy^{-1}x.$$

Since $\psi \circ \varphi$ is an automorphism of G we have that A and $x^{-1}yBy^{-1}x$ generate G. It follows from Lemma 26 that there is an element a in A such that $x^{-1}yBy^{-1}x = aBa^{-1}$. Hence,

$$\varphi(A) = (xa)A(xa)^{-1} \quad \text{and} \quad \varphi(B) = (xa)B(xa)^{-1},$$

and therefore Λ_T is contained in $\mathrm{Aut}_T(G)$. □

Notice that if A and B are indecomposable, not infinite cyclic and non isomorphic to each other, then Λ_T coincides with $\mathrm{Aut}(G)$ (see [65, p. 152]). As a consequence of Proposition 27, we obtain the following result:

Theorem 28. *Let G be a group that splits as a free product $A * B$ with A and B indecomposable, not infinite cyclic and non isomorphic to each other. Then, the induced action of* $\mathrm{Inn}(G)$ *on T extends to a G-compatible action of* $\mathrm{Aut}(G)$ *on T.*

Remark 29. *If $G = A * A$, the same argument proves that* $\mathrm{Aut}(G)$ *contains a subgroup of index 2 which extends the action of G on its Bass-Serre tree. The group* $\mathrm{Aut}(G)$ *itself acts on the first barycentric division of the Bass-Serre tree of G.*

1.3.3 Edge stabilizer conditions

In order to study a more general situation, we will suppose that G splits over a subgroup C and that the action of G on its Bass-Serre tree T satisfies the following conditions:

B1 The stabilizer of an edge is non-trivial and properly contained in the stabilizers of its vertices. Moreover, the normalizer of an edge stabilizer is contained in the stabilizer of one of its vertices.

B2 The stabilizers of two edges e and e' with a common vertex v are either identical or their intersection is trivial. If they are identical there is an element g in the stabilizer of v such that $e' = g \cdot e$. In particular g is in the normalizer of the stabilizer of e.

Let us fix an edge $e = (v_1, v_2)$ of T. By condition **B1** we may assume that the normalizer of G_e is contained in G_{v_1}. For any two vertices v and v' of T we will denote by $d(v, v')$ the number of edges in the path $[v, v']$. Let G_0 be the kernel of the map

$$\begin{array}{ccc} G & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ g & \mapsto & \begin{cases} 0 & \text{if } d(v_1, g \cdot v_1) \text{ is even;} \\ 1 & \text{if } d(v_1, g \cdot v_1) \text{ is odd.} \end{cases} \end{array}$$

Note that for every vertex v of T and for all g in G_0, $d(v, g \cdot v)$ is even. If G splits as an amalgamated product then $G = G_0$. If G splits as an HNN extension then G_0 is a subgroup of G of index 2 which defines a cohomology class in $H^1(G, \mathbb{Z}/2\mathbb{Z})$. More precisely, if G splits as the HNN extension $A*_{C,\theta}$, we have that G_0 is the kernel of the map $\eta : G \to \mathbb{Z}/2\mathbb{Z}$ such that $\eta(t) = 1$ and $\eta(a) = 0$ for all a in A.

Lemma 30. *Let $\varphi : G \to G$ be an automorphism of G. Suppose that for $i = 1, 2$ there is an element g_i in G_0 such that $\varphi(G_{v_i}) = g_i G_{v_i} g_i^{-1}$.*

1. *If G_e is contained in $\varphi(G_{v_1})$, then g_1 is an element of G_{v_1}.*

2. *If G_e is contained in $\varphi(G_{v_2})$,then there is an element h in the normalizer of G_e such that hg_2 is an element of G_{v_2}.*

Proof.

1. If, to obtain a contradiction, we suppose that g_1 is not an element of G_{v_1}, then the path $[v_1, g_1 \cdot v_1]$ is non-trivial. Note that the stabilizer of $g_1 \cdot v_1$ is precisely $\varphi(G_{v_1}) = g_1 G_{v_1} g_1^{-1}$. Therefore, since G_e is contained in both G_{v_1} and $\varphi(G_{v_1})$, we get that G_e stabilizes the path $[v_1, g_1 \cdot v_1]$. In particular, it stabilizes the first two edges of this path: $e' = (v_1, v')$ and $e'' = (v', v'')$. By condition **B2**, the stabilizers of e' and of e'' are equal to G_e and there is a non-trivial element g in the stabilizer of v' such that $g \cdot e' = e''$. Using the fact that the stabilizers of e, e' and e'' are equal, we conclude that g is in the normalizer of G_e and therefore in G_{v_1}. Hence g stabilizes e' which contradicts the fact that the path $[v_1, g_1 \cdot v_1]$ is without backtracking.

2. Suppose that g_2 is not an element of G_{v_2}. Let e' and e'' be the first two edges of the path $[v_2, g_2 \cdot v_2]$. As before, the stabilizers of e, e' and e'' are equal. The proof of the first part of this lemma implies that $e = e'$ and that the path $[v_2, g_2 \cdot v_2]$ can not contain more than two edges. Hence this path has exactly two edges and the vertex between v_2 and $g_2 \cdot v_2$ is v_1. Finally, condition **B2** implies that there is an element h in the stabilizer of v_1 such that $h \cdot (g_2 \cdot v_2) = v_2$. □

Corollary 31. *The action of G on T is faithful and the center $Z(G)$ of G is trivial.*

Proof. Let x be an element of G that fixes every vertex of T and let g be an element of $G_{v_2} \setminus G_e$. If we denote by φ the inner automorphism of G given by the conjugation by g then we have that

$$\varphi(G_{v_1}) = gG_{v_1}g^{-1}, \quad \varphi(G_{v_2}) = G_{v_2}, \quad \text{and} \quad \varphi(G_e) = gG_eg^{-1}.$$

Since x fixes every vertex of T, it fixes in particular the path $[v_1, g \cdot v_1]$ which is the path whose edges are e and $g \cdot e$. If we suppose that x is not the identity we get that $G_e \cap G_{g \cdot e}$ is non-trivial and by condition **B2** we get that $G_e = G_{g \cdot e} = gG_eg^{-1}$. Therefore G_e is contained in $\varphi(G_{v_1})$ and by Lemma 30 g is an element of G_{v_1}. This would imply g is contained in $G_{v_1} \cap G_{v_2} = G_e$ which contradicts the choice of g. Hence, we conclude that x is the identity. This implies that $Z(G)$ is trivial since Lemma 16, Lemma 18 and condition **B1** imply that $Z(G)$ acts trivially on T. □

In the following, we will denote by Λ_T^0 the subgroup of $\mathrm{Aut}(G)$ given by the following property: an automorphism $\varphi : G \to G$ is an element of Λ_T^0 if for any edge $e = (v_1, v_2)$ of T, there are elements x, y, z in the same left coset of G_0 in G such that

$$\varphi(G_{v_1}) = xG_{v_1}x^{-1}, \quad \varphi(G_{v_2}) = yG_{v_2}y^{-1}, \quad \text{and} \quad \varphi(G_e) = zG_ez^{-1}.$$

Note that if $xG_{v_1}x^{-1} = aG_{v_1}a^{-1}$ for some a in G, by condition **B1** and Lemma 18 we have that $a^{-1}x$ is in G_{v_1}. Therefore, since G_{v_1} is contained in G_0 we get that $xG_0 = aG_0$ and thereby the left coset xG_0 is well defined. Similarly, we get that the left cosets yG_0 and zG_0 are well defined.

The main result of this section is the following:

Theorem 32. *Let G be a group that splits over a subgroup C and let T be its Bass-Serre tree. If the action of G on T satisfies conditions **B1** and **B2**, then the induced action of $\mathrm{Inn}(G)$ on T extends to a G-compatible action of Λ_T^0 on T.*

Proof. By Lemma 20 and Lemma 23 it suffices to show that Λ_T^0 is contained in $\mathrm{Aut}_T(G)$: Let φ be an automorphism of G contained in Λ_T^0 and let x, y, z be elements in the same left coset of G_0 in G such that

$$\varphi(G_{v_1}) = xG_{v_1}x^{-1}, \quad \varphi(G_{v_2}) = yG_{v_2}y^{-1}, \quad \text{and} \quad \varphi(G_e) = zG_ez^{-1}.$$

Then G_e is contained in $z^{-1}xG_{v_1}x^{-1}z$ and $z^{-1}yG_{v_2}y^{-1}z$. Note that $z^{-1}x$ and $z^{-1}y$ are elements in G_0. By the first part of Lemma 30 $z^{-1}x$ is an element of G_{v_1}. By the second part of Lemma 30, there is an element h in the normalizer of G_e (thereby in G_{v_1}) such that $hz^{-1}y$ is in G_{v_2}. Hence we get that

$$\varphi(G_{v_1}) = zh^{-1}G_{v_1}hz^{-1}, \quad \varphi(G_{v_2}) = zh^{-1}G_{v_2}hz^{-1}, \quad \text{and} \quad \varphi(G_e) = zh^{-1}G_ehz^{-1}.$$

Therefore Λ_T^0 is contained in $\mathrm{Aut}_T(G)$. □

Lemma 33. *The group Λ_T^0 is a subgroup of Λ_T of index at most 2.*

Since $\Lambda_T = \Lambda_T^0$ when G splits as an amalgamated product, it suffices to show that this is true when G splits as an HNN extension. With this aim we will suppose for the rest of this subsection that G splits as the amalgamated product $G_{v_1} *_{G_e, \theta}$ where $\theta : G_e \to G_{v_1}$ is the monomorphism given by the conjugation of an element t in $G \setminus G_{v_1}$. To prove Lemma 33 we will need the following result:

Lemma 34. *If φ is an automorphism of G contained in Λ_T then $\varphi(G_0) = G_0$.*

Proof. Note that G_0 can be obtained as the kernel of the map $\eta : G \to \mathbb{Z}/2\mathbb{Z}$ such that $\eta(t) = 1$ and $\eta(g) = 0$ for all g in G_{v_1}. This follows from the fact that for all x in G the set of vertices of T joined to $x \cdot v_1$ is

$$\{xgt^{\epsilon} | g \in G_{v_1} \quad \text{and} \quad \epsilon = \pm 1\}.$$

Every element in G can be written as

$$g_1 t^{n_1} g_2 t^{n_2} \cdots g_k t^{n_k} g_{k+1},$$

where g_i is in G_{v_1} for all $i = 1, \ldots, n+1$ and n_i is a non-zero integer for all $i = 1, \ldots, n$.

Hence, if φ is an automorphism of G contained in Λ_T and x is an element of G such that $\varphi(G_{v_1}) = xG_{v_1}x^{-1}$, we get that for all $i = 1, \ldots, n+1$ there is an element g'_i in G_{v_1} such that $\varphi(g_i) = xg'_i x^{-1}$. Therefore,

$$\varphi(g_1 t^{n_1} g_2 t^{n_2} \cdots g_k t^{n_k} g_{k+1}) = (xg'_1 x^{-1} \varphi(t)^{n_1} \cdots \varphi(t)^{n_k} x g'_{k+1} x^{-1}).$$

Hence we get that $\eta \circ \varphi(g_1 t^{n_1} g_2 t^{n_2} \cdots g_k t^{n_k} g_{k+1}) = \eta(\varphi(t)^{n_1 + \cdots + n_k})$. This implies that $\eta(\varphi(t)) = 1$, otherwise $\varphi(G)$ would be contained in G_0 which is not possible since φ is an automorphism of G and G_0 is a subgroup of index 2. We conclude that $\eta = \eta \circ \varphi$ and thereby $\varphi(G_0)$ is contained in G_0. The same argument shows that $\varphi^{-1}(G_0)$ is contained in G_0 which implies the result. □

Proof of Lemma 33. Let φ be an automorphism of G contained in Λ_T and let x, y, z be elements in G such that

$$\varphi(G_{v_1}) = xG_{v_1}x^{-1}, \quad \varphi(G_{v_2}) = yG_{v_2}y^{-1}, \quad \text{and} \quad \varphi(G_e) = zG_e z^{-1}. \tag{1.3.2}$$

First of all notice that the left cosets xG_0 and yG_0 are equal: since $G_{v_2} = tG_{v_1}t^{-1}$ then

$$\varphi(G_{v_2}) = \varphi(t)\varphi(G_{v_1})\varphi(t^{-1}) = \varphi(t)xt^{-1}G_{v_2}tx^{-1}\varphi(t^{-1}).$$

By Lemma 34 and its proof, t and $\varphi(t)$ are not contained in G_0. This implies that x is in G_0 if and only if $\varphi(t)xt^{-1}$ is in G_0, if and only if y is in G_0.

Now, let $\beta : \Lambda_T \to \mathbb{Z}/2\mathbb{Z}$ be the function such that for an automorphism φ as in (1.3.2)

$$\begin{array}{ccc} \Lambda_T & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ \varphi & \mapsto & \begin{cases} 0 & \text{if } z^{-1}x \in G_0; \\ 1 & \text{if } z^{-1}x \notin G_0. \end{cases} \end{array}$$

Let us see that β is a homomorphism:

• It is well defined: let φ be a automorphism of G as in (1.3.2). If there are elements a, c in G such that

$$\varphi(G_{v_1}) = aG_{v_1}a^{-1} \quad \text{and} \quad \varphi(G_e) = cG_e c^{-1},$$

then by Lemma 18, $a^{-1}x$ is in $G_{v_1} \subset G_0$ and by condition **B1** $c^{-1}z$ is in the normalizer of G_e which is contained in $G_{v_1} \subset G_0$. Hence $xG_0 = aG_0$ and $zG_0 = cG_0$ which implies that $z^{-1}xG_0 = c^{-1}aG_0$. Therefore β is well defined.

• It is an homomorphism: suppose that φ and ψ are automorphisms of G contained in Λ_T and let x, z, a, c be elements in G such that

$$\varphi(G_{v_1}) = xG_{v_1}x^{-1} \quad \text{and} \quad \varphi(G_e) = zG_e z^{-1},$$

$$\psi(G_{v_1}) = aG_{v_1}a^{-1} \quad \text{and} \quad \psi(G_e) = cG_e c^{-1}.$$

Then we get that

$$\psi \circ \varphi(G_{v_1}) = \psi(x)aG_{v_1}a^{-1}\psi(x^{-1}) \quad \text{and} \quad \psi \circ \varphi(G_e) = \psi(z)cG_e c^{-1}\psi(z^{-1}).$$

Note that $c^{-1}\psi(z^{-1}x)a = c^{-1}(\psi(z^{-1}x))c(c^{-1}a)$ and therefore by Lemma 34, $c^{-1}\psi(z^{-1}x)a$ is in G_0 if and only if $(z^{-1}x)(c^{-1}a)$ is in G_0 which implies that $\beta(\psi \circ \varphi) = \beta(\psi)\beta(\varphi)$.

Finally we conclude that the kernel of β is a subgroup of Λ_T of index at most 2 which is contained in Λ_T^0. □

1.3.4 Surface groups

Let S be a closed surface of genus $g \geq 2$ and let γ be a simple closed curve in S. We know from Section 1.2.3 that the fundamental group of S splits over the cyclic subgroup C generated by the homotopy class of γ. In the following, we will omit the base point of the fundamental group of S to simplify notation.

Remark 35. *The group of automorphisms of $\pi_1(S)$ that preserves the conjugacy class of γ is a subgroup of index at most 2 of the subgroup of $\pi_1(S)$ made of the automorphisms $\varphi : \pi_1(S) \to \pi_1(S)$ such that*

$$\varphi(C) = xCx^{-1} \tag{1.3.3}$$

for some x in $\pi_1(S)$. This follows from the fact that if an automorphism φ satisfies (1.3.3) then $\varphi([\gamma])$ must be equal to either $x[\gamma]x^{-1}$ or to $x[\gamma]^{-1}x^{-1}$.

Lemma 36. *Let γ be a simple closed curve in S and let T be the Bass-Serre tree of $\pi_1(S)$ associated to γ. The group of automorphisms of $\pi_1(S)$ that preserves the conjugacy class of C contains $\mathrm{Aut}_T(\pi_1(S))$ as a subgroup of index at most 2. Hence Λ_T contains $\mathrm{Aut}_T(\pi_1(S))$ as a subgroup of index at most 2.*

Proof. Let Γ be the subgroup of $\mathrm{Aut}(\pi_1(S))$ that preserves the conjugacy class of C. We have the inclusions:

$$\mathrm{Aut}_T(\pi_1(S)) \subset \Lambda_T \subset \Gamma.$$

Hence the second point of the lemma follows from the first one. By the Dehn-Nielsen-Baer Theorem (see [32]) we have an isomorphism between the (extended) mapping class group of S and the outer automorphism group of $\pi_1(S)$

$$\mathrm{MCG}^{\pm}(S) = \mathrm{Diff}^{\pm}(S)/\mathrm{Diff}_0(S) \to \mathrm{Out}(\pi_1(S)) = \mathrm{Aut}(\pi_1(S))/\mathrm{Inn}(\pi_1(S)),$$

which takes the class of a diffeomorphism $f : S \to S$ to the automorphism induced by f on $\pi_1(S)$ (well-defined up to conjugacy). Let us fix an automorphism φ of $\pi_1(S)$ contained in Γ and let $f : S \to S$ be a diffeomorphism inducing φ.

We fix a base point x_0 on γ and a collar neighborhood U_γ of γ. Since the curve $f \circ \gamma$ is isotopic to either γ or $\bar{\gamma}$, one can replace f by a diffeomorphism f_1 isotopic to f such that f_1 preserves γ globally and fixes x_0. Let $\mathrm{Diff}(S, \gamma)$ be the group of diffeomorphisms of S preserving γ globally, preserving each connected component of $U_\gamma \setminus \gamma$ and fixing x_0. Then, a diffeomorphism contained in $\mathrm{Diff}(S,\gamma)$ induces an automorphism of $\pi_1(S)$ that preserves the subgroup C and the fundamental groups of the connected components of $S \setminus \gamma$. In particular, this induced automorphism lies in $\mathrm{Aut}_T(\pi_1(S))$.

The elements of Γ which are induced by diffeomorphisms of $\mathrm{Diff}(S,\gamma)$ form a subgroup Γ_0 of index at most 2. Hence if $\varphi \in \Gamma_0$ we obtain that $\varphi = f_* \circ I$ for some $I \in \mathrm{Inn}(\pi_1(S))$ and for some $f \in \mathrm{Diff}(S,\gamma)$. Therefore $\varphi \in \mathrm{Aut}_T(\pi_1(S))$. □

Since S is a closed surface of genus $g \geq 2$, given a non-nullhomotopic simple closed curve γ in S, we get that the connected components obtained after cutting S along γ are surfaces of positive genus with boundary components. Therefore, the fundamental group of such surfaces are free groups of rank greater or equal than 2, containing the cyclic subgroup generated by the homotopy class of γ. Hence, by Lemmas 16 and 18 we get that the stabilizer of any vertex of T is its own normalizer. Hence, we obtain the following result as a consequence of Lemmas 23 and 36.

Proposition 37. *The action of $\pi_1(S)$ on T extends to a subgroup of index at most 2 of the group of automorphisms of $\pi_1(S)$ that preserves the conjugacy class of γ. Moreover, this extension is $\pi_1(S)$-compatible.*

1.3.5 Maximal families of edge stabilizers

Definition 38. Let $\mathscr{F}$ be a family of subgroups of G. We say that $\mathscr{F}$ is maximal if for any 2 subgroups H, K in $\mathscr{F}$, $K < H$ implies $K = H$.

Let G be a group acting on a tree T and let v be a vertex of T. We say that an edge of T is *v-incident* if one of its vertices is v. We will be interested in the following conditions:

C1 The stabilizer of an edge of T is properly contained in the stabilizers of its vertices.

C2 For any vertex v in T the family of stabilizers of v-incident edges is maximal.

Remark 39. *Recall that condition **C1** implies that the stabilizer of any vertex is its own normalizer and therefore the center of G acts trivially on T (see Remark 21).*

Here we will see that if a group G splits as an amalgamated product or an HNN extension and the action of G on its Bass-Serre tree satisfies conditions **C1** and **C2**, then the induced action of $\mathrm{Inn}(G)$ on T extends to a larger group of automorphisms of G. The case of an amalgamated product is due to Karras, Pietrowski and Solitar (see [44]) and the case of an HNN extension is due to Pettet (see [59]). These results generalize the results of Sections 1.3.2 and 1.3.3.

Theorem 40 (Karrass, Pietrowski, Solitar, [44])**.** *Let G be a group that splits as an amalgamated product $A *_C B$ and let T be its Bass-Serre tree. If the action of G on T satisfies conditions **C1** and **C2**, then the induced action of $\mathrm{Inn}(G)$ on T extends to a G-compatible action of Λ_T on T.*

Theorem 41 (Pettet, [59])**.** *Let G be a group that splits as an HNN extension $A*_{C,\theta}$ and let T be its Bass-Serre tree. If the action of G on T satisfies conditions **C1** and **C2**, then the induced action of $\mathrm{Inn}(G)$ on T extends to a subgroup of Λ_T of index at most 2. Moreover, this extension is G-compatible.*

Remark 42. *The original statements of Theorems 40 and 41 only give information about the decomposition of Λ_T as an amalgamated product in the case of Theorem 40, and as an HNN extension in the case of Theorem 41, but we can deduce from the proofs of these results that such decompositions are induced by an action of Λ_T on T that extends the one of G on T.*

In [59], Pettet gives a more general result than the one stated above. Pettet studied groups that split as the fundamental group of a graph of groups with a more restrictive condition on the edge stabilizers of its Bass-Serre tree (*edge group incomparability hypothesis*), which coincides with the maximality of the families of v-incident edge stabilizers when G splits as an HNN extension. In [35], Gilbert, Howie, Metaftsis and Raptis studied the particular situation when the edge stabilizers are cyclic. In particular they proved the following result on Baumslag-Solitar groups:

Theorem 43 (Gilbert, Howie, Metaftsis, Raptis)**.** *Let G be the Baumslag-Solitar group*

$$\langle x, t | tx^p t^{-1} = x^q \rangle,$$

where p, q are integers with $p, q > 1$ and such that neither one is a multiple of the other. Then the induced action of $\mathrm{Inn}(G)$ *on T extends to a G-compatible action of* $\mathrm{Aut}(G)$ *on T.*

Remark 44. *If G splits as a free product $A * B$, then for any vertex v in its Bass-Serre tree T, the family of stabilizers of v-incident edges is maximal. Hence, one can recover Theorem 28 as a consequence of Theorem 40.*

1.3.6 One-ended hyperbolic groups

Interesting examples of groups with a non-trivial splitting are given by one-ended hyperbolic groups whose group of outer automorphisms is infinite. In Section 1.3.4 we saw that a surface group splits over a cyclic subgroup. The last assertion holds as well for a group which is virtually a surface group. When the group is a one-ended hyperbolic group which is not virtually a surface group, the theory of JSJ decompositions guarantees the existence of a non-trivial splitting over a virtually cyclic subgroup under the assumption that its group of outer automorphisms is infinite (see for instance Theorem 1.4 in [48]). The notion of JSJ decomposition was introduced in [63] by Sela for one-ended hyperbolic groups without torsion. In [11], Bowditch constructed a slightly different JSJ decomposition for one-ended hyperbolic groups (not necessarily torsion-free) by studying cut points on the boundary of the group. Both decompositions are "canonical" and they coincide when the group is torsion-free. In the rest of this section, G will denote a one-ended torsion-free hyperbolic group.

Theorem 45 (Sela, Bowditch)**.** *Suppose that* $\mathrm{Out}(G)$ *is an infinite group and that G is not virtually a surface group. Then there is a non-trivial splitting of G, given by a minimal action of G on a tree $\mathscr{T}$ without inversions and finite quotient $\mathscr{T}/G$. Moreover, the edge stabilizers are all cyclic subgroups and the set of vertex stabilizers is* $\mathrm{Aut}(G)$*-invariant. In particular, there is a finite index subgroup of* $\mathrm{Aut}(G)$ *that preserves this splitting i.e. that preserves the conjugacy class of each vertex stabilizer and each edge stabilizer.*

This splitting of G is called the JSJ splitting. The adjacency relation on the set of vertices induces a well-defined binary relation on the set of vertex stabilizers. The JSJ splitting is called canonical since $\mathrm{Aut}(G)$ preserves the set of vertex stabilizers with the induced binary relation. In fact, by considering the action of G on ∂G, Bowditch proved in [11] that the extended action of $\mathrm{Aut}(G)$ on $\mathscr{T}$ is G-compatible. We refer to [11] and [48] for a detailed description of this splitting. As in the case of a group that splits over a subgroup C, we obtain that the action of G on $\mathscr{T}$ is minimal and without fixed points on the boundary.

1.4 Kähler extensions and actions on trees

In this section we will prove our main results.

1.4.1 Applying Gromov and Schoen's Theorem

Let Γ be a Kähler group that fits in a short exact sequence

$$1 \longrightarrow G \longrightarrow \Gamma \xrightarrow{P} Q \longrightarrow 1, \tag{1.4.1}$$

where G is a finitely generated group with trivial center acting on a tree T. Suppose that T is not isomorphic to a line nor a point and that the action of G on T is minimal, faithful and without fixed points on the boundary. Note that since the center of G is trivial, the action of G on T coincides with the induced action of $\mathrm{Inn}(G)$ on T.

Let $\overline{\Gamma}$ be the image of the morphism $\Gamma \to \mathrm{Aut}(G)$ induced by the conjugation action of Γ on G. The main result of this section is the following:

Theorem A. *Suppose that there is a finite index subgroup $\overline{\Gamma}_0$ of $\overline{\Gamma}$ containing $\mathrm{Inn}(G)$ such that the action of G on T can be extended to a G-compatible action of $\overline{\Gamma}_0$ on T. Then G is virtually a surface group. Moreover, there is a finite index subgroup Γ_1 of Γ containing G such that the restricted short exact sequence*

$$1 \longrightarrow G \longrightarrow \Gamma_1 \xrightarrow{P} P(\Gamma_1) \longrightarrow 1$$

splits as a direct product.

To prove Theorem A we will need a result of Gromov and Schoen [37] on Kähler groups acting on trees (see also Sun [71]). For the reader's convenience we recall some properties of holomorphic maps between compact complex manifolds and closed Riemann surfaces. A surjective holomorphic map with connected fibers $f : X \to S$ from a compact complex manifold X to a closed Riemann surface S induces an orbifold structure as follows. For every point p in S, let $m(p)$ be the greatest common divisor of the multiplicities of the irreducible components of the fiber $f^{-1}(p)$ and let Δ be the set of points in S such that $m(p) \geq 2$. Note that Δ is finite since it is contained in the set of critical values of f. Hence, if $\Delta = \{p_1, \dots, p_k\}$, by assigning the multiplicity $m_i = m(p_i)$ to each p_i, the orbifold induced by f is given by S endowed with the set of marked points $\{(p_i, m_i) \mid i = 1, \dots, k\}$, which is denoted by

$$\Sigma = \{S; (p_1, m_1), \dots, (p_k, m_k)\}.$$

For all $i = 1, \dots, k$ let γ_i be a loop around p_i given by the boundary of a small enough disk such that p_i is the only singular value contained in the disk. If we denote by c_i the homotopy class of γ_i in $\pi_1(S \setminus \Delta)$, the fundamental group of the orbifold Σ is given by

$$\pi_1^{orb}(\Sigma) = \pi_1(S \setminus \Delta) / \ll c_1^{m_1}, \dots, c_k^{m_k} \gg,$$

where $\ll c_1^{m_1}, \ldots, c_k^{m_k} \gg$ is the normal closure of $\{c_1^{m_1}, \ldots, c_k^{m_k}\}$ in $\pi_1(S \setminus \Delta)$.

The following lemma is a well-known result. A more general statement of this result can be found in [26].

Lemma 46. *Let X be a compact complex manifold, S be a closed Riemann surface and $f : X \to S$ be a surjective holomorphic map whose generic fiber F is connected. If we denote by $\iota : F \hookrightarrow X$ the inclusion map and by N the image of the induced homomorphism on fundamental groups $\iota_* : \pi_1(F) \to \pi_1(X)$, we obtain a short exact sequence*

$$1 \longrightarrow N \longrightarrow \pi_1(X) \xrightarrow{\Pi} \pi_1^{orb}(\Sigma) \longrightarrow 1.$$

Since F is compact we get that N is finitely generated. The kernel of the map $\rho : \pi_1^{orb}(\Sigma) \to \pi_1(S)$ is isomorphic to $\ker(f_*)/N$ and we have the commutative diagram

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\quad f_* \quad} & \pi_1(S) \\ & {}_{\Pi}\searrow \quad \nearrow_{\rho} & \\ & \pi_1^{orb}(\Sigma) & \end{array}$$

The result of Gromov and Schoen is the following.

Theorem 47 (Gromov-Schoen)**.** *Let X be a compact Kähler manifold whose fundamental group Γ acts on a tree which is not isomorphic to a line nor a point. Suppose that the action is minimal with no fixed points on the boundary. Then there is a surjective holomorphic map with connected fibers from X to a closed hyperbolic orbifold inducing the short exact sequence*

$$1 \longrightarrow N \longrightarrow \Gamma \xrightarrow{\Pi} \pi_1^{orb}(\Sigma) \longrightarrow 1,$$

such that the restriction of the action to N is trivial.

Proof of Theorem A. Let $\overline{\Gamma}_0$ be the finite index subgroup of $\overline{\Gamma}$ satisfying the hypothesis of the theorem and let Γ_0 be the preimage of $\overline{\Gamma}_0$ under the morphism $\Gamma \to \mathrm{Aut}(G)$. Then, Γ_0 is a finite index subgroup of Γ containing G such that the action of G on T can be extended to Γ_0. By Theorem 47 there is a short exact sequence

$$1 \longrightarrow N \longrightarrow \Gamma_0 \xrightarrow{\Pi} \pi_1^{orb}(\Sigma) \longrightarrow 1,$$

where the restriction of the action of Γ_0 on T to the subgroup N is trivial and $\pi_1^{orb}(\Sigma)$ is virtually isomorphic to a surface group. Now, let $\lambda : G \to \pi_1^{orb}(\Sigma)$ be the morphism given by the restriction of Π to G. This morphism is injective since $N \cap G$ is trivial. The latter assertion follows from the faithfulness of the action of G on T. We claim that the subgroup $\Gamma_1 = \Pi^{-1}(\lambda(G))$ is the subgroup we are looking for. First of all, since for a normal subgroup of $\pi_1^{orb}(\Sigma)$ being finitely generated is equivalent to having finite index, we get that $\lambda(G)$ has finite index in $\pi_1^{orb}(\Sigma)$. This implies that Γ_1 is a finite index subgroup of Γ_0 (and thereby of Γ) and that G is virtually isomorphic to a surface group. To end the proof, it suffices to observe that the morphism

$$\begin{aligned} \eta : \Gamma_1 \quad &\to \quad G \times P(\Gamma_1) \\ x \quad &\mapsto (\lambda^{-1} \circ \Pi(x), P(x)) \end{aligned}$$

is bijective. The injectivity follows from the fact that the restriction of $\lambda^{-1} \circ \Pi$ to G is the identity. For the surjectivity, if (x_0, q_0) is an element of $G \times P(\Gamma_1)$, by taking y in Γ_1 such that $P(y) = q_0$ and x in G such that $\lambda(x) = \Pi(y)$, we get that $\eta(x_0 x^{-1} y) = (x_0, q_0)$. □

Remark 48. *In the latter proof, we used the fact that if the action of G on T is minimal without fixed points on the boundary, then any extension of this action has these properties as well.*

1.4.2 Applications

Here we will apply Theorem A and the results of Section 1.3 to prove Theorems B, C, D and E. For this, we need to check that the action with which we start with is minimal, faithful and without fixed points on the boundary. One can verify from the discussion in Section 1.2.2, that for a surface group (see also Section 1.2.3), a free product of two groups and a Baumslag-Solitar group as in Theorem D, the actions on their respective Bass-Serre tree are minimal and without fixed points on the boundary of the tree. For one-ended hyperbolic groups which are not virtually a surface group, this is a consequence of the theory of JSJ decompositions (see [11, 63]). Hence, we only need to show faithfulness in all cases.

Proposition 49. *Let S be a closed surface of genus $g \geq 2$ and let C be the cyclic subgroup of $\pi_1(S)$ generated by the homotopy class of a simple closed curve γ. Then the action of $\pi_1(S)$ on the Bass-Serre tree associated to the splitting of $\pi_1(S)$ over C is faithful.*

Proof. Let T be the Bass-Serre tree associated to the splitting of $\pi_1(S)$ over C. Recall that T coincides with the dual tree associated to γ (see Section 1.2.4). This identification is given by a bijection between the edges of T and a set of disjoint bi-infinite geodesics in $\mathbb{H}^2$. Hence, an edge stabilizer of T coincides with the stabilizer of a bi-infinite geodesic in $\mathbb{H}^2$ under the action of $\pi_1(S)$ on $\mathbb{H}^2$. From this, we conclude that the intersection of the stabilizers of two different edges of T is trivial, which implies the faithfulness of the action of G on T. □

Proof of Theorem B. Let γ be the simple closed curve in S, whose conjugacy class is preserved by the conjugation action of Γ on $\pi_1(S)$. Let C be the cyclic subgroup of $\pi_1(S)$ generated by the homotopy class of γ and let T be the Bass-Serre tree associated to the splitting of $\pi_1(S)$ over C. Therefore, the image of $\Gamma \to \mathrm{Aut}(\pi_1(S))$ denoted by $\overline{\Gamma}$ is contained in the subgroup of automorphisms of $\pi_1(S)$ that preserves the conjugacy class of γ. By Proposition 37, the action of $\pi_1(S)$ on T can be extended to a $\pi_1(S)$-compatible action of a subgroup of $\overline{\Gamma}$ of index at most 2 on T. By Proposition 49 we have that the action of $\pi_1(S)$ on T is faithful and the result follows from Theorem A. □

Remark 50. *Let G be a group that splits over a subgroup C and let T be its Bass-Serre tree. Suppose that for an edge of T, its stabilizer is properly contained in its vertex stabilizers. Suppose as well that for any vertex v of T, the family of stabilizers of v-incident edges is maximal. Let Γ be a Kähler group containing G as a normal subgroup such that the conjugation action of Γ on G preserves the conjugacy classes of edge stabilizers and vertex stabilizers of T. If the action of G on its Bass-Serre tree is faithful, then as a consequence of Theorems 40, 41 and A we have that G must be virtually a surface group.*

Theorem C follows from Theorems 28 and A, and the following proposition.

Proposition 51. *If a surface group is embedded in a free product $A * B$, then it is embedded in A or it is embedded in B (up to conjugacy). In particular, a non-trivial free product of two groups is not virtually a surface group.*

Proof. First, we will prove that a non-trivial free product is not isomorphic to a surface group. Let us assume by contradiction that the free product $C*D$ is isomorphic to a surface group. Recall that an infinite index subgroup of a surface group is a free group. Hence, since C and D are both infinite index subgroups of $C*D$, we obtain that they are both free groups. Thus, the free product $C*D$ is also a free group, which is a contradiction. Note that the same argument shows that a surface group is not isomorphic to a free product with more than two free factors. The general case follows from the Kurosh subgroup Theorem (see [65, p. 151]), which states that a subgroup of the free product $A*B$ is given by the free product of a free group with subgroups of conjugates of A and B. Hence, if a surface group embeds in $A*B$, by the Kurosh subgroup Theorem and the previous argument it embeds as a subgroup of a conjugate of A or as a subgroup of a conjugate of B. □

Theorem D follows from Theorem 43 and A, and the following two propositions:

Proposition 52. *The Baumslag-Solitar group*

$$G(p,q)=\langle x,t|tx^pt^{-1}=x^q\rangle$$

is not virtually a surface group.

Proof. If $|p|\neq|q|$ and if we suppose that $G(p,q)$ is a surface group, we would have that $G(p,q)$ acts on $\mathbb{H}^2$. Hence, if we denote by ℓ the translation length of x in $\mathbb{H}^2$, as a consequence of the relation $tx^pt^{-1}=x^q$, we would have that $|p|\ell=|q|\ell$ which is impossible. If H is a finite index subgroup of $G(p,q)$, there exists an integer k such that x^k and t^k are contained in H. From the unique relation of the Baumslag-Solitar group, we deduce that $tx^{np}t^{-1}=x^{nq}$ for every integer n. Then, an argument by induction shows that

$$t^kx^{k\cdot p^k}t^{-k}=x^{k\cdot q^k}.$$

Hence, if we suppose that H is a surface group and we denote by ℓ the translation length of x^k in $\mathbb{H}^2$, we would have that $|p^k|\ell=|q^k|\ell$, which is impossible.

If $|p|=|q|$, the subgroup generated by x^p is normal in $G(p,q)$. Hence its intersection with any finite index subgroup H of $G(p,q)$ is normal in H. This cannot happen if H is a surface group since such a group does not admit a non-trivial Abelian normal subgroup. □

Proposition 53. *The Baumslag-Solitar group*

$$G(p,q)=\langle x,t|tx^pt^{-1}=x^q\rangle,$$

where p,q are different integers with $p,q>1$, acts faithfully on its Bass-Serre tree.

Proof. The kernel N of the action of $G(p,q)$ on its Bass-Serre tree is contained in every edge stabilizer. In particular, it is contained in $\langle x^p\rangle$. Assume by contradiction that N is non-trivial. Then, N is generated by x^{kp} for some nonzero integer k. Since N is a normal subgroup of $G(p,q)$, we get that $tx^{kp}t^{-1}=x^{kq}$ is an element of $\langle x^{kp}\rangle$. Therefore $kq=mkp$ and p divides q. By symmetry, we obtain that q divides p and thereby $p=q$, which is a contradiction. □

Finally, Theorem E follows from Theorems 45 and A, and the fact that the action of a one-ended hyperbolic group with infinite group of outer automorphisms acts faithfully on its JSJ tree. The latter assertion follows from a similar argument to the one of Proposition 49 (in this case the intersection of the stabilizers of two different edges without common vertices is trivial).

1.5 Surface groups and Kähler groups

In this section we use Theorem B to study Kähler extensions of solvable groups by surface groups. First, we focus on the simpler case of Kähler extensions of Abelian groups by surface groups, which gives an alternative proof of Bregman and Zhang's result. At the end of this section we present a slightly more general statement of Theorem B.

1.5.1 Kähler extension of an Abelian group by a surface group

Corollary 54 (Bregman-Zhang). *Let S be a closed surface of genus $g \geq 2$, Γ be a Kähler group and k be a positive integer such that there is a short exact sequence*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \xrightarrow{P} \mathbb{Z}^k \longrightarrow 1.$$

Then there is a finite index subgroup Γ_1 of Γ containing $\pi_1(S)$ such that the restricted short exact sequence

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma_1 \xrightarrow{P} P(\Gamma_1) \simeq \mathbb{Z}^k \longrightarrow 1$$

splits as a direct product.

Note that k must be even since Γ_1 is a Kähler group. Before proving Corollary 54, we will recall some definitions and facts about the mapping class group of a closed oriented surface S (these can be found with more details in [9]). Let $\mathscr{S}(S)$ denote the set of isotopy classes of simple closed curves in S. If τ is an element of the mapping class group of S and $\mathscr{A}$ is a subset of $\mathscr{S}(S)$, we denote by $\tau(\mathscr{A})$ the set

$$\{\tau(\alpha) | \alpha \in \mathscr{A}\},$$

where $\tau(\alpha)$ denotes the isotopy class of $t(a)$ for $t \in \tau$ and $a \in \alpha$. A subset $\mathscr{A}$ of $\mathscr{S}(S)$ is said admissible if there is a set of disjoint simple closed curves that represent all the isotopy classes of $\mathscr{A}$. Notice that an admissible subset of $\mathscr{S}(S)$ must be finite. An element τ of the mapping class group of S is said to be

1. *reducible*: if there is a non-empty admissible set $\mathscr{A}$ such that $\tau(\mathscr{A}) = \mathscr{A}$,

2. *pseudo-Anosov*: if for any isotopy class α in $\mathscr{S}(S)$ and for every nonzero integer n, $\tau^n(\alpha)$ is different from α (this is one definition of a pseudo-Anosov mapping class among many others).

For a closed oriented surface S, the Nielsen-Thurston classification Theorem states the following (see [32] Chapter 13).

Theorem 55 (Nielsen-Thurston). *Every element of the mapping class group of S is either of finite order, reducible or pseudo-Anosov. Furthermore, pseudo-Anosov mapping classes are neither periodic nor reducible.*

To prove Corollary 54 we will need the Nielsen-Thurston classification Theorem and two lemmas. The first lemma is due to Birman, Lubotzky and McCarthy (see Lemma 3.1 in [9]).

Lemma 56 (Birman-Lubotzky-McCarthy)**.** *Let A be an Abelian subgroup of the mapping class group of a closed oriented surface of genus $g \geq 2$ generated by reducible elements $\{\tau_1, \dots, \tau_k\}$. Then, there is a non-empty canonical admissible set $\mathscr{A}$ such that $\tau_i(\mathscr{A}) = \mathscr{A}$ for all $i = 1, \dots, k$.*

The second lemma is the following. It will be a consequence of Thurston's hyperbolization Theorem (see [58]), together with a classical result of Carlson and Toledo [22].

Lemma 57. *Let S be a closed surface of genus $g \geq 2$, Γ be a Kähler group and k be a positive integer such that there is a short exact sequence*

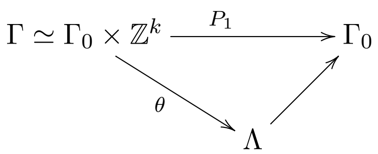
$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \xrightarrow{P} \mathbb{Z}^k \longrightarrow 1.$$

Then the image of the monodromy map $\psi : \mathbb{Z}^k \to \mathrm{MCG}(S)$ does not contain pseudo-Anosov elements.

Proof. Let us assume by contradiction that the image of ψ contains a pseudo-Anosov element τ. It is well known that the cyclic subgroup generated by a pseudo-Anosov element is a finite index subgroup of its centralizer (see [51]). Hence, by passing to a finite index subgroup of Γ if necessary (which is also a Kähler group), we may assume that the image of ψ is the cyclic subgroup generated by τ. Let $\{e_1, \dots, e_k\}$ be a basis of $\mathbb{Z}^k$ such that $\{e_1, \dots, e_{k-1}\}$ generates the kernel of ψ and $\psi(e_k) = \tau$. We denote by Γ_0 the subgroup of Γ generated by $\pi_1(S)$ and $P^{-1}(e_k)$.

Claim 58. *Γ is isomorphic to $\Gamma_0 \times \mathbb{Z}^{k-1}$.*

Claim 58 implies the result since by Thurston's hyperbolization Theorem (see [58]), Γ_0 is a cocompact lattice in the group of orientation preserving isometries of the hyperbolic 3-space and by Carlson and Toledo's Theorem (see [22]), the projection of Γ onto the cocompact lattice Γ_0 factors through a surface group Λ. Hence, there is a commutative diagram



where P_1 is the projection onto Γ_0. This leads to the desired contradiction, since $\theta(\Gamma_0)$ is a subgroup of Λ isomorphic to Γ_0 (as a consequence of the fact that $P_1 \restriction_{\Gamma_0}$ is the identity map), and every subgroup of Λ is the fundamental group of a (closed or open) surface, but Γ_0 is neither free nor isomorphic to the fundamental group of a closed oriented surface since $H_3(\Gamma_0, \mathbb{Z})$ is non-trivial. ☐

Proof of Claim 58. Observe that for each e_i in the kernel of ψ, there is a unique element g_i in Γ such that g_i centralizes $\pi_1(S)$ and $P(g_i) = e_i$. Since the commutator $[g_i, g_j]$ is an element of $\pi_1(S)$ which centralizes $\pi_1(S)$ we get that the group K generated by $\{g_1, \dots, g_{k-1}\}$ is isomorphic to $\mathbb{Z}^{k-1}$ and by construction $[K, \pi_1(S)]$ is trivial. Using the fact that $\pi_1(S)$ is a normal subgroup of Γ and that K centralizes $\pi_1(S)$ we obtain that $[K, \Gamma]$ centralizes $\pi_1(S)$. Finally, since $[K, \Gamma]$ is in the kernel of P (which is $\pi_1(S)$) we conclude that $[K, \Gamma]$ is in the center of $\pi_1(S)$ which is trivial. Therefore the subgroups K and Γ_0 commute and the result follows from observing that $\Gamma = K\Gamma_0$ and $K \cap \Gamma_0$ is trivial. ☐

Now, we give our proof of Corollary 54:

Proof of Corollary 54. By Nielsen-Thurston's classification Theorem and Lemma 57, up to passing to a finite index subgroup of $\mathbb{Z}^k$ we have that the image of the monodromy map $\psi : \mathbb{Z}^k \to \mathrm{MCG}(S)$ is generated by reducible elements. Hence, by Lemma 56 there is a non-empty canonical admissible set $\mathscr{A}$ of $\mathscr{S}(S)$ such that $\psi(\mathbb{Z}^k)$ preserves $\mathscr{A}$. Since a finite index subgroup of $\mathbb{Z}^k$ (which is isomorphic to $\mathbb{Z}^k$) acts trivially on $\mathscr{A}$, we can apply Theorem B which concludes the proof. □

1.5.2 Kähler extension by a surface group whose monodromy map has solvable image

Corollary 59. *Let S be a closed surface of genus $g \geq 2$ and Γ be a Kähler group such that there is a short exact sequence*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \xrightarrow{P} Q \longrightarrow 1.$$

If the image of the monodromy map $\psi : Q \to \mathrm{MCG}(S)$ is solvable, then there are finite index subgroups Γ_1 of Γ and Q_1 of Q such that $\pi_1(S)$ is contained in Γ_1 and the restricted short exact sequence

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma_1 \longrightarrow Q_1 \longrightarrow 1$$

splits as a direct product.

According to a result of Birman, Lubotzky and McCarthy (see [9, Theorem B]), every solvable subgroup of the mapping class group of a closed oriented surface of genus $g \geq 2$ is virtually Abelian. Hence, to prove Corollary 59 it suffices to prove a result analogous to Lemma 57 for the case when the monodromy map $\psi : Q \to \mathrm{MCG}(S)$ has Abelian image.

Lemma 60. *Let S be a closed surface of genus $g \geq 2$ and Γ be a Kähler group such that there is a short exact sequence*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \xrightarrow{P} Q \longrightarrow 1.$$

If the image of the monodromy map $\psi : Q \to \mathrm{MCG}(S)$ is Abelian, then it does not contain pseudo-Anosov elements.

Proof. The proof is by contradiction. Let us suppose that the image of ψ contains a pseudo-Anosov element τ. As in Lemma 57, up to passing to a finite index subgroup of Γ we may assume that the image of ψ is the cyclic subgroup generated by τ. Let t be an element in Γ such that $\psi \circ P(t) = \tau$ and let Γ_0 be the subgroup of Γ generated by $\pi_1(S)$ and t. The result will follow from exhibiting a homomorphism $\Gamma \to \Gamma_0$ whose restriction to Γ_0 is the identity, since Thurston's hyperbolization Theorem and Carlson and Toledo's Theorem will lead us to a contradiction as in Lemma 57.

Now, to construct this homomorphism, observe that since each element in the kernel of ψ has a unique lift in Γ that centralizes $\pi_1(S)$, there is a subgroup K of Γ isomorphic to the kernel of ψ that centralizes $\pi_1(S)$. Indeed, K is the centralizer of $\pi_1(S)$ in Γ. Hence, the subgroup $[\Gamma, K]$ centralizes $\pi_1(S)$ and since its image under P is contained in the kernel of ψ, we get that $[\Gamma, K]$ is contained in K and therefore K is a normal subgroup of Γ. Finally, since the kernel of $\psi \circ P$ is isomorphic to $K \times \pi_1(S)$, we obtain that Γ is isomorphic to the semi-direct product $(K \times \pi_1(S)) \rtimes \langle t \rangle$, and thereby there is a well-defined homomorphism

$$\begin{aligned} \Gamma &\to \Gamma_0 \\ kxt^m &\mapsto xt^m, \end{aligned}$$

where k is in K, x is in $\pi_1(S)$ and m is in $\mathbb{Z}$. To verify that it is a well-defined map we just need to observe that for $k_1x_1t^{m_1}$ and $k_2x_2t^{m_2}$ in Γ we have that

$$k_1x_1t^{m_1}k_2x_2t^{m_2} = (k_1t^{m_1}k_2t^{-m_1})(x_1t^{m_1}x_2t^{-m_1})t^{m_1+m_2},$$

where we notice that $t^{m_1}k_2t^{-m_1}$ is in K and $t^{m_1}x_2t^{-m_1}$ is in $\pi_1(S)$. □

The following result is a consequence of a theorem due to Ivanov which states the following. Let S be a closed oriented surface and Γ be an infinite subgroup of $\mathrm{MCG}(S)$ that does not preserve any admissible set $\mathscr{A}$ of $\mathscr{S}(S)$. Then, Γ is either virtually a cyclic group generated by a pseudo-Anosov element, or it contains a free group generated by 2 pseudo-Anosov elements (see [41, Theorem 2]).

Corollary 61. *Let S be a closed surface of genus $g \geq 2$ and Γ a a Kähler group such that there is a short exact sequence*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \xrightarrow{P} Q \longrightarrow 1.$$

If the image of the monodromy map $\psi : Q \to \mathrm{MCG}(S)$ is infinite, then it contains a free subgroup generated by 2 pseudo-Anosov elements.

Proof. Recall that an extension as above is virtually trivial if and only if the monodromy subgroup of the short exact sequence is finite. Therefore, by Theorem B, $\psi(Q)$ does not preserve any admissible set $\mathscr{A}$ of $\mathscr{S}(S)$. Finally, the result follows from Lemma 60 and Ivanov's result recalled before the statement. □

1.5.3 More restrictions on Kähler extensions by a surface group

A classical result due to Scott (see [64]) allows us to extend Theorem B to any closed curve in S.

Theorem 62 (Scott)**.** *Let S be a topological surface and let x be an element of $\pi_1(S)$. Then, there is a finite covering map $S' \to S$ such that x is in $\pi_1(S')$ and can be represented by a simple closed curve in S'.*

Lemma 63. *Let G be a group and H be a finitely generated normal subgroup of G. If Λ is a finite index subgroup of H, then the normalizer of Λ in G is a finite index subgroup of G.*

Proof. Let n be the index of Λ in H. Then G acts on the set of subgroups of H of index n by conjugation. This set is finite and therefore the stabilizer of any of these subgroups gives a finite index subgroup of G. Finally, notice that the stabilizer of Λ is precisely the normalizer of Λ in G and the result follows. □

Theorem 64. *Let Γ be a Kähler group such that there is a short exact sequence*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \xrightarrow{P} Q \longrightarrow 1,$$

with S a closed surface of genus $g \geq 2$. If the conjugation action of Γ on $\pi_1(S)$ preserves the conjugacy class of a non-trivial element of $\pi_1(S)$, then there is a finite index subgroup Γ_1 of Γ containing a finite index subgroup Λ of $\pi_1(S)$ which is normal in Γ_1 such that the extension

$$1 \longrightarrow \Lambda \longrightarrow \Gamma_1 \longrightarrow \Gamma_1/\Lambda \longrightarrow 1$$

splits as a direct product.

Proof. Let γ be the closed curve in S whose conjugacy class is preserved by the action of Γ. We may assume that γ is not simple. By Theorem 62, there is a finite covering map $S' \to S$, such that γ lifts to a simple closed curve in S'. Let Λ be the fundamental group of S' and let Γ' be the normalizer of Λ in Γ. By Lemma 63, Γ' is a finite index subgroup of Γ and therefore Γ' is a Kähler group. Hence, the short exact sequence

$$1 \longrightarrow \Lambda \longrightarrow \Gamma' \longrightarrow \Gamma'/\Lambda \longrightarrow 1$$

satisfies the hypothesis of Theorem B, which implies the existence of a subgroup Γ_1 of Γ' of finite index (and therefore a finite index subgroup of Γ) such that the exact sequence

$$1 \longrightarrow \Lambda \longrightarrow \Gamma_1 \longrightarrow \Gamma_1/\Lambda \longrightarrow 1,$$

splits as a direct product. □

Chapter 2

Irrational pencils and Betti numbers

We study irrational pencils with isolated critical points on compact aspherical complex manifolds. We prove that if the set of critical points is nonempty, the homology of the kernel of the morphism induced by the pencil on fundamental groups is not finitely generated. This generalizes a result by Dimca, Papadima and Suciu. By considering self-products of the Cartwright-Steger surface, this allows us to build new examples of smooth projective varieties whose fundamental group has a non-finitely generated homology.

2.1 Exotic finiteness properties and irrational pencils

2.1.1 Some history

In Section 2.2 we will define the topological finiteness property $\mathscr{F}_n$ and the homological finiteness property FP_n. As we saw in the introduction, Dimca, Papadima and Suciu constructed the first examples of Kähler groups satisfying an exotic finiteness property, *i.e.* Kähler groups that are of type $\mathscr{F}_{n-1}$ but not of type FP_n. Let us introduce some notations before recalling their results.

Notations. Let X be a (connected) compact complex manifold of complex dimension $n \geq 2$ and S a closed Riemann surface of positive genus. Recall that a surjective holomorphic map with connected fibers $f : X \to S$ is called an *irrational pencil*. For such a map, we will always denote by Λ the kernel of the induced homomorphism $f_* : \pi_1(X) \to \pi_1(S)$.

The results of Dimca, Papadima and Suciu [31] are the following:

Theorem 65 (Dimca, Papadima and Suciu)**.** *If $n \geq 3$ and if $f : X \to S$ is an irrational pencil with isolated critical points, then the fundamental group of a smooth fiber of f embeds into that of X and coincides with the kernel Λ of the induced homomorphism $f_* : \pi_1(X) \to \pi_1(S)$.*

Theorem 66 (Dimca, Papadima and Suciu)**.** *Let $X = \Sigma_1 \times \cdots \times \Sigma_n$ be a direct product of n Riemann surfaces of genus greater than 1 and let S have genus 1. If $f : X \to S$ is an irrational pencil with isolated critical points then the group $H_n(\Lambda, \mathbb{Q})$ has infinite dimension.*

Lemma 67. *Let X and Y be compact complex manifolds and let S be a closed Riemann surface of genus 1. If $f : X \times Y \to S$ is a holomorphic map, then there exist holomorphic maps $f_1 : X \to S$ and $f_2 : Y \to S$ such that*

$$f(x, y) = f_1(x) + f_2(y).$$

Proof. We view S as the quotient of $\mathbb{C}$ by a lattice Λ, so that tangent vectors to S can be identified with complex numbers. Given a point y in Y and a tangent vector v in T_yY, we have that the holomorphic map

$$\begin{array}{rcl} X & \longrightarrow & \mathbb{C} \\ x & \mapsto & df_{(x,y)}(v,0) \end{array}$$

is constant. This follows from the compactness of X. If now we fix a point x_0 in X, the group structure on S allows us to define the holomorphic map

$$\begin{array}{rcl} X \times Y & \longrightarrow & S \\ (x,y) & \mapsto & f(x,y) - f(x_0,y). \end{array}$$

We deduce from the above argument that this map only depends on x and therefore

$$f(x,y) = f_1(x) + f(x_0,y),$$

which implies the result. □

Remark 68. *Lemma 67 implies that any irrational pencil $f : X \to S$ as in Theorem 66 is the sum of holomorphic maps $f_i : \Sigma_i \to S$, i.e.*

$$f(p_1,\ldots,p_n) = f_1(p_1) + \cdots + f_n(p_n).$$

If all the f_i's are nonconstant and $n \geq 2$, f has connected fibers if and only if it is π_1-surjective, see Lemma 2.1 in [49]. The set of critical points of f is the product of the critical sets of the f_i's. Hence f has isolated critical points if and only if all the f_i's are nonconstant.

Let $\widehat{X} \to X$ the covering space of X such that $\pi_1(\widehat{X}) \simeq \Lambda$. Under the hypothesis of Theorem 65, the authors proved that the inclusion of a smooth fiber of f in $\widehat{X}$ induces isomorphisms on the homotopy groups of dimension $0,\ldots,n-2$ (see Lemma 3.3 in [31]). Hence, if $f : X \to S$ is an irrational pencil as in Theorem 66, we get that the group Λ is of type $\mathscr{F}_{n-1}$ and by Theorem 66 it cannot be of type FP_n as the group $H_n(\Lambda,\mathbb{Q})$ has infinite dimension (see Proposition 87).

2.1.2 Finiteness properties for arbitrary pencils

Let $f : X \to S$ be an irrational pencil with $\dim_{\mathbb{C}} X = n \geq 2$. As before, let Λ denote the kernel of the morphism $f_* : \pi_1(X) \to \pi_1(S)$ and let $\widehat{X} \to X$ be the covering space such that $\pi_1(\widehat{X}) \simeq \Lambda$. We assume that the critical points of f are isolated and that f is not a submersion; its critical set is then a nonempty finite set. Our main results are the following:

Theorem F. *The homology group $H_n(\widehat{X},\mathbb{Q})$ has infinite dimension.*

Theorem G. *If X is aspherical, the group $H_n(\Lambda,\mathbb{Q})$ has infinite dimension. In particular Λ is not of type* FP_n.

If X is aspherical, the space $\widehat{X}$ is a classifying space of the group Λ (see Remark 110). Hence Theorem G follows from Theorem F. The main ingredient of the proof of Theorem F is the study of the topology of the covering space $\widehat{X}$ (see Section 2.4.3). This method, which can be seen as a complex analog of Bestvina and Brady's work (see [6]) appears already in the articles [31, 43].

When $n \geq 3$, the group Λ is isomorphic to the fundamental group of a generic fiber of f, thanks to Theorem 65. Hence, when $n \geq 3$ and X is aspherical and Kähler (resp. projective), Λ is a Kähler (resp. projective) group of type $\mathscr{F}_{n-1}$ but not of type FP_n.

2.1.3 Related results

2.1.3.a Direct products of Riemann surfaces

In the special case when X is a product of Riemann surfaces and S has genus 1, Theorem G reduces to Theorem 66 above. Dimca, Papadima and Suciu's proof was based on the notion of *characteristic varieties*. However, the proof we will present in Section 2.4.4 is more direct and applies in full generality: X can be any aspherical complex manifold and the genus of S need not be equal to 1. On the other hand the article [31] also studies the finiteness properties of arbitrary normal coabelian subgroups of direct products of fundamental groups of closed surfaces, not necessarily coming from irrational pencils.

If X is a product of Riemann surfaces, the fact that Λ is not of type FP_n can be deduced from the work of Bridson, Howie, Miller and Short [13]. See [49, 50] for further results which rely on properties of subgroups of direct products of surface groups.

2.1.3.b Irrational pencils with nondegenerate critical points

Under similar assumptions as in Theorem G, Biswas, Mj and Pancholi proved the following result (see Theorem 7.3 in [10]), whose conclusion is weaker than the one of Theorem G.

Theorem 69 (Biswas, Mj and Pancholi)**.** *Let $f : X \to S$ be an irrational pencil which is not a submersion and whose critical points are isolated. If X is an aspherical complex manifold of dimension greater or equal than 3, then the fundamental group of the generic fiber of f is not of type* FP*.*

See Section 2.2.3 for the definition of the homological finiteness property FP.

2.1.3.c Aspherical compact Kähler surfaces

We recall the result of Kapovich on irrational pencils of aspherical compact Kähler surfaces stated in the introduction (see [43]).

Theorem 70. *Let X be an aspherical compact Kähler surface, S a compact Riemann surface of positive genus and $f : X \to S$ an irrational pencil whose singular fibers have multiplicity one. Then f is a submersion or the kernel of the induced map on fundamental groups is not finitely presented.*

In the case of maps with isolated critical points, Theorem G gives a slight strengthening of Kapovich's result since a finitely presented group must have finitely generated second homology group. A detailed proof of Theorem 70 will be given in Section 2.6. We will see as well in this section that the hypothesis on singular fibers implies that the kernel of $f_* : \pi_1(X) \to \pi_1(S)$ is finitely generated, thus showing that $\pi_1(X)$ is not coherent if f is not a submersion.

2.2 Homology of a group, classifying spaces and finiteness properties

2.2.1 Homology of a group

Here we summarize some definitions and results about the homology of a group. See [16] for an introduction to this subject.

Definition 71. Let G be a group. The integral group ring $\mathbb{Z}G$ is defined as the free $\mathbb{Z}$-module generated by the elements of G.

An element of $\mathbb{Z}G$ is uniquely expressed as

$$\sum_{g \in G} n(g)g,$$

where $n(g)$ is an integer and $n(g) = 0$ for all but finitely many g in G.

Definition 72. A $\mathbb{Z}G$-module is an Abelian group A together with an action of G on A.

Given two $\mathbb{Z}G$-modules A and B, a *homomorphism of* $\mathbb{Z}G$*-modules* is a homomorphism of Abelian groups $\varphi : A \to B$ such that for all g in G and for all a in A, $\varphi(g \cdot a) = g \cdot \varphi(a)$.

Example 73. *Let G be a group. The trivial action of G on $\mathbb{Z}$ induces a structure on $\mathbb{Z}$ of $\mathbb{Z}G$-module.*

A *free* $\mathbb{Z}G$*-module* is a $\mathbb{Z}G$-module of the form $\oplus_{i \in I}\mathbb{Z}G$. We will also use the notion of projective $\mathbb{Z}G$-module, we refer the reader to [16, p. 21] for the definition.

Definition 74. Let G be a group. A projective (resp. free) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ is an exact sequence of $\mathbb{Z}G$-modules

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0,$$

such that P_i is a projective (resp. free) $\mathbb{Z}G$-module for every $i \geq 0$.

Let G be a group, A be a $\mathbb{Z}G$-module and let us consider the subgroup of A generated by the elements of the form $g \cdot a - a$, where g is an element of G and a is an element of A. The quotient of A by this subgroup is called *the group of co-invariants of* A and it is denoted by A_G. Since the group of co-invariants of A is obtained from A by identifying each element of A with all the elements of its G-orbit, we obtain that A_G is a $\mathbb{Z}G$-module endowed with a trivial G-action.

Remark 75. *If A is a free $\mathbb{Z}G$-module with basis $\{a_i\}_{i \in I}$, and if we denote by $\bar{a}_i$ the image of a_i in A_G, then A_G is a free Abelian group with basis $\{\bar{a}_i\}_{i \in I}$.*

Definition 76. Let G be a group and let

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The n-th homology group of G is defined as

$$H_n(G) = \frac{\ker\left((P_n)_G \to (P_{n-1})_G\right)}{\operatorname{Im}((P_{n+1})_G \to (P_n)_G)}$$

Every $\mathbb{Z}G$-module has a projective resolution. Then, one can always define the homology groups of G. Moreover, these groups do not depend on the choice of the projective resolution (see for instance Chapter 1 of [16]). If M is a $\mathbb{Z}G$-module, one can define the homology groups of G with coefficients in M in a similar way as before, by considering a projective resolution of M over $\mathbb{Z}G$. Then, $H_n(G)$ is the n-th homology group of G with coefficients in $\mathbb{Z}$. In this chapter we will work with the homology groups of a group G with coefficients in $\mathbb{Q}$, where the trivial action of G on $\mathbb{Q}$ induces a structure of $\mathbb{Z}G$-module on $\mathbb{Q}$.

2.2.2 Classifying space

Definition 77. A *classifying space of a discrete group G* (or a $K(G,1)$ *space*) is a topological space X such that

1. X is path connected.
2. The fundamental group of X is isomorphic to G.
3. The universal covering space of X is contractible.

The last condition implies that X is aspherical, *i.e.* that $\pi_k(X)$ is trivial for $k \geq 2$. In the context of CW-complexes (see [40] for an introduction to this subject), being aspherical and having a contractible universal covering space are equivalent conditions. This is a consequence of a theorem of Whitehead. Combined with a theorem of Hurewicz it implies that for a CW-complex, being aspherical is equivalent to having a universal covering space with trivial integral homology. See [40] for all the details of these statements. Throughout this text we will work with smooth manifolds and CW-complexes, and we will use these equivalences without recalling them.

Example 78. *Since the universal covering space of $\mathbb{S}^1$ is $\mathbb{R}$ which is contractible, we get that $\mathbb{S}^1$ is a classifying space of $\mathbb{Z}$.*

Example 79. *If X is a classifying space of G and Y is a classifying space of H, then $X \times Y$ is a classifying space of $G \times H$.*

The following example shows that given a group G, one can always choose a classifying space of G which is a CW-complex. In this case the homotopy type of such a space is uniquely defined.

Example 80. *Given a presentation of a group $G = \langle g_1, g_2, \dots | r_1, r_2, \dots \rangle$ one can construct a classifying space of G which is a* CW*-complex as follows. The* 1*-skeleton X^1 is given by a wedge-sum of circles, one for each generator g_i. Hence, each generator g_i of G defines a loop γ_i in X^1. Now, each relation $r_i = g_{i_1} g_{i_2} \cdots g_{i_k}$ defines a cellular map $\mathbb{S}^1 \to X^1$ whose image is the concatenation of the loops $\gamma_{i_1}, \gamma_{i_2}, \dots, \gamma_{i_k}$. By attaching a* 2*-cell to X_1 for each of this cellular maps, one obtains a* CW*-complex X^2 of dimension* 2 *whose fundamental group is isomorphic to G. Now, one can attach a* 3*-cell to X^2 for each generator of $\pi_2(X^2)$ to obtain a* CW*-complex X^3 of dimension* 3*, whose fundamental group is isomorphic to G and such that $\pi_2(X^3)$ is trivial. One can iterate this process as follows. Once the* CW*-complex X^k of dimension k is constructed, one can attach a $(k+1)$-cell to X^k for each generator of $\pi_k(X^k)$ to obtain a* CW*-complex X^{k+1} of dimension $k+1$ whose fundamental group is isomorphic to G and whose homotopy groups of dimension $2, \dots, k$ are trivial. The limit space obtained in this way is the desired space.*

Let G be a group and let X be a classifying space for G which is a CW-complex. The universal covering space $\widehat{X} \to X$ of X inherits a CW-structure and the action of G on $\widehat{X}$ freely permutes the cells of $\widehat{X}$. In particular, G acts on the set of n-cells of $\widehat{X}$. Therefore, the free Abelian group generated by the set of n-cells of $\widehat{X}$ denoted by $C_n(\widehat{X})$ is a free $\mathbb{Z}G$-module with one basis element for each G-orbit of n-cells.

Therefore, one can define the $\mathbb{Z}G$-module $C_n(\widehat{X})$ as the free Abelian group generated by the set of n-cells of $\widehat{X}$. The action of G on this set extends to a $\mathbb{Z}$-linear action of G on $C_n(\widehat{X})$. By construction, one obtains that the set of n-cells of X is a basis of $C_n(\widehat{X})$ as a $\mathbb{Z}G$-module.

Definition 81. The canonical augmentation map $\epsilon : C_0(\widehat{X}) \to \mathbb{Z}$ is the map of $\mathbb{Z}G$-modules that sends every 0-cell of $\widehat{X}$ to 1.

For each positive integer n, there exists a boundary map $C_n(\widehat{X}) \to C_{n-1}(\widehat{X})$ which is a map of $\mathbb{Z}G$-modules. With the canonical augmentation map, this defines a chain complex of $\mathbb{Z}G$-modules

$$\cdots \to C_n(\widehat{X}) \to C_{n-1}(\widehat{X}) \to \cdots \to C_0(\widehat{X}) \to \mathbb{Z} \to 0.$$

This sequence is called the *augmented cellular chain complex of the universal cover of* X. We refer to [40, p. 137-139] for a precise definition of the boundary maps $C_n(\widehat{X}) \to C_{n-1}(\widehat{X})$.

Proposition 82. *Let G be a group and let X be a classifying space of G. Then the augmented cellular chain complex of the universal covering space of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

By the latter result, one can compute the homology groups of G by using the augmented cellular chain complex of the universal covering space of X. Moreover, we have the following result (see Chapter 2 of [16]).

Proposition 83. *Let G be group. If X is a classifying space of G which is a* CW*-complex, then*

$$H_*(G) \simeq H_*(X).$$

We have an equivalent result of the latter proposition for homology groups with coefficients in $\mathbb{Q}$.

2.2.3 Finiteness properties

A group G is finitely generated if there exists a free group F of finite rank and a surjective homomorphism $F \to G$. Equivalently, G is finitely generated if there exists a finite subset S of G such that the smallest subgroup of G containing S is equal to G. If in addition there exists a finite subset R of F whose normal closure (*i.e.* the smallest normal subgroup of F containing R) is the kernel of the map $F \to G$, we say that G is finitely presented. These are the classical finiteness properties in group theory. Here we present more finiteness properties of groups.

2.2.3.a Homological

Definition 84. A group G is of type

1. FP, if there exists a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_i is finitely generated for all $i \geq 0$ and trivial for all but finitely many $i \geq 0$.

2. FP_n, if there exists a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_i is finitely generated for all $i \leq n$.

3. FP_∞, if there exists a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_i is finitely generated for all $i \geq 0$.

As a consequence of a generalization of Schanuel's Lemma (see Chapter $VIII$ Section 4 of [16]), we have the following result.

Proposition 85. *A group G is of type* FP_n *if and only if there exists a free resolution of* $\mathbb{Z}$ *over* $\mathbb{Z}G$

$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0,$$

such that F_i is of finite rank for all $i \leq n$.

2.2.3.b Topological

Definition 86. A group G is of type

1. $\mathscr{F}$, if there exists a classifying space for G which is a finite CW-complex.
2. $\mathscr{F}_n$, if there exists a classifying space for G which is a CW-complex with finite n-skeleton.
3. $\mathscr{F}_\infty$, if there exists a classifying space or G which is a CW-complex of finite type, (*i.e.*, having a finite number of cells of each dimension).

The topological finiteness condition $\mathscr{F}_n$ and $\mathscr{F}_\infty$ were introduced by Wall [76]. An important consequence of Proposition 82, is that the topological finiteness condition $\mathscr{F}_n$ (resp. $\mathscr{F}_\infty$) implies the homological finiteness condition FP_n (resp. FP_∞). One can verify this as follows: let G be a group of type $\mathscr{F}_n$ and let X be a classifying space of G with finite n-skeleton. Then, by considering the augmented cellular chain complex of the universal covering space of X, we obtain a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Since $C_i(\widehat{X})$ is a free $\mathbb{Z}G$-module of finite rank for all $i \leq n$ (it is generated by the i-cells of X), we conclude that G is of type FP_n.

Finally, observe that if G is of type FP_n, by Proposition 85, there is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0,$$

such that F_i is of finite rank for all $i \leq n$. For such a resolution we obtain that $(F_i)_G$ is a finitely generated free Abelian group for all $i \leq n$ (see Remark 75). Therefore, the kernel of $(F_i)_G \to (F_{i-1})_G$ is finitely generated. This implies that $H_i(G)$ is finitely generated for all $i \leq n$.

We can sum up the above discussion in the following result.

Proposition 87. *Let G be a group.*
- *If G is of type $\mathscr{F}_n$, then G is of type* FP_n
- *If G is of type* FP_n*, then $H_i(G)$ is finitely generated for all $i \leq n$.*

Remark 88. *Proposition 87 implies that if G is of type* FP_n*, then $H_i(G, \mathbb{Q})$ has finite dimension for all $i \leq n$.*

Proposition 89. *If G is a finitely presented group, the finiteness conditions $\mathscr{F}_n$ (resp. $\mathscr{F}_\infty$) and* FP_n *(resp.* FP_∞*) are equivalent.*

See Chapter 8 Section 7 of [16] for a proof of Proposition 89.

Given a group G one can ask whether it satisfies a topological or a homological finiteness property. We refer to [6, 7, 13, 69] for important works on these notions. For a group G, properties $\mathscr{F}_1$ and $\mathscr{F}_2$ are respectively equivalent to being finitely generated and finitely presented. Since the class of finitely generated groups is uncountable (see for instance [39, p. 60-61]) and the class of finitely presented groups is countable, there are many examples of groups which are finitely generated but not finitely presented. It is natural to ask whether one can construct groups of type $\mathscr{F}_{n-1}$ which are not of type $\mathscr{F}_n$. In this context, Stallings [69] constructed the first example of a group of type $\mathscr{F}_2$ which is not of type $\mathscr{F}_3$. This construction was generalized by Bieri [8] to construct groups which are of type $\mathscr{F}_{n-1}$ but not of type $\mathscr{F}_n$. These groups can be described as follows.

Example 90 (Stallings-Bieri)**.** *Let F_2 be the free group with two generators and let us consider the n-fold direct product F_2^n of F_2 with itself. Then the kernel of the map $F_2^n \to \mathbb{Z}$ that sends all generators to 1 is a group of type $\mathscr{F}_{n-1}$ but not of type $\mathscr{F}_n$.*

There are not so many constructions of groups which are of type $\mathscr{F}_{n-1}$ but not of type FP_n (or $\mathscr{F}_n$). In this chapter we provide a method to construct examples of groups with these properties using complex geometry.

Remark 91. *If X is a* CW*-complex with finite n-skeleton such that $\pi_k(X)$ is trivial for all $k = 2, \ldots, n-1$, then $\pi_1(X)$ is a group of finiteness type $\mathscr{F}_n$. This follows from the fact that we can attach cells of dimension greater than n to obtain a space X' whose fundamental group is isomorphic to that of X and such that $\pi_k(X')$ is trivial for all $k \geq 2$ (see Example 80). Hence X' is an aspherical* CW*-complex with finite n-skeleton such that $\pi_1(X) \simeq \pi_1(X')$.*

We conclude this section with the following discussion on commensurability that clarifies the fact that the results of Dimca, Papadima and Suciu [31] gave a negative answer to Kollár's question (discussed in the introduction of this text).

Definition 92. Two groups G and H are commensurable if there exist subgroups $G_1 < G$ and $H_1 < H$ of finite index such that G_1 and H_1 are isomorphic.

Two groups G and H are commensurable up to finite kernels if there exists a finite sequence

$$G = G_1 \to G_2 \leftarrow G_3 \to \cdots \leftarrow G_n = H,$$

where each arrow indicates a homomorphism of groups with finite kernel and with image of finite index.

Proposition 93. *The property of being of type $\mathscr{F}_n$ or FP_n is invariant by the commensurability relation up to finite kernels.*

For a proof of this statement see Corollary 7.2.4 and Theorem 7.2.21 in [34, Chapter 7] for condition $\mathscr{F}_n$, and Proposition 2.7 in [31] for condition FP_n.

2.3 Isolated critical points of holomorphic maps

In this section we recall some definitions and results about holomorphic maps with isolated critical points. Section 2.3.1 can be found with more details in Chapter 14 of [75] and Section 2.3.2 in Appendix B of [52].

Let X be a complex manifold and let $f : X \to \mathbb{C}$ be a holomorphic map.

Definition 94. A point x in X is called a *critical point* of f if the differential map $df_x : T_xX \to \mathbb{C}$ is identically zero, otherwise it is called a regular point. A critical point x of f is called isolated if there exists a neighborhood U of x such that x is the only critical point of f contained in U. The preimage $F_t = f^{-1}(t)$ of a point t in $\mathbb{C}$, is called a singular fiber if it contains a critical point of f, otherwise it is called a regular fiber.

2.3.1 Vanishing sphere of a nondegenerate critical point

Let x in X be a critical point of a holomorphic map $f : X \to \mathbb{C}$ and let $(z_1, \ldots, z_n)$ be a holomorphic coordinate system in a neighborhood of x. The *Hessian of f at x* is defined as the $\mathbb{C}$-bilinear form on T_xX such that

$$\mathrm{H}ess_x f\left(\frac{\partial}{\partial z_i}|x, \frac{\partial}{\partial z_j}|x\right) = \frac{\partial^2 f}{\partial z_i \partial z_j}(x),$$

which is well defined since df_x is zero.

Definition 95. Let $f : X \to \mathbb{C}$ be a holomorphic map. A critical point x of f is called nondegenerate if the Hessian of f at x is a nondegenerate quadratic form, *i.e.* if the Hessian matrix $\left(\frac{\partial^2 f}{\partial z_i \partial z_j}\right)_{i,j}$ has nonzero determinant at x, otherwise it is called degenerate.

The following result known as the holomorphic Morse Lemma characterizes the nondegenerate critical points of a holomorphic map.

Lemma 96. *(holomorphic Morse lemma) Let $f : X \to \mathbb{C}$ be a holomorphic map and let x in X be a nondegenerate critical point of f. Then, there exist holomorphic coordinates $(z_1, \ldots, z_n)$ centered at x such that f can be written in these coordinates as*

$$f(z) = f(x) + \sum_{i=1}^{n} z_i^2.$$

For a proof of the latter result see [75, p. 46]. Now, consider the homogeneous polynomial of degree 2

$$\begin{aligned} f : \quad \mathbb{C}^n &\longrightarrow \mathbb{C} \\ (z_1, \ldots, z_n) &\mapsto \sum_{i=1}^{n} z_i^2. \end{aligned} \tag{2.3.1}$$

For such a function, $\bar{0} \in \mathbb{C}^n$ is the only critical point of f. Hence, the central fiber $F_0 = f^{-1}(0)$ is the only singular fiber of f and for any t in $\mathbb{C}^*$, the regular fiber $F_t = f^{-1}(t)$ is a complex $(n-1)$-manifold. Let ϵ be a positive real number and for any z in $\mathbb{C}^n$ let us write $z = x + iy$ with x, y in $\mathbb{R}^n$. Then the fiber F_ϵ has the form

$$F_\epsilon = \{z = x + iy \in \mathbb{C}^n \,|\, |x|^2 = \epsilon + |y|^2, x \perp y\},$$

where $x \perp y$ means that x and y are orthogonal for the usual inner product of $\mathbb{R}^n$. This fiber contains the real $(n-1)$-sphere of radius $\sqrt{\epsilon}$

$$\mathbf{S}_\epsilon = \{z = x + iy \in \mathbb{C}^n \,|\, |x|^2 = \epsilon, y = \bar{0}\},$$

which contracts to the origin in $\mathbb{C}^n$ as ϵ tends to 0.

A similar situation occurs for any fiber F_t. We use the following notation for the action of $\mathbb{C}^*$ on $\mathbb{C}^n$:

$$t \cdot (z_1, \ldots, z_n) = (tz_1, \ldots, tz_n).$$

Since

$$f\left(t \cdot (z_1, \ldots, z_n)\right) = t^2 f(z_1, \ldots, z_n),$$

we get that this action preserves the set of fibers $\{F_t\}_{t\in\mathbb{C}}$. In particular the singular fiber F_0 is invariant under this action. From here, we deduce that for all t in $\mathbb{C}^*$, the fibers F_t and $F_{|t|}$ are diffeomorphic and F_t contains the real $(n-1)$-sphere

$$\mathbf{S}_t = \frac{\sqrt{t}}{|\sqrt{t}|} \cdot \mathbf{S}_{|t|},$$

which contracts to the origin as $|t|$ tends to 0. Let $T\mathbb{S}^{n-1}$ be the tangent space of the unitary $(n-1)$-sphere. Then, for any positive real number ϵ, the map

$$\begin{array}{ccc} F_\epsilon & \longrightarrow & T\mathbb{S}^{n-1} \\ z = x + iy & \mapsto & \left(\frac{x}{|x|}, y\right) \end{array}$$

is a diffeomorphism that sends $\mathbf{S}_\epsilon$ to $\mathbb{S}^{n-1}$, seen as the zero section of $T\mathbb{S}^{n-1}$. Hence, the homology class of $\mathbf{S}_\epsilon$ is a generator of $H_{n-1}(F_\epsilon, \mathbb{Z})$ and in general the homology class of $\mathbf{S}_t$ is a generator of $H_{n-1}(F_t, \mathbb{Z})$.

Definition 97. The homology class of $\mathbf{S}_t$ in $H_{n-1}(F_t, \mathbb{Z})$ is called the vanishing sphere of the homogeneous polynomial (2.3.1).

Let B be the ball in $\mathbb{C}^n$ of radius r centered at $\bar{0}$. Then the image of B under the homogeneous polynomial (2.3.1) is exactly the disk of radius r^2 centered at 0 in $\mathbb{C}$ that we will denote by D. The following results allow us to understand the topology close to the origin of the singular fiber of such a function.

Lemma 98. *Let Δ be a disk in $\mathbb{C}$ of small radius with respect to the radius of B. Then the restriction of f to the boundary of B $f\restriction_{\partial B}: \partial B \to D$ is a submersion along $f^{-1}(\Delta) \cap \partial B$.*

This lemma implies that $f^{-1}(\Delta)\cap B$ is a manifold with corners whose boundary consists of the union of $f^{-1}(\Delta)\cap\partial B$ and $f^{-1}(\partial\Delta)\cap B$, which are manifolds with boundary meeting transversally along their boundaries.

The compactness of ∂B implies that $f\restriction_{\partial B}: \partial B \to D$ is a proper map. Therefore by Ehresmann's fibration Theorem we get that $f^{-1}(\Delta) \cap \partial B$ deformation retracts onto the intersection of a regular fiber of f with the boundary of B More precisely, if t is a complex number contained in $\Delta \setminus \{0\}$, then $f^{-1}(\Delta) \cap \partial B$ deformation retracts onto $f^{-1}(t) \cap \partial B$.

The cone over $\mathbf{S}_\epsilon$ with vertex at $\bar{0}$, is the real n-ball of radius $\sqrt{\epsilon}$

$$\mathbf{B}_\epsilon = \{z = x + iy \in \mathbb{C}^n \mid |x|^2 \leq \epsilon, y = \bar{0}\}.$$

Using the action of $\mathbb{C}^*$ on the set of fibers of f, one can define the cone over $\mathbf{S}_t$ with vertex at $\bar{0}$ as the real n-ball of radius $\sqrt{\epsilon}$

$$\frac{\sqrt{t}}{|\sqrt{t}|}\mathbf{B}_{|t|}.$$

Proposition 99. *Let Δ be a disk in $\mathbb{C}$ of small radius with respect to the radius of B and let t be a complex number contained in $\Delta\setminus\{0\}$. Then $f^{-1}(\Delta)\cap B$ deformation retracts onto $(f^{-1}(t) \cap B) \cup \mathbf{B}_t$. Moreover this retraction by deformation can be chosen to be induced by a retraction by deformation of $f^{-1}(\Delta) \cap \partial B$ onto $f^{-1}(t) \cap \partial B$.*

A global version of the last proposition is the following result.

Theorem 100. *Let X be a complex n-manifold, and $f : X \to \mathbb{C}$ a proper holomorphic map. Suppose that there exists a point x in X such that $f : X \setminus \{x\} \to \mathbb{C}$ is a submersion and x is a nondegenerate critical point of f. If $t \in \mathbb{C}$ is a regular value of f sufficiently close to $f(x)$, then X deformation retracts onto the union of the fiber $F_t = f^{-1}(t)$ with a real n-ball glued to F_t along a vanishing sphere of f contained in F_t.*

The proof of Theorem 100 can be found in [75] and it essentially an application of the holomorphic Morse Lemma and Proposition 99. Indeed, by Proposition 99, the n-ball glued to F_t is given by the cone over the vanishing sphere of f contained in F_t with vertex at x. We can deduce from this the following result.

Corollary 101. *Under the hypothesis of Theorem 100, given a neighborhood U of the nondegenerate critical point x, we can choose a regular value t sufficiently close to $f(x)$, so that X deformation retracts onto the union of the fiber $F_t = f^{-1}(t)$ with a real n-ball contained in U, glued to F_t along a vanishing sphere of f contained in F_t.*

2.3.2 Milnor number and degenerate critical points

Let $g_1, \ldots, g_n : (\mathbb{C}^n, \bar{0}) \to (\mathbb{C}, 0)$ be holomorphic map-germs and let us write $g = (g_1, \ldots, g_n)$. Let S be a small enough sphere centered at $\bar{0}$ and suppose that $\bar{0}$ is the only solution of

$$g_1(z) = g_2(z) = \cdots = g_n(z) = 0 \tag{2.3.2}$$

in the ball bounded by S. The *multiplicity of g at $\bar{0}$* is defined as the degree of the mapping

$$\begin{array}{ccc} S & \longrightarrow & \mathbb{S}^{2n-1} \\ z & \mapsto & \dfrac{g(z)}{|g(z)|}. \end{array}$$

We recall some important results on this notion. We refer to [52] for a detailed introduction to this subject and for the proofs of these results.

Lemma 102. *Suppose that $\bar{0}$ is an isolated solution of (2.3.2). If the Jacobian matrix $\left(\frac{\partial g_i}{\partial z_j}\right)_{i,j}$ has nonzero determinant at $\bar{0}$, then the multiplicity of $\bar{0}$ is equal to 1, otherwise it is strictly greater than 1.*

Lemma 103. *Let $\bar{0}$ be a an isolated solution of (2.3.2) of multiplicity μ and let B be a ball in $\mathbb{C}^n$ centered at $\bar{0}$ which does not contain any other solution of (2.3.2). Then, for almost all points ℓ in $\mathbb{C}^n$ sufficiently close to the origin, the equation $g(z) = \ell$ has precisely μ different isolated solutions of multiplicity 1 within B.*

Remark 104. *More precisely, the latter lemma holds if ℓ is a regular value of g, close enough to the origin. Sard's Theorem ensures that there are plenty of regular values.*

Suppose that $\bar{0}$ is an isolated critical point of a holomorphic map-germ $f : (\mathbb{C}^n, \bar{0}) \to (\mathbb{C}, 0)$. *The Milnor number μ of* $\bar{0}$ is defined as the multiplicity at $\bar{0}$ of

$$g = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right),$$

which measures the degeneracy of $\bar{0}$ as a critical point of f.

If f is defined in a complex manifold and x is an isolated critical point of f, by choosing a holomorphic coordinate system in a neighborhood U of x, we can identify U with an open ball B centered at the origin of $\mathbb{C}^n$. Under this identification $\bar{0}$ is an isolated critical point of $f\restriction_B\colon B \to \mathbb{C}$ with Milnor number equal to $\mu \geq 1$. The following remarks are an interpretation of Lemmas 102 and 103.

Remark 105. *The critical point x is nondegenerate if and only if μ is equal to* 1.

Remark 106. *If we perturb $f\restriction_B\colon B \to \mathbb{C}$ by subtracting a linear map $\ell_1 z_1 + \cdots + \ell_n z_n$ from $f\restriction_B$, which is a regular value of $df : B \to (\mathbb{C}^n)^*$ sufficiently close to the origin, then* 0 *splits up into μ nondegenerate critical points contained in B.*

2.4 Irrational pencils and fiber products

2.4.1 Fiber product

Let X and Y be topological spaces, $f : X \to Y$ a continuous map and $q : \widehat{Y} \to Y$ a covering map of Y. Then the fiber product of $f : X \to Y$ and $q : \widehat{Y} \to Y$ is defined as

$$\widehat{X} = \{(x, \hat{y}) \in X \times \widehat{Y} \mid f(x) = q(\hat{y})\}.$$

If we denote by $\widehat{f} : \widehat{X} \to \widehat{Y}$ and $\widehat{q} : \widehat{X} \to X$ the natural projections, we get that the diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{f}} & \widehat{Y} \\ {\scriptstyle \widehat{q}}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{2.4.1}$$

commutes. Moreover, $\widehat{f} : \widehat{X} \to \widehat{Y}$ is a continuous map whose fibers are isomorphic to those of f: let $\hat{y}$ be an element of $\widehat{Y}$, then

$$\begin{aligned} \widehat{f}^{-1}(\hat{y}) &= \{(x, \hat{y}) \in X \times \widehat{Y} \mid f(x) = q(\hat{y})\} \\ &= f^{-1}(q(\hat{y})) \times \{\hat{y}\}. \end{aligned}$$

Notice that if the continuois map f is given by an inclusion $X \hookrightarrow Y$, then the fiber product $\widehat{X}$ is isomorphic to $q^{-1}(X)$.

Let x_0, y_0 and $\hat{y}_0$ be base points of X, Y and $\widehat{Y}$ respectively such that $f(x_0) = y_0$ and $q(\hat{y}_0) = y_0$. We will denote by $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $q_* : \pi_1(\widehat{Y}, \hat{y}_0) \to \pi_1(Y, y_0)$ the induced maps on fundamental groups of f and q respectively. In general, the fiber product $\widehat{X}$ is not connected. The following result gives a condition that guarantees the connectedness of the fiber product.

Proposition 107. *If $q : \widehat{Y} \to Y$ is the universal covering map of Y, then the induced map on fundamental groups $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is surjective if and only if the fiber product $\widehat{X}$ is connected.*

Lemma 108. *The natural projection $\widehat{q} : \widehat{X} \to X$ is a covering map and the fundamental group of $\widehat{X}$ based at $(x_0, \hat{y}_0)$ is isomorphic to $f_*^{-1}(q_*(\pi_1(\widehat{Y}, \hat{y}_0)))$.*

Proof. Let x be an element of X and let U be an open neighborhood of $f(x)$ evenly covered by $q : \widehat{Y} \to Y$. We write $q^{-1}(U) = \cup_{i\in I} U_i$, where U_i is homeomorphic to U for all i in I. Let $V = f^{-1}(U)$ and let $V_i = \widehat{X} \cap (V \times U_i)$ for all i in I. Then, $\widehat{q}^{-1}(V) = \cup_{i\in I} V_i$, where V_i is homeomorphic to V for all i in I. Hence V is an open neighborhood of x evenly covered by $\widehat{q} : \widehat{X} \to X$ and we obtain that $\widehat{q} : \widehat{X} \to X$ is a covering map. Since $q \circ \widehat{f} = f \circ \widehat{q}$, we get that

$$q_*(\widehat{f}_*(\pi_1(\widehat{X}, (x_0, \hat{y}_0)))) = f_*(\widehat{q}_*(\pi_1(\widehat{X}, (x_0, \hat{y}_0)))),$$

which implies that the image of $\widehat{q}_*$ is contained in $f_*^{-1}(q_*(\pi_1(\widehat{Y}, \hat{y}_0)))$. To prove the other inclusion, let $\alpha : [0,1] \to X$ be closed curve based at x_0 such that $f_*([\alpha])$ is contained in $q_*(\pi_1(\widehat{Y}, \hat{y}_0))$. Now, let $\gamma : [0,1] \to \widetilde{Y}$ be a lifting of $f \circ \alpha$ based at $\hat{y}_0$. Hence, for all t in $[0,1]$ we get that $q \circ \gamma(t) = f \circ \alpha(t)$. Therefore, (α, γ) defines a closed curve in $\widetilde{X}$ such that $\widetilde{q}_*([(\alpha, \gamma)]) = [\alpha]$, which concludes the proof. ☐

Corollary 109. *If X and Y are compact complex manifolds and $f : X \to Y$ is a holomorphic map, then $\widehat{f} : \widehat{X} \to \widehat{Y}$ is a proper holomorphic map.*

Proof. To see that $\widehat{f}$ is proper, let K be a compact subset of $\widehat{Y}$. Then $\widehat{f}^{-1}(K)$ is a closed subset of $\widehat{X}$ contained in $X \times K$. Since the latter space is compact we obtain that $\widehat{f}^{-1}(K)$ is compact as well. ☐

Remark 110. *If X is an aspherical* CW*-complex or an aspherical smooth manifold, then the fiber product $\widehat{X}$ constructed in* (2.4.1) *is a classifying space of $f_*^{-1}(q_*(\pi_1(Y)))$. In particular, if $q : \widehat{Y} \to Y$ is the universal covering space of Y, we obtain that $\widehat{X}$ is a classifying space of* $\ker(f_*)$.

2.4.2 Irrational pencils

Let $f : X \to S$ be an irrational pencil, let $\widehat{S} \to S$ be the universal covering space of S and let $\widehat{X}$ be the fiber product of $f : X \to S$ and $\widehat{S} \to S$. As we saw previously we obtain a commutative diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{f}} & \widehat{S} \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & S, \end{array} \tag{2.4.2}$$

where $\widehat{X} \to X$ is a covering map such that $\pi_1(\widehat{X})$ is isomorphic to the kernel of $f_* : \pi_1(X) \to \pi_1(S)$ and $\widehat{f} : \widehat{X} \to \widehat{S}$ is a holomorphic proper map. In the following, $\widehat{X}$ will be called the *universal fiber product of the irrational pencil* $f : X \to S$ and $\widehat{f} : \widehat{X} \to \widehat{S}$ a *lift of the irrational pencil.*

We identify $\widehat{S}$ with $\mathbb{C}$ or the unit disk of $\mathbb{C}$. If we suppose that f is not a submersion and has isolated critical points, then its critical set $Crit(f)$ is a nonempty discrete subset of X. Moreover, we can deduce the following.

• The set of critical values of the lift $\widehat{f} : \widehat{X} \to \widehat{S}$ is an infinite discrete subset of $\widehat{S}$: since the diagram (2.4.2) commutes, we get that the set of critical values of $\widehat{f}$ is infinite; it is precisely the preimage of the set of critical values of f under the covering map $\widehat{S} \to S$. Finally, we observe that this set is discrete, since it is given by the orbit of a finite number of points (one for each critical value of f) under the action of the fundamental group of S.

• The set of critical points of $\widehat{f}$ is an infinite discrete subset of $\widehat{X}$. The argument is the same as above since the critical set of $\widehat{f}$, denoted by $Crit(\widehat{f})$, is the preimage of $Crit(f)$ under the covering map $\widehat{X} \to X$.

2.4.3 Topology of the universal fiber product of an irrational pencil

The following result shows that we can perturb a lift of an irrational pencil with isolated critical points to obtain a C^∞ map with nondegenerate critical points. It is an application of the results of Section 2.3.

Proposition 111. *Let $\widehat{f} : \widehat{X} \to \widehat{S}$ be a lift of an irrational pencil with isolated critical points. Then, there exists a C^∞ map $\widehat{f}_0 : \widehat{X} \to \widehat{S}$, C^∞-close to $\widehat{f}$, which is holomorphic in a neighborhood of its critical set and such that each critical point of $\widehat{f}_0$ is nondegenerate.*

Proof. To construct the map $\widehat{f}_0$ we will perturb $\widehat{f}$ in a neighborhood of each degenerate critical point of $\widehat{f}$ as follows. Recall that we identified $\widehat{S}$ with $\mathbb{C}$ or the unit disk of $\mathbb{C}$. For each point q in the critical set $Crit(\widehat{f})$ of $\widehat{f}$, we pick a neighborhood U_q such that the open sets

$$(U_q)_{q \in Crit(\widehat{f})}$$

are disjoint. By choosing a holomorphic coordinate system in U_q, we can identify U_q with an open ball $B(0, \varepsilon_q)$ centered at the origin of $\mathbb{C}^n$ of radius ε_q. For each point q in $Crit(\widehat{f})$, we consider the restriction $f : B(0, \varepsilon_q) \to \widehat{S}$. If q is a degenerate critical point of $\widehat{f}$, by Remark 106, we can choose a small enough linear form ℓ_q on $\mathbb{C}^n$ such that the map $\widehat{f} - \ell_q : B(0, \varepsilon_q) \to \widehat{S} \subset \mathbb{C}$ still takes values in $\widehat{S}$ and has finitely many critical points contained in $B(0, \frac{\varepsilon_q}{2})$ which are all nondegenerate. The number of its critical points is exactly the Milnor number of the critical point q for the map $\widehat{f}$. If q is a nondegenerate critical point we take $\ell_q = 0$. Under the identification of U_q with the open ball $B_q = B(0, \varepsilon_q)$, we denote by V_q the open subset of U_q corresponding to $B(0, \frac{\varepsilon_q}{2})$. Finally, one builds the map $\widehat{f}_0$ by declaring that $\widehat{f}_0$ is equal to $\widehat{f} - \ell_q$ on V_q, to $\widehat{f}$ outside the union of the open sets $(U_q)_{q \in Crit(\widehat{f})}$ and to a deformation between $\widehat{f}$ and $\widehat{f} - \ell_q$ on $U_q - V_q$. □

We would like to point out the following facts about the morsification $\widehat{f}_0 : \widehat{X} \to \widehat{S}$:
• All the critical points of $\widehat{f}_0$ are contained in one of the V_q's.
• The construction of $\widehat{f}_0 : \widehat{X} \to \widehat{S}$ can be done in an equivariant way for the action of $\pi_1(X)$, so that the map $\widehat{f}_0$ descends to a map $f_0 : X \to S$ homotopic to f.

We will now work with the map $\widehat{f}_0$ instead of $\widehat{f}$. We study the topology of $\widehat{X}$ by viewing it as the increasing union of a well-chosen sequence of compact subsets (see [31] for a very similar construction). With that purpose, we are looking for a sequence of closed disks $(D_k)_{k \geq 1}$ of $\widehat{S}$ such that
• $\widehat{S} = \cup_{k \geq 1} D_k$.
• The disk D_k is contained in the interior of the disk D_{k+1}.
• No critical value of $\widehat{f}_0$ is contained in the boundary of D_k.
• $D_{k+1} \setminus D_k$ contains exactly one critical value of $\widehat{f}_0$ for all $k \geq 1$.
• D_1 contains no critical value.
For such a sequence $(D_k)_{k \geq 1}$, we set

$$X_k := \widehat{f}_0^{-1}(D_k).$$

Hence X_1 retracts onto a smooth fiber of $\widehat{f}_0$. For $k \geq 2$, the topology of the space X_k is described thanks to the following proposition.

Proposition 112. *The space X_{k+1} has the homotopy type of a space obtained from X_k by gluing to it a finite number $m_k > 0$ of n-dimensional cells.*

This proposition is well-known (see e.g. Lemma 3.3 in [31] for a related statement, although in that lemma the authors work with the original map $\widehat{f}$ instead of our perturbation $\widehat{f}_0$). Before giving the proof of this result we give a construction of the sequence $(D_k)_{k\geq 1}$.

We endow $\widehat{S}$ with its classical Riemannian metric of constant sectional curvature and we denote by Ω the set of critical values of $\widehat{f}_0$. Each pair of points (z, w) in Ω defines the following subsets of $\widehat{S}$.

- $\ell_1(z, w) :=$ the bi-infinite geodesic whose points are equidistant from z and w.
- $\ell_2(z, w) :=$ the bi-infinite geodesic passing through z and w.

We write

$$L = \bigcup_{\substack{z,w\in\Omega \\ z\neq w}} \ell_1(z, w) \cup \ell_2(z, w)$$

and we pick a point z_0 in $\widehat{S}$ which is not contained in L.

- The point z_0 is a regular value of $\widehat{f}_0$ such that all the points in Ω are at different distances from z_0.
- Any geodesic passing through z_0 contains at most 1 point of Ω.

Hence, if we denote by $d(\cdot, \cdot)$ the distance induced by the Riemannian metric of $\widehat{S}$, we can enumerate the critical values of $\widehat{f}_0$ in such a way that $d(z_0, z_i) < d(z_0, z_j)$ if $i < j$. Finally, let

$$r_1 = \frac{d(z_0, z_1)}{2} \quad \text{and} \quad r_k = \frac{d(z_0, z_k) + d(z_0, z_{k+1})}{2}.$$

Then the set of disks $(D_k = D(z_0, r_k))_{k\geq 1}$ centered at z_0 of radius r_k has the desired properties. Notice that if we allow the disks $(D_k)_{k\geq 1}$ to have different centers, the construction of such a sequence of disks is simpler.

Proof of Proposition 112. Let $c : [0, 1] \to \mathrm{Int}(D_{k+1})$ be an embedded arc going from a boundary point $c(0)$ of D_k to the unique critical value contained in $D_{k+1} - D_k$. We assume that $c(t) \notin D_k$ for $t > 0$. If we define the sets $(X_k)_{k\geq 1}$ by using the latter construction of disks $(D_k)_{k\geq 1}$, this can easily be done by using the unique geodesic passing through z_0 and z_k. Let D^* be a small disk centered at $c(1)$ and contained in $\mathrm{Int}(D_{k+1} \setminus D_k)$ (see Figure 2.4.1).

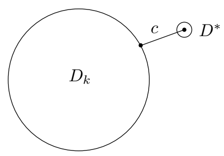


Figure 2.4.1: The disks D_k and D^*

Since the restriction of $\widehat{f}_0$ to the preimage of the set of regular values is a locally trivial fibration, X_{k+1} deformation retracts onto $X_k \cup \widehat{f}_0^{-1}(c([0, 1]) \cup D^*)$. Let $m_k > 0$ be the number of critical points in the level $\widehat{f}_0^{-1}(c(1))$ and let $x_1, \ldots, x_{m_k}$ be the corresponding critical points. For each $i = 1, \ldots, m_k$, let U_i be a small neighborhood of x_i such that the sets $(U_i)_{i=1}^{m_k}$ are disjoint. According to Corollary 101, we can fix a regular value $t = c(1-\delta)$ (for a small $\delta > 0$) of $\widehat{f}_0$ such that $\widehat{f}_0^{-1}(D^*)$ deformation retracts onto the union of the generic fiber $\widehat{f}_0^{-1}(t)$ with m_k n-balls $B_1, \ldots, B_k$, where the ball B_i is contained in U_i and it is glued to $\widehat{f}_0^{-1}(t)$ along a vanishing sphere S_i of $\widehat{f}_0$ contained in $\widehat{f}_0^{-1}(t)$.

By fixing a trivialization of the fibration

$$\widehat{f}_0 : \widehat{f}_0^{-1}(c([0,1[)) \to c([0,1[),$$

each sphere S_i can be identified to a sphere $S_i^* \subset \widehat{f}_0^{-1}(c(0))$ in such a way that the $(S_i^*)_{1\leq i\leq m_k}$ are disjoint. Hence X_{k+1} retracts onto the space obtained from X_k by gluing a ball to each of the spheres S_i^*. This proves the result. □

2.4.4 Growth of the n-th Betti number

Here we give a proof of Theorem F

Observation. Proposition 112 implies that the sequence $b_{n-1}(X_k) = \dim_{\mathbb{Q}} H_{n-1}(X_k, \mathbb{Q})$ is decreasing with k.

Let k_0 be a large enough integer such that the sequence

$$(b_{n-1}(X_k))_{k\geq k_0}$$

is constant. We will now prove that the sequence $(b_n(X_k))_{k\geq k_0}$ is strictly increasing with k and that each map

$$H_n(X_k, \mathbb{Q}) \to H_n(X_{k+1}, \mathbb{Q}) \tag{2.4.3}$$

induced by the inclusion $X_k \hookrightarrow X_{k+1}$ is injective for $k \geq k_0$. This immediately implies Theorem F since the vector space $H_n(\widehat{X}, \mathbb{Q})$ is the direct limit of the $H_n(X_k, \mathbb{Q})$'s.

We use the notation from the proof of Proposition 112. Let $k \geq k_0$. We know that X_{k+1} has the homotopy type of a space W_{k+1} obtained by gluing a ball to each of the spheres

$$S_1^* \sqcup \ldots \sqcup S_{m_k}^* \subset X_k.$$

We write:

$$W_{k+1} = X_k \cup B_1 \cup \ldots \cup B_{m_k} \tag{2.4.4}$$

where each B_j is homeomorphic to an n-dimensional ball with $B_j \cap B_l = \emptyset$ for $l \neq j$ and $X_k \cap B_j$ is equal to the boundary of B_j (or to the sphere S_j^* depending on whether one views it inside X_k or B_j). Since the inclusion of X_k into X_{k+1} induces an isomorphism on $(n-1)$-dimensional homology groups, the same occurs for each inclusion $X_k \hookrightarrow W_{k+1}$. We now apply the Mayer-Vietoris exact sequence to the decomposition of W_{k+1} given in (2.4.4).

We obtain (all homology groups being with $\mathbb{Q}$ coefficients):

$$H_n(\sqcup_{j=1}^{m_k} \partial B_j) \longrightarrow H_n(X_k) \oplus H_n(\sqcup_{j=1}^{m_k} B_j) \longrightarrow H_n(W_{k+1}) \longrightarrow \tag{2.4.5}$$

$$\longrightarrow H_{n-1}(\sqcup_{j=1}^{m_k} \partial B_j) \longrightarrow H_{n-1}(X_k) \oplus H_{n-1}(\sqcup_{j=1}^{m_k} B_j) \longrightarrow H_{n-1}(W_{k+1}).$$

Since the groups $H_n(\sqcup_{j=1}^{m_k} \partial B_j)$, $H_n(\sqcup_{j=1}^{m_k} B_j)$ and $H_{n-1}(\sqcup_{j=1}^{m_k} B_j)$ are zero we obtain:

$$\{0\} \longrightarrow H_n(X_k) \longrightarrow H_n(W_{k+1}) \longrightarrow H_{n-1}(\sqcup_{j=1}^{m_k} \partial B_j) \longrightarrow H_{n-1}(X_k) \longrightarrow H_{n-1}(W_{k+1}). \tag{2.4.6}$$

The last arrow on the right hand side being an isomorphism, this implies that the following sequence is exact:

$$\{0\} \longrightarrow H_n(X_k) \longrightarrow H_n(W_{k+1}) \longrightarrow H_{n-1}(\sqcup_{j=1}^{m_k} \partial B_j) \longrightarrow \{0\}. \tag{2.4.7}$$

This implies that each inclusion $X_k \hookrightarrow W_{k+1}$ (and hence the inclusion $X_k \hookrightarrow X_{k+1}$) induces an injective map on $H_n(\cdot, \mathbb{Q})$ and that $b_n(X_k) = b_n(X_{k+1}) + m_k$. Note that $m_k > 0$. This is the desired result and concludes the proof of Theorem F.

Remark 113. *Assume that X is aspherical. Then the space $\widehat{X}$ is a classifying space of the kernel of $f_* : \pi_1(X) \to \pi_1(S)$. As we said in the introduction, Dimca, Papadima and Suciu proved in [31] that the inclusion of the smooth generic fiber of f in $\widehat{X}$ induces isomorphisms on the homotopy groups of dimension $0, \ldots, n-2$ (see Lemma 3.3 and Corollary 5.8 in [31]). This essentially follows from Proposition 112. This implies that the kernel of $f_* : \pi_1(X) \to \pi_1(S)$ is of type $\mathscr{F}_{n-1}$ (hence FP_{n-1}), although it is not of type FP_n.*

2.5 New Kähler groups with exotic finiteness properties

In this section we will construct new examples of Kähler groups with exotic finiteness properties using the Cartwright-Steger surface. We recall the definition of the complex hyperbolic space to introduce this complex surface.

2.5.1 Complex hyperbolic space

Let us denote by $\langle \cdot , \cdot \rangle : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$ the product given by

$$\langle (z_1, \ldots, z_{n+1}), (w_1, \ldots, w_{n+1}) \rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n - z_{n+1}\bar{w}_{n+1}.$$

Observe that the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ given by the scalar multiplication, preserves the set

$$\{z \in \mathbb{C}^{n+1} \mid \langle z , z \rangle < 0\}.$$

Definition 114. The complex hyperbolic space of dimension n, denoted by $\mathbb{H}^n_{\mathbb{C}}$, is defined as the subset of $\mathbb{CP}^n$ given by

$$\{[z] = [z_1 : \cdots : z_{n+1}] \mid \langle z , z \rangle < 0\}.$$

If $[z] = [z_1 : \cdots : z_{n+1}]$ is a point in $\mathbb{H}^n_{\mathbb{C}}$, then $z_{n+1} \neq 0$, otherwise $\langle z , z \rangle$ would be non-negative. Moreover, the usual coordinate chart (U_{n+1}, φ_{n+1}) of $\mathbb{CP}^n$ given by the homogeneous coordinate $U_{n+1} = \{[z_1 : \cdots : z_{n+1}] \mid z_{n+1} \neq 0\}$ and the map

$$\begin{array}{rccc} \varphi_{n+1} : & U_{n+1} & \longrightarrow & \mathbb{C}^n \\ & [z_1 : \cdots : z_{n+1}] & \mapsto & \left(\dfrac{z_1}{z_{n+1}}, \ldots, \dfrac{z_n}{z_{n+1}} \right), \end{array}$$

identifies $\mathbb{H}^n_{\mathbb{C}}$ with the unit ball of $\mathbb{C}^n$.

Under this identification we can endow $\mathbb{H}^n_{\mathbb{C}}$ with a Kähler metric as follows. Let z be a point in the unit ball B of $\mathbb{C}^n$ and let $v, w \in T_zB$. We write

$$g_z(u, v) = \frac{1}{1 - |z|^2} \left((u, v)_{\mathbb{C}^n} + \frac{(u, z)_{\mathbb{C}^n}(z, w)_{\mathbb{C}^n}}{1 - |z|^2} \right), \tag{2.5.1}$$

where $(u, v)_{\mathbb{C}^n} = u_1\bar{v}_1 + \cdots + u_n\bar{v}_n$ is the usual Hermitian product in $\mathbb{C}^n$. The metric defined in (2.5.1) is called *the hyperbolic metric*.

Definition 115. The unitary group $\mathrm{U}(n, 1)$ of signature $(n, 1)$ is defined as the subgroup of $\mathrm{Gl}(n + 1, \mathbb{C})$ that preserves $\langle \cdot , \cdot \rangle$.

The projective unitary group $\mathrm{PU}(n, 1)$ which is defined as the quotient of $\mathrm{U}(n, 1)$ by its center, acts on $\mathbb{H}^n_{\mathbb{C}}$ by holomorphic isometries. Furthermore, we have the following result.

Proposition 116. *The group of holomorphic isometries of* $\mathbb{H}^n_{\mathbb{C}}$ *is* $\mathrm{PU}(n, 1)$.

Hence for any torsion-free discrete subgroup Γ of $\mathrm{PU}(n, 1)$, the quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Kähler manifold.

2.5.2 Examples built from self-products of the Cartwright-Steger surface

The *Cartwright-Steger surface* is a smooth compact complex surface which is a quotient of $\mathbb{H}^2_{\mathbb{C}}$ by a discrete subgroup of $\mathrm{PU}(2, 1)$. It is characterized (up to changing the sign of the complex structure) by the fact that its Euler characteristic is equal to 3 and its first Betti number is equal to 2. It was discovered in [24] in the context of the classification of fake projective planes. It was further studied by several authors, see for instance [23, 70, 74].

We now denote by Y the Cartwright-Steger surface and by $h : Y \to E$ its Albanese map, whose target is an elliptic curve since $b_1(Y) = 2$. Cartwright, Koziarz and Yeung [23] have proved that the map h has isolated critical points and Koziarz and Yeung later proved that these critical points are nondegenerate [46] (see also Rito [61]). We can thus consider the product Y^b of Y with itself b times and the map

$$h + \cdots + h : Y^b \to E. \tag{2.5.2}$$

As we said in the introduction, this provides natural examples to which one can apply Theorems F and G. If we denote by $\Gamma < \mathrm{PU}(2, 1)$ the fundamental group of the Cartwright-Steger surface (we refer the reader to [23] for a detailed description of this lattice), we obtain from Theorem G together with the latter construction the following result:

Theorem H. *The direct product of b copies of Γ contains a coabelian normal subgroup N which is of type FP_{2b-1} but satisfies that $H_{2b}(N, \mathbb{Q})$ has infinite dimension.*

The group N appearing above is the kernel of the morphism on fundamental groups induced by the map (2.5.2). Recall that by Remark 113, we have that N is of type FP_{2b-1}.

Besides considering the products $Y \times \cdots \times Y$, one can also build more examples by combining the construction by Dimca, Papadima and Suciu and our construction. We fix a family of ramified covers $p_i : \Sigma_i \to E$ of the elliptic curve E $(1 \leq i \leq a)$, where each Σ_i has negative Euler characteristic. We then consider the map

$$f : \Sigma_1 \times \cdots \times \Sigma_a \times Y \times \cdots \times Y \to E$$

(where there are $b \geq 1$ copies of Y) which is the sum of the p_i's and of the map h on each copy of Y. All the results until the end of this section also apply when $a = 0$, i.e. when one studies the map $f = h + \cdots + h$ as in (2.5.2).

The map f has a finite non-empty set of critical points and connected fibers. This last point follows from the fact that $h : Y \to E$ has connected fibers. We denote by Λ the kernel of the map induced by f on fundamental groups. Theorem 65 and Theorem G imply that the $\mathbb{Q}$-vector space $H_{a+2b}(\Lambda, \mathbb{Q})$ has infinite dimension and that Λ is projective if and only if $2b + a \geq 3$. The following proposition shows that the group Λ is of a different nature compared to the examples from [31, 49].

Proposition 117. *No finite index subgroup of Λ embeds in a direct product of surface groups.*

By a *surface group* we mean here the fundamental group of an oriented surface of finite type (open or closed). Hence a surface group is either free or the fundamental group of a closed oriented surface. To prove Proposition 117, we will make use of the following theorem due to Bridson, Howie, Miller and Short [13].

Theorem 118. *Let $F_1, \ldots, F_m$ be surface groups. Let G be a subgroup of the direct product $F_1 \times \cdots \times F_m$. If G is of type FP_m, then G is virtually isomorphic to a direct product of the form $H_1 \times \cdots \times H_k$ where $k \leq m$ and each H_i is a surface group. In particular G is of type FP_∞.*

Besides this theorem, we will also use the fact that the property of being FP_m (for $m \in \mathbb{N} \cup \{\infty\}$) is invariant under commensurability (see Proposition 93).

Proof of Proposition 117. The group Λ sits inside the direct product

$$\pi_1(\Sigma_1) \times \cdots \times \pi_1(\Sigma_a) \times \pi_1(Y) \times \cdots \times \pi_1(Y). \tag{2.5.3}$$

The *factors* of this direct product are the subgroups of the form

$$\{1\} \times \cdots \times \pi_1(\Sigma_j) \times \cdots \times \{1\}$$

or

$$\{1\} \times \cdots \times \pi_1(Y) \times \cdots \times \{1\}.$$

Each factor intersects Λ non-trivially. This implies that Λ contains copies of the group $\mathbb{Z}^{a+b}$. The Gromov hyperbolicity of each factor implies that $\mathbb{Z}^{a+b+1}$ does not embed in Λ. Now suppose that a finite index subgroup Λ_1 of Λ embeds in a direct product of surface groups $F_1 \times \cdots \times F_m$. By taking m to be minimal, we may assume that $L_i = \Lambda_1 \cap F_i$ is non-trivial for $i = 1, \ldots, m$. Otherwise Λ_1 embeds in a direct product of $m-1$ surface groups. Since Λ_1 does not contain any non-trivial Abelian normal subgroup, this implies that each F_i is non-Abelian, hence hyperbolic. A similar argument as before then shows that Λ_1 contains copies of $\mathbb{Z}^m$ but no copy of $\mathbb{Z}^{m+1}$. Hence $m = a + b$. By Remark 113, the group Λ is of type FP_{2b+a-1}, hence Λ_1 is. Since $2b + a - 1 \geq m = a + b$, Λ_1 is of type FP_m and Theorem 118 implies that Λ_1 is of type FP_∞. This contradicts the fact that Λ_1 is not of type FP_{a+2b}. □

Finally, we compute, in some cases, the first Betti number of the group $\Lambda = \ker(f_*)$. A similar computation appears in [50, §7], which applies to some of the examples built in [31, 49, 50].

Proposition 119. *Assume that $a + 2b \geq 3$. Assume furthermore that $b \geq 2$ or that $b = 1$ and that the map $\pi_1(\Sigma_j) \to \pi_1(E)$ is surjective for some $j \in \{1, \ldots, a\}$. Then the first Betti number of Λ is equal to:*

$$b_1(\Sigma_1 \times \cdots \times \Sigma_a \times Y^b) - 2.$$

Proof. We consider the surjective homomorphism

$$\Lambda \to \pi_1(\Sigma_1) \times \cdots \times \pi_1(\Sigma_a) \times \pi_1(Y)^{b-1}$$

obtained by considering the inclusion of Λ in the direct product (2.5.3) and by projecting onto the first $a + b - 1$ factors. Its kernel N consists of elements of the form

$$(1, \ldots, 1, g) \in \pi_1(\Sigma_1) \times \cdots \times \pi_1(\Sigma_a) \times \pi_1(Y) \times \cdots \times \pi_1(Y)$$

where $g \in \ker(h_*)$; it is isomorphic to $\ker(h_*)$. Hence we have the following exact sequence:

$$0 \to N \to \Lambda \to \pi_1(\Sigma_1) \times \cdots \times \pi_1(\Sigma_a) \times \pi_1(Y)^{b-1} \to 0.$$

It induces the following short exact sequence (see [16, VII.6], all homology groups are taken with $\mathbb{Z}$ coefficients):

$$H_1(N)_\Lambda \to H_1(\Lambda) \to H_1(\pi_1(\Sigma_1) \times \cdots \times \pi_1(\Sigma_a) \times \pi_1(Y)^{b-1}) \to 0. \quad (2.5.4)$$

Here $H_1(N)_\Lambda$ is the group of coinvariants of $H_1(N)$ for the Λ-action. It is isomorphic to the quotient of N by the group $[N, \Lambda]$ generated by commutators of elements of Λ and of N. Note that if $x = (1, \ldots, 1, g) \in N$ and $y = (y_1, \ldots, y_a, h_1, \ldots, h_b) \in \Lambda$ then

$$xyx^{-1}y^{-1} = (1, \ldots, 1, gh_b g^{-1} h_b^{-1}).$$

Hence when we identify N with $\ker(h_*)$, $[N, \Lambda]$ is identified with the group $[\ker(h_*), \pi_1(Y)]$ (we are using here that $b \geq 2$ or that one of the $\pi_1(\Sigma_j)$ surjects onto $\pi_1(E)$). In particular the groups $H_1(N)_\Lambda$ and $H_1(\ker(h_*))_{\pi_1(Y)}$ are isomorphic. Now the short exact sequence

$$0 \to \ker(h_*) \to \pi_1(Y) \to \mathbb{Z}^2 \to 0$$

induces the short exact sequence (see [16, VII.6] again):

$$H_2(\mathbb{Z}^2) \to H_1(\ker(h_*))_{\pi_1(Y)} \to H_1(Y) \to H_1(\mathbb{Z}^2) \to 0.$$

Since the map

$$H_1(Y) \otimes \mathbb{Q} \to H_1(\mathbb{Z}^2) \otimes \mathbb{Q}$$

is an isomorphism, we obtain that $H_1(\ker(h_*))_{\pi_1(Y)} \otimes \mathbb{Q}$ has dimension at most 1. Hence $H_1(N)_\Lambda \otimes \mathbb{Q}$ has dimension at most 1. Since Λ is Kähler and hence has even first Betti number, this implies that the first arrow in (2.5.4) has finite image. Hence $H_1(\Lambda) \otimes \mathbb{Q}$ and $H_1(\pi_1(\Sigma_1) \times \cdots \times \pi_1(\Sigma_a) \times \pi_1(Y)^{b-1}) \otimes \mathbb{Q}$ are isomorphic. This gives the desired result. □

Remark 120. *By considering the case where $a \in \{0, 1\}$, our construction provides for each $n \geq 2$ an example of a* CAT(0) *group G containing a subgroup of type* FP_{n-1} *but not* FP_n *and such that G does not contain free Abelian subgroups of rank greater than $\left\lfloor \frac{n+1}{2} \right\rfloor$. See [12, 47] for related results and motivation. The article [47] produces other examples with a smaller bound on the rank of Abelian subgroups. More precisely, for each positive integer n, Kropholler gives in [47] an example of a group of type $\mathscr{F}_{n-1}$ but not of type $\mathscr{F}_n$, which does not contain free Abelian groups of rank greater than $\left\lceil \frac{n}{3} \right\rceil$ and which is a subgroup of a* CAT(0) *group.*

2.6 Irrational pencils on aspherical compact complex surfaces

In this section we give a detailed proof of Theorem 70 due to Kapovich. There are no original results here, this is an expository section.

2.6.1 Milnor's Fibration Theorem

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a nonconstant holomorphic map that vanishes at the origin and let S_ε be the real $(2n-1)$-sphere of radius ε centered at the origin. We write

$$V = f^{-1}(0) \quad \text{and} \quad K = V \cap S_\varepsilon.$$

Assume that $\bar{0}$ is a critical point of f (not necessarily isolated). Then, Milnor's Fibration Theorem states the following.

Theorem 121. *There exists ε_0 such that for all $\varepsilon \leq \varepsilon_0$, the space $S_\varepsilon \setminus K$ is a smooth fiber bundle over $\mathbb{S}^1$ with projection mapping*

$$\begin{aligned} S_\varepsilon \setminus K &\longrightarrow \mathbb{S}^1 \\ z \quad &\mapsto \frac{f(z)}{|f(z)|}. \end{aligned}$$

The following results give a description of the topology of the fibers of Milnor's Fibration Theorem. The first one is a result of Milnor (see [52]) and the second one of A'Campo (see [1])

Theorem 122. *The fiber of Milnor's Fibration Theorem is parallelizable and has the homotopy type of a* CW*-complex of dimension $n-1$.*

Theorem 123. *If the fiber of Milnor's Fibration Theorem has the homotopy type of a point, then 0 is regular point of f.*

2.6.2 Reeb Stability Theorem

We recall some facts on the holonomy of foliations that we will need later.

Let X be a smooth manifold endowed with a foliation $\mathcal{F}$ and let L be a leaf of $\mathcal{F}$. Given a simple closed curve γ in L, the *holonomy* of the foliation describes the behavior of the foliation in a neighborhood of γ. Given a base point x in L and a transverse space D to the foliation containing x, the holonomy of the foliation is given by a surjective map

$$\pi_1(L, x) \to G,$$

where G is a subgroup of the germ of diffeomorphisms of D (see [21] for the details on this subject). The following result is a version of Reeb Stability Theorem (see §2.4 of [21])

Theorem 124. *Let $(X, \mathcal{F})$ be a foliated manifold. If there is a compact leaf L in $\mathcal{F}$ with trivial germinal holonomy group, then there exists a neighborhood of L in X that is a union of leaves that are homeomorphic to L.*

Remark 125. *If we suppose L to be contained in a compact subset of X instead of being compact, the conclusion of Theorem 124 holds. In the proof of Theorem 124 the compactness of L guarantees that there is a finite number of plaques covering L which can be done as well if L is contained in a compact subset of X. The rest of the proof is not affected by this change of hypothesis.*

2.6.3 Irrational pencils with non-isolated critical points

Let $f : X \to S$ be a holomorphic map with connected fibers from a complex manifold of dimension n to a Riemann surface. If s is a singular value of f such that the critical points of the singular fiber $F_s = f^{-1}(s)$ are non-isolated, then F_s decomposes as the union of a finite number of irreducible components $Z_1, \ldots, Z_k$. Given a smooth point p in an irreducible component Z_j, there exist local coordinates $(z_1, \ldots, z_n)$ centered at p such that f is locally given by the map

$$(z_1, \ldots, z_n) \mapsto z_1^{m_j}.$$

The integer m_j does not depend on the choice of the smooth point p in Z_j and it is called *the multiplicity of the irreducible component* Z_j. The multiplicity $m(s)$ of the singular fiber $f^{-1}(s)$ is defined as the greatest common divisor of $m_1, \ldots, m_k$.

2.6.3.a Irrational pencils with singular fibers of multiplicity one

Let $\widehat{X}$ be the universal fiber product of an irrational pencil $f : X \to S$ with singular fibers of multiplicity one and let $\widehat{f} : \widehat{X} \to \widehat{S}$ be a lift of f (see Section 2.4.2). As we saw in Section 108, $\widehat{f}$ and f have the same fibers and $\widehat{f}$ is locally defined as f (see Corollary 109). Therefore, $\widehat{f} : \widehat{X} \to \widehat{S}$ has connected fibers and all the singular fibers have multiplicity 1.

As in Section 2.4.3, let $(D_k)_{k\geq 1}$ be a sequence of closed disks of $\widehat{S}$ such that

- $\widehat{S} = \cup_{k\geq 1} D_k$.
- The disk D_k is contained in the interior of the disk D_{k+1}.
- No critical value of $\widehat{f}_0$ is contained in the boundary of D_k.
- $D_{k+1} \setminus D_k$ contains exactly one critical value for all $k \geq 1$.
- D_1 contains no critical value.

For such a sequence $(D_k)_{k\geq 1}$, we set

$$X_k := \widehat{f}^{-1}(D_k).$$

Let F be a regular fiber whose image is contained in D_1. In Chapter 1 Section 1.4.1, we saw that a surjective holomorphic map with connected fibers from a compact complex manifold to a closed Riemann surface induces an orbifold structure on the Riemann surface. The latter assertion holds as well for a proper holomorphic map with connected fibers from a non-compact complex manifold to a Riemann surface. In this case, the orbifold structure is given by a discrete set of marked points on the Riemann surface that might be infinite. Hence, the lift $\widehat{f} : \widehat{X} \to \widehat{S}$ induces an orbifold structure Σ on $\widehat{S}$ and we obtain an exact sequence

$$\pi_1(F) \longrightarrow \pi_1(\widehat{X}) \longrightarrow \pi_1^{orb}(\Sigma) \longrightarrow 1.$$

Since the singular fibers of $\widehat{f}$ have multiplicity one, we get that $\pi_1^{orb}(\Sigma) = \pi_1(\widehat{S})$, which is trivial since $\widehat{S}$ is simply connected. Therefore the map $\pi_1(F) \to \pi_1(\widehat{X})$ induced by the inclusion $i : F \hookrightarrow \widehat{X}$, is surjective. The compactness of F implies that $\pi_1(F)$ is finitely generated. We fix some loops $\gamma_1, \ldots, \gamma_m : [0,1] \to F$ with the same base point such that the set of homotopy classes $\{[\gamma_1], \ldots, [\gamma_m]\}$ generates $\pi_1(F)$. Hence, the set $\{i_*[\gamma_1], \ldots, i_*[\gamma_m]\}$ generates $\pi_1(\widehat{X})$. Notice that F is contained in X_k for all positive integer k and the same argument as before shows that $\pi_1(F)$ generates $\pi_1(X_k)$.

Remark 126. *Let $\langle x_1, \ldots, x_m | r_1, r_2, \ldots \rangle$ be a presentation of $\pi_1(\widehat{X})$ (not necessarily finite), such that x_j corresponds to $i_*[\gamma_j]$ and let $r = x_{j_1} \cdots x_{j_d}$ be a relation of such presentation. Then, there exists a positive integer N such that the loop given by the concatenation of $\gamma_{j_1} \cdots \gamma_{j_d}$ (which corresponds to r) is nullhomotopic in X_k for all $k \geq N$.*

Lemma 127. *If the kernel of the map $f_* : \pi_1(X) \to \pi_1(S)$ is finitely presented, there exists a positive integer N such that for any $k \geq N$ the inclusion map $X_k \hookrightarrow \widehat{X}$ induces an isomorphism on fundamental groups.*

Proof. Let $\langle x_1, \ldots, x_m | r_1, \ldots, r_l \rangle$ be a presentation of $\pi_1(\widehat{X})$, where x_j corresponds to $i_*[\gamma_j]$. Since there are finitely many relations, by Remark 126, for all relation $r \in \{r_1, \ldots, r_l\}$ written as $r = x_{j_1} \cdots x_{j_d}$, the loop given by $\gamma_{j_1} \cdots \gamma_{j_d}$ is nullhomotopic in X_k for all $k \geq N$.

Now, we fix a positive integer k such that $k \geq N$ and let $i : F \hookrightarrow \widehat{X}$, $i_1 : F \hookrightarrow X_k$ and $i_2 : X_k \hookrightarrow \widehat{X}$ be the inclusion maps. Then, we have that the diagram

$$\begin{array}{ccc} F & \xrightarrow{Id} & F \\ {\scriptstyle i_1}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ X_k & \underset{i_2}{\hookrightarrow} & \widehat{X} \end{array}$$

commutes. Hence $\pi_1(X_k)$ is generated by the set $\{i_{1*}[\gamma_1], \ldots, i_{1*}[\gamma_m]\}$ and $i_{2*} : \pi_1(X_k) \to \pi_1(\widehat{X})$ sends $i_{1*}[\gamma_j]$ to $i_*[\gamma_j]$.

We claim that i_{2*} is an isomorphism. By construction we have that i_{2*} is surjective since F is contained in X_k. To prove the injectivity of i_{2*}, let F_m be the free group with m generators $\{x_1, \ldots, x_m\}$, and let $\theta : F_m \to \pi_1(X_k)$ be the morphism that sends x_j to $i_{1*}[\gamma_j]$. Now, consider the commutative diagram

$$\begin{array}{ccc} F_m & \xrightarrow{Id} & F_m \\ {\scriptstyle \theta}\big\downarrow & & \big\downarrow{\scriptstyle i_{2*}\circ\theta} \\ \pi_1(X_k) & \underset{i_{2*}}{\longrightarrow} & \pi_1(\widehat{X}). \end{array} \tag{2.6.1}$$

Let $\gamma : [0,1] \to F$ be such that $i_{1*}([\gamma])$ is in the kernel of i_{2*} and let x be an element in F_m such that $\theta(x) = i_{1*}([\gamma])$. By construction, we get that $i_{2*} \circ \theta : F_m \to \pi_1(\widehat{X})$ sends x_j to $i_*([\gamma_j])$. The kernel of $i_{2*} \circ \theta$ is the subgroup of F_m whose elements are of the form

$$\prod_{t=1}^{d} (g_t \, r_{j_t}^{\pm 1} \, g_t^{-1}) \quad \text{with} \quad g_t \in F_m \quad \text{and} \quad r_{j_t} \in \{r_1, \ldots, r_l\}. \tag{2.6.2}$$

Hence by the commutativity of diagram (2.6.1) we get that x is of the form (2.6.2). Finally, observe that for any relation $r \in \{r_1, \ldots, r_l\}$ we have that $\theta(r) = 1$. This follows from the fact that the simple closed curve given by concatenating some of the $\gamma_j' s$ whose homotopy class represents the relation r is nullhomotopic in X_k. Therefore x is in the kernel of θ, which implies the injectivity of i_{2*}. □

2.6.3.b Foliations induced by the fibers of an irrational pencil

Let $\{p_1, \ldots, p_l\}$ be the set of singular points of F_s. Via the choice of some holomorphic coordinates, for each singular point p_i, we pick a closed neighborhood $\mathcal{B}_i(\varepsilon)$ of p_i that we identify with a closed ball $B_i(0,\varepsilon)$ centered at the origin of $\mathbb{C}^n$, and a closed neighborhood of s that we identify with a closed disk centered at the origin such that f is given in these coordinates by a holomorphic map that vanishes at the origin. Since the singular points of F_s are finitely many, there exists a small enough ε_0 such that for all $i = 1, \dots, l$ and for all $\varepsilon \leq \varepsilon_0$, the boundary of $B_i(0,\varepsilon)$ satisfies the conclusion of Milnor's Fibration Theorem.

From now on, we will assume that there is at least one singular point in F_s. We fix a Riemannian metric on X. Let Y_0 be a connected component of $F_s \setminus \cup_{i=1}^{l} \mathcal{B}_i(\varepsilon_0)$ and let $NY_0 \to Y_0$ be the associated normal bundle, *i.e.* NY_0 is orthogonal complement of TY_0 in TX for the fixed metric on X.

Lemma 128. *There exists an open neighborhood U of the* 0*-section of the normal bundle $NY_0 \to Y_0$ and an open neighborhood V of Y_0 such that the exponential map* $\exp : U \to V$ *is a diffeomorphism and the intersection of the fibers of f with V are connected.*

Proof. Let $\varepsilon < \varepsilon_0$ and let Y be the connected component of $F_s \setminus \cup_{i=1}^{l} \mathcal{B}_i(\varepsilon)$ containing Y_0. We denote by $U(r)$ the open r-neighborhood of the 0-section $NY \to Y$. Since $\exp : NY \to X$ is a local diffeomorphism in every point of the 0-section of $NY \to Y$, we get that up to increasing ε (preserving the inequality $\varepsilon < \varepsilon_0$), there exists a positive number $r > 0$ such that $\exp : U(r) \to X$ is a submersion. Otherwise, there would exist a decreasing sequence of positive numbers $(r_n)_{n\geq 1}$ converging to zero and a sequence $(x_n)_{n\geq 1}$ in NY of singular points of $\exp : NY \to X$ such that x_n is contained in $U(r_n)$. If we consider a compact set K such that

$$Y \subset K \subset F_s \setminus \{p_1, \dots, p_l\},$$

we obtain that there exists a point y in the 0-section of $NY \to Y$ such that (x_n) converges to y, a contradiction to the existence of a neighborhood of y where exp is a diffeomorphism. By a similar result, we can also assume that $\exp : U(r) \to X$ is injective. Hence, if we denote by $V(r)$ the image of $U(r)$ under exp we obtain that $\exp : U(r) \to V(r)$ is a diffeomorphism. Finally, since the restriction of f to the open set $V(r)$ is continuous, we can find an open subset V of $V(r)$ such that the fibers of F intersecting V are connected. □

Lemma 129. *There is a neighborhood U of Y_0 such that if a generic fiber F of f intersects U, then $F \cap U$ is a connected finite covering space of Y_0.*

Proof. Let U and V be open neighborhoods of the 0-section of $NY_0 \to Y_0$ and of Y_0 respectively, such that the conclusion of Lemma 128 holds. Let y be a base point in Y_0 and let D be a holomorphic disk centered in y, contained in V and transverse to the fibers of f. We can identify D to the unit disk of $\mathbb{C}$ in such a way that the restriction of f to D corresponds to the function $z \mapsto z^k$, where k is the multiplicity of the irreducible component of F_s containing Y_0. Let us consider the foliation of V given by the fibers of f. Hence, the holonomy associated to this foliation is given by a morphism $\rho : \pi_1(Y_0, y) \to G$, where G is a subgroup of the germ of diffeomorphisms of D, whose elements are of the form $z \mapsto \xi z$, where ξ is a k-th root of unity.

Let $q : (\widehat{Y}, \hat{y}) \to (Y_0, y)$ be the finite covering map such that $\pi_1(\widehat{Y}, \hat{y})$ is isomorphic to the kernel of ρ, and let $N\widehat{Y} \to \widehat{Y}$ be the pullback bundle induced by q, *i.e.*, we have the commutative diagram

$$\begin{array}{ccc} N\widehat{Y} & \longrightarrow & \widehat{Y} \\ \Big\downarrow{\scriptstyle \widehat{q}} & & \Big\downarrow{\scriptstyle q} \\ NY_0 & \longrightarrow & Y_0. \end{array}$$

The foliation in V given by the fibers of f induces a foliation $\mathcal{F}$ in U, which induces a foliation $\widehat{\mathcal{F}}$ in $\widehat{U} := \widehat{q}^{-1}(U)$ (via the covering $\widehat{q}$). Let L and $\widehat{L}$ be the leafs in $\mathcal{F}$ and $\widehat{\mathcal{F}}$ given by the 0-sections of $NY_0 \to Y_0$ and $N\widehat{Y} \to \widehat{Y}$ respectively.

Since $\widehat{q}$ is a covering map, we get that for every point $\hat{y}$ in $\widehat{L}$ there is a holomorphic disk transverse to the foliation. We deduce that the holonomy of L is precisely given by ρ and the germinal holonomy group of $\widehat{L}$ is trivial. By Reeb Stability Theorem (see Remark 125) we obtain that there is a neighborhood of $\widehat{L}$ that is a union of leaves that are homeomorphic to $\widehat{L}$. Then, the result follows from the fact that $\widehat{U} \to U$ is a finite covering map and that $\exp : U \to V$ is a diffeomorphism. □

2.6.3.c Irrational pencils on aspherical compact Kähler surfaces

Suppose now that X is an aspherical compact Kähler surface and let $f : X \to S$ be an irrational pencil with a singular value s. We resume with the notation of Section 2.6.3.b.

Lemma 130. *If Y_0 is a connected component of $F_s \setminus \cup_{i=1}^{l} \mathcal{B}_i(\varepsilon_0)$. Then the map $\pi_1(Y_0) \to \pi_1(X)$ induced by the inclusion $Y_0 \hookrightarrow X$ has infinite image.*

Proof. For every irreducible component Z of F_s, let $Z^* \to Z$ be its desingularization map (see [36] p. 498). Then, the map $\pi_1(Z^*) \to \pi_1(X)$ induced by the composition $Z^* \to Z \hookrightarrow X$, has infinite image. Otherwise, if we denote by $\widetilde{X} \to X$ the universal covering space of X and by $\widetilde{Z^*}$ the fiber product of $Z^* \to X$ and $\widetilde{X} \to X$, we would obtain a finite covering map $\widetilde{Z_0^*} \to Z^*$, where $\widetilde{Z_0^*}$ is a connected component of $\widetilde{Z^*}$ (by Lemma 108 $\pi_1(\widetilde{Z_0^*}) \simeq \ker(\pi_1(Z^*) \to \pi_1(X))$). Since the map $Z^* \to X$ is a nonconstant holomorphic function, we would obtain that there is a nonconstant holomorphic map from the closed Riemann surface $\widetilde{Z_0^*}$ to $\widetilde{X}$, which contradicts the asphericity of X.

Finally, suppose that Y_0 is contained in the irreducible component Z of F_s and let Δ denote the singular points of Z. Hence, there exists a retraction of $Z \setminus \Delta$ onto Y_0. Therefore, the composition of the inclusions $Y_0 \hookrightarrow Z \setminus \Delta \hookrightarrow Z^*$ induces a surjective map on fundamental groups and the result follows directly. □

Remark 131. *In the proof of the latter lemma, we used the fact that the universal covering space of an aspherical compact Kähler surface cannot contain the image of a closed Riemann surface under a nonconstant holomorphic map. This follows from the fact that for a Kähler surface X, the integral of the Kähler form over any analytic compact curve is always positive, which cannot occur if $H_2(X)$ is trivial.*

A direct consequence of Lemmas 129 and 130 is the following result.

Corollary 132. *There exists a neighborhood of F_s such that for any regular fiber F of $f : X \to S$ contained in such a neighborhood and any connected component Y of $F \setminus \cup_{i=1}^{l} \mathcal{B}_i(\varepsilon_0)$, the map on fundamental groups induced by the inclusion $Y \hookrightarrow X$ has infinite image.*

2.6.4 Proof of Kapovich's Theorem

In this section we give the proof of Theorem 70 for the general case.

Let X be an aspherical compact Kähler surface and let $f : X \to S$ be an irrational pencil whose singular fibers are of multiplicity one. We resume with the construction of Section 2.4.3. Let us assume by contradiction that f is not a submersion and that the kernel of $f_* : \pi_1(X) \to \pi_1(S)$ is finitely presented. By Lemma 127, we know that there exists a positive integer k_0 such that for all $k \geq k_0$, the inclusion map $X_k \hookrightarrow \widehat{X}$ induces an isomorphism on fundamental groups. We fix $k \geq k_0$.

As in the proof of Proposition 112, let $c : [0,1] \to \mathrm{Int}(D_{k+1})$ be an embedded arc going from a boundary point $c(0)$ of D_k to the unique critical value contained in $D_{k+1} - D_k$. We assume that $c(t) \notin D_k$ for $t > 0$ and let D^* be a small disk centered at $c(1)$ and contained in $\mathrm{Int}(D_{k+1} \setminus D_k)$. We may assume as well that $f^{-1}(D^*)$ is contained in a neighborhood of F_s satisfying the conclusion of Corollary 132.

We denote by $p : \widetilde{X} \to X$ the universal covering space of X and we write,

$$U = p^{-1}(f^{-1}(D_k \cup c[0,1[)) \quad \text{and} \quad V = p^{-1}(f^{-1}(D^*)).$$

The desired contradiction to prove Theorem 70 will follow from studying the following fragment of the Mayer-Vietoris sequence associated to U and V.

$$H_2(U \cap V) \longrightarrow H_2(U) \oplus H_2(V) \longrightarrow H_2(U \cup V) \longrightarrow \tag{2.6.3}$$

$$\longrightarrow H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(U \cup V).$$

Let $\widehat{X}$ be the universal fiber product of f. Since $p : \widetilde{X} \to X$ is the universal covering space of X and $\widehat{q} : \widehat{X} \to X$ is a covering space of X, there exists a covering map $p_1 : \widetilde{X} \to \widehat{X}$ such that the diagram

$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow{\scriptstyle p_1} & & \\ \widehat{X} & \xrightarrow{\widehat{f}} & \widehat{S} \\ \downarrow{\scriptstyle \widehat{q}} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & S \end{array}$$

commutes and $p = \widehat{q} \circ p_1$. We obtain from this that
- $U \cap V$ deformation retracts onto $\widetilde{F} := p^{-1}(F)$, where F is a regular fiber of f, whose image is contained in D^*.
- U deformation retracts onto $\widetilde{X}_k := p_1^{-1}(X_k)$.
- $\widetilde{X}_{k+1} := p_1^{-1}(X_{k+1})$ deformation retracts onto $U \cup V$.

By Proposition 107, we get that $\widetilde{F}$ is a connected surface contained in $\widetilde{X}$, since the inclusion $F \hookrightarrow X$ induces a surjective map on fundamental groups. The asphericity of X implies that $\widetilde{F}$ is non-compact (see Remark 131). Therefore, $H_2(\widetilde{F})$ is trivial. We next observe that $\widetilde{X}_k$ and $\widetilde{X}_{k+1}$ are simply connected, since the inclusions $X_k, X_{k+1} \hookrightarrow \widehat{X}$ induce isomorphisms on fundamental groups. Hence (2.6.3) can be simplified to the exact sequence

$$0 \longrightarrow H_2(\widetilde{X}_k) \oplus H_2(V) \longrightarrow H_2(\widetilde{X}_{k+1}) \longrightarrow H_1(\widetilde{F}) \longrightarrow H_1(V) \longrightarrow 0. \tag{2.6.4}$$

Lemma 133. *The map $H_2(\widetilde{X}_k) \to H_2(\widetilde{X}_{k+1})$ induced by the inclusion $\widetilde{X}_k \hookrightarrow \widetilde{X}_{k+1}$ is injective and $H_2(\widetilde{X}_{k+1})$ is non-trivial.*

The injectivity of $H_2(\widetilde{X}_k) \to H_2(\widetilde{X}_{k+1})$ follows directly from the exactness of (2.6.4). Then Lemma 133 follows from the following result (by the exactness of (2.6.4)).

Lemma 134. *The map $H_1(\widetilde{F}) \to H_1(V)$ induced by the inclusion $\widetilde{F} \hookrightarrow V$ has non-trivial kernel.*

Proof. Let C be a connected component of $p^{-1}(F \cap \mathcal{B}_i(\varepsilon_0))$ for some $i = 1, \dots, l$ (which is homeomorphic to $F \cap \mathcal{B}_i(\varepsilon_0)$). By Theorem 122, we know that C has the homotopy type of a CW-complex of dimension 1, and by Theorem 123, C is not simply connected. Hence, C is a compact surface with non-empty boundary and it is not a disk.

Now, let $H_1(C) \to H_1(\widetilde{F})$ be the map on homology groups induced by the inclusion $C \hookrightarrow \widetilde{F}$. Lemma 134 follows from the following two facts

(a) The image of $H_1(C)$ in $H_1(\widetilde{F})$ is contained in the kernel of the map $H_1(\widetilde{F}) \to H_1(V)$.

(b) The map $H_1(C) \to H_1(\widetilde{F})$ is non-trivial.

Notice that (a) follows directly from the fact that C is contained in the ball $\mathcal{B}_i(\varepsilon_0)$. From now on, to prove (b) we will be working with the surface $\widetilde{F}$ as our total space. If the surface C has a handle, (b) follows immediately since there exists an element in $H_1(C)$ which has non-trivial image in $H_1(F)$ in this case (which is stronger than having non-trivial image in $H_1(\widetilde{F})$). If C is a planar surface the last argument does not work. Using the Mayer-Vietoris sequence for the sets (or rather small enough neighborhoods of these sets) C and $(\widetilde{F} \setminus C)$, we get the exact sequence

$$H_1(\partial C) \xrightarrow{(i_*, -j_*)} H_1(C) \oplus H_1(\widetilde{F} \setminus C) \xrightarrow{k_* + l_*} H_1(\widetilde{F}), \tag{2.6.5}$$

where, i, j, k, l are the respective inclusions.

Claim 135. *All the connected components of $\widetilde{F} \setminus C$ are non-compact.*

We now conclude the proof by using Claim 135. If ∂C has n connected components $\{a_1, \dots, a_n\}$, then $H_1(\partial C)$ is a free Abelian group generated by the a_i's with the unique relation

$$\sum_{i=1}^{n} i_* a_i = 0.$$

Let W be a connected component of $\widetilde{F} \setminus C$. Up to reordering the set $\{a_1, \dots, a_n\}$, we can suppose that $\{a_1, \dots, a_l\}$ are the common boundary components of C and W for some $l \leq n$. Let us assume by contradiction that the map $k_* : H_1(C) \to H_1(\widetilde{F})$ is trivial. Then $(i_* a_1, 0)$ lies in $\ker(k_* + j_*) = \operatorname{Im}(i_*, -j_*)$, and by the exactness of 2.6.5 there is a non-zero element a in $H_1(\partial C)$ such that $(i_* a, -j_* a) = (i_* a_1, 0)$. Thus $a - a_1$ lies in $\ker(i_*)$, which is generated by $a_1 + \cdots + a_n$, *i.e.* $a = a_1 + \lambda(a_1 + \cdots + a_n)$. Since $H_1(\widetilde{F} \setminus C) = H_1(W) \oplus H_1(\widetilde{F} \setminus (C \cup W))$ and $j_* a = 0$, we deduce that $j_*(a_1 + \lambda(a_1 + \cdots + a_l)) = 0$. Finally, since W is non-compact, $\{j_* a_1, \dots, j_* a_l\}$ form a linearly independent set and thereby $\lambda = -1$ and $l = 1$. A contradiction to the fact that a is non-zero. ☐

Proof of Claim 135. Let W be a connected component of $\widetilde{F} \setminus C$. Recall that C is a connected component of $p^{-1}(F \cap \mathcal{B}_i(\varepsilon_0))$ which indeed is homeomorphic to $F \cap \mathcal{B}_i(\varepsilon_0)$. Now, let Y be a connected component of $F \setminus \cup_{i=1}^{l} \mathcal{B}_i(\varepsilon_0)$ and let $\widetilde{Y}$ be a connected component of $p^{-1}(Y)$ contained in W. Hence, we get the commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow p \\ Y & \longrightarrow & X. \end{array}$$

By Corollary 132, the map $\pi_1(Y) \to \pi_1(X)$ has infinite image, and therefore $p_*(\pi_1(\widetilde{Y}))$ is an infinite index subgroup of $\pi_1(Y)$. This implies that $\widetilde{Y} \to Y$ is an infinite covering map. We deduce from this that $\widetilde{Y}$ is non-compact and thus, so is W. □

Now, we can give the proof of Theorem 70

Proof of Theorem 70. On the one hand, by Lemma 133, for large enough k, the map $H_2(\widetilde{X}_k) \to H_2(\widetilde{X}_{k+1})$ induced by the inclusion $\widetilde{X}_k \hookrightarrow \widetilde{X}_{k+1}$ is injective. Since $H_2(\widetilde{X})$ is the direct limit of $H_2(\widetilde{X}_k)_{k\geq 1}$, we get that for large enough k, the map $H_2(\widetilde{X}_k) \to H_2(\widetilde{X})$ is injective. From this and the asphericity of X, we deduce that for large enough k, $H_2(\widetilde{X}_k)$ is trivial.

On the other hand, by Lemma 133, one can find large enough k such that $H_2(\widetilde{X}_k)$ is non-trivial, which is the desired contradiction. □

Bibliography

[1] N. A'Campo, *Le nombre de Lefschetz d'une monodromie*, Nederl. Akad. Wetensch. Proc. Ser. A **76** = Indag. Math. **35**, (1973), 113–118. – cited on p. 71

[2] J. Amorós, M. Burger, K. Corlette, D. Kotschick and D. Toledo, *Fundamental groups of compact Kähler manifolds*, Mathematical Surveys and Monographs. **44**, American Mathematical Society, Providence, RI, (1996). – cited on p. 1, 9

[3] D. Arapura, *Fundamental groups of smooth projective varieties*, Current topics in complex algebraic geometry (Berkeley, CA, 1992/93), Math. Sci. Res. Inst. Publ, Cambridge Univ. Press, Cambridge. **28**, (1995), 1–16. – cited on p. 1, 9

[4] D. Arapura, P. Bressler and M. Ramachandran, *On the fundamental group of a compact Kähler manifold*, Duke Mathematical Journal. **68**, No. 3, (1992), 477–488. – cited on p. 2, 10

[5] D. Arapura and M. Nori, *Solvable fundamental groups of algebraic varieties and Kähler manifolds*, Compositio Mathematica. **116**, No. 2, (1999), 173–188. – cited on p. 2, 10

[6] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129**, No. 3 (1997), 445–470. – cited on p. 52, 57

[7] R. Bieri, *Homological dimension of discrete groups*, Queen Mary College Mathematics Notes, Mathematics Department, Queen Mary College, London (1976). – cited on p. 7, 15, 57

[8] R. Bieri, *Normal subgroups in duality groups and in groups of cohomological dimension* 2, Journal of Pure and Applied Algebra. **7**, No. 1, (1976), 35–51. – cited on p. 58

[9] J. S. Birman, A. Lubotzky and J. McCarthy, *Abelian and solvable subgroups of the mapping class groups*, Duke Mathematical Journal. **50**, No. 4, (1983), 1107–1120. – cited on p. 46, 48

[10] I. Biswas, M. Mj and D. Pancholi, *Homotopical height*, Internat. J. Math. **25**, No. 13 (2014). – cited on p. 53

[11] B. H. Bowditch, *Cut points and canonical splittings of hyperbolic groups*, Acta Mathematica. **180**, No. 2, (1998), 145–186. – cited on p. 4, 12, 18, 41, 42, 44

[12] N. Brady, *Branched coverings of cubical complexes and subgroups of hyperbolic groups*, J. London Math. Soc.(2) **60**, No. 2 (1999), 461–480. – cited on p. 70

[13] M. R. Bridson, J. Howie, C. F. Miller III and H. Short, *The subgroups of direct products of surface groups*, Dedicated to John Stallings on the occasion of his 65th birthday, Geom. Dedicata **92** (2002), 95–103. – cited on p. 53, 57, 69

[14] M. R. Bridson and C. Llosa Isenrich, *Kodaira fibrations, Kähler groups, and finiteness properties*, Trans. Amer. Math. Soc. **372**, No. 8 (2019), 5869–5890. – cited on p. 3, 7, 11, 15

[15] C. Bregman and L. Zhang, *On Kähler extensions of abelian groups*, arXiv e-prints, (2016). – cited on p. 3, 11

[16] K. Brown, *Cohomology of groups*, Graduate Texts in Mathematics. **87**, Springer-Verlag, New York-Berlin, (1982). – cited on p. 53, 54, 56, 57, 70

[17] A. Brudnyi, *Solvable matrix representations of Kähler groups*, Differential Geometry and its Applications. **19**, No. 2, (2003), 167–191. – cited on p. 2, 10

[18] M. Burger, *Fundamental groups of Kähler manifolds and geometric group theory*, Séminaire Bourbaki. Vol. 2009/2010. Exposés 1012–1026, Astérisque, No. 1022, (2011), 305–321. – cited on p. 1, 9

[19] F. Campana, *Ensembles de Green-Lazarsfeld et quotients résolubles des groupes de Kähler*, Journal of Algebraic Geometry. **10**, No. 4, (2001), 599–622. – cited on p. 2, 10

[20] F. Campana, *Remarques sur les groupes de Kähler nilpotents*, Annales Scientifiques de l'École Normale Supérieure. Quatrième Série. **28**, No. 3, (1995), 307–316. – cited on p. 2, 10

[21] A. Candel and L. Conlon, *Foliations. I*, Graduate Studies in Mathematics. **27**, American Mathematical Society, Providence, RI, (2000). – cited on p. 71

[22] J. A. Carlson and D. Toledo, *Harmonic mappings of Kähler manifolds to locally symmetric spaces*, Institut des Hautes Études Scientifiques. Publications Mathématiques, No. 69, (1989), 173–201. – cited on p. 2, 9, 47

[23] D. I. Cartwright, V. Koziarz and S. K. Yeung, *On the Cartwright-Steger surface*, J. Algebraic Geom. **26**, No. 4 (2017), 655–689. – cited on p. 8, 16, 68

[24] D. I. Cartwright and T. Steger, *Enumeration of the 50 fake projective planes*, C. R. Math. Acad. Sci. Paris **348**, No. 1-2 (2010), 11–13. – cited on p. 8, 16, 68

[25] F. Catanese, *Fibred Kähler and quasi-projective groups*, Special issue dedicated to Adriano Barlotti, Advances in Geometry, (2003), S13–S27. – cited on p. 6, 14

[26] F. Catanese, J. Keum and K. Oguiso, *Some remarks on the universal cover of an open $K3$ surface*, Mathematische Annalen, **325**, No. 2, (2003), 279–286. – cited on p. 43

[27] B. Claudon, A. Höring and H-Y. Lin, *The fundamental group of compact Kähler threefolds*, Geometry & Topology. **23**, No. 7, (2019), 3233–3271. – cited on p. 1, 9

[28] M. Dehn, *Papers on group theory and topology*, Translated from the German and with introductions and an appendix by John Stillwell, With an appendix by Otto Schreier, Springer-Verlag, New York, (1987). – cited on p. 4, 11

[29] T. Delzant, *L'invariant de Bieri-Neumann-Strebel des groupes fondamentaux des variétés kählériennes*, Mathematische Annalen. **348**, No. 1, (2010), 119–125. – cited on p. 2, 6, 7, 10, 14, 15

[30] A. Dimca, *Singularities and topology of hypersurfaces*, Universitext, Springer-Verlag, New York (1992). – cited on p. 7, 15

[31] A. Dimca, S. Papadima and A. Suciu, *Non-finiteness properties of fundamental groups of smooth projective varieties*, J. Reine Angew. Math. **629** (2009), 89–105. – cited on p. 3, 7, 8, 11, 15, 16, 51, 52, 53, 58, 64, 65, 67, 68, 69

[32] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series. **49**, Princeton University Press, Princeton, NJ, (2012). – cited on p. 27, 39, 46

[33] S. Friedl and S. Vidussi *Virtual algebraic fibrations of Kähler groups*, preprint, (2019), arXiv:1704.07041. – cited on p. 6, 14

[34] R. Geoghegan, *Topological methods in group theory*, Graduate Texts in Mathematics. **243**, Springer, New York, (2008). – cited on p. 58

[35] N. D. Gilbert, J. Howie, V. Metaftsis and E. Raptis, *Tree actions of automorphism groups*, Journal of Group Theory. **3**, No. 2, (2000), 213–223. – cited on p. 4, 12, 17, 18, 31, 41

[36] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, (1978). – cited on p. 75

[37] M. Gromov and R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, Institut des Hautes Études Scientifiques. Publications Mathématiques, No. 76, (1992), 165–246. – cited on p. 4, 12, 18, 42

[38] M. Gromov, *Sur le groupe fondamental d'une variété kählérienne*, Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique. **308**, No. 3, (1989), 67–70. – cited on p. 2, 10

[39] P. Harpe *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, (2000). – cited on p. 57

[40] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, (2002). – cited on p. 55, 56

[41] N. V. Ivanov, *Subgroups of Teichmüller modular groups*, Translations of Mathematical Monographs. **115**, Translated from the Russian by E. J. F. Primrose and revised by the author, American Mathematical Society, Providence, RI, (1992). – cited on p. 49

[42] M. Kapovich, *Noncoherence of arithmetic hyperbolic lattices*, Geometry & Topology. **17**, No. 1, (2013), 39–71. – cited on p. 6, 14

[43] M. Kapovich, *On normal subgroups in the fundamental groups of complex surfaces*, preprint arXiv:9808085 (1998). – cited on p. 6, 14, 52, 53

[44] A. Karrass, A. Pietrowski and D. Solitar, *Automorphisms of a free product with an amalgamated subgroup*, Contributions to group theory, Contemp. Math., Amer. Math. Soc., Providence, RI. **33**, (1984), 328–340. – cited on p. 17, 18, 31, 40

[45] J. Kollár, *Shafarevich maps and automorphic forms*, M. B. Porter Lectures, Princeton University Press, Princeton N.J. (1995). – cited on p. 7, 15

[46] V. Koziarz and S. K. Yeung, *Stability of the Albanese fibration on the Cartwright-Steger surface*, preprint 2019. – cited on p. 8, 16, 68

[47] R. Kropholler, *Almost hyperbolic groups with almost finitely presented subgroups*, preprint arXiv:1802.01658, (2018). – cited on p. 70

[48] G. Levitt, *Automorphisms of hyperbolic groups and graphs of groups*, Geometriae Dedicata. **114**, (2005), 49–70. – cited on p. 41, 42

[49] C. Llosa Isenrich, *Branched covers of elliptic curves and Kähler groups with exotic finiteness properties*, Ann. Inst. Fourier (Grenoble) **69**, No. 1 (2019), 335–363. – cited on p. 3, 7, 8, 11, 15, 16, 52, 53, 68, 69

[50] C. Llosa Isenrich, *Kähler groups and subdirect products of surface groups*, Geometry & Topology **24**, No. 2 (2020), 971–1017. – cited on p. 3, 7, 8, 11, 15, 16, 53, 69

[51] J. McCarthy, *Normalizers and centralizers of pseudo-Anosov mapping classes*, https://users.math.msu.edu/users/mccarthy/publications/normcent.pdf. – cited on p. 47

[52] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J., University of Tokyo Press, Tokyo (1968). – cited on p. 58, 61, 71

[53] A. Minasyan and D. Osin, *Acylindrical hyperbolicity of groups acting on trees*, Mathematische Annalen. **362**, No. 3-4, (2015), 1055–1105. – cited on p. 23

[54] T. Napier and M. Ramachandran, *Hyperbolic Kähler manifolds and proper holomorphic mappings to Riemann surfaces*, Geometric and Functional Analysis. **11**, No. 2, (2001), 382–406. – cited on p. 7, 15

[55] F. Nicolás and P. Py *Irrational pencils and Betti numbers*, preprint (2020), arXiv:2006.09566. – cited on p. 8, 15

[56] F. Nicolás, *On finitely generated normal subgroups of Kähler groups*, preprint, (2021), arXiv:2102.01128. – cited on p. 4, 12

[57] J. Nielsen, *Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen*, Acta Mathematica. **50**, No. 1, (1927), 189–358. – cited on p. 4, 11

[58] J. P. Otal, *Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3*, Astérisque, No. 235, (1996). – cited on p. 47

[59] M. R. Pettet, *The automorphism group of a graph product of groups*, Communications in Algebra. **27**, No. 10, (1999), 691–4708. – cited on p. 4, 12, 17, 18, 31, 40, 41

[60] P. Py, *Some noncoherent, nonpositively curved Kähler groups*, L'Enseignement Mathématique. **62**, No. 1-2, (2016), 171–187. – cited on p. 6, 14

[61] C. Rito *Surfaces with canonical map of maximum degree*, preprint, (2020), arXiv:1903.03017. – cited on p. 8, 16, 68

[62] O. Schreier, *Die Untergruppen der freien Gruppen*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg. **5**, No. 1, (1927), 161–183. – cited on p. 6, 14

[63] Z. Sela, *Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank* 1 *Lie groups. II*, Geometric and Functional Analysis. **7**, No. 3, (1997), 561–593. – cited on p. 4, 12, 18, 41, 44

[64] P. Scott, *Subgroups of surface groups are almost geometric*, Journal of the London Mathematical Society. Second Series. **17**, No. 3, (1978), 555–565. – cited on p. 49

[65] P. Scott and T. Wall, *Topological methods in group theory*, Homological group theory (Proc. Sympos., Durham, 1977), London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge-New York. **36**, (1979), 137–203. – cited on p. 17, 19, 21, 35, 45

[66] J. P. Serre, *Trees*, Springer Monographs in Mathematics, Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation, Springer-Verlag, Berlin, (2003). – cited on p. 17, 19, 20, 21

[67] J. P. Serre, *Sur la topologie des variétés algébriques en caractéristique* p, Symposium internacional de topología algebraica International symposium on algebraic topology, Universidad Nacional Autónoma de México and UNESCO, Mexico City, (1958), 24–53. – cited on p. 1, 9

[68] H. Shiga, *On monodromies of holomorphic families of Riemann surfaces and modular transformations*, Mathematical Proceedings of the Cambridge Philosophical Society. **122**, No. 3, (1997), 541–549. – cited on p. 3, 11, 17

[69] J. R. Stallings, *A finitely presented group whose 3-dimensional integral homology is not finitely generated*, Am. J. Math. **85**, (1963), 541–543. – cited on p. 57, 58

[70] M. Stover, *On general type surfaces with* $q = 1$ *and* $c_2 = 3p_g$, Manuscripta Math. **159**, No. 1-2, (2019), 171–182. – cited on p. 68

[71] X. Sun, *Regularity of harmonic maps to trees*, American Journal of Mathematics. **125**, No. 4, (2003), 737–771. – cited on p. 42

[72] D. Toledo, *Examples of fundamental groups of compact Kähler manifolds*, The Bulletin of the London Mathematical Society. **22**, No. 4, (1990), 339–343. – cited on p. 1, 9

[73] D. Toledo, *Projective varieties with non-residually finite fundamental group*, Institut des Hautes Études Scientifiques. Publications Mathématiques, No. 77, (1993), 103–119. – cited on p. 2, 10

[74] S. Vidussi, *The slope of surfaces with Albanese dimension one*, Math. Proc. Cambridge Philos. Soc. **167**, No. 2 (2019), 355–360. – cited on p. 68

[75] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés **10**, Société Mathématique de France, Paris (2002). – cited on p. 1, 9, 58, 59, 61

[76] C. T. C. Wall, *Finiteness conditions for CW-complexes*, Ann. of Math. (2) **81** (1965), 56–69. – cited on p. 57

Prénom NOM

TITRE de la thèse

Résumé

Dans cette thèse nous nous sommes intéressés à l'étude des sous-groupes normaux de type fini des groupes kählériens. Nous étudions les groupes de type fini munis d'une action sur un arbre qui admettent un plongement dans un groupe kählérien comme sous-groupes normaux et dont l'action sur l'arbre peut être virtuellement étendue au groupe kählérien. L'un des ingrédients principaux de cette étude est un résultat de Gromov et Schoen sur les groupes kählériens agissant sur un arbre. Nous donnons également de nouveaux exemples de groupes kählériens qui apparaissent comme des sous-groupes normaux de groupes kählériens précédemment connus. Les nouveaux exemples qui apparaissent de cette manière sont liés aux propriétés de finitude dans la théorie des groupes. Notre principal outil pour construire ces exemples est l'étude des pinceaux irrationnels avec des points critiques isolés sur des variétés complexes compactes asphériques.

Résumé en anglais

In this thesis, we focus on the study of finitely generated normal subgroups of Kähler groups. We present some restrictions on finitely generated groups that can occur as normal subgroups of Kähler groups. We study finitely generated groups acting on a tree that admit an embedding into a Kähler group as a normal subgroup, and whose action on the tree can be virtually extended to the Kähler group. One of the main ingredients of this study is a result of Gromov and Schoen about Kähler groups acting on trees. We also give new examples of Kähler groups which occur as normal subgroups of previously known Kähler groups. The new examples that occur in this way are related to finiteness properties in group theory. Our main tool to construct these examples is the study of irrational pencils with isolated critical points on compact aspherical complex manifolds.