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Ergodic actions of Torelli groups on character varieties and pure modular groups on relative character varieties and topological dynamics of modular groups

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"Aujourd'hui on raconte une histoire abracadabrantesque."
Jacques Chirac

Introduction en français

Soient Γ un groupe de type fini et G un groupe de Lie semi-simple. L'*espace des représentations* $\mathrm{Hom}(\Gamma, G)$ est l'ensemble de tous les homomorphismes $\Gamma \to G$ que nous munissons de la topologie compact-ouverte.
Le groupe G agit par conjugaison sur $\mathrm{Hom}(\Gamma, G)$. En effet, pour $g \in G$ et $\rho : \pi_1\Sigma \to G$, nous définissons la représentation $g \cdot \rho$ comme étant le morphisme $\gamma \mapsto g\rho(\gamma)g^{-1}$. Le quotient de $\mathrm{Hom}(\Gamma, G)$ par l'action de G n'est généralement pas séparé.
La *variété des caractères* est le quotient Hausdorff $\mathrm{Hom}(\Gamma, G)//G$ et sera notée par $\mathcal{X}(\Gamma, G)$. Dans le cas plus général où G est un groupe algébrique complexe, nous pourrions définir la variété des caractères comme le quotient GIT de l'espace des représentations par l'action de G. Cependant, si G est un groupe réductif, le quotient GIT et le quotient Hausdorff coïncident. Dans cette thèse, G est supposé être un groupe de Lie réel semi-simple, cette question n'intervient pas. La variété des caractères contient des points singuliers :

Définition 1. *Une représentation est* régulière *si son centralisateur, qui est le stabilisateur pour l'action par conjugaison de G, est discret.*

Soit $\mathcal{R}\mathrm{ep}(\Gamma, G)$ le sous-ensemble de $\mathcal{X}(\Gamma, G)$ formé des classes de conjugaison de représentations régulières. C'est un sous-espace dense, ouvert et lisse. Dans **Goldman 1**, lorsque Γ est le groupe fondamental d'une surface Σ orientée, compacte, connexe et de genre $g \geq 2$, Goldman construit une structure symplectique sur $\mathcal{R}\mathrm{ep}(\pi_1\Sigma, G)$ issue de la forme de Killing sur G. La forme symplectique de Goldman induit une mesure de Radon λ sur $\mathcal{R}\mathrm{ep}(\pi_1\Sigma, G)$. Cette mesure s'étend naturellement en une mesure de Radon sur $\mathcal{X}(\pi_1\Sigma, G)$, que nous noterons encore λ, telle que $\mathcal{R}\mathrm{ep}(\pi_1\Sigma, G)$ est de mesure pleine pour λ. Cette mesure a de plus le bon goût d'être dans la classe de Lebesgue de $\mathcal{X}(\pi_1\Sigma, G)$.

La variété des caractères encodent des structures géométriques et est l'espace des modules d'objets géométriques :

Exemple 1. *Supposons que Γ est un groupe de surface et que $G = \mathrm{PSL}_2(\mathbf{R})$, c'est à dire que G est le groupe d'isométrie du demi-plan de Poincaré. D'après les travaux de Goldman, le second point du théorème 5 précisément, la variété des caractères $\mathcal{X}(\pi_1\Sigma, \mathrm{PSL}_2(\mathbf{R}))$ contient alors deux composantes connexes formées de classes de conjugaison d'holonomies de structures hyperboliques sur Σ. En particulier, ces composantes sont toutes deux homéomorphes à l'espace de Teichmüller de Σ.*

Exemple 2. *Supposons que Γ est le groupe fondamental d'une surface de Riemann Σ et que $G = \mathrm{SU}(n)$. Le théorème de Narasimhan-Seshadri assure l'existence d'une bijection entre $\mathcal{R}\mathrm{ep}(\pi_1\Sigma, \mathrm{SU}(n))$ et l'espace des modules de fibrés vectoriels holomorphes de rang n, semi-stables et de degré 0 sur Σ, voir Donaldson 2018.*

Ces deux exemples suggèrent la richesse géométrique de l'étude des variétés de caractères.

Dans une direction plus dynamique, le groupe des automorphismes de Γ agit par pré-composition sur l'espace des représentations. Précisément, pour $\psi \in \mathrm{Aut}(\Gamma)$ et $\rho \in \mathrm{Hom}(\Gamma, G)$, nous définissons $\psi \cdot \rho$ comme la représentation $\gamma \mapsto \rho(\psi^{-1}(\gamma))$. De plus, l'action du groupe des automorphismes intérieurs de Γ est contenue dans l'action par conjugaison de G sur $\mathrm{Hom}(\Gamma, G)$. En effet, soit $\psi_x : \gamma \mapsto x\gamma x^{-1}$ un automorphisme intérieur et $\rho \in \mathrm{Hom}(\Gamma, G)$, alors $\psi_x \cdot \rho : \gamma \mapsto \rho(x^{-1})\rho(\gamma)\rho(x)$ est conjuguée à ρ. Nous obtenons ainsi une action de $\mathrm{Out}(\Gamma)$, défini comme le quotient $\frac{\mathrm{Aut}(\Gamma)}{\mathrm{Inn}(\Gamma)}$, sur la variété des caractères via la formule

$$[\psi] \cdot [\rho] = [\psi \cdot \rho].$$

Cette action préserve la mesure de Goldman et implique une dynamique mesurable sur la variété des caractères. L'objet de cette thèse est d'étudier les propriétés dynamiques de cette action.

Dans les travaux que cette thèse présente, Γ est le groupe fondamental d'une surface fermée, connexe et compacte Σ. Le théorème de Dehn-Nielsen-Baer, voir Farb and Margalit 2011, nous assure alors de l'existence d'un isomorphisme entre $\mathrm{Out}(\Sigma)$ et le groupe modulaire de Σ, que nous noterons par $\mathrm{Mod}(\Sigma)$. Ce groupe est le groupe des difféomorphismes de Σ à isotopie près. Nous noterons par $\mathrm{Mod}^+(\Sigma)$ le sous-groupe des classes d'isotopies de difféomorphismes directs de Σ. Ce point de vue permet une description topologique de $\mathrm{Out}(\Sigma)$ et nous procure une approche géométrique de son action. Introduisons certains sous-groupes particuliers du groupe modulaire :

- Le *groupe de Torelli* est le noyau de l'action de $\mathrm{Mod}^+(\Sigma)$ sur le premier groupe d'homologie de Σ, $\mathrm{H}_1(\Sigma, \mathbf{Z})$.

- Le *premier sous-groupes de Johnson* est le sous-groupe de $\mathrm{Mod}^+(\Sigma)$ engendré par les twists de Dehn le long de courbes séparantes.

Remarquons que le premier sous-groupe de Johnson est inclus dans le groupe de Torelli.

Cas compact

Une action d'un groupe H sur un espace mesuré (X, μ), dont nous supposons la $H-$invariance de μ, est *ergodique* si tout sous-ensemble mesurable et H-invariant de X est de mesure nulle ou a un complémentaire de mesure nulle (nous dirons que l'ensemble est de mesure pleine si cette condition est vérifiée). En particulier, si H' est un sous-groupe de H et si H agit sur un espace mesuré en préservant la mesure, l'ergodicité de l'action de H' implique l'ergodicité de l'action de H. L'exemple classique d'une action ergodique est celui d'une translation irrationnelle du tore $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. Le groupe engendré par le vecteur $(a_1, \ldots, a_n) \in \mathbf{R}^n$ agit par translation sur $\mathbf{T}^n$ et préserve sa mesure de Haar. L'action de $\langle (a_1, \ldots, a_n) \rangle$ sur $\mathbf{T}^n$ est ergodique par rapport à la mesure de Haar si et seulement si les réels $a_1, \ldots, a_n, 1$ sont linéairement indépendants sur $\mathbf{Q}$.

Théorème 1. *(Goldman, Pickrell-Xia) Soit G un groupe de Lie compact et soit $\Gamma = \pi_1\Sigma$ un groupe de surface. Le groupe modulaire $\mathrm{Mod}^+(\Sigma)$ agit ergodiquement sur les composantes connexes de la variété des caractères $\mathcal{X}(\pi_1\Sigma, G)$ par rapport à la mesure symplectique de Goldman.*

Ce théorème a été démontré pour la première fois par Goldman dans **Goldman 1** lorsque G est localement isomorphe à SU(2) puis Pickrell et Xia ont traité le cas général dans **Pickrell-Xia 1**. Certaines questions peuvent être posées à partir de l'ergodicité, en particulier la possibilité d'avoir des propriétés plus précises : trouver des sous-groupes propres de $\mathrm{Mod}^+(\Sigma)$ qui agissent ergodiquement sur les variétés de caractères.

Funar et Marché ont prouvé dans **Funar-Marche** le résultat suivant :

Théorème 2. *(Funar-Marché) Le premier sous-groupe de Johnson agit ergodiquement sur la variété de caractères $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ par rapport à la mesure de Goldman.*

En particulier :

Théorème 3. *Le sous-groupe de Torelli agit ergodiquement sur la variété de caractères $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ par rapport à la mesure de Goldman.*

La preuve utilise des développements de Taylor des fonctions de traces associées aux courbes séparantes en la représentation triviale.

Nous proposons une nouvelle preuve du théorème 3 et généralisons celle-ci au cas des groupes de Lie semi-simples et compacts :

Théorème A. *Soient une surface fermée, compacte, connexe et orientée Σ de genre $g \geq 3$ et G un groupe de Lie connexe, semi-simple et compact. Le groupe de Torelli agit ergodiquement sur les composantes connexes de la variété de caractères $\mathcal{X}(\pi_1\Sigma, G)$ par rapport à la mesure de Goldman.*

Pour prouver le théorème A, nous montrons que toute fonction mesurable $f : \mathcal{X}(\pi_1\Sigma, G) \to \mathbf{R}$ qui est invariante par l'action du groupe de Torelli est presque partout invariante par le groupe modulaire. Pour ce faire, nous introduisons pour chaque paire de courbes co-bordantes, un sous-ensemble de $\mathcal{X}(\pi_1\Sigma, G)$ pour lequel nous prouvons qu'il est de mesure pleine. Sur cet ensemble, chaque fonction invariante (sous l'action du groupe de Torelli) est presque partout invariante sous l'action des twists de Dehn des courbes de la paire. Il suffit donc de restreindre une fonction qui est invariante par le groupe de Torelli à une intersection dénombrable de ces sous-ensembles pour qu'elle soit presque partout invariante par le groupe modulaire.

En corollaire, quitte à remplacer G par un produit fini $G \times \cdots \times G$ et identifier $\mathcal{X}(\pi_1\Sigma, G^k)$ à $\mathcal{X}(\pi_1\Sigma, G)^k$, nous obtenons :

Théorème B. *Soient Σ et G respectivement une surface et un groupe vérifiant l'hypothèse du théorème A. Alors, pour tout $k \geq 1$, le groupe de Torelli* $\mathrm{Tor}(\Sigma)$ *agit ergodiquement sur chaque composante connexe du produit $\mathcal{X}(\Gamma, G)^k$. En particulier, sur les composantes de $\mathcal{X}(\pi_1\Sigma, G)$, l'action du groupe de Torelli est faiblement mélangeante.*

Cas de $\mathrm{PSL}_2(\mathbf{R})$

Soit Σ une surface connexe, orientée et fermée de genre $g \geq 2$. Son groupe fondamental admet la présentation finie :

$$\langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle.$$

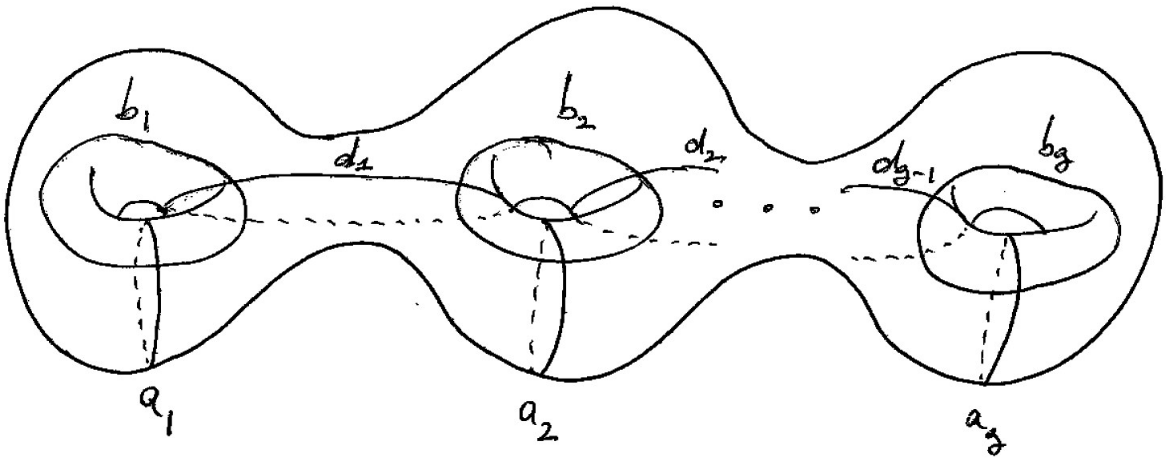

Figure 1

Composantes connexes pour les surfaces fermées

Dans ce paragraphe, nous introduisons la classe d'Euler. C'est un invariant associé à chaque représentation et qui permet de classifier les composantes connexes de $\mathcal{X}(\pi_1\Sigma, \mathrm{PSL}_2(\mathbf{R}))$ (que nous noterons désormais par $\mathcal{X}$ dans le cas de $\mathrm{PSL}_2(\mathbf{R})$ pour éviter des notations trop lourdes).

Soit $\rho : \pi_1\Sigma \to \mathrm{PSL}_2(\mathbf{R})$ une représentation. Pour chaque générateur $x \in \{a_1, b_1, \ldots, a_g, b_g\}$, on choisit un relevé $\widetilde{\rho(x)}$ dans le revètement universel $\widetilde{\mathrm{PSL}_2(\mathbf{R})}$ de $\mathrm{PSL}_2(\mathbf{R})$. Le produit

$$\prod_{i=1}^{g} [\widetilde{\rho(a_i)}, \widetilde{\rho(b_i)}]$$

se projette sur l'identité. On en déduit que $\prod_{i=1}^{g} [\widetilde{\rho(a_i)}, \widetilde{\rho(b_i)}]$ est un élément de $\pi_1(\mathrm{PSL}_2(\mathbf{R})) \cong \mathbf{Z}$. Cet entier ne dépend que de ρ et en aucun cas du choix des relevés. Nous le noterons donc $\mathrm{eu}(\rho)$. L'application $\mathrm{eu} : \mathrm{Hom}(\pi_1\Sigma, \mathrm{PSL}_2(\mathbf{R})) \to \mathbf{Z}$ est de plus invariante par l'action de $\mathrm{PSL}_2(\mathbf{R})$ par conjugaison et descend donc comme une application

$$\mathrm{eu} : \mathcal{X} \to \mathbf{Z}.$$

Dans Milnor 1957/58, Milnor montre que cet invariant est borné :

Théorème 4. *(Milnor 1957/58) Pour tous les* $[\rho] \in \mathcal{X}$ *:*

$$|\mathrm{eu}([\rho])| \leq |\chi(\Sigma)|.$$

Une représentation dont la classe d'Euler est égale à $\pm|\chi(\Sigma)|$ est appelée *maximale.*
Dans W. M. Goldman 1988, Goldman a classifié les composantes connexes par la classe d'Euler et dans W. M. Goldman 1982 il a donné une caractérisation des représentations maximales :

Théorème 5. *(W. M. Goldman 1988 et W. M. Goldman 1982, Goldman)*

1. *Pour tout* $k \in \mathbf{Z} \cap [\chi(\Sigma), -\chi(\Sigma)]$, *le sous-espace* $\mathrm{eu}^{-1}(k)$ *est non vide et les composantes connexes de* $\mathcal{X}$ *sont exactement les* $\mathrm{eu}^{-1}(k)$ *pour* $k \in \mathbf{Z} \cap [\chi(\Sigma), -\chi(\Sigma)]$. *Nous les noterons par* $\mathcal{X}^k$ *et par* $\mathcal{R}\mathrm{ep}^k$ *leurs intersections avec* $\mathcal{R}\mathrm{ep}$.

2. *Une représentation* $\rho : \pi_1\Sigma \to \mathrm{PSL}_2(\mathbf{R})$ *est maximale si et seulement si c'est l'holonomie d'une structure hyperbolique sur* Σ.

Le deuxième point de ce théorème montre que $\mathcal{X}^{\pm|\chi(\Sigma)|}$ est homéomorphe à l'espace de Teichmüller de Σ.

Action du groupe modulaire pour les surfaces fermées

Il est bien connu que l'action du groupe modulaire sur l'espace de Teichmüller de Σ est *proprement discontinue* : pour tout sous-ensemble compact K de l'espace de Teichmüller de Σ, l'ensemble

$$\big\{[\phi] \in \mathrm{Out}(\pi_1\Sigma) \mid [\phi] \cdot K \cap K \neq \emptyset\big\}$$

est fini.

Le comportement de cette action sur les composantes non-maximales est diamétralement différent. L'existence d'éléments $\gamma \in \pi_1\Sigma$ qui sont envoyés sur des isométries non-hyperboliques permet de s'approcher des arguments utilisés dans le cas compact, voir Goldman and Xia 2011. Dans W. Goldman 2006, Goldman a conjecturé :

Conjecture 1. *(Conjecture 3.1, W. Goldman 2006) Soit* $k \in \mathbf{Z} \cap \big]\chi(\Sigma), -\chi(\Sigma)\big[$. *Le groupe modulaire agit ergodiquement sur* $\mathcal{X}^k$ *par rapport à la mesure* λ.

En reliant cette conjecture à la condition de Bowditch, voir Bowditch 1998, et dans la voie de cette conjecture, Marché et Wolff introduisent dans Marché and Wolff 2016, pour $k \in \mathbf{Z} \cap \big]\chi(\Sigma), -\chi(\Sigma)\big[$, le sous-espace

$$\mathcal{NH}^k := \big\{[\rho] \in \mathcal{R}\mathrm{ep}^k \mid \exists \gamma \in \pi_1\Sigma \text{ simple}\,, \mathrm{tr}\big(\rho(\gamma)\big) \in [-2, 2]\big\}.$$

Ils ont prouvé ce qui suit :

Théorème 6. *(Théorème 1.5 et Théorème 1.6, Marché and Wolff 2016)*

1. *Supposons que Σ est de genre 2. Alors l'action de* $\mathrm{Mod}^+(\Sigma)$ *sur* $\mathcal{X}^{\pm 1}$ *est ergodique par rapport à* λ*.*

2. *Supposons que $g \geq 3$ et soit k un entier tel que $|k| < |\chi(\Sigma)|$, alors l'action de* $\mathrm{Mod}^+(\Sigma)$ *sur* $\mathcal{NH}^k$ *est ergodique par rapport à* λ*.*

En genre 2 et pour la classe d'Euler 0, la composante $\mathcal{X}^0$ se décompose en deux sous-espaces disjoints $\mathcal{X}^0_-$ et $\mathcal{X}^0_+$ de mesures non nulles et sur lesquels le groupe modulaire agit ergodiquement par rapport à λ, Marché and Wolff 2016 et Marché and Wolff 2015.

Cas des variétés de caractères relatives dans $\mathrm{PSL}_2(\mathbf{R})$

Soit $\dot{\Sigma}$ une surface connexe, orientée et compacte de genre $g \geq 1$ et à $n > 0$ composantes de bord, telle que $\chi(\dot{\Sigma}) = 2 - 2g - n < 0$. Le groupe fondamental $\pi_1\dot{\Sigma}$ de $\dot{\Sigma}$ est le groupe libre :

$$\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots c_n | \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{n} c_j = 1\rangle$$

où $c_1, \ldots, c_n$ sont des courbes sur $\dot{\Sigma}$ qui sont homotopes à un cercle autour de chaque composante de bord.

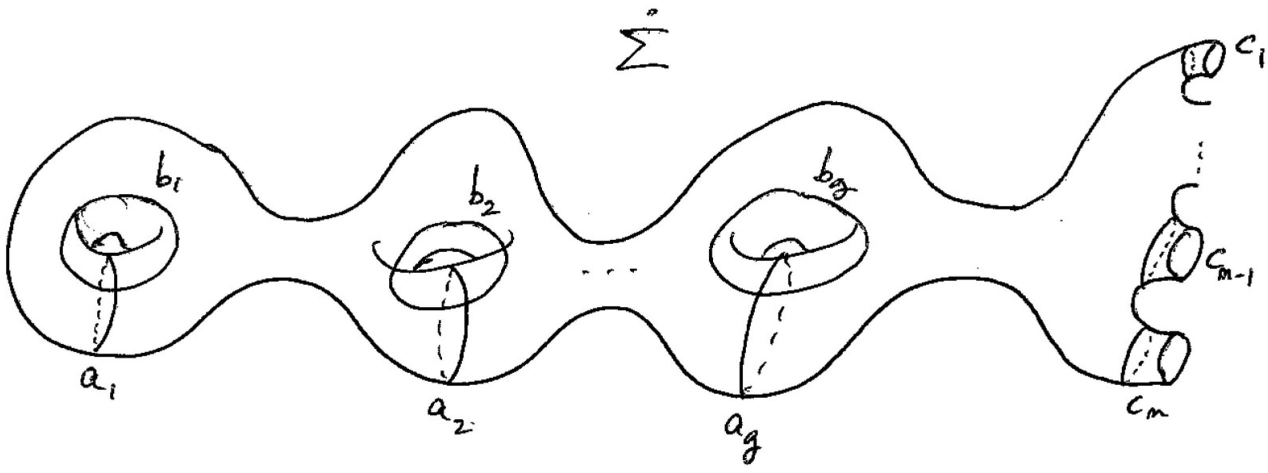

Figure 2

Définition 2. *La* variété de caractères $\mathcal{X}(\dot{\Sigma})$ *est le quotient Hausdorff*

$$\mathrm{Hom}\big(\dot{\Sigma}, \mathrm{PSL}_2(\mathbf{R})\big)//\mathrm{PSL}_2(\mathbf{R}).$$

Une représentation $\rho : \pi_1\dot{\Sigma} \to \mathrm{PSL}_2(\mathbf{R})$ est appelée *Zariski-dense* si son image est Zariski-dense dans $\mathrm{PSL}_2(\mathbf{R})$. En particulier, une représentation Zariski-dense est régulière. Le sous-espace $\mathcal{R}\mathrm{ep}(\dot{\Sigma})$ des classes de représentations Zariski-denses est un sous-ensemble de mesure pleine, ouvert et dense de $\mathcal{X}(\dot{\Sigma})$. Cet espace est partitionné par le comportement sur les composantes de bord. Précisément, soient $\mathcal{C}_1, \ldots, \mathcal{C}_n$ des classes de conjugaison dans $\mathrm{PSL}_2(\mathbf{R})$ et $\underline{\mathcal{C}} := (\mathcal{C}_1, \ldots, \mathcal{C}_n)$, on définit l'espace $\mathrm{Hom}\big(\dot{\Sigma}, \mathrm{PSL}_2(\mathbf{R}), \underline{\mathcal{C}}\big)$ des représentations $\rho : \pi_1\dot{\Sigma} \to \mathrm{PSL}_2(\mathbf{R})$ telles que pour tout $i \in \{1, \ldots, n\}$,

$$\rho(c_i) \in \mathcal{C}_i.$$

Le groupe $\mathrm{PSL}_2(\mathbf{R})$ agit sur l'espace $\mathrm{Hom}\big(\dot{\Sigma}, \mathrm{PSL}_2(\mathbf{R}), \underline{\mathcal{C}}\big)$ par conjugaison.

Définition 3. *La* variété de caractères relative *associée à $\underline{\mathcal{C}}$, notée $\mathcal{X}(\dot{\Sigma}, \underline{\mathcal{C}})$, est le quotient Hausdorff*

$$\mathrm{Hom}\big(\dot{\Sigma}, \mathrm{PSL}_2(\mathbf{R}), \underline{\mathcal{C}}\big) /\!/ \mathrm{PSL}_2(\mathbf{R}).$$

Notons $\mathcal{R}\mathrm{ep}(\dot{\Sigma}, \underline{\mathcal{C}})$ le sous-espace des classes de représentations Zariski-denses de $\mathcal{X}(\dot{\Sigma}, \underline{\mathcal{C}})$. Cet ensemble est une variété lisse de dimension $6g - 6 + 2n$, voir Mondello 2017.

Définition 4. *Soit $\rho : \pi_1\dot{\Sigma} \to \mathrm{PSL}_2(\mathbf{R})$ et fixons une relevé $\widetilde{\rho} : \pi_1\dot{\Sigma} \to \widetilde{\mathrm{PSL}_2(\mathbf{R})}$. Pour chaque $i \in \{1, \ldots, n\}$, notons r_i le nombre de translation de $\widetilde{\rho}(c_i)$. La* classe d'Euler *de ρ est le nombre* $\mathrm{eu}(\rho)$ *défini par*

$$-\sum_{i=1}^{n} r_i.$$

La classe d'Euler d'une représentation ne dépend pas des choix des relevés et est constant sur sa classe de conjugaison. On définit ainsi la classe d'Euler d'une classe de représentations $[\rho]$ par la classe d'Euler d'un représentant de cette classe. Cela donne une application

$$\mathrm{eu} : \mathcal{X}(\dot{\Sigma}) \to \mathbf{R}$$

qui induit par restriction

$$\mathrm{eu}_{\underline{\mathcal{C}}} : \mathcal{X}(\dot{\Sigma}, \underline{\mathcal{C}}) \to \mathbf{R}$$

qui vérifie la propriété $\mathrm{eu}_{\underline{\mathcal{C}}}([\rho]) + \|\{\mathbf{r}\}\|_1 \in \mathbf{Z}$ où $\{\mathbf{r}\}$ est le vecteur des parties fractionnaires des r_i. La préimage de $\mathrm{eu}_{\underline{\mathcal{C}}}^{-1}(e)$, pour $e \in \mathbf{R}$, sera notée $\mathcal{X}^e(\dot{\Sigma}, \underline{\mathcal{C}})$. Dans Mondello 2017, Gabriele Mondello explicite les composantes connexes de $\mathcal{X}(\dot{\Sigma}, \underline{\mathcal{C}})$.

Théorème 7. *(Théorème 2.20, Mondello 2017)*

1. *L'image de* $\mathrm{eu} : \mathcal{X}(\dot{\Sigma}) \to \mathbf{R}$ *est l'intervalle* $\big[\chi(\dot{\Sigma}), -\chi(\dot{\Sigma})\big]$. *Si* $\mathrm{eu}(\rho) = -\chi(\dot{\Sigma})$, *alors tous les $\rho(c_i)$ sont hyperboliques et ρ est la monodromie d'une structure hyperbolique sur $\dot{\Sigma}$ avec des composantes de bord géodésiques.*

2. *Supposons $e > 0$. Alors l'espace $\mathcal{X}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ est non vide si et seulement si*

$$e + \|\{\mathbf{r}\}\|_1 + s_0 + s_- \in \mathbf{Z} \cap \big]0, -\chi(\dot{\Sigma})\big],$$

où s_0 est le cardinal de $\{i \in \{1, \ldots, n\} \mid C_i = \mathrm{id}\}$ et s_- est le cardinal de

$$\{i \in \{1, \ldots, n\} | C_i \text{ unipotent negatif}\}$$

où une matrice unipotente négative est conjuguée à la matrice $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. *Dans ce cas, l'espace* $\mathcal{X}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ *est connexe et lisse.*

Le *groupe modulaire pur* de $\dot{\Sigma}$, noté $\mathrm{PMod}(\dot{\Sigma})$ est le groupe des classes d'isotopies de difféomorphismes de $\dot{\Sigma}$ qui agissent trivialement sur $\partial\dot{\Sigma}$. Il agit naturellement sur les variétés de caractères relatives de manière similaire à celle décrite pour l'action du groupe modulaire dans le cas des surfaces fermées et préserve les composantes connexes de la variété des caractères relative ainsi que la mesure de Goldman.

Résultats

Nous démontrons un analogue du théorème 6 de Marché-Wolff pour les surfaces à bord : soient $\mathcal{C}_1, \ldots, \mathcal{C}_n$ des classes de conjugaison d'isométries elliptiques et soit $e = \pm(k - ||\{\mathbf{r}\}||_1)$ avec $k \in \mathbf{Z} \cap \left[||\{\mathbf{r}\}||_1, -\chi(\dot{\Sigma})\right[$. Alors le sous-espace

$$\mathcal{NH}^e(\dot{\Sigma}, \underline{\mathcal{C}}),$$

des classes de représentations $[\rho]$ qui admettent une courbe simple $\gamma \subset \dot{\Sigma}$ non homotope à une composante de $\partial\dot{\Sigma}$, telle que $\rho(\gamma)$ est non-hyperbolique et telle que $\rho(c_i) \in \mathcal{C}_i$, est non-vide. De plus :

Théorème C. *Le groupe modulaire pur* $\mathrm{PMod}(\dot{\Sigma})$ *agit ergodiquement sur l'espace* $\mathcal{NH}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ *par rapport à la mesure de Goldman.*

La preuve de ce théorème utilise essentiellement la même stratégie que la preuve du théorème 6.6 de Marché and Wolff 2016.

Nous conjecturons que le résultat dynamique du théorème 6 peut être renforcé par l'existence d'un sous-groupe propre agissant ergodiquement sur $\mathcal{NH}^k$.

Conjecture A. *Si $g \geq 3$ et k est un entier tel que $|k| \leq 2g - 5$, alors le groupe* $\mathrm{Tor}(\Sigma)$ *agit ergodiquement sur les sous-espaces $\mathcal{NH}^k$ par rapport à λ.*

Ce résultat serait une version non-compacte du théorème A. Nous developpons une stratégie en section **??** et expliquons en quoi notre proposition est reliée à un analogue de la conjecture de Goldman.

Dynamique topologique

L'exemple de l'action des translations sur les tores nous donne une intuition dynamique des orbites. Dans le cas du cercle, on voit facilement que l'action d'une rotation d'angle irrationnel n'a que des orbites denses. Ce phénomène chaotique peut être vu en dimension supérieure et est un résultat général provenant de l'ergodicité :

Proposition 1. *Soit H un groupe topologique agissant sur un espace borélien (X, μ) et qui préserve la mesure μ. Supposons que chaque ouvert non-vide a une mesure non nulle et que la topologie de X est engendrée par une base dénombrable d'ouverts. Si l'action de H sur (X, μ) est ergodique, alors presque toutes les orbites sont denses.*

Malheureusement, ce fait est un résultat probabiliste et aucune description des points de X pour lesquels l'orbite est dense n'est donnée en général.

Dans W. Goldman 2006, Goldman pose la question suivante :

Question 1. *Pourrait-on trouver une condition sur une représentation $\rho : \pi_1\Sigma \to \mathrm{SU}(2)$ pour que son orbite pour l'action de $\mathrm{Mod}^+(\Sigma)$ soit dense ?*

Dans Previte and Xia 2000 et dans Previte and Xia 2002, Previte et Xia donnent une condition nécessaire et suffisante :

Théorème 8. *Une représentation $\rho : \pi_1\Sigma \to \mathrm{SU}(2)$ a une $\mathrm{Mod}^+(\Sigma)-$orbite dense si et seulement si ρ a une image dense dans $\mathrm{SU}(2)$.*

Avec Gianluca Faraco, nous traitons le cas abélien. Un groupe de Lie compact, abélien et connexe est isomorphe à un tore de dimension n. En utilisant le théorème de Ratner, voir Ratner 1991, nous prouvons dans Bouilly and Faraco 2021 :

Théorème D. *(Bouilly and Faraco 2021) Une représentation $\rho : \pi_1\Sigma \to \mathbf{T}^n$ a une orbite dense sous l'action de $\mathrm{Mod}^+(\Sigma)$ si et seulement si ρ a une image dense dans $\mathbf{T}^n$.*

Nous conjecturons qu'un tel résultat est vrai lorsque $G = \mathrm{SU}(3)$ et Σ est un tore troué.

Lien avec le théorème d'approximation de Kronecker

Le résultat dynamique fourni par le théorème D trouve une application en théorie géométrique des nombres. Un théorème important de cette théorie est le théorème de Kronecker concernant l'approximation diophantienne inhomogène.

Théorème E. *Pour $g \geq 1$. Soit $b^{(i)} = \left(b_1^{(i)}, \ldots, b_{2g}^{(i)}\right)$, avec $i = 1, \ldots, n$, des vecteurs de $\mathbf{R}^{2g}$ tel que $b^{(1)}, \ldots, b^{(n)}$, $\pi e_1, \ldots, \pi e_{2g}$ sont linéairement indépendants sur $\mathbf{Q}$ dans l'espace vectoriel $\mathbf{R}^{2g}$. Soit $A \in \mathrm{M}\big(n, 2g; \mathbf{R}\big)$ une matrice réelle et ε un nombre positif. Alors il existe un élément $K \in \mathrm{Sp}_{2g}(\mathbf{Z})$ tel que*

$$\Big\| A - B\,K \Big\| < C\varepsilon \bmod 2\pi. \tag{1}$$

où C est une constante ne dépendant que de m et n, $\{e_1, \ldots, e_{2g}\}$ est la base canonique de $\mathbf{R}^{2g}$ et la norme est n'importe quelle norme sur $\mathrm{M}\big(n, 2g; \mathbf{R}\big)$.

Introduction

Let Γ be a finitely-generated group and G be a semi-simple Lie group. The *representation space* $\mathrm{Hom}(\Gamma, G)$ is the set of all homomorphisms $\Gamma \to G$ we endow with the compact-open topology.
The group G acts by conjugation on the set $\mathrm{Hom}(\Gamma, G)$. Precisely, for $g \in G$ and $\rho : \pi_1\Sigma \to G$, we define by $g \cdot \rho$ the representation $\gamma \mapsto g\rho(\gamma)g^{-1}$. The quotient of $\mathrm{Hom}(\Gamma, G)$ by the action of G, is generally not Hausdorff.
The *character variety* is the Hausdorff quotient $\mathrm{Hom}(\Gamma, G)//G$ and will be denoted by $\mathcal{X}(\Gamma, G)$. In the more general case when G is a complex algebraic group, we could define the character variety as the GIT quotient of the representation space by G. However, if G is reductive the GIT quotient and the Hausdorff quotient coincide. In this thesis, G is assumed to be a semi-simple real Lie group. The character variety may be singular at some points.

Definition 1. *A representation is* regular *if its centralizer, which is its stabilizer under the conjugation action of G, is discrete.*

Let $\mathcal{R}\mathrm{ep}(\Gamma, G)$ be the subset of $\mathcal{X}(\Gamma, G)$ of classes of regular representations. It is a dense and open subset of the character variety. This subspace is moreover smooth. In **Goldman 1**, when Γ is the fundamental group of a connected, compact and oriented surface Σ of genus $g \geq 2$, Goldman constructs a symplectic structure on $\mathcal{R}\mathrm{ep}(\Gamma, G)$ which comes from the Killing form on G. The Goldman symplectic form gives rise to a Radon measure λ on $\mathcal{R}\mathrm{ep}(\Gamma, G)$, which we extend naturally to a Radon measure on $\mathcal{X}(\Gamma, G)$, we will denote by λ too, and such that $\mathcal{R}\mathrm{ep}(\Gamma, G)$ has full measure. This measure lies in the Lebesgue class of $\mathcal{X}(\Gamma, G)$.

Geometrically, character varieties often encode structures and are moduli spaces of geometric objects. The two following examples explain this geometric point of view.

Example 1. *Assume that $\Gamma = \pi_1\Sigma$ for a closed, connected and oriented surface Σ of genus $g \geq 2$ and that $G = \mathrm{PSL}_2(\mathbf{R})$. Goldman works and precisely second point of Theorem 5 imply that the character variety $\mathcal{X}(\pi_1\Sigma, \mathrm{PSL}_2(\mathbf{R}))$ contains two connected components of classes of representations which are holonomies of hyperbolic structures on Σ. In particular, these components are both isomorphic to the Teichmüller space of Σ.*

Example 2. *Assume that Γ is the fundamental group of a Riemann surface Σ and that $G = \mathrm{SU}(n)$. Then, by Narasimhan-Seshadri theorem, the character variety $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n))$ is in bijection with the moduli space of holomorphic vector bundles of rank n, semi-stable and of degree 0 on Σ, see Donaldson 2018.*

These two examples explain the geometric richness of the study of character varieties and the reasons of this point of view.

The automorphisms group of Γ acts by pre-composition on the representation space: precisely, for $\psi \in \mathrm{Aut}(\Gamma)$ and $\rho \in \mathrm{Hom}(\Gamma, G)$, we define $\psi \cdot \rho$ as the representation $\gamma \mapsto \rho(\psi^{-1}(\gamma))$. Moreover the action of the group of inner automorphisms of Γ is contained in the action by conjugation of G on

$\mathrm{Hom}(\Gamma, G)$ in the following sense: let $\psi_x : \gamma \to x\gamma x^{-1}$ be an inner automorphism and $\rho \in \mathrm{Hom}(\Gamma, G)$, then $\psi_x \cdot \rho : \gamma \mapsto \rho(x^{-1})\rho(\gamma)\rho(x)$ which is in the conjugacy class of ρ. We hence have an action of the group $\mathrm{Out}(\Gamma)$, which is the quotient $\frac{\mathrm{Aut}(\Gamma)}{\mathrm{Inn}(\Gamma)}$, on the character variety via the formula

$$[\psi] \cdot [\rho] = [\psi \cdot \rho].$$

This action preserves the Goldman measure and define a measurable dynamic on the character varieties.

In a lot of case we treat in these works, Γ is the fundamental group of a closed, connected and oriented surface Σ. The Dehn-Nielsen-Baer theorem ensures of an isomorphism between $\mathrm{Out}(\Sigma)$ and the mapping class group of Σ, we will denote by $\mathrm{Mod}^+(\Sigma)$, see Farb and Margalit 2011. This group is the quotient of the group of positive diffeomorphisms of Σ by the equivalence relation of isotopy. This point of view allows to describe the group $\mathrm{Out}(\Sigma)$ well and to have a geometric approach of this action.

We introduce briefly two remarkable subgroups of the mapping class group:

- The *Torelli subgroup* of $\mathrm{Mod}^+(\Sigma)$ is the kernel of the action of $\mathrm{Mod}^+(\Sigma)$ on the homological subspace $\mathrm{H}_1(\Sigma, \mathbf{Z})$.
- The *first Johnson subgroups* is the subgroup of $\mathrm{Mod}^+(\Sigma)$ generated by Dehn twists along separating and simple curves.

Denoting by $\mathrm{tw}_c \in \mathrm{Mod}^+(\Sigma)$ the Dehn twist along c, the Torelli group is generated by the mapping classes of the form $\mathrm{tw}_c \mathrm{tw}_{c'}^{-1}$, for two cohomologuous curves c and c' and by the Dehn twists along separating simples curves.

Compact case

An action of a group H on a measurable space (X, μ), where we assume the measure μ to be $H-$invariant, is *ergodic* if every measurable and $H-$invariant subset of X has null measure or is of full measure, this means that its complement has null measure. In particular, if H' is a subgroup of H and if H acts on a measurable set, preserving the measure, then, if the action of H' is ergodic, the action of H is ergodic. The most classical example of an ergodic action is the case of the $n-$torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. The group generated by a vector $(a_1, \ldots, a_n) \in \mathbf{R}^n$ acts by translation of $\mathbf{T}^n$ and preserve the Haar measure. The action of $\langle (a_1, \ldots, a_n) \rangle$ on $\mathbf{T}^n$ is ergodic with respect to the Haar measure if and only if the real numbers $a_1, \ldots, a_n, 1$ are linearly independent over $\mathbf{Q}$.

Theorem 1. *(Goldman, Pickrell-Xia) Let G be a compact Lie group and let $\Gamma = \pi_1\Sigma$ be the fundamental group of a closed, connected and oriented surface Σ. Then the modular group $\mathrm{Mod}^+(\Sigma)$ acts ergodically on the character variety $\mathcal{X}(\pi_1\Sigma, G)$ with respect to the Goldman measure.*

This theorem was first proved by Goldman in **Goldman 1** when G is locally isomorphic to $\mathrm{SU}(2)$ and Pickrell-Xia treated the general case in **Pickrell-Xia 1**. Some questions can be asked from the ergodicity, especially the possibility to have stronger properties: find proper subgroups of $\mathrm{Mod}^+(\Sigma)$ which acts ergodically on character varieties.

Funar and Marché proved in **Funar-Marche** the following:

Theorem 2. *(Funar-Marché) The first Johnson subgroup act ergodically on the character variety $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ with respect to the Goldman measure.*

From this theorem and the inclusion of the first Johnson subgroup in the Torelli subgroup, we can prove directly the following corollary:

Theorem 3. *The Torelli subgroup acts ergodically on the character variety $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ with respect to the Goldman measure.*

The proof use Taylor series at the trivial representation of traces functions associated to separating curves.

We propose a new proof of theorem 3 and generalize this one for semi-simple and compact Lie groups:

Theorem A. *Let Σ be a closed, compact, connected and oriented surface of genus $g \geq 3$ and let G be a connected, semi-simple and compact Lie group. Then the Torelli group acts ergodically on each connected component of the character variety $\mathcal{X}(\pi_1\Sigma, G)$ w.r.t Goldman measure.*

To prove Theorem A, we show that any measurable function $f : \mathcal{X}(\pi_1\Sigma, G) \to \mathbf{R}$ which is invariant by the Torelli subgroup is almost everywhere invariant by the mapping class group. To do that we introduce for each bounding pair, a subset of $\mathcal{X}(\pi_1\Sigma, G)$ for which we prove that it has full measure. On this full measure set, every invariant function (under the Torelli group action) is almost-everywhere invariant under the action of the Dehn twists of the curves of the pair. It is so sufficient to restrict a function which is invariant by Torelli group to an intersection of enough of these subsets to be mapping class group invariant.

As corollary, replacing G by a finite product $G \times \cdots \times G$ and identifying $\mathcal{X}(\pi_1\Sigma, G^k)$ with $\mathcal{X}(\pi_1\Sigma, G)^k$, we obtain:

Theorem B. *Let Σ and G be a surface and a group verifying the hypothesis of theorem A. Then, for all $k \geq 1$, the Torelli group $\mathrm{Tor}(\Sigma)$ acts ergodically on each connected component of the product $\mathcal{X}(\pi_1\Sigma, G)^k$. In particular on these components, the action of the Torelli group is weakly mixing.*

Non-compact Lie group case

Let Σ be a closed surface of genus $g \geq 2$. Its fundamental group is finitely generated and admits the classical presentation

$$\pi_1\Sigma \cong \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g}[a_i, b_i] = 1\rangle.$$

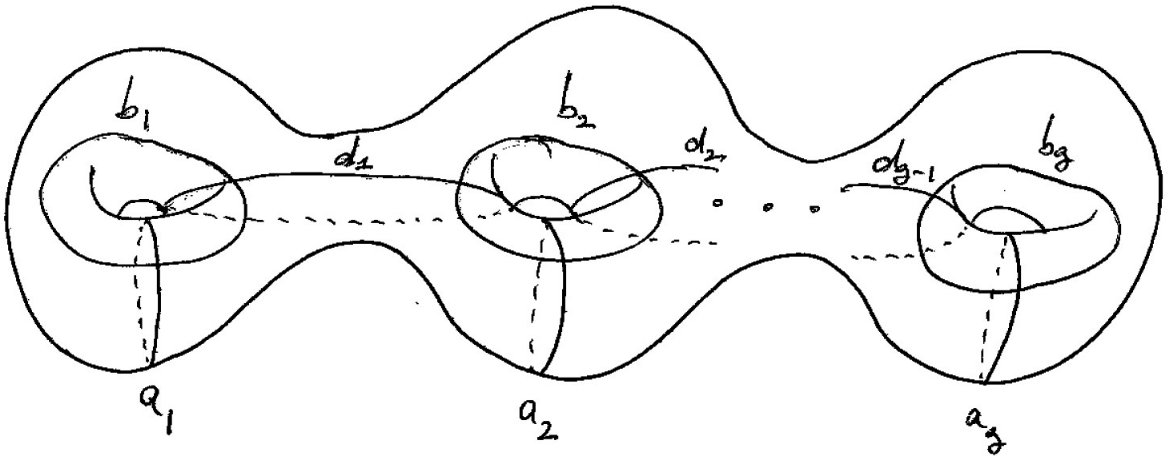

Figure 3

The character variety has singular points but contains a dense and open subset of regular points. Precisely, a representation $\rho : \pi_1\Sigma \to \mathrm{PSL}_2(\mathbf{R})$ is *Zariski-dense* if its image is a Zariski-dense subgroup of $\mathrm{PSL}_2(\mathbf{R})$. Let $\mathrm{Hom}_{\mathrm{ZD}}\big(\Sigma, \mathrm{PSL}_2(\mathbf{R})\big)$ be the subset of $\mathrm{Hom}\big(\Sigma, \mathrm{PSL}_2(\mathbf{R})\big)$ of Zariski-dense representations and let $\mathcal{R}\mathrm{ep}$ be its projection in the character variety. In W. M. Goldman 1984, Goldman proved that $\mathcal{R}\mathrm{ep}$ is a smooth manifold of dimension $6g-6$. Goldman constructed a symplectic form on $\mathcal{R}\mathrm{ep}$, we will denote by ω. This symplectic structure gives rise to a Radon measure, which will be denoted by λ. We extend this Radon measure to a Radon measure $\mathcal{X}$, we will denote again by λ, such that:

$$\lambda\big(\mathcal{X} \setminus \mathcal{R}\mathrm{ep}\big) = 0.$$

Connected components for the closed case

In this paragraph, we introduce the Euler class. It is an invariant associated to each representation and which is known to classify the connected components of $\mathcal{X}$.

Let $\rho : \pi_1\Sigma \to \mathrm{PSL}_2(\mathbf{R})$ be a representation. For each generator $x \in \{a_1, b_1, \dots, a_g, b_g\}$, we choose a lift $\widetilde{\rho(x)}$ in the universal cover $\widetilde{\mathrm{PSL}_2(\mathbf{R})}$ of $\mathrm{PSL}_2(\mathbf{R})$. Hence the product

$$\prod_{i=1}^{g} [\widetilde{\rho(a_i)}, \widetilde{\rho(b_i)}]$$

projects on the identity. We hence deduce that $\prod_{i=1}^{g} [\widetilde{\rho(a_i)}, \widetilde{\rho(b_i)}]$ lies in $\pi_1\big(\mathrm{PSL}_2(\mathbf{R})\big) \cong \mathbf{Z}$. This integer only depends on ρ and not on the choice of the lifts. We hence will denote it by $\mathrm{eu}(\rho)$. The map $\mathrm{eu} : \mathrm{Hom}\big(\Sigma, \mathrm{PSL}_2(\mathbf{R})\big) \to \mathbf{Z}$ is moreover invariant by the action of $\mathrm{PSL}_2(\mathbf{R})$ by conjugation and thus descends to a map

$$\mathrm{eu} : \mathcal{X} \to \mathbf{Z}.$$

In Milnor 1957/58, Milnor shows that this invariant is bounded:

Theorem 4. *(Milnor 1957/58) For all $[\rho] \in \mathcal{X}$:*

$$|\mathrm{eu}([\rho])| \leq |\chi(\Sigma)|.$$

A representation whose Euler class is equal to $\pm|\chi(\Sigma)|$ is called *maximal.*
In W. M. Goldman 1988, Goldman classified the connected components by Euler class and in W. M. Goldman 1982 he gave a characterisation of maximal representations:

Theorem 5. *(W. M. Goldman 1988 and W. M. Goldman 1982)*

1. *For all $k \in \mathbf{Z} \cap \big[\chi(\Sigma), -\chi(\Sigma)\big]$, the subspace $\mathrm{eu}^{-1}(k)$ is non-empty and the connected components of $\mathcal{X}$ are exactly the $\mathrm{eu}^{-1}(k)$ for $k \in \mathbf{Z} \cap \big[\chi(\Sigma), -\chi(\Sigma)\big]$. We will denote them by $\mathcal{X}^k$ and by $\mathcal{R}\mathrm{ep}^k$ their intersections with $\mathcal{R}\mathrm{ep}$.*
2. *A representation $\rho : \pi_1\Sigma \to \mathrm{PSL}_2(\mathbf{R})$ is maximal if and only if it is the holonomy of a hyperbolic structure on Σ.*

The second point of Theorem 5 implies that $\mathcal{X}^{\pm|\chi(\Sigma)|}$ is homeomorphic to the Teichmüller space of Σ.

Action of the modular group in the closed surface case

The *modular group* of Σ is the quotient group of positive diffeomorphisms of Σ up to isotopy. Dehn-Nielsen-Baer Theorem (Theorem 8.1, Farb and Margalit 2011) gives an isomorphism between the modular group of the surface Σ and the outer automorphisms group:

$$\mathrm{Out}(\pi_1\Sigma) := \frac{\mathrm{Aut}(\pi_1\Sigma)}{\mathrm{Inn}(\pi_1\Sigma)}.$$

It is well known that the action of the modular group on the Teichmüller space of Σ is *properly discontinuous*: for all compact subset K of the Teichmüller space of Σ, the set

$$\big\{[\phi] \in \mathrm{Out}(\pi_1\Sigma) \mid [\phi] \cdot K \cap K \neq \emptyset\big\}$$

is finite.

The behavior of this action on non-maximal components seems completely opposite. In W. Goldman 2006, Goldman conjectured:

Conjecture 1. *(Conjecture 3.1, W. Goldman 2006) Let $k \in \mathbf{Z} \cap \big]\chi(\Sigma), -\chi(\Sigma)\big[$. Then the modular group acts ergodically on $\mathcal{X}^k$ with respect to the measure λ.*

Related to the Bowditch condition Bowditch 1998 and in the way of this conjecture, Marché and Wolff introduced in Marché and Wolff 2016, for $k \in \mathbf{Z} \cap \big(\chi(\Sigma), -\chi(\Sigma)\big)$, the subspace

$$\mathcal{NH}^k := \big\{[\rho] \in \mathcal{R}\mathrm{ep}^k \mid \exists \gamma \in \pi_1\Sigma \text{ simple}\,, \mathrm{tr}\big(\rho(\gamma)\big) \in [-2, 2]\big\}.$$

They proved the following:

Theorem 6. *(Theorem 1.5 and Theorem 1.6, Marché and Wolff 2016)*

1. *Assume* Σ *has genus* 2. *Then the action* $\mathrm{Mod}^+(\Sigma)$ *on* $\mathcal{X}^{\pm 1}$ *is ergodic with respect to* λ.
2. *Assume* $g \geq 3$ *and* k *is an integer such that* $|k| < |\chi(\Sigma)|$, *then the action of* $\mathrm{Mod}^+(\Sigma)$ *on* $\mathcal{NH}^k$ *is ergodic with respect to* λ.

In genus 2 and for Euler class 0, the component $\mathcal{X}^0$ decomposes in two disjoint subspaces $\mathcal{X}^0_-$ and $\mathcal{X}^0_+$ of non-zero measure and on which the modular group acts ergodically with respect to λ, see Marché and Wolff 2016 and Marché and Wolff 2015.

Relative $\mathrm{PSL}_2(\mathbf{R})$-character varieties

Let $\dot{\Sigma}$ be a connected, oriented and compact surface of genus $g \geq 1$ and with $n > 0$ punctures, such that $\chi(\dot{\Sigma}) = 2 - 2g - n < 0$. Fix $c_1, \dots, c_n$ be curves on $\dot{\Sigma}$ which are homotopic to a circle around each puncture.

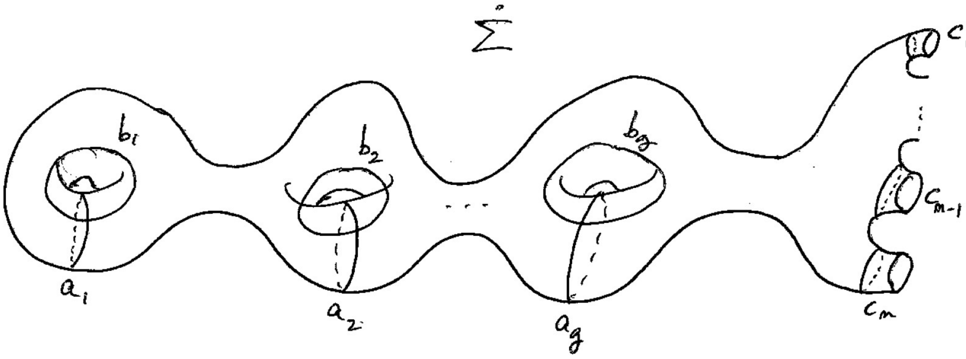

Figure 4

The fundamental group $\pi_1\dot{\Sigma}$ is the free group:

$$\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots c_n | \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{n} c_j = 1 \rangle.$$

The representation space $\mathrm{Hom}\big(\dot{\Sigma}, \mathrm{PSL}_2(\mathbf{R})\big)$ is the space of morphisms $\rho : \pi_1\dot{\Sigma} \to \mathrm{PSL}_2(\mathbf{R})$. The group $\mathrm{PSL}_2(\mathbf{R})$ acts on the representation space by conjugation.

Definition 2. *The* character variety $\mathcal{X}(\dot{\Sigma})$ *is the Hausdorff quotient*

$$\mathrm{Hom}\big(\dot{\Sigma}, \mathrm{PSL}_2(\mathbf{R})\big) // \mathrm{PSL}_2(\mathbf{R}).$$

A representation $\rho : \pi_1\dot{\Sigma} \to \mathrm{PSL}_2(\mathbf{R})$ is called *Zariski-dense* if its image is Zariski-dense in $\mathrm{PSL}_2(\mathbf{R})$. The subspace $\mathcal{R}\mathrm{ep}(\dot{\Sigma})$ of classes of Zariski-dense representations is a dense, open and full measure subset of $\mathcal{X}(\dot{\Sigma})$. This space is partitioned by behavior on the boundary components. Precisely we impose conditions on the boundary: for $\mathcal{C}_1, \dots, \mathcal{C}_n$ be n conjugacy classes in $\mathrm{PSL}_2(\mathbf{R})$ and $\underline{\mathcal{C}} := (\mathcal{C}_1, \dots, \mathcal{C}_n)$, we define the space $\mathrm{Hom}\big(\dot{\Sigma}, \mathrm{PSL}_2(\mathbf{R}), \underline{\mathcal{C}}\big)$ of representations $\rho : \pi_1\dot{\Sigma} \to \mathrm{PSL}_2(\mathbf{R})$ such that for all $i \in \{1, \dots, n\}$,

$$\rho(c_i) \in \mathcal{C}_i.$$

The groupe $\mathrm{PSL}_2(\mathbf{R})$ acts on the space $\mathrm{Hom}\big(\dot{\Sigma}, \mathrm{PSL}_2(\mathbf{R}), \underline{\mathcal{C}}\big)$ by conjugation.

Definition 3. *The* relative character variety *associated with $\underline{\mathcal{C}}$, denoted by $\mathcal{X}(\dot{\Sigma}, \underline{\mathcal{C}})$, is the Hausdorff quotient*

$$\mathrm{Hom}\big(\dot{\Sigma}, \mathrm{PSL}_2(\mathbf{R}), \underline{\mathcal{C}}\big) // \mathrm{PSL}_2(\mathbf{R}).$$

Let us denote by $\mathcal{R}\mathrm{ep}(\dot{\Sigma}, \underline{\mathcal{C}})$ the subspace of classes of Zariski-dense representations of $\mathcal{X}(\dot{\Sigma}, \underline{\mathcal{C}})$. This set is a smooth manifold of dimension $6g - 6 + 2n$.

For each $C_i \in \mathrm{PSL}_2(\mathbf{R})$, we choose a lift $\widetilde{C}_i \in \widetilde{\mathrm{PSL}_2(\mathbf{R})}$, we denote by r_i the *translation number* of $\widetilde{C}_i$ and by $\mathbf{r}$ the vector $(r_1, \dots, r_n)$. We define, for the vector $\mathbf{r} \in \mathbf{R}^n$, the vector $\{\mathbf{r}\}$ of fractional parts of the coordinates of $\mathbf{r}$. This means that $\{\mathbf{r}\} = \big(\{r_1\}, \dots, \{r_n\}\big)$ with for each $i \in \{1, \dots, n\}$, $\{r_i\} = r_i - \lfloor r_i \rfloor$.

Definition 4. *Let $\rho : \pi_1\dot{\Sigma} \to \mathrm{PSL}_2(\mathbf{R})$ and fix a lift $\widetilde{\rho} : \pi_1\dot{\Sigma} \to \widetilde{\mathrm{PSL}_2(\mathbf{R})}$. For each $i \in \{1, \dots, n\}$, let r_i be the translation number of any lift $\widetilde{\rho}(c_i)$. The* Euler number *of ρ is the number* $\mathrm{eu}(\rho)$ *defined by*

$$-\sum_{i=1}^{n} r_i.$$

The Euler number of a representation does not depend of the choices of the lifts and is constant on each conjugacy class of it. We thus define the Euler number of a class of representations $[\rho]$ by the Euler number of a representative of this class. This gives a map

$$\mathrm{eu} : \mathcal{X}(\dot{\Sigma}) \to \mathbf{R}$$

which induces the restriction map

$$\mathrm{eu}_{\underline{\mathcal{C}}} : \mathcal{X}(\dot{\Sigma}, \underline{\mathcal{C}}) \to \mathbf{R}$$

such that $\mathrm{eu}_{\underline{\mathcal{C}}}([\rho]) + \|\{\mathbf{r}\}\|_1 \in \mathbf{Z}$ with $\{\mathbf{r}\}$ be the vector of fractionnal parts of the r_i. The preimage of $\mathrm{eu}_{\underline{\mathcal{C}}}^{-1}(e)$, for $e \in \mathbf{R}$, will be denoted by $\mathcal{X}^e(\dot{\Sigma}, \underline{\mathcal{C}})$. In Mondello 2017, Mondello described the topology of $\mathcal{X}(\dot{\Sigma}, \underline{\mathcal{C}})$.

Theorem 7. *(Theorem 2.20, Mondello 2017)*

1. *The image of* $\mathrm{eu} : \mathcal{X}(\dot{\Sigma}) \to \mathbf{R}$ *is the interval* $\big[\chi(\dot{\Sigma}), -\chi(\dot{\Sigma})\big]$. *If* $\mathrm{eu}(\rho) = -\chi(\dot{\Sigma})$, *then all the $\rho(c_i)$ are hyperbolic and ρ is the holonomy of a hyperbolic structure on $\dot{\Sigma}$ with geodesic boundary components.*

2. *Assume $e > 0$. Then the space $\mathcal{X}^e(\dot\Sigma, \underline{\mathcal{C}})$ is non-empty if and only if $e + \|\{\mathbf{r}\}\|_1 + s_0 + s_- \in \mathbf{Z} \cap \left]0, -\chi(\dot\Sigma)\right]$, where s_0 is the cardinal of $\{i \in \{1, \ldots, n\} \mid C_i = \mathrm{id}\}$ and s_- is the cardinal of*

$$\{i \in \{1, \ldots, n\} | C_i \text{ negative unipotent}\}$$

where a matrix is negative unipotent if it is conjugated to the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. In this case, the space $\mathcal{X}^e(\dot\Sigma, \underline{\mathcal{C}})$ is connected and smooth.

This work will always consider conjugacy classes of elliptic isometries. In this situation the number s_0 and s_- are both zero.

The *pure mapping class group* of $\dot\Sigma$, denoted by $\mathrm{PMod}(\dot\Sigma)$, is the group of isotopy classes of diffeomorphisms of $\dot\Sigma$ whose action on $\partial\dot\Sigma$ is trivial. It acts naturally on the relative character varieties with an action which is similar than the one of mapping class group on character varieties in closed case and preserves the connected components of relative character varieties and Goldman measure.

Statements

Fix $\underline{\mathcal{C}} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)$ with each $\mathcal{C}_i$ be a conjugacy class of an elliptic isometry of angle υ_i in $\mathrm{PSL}_2(\mathbf{R})$. The number $\|\{\mathbf{r}\}\|_1 = \sum_{i=1}^n |\upsilon_i - \lfloor \upsilon_i \rfloor|$ depends only of the $\mathcal{C}_i$.

We will denote by $\mathcal{NH}(\dot\Sigma, \underline{\mathcal{C}})$ the space we define to be the subspace of $\mathcal{R}\mathrm{ep}(\dot\Sigma, \underline{\mathcal{C}})$ whose elements are the classes of representations $[\rho]$ for which there existes a simple and closed curve $\gamma \subset \dot\Sigma \setminus \partial\dot\Sigma$ such that $\rho(\gamma)$ is a non-hyperbolic isometry of $\mathbf{H}^2$. For $e \in \mathbf{R}$, we denote by $\mathcal{NH}^e(\dot\Sigma, \underline{\mathcal{C}})$ the intersection

$$\mathcal{NH}(\dot\Sigma, \underline{\mathcal{C}}) \bigcap \mathcal{R}\mathrm{ep}^e(\dot\Sigma, \underline{\mathcal{C}}).$$

We prove the theorem:

Theorem C. *For $e = \pm(k - \|\{\mathbf{r}\}\|_1)$ with $k \in \left]\|\{\mathbf{r}\}\|_1, -\chi(\dot\Sigma)\right[\cap \mathbf{Z}$, the pure mapping class group acts ergodically on $\mathcal{NH}^e(\dot\Sigma, \underline{\mathcal{C}})$.*

The proof of Theorem C is an adaptation to the case of surfaces with boundaries of the proof of Theorem 1.6 of Marché and Wolff 2016.

We conjecture the dynamic of the modular group can be ensured, stating that a proper subgroup acts ergodically for the case of closed surfaces:

Conjecture A. *Let Σ be a closed surface of genus $g \geq 3$. If k is an integer such that $|k| \leq 2g - 5$, then the group $\mathrm{Tor}(\Sigma)$ acts ergodically on the subspaces $\mathcal{NH}^k$.*

The section **??** is devoted to explain a strategy and a large part of proofs in the way of this conjecture. A positive answer to a question which is analogue to the Goldman conjecture, see Conjecture **??**, could prove Conjecture A.

Topological dynamic

The example of translations on torus gives us a dynamical intuition of the orbits. In the case of the circle, we easily see that the action of a rotation of irrational angle has only dense orbits. This chaotic phenomena can be seen in higher dimension and is a general result coming from ergodicity:

Proposition 1. *Let H be a topological group acting on an Borelian space (X, μ) and which preserves the measure μ. Assume that each non-empty and open subset has non-zero measure and that the topology of X is countably generated. If the action of H on (X, μ) is ergodic, then almost every orbit is dense.*

Unfortunately, this fact is a probabilistical result and no description of point of X for which the orbit is dense is given in general.

In W. Goldman 2006, Goldman ask the following:

Question 1. *Could we find a condition on a representation $\rho : \pi_1\Sigma \to \mathrm{SU}(2)$ to have that its $\mathrm{Mod}^+(\Sigma)$-orbit is dense ?*

In Previte and Xia 2000 and in Previte and Xia 2002, Previte-Xia gives a necessary and sufficient condition:

Theorem 8. *A representation $\rho : \pi_1\Sigma \to \mathrm{SU}(2)$ has a dense $\mathrm{Mod}^+(\Sigma)$-orbit if and only if ρ has a dense image in $\mathrm{SU}(2)$.*

We start to prove a similar statement with the Abelian case. A compact, Abelian and connected Lie group is isomorphic to a $n-$dimensional torus. Using Ratner theorem, see Morris 2005, we prove:

Theorem D. *A representation $\rho : \pi_1\Sigma \to \mathbf{T}^n$ has a dense $\mathrm{Mod}^+(\Sigma)-$orbit if and only if ρ has a dense image in $\mathbf{T}^n$.*

We conjecture such a result is true when $G = \mathrm{SU}(3)$ and Σ is a one-holded torus.

Connection with the Kronecker's Approximation Theorem

The dynamical result provided by Theorem D finds an application on the theory of geometry of numbers. An important theorem in this topic is the Kronecker's theorem concerning inhomogeneous Diophantine approximation, see section 4 below for the precise statement.

Theorem E. *Let $g \geq 1$. Let $b^{(i)} = \big(b_1^{(i)}, \ldots, b_{2g}^{(i)}\big)$, with $i = 1, \ldots, n$, be vectors of $\mathbf{R}^{2g}$ such that $b^{(1)}, \ldots, b^{(n)}$, $\pi e_1, \ldots, \pi e_{2g}$ are linearly independent over $\mathbf{Q}$ in the vector space $\mathbf{R}^m$. Let $A \in \mathrm{M}\big(n, 2g; \mathbf{R}\big)$ be a real matrix and let ε be a positive number. Then there is an element $K \in \mathrm{Sp}_{2g}(\mathbf{Z})$ such that*

$$\Big\| A - B\,K \Big\| < C\varepsilon \bmod 2\pi. \tag{2}$$

where C is a constant depending only on $2g$ and n, $\{e_1, \ldots, e_{2g}\}$ is the canonical basis of $\mathbf{R}^{2g}$ and the norm is any norm on $\mathrm{M}\big(n, 2g; \mathbf{R}\big)$.

Contents

1 Background

Let Σ be a connected, oriented closed surface of genus $g \geq 2$

1.1 The Torelli group

For more details on this part, see Farb and Margalit 2011. The group $\mathrm{Mod}^+(\Sigma)$ is generated by the $3g-1$ Dehn twists $\mathrm{tw}_{a_1}, \mathrm{tw}_{b_1} \dots, \mathrm{tw}_{a_g}, \mathrm{tw}_{b_g}, \mathrm{tw}_{d_1}, \dots, \mathrm{tw}_{d_{g-1}}$ where, for $i = 1, \dots, g$, the curves a_i, b_i on Σ are given by the presentation of the fundamental group of Σ :

$$\pi_1\Sigma = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1\rangle$$

and, for $j = 1, \dots, g-1$, the curves d_j are the products $a_j^{-1}a_{j+1}$. See Figure 2.
The first homology group $\mathrm{H}_1(\Sigma, \mathbf{Z}) \cong \mathbf{Z}^{2g}$ is freely generated by the curves $[a_1], [b_1], \dots, [a_g], [b_g]$ and is a lattice in the real homology group $\mathrm{H}_1(\Sigma, \mathbf{R})$.

Definition 1.1. *The* Torelli group $\mathrm{Tor}(\Sigma)$ *is the kernel of the action of the mapping class group on* $\mathrm{H}_1(\Sigma, \mathbf{Z})$.

An explicit generating set of the Torelli group is the set

$$\{\mathrm{tw}_c, \mathrm{tw}_a\mathrm{tw}_b^{-1} \mid c \text{ separating, } a, b \,\text{cohomologous}\}.$$

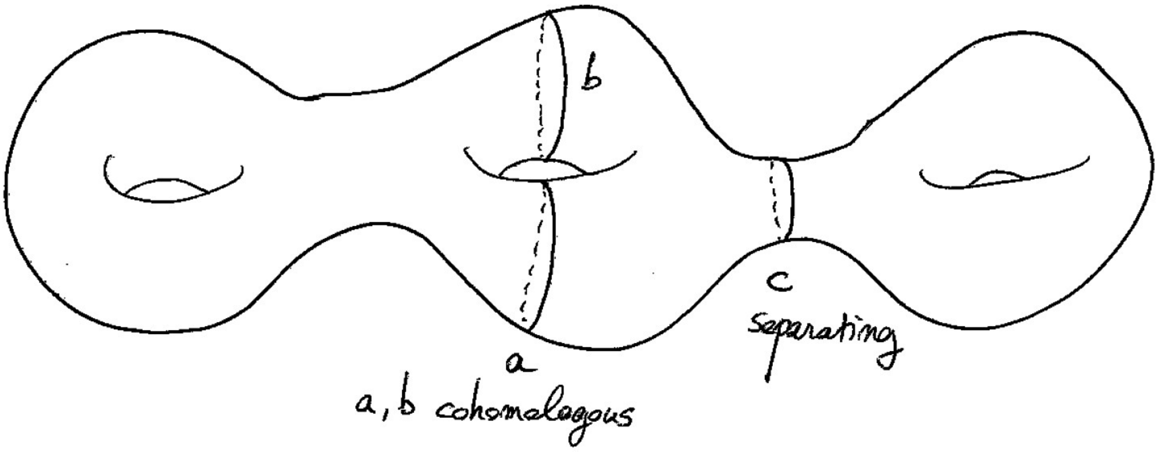

Figure 5

1.2 Borel cross sections

Let X be a topological set on which a topological group H acts. The set X is endowed with its Borelian σ-algebra and a Borelian measure μ. A function $X_1 \to X_2$ between two measured sets is called bimeasurable if it is measurable, invertible and has a measurable inverse. If such a function exists, we then say that X_1 and X_2 are bimeasurable. The quotient set X/H carries the quotient topology such that the canonical projection $X \to X/H$ is continuous. A *Borel cross section* is a subset $\mathcal{S} \subset X$ which intersects exactly once the orbit $H.x$, for every element $x \in X$. The following theorem is proven by Edward Effros in Effros 1965. For our purposes we shall need the following partial statement:

Theorem 1.2. *(Theorem 2.1, Effros 1965) If X is a separable, complete, metrizable and locally compact topological space and if H is a topological group acting continuously on X, then the following conditions are equivalent :*

- *Every orbit $H.x$ is locally closed.*
- *Every orbit $H.x$ is locally compact.*
- *There exists a Borel cross section $\mathcal{S}$ for the orbits of H in X.*

A key corollary of this result is given by James Bondar in Bondar 1976:

Theorem 1.3. *(Theorem 2, Bondar 1976) If X and H verify the hypothesis of the previous theorem and if furthermore the action of H on X is free and one of the conditions of the previous statement holds, then the Borel cross section $\mathcal{S}$ is bimeasurable to the quotient space X/H and there exists a Borelian measure $\mu_{\mathcal{S}}$ on $\mathcal{S}$ such that for every measurable function $f : X \to \mathbf{R}$ one has:*

$$\int_X f d\mu = \int_{\mathcal{S}} \Big(\int_H f(h.s) d\mu_H(h) \Big) d\mu_{\mathcal{S}}(s),$$

where μ_H is the Haar-measure on H.

1.3 The mapping class group action on the SU(2)-characters

In the compact case and in rank 1 case we consider, the Lie algebra of SU(2) is the Lie algebra $\mathfrak{su}(2)$ of traceless skew-Hermitian complex 2×2-matrices. Let $f : \mathrm{SU}(2) \to \big[-2, 2\big]$ denote the trace function. Its *variation function* F is defined as the unique function $F : \mathrm{SU}(2) \to \mathfrak{su}(2)$ such that, for all $x \in \mathrm{SU}(2)$ and $X \in \mathfrak{su}(2)$,

$$\frac{d}{dt}_{|t=0} f(x \cdot \exp(tX)) = \langle F(x), X \rangle.$$

Goldman proved, in W. M. Goldman 1986, that:

$$F(x) = x - \frac{\mathrm{tr}(x)}{2}\mathrm{id}.$$

Following Goldman and Xia 2011, we let:

$$\zeta^t : \mathrm{SU}(2) \to \mathrm{SU}(2)$$

defined as the map sending $x \in \mathrm{SU}(2)$ on $\exp(t \cdot F(x))$ For $x \in \mathrm{SU}(2)$, the map $t \mapsto \zeta^t(x)$ is a one-parameter subgroup of $\mathrm{SU}(2)$.

1.3.1 Separating curve

If a curve α on Σ is separating, then $\Sigma \setminus \alpha$ can be written as the disjoint union $\Sigma_1 \sqcup \Sigma_2$ and the fundamental group $\pi_1\Sigma$ is the amalgamated product

$$\pi_1\Sigma_1 \underset{\langle\alpha\rangle}{*} \pi_1\Sigma_2.$$

The data of two representations $\rho_1 : \pi_1\Sigma_1 \to \mathrm{SU}(2)$ and $\rho_2 : \pi_1\Sigma_2 \to \mathrm{SU}(2)$, such that

$$\rho_1(\alpha) = \rho_2(\alpha),$$

allows us to construct by amalgamation a unique representation $\rho : \pi_1\Sigma \to \mathrm{SU}(2)$ defined by $\rho_{|\pi_1\Sigma_1} = \rho_1$ and $\rho_{|\pi_1\Sigma_2} = \rho_2$.

Let $\xi^t_\alpha : \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2)) \to \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ be the twist flow defined, for $\gamma \in \pi_1\Sigma$, by

$$\xi^t_\alpha\rho(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma_1) \\ \zeta^t(\rho(\alpha))\rho(\gamma)\zeta^{-t}(\rho(\alpha)) & \text{if } \gamma \in \pi_1(\Sigma_2) \end{cases}.$$

This flow is well defined since $\zeta^t(\rho(\alpha))$ is the exponential of a polynomial in $\rho(\alpha)$ and thus commutes with $\rho(\alpha)$.

1.3.2 Non-separating curve

If a curve α on Σ is non-separating, the fundamental group $\pi_1\Sigma$ is the HNN-extension:

$$\Big(\pi_1(\Sigma|\alpha) * \langle\beta\rangle\Big)/\langle\beta\alpha^-\beta^{-1}(\alpha^+)^{-1}\rangle$$

where $\alpha^\pm$ represent the boundary components of $\Sigma \setminus \alpha$. Hence the data of a representation

$$\rho_0 : \pi_1(\Sigma|\alpha) \to \mathrm{SU}(2)$$

and a matrix $B \in \mathrm{SU}(2)$ such that

$$B\rho_0(\alpha^-)B^{-1} = \rho_0(\alpha^+)$$

defines a unique representation $\rho : \pi_1\Sigma \to \mathrm{SU}(2)$ such that $\rho_{|\pi_1(\Sigma|\alpha)} = \rho_0$ and $\rho(\beta) = B$. Let $\xi_\alpha^t : \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2)) \to \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ be the twist flow defined, for $\gamma \in \pi_1\Sigma$, by

$$\xi_\alpha^t\rho(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\ \zeta^t(\rho(\alpha))\rho(\beta) & \text{if } \gamma = \beta \end{cases}.$$

This flow is well defined since the relation

$$\xi_\alpha^t\rho(\beta)\xi_\alpha^t\rho(\alpha^-)\xi_\alpha^t\rho(\beta)^{-1} = \xi_\alpha^t\rho(\alpha^+)$$

is satisfied.

1.3.3 The Dehn twists and the flows

For a simple and closed curve α, consider the trace function $f_\alpha : \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2)) \to [-2, 2]$ associated with the curve α, that is:

$$f_\alpha([\rho]) = \mathrm{tr}(\rho(\alpha)).$$

In W. M. Goldman 1986, Goldman proved that the flow ξ_α is the Hamiltonian flow of the trace function f_α, that means:

$$df_\alpha X = \omega_G\left(\frac{d}{dt}_{|t=0}\xi_\alpha^t, X\right)$$

for all $X \in T\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ and where ω_G is the symplectic form constructed by Goldman in W. M. Goldman 1984.

If $x \in \mathrm{SU}(2)$, then there exists $g \in \mathrm{SU}(2)$ such that $x = g\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}g^{-1}$ with $\theta = \cos^{-1}\left(\frac{f(x)}{2}\right)$. We then compute $F(x) = x - \frac{f(x)}{2}\mathrm{id}$ which we can write:

$$F(x) = g\begin{pmatrix} i\sin(\theta) & 0 \\ 0 & -i\sin(\theta)\end{pmatrix}g^{-1},$$

and so, by definition

$$\zeta^t(x) = g\begin{pmatrix} e^{it\sin(\theta)} & 0 \\ 0 & e^{-it\sin(\theta)}\end{pmatrix}g^{-1}.$$

In particular, if $x = \pm\mathrm{id}$, then for all $t \in \mathbf{R}$ we have $\zeta^t(x) = \mathrm{id}$ and for $x \neq \pm\mathrm{id}$, we have the equality $\zeta^t(x) = \mathrm{id}$, if and only if $t \in \frac{2\pi}{\sin(\theta)}\mathbf{Z}$. Thus we state the following (see Katok and Hasselblatt 1995):

Lemma 1.4. *If $x \neq \pm\mathrm{id}$, then x belongs to the one-parameter subgroup $\{\zeta^t(x)\}_{t\in\mathbf{R}}$ and more precisely, $x = \zeta^{s(x)}(x)$ for:*

$$s(x) = \frac{\theta}{\sin(\theta)}.$$

For $x \in \mathrm{SU}(2)$ such that $\theta \notin \pi\mathbf{Q}$, the subgroup $\langle x\rangle$ is dense in the circle $\{\zeta^t(x)|t \in \mathbf{R}\} \cong \mathbf{S}^1$ and acts ergodically on it with respect to the Lebesgue measure.

Furthermore, the Dehn twist tw_α acts on ρ via the relation

$$\mathrm{tw}_\alpha.\rho = \xi_\alpha^{s(\rho(\alpha))}\rho.$$

Thus, if $\cos^{-1}\left(\frac{f(\rho(\alpha))}{2}\right) \times \frac{1}{\pi}$ is irrational, then the orbit $\langle \mathrm{tw}_\alpha \rangle.\rho$ is dense in the circle defined by the orbit $\{\xi_\alpha^t \rho\}_{t\in\mathbf{R}}$. The Hamiltonian flow gives an action of the circle $\mathrm{U}_\alpha := \mathbf{S}^1$ on the character variety and a free action on the subspace consisting of classes of representations $[\rho]$ such that $\rho(\alpha) \neq \pm\mathrm{id}$.

1.4 Action of the modular group on $\mathrm{PSL}_2(\mathbf{R})$-character varieties and Hamiltonian flows

Similarly to the previous part, the Lie algebra $\mathfrak{sl}_2(\mathbf{R})$ of $\mathrm{PSL}_2(\mathbf{R})$ is the space of traceless real 2×2 matrices. Let $\exp = \mathfrak{sl}_2(\mathbf{R}) \to \mathrm{PSL}_2(\mathbf{R})$ be the exponential map. The Lie algebra $\mathfrak{sl}_2(\mathbf{R})$ is endowed with its Killing form we will denote by $\mathrm{K}(\,.\,,\,.\,)$. Let $f : \mathrm{PSL}_2(\mathbf{R}) \to \mathbf{R}$ be a smooth function. Then there exists a unique map $F : \mathrm{PSL}_2(\mathbf{R}) \to \mathfrak{sl}_2(\mathbf{R})$, called the variation function associated to f, such that for all $g \in \mathrm{PSL}_2(\mathbf{R})$ and all $X \in \mathfrak{sl}_2(\mathbf{R})$,

$$\frac{d}{dt}_{|t=0} f(g \cdot \exp(tX)) = \mathrm{K}(F(g), X).$$

In W. M. Goldman 1986, Goldman computed that the variation function of the trace function $f := \mathrm{tr}$ is given, for all $g \in \mathrm{PSL}_2(\mathbf{R})$, by the formula $F(g) = g - \frac{\mathrm{tr}(g)}{2}\mathrm{id}$. We introduce the flow

$$\begin{array}{cccc} \zeta^t & : & \mathrm{PSL}_2(\mathbf{R}) & \to & \mathrm{PSL}_2(\mathbf{R}) \\ & & g & \mapsto & \exp(tF(g)) \end{array}.$$

Let $g \in \mathrm{PSL}_2(\mathbf{R})$ be an elliptic isometry and let θ its angle. We assume $\sin(\theta) \neq 0$. The isometry g has the form $g = h \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} h^{-1}$ for some $h \in \mathrm{PSL}_2(\mathbf{R})$.
We directly compute that

$$F(g) = h \begin{pmatrix} 0 & -\sin(\theta) \\ \sin(\theta) & 0 \end{pmatrix} h^{-1}$$

and hence that

$$\zeta^t(g) = h \begin{pmatrix} \cos(t\sin(\theta)) & -\sin(t\sin(\theta)) \\ \sin(t\sin(\theta)) & \cos(t\sin(\theta)) \end{pmatrix} h^{-1}.$$

In particular, $\zeta^t(g) = \mathrm{id}$ if and only if $t \in \frac{2\pi}{\sin\theta}\mathbf{Z}$ and the matrix g is contained in the flow $\{\zeta^t(g)\}_{t\in\mathbf{R}}$. More precisely $g = \zeta^{s(g)}(g)$ for

$$s(g) = \frac{\theta}{\sin\theta}$$

and moreover:

Lemma 1.5. *If g is an elliptic isometry of infinite order then the subgroup $\langle g \rangle$ is dense in the circle*

$$\{\zeta^t(g) \mid t \in \mathbf{R}\} \cong \mathbf{S}^1$$

and acts ergodically on it.

1.4.1 Non-separating curve

If a curve α is non-separating, the fundamental group $\pi_1\Sigma$ is the HNN-extension:

$$\left(\pi_1(\Sigma \setminus \alpha) \coprod \langle \beta \rangle\right) / \left(\beta\alpha^-\beta^{-1} = \alpha^+\right)$$

where $\alpha^\pm$ are the boundary components of $\Sigma \setminus \alpha$. Hence the data of a representation

$$\rho_0 : \pi_1(\Sigma \setminus \alpha) \to \mathrm{PSL}_2(\mathbf{R})$$

and a matrix $B \in \mathrm{PSL}_2(\mathbf{R})$ such that

$$B\rho_0(\alpha^-)B^{-1} = \rho_0(\alpha^+)$$

allows to construct a representation $\rho : \pi_1\Sigma \to \mathrm{PSL}_2(\mathbf{R})$ defined by $\rho_{|\pi_1(\Sigma\setminus\alpha)} = \rho_0$ and $\rho(\beta) = B$.

We define the flow $\xi_\alpha^t : \mathcal{R}\mathrm{ep}^k \to \mathcal{R}\mathrm{ep}^k$ by:

$$\xi_\alpha^t\rho(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma \setminus \alpha) \\ \zeta^t(\rho(\alpha))\rho(\beta) & \text{if } \gamma = \beta \end{cases}.$$

This flow is well defined since the relation

$$\xi_\alpha^t\rho(\beta)\xi_\alpha^t\rho(\alpha^-)\xi_\alpha^t\rho(\beta)^{-1} = \xi_\alpha^t\rho(\alpha^+).$$

is satisfied.

1.4.2 Separating curve

If α is separating, then $\Sigma \setminus \alpha$ is the disjoint union $\Sigma_1 \coprod \Sigma_2$ and its fundamental group Γ is the amalgamated product:

$$\pi_1\Sigma_1 \underset{\langle\alpha\rangle}{*} \pi_1\Sigma_2$$

Similarly to the HNN-extension, the data of two representations $\rho_1 : \pi_1\Sigma_1 \to \mathrm{PSL}_2(\mathbf{R})$ and $\rho_2 : \pi_1\Sigma_2 \to \mathrm{PSL}_2(\mathbf{R})$ such that

$$\rho_1(\partial\Sigma_1) = \rho_2(\partial\Sigma_2)$$

allows to construct the representation $\rho : \pi_1\Sigma \to \mathrm{PSL}_2(\mathbf{R})$ defined by $\rho_{|\partial\Sigma_1} = \rho_1$ and $\rho_{|\partial\Sigma_2} = \rho_2$.

We let the twist flow $\xi_\alpha^t : \mathcal{R}\mathrm{ep}^k \to \mathcal{R}\mathrm{ep}^k$ by :

$$\xi_\alpha^t \rho(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma_1) \\ \zeta^t(\rho(\alpha))\rho(\gamma)\zeta^{-t}(\rho(\alpha)) & \text{if } \gamma \in \pi_1(\Sigma_2) \end{cases}.$$

This flow is well defined because $\zeta^t(\rho(\alpha))$ commutes with $\rho(\alpha)$ as the exponential of a polynomial in $\rho(\alpha)$.

In W. M. Goldman 1986, Goldman proved that the flow $\{\xi_\alpha^t\}$ we defined is the Hamiltonian flow of the trace function

$$f_\alpha : \rho \mapsto \mathrm{tr}\big(\rho(\alpha)\big).$$

Theses flows defines action of the circle $\mathrm{U}_\alpha := \mathbf{S}^1$ on the subspace of classes of representations which send α on an elliptic element. A corollary of Lemma 1.5 is the following:

Lemma 1.6. *Let $\rho : \pi_1\Sigma \to \mathrm{PSL}_2(\mathbf{R})$ and let α be a simple closed curve such that $\rho(\alpha)$ is elliptic of infinite order. Then the action of $\langle \mathrm{T}_\alpha \rangle$ on the circle $\mathrm{U}_\alpha \cdot [\rho]$ is ergodic with respect to the Lebesgue measure.*

Remark 1.7. *If $\alpha_1, \dots, \alpha_\ell$ are simple curves and if we assume there are pairewise disjoint, then the flows $\xi_{\alpha_1}^{t_1}, \dots, \xi_{\alpha_\ell}^{t_\ell}$ commute. Their action on ρ give a topological torus orbit $\mathrm{U}_{\alpha_1} \times \cdots \times \mathrm{U}_{\alpha_\ell} \cdot \rho$.*

2 Torelli group action on compact character varieties

In this section, we prove Theorem A.

2.1 Ergodicity for the case of $\mathrm{SU}(2)$

The strategy in proving the ergodicity of the Torelli group action is to find a full measure subset of the character variety on which every measurable and $\mathrm{Tor}(\Sigma)$-invariant function is almost everywhere invariant by the mapping class group action.

2.1.1 Ergodicity of translation actions

In this subsection, we generalize Lemma 1.4. We remark that if two simple curves c_1 and c_2 are disjoint, then the associated flows $\xi_{c_1}^t$ and $\xi_{c_2}^s$ commute. Hence, under this assumption, the actions of these flows on a representation ρ give a topological torus orbit $\{\xi_{c_1}^t.\xi_{c_2}^s\rho\}_{t,s\in\mathbf{R}}$ obtained by the action of $\mathrm{U}_{c_1}\times\mathrm{U}_{c_2}$ on the characters which are not $\pm\mathrm{id}$ evaluated in c_1, c_2.

Lemma 2.1. *Let $[\rho]$ be a class of representations in $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ and suppose that there exist simple closed curves $c_1, \ldots, c_\ell$ on Σ which are pairwise disjoint and such that*

$$\theta_1 = \cos^{-1}\left(\frac{f(\rho(c_1))}{2}\right), \ldots, \theta_\ell = \cos^{-1}\left(\frac{f(\rho(c_\ell))}{2}\right), \pi$$

are linearly independent over $\mathbf{Q}$. Then the action of $h = \mathrm{tw}_{c_1}\ldots\mathrm{tw}_{c_\ell}$ on the orbit $\left(\mathrm{U}_{c_1}\times\cdots\times\mathrm{U}_{c_\ell}\right)\cdot[\rho]$ is ergodic with respect to the Lebesgue measure on this torus orbit.

In particular, every orbit for the action of h is dense in the topological torus

$$\left(\mathrm{U}_{c_1}\times\cdots\times\mathrm{U}_{c_\ell}\right)\cdot[\rho].$$

Remark 2.2. *With the notation of the Lemma 2.1 and if $[\rho]$ satisfies the hypothesis of this lemma, then for all $i = 1, \ldots, \ell$, the matrix $\rho(c_i)$ is not $\pm\mathrm{id}$ and the torus orbit $\mathrm{U}_{c_1}\times\cdots\times\mathrm{U}_{c_\ell}\cdot[\rho]$ is the torus obtained as the quotient:*

$$\mathbf{R}^\ell/\Lambda$$

where Λ is the lattice $\frac{2\pi}{\sin\theta_1}\mathbf{Z}\oplus\cdots\oplus\frac{2\pi}{\sin\theta_\ell}\mathbf{Z}$. By definition of the torus $\mathbf{R}^\ell/\Lambda$, the action of $\mathrm{U}_{c_1}\times\cdots\times\mathrm{U}_{c_\ell}$ on the character variety is free.

The following is a classical result needed for proving Lemma 2.1 and for which a proof may be find in Katok and Hasselblatt 1995. We denote by $\mathbf{T}$ the torus $\mathbf{R}/2\pi\mathbf{Z}$. we precise that $\mathbf{T}$ is isomorphic to a maximal torus of $\mathrm{SU}(2)$.

Lemma 2.3. *Let $t = (t_1, \dots, t_\ell) \in \mathbf{R}^\ell$ such that $t_1, \dots, t_\ell, \pi$ are linearly independent over $\mathbf{Q}$ and let $f_t : \mathbf{T}^\ell \to \mathbf{T}^\ell$ be the translation of vector t. Then the action of $\langle f_t \rangle$ on $\mathbf{T}^\ell$ is ergodic with respect to the Lebesgue measure.*

Proof of the lemma 2.1. The Dehn twists tw_{c_k} act as a translation of the torus orbit $\mathrm{U}_{c_1} \times \cdots \times \mathrm{U}_{c_\ell} \cdot [\rho]$. The orbit $h^{\mathbf{Z}} \cdot [\rho]$ is the orbit $\langle \xi_{c_1}^{t_1} \dots, \xi_{c_\ell}^{t_\ell} \rangle \cdot [\rho]$ with for all $k \in \{1, \dots, \ell\}$:

$$t_k = \frac{\theta_k}{2\sin(\theta_k)}.$$

For such a t_k, the action of $\xi_{c_k}^{t_k}$ is given by the multiplication by a matrix conjugated to

$$\begin{pmatrix} e^{i\theta_k} & 0 \\ 0 & e^{-i\theta_k} \end{pmatrix}.$$

We deduce from this that the action of h on $\mathrm{U}_{c_1} \times \cdots \times \mathrm{U}_{c_\ell} \cdot [\rho]$ is given by the translation of the vector $(\theta_1, \dots, \theta_\ell)$. Then the lemma 2.3 shows that this action is ergodic with respect to the Lebesgue measure since $\pi, \theta_1, \dots, \theta_\ell$ are linearly independent over $\mathbf{Q}$. □

2.1.2 A full measure set

A multicurve m is the union of a finite number of simple, closed and pairwise disjoint curves. Let us denote by $\mathrm{MC}(\Sigma)$ the set of multicurves m whose curves are simple, closed and non-separating and such that m is the boundary of a subsurface of Σ. By $\mathrm{MC}_0(\Sigma)$ we will denote its subset of bounding pair of Σ.

Definition 2.4. *Let $m = c_1 \cup \cdots \cup c_\ell \in \mathrm{MC}(\Sigma)$ be a mutlicurve. We say that a class $[\rho] \in$* $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ satisfies the condition (M_m) *if the real numbers :*

$$\pi, \theta_1 = \cos^{-1}\left(\frac{\mathrm{tr}(\rho(c_1))}{2}\right), \dots, \theta_\ell = \cos^{-1}\left(\frac{\mathrm{tr}(\rho(c_\ell))}{2}\right)$$

are linearly independent over $\mathbf{Q}$.

Following the previous definition, for $m \in \mathrm{MC}_0(\Sigma)$, we now consider the set :

$$\mathcal{M}_m(\pi_1\Sigma, \mathrm{SU}(2)) = \{[\rho] \in \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2)) \mid [\rho] \text{ statisfies the condition } (\mathrm{M}_m)\}$$

The aim of this section is to prove the proposition :

Proposition 2.5. *For $m \in \mathrm{MC}_0(\Sigma)$, the set $\mathcal{M}_m(\pi_1\Sigma, \mathrm{SU}(2))$ has full measure in $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$.*

Before to prove this proposition, we note that for a curve γ, the angle of $\rho(\gamma)$, expressed by the formula

$$\cos^{-1}\left(\frac{f(\rho(\gamma))}{2}\right),$$

defines a function $\theta_\gamma : \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2)) \to \mathbf{S}^1$. To simplify notation, as previously, for a multicurve $m = c_1 \cup c_2 \in \mathrm{MC}_0(\Sigma)$, we will denote by θ_1 and θ_2 the functions θ_{c_1} and θ_{c_2}. Moreover, $c_1 \cup c_2$ separate Σ in two subsurface Σ_1 and Σ_2 of respective genus g_1 and g_2 and with two boundary components. Let Σ_1' and Σ_2' be the two subsurfaces of Σ_1 and Σ_2 which have genus g_1 and g_2 and with one boundary component which for a pair of pant with c_1 and c_2. See next picture.

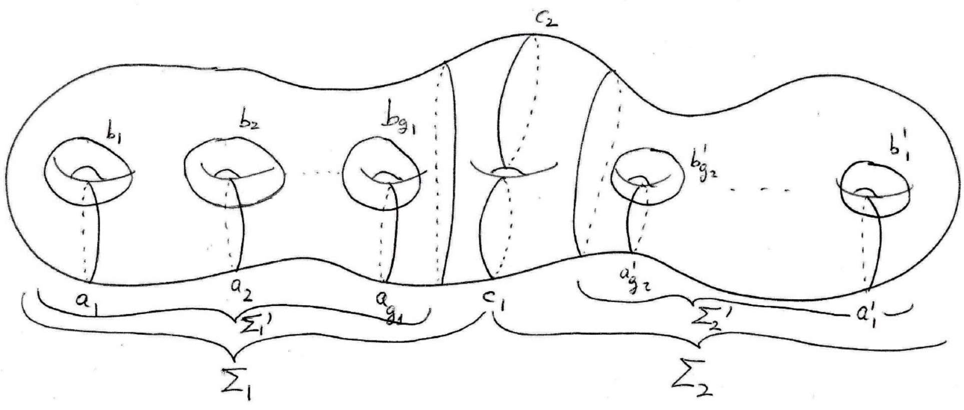

Figure 6

Let $\mathcal{N}_m$ be the subset of $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ of conjugacy classes of representations ρ such that the centralizers $\mathrm{Z}_{\mathrm{SU}(2)}\big(\rho_{|\pi_1\Sigma_1'}\big)$ and $\mathrm{Z}_{\mathrm{SU}(2)}\big(\rho_{|\pi_1\Sigma_2'}\big)$ are discrete in $\mathrm{SU}(2)$. This subspace has full measure because if one of the centralizers is non discrete, the image of at least one restriction is contained in a parabolic subgroup. Hence the complement of $\mathcal{N}_m$ is a finite union of submanifolds and hence has measure zero.

The complement of $\mathcal{M}_m(\pi_1\Sigma, \mathrm{SU}(2))$ is the set:

$$\bigcup_{(q_0,q_1,q_2)\in\mathbf{Z}^3\backslash\{0\}} \Big\{[\rho] \in \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2)) \mid q_1\theta_1(\rho) + q_2\theta_2(\rho) = q_0\pi\Big\}.$$

Let q denote the vector (q_1, q_2). If $q_0 \neq 0$ and $q = 0$, the relation $q_1\theta_1(\rho) + q_2\theta_2(\rho) = q_0\pi$ is empty. Thus, we only have to consider the case $q \neq 0$. The proposition will be hence proved if for every $(q_0, q_1, q_2) \in \mathbf{Z}^3\backslash\{0\}$ with $q \neq 0_{\mathbf{Z}^2}$, the set:

$$\big\{[\rho] \in \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2)) \mid \underbrace{q_1\theta_1(\rho) + q_2\theta_2(\rho)}_{=\psi_{m,q}([\rho])} = q_0\pi\big\} = \psi_{m,q}^{-1}(q_0\pi)$$

has null measure. Fix $(q_0, q_1, q_2) \in \mathbf{Z}^3\backslash\{0\}$ with $q \neq 0_{\mathbf{Z}^2}$.

Lemma 2.6. *The map $\psi_{m,q}$ is a submersion on every class of representation $[\rho] \in \mathcal{N}_m$.*

By Goldman work W. M. Goldman 1984 part 3.7, the map $\mathcal{E}_1 : \mathrm{SU}(2)^{2g_1+1} \to \mathrm{SU}(2)^2$ defined by :

$$\mathcal{E}_1(A_1, \ldots, B_{g_1}, C) = \Big(\prod_{j=1}^{g_1}[A_j, B_j], C\Big)$$

is a submersion at the point $\{A_1, \dots, B_{g_1}\}$ if the centralizer $\mathrm{Z}_{\mathrm{SU}(2)}\big(\{A_1, \dots, B_{g_1}\}\big)$ is discrete. For the same reasons, the map $\mathcal{E}_2 : \mathrm{SU}(2)^{2g_2+1} \to \mathrm{SU}(2)^2$ defined by:

$$\mathcal{E}_2(A_1, \dots, B_{g_2}, C) = \Big(\prod_{j=1}^{g_2}[A_j, B_j], C\Big)$$

is a submersion at the point $\{A_1, \dots, B_{g_2}\}$ if the centralizer $\mathrm{Z}_{\mathrm{SU}(2)}\big(\{A_1, \dots, B_{g_2}\}\big)$ is discrete.

Hence the evaluation $\mathrm{Ev}_{\Sigma_1} : \mathrm{Hom}\big(\pi_1\Sigma_1, \mathrm{SU}(2)\big) \to \mathrm{SU}(2)^2$ defined by:

$$\mathrm{Ev}_{\Sigma_1}(\rho) = (\rho(c_1), \rho(c_2))$$

is a submersion at every ρ such that the centralizer $\mathrm{Z}_{\mathrm{SU}(2)}\big(\rho_{|\pi_1\Sigma_1'}\big)$ is discrete since Ev_{Σ_1} is the composition of $\mathcal{E}_1$ with the diffeomorphism:

$$\begin{array}{ccc} \mathrm{SU}(2)^2 & \to & \mathrm{SU}(2)^2 \\ (A, B) & \mapsto & (B^{-1}A, B) \end{array}.$$

Since the trace function $\mathrm{tr} : \mathrm{SU}(2) \to \big[-2, 2\big]$ is a submersion at each matrix $A \neq \pm\mathrm{id}$, the application $(f_{c_1}, f_{c_2}) : \mathrm{Hom}\big(\pi_1\Sigma, \mathrm{SU}(2)\big) \to \big[-2, 2\big] \times \big[-2, 2\big]$ is a submersion at each representation ρ such that both $\mathrm{Z}_{\mathrm{SU}(2)}\big(\rho_{|\pi_1\Sigma_1'}\big)$ and $\mathrm{Z}_{\mathrm{SU}(2)}\big(\rho_{|\pi_1\Sigma_2'}\big)$ are discrete. In particular, under this condition, the linear form $d_{[\rho]}f_{c_1}$ and $d_{[\rho]}f_{c_2}$ are non-colinear.

Proof of Lemma 2.6. Let ρ such that $\rho(c_1)$ and $\rho(c_2)$ are not equal to $\pm\mathrm{id}$. We then have that both $\sin(\theta_1(\rho))$ and $\sin(\theta_2(\rho))$ are non-zero. We hence compute that:

$$d_{[\rho]}\psi_{m,q} = \frac{-q_1}{\sin(\theta_1(\rho))} d_{[\rho]}f_{c_1} + \frac{-q_2}{\sin(\theta_2(\rho))} d_{[\rho]}f_{c_2}.$$

If ${}_{[\rho]}\psi_{m,q} = 0$, then we deduce that $d_{[\rho]}f_{c_1}$ and $d_{[\rho]}f_{c_2}$ are colinear. But it is impossible if $[\rho] \in \mathcal{N}_m$. Then $\psi_{m,q}$ is a submersion at each point of $\mathcal{N}_m$. $\square$

2.1.3 Proof of the ergodicity

In order to prove Theorem A for SU(2), we will consider a measurable function which is $\mathrm{Tor}(\Sigma)$-invariant and prove that, up to a restrict this function to a full measure subset, it is invariant under the action of the Dehn twists which generate the mapping class group.

Let $F : \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2)) \to \mathbf{R}$ be a measurable function and assume that F is $\mathrm{Tor}(\Sigma)$-invariant. Let $x \in \{a_1, b_1, \dots, a_g, b_g, d_1, \dots, d_{g-1}\}$ be a curve of the generating set of the mapping class group, fix a bounding pair $m_x = x \cup c_2$ in $\mathrm{MC}_0(\Sigma)$ and denote by h the product of Dehn twists $\mathrm{tw}_x \cdot \mathrm{tw}_{c_2}$. The set

$$\mathcal{M}_{m_x}(\pi_1\Sigma, \mathrm{SU}(2))$$

has full measure by Proposition 2.5. As the orbits $\mathrm{U}_x \times \mathrm{U}_{c_2} \cdot [\rho]$ are tori and so are compact in $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$, Theorem 1.2 ensures the existence of a Borel cross section of $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ for the action of $\mathrm{U}_x \times \mathrm{U}_{c_2} = \mathbf{T}^3$. We will denote this section by $\mathcal{S}$. Since $\mathbf{T}^2$ and $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$ verify the assumption of Theorem 1.3, then the section $\mathcal{S}$ is bimeasurable to the quotient

$$\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))/\mathbf{T}^2$$

and the measure λ decomposes, for all function $f : \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2)) \to \mathbf{R}^+$, by the formula :

$$\int_{\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))} f d\lambda = \int_S \left(\int_{\mathbf{T}^2} f(t.s) d\mu_{\mathbf{T}^2}(t) \right) d\mu_{\mathcal{S}}(s)$$

where $\mu_{\mathbf{T}^2}$ is the Haar measure on the tori $\mathbf{T}^2$ given by Theorem 1.3 and $\mu_{\mathcal{S}}$ is a measure on $\mathcal{S}$ given by the same theorem.

The function F induces a measurable function $\widetilde{F} : \mathcal{S} \times \mathbf{T}^2 \to \mathbf{R}$ defined by :

$$\widetilde{F}(s,t) = F(t \cdot s).$$

Fix $[\rho] \in \mathcal{M}_{m_x}(\pi_1\Sigma, \mathrm{SU}(2))$, denote by $\overline{[\rho]}$ its projection in $\mathcal{S}$ and let $\widetilde{F}_{|\{\overline{[\rho]}\}\times\mathbf{T}^2} : \mathbf{T}^2 \to \mathbf{R}$. Such a function is measurable and invariant by the action of $\langle h \rangle$ since F is. Since $[\rho] \in \mathcal{M}_{m_x}(\pi_1\Sigma, \mathrm{SU}(2))$, the action of the translation $\langle h \rangle$ on $\mathbf{T}^2$ is ergodic with respect to $\mu_{\mathbf{T}^2}$. It implies that $\widetilde{F}_{|\{\overline{[\rho]}\}\times\mathbf{T}^2}$ is almost everywhere constant and hence that the restriction $F_{|\mathbf{T}^2.[\rho]}$ is almost everywhere constant. Since the Dehn twist tw_x acts as a translation of the torus on $\mathrm{U}_x \times \mathrm{U}_{c_2} \cdot [\rho]$, we deduce that on a full measure subset of $\mathrm{U}_x \times \mathrm{U}_{c_2} \cdot [\rho]$ the function F and $F \circ \mathrm{tw}_x$ are equal. This fact is true for almost every $[\rho] \in \mathcal{M}_{m_x}(\pi_1\Sigma, \mathrm{SU}(2))$ which has full measure by Proposition 2.5. It follows that $F_{|\mathcal{M}_{m_x}(\pi_1\Sigma, \mathrm{SU}(2))}$ is almost everywhere invariant by the Dehn twist tw_x.

We then deduce that on the space

$$\bigcap_{x \in \{a_1, b_1, \ldots, a_g, b_g, d_1, \ldots, d_{g-1}\}} \mathcal{M}_{m_x}(\pi_1\Sigma, \mathrm{SU}(2)),$$

which has full measure in $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$, the function F is almost everywhere invariant by the Dehn twists tw_x, for all $x \in \{a_1, \ldots, b_g, d_1, \ldots, d_{g-1}\}$. It implies that F is almost everywhere invariant under the action of $\mathrm{Mod}(\Sigma)$, which is known to be ergodic from Theorem 1. Hence F is almost everywhere constant and this proves the ergodicity of the Torelli group action on $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(2))$.

2.2 Ergodicity for $G = \mathrm{SU}(n)$

This part is devoted to the proof of Theorem A in the more general case $G = \mathrm{SU}(n)$, for $n \geq 2$. We apply the same strategy as in the previous section, adapting the tools developed there. Let us introduce the notation we are going to use.

2.2.1 Torus actions on $\mathcal{M}^{\alpha\text{-reg}}(\pi_1\Sigma, \mathrm{SU}(n))$

For a matrix $A \in \mathrm{SU}(n)$ with distinct eigenvalues, denote by $\lambda_1(A), \dots, \lambda_n(A)$ its eigenvalues and $\theta_1(A), \dots, \theta_n(A)$ their arguments which can be expressed up to the sign as

$$\cos^{-1}\left(\frac{\lambda_i(A) + \lambda_i(A)^{-1}}{2}\right)$$

with the normalisation $0 \leq \theta_1(A) \leq \cdots \leq \theta_n(A) < 2\pi$. The group $\mathrm{SU}(n)$ has rank $n-1$ and every maximal torus is conjugate to the group :

$$\left\{ \begin{pmatrix} e^{i\theta_1} & 0 & \dots & 0 \\ 0 & e^{i\theta_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{i\theta_n} \end{pmatrix}, (\theta_1, \dots, \theta_n) \in \left[0, 2\pi\right[\text{ and } \prod_{k=1}^{n} e^{i\theta_k} = 1 \right\}$$

which is isomorphic to $\mathbf{T}^{n-1}$.

A matrix $A \in \mathrm{SU}(n)$ is called *regular* if its eigenvalues are simple. Similarly, for a curve α, a character $[\rho] \in \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n))$ is called $\alpha-$*regular* if the matrix $\rho(\alpha)$ is regular. We will denote by $\mathcal{M}^{\alpha\text{-reg}}(\pi_1\Sigma, \mathrm{SU}(n))$ the subsets of $\alpha-$regular $\mathrm{SU}(n)$-characters. For any curve α, the subset $\mathcal{M}^{\alpha\text{-reg}}(\pi_1\Sigma, \mathrm{SU}(n))$ is an open subset of $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n))$ and has full measure. We will define actions of $\mathbf{T}^{n-1}$ on the character variety $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n))$. Let $z = (z_1, ..., z_{n-1}) \in \mathbf{T}^{n-1}$ and h_z be the associated diagonal matrix :

$$\mathrm{diag}\big(z_1, \dots, z_{n-1}, \frac{1}{z_1 \cdots z_{n-1}}\big) \in \mathrm{SU}(n).$$

Let $A \in \mathrm{SU}(n)$ be a regular matrix. There exists a unique decomposition $[e_1] \oplus \cdots \oplus [e_n]$ of $\mathbf{C}^n$ in lines such that:

$$Ae_i = \lambda_i(A)e_i,$$

for all $i \in \{1, \dots, n\}$.

Let α be a simple and closed curve and let $[\rho] \in \mathcal{M}^{\alpha\text{-reg}}(\pi_1\Sigma, \mathrm{SU}(n))$. With the same notation as in part 1.3 and with respect to the unique basis of $\mathbf{C}^n$ in which $\rho(\alpha) = h_{(\lambda_1(\rho(\alpha)), \cdots \lambda_{n-1}(\rho(\alpha)))}$, we define for $z \in \mathbf{T}^{n-1}$ and for α non-separating, the representation $z \cdot \rho$ by:

$$z \cdot \rho(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma \setminus \alpha) \\ h_z\rho(\beta) & \text{if } \gamma = \beta \end{cases}.$$

In the case when α is separating, the representation $z \cdot \rho$ by:

$$z \cdot \rho(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma_1) \\ h_z\rho(\gamma)h_z^{-1} & \text{if } \gamma \in \pi_1(\Sigma_2) \end{cases}.$$

Therefore, the action of the torus $\mathrm{U}_\alpha := \mathbf{T}^{n-1}$ on the subspace of $\alpha-$regular characters depends on α. Moreover the action of Dehn twists along α on $\mathcal{M}^{\alpha\text{-reg}}(\pi_1\Sigma, \mathrm{SU}(n))$ is given by:

$$\mathrm{tw}_\alpha \cdot [\rho] = \lambda(\rho(\alpha)) \cdot [\rho],$$

where $\lambda(\rho(\alpha)) = (\lambda_1(\rho(\alpha)), \ldots, \lambda_{n-1}(\rho(\alpha)))$. A direct computation shows that if α and β are disjoint curves, then the actions of the tori U_α and U_β on $\mathcal{M}^{\alpha\text{-reg}}(\pi_1\Sigma, \mathrm{SU}(n)) \cap \mathcal{M}^{\beta\text{-reg}}(\pi_1\Sigma, \mathrm{SU}(n))$ commute. This hence defines an action of $\mathrm{U}_\alpha \times \mathrm{U}_\beta$ on the space of $\alpha-$regular and $\beta-$regular characters. We thus obtain an analogue of Lemma 2.1:

Lemma 2.7. *Let $[\rho] \in \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n))$ and suppose that there exist $c_1, \ldots, c_\ell$ pairwise disjoints, simple and closed curves on Σ such that:*

$$\theta_1(\rho(c_1)), \ldots, \theta_{n-1}(\rho(c_1)), \ldots, \theta_1(\rho(c_\ell)), \ldots, \theta_{n-1}(\rho(c_\ell)), \pi$$

are linearly independent over $\mathbf{Q}$. Then the action of $h = \mathrm{tw}_{c_1} \cdots \mathrm{tw}_{c_\ell}$ on the orbit $\mathrm{U}_{c_1} \times \cdots \times \mathrm{U}_{c_\ell} \cdot [\rho]$ is ergodic with respect to the Lebesgue measure on this torus orbit.

Proof. As in the proof of Lemma 2.1, the action of the Dehn twists tw_{c_i} is an action by translation on the torus. The flows commute on the character variety because the curves are disjoint and the orbit is given by the formula

$$h^k \cdot [\rho] = (\lambda_1(\rho(c_1))^k, \ldots \lambda_{n-1}(\rho(c_\ell))^k, \ldots, \lambda_1(\rho(c_\ell))^k, \ldots \lambda_{n-1}(\rho(c_\ell))^k) \cdot [\rho].$$

Hence the condition on the $\theta_i(\rho(c_k))$ implies the desired ergodicity by the lemma 2.3. □

2.2.2 A full measure set and the ergodicity

In this subsection, we define a full measure subspace of the character variety with the conditions of the previous lemma and conclude with the ergodicity of the Torelli group action.

Definition 2.8. *Let $m = c_1 \cup \cdots \cup c_\ell$ be a multicurve of simple closed and non-separating curves. We say that a class of a representation $[\rho] \in \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n))$ satisfies the* condition (M_m) *if*

$$\theta_1(\rho(c_1)) \ldots, \theta_{n-1}(\rho(c_1)), \ldots, \theta_1(\rho(c_\ell)), \ldots \theta_{n-1}(\rho(c_\ell)), \pi$$

are linearly independent over $\mathbf{Q}$.

Remark 2.9. *If a class of a representation $[\rho]$ satisfies the condition (M_m) for some $m = c_1 \cup \cdots \cup c_\ell \in \mathrm{MC}(\Sigma)$ then $[\rho]$ is c_i-regular for all $i \in \{1, \ldots, \ell\}$. To simplify notation, for a multicurve $m = c_1 \cup \cdots \cup c_\ell$, we will denote by $\mathcal{M}^{m-\mathrm{reg}}(\pi_1\Sigma, \mathrm{SU}(n))$ the intersection*

$$\bigcap_{i=1}^{\ell} \mathcal{M}^{c_i-\mathrm{reg}}(\pi_1\Sigma, \mathrm{SU}(n))$$

which has full measure.

We define, for $m \in \mathrm{MC}_0(\Sigma)$, the set:

$$\mathcal{M}_m(\pi_1\Sigma, \mathrm{SU}(n)) = \big\{[\rho] \in \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n)) \mid [\rho] \text{ statisfies the condition } (\mathrm{M}_m)\big\}.$$

Note that Remark 2.9 assures that $\mathcal{M}_m(\pi_1\Sigma, \mathrm{SU}(n))$ is contained in $\mathcal{M}^{m-\mathrm{reg}}(\pi_1\Sigma, \mathrm{SU}(n))$.

Proposition 2.10. *For all $m = c_1 \cup c_2 \in \mathrm{MC}_0(\Sigma)$, the set $\mathcal{M}_m(\pi_1\Sigma, \mathrm{SU}(n))$ has full measure in the character variety.*

A previously, $c_1 \cup c_2$ separate Σ in two subsurface Σ_1 and Σ_2 of respective genus g_1 and g_2 and with two boundary components. Let Σ_1' and Σ_2' be the two subsurfaces of Σ_1 and Σ_2 which have genus g_1 and g_2 and with one boundary component which for a pair of pant with c_1 and c_2. See Figure 6.

We can write its complement as the set:

$$\bigcup_{q=(q_1^1,\dots,q_{n-1}^1,\dots,q_1^2,\dots q_{n-1}^2)\in\mathbf{Z}^{2(n-1)}\setminus\{0\},q_0\in\mathbf{Z}} \left\{[\rho] \in \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n)) \mid \sum_{k=1}^{2}\sum_{i=1}^{n-1} q_i^k \theta_i(\rho(c_k)) = q_0\pi\right\}.$$

We will show next that each set in the previous union is a codimension 1 submanifold and hence this union has null measure.

Let $\psi_{m,q} : \mathcal{M}^{m-\mathrm{reg}}(\pi_1\Sigma, \mathrm{SU}(n)) \to \mathbf{R}$ be defined by the formula :

$$\psi_{m,q}([\rho]) = \sum_{k=1}^{2}\sum_{i=1}^{n-1} q_i^k \theta_i(\rho(c_k)).$$

For $\pi_i^k : [\rho] \mapsto \lambda_i(\rho(c_k)) + \lambda_i(\rho(c_k))^{-1}$, we may write $\theta_i(\rho(c_k)) = \cos^{-1}(\frac{\pi_i^k([\rho])}{2})$ up to the sign. The differential of $\psi_{m,q}$ is the computed to be

$$d_{[\rho]}\psi_{m,q} = \sum_{k=1}^{2}\sum_{i=1}^{n-1} -\frac{q_i^k}{2\sin(\theta_i(\rho(c_k)))} d_{[\rho]}\pi_i^k.$$

Let $\mathcal{N}_m$ be the subset of $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n))$ of conjugacy classes of representations ρ such that the centralizers $\mathrm{Z}_{\mathrm{SU}(n)}(\rho_{|\pi_1\Sigma_1'})$ and $\mathrm{Z}_{\mathrm{SU}(n)}(\rho_{|\pi_1\Sigma_2'})$ are discrete in $\mathrm{SU}(n)$. By the same reason we explained for the case of $\mathrm{SU}(2)$, we have:

Lemma 2.11. *The subset $\mathcal{N}_m$ has full measure in $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n))$*

We will show the next Lemma as for $\mathrm{SU}(2)$.

Lemma 2.12. *The map $\psi_{m,q}$ is a submersion on every class of representation $[\rho] \in \mathcal{N}_m$.*

By Goldman work W. M. Goldman 1984 part 3.7, the map $\mathcal{E}_1 : \mathrm{SU}(n)^{2g_1+1} \to \mathrm{SU}(n)^2$ defined by :

$$\mathcal{E}_1(A_1, \dots, B_{g_1}, C) = \big(\prod_{j=1}^{g_1}[A_j, B_j], C\big)$$

is a submersion at the point $\{A_1, \ldots, B_{g_1}\}$ if the centralizer $\mathrm{Z}_{\mathrm{SU}(n)}\big(\{A_1, \ldots, B_{g_1}\}\big)$ is discrete. For the same reasons, the map $\mathcal{E}_2 : \mathrm{SU}(n)^{2g_2+1} \to \mathrm{SU}(n)^2$ defined by:

$$\mathcal{E}_2(A_1, \ldots, B_{g_2}, C) = \big(\prod_{j=1}^{g_2}[A_j, B_j], C\big)$$

is a submersion at the point $\{A_1, \ldots, B_{g_2}\}$ if the centralizer $\mathrm{Z}_{\mathrm{SU}(n)}\big(\{A_1, \ldots, B_{g_2}\}\big)$ is discrete.

Hence the evaluation $\mathrm{Ev}_{\Sigma_1} : \mathrm{Hom}\big(\pi_1\Sigma_1, \mathrm{SU}(n)\big) \to \mathrm{SU}(n)^2$ defined by:

$$\mathrm{Ev}_{\Sigma_1}(\rho) = (\rho(c_1), \rho(c_2))$$

is a submersion at every ρ such that the centralizer $\mathrm{Z}_{\mathrm{SU}(n)}\big(\rho_{|\pi_1\Sigma_1'}\big)$ is discrete since Ev_{Σ_1} is the composition of $\mathcal{E}_1$ with the diffeomorphism:

$$\begin{array}{ccc} \mathrm{SU}(n)^2 & \to & \mathrm{SU}(n)^2 \\ (A,B) & \mapsto & (B^{-1}A, B) \end{array}.$$

Since the functions $\lambda_i + \lambda_i^{-1} : \mathrm{SU}(n) \to \big[-2,2\big]$ are submersions at each regular matrix A, the application $(\pi_i^k)_{k=1,2\,;\,i=1,\ldots,n-1} : \mathrm{Hom}\big(\pi_1\Sigma, \mathrm{SU}(n)\big) \to \big[-2,2\big]^{2(n-1)}$ is a submersion at each representation ρ such that both $\mathrm{Z}_{\mathrm{SU}(n)}\big(\rho_{|\pi_1\Sigma_1'}\big)$ and $\mathrm{Z}_{\mathrm{SU}(n)}\big(\rho_{|\pi_1\Sigma_2'}\big)$ are discrete. In particular, for the first $q_{i_0}^{k_0} \neq 0$, there exists a vector field X such that $d_{[\rho]}\pi_i^k X = \delta_{i_0}^{k_0}$. For such a vector fiel:

$$d_{[\rho]}\psi_{m,q}X = \frac{-q_{i_0}^{k_0}}{\sin(\theta_{i_0}(\rho(c_{k_0})))} d_{[\rho]}\pi_{i_0}^{k_0} X \neq 0.$$

This concludes the proof of Lemma 2.12 and thus of Proposition 2.10.

The proof of the ergodicity uses the same arguments as in case of $\mathrm{SU}(2)$.

Let $F : \mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n)) \to \mathbf{R}$ be a measurable and $\mathrm{Tor}(\Sigma)$-invariant function. For each curve x in the set $\{a_1, b_1, \ldots, a_g, b_g, d_1, \ldots, d_{g-1}\}$, we fix a bounding pair $m_x = x \cup c_2$.

Replacing the torus $\mathbf{T}^2$ we used for the case $\mathrm{SU}(2)$ by the torus $\mathbf{T}^{2(n-1)}$, we conclude using the same methods that on the space

$$\bigcap_{x \in \{a_1, \ldots, b_g, d_1, \ldots, d_{g-1}\}} \mathcal{M}_{m_x}(\pi_1\Sigma, \mathrm{SU}(n)),$$

which has full measure from Proposition 2.10, the function F is almost everywhere invariant by the Dehn twists tw_x for all $x \in \{a_1, b_1, \ldots, a_g, b_g, d_1, \ldots, d_{g-1}\}$. This implies that it is almost everywhere invariant by the mapping class group. Theorem 1 now shows that F is constant on a full measure subset of $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n))$. This proves the ergodicity of the Torelli group action on $\mathcal{X}(\pi_1\Sigma, \mathrm{SU}(n))$.

2.3 Ergodicity for semi-simple, connected and compact Lie groups

In this section we generalize the proofs of the ergodicity of the Torelli group on character varieties with values in a semi-simple, connected and compact Lie group. We will use the same strategy as for compact Lie groups $\mathrm{SU}(n)$ but need to replace the tools we used by their appropriate analogues in a more general sense.

2.3.1 Preliminaries on compact Lie group theory

Let G be a semi-simple, connected and compact Lie group with Lie algebra $\mathfrak{g}$. A maximal torus is a connected and abelian subgroup of G which is maximal in these properties. Such a subgroup exists, so fix $T < G$ be a maximal torus and let $\mathfrak{t}$ be its Lie algebra. It is an abelian subalgebra of $\mathfrak{g}$. The subgroup T is isomorphic to an r-dimensional torus $\mathbf{T}^r$ and its Lie algebra $\mathfrak{t}$ is isomorphic to the commutative Lie algebra $\mathbf{R}^r$. We will so use the existence of coordinates on $\mathfrak{t}$ via this isomorphism. Precisely, for $i \in \{1, \ldots, r\}$ and $t \in T$, we denote by $\lambda_i(t)$ the projection on the i-th factor of $t \in T \cong \mathbf{T}^r$. It is a well known fact that every element of G is contained in a maximal torus. We have however the more precise result (see Bröcker and tom Dieck 1995 for more details):

Theorem 2.13. *Every $k \in G$ is conjugated to an element of T. Moreover, all the maximal tori are conjugated and hence are isomorphic to $\mathbf{T}^r$.*

Remark that a maximal torus can contain two conjugated elements. The integer r is called the *rank* of the group G. The *Weyl group* associated to T is the group $\mathrm{N}_G(T)/T$, where the subgroup $\mathrm{N}_G(T)$ is the normalizer of T in G.

Proposition 2.14. *(Bourbaki 1981) The Weyl group associated to a maximal torus $T < G$ is finite.*

A weight of T is a real and irreducible representation. Let ω be a weight of T and $\sigma : G \to \mathrm{Aut}(V)$ be a representation. The sum of all invariant subspaces of $\sigma|_T$ isomorphic to ω is called the *weight space* associated to ω of σ. Define, for $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbf{Z}^r$, the linear form:

$$\begin{array}{cccc} \Theta^*_{\mathbf{n}} : & \mathfrak{t} & \to & \mathbf{R} \\ & (x_1, \ldots, x_r) & \mapsto & n_1 x_1 + \cdots + n_r x_r \end{array},$$

where we use the coordinates on $\mathfrak{t}$ given by the isomorphism $\mathfrak{t} \cong \mathbf{R}^r$ coming from

$$T \cong \big(\mathbf{R}/\mathbf{Z}\big)^r.$$

It is then well known that the weights of T are either the trivial one-dimensional representation or the representations $\Theta_{\mathbf{n}} : T \cong \mathbf{T}^r \to \mathrm{SO}_2(\mathbf{R})$, for $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbf{Z}^r \backslash \{0\}$, defined by :

$$\Theta_{\mathbf{n}}([x_1, \ldots, x_r]) = \begin{pmatrix} \cos(2\pi\Theta^*_{\mathbf{n}}(x_1, \ldots, x_r)) & -\sin(2\pi\Theta^*_{\mathbf{n}}(x_1, \ldots, x_r)) \\ \sin(2\pi\Theta^*_{\mathbf{n}}(x_1, \ldots, x_r)) & \cos(2\pi\Theta^*_{\mathbf{n}}(x_1, \ldots, x_r)) \end{pmatrix}.$$

Definition 2.15. *A linear form* $\alpha \in \mathfrak{t}^*$ *is a* root *of* G *if there exists* $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbf{Z}^r \backslash \{0\}$ *such that* $\alpha = \Theta_{\mathbf{n}}^*$ *and if the weight space of the adjoint representation* $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ *associated to* $\Theta_{\mathbf{n}}$ *is non-trivial.*

We denote by Δ the set of roots of G. Since G is a semi-simple Lie group, the Killing form $\langle ., . \rangle$ is a scalar product and the subspace $\mathfrak{t} < \mathfrak{g}$ becomes a Euclidean space. Using the induced isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$, we can see Δ as a subset of $\mathfrak{t}$ and for $\alpha \in \Delta$, we define the reflection

$$r_\alpha : \beta \mapsto \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

It is well known that the Weyl group associated to T is isomorphic to the subgroup of $\mathrm{GL}(\mathfrak{t})$:

$$\langle r_\alpha | \alpha \in \Delta \rangle.$$

The *alcoves* of $\mathfrak{t}$ are the connected components of

$$\mathfrak{t} \backslash \bigcup_{\alpha \in \Delta, n \in \mathbf{N}} \ker(r_\alpha - n\mathrm{id}).$$

The Weyl group acts simply transitively on the images of the alcoves in T.

Definition 2.16. *An element* $k \in G$ *is called* regular*, if it is contained in a unique maximal torus.*

Let M be the image by the exponential map $\mathfrak{t} \to T$ of an alcove of $\mathfrak{t}$. Such M will be called an alcove of T and let $k \in G$ be a regular element. There exists a unique class $\overline{g_k}$ of $G/\mathrm{Z}_G(k)$, with $\mathrm{Z}_G(k)$ the centralizer of k in G, such that

$$g_k k g_k^{-1} \in M.$$

Example 2.17. *For* $G = \mathrm{SU}(n)$ *and* T *the set of diagonal matrices, the roots of* G *are given by* $\lambda_i - \lambda_k$ *for* $i, k \in \{1, \ldots, n\}$ *such that* $i \neq k$ *and where the* λ_i *are the eigenvalues. The Weyl group is then the symmetric group* $\mathfrak{S}_n$*.*

2.3.2 Density of some orbits

Let G be a semi-simple, connected and compact Lie group of rank r, let T be a maximal torus and M be an alcove of T.

Let $\alpha \in \pi_1\Sigma$ be a simple curve. A character $[\rho] \in \mathcal{X}(\pi_1\Sigma, G)$ is called $\alpha-$*regular* if $\rho(\alpha)$ is regular. Then there exists a unique class $\overline{g_{\rho(\alpha)}}$ in the quotient $G/\mathrm{Z}_G(\rho(\alpha))$, such that

$$g_{\rho(\alpha)} \rho(\alpha) g_{\rho(\alpha)}^{-1} \in M.$$

The set of $\alpha-$regular characters is an open subset of $\mathcal{X}(\pi_1\Sigma, G)$ and has full measure. The maximal torus T acts on the space $\mathrm{Hom}^{\alpha-\mathrm{reg}}(\pi_1\Sigma, G)$ of $\alpha-$regular representations via the action, defined as

follows:
if α is non-separating, for $t \in T$ and $\rho \in \mathrm{Hom}^{\alpha-\mathrm{reg}}(\pi_1\Sigma, G)$,

$$t \cdot \rho(\gamma) = \begin{cases} g_{\rho(\alpha)}\rho(\gamma)g_{\rho(\alpha)}^{-1} & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\ tg_{\rho(\alpha)}\rho(\beta)g_{\rho(\alpha)}^{-1} & \text{if } \gamma = \beta \end{cases}$$

and if α is separating:

$$t \cdot \rho(\gamma) = \begin{cases} g_{\rho(\alpha)}\rho(\gamma)g_{\rho(\alpha)}^{-1} & \text{if } \gamma \in \pi_1(\Sigma_1) \\ tg_{\rho(\alpha)}\rho(\gamma)g_{\rho(\alpha)}^{-1}t^{-1} & \text{if } \gamma \in \pi_1(\Sigma_2). \end{cases}.$$

Since this action commutes with the conjugation action of G on the representations, we thus defined an action of the maximal torus $\mathrm{U}_\alpha := T$ on the $\alpha-$regular characters, which only depends on the curve α. As for the case of $\mathrm{SU}(n)$, we have an expression of the tw_α action on the subspace of $\alpha-$regular characters, via the formula

$$\mathrm{tw}_\alpha \cdot [\rho] = (g_{\rho(\alpha)}\rho(\alpha)g_{\rho(\alpha)}^{-1}) \cdot [\rho].$$

We then note the important fact that for two disjoint curves α and β, the actions of the maximal tori U_α and U_β on $\mathcal{M}^{\alpha-\mathrm{reg}}(\pi_1\Sigma, G) \cap \mathcal{M}^{\beta-\mathrm{reg}}(\pi_1\Sigma, G)$ commute. It hence implies an action of the product $\mathrm{U}_\alpha \times \mathrm{U}_\beta$ on the previous intersection. We define the map $t_\alpha : \mathrm{Hom}^{\alpha-\mathrm{reg}}(\pi_1\Sigma, G) \to M$ by

$$t_\alpha(\rho) = g_{\rho(\alpha)}\rho(\alpha)g_{\rho(\alpha)}^{-1}.$$

For every $i \in \{1, \ldots, r\}$, the projection λ_i induces a function on the space $\mathrm{Hom}^{\alpha-\mathrm{reg}}(\pi_1\Sigma, G)$ which we will denote by $\lambda_{i,\alpha}$. If we conjugate ρ by $g \in G$, we obtain by the uniqueness of $g_{\rho(\alpha)}$ up to the centralizer of $\rho(\alpha)$, that there exists $z \in \mathrm{Z}_G(\rho(\alpha))$ such that

$$g_{g\rho(\alpha)g^{-1}}g = g_{\rho(\alpha)}z$$

and hence we obtain that t_α is invariant under the conjugation action of G on the representation variety $\mathrm{Hom}^{\alpha-\mathrm{reg}}(\pi_1\Sigma, G)$ and descends to a map

$$\mathcal{M}^{\alpha-\mathrm{reg}}(\pi_1\Sigma, G) \to M,$$

which we will denote by t_α again.

An element $k \in G$ is said to be *generic* if $\langle g_k k g_k^{-1} \rangle$ is dense in T.

Claim 2.18. *An element $k \in G$ is generic if and only if for all non-trivial character $\chi : T \to \mathbf{S}^1$,*

$$\chi(g_k k g_k^{-1}) \neq 1.$$

Proof. Let $\phi : T \to \mathbf{T}^r$ be an isomorphism which identifies T with the $r-$torus. Then the map $\chi(\phi^{-1})$ is a non-trivial character of the torus $\mathbf{T}^r$. Since a character of $\mathbf{T}^r$ is induced by a linear form of $\mathbf{R}^r$ which maps $\mathbf{Z}^r$ on $\mathbf{Z}$, it as the form $(x_1, \ldots, x_r) \mapsto n_1x_1 + \cdots + n_rx_r$ with the $n_i \in \mathbf{Z}$. An element $k \in T$ is generic if and only if $\phi(k)$ is generic, that means if it generates a dense subgroup in $\mathbf{T}^r$. We then have that $\phi(k)$ is generic, if and only if $\chi(\phi^{-1})(\phi(k)) \neq 1$. $\square$

In particular, a generic element of k is regular.

Lemma 2.19. *Let $\rho : \pi_1\Sigma \to G$ be a representation and α be a simple closed curve such that $\rho(\alpha)$ is generic. Then the orbit*

$$\langle \mathrm{tw}_\alpha \rangle \cdot [\rho]$$

is dense in the torus orbit $\mathrm{U}_\alpha \cdot [\rho]$. Moreover, the action of $\langle \mathrm{tw}_\alpha \rangle$ on $\mathrm{U}_\alpha \cdot [\rho]$ is ergodic with respect to the Lebesgue measure.

Definition 2.20. *Let $m = c_1 \cup \cdots \cup c_\ell$ be a multicurve on Σ. A class of a representation $[\rho] \in \mathcal{X}(\pi_1\Sigma, G)$ is called $m-$*regular*, if the elements $\rho(c_1), \ldots, \rho(c_\ell)$ are regular and we denote by $\mathcal{M}^{m-\mathrm{reg}}(\pi_1\Sigma, G)$ the set all of $m-$regular characters. Precisely, we set*

$$\mathcal{M}^{m-\mathrm{reg}}(\pi_1\Sigma, G) := \bigcap_{i=1}^{\ell} \mathcal{M}^{c_i-\mathrm{reg}}(\pi_1\Sigma, G).$$

Since the curves $c_1, \ldots, c_\ell$ are disjoint, the actions of the tori $\mathrm{U}_{c_1}, \ldots, \mathrm{U}_{c_\ell}$ on $\mathcal{M}^{m-\mathrm{reg}}(\pi_1\Sigma, G)$ commute. In this general setting and similarly to Lemmas 2.1 and 2.7, we state:

Lemma 2.21. *Let $[\rho] \in \mathcal{X}(\pi_1\Sigma, G)$ and suppose that there exist $c_1, \ldots, c_\ell$ pairwise disjoint, simple and closed curves of Σ such that for all non-trivial characters $\chi : T^\ell \to \mathbf{S}^1$:*

$$\chi(t_{c_1}([\rho]), \ldots, t_{c_\ell}([\rho])) \neq 1.$$

Then, if we set $h := \mathrm{tw}_{c_1} \cdots \mathrm{tw}_{c_\ell}$, the action of $\langle h \rangle$ on $\mathrm{U}_{c_1} \times \cdots \times \mathrm{U}_{c_\ell} \cdot [\rho]$ is ergodic with respect to the Lebesgue measure.

To simplify notation we shall denote by T^ℓ the product $\mathrm{U}_{c_1} \times \cdots \times \mathrm{U}_{c_\ell}$.

Proof of the lemma 2.21. Let ϕ be the isomorphism $T \cong \mathbf{T}^r$ and let $\chi : T^\ell \to \mathbf{S}^1$ be a non-trivial character. Then the composition $\chi \circ (\phi^{-1}, \ldots, \phi^{-1})$ is a non-trivial character of $\mathbf{T}^{r\ell}$, which we may identify with $\mathbf{R}^{r\ell}/\mathbf{Z}^{r\ell}$. The action of h is then given by the translation of the vector

$$(\theta_1(\rho(c_1)), \ldots, \theta_r(\rho(c_1)), \ldots, \theta_1(\rho(c_\ell)), \ldots, \theta_r(\rho(c_\ell))),$$

where $\theta_i(\rho(c_k))$ is the argument of $\lambda_i(\rho(c_k))$. Then, by Lemma 2.3, the action of h is ergodic on the torus orbit $\mathrm{U}_{c_1} \times \cdots \times \mathrm{U}_{c_\ell} \cdot [\rho]$ with respect to the Lebesgue measure if and only if

$$\theta_1(\rho(c_1)), \ldots, \theta_r(\rho(c_1)), \ldots, \theta_1(\rho(c_\ell)), \ldots, \theta_r(\rho(c_\ell)) \text{ and } 1$$

are linearly independent over $\mathbf{Q}$. As a character of a torus is given by a linear form of $\mathbf{R}^{r\ell}$ with integer coefficients, this condition is equivalent to the fact that for all non-trivial characters χ' of $\mathbf{T}^{r\ell}$,

$$\chi'(\theta_1(\rho(c_1)), \ldots, \theta_r(\rho(c_1)), \ldots, \theta_1(\rho(c_\ell)), \ldots, \theta_r(\rho(c_\ell))) \neq 1.$$

The proof of the lemma is then complete. □

2.3.3 Proof of the ergodicity

We will adapt the previous proofs of ergodicity with the condition of Lemma 2.21.

Definition 2.22. *A class of a representations $[\rho] \in \mathcal{X}(\pi_1\Sigma, G)$ satisfies the* condition (M_m)*, if for all non-trivial characters $\chi : T^\ell \to \mathbf{S}^1$,*

$$\chi(t_{c_1}([\rho]), \ldots, t_{c_\ell}([\rho])) \neq 1.$$

We thus introduce

$$\mathcal{M}_m(\pi_1\Sigma, G) = \big\{[\rho] \in \mathcal{X}(\pi_1\Sigma, G) \mid [\rho] \text{ satisfies condition } (\mathrm{M}_m)\big\}.$$

Remark 2.23. *The set $\mathcal{M}_m(\pi_1\Sigma, G)$ is contained in $\mathcal{M}^{m-\mathrm{reg}}(\pi_1\Sigma, G)$.*

We hence prove the following:

Proposition 2.24. *For all $m = c_1 \cup c_2 \in \mathrm{MC}_0(\Sigma)$, the space $\mathcal{M}_m(\pi_1\Sigma, G)$ has full measure in the character variety.*

As in the previous cases, we prove that the set $\mathcal{M}_m(\pi_1\Sigma, G)$ is the complement of a countable union of submanifolds of codimension 1. The strategy we use is the same as in Propositions 2.5 and 2.10. For $m = c_1 \cup c_2 \cup c_3 \in \mathrm{MC}_0(\Sigma)$, we write the complement of the set of characters which satisfy condition (M_m) as the union:

$$\bigcup_{\substack{\chi : T^2 \to \mathbf{S}^1 \\ \text{non-trivial character}}} \Big\{[\rho] \in \mathcal{X}(\pi_1\Sigma, G) \mid \chi(t_{c_1}([\rho]), t_{c_2}([\rho])) = 1\Big\}.$$

We will hence prove that for all non-trivial characters $\chi : T^2 \to \mathbf{S}^1$, the set

$$\Big\{[\rho] \in \mathcal{X}(\pi_1\Sigma, G) \mid \chi(t_{c_1}([\rho]), t_{c_2}([\rho])) = 1\Big\}$$

has null measure, being the preimage of 1 by the map $\psi_{\chi,m} := \chi(t_{c_1}(\cdot), t_{c_2}(\cdot))$ on $\mathcal{M}^{m-\mathrm{reg}}(\pi_1\Sigma, G)$ which we will show to be a submersion. This is the goal of the following lemmas.

Let $\mathcal{N}_m$ be the subset of $\mathcal{X}(\pi_1\Sigma, G)$ of conjugacy classes of representations ρ such that the centralizers $\mathrm{Z}_G\big(\rho_{|\pi_1\Sigma'_1}\big)$ and $\mathrm{Z}_G\big(\rho_{|\pi_1\Sigma'_2}\big)$ are discrete in G.

With the same explanation than in the case of $\mathrm{SU}(2)$, we have the following fact:

Lemma 2.25. *The subset $\mathcal{N}_m$ has full measure in $\mathcal{X}(\pi_1\Sigma, G)$*

We will hence prove that $\mathcal{N}_m \cap \mathcal{M}_m(\pi_1\Sigma, G)$ has full measure.

Lemma 2.26. *For all non-trivial characters $\chi : T^2 \to \mathbf{S}^1$, the map $\psi_{\chi,m}$ is a submersion at each point of $\mathcal{N}_m$.*

Proof. It suffices, for $[\rho] \in \mathcal{X}(\pi_1\Sigma, G)$, to find a vector $X \in T_{[\rho]}\mathcal{X}(\pi_1\Sigma, G)$ such that

$$d_{[\rho]}\psi_{\chi,m}X \neq 0.$$

Write

$$d\psi_{\chi,m}X = d\chi(dt_{c_1}X, dt_{c_2}X)$$

where $d_{[\rho]}t_{c_k}X$ is

$$d\phi^{-1}_{\phi(t_{c_k}([\rho]))}\Big(d_{[\rho]}\lambda_{1,c_k}X, \dots, d_{[\rho]}\lambda_{r,c_k}X\Big),$$

for ϕ be the isomorphism $T \cong \mathbf{T}^r$ we use in the proof of Lemma 2.21.

With the same notations as the cases of $\mathrm{SU}(n)$, by Goldman work W. M. Goldman 1984 part 3.7, the map $\mathcal{E}_1 : G^{2g_1+1} \to G^2$ defined by :

$$\mathcal{E}_1(A_1, \dots, B_{g_1}, C) = \big(\prod_{j=1}^{g_1}[A_j, B_j], C\big)$$

is a submersion at the point $\{A_1, \dots, B_{g_1}\}$ if the centralizer $\mathrm{Z}_G\big(\{A_1, \dots, B_{g_1}\}\big)$ is discrete. For the same reasons, the map $\mathcal{E}_2 : G^{2g_2+1} \to G^2$ defined by:

$$\mathcal{E}_2(A_1, \dots, B_{g_2}, C) = \big(\prod_{j=1}^{g_2}[A_j, B_j], C\big)$$

is a submersion at the point $\{A_1, \dots, B_{g_2}\}$ if the centralizer $\mathrm{Z}_G\big(\{A_1, \dots, B_{g_2}\}\big)$ is discrete.

Hence the evaluation $\mathrm{Ev}_{\Sigma_1} : \mathrm{Hom}\big(\pi_1\Sigma_1, G\big) \to G^2$ defined by:

$$\mathrm{Ev}_{\Sigma_1}(\rho) = (\rho(c_1), \rho(c_2))$$

is a submersion at every ρ such that the centralizer $\mathrm{Z}_G\big(\rho_{|\pi_1\Sigma_1'}\big)$ is discrete since Ev_{Σ_1} is the composition of $\mathcal{E}_1$ with the diffeomorphism:

$$\begin{array}{ccc} G^2 & \to & G^2 \\ (A, B) & \mapsto & (B^{-1}A, B) \end{array}.$$

Since the functions $\lambda_i : M \to \mathbf{R}$ are submersions at each regular matrix point, the application $(\lambda_{i,c_k})_{k=1,2\,;\,i=1,\dots,r} : \mathrm{Hom}\big(\pi_1\Sigma, G\big) \to \mathbf{R}^{2r}$ are submersions at each representation ρ such that both $\mathrm{Z}_G\big(\rho_{|\pi_1\Sigma_1'}\big)$ and $\mathrm{Z}_G\big(\rho_{|\pi_1\Sigma_2'}\big)$ are discrete. In particular, for the first indexes k_0 and i_0 such that $d\chi e_{i_0}$, with $(e_i)_i$ is the k_0-th copy of $\mathbf{R}^r$, there exists a vector field X such that $dt_{c_k} X = \delta^k_{k_0} e_{i_0}$. This allows to conclude that $\psi_{\chi,m}$ is a submersion. □

The set of characters $T^2 \to \mathbf{S}^1$ is countable because such a character is given by a linear form

$$\tilde{\chi} : \mathbf{R}^{2r} \to \mathbf{R},$$

such that $\tilde{\chi}(\mathbf{Z}^{2r}) \subset \mathbf{Z}$. Hence there is a countable number of possibilities to obtain characters of the $2r$-torus, looking the image by $\tilde{\chi}$ of the canonical basis. Since the complement of $\mathcal{M}_m(\pi_1\Sigma, G)$ is a countable union of codimension 1 submanifolds and hence a countable union of null measure sets, we conclude to the statement of Proposition 2.24.

The proof of the ergodicity is completed using the same arguments as in the previous cases. We thus prove that all $\mathrm{Tor}(\Sigma)$-invariant and measurable functions $\mathcal{X}(\pi_1\Sigma, G) \to \mathbf{R}$ can be restricted to a full measure set on which it will be invariant under the generators of the mapping class group and then Theorem 1 allows to conclude that such a function is almost everywhere constant, that is the Torelli group action on $\mathcal{X}(\pi_1\Sigma, G)$ is ergodic. Therefore, we derived the Theorem A.

3 Modular action on relative $\mathrm{PSL}_2(\mathbf{R})$-character varieties

This section is devoted to the proof of Theorem C. We adapt the strategy and the tools which was used by Marché and Wolff 2016.

We denote by $\mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ the subspace of $\mathcal{NH}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ of classes of representations which admit a simple, closed and non-separating curve $\gamma \subset \dot{\Sigma}$ which is not homotopic to a component of $\partial\dot{\Sigma}$ and such that the isometry $\rho(\gamma)$ is elliptic with infinite order.

The next results are steps in order to prove Theorem C. The proofs of these proposition will be found in the next pages.

Proposition 3.1. *For $e \in \mathbf{R}_{>0}$, the subset $\mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ is non-empty if and only if the integer k verifies the inequality $\|\{\mathbf{r}\}\|_1 < |k| \leq -\chi(\dot{\Sigma}) - 1$.*

Proposition 3.1 use Mondello Theorem 7. Indeed, fixing a curve γ verifying the assumptions asking in the definition of $\mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ and a conjugacy class of a elliptic isometry of infinite order, we study the existence of representations of $\dot{\Sigma} \setminus \gamma$ with conditions on boundaries.

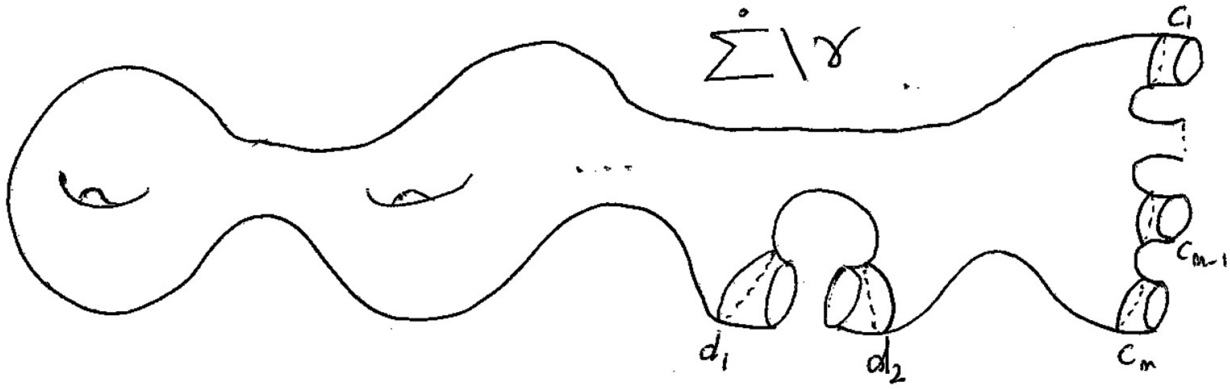

Figure 7

Following Proposition 3.1, fix definitively $k \in \left]\|\{\mathbf{r}\}\|_1, -\chi(\dot{\Sigma})\right[\cap \mathbf{Z}$ and hence $e = k - \|\{\mathbf{r}\}\|_1$.

Proposition 3.2. *The subspace $\mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ is connected and has full measure in $\mathcal{NH}^e(\dot{\Sigma}, \underline{\mathcal{C}})$.*

Recall for $\gamma \in \pi_1\dot{\Sigma}$, we denote by $f_\gamma : \mathcal{R}\mathrm{ep}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \to \mathbf{R}$ the trace function $[\rho] \mapsto \mathrm{tr}\big(\rho(\gamma)\big)$. Such a function admits a Hamiltonian vector field denoted by X_γ.

Let $\mathcal{U}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ be the open subset of classes of representations $[\rho]$ for which there is $N = 6g - 6 + 2n$ curves $\gamma_1, \ldots, \gamma_N \in \pi_1\dot{\Sigma}$ such that for all $i \in \{1, \ldots, N\}$ the isometry $\rho(\gamma_i)$ is elliptic and:

$$\langle df_{\gamma_i} \mid i = 1, \ldots, N\rangle = \mathrm{T}^*_{[\rho]}\mathcal{R}\mathrm{ep}^k(\dot{\Sigma}, \underline{\mathcal{C}})$$

Proposition 3.3. *The space $\mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ is contained in $\mathcal{U}^e(\dot{\Sigma}, \underline{\mathcal{C}})$.*

Assuming the Propositions 3.1, 3.2 and 3.3, we prove the ergodicity of the pure mapping class group:

Proposition 3.4. *Let $f : \mathcal{U}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \to \mathbf{R}$ be a measurable and* $\mathrm{Mod}(\dot{\Sigma})$*-invariant. Then each $[\rho] \in \mathcal{U}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ is contained in a neighborhood $\mathcal{V}_{[\rho]}$ such that the restriction of f to $\mathcal{V}_{[\rho]}$ is almost everywhere constant.*

The proof is essentially the same than the one of Proposition 6.5 of Marché and Wolff 2016.

Proof. Let $[\rho] \in \mathcal{U}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ and let $\gamma_1, \ldots \gamma_N$ be the curves given by the definition of $\mathcal{U}^e(\dot{\Sigma}, \underline{\mathcal{C}})$. Let us denote by ϕ_i its Hamiltonian flow of the vector field X_{γ_i}. This flow is 2π-periodic and the action of tw_{γ_i} is given by:

$$\mathrm{tw}_{\gamma_i} \cdot [\varphi] = \phi_i^{\theta_{\gamma_i}} \cdot [\varphi].$$

We deduce that for every $[\varphi]$ such that $\varphi(\gamma_i)$ elliptic, if $\theta_{\gamma_i}([\varphi]) \notin \pi\mathbf{Q}$, an invariant function $f : \mathcal{U}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \to \mathbf{R}$ satisfies the equality $f(\phi_i^\theta \cdot [\varphi]) = f([\varphi])$, for almost-every $\theta \in \mathbf{S}^1$. Let $\mathcal{V}_{[\rho]}$ be an open neighborhood such that every $[\varphi] \in \mathcal{V}_{[\rho]}$ sends the γ_i on elliptic isometries and satisfies the condition:

$$\langle X_{\gamma_1}, \ldots, X_{\gamma_N} \rangle = \mathrm{T}_{[\varphi]} \mathcal{R}\mathrm{ep}^k(\dot{\Sigma}, \underline{\mathcal{C}}).$$

Up to reducing $\mathcal{V}_{[\rho]}$, we can assume that the flows ϕ_i act transitively on $\mathcal{V}_{[\rho]}$ and such that $f_{|\mathcal{V}_{[\rho]}}$ is almost-everywhere constant on almost of their orbits. We hence deduce that $f_{|\mathcal{V}_{[\rho]}}$ is almost-everywhere constant. $\square$

The ergodicity of $\mathrm{PMod}(\dot{\Sigma})$ on $\mathcal{NH}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ comes from Proposition 3.4 and the fact that $\mathcal{U}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ has full measure in $\mathcal{NH}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ and is connected since it is open and since it contains the full measure and connected subset $\mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}})$.

3.1 Condition on e to have $\mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \neq \emptyset$

We already now from Mondello 2017 that the set $\mathcal{X}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ is non-empty if and only if $e = k - \|\{\mathbf{r}\}\|_1$ with $k \in \mathbf{Z} \cap \big]\|\{\mathbf{r}\}\|_1, -\chi(\dot{\Sigma})\big]$. Consider Σ' a connected, compact and oriented surface of genus $g-1$ and with $n+2$ boundary components, denoted by $c_1, \ldots, c_n, d_1, d_2$. Let $\mathcal{A} \subset \mathrm{PSL}_2(\mathbf{R})$ be a conjugacy class of an elliptic isometry of angle $\upsilon_{\mathcal{A}} \notin \mathbf{Z} \cup \pi\mathbf{Q}$ and hence of translation number $\frac{\upsilon_{\mathcal{A}}}{\pi}$. Denote by $\underline{\mathcal{C}}' = (\mathcal{C}_1, \ldots, \mathcal{C}_n, \mathcal{A}, \mathcal{A}^{-1})$ and by $\mathcal{R}\mathrm{ep}(\Sigma', \underline{\mathcal{C}}')$ the relative character variety associated to Σ' and $\underline{\mathcal{C}}'$. Since $|\frac{\upsilon_{\mathcal{A}}}{\pi} - \lfloor\frac{\upsilon_{\mathcal{A}}}{\pi}\rfloor| + |-\frac{\upsilon_{\mathcal{A}}}{\pi} - \lfloor-\frac{\upsilon_{\mathcal{A}}}{\pi}\rfloor| = 1$ and $\chi(\dot{\Sigma}) = \chi(\Sigma')$, Theorem 7 shows that for $e \in \mathbf{R}_{>0}$, the pre-image

$$\mathcal{R}\mathrm{ep}^e(\Sigma', \underline{\mathcal{C}}')$$

is non-empty if and only if $e = k - 1 - \|\{\mathbf{r}\}\|_1$ with $k - 1 \in \mathbf{Z} \cap \big]0, -\chi(\dot{\Sigma})\big]$. In other words if and only if $e = k - \|\{\mathbf{r}\}\|_1$ with $k \in \mathbf{Z} \cap \big[0, -\chi(\dot{\Sigma})\big[$. The assumption $e > 0$ imposes $k > \|\{\mathbf{r}\}\|_1$. Under this condition, $\mathcal{R}\mathrm{ep}^e(\Sigma', \underline{\mathcal{C}}')$ is connected and smooth.

The fundamental group $\pi_1\dot{\Sigma}$ is the HNN-extension

$$[\pi_1\Sigma' * \langle b \rangle] / \langle b d_1 b^{-1} = d_2 \rangle.$$

Let $[\rho'] \in \mathcal{R}\mathrm{ep}^e(\Sigma', \underline{\mathcal{C}}')$. Since $\rho'(d_1) \in \mathcal{A}$ and $\rho'(d_2) \in \mathcal{A}^{-1}$, there exists $B \in \mathrm{PSL}_2(\mathbf{R})$ which satisfies the relation

$$B\rho'(d_1)B^{-1} = \rho'(d_2)^{-1}.$$

The data of such a ρ' and such a B gives a representation $\rho : \pi_1\dot{\Sigma} \to \mathrm{PSL}_2(\mathbf{R})$ which send the curve obtained by gluing d_1 and d_2^{-1} into the conjugacy class $\mathcal{A}$. We thus have that for $e > 0$:

$$\mathcal{E}\mathcal{I}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \neq \emptyset \iff e = k - \|\{\mathbf{r}\}\|_1 \,\text{with}\, \|\{\mathbf{r}\}\|_1 < k \leq -\chi(\dot{\Sigma}) - 1$$

which is the sought for condition in Proposition 3.1.

3.2 Arc-connectedness of relative character varieties

Let us consider some arc-connectedness properties for relative character varieties.

Proposition 3.5. *Let Σ' be a compact, connected and oriented surface of genus $g-1$ and with $n+1$ boundary components denoted by $c_1, \ldots, c_n, c$. Let ℓ be a real number such that $\mathcal{R}\mathrm{ep}^\ell(\Sigma')$ and the next boundary conditions are non-empty. Let $A : \left[0,1\right] \to \mathrm{PSL}_2(\mathbf{R})$ be a continuous path whose image is contained in the subset of hyperbolic isometries and let $\rho : \pi_1\Sigma' \to \mathrm{PSL}_2(\mathbf{R})$ be a representation of Euler class ℓ such that $\rho(c_i) \in \mathcal{C}_i$ for all $i = 1, \ldots, n$ and $\rho(c) = A(0)$. Then there exists a continuous path $\widetilde{A} : \left[0,1\right] \to \mathrm{Hom}(\Sigma', \mathrm{PSL}_2(\mathbf{R}))$ such that $\widetilde{A}(0) = \rho$ and for all $t \in \left[0,1\right]$, $\mathrm{eu}(\widetilde{A}(t)) = \ell$ and*

$$\widetilde{A}_t(c) = A(t),\ \widetilde{A}_t(c_1) \in \mathcal{C}_1, \ldots,\ \widetilde{A}_t(c_n) \in \mathcal{C}_n.$$

We use a pairs of pants decomposition of Σ'. The proposition 3.5 will hence follow from the three next lemmas.

The first one was proved by Goldman in W. M. Goldman 1988:

Lemma 3.6. *Let Σ_0 be a one-holded torus. Let $k = -1, 0$ or 1, $A : \left[0,1\right] \to \mathrm{PSL}_2(\mathbf{R})$ be a continuous path whose image is contained in the subset of hyperbolic isometries and fix $\rho : \pi_1\Sigma_0 \to \mathrm{PSL}_2(\mathbf{R})$ of Euler class k and such that $\rho(\partial\Sigma_0) = A(0)$. Then there exists a continuous path of representations $\widetilde{A} : \left[0,1\right] \to \mathrm{Hom}(\pi_1\Sigma_0, \mathrm{PSL}_2(\mathbf{R}))$ such that $\widetilde{A}(0) = \rho$ and for all $t \in \left[0,1\right]$, $\mathrm{eu}(\widetilde{A}(t)) = k$ and*

$$\widetilde{A}_t(\partial\Sigma_0) = A(t).$$

Such an extension by representations is true for pair of pants and for $n+1$-punctured sphere with elliptic conditions on some boundaries. It is the aims of next lemmas:

Lemma 3.7. *Let $\mathcal{P}$ be a pair of pants with as boundary components the curves d, d_1, d_2. Let $A : \left[0,1\right] \to \mathrm{PSL}_2(\mathbf{R})$ be a continuous path whose image is contained in the subset of hyperbolic isometries. Then there exists a continuous path of representations $\widetilde{A} : \left[0,1\right] \to \mathrm{Hom}(\pi_1\mathcal{P}, \mathrm{PSL}_2(\mathbf{R}))$ such that for all $t \in \left[0,1\right]$,*

$$\widetilde{A}_t(d) = A(t)\, and\, for\, all\, i \in \{1,2\},\ \widetilde{A}_t(d_i)\, hyperbolic.$$

Proof. We can easily find for all $t \in [0,1]$, an isometry $D_1(t) \in \mathrm{PSL}_2(\mathbf{R})$ such that $A(t)D_1(t)$ is hyperbolic and such that the map $t \mapsto D_1(t)$ is continuous. We hence choose $D_2(t)$ to be the isometry $D_1(t)^{-1}A(t)^{-1}$. Hence we define the expected path $\widetilde{A} : [0,1] \to \mathrm{Hom}(\pi_1\mathcal{P}, \mathrm{PSL}_2(\mathbf{R}))$ by the data

$$\widetilde{A}(t)(c) = A(t), \widetilde{A}(t)(d_1) = D_1(t) \text{ and } \widetilde{A}(t)(d_2) = D_2(t).$$

$\square$

As it is state previously, fixing a representation ρ of the pair of pant $\mathcal{P}$ with Euler class ± 1 and with $\rho(d) = A(0)$ corresponds to fix D_1, D_2 two hyperbolic isometries such that $A(0)D_1D_2 = \mathrm{id}$. Hence we the same reasoning than the proof of Lemma 3.7, fixing $D_1(0) = D_1$, we can extend the path A in a path of representations $\widetilde{A}$ as expected in the lemma with the condition $\widetilde{A}(0) = \rho$.

Lemma 3.8. *Let $\mathcal{S}_{n+1}$ be a $n+1$-sphere with as boundary components the curves $s, c_1, \ldots, c_n$. Let τ be a real number such that $\mathcal{R}\mathrm{ep}^\tau(\mathcal{S}_{n+1})$ and the next boundary conditions are non-empty. Let $A : [0,1] \to \mathrm{PSL}_2(\mathbf{R})$ be a continuous path whose image is contained in the subset of hyperbolic isometries and fix $\rho : \pi_1\mathcal{S}_{n+1} \to \mathrm{PSL}_2(\mathbf{R})$ such that $\mathrm{eu}(\rho) = \tau$ and $A(0) = \rho(s)$. Then there exists a continuous path of representations $\widetilde{A} : [0,1] \to \mathrm{Hom}(\pi_1\mathcal{S}_{n+1}, \mathrm{PSL}_2(\mathbf{R}))$ such that $\widetilde{A}(0) = \rho$ and for all $t \in [0,1]$, $\mathrm{eu}(\rho) = \tau$*

$$\widetilde{A}_t(s) = A(t),\ \widetilde{A}_t(c_1) \in \mathcal{C}_1, \ldots, \widetilde{A}_t(c_n) \in \mathcal{C}_n.$$

Fix ρ with the hypothesis of the Lemma 3.8 is equivalent to fix $C_i \in \mathcal{C}_i$ for all $i \in \{1, \ldots, n\}$ with $A(0)C_1 \ldots C_n = \mathrm{id}$. We hence fix these isometries. Up to restrict our reasoning to a Zariski-open subset of the representation space, we can assume that the commutator $[C_\ell, C_i] \neq \mathrm{id}$ for all $\ell, i \in \{1, \ldots, n\}$ distinct.

Proof. We are looking for continuous paths $g_i : [0,1] \to \mathrm{PSL}_2(\mathbf{R})$ such that for all $t \in [0,1]$, we have

$$A(t) \times g_1(t)C_1g_1(t)^{-1} \times \cdots \times g_n(t)C_ng_n(t)^{-1} = \mathrm{id}.$$

Let $K : \mathrm{PSL}_2(\mathbf{R})^n \to \mathrm{PSL}_2(\mathbf{R})$ be the map defined by

$$K(g_1, \ldots, g_n) = g_1C_1g_1^{-1} \times \cdots \times g_nC_ng_n^{-1}.$$

We will prove the claim.

Claim 3.9. *The map K is a submersion at the point $(g_1, \ldots, g_n)$ such that each g_ℓ does not send the fix point of C_ℓ on the fix points of the other C_i.*

As in W. M. Goldman 1984, for $j \in \{1, \ldots, g-1\}$, let Π_j the product

$$\prod_{k=1}^{j} g_kC_kg_k^{-1}$$

and by convention we let $\Pi_0 = \mathrm{id}$. By Fox calculus, we compute that for all $\ell = 1, \ldots, n$:

$$\frac{\partial}{\partial g_\ell} K(g_1, \ldots, g_n) = \Pi_{\ell-1}(\mathrm{id} - g_\ell C_\ell g_\ell^{-1})$$

We refer to Fox 1953 and W. M. Goldman 1984, for basics on Fox calculus. Since:

$$d_{(g_1,\ldots,g_n)} K = \sum_{\ell=1}^{n} \mathrm{Ad}\big(\frac{\partial}{\partial g_\ell} K(g_1, \ldots, g_n)\big) d_{g_\ell},$$

we have that the orthogonal space $d_{(g_1,\ldots,g_n)} K\big(\mathfrak{sl}_2(\mathbf{R})^n\big)^\perp$ is the intersection:

$$\bigcap_{\ell=1}^{n} \mathrm{Ker}\bigg(\mathrm{id} - \mathrm{Ad}\big(\Pi_{\ell-1} g_\ell C_\ell g_\ell^{-1} \Pi_{\ell-1}^{-1}\big)\bigg).$$

Each kernel of the intersection is the centralizer in $\mathfrak{sl}_2(\mathbf{R})$ of the element $\Pi_{\ell-1} g_\ell C_\ell g_\ell^{-1} \Pi_{\ell-1}^{-1}$. Up to change the representative in the conjugacy class of $\mathcal{C}_1$, we can assume that $g_1 = \mathrm{id}$. Hence we deduce that $d_{(g_1,\ldots,g_n)} K\big(\mathfrak{sl}_2(\mathbf{R})^n\big)^\perp$ is the intersection

$$\bigcap_{\ell=1}^{n} \mathrm{Z}_{\mathfrak{sl}_2(\mathbf{R})}\big(\Pi_{\ell-1} g_\ell C_\ell g_\ell^{-1} \Pi_{\ell-1}^{-1}\big).$$

Finally, since $\{g_1 C_1 g_1^{-1}, \ldots, \Pi_{n-1} g_n C_n g_n^{-1} \Pi_{n-1}^{-1}\}$ generates the same subgroup than

$$\{g_1 C_1 g_1^{-1}, \ldots, g_n C_n g_n^{-1}\},$$

we conclude that if for all ℓ, g_ℓ does not send the fix point of C_ℓ on the fix point of a C_i, the orthogonal space $d_{(g_1,\ldots,g_n)} K\big(\mathfrak{sl}_2(\mathbf{R})^n\big)^\perp$ is trivial. Then the rank of K at the point $(g_1, \ldots, g_n)$ for which each g_ℓ does not send the fix point of C_ℓ on the fix point of another C_i, is maximal. This proves the claim. In particular K is a submersion at the point $(\mathrm{id}, \ldots, \mathrm{id})$.

Then for each $t \in \big[0, 1\big]$, the preimage $K^{-1}(A(t)^{-1})$ is a codimension 1 submanifold of $\mathrm{PSL}_2(\mathbf{R})^n$. There exists then a finite covering

$$\big[0, 1\big] = \cup_{j=0,\ldots,\kappa-1}\big[t_j, t_{j+1}\big]$$

with $t_0 = 0$ and $t_\kappa = 1$ such that on each $\big[t_j, t_{j+1}\big]$, by Claim 3.9, there is paths $g_\ell^j : \big[t_j, t_{j+1}\big] \to \mathrm{PSL}_2(\mathbf{R})$, for $\ell = 1, \ldots, n$ such that for all $t \in \big[t_j, t_{j+1}\big]$, the relation

$$A(t) \times g_1^j(t) C_1 g_1^j(t)^{-1} \times \cdots \times g_n^j(t) C_n g_n^j(t)^{-1} = \mathrm{id}$$

is verified. In addition we have that for all $\ell \in \{1, \ldots, n\}$, $g_\ell^0(0) = \mathrm{id}$.

At each t_j, since K_{t_j} is a submersion, there is a path which connect $g_\ell^{j-1}(t_j-\varepsilon)$ and $g_\ell^j(t_j+\varepsilon)$. So up to a reparametrization, there exists continuous paths $g_\ell : [0,1] \to \mathrm{PSL}_2(\mathbf{R})$ for $\ell = 1,\dots,n$ verifying the expected condition.

We hence define $\widetilde{A}_t$ as the representation $\pi_1\mathcal{S}_{n+1} \to \mathrm{PSL}_2(\mathbf{R})$ defined by $\widetilde{A}_t(s) = A(t)$, $\widetilde{A}_t(c_1) = g_1(t)C_1g_1(t)^{-1}, \dots, \widetilde{A}_t(c_n) = g_n(t)C_ng_n(t)^{-1}$. $\square$

To conclude Proposition 3.5, we decompose Σ' by pair of pants $\mathcal{P}_i$ with $i = 1, ..., p$ and $c \subset \partial\mathcal{P}_1$, g one-holded tori and a $n+1$-sphere. By Lemma 3.7 we can find a path of representations of the $g+1$-sphere by restricting ρ to each $\mathcal{P}_i$ and gluing the extension we obtain. We extend this path extend to one-holded tori glued to $g-1$ boundary components by Lemma 3.6 and to the $n+1$-sphere by Lemma 3.8. This gives a paths of representations of Σ' which extend the path A and with initial condition ρ.

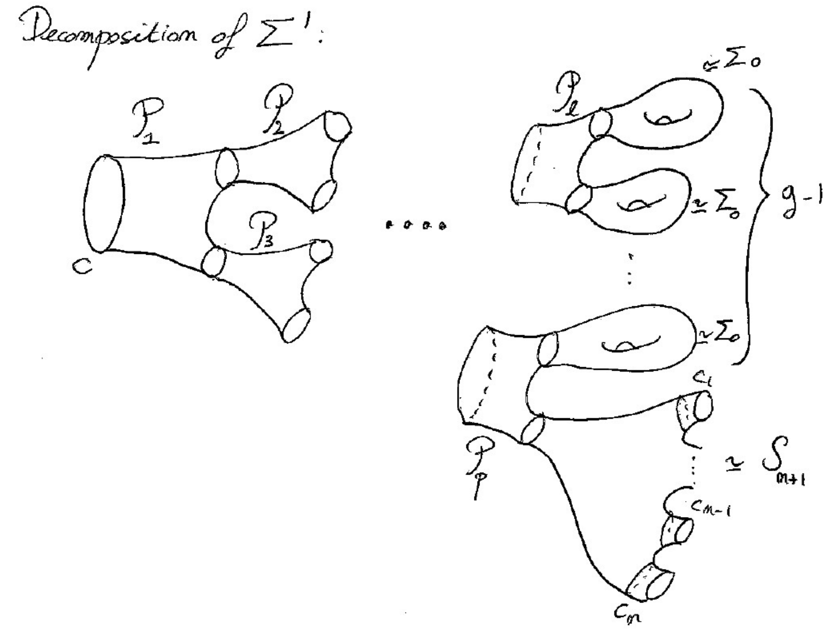

Figure 8

3.3 Connectedness of $\mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}})$

We will follow the same strategy than Marché and Wolff 2016, section 6. We denote by $\mathcal{S}_1$ the set of pairs of simple, closed and non-separating curves (a,b) with geometric intersection number $i(a,b) = 1$. As in Marché and Wolff 2016, we introduce, for $(a,b) \in \mathcal{S}_1$, the set $\mathcal{EI}^e_{(a,b)}(\dot{\Sigma}, \underline{\mathcal{C}})$ define as the set:

$$\big\{[\rho] \in \mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \mid \rho(a), \rho(b) \,\text{elliptic}, \theta_a(\rho) \,\text{or}\, \theta_b(\rho) \notin \pi\mathbf{Q}, [\rho(a), \rho(b)] \neq \mathrm{id}\big\}.$$

In particular, if $[\rho] \in \mathcal{EI}^e_{(a,b)}(\dot{\Sigma}, \underline{\mathcal{C}})$, then $[\rho(a), \rho(b)]$ is an hyperbolic isometry.

Lemma 3.10. *(Lemma 6.8, Marché and Wolff 2016) Let $A \in \mathrm{PSL}_2(\mathbf{R})$ an elliptic isometry of infinite order and $B \in \mathrm{PSL}_2(\mathbf{R})$. Then there exists an integer $n \in \mathbf{Z}$ such that A^nB is an elliptic isometry.*

We refer to Marché and Wolff 2016 for a proof of Lemma 3.10.

Lemma 3.11. *We have the equality* $\mathcal{EI}^e(\dot\Sigma,\underline{\mathcal{C}}) = \bigcup_{(a,b)\in\mathcal{S}_1}\mathcal{EI}^e_{(a,b)}(\dot\Sigma,\underline{\mathcal{C}})$

Proof. Let $[\rho]\in\mathcal{EI}^e(\dot\Sigma,\underline{\mathcal{C}})$. There is a simple, closed and non-separating curve a such that $\rho(a)$ is an elliptic isometry of infinite order. Since $[\rho]$ is Zariski-dense, we can find a non-separating curve b such that $i(a,b,)=1$ and $[\rho(a),\rho(b)]\neq \mathrm{id}$. Lemma 3.10 implies that there is an integer $n\in\mathbf{Z}$ such that $\rho(a^nb)$ is elliptic. Moreover we compute $[\rho(a),\rho(a^nb)] = \rho(a^n)[\rho(a),\rho(b)]\rho(a^n)^{-1}\neq \mathrm{id}$. Then $[\rho]\in\mathcal{EI}^e_{(a,a^nb)}(\dot\Sigma,\underline{\mathcal{C}})$. The other inclusion is trivial. □

Lemma 3.12. *For all* $(a,b)\in\mathcal{S}_1$*, the set* $\mathcal{EI}^e_{(a,b)}(\dot\Sigma,\underline{\mathcal{C}})$ *is connected.*

Proof. Let $(a,b)\in\mathcal{S}_1$ and let Σ_0 be a one-holded torus containing both a and b and let Σ' be the surface of genus $g-1$ and with $n+1$ boundary components $c,c_1,\dots,c_n$ such that the surface $\dot\Sigma$ is the connected sum

$$\Sigma_0\sharp\Sigma'.$$

Let $\mathcal{M}_0$ be the set of classes of representations $\pi_1\Sigma_0\to\mathrm{PSL}_2(\mathbf{R})$ sending a and b both on elliptic isometries which do not commute. We will denote by $x_{\rho(\sigma)}$ the fix point of $\rho(\sigma)$ for $\sigma\in\{a,b\}$. The map $\mathcal{M}_0\to(0,2\pi)\times(0,2\pi)\times(0,+\infty)$ defined by:

$$[\rho]\mapsto\big(\theta_a(\rho),\theta_b(\rho),\mathrm{d}_{\mathbf{H}^2}(x_{\rho(a)},x_{\rho(b)})\big)$$

is a homeomorphism. Let $\mathcal{M}_0'$ be the subset of $\mathcal{M}_0$ of classes of representations which send a or b on an elliptic isometry of infinite order. Since in $(0,2\pi)\times(0,2\pi)$, the pairs (x,y) such that $x\notin\pi\mathbf{Q}$ or $y\notin\pi\mathbf{Q}$ form a connected space, the subspace $\mathcal{M}_0'$ is connected. We now prove the connectedness of $\mathcal{EI}^e_{(a,b)}(\dot\Sigma,\underline{\mathcal{C}})$. Let $[\rho],[\rho']\in\mathcal{EI}^e_{(a,b)}(\dot\Sigma,\underline{\mathcal{C}})$ and let ρ_0 and ρ_0' be the restrictions to $\pi_1\Sigma_0$ of ρ and ρ'. The classes $[\rho_0]$ and $[\rho_0']$ lie in $\mathcal{M}_0'$. Then, by connectedness of $\mathcal{M}_0'$, there exists a continuous path $A_0:\big[0,1\big]\to\mathcal{M}_0'$ such that $A_0(0)=[\rho_0]$ and $A_0(1)=[\rho_0']$. this path induces a continuous path $A:\big[0,1\big]\to\mathrm{PSL}_2(\mathbf{R})$, defined by $A(t)=A_0(t)([a,b])$, whose image is included in the set of hyperbolic isometries. Proposition 3.5 allows to conclude to the existence of a continuous path $\widetilde{A}$ in $\mathcal{EI}^e_{(a,b)}(\dot\Sigma,\underline{\mathcal{C}})$ such that $\widetilde{A}(0)=[\rho]$ and $\widetilde{A}(1)([a,b])=\rho'([a,b])$. In other words, $\widetilde{A}(1)_{|\pi_1\Sigma'}$ and $\rho'_{|\pi_1\Sigma'}$ lie in the same relative character variety and have the same Euler class. By Mondello Theorem 7, there exists a continuous path joining $\widetilde{A}(1)_{|\pi_1\Sigma'}$ and $[\rho'_{|\pi_1\Sigma'}]$and thus there is a path joining $\widetilde{A}(1)$ and $[\rho]$. We then found a path joining $[\rho]$ and $[\rho']$ in $\mathcal{EI}^e_{(a,b)}(\dot\Sigma,\underline{\mathcal{C}})$, which conclude the connectedness of this space. □

We introduce now the equivalence relation on $\mathcal{S}_1$ generated by the relation

$$(a,b)\sim(a',b')\iff\mathcal{EI}^e_{(a,b)}(\dot\Sigma,\underline{\mathcal{C}})\cap\mathcal{EI}^e_{(a',b')}(\dot\Sigma,\underline{\mathcal{C}})\neq\emptyset.$$

Lemma 3.13. *Let* $(a,b)\in\mathcal{S}_1$*. For all* $n\in\mathbf{Z}$*,* $(a,b)\sim(a,a^nb)$*.*

Proof. Let Σ_0 and Σ' as in the previous proof. We will construct a class of representations in $\mathcal{EI}^e_{(a,b)}(\dot{\Sigma},\underline{\mathcal{C}}) \cap \mathcal{EI}^e_{(a,a^nb)}(\dot{\Sigma},\underline{\mathcal{C}})$ starting by construct it on Σ_0 and extend it by Proposition 3.5 to Σ' (and thus to $\dot{\Sigma}$). Let A and B be the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1-\varepsilon & \varepsilon \end{pmatrix}$ for $\varepsilon \in \big[0,1\big]$. We then compute that A^nB is elliptic. Up to change ε, we can assume that $[A,B]$ is hyperbolic. We construct $\rho_0 : \langle a,b\rangle \to \mathrm{PSL}_2(\mathbf{R})$ defined by $\rho_0(a) = A$ and $\rho_0(b) = B$. Apply Proposition 3.5 to the constant path $t \mapsto [A,B]$ allows to construct a class of representations $[\rho]$ such that $\rho_{|\pi_1\Sigma_0} = \rho_0$ and then

$$[\rho] \in \mathcal{EI}^e_{(a,b)}(\dot{\Sigma},\underline{\mathcal{C}}) \cap \mathcal{EI}^e_{(a,a^nb)}(\dot{\Sigma},\underline{\mathcal{C}}).$$

□

The two following lemmas come from Marché and Wolff 2016 (Lemmas 6.10 and 6.11).

Lemma 3.14. *(Marché and Wolff 2016, Lemma* 6.10*) Let $(a,b) \in \mathcal{S}_1$ and c be a simple, closed and non-separating which is disjoint of b such that $(a,c) \in \mathcal{S}_1$. Then $(a,b) \sim (a,c)$.*

Proof. Let $[\rho] \in \mathcal{EI}^e_{(a,b)}(\dot{\Sigma},\underline{\mathcal{C}})$ and assume that $\rho(a)$ has infinite order. By lemma 3.10, there exists an integer $n \in \mathbf{Z}$ such that $\rho(a^nc)$ is elliptic. We necessarily have that $\mathrm{tw}_a^n \cdot \rho\,(a) = \rho(a)$ is an elliptic isometry of infinite order and the isometries $\mathrm{tw}_a^n \cdot \rho\,(a^{-n}b) = \rho(b)$ and $\mathrm{tw}_a^n \cdot \rho\,(c)$ are elliptic isometry. We hence have that $\mathrm{tw}_a^n \cdot [\rho] \in \mathcal{EI}^e_{(a,a^{-n}b)}(\dot{\Sigma},\underline{\mathcal{C}}) \cap \mathcal{EI}^e_{(a,c)}(\dot{\Sigma},\underline{\mathcal{C}})$ and so $(a,a^{-n}b) \sim (a,c)$. By Lemma 3.13 we conclude that $(a,b) \sim (a,c)$. □

Lemma 3.15. *(Marché and Wolff 2016,Lemma* 6.11*) Let b be a simple, closed and non-separating curve and let a and a' be two curves such that (a,b) and (a',b) are both in $\mathcal{S}_1$. Then $(a,b) \sim (a',b)$.*

Proof. If a' lies in the one-holded torus obtained by thickening $a \cup b$, we apply Lemma 3.14 to concude. If not, we process by induction on $i(a,a')$:

- If $i(a,a') = 0$, we apply Lemma 3.14.
- If $i(a,a') = n > 0$, starting from the lower intersection point of a' and b, we follow a' until the upper line of a' and go directly to hit b. This constructs a new curve a'' which do not intersect a'. Then $(a'',b) \sim (a',b)$ and by induction hypothesis, $(a,b) \sim (a'',b)$.

□

The conclusion of the connectedness of $\mathcal{EI}^e(\dot{\Sigma},\underline{\mathcal{C}})$ of Proposition 2.5 use the connectedness of 1-skeleton of curves complex. Let $(a,b),(a',b') \in \mathcal{S}_1$. There exists non-separating curves $b_0 = b,\dots,b_n = b'$ such that for $i \in \{0,\dots,n\}$, the curves b_i and b_{i+1} are disjoint. In addition, we can find non-separating curves $a_0 = a,\dots,a_n = a'$ such that (a_i,b_i) and (a_i,b_{i+1}) are in $\mathcal{S}_1$. Lemma 3.14 assures that $(a_i,b_i) \sim (a_i,b_{i+1})$ and hence Lemma 3.15 that $(a_i,b_i) \sim (a_{i+1},b_{i+1})$ for all $i \in \{0,\dots,n-1\}$. By Lemmas 3.11 and 3.12 we conclude that $\mathcal{EI}^e(\dot{\Sigma},\underline{\mathcal{C}})$ is connected.

3.4 $\mathcal{EI}^e(\dot\Sigma,\underline{\mathcal{C}}) \subset \mathcal{U}^e(\dot\Sigma,\underline{\mathcal{C}})$

Let $[\rho] \in \mathcal{EI}^e(\dot\Sigma,\underline{\mathcal{C}})$ and take a non-separating curve γ such that $\rho(\gamma)$ is an elliptic isometry of infinite order and of angle θ. We will denote by $\mathcal{F}_{[\rho]}$ the vector subspace of $\mathrm{T}^*_{[\rho]}\mathcal{R}\mathrm{ep}(\dot\Sigma,\underline{\mathcal{C}})$ spanned by the linear forms df_σ for σ all the non-separating curves which are send on a elliptic isometry by ρ. The following lemmas will use the well-known relation which holds for all $A, B \in \mathrm{SL}_2(\mathbf{R})$:

$$\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\mathrm{tr}(B).$$

These lemmas and their proofs are done in Marché and Wolff 2016 and we will reproduce the proofs for the reader's convenience.

Lemma 3.16. *Let δ be a non-separating curve such that $i(\gamma,\delta) = 1$. Then $df_\delta \in \mathcal{F}_{[\rho]}$.*

Proof. We will denote by δ_n the curve $\gamma^n\delta = \mathrm{tw}^n_\gamma(\delta)$. The trace relation gives directly that for all $n \in \mathbf{Z}$, the equality

$$f_{\delta_{n+1}} + f_{\delta_{n-1}} = f_\gamma f_{\delta_n}$$

holds. We deduce from this equality the relation

$$df_{\delta_{n+1}} + df_{\delta_{n-1}} = f_\gamma df_{\delta_n} + f_{\delta_{n+1}} df_\gamma = f_\gamma df_{\delta_n} \operatorname{mod} \mathcal{F}_{[\rho]}.$$

Then, modulo $\mathcal{F}_{[\rho]}$, the sequence of linear forms $\left(df_{\delta_n}\right)_n$ satisfies the order two recursive equation $u_{n+1}+u_{n-1} = 2\cos(\theta)u_n$ and there hence exists linear forms Λ, Θ in the cotangent space $\mathrm{T}^*_{[\rho]}\mathcal{R}\mathrm{ep}(\dot\Sigma,\underline{\mathcal{C}})$ such that for all $n \geq 0$:

$$df_{\delta_n} = \cos(n\theta)\Lambda + \sin(n\theta)\Theta \operatorname{mod} \mathcal{F}_{[\rho]}.$$

By Lemma 3.10, there exists infinitly many integers $n \in \mathbf{Z}$ such that $\rho(\delta_n)$ is elliptic. For these n, the linear form df_{δ_n} is null up to $\mathcal{F}_{[\rho]}$. We hence deduce, since $\theta \notin \pi\mathbf{Q}$, that Λ and Θ are both the trivial linear form modulo $\mathcal{F}_{[\rho]}$. Then, for all integer $n \in \mathbf{Z}$, $df_{\delta_n} \in \mathcal{F}_{[\rho]}$. In particular for $n = 0$. ☐

Lemma 3.17. *Let δ be a curve such that $i(\gamma,\delta) = 0$. Then $df_\delta \in \mathcal{F}_{[\rho]}$.*

Proof. If δ is homotopic to γ, that is clear that $df_\delta \in \mathcal{F}_{[\rho]}$. If not, assume first that δ is non-separating. Then there is a curve σ, such that $i(\gamma,\sigma) = i(\delta,\sigma) = 1$. We have the formula

$$f_{\delta\sigma} + f_{\delta\sigma^{-1}} = f_\delta f_\sigma$$

whose we deduce

$$df_{\delta\sigma} + df_{\delta\sigma^{-1}} = f_\delta df_\sigma + f_\sigma df_\delta.$$

Moreover, Lemma 3.16 ensures that $df_{\delta\sigma}, df_{\delta\sigma^{-1}},$ and $df_\sigma \in \mathcal{F}_{[\rho]}$. We hence conclude that $df_\delta \in \mathcal{F}_{[\rho]}$.

If we now assume that δ is separating, fix a base point on δ. then there is a curve σ such that $i(\gamma,\sigma)=1$ and $i(\delta,\sigma)=2$. We can decompose σ as a product $\sigma_1\sigma_2$ with $i(\gamma,\sigma_1)=1$. We hence have that

$$f_\sigma f_\delta = f_{\sigma_1\sigma_2} f_\delta = f_{\sigma_1} f_{\sigma_2\delta} - f_{\sigma_1\delta^{-1}\sigma_2^{-1}} + f_{\sigma_2} f_{\sigma_1\delta^{-1}} - f_{\sigma_1\sigma_2^{-1}\delta^{-1}}.$$

Since the curves σ_2 and $\sigma_2\delta$ are non-separating and do not intersect γ, by the non-separating case, we have that $df_{\sigma_2}, df_{\sigma_2\delta} \in \mathcal{F}_{[\rho]}$. Since the curves σ_1, $\sigma_1\delta^{-1}\sigma_2^{-1}$, $\sigma_1\delta^{-1}$, $\sigma_1\sigma_2^{-1}\delta^{-1}$ and $\sigma_1\sigma_2$ are non-separating and intersect γ once, Lemma 3.16 ensures that the differential of traces corresponding to these curves are in $\mathcal{F}_{[\rho]}$. We hence prove, by derivation, that $df_\delta \in \mathcal{F}_{[\rho]}$. □

Lemma 3.18. *There exists a curve δ such that $i(\gamma,\delta)=1$ and such that the Poisson bracket $\{f_\gamma, f_\delta\} := df_\gamma X_\delta \neq 0$. In particular, the map f_γ is a submersion at each class of representation of $\mathcal{R}\mathrm{ep}^e(\dot\Sigma,\underline{\mathcal{C}})$ which send γ on elliptic isometry of infinite order and the subset*

$$f_\gamma^{-1}\big(2\cos\theta\big)(\underline{\mathcal{C}})^e := \mathcal{R}\mathrm{ep}^e(\dot\Sigma,\underline{\mathcal{C}}) \cap f_\gamma^{-1}\big(2\cos\theta\big)$$

is a codimension 1-submanifold of $\mathcal{R}\mathrm{ep}^e(\dot\Sigma,\underline{\mathcal{C}})$.

Proof. Let $\rho(\gamma)=A(\theta)$ and $\rho(\delta)=B$. Then for all $n\in\mathbf{Z}$, the Poisson bracket $\{f_\gamma, f_{\delta_n}\}=0$ if and only if $\mathrm{tr}\big(A'(0)A(n\theta)B\big)=0$. This is impossible since $\theta\notin\pi\mathbf{Q}$. □

We will denote by $\mathcal{A}$ the conjugacy class corresponding to the matrices of trace $2\cos\theta$ and by $\underline{\mathcal{C}}' = (\mathcal{C}_1,\dots,\mathcal{C}_n,\mathcal{A},\mathcal{A}^{-1})$.

We show now that $\mathcal{F}_{[\rho]}$ coincides with the space $\mathrm{T}^*_{[\rho]}\mathcal{R}\mathrm{ep}^e(\dot\Sigma,\underline{\mathcal{C}})$. Let $X\in\mathrm{T}_{[\rho]}\mathcal{R}\mathrm{ep}^e(\dot\Sigma,\underline{\mathcal{C}})$ be a vector orthogonal to $\mathcal{F}_{[\rho]}$. We have $df_\gamma X=0$ whose we deduce that X is tangent to $f_\gamma^{-1}\big(2\cos\theta\big)(\underline{\mathcal{C}})^e$. Let $r: f_\gamma^{-1}\big(2\cos\theta\big)(\underline{\mathcal{C}})^e \to \mathcal{R}\mathrm{ep}^e(\dot\Sigma\setminus\gamma,\underline{\mathcal{C}}')$ be the restriction map. It is well defined since the image by r of a Zariski-dense representation is Zariski-dense. Since the cotangent space of $\mathcal{R}\mathrm{ep}^e(\dot\Sigma\setminus\gamma,\underline{\mathcal{C}}')$ is generated by all the linear form df_σ with σ disjoint from γ and $\partial\dot\Sigma$, Lemma 3.17 implies that $drX=0$ whose we deduce that $X\in\langle X_\gamma\rangle$. There exists $u\in\mathbf{R}$ such that $X=uX_\gamma$. Let δ be a curve given by Lemma 3.18. Lemma 3.16 show that $df_\delta\in\mathcal{F}_{[\rho]}$ and hence $df_\delta X=0$. But $df_\delta X = u\times df_\delta X_\gamma = -u\times\{f_\gamma,f_\delta\}$. Since $\{f_\gamma,f_\delta\}\neq 0$, we have $u=0$. This proves Proposition 3.3.

3.5 $\mathcal{EI}^e(\dot\Sigma,\underline{\mathcal{C}})$ has full measure in $\mathcal{NH}^e(\dot\Sigma,\underline{\mathcal{C}})$

The proof of Marché and Wolff 2016 holds in our case and we write these to explain how to adapt the proof to boundary case. Let $\mathcal{N}^e(\dot\Sigma,\underline{\mathcal{C}})$ be the subset of $\mathcal{X}^e(\dot\Sigma,\underline{\mathcal{C}})$ of classes of representations which send simple and closed curves, not homotopic to a boundary component, either on elliptic isometries of infinite order or hyperbolic isometries. It is clearly a full measure subset because its complementary subset is the union:

$$\bigcup_\gamma \bigcup_{t\in 2\cos(\pi\mathbf{Q})} f_\gamma^{-1}(t)$$

which is a countable union of subset of measure zero.

We will then show that $\mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \cap \mathcal{N}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ has full measure in $\mathcal{NH}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \cap \mathcal{N}^e(\dot{\Sigma}, \underline{\mathcal{C}})$. If $[\rho] \in \mathcal{NH}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \cap \mathcal{N}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ then there exists a simple and closed curve $\gamma \notin \dot{\Sigma} \setminus \partial\dot{\Sigma}$ such that $\rho(\gamma)$ is elliptic of infinite order. If γ is freely homotopic to a non-separating curve, then $[\rho] \in \mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \cap \mathcal{N}^e(\dot{\Sigma}, \underline{\mathcal{C}})$. It is so sufficient to treat the case of γ is freely homotopic to a separating curve.

Lemma 3.19. *Let Σ' be a compact, connected and oriented surface of genus $g \geq 1$ and with $k \in \{1, \dots, n+1\}$ boundary components denoted by $c, c_1, \dots, c_{k-1}$. Let $\rho : \pi_1\Sigma' \to \mathrm{PSL}_2(\mathbf{R})$ such that:*

- *The condition $\rho(c_i) \in \mathcal{C}_i$ holds.*
- *No simple and closed curve other than $c, c_1, \dots, c_{k-1}$ is send by ρ to an elliptic isometry of finite order, or on a parabolic isometry or on the identity.*
- *$\rho(c)$ is an elliptic isometry of infinite order of fixed point $x_0 \in \mathbf{H}^2$.*
- *Every non-separating curve of $\Sigma' \setminus \partial\Sigma'$ is sent on an hyperbolic isometry.*

Then there exists a real number $D > 0$ and a sequence of non-separating and simple loops $(\gamma_n)_n$ such that $\rho(\gamma_n)$ is hyperbolic with displacement $\lambda\big(\rho(\gamma_n)\big) = D$ and axis A_n such that

$$\mathrm{d}_{\mathbf{H}^2}\big(x_0, \mathrm{A}_n\big) \xrightarrow[n\to\infty]{} +\infty.$$

For $f \in \mathrm{PSL}_2(\mathbf{R})$, we will denote by A_f its invariant axis if f is hyperbolic and by x_f its fixed point if f is elliptic.

Proof.

- If $g \geq 2$, we refer to the proof of Lemma 6.16 inMarché and Wolff 2016.
- If $g = 1$, let a, b as on the next picture. By assumption, $\rho(a)$ and $\rho(b)$ are hyperbolic. If $A_{\rho(a)}$ and $A_{\rho(b)}$ are disjoint, then the sequence $(\gamma_n)_n := (a^n b a^{-n})_n$ satisfies the conclusion of Lemma 3.19. If $A_{\rho(a)}$ and $A_{\rho(b)}$ intersect, then, by assumption on ρ, $[\rho(a), \rho(b)]$ is either elliptic of infinite order or hyperbolic. In the first case, we can apply the case of disjoint axes to $\rho(a)$ and $\rho([a, b]^k \cdot b \cdot [a, b]^{-k})$ for a suitable $k \in \mathbf{Z}$. In the second one, $\rho(a)$ and $\rho([a, b])$ have disjoint axes. We so apply the corresponding case to $\rho(a)$ and $\rho([a, b])$.

$\square$

As in Marché and Wolff 2016, a triplet $(x, y, z) \in \big(\mathbf{R}_{\geq 0}\big)^3$ satisfies the *condition* (Hex), if there exists a right-angled hexagon $\mathrm{H} \subset \mathbf{H}^2$ with three consecutive sides of respective lengths x, y and z.

Remark 3.20. *For all sequences $(x_n)_n, (y_n)_n, (z_n)_n \in \big(\mathbf{R}_{\geq 0}\big)^{\mathbf{N}}$ which do not accumulate on zero and such that either $y_n \xrightarrow[n\to\infty]{} +\infty$ or $x_n, z_n \xrightarrow[n\to\infty]{} +\infty$, then for all n enough large, (x_n, y_n, z_n) satisfies* (Hex). *Moreover,* (Hex) *is an open condition and if (x, y, z) satisfies* (Hex)*, then for all $x' \geq x, y' \geq y$ and $z' \geq z$, (x', y', z') satisfies* (Hex).

Lemma 3.21. *Let $[\rho] \in \mathcal{X}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ such that ρ send a separating curve c on an elliptic isometry of infinite order. Assume $\rho \in \mathcal{N}^e(\dot{\Sigma}, \underline{\mathcal{C}})$. Then there exists a non-separating curve which is send by ρ on an elliptic element.*

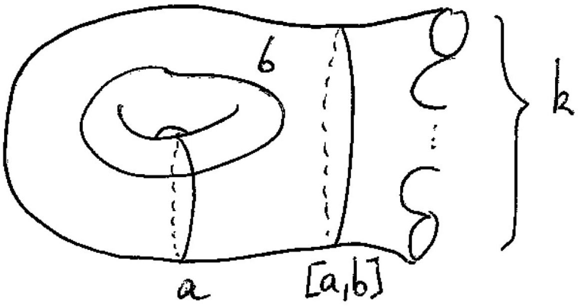

Figure 9

Proof. Let Σ_1 and Σ_2 be the tow connected components of $\dot{\Sigma} \setminus c$. One of them, Σ_1 for example, verifies the hypothesis of Lemma 3.19. This Lemma guarantees the existence of two non-separating curves $a \subset \Sigma_1$ and $b \subset \Sigma_2$ such that $\big(\lambda\big(\rho(a)\big), |D_a - D_b|, \lambda\big(\rho(a)\big)\big)$ satisfies $\big(\text{Hex}\big)$, where $\lambda\big(\rho(a)\big) = \inf_{x \in \mathrm{H}^2} \mathrm{d}_{\mathbf{H}^2}\big(x, \rho(a)x\big)$ and $D_a = \mathrm{d}_{\mathbf{H}^2}\big(x_0, \mathrm{A}_{\rho(a)}\big)$. Indeed, by Remark 3.20, we fix b arbitrarily and we choose $a = \gamma_N$ for N enough large, with the γ_n are given by Lemma 3.19. We may assume, up to switch b by b^{-1}, that ab is non-separating. Denote $A = \rho(a)$, $B = \rho(b)$ and $C = \rho(c)$. Up to conjugate b by a good power of c, we may assume that x_0 is close to the perpendicular common line to $\mathrm{A}_{\rho(a)}$ and $\mathrm{A}_{\rho(b)}$. Either the orientations of $\mathrm{A}_{\rho(a)}$ and $\mathrm{A}_{\rho(b)}$ agree or not. If they agree, since $\big(\lambda\big(\rho(a)\big), |D_a - D_b|, \lambda\big(\rho(a)\big)\big)$ satisfies $\big(\text{Hex}\big)$, as in Marché and Wolff 2016, the element AC^NBC^{-N} is elliptic for a suitable $N \in \mathbf{Z}$. If they disagree, we apply the case where the orientations agree to the isometries A and C^NBC^{-N} for a suitable N and we hence conclude.

$\square$

The Lemma 3.21 allows to conclude that $\mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ has full measure in $\mathcal{NH}^e(\dot{\Sigma}, \underline{\mathcal{C}})$. Indeed, $\mathcal{N}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ has full measure in the character variety and we showed that $\mathcal{EI}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \cap \mathcal{N}^e(\dot{\Sigma}, \underline{\mathcal{C}})$ is equal to $\mathcal{NH}^e(\dot{\Sigma}, \underline{\mathcal{C}}) \cap \mathcal{N}^e(\dot{\Sigma}, \underline{\mathcal{C}})$.

4 Toplogical dynamic for modular action on representation spaces in Abelian compact Lie groups; with Gianluca Faraco

The proof of Theorem D strongly relies on the explicit knowledge of the objects involved. In fact, the n-torus has a well-known description and, thanks to the abelian property, the character variety coincides with the representation space since the action of $\mathbf{T}^n$ by conjugation is trivial. Even better, the representation space can be identified with a torus of suitable dimension, hence the description of the representation space – and then of the character variety – is very explicit. The main difficulties in the abelian case concern questions coming from number theory and ergodic theory.

4.1 Strategy of the proof and related results

Each given representation $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$ induces a homological representation $\overline{\rho} : \mathrm{H}_1(\Sigma, \mathbf{Z}) \longrightarrow \mathbf{T}^n$. The map associating to any representation ρ its homological representation defines a bijection between the representation space and the homological representation space. This essentially follows because the commutator group $[\pi_1\Sigma, \pi_1\Sigma]$ is trivially a subgroup of $\mathsf{ker}\rho$ in the abelian case, and such a property is no longer true for a generic non-abelian Lie group. There is also a well-defined action of the symplectic group $\mathrm{Sp}_{2g}(\mathbf{Z})$ on the space $\mathrm{Hom}\big(\mathrm{H}_1(\Sigma, \mathbf{Z}), \mathbf{T}^n\big)$ by precomposition. Given a representation $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$ and its induced representation $\overline{\rho} : \mathrm{H}_1(\Sigma, \mathbf{Z}) \longrightarrow \mathbf{T}^n$, the $\mathrm{Mod}(\Sigma)$-orbit of ρ coincides with the $\mathrm{Sp}_{2g}(\mathbf{Z})$-orbit of $\overline{\rho}$. As an immediate consequence we obtain an equivalent version of the main theorem D, namely we have the following.

Theorem F. *Let Σ be a surface of genus $g \geq 1$ and let $\overline{\rho} : \mathrm{H}_1(\Sigma, \mathbf{Z}) \longrightarrow \mathbf{T}^n$ be a representation. Then the image of ρ is dense in $\mathbf{T}^n$ if and only if the symplectic group orbit $\mathrm{Sp}_{2g}(\mathbf{Z}) \cdot \overline{\rho}$ is dense in the homological representation space.*

Along the way of our investigation we shall remark that the action of the Torelli group $\mathrm{Tor}(\Sigma)$ on the representation space $\mathrm{Hom}\big(\pi_1\Sigma, \mathbf{T}^n\big)$ is trivial which contrasts with Theorem A.
Given a representation, we are then reduced to consider its $\mathrm{Sp}_{2g}(\mathbf{Z})$-orbit of instead of its modular orbits. This makes the study of orbits more understandable because the symplectic group is linear. We shall make the action even more explicit by identifying a representation with a matrix in the space $\mathrm{M}\big(n, 2g;\, \mathbf{T}\big)$. After these reductions, we shall see that we are in the position to apply Ratner's Theorem for studying orbit closures. In particular, we shall derive Theorem D.

Remark 4.1. *When the genus of Σ is 1, the reader may notice that theorems D and F are not only equivalent but actually the same statement in the strict sense. Indeed, in this very particular case the following equalities $\pi_1\Sigma = \mathrm{H}_1(\Sigma, \mathbf{Z})$ and $\mathrm{Mod}(\Sigma) = \mathrm{SL}_2(\mathbf{Z}) = \mathrm{Sp}_{2g}(\mathbf{Z})$ holds.*

The strategy we propose for Theorem D is different to the one developed by Previte-Xia to show their main theorem Previte and Xia 2000. Let us briefly give some more details. Given a representation

$\rho : \pi_1\Sigma \longrightarrow \mathrm{SU}(2)$ with a dense image - Previte-Xia defined such a representation *generic* (see Previte and Xia 2000) - they firstly found a handle Σ_0, namely a one-holed torus, such that the restriction of ρ to $\pi_1\Sigma_0$ is dense. After obtaining a dense handle, they proceed to demonstrate the base density theorem for the $(n+2g-2)$-holed torus. A similar process in the abelian case is not possible because dense handles do not always exist, see Example 4.24. In the light of Proposition 4.10, we shall bypass this issue by looking at the $\mathrm{Sp}_{2g}(\mathbf{Z})$-action on the representation space as described above.

The present section is organised as follow. In subsection 4.2 we begin with a description of the $\mathbf{T}^n$-character variety and then subsequently introduce the homological representation space and show the identification with the character variety. We finally describe the action of the symplectic group $\mathrm{Sp}_{2g}(\mathbf{Z})$ on the homological representation space. As a consequence, we shall derive the Proposition 4.10 and the equivalence of Theorems D and F. In subsection 4.7 we shall give a complete characterisation of dense representations in the n-dimensional torus by proving Theorem 4.21. In subsection 4.8 we shall finally derive our main Theorem D. In the last section, we prove Proposition 4.9 and indeed Theorem E establishing the connection of our dynamical result with the Kronecker's Approximation Theorem. We finally conclude with a serie of appendix on which we shall discuss some further aspects related to our project. Appendix 4.10 we discuss about a direct approach to our problem which works for a fairly general class of representations. In Appendix 4.10.3 we digress a little by providing a brief description of the relative $\mathbf{T}^n$-character variety for surfaces with one puncture and then we claim that our main results extend to one-punctured surfaces.

4.2 $\mathbf{T}^n$-character variety

In this work we are interested in characterising the orbits of the $\mathrm{Mod}(\Sigma)$-action on $\mathrm{Hom}(\pi_1\Sigma, G)/G$ where G is a compact, connected and abelian Lie group. It is classical to see that any such a group is isomorphic to the n-dimensional torus $\mathbf{T}^n$. The specific interest for the abelian case comes from its connection with abstract harmonic analysis, the geometry of numbers and the theory of group actions on homogeneous spaces (connections with Ratner's Theorem, see section 4.8).

In the introduction we have given a very brief view of the character variety for a generic compact Lie group G. In this section we specialise the discussion for compact and connected abelian Lie groups. From the Lie theory, any such a group is known to be a n-dimensional torus, namely the product of n copies of the unit circle $\mathbf{S}^1$. In the present work $\mathbf{S}^1$ is seen as $\{e^{i\theta} \mid \theta \in [0, 2\pi[\}$ where $[0, 2\pi[$ carries the quotient topology obtained identifying the boundary points of the closed interval $[0, 2\pi]$. Consequently, the n-torus $\mathbf{T}^n$ is defined as $\{(e^{i\theta_1}, \dots, e^{i\theta_n}) \mid \theta_i \in [0, 2\pi[, \text{ for any } i = 1, \dots, n\}$ endowed with the product topology.

Let Σ be a closed surface and let $a_1, b_1, \dots, a_g, b_g$ be any standard generating system of the fundamental group. The choice of a representation $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$ amounts to choose for each generator an element of $\mathbf{T}^n$ such that these elements satisfy the condition imposed by the presentation of the fundamental group ofΣ. However, the abelian property of $\mathbf{T}^n$ implies that the condition $[A_1,\, B_1]\cdots[A_g,\, B_g]=1$ is automatically satisfied for any choice of $2g$ elements in $(A_1, B_1, \dots, A_g, B_g)\in \mathbf{T}^n$. Thus, the representation space can be identified with the group $\big(\mathbf{T}^n\big)^{2g}\cong \mathbf{T}^{2ng}$. Even more, thanks again to the abelian property, the action of $\mathbf{T}^n$ on $\mathrm{Hom}\big(\pi_1\Sigma, \mathbf{T}^n\big)$ by post-composition with inner automorphisms of $\mathbf{T}^n$ is trivial. As a consequence, the $\mathbf{T}^n$-character variety coincides with the representation space.

Remark 4.2. *Let Σ be any surface of genus $g\ge 2$. The representation space $\mathrm{Hom}\big(\pi_1\Sigma, \mathbf{T}^n\big)$ splits as the direct sum of g copies of $\mathrm{Hom}\big(\pi_1 T, \mathbf{T}^n\big)$ where T denote the 2-torus. The basis $\big\{a_1, b_1, \dots, a_g, b_g\big\}$ of $\pi_1\Sigma$ we fixed satisfies moreover to the equalities $i\big(a_i, a_j\big)=i\big(b_i, b_j\big)=0$, and $i\big(a_i, b_j\big)=\delta_{ij}$ for all i,j with $1\le i,j\le g$. We may associate to any representation $\rho:\pi_1\Sigma\longrightarrow\mathbf{T}^n$ the g-tuple of representations $\big(\rho_1,\dots,\rho_g\big)$ where ρ_i is the restriction of ρ to the handle generated by a_i, b_i. Such a mapping defines then an isomorphism*

$$\mathrm{Hom}\big(\pi_1\Sigma, \mathbf{T}^n\big)\cong\bigoplus_{i=1}^{g}\mathrm{Hom}\big(\langle a_i, b_i\rangle, \mathbf{T}^n\big).$$

which depends on the basis chosen. This decomposition is a consequence of the fact that a surface of genus g is the connected sum of a surface of genus $g-1$ and a torus T along with the property that each representation ρ sends all simple closed separating curves to the identity. A recursive argument leads to the desire conclusion.

4.3 Homological representations

Let $\mathrm{H}_1(\Sigma,\mathbf{Z})$ be the first homology group. The close connection between the objects $\pi_1\Sigma$ and $\mathrm{H}_1(\Sigma,\mathbf{Z})$ is well-known, indeed the latter is known to be isomorphic to the abelianization of $\pi_1\Sigma$. As we have seen above, the representation space $\mathrm{Hom}\big(\pi_1\Sigma, \mathbf{T}^n\big)$ naturally identifies with the $2gn$-dimensional torus assigning to any representation ρ the $2g$-tuple $\big(\rho(a_1),\rho(b_1),\dots,\rho(a_g),\rho(b_g)\big)$, where $a_1, b_1,\dots,a_g,b_g$ is a basis for $\pi_1\Sigma$. Every representation ρ fails to be injective and its kernel $\mathsf{ker}(\rho)$ always contains the subgroup generated by the commutators since the target is abelian. Therefore, ρ boils down to a representation

$$\overline{\rho}:\mathrm{H}_1(\Sigma,\mathbf{Z})\cong\frac{\pi_1\Sigma}{\big[\pi_1\Sigma,\pi_1\Sigma\big]}\longrightarrow\mathbf{T}^n,\qquad \overline{\rho}\big(\big[\gamma\big]\big):=\rho(\gamma).$$

In fact, let $\gamma\in\pi_1\Sigma$ and let $\big[\gamma\big]$ be its image via the canonical projection $p:\pi_1\Sigma\longrightarrow \mathrm{H}_1(\Sigma,\mathbf{Z})$. Let $\gamma\big[\sigma_1,\sigma_2\big]$ be a representative of $\big[\gamma\big]$. Since the following chain of equalities holds

$$\rho\big(\gamma\big[\sigma_1,\sigma_2\big]\big)=\rho\big(\gamma\big)\rho\big(\big[\sigma_1,\sigma_2\big]\big)=\rho\big(\gamma\big),$$

the representation $\overline{\rho}$ is well-defined and the image does not depend on the choice of the representative. Furthermore, the image of ρ agrees with the image $\overline{\rho}$ by contruction.

Definition 4.3. *We define the* homological representation space *as the set* $\mathrm{Hom}\big(\mathrm{H}_1(\Sigma,\mathbf{Z}),\mathbf{T}^n\big)$ *of representations of* $\mathrm{H}_1(\Sigma,\mathbf{Z})$ *in* $\mathbf{T}^n$ *endowed with the compact-open topology.*

Lemma 4.4. *The homological representation space* $\mathrm{Hom}\big(\mathrm{H}_1(\Sigma,\mathbf{Z}),\mathbf{T}^n\big)$ *identifies with the* $2gn$*-dimensional torus* $\mathbf{T}^{2gn}$*.*

Proof. To any representation $\overline{\rho}$ we can assign the $2g$-tuple $\big(\overline{\rho}([a_1]),\overline{\rho}([b_1]),\dots,\overline{\rho}([a_g]),\overline{\rho}([b_g])\big)$, where the collection $\big[a_i\big],\big[b_i\big]$, $1\leq i\leq g$ is a fixed basis of the homology group $\mathrm{H}_1(\Sigma,\mathbf{Z})$. Conversely, given a $2g$-tuple $\big(v_1,w_1,\dots,v_g,w_g\big)\in\big(\mathbf{T}^n\big)^{2g}$, as $\mathbf{T}^n$ is abelian, the universal property of free abelian groups implies the existence of a unique group homomorphism from $\mathrm{H}_1(\Sigma,\mathbf{Z})$ into the n-torus $\mathbf{T}^n$ which sends $[a_i]$ to v_i and $[b_i]$ to w_i, for every $i=1,\dots,g$. □

The implications of this lemma are quite simple, but of crucial importance. Upon choosing a standard generating system for $\pi_1\Sigma$; the representation space $\mathrm{Hom}\big(\pi_1\Sigma,\mathbf{T}^n\big)$ identifies with the homological representation space $\mathrm{Hom}\big(\mathrm{H}_1(\Sigma,\mathbf{Z}),\mathbf{T}^n\big)$ and the identification is explicitely given by the association $\rho\mapsto\overline{\rho}$. According to this property, we derive the following lemma.

Lemma 4.5. *Let* $\rho_1,\rho_2:\pi_1\Sigma\longrightarrow\mathbf{T}^n$ *be two representations. Then* $\rho_1\equiv\rho_2$ *if and only if* $\overline{\rho}_1\equiv\overline{\rho}_2$*.*

Proof. This is just a matter of definitions given so far. The necessary condition follows trivially. The sufficient condition follows from $\overline{\rho}\big(\big[\gamma\big]\big)=\rho(\gamma)$ for any $\gamma\in\pi_1\Sigma$. □

4.4 Actions of the symplectic group $\mathrm{Sp}_{2g}(\mathbf{Z})$

In this section we are going to describe the action of the symplectic group $\mathrm{Sp}_{2g}(\mathbf{Z})$ both on the representation space and on the homological representation space.

4.4.1 The symplectic group $\mathrm{Sp}_{2g}(\mathbf{Z})$.

We begin with recalling some standard notions. The algebraic intersection number

$$\cap:\mathrm{H}_1(\Sigma,\mathbf{Z})\times\mathrm{H}_1(\Sigma,\mathbf{Z})\longrightarrow\mathbf{Z}$$

extends uniquely to a nondegenerate, alternating bilinear map

$$\cap:\mathrm{H}_1(\Sigma,\mathbf{R})\times\mathrm{H}_1(\Sigma,\mathbf{R})\longrightarrow\mathbf{R}$$

which realises $\mathrm{H}_1(\Sigma,\mathbf{R})$ as a symplectic vector space.

Definition 4.6. *A collection of elements* $[a_i], [b_i]$, $1 \leq i \leq g$ *of* $\mathrm{H}_1(\Sigma, \mathbf{Z}) < \mathrm{H}_1(\Sigma, \mathbf{R})$ *such that*

$$[a_i] \cap [a_j] = [b_i] \cap [b_j] = 0, \quad [a_i] \cap [b_j] = \delta_{ij}$$

for all i, j *with* $1 \leq i, j \leq g$ *is called a* symplectic basis of the group $\mathrm{H}_1(\Sigma, \mathbf{Z})$ *or a basis for the symplectic vector space* $(\mathrm{H}_1(\Sigma, \mathbf{Z}), \cap)$. *We define a collection of curves* a_i, b_i *such that* $\{[a_i], [b_i]\}$ *is a symplectic basis as* geometric symplectic basis *for* $\pi_1\Sigma$.

The matrix associated to the antisymmetric bilinear form $\cap$ on the basis $[a_i], [b_i]$ is the $2g \times 2g$ blockwise diagonal matrix

$$J = \begin{pmatrix} J_o & & \\ & \ddots & \\ & & J_o \end{pmatrix} \quad \text{with} \quad J_o = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The symplectic linear group $\mathrm{Sp}_{2g}(\mathbf{R})$ is defined as the group of invertible matrices A satisfying the relation $AJA^t = J$ and we denote by $\mathrm{Sp}_{2g}(\mathbf{Z})$ the subgroup of those matrices with integer coefficients.

Remark 4.7. *Here, the symplectic group* $\mathrm{Sp}_{2g}(\mathbf{R})$ *is the subgroup of* $\mathrm{SL}_{2g}(\mathbf{R})$ *of matrices preserving the alternating* 2*-form* $\omega = e_1 \wedge e_2 + \cdots + e_{2g-1} \wedge e_{2g}$. *By Remark 4.2,* $\mathrm{Sp}_{2g}(\mathbf{R})$ *contains the g-times product* $\mathrm{SL}_2(\mathbf{R}) \times \cdots \times \mathrm{SL}_2(\mathbf{R})$ *as a proper subgroup. In turns, the group* $\mathrm{Sp}_{2g}(\mathbf{R})$ *contains the g-times product* $\mathrm{SL}_2(\mathbf{Z}) \times \cdots \times \mathrm{SL}_2(\mathbf{Z})$ *as a proper subgroup. This property will be useful in the sequel.*

An orientation preserving homeomorphism induces an isomorphism in homology which preserves the intersection form $\cap$ defined above. Since isotopic homeomorphisms induce the same map in homology, there is a representation

$$\mu : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Aut}^+\big(\mathrm{H}_1(\Sigma, \mathbf{Z})\big) \cong \mathrm{SL}_{2g}(\mathbf{Z}).$$

As each homeomorphism preserves the intersection form $\cap$, the image of μ lies also inside $\mathrm{Sp}_{2g}(\mathbf{R})$. Therefore the image of μ lies inside $\mathrm{SL}_{2g}(\mathbf{Z}) \cap \mathrm{Sp}_{2g}(\mathbf{R})$. The representation $\mu : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Sp}_{2g}(\mathbf{Z})$ – usually called symplectic representation of $\mathrm{Mod}(\Sigma)$. The Torelli group is the kernel of μ.

Remark 4.8. *In the genus one case the Torelli subgroup is trivial. Indeed,* $\mathrm{Mod}(T) \cong \mathrm{SL}_2(\mathbf{Z}) = \mathrm{Sp}_2(\mathbf{Z})$.

4.4.2 Comparison of the $\mathrm{Mod}(\Sigma)$-orbits with the $\mathrm{Sp}_{2g}(\mathbf{Z})$-orbits.

We now consider the effect of changing the basis of $\mathrm{H}_1(\Sigma, \mathbf{Z})$ pre-composing any homological representation with an automorphism $\phi \in \mathsf{Aut}^+\big(\mathrm{H}_1(\Sigma, \mathbf{Z})\big)$ such that any representation $\overline{\rho}$ is sent to $\overline{\rho} \circ \phi^{-1}$. We can, therefore, consider $\mathrm{SL}_{2g}(\mathbf{Z})$-action on the space $\mathrm{Hom}\big(\mathrm{H}_1(\Sigma, \mathbf{Z}), \mathbf{T}^n\big)$. Of course, this action restricts to an action of the symplectic group $\mathrm{Sp}_{2g}(\mathbf{Z})$ on the same space. We are interested in studying the $\mathrm{Sp}_{2g}(\mathbf{Z})$-orbits in the homological representation space. The main goal of this section is proving the following claim.

Proposition 4.9. *Let $\rho_1, \rho_2 : \pi_1\Sigma \longrightarrow \mathbf{T}^n$ be two representations and let $\overline{\rho}_1, \overline{\rho}_2 : \mathrm{H}_1(\Sigma, \mathbf{Z}) \longrightarrow \mathbf{T}^n$ be the induced representations. Suppose there is $\phi \in \mathrm{Mod}(\Sigma)$ such that $\rho_2 = \rho_1 \circ \phi$. Then $\overline{\rho}_2 = \overline{\rho}_1 \circ \mu(\phi)$, where μ is the symplectic representation of* $\mathrm{Mod}(\Sigma)$.

Proof. Let $\phi : \pi_1\Sigma \longrightarrow \pi_1\Sigma$ be any element of $\mathrm{Out}(\pi_1\Sigma)$. As the image of any commutator is also a commutator, the mapping ϕ boils down to a isomorphism in homology $\mu(\phi) : \mathrm{H}_1(\Sigma, \mathbf{Z}) \longrightarrow \mathrm{H}_1(\Sigma, \mathbf{Z})$. Two mappings ϕ_1 and ϕ_2 boil down to the same isomorphism in homology if and only if $\phi_2 \circ \phi_1^{-1}$ descends to the identity map in homology, that is $\phi_2 \circ \phi_1^{-1}$ is an element of the Torelli subgroup. Therefore, the association $\phi \longmapsto \mu(\phi)$ defines the symplectic representation μ seen above. Look at now the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_1(\Sigma, \mathbf{Z}) \arrow[rr, "\overline{\rho}_2 = \overline{\rho_1 \circ \phi}"] & & \mathbf{T}^n \\
\pi_1\Sigma \arrow[u, "p"] \arrow[r, "\phi"] \arrow[d, "p"] & \pi_1\Sigma \arrow[r, "\rho_1"] \arrow[d] & \mathbf{T}^n \arrow[u, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\
\mathrm{H}_1(\Sigma, \mathbf{Z}) \arrow[r, "\mu(\phi)"] & \mathrm{H}_1(\Sigma, \mathbf{Z}) \arrow[r, "\overline{\rho}_1"] & \mathbf{T}^n
\end{tikzcd} \tag{3}$$

where p is the canonical projection. As $\rho_2 = \rho_1 \circ \phi$ by assumption, it turns out $\overline{\rho}_2 = \overline{\rho_1 \circ \phi} = \overline{\rho}_1 \circ \mu(\phi)$ as desired. □

4.4.3 Direct consequences.

Proposition 4.9 leads to some interesting consequences that we are going to show. The first one concerns the action of the Torelli subgroup $\mathrm{Tor}(\Sigma)$ on the representation space $\mathrm{Hom}(\pi_1\Sigma, \mathbf{T}^n)$.

Proposition 4.10. *The action of the Torelli group $\mathrm{Tor}(\Sigma)$ on the representation space $\mathrm{Hom}(\pi_1\Sigma, \mathbf{T}^n)$ is trivial.*

Proof. Let $\rho_1 \in \mathrm{Hom}(\pi_1\Sigma, \mathbf{T}^n)$ be any representation and let $\phi \in \mathrm{Tor}(\Sigma)$. Set $\rho_2 = \phi \cdot \rho_1 = \rho_1 \circ \phi^{-1}$. Proposition 4.9 implies that $\overline{\rho}_1 = \overline{\rho}_2$ because $\mu(\phi) = 1$. We now invoke Lemma 4.5 to conclude $\rho_1 = \rho_2$, namely the action of ϕ is trivial. □

The n-torus $\mathbf{T}^n$ is a compact and connected Lie group and hence mapping class group $\mathrm{Mod}(\Sigma)$ acts ergodically on the representation space - see Theorem 1 in the introduction. As the action of the Torelli subgroup $\mathrm{Tor}(\Sigma)$ is trivial, the action of the quotient group is also well-defined and the following holds.

Proposition 4.11. *The action of $\mathrm{Sp}_{2g}(\mathbf{Z}) \cong \dfrac{\mathrm{Mod}(\Sigma)}{\mathrm{Tor}(\Sigma)}$ on $\mathrm{Hom}(\pi_1\Sigma, \mathbf{T}^n)$ is ergodic with respect to the Haar measure.*

As the homological representation spaces identifies with the representation space, it also carries a finite measure. Calling $\imath$ the identifying map, this finite measure can be seen as the pull-back measure $\imath^*\mu_\Sigma$, where μ_Σ is the finite measure carried by the representation space.

Corollary 4.12. *The action of the symplectic group* $\mathrm{Sp}_{2g}(\mathbf{Z})$ *on the space* $\mathrm{Hom}\big(\mathrm{H}_1(\Sigma,\mathbf{Z}),\mathbf{T}^n\big)$ *is ergodic with respect to the finite measure* $\imath^*\mu_\Sigma$.

As a final consequence we have the following characterisation.

Proposition 4.13. *Let* $\rho:\pi_1\Sigma\longrightarrow\mathbf{T}^n$ *be a representation and let* $\overline{\rho}:\mathrm{H}_1(\Sigma,\mathbf{Z})\longrightarrow\mathbf{T}^n$ *the homological representation induced by* ρ*. Then the mapping class group orbit* $\mathrm{Mod}(\Sigma)\cdot\rho$ *is dense if and only if the symplectic group orbit* $\mathrm{Sp}_{2g}(\mathbf{Z})\cdot\overline{\rho}$ *is dense.*

Proof. Proposition 4.9 implies that the mapping class group orbit of ρ coincides with the symplectic group orbit of $\overline{\rho}$ via the identification $\rho\mapsto\overline{\rho}$. Therefore, one orbit is dense if and only if the other is dense. $\square$

Corollary 4.14. *Let* Σ *be a surface of genus* $g\geq 1$*. Then Theorem D holds if and only if Theorem F holds.*

4.5 The matrix presentation

The n-torus $\mathbf{T}^n$ can be seen also as the quotient of $\mathbf{R}^n$ by the standard action of the lattice $2\pi\mathbf{Z}^n$, indeed the exponential map provides an identification between $\mathbf{R}^n/2\pi\mathbf{Z}^n$ and the n-torus described above. We shall define the map

$$\mathsf{exp}:\mathbf{R}^n\longrightarrow\mathbf{T}^n\qquad\big(\theta_1,\dots,\theta_n\big)\longmapsto\big(e^{i\theta_1},\dots,e^{i\theta_n}\big)\tag{4}$$

as the *canonical projection.* We define $2\pi\mathbf{Z}^n$ as the *the standard lattice* $-$ notice that this lattice is 2π times the usual standard lattice.

Fix a set of generators $\big\{a_1,b_1,\dots,a_g,b_g\big\}$ and let us consider $\mathbf{T}^n$ as the quotient of $\mathbf{R}^n$ by the action of the standard lattice. Let $\rho:\pi_1\Sigma\longrightarrow\mathbf{T}^n$ be any representations and set

$$\rho(a_i)=\big(e^{i\theta_{1,2i-1}},\dots,e^{i\theta_{n,2i-1}}\big)$$

$$\rho(b_i)=\big(e^{i\theta_{1,2i}},\dots,e^{i\theta_{n,2i}}\big)$$

for any $i=1,\dots,n$. The elements $\rho(a_1),\rho(b_1),\dots,\rho(a_g),\rho(b_g)$ generate the image of the representation ρ. Any element $\gamma\in\pi_1\Sigma$ may be seen as a word in the letters a_i,b_i for $i=1,\dots,2g$. Hence, by the

Abelian property, $\rho(\gamma) = \rho\big(w(a_1, b_1, \dots a_g, b_g)\big)$ is equal to $\rho(a_1)^{k_1} \cdots \rho(b_g)^{k_{2g}}$ for some $k_1, \dots, k_{2g} \in \mathbf{Z}$. In particular, the element $\rho(\gamma)$ can be computed with the following matrix multiplication

$$\begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,i} & \cdots & \theta_{1,2g} \\ \vdots & & & & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,i} & \cdots & \theta_{n,2g} \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ k_i \\ \vdots \\ k_{2g} \end{pmatrix}.$$

Definition 4.15. *Let Θ_ρ be the matrix having as entries the values $\theta_{i,j} \in \big[0, 2\pi\big[$ with $i = 1, \dots, n$ and $j = 1, \dots, 2g$. We define Θ_ρ as* the matrix associated to ρ *with respect the basis $\big\{a_1, b_1, \dots, a_g, b_g\big\}$ and the standard lattice $2\pi\mathbf{Z}^n$.*

In what follows, we shall often identify a representation ρ with its associated matrix. Let us briefly see the reason why we are legitimated to do that. Consider the topological vector space $\mathrm{M}\big(n, 2g; \mathbf{R}\big)$ and introduce an equivalence relation where $A \sim B$ if and only if $A - B = 2\pi H \in \mathrm{M}\big(n, 2g; 2\pi\mathbf{Z}\big)$. The mapping $\imath$ associating to any ρ its associated matrix Θ_ρ provides an homeomorphism between the representation space $\mathrm{Hom}\big(\pi_1\Sigma, \mathbf{T}^n\big)$ and the quotient space $\mathrm{M}\big(n, 2g, \mathbf{T}\big)$. Moreover, the post-composition of the mapping $\rho \mapsto \overline{\rho}$ with $\imath^{-1}$ defines a homeomorphism between the spaces $\mathrm{Hom}\big(\mathrm{H}_1(\Sigma, \mathbf{Z}), \mathbf{T}^n\big)$ and $\mathrm{M}\big(n, 2g, \mathbf{T}\big)$.

Given a representation ρ, the matrix Θ_ρ depends on the choice of a set of generators for $\pi_1\Sigma$ and also on the choice of a lattice $\Lambda < \mathbf{R}^n$. Let us see how these choices affect definition 4.15. We begin describing the effect of changing the set of generators of $\pi_1\Sigma$.

4.5.1 The effect of changing basis.

Given two basis $\mathcal{B} = \big\{a_1, b_1, \dots, a_g, b_g\big\}$ and $\mathcal{B}' = \big\{a_1', b_1', \dots, a_g', b_g'\big\}$ of $\pi_1\Sigma$, we define Θ_ρ and Θ_ρ' the matrices associated to $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$ with respect to $\mathcal{B}$ and $\mathcal{B}'$ respectively. Every generator a_l' and b_l' is a finite word in the letters $a_1, b_1, \dots, a_g, b_g$, so there are integers a_{ij} with $i, j \in \big\{1, \dots, 2g\big\}$ such that

$$\rho(a_l') = \rho(a_1)^{a_{2l-1\,1}} \cdots \rho(b_g)^{a_{2l-1\,2g}} \quad \text{and} \quad \rho(b_l') = \rho(a_1)^{a_{2l\,1}} \cdots \rho(b_g)^{a_{2l\,2g}}.$$

Setting A as the integral matrix $\big(a_{ij}\big)$ with $i, j \in \big\{1, \dots, 2g\big\}$, a direct computation shows that Θ_ρ' equals $\Theta_\rho \cdot A$. Likewise, a_l, b_l are also finite words in the letters $a_1', b_1', \dots, a_g', b_g'$. Hence, there exist integers b_{ij} such that

$$\rho(a_l) = \rho(a_1')^{b_{2l-1\,1}} \cdots \rho(b_g')^{b_{2l-1\,2g}} \quad \text{and} \quad \rho(b_l) = \rho(a_1')^{b_{2l\,1}} \cdots \rho(b_g')^{b_{2l\,2g}}.$$

Setting B as the integral matrix $\big(b_{ij}\big)$ with $i, j \in \big\{1, \dots, 2g\big\}$, the same computation implies Θ_ρ equals $\Theta'_\rho \cdot B$.
It is worth noticing $\Theta_\rho = \Theta_\rho \cdot AB$ and the matrices A, B satisfy the equation $AB = \mathrm{I}_{2g}$ implying that A, B are unimodular. As the matrix Θ_ρ can be singular we cannot directly deduce that $AB = \mathrm{I}_{2g}$, hence let us give a glimpse of why this is true.

Instead of working in $\pi_1\Sigma$ we are going to look at the situation in the first homology group $\mathrm{H}_1(\Sigma, \mathbf{Z}) \cong \mathbf{Z}^{2g}$. Let $a_1 = w\big(a'_1, b'_1, \dots, a'_g, b'_g\big)$, in particular

$$\big[a_1\big] = \big[a'_1\big]^{b_{1\,1}}\big[b'_1\big]^{b_{1\,2}} \cdots \big[a'_g\big]^{b_{1\,2g-1}}\big[b'_g\big]^{b_{1\,2g}}$$

where b_{1j} with $j = 1, \dots, 2g$ are as above. On the other hand, any $\big[a'_l\big]$ and $\big[b'_l\big]$ is of the form

$$\big[a'_l\big] = \big[a_1\big]^{a_{2l-1\,1}}\big[b_1\big]^{a_{2l-1\,2}} \cdots \big[a_g\big]^{a_{2l-1\,2g-1}}\big[b_g\big]^{a_{2l-1\,2g}},$$

$$\big[b'_l\big] = \big[a_1\big]^{a_{2l\,1}}\big[b_1\big]^{a_{2l\,2}} \cdots \big[a_g\big]^{a_{2l\,2g-1}}\big[b_g\big]^{a_{2l\,2g}}.$$

where a_{ij} are as above. Replacing each $\big[a'_l\big]$ and $\big[b'_l\big]$ inside $\big[w\big] = \big[a_1\big]$, for any $l = 1, \dots, g$, we obtain

$$\big[a_1\big] = \big[a_1\big]^{k_1}\big[b_1\big]^{k_2} \cdots \big[a_g\big]^{k_{2g-1}}\big[b_g\big]^{k_{2g}}.$$

As $\mathbf{Z}^{2g}$ is torsion-free, we may deduce that $k_1 = 1$ and $k_2 = \cdots = k_{2g} = 0$. On the other hand, it is straightforward to see that $k_m = \sum_{r=1}^{2g} b_{1r}a_{rm}$. Applying the same reasoning to any other generator we get the desire conclusion.

Remark 4.16. *The matrices A and B found above may not have any geometrical meaning. Indeed, for closed surfaces the action of* $\mathrm{Aut}(\pi_1 S)$ *is not transitive on the set of basis of $\pi_1\Sigma$ and then two different basis may not be related by any automorphisms of $\pi_1\Sigma$. This means that not all matrices in* $\mathrm{SL}_{2g}(\mathbf{Z})$ *have a geometrical interpretation. As we shall see, a matrix has a geometrical meaning, that is induced by a homeomorphism of Σ, if and only if it is symplectic - see Proposition 4.11 above.*

4.5.2 The effect of changing the basis of the lattice.

We begin noticing that the j-th column of the matrix Θ_ρ corresponds to the vector of coordinates of a lift of the j-th generator of $\rho\big(\pi_1\Sigma\big)$ with respect to the standard lattice. Given any lattice Λ with basis $\big\{v_1, \dots, v_n\big\}$ there is a matrix $g \in \mathrm{GL}_n(\mathbf{R})$ such that $\Lambda = g \cdot \big(2\pi\mathbf{Z}^n\big)$. In particular $g\big(e_i\big) = v_i$. Change the basis means to change the coordinates of the vectors forming the columns of the matrix Θ_ρ. Therefore, with respect to the lattice Λ, the matrix associated to ρ has the following form $g\Theta_\rho$. In the sequel we shall need to consider the matrix Θ_ρ with respect to a lattice Λ different to the standard one. We therefore extend the notation in the following way: We denote by $\Theta_\rho(\Lambda)$ the matrix associated to with respect to Λ. We shall use again the notation Θ_ρ when the lattice is the standard one.

4.5.3 The $\mathbf{Z}$-row rank of the associated matrix.

We now introduce the following numerical invariant concerning the associated matrix Θ_ρ. As we shall see, such an invariant give us a way to characterise dense representations in $\mathbf{T}^n$ completely.

Definition 4.17. *Let $M \in \mathrm{M}\big(n, m; \mathbf{R}\big)$. We define the $\mathbf{Z}$-row rank of M as the dimension of the $\mathbf{Z}$-module generated by the rows of M. We shall denote it as $rk_{\mathbf{Z}}\big(M\big)$.*

We remark that the $\mathbf{Z}$-row rank is not invariant by transposition.

Lemma 4.18. *Let $M \in \mathrm{M}\big(n, m; \mathbf{R}\big)$. The $\mathbf{Z}$-row rank $\mathrm{rk}_{\mathbf{Z}}\big(M\big)$ of M is invariant under the left-action of $\mathrm{SL}_n(\mathbf{Z})$. Similarly, $\mathrm{rk}_{\mathbf{Z}}\big(M\big)$ is invariant under the right-action of $\mathrm{SL}_m(\mathbf{Z})$.*

Proof. We begin showing the first claim. Let $k = \mathrm{rk}_{\mathbf{Z}}\big(M\big) \leq n$. Define Z as the subset of $\mathbf{Z}^n$ of those vectors v such that $vM = 0$. Notice that Z is a $\mathbf{Z}$-module of rank $n-k$. Let A be any matrix in $\mathrm{SL}_n(\mathbf{Z})$ and compute AM. It is easy to check that the j-th row is given by the linear combination $\sum_{i=1}^n a_{ji}\big(m_{i1}, \dots, m_{im}\big)$. Suppose there is a vector $\mu = \big(\mu_1, \dots, \mu_n\big)$ such that $\mu AM = 0$, then a straightforward computation shows that $\mu A \in Z$, that is $\mu = vA^{-1}$ for some $v \in Z$. The subset $Z \cdot A^{-1}$ of $\mathbf{Z}^n$ is thence the set of vectors μ such that $\mu AM = 0$ and it has dimension $n-k$ over $\mathbf{Z}$. Therefore $\mathrm{rk}_{\mathbf{Z}}\big(AM\big) = k$. Since the $\mathbf{Z}$-module generated by the rows of M does not change by the action of $\mathrm{SL}_m(\mathbf{Z})$, we obtain the invariance by the right-action of $\mathrm{SL}_m(\mathbf{Z})$ of the $\mathbf{Z}$-row rank of M. □

Given a representation $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$, the following claims are direct consequences of the lemma above applied to the matrix Θ_ρ.

Corollary 4.19. *Let $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$ be a representation and let Θ_ρ be the matrix associated to ρ with respect to some basis of $\pi_1\Sigma$. The $\mathbf{Z}$-row rank of Θ_ρ is well-defined and it does not depend on any choice of a basis for $\pi_1\Sigma$ either on the choice of any lattice.*

Let $v_1 \dots, v_k$ be vectors in $\mathbf{R}^n$. In the sequel, we shall say that a $\mathbf{Z}$-module generated by $v_1 \dots, v_k$ is $\pi\mathbf{Q}$-free if and only if $\langle v_1, \dots, v_k\rangle_{\mathbf{Z}} \cap \pi\mathbf{Q}^n = \big\{(0, \dots, 0)\big\}$. Keeping this definition in mind we finally state the following consequence.

Proposition 4.20. *Let $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$ be a representation and let Θ_ρ be the matrix associated to ρ with respect to some basis of $\pi_1\Sigma$. The $\mathbf{Z}$-module $\langle \Theta_j \mid j = 1, \dots, n\rangle_{\mathbf{Z}}$ generated by the rows of Θ_ρ is $\pi\mathbf{Q}$-free if and only if the $\mathbf{Z}$-module $\langle \big(A\Theta_\rho\big)_j \mid j = 1, \dots, n\rangle_{\mathbf{Z}}$ generated by the rows of $A\Theta_\rho$ is $\pi\mathbf{Q}$-free, where $A \in \mathrm{SL}_n(\mathbf{Z})$. Similarly, for $B \in \mathrm{SL}_{2g}(\mathbf{Z})$, $\langle \Theta_j \mid j = 1, \dots, n\rangle_{\mathbf{Z}}$ is $\pi\mathbf{Q}$-free if and only if the $\mathbf{Z}$-module $\langle \big(\Theta_\rho B\big)_j \mid j = 1, \dots, n\rangle_{\mathbf{Z}}$ is $\pi\mathbf{Q}$-free since the tow $\mathbf{Z}$-modules are equal.*

Proof of Proposition 4.20. Look at the matrix $A\Theta_\rho$ and suppose there are $\lambda_1, \dots, \lambda_n \in \mathbf{Z}$, not all zero, such that

$$\sum_{j=1}^n \lambda_j \Big(\sum_{i=1}^n a_{ji}\big(\theta_{i,1}, \dots, \theta_{i,2g}\big)\Big) \in \pi\mathbf{Q}^{2g}.$$

A simple manipulation of the formula above shows that

$$\sum_{j=1}^{n} \lambda_j \Big(\sum_{i=1}^{n} a_{ji} \big(\theta_{i,1}, \ldots, \theta_{i,2g}\big) \Big) = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \lambda_j a_{ji} \Big) \big(\theta_{i,1}, \ldots, \theta_{i,2g}\big),$$

implying the existence of some $\mu_1, \ldots, \mu_n \in \mathbf{Z}$, not all zero, such that $\sum_{j=1}^{n} \mu_j \big(\theta_{i,1}, \ldots, \theta_{i,2g}\big) \in \pi \mathbf{Q}^{2g}$. The proof of the second claim works similarly: Suppose there are $\lambda_1, \ldots, \lambda_n \in \mathbf{Z}$, not all zero, such that

$$\sum_{i=1}^{n} \lambda_i \Big(\sum_{j=1}^{2g} \theta_{i,j} \big(b_{j1}, \ldots, b_{j2g}\big) \Big) \in \pi \mathbf{Q}^{2g}.$$

The same manipulation shows that

$$\sum_{i=1}^{n} \lambda_i \Big(\sum_{j=1}^{2g} \theta_{i,j} \big(b_{j1}, \ldots, b_{j2g}\big) \Big) = \sum_{j=1}^{2g} \Big(\sum_{i=1}^{n} \lambda_i \theta_{i,j} \Big) \big(b_{i1}, \ldots, b_{i2g}\big),$$

implying that $\sum_{i=1}^{n} \lambda_i \theta_{i,j} \in \pi \mathbf{Q}$ for any $j = 1, \ldots, 2g$. That is $\big(\lambda_1, \ldots, \lambda_n\big)\Theta_\rho \in \pi \mathbf{Q}^{2g}$. □

4.6 Remarks and comments on the modular action

In this section we collect a couple of final remarks about the $\mathrm{Sp}_{2g}(\mathbf{Z})$-action.

4.6.1 Explicit description of the modular action

The action of the mapping class group on the representation space is defined by pre-composition of any representation with an automorphism $\phi \in \mathsf{Out}(\pi_1\Sigma)$ such that any representation ρ is sent to $\rho \circ \phi^{-1}$. Since the Torelli subgroup acts trivially on the representation space by our Proposition 4.10, the action of mapping class group boils down to an action of the group $\mathrm{Sp}_{2g}(\mathbf{Z})$ which agrees with the $\mathrm{Sp}_{2g}(\mathbf{Z})$-action on $\mathrm{Hom}\big(\mathrm{H}_1(\Sigma, \mathbf{Z}), \mathbf{T}^n\big)$. In section 4.5, we have identified the representation space with $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$ by using the mapping $\imath$ associating to any representation ρ its matrix Θ_ρ. We use such a mapping to transfer the action of $\mathrm{Mod}(\Sigma)$ on $\mathrm{Hom}\big(\pi_1\Sigma, \mathbf{T}^n\big)$ to an action of $\mathrm{Sp}_{2g}(\mathbf{Z})$ on $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$. Since any $\phi \in \mathrm{Tor}(\Sigma)$ leaves ρ fixed, the matrix associated to $\rho' = \phi \cdot \rho = \rho \circ \phi^{-1}$ agrees with Θ_ρ, this is a consequence of Proposition 4.10. Any coset $\phi\, \mathrm{Tor}(\Sigma)$ defines a unique matrix A in $\mathrm{Sp}_{2g}(\mathbf{Z})$. In the light of the discussion given at subsection 4.5.1, the matrix associated to $\rho' = \phi \cdot \rho$ is $\Theta_{\rho'} = \Theta_\rho\, A^{-1}$. Therefore, the action of $\mathrm{Sp}_{2g}(\mathbf{Z})$ on $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$ is defined as $A \cdot \Theta_\rho = \Theta_\rho\, A^{-1}$. As the mapping $\imath$ is a homeomorphism, it is clear that $\mathrm{Mod}(\Sigma)$-orbit of ρ is dense in the representation space if and only if the $\mathrm{Sp}_{2g}(\mathbf{Z})$-orbit of Θ_ρ is dense in $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$.

4.6.2 The modular action commutes with the change of lattice

Given a representation $\rho : \pi_1\Sigma \to \mathbf{T}^n$, the main goal of the present paper is to study its orbit under the action of the mapping class group. This reduces to study the orbit of the matrix Θ_ρ naturally attached to ρ in the space $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$. However the matrix Θ_ρ depends on the lattice chosen and hence the orbit also. The aim of this paragraph is to point out that this is not the case; indeed the change of lattice commutes with the modular action. Each element Θ in the space $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$ can be thought as the datum of n vectors $\Theta_i \in \mathbf{T}^{2g}$ corresponding to the rows of Θ. By adopting this point of view, the space $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$ identifies with $\mathbf{T}^{2g} \times \cdots \times \mathbf{T}^{2g}$. There is a left action of the group

$$G = \left\{ \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} : A \in \mathrm{Sp}_{2g}(\mathbf{Z}) \right\} \cong \mathrm{Sp}_{2g}(\mathbf{Z}) < \mathrm{SL}_{2gn}(\mathbf{Z})$$

on the $2gn$-dimensional torus induced by the natural right action of the symplectic group in the matrix space $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$. Using this new perspective, one can easily verify that any change of lattice commutes with the $\mathrm{Sp}_{2g}(\mathbf{Z})$ action. Indeed, any change of lattice $h \in \mathrm{SL}_n(\mathbf{Z})$ can be seen as an element of the group H defined as

$$H = \left\{ \begin{pmatrix} h_{1\,1}\mathrm{I}_{2g} & \cdots & h_{1\,n}\mathrm{I}_{2g} \\ \vdots & \ddots & \vdots \\ h_{n\,1}\mathrm{I}_{2g} & \cdots & h_{n\,n}\mathrm{I}_{2g} \end{pmatrix} \;\middle|\; \text{where } \begin{pmatrix} h_{1\,1} & \cdots & h_{1\,n} \\ \vdots & & \vdots \\ h_{n\,1} & \cdots & h_{n\,n} \end{pmatrix} \in \mathrm{SL}_n(\mathbf{Z}) \right\} \cong \mathrm{SL}_n(\mathbf{Z}) < \mathrm{SL}_{2gn}(\mathbf{Z})$$

Since H commutes with the group G defined above, the $\mathrm{Sp}_{2g}(\mathbf{Z})$ action commutes with the change of lattice.

4.7 Characterising dense representations

In this section we provide a complete characterisation of representations with dense image by providing necessary and sufficient conditions. We have seen in the previous section that, upon choosing a basis of the fundamental group and a lattice, each representation is represented by a well-define matrix Θ_ρ. Along this section we fix an arbitrary basis for the fundamental group and we consider $\mathbf{T}^n$ as the quotient of $\mathbf{R}^n$ with the standard lattice.

Theorem 4.21. *Let $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$ be a representation. Then ρ has dense image in $\mathbf{T}^n$ if and only if $\mathrm{rk}_{\mathbf{Z}}\big(\Theta_\rho\big) = n$ and the rows of Θ_ρ generate a $\pi\mathbf{Q}$-free $\mathbf{Z}$-module.*

Notice that the necessary condition means that the $\mathbf{Z}$-module generated by the rows of the matrix Θ_ρ does not intersect $\pi\mathbf{Q}^{2g}$ and is equivalent to say that row rank over $\mathbf{Z}$ of the matrix

$$\begin{pmatrix} \Theta_\rho \\ \pi \cdot \mathrm{I}_{2g} \end{pmatrix} = \begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,2g} \\ \vdots & & \vdots \\ \theta_{n,1} & \ldots & \theta_{n,2g} \\ \pi & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi \end{pmatrix} \tag{5}$$

is maximal, namely $2g+n$. Before proving the Theorem, we need a preliminar Lemma.

Lemma 4.22. *Suppose that $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$ is dense. Then each representation $\rho_k = \pi_k \circ \rho$, where π_k is the projection to k^{th} factor, is dense.*

Proof of Lemma 4.22. Suppose there is k for which the representation ρ_k is not dense. Then there is an open subset $A \subset \mathbf{S}^1$ such that $A \cap \rho_k\big(\pi_1\Sigma\big) = \emptyset$. Suppose without loss of generality that $k=1$. Then $\big(A \times \mathbf{T}^{n-1}\big) \cap \rho\big(\pi_1\Sigma\big) = \emptyset$. In particular ρ is not dense, hence a contradiction. $\square$

Proof of Theorem 4.21. Assume ρ has a dense image and suppose the $\mathbf{Z}$-module generated by the rows intersect $\pi\mathbf{Q}^{2g}$ that is $\mathrm{rk}_{\mathbf{Z}} \begin{pmatrix} \Theta_\rho \\ \pi \cdot \mathrm{I}_{2g} \end{pmatrix} < 2g+n$. Thus, there is a row Θ_i of Θ such that:

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} -\lambda_j \Theta_j + \big(\lambda_{n+1}\pi, \ldots, \lambda_{n+2g}\pi\big) = \lambda_i \Theta_i$$

with λ_i different to zero. Such a summation can be rewritten as:

$$\sum_{j=1}^{n} \lambda_j \Theta_j = \big(\lambda_{n+1}\pi, \ldots, \lambda_{n+2g}\pi\big) \tag{6}$$

for some $\lambda_j \in \mathbf{Z}$ and not all zero. Consider the matrix $M \in \mathrm{M}\big(n, \mathbf{Z}\big) \cap \mathrm{GL}_n(\mathbf{R})$ defined as:

$$M = I_n + (\lambda_i - 1)E_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda_j E_{ij} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ -\lambda_1 & -\lambda_2 & \cdots & \lambda_i & \cdots & -\lambda_n \\ \cdots & \cdots & \cdots & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

where the $E_{ij} = \big(e_{kl}\big)$ are the matrices with coefficients $e_{kl} = \delta_{ki}\delta_{lj}$. The matrix M defines a linear homeomorphism, say f_M of $\mathbf{R}^n$ with respect to the canonical basis because $\det M = \lambda_i$ which is different to zero. The mapping f_M sends $\mathbf{Z}^n$ to itself and descends to a finite-degree covering

$\overline{f}_M : \mathbf{T}^n \longrightarrow \mathbf{T}^n$ - in fact the degree coincides with the determinant of M. In particular the following equation holds: $\pi \circ f_M = \overline{f}_M \circ \exp$, where $\exp : \mathbf{R}^n \longrightarrow \mathbf{T}^n$ denotes as usual the canonical projection. Consider now the $\mathbf{Z}$-module $\langle \Theta_j \mid j = 1, \ldots, n \rangle_{\mathbf{Z}}$ generated by the rows of Θ_ρ. By Equation 6, a straightforward computation shows that its image via the mapping f_M is the $\mathbf{Z}$-module generated by the vectors

$$\left\langle \begin{pmatrix} \theta_{1,1} \\ \cdots \\ \lambda_{n+1}\pi \\ \cdots \\ \theta_{n,1} \end{pmatrix}, \ldots, \begin{pmatrix} \theta_{1,2g} \\ \cdots \\ \lambda_{n+2g}\pi \\ \cdots \\ \theta_{n,2g} \end{pmatrix} \right\rangle_{\mathbf{Z}}$$

Let us point out the following fact: As ρ is assumed to be dense in the torus, the image via the canonical projection in $\mathbf{T}^n$ of $\mathbf{Z}$-module generated by the rows of Θ_ρ fills a dense subset of $\mathbf{T}^n$, namely the image of ρ. As M commutes with the action of $2\pi\mathbf{Z}^n$ and pass through to the quotient as a finite-degree covering map of the $\mathbf{T}^n$, the $\mathbf{Z}$-module $M \cdot \langle \Theta_j \mid j = 1, ..., n \rangle_{\mathbf{Z}}$ is mapped on a dense subset of the torus. On the other hand, the projection of the i-th factor is discrete. Lemma 4.22 implies the desire contradiction.

We now prove the opposite implication and again we argue by contradiction. Suppose ρ does not have a dense image in the n-torus, then its closure is a k-dimensional sub-manifold, say S_0, of dimension $k < n$. We note that S_0 may not be connected in general. Indeed any closed subgroup of $\mathbf{T}^n$ is homeomorphic to $\mathbf{T}^d \times \frac{\mathbf{Z}}{m_1\mathbf{Z}} \times \cdots \times \frac{\mathbf{Z}}{m_{n-d}\mathbf{Z}}$. Assume first S_0 be connected; we shall deduce the general case later on. The subspace S_0 lifts to a linear subspace $\widetilde{S_0}$ of $\mathbf{R}^n$ which of course contains the $\mathbf{Z}$-module $\langle \Theta^j \mid j = 1, ..., 2g \rangle_{\mathbf{Z}}$ generated by the columns of Θ_ρ. We now invoke the following lemma.

Lemma 4.23. *There is $g \in \mathrm{SL}_{2g}(\mathbf{Z})$ such that :*

$$g \cdot \langle \Theta^j \mid j = 1, \ldots, 2g \rangle_{\mathbf{Z}} < \left\langle e_1, \ldots, e_k \right\rangle_{\mathbf{R}}$$

where the e_i's are the vectors of the canonical basis of $\mathbf{R}^n$.

Assume the lemma holds. The $\mathbf{Z}$-module $g \cdot \langle \Theta^j \mid j = 1, ..., 2g \rangle_{\mathbf{Z}}$ is contained in the first factor of $\mathbf{T}^n = \mathbf{T}^k \times \mathbf{T}^{n-k}$ and then Θ_ρ cannot have maximal row rank over $\mathbf{Z}$. As a consequence the matrix given in the equation 5 cannot have maximal row rank over $\mathbf{Z}$. The general case follows by applying the same reasoning to the component S_0^o of S_0 containing the identity which contains a finite-index $\mathbf{Z}$-module of $\langle \Theta^j \mid j = 1, ..., 2g \rangle_{\mathbf{Z}}$. In the general case, $g \cdot \langle \Theta^j \mid j = 1, ..., 2g \rangle_{\mathbf{Z}}$ is contained in $\mathbf{T}^n = \mathbf{T}^k \times F$, where F is isomorphic to the finite group $\frac{\mathbf{Z}}{m_1\mathbf{Z}} \times \cdots \times \frac{\mathbf{Z}}{m_{n-d}\mathbf{Z}}$. Let us proceed with the proof of Lemma 4.23.

Proof of Lemma 4.23. If $\widetilde{S_0}^{o}$ is contained in $\left\langle e_{\sigma(1)}, \ldots, e_{\sigma(k)} \right\rangle_{\mathbf{R}}$ for some $\sigma \in \mathfrak{S}_n$ then it is sufficient to rename the coordinates. This corresponds to a matrix g obtained by product of elementary matrices.

Assume $\widetilde{S_0}^o$ is not contained in any such a space. Let x_i be the intersection of $\widetilde{S_0}^o$ with the affine space $e_i + \mathbf{R}^{n-k}$ and let d_i its Euclidean distance to $\mathbf{R}^k$. Then x_i has the following form:

$$x_i = \big(0, \ldots, 1, \ldots, 0, t_1, \ldots, t_{n-k}\big)$$

where $t_i \in \mathbf{Q}$. In fact, if this had been not true then S_0^o would have been a dense subspace of dimension $k+1$ in the torus. As a consequence $d_i^2 \in \mathbf{Q}$ for any $i = 1, \ldots, k$ and $\widetilde{S_0}^o$ is described by $n-k$ equations with integer coefficients. Look at the set $\widetilde{S_0}^o \cap \mathbf{Z}^n$. This is a lattice in $\widetilde{S_0}^o$ and there is a basis $v_1, \ldots, v_k$ made of integer vectors. We invoke Cassels 1997, Corollary 3, pag.14 to claim the existence of $n-k$ vectors $v_{k+1}, \ldots, v_n$ such that the vectors $v_1, \ldots, v_n$ gathered together form a basis for $\mathbf{Z}^n$. Since $\mathrm{SL}_{2g}(\mathbf{Z})$ acts transitively on the space of lattices, there is g such that

$$g \cdot \langle \Theta^j \mid j = 1, \ldots, 2g\rangle_{\mathbf{Z}} < g \cdot \widetilde{S_0} = \big\langle e_1, \ldots, e_k \big\rangle_{\mathbf{R}}.$$

This concludes the proof of Lemma 4.23 and indeed the proof of Theorem 4.21. □

From the proof we deduce that the row rank of the matrix Θ_ρ has a very explicit geometric interpretation, in fact it coincides with the dimension of the subspace containing the image of ρ. Of course, the proof does not depend on the presentation of $\pi_1\Sigma$ either on the lattice chosen. Let us prove these facts.

Independence on the chosen basis. Let $\langle \Theta^j \mid j = 1, \ldots, 2g\rangle_{\mathbf{Z}}$ be the $\mathbf{Z}$-module generated by the columns of Θ_ρ. In section 4.5.1 we have seen that the change of basis corresponds to multiply on the right the matrix Θ_ρ with a matrix $A \in \mathrm{SL}_{2g}(\mathbf{Z})$. Since the row rank of Θ_ρ is invariant under the action by right-multiplication of $\mathrm{SL}_{2g}(\mathbf{Z})$, the matrices Θ_ρ and $\Theta_\rho A$ have the same row rank. Furthermore, in the light of Proposition 4.20, $\pi\mathbf{Q}$-freedom is also invariant under the right action of $\mathrm{SL}_{2g}(\mathbf{Z})$. On the other hand, let Ξ be the closure of the subspace of $\mathbf{T}^n$ generated by the columns of Θ_ρ. Its lift $\widetilde{S}$ is a linear subspace of $\mathbf{R}^n$ described by $n-k$ equations. As the columns of $\Theta_\rho A$ satisfy the same equations, the image of unaffected by the change of basis. This proves the independence on the basis chosen. □

Independence on the lattice chosen. Given two lattices Λ_1 and $\Lambda_2 = A \cdot \Lambda_1$ for some element $A \in \mathrm{SL}_n(\mathbf{Z})$. Such a map A descends to a homeomorphism of the n-torus and hence the $\mathbf{Z}$-module $\langle \Theta^j \mid j = 1, \ldots, 2g\rangle_{\mathbf{Z}}$ projects to a dense subset of the torus if and only if its image via A projects to dense subset as well. On the other hand, the row rank of the associated matrix $\Theta_\rho(\Lambda_1)$ equals the one of $\Theta_\rho(\Lambda_2)$ because the row rank is invariant under the action by left-multiplication of $\mathrm{SL}_n(\mathbf{Z})$. Again, Proposition 4.20 implies $\pi\mathbf{Q}$-freedom is invariant under the left action of $\mathrm{SL}_n(\mathbf{Z})$. Hence the conclusion. □

We finally provide a couple of explicit examples.

Example 4.24. *Let Σ be a surface of genus 2, and let $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^2 \cong \mathbf{S}^1 \times \mathbf{S}^1$ be the representation such that $\rho(a_1) = \rho(a_2) = \big(e^{i\varphi}, e^{i\varphi}\big)$, where $\varphi \in \mathbf{R} \setminus \pi\mathbf{Q}$, and $\rho(b_1) = \rho(b_2) = (1, 1)$.*

The matrix Θ has the following form

$$\begin{pmatrix} \varphi & 0 & \varphi & 0 \\ \varphi & 0 & \varphi & 0 \end{pmatrix}$$

If $\gamma \in \pi_1\Sigma$, then $\rho(\gamma) = \rho(a_1)^{k_1}\rho(b_1)^{k_2}\rho(a_2)^{k_3}\rho(b_2)^{k_4}$ with $k_i \in \mathbf{Z}$. Consider the vector $v = (k_1, k_2, k_3, k_4)$, then $\Theta \cdot v = \big((k_1 + k_3)\varphi, (k_1 + k_3)\varphi\big)$. The image of ρ is densely contained the main diagonal. Both projections are dense in $\mathbf{S}^1$, but the image does not fill $\mathbf{T}^2$. Notice that the row rank of Θ over $\mathbf{Z}$ is 1 as the dimension of the smallest subspace containing $\rho\big(\pi_1\Sigma\big)$.

Example 4.25. *Let Σ be a surface of genus 2, and let $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^2 \cong \mathbf{S}^1 \times \mathbf{S}^1$ be the representation such that $\rho(a_1) = \rho(a_2) = \big(e^{i\varphi}, 1\big)$ and $\rho(b_1) = \rho(b_2) = \big(1, e^{i\varphi}\big)$ with $\varphi \in \mathbf{R} \setminus \pi\mathbf{Q}$.*

The matrix Θ has the following form

$$\begin{pmatrix} \varphi & 0 & \varphi & 0 \\ 0 & \varphi & 0 & \varphi \end{pmatrix}$$

If $\gamma \in \pi_1\Sigma$, then $\rho(\gamma) = \rho(a_1)^{k_1}\rho(b_1)^{k_2}\rho(a_2)^{k_3}\rho(b_2)^{k_4}$ with $k_i \in \mathbf{Z}$. Consider the vector $v = (k_1, k_2, k_3, k_4)$, then $\Theta \cdot v = \big((k_1 + k_3)\varphi, (k_2 + k_4)\varphi\big)$. The image of ρ densely fills the torus. Notice that the rank of Θ is 2 in this case and both projections are dense.

4.8 $\mathrm{Sp}_{2g}(\mathbf{Z})$-action and orbit closures

The symplectic group $\mathrm{Sp}_{2g}(\mathbf{Z})$ acts on the homological representation space $\mathrm{Hom}\big(\mathrm{H}_1(\Sigma, \mathbf{Z}), \mathbf{T}^n\big)$ by precomposition. We have seen in section 4.5 that, up to a choice of a symplectic basis, this latter space identifies with the space $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$. In this section we would like to study the orbit closures of an element of $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$ under the action of $\mathrm{Sp}_{2g}(\mathbf{Z})$. The first thing we notice is that a subset $\Omega \subset \mathrm{M}\big(n, 2g; \mathbf{R}\big)$ is invariant under the action $\mathrm{Sp}_{2g}(\mathbf{Z}) \ltimes \mathrm{M}\big(n, 2g; 2\pi\mathbf{Z}\big)$ if and only if its projection onto $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$ is $\mathrm{Sp}_{2g}(\mathbf{Z})$-invariant. This simple remark legitimates us to study the orbit closures on the universal cover, that is $\mathrm{M}\big(n, 2g; \mathbf{R}\big)$.

Let us consider the group $G = \mathrm{Sp}_{2g}(\mathbf{R}) \ltimes \mathrm{M}\big(n, 2g; \mathbf{R}\big)$. Given two elements $(A,\, a)$ and $(B,\, b)$, their product is defined as follows $(A,\, a) \cdot (B,\, b) = (AB,\, bA^{-1} + a)$. The group G acts transitively on the space $\mathrm{M}\big(n, 2g; \mathbf{R}\big)$ with the action being defined as $p \cdot (A,\, a) = pA^{-1} + a$ – indeed a point $p \in \mathrm{M}\big(n, 2g; \mathbf{R}\big)$ may be regarded as the couple (I, p). The space $\mathrm{M}\big(n, 2g; \mathbf{R}\big)$ naturally identifies with the G/P, where P is the stabiliser of any point. It is straighforward to check that the stabiliser of the zero matrix is nothing but $\mathrm{Sp}_{2g}(\mathbf{R})$. The subgroup $\Gamma = \mathrm{Sp}_{2g}(\mathbf{Z}) \ltimes \mathrm{M}\big(n, 2g; 2\pi\mathbf{Z}\big)$ is a lattice in G and acts in the obvious way on G/S. Under these conditions we are in the right position to apply Ratner's Theorem, see **RM**, which we state as follows according to our setting.

Ratner's Theorem. *Let G, P, Γ as above, let $p \in \mathrm{M}\big(n, 2g; \mathbf{R}\big) = G/P$ and let $\gamma \in G$ such that $p = \gamma\, P$. Then there is a closed and connected subgroup H_γ such that the following holds.*

- $P_\gamma = \gamma\, P\, \gamma^{-1} \leq H_\gamma$,
- $\Gamma \cap H_\gamma$ *is a lattice in* H_γ *and*
- $\overline{\Gamma \cdot p} = \Gamma\, H_\gamma\, p$.

Notice that γ can be taken as $(I,\, p)$. Since our goal here is to classify the closures of Γ-orbits of any point in the space $\mathrm{M}\big(n, 2g; \mathbf{R}\big)$, we just need to figure out which subgroups of G may be provided by Ratner's Theorem. To this purpose, let us consider the projection $\Phi : \mathrm{Sp}_{2g}(\mathbf{R}) \ltimes \mathrm{M}\big(n, 2g; \mathbf{R}\big) \longrightarrow \mathrm{Sp}_{2g}(\mathbf{R})$. Given a point p in $\mathrm{M}\big(n, 2g; \mathbf{R}\big)$, the group H_γ is isomorphic to the semidirect product $H_\gamma = P_\gamma \ltimes K_\gamma$, where K_γ is defined as $\ker \Phi \cap H_\gamma$, that is the kernel of the mapping Φ restricted to H_γ. Notice that the image of H_γ under the mapping Φ is the whole group $\mathrm{Sp}_{2g}(\mathbf{R})$ because $H_\gamma \geq P_\gamma \cong \mathrm{Sp}_{2g}(\mathbf{R})$. In particular, $H_\gamma \cong \mathrm{Sp}_{2g}(\mathbf{R}) \ltimes K_\gamma$.

Let us proceed on understanding K_γ. The first thing we notice is that any change of lattice $h \in \mathrm{SL}_{2g}(\mathbf{Z})$ extends to a homeomorphism ϕ_h of G defined as

$$\phi_h : G \longrightarrow G, \qquad \phi_h(A,\, a) = (A, ha).$$

This is an automorphism of G and its restriction to $\ker \Phi$, where Φ is the projection just defined above, is linear and corresponds to a change of lattice in $\mathrm{M}\big(n, 2g; \mathbf{R}\big)$. In particular, the relation

$$\Gamma \cdot (h\, p) = \phi_h\big(\Gamma \cdot p\big) \tag{7}$$

holds for any $p \in \mathrm{M}\big(n, 2g; \mathbf{R}\big)$. As a consequence of Lemma 4.23, there is an element $h \in \mathrm{SL}_{2g}(\mathbf{Z})$ such that $h \cdot p$ is of the following form

$$\begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,2g} \\ \vdots & & \vdots \\ \theta_{k,1} & \cdots & \theta_{k,2g} \\ \pi\, q_{k+1,1} & \cdots & \pi\, q_{k+1,2g} \\ \vdots & & \vdots \\ \pi\, q_{n,1} & \cdots & \pi\, q_{n,2g} \end{pmatrix} = \begin{pmatrix} \Theta_o \\ \pi Q \end{pmatrix} \tag{8}$$

where

- $\Theta_o \in \mathrm{M}\big(k, 2g; \mathbf{R}\big)$ for some $0 \leq k \leq n$,
- $\pi Q \in \mathrm{M}\big(n-k, 2g; \pi\mathbf{Q}\big)$,
- $\big\langle (\theta_{i,1}, \ldots, \theta_{i,2g}) \,:\, i = 1, \ldots, k \big\rangle$ is $\pi\mathbf{Q}$-free, and thus

- the vectors $\{(\theta_{i,1},\ldots,\theta_{i,2g})\}_{i=1,\ldots,k}$ are linearly independent over $\mathbf{Z}$;

and hence it is sufficient to study K_γ for $\gamma=(I,p)$ and p is a matrix given in Equation 8. Furthermore, it will be sufficient to study the closures of Γ-orbits for matrices in these form. The second thing we notice is that K_γ is a linear subspace of $\mathrm{M}\big(n,2g;\mathbf{R}\big)$ invariant under the action of P_γ by conjugation. Indeed, suppose $\gamma=(I,\,p)$, let $q=(I,\,q)\in K_\gamma$ be any point and let $(A,p-pA^{-1})$, we can write as the element $(I,p)\cdot(A,\mathrm{id})\cdot(I,-p)$, be a generic element of P_γ. Then

$$(A,p-pA^{-1})\cdot(I,\,q)\cdot(A^{-1},p-pA)=(I,\,qA^{-1})\in K_\gamma$$

as claimed. The following Lemma implies Theorem D for representations of closed surface groups into the unit circle $\mathbf{S}^1$.

Lemma 4.26. *Let $\overline{\theta}=\big(\theta_1,\ldots,\theta_{2g}\big)\in\mathbf{R}^{2g}$. If $\overline{\theta}\in\pi\mathbf{Q}^{2g}$, then $\big(\mathrm{Sp}_{2g}(\mathbf{Z})\ltimes 2\pi\mathbf{Z}^{2g}\big)\cdot\overline{\theta}$ is discrete in $\mathbf{R}^{2g}$. Otherwise, if $\overline{\theta}\in\mathbf{R}^{2g}\setminus\pi\mathbf{Q}^{2g}$, then $\mathrm{Sp}_{2g}(\mathbf{Z})\cdot\overline{\theta}$ is dense in $\mathbf{T}^{2g}$ and hence $\big(\mathrm{Sp}_{2g}(\mathbf{Z})\ltimes 2\pi\mathbf{Z}^{2g}\big)\cdot\overline{\theta}$ is dense in $\mathbf{R}^{2g}$.*

Proof. Let Λ be the subgroup of $\mathbf{R}$ generated by the entries of $\overline{\theta}$ and consider Λ^{2g}. The first claim is easy to establish. In this case Λ^{2g} is a lattice in $\mathbf{R}^{2g}$ containing $2\pi\mathbf{Z}^{2g}$ and preserved by $\mathrm{Sp}_{2g}(\mathbf{Z})$. Now observe that the $\mathrm{Sp}_{2g}(\mathbf{Z})$-orbit of $\overline{\theta}$ is contained in Λ^{2g}. Suppose $\overline{\theta}\in\mathbf{R}^{2g}\setminus\pi\mathbf{Q}^{2g}$, hence there exists $\theta_i\in\mathbf{R}\setminus\pi\mathbf{Q}$. There is an element in $\mathrm{Sp}_{2g}(\mathbf{R})$ such that all the entries are $\mathbf{R}\setminus\pi\mathbf{Q}$. We may assume $\theta_i\in\big[0,2\pi\big[$ for all $i=1,\ldots,2g$. Let $\theta^*\in\mathbf{T}^{2g}$ be any point. For each couple $(\theta_{2i-1},\,\theta_{2i})$, where $i=1,\ldots,g$, there are two integers $k_i,\,h_i$ such that the couple $\big(k_i\,\theta_{2i-1}+\theta_{2i},\,(k_i\,h_i-1)\theta_{2i-1}+h_i\,\theta_{2i}\big)$ is closed to $(\theta^*_{2i-1},\,\theta^*_{2i})$. Therefore the $\mathrm{Sp}_{2g}(\mathbf{R})$-orbit of $\overline{\theta}$ is dense in $\mathbf{T}^{2g}$ and hence $\mathrm{Sp}_{2g}(\mathbf{Z})\ltimes 2\pi\mathbf{Z}^{2g}\cdot\overline{\theta}$ is dense in $\mathbf{R}^{2g}$ as desired. $\square$

Before proving the general case we need the following proposition on which we describe the group K_γ.

Proposition 4.27. *Let $p\in X$ be any point in the form given in the equation* (8) *and let k the number of lines not in $\pi\mathbf{Q}^{2g}$. Let H_γ be the group provided by Ratner's Theorem, where $\gamma=(I,\,p)$. Then K_γ is trivial or $K_\gamma\cong\mathrm{M}\big(k,2g;\mathbf{R}\big)$.*

Proof of Proposition 4.27. Let p be any point in $\mathrm{M}\big(n,2g;\mathbf{R}\big)$. Assume p be different from the zero matrix for which the claim trivially holds. Let us begin with the case $p=\pi Q\in\mathrm{M}\big(n,2g;\pi\mathbf{Q}\big)$, that means $k=0$. We claim K_γ to be trivial. Let $\gamma=(I,p)$ and let H_γ be the group provided by Ratner's Theorem. The orbit $\Gamma\cdot p$ lies in the subgroup of $\mathrm{M}\big(n,2g;\mathbf{R}\big)$ generated by the matrices $\pi q_{ij}\,E_{ij}$, where πq_{ij} are the entries of p, which is discrete and closed. This means that $\overline{\Gamma\cdot p}=\Gamma\cdot p$ and implies H_γ is the stabiliser of p. Therefore $H_\gamma=P_\gamma$ and hence K_γ is trivial. Notice that this argument generalises the first case of the previous Lemma 4.26. Let us now assume $k>0$. The linear space K_γ is completely determined by Θ_o, indeed the block πQ does not give any contribution. In this case, the orbit $\Gamma\cdot p$ is no longer closed and the $\mathrm{Sp}_{2g}(\mathbf{Z})$-orbit of p is contained in some linear subspace of $\mathrm{M}\big(k,2g;\mathbf{R}\big)$ of dimension $2g\,l$, where l is the dimension of the linear space generated by the rows of Θ_o. Hence K_γ contains V as a proper subspace. We can notice that V is $\mathrm{Sp}_{2g}(\mathbf{Z})$-invariant but $V \cap \mathrm{M}\big(k, 2g; 2\pi\mathbf{Z}\big)$ is not a lattice because the $\mathbf{Z}$-module generated by the rows of Θ_o is $\pi\mathbf{Q}$-free. For the same reason, the minimal linear space containing V and a lattice is $\mathrm{M}\big(k, 2g; \mathbf{R}\big)$, hence $K_\gamma = \mathrm{M}\big(k, 2g; \mathbf{R}\big)$. □

From the proof of Proposition 4.27 we can deduce the following corollary.

Corollary 4.28. *Let $p \in \mathrm{M}\big(n, 2g; \mathbf{R}\big)$ be any point in the form given in the equation (8) and let k the number of lines not in $\pi\mathbf{Q}^{2g}$. There exists a closed connected subgroup $H \leq \mathbf{T}^n$ of dimension k such that $\overline{\Gamma \cdot p}$ projects to a finite union of inhomogeneous torii of dimension k corresponding to cosets of H. In particular, the modular orbit of a dense representation $\rho : \mathrm{H}_1(\Sigma, \mathbf{Z}) \longrightarrow \mathbf{T}^n$ is dense in the representation space.*

This corollary implies Theorem F and indeed Theorem D. In the appendix, we shall study the modular orbits by applying a direct approch without rely on Ratner's Theory.

4.9 An application: Approximation result

The aim of this final section consists in showing Proposition 4.9 and indeed Theorem E. Let us begin by recalling the statement of Kronecker's Theorem as formulated in Hewitt and Ross 1963, Section 26.19(e). The reader may also consult Bekka and Mayer 2000, Section 1.12(iii) for another one-dimensional version of Kronecker's theorem.

Kronecker's Approximation Theorem. *Let $b^{(i)} = \big(b_1^{(i)}, \dots, b_m^{(i)}\big)$, with $i = 1, \dots, n$, be vectors of $\mathbf{R}^m$ such that $b^{(1)}, \dots, b^{(n)}$, $\pi e_1, \dots, \pi e_m$ are linearly independent over $\mathbf{Q}$ in the vector space $\mathbf{R}^m$ (where the e_j's form the canonical basis of $\mathbf{R}^m$). Let $a_1, \dots, a_n$ be any real numbers and let ε be a positive number. Then there is an element $\big(k_1, \dots, k_m\big) \in \mathbf{Z}^m$ such that*

$$\Big| a_i - \sum_{l=1}^{m} k_l b_l^{(i)} \Big| < \varepsilon \mod 2\pi \tag{9}$$

for every $i = 1, \dots, n$.

For a real a, the expression $|a| < \varepsilon$ mod 2π means that $|a - 2k\pi| < \varepsilon$ for some integer k. From the equation (9) above, one can easily infer the equivalent estimate

$$\Big\| \big(a_1, \dots, a_n\big)^t - 2\pi\big(h_1, \dots, h_n\big)^t - B\,\big(k_1, \dots, k_m\big)^t \Big\| < C\varepsilon \tag{10}$$

where $\big(h_1, \dots, h_n\big) \in \mathbf{Z}^n$, B is the matrix having $b^{(i)}$'s as rows, C is a real constant depending only on n and $\|\cdot\|$ is any norm on $\mathbf{R}^n$. Kronecker's theorem generalises to simultaneous approximation of l given real vectors $a^{(j)} = \big(a_{1j}, \dots, a_{nj}\big)^t$ where $j = 1, \dots, l$. Indeed, for any $\varepsilon > 0$ there is a matrix $K \in \mathrm{M}\big(m, l; \mathbf{Z}\big)$ such that

$$\Big\| A - 2\pi H - B\,K \Big\| < C\varepsilon \tag{11}$$

where A is the matrix having $a^{(j)}$'s as columns, $H \in \mathrm{M}\big(n, l; \mathbf{Z}\big)$ and C is a constant depending only on l, n. That is

$$\Big\| A - B\,K \bmod 2\pi \Big\| < \varepsilon. \tag{12}$$

Let Σ be a closed surface of genus greater than zero, let $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$ be a representation and let Θ_ρ be the associated matrix in the sense of Definition 4.15.

Proposition 4.9. *The following are equivalent.*

1. $\mathrm{Mod}(\Sigma) \cdot \rho$ *is dense in the representation space.*
2. *For any matrix* $A \in \mathrm{M}\big(n, 2g; \mathbf{R}\big)$ *and any* $\varepsilon > 0$ *there is a matrix* $g \in \mathrm{Sp}_{2g}(\mathbf{R})$ *such that*

$$\big\| A - \Theta_\rho\, g \bmod 2\pi \big\| < \varepsilon.$$

Proof of Proposition 4.9. Each representation is identified with its associated matrix and the representation space with $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$. Suppose $\mathrm{Mod}(\Sigma)\cdot\Theta_\rho$ is not dense in the representation space. Then there is an open set U such that $\mathrm{Mod}(\Sigma) \cdot \Theta_\rho \cap U = \phi$. Let A be any matrix in U and ε a strictly positive real number such that the open ball $B_\varepsilon(A) \subset U$. Then, for any $g \in \mathrm{Mod}(S)$ the following estimate $\big\| A - \Theta_\rho\, g \big\| > \varepsilon \bmod 2\pi$ holds. As the action of the Torelli subgroup is trivial by Proposition 4.10, the action of the mapping class group coincides with the action of $\mathrm{Sp}_{2g}(\mathbf{Z})$. Therefore, Theorem E implies Theorem D.
Suppose $\mathrm{Mod}(\Sigma) \cdot \Theta_\rho$ dense in the representation space. Then, for any $A \in \mathrm{M}\big(n, 2g; \mathbf{T}\big)$ and for any $\varepsilon > 0$ the mapping class group orbit intersects the open set $B_\varepsilon(A) \subset \mathrm{M}\big(n, 2g; \mathbf{T}\big)$, *i.e.* there is an element $g \in \mathrm{Mod}(S)$ such that $g^{-1} \cdot \Theta_\rho = \Theta_\rho\, g \in B_\varepsilon(A)$. In particular, $\big\| A - \Theta_\rho\, g \bmod 2\pi \big\| < \varepsilon$. Once again, by Proposition 4.10, the matrix g can be taken in $\mathrm{Sp}_{2g}(\mathbf{R})$ and so Theorem D implies Theorem E as desired. $\square$

4.10 Dense Orbits and further discussion

In this subsection we are going to prove Theorem D for almost every representation without relying on Ratner's Theorem. We begin consider the genus one case and we shall use it to extend the discussion to surfaces of arbitrary genus.

4.10.1 Direct proof of Theorem F for almost every representations

The set of matrices $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$ contains, as a proper subset, the space $\mathcal{D}$ of all of those matrices of the following form

$$\begin{pmatrix} \theta_1 & \theta_2 & \cdots & \theta_{2i-1} & \theta_{2i} & \cdots & \theta_{2g-1} & \theta_{2g} \\ \lambda_2\theta_1 & \lambda_2\theta_2 & \cdots & \lambda_2\theta_{2i-1} & \lambda_2\theta_{2i} & \cdots & \lambda_2\theta_{2g-1} & \lambda_2\theta_{2g} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \lambda_j\theta_1 & \lambda_j\theta_2 & \cdots & \lambda_j\theta_{2i-1} & \lambda_j\theta_{2i} & \cdots & \lambda_j\theta_{2g-1} & \lambda_j\theta_{2g} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \lambda_n\theta_1 & \lambda_n\theta_2 & \cdots & \lambda_n\theta_{2i-1} & \lambda_n\theta_{2i} & \cdots & \lambda_n\theta_{2g-1} & \lambda_n\theta_{2g} \end{pmatrix} \tag{13}$$

where $\big(\theta_1, \theta_2, \ldots, \theta_{2g-1}, \theta_{2g}\big) \in \mathbf{R}^{2g} \setminus \pi\mathbf{Q}^{2g}$ is the lift of $\big(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_{2g-1}}, e^{i\theta_{2g}}\big) \in \mathbf{T}^{2g}$ contained in $\big[0, 2\pi\big[^{2g}$ and the reals $\{1, \lambda_2, \ldots, \lambda_n\} \subset \mathbf{R}$ are linearly independent over $\mathbf{Q}$.

Lemma 4.29. *$\mathcal{D}$ is dense in* $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$.

Proof. Let $\lambda_2, \ldots, \lambda_n$ real numbers such that $1, \lambda_2, \ldots, \lambda_n$ are linearly independent over $\mathbf{Q}$. Consider the mapping $\varphi : \mathbf{R}^{2g} \longrightarrow \mathbf{T}^{2gn}$ defined as

$$\Big(\theta_1, \theta_2, \ldots, \theta_{2g-1}, \theta_{2g}\Big) \mapsto \Big(\big(e^{i\theta_1}, \ldots, e^{i\theta_{2g}}\big), \big(e^{i\lambda_2\theta_1}, \ldots, e^{i\lambda_2\theta_{2g}}\big), \ldots, \big(e^{i\lambda_n\theta_1}, \ldots, e^{i\lambda_n\theta_{2g}}\big)\Big).$$

This mapping factors through a mapping $\overline{\varphi} : \mathbf{R}^{2g} \longrightarrow \mathbf{R}^{2gn}$ such that $\varphi = \exp \circ \overline{\varphi}$ and exp is the exponential mapping thought as in equation (4) introduced in section 4.5. The image of $\overline{\varphi}$ is a $2g$-dimensional linear subspace. As $1, \lambda_2, \ldots, \lambda_n$ are linearly independent over $\mathbf{Q}$, then the projection via the exponential mapping is dense in $\mathbf{T}^{2gn}$. The space $\mathcal{D}$ is defined as the union of the images for each possible subset $\{\lambda_2, \ldots, \lambda_n\} \subset \mathbf{R}$ such that $1, \lambda_2, \ldots, \lambda_n$ are linearly independent over $\mathbf{Q}$. Therefore $\mathcal{D}$ is dense. $\square$

The following lemma is easy to establish.

Lemma 4.30. *$\mathcal{D}$ is* $\mathrm{Sp}_{2g}(\mathbf{Z})$*-invariant.*

Let us consider first surfaces of genus one. Let T be the torus, let $\rho : \pi_1 T \longrightarrow \mathbf{T}^n$ be a dense representation and let Θ_ρ be its associated matrix with respect to some basis $\big\{a, b\big\}$ and the standard lattice of $\mathbf{R}^n$. Let Ω_ρ be the $\mathrm{SL}_2(\mathbf{Z})$-orbit of Θ_ρ in $\mathrm{M}\big(n, 2; \mathbf{T}\big)$. The associated matrix Θ_ρ has the following form:

$$\Theta_\rho = \begin{pmatrix} \theta_1 & \theta_2 \\ \vdots & \vdots \\ \lambda_i\theta_1 & \lambda_i\theta_2 \\ \vdots & \vdots \\ \lambda_n\theta_1 & \lambda_n\theta_2 \end{pmatrix} \tag{14}$$

where $\big(\theta_1,\, \theta_2\big) \in \mathbf{R}^2$ is the lift of $\big(e^{i\theta_1},\, e^{i\theta_2}\big) \in \mathbf{T}^2$ contained in $\big[0, 2\pi\big[^2$ and $\lambda_i \in \mathbf{R}$, for any $i = 2, \ldots, n$, are linearly independent over $\mathbf{Q}$. Set

$$\overline{\Theta}_\rho = \begin{pmatrix} \Theta_\rho \\ \pi \cdot \mathrm{I}_2. \end{pmatrix}$$

Since the representation ρ is assumed to have a dense image, the matrix $\overline{\Theta}_\rho$ has maximal row rank, that is $\mathrm{rk}_{\mathbf{Z}} = n + 2$. This implies the following properties of the matrix Θ_ρ above.

14.i The real numbers θ_1 and θ_2 cannot be both elements of $\pi\mathbf{Q}$. If this were the case, the row rank of the matrix $\overline{\Theta}_\rho$ would fail to be maximal, contradicting our assumptions. In the case one of them is an element of $\pi\mathbf{Q}$, we can change the basis in such a way they are both elements of $\mathbf{R} \setminus \pi\mathbf{Q}$. Indeed, assume without loss of generality that $\theta_2 \in \pi\mathbf{Q}$. The Dehn twist tw_a along a maps the curve b to ab and hence $\rho(b)$ is mapped to $\rho(ab)$. The second column of Θ_ρ changes accordingly and the element of place $(1, 2)$ of $\Theta_{\mathrm{tw}_a \cdot \rho}$ is nothing else that $\theta_1 + \theta_2$. As $\theta_1 \notin \pi\mathbf{Q}$ the same necessarily holds for $\theta_1 + \theta_2$. In what follows, we shall assume both $\theta_1, \theta_2 \notin \pi\mathbf{Q}$.

14.ii The real numbers $\pi, \theta_1, \ldots, \lambda_i\theta_1, \ldots, \lambda_n\theta_1$ are linearly independent over $\mathbf{Q}$. Indeed, if this were not the case then one can easily check that $\overline{\Theta}_\rho$ has not maximal rank. This implies that the subgroup of $\mathbf{T}^n$ generated by the vector $\big(\theta_1, \ldots, \lambda_i\theta_1, \ldots, \lambda_n\theta_1\big)$ is dense in $\mathbf{T}^n$, see Bekka and Mayer 2000, Exercise 1.13. The same holds also for the real numbers $\pi, \theta_2, \ldots, \lambda_i\theta_2, \ldots, \lambda_n\theta_2$.

14.iii The real numbers $1, \lambda_2, \ldots, \lambda_n$ are linearly independent over $\mathbf{Q}$. If this were not the case, then there would be $a_1, \ldots, a_n \in \mathbf{Q}$ such that $a_1 + a_2\lambda_2 + \cdots + a_n\lambda_n = 0$. In particular,

$$\sum_{i=1}^{n} a_i\lambda_i\big(\theta_1, \theta_2\big) = 0,$$

with $\lambda_1 = 1$. That is the rows of Θ_ρ are linearly dependent over $\mathbf{Q}$. Therefore the row rank cannot be maximal, a contradiction. In particular, $\lambda_i \notin \mathbf{Q}$ for every $i = 2, \ldots, n$. In what follows, we shall sometimes refer to 1 as λ_1 for simplify the formulas.

We begin with considering the $\mathrm{SL}_2(\mathbf{Z})$ action on the space $\mathrm{M}\big(n, 2; \mathbf{R}\big)$ seen as the universal cover of $\mathrm{M}\big(n, 2; \mathbf{T}\big)$, see also Remark 4.15. Given the matrix Θ_ρ as in 14, there is a unique lift, say $\Theta(\rho)$, in $\mathrm{M}\big(n, 2; \mathbf{R}\big)$ which is still of the form of Equation 14. Notice that such a matrix is the unique one who has all the entries in the interval $[0,\, 2\pi)$. Let us finally denote with $\Omega(\rho)$ the $\mathrm{SL}_2(\mathbf{Z})$-orbit of $\Theta(\rho)$ in $\mathrm{M}\big(n, 2; \mathbf{R}\big)$.

Since $\Theta(\rho)$ is of the form of Equation 14, an easy computation shows that the matrix $A \cdot \Theta(\rho) \in \Omega(\rho)$ is still of the form 14 for any $A \in \mathrm{SL}_2(\mathbf{Z})$, that is the i-th row of $A \cdot \Theta(\rho)$ is λ_i-times $\big(\theta_1,\, \theta_2\big)A^{-1}$.

Therefore, we can deduce that $\Omega(\rho)$ is contained in some proper linear subspace S of $\mathbf{R}^{2n}$. In fact, the coefficients of any matrix $A \cdot \Theta(\rho) = (\theta_{i,j})_{i,j} \in \Omega(\rho)$ satisfy the following homogeneous linear system

$$\mathcal{S}: \begin{cases} \theta_{2,1} - \lambda_2 \theta_{1,1} &= 0 \\ \quad \vdots \\ \theta_{n,1} - \lambda_n \theta_{1,1} &= 0 \\ \theta_{2,2} - \lambda_2 \theta_{1,2} &= 0 \\ \quad \vdots \\ \theta_{n,2} - \lambda_n \theta_{1,2} &= 0 \end{cases} \tag{15}$$

in $2n-2$ equations and $2n$ variables. Hence, S is defined as the full space of solutions of the linear system $\mathcal{S}$. Let us consider then the subspace S. Since each of the λ_i is taken as an element of $\mathbf{R} \setminus \mathbf{Q}$, the subspace S meets the lattice $\mathrm{M}\big(n, 2; 2\pi\mathbf{Z}\big)$ only at the origin. Therefore, the projection of the subspace S into the space $\mathrm{M}\big(n, 2; \mathbf{T}\big)$ densely fills a closed subspace K of $\mathrm{M}\big(n, 2; \mathbf{T}\big)$. We finally claim that K cannot be a proper subspace. To this end, we begin with noting that, due to the nature of the linear system $\mathcal{S}$, the subspace S splits as the direct product $V_1 \times V_2$ inside the space $\mathbf{R}^n \times \mathbf{R}^n \cong \mathrm{M}\big(n, 2; \mathbf{R}\big)$. Therefore, the image of S into the space $\mathrm{M}\big(n, 2; \mathbf{T}\big)$ lies inside a closed subgroup of the form $H_1 \times H_2$, where $H_i < \mathrm{M}\big(n, 1; \mathbf{T}\big) \cong \mathbf{T}^n$, for $i = 1, 2$. Notice that K is a proper subgroup of $\mathrm{M}\big(n, 2; \mathbf{T}\big)$ if and only if H_i is a proper subgroup of $\mathrm{M}\big(n, 1; \mathbf{T}\big)$. Therefore the proof of the final claim boils down to show that H_i cannot be a proper subgroup for both $i = 1, 2$. As the group H_1 contains the vector $\exp\big(\theta_1, \lambda_2\theta_1, \dots, \lambda_n\theta_1\big)$, then it contains also the subgroup $\big\{ \exp\big(t\big(\theta_1, \lambda_2\theta_1, \dots, \lambda_n\theta_1\big)\big) \mid t \in \mathbf{Z}\big\}$ and thus its closure which we know to be equal to the full space $\mathbf{T}^n$. In the same fashion, we can prove $H_2 = \mathbf{T}^n$. Therefore $K = \mathrm{M}\big(n, 2; \mathbf{T}\big)$ and the $\mathrm{SL}_2(\mathbf{Z})$-orbit of Θ_ρ is dense in $\mathrm{M}\big(n, 2; \mathbf{T}\big)$ as desired.

The general case for surfaces of genus greater than one works similarly. Given any matrix of the form as in equation (13), up to change the matrix with any element of $\mathrm{Sp}_{2g}(\mathbf{Z})$ we may assume, without loss of generality, that at least one of $\theta_{2i-1}, \theta_{2i} \notin \pi\mathbf{Q}$. Under this condition all the observations of subsection 5.10.1 hold for each pair of columns

$$\begin{pmatrix} \theta_{2i-1} & \theta_{2i} \\ \vdots & \vdots \\ \lambda_j\theta_{2i-1} & \lambda_j\theta_{2i} \\ \vdots & \vdots \\ \lambda_n\theta_{2i-1} & \lambda_n\theta_{2i} \end{pmatrix} \tag{16}$$

Therefore the action of the g-times product $\mathrm{SL}_2(\mathbf{Z}) \times \cdots \times \mathrm{SL}_2(\mathbf{Z}) < \mathrm{Sp}_{2g}(\mathbf{Z})$ provides a dense orbit inside the space $\mathrm{M}\big(n, 2g; \mathbf{T}\big)$ as desired.

4.10.2 Finding curve generating dense subgroups

All the representations ρ considered in subsection 4.10.1 above are characterized by the following property: each column of the associated matrix Θ_ρ generates a dense subgroup of $\mathbf{T}^n$. Actually, for any such a representation one can find infinitely many curves whose image generates a dense subgroup in $\mathbf{T}^n$. This lead us to ask: given a dense representation $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$, can we find a simple closed curve γ such that $\langle\rho(\gamma)\rangle$ is dense in $\mathbf{T}^n$? Remind that a vector $\big(e^{i\theta_1}, \dots, e^{i\theta_n}\big) \in \mathbf{T}^n$ generates a dense subgroup if and only if $\pi, \theta_1, \dots, \theta_n$ are linearly independent over $\mathbf{Q}$. As a corollary of Lemma we deduce the following Lemma.

Lemma 4.31. *Let $\rho : \pi_1\Sigma \longrightarrow \mathbf{S}^1$ be a dense representation. Then there always exists a simple closed curve γ such that $\overline{\langle\rho(\gamma)\rangle} = \mathbf{S}^1$.*

However, for $n \geq 2$, the scenario changes completely. Indeed, for any n we can find examples of dense representations which do not have any curve generating a dense subgroup in $\mathbf{T}^n$.

Example 4.32. Let Σ be a surface of genus g and $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^2$ be the representation associated to the matrix $\Theta_\rho \in \mathrm{M}\big(2, 2g; \mathbf{T}\big)$ defines as follow.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \in \mathrm{M}\big(2, 2g; \mathbf{T}\big).$$

One can show that $\overline{\Theta}_\rho$ has maximal rank and hence ρ is a dense representation. However, no curve is applied by ρ to a vector generating a dense subgroup.

Example 4.33. Let Σ be a surface of genus g and $\rho : \pi_1\Sigma \longrightarrow \mathbf{T}^n$ be the representation associated to the matrix $\Theta_\rho \in \mathrm{M}\big(n, 2g; \mathbf{T}\big)$ defines as follow.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \theta_3 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & & & & & & & & \vdots \\ \theta_n & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \in \mathrm{M}\big(n, 2g; \mathbf{T}\big),$$

where $1, \theta_3, \dots, \theta_n$ are linearly independent over $\mathbf{Q}$. One can show that $\overline{\Theta}_\rho$ has maximal rank and hence ρ is a dense representation. However, no curve is applied by ρ to a vector generating a dense subgroup.

4.10.3 Surfaces with one puncture

Let us now discuss the case of the one-holed torus Σ. We shall denote $\pi_1\Sigma \cong \langle a, b\rangle$ the fundamental group of Σ. Also in this case the choice of a representation consists in choosing for each generator an element of $\mathbf{T}^n$. The representation space $\mathrm{Hom}\big(\pi_1\Sigma, \mathbf{T}^n\big)$ trivially identifies with the space $\mathbf{T}^n \times \mathbf{T}^n$. For each choice of an element $\mathbf{c}$, the relative representation variety $\mathrm{Hom}_{\mathbf{c}}\big(\pi_1\Sigma, \mathbf{T}^n\big)$ is defined as the preimage of $\mathbf{c}$ via the commutator map $k : \mathbf{T}^n \times \mathbf{T}^n \to \mathbf{T}^n$. Thus, as a consequence of the abelian property, the relative representation space is empty for any $\mathbf{c} \neq (1, \dots, 1)$ and coincides with the full representation variety when $\mathbf{c} = (1, \dots, 1)$. Once again, the action of $\mathbf{T}^n$ by inner automorphisms is trivial and hence the character variety trivially coincides with the representation space. As a consequence, the space $\mathrm{Hom}\big(\pi_1\Sigma, \mathbf{T}^n\big)$ naturally identifies with the space $\mathrm{Hom}\big(\pi_1 T, \mathbf{T}^n\big)$. The equalities $\mathrm{Mod}(T) = \mathrm{Mod}(\Sigma) = \mathrm{SL}(2, \mathbf{Z})$ are well-known and the actions of $\mathrm{Mod}(T)$ and $\mathrm{Mod}(\Sigma)$ on the representation spaces associated to T and Σ respectively coincide. Therefore, we have the following proposition.

Proposition 4.34. *Theorem D and Theorem F hold for the torus T if and only if they hold for the one-holed torus Σ.*

More generally, the main results of the present work extend to surfaces of higher genus and with one boundary component. Indeed, let $\Sigma_{g,1}$ be a surface a surface of genus g and one boundary component. We have already seen above that this is true for the one-holed torus Σ, Proposition 4.34. The general claim follows because, as a consequence of the abelian property of $\mathbf{T}^n$, one can establish an identification between the representations spaces $\mathrm{Hom}\big(\pi_1\Sigma, \mathbf{T}^n\big)$ and $\mathrm{Hom}\big(\pi_1\big(\Sigma_{g,1}\big), \mathbf{T}^n\big)$. Since the mapping class group coincides with the pure mapping class group for one-puncture surfaces the following proposition also holds.

Proposition 4.35. *Theorem D and Theorem F hold for a closed surface of genus g if and only if they hold for the one-holed surface of genus g.*

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L'objectif de cette thèse est de répondre à des problématiques dynamiques quant à l'action du groupe modulaire sur les variétés de caractères de surfaces.
Nous étudions d'abord le cas des variétés de caractères à valeurs dans des groupes de Lie semi-simples et compacts. L'action du groupe modulaire est alors connue pour être ergodique. Nous renforçons cette propriété dynamique en démontrant que le sous-groupe de Torelli agit ergodiquement sur ces variétés de caractères. Ce fait généralise un corollaire d'un résultat de Funar-Marché pour les variétés de caractères à valeurs dans SU(2).
Ensuite nous adaptons les arguments de la preuve d'un théorème de Marché-Wolff au cas des surfaces à bord. Le groupe modulaire pur agit sur les variétés de caractères relatives. En imposant des conditions aux bords et en considérant les classes d'Euler qui le permettent, nous introduisons un sous-espace analogue à celui de Marché-Wolff et montrons que le groupe modulaire pur agit ergodiquement sur celui-ci.
Enfin, l'ergodicité des actions considérées implique que presque toutes les orbites sont denses. Prévite-Xia donne une condition nécessaire et suffisante pour que la classe de conjugaison de $\rho : \pi_1\Sigma \to \mathrm{SU}(2)$ ait une orbite dense dans la variété des caractères de Σ à valeurs dans SU(2). Avec Gianluca Faraco, nous démontrons un résultat analogue pour l'espace des représentations d'une surface dans un groupe de Lie connexe, compact et Abélien. Nous montrons ainsi qu'une représentation a une orbite dense si et seulement si son image est dense et relions cet énoncé à des problèmes d'approximations diophantiennes.

The aim of this thesis is to answer to some dynamical questions about the modular group action on character varieties of surfaces.
We first study the case of character varieties with values in a semi-simple and compact Lie groups. The modular group action is then known to be ergodic. We improve this dynamical property in proving that the action of the Torelli subgroup is ergodic on such varieties. This fact generalizes a result of Funar-Marché for character varieties in SU(2).
We then adapt the arguments of the proof of Marché-Wolff in the case of surfaces with boundaries. The pure modular group acts on the relative character varieties. Fixing conditions on the boundary and taking the Euler classes which allows it, we introduce a subspace which is analogue to the one of Marché-Wolff and prove that the pure modular group acts ergodically on it.
Finally, the ergodicity of the actions we considered implies that almost every orbit is dense. Prévite-Xia gives a necessary and sufficient sufficient on a representation $\rho : \pi_1\Sigma \to \mathrm{SU}(2)$ such that its orbits is dense in the character variety in SU(2). With Gianluca Faraco, we prove an analogue result for the representations space in a compact, connected and Abelian Lie group. We hence show that a representation has a dense orbit if and only if its image is dense and relate this statement to problems of Diophantine approximations.

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