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Enumerative study of intervals in lattices of Tamari type

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Résumé

Le treillis de Tamari est un ordre partiel sur les objets comptés par les nombres de Catalan. Plusieurs descriptions de ce treillis existent et donnent lieu à différentes familles de généralisations. Dans cette thèse, on étudie ces différents ordres partiels et notamment leurs intervalles, en particulier d'un point de vue énumératif.

Après une première partie préliminaire, une seconde partie concerne concerne l'étude de la sous-famille des intervalles linéaires dans le treillis de Tamari et ses différentes généralisations. On définit en particulier les familles des ordres alt-Tamari et alt ν-Tamari. On prouve bijectivement des résultats d'équidistribution de ces intervalles linéaires, que l'on énumère dans le cas des treillis alt-Tamari.

Une troisième partie se penche sur une conjecture de Stump, Thomas et Williams selon laquelle les treillis m-Cambriens en type A linéaire et m-Tamari auraient le même nombre d'intervalle. On présente et généralise l'étude dans le cas m-Tamari, puis on étudie les treillis m-Cambriens, dont on propose une nouvelle description conjecturale.

Abstract

The Tamari lattice is a partial order on objects counted by the Catalan numbers. There are several descriptions of this lattice, which lead to different families of generalizations. In this manuscript, we study these different partial orders and their intervals, especially from an enumerative perspective.

After a first preliminary part, a second part focuses on the study of the subfamily of linear intervals in the Tamari lattice and its generalizations. We define in particular the new families of alt-Tamari and alt ν-Tamari orders. We prove bijectively some equidistributivity results of these linear intervals, that we enumerate in the case of the alt-Tamari lattices.

A third part is motivated by a conjecture of Stump, Thomas and Williams, according to which the m-Cambrian lattices in linear type A and the m-Tamari lattices would have the same number of intervals. We present and generalize the study in the m-Tamari case, then we study the m-Cambrian lattices, for which we propose a new conjectural description.

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[0] If you think you would deserve to be thanked, and you have not, please add your name: I am specially thankful to . . . .

[1] Thanks to Quentin and Victoria for this tip.

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[2] P.-S.: I love you more!

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[0] Si vous pensez que je vous mériteriez d'être remercié-e et que vous ne l'avez pas été, veuillez ajouter votre nom : je suis particulièrement reconnaissant à ....

[1] Merci à Quentin et Victoria pour cette astuce.

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[2] P.-S. : Je t'aime plusssss !

Contents

Introduction en Français
Étude énumérative des intervalles dans les treillis de type Tamari

Ce manuscrit s'inscrit dans le champ de la combinatoire algébrique et énumérative. Ce domaine s'intéresse à l'étude de structures algébriques discrètes, comptant certaines quantités pour en tirer des informations algébriques, et inversement en utilisant des outils algébriques pour compter des objets d'intérêt. Ce travail est en particulier dédié à l'étude de certains ensembles partiellement ordonnés liés au treillis de Tamari, plus spécifiquement du point de vue de leurs intervalles et de leur énumération. Ce document s'organise en trois parties, la première étant largement introductive au domaine et aux objets étudiés, tandis que les deuxième et troisième parties sont plus centrées sur les principales contributions de ce travail.

Les **ensembles partiellement ordonnés**, souvent appelés simplement *posets* ou *ordres partiels*, sont des structures très naturelles et constituent des outils utiles apparaissant dans la plupart des domaines des mathématiques. Un ordre partiel sur un ensemble E est une relation binaire réflexive, transitive et antisymétrique sur E. Les ordres naturels sur $\mathbb{N}$, $\mathbb{Q}$ et $\mathbb{R}$ sont des exemples d'ordres totaux, et l'ordre de divisibilité sur les entiers naturels, les ensembles de parties d'un ensemble munis de l'ordre d'inclusion sont des exemples très courants de posets. De nombreux autres exemples sont présentés dans ce manuscrit, notamment dans la Section 2.2 et le Chapitre 4.

Une fois définie la notion d'ordres partiels, il est très naturel de s'intéresser aux morphismes qui respectent la structure d'ordre, c'est-à-dire aux fonctions croissantes, et en particulier aux isomorphismes, qui sont les morphismes inversibles. On peut se demander si deux ordres partiels *a priori* différents correspondent en fait à "la même" structure, c'est-à-dire s'ils sont *isomorphes*, ce qui n'est pas toujours simple à déterminer. Si deux ordres ne sont pas isomorphes, on peut les comparer d'autres manières, pour évaluer à quel point ils sont similaires, quelles propriétés ou quantités ils partagent.

Une première quantité très naturelle à considérer sur des ensembles partiellement ordonnés finis est leur **cardinalité**, c'est-à-dire le nombre d'éléments, et en quelque sorte si les deux ordres partiels peuvent être définis sur les mêmes objets. Une seconde quantité très naturelle est le nombre de relations, ou d'**intervalles**, qui sont des paires d'éléments comparables. De même que la cardinalité, ce nombre est un invariant additif sous les sommes d'ordres partiels (ou unions disjointes) et multiplicatif sous les produits cartésiens. C'est aussi la dimension d'une algèbre naturellement associée à l'ordre partiel, à savoir l'algèbre d'incidence, qui peut aussi être utilisée comme outil algébrique pour étudier l'ordre partiel lui-même. Par exemple, cela mène à la notion d'équivalence dérivée, qui est une notion plus faible que l'isomorphisme entre ordres partiels. Pour chaque poset, on peut associer la catégorie des modules finis (ou représentations) de son algèbre d'incidence, et deux posets sont dits **dérivés équivalents** si les catégories dérivées de leurs catégories de modules respectives sont équivalentes.

L'étude des intervalles dans un ensemble partiellement ordonné peut prendre diverses formes, depuis le (pas si) simple comptage de leur nombre total, ou d'éléments dans certains sous-ensembles d'intervalles, jusqu'à la distribution de certaines statistiques sur les intervalles, comme nous le verrons tout au long de ce travail. On peut aussi considérer des intervalles pondérés, des ordres partiels sur l'ensemble des intervalles d'un poset donné, des informations topologiques ou des structures plus algébriques comme l'algèbre d'incidence. Ce travail est fréquemment motivé par certaines coïncidences énumératives observées, qui peuvent cacher des raisons sous-jacentes plus profondes. Par exemple, le nombre d'intervalles peut admettre une formule close qui se trouve compter également d'autres objets combinatoires, suggérant une possible bijection entre eux, ou bien coïncider avec le nombre d'intervalles dans un autre poset qui ne lui est pas nécessairement isomorphe.

Le treillis de Tamari est un exemple pour ces coïncidences énumératives. La formule pour le nombre de ses intervalles compte également certains types de cartes combinatoires. Une correspondance bijective entre les deux ensembles a été construite dans [BB09]. Cette coïncidence semble être plus profonde que cela, car les sous-ensembles des intervalles synchrones d'une part et nouveaux d'autre part, admettent tous deux des formules produit closes, en bijection avec certaines familles de cartes combinatoires, comme exposé dans [FPR17, Fan21]. D'autre part, le nombre d'intervalles dans le treillis de Tamari semble aussi correspondre à la dimension de certains sous-espaces de coinvariants diagonaux, c'est-à-dire dans certaines représentations du groupe symétrique, comme décrit un peu plus tard dans l'introduction.

0.1 Les objets Catalan

Les nombres de Catalan sont une suite d'entiers naturels $(C_n)_{n\in\mathbb{N}}$ qui sont connus pour compter de nombreuses familles d'objets combinatoires, et qui apparaissent dans de nombreuses branches de la combinatoire, et plus généralement des mathématiques. Ils constituent l'une des plus longues entrées de l'Encyclopédie en ligne des suites d'entiers (OEIS) et commencent par:

$$[OEIS A007767] \quad C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, C_6 = 132, \ldots$$

R. Stanley a décrit 214 familles d'objets combinatoires comptés par les nombres de Catalan, que nous pouvons appeler objets Catalan, dans son célèbre livre [Sta15], ainsi que de nombreuses bijections entre certaines de ces familles. Ils seraient apparus pour la première fois dans les travaux d'un mathématicien mongol, Sharabiin Myangat, dans les années 1730 et plus tard dans ceux de Leonhard Euler dans les années 1750 comme le nombre de triangulations d'un polygone. Ils ont été nommés d'après un mathématicien français et belge, Eugène Charles Catalan, qui a donné en 1838 la formule close suivante pour les nombres de Catalan :

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Ces nombres ont des propriétés très intéressantes et riches ; ils ont été très étudiés et de nombreuses généralisations ont été proposées depuis. Le travail présenté dans ce manuscrit repose sur des structures d'ordre partiel sur certains objets Catalan ou des généralisations de ceux-ci.

0.2 Les ordres partiels et les treillis

Les ordres partiels sont les objets d'intérêt majeur dans ce travail. Ils sont définis comme un ensemble (la plupart du temps fini, dans les cas qui nous intéressent) muni d'une relation binaire réflexive, transitive et antisymétrique. Pour un traitement plus complet de la théorie des ordres partiels, nous renvoyons le lecteur à [DP02, Wac07].

On représente souvent les ordres partiels (finis) par leur **diagramme de Hasse**, qui est un graphe orienté acyclique, dont les sommets sont les éléments de l'ensemble et les arcs représentent les relations de couverture, c'est-à-dire les relations minimales dans l'ordre partiel. Formellement, une relation de couverture est une paire d'éléments (x, y) telle que $x < y$ et qu'il n'existe pas d'élément z tel que $x < z < y$.

Dans un ensemble partiellement ordonné $(P, \leq)$, une **chaîne** de longueur k, ou k-chaîne, est une suite strictement croissante $x_0 < x_1 < \cdots < x_k$ et une k-multichaîne est une suite faiblement croissante $x_0 \leq x_1 \leq \cdots \leq x_k$. Si $x \leq y$, on appelle **intervalle** $[x, y]$ le sous-ensemble des éléments compris entre x et y, c'est-à-dire $[x, y] = \{z \in P \mid x \leq z \leq y\}$. On confondra souvent l'intervalle $[x, y]$ avec la 2-multichaîne (x, y), la donnée de l'un étant équivalente à la donnée de l'autre. On définit également la **hauteur** d'un intervalle comme la longueur maximale d'une chaîne contenue dans cet intervalle.

Ce travail se concentre principalement sur les intervalles dans les ordres partiels liés au treillis de Tamari, en particulier d'un point de vue énumératif. La Partie II se concentrera sur les **intervalles linéaires** dans ces treillis, c'est-à-dire les paires $x \leq y$ telles que l'intervalle $[x, y]$ soit totalement ordonné ou dit autrement, une chaîne. La Partie III quant à elle portera essentiellement sur l'étude des intervalles dans deux de ces familles, à savoir les treillis m-Tamari et m-cambrien en type A linéaire, sur la base d'une conjecture de [STW18], selon laquelle le nombre total d'intervalles de ces ensembles partiellement ordonnés coïnciderait dans les deux familles.

La plupart des ordres partiels considérés dans ce travail sont en fait des **treillis**, d'où en particulier leur nom, bien que ce travail n'exploite pas particulièrement cette propriété. Un treillis est un ordre partiel dans lequel tout couple d'éléments (x, y) admet une borne inférieure $x \wedge y$ et une borne supérieure $x \vee y$, appelés respectivement leur inf et leur sup. La borne inférieure est, si elle existe, le plus grand des minorants, et de façon duale, la borne supérieure est le plus petit des majorants, s'il y en a un.

0.3 Le treillis de Tamari

Parmi le vaste monde des ordres partiels vit le treillis de Tamari, objet central de ce travail. Il s'agit d'un ordre défini sur des objets comptés par les nombres de Catalan, et qui a beaucoup inspiré la recherche dans divers domaines des mathématiques, en particulier en combinatoire.

De même qu'il existe de nombreuses familles d'objets Catalan, le treillis de Tamari peut être décrit de façon plutôt naturelle sur plusieurs de ces familles. Il existe un treillis de Tamari Tam_n sur les objets Catalan de taille n pour tout entier strictement positif n, mais nous les désignerons simplement par “le” treillis de Tamari. Dans ce qui suit, par convention, nous ne considérerons généralement pas le treillis de Tamari de taille 0, bien que nous puissions considérer des objets Catalan de taille 0.

Cet ordre partiel est nommé d'après le chercheur israélien Dov Tamari, qui l'a défini et étudié pour la première fois en interprétant la relation d'associativité comme une relation d'ordre (voir [Tam62]). Plus précisément, une opération binaire $\cdot : A \times A \to A$ est associative si elle satisfait $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ pour tout $a, b, c \in A$. En considérant l'ensemble des parenthésages d'un produit de $n + 1$ éléments, au nombre de $C(n)$, on peut définir une relation d'ordre sur ces parenthésages en autorisant à transformer un sous-produit de la forme $a \cdot (b \cdot c)$ en $(a \cdot b) \cdot c$, mais pas dans l'autre sens. Ce faisant, on obtient un ordre partiel qui est notre première instance du treillis de Tamari, qui possède des propriétés très intéressantes, notamment celle d'être un treillis [HT72].

Il existe de nombreuses descriptions du treillis de Tamari, sur les parenthésages, des vecteurs d'entiers, des arbres binaires, des triangulations, des chemins, des modules basculants, des complexes de sous-mots, pour ne citer que celles-ci. De cette riche variété de descriptions ont émergé de nombreuses généralisations, dont beaucoup sont présentées et étudiées dans ce travail, et en particulier dans le Chapitre 4.

- Tout d'abord, une description du treillis de Tamari sur les chemins de Dyck peut être généralisée aux chemins de m-Dyck, puis à l'ensemble des chemins faiblement au-dessus d'un chemin donné ν, aussi appelés ν-chemins. Ces deux généralisations sont respectivement appelées **treillis m-Tamari** ([Ber12]) et **treillis ν-Tamari** ([PRV17]).

- Très récemment, comme présenté dans ce travail et comme sujet central des Chapitres 5 et 6, les nouvelles familles de **treillis alt-Tamari** et **treillis alt ν-Tamari** ont été introduites et étudiées, en particulier du point de vue de leurs intervalles linéaires.

- Le treillis de Tamari apparaît également naturellement comme le cas du type A linéaire des **posets de modules basculants** ([RS91, HU05]) et des **treillis cambriens** ([Rea06]). Ces deux structures d'ordres partiels peuvent être définies dans le contexte des groupes de Coxeter, une fois fixé un *élément de Coxeter*. Les treillis cambriens ont par la suite été généralisés en les treillis m-cambriens dans l'article [STW18], ce qui est le sujet principal du Chapitre 9.

- Pour finir, une dernière famille de généralisations du treillis de Tamari présentée dans ce travail est celle des **treillis permusylvestres** ([PP18]), qui englobe le treillis de Tamari ainsi que l'ordre faible sur le groupe symétrique, tous les treillis cambriens de type A et le treillis booléen.

Toutes ces familles sont présentées de façon schématisée dans les Figures 1 et 2. La première figure présente le treillis de Tamari, en gris au milieu, et chacune des familles comme une généralisation de celui-ci, dans une direction (et couleur) différente. La seconde présente ces mêmes objets comme un ordre partiel en indiquant plus précisément les relations d'inclusion entre les familles.

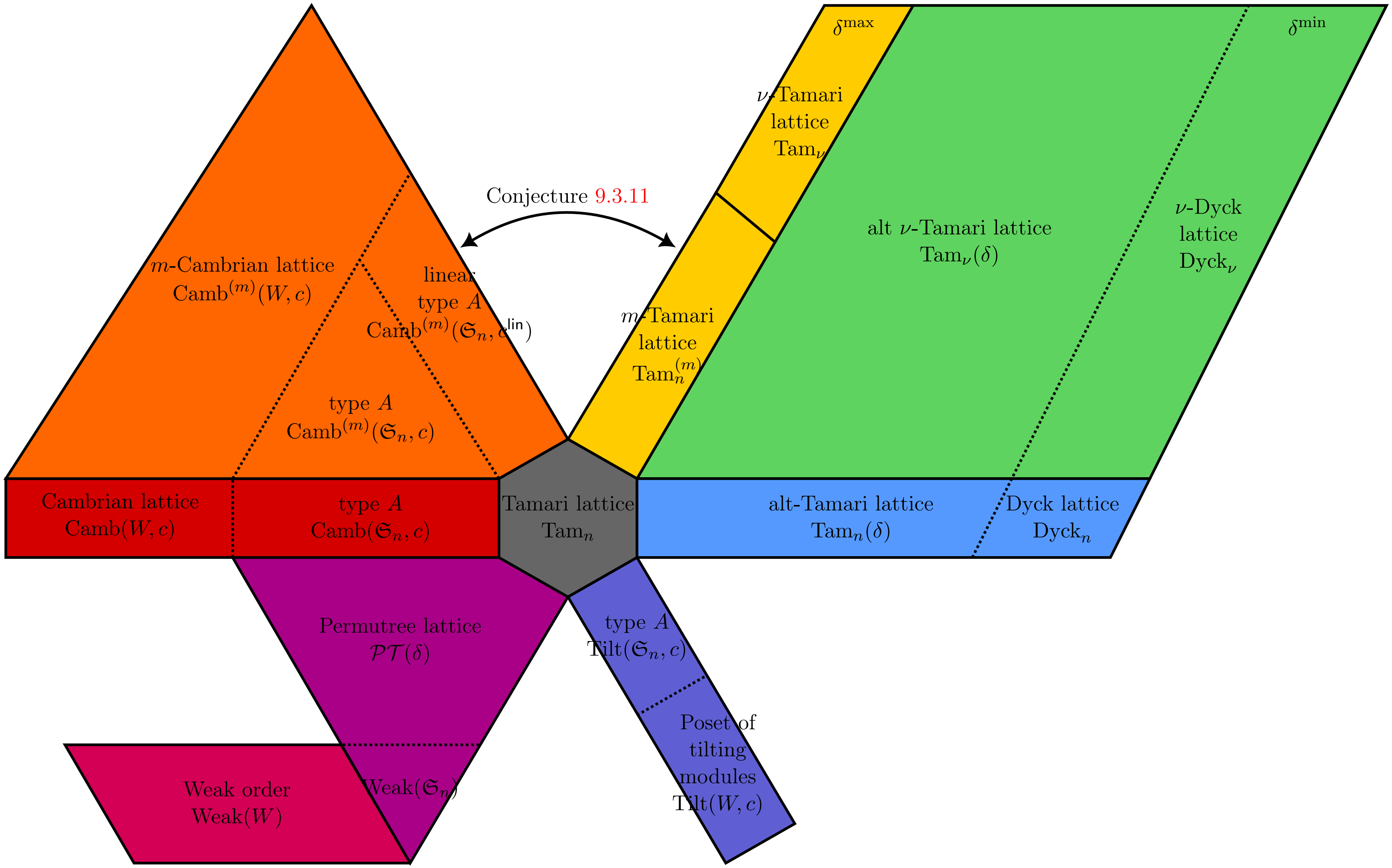


Figure 1: Familles de généralisations du treillis de Tamari.

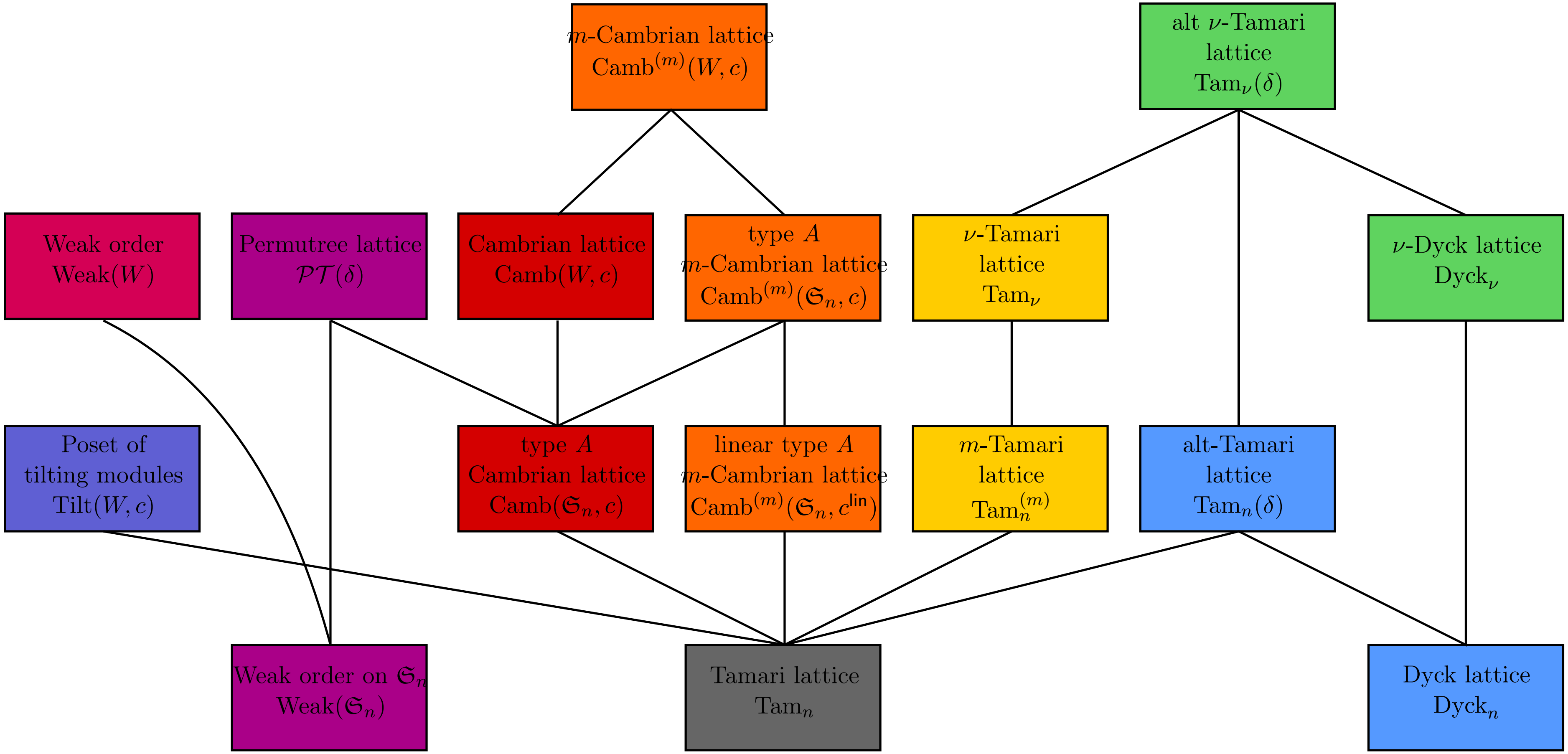


Figure 2: Ordre des familles de généralisations du treillis de Tamari.

0.4 L'énumération des intervalles

Une direction de recherche qui a reçu beaucoup d'attention ces dernières années concerne le nombre d'intervalles dans le treillis de Tamari et ses généralisations. Les intervalles du treillis de Tamari ont d'abord été comptés par F. Chapoton dans [Cha06], en utilisant une approche par séries génératrices, ce qui a permis d'obtenir une formule close très élégante :

$$\frac{2\,(4n+1)!}{(n+1)!\,(3n+2)!}.$$

Cette formule est aussi apparue comme énumérant les cartes cubiques planaires 3-connexes et (dualement) les triangulations 3-connexes, comme prouvé par W. T. Tutte (voir [CS03, Tut62]). Une bijection a ensuite été trouvée dans [BB09, Theorem 4.1], ce qui explique cette coïncidence, surprenante en premier lieu. De plus, ces nombres semblent aussi correspondre à la dimension de la composante alternée dans l'étude des coinvariants diagonaux, dans le cas trivarié, comme expliqué ci-après (voir également [Hai94, BPR12]).

0.4.1 Une motivation par les représentations

Bien que ce travail ne soit pas centré sur la théorie des représentations, elle transparaît tout au long de ce manuscrit comme une motivation initiale avec les coinvariants diagonaux, comme une manière de représenter certains objets combinatoires (notamment les "complexes d'amas", ou associaèdres généralisés duaux), ou comme un outil pour étudier les ordres partiels considérés, en particulier via la notion d'équivalence dérivée. Précisément, la connexion établie conjecturalement entre le nombre d'intervalles du treillis de Tamari et les représentations du groupe symétrique ont été la principale motivation pour introduire le treillis m-Tamari dans [Ber12].

L'action du groupe symétrique par permutation des indices sur l'ensemble des polynômes en n variables a d'excellentes propriétés. En particulier, l'ensemble des polynômes symétriques, ou invariants, forme une algèbre polynomiale dont les générateurs sont les polynômes symétriques élémentaires. Dans ce cas, les **coinvariants**, c'est-à-dire le quotient de l'espace total par l'idéal engendré par les polynômes symétriques non constants, est connu pour être isomorphe à la **représentation régulière** de $\mathfrak{S}_n$. En fait, on peut considérer le sous-espace des polynômes harmoniques, qui est un sous-espace supplémentaire de l'espace des polynômes symétriques, et en fait son complément orthogonal pour un certain produit scalaire naturel. Le quotient des coinvariants et l'espace des polynômes harmoniques sont isomorphes, leur dimension totale est $n!$ et la multiplicité de la représentation signe est 1.

Vient ensuite l'idée de considérer l'*action diagonale* du groupe symétrique sur plusieurs ensembles de n variables. Dans ce cas, on peut toujours considérer les coinvariants comme le quotient de l'espace total par l'idéal général par les polynômes invariants non constants. Cet espace est à nouveau isomorphe au sous-espace des polynômes harmoniques.

Dans le cas bivarié, c'est-à-dire avec deux jeux de variables, M. Haiman a conjecturé dans [Hai94] et prouvé dans [Hai02] que la dimension du sous-espace des polynômes harmoniques est égale à $(n+1)^{n-1}$, qui est le nombre de **fonctions de parking**. Il a également été prouvé dans [Hai02] que le sous-espace des **polynômes harmoniques alternés**, qui correspond à la composante isotypique de la représentation signe, a pour dimension le nombre de Catalan C_n. Certaines formules ont également été conjecturées pour le cas trivarié, notamment pour la dimension de la composante alternée. De plus, la formule conjecturale s'est curieusement avérée coïncider avec celle énumérant les intervalles dans le treillis de Tamari.

Par la suite, une version "supérieure" de ces polynômes harmoniques et polynômes harmoniques alternés a été introduite dans [BPR12], ainsi que des conjectures sur leurs dimensions dans les cas bivarié et trivarié. En particulier, ce travail a conduit à la définition du treillis de Tamari sur les chemins de Dyck, ainsi qu'à sa généralisation en le treillis m-Tamari, défini sur les chemins de m-Dyck (voir [Ber12, BPR12]). Dans le cas trivarié, la dimension de la composante alternée des polynômes harmoniques supérieurs a été conjecturée être égale au nombre d'intervalles dans le treillis m-Tamari, avec également une formule produit élégante. La dimension du m-ème sous-espace des polynômes harmoniques supérieurs a également été conjecturée être égale au nombre d'intervalles décorés dans le treillis m-Tamari, où une fonction de parking est associée à l'élément du haut de chaque intervalle, encore avec une formule produit conjecturale.

La première formule, pour énumérer les intervalles m-Tamari, a été prouvée dans [BMFPR11], et la seconde, pour énumérer les intervalles décorés, a été obtenue par la suite dans [BMCPR13].

En plus de cela, une autre généralisation du treillis de Tamari impliquant un entier $m \geq 1$ a été introduite dans [STW18], à savoir le **treillis cambrien de type A linéaire**. Cet ensemble partiel est lui aussi défini sur des objets Fuß-Catalan, mais n'est pas isomorphe au treillis m-Tamari, pour $m \geq 2$. Par exemple, il est auto-dual, tandis que le treillis m-Tamari ne l'est pas. En revanche, son nombre d'intervalles a été conjecturé coïncider avec celui des intervalles m-Tamari, ainsi que son nombre d'intervalles décorés (voir [STW18, Section 6.10]).

Ces conjectures sont la motivation principale pour les considérations de la Partie III.

0.4.2 Le sous-ensemble des intervalles linéaires

Une nouvelle direction de recherche a été ouverte avec l'idée de F. Chapoton de considérer le sous-ensemble des **intervalles linéaires** dans un poset. Ce sous-ensemble est défini comme l'ensemble des intervalles totalement ordonnés, ou de manière équivalente dont le diagramme de Hasse est un chemin. Ce sont, en un sens, les intervalles les plus simples.

De manière générale, les intervalles de hauteur 0 et 1 sont toujours linéaires, ils correspondent respectivement aux intervalles de la forme $[x, x]$, appelés *intervalles triviaux* et aux relations de couverture. Dans certains cas, l'étude des intervalles linéaires n'est pas très intéressante, notamment dans les ordres partiels qui ne possèdent pas d'intervalles linéaires de longueur 2 ou plus. Ces ordres partiels sont appelés 2-épais ([Bjö81]), et des exemples de tels ordres partiels sont donnés par les treillis booléens, les treillis de partitions d'un ensemble, ou n'importe quel treillis géométrique.

Il est particulièrement surprenant que la distribution des intervalles linéaires dans les ordres partiels liés au treillis de Tamari semble se comporter très bien ; la Partie II est consacrée à cette étude. D'une part, le nombre d'intervalles linéaires de longueur k dans le treillis de Tamari est presque un nombre binomial. D'autre part, la distribution des intervalles linéaires selon leur longueur dans le treillis de Dyck est exactement la même que dans le treillis de Tamari.

Ces deux coïncidences remarquables ont conduit F. Chapoton à conjecturer une formule pour la distribution des intervalles linéaires dans le treillis de Tamari, ainsi que l'existence d'une nouvelle famille d'ordres partiels qui contiendrait les treillis de Tamari et de Dyck, et dont tous les membres auraient la même distribution d'intervalles linéaires selon leur longueur. Ces conjectures ont initié le travail présenté dans le Chapitre 5, où nous prouvons ces conjectures initiales et plus encore. Nous introduisons en particulier les **treillis alt-Tamari**, prouvons le résultat sur la distribution des intervalles linéaires et énumérons leurs intervalles linéaires par des méthodes uniformes, résultats principalement issus d'un premier article prépublié [Che22].

En plus des treillis cambriens et des posets de modules basculants en type A, les treillis alt-Tamari semblent être une troisième famille d'ordres partiels qui contiennent (et généralisent) le treillis de Tamari, et qui partagent de très bonnes propriétés structurelles. D'une part, nous conjecturons que la distribution des intervalles linéaires dans les treillis cambriens et les posets de modules basculants en type A coïncide également avec celle du treillis de Tamari. D'autre part, S. Ladkani a prouvé un résultat d'équivalence dérivée pour les treillis cambriens et les posets de modules basculants (voir [Lad07a, Lad07b]) et nous conjecturons qu'il en est de même dans la famille des treillis alt-Tamari.

Généralisant davantage les treillis m-Tamari, L.-F. Préville-Ratelle et X. Viennot ont défini le treillis ν-Tamari dans [PRV17] comme un ordre partiel sur les ν-chemins, où ν est un chemin quelconque formé de pas unité de type nord ou est. Ils ont prouvé que non seulement cette famille d'ordres partiels contient le treillis de Tamari comme cas particulier, mais que chaque treillis ν-Tamari est en fait isomorphe à un certain intervalle dans un treillis de Tamari plus grand, introduisant la notion de **canopée** d'un arbre. En fait, chaque treillis de Tamari de taille n peut être partitionné en intervalles en regroupant les éléments selon leur canopée. Pour une canopée fixée, l'ensemble des arbres correspondants forme un intervalle, qui est isomorphe à un treillis ν-Tamari pour un certain chemin ν de longueur $n-1$, avec une correspondance bijective entre canopées et chemins.

Surprenamment, les intervalles dont les éléments du bas et du haut partagent la même canopée, appelés **intervalles synchrones**, sont comptés par une formule produit fermée qui ressemble à celle du nombre total d'intervalles, comme prouvé dans [FPR17]. Les auteurs exhibent également une bijection entre les intervalles synchrones et les cartes planaires non séparables, renforçant le lien entre les intervalles de type Tamari et les cartes combinatoires. Il est à noter que l'on peut également définir des notions d'intervalles *nouveaux* ou *modernes* dans le treillis de Tamari, qui ont été prouvés dans [Fan21] être en bijection avec encore une autre famille de cartes, à savoir les cartes biparties.

Le treillis de Dyck peut lui aussi s'étendre très naturellement à l'ensemble des ν-chemins, en considérant la relation d'être faiblement au-dessus, ce que l'on appellera le treillis ν-Dyck. On peut de plus observer que la distribution des intervalles linéaires selon leur longueur dans le treillis ν-Dyck coïncide encore avec celle du treillis ν-Tamari pour le même chemin ν. Ces observations suggéraient l'existence d'une généralisation du treillis de Tamari dans les directions conjointes des treillis alt-Tamari et ν-Tamari. C'est précisément l'objet d'un travail en collaboration avec Cesar Ceballos, dans lequel nous avons défini le **treillis alt ν-Tamari**, généralisant à la fois les treillis alt-Tamari et ν-Tamari, et prouvé bijectivement que la distribution de leurs intervalles linéaires ne dépendait que du chemin ν. Ce travail a fait l'objet d'un second article prépublié [CC23] et est exposé dans le Chapitre 6.

De même que pour les intervalles synchrones, il semble que la distribution des intervalles synchrones et linéaires à la fois dans le treillis de Tamari admet une jolie formule close. Pour le dire autrement, même si les distributions individuelles des intervalles linéaires dans les treillis ν-Tamari ne semble pas particulièrement élégante, leur somme sur tous les chemins ν de longueur fixée semble l'être et admettre une formule très simple. Nous nous attendons également à un résultat d'équivalence dérivée au sein de la famille des treillis alt ν-Tamari, pour un chemin ν fixé, ce qui témoignerait davantage de la richesse et de l'élégance structurelles de ces nouveaux objets.

Pour finir dans cette direction de recherche, l'étude des intervalles linéaires semble très prometteuse, notamment pour les ordres partiels liés au treillis de Tamari. En particulier, d'autres exemples d'ordres partiels, comme l'ordre faible sur le groupe symétrique, semblent admettre des distributions élégantes d'une part, et d'autre part des résultats d'équidistribution et d'équivalence dérivée semblent être vrais dans d'autres familles d'ordres partiels, et spécialement dans les treillis m-cambriens ou bien dans les treillis permusylvestres.

0.5 Résumé détaillé du manuscrit

Ce manuscrit est organisé en trois parties. La première est une introduction générale au domaine de recherche, tandis que les deux autres parties contiennent les principales contributions de ce travail. Toutes deux sont centrées sur les ordres partiels liés au treillis de Tamari, avec un accent particulier sur leurs intervalles, d'un point de vue énumératif.

0.5.1 Contributions et attributions

Les Chapitres 1, 2 et 3 contiennent des rappels et définitions déjà établis dans la littérature. Le Chapitre 4 présente les différentes généralisations du treillis de Tamari apparaissant dans ce travail. Les définitions des treillis alt-Tamari et alt ν-Tamari sont nouvelles mais présentées dans la partie suivante.

La Partie II est consacrée à l'étude des intervalles linéaires dans les ordres partiels en lien avec le treillis de Tamari.

La plupart des résultats du Chapitre 5 ont été publiés sous forme préliminaire dans [Che22], retravaillé pour s'adapter davantage à ce manuscrit.

Les résultats de la Section 5.5 ont été obtenus lors de discussions avec Vincent Pilaud et sont inédits.

Les résultats du Chapitre 6 étendent ceux du Chapitre 5 et ont été obtenus en collaboration avec Cesar Ceballos. Ils ont été d'abord soumis sous la forme d'un abstract étendu à la conférence FPSAC Davis 2023, accepté comme poster, puis publiés sous forme préliminaire [CC23]. Les deux auteurs ont contribué de manière égale à ce travail.

Le Chapitre 7 contient des résultats inédits. L'énumération des intervalles linéaires dans l'ordre faible en Section 7.1 a été conjecturée par Frédéric Chapoton et obtenue lors de discussions avec Viviane Pons et Vincent Pilaud.

Le cas du poset "Dyck glouton" de la Section 7.2 a été considéré par Philippe Nadeau, qui a observé expérimentalement que la distribution de leurs intervalles linéaires était identique à celle du treillis de Tamari.

Les conjectures des treillis permusylvestres dans la Section 7.3.4 ont été éprouvées à l'aide de Daniel Tamayo.

Les conjectures des distributions des intervalles linéaires dans le poset de Pallo et dans les ordres cambriens et des modules basculants en type B et D sont dues à Frédéric Chapoton.

La Partie III est consacrée à l'étude des treillis m-Tamari et m-cambriens, ainsi que de leurs intervalles.

Les résultats des Sections 8.1 et 8.2 sont déjà connus [Cha06, BMFPR11], de même que certains résultats cités, notamment sur les intervalle-posets [Pon19]. Les autres conjectures et résultats du Chapitre 8 sont inédits. L'encodage des partitions d'entiers par des familles de variables pour formuler le Théorème 8.2.21 a été suggéré par Houcine Ben Dali.

Le Chapitre 9 est consacré à l'étude des treillis m-cambriens. Leurs définitions rappelées en Sections 9.1 et 9.2 sont issues de [STW18]. Les conjectures raffinant celles de cet article sont nouvelles et ont été obtenues lors de cette thèse sous la direction de Frédéric Chapoton et de Christian Stump.

L'autre contribution majeure de ce chapitre est la nouvelle définition, encore conjecturale, des treillis m-cambriens en Section 9.4. Celle-ci repose sur l'Assertion 1 dont la preuve n'est pas complète. Cette nouvelle définition a été proposée après des discussions avec Corentin Henriet et Wenjie Fang, et fait depuis mai 2023 l'objet d'un projet de recherche, qui vise à établir la définition et notamment à en exploiter les résultats en type A linéaire.

0.5.2 Partie I : Préliminaires

La Partie I vise à introduire les principaux objets apparaissant dans ce manuscrit, ainsi que les outils utilisés par la suite. Son but est de rassembler les différentes définitions, notations et les résultats connus de la littérature, et d'esquisser le contexte historique et actuel du domaine de recherche dans lequel s'inscrit ce manuscrit. La plupart du contenu de cette partie n'est pas nouveau, hormis les définitions des treillis alt-Tamari et alt ν-Tamari. Cette partie est conçue pour aider un lecteur à se familiariser avec le domaine et les différentes notions impliquées.

Dans le Chapitre 1, nous introduisons les nombres et objets de Catalan, et les utilisons comme une opportunité pour manipuler les fonctions génératrices, la décomposition combinatoire et l'inversion de Lagrange.

Dans un premier temps, nous définissons les **classes combinatoires**. Un lecteur souhaitant se familiariser avec ces notions peut se référer à [FS09]. Les classes combinatoires et leurs **séries génératrices** sont de puissants outils pour énumérer des objets combinatoires via le calcul symbolique. En particulier, en établissant des équations sur les classes ou leurs séries, on peut prouver que des ensembles sont en bijection sans la construire explicitement, déduire des formules explicites via notamment l'inversion de Lagrange, ou utiliser par exemple des outils de l'analyse complexe pour obtenir des résultats asymptotiques.

Une classe combinatoire est définie comme un ensemble muni d'une statistique de taille, qui est une fonction à valeurs dans $\mathbb{N}$ telle que chaque valeur est atteint un nombre fini de fois. On peut définir la somme (union disjointe) et le produit (cartésien) de classes combinatoires, qui se traduisent respectivement comme somme et produit au niveau des séries génératrices. On peut également ajouter davantage de statistiques et en tenir compte dans les séries génératrices. Un des principaux résultats de cette section est la formule d'inversion de Lagrange, rappelée dans le Theorem 1.1.4.

Dans un second temps, nous définissons plusieurs objets Catalan et établissons des bijections entre eux. Historiquement, L. Euler s'est intéressé aux triangulations d'un polygone, introduisant une décomposition récursive que l'on rappelle dans la Section 1.2. Nous profitons de cet exemple pour illustrer l'utilisation d'équations sur les classes combinatoires et l'inversion de Lagrange.

Nous présentons ensuite d'autres objets Catalan apparaissant dans ce travail, parmi lesquels les arbres binaires plans enracinés, les chemins de Dyck, et les partitions non croisées. Les bijections présentées apparaissent pour la plupart dans la suite du document, et sont choisies pour que les conventions et notations soient cohérentes avec la suite.

Dans un troisième temps, nous présentons des généralisations des nombres et objets Catalan, notamment les nombres Fuß-Catalan, qui comptent les $(m+2)$-angulations, les arbres $(m+1)$-aires (plans enracinés), les chemins m-Dyck, et les partitions non croisées m-divisibles. Pour tous ces objets, m est un nombre entier strictement positif et le cas $m = 1$ correspond au cas précédent. Nous présentons également les ν-chemins, qui généralisent les chemins de Dyck et m-Dyck.

Le Chapitre 2 présente les ordres partiels, la structure algébrique centrale dans cette thèse, ainsi que les treillis. Comme évoqué précédemment, une relation d'ordre est une relation binaire transitive, réflexive et antisymétrique. Un treillis est un ordre partiel tel que toute paire d'éléments possède un inf (borne inférieure) et un sup (borne supérieure). Nous présentons quelques exemples classiques, et définissons les notions essentielles dans ce travail sur les ordres partiels, en particulier les intervalles et les chaînes. On définit également les notions de sous-treillis et de treillis quotient, qui apparaissent dans les définitions des treillis permusylvestres et cambriens.

Les ordres partiels entretiennent également des liens forts avec la topologie. C'est l'objet de la Section 2.1.3, qui présente les complexes simpliciaux, et comment attacher à un ordre partiel un complexe simplicial, en l'occurrence son complexe d'ordre. On peut également associer à un tel complexe l'ordre partiel sur ses faces induit par l'inclusion. Ces deux constructions permettent d'associer à un ordre partiel un espace topologique. On présente également la fonction de Möbius, définie sur les intervalles. Elle est un invariant de l'intervalle et sa valeur est en fait la caractéristique d'Euler réduite du complexe d'ordre associé à l'intervalle. La fonction de Möbius permet de définir l'inversion de Möbius. D'autres propriétés topologiques du complexe d'ordre peuvent provenir de notions combinatoires sur l'ordre partiel. Par exemple, on peut prouver que l'ordre est EL-épluchable pour prouver que le complexe est épluchable, ce qui implique des propriétés fortes, comme d'être Cohen-Macaulay. Ces propriétés sont très riches et intéressantes, bien qu'elles dépassent le cadre de ce manuscrit.

Pour finir, nous présentons quelques exemples d'ordres partiels centraux dans notre étude, notamment les treillis de Dyck et ν-Dyck, le treillis de Tamari, défini tant sur les arbres que sur les chemins de Dyck, le treillis des partitions non croisées. On présente aussi l'ordre faible sur le groupe symétrique et toute la famille des treillis permusylvestres, qui contient tant l'ordre faible que le treillis de Tamari, mais également tous les treillis cambriens de type A, définis plus tard.

Le Chapitre 3 vise à définir les groupes de Coxeter et toutes les notions liées qui apparaissent dans ce manuscrit. La théorie des groupes de Coxeter est très riche et variée, liant différentes disciplines mathématiques. Plusieurs points de vue peuvent être adoptés pour parler de ces groupes, certains plus algébriques, d'autres plus géométriques. Ils ont été définis par H. S. M. Coxeter comme une généralisation abstraite des groupes de réflexions réelles, c'est-à-dire de groupes engendrés par des réflexions orthogonales dans un espace euclidien. Le membre le plus éminent de cette famille est sans aucun doute le groupe symétrique, que l'on appelle aussi le groupe de Coxeter de type A. Un slogan général est que les résultats qui sont vrais sur le groupe symétrique ont souvent une contrepartie dans le monde des groupes de Coxeter, en particulier pour ceux qui sont finis. Ces derniers ont tous été classifiés par H. S. M. Coxeter [Cox34, Cox35]. À chaque tel groupe, on peut associer un graphe de Coxeter, qui encode une présentation de Coxeter de ce groupe.

Un groupe de Coxeter est un groupe présenté par générateurs et relations, tels que les générateurs sont d'ordre 2 et deux générateurs différents sont éventuellement liés par une relation de tresse, c'est-à-dire que leur produit est d'ordre fini (au moins égal à deux), le cas échéant. Les générateurs sont appelés les **réflexions simples** et leurs conjugués forment l'ensemble des **réflexions** du groupe. Le groupe de Coxeter de type A correspond au groupe symétrique, engendré par les transpositions simples $(i, i+1)$.

Un point de vue plus géométrique et très important est celui des **systèmes de racines**, qui sont des ensembles de vecteurs Φ dans un espace euclidien, satisfaisant certaines propriétés. En particulier, la réflexion orthogonale par rapport à une racine doit stabiliser Φ, et l'opposé de chaque racine est une racine également, la seule qui lui soit colinéaire. À un système de racines, on peut associer le groupe engendré par les réflexions orthogonales aux racines du système, et réciproquement, à un groupe de Coxeter, on peut associer un système de racines, dans un espace *ad hoc*. Par ailleurs, on peut séparer les racines en une partie positive Φ^+ et une partie négative Φ^-, et définir un ensemble de racines simples Δ, qui sont les racines positives extrémales, c'est-à-dire n'étant pas dans le cône engendré par les autres racines positives.

En voyant le groupe comme un quotient du monoïde des mots en les générateurs simples, l'on peut définir une notion de longueur de Coxeter comme la taille minimale d'un mot qui représente l'élément. En particulier, tout groupe de Coxeter fini possède un unique élément le plus long, noté $w_\circ$. En le voyant comme agissant sur le système de racines, on peut définir des ensembles d'inversions, comme l'ensemble des racines positives envoyées sur une racine négative. Ces points de vue coïncident, en ce que la taille de l'ensemble d'inversions correspond à la longueur de Coxeter de l'élément, et que l'on peut retrouver l'ensemble d'inversions à partir d'un mot réduit. La complémentarité de ces différentes notions se montrera très utile par la suite.

Les groupes de Coxeter sont naturellement munis de plusieurs structures d'ordre partiel présentées en Section 3.3, parmi lesquelles l'ordre faible et l'ordre absolu. L'**ordre faible** (droit) est une généralisation de celui présenté précédemment sur le groupe symétrique. Il peut se définir comme le fait d'être un préfixe. Dit autrement, un élément u est plus petit qu'un élément w pour l'ordre faible s'il existe une expression réduite pour w dont un préfixe est une expression réduite pour u.

L'ordre absolu est défini de façon "duale" en considérant l'ensemble des réflexions comme système de générateurs, et en définissant une notion de longueur absolue de manière tout à fait similaire.

La Section 3.4 présente ensuite les treillis cambriens, définis par N. Reading dans [Rea06, Rea07b, Rea07a], de trois points de vue différents et complémentaires. Chaque définition repose sur le choix d'un **élément de Coxeter**, c'est-à-dire d'un produit des générateurs simples, où chacun apparaît une et une seule fois, voire d'un mot réduit pour cet élément, que l'on appelle **mot de Coxeter**.

1. La première construction repose la notion de **c-triabilité**. Le choix d'un mot de Coxeter donne pour chaque élément un mot réduit privilégié, appelée écriture c-triée. Celle-ci peut se trouver de façon gloutonne en lisant le mot infini c^{∞}, ajoutant chaque lettre lorsqu'il est possible de l'ajouter pour créer un mot réduit pour l'élément considéré. L'élément est alors c-triable si son écriture c-triée vérifie une propriété simple. Le treillis cambrien correspondant est alors défini comme la restriction de l'ordre faible à l'ensemble des éléments c-triables.

2. La seconde construction nécessite la notion de **complexe de sous-mot**, définie dans [KM04, KM05] et généralisée dans [STW18]. Étant donné un mot Q en les générateurs simples, un élément $w \in W$ et une longueur a, on peut considérer le complexe simplicial $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ dont les facettes sont les complémentaires des sous-mots de Q qui forment un mot de longueur a qui représente l'élément w. On peut alors définir un ordre partiel sur les facettes d'un tel complexe. Un flip croissant consiste à supprimer un indice d'une facette et le remplacer par un indice plus grand. Le poset de flips est défini comme la clôture transitive des flips croissants. Le treillis cambrien correspond alors à un certain choix de complexe de sous-mot, c'est-à-dire au mot $\mathsf{Q} = \mathsf{c}w_\circ(c)$, où $w_\circ(\mathsf{c})$ est l'écriture c-triée de l'élément le plus long $w_\circ$, à $w = w_\circ$ et à $a = \ell_{\mathcal{S}}(w_\circ)$.

3. La troisième construction repose sur les **partitions non-croisées**. En remarquant que tout élément de Coxeter c est nécessairement maximal dans l'ordre absolu, on peut d'abord définir le treillis des partitions non-croisées. On peut également définir la notion de **delta-suite**, qui consiste en une factorisation réduite de l'élément de Coxeter c. En particulier, à une partition non-croisée w, on peut associer la delta-suite (cw^{-1}, w). On peut alors définir un ordre partiel sur les delta-suites, en définissant une notion de flip croissant et le treillis cambrien est alors obtenu comme la clôture transitive de ces flips croissants.

Ces trois définitions donnent en fait lieu à trois structures isomorphes. Pour traduire de la première à la troisième, on peut utiliser la notion d'**ensemble de sauts** d'un élément c-triable, et pour passer de la deuxième à la troisième, associer à chaque facette sa **configuration de racines**. Chaque définition donnera lieu à une "m-éralisation" pour définir les treillis m-cambriens, dans le Chapitre 9.

Le Chapitre 4 présente toutes les généralisations du treillis de Tamari qui apparaissent dans ce manuscrit, comme illustré sur les Figures 1 et 2.

Le treillis de Tamari peut se définir sur les chemins de Dyck. Cette construction peut alors se généraliser sur les chemins de m-Dyck, qui sont des chemins de Dyck dont les tailles des montées sont des multiples de m. Le treillis m-Tamari de taille n se défini alors comme la restriction du treillis de Tamari de taille mn aux m-chemins de Dyck. Cette définition est due à F. Bergeron dans [Ber12], pour des motivations liées à la théorie des représentations.

Cette définition peut encore être généralisée en constatant que les chemins de Dyck sont l'ensemble des chemins qui restent au-dessus de la diagonale, c'est-à-dire faiblement au-dessus du chemin $(NE)^n$. En remplaçant cette frontière par n'importe quel chemin ν consistant en des pas nord et est, L.-F. Préville-Ratelle et X. Viennot ont défini les treillis ν-Tamari dans [PRV17]. Cet ordre partiel peut être défini sur des ν-chemins, mais également sur une famille d'arbres appelés ν-arbres, sur lesquels une rotation essentiellement identique à celle du treillis de Tamari peut être définie. Cette seconde définition est due à C. Ceballos, A. Padrol, et C. Sarmiento dans [CPS20].

Un isomorphisme explicite entre les deux ordres partiels est donné par un algorithme de chasse horizontale.

Les treillis alt-Tamari sont une nouvelle famille de treillis définie sur les chemins de Dyck. C'est une famille qui contient les treillis de Dyck et de Tamari, dont les relations de couverture sont assez semblables, en ce qu'elles consistent à échanger le pas descendant d'une vallée avec un morceau de l'expédition qui le suit directement. Chaque ordre partiel de la famille dépend du choix d'un vecteur d'incrément $\delta \in \{0,1\}^n$, et l'ordre alt-Tamari correspondant est défini comme la clôture transitive de certaines opérations appelées δ-rotations. Cette famille est définie dans le Chapitre 5.

Cette définition peut également être transportée sur l'ensemble des ν-chemins, ce qui donne lieu à la définition des treillis alt ν-Tamari. Cette famille généralise à la fois les treillis alt-Tamari et ν-Tamari et sera l'objet principal du Chapitre 6. Cet ordre partiel peut être défini sur des chemins, mais également sur certaines familles d'arbres appelés (δ, ν)-arbres.

Dans les Chapitres 5 et 6, nous montrons en particulier que chaque ordre alt ν-Tamari est en fait isomorphe à un intervalle dans un certain treillis $\breve{\nu}$-Tamari, pour un bon choix de chemin $\breve{\nu}$. Ce sont en conséquence bien des treillis. Nous prouvons également que les distributions des intervalles linéaires dans les treillis alt-Tamari et alt ν-Tamari ne dépendent pas du vecteur d'incrément δ.

On peut également définir les treillis cambriens, qui seront généralisés plus tard en les treillis m-cambriens. On peut aussi définir les posets des modules basculants, définis dans [RS91, HU05]. Tous ces ordres partiels dépendent du choix d'un élément de Coxeter dans un groupe de Coxeter, ou de manière équivalente, d'une orientation du graphe de Coxeter.

On peut enfin définir les treillis permusylvestres comme une famille de treillis qui contient le treillis de Tamari, l'ordre faible sur le groupe symétrique, les treillis cambriens de type A et le treillis booléen. Tous sont en fait des treillis quotients de l'ordre faible, et peuvent être définis sur une famille d'arbres orientés, sur lesquels on définit une notion de rotation, et un ordre partiel comme clôture transitive de ces rotations. Cette famille est définie dans [PP18].

0.5.3 Partie II : Intervalles linéaires

Le point d'intérêt principal de la Partie II est l'étude de la distribution des intervalles linéaires dans le treillis de Tamari et dans les ordres partiels associés. Les intervalles linéaires sont ceux qui sont totalement ordonnés. Cette partie consiste principalement de travaux originaux et se base en particulier sur les deux articles prépubliés [Che22, CC23]. Les deux articles ont été retravaillés et adaptés pour s'insérer comme les deux premiers chapitres de cette partie. Les preuves et résultats principaux sont restés inchangés, mais d'autres résultats ont été ajoutés, notamment dans la Section 5.5. Les résultats et conjectures du Chapitre 7 sont également inédits et le sujet de recherches en cours, notamment sur les permutarbres.

Dans le Chapitre 5, nous étudions dans un premier temps les treillis de Dyck et de Tamari. F. Chapoton a observé que le nombre d'intervalles linéaires d'une longueur donnée était le même dans les deux ensembles partiellement ordonnés, et que par ailleurs, ce nombre était le double d'un coefficient binomial. Les deux ordres partiels peuvent être définis sur les chemins de Dyck, de sorte que chaque paire d'éléments comparables pour le treillis de Tamari soit aussi comparable pour le treillis de Dyck, dit autrement, de sorte que le treillis de Dyck soit une extension du treillis de Tamari. Malgré cette identification, les intervalles linéaires pour le treillis de Tamari ne restent pas nécessairement linéaires, ni de la bonne longueur.

On commence par étudier indépendamment les intervalles linéaires dans les deux ordres partiels. Dans le treillis de Dyck, on remarque que les intervalles linéaires non triviaux peuvent être séparés en deux familles que l'on appelle intervalles droits et intervalles gauches. Par ailleurs, l'involution miroir sur les chemins de Dyck, qui est un automorphisme du treillis, échange les intervalles droits et gauches. Dans le treillis de Tamari, défini sur les arbres binaires, on remarque également que les intervalles linéaires non triviaux se comprennent comme des intervalles que l'on appelle gauches et droits. Il existe également une involution miroir sur les arbres binaires, qui est toutefois un anti-automorphisme du treillis de Tamari, mais qui échange également les intervalles gauches et droits.

Dans un second temps, on montre que dans les deux ordres partiels, les intervalles gauches de longueur ℓ peuvent être compris comme un objet Catalan marqué avec une suite de ℓ objets Catalan, et de même pour les intervalles droits. Comme les séries génératrices des intervalles linéaires de longueur fixée sont solutions de la même équation fonctionnelle pour les deux ensembles ordonnés, on en déduit que la distribution de leurs intervalles linéaires coïncide en effet, confirmant les observations de F. Chapoton. On peut aussi utiliser l'inversion de Lagrange pour trouver une formule explicite pour le nombre d'intervalles linéaires de longueur ℓ.

En fait, on peut remarquer que les deux ordres partiels sont tout de même assez similaires du point de vue des relations de couverture, comme expliqué plus tôt. En effet, dans le treillis de Tamari, elles consistent à échanger le pas descendant d'une vallée avec l'expédition qui démarre à cette vallée. Dans le treillis de Dyck, elles consistent à envoyer le pas descendant de la vallée après le pas montant qui le suit, c'est-à-dire le premier pas de l'expédition qui démarre à cette vallée.

On définit en fait dans la Section 5.4 une famille d'ordres partiels qui contient les deux, en prescrivant des relations de couverture qui consistent à échanger le pas descendant d'une vallée avec un morceau de l'expédition qui le suit directement. Pour ce faire, on se fixe un vecteur d'incrément $\delta \in \{0,1\}^n$, où δ_i prescrit que le i-ème pas montant fait augmenter la δ-altitude de δ_i, et chaque pas descendant la fait diminuer de 1. Une δ-rotation consiste alors à échanger le pas descendant d'une vallée avec le plus petit sous-mot non vide qui démarre à cette vallée et dont la différence de δ-altitude entre les deux bouts est nulle. On retrouve donc les treillis de Dyck et de Tamari pour les valeurs extrémales de δ.

Une fois démontré que ces ordres partiels sont bien définis, on peut constater à nouveau que leurs intervalles linéaires non triviaux se répartissent en deux familles, où un pas montant est envoyé plusieurs fois vers la droite (appelés intervalles droits), ou bien une δ-excursion plusieurs fois vers la gauche (appelés intervalles gauches). On peut d'autre part décomposer les intervalles droits ou gauches similairement à ceux du treillis de Dyck. On en déduit une bijection entre les intervalles gauches (resp. droits) de n'importe quels ordres alt-Tamari sur les chemins de taille n, ce qui prouve qu'ils sont équidistribués.

On peut également montrer que l'ordre alt-Tamari associé au vecteur δ est une extension de celui associé à δ' dès lors que $\delta \leq \delta'$ composante par composante. La preuve que ces ordres partiels sont des treillis est donnée dans le chapitre suivant.

Finalement, dans la Section 5.5, on présente une explication bijective au fait que chaque treillis alt-Tamari contienne $\binom{2n-\ell}{n+1}$ intervalles droits de longueur ℓ. On construit en fait explicitement une bijection entre les intervalles droits de longueur ℓ et les chemins dans une grille rectangulaire de taille $(n+1) \times (n-\ell-1)$. L'idée est de prendre un tel intervalle et de supprimer $\ell+1$ pas montants (ou descendants), et d'ajouter un pas descendant (ou montant), de sorte que la transformation soit bijective, et c'est précisément ce qui est fait dans cette section.

Le Chapitre 6 est une généralisation des résultats du chapitre précédent au contexte des ν-chemins. Il se base sur les travaux de la prépublication [CC23]. Cette partie est le fruit d'une collaboration avec Cesar Ceballos, qui a été initiée après une présentation des résultats du Chapitre 5 et de quelques conjectures ouvertes. Ces questions ont bénéficié de l'expertise de Cesar sur le treillis ν-Tamari et de son idée de structures arborescentes pour attaquer le problème. Les deux auteurs ont contribué de façon égale à ce travail, qui a été accepté comme poster à la conférence FPSAC 2023.

Dans un premier temps, on étudie les intervalles linéaires dans les treillis ν-Dyck et ν-Tamari. On constate que l'on peut généraliser les définitions des intervalles gauches et droits du treillis de Dyck aux treillis ν-Dyck, et qu'ils décrivent à nouveau tous les intervalles linéaires non triviaux. On peut également utiliser la description alternative du treillis ν-Tamari sur les ν-arbres pour généraliser de même la notion d'intervalles gauches et droits aux treillis ν-Tamari. Ceux-ci décrivent encore tous les intervalles linéaires non triviaux.

Dans un second temps, on définit les ordres alt ν-Tamari, qui généralisent à la fois les treillis alt-Tamari et ν-Tamari. De même que les ordres alt-Tamari, ils dépendent du choix d'un vecteur d'incrément δ, et les treillis ν-Dyck et ν-Tamari apparaissent comme cas extrêmes.

On définit d'abord ces ordres partiels sur les ν-chemins, en introduisant une notion de δ-altitude et de δ-excursion, comme la clôture transitive de δ-rotations. Similairement aux ordres alt-Tamari du chapitre précédent, étant donné un chemin $\nu = (\nu_0, \ldots, \nu_m)$, où ν_i est le nombre de pas est d'ordonnée i, on fixe un entier $\delta_i \in [0, \nu_i]$ pour chaque i, et on définit que le i-ème pas montant fait augmenter la δ-altitude de δ_i. Chaque pas descendant la fait diminuer de 1. La δ-excursion d'un pas montant est définie comme le plus petit sous-chemin qui démarre par ce pas montant et dont la variation de δ-altitude est nulle. Une δ-rotation $P \lessdot_\delta Q$ consiste alors en l'envoi du pas descendant d'une vallée de P après la δ-excursion qui le suit directement.

On peut en fait définir un chemin $\breve{\nu}$ à partir des δ_i, de sorte que le treillis alt ν-Tamari $\mathrm{Tam}_\nu(\delta)$ coïncide avec la restriction du treillis $\breve{\nu}$-Tamari à l'ensemble des ν-chemins, qui forme en fait un intervalle. Cela prouve en particulier que les ordres alt ν-Tamari sont des treillis. On peut alors appliquer l'algorithme de chasse horizontale aux ν-chemins, considérés en tant que $\breve{\nu}$-chemins. Son image est une sous-famille des $\breve{\nu}$-arbres, que l'on appelle les (δ, ν)-arbres. L'ordre alt ν-Tamari peut alors être défini sur les (δ, ν)-arbres et l'étude des intervalles linéaires du treillis $\breve{\nu}$-Tamari se transpose à ce cas.

Enfin, une fois étudiée la structure des intervalles linéaires dans les treillis alt ν-Tamari, on prouve le résultat principal de ce chapitre, à savoir que la distribution des intervalles gauches et droits ne dépend pas du vecteur d'incrément δ. En particulier, tous les treillis alt ν-Tamari définis pour un même chemin ν possèdent le même nombre d'intervalles linéaires.

Ce résultat est prouvé bijectivement, en utilisant la chasse horizontale pour les intervalles gauches et une bijection de chasse verticale que l'on définit pour les intervalles droits. L'étude passe par la définition de vecteurs de colonnes et de lignes, qui consistent à compter le nombre de nœuds des (δ, ν)-arbres sur ses colonnes et ses lignes, respectivement. On montre que ces vecteurs contiennent les informations des intervalles droits et gauches, respectivement, puis que les bijections présentées préservent l'un ou l'autre de ces vecteurs.

Bien que pour un chemin ν général, on ne puisse pas espérer une formule produit aussi jolie que pour le cas des treillis alt-Tamari pour la distribution des intervalles linéaires, des formules explicites semblent émerger lorsque l'on somme ces distributions sur tous les chemins ν d'une longueur fixée. Enfin, en plus de l'équidistribution des intervalles linéaires, nous nous attendons à un résultat d'équivalence dérivée pour les catégories de modules associées aux treillis alt ν-Tamari.

Le Chapitre 7 est une collection de résultats explorant cette étude des intervalles linéaires dans d'autres ordres partiels, en particulier dans ceux ayant des liens avec le treillis de Tamari. Ce chapitre est basé sur des travaux non publiés et des discussions avec différents chercheurs et chercheuses.

On calcule d'abord la distribution des intervalles linéaires dans l'ordre faible sur le groupe symétrique. Ce résultat a été obtenu après des discussions avec Viviane Pons et Vincent Pilaud. En écrivant les permutations selon la notation en une ligne, les relations de couverture consistent à échanger deux entrées consécutives $\sigma_i < \sigma_{i+1}$ d'une permutation σ de sorte à ajouter une inversion. Les intervalles linéaires peuvent alors être compris comme étant soit des intervalles droits, où σ_i est ainsi déplacé plusieurs fois vers la droite, soit des intervalles gauches, où σ_{i+1} est ainsi déplacé plusieurs fois vers la gauche. Cette description permet un dénombrement explicite très simple des intervalles linéaires.

Inspirés par des travaux récents [Der23, BMC23], on introduit une version "gloutonne" des treillis alt ν-Tamari, que l'on appelle les ordres alt ν-Tamari gloutons. L'idée est de définir des rotations "gloutonnes" qui consistent à échanger le pas descendant d'une vallée avec toutes les δ-excursions consécutives qui le suivent directement, au lieu de seulement la première.

On peut encore séparer les intervalles linéaires des ordres obtenus en des intervalles "gauches" ou "droits". La distribution des intervalles gauches est la même que dans les treillis alt ν-Tamari correspondants, mais ce n'est en général pas le cas pour les intervalles droits. Toutefois, dans le cas des chemins de Dyck, donc de $\nu = (NE)^n$, et pour $\delta_i = 0$ pour tout i, on obtient en effet un treillis dont la distribution des intervalles linéaires coïncide avec celle des treillis alt-Tamari, ce que l'on prouve bijectivement. Ce dernier cas apparaît comme un intervalle dans un treillis défini récemment par P. Nadeau et V. Tewari [NT23, Section 5].

On présente également plusieurs formules conjecturales dues à Frédéric Chapoton pour la distribution des intervalles linéaires dans plusieurs ensembles partiellement ordonnés. En particulier, on présente l'ordre de peigne de Pallo, défini dans [Pal03] comme la clôture transitive du sous-ensemble des relations de couverture du treillis de Tamari où la rotation s'effectue sur une vallée qui est aussi un contact. Une formule simple semble émerger pour les intervalles linéaires, de même que dans une généralisation naturelle de cet ordre aux chemins m-Dyck (voir [AC18]).

On se penche également sur les ordres cambriens et des modules basculants. Tous les deux sont définis sur des groupes de Coxeter, et dépendent donc du choix d'un élément de Coxeter. On conjecture d'une part que dans les deux cas, la distribution des intervalles linéaires est indépendante du choix de l'élément de Coxeter. Ceci implique en particulier qu'en type A, cette distribution est donnée par les formules du Théorème 5.3.1. D'autre part, on fournit dans la Conjecture 7.3.7 des formules présumées pour les distributions des intervalles linéaires dans les treillis cambriens et des modules basculants en type B et D.

Enfin, cette indépendance du choix de l'élément de Coxeter pour l'énumération des intervalles linéaires semble toujours vraie dans les treillis m-cambriens. Le résultat d'équivalence dérivée mentionné précédemment et prouvé pour les ordres cambriens et des modules basculants semble également tenir pour les treillis m-cambriens.

Enfin, pour clore ce chapitre et cette partie, on s'intéresse aux treillis permusylvestres. Chacun de ces ordres dépend du choix préalable d'une décoration $\delta \in \left\{\{0,1\}^2\right\}^n$. Cette famille de treillis contient à la fois l'ordre faible (pour $\delta = (0,0)^n$) et les treillis cambriens en type A (pour $\delta \in \{(0,1),(1,0)\}^n$).

On peut définir des intervalles droits et gauches dans ces ensembles partiellement ordonnés, de sorte à retrouver les définitions précédentes dans les cas de l'ordre faible et de Tamari. On prouve que tout intervalle linéaire est soit un intervalle droit, soit un intervalle gauche, et on conjecture que ceux-ci sont en effet linéaires. Une preuve de cette dernière assertion semble pouvoir être obtenue en utilisant la représentation cubique de ces treillis, définie par D. Tamayo dans [Tam23].

Enfin, on conjecture que la distribution des intervalles linéaires dans ces treillis ne change pas lorsque l'on modifie une décoration δ en transformant une entrée $(0,1)$ en $(1,0)$. Il semble également qu'une telle transformation donne lieu à deux posets dérivés équivalents. Des tests ont été réalisés avec Daniel Tamayo en calculant les distributions des intervalles linéaires et des polynômes de Coxeter sur tous les treillis permusylvestres pour $n \leq 6$.

0.5.4 Part III: m-éralisations

La Partie III est dédiée à l'étude de deux "m-éralisations" du treillis de Tamari, à savoir les treillis m-Tamari et m-cambriens, où $m \geq 1$ est un entier. Le treillis de Tamari apparaît d'une part comme le treillis 1-Tamari, et d'autre part comme le treillis cambrien en type A linéaire.

Cela donne en particulier deux familles de treillis à deux paramètres entiers $n, m \geq 1$ qui généralisent le treillis de Tamari, d'un côté le treillis m-Tamari $\mathrm{Tam}_n^{(m)}$, et de l'autre le treillis m-cambrien en type A linéaire $\mathrm{Camb}^{(m)}(\mathfrak{S}_n, c^{\mathsf{lin}})$.

Pour $m, n \geq 1$ fixés, ces deux ordres partiels ne sont en général pas isomorphes. Cependant, une conjecture formulée dans [STW18, Section 6.10] prédit que les deux treillis auraient le même nombre d'intervalles, et même d'intervalles décorés par certains paramètres. Leur énumération semble par ailleurs coïncider avec les dimensions de l'espace des polynômes harmoniques ou harmoniques alternés dans l'étude des coinvariants diagonaux supérieurs trivariés, comme mentionné plus tôt. Cette conjecture a été le point de départ de ce travail, et nous présentons ici les principaux résultats obtenus dans cette direction, basés sur des travaux originaux non encore prépubliés.

Le Chapitre 8 s'intéresse d'abord aux intervalles dans les treillis m-Tamari, et notamment à leur énumération.

Dans son article [Cha06], F. Chapoton a étudié les intervalles du treillis de Tamari. Il a décrit une décomposition récursive des intervalles en introduisant un paramètre dit catalytique, qui permet d'écrire une équation sur la série génératrice des intervalles. Cette équation peut être résolue pour obtenir une formule explicite pour le nombre d'intervalles. Ce résultat a ensuite été étendu au cas des treillis m-Tamari dans [BMFPR11].

Le début de ce chapitre reproduit brièvement la preuve de [BMFPR11], en présentant certaines adaptations dans les notations et conventions. La discussion initiale se concentre sur le cas plus simple du treillis de Tamari, pour $m = 1$, où l'on met en évidence un processus d'expansion et collage pour construire chaque intervalle à partir d'intervalles plus petits. Nous expliquons ensuite les changements nécessaires pour le cas général $m \geq 1$, qui consistent principalement à exécuter m processus d'expansion et collage consécutifs.

Nous présentons ensuite de nombreuses statistiques que nous utilisons pour décorer — et donc distinguer — les intervalles m-Tamari. Ces statistiques semblent intéressantes d'un point de vue énumératif sur les intervalles m-Tamari et ont naturellement émergé dans notre travail comparatif avec les intervalles cambriens en type A linéaire. Elles sont en fait aussi apparues dans les travaux de L.-F. Préville-Ratelle [PR12] et de V. Pons [Pon19].

Nous présentons les degrés entrants et sortants, supérieurs et inférieurs. Ces degrés correspondent au nombre de façons de réduire ou d'étendre un intervalle, par le haut ou par le bas, en changeant une borne de l'intervalle via une relation de couverture. Cela correspond aux flèches entrantes ou sortantes d'un intervalle dans le diagramme de Hasse, ou encore à regarder des relations de couvertures dans un ordre partiel sur les intervalles, où deux intervalles $[x, y]$ et $[x', y']$ sont comparables si $x \leq x'$ et $y \leq y'$. On conjecture que la série génératrice des intervalles m-Tamari décorés de ces quatre statistiques présente une symétrie d'ordre trois lorsque l'on oublie le degré entrant supérieur. Le cas particulier du treillis de Tamari, où $m = 1$, a été prouvé dans [Cha18].

Nous nous intéressons également à la hauteur d'un intervalle, à savoir la longueur maximale d'une chaîne au sein de l'intervalle. Nous présentons deux algorithmes gloutons qui, partant d'une paire (x, y) de chemins de Dyck, produisent une chaîne maximale de x à y si $x \leq y$ dans le treillis de Tamari (et déterminent que $x \not\leq y$ sinon), ainsi qu'un moyen de calculer cette hauteur.

Nous attachons aussi deux partitions d'un entier aux intervalles ($m-$)Tamari. La première, appelée partition des montées, est la partition dont les parts sont les tailles des montées de l'élément maximal de l'intervalle. La taille de la première montée est marquée et correspond à la statistique de la *montée initiale*. La seconde, appelée partition des contacts, est la partition dont les parts sont les tailles des suites d'excursions consécutives de l'élément minimal de l'intervalle. Les excursions à hauteur 0 forment une part marquée qui correspond à la statistique des *contacts* de l'intervalle.

Nous présentons ensuite deux raffinements de l'équation fonctionnelle obtenue dans [BMFPR11]. La première équation est obtenue sur la série génératrice décorée où l'on tient compte des degrés sortants et de la hauteur des intervalles, en plus des contacts et de la montée initiale. La seconde équation raffine davantage la précédente, en tenant compte des deux partitions attachées aux intervalles et de la hauteur, au moyen de familles infinies de variables. Cette dernière équation (8.13) pourrait être utile pour relier les intervalles m-Tamari à certains sous-ensembles de cartes ou de constellations, étant donné que les opérateurs qui y apparaissent ressemblent à ceux présents dans les travaux de G. Chapuy et M. Dołęga [CD22].

Ce chapitre se termine avec la présentation des intervalle-posets, définis par G. Châtel et V. Pons dans [CP15]. Ceux-ci sont en bijection avec les intervalles du treillis de Tamari. De plus, on peut attacher à chaque intervalle-poset deux partitions et un nombre d'inversions Tamari, de telle sorte que la distribution jointe de ces décorations soit la même que celle des intervalles m-Tamari décorés des partitions de montées et de contacts, ainsi que de leur hauteur. Cette équidistribution est un outil crucial dans l'article [Pon19], qui démontre notamment que la distribution jointe des contacts et de la montée initiale des intervalles m-Tamari (et en fait la distribution jointe de leurs deux partitions) est symétrique, confirmant l'observation faite dans [BMFPR11].

On présente finalement une sous-famille des intervalle-posets, que nous appelons les intervalle-posets m-Tamari[1], qui sont ceux dont les deux partitions ont toutes leurs parts divisibles par m. Ces dernières sont en bijection avec les intervalles m-Tamari, comme prouvé dans [CP15].

Les intervalle-posets m-Tamari sont particulièrement intéressants car on peut y lire des statistiques comme sur les intervalles m-Tamari, mais ils possèdent une involution naturelle supplémentaire, appelée l'involution de complément.

Le Chapitre 9, dernier chapitre de ce travail, est dédié aux treillis m-cambriens. Nous présentons notamment le contexte théorique nécessaire à cette étude, et en particulier la définition du monoïde positif d'Artin. En effet, l'idée clé développée dans [STW18] pour généraliser du treillis de cambrien au treillis m-cambrien est de transporter les définitions et propriétés du groupe de Coxeter au monoïde d'Artin, où beaucoup de résultats subsistent.

Un groupe de Coxeter est un groupe qui admet une présentation par générateurs et relations, où les générateurs sont d'ordre 2, et les autres relations sont des relations de tresse. Le groupe d'Artin est le groupe obtenu en supprimant la condition d'ordre 2 sur les générateurs et le monoïde positif d'Artin est le sous-monoïde engendré par les générateurs du groupe d'Artin. Ce dernier peut être vu comme le monoïde des mots en les générateurs, soumis aux relations de tresse. Le groupe de Coxeter s'identifie naturellement au sous-ensemble du monoïde d'Artin des mots qui correspondent à des $\mathcal{S}$-mots réduits.

Dans le monoïde d'Artin, on peut définir une notion de longueur et un ordre faible droit, correspondant au fait d'être un préfixe. En prenant l'ensemble des racines colorées, correspondant aux racines positives décorées d'un entier naturel, on peut également définir les inversions colorées d'un élément du monoïde. Les racines de couleur 0 ou 1 jouent respectivement le rôle des racines positives et négatives, et pour un élément du groupe de Coxeter, son ensemble d'inversions dans le groupe de Coxeter ou dans le monoïde d'Artin coïncident alors.

On peut définir la factorisation de Garside d'un élément et la longueur de Garside comme le nombre de facteurs dans cette factorisation. Le groupe de Coxeter s'identifie donc comme l'ensemble des éléments de longueur de Garside au plus 1. On peut ainsi m-éraliser ceci en considérant l'ensemble des éléments de longueur de Garside au plus m. On obtient ainsi l'ordre faible m-éralisé en restreignant l'ordre faible à ces éléments de longueur de Garside au plus m, ou de façon équivalente pour un groupe de Coxeter fini, à l'intervalle $[\boldsymbol{e}, \boldsymbol{w}_\circ^m]$, où $\boldsymbol{w}_\circ$ est l'élément du monoïde correspondant à l'élément le plus long $w_\circ$ du groupe de Coxeter, sous l'identification canonique.

[1] Ceux-ci sont appelés *intervalle-posets montée-contact-m-divisibles* dans [Pon19].

Nous donnons ensuite les trois définitions de [STW18] des treillis m-cambriens, ainsi qu'une façon de passer de l'une à l'autre. Chacune dépend toujours du choix d'un élément de Coxeter c, voire d'un mot de Coxeter c pour cet élément de Coxeter.

1. La première construction repose la notion de **c-triabilité**. Celle-ci se généralise naturellement au monoïde d'Artin, où le choix d'un mot de Coxeter donne pour chaque élément une écriture c-triée, conduisant à la même notion de c-triabilité. Le treillis m-cambrien correspondant est alors défini comme la restriction de l'ordre faible m-éralisé à l'ensemble des éléments c-triables.

2. La seconde construction est aussi définie sur les **complexes de sous-mot**, bénéficiant de la généralisation de [STW18]. Des généralisations différentes de complexes de sous-mot pourraient être considérées dans un monoïde d'Artin, mais celles-ci se recoupent toutes dans le cas d'un complexe c-initial. Le treillis m-cambrien correspond alors au complexe de sous-mot $\mathrm{SC}_{\mathcal{S}}(\mathsf{c}{w_\circ}^m(c), {w_\circ}^m, m\,\ell_{\mathcal{S}}(w_\circ))$.

3. La troisième description est donnée sur les **partitions non-croisées m-éralisées**, c'est-à-dire sur les m-multichaînes dans l'intervalle $[1, c]$ pour l'ordre absolu, ou de façon équivalente, aux m-delta suites, correspondant aux $\mathcal{R}$-factorisations de c en produits de $m+1$ éléments du groupe de Coxeter. On peut alors définir un ordre partiel sur les m-delta suites, en définissant une notion de flip croissant comme dans le cas $m = 1$. Le treillis m-cambrien est alors obtenu comme la clôture transitive de ces flips croissants.

De même que dans le cas des treillis cambriens, on peut définir des notions d'ensemble de sauts et de configuration de racines, comme détaillé dans [STW18], pour passer d'une des deux premières définitions à la troisième.

Dans le cas particulier du type A linéaire, nous présentons des statistiques sur les intervalles qui permettent de raffiner la conjecture d'équidistribution des intervalles m-Tamari et m-cambriens en type A linéaire. Ce dernier peut par exemple se décrire sur des $(m+2)$-angulations, et dans cette description, l'élément minimal a toutes ses diagonales qui démarrent au même sommet, appelées diagonales initiales et l'élément maximal a toutes ses diagonales qui se terminent au même sommet, appelées diagonales finales. La distribution jointe de ces deux statistiques est conjecturalement la même que celle des contacts et de la montée initiale sur les intervalles m-Tamari.

Plus généralement, en type A linéaire, on peut définir des partitions initiales et finales. La partition initiale est obtenue sur une m-delta suite en prenant le type cyclique du premier facteur de l'élément du bas. La partition finale est obtenue de façon similaire sur le dernier facteur de l'élément du haut.

Nous conjecturons que la distribution des intervalles m-Tamari décorés de la hauteur et des partitions des montées et des contacts est la même que celle des intervalles m-cambriens en type A linéaire également décorés de la hauteur et des partitions initiale et finale. Cette conjecture raffine celle de [STW18]. Mieux, nous conjecturons une bijection entre les intervalles m-cambriens ainsi décorés et les m-Tamari intervalle-posets, qui identifierait de plus l'involution naturelle des intervalles m-cambriens de type A linéaire et l'involution de complément des m-Tamari intervalle-posets.

Le reste du chapitre se concentre sur une nouvelle définition conjecturale des treillis m-cambriens en général, sur la base d'un projet commun en cours avec Corentin Henriet et Wenjie Fang. La preuve n'est pas complète, mais plusieurs approches sont présentées et le résultat est appuyé par des expérimentations informatiques.

Cette description serait une définition très pratique car elle donnerait un critère de comparaison simple sur les m-multichaînes de partitions non-croisées, elles-mêmes simples à décrire. On définit que deux m-multichaînes de partitions non-croisées $w_1 \leq_{\mathcal{R}} \leq_{\mathcal{R}} w_m$ et $w'_1 \leq_{\mathcal{R}} \leq_{\mathcal{R}} w'_m$ si et seulement si $w_i \leq_{\mathcal{R}} w'_{i+1}$ dans l'ordre absolu pour tout $1 \leq i < m$ et $w_i \leq w'_i$ dans l'ordre 1-cambrien pour tout $1 \leq i \leq m$. On prouve que cette définition est équivalente à la bonne définition d'un algorithme glouton qui produit l'unique chaîne c-croissante entre deux m-multichaînes de partitions non-croisées, si elle existe.

Il reste à prouver qu'une telle chaîne existe toujours entre deux éléments comparables dans le treillis m-cambrien, ce qui repose, dans notre preuve incomplète, sur l'Assertion 1.

Cette nouvelle définition conjecturale permet de passer d'un intervalle dans le treillis m-cambrien à une m-multichaîne dans un nouvel ordre partiel sur les intervalles du treillis 1-cambrien. En particulier, dans le cas du type A linéaire, cela donnerait une bijection entre les intervalles m-cambriens et des suites d'intervalles de Tamari qui satisfont certains critères de comparaison, ou de façon équivalente, certaines suites d'intervalle-posets.

Nous présentons finalement une manière de traduire un intervalle 1-cambrien en type A linéaire en un intervalle-poset via une notion de diagramme d'arches, sur lesquels les critères de comparaison dans les partitions non-croisées et dans le treillis 1-cambrien sont facilement lisibles. En particulier, il est à noter que ces objets sont par nature très proches des m-Tamari intervalle-posets.

Introduction
Tamarea: What's nu?

This manuscript falls within the realm of algebraic and enumerative combinatorics. This domain is concerned with the study of discrete algebraic structures, using the counting of some quantities to derive algebraic information, and conversely using algebraic tools to count interesting objects. This work in particular is dedicated to the study of some partially ordered sets related to the Tamari lattice, and more specifically from the perspective of their intervals and their enumeration. This document is organized in three parts, the first being widely introductory to the domain and the objects studied, while the second and third parts are more focused on the main contributions of this work.

Partially ordered sets, often called simply posets, are very natural structures and useful tools that appear in many domains in mathematics. A partial order on a set E is a reflexive, transitive and antisymmetric binary relation on E. Very common examples of posets are totally ordered sets as the natural orders on $\mathbb{N}$, $\mathbb{Q}$, or $\mathbb{R}$, the divisibility order on the natural numbers, power sets (i.e. the set $\mathcal{P}(E)$ of subsets of E with the inclusion order), and many more.

When studying partial orders, one may be interested in comparing in particular when two *a priori* different posets correspond in fact to "the same" structure, i.e. if they are *isomorphic*. If not, one may try to compare them in a weaker sense, to explore how similar they are, what properties or quantities they have in common.

A first very natural quantity when it comes to finite posets is the cardinality, that is to say the number of elements, and somehow whether the two posets can be defined on the same objects. Another very natural quantity is the number of relations, or **intervals**, which are pairs of comparable elements. As the cardinality, this number is an invariant that is additive under sums and multiplicative under Cartesian products of posets. It also is the dimension of an algebra that is naturally attached to the poset, the incidence algebra, which can also be used as an algebraic tool to study the partial order itself. For instance, this leads to the notion of derived equivalence, which is a weaker notion of isomorphism between posets.

The study of intervals in a poset can take various approaches, from the (not so) simple counting of their total number, or of elements in some subsets of intervals, to the distribution of some statistics on intervals, as we shall see along this work. We can also consider weighted intervals, poset structures on the set of intervals in a given poset, topological information, or more algebraic structures. This work is often motivated by the observation of some enumerative coincidences that may hide deep underlying reasons. For instance, the number of intervals in a poset may admit nice formulas, count other families of objects, hinting at a possible bijection between them, or coincide with the number of intervals in another poset, not necessarily isomorphic. This is the case for the Tamari lattice, where the number of intervals admits a closed product formula that also counts some combinatorial maps. It seems furthermore to coincide with the dimension of some subspaces of diagonal coinvariants, that is to say in some representation of the symmetric group.

0.6 The Tamari lattice

Among this vast world of posets lives the Tamari lattice, which is central in this work. It is a partial order defined on objects counted by Catalan numbers, and it has inspired a vast amount of research in various fields of mathematics.

The Catalan numbers are one of the most famous and studied sequence of integers, and Chapter 1 will provide a more extensive treatment of them. They are known to count numerous families of objects, for instance Dyck paths or binary trees, and appear in a lot of different contexts, as for instance the dimensions of the alternating component of the bivariate diagonal coinvariants of the symmetric group (see [Hai01, Hai02]).

In many places where the Catalan numbers appear, it is possible to give a nice description of the Tamari lattice. There is one such Tamari lattice Tam_n on objects of size n for every positive integer n, but we will refer to them as "the" Tamari lattice. This poset is named after Dov Tamari, who first defined and studied them by interpreting the associativity relation as an order relation (see [Tam62, HT72]).

The rich variety of descriptions of the Tamari lattice led to many generalizations, many of which are presented and studied in this work, and in particular in Chapter 4.

Firstly, a description of the poset on Dyck paths can be generalized to m-Dyck paths, and then to the set of paths weakly above a given path ν, also called ν-paths. These two generalizations are respectively referred to as the **m-Tamari lattice** ([Ber12]) and the **ν-Tamari lattice** ([PRV17]).

Very recently, as presented in this work and as the central concern of Chapters 5 and 6, the new families of **alt-Tamari** and **alt ν-Tamari lattices** have been introduced and studied, especially from the perspective of their linear intervals.

The Tamari lattice also appears naturally as the linear type A cases of the **posets of tilting modules** ([RS91, HU05]) and the **Cambrian lattices** ([Rea06]), which are two partial order structures that can be defined in the context of Coxeter groups, once fixed a *Coxeter element*. The latter were further generalized in the **m-Cambrian lattices** ([STW18]), which are the main focus of Chapter 9.

Finally, a last family of generalizations of the Tamari lattice is the one of **permutree lattices** ([PP18]), which encompasses the Tamari lattice as well as the weak order on the symmetric group and all type A Cambrian lattices.

All these families and how they relate to each other are presented in Figures 1 and 2.

0.7 Enumerating intervals

One direction of research which has received a lot of attention in recent years regards the number of intervals in the Tamari lattice and its generalizations. They were first enumerated by F. Chapoton in [Cha06] using a generating function approach, providing a very nice closed product formula:

$$\frac{2\,(4n+1)!}{(n+1)!\,(3n+2)!}.$$

These numbers also appear to count rooted cubic 3-connected planar maps and (dually) 3-connected triangulations, enumerated by W. T. Tutte (see [CS03, Tut62]). A bijection was then found in [BB09, Theorem 4.1], which explained this surprising coincidence. Moreover, these numbers also seem to correspond to the dimension of the alternating component in the study of diagonal coinvariants in the trivariate case (see [Hai94, BPR12]).

0.7.1 A representation motivation

Though we are not exploring this direction in this work, it is worth mentioning the connection between the enumeration of intervals in the Tamari lattice and the representation theory of the symmetric group, especially as it has been the main motivation for introducing the m-Tamari lattice.

The action of the symmetric group by permutation of indices on the set of polynomials in n variables is known to behave extremely well, and in particular, the set of symmetric polynomials—or invariants—is a polynomial algebra with generators the elementary symmetric polynomials. In this case, the coinvariants, that is to say the quotient of the total space by the ideal generated by non-constant symmetric polynomials, is known to be isomorphic to the *regular representation* of $\mathfrak{S}_n$. In fact, one may consider the subspace of harmonic polynomials, which is a supplementary subspace of the space of symmetric polynomials, and in fact its orthogonal complement for some natural scalar product. In particular, the coinvariant and the harmonic subspace are isomorphic, moreover their total dimension is $n!$ and the multiplicity of the sign representation is 1.

Then comes the idea to consider the *diagonal* action of the symmetric group on several sets of n variables. In this case, one still may consider the coinvariants as the quotient of the total space by the ideal generated by non-constant symmetric polynomials, and this space is isomorphic to the subspace of harmonic polynomials. In the bivariate case, that is to say with two sets of variables, it was conjectured in [Hai94] and proven in [Hai02] that the dimension of the harmonic polynomials is equal to $(n+1)^{n-1}$, which is the number of **parking functions**. It was also proven there that the subspace of **alternating diagonal harmonics**, which corresponds to the isotypic component of the sign representation, has dimension the Catalan number C_n. Some formulas were also conjectured for the trivariate case, and in particular the dimension of the alternating component was conjectured to be equal to the number of intervals in the Tamari lattice.

Later on, *higher* versions of the diagonal harmonics and alternating diagonal harmonics were introduced in [BPR12], along with conjectures on their dimensions in the bivariate and trivariate cases. In particular, this work led to the definition of the Tamari lattice on the Dyck paths and to the generalization of it to the **m-Tamari lattice** on m-Dyck paths (see [Ber12, BPR12]). In particular, in the trivariate case, the dimension of the m-th higher alternating diagonal harmonics was conjectured to be equal to the number of intervals in the m-Tamari lattice, with a nice product formula. Moreover, the dimension of the m-th higher diagonal harmonics was conjectured to be equal to the number of decorated intervals in the m-Tamari lattice, where a parking function is associated to the top element of each interval, again with a conjectured product formula. The first formula enumerating m-Tamari intervals was proven in [BMFPR11], and the second one enumerating decorated intervals was subsequently obtained in [BMCPR13].

On top of this, another generalization of the Tamari lattice involving an integer $m \geq 1$ was introduced in [STW18], namely the **linear type A m-Cambrian lattice**. This poset is not isomorphic to the m-Tamari lattice, for general m (for instance it is self dual, while the m-Tamari lattice is not), but its number of intervals was conjectured to coincide with the number of m-Tamari intervals, as well as its number of decorated intervals (see [STW18, Section 6.10]). These conjectures are the main motivation for the considerations in Part III.

0.7.2 The subset of linear intervals

A new direction of research was opened after F. Chapoton's idea to consider the subset of **linear intervals** in a poset. This subset is defined as the set of intervals that are totally ordered, or equivalently whose Hasse diagram is a path. In a way, these are the most simple intervals.

What is particularly surprising is that the distribution of linear intervals—that is to say the number of linear intervals of each length—in posets related to the Tamari lattices seems to behave very well, and Part II will be devoted to this study. On the one hand, the number of linear intervals of a given length in the Tamari lattice is almost a binomial number. On the other hand, the distribution of linear intervals in the Dyck lattice is exactly the same as in the Tamari lattice. The Dyck lattice is perhaps the most natural partial order on Dyck paths, where a path is greater than another if it is weakly above. These remarkable coincidences led F. Chapoton to conjecture a formula for the distribution of linear intervals, as well as the existence of a new family of posets that would contain the Tamari lattice and the Dyck lattice, and whose linear intervals would all have the same distribution. This initiated the work presented in Chapter 5, where we prove these initial conjectures and more. In particular, we introduce the **alt-Tamari lattices**, and prove the conjectured result on the distribution of their linear intervals, as well as their enumeration.

Together with the type A Cambrian lattices and posets of tilting modules, the alt-Tamari lattices seem to be a third family of posets generalizing the Tamari lattice and sharing very nice structural properties. In particular, on the one hand, we conjecture that the distribution of linear intervals in the type A Cambrian and tilting posets is also the same as in the Tamari lattice. On the other hand, we conjecture a derived-equivalence result among the alt-Tamari lattices, as has been proven by S. Ladkani for the type A Cambrian lattices and the posets of tilting modules (see [Lad07a, Lad07b]).

Further extending the m-Tamari lattice, L.-F. Préville-Ratelle and X. Viennot introduced the ν-Tamari lattice in [PRV17] as a poset on ν-paths, where ν is a given north-east lattice path. They proved that not only this family of posets contains the Tamari lattice as a special case, but that each ν-Tamari lattice is in fact isomorphic to an interval in some bigger Tamari lattice, introducing the notion of canopy of a tree. In fact, each Tamari lattice of size n can be split into disjoint intervals according to the canopies, each of which corresponding bijectively to a ν-Tamari lattice for one path ν of length $n-1$. Interestingly, intervals such that the bottom and top elements share the same canopy, called *synchronized intervals* are counted by a closed product formula resembling the one for the total number of intervals, as proven in [FPR17]. The authors also exhibit a bijection between synchronized intervals and nonseparable planar maps, strengthening the connection between intervals of Tamari type and maps. Note that one can also define a notion of *new* intervals in the Tamari lattice, that were proven in [Fan21] to be in bijection with yet another family of maps, namely bipartite maps.

The Dyck lattice can also be naturally generalized to the set of ν-paths as the relation of being weakly above. Moreover, we observed that the distributions of linear intervals still coincide in the ν-Dyck and in the ν-Tamari lattices. In a joint work with Cesar Ceballos, we introduced the **alt ν-Tamari lattices**, which are the joint generalization of the alt-Tamari lattices and the ν-Tamari lattices, and we prove bijectively that their linear intervals all share the same distribution. This work is exposed in Chapter 6.

As well as the number of synchronized intervals, it seems that the distribution of intervals in the Tamari lattice that are both synchronized and linear admits a neat enumeration formula. Saying it otherwise, even if the individual distributions of linear intervals in the ν-Tamari lattices does not seem to admit a nice formula, it seems to admit one when summed over all paths ν of a given length. We also expect the derived-equivalence result to hold within the family of alt ν-Tamari lattices, which would provide further evidence that these new objects showcase an elegant and rich structure.

Lastly, investigating this direction of research seems very promising, in particular in posets related to the Tamari lattice, as other examples occur where the distribution of linear intervals admits nice formulas or behavior. In particular, equidistribution of linear intervals and derived-equivalence seems to hold in the permutree lattice family, as well as in the m-Cambrian lattices.

0.8 Overview of the manuscript

The manuscript is organized in three parts. The first one is a general introduction to the research area, while the second and third parts consist of the main contributions of this work, both centered on partial orders related to the Tamari lattice, with a particular focus of their intervals, from an enumerative perspective.

0.8.1 Contributions and attributions

Part I mostly recalls definitions and results already established in the literature, and also serves as an opportunity to present the state of the art. The alt-Tamari and alt ν-Tamari lattices are new however, and are defined in the next part.

Part II is focused on the study of linear intervals in posets related to the Tamari lattice.

Most results of Chapter 5 were published in preliminary form in [Che22], reworked to fit better in this manuscript. The bijections of Section 5.5 were obtained with Vincent Pilaud and are unpublished.

Results in Chapter 6 extend those of the previous chapter, and were obtained in a joint work with Cesar Ceballos. They were first submitted as an extended abstract to the FPSAC Davis conference, and accepted as a poster. A long version was then prepublished as well [CC23]. Both authors contributed equally to this work.

Chapter 7 contains unpublished results. The enumeration of linear intervals in the weak order presented in Section 7.1 was conjectured by Frédéric Chapoton and the proof was obtained with discussions with Viviane Pons and Vincent Pilaud.

The "greedy Dyck" poset in Section 7.2 was considered by Philippe Nadeau, who observed experimentally that its distribution of linear intervals was identical to the one of the Tamari lattice.

Conjectures concerning the permutree lattices in Section 7.3.4 were tested thanks to data of Daniel Tamayo.

Conjectures on distributions of linear intervals in Pallo's comb poset and in type B and D Cambrian and tilting modules posets are due to Frédéric Chapoton.

Part III is dedicated to the study of the m-Tamari and m-Cambrian lattices, and especially of their intervals.

Results in Sections 8.1 and 8.2 are already known [Cha06, BMFPR11], as well as some cited results, especially those about interval-posets [Pon19]. Other results and conjectures in Chapter 8 are unpublished. Encoding of integer partitions with infinite families of variables to formulate Theorem 8.2.21 was suggested by Houcine Ben Dali.

Chapter 9 is focused on the study of m-Cambrian lattices. Their definition is due to [STW18] and recalled in Sections 9.1 and 9.2. Some conjectures refining those of [STW18] are formulated in Section 9.3 and have not been published.

The other main contribution of this chapter is the—still conjectural—new definition of the m-Cambrian lattices in Section 9.4. It relies on Assumption 1, whose proof is incomplete. This new definition was proposed after discussions with Corentin Henriet et Wenjie Fang, and is the object of an ongoing research project, started in May 2023, which aims at first proving the definition and trying to use it especially for the study of the linear type A case.

0.8.2 Part I: Preliminaries

Part I aims at introducing the main objects appearing in this manuscript as well as the tools used later. Its purpose is to gather the different definitions, notations and known results of the literature, as well as to outline the historical and current research landscape and how this works fits within it. Most of the content of this part is not original, and is designed to familiarize a reader new to the domain.

In Chapter 1, we first introduce the Catalan numbers and objects, and use them as an opportunity to manipulate generating functions, combinatorial decomposition and Lagrange inversion. We present in particular triangulations, binary trees, Dyck paths, and noncrossing partitions, as well as bijections between them. These objects and bijections are central in the present study and will be used throughout the manuscript. We then define a few generalizations of the Catalan numbers and objects, namely Fuß-Catalan analogs of the presented Catalan objects, and also ν-paths and how to count them.

Chapter 2 presents the core algebraic structure of this thesis, namely partially ordered sets and lattices. This portion provides a range of tools and results on posets that will be useful in later chapters, while also shedding light on some important aspects of these notions in other domains. Many examples of posets that are central in this study are presented, both for illustration of posets and for a first contact with posets related to the Tamari lattice.

Chapter 3 is dedicated to another algebraic structure of major importance in this work, namely Coxeter groups. It provides background material on these groups whose most representative member is without a doubt the symmetric group. These groups are also naturally endowed with several partial orders that we are interested in, and especially the weak order. This is also the context of a generalization of the Tamari lattice, namely the Cambrian lattices, which can be defined in several ways. We present three definitions, using sortable elements, subword complexes and noncrossing partitions, and we explain how to navigate between them. Their generalization into the m-Cambrian lattices will however be introduced only in Chapter 9, as it is more sophisticated and will appear mostly there.

Finally, Chapter 4 is a general tour on most posets of the Tamari family that we encounter in this work. It presents or recalls the different generalizations of the Tamari lattice that we are interested in, including the m-Tamari and ν-Tamari lattices, the new families of alt-Tamari and alt ν-Tamari lattices, the Cambrian and m-Cambrian lattices, the posets of tilting modules and the permutree lattices. Some more precise definitions are postponed to later chapters and are only outlined here.

0.8.3 Part II: Linear intervals

The focal point of Part II is the study of the distribution of linear intervals with respect to their length in the Tamari lattice and related posets. Linear intervals are intervals that are totally ordered. These chapters consist mostly of original work and are in particular based on the two prepublished articles [Che22, CC23]. Both articles content has been reworked and adapted to fit better in this manuscript, some additional results are given, but the main proofs and results are mostly unchanged.

This part also contains some unpublished results, especially in Chapter 7, which are the subject of ongoing research, especially regarding permutrees (see Section 7.3.4).

In Chapter 5, we prove that the Dyck and the Tamari lattices possess the same distribution of linear intervals with regard to their length. Both posets can be defined on Dyck paths in a way that each interval $[P, Q]$ in the Tamari lattice also defines an interval $[P, Q]$ in the Dyck lattice, i.e. the Dyck lattice is an extension of the Tamari lattice. Despite this identification, linear intervals do not necessarily remain linear, nor of the correct length.

We study the linear intervals in both posets and prove that their nontrivial linear intervals can be classified into two classes, which we call **left** and **right intervals**, and which behave symmetrically under natural symmetries on each side. We also define the **alt-Tamari lattices**, which are a new family of posets containing the Tamari lattice and the Dyck lattice as extreme cases. The proof that they are lattices is postponed to the next chapter. Each alt-Tamari lattice is defined after a choice of **increment vector** $\delta \in \{0,1\}^n$, and we prove through a bijective decomposition of linear intervals that their distribution does not depend on the choice of δ. We also prove a few structural results on the family, namely that whenever two increment vectors are componentwise comparable, then the alt-Tamari lattice corresponding to the smaller one is an extension of the other.

The last section of this chapter is based on discussions with Vincent Pilaud, and presents bijections between linear intervals and paths in a rectangle, as both are counted by binomial coefficients.

Chapter 6 is a generalization of the results of the previous chapter, in the framework of ν-paths. It is based on a joint work with Cesar Ceballos, that started after a presentation in a conference of the results of Chapter 5 and of some open conjectures. These questions benefited greatly from Cesar's expertise on the ν-Tamari lattice and fruitful ideas of using tree-like structures to attack the problem. This work was accepted as a poster presentation to the conference in Davis of Formal Power Series and Algebraic Combinatorics (FPSAC) 2023.

In this part, we extend the definition of the alt-Tamari lattices to the context of ν-paths. Again, these posets depend on the choice of an increment vector δ, and the ν-Dyck and ν-Tamari lattices appear as extreme cases. We define them on paths and on trees, and prove that they are lattices. We then analyze their linear intervals, and show bijectively that their distribution in the alt ν-Tamari lattices does not depend on the choice of the increment vector δ. Though there is not so much hope for a product formula for the distribution of linear intervals for a given path ν, we have some evidence that the sum of the distributions of linear intervals over all ν-Tamari lattices for all paths ν of length $n-1$ would admit a neat product formula, though we do not explore this direction. Finally, we expect a derived equivalence result to hold in this context as well.

Chapter 7 is a collection of results further exploring this direction of linear intervals, in particular in posets related to the Tamari lattice.

We present a "greedy" version of the alt ν-Tamari lattices, inspired by [Der23, BMC23]. We prove that in all this family, nontrivial linear intervals can still be understood as either left or right intervals. In a very specific case that we call the greedy Dyck lattice, we prove that we recover the same distribution of linear intervals as in the Dyck and the Tamari lattices.

We also present a few conjectural formulas for the distributions of linear intervals in several posets, namely Pallo's comb poset [Pal03, CSS14], and type B and D Cambrian lattices and posets of tilting modules. We expect in fact that for both the Cambrian lattices and posets of tilting modules, the distribution of linear intervals should be independent of the choice of the Coxeter element, and that this result should extend to the m-Cambrian lattices.

In this chapter, we also compute the distribution of linear intervals in the weak order on $\mathfrak{S}_n$. We then examine linear intervals in permutree lattices. Each permutree lattice also depends on a choice of decoration, and we prove that in all cases, their linear intervals can be classified as trivial, left, and right intervals. This last point was obtained thanks to fruitful conversations with Daniel Tamayo and some of his recent work. We conjecture that the distribution should not be modified under some small changes of decorations, and also that a derived equivalence result should hold in this context as well.

0.8.4 Part III: m-eralizations

The main focus of Part III is to study two "m-eralizations" of the Tamari lattice, namely the m-Tamari and the m-Cambrian lattices, where $m \geq 1$ is an integer. The Tamari lattice can be described as the 1-Tamari lattice and as the linear type A 1-Cambrian lattice. For general $n, m \geq 1$, the m-Tamari and the linear type A m-Cambrian lattice are not isomorphic. However, it was conjectured in [STW18, Section 6.10] that both have the same number of intervals and even decorated intervals. These are furthermore conjectured to coincide with the dimensions of the higher diagonal harmonics in the trivariate case, as mentioned earlier. This conjecture was the starting point of this work, and we present here the main results obtained in this direction, based on original work not yet prepublished.

In [Cha06], F. Chapoton enumerated the intervals in the Tamari lattice, and this result was later extended to the m-Tamari lattice in [BMFPR11]. In both cases, a combinatorial decomposition of intervals allowed to write a functional equation on the generating function of intervals, which could be solved afterwards.

We begin Chapter 8 by briefly reproducing and adapting the proof of [BMFPR11] to slightly different conventions and notations. Our initial discussion focuses on the simpler case of the Tamari lattice for $m = 1$, in which we underline an expansion-gluing process to build each interval starting with smaller ones. We explain subsequently the changes required for general $m \geq 1$, which consists mainly of executing m consecutive expansion-gluing processes in a row.

We then present many statistics that we decorate m-Tamari intervals with, that emerged in our comparative work with linear type A m-Cambrian intervals. These statistics were in fact already present in the work of L.-F. Préville-Ratelle (see [PR12]) and subsequently of V. Pons in [Pon19]. We prove two refinements of the functional equation obtained in [BMFPR11], with two different sets of statistics. We may expect that the functional equation (8.13) could be useful to connect m-Tamari intervals and some subset of maps or constellations, as the operators that are involved are resembling those appearing in the work of G. Chapuy and M. Dołęga in [CD22].

We finish this chapter by presenting Tamari interval-posets, which were defined by G. Châtel and V. Pons in [CP15] and proven to be in bijection with intervals in the Tamari lattice. A subset of them, that we call here m-Tamari interval-posets is particularly interesting, as it is in bijection with m-Tamari intervals, in a way that all statistics are transported nicely, with an additional involution on the family. We formulate several conjectures on the distribution of these objects.

Chapter 9, the last chapter of this work, is dedicated to the m-Cambrian lattices. We first present the theoretical background needed for this study, and in particular the definition of the Artin monoid. Indeed, the key idea developed in [STW18] to generalize from the Cambrian to the m-Cambrian lattices is to translate the definitions and properties from the Coxeter group to the Artin monoid, where much subsists. In particular, we can still define a weak order, and most notions of the Coxeter group have an analog in the Artin monoid.

We give the three definitions of [STW18] of the m-Cambrian lattices, which generalize the three descriptions in Chapter 3 of the Cambrian lattices, and we describe how to translate between them. We also define statistics on the intervals in linear type A, in order to state the original conjectures of C. Stump, N. Williams and H. Thomas, namely that (decorated) intervals in the linear type A m-Cambrian lattice would be counted by the same numbers as in the m-Tamari lattice. We also present refined versions of these statistics, and in particular one relating intervals in $\mathrm{Camb}^{(m)}(\mathfrak{S}_n, c^{\mathsf{lin}})$ and m-Tamari interval-posets.

The rest of the chapter is dedicated to giving a tentative new definition of the m-Cambrian lattices. This is a joint work that started in a conference with Corentin Henriet and Wenjie Fang. The proof is not complete, but this seems promising as the result is supported by experimentation. Moreover, this would be a very handy definition as it would give a simple criterion to compare two m-eralized noncrossing partitions, themselves easy to describe. In fact, one would only need to know how to compare two noncrossing partitions in the 1-Cambrian lattice $\mathrm{Camb}_{\mathrm{NC}}(W, c)$ and in the noncrossing partition lattice $\mathrm{NCL}(W, c)$.

This definition gives a new poset structure on intervals in the Cambrian lattice which, to the best of our knowledge, has never been considered before, even for the Tamari lattice in the linear type A case. This new definition might be a key step in a proof of the conjecture relating m-Tamari and m-Cambrian intervals, at the center of this work, and could possibly have much more to offer.

Part I

Tools and structures

Chapter 1

Catalan objects

The **Catalan numbers** are a sequence of natural numbers $(C_n)_{n\in\mathbb{N}}$ that are known to count many families of objects that appear in many branches of combinatorics and more generally of mathematics. They constitute one of the longest entries of the Online Encyclopedia of Integer Sequences (OEIS) and start with:

$$[OEIS\,A007767] \quad C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, C_6 = 132, \ldots$$

R. Stanley has described 214 families of combinatorial objects counted by the Catalan numbers, that we can call Catalan objects, in his famous book [Sta15], as well as many bijections between some of these families. They first appeared in the work of a Mongolian mathematician Sharabiin Myangat in the 1730s and later in Leonhard Euler's work in the 1750s as the number of triangulations of a polygon. They are named after a French and Belgian mathematician, Eugène Charles Catalan who gave in 1838 the following closed formula for the Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1.1}$$

These numbers have very interesting and rich properties and have extensively been studied and generalized ever since. The work presented in this manuscript relies on partial order structures on some Catalan objects or generalizations of these. Hence, this chapter aims to introduce these objects as well as some useful tools and techniques appearing later.

1.1 Combinatorial classes and generating functions

A very useful tool in combinatorics is the notion of **combinatorial classes** and their **generating functions** (or series) as defined in [FS09] which are powerful methods for counting objects using symbolic calculus. They can in particular prove that two sets are in bijection without explicitly constructing the bijection and enable for instance the use of complex analysis to understand asymptotic behaviors.

Definition 1.1.1. A **combinatorial class** (or simply class) is a set $\mathcal{A}$ of objects together with a **size** statistic $|\cdot| : \mathcal{A} \to \mathbb{N}$ such that there are finitely many objects of each size, i.e. for every $n \in \mathbb{N}$, the set $\mathcal{A}_n = \{a \in \mathcal{A} \mid |a| = n\}$ is finite.

The counting sequence $(A_n)_{n\in\mathbb{N}}$ of the class $\mathcal{A}$ is the sequence of cardinal numbers of the sets $\mathcal{A}_n$. Two classes $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent* if their counting sequences are equal. The (ordinary) **generating function** of the class $\mathcal{A}$ is the formal series

$$A(t) = \sum_{n\geq 0} A_n t^n = \sum_{a\in\mathcal{A}} t^{|a|}.$$

We will treat the generating functions as *formal power series* in $R[[t]]$ where R is a commutative ring (for instance $R = \mathbb{Z}$), and we will always use the formal variable t to keep track of the size statistic. When dealing with combinatorial classes, the size is generally an intrinsic characteristic of the objects that we count, as for instance the number of nodes of a binary tree, the number of letters of a word, or the number of steps of a path. As expressed above, going from the class to its generating function corresponds to giving a weight t^n to each object of size n and to summing the weights of all objects of the class. In what follows, we use the notation $[t^n]S(t)$ to denote the **extraction** of the coefficient of t^n in the series $S(t)$. The first very interesting property of combinatorial classes is that the operations of (disjoint) unions and Cartesian products can be naturally defined and translate respectively to the operations of addition and multiplication at the level of the series.

Definition 1.1.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two classes with generating functions $A(t)$ and $B(t)$, respectively.

The **union** (or *sum*) of $\mathcal{A}$ and $\mathcal{B}$ is the class $\mathcal{C} = \mathcal{A} \cup \mathcal{B} = \mathcal{A} + \mathcal{B}$ whose objects are the disjoint union of the objects of $\mathcal{A}$ and $\mathcal{B}$, each object inheriting of its size, i.e. $\mathcal{C}_n = \mathcal{A}_n \sqcup \mathcal{B}_n$ for every $n \in \mathbb{N}$.

Similarly, the **(Cartesian) product** of $\mathcal{A}$ and $\mathcal{B}$ is the class $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ whose objects are pairs (e, f) of an object $e \in \mathcal{A}$ and an object $f \in \mathcal{B}$, whose size is the sum of the sizes of e and f, i.e. $\mathcal{C}_n = \bigsqcup_{i=0}^{n} \mathcal{A}_i \times \mathcal{B}_{n-i}$ for every $n \in \mathbb{N}$.

Proposition 1.1.3. *The generating functions of the disjoint union $\mathcal{A} \sqcup \mathcal{B}$ is $A(t) + B(t)$ and the generating functions of the Cartesian product $\mathcal{A} \times \mathcal{B}$ is $A(t) \cdot B(t)$.*

Other classical constructions exist, as for instance k-tuples, marking, or sequences, which can be translated to the level of the generating functions as well. We will usually denote by $\mathcal{E}$ the combinatorial class consisting of a single object of size 0, whose generating function is $E(t) = 1$, which is the neutral element for the product.

- If $\mathcal{A}$ is a class and $k \in \mathbb{N}$, the class $\mathcal{A}^k$ is the class of **k-tuples** of objects of $\mathcal{A}$ (with sizes summing up), and we have the equality $A^k(t) = (A(t))^k$.
- If $\mathcal{A}$ is a class, we denote by $\mathcal{A}_\bullet$ the class of marked objects of $\mathcal{A}$, i.e. objects of $\mathcal{A}$ with a distinguished "element of size". More precisely, $\mathcal{A}_\bullet = \bigcup_n \mathcal{A}_n \times [n]$, where $[n]$ is the set $\{1, 2, \ldots, n\}$, and we have the equality $A_\bullet(t) = t\frac{\partial A}{\partial t}$.
- If $\mathcal{A}$ is a class, the class of **sequences** of objects of $\mathcal{A}$ is the class $\mathcal{A}^*$ whose objects are finite sequences of objects of $\mathcal{A}$, which can be written as $\mathcal{E} + \sum_{k\geq 1} \mathcal{A}^k$, and we have the equality $A^*(t) = \frac{1}{1-A(t)}$.

There also exists the notion of *exponential generating function*, which is particularly useful when objects of size n are naturally *labelled* with integers in $[n]$, and an even more general notion of *species* where each set $\mathcal{A}_n$ is equipped with an action of the symmetric group $\mathfrak{S}_n$. These notions will however not really appear in the present work, though the objects present strong links with representation and operad theories.

The translation of these operations into operations on generating functions gives rise to equations on these formal series, which usually admit a unique solution in the ring of formal power series. In fact, one will typically produce such a functional equation on the generating function using combinatorial arguments, in the same spirit as double counting arguments. One can then use the uniqueness of the solution to prove the equivalence of two classes, or to determine the counting sequence by solving the equation, using for instance tools such as **Lagrange inversion formula**.

Theorem 1.1.4 (Lagrange Inversion Formula)**.** *Let $A(t) \in R[[t]]$ and $F(x) \in R[[x]]$ be two formal power series satisfying $A(t) = tF(A(t))$. Then we have*

$$[t^n]A(t) = \frac{1}{n}[x^{n-1}]F(x)^n. \tag{1.2}$$

More generally, if $\phi(x) \in R[[x]]$, we also have

$$[t^n]\phi(A(t)) = \frac{1}{n}[x^{n-1}]\phi'(x)F(x)^n. \tag{1.3}$$

Note in particular that the solution in $R[[t]]$ to the equation $y = tF(y)$ exists and is unique.

In what follows, we will usually use decorations on the objects of a combinatorial class, as one or several statistics, in addition to the size. Adding a decoration can be seen as a refinement, as some objects whose size statistic is the same may have a different decoration. We will refer to the "distribution" of the statistics as the number of objects associated to each possible decoration of each size.

Definition 1.1.5. A **statistic** on a combinatorial class $\mathcal{A}$ is a function $f : \mathcal{A} \to \mathbb{N}$ (integer statistic), or more generally any function $f : \mathcal{A} \to E$ where E is some set, for instance the set of integer partitions.

If $f_1, \ldots, f_k$ are k integer statistics on $\mathcal{A}$, one can consider the *(decorated) generated function* $A(t; x_1, \ldots, x_k)$ of $\mathcal{A}$ as the formal series in $R[x_1, \ldots, x_k][[t]]$ defined as

$$A(t; x_1, \ldots, x_k) = \sum_{a \in \mathcal{A}} t^{|a|} x_1^{f_1(a)} \ldots x_k^{f_k(a)}.$$

These decorated generating functions can for instance *refine* the counting of objects of a class by distinguishing objects of the same size according to some statistics, as for instance the number of valleys of a Dyck path, which produces the Narayana numbers, summing up to the Catalan numbers. Adding statistics to the objects is sometimes crucial to write functional equations on the class, using them as **catalytic parameters**. Additionally, two classes $\mathcal{A}$ and $\mathcal{B}$ equipped with decorations in the same set E can have the same distribution (or equivalently the same decorated generating function), which is stronger than the equivalence of the classes, and hints on the existence of a natural bijection between them, which would respect the decorations. This will appear especially in Part III with statistics on intervals in the m-Tamari lattices and the m-Cambrian lattices that we conjecture to be equidistributed.

1.2 Triangulations of a polygon

The Catalan numbers $C_n := \dfrac{1}{n+1}\dbinom{2n}{n} = \dfrac{1}{2n+1}\dbinom{2n+1}{n}$ appeared in Euler's work as the number of triangulations of a polygon. We present here briefly this approach and one way to obtain Equation (1.1).

Definition 1.2.1. A *diagonal* of a (convex) polygon is a segment joining two of its vertices. A **triangulation** of a polygon is a way to decompose it into triangles using pairwise noncrossing diagonals. Let $\mathcal{C}$ be the combinatorial class of triangulations of a polygon, where the **size** of the triangulation is the number of triangles. Equivalently, a triangulation of an $(n+2)$-sided polygon is of size n.

The **Catalan numbers** are the counting sequence of the class $\mathcal{C}$.

As an example, see Figure 1.1. Note that the polygon is fixed, and we do not identify two triangulations when one can be obtained from the other by a rotation of the figure. We can consider for instance the vertices of the polygon numbered from 1 to $n+2$ counterclockwise.

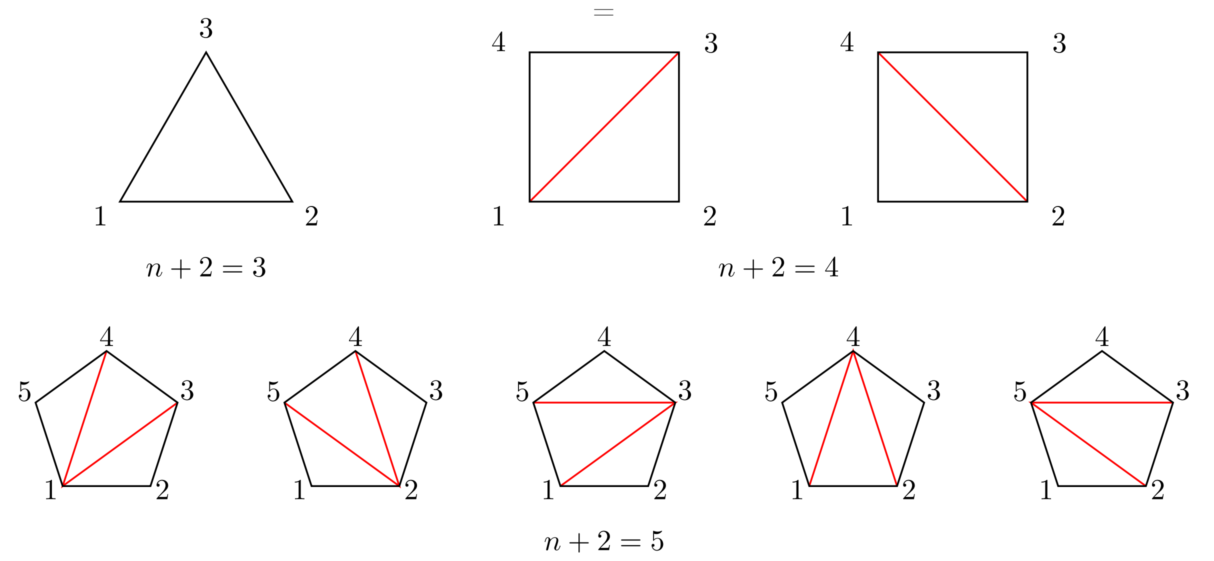


Figure 1.1: Triangulations of the triangle, the square, and the pentagon.

One classical way to count the number of such triangulations of a given size is to decompose a triangulation into smaller ones. Here is a way to proceed. Let T be a triangulation of size $n+1$, that is to say of an $(n+3)$-sided polygon, whose vertices are numbered from 1 to $n+3$ counterclockwise. In particular, there is a triangle whose vertices are 1, $k+2$ and $n+3$ for some $0 \le k \le n$. This triangle cuts the polygon into two polygons, one with vertices from 1 to $k+2$ and one with vertices from $k+2$ to $n+3$. This gives respectively a triangulation of size k and a triangulation of size $n-k$ (whose vertex labels are shifted by $k+1$). This decomposition is possible for any triangulation except the one of size 0. Conversely, any pair of triangulations of size k and $n-k$ can be glued together to form a triangulation of size $n+1$, adding the triangle with vertices 1, $k+2$ and $n+3$. We can thus write the following equation on $\mathcal{C}$:

$$\mathcal{C} = \mathcal{E} + \mathcal{C}_1 \times \mathcal{C} \times \mathcal{C}, \tag{1.4}$$

where $\mathcal{C}_1$ is the class consisting of only the triangle of size 1. This yields the following equation on the generating function $C(t)$ of $\mathcal{C}$:

$$C(t) = 1 + tC(t)^2. \tag{1.5}$$

Defining now $B(t) = C(t) - 1$, one can write

$$B = t(B+1)^2 \tag{1.6}$$

and apply the Lagrange inversion formula of Theorem 1.1.4 with $F(x) = (x+1)^2$ to obtain that

$$\begin{aligned} B_n &= \frac{1}{n}[t^{n-1}](F(x))^n \\ &= \frac{1}{n}[x^{n-1}](x+1)^{2n} \\ B_n &= \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}. \end{aligned} \tag{1.7}$$

One then notes that $C_0 = 1$ and $C_n = B_n$ and that the right-hand side of Equation (1.7) evaluates to 1 for $n = 0$, and thus we finally obtain the desired formula (1.1).

1.3 Other Catalan objects

Families enumerated by Catalan numbers are very numerous. In this section, we define a few Catalan objects that we will need further and describe some various bijections between them. In particular, planar rooted binary trees appear very naturally as a "dualized version" of triangulations of a polygon and planar rooted forests (or trees) come with them. We also define Dyck paths or ballot paths, which appear also very naturally.

1.3.1 Planar trees and binary trees

Definition 1.3.1. A **tree** is a finite connected acyclic graph. It is **rooted** when it has a distinguished vertex called the *root*. A tree is **directed** when its edges are oriented. In this case, if $i \to j$ is an edge, we say that i is a **parent** of j and j is a **child** of i.

For an undirected rooted tree, every vertex has a **height**, which is its distance to the root, i.e. the number of edges of the shortest path connecting it to the root. The tree can be naturally endowed with an orientation of its edges, from their vertex of smallest depth to their vertex of largest depth. Every vertex but the root has then a unique ingoing edge. A vertex with no outgoing edge is called a **leaf** and a **node** otherwise. We will usually draw the trees with the root at the bottom and without arrows on the edges (going up according to the height).

A **planar rooted tree** is a rooted tree with a linear left-to-right ordering of the children of each vertex. The size of a planar rooted tree is the number of edges.

A **planar rooted forest** is a list (ordered from left to right) of planar rooted trees or equivalently a planar rooted tree whose root has been erased. The size of a planar rooted tree forest is its number of vertices.

A **planar rooted binary tree** is a planar rooted tree such that each node has exactly two children. Its size is the number of nodes.

In what follow all trees will be planar and rooted unless stated otherwise, and most will be binary trees. They will be equipped with the natural orientation of their edges. If the context is clear, we will call them binary trees or simply trees. The binary tree of size 0, reduced to a vertex, will be considered as a trivial tree. For $n \geq 0$, let us denote by $\mathcal{Y}_n$ the set of all binary trees of size n.

Proposition 1.3.2. *The classes of planar rooted trees, planar rooted forests and planar rooted binary trees are Catalan classes, i.e. their counting sequence is the sequence of Catalan numbers.*

Proof. Triangulations and (planar rooted) binary trees are in bijection by dualizing the triangulation. More precisely, one puts a node inside each triangle and connects two nodes if the corresponding triangles share an edge. The root node is the one in the triangle with vertices 1, $k+2$ and $n+2$. Then for every external edge of the $(n+2)$-gon, except the edge between 1 and $n+2$, one adds a leaf to the node corresponding to the triangle containing this edge. This bijection preserves the size.

Alternatively, one can provide a decomposition of binary trees similar to the one of triangulations, providing the same equation on the class of trees as in Equation (1.4). Indeed any nontrivial binary tree consists of a root node with two children, each being a binary tree, which is precisely of the prescribed form.

Now there is a very classical bijection between (planar rooted) binary trees and (planar rooted) forests. We take a binary tree T with n nodes, and we build a forest F with n vertices, one for each node of the tree. All nodes from bottom to top on the rightmost branch of T will be roots of the trees in F from left to right. Then, if a node u is the right child of a node v in T, the vertex in F corresponding to u is the leftmost child of the vertex corresponding to v in F. If a node u is the left child of a node v in T, then the vertex in F corresponding to u is the right sibling of v in F. This bijection preserves the size.

Finally, there is an immediate bijection between planar rooted forests and planar rooted trees by adding a root node to the forest and connecting it to all the roots of the trees in the forest. If the forest had n vertices, then the resulting planar tree has $n+1$ vertices and thus n edges.

These bijections are illustrated in Figure 1.2. □

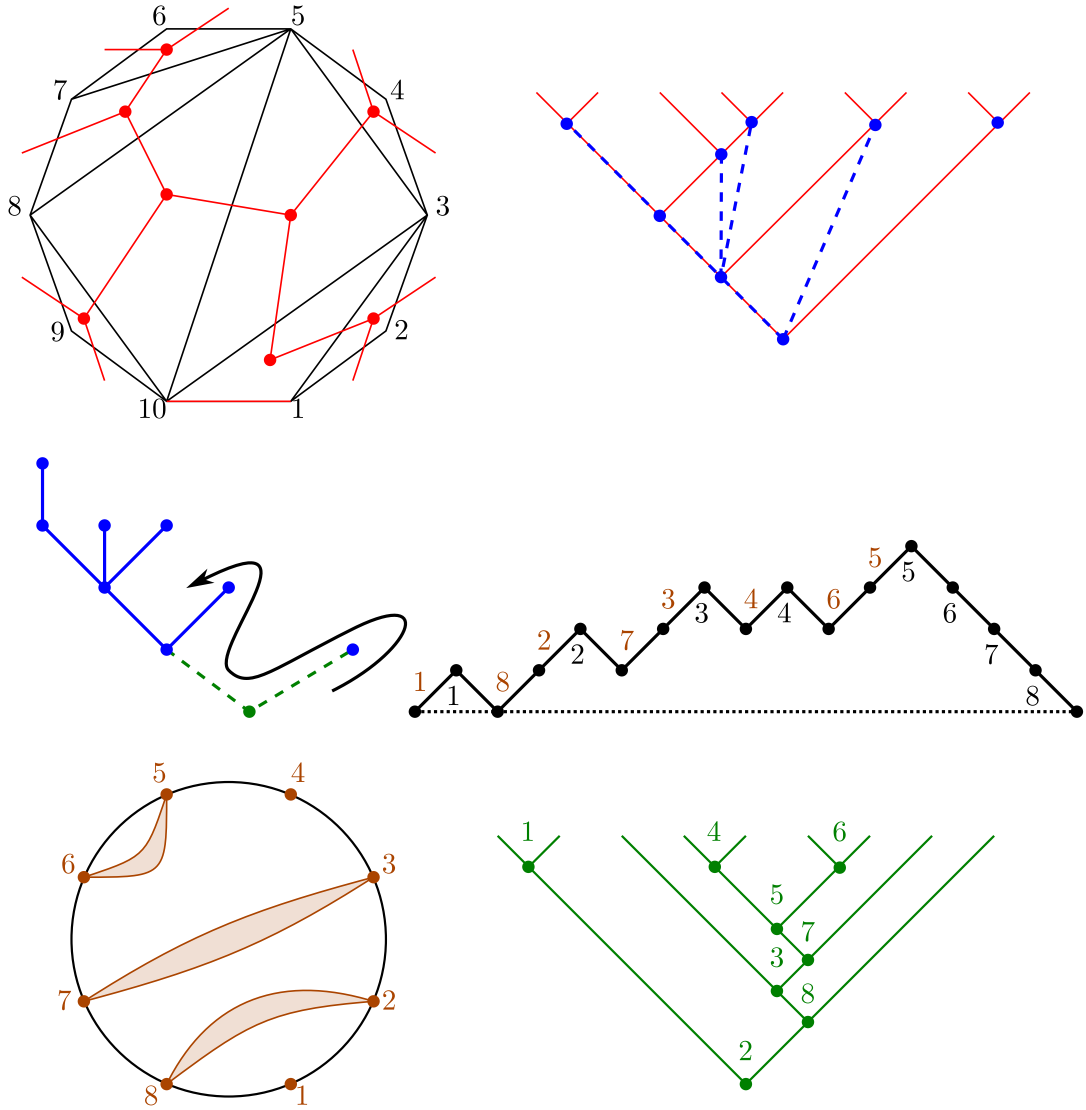


Figure 1.2: A series of bijections between Catalan objects. Top left: from triangulations to binary trees (in red) by duality. Top right: from binary trees to planar forests (in blue) by transforming right children into right siblings. Middle left: from planar forests to planar trees by adding a root. Middle right: from planar trees to Dyck paths by walking around the tree. Bottom left: from Dyck paths to noncrossing partitions (in brown) by grouping the up steps according to rises. Bottom right: from noncrossing partitions to binary trees—whose nodes are labeled in in-order—taking parts as nodes on the same right branch.

There exist some useful operations on binary trees, and in particular two that we use later on, namely *grafting* and *plugging*. We can graft a tree T' on a chosen leaf of another tree T. This is done by identifying the root node r of the tree T' with the chosen leaf of T, producing a tree whose size is the sum of the sizes of T and T', as drawn in Figure 1.3. Grafting a tree of size 0 does nothing.

We can also plug a binary tree T' into a chosen edge $a \to b$ of another binary tree T. To do so, we create a new node n on the selected edge of T, and we connect this node with the root

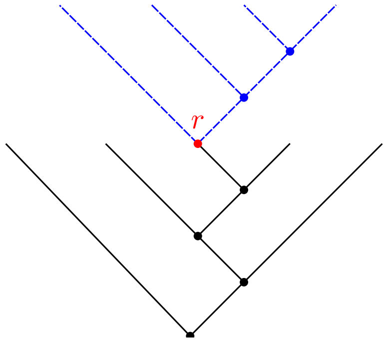


Figure 1.3: Grafting a (dashed blue) tree on another one. The leaves of the trees are at the top of each tree, and the root at the bottom. The nodes are highlighted with dots, but we usually do not draw them.

of T'. If the node b was the left child of a, the tree T' will be the right subtree of n, and vice versa. If T and T' are respectively of size m and m', plugging T' into an edge of T will produce a tree of size $m + m' + 1$. In Figure 1.4, we take the same two trees T and T' as in the example of grafting, but this time, we plug T' into the red dotted edge of T. Note that plugging a tree of size 0 on an edge of a tree T produces a tree that is different from T, with one more vertex.

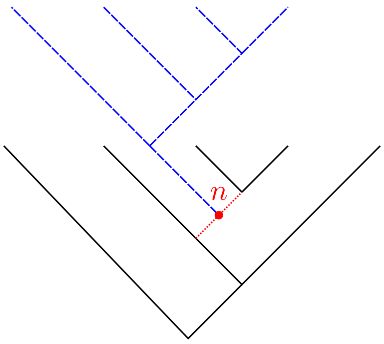


Figure 1.4: Plugging a tree into the selected (red dotted) edge of another one. This creates a new node n in the selected edge.

In the following, we will also need a special family of trees, namely **left and right combs**, recursively defined as follows: $\ell_1 = r_1 =: \mathsf{V}$ is the only tree of size 1 and for $n \geq 2$, ℓ_n (resp. r_n) is obtained as the grafting of V on the leftmost (resp. rightmost) leaf of ℓ_{n-1} (resp. r_{n-1}). In Figure 5.6, the first tree of the first row is the right comb r_4 and the last tree of the second row is the left comb ℓ_5.

1.3.2 Dyck paths and ballot paths

Dyck paths constitute another Catalan family that is central in this work.

Definition 1.3.3. A **Dyck path** of size n is a path in $\mathbb{N}^2$ consisting of up steps $(1, 1)$ and down steps $(1, -1)$, starting from $(0, 0)$ and ending at $(2n, 0)$.

A **ballot path** of size n is a path in $\mathbb{N}^2$ consisting of north and east unit steps, starting from $(0, 0)$, ending at (n, n), and always remaining weakly above the diagonal $\{y = x\}$.

A **Dyck word** or *balanced word* of size n is a word in letters (and) with n pairs of parentheses such that a closing parenthesis always closes the last opening parenthesis and all parentheses are closed at the end. This can be rephrased as the condition that each letter appears n times and all prefixes of the word contain at least as many opening parentheses as closing parentheses.

A **bracketing** of a product of $n+1$ terms is a way to put parentheses in a product of $n+1$ terms in order to group the sequence of objects into pairs of operands of a binary product.

The three classes are in direct bijection, by identifying up steps, north steps and opening parentheses, on the one hand, and down steps, east steps and closing parentheses, on the other hand. Alternatively, we will use letters u and d (for up and down steps) or letters N and E (for north and east steps) instead of brackets. Thus, we may consider Dyck paths as ballot paths or as Dyck words if this is more helpful.

There is also a neat way of obtaining a Dyck path from a bracketing of a product of $n+1$ terms, which consists in reading the bracketing from left to right and drawing an up step for each opening parenthesis and a down step for each product symbol.

Let us denote by $\mathcal{Z}$ the combinatorial class of Dyck paths.

Proposition 1.3.4. *The class $\mathcal{Z}$ is a Catalan class.*

Proof. One way to prove this is to remark that the bracketing of an expression of $n+1$ elements can naturally be seen as a binary tree, each node corresponding to a product sign whose left and right operands produce the left and right subtrees of the node. Then, the bijection described above between Dyck paths and bracketings of $n+1$ elements gives the result.

We gave a recipe to transform a bracketing of a product of $n+1$ terms into a balanced word of size n, but it was not completely clear that this transformation is indeed bijective as the inverse transformation is not totally straightforward. Let us describe a useful bijection from binary trees to Dyck paths. This is the mirrored version of the bijection described in [BB09].

First, draw the planar binary tree with the root at the bottom, such that each edge goes either to the left or to the right. The idea is to walk around the planar binary tree counterclockwise, starting from the root and to remove its leaves one by one and to build the Dyck path step by step accordingly. More precisely, each time we encounter a leaf, we turn around it and remove it from the tree, and we insert the edge as a step in a Dyck path without changing its direction: as an up step if it was a right child and as a down step if it was a left child. When both outgoing edges of a node are removed, the vertex becomes a leaf itself (and its incoming edge is immediately removed as the circuit goes on).

Another bijection is even easier to describe starting from a planar tree, drawn with the root at the bottom: one also walks counterclockwise around it, starting from the root. This time, whenever we go along an edge, we insert an up step if they are going up, and a down step if they are going down. Saying it otherwise, drawing each node at its respective height, the Dyck path is recording the height of the nodes encountered during the circuit. Thus, it is clearly a Dyck path and the transformation is clearly a bijection. This last bijection is illustrated in Figure 1.2. □

Now let us define some notions that we need later on.

Definition 1.3.5. A **factor** B of a Dyck path (or a ν-path defined later) A will be a portion of **consecutive** letters of A. We will denote $B \subset A$.

The **altitude** of a step of a Dyck path is defined as the second coordinate of the vertex at which it starts, or as Dyck words, as the number of u minus the number of d in the prefix preceding this step.

Given a Dyck path, we define a **valley** as a down step followed by an up step, and a **peak** as an up step followed by a down step. A **contact** is a vertex point at altitude 0, which includes the initial and final points of the path. A **rise** is a maximal sequence of consecutive up steps and a **fall** is a maximal sequence of consecutive down steps.

There is a natural matching of up and down steps, which is precisely the pairs of opening and closing brackets in a balanced word. Given an up step, we can define the **excursion** of this step as the portion of the path that starts at this up step and ends at the first down step that ends at the same altitude. In other words, it is the smallest factor starting at this letter u that is a Dyck word. Note that an excursion has no internal contact.

1.3.3 Noncrossing partitions

Another Catalan class that appears in this work is the one of noncrossing partitions. These are set partitions of a finite set whose parts are pairwise noncrossing when drawn on a circle. They have many symmetries and appear for instance in free probability theory.

Definition 1.3.6. A **(set) partition** of a set E is a set of nonempty pairwise disjoint subsets of E whose union is E. The subsets are called the **blocks** of the partition. A **noncrossing partition** of size n is a partition of the set $[n]$ such that the blocks are pairwise noncrossing, that is to say, if $a < b < c < d$ are such that a and c are in the same block and b and d are in the same block, then all four are in the same block.

Proposition 1.3.7. *The class of noncrossing partitions is a Catalan class.*

Proof. A bijective proof consists in taking a binary tree of size n and to number its nodes in in-order, that is to say give the unique bijection from nodes to $[n]$ such that for each node, all labels appearing in its left subtree are smaller than its own label and all labels appearing in its right subtree are bigger than its own label. Then, grouping each node with its right child if any produces a noncrossing partition of $[n]$.

This transformation is bijective and an example is given in Figure 1.2. The blocks of the partitions tell us which nodes are on the same branch going to the right, in increasing order. For example, if $\{2, 8\}$ is a part, then 8 is the right child of 2. Then, starting with the part containing the biggest label, the noncrossing condition ensures that there is exactly (and only) one way to insert the next block containing the biggest label in the tree such that the labels are read increasingly in in-order. For instance, if the next part is $\{3, 7\}$, the nodes must appear between 2 and 8 in in-order, so 3 is the left child of 8.

Alternatively, one can label the vertices of a planar tree in post-order, that is to say in such a way that each vertex has a greater label than all of its descendants, and smaller than all nodes appearing in the subtree of its right sibling if any. This is the same as turning around the tree clockwise starting from the root and labeling each node in the order of their last visit. Hence, the root is labelled $n + 1$. Then, the noncrossing partition is built by grouping the children of each node in the same block.

Another useful bijection can be given from a Dyck path to a noncrossing partition. To do so, we label each down step of the path from right to left. Then, each up step is labelled with the label of the down step that closes the excursion starting at this up step. The set partition is obtained by grouping together the labels of each rise. It is always a noncrossing partition of $[n]$. Indeed, if $a < b < c$ are three integers, and a and c are in the same block, then either b is also in the same block, or the excursion ending at the down step b starts after the down step a and ends before the down step d. In the latter case, b can not be in the same part as d. This transformation is also illustrated in Figure 1.2. □

It is convenient to draw a noncrossing partition on a circle, with vertices labelled from 1 to n clockwise. Then, each block of the partition is drawn as the convex hull of its elements, and in this case, a noncrossing partition is exactly a partition whose blocks in this drawing are pairwise noncrossing. A very natural and useful operation is called the **Kreweras complement**, and is illustrated in Figure 1.5.

Definition 1.3.8. Consider a noncrossing partition of size n drawn on the circle. Put on the circle a point $1'$ between the vertices n and 1, and a point i' between the vertices $i - 1$ and i. Then, the **Kreweras complement** of the noncrossing partition is the coarsest partition on the set of primed vertices such that no two parts are crossing, i.e. two primed numbers are in the same part of the complement if the chord between then does not cross any part of the partition we started with. It is indeed a noncrossing partition.

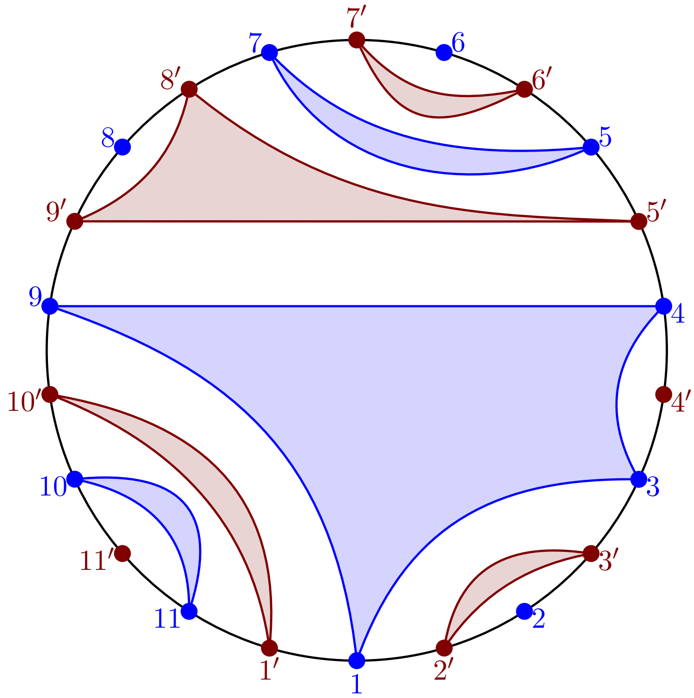


Figure 1.5: The Kreweras complement of the partition $\{\{1,3,4,9\},\{2\},\{5,7\},\{6\},\{8\},\{10,11\}\}$ is the partition $\{\{1,10\},\{2,3\},\{4\},\{5,8,9\},\{6,7\},\{11\}\}$.

Remark 1.3.9. Applying twice the Kreweras complement operation is a "rotation" of the partition, that is to say, changing a label i into $i+1$ (and n is changed into 1). Another way to say this is that twice the Kreweras complement corresponds to the action of the long cycle $(1,2,\ldots,n)$ on the vertices. It is clearly bijective.

1.4 Generalizations

As the Catalan numbers count many interesting objects, many generalizations of these objects have been studied and are very fascinating in their own right. These numbers admit correspondingly many formulas. Some of these generalizations appear crucially in this work, as we define and study very appealing partial orders on objects of the "Cataland". Among these, we can mention in particular Fuß-Catalan and ν-Catalan objects and numbers that we present here, but there also exist rational Catalan numbers, (Fuß-)Catalan numbers "of other Coxeter type", super-Catalan numbers, factorial-Catalan numbers, q,t-Catalan numbers, and more!

1.4.1 Fuß-Catalan objects

A first generalization of the Catalan numbers is the one of Fuß-Catalan numbers. These can be obtained in several ways, by considering for instance $(m+1)$-ary trees, m-divisible forests, m-Dyck paths, m-divisible noncrossing partitions or $(m+2)$-angulations. In all cases, m is a positive integer parameter, and the case $m=1$ yields the classical Catalan objects defined previously. Interestingly, all these objects, seemingly quite different, are still in bijection with each other.

Definition 1.4.1.

- A (planar rooted) **$(m+1)$-ary tree** is a planar rooted tree such that each node has exactly $m+1$ children. The size of an $(m+1)$-ary tree is the number of nodes.

- A (planar rooted) **m-divisible forest** of size n is a planar rooted forest with mn vertices such that each node has a multiple of m number of children. Note that this implies that the forest consists of a multiple of m trees.

- An **m-Dyck path** of size n is a Dyck path of size mn whose rises' lengths are all multiples of m. An **m-ballot path** of size n is a path from $(0,0)$ to (mn,n) using unit north and east steps and staying weakly above the line $\{y = x/m\}$.

- An **m-divisible noncrossing partition** of size n is a noncrossing partition of size mn whose blocks all have a size multiple of m.

- An **$(m+2)$-angulation** is the decomposition of a polygon into $(m+2)$-gons, and its size is the number of $(m+2)$-gons. An $(m+2)$-angulation of size n is necessarily the decomposition of an $(mn+2)$-gon.

Proposition 1.4.2. *All these combinatorial classes are equivalent.*

Proof. The proof is in fact fairly easy: most of the bijections described previously generalize very well to the case $m > 1$.

One can for instance produce an $(m+1)$-ary tree from an $(m+2)$-angulation by duality.

One can then produce an m-ballot path from the $(m+1)$-ary tree by removing the leaves of the tree one by one in a circuit around an $(m+1)$-ary tree, inserting a north step for each node when removing its first leaf, and an east step for all other leaves.

Then, transforming each north step into m consecutive north steps gives an m-Dyck path, whose associated noncrossing partition (as explained in the proof of Proposition 1.3.7) is m-divisible.

Finally, we can build a forest from the m-divisible partition whose trees' roots are the elements of the part containing 1; and for each other part, one declares that its elements are the children of the vertex i, where $i+1$ is the smallest element of the part. □

Calling the common counting sequence $\left(C_n^{(m)}\right)_{n\in\mathbb{N}}$ of these classes the **Fuß-Catalan numbers**, we can use the same technique of Lagrange inversion to obtain a closed formula.

Proposition 1.4.3. *The Fuß-Catalan generating function is solution of the equation:*

$$F(t) = 1 + tF(t)^{m+1}. \tag{1.8}$$

The Fuß-Catalan numbers are given by the formula:

$$C_n^{(m)} = \frac{1}{mn+1}\binom{(m+1)n}{n} = \frac{1}{(m+1)n+1}\binom{(m+1)n+1}{n}. \tag{1.9}$$

The equation is for instance obtained by taking an $(m+1)$-ary tree and if not trivial, removing the root, whose $(m+1)$ children are themselves $(m+1)$-ary trees. Then, Lagrange inversion gives the formula.

The *rational Catalan* numbers mentioned above come from fixing two coprime integers a and b and taking north-east paths from $(0,0)$ to (a,b) staying weakly above the line $\{ay = bx\}$. The number of such paths still has a very nice closed formula, namely $\frac{1}{a+b}\binom{a+b}{a}$ [Biz54].

1.4.2 ν-Catalan objects

Seeing the Dyck paths as ballot paths, it is very natural to generalize these to the sets of paths staying weakly above some boundary path. This further generalizes m-Dyck paths (and rational Dyck paths). However, the number of ν-paths does not admit a nice product formula in general.

If ν is a path in the plane using north and east unit steps, we may represent it as a word in the letters E and N for east and north steps respectively. We may as well represent such a path ν from

$(0,0)$ to (m,n) as a sequence of nonnegative integers $(\nu_0, \nu_1, \dots, \nu_n)$, where $n \in \mathbb{N}$ is the number of north steps of ν, ν_0 is the number of initial east steps, and $\nu_i \geq 0$ is the number of consecutive east steps immediately following the i-th north step of ν. In particular, $m = \nu_0 + \cdots + \nu_n$ is the total number of east steps. For instance, the path $ENEENNENEEE$ would correspond to the sequence $(1,2,0,1,3)$, while $ENEENN$ corresponds to $(1,2,0,0)$.

Definition 1.4.4. Let ν be a path on the plane using north and east unit steps. A ν-**path** is a lattice path using north and east steps, with the same endpoints as ν, that stays weakly above ν.

Alternatively, $\mu = (\mu_0, \dots, \mu_n)$ is a ν-path if and only if $\sum_{i=0}^{j} \mu_i \leq \sum_{i=0}^{j} \nu_i$ for all $0 \leq j \leq n$, with equality for $j = n$.

We denote by F_ν the Ferrers diagram that lies weakly above ν in the smallest rectangle containing ν. This is the diagram of a partition λ_ν that can be easily obtained from $\nu = (\nu_0, \nu_1, \dots, \nu_n)$ by taking the initial sums of the (ν_i), namely $\lambda_\nu = (\sum_{k=1}^{n-i} \nu_k)_{0 \leq i \leq n-1}$ (the resulting partition may contain parts equal to 0). For instance, if $\nu = (1,2,0,1,3)$, then $\lambda_\nu = (4,3,3,1)$. Then, a ν-path corresponds to the boundary of a partition whose Ferrers diagram is contained in F_ν.

There is no product formula for the number of ν-paths as there was for Fuß-Catalan or rational Catalan numbers. For instance, if ν is the path $ENEENENN$, or equivalently $(1,2,1,0,0)$, then the number of ν-paths is 53, which is a prime number. However, there is an algorithmic way to compute these numbers, as well as a determinantal formula for the number of ν-paths.

Let us draw the Ferrers diagram of ν as a collection of boxes in the plane. Let L_ν denote the set of lattice points inside F_ν. For each lattice point p of L_ν, we will count the number $P(p)$ of paths using east and north steps that start at $(0,0)$ and end at this point, and that stay weakly above ν. In each box of F_ν, we write the number corresponding to its upper right corner. For this purpose, we also write these numbers in the boxes immediately below F_ν or left to this shape.

Observe that:

1. $P(p) = 1$ for all points p of coordinates $(a,0)$ or $(0,b)$,

2. The number in each box is the sum of the numbers in the two boxes immediately below and left to it (an empty box is counted as containing 0). Indeed, such a path from $(0,0)$ to p has to end with a north or an east step.

Then, starting from the bottom left corner of F_ν with $P((0,0)) = 1$, one can fill each box from bottom to top, from left to right, with the sum of the numbers below and left to it. The number of ν-paths is then the number in the top right corner of F_ν. An example for $\nu = (1,2,1,0,0)$ is given in Figure 1.6.

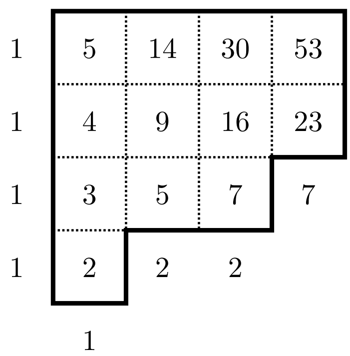


Figure 1.6: Counting ν-paths for $\nu = (1,2,1,0,0)$.

Theorem 1.4.5 ([Kre65]). *Let ν be a north-east lattice path and $\lambda = \lambda_\nu = (\lambda_1, \dots, \lambda_k)$ be its corresponding integer partition.*

The **ν-Catalan number**, *denoted C_ν and defined as the number of ν-paths, is given by the following determinantal formula:*

$$C_\nu = \det\left(\binom{\lambda_j + 1}{j - i + 1}\right)_{1 \le i,j \le k}. \tag{1.10}$$

As an example, if $\nu = (1,2,1,0,0)$, then $\lambda = (4,4,3,1)$, and the number of ν-paths is 53, which is indeed the determinant of the following matrix:

$$\begin{pmatrix} \binom{5}{1} & \binom{5}{2} & \binom{4}{3} & \binom{2}{4} \\ \binom{5}{0} & \binom{5}{1} & \binom{4}{2} & \binom{2}{3} \\ \binom{5}{-1} & \binom{5}{0} & \binom{4}{1} & \binom{2}{2} \\ \binom{5}{-2} & \binom{5}{-1} & \binom{4}{0} & \binom{2}{1} \end{pmatrix} = \begin{pmatrix} 5 & 10 & 4 & 0 \\ 1 & 5 & 6 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Chapter 2

Posets and lattices

Partially ordered sets, or posets for short, are objects of major interest in this manuscript. This chapter aims at introducing the main tools used later, as well as some central examples.

2.1 Definitions and tools

2.1.1 Partially ordered sets

A binary relation on a set E is a subset of $E \times E$. A pair $(x, y) \in R \subset E^2$ is called a relation and often denoted as xRy. The most common and studied relations on a set are those satisfying certain properties, namely equivalence and partial order relations. The latter are the main objects of interest in this manuscript. For a more comprehensive overview on posets and lattices, we refer the reader to [DP02, Wac07].

Definition 2.1.1. A **partial order** is a binary relation $\leq$ on P which is:

- **reflexive**, i.e. $x \leq x$ for all $x \in P$,
- **transitive**, i.e. $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in P$,
- **antisymmetric**, i.e. $x \leq y$ and $y \leq x$ imply $x = y$ for all $x, y \in P$.

We say that $(P, \leq)$ (or simply P if the relation is clear) is a **partially ordered set**, or **poset** for short. We may note $x < y$ when $x \leq y$ and $x \neq y$.

An **equivalence relation** $\sim$ on a set E is a *reflexive*, *transitive*, and *symmetric* binary relation on E.

Example 2.1.2. As classical examples of posets, one can mention:

- the natural order $\leq$ on $\mathbb{Z}$,
- the divisibility relation on $\mathbb{N}$, where $a|b$ if a divides b,
- the **boolean lattice** $\mathcal{B}_n$, which is the poset of subsets of $[n]$ ordered by inclusion,
- the **partition lattice** $\mathcal{P}_n$, which is the poset of partitions of the set $[n]$ with coarsening order, that is to say that a partition π' is greater than a partition π if π' can be obtained by fusing together some parts of π.

Another useful example that is worth defining in the context of this work is the lexicographic order.

Definition 2.1.3. Let $(P, \leq)$ be any poset. A word in the alphabet P is a (maybe empty) finite sequence of elements of P. The set of words in P will usually be denoted P^*. The **lexicographic order** on P^* is the order (still denoted $\leq$) defined by $(x_1, \ldots, x_p) \leq (y_1, \ldots, y_q)$ if:

1. either $p \leq q$ and $x_i \leq y_i$ for all $1 \leq i \leq p$,
2. or $x_i \neq y_i$ for some i and $x_i < y_i$ for the smallest such i.

Several fundamental notions concerning general partial orders arise naturally and are recalled hereafter.

Definition 2.1.4.

- If Q is a subset of P, we define the **subposet** $(Q, \leq)$ to be the restriction of $(P, \leq)$ to the elements of Q, namely $\forall x, y \in Q$, $x \leq y$ in Q if and only if $x \leq y$ in P. We may refer to the *induced* poset structure on the subset Q.
- A **morphism of posets** from $(P_1, \leq_1)$ to $(P_2, \leq_2)$ is an increasing map $f \colon P_1 \to P_2$, i.e. a map such that $x \leq_1 y$ implies $f(x) \leq_2 f(y)$.
  - We say that P_2 is a **refinement** or an **extension** of P_1 when f is bijective. In that case, we may consider that the two posets are defined on the same ground set, and all relations of P_1 are also relations of P_2, but P_2 may have more relations.
  - If f is bijective and its inverse is also an increasing map, then we say that f is an **isomorphism of posets**.
  - One may define similarly **antimorphisms** and **anti-isomorphism** of posets with decreasing maps.
- Two elements $x, y \in P$ are said to be *comparable* if $x \leq y$ or $y \leq x$. If all pairs of elements of P are comparable, we say that P is a *total order* or **linear order**.
- If $x \leq y$, the (closed) **interval** $[x, y]$ is defined as the subset $\{z \in P | x \leq z \leq y\}$.
  - An interval $[x, y]$ is called **trivial** when $x = y$.
  - If $x < y$ and the interval $[x, y]$ is reduced to the set $\{x, y\}$, we say that it is a **covering relation**, and we denote it by $x \lessdot y$. In that case, we say that y covers x, or x is covered by y. We also say that y is an upper cover of x and x is a lower cover of y.
  - An interval $[x, y]$ is called **linear** when the subposet $([x, y], \leq)$ is a linear order. It is then of **length** ℓ if it has cardinality $\ell + 1$.
- A weakly increasing sequence $x_0 \leq x_1 \leq \cdots \leq x_k$ is called a **k-multichain**, or multichain of length k, from x_0 to x_k. It is a **k-chain**, or chain of length k when the sequence is strictly increasing.
  - A chain is said to be *saturated* if for all $0 \leq i \leq k-1$, $x_i \lessdot x_{i+1}$ is a covering relation.
  - Any subset of a chain constitutes itself a chain and is then called a *subchain*.
  - A chain is said to be *maximal* if it is not contained in any other chain. A maximal chain is saturated, and a chain from x to y is saturated if and only if it is maximal in the interval $[x, y]$ as a subposet.
- The **height** $h(I)$ of an interval I is the maximal length of a chain within I.
- A **(lower) ideal** of a poset $(P, \leq)$ is a subset $Q \subset P$ such that $x \leq y$ implies $x \in Q$ for all $y \in Q$, that is to say a subset of P that is closed under "going down". Dually, an **upper ideal** (or *filter*) is closed under "going up", or equivalently the complement of a lower ideal.

As a remark, there is a one-to-one correspondence between intervals $[x, y]$ and 2-multichains (x, y), sending each interval to the pair consisting of its bottom and top elements. We may by extension refer to the pair (x, y) as the interval $[x, y]$.

Remark 2.1.5. An interval $[x, y]$ is linear if and only if it is a chain, or equivalently if it contains a unique saturated chain. The length of a linear interval is thus also the height of the interval.

In all posets, intervals of height 0 are exactly trivial intervals, and those of height 1 are exactly covering relations and they are always linear.

One way to visualize a poset P is to draw its **Hasse diagram**, which is the oriented graph whose vertices are elements of P and edges are covering relations of P, oriented from the smallest to the largest element. In all drawings, we will draw arrows from bottom to top and generally omit the arrows at the top of the edge. A poset is called *locally finite* if all its intervals are finite. When it is the case, two elements are comparable if and only if they are connected in the Hasse diagram. In this discussion, most if not all posets will be finite (and thus locally finite).

Several constructions with posets can be made, as the dual, the sum or the Cartesian product.

Definition 2.1.6.

- If $(P, \leq)$ is a poset, its **dual** is the poset $(P, \leq^*)$, where $y \leq^* x$ if and only if $x \leq y$. When a poset is isomorphic to its dual, it is called **self-dual**.

- If $(P_i, \leq_i)_{i \in I}$ is a family of posets, their **sum** is the poset $\left(\bigsqcup_{i \in I} P_i, \leq\right)$ on the (disjoint) union of the sets $(P_i)_{i \in I}$, where $x \leq y$ if and only if $x \leq_i y$ for some i (which implies that both x and y are in P_i for some i). The Hasse diagram of the sum is the disjoint union of the Hasse diagrams of the posets.

- If $(P_i, \leq_i)_{i \in I}$ is a (finite) family of posets, their **Cartesian product** is the poset $\left(\prod_{i \in I} P_i, \leq\right)$ on the Cartesian product of the sets $(P_i)_{i \in I}$, where $x \leq y$ if and only if all comparisons hold componentwise. The Hasse diagram of the product is the Cartesian product of the Hasse diagrams of the posets.

Remark 2.1.7. The Cartesian product of two intervals in P and Q gives obviously an interval in $P \times Q$, but the converse is also true, namely every interval of $P \times Q$ comes from the Cartesian product of two intervals.

Thus, the number of elements, or **cardinal** of a poset and its number of relations, or **relation number** are two invariants (i.e. preserved by poset isomorphisms) of finite posets that are additive with respect to the sum and multiplicative with respect to the Cartesian product.

Lastly, a very common way to construct a poset is to take the (reflexive) transitive closure of a binary relation on a set E, provided the result is indeed an antisymmetric relation.

Definition 2.1.8. Let R be a relation on a set E. The **transitive closure** of R is the smallest reflexive transitive relation containing R.

Remark 2.1.9.

- Two elements x and y are comparable in the transitive closure of R whenever there exists a finite sequence $x = x_0 R x_1 R \dots R x_n = y$ of relations in R. In other words, the relation R defines a directed graph with vertices E and the relation $\leq$ is given by paths on the graph.

- The transitive closure of a relation R is antisymmetric (and thus a partial order) whenever there is no cycle $x_1 R x_2 R \dots R x_n R x_1$ of relations in R, with $n > 1$.

- If $\leq$ is the transitive closure of R and is antisymmetric, then its covering relation are necessarily of the form xRy. However, not all relations of R are necessarily covering relations of $\leq$. In fact, xRy is a covering relation of $\leq$ if and only if xRy is the only sequence from x to y of nontrivial relations in R.

2.1.2 Lattices

In this study, most posets will have additional structure, namely most of them will be lattices or semilattices. To introduce these notions, we will first need the notions of lower and upper bounds, and of "infimum" and "supremum", usually called **meet** and **join**.

Definition 2.1.10. An element x of a poset $(P, \leq)$ is called minimal (resp. maximal) if there is no $y \in P$ such that $y < x$ (resp. $x < y$).

A poset $(P, \leq)$ is called **maximally bounded** if it has a unique maximal element, that we then denote by $\hat{1}$, and **minimally bounded** if it has a unique minimal element, then denoted by $\hat{0}$. It is *bounded* when minimally and maximally bounded. One can always add a minimal element $\hat{0}$ and a maximal element $\hat{1}$ to a poset P to make it bounded (even when P already had such bottom or top elements), and the resulting poset is called the **bounded extension** $\hat{P}$ of P.

Let now $x, y \in P$.

- A lower bound (resp. upper bound) of x and y is an element $z \in P$ such that $z \leq x$ and $z \leq y$ (resp. $x \leq z$ and $y \leq z$).

- The set of lower (resp. upper) bounds of x and y is a lower (resp. upper) ideal of P. If it has a unique maximal (resp. minimal) element, this element is called the **meet** (resp. **join**) of x and y, denoted by $x \wedge y$ (resp. $x \vee y$). The meet and the join are sometimes called infimum and supremum, or greatest lower bound and least upper bound. For a (nonempty) subset $S \subset P$, we denote by $\bigwedge S$ (resp. $\bigvee S$) the meet (resp. join) of all elements of S if it exists.

- A **meet-semilattice** (resp. **join-semilattice**) is a poset where every pair of elements (or equivalently every finite subset S) has a meet (resp. a join). A **lattice** is a poset that is both a meet-semilattice and a join-semilattice.

In fact, one can define a meet- or a join-semilattice as a set with a binary operation $\wedge$ or $\vee$ satisfying certain axioms as $x \leq y$ if and only if $x \wedge y = x$ or $x \vee y = y$, and these axioms can be shown to be equivalent to the definition above. While we will not use this more algebraic setup, this interpretation gives however a natural condition for a subset of elements to constitute a "sublattice". We require such a subset to be closed under meets and joins, and this condition is stronger than the more naive condition of being a subposet that admits a lattice structure. In other words, we want the lattice structure of the subposet to come from the lattice structure of the whole poset.

Definition 2.1.11. Let P be a lattice and Q be a subposet of P. It is a **sublattice** of P if it is stable under meets and joins, i.e. if $x, y \in Q$ implies $x \wedge y \in Q$ and $x \vee y \in Q$. In other words, Q is a lattice and the meets and the joins of elements of Q agree when computed in P or in Q.

Similarly, a **lattice morphism** is a poset morphism between two lattices that preserves meets and joins.

Proposition 2.1.12. *An interval $[x, y]$ in a lattice $(P, \leq)$ is a lattice for the induced order. It is in fact a sublattice of P.*

Proposition 2.1.13 ([Sta12, Proposition 3.3.1])**.** *A meet-semilattice with a unique maximal element $\hat{1}$ is a lattice. Dually, a join-semilattice with a unique minimal element $\hat{0}$ is a lattice.*

Subsets, unions and Cartesian products of sets translate very naturally to the world of posets, as we have explained. Another very common construction is to consider quotients of sets, and as such, one can naturally try to extend these into "quotients" of posets and lattices. If $\sim$ is an equivalence relation on the ground set P of a poset $(P, \leq)$, one may want to define a poset structure on the quotient $P/\sim$. The most natural choice would be to fix $[x] \leq [y]$ as soon as $x \leq y$, that is to say the transitive closure of the induced relation, but this does not always produce a poset structure. For instance, if $1 < 2 < 3$ and $1 \sim 3 \not\sim 2$, then $[1] \leq [2]$ and $[2] \leq [3] = [1]$ but $[1] \neq [2]$. One may alternatively say that Q is a quotient of P when there is a surjective morphism of posets $f\colon P \to Q$, whose fibers are then exactly the equivalence classes of $\sim$. In that case indeed, the transitive closure of the induced relation on $P/\sim$ is isomorphic to Q.

The question of defining quotients is in fact quite hard to answer in general for posets, and several answers have been considered in the literature, as explained in the survey article [Wil23]. However, as well as for defining sublattices, the very natural condition of preserving meet and join operations gives rise to quotient sets that indeed inherit the lattice structure. Such an equivalence relation is called a **lattice congruence** and the result is called a **lattice quotient**.

Definition 2.1.14. Given a lattice $(P, \leq)$, a **lattice congruence** on P is an equivalence relation $\sim$ on P such that for all $x_1, x_2, y_1, y_2 \in P$, if $x_1 \sim x_2$ and $y_1 \sim y_2$, then $x_1 \wedge y_1 \sim x_2 \wedge y_2$ and $x_1 \vee y_1 \sim x_2 \vee y_2$.

Proposition 2.1.15. *Let $\sim$ be a lattice congruence on $(P, \leq)$. Let $\leq_\sim$ be the binary relation on $P/\sim$ defined by $[x] \leq_\sim [y]$ if and only if there exists $x' \sim x$ and $y' \sim y$ such that $x' \leq y'$. Then $(P/\sim, \leq_\sim)$ is a lattice, and we have $[x] \wedge [y] = [x \wedge y]$ and $[x] \vee [y] = [x \vee y]$.*

In fact, a useful characterization of lattice congruences in the case of finite posets is given by the following proposition of N. Reading.

Proposition 2.1.16 ([Rea02, Section 2])**.** *An equivalence relation $\sim$ on a finite lattice P is a lattice congruence if and only if the following conditions hold:*

1. *each equivalence class of $\sim$ is an interval of P,*
2. *the mapping $\pi_\uparrow$ mapping an element $x \in P$ to the top element of its equivalence class is order-preserving,*
3. *the mapping $\pi_\downarrow$ mapping an element $x \in P$ to the bottom element of its equivalence class is order-preserving.*

In particular, in this case, $P/\sim$ can be seen not only as a lattice quotient of P but also as the two (isomorphic) sublattices of P consisting of the restrictions of P to the images of $\pi_\uparrow$ and $\pi_\downarrow$.

2.1.3 Möbius function and topology

Posets can be studied from an algebraic point of view, for instance by considering the incidence algebra of a poset as we will explain later, or as mentioned above by considering the algebraic structure of lattices. Another fruitful approach is topological and consists in attaching a simplicial complex to a poset, called its **order complex**, and in the other way around, considering the poset of faces of a simplicial complex, called its **face poset**. These operations are not inverse to each other, but they do not lose any topological information, in the sense that the order complex of the face poset of a simplicial complex is homeomorphic to the original simplicial complex. The interested reader may refer to [Wac07].

Definition 2.1.17. An **abstract simplicial complex** Δ on a finite set V of vertices is a (nonempty) collection of subsets of V stable by subsets and containing all singletons. The elements of Δ are called **faces** and the maximal faces with respect to inclusion are called **facets**. The **dimension** of a face F is the cardinality of F minus one and the dimension of Δ is the maximal dimension of its faces.

The **face poset** $P(\Delta)$ of Δ is the poset $(\Delta \setminus \{\emptyset\}, \subset)$ of nonempty faces of Δ, ordered by inclusion. One may also consider the **face lattice** $L(\Delta)$ which is the bounded extension of $P(\Delta)$.

The **order complex** $\Delta(P)$ of a poset P is the simplicial complex whose faces are the chains of P (including the empty chain).

To any abstract simplicial complex, one may attach a *geometric simplicial complex*, which is a finite collection of simplices such that the intersection of any two simplices of the collection is a common face of each. Moreover, such a *realization* of an abstract simplicial complex is unique up to homeomorphism as a topological space. Thus, topological notions can be studied on abstract simplicial complexes. Examples of order complexes and face posets are given in Figure 2.1.

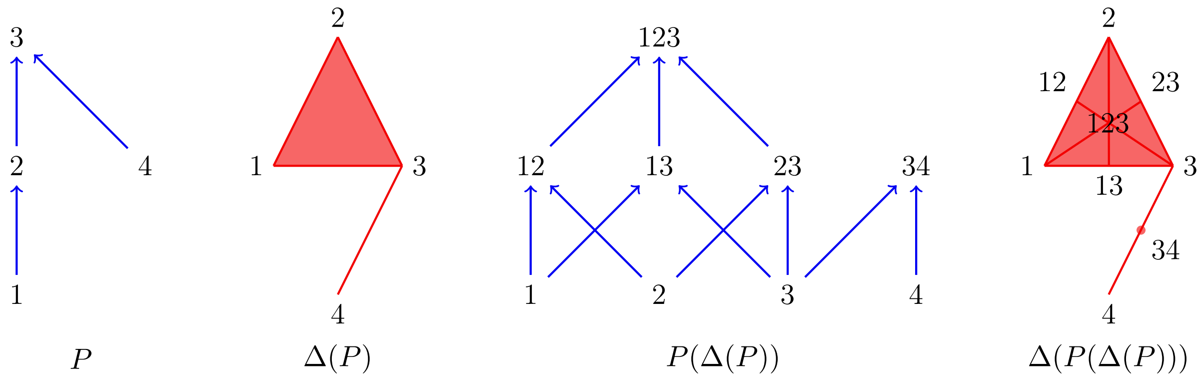


Figure 2.1: A poset P, its order complex $\Delta(P)$, the face poset $P(\Delta(P))$, and the order complex $\Delta(P(\Delta(P)))$. Note that $\Delta(P(\Delta(P)))$ is the barycentric subdivision of $\Delta(P)$.

Proposition 2.1.18. *If Δ is an abstract simplicial complex on V, then $\Delta(P(\Delta))$ is the* **barycentric subdivision** *of Δ and is thus homeomorphic to Δ.*

This result implies that no topological information is lost when translating between posets and simplicial complexes. Thanks to this, one may transport topological information of the order complex to the world of posets. We may thus refer to the homotopy type of a poset as the homotopy type of its order complex, and so on. The order complex and its topological information is thus another invariant that may be attached to posets, as well as the order complexes of their intervals.

Remark 2.1.19.

- The order complex of a poset and of its dual are identical.
- If a poset P has a unique maximal (or minimal) element, then $\Delta(P)$ is a cone (i.e. as a topological space, it is of the form $(X \times [0,1])/(X \times \{0\})$), and it is thus contractible. Thus, we usually remove the top and bottom elements of a bounded poset to study the more interesting topology of the remaining poset $P \setminus \{\hat{0}, \hat{1}\}$ called its **proper part** and denoted by $\overline{P}$. The degenerate case when P is a singleton will be handled as the degenerate empty complex for the order complex of the proper part.
- In particular, as an interval is bounded, all order complexes of intervals are contractible. Thus, when referring to the homotopy type of an interval, one will always consider its proper part, i.e. the open interval.

A combinatorial object that can be attached to a poset P is the **Möbius function**, denoted μ_P (or simply μ), which is a function on intervals of a poset, defined recursively. It generalizes the classical Möbius function on integers, when viewing them with the divisibility order. We can then define the Möbius invariant of bounded posets, which turns out to be equal to the reduced Euler characteristic of their proper part.

Definition 2.1.20. The **Möbius function** μ_P of a poset P is the function on intervals of P defined recursively by:

$$\mu_P(x,x) = 1, \quad \forall x \in P \tag{2.1}$$

$$\mu_P(x,y) = -\sum_{x \leq z < y} \mu(x,z), \quad \forall x < y \in P. \tag{2.2}$$

If P is a bounded poset, its **Möbius invariant** is defined as $\mu(P) = \mu_P(\hat{0}, \hat{1})$.

Proposition 2.1.21. *Let P be a bounded poset. Then $\mu_P(\hat{0},\hat{1}) = \tilde{\chi}(\overline{P})$, where $\tilde{\chi}$ is the reduced Euler characteristic—which can be defined as the alternating sum of its reduced Betti numbers, or ranks of the reduced homology groups.*

This is interesting as it relates the homotopy type of the order complex with combinatorial data readable and computable on the poset. If for instance we know that the homotopy type is a wedge of spheres (which for instance is the case for geometric semilattices), then we know that the Möbius invariant is the alternate sum of their dimensions.

An alternative definition of the Möbius function is given with the incidence algebra of a poset, which is the occasion to introduce this algebraic object that we already mentioned. For all what follows, we fix a ground field k or a commutative unitary ring R.

Definition 2.1.22. Let P be a (locally finite) poset. Its **incidence algebra** $I(P)$ is the k-vector space (or R-module) of functions on the intervals of P with values in k (or in R), with pointwise addition and scalar multiplication and whose product is given by the convolution $f \star g(a,b) = \sum_{a \leq c \leq b} f(a,c)g(c,b)$.

Remark 2.1.23. When P is finite, one can think of this algebra as upper triangular matrices indexed by elements of the poset, such that every coefficient in position (x,y) where $[x,y]$ is not an interval is equal 0. The product of two such matrices is then the usual matrix product.

It is also useful in this case to think of the incidence algebra as a vector space whose basis is given by intervals of P and the product is given by the formula $[a,b][c,d] = \delta_{b,c}[a,d]$, where $\delta_{b,c}$ is the Kronecker symbol, equal to 1 if $b = c$ and 0 otherwise. This is equivalent to taking the algebra of paths in the Hasse diagram, and to quotient it by the simple equivalence relation where two paths are equivalent if their endpoints are the same.

Proposition 2.1.24. *The incidence algebra $I(P)$ of P is a unitary algebra whose neutral element for the convolution product is the Kronecker symbol $(a,b) \mapsto \delta_{a,b}$.*

The particular function constant to 1 in this algebra is denoted ζ_P and is invertible. Its inverse is exactly the previously defined Möbius function μ_P.

The notations actually arise from the very well-known Riemann zeta function seen as a Dirichlet series (with coefficients constant to 1). Dirichlet series are series of the form $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ and they form an algebra, with the sum and convolution of series. Their invertible elements are exactly the series such that a_0 is invertible.

The algebra of Dirichlet series can be seen as a subalgebra of the incidence algebra of the poset of natural numbers with divisibility order, formed of functions f such that $f(a,b) = f(ka,kb)$ for all $k \geq 1$. For such a function f, the value $f(a,b) = f(1,b/a)$ only depends on the integer ratio b/a. The inverse series of the Riemann zeta function has the classical Möbius function on integers as coefficients.The well-known result of the Möbius inversion formula generalizes to this context and is a very powerful tool.

Theorem 2.1.25 (Möbius inversion)**.** *Let P be a poset such that for each element $y \in P$, the* principal order ideal *of y, namely $\{x \in P \mid x \leq y\}$ is finite. Let f and g be two functions on the elements of P (with values in any unitary ring R). Then we have the following equivalence:*

$$g(y) = \sum_{x \leq y} f(x) \iff f(y) = \sum_{x \leq y} g(x)\mu(x,y). \tag{2.3}$$

Remark 2.1.26.

- This theorem is in fact a direct consequence of Proposition 2.1.24, once a function f on P is considered as a function on the intervals of P by setting $f(x, y) = \delta_{x,\hat{0}} f(y)$, adding an element $\hat{0}$ if necessary.
- The poset of integers with divisibility order gives the classical Möbius inversion formula.
- The boolean poset B_n of subsets of $[n]$ with inclusion gives the well-known inclusion-exclusion principle.
- A dual statement can be stated for posets such that the set $\{y \in P \mid x \leq y\}$ is finite for all $x \in P$. In that case, we can write $g(x) = \sum_{x \leq y} f(y) \iff f(x) = \sum_{x \leq y} \mu(x, y) g(y)$.

Lastly for this section, we will introduce the notions of **vertex-decomposable** and **shellable** complexes. These notions have been extensively studied, in particular because of deep topological and algebraic implications in the case of pure complexes, as having the homotopy type of a wedge of spheres, or the Cohen-Macaulayness of some polynomial ring attached to the complex. However, these notions are beyond the scope of this thesis.

Definition 2.1.27. A simplicial complex is **pure** if all its facets have the same dimension. In particular, on the poset side, this means that all maximal chains have the same length.

The **deletion** of a vertex i of an abstract simplicial complex Δ is the simplicial complex $\mathrm{del}_i(\Delta)$ whose faces are the faces of Δ not containing i, i.e. $\mathrm{del}_i(\Delta) = \{F \in \Delta \mid i \notin F\}$.

The **link** of a vertex i of Δ is the simplicial complex $\mathrm{lk}_i(\Delta)$ obtained as the deletion of i in the complex generated by the complement of $\mathrm{del}_i(\Delta)$, i.e. $\mathrm{lk}_i(\Delta) = \{F \mid i \notin F, F \cup \{i\} \in \Delta\}$.

A **shelling order** of a simplicial complex Δ is a linear order $F_1, \dots, F_n$ of the facets of Δ such that the for all $2 \leq k \leq n$, the simplicial complex $\langle F_1, \dots, F_{k-1} \rangle \cap \langle F_k \rangle$ is pure of dimension $\dim(F_k) - 1$, where $\langle (F_i)_{i \in I} \rangle$ is the complex consisting of all faces of all the $(F_i)_{i \in I}$. A simplicial complex is **shellable** if it admits a shelling order.

A pure simplicial complex of dimension d is **vertex-decomposable** if it is either empty or it contains a vertex i such that both $\mathrm{del}_i(\Delta)$ and $\mathrm{lk}_i(\Delta)$ are vertex-decomposable, pure, and of respective dimensions d and $d - 1$.

In particular, if a simplicial complex is vertex-decomposable, then there is at least one total order on its vertices such that the recursive deletions of vertices in this order give vertex-decomposable complexes. This order then also gives a lexicographic order on the facets of the complex, which turns out to be a shelling order, as proven in [BW97, Theorem 11.3].

Theorem 2.1.28. *If Δ is a vertex-decomposable simplicial complex, then Δ is shellable. The lexicographic order on the facets of Δ is a shelling order.*

Theorem 2.1.29. *A shellable simplicial complex Δ has the homotopy type of a wedge of spheres, where for each i, the number of i-spheres is the number of i-facets whose entire boundary is contained in the union of the earlier facets.*

The general case is due to B. Björner and M. Wachs. The pure case was already known and if Δ is pure of dimension d, then it has the homotopy type of a wedge of d-dimensional spheres. If it is the order poset of a (pure) poset, then computing the Möbius invariant of the poset gives exactly the number of spheres. More precise results for some posets have been studied and for instance, it is known that each interval of the Tamari lattice is either contractible or has the homotopy type of exactly one sphere. Various notions of shellability have been studied, as the more combinatorial one of EL-shellability (for "edge-lexicographic") first introduced by A. Björner in [Bjö80] for pure posets, then extended for general bounded posets [BW96, Section 5]. This is a property directly readable on the poset, and it implies that the order complex of its proper part is shellable. The idea is to associate a label to each covering relation of the poset, that is to say to every edge of its Hasse diagram, with good properties.

Definition 2.1.30. Let P be a bounded poset. An **edge-labeling** of P is a map $\lambda\colon \{x \lessdot y \mid x, y \in P\} \to \Lambda$, where Λ is some poset. If $\lambda(x \lessdot y) = a$, we may then denote with $x \xrightarrow{a} y$.

Given an edge-labeling λ of P, we say that a saturated chain $c = x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} \dots \xrightarrow{a_k} x_k$ is **rising** if the sequence $\lambda(c) = (a_1, a_2, \dots, a_k)$ is strictly increasing in Λ.

The poset P is called **EL-shellable** if there exists an edge-labeling λ of P such that:

1. every interval $[x, y]$ of P contains a unique rising chain,
2. this unique rising chain c is lexicographically smaller than any other maximal chain in $[x, y]$, that is to say $\lambda(c) < \lambda(c')$ in the lexicographic order on $\Lambda*$.

Theorem 2.1.31 ([BW96, Theorem 5.8])**.** *If a bounded poset P is EL-shellable, then $\Delta(\overline{P})$ is shellable. Any order on the facets that extends the lexicographic order on the rising chains is a shelling order.*

In fact, such an edge-labeling gives more information, namely the homotopy type of an interval can be read by looking at the falling chains and their length, that is to say (saturated) chains such that every label is *not* smaller that the next one, each one contributing to an ℓ-2 dimensional sphere, where ℓ is the length of the falling chain [BW96, Theorem 5.9].

2.2 Central examples

2.2.1 The Dyck and ν-Dyck lattices

We defined the class $\mathcal{Z}$ of Dyck paths in Section 1.3.2. We will now define a first order relation on the set $\mathcal{Z}_n$ of Dyck paths of length n. This poset, called the **Dyck lattice** (or Stanley lattice), is perhaps the most natural poset on $\mathcal{Z}_n$, where comparison relations are "being below" [Sta12, BB09]. It has very nice properties, as it is indeed a lattice, furthermore graded by the "area". Moreover, it will be naturally extended to the set of ν-paths for any path ν. In fact, viewing such paths as the boundary of the Young diagram of a partition, this poset will simply be the containment order on the set of partitions lying in a given shape.

Definition 2.2.1. The **Dyck lattice** Dyck_n, for $n \geq 1$, is the poset on Dyck paths of size n where $P \leq Q$ if P is always under Q when we draw them together. In other words, $P \leq Q$ if for every $1 \leq k \leq n$, the altitude of the k-th up step of P is less than or equal to the altitude of the k-th up step of Q.

Remark 2.2.2. Covering relations in Dyck_n can be defined as transforming a valley du of a path P in a peak ud.

The path of size 0 will be considered as a trivial path, but we will usually not consider the Dyck lattice of size 0 and its unique trivial interval.

There is an involution on Dyck paths that exchanges a path with its mirror image, that can be seen as a vertical symmetry on the drawings. More precisely, it reverses up and down steps and the order of the steps.

Remark 2.2.3. The mirror involution on Dyck paths is an order-preserving involution on the Dyck lattice.

The number of intervals in the Dyck lattice can be counted by a closed formula [dSCV86], using the classical determinant formula for counting nonintersecting lattice paths, known as the Lindström-Gessel-Viennot lemma [Lin73, GV85]. Indeed, lifting the top path of an interval gives two noncrossing paths.

Theorem 2.2.4. *The number of intervals in the Dyck lattice* Dyck_n *is equal to:*

$$\frac{6(2n)!\,(2n+2)!}{n!\,(n+1)!\,(n+2)!\,(n+2)!}. \tag{2.4}$$

Now, we can straightforwardly generalize this partial order structure to the set of ν-paths for any path ν.

Definition 2.2.5. The **ν-Dyck lattice** Dyck_ν is the poset on ν-paths where $P \leq Q$ if Q is weakly above P.

Equivalently, if $P = (P_0, \dots, P_n)$ and $Q = (Q_0, \dots, Q_n)$ are two ν-paths, then $P \leq Q$ if $\sum_{i=0}^{j} Q_i \leq \sum_{i=0}^{j} P_i$ for all $0 \leq j \leq n$.

An example of the ν-Dyck lattice for $\nu = ENEEN$ is illustrated in Figure 2.2.

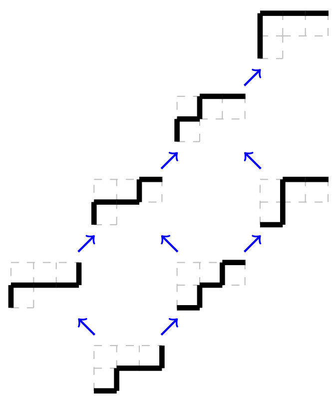


Figure 2.2: The ν-Dyck lattice for $\nu = ENEEN = (1, 2, 0)$.

Remark 2.2.6.

- The case where ν is $(NE)^n$ coincides with the classical Dyck lattice on Dyck paths of size n.
- Covering relations $P \lessdot Q$ in the ν-Dyck lattice also consist of transforming a valley EN in a peak NE in some ν-path P.
- The (ν-)Dyck lattice is indeed a lattice, meets and joins are easy to compute as they are respectively unions and intersections of Ferrers diagrams (viewed as collection of boxes). It is graded by the area, which is the number of "boxes under the path", that is to say of missing boxes of the Ferrers diagram of ν, each covering relation removing one of them.
- The number of intervals in the ν-Dyck lattice does not admit in general such a nice closed product formula as in the Dyck lattice. However, the very same technique of lifting the second path allows using again the Lindström-Gessel-Viennot lemma to express it as the determinant of a 2×2 matrix of ν-Catalan numbers.

2.2.2 The Tamari lattice

The Tamari lattice is another poset defined on Catalan objects, central in all this study, as it gave its name to this thesis. It is a lattice as well, though it is not graded. It can also be defined on Dyck paths but for historical reasons, we will first define it on binary trees. This partial order is named after Dov Tamari, who first defined and studied it as a poset on bracketings, by orienting associativity relations [Tam62]. More precisely, instead of allowing the transformation of an expression $(a \cdot (b \cdot c)) = ((a \cdot b) \cdot c)$ in both ways, if one allows moving parentheses only to the left, then we obtain a poset, the Tamari lattice, that turns out to have very good properties, among which the fact that it is a lattice [HT72]. This translates very well on the set $\mathcal{Y}_n$ of binary trees through the easy bijection between bracketings and binary trees, where an "associativity relation" can be understood as "unplugging" a subtree and plugging it in the opposite branch.

Definition 2.2.7. Let T be a binary tree with a node s whose right child is not a leaf. Let B be the left subtree of s, and C and D be the left and right subtrees of the right child of s.

Let T' be the tree obtained from T by unplugging the subtree C and plugging it in the left outgoing edge of s, such that the node s has now the tree D as right subtree and a left child whose left subtree is B and right subtree is C. Such a transformation from T to T' is called a **left rotation** (at the node s) and is denoted $T \lessdot T'$.

The **Tamari poset** Tam_n of size n is the partial order on $\mathcal{Y}_n$ obtained as the transitive closure of left rotations.

Theorem 2.2.8 ([HT72])**.** *The Tamari poset is a lattice.*

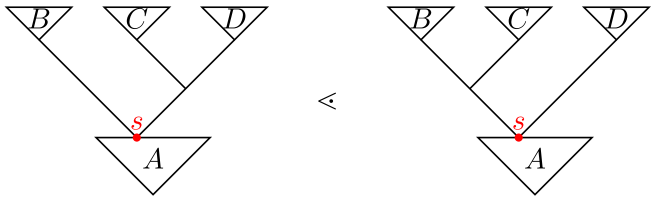


Figure 2.3: A covering relation in the Tamari lattice.

An example of left rotation in the Tamari lattice is illustrated in Figure 2.3. The Tamari lattice on trees of size 3 is illustrated in Figure 2.4. As for the Dyck lattice, we will usually not consider the Tamari lattice of size 0 and its unique trivial interval.

There is also an involution on trees that exchanges a tree with its mirror image, which can be seen as a vertical symmetry on the drawings. More precisely, for every node, it exchanges its left and right children. As an example, the right comb r_n and the left comb ℓ_n are exchanged through this involution.

Remark 2.2.9. The mirror involution on trees is an order-reversing involution on the Tamari lattice.

Proof. Let $T \lessdot T'$ be a covering relation, and S and S' be the respective mirror images of T and T'. Then $S' \lessdot S$ is a covering relation in the Tamari lattice. $\square$

Remark 2.2.10. Note that the bijection between Dyck paths and binary trees that we described in the proof of Proposition 1.3.4 does *not* transport the mirror involution on Dyck paths to the mirror involution on binary trees.

The image of the mirror involution on Dyck paths under this bijection is neither an order preserving nor an order reversing involution on the Tamari lattice, and similarly, the image of the mirror involution on binary trees under this bijection is neither an order preserving nor an order reversing involution on the Dyck lattice.

However, this bijection does transport the Tamari lattice nicely on the set of Dyck paths, saying it otherwise, it is possible to define a poset on Dyck paths in a nice combinatorial way, that will be isomorphic to the Tamari lattice under this bijection between Dyck paths and binary trees.

Definition 2.2.11. Let P be a Dyck path of size n with a valley du. Let C be the excursion of this up step u, such that we can write $P = AdCB$.

Let Q be the path obtained from P by exchanging the down step d of this valley with the excursion following it, namely $Q = ACdB$. Such a transformation from P to Q is called a **rotation** and is denoted $P \lessdot Q$.

Proposition 2.2.12. *The bijection from binary trees to Dyck paths described in the proof of Proposition 1.3.4 transforms a right rotation in the Tamari lattice to a rotation on Dyck paths as described above.*

Thus, the Tamari lattice is isomorphic to the transitive closure of rotations on Dyck paths.

Proof. Let $T \lessdot T'$ be a left rotation in the Tamari lattice. Let P be the Dyck path corresponding to T.

First remark that each edge of T going to the right corresponds to an up step u of P, and the "sibling" edge going to the left out of the same node is exactly its matching down step d, namely the last step of the excursion starting at u. Moreover, the steps of the excursion correspond exactly to all edges of the two subtrees of the node u.

Now, when removing an edge going to the left out of a node, this node becomes a leaf itself, that is immediately removed afterwards. Thus, valleys in the path correspond exactly to nontrivial right children.

Thanks to these observations, we can see that the two edges out of the node s in T correspond to an excursion C in the Dyck path D, that comes immediately after the down step d corresponding to the edge going to the left out of the right child of s. Proceeding to the left rotation at the node s indeed translates into sending this down step d after the excursion C. $\square$

This alternative description of the Tamari lattice on Dyck paths was introduced by F. Bergeron and L.-F. Préville-Ratelle [BPR12]. An example can be found in Figure 2.4.

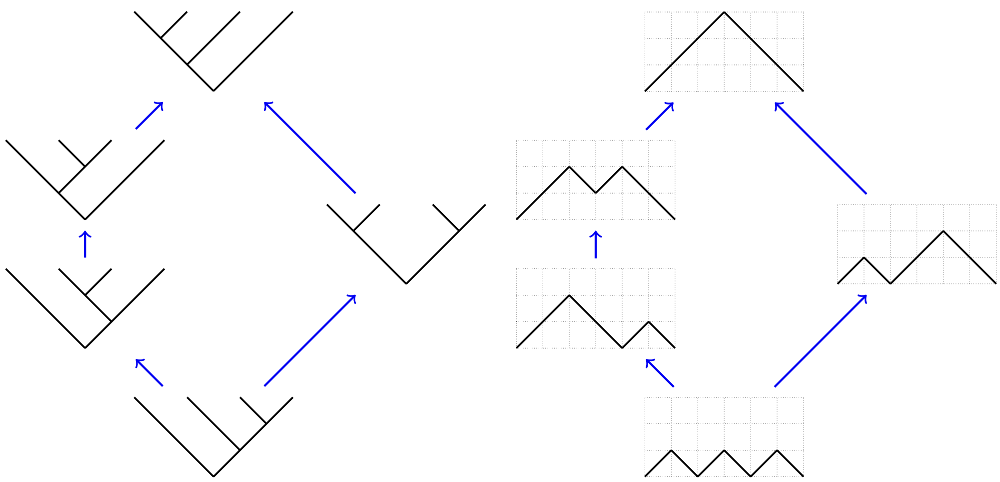


Figure 2.4: The Tamari lattice on trees of size 3 and on Dyck paths of size 3.

Remark 2.2.13. From the definition, it is immediate to see that any rotation of a Dyck path produces a path that remains weakly above it. Thus, any interval $[P, Q]$ in the Tamari lattice defined this way is an interval in Dyck_n and thus, the Dyck lattice is an extension of the Tamari lattice.

Using a catalytic variable, F. Chapoton could produce a functional equation on the generating function of intervals in the Tamari lattices, that he solved, providing a closed formula for the number of intervals in the Tamari lattice [Cha06]. As it is of central interest in our study, the proof is quickly reproduced in Chapter 8.

Theorem 2.2.14. *The number of intervals in the Tamari lattice* Tam_n *is equal to:*

$$\frac{2(4n+1)!}{(n+1)!\,(3n+2)!}. \tag{2.5}$$

This formula also counts rooted cubic 3-connected maps, which are a special family of maps, i.e. graphs embedded in a surface.

2.2.3 The noncrossing partition lattice

There is another poset that can be naturally defined on Catalan objects, namely the **noncrossing partition lattice**, or Kreweras lattice. It is the most natural poset on the set of noncrossing partitions, and it is also a lattice, and graded as well (by the number of blocks). It can also be described on Dyck paths and it turns out to be in turn extended by the Tamari lattice.

Definition 2.2.15. The **noncrossing partition lattice** NCL_n of size n is the restriction of the partition lattice $\mathcal{P}_n$ to the set of noncrossing partitions (with the coarsening order).

The poset NCL_n turns out to be a lattice itself, but *not* a sublattice of $\mathcal{P}_n$ since the join of two noncrossing partitions is not necessarily the same in both. For instance in $\mathcal{P}_4$, the join of $\{\{1\},\{2,4\},\{3\}\}$ and $\{\{1,3\},\{2\},\{4\}\}$ is $\{\{1,3\},\{2,4\}\}$, which is not noncrossing. In NCL_4, their join is $\{\{1,2,3,4\}\}$. The noncrossing partition lattice NCL_n is self-dual and graded by n minus the number of blocks.

Proposition 2.2.16. *The Kreweras complement described in Definition 1.3.8 is an order-reversing involution on the noncrossing partition.*

Proposition 2.2.17. *The noncrossing partition lattice is indeed a lattice. The meet of two noncrossing partitions is the same when it is computed in* $\mathcal{P}_n$ *and in* NCL_n*. It is graded as is the partition lattice.*

Definition 2.2.18. Let P be a Dyck path with a valley du. Let B be any factor of P starting at this up step u that is also a Dyck path, and P' the path obtained by swapping P with the entire fall preceding it. Calling such a transformation of a Dyck path a "Kreweras rotation" lets us define the "Kreweras lattice" as the transitive closure of Kreweras rotations.

Proposition 2.2.19 ([BB09, Proposition 2.3])**.** *The "Kreweras lattice" defined on Dyck paths is isomorphic to the noncrossing partition lattice.*

The number of intervals in the noncrossing partition lattice is also counted by a closed formula, proved inductively by G. Kreweras [Kre72] and bijectively by P. H. Edelman [Ede82]. These numbers are in fact the Fuß-Catalan numbers for $m = 2$, that is to say the number of ternary trees for instance.

Theorem 2.2.20. *The number of intervals in the noncrossing partition lattice* NCL_n *is equal to*

$$\frac{1}{2n+1}\binom{3n}{n}. \tag{2.6}$$

2.2.4 The weak order on the symmetric group

The set $\mathfrak{S}_n$ of permutations of the interval $[n] = \{1,\dots,n\}$ can be endowed with a very interesting partial order, called the **right weak order**, also called *right weak Bruhat order* or *right permutahedron order*. There also exists a *left weak order*, which is a distinct order but is in fact isomorphic to the right weak order through the inverse map. These posets seems to have been first considered in the 60s in the context of statistics. They can in fact be generalized to weak orders on Coxeter groups, as we will see in Section 3.3.2.

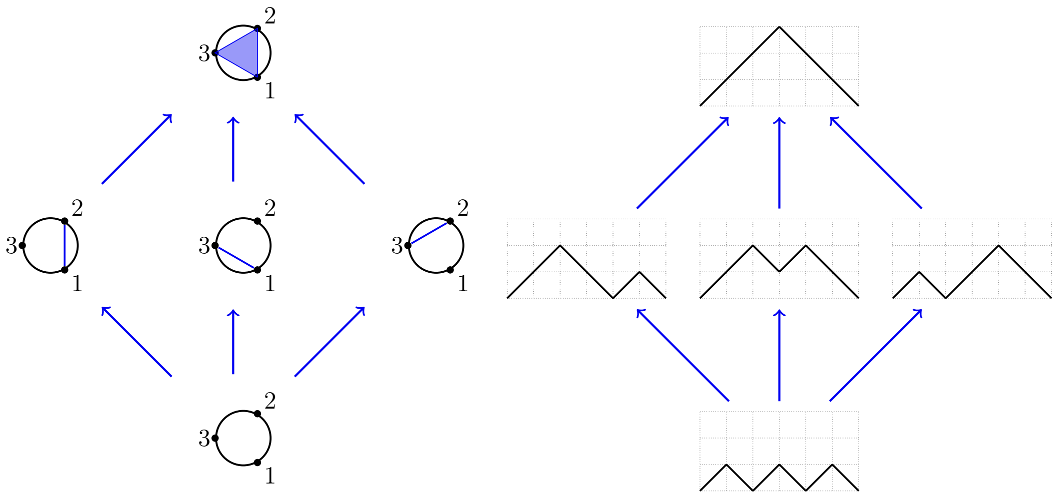


Figure 2.5: The Kreweras lattice on (noncrossing) partitions of size 3 and on Dyck paths of size 3.

We can write a permutation $\sigma \in \mathfrak{S}_n$ as the sequence $(\sigma_1, \ldots, \sigma_n) = (\sigma(1), \ldots, \sigma(n))$ of its images, which is called the one-line notation of σ. The right weak order can be defined as the transitive closure of the relations $\lessdot_r$, where $\sigma \lessdot_r \sigma'$ if σ' can be obtained from σ by exchanging two consecutive entries $\sigma_i < \sigma_{i+1}$.

Definition 2.2.21. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\sigma' = (\sigma'_1, \ldots, \sigma'_n)$ be two permutations in $\mathfrak{S}_n$.

We write $\sigma \lessdot_r \sigma'$ (or $\sigma \xrightarrow{i}_r \sigma'$) when there exists $i \in [n-1]$ such that $\sigma'_{i+1} = \sigma_i < \sigma_{i+1} = \sigma'_i$ and for all $k \notin \{i, i+1\}, \sigma_k = \sigma'_k$. The **right weak order** $\leq_r$ on $\mathfrak{S}_n$ is the transitive closure of the relations $\lessdot_r$, and we denote this poset as $\mathrm{Weak}(\mathfrak{S}_n)$.

Using the group structure on $\mathfrak{S}_n$ given by $\sigma\sigma'(i) = \sigma(\sigma'(i))$, we can rewrite $\sigma' = \sigma s_i$, with s_i the transposition $(i, i+1)$, with an extra condition to decide whether $\sigma \lessdot_r \sigma'$ or the other way around. The extra condition can be expressed in terms of *length* of elements as we will see later in the general case of Coxeter groups, or using *inversions*, that can be defined quite naturally in the symmetric group but will be later generalized as well, and whose number will correspond to the length. The term "right" comes from the fact that we are multiplying elements of the group on the right by transpositions of the form s_i. The "left" analogue exists as well, where we multiply on the left but will not be defined in this section. This poset has very good properties and has been studied extensively.

Definition 2.2.22. Let $\sigma \in \mathfrak{S}_n$ and $1 \leq i < j \leq n$. The pair (i, j) is an **inversion** of σ if $\sigma^{-1}(i) > \sigma^{-1}(j)$.

The **inversion set** of σ is the set $\mathrm{inv}(\sigma) = \{(i, j) \mid 1 \leq i < j \leq n \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j)\}$.

In other words, the set of inversions of the permutation σ is the set of pairs of integers that appear in the "wrong" order in the one-line notation of σ. For instance, the permutation $\sigma = (3, 1, 4, 2)$ (that we may write 3142 for short) has inversion set $\mathrm{inv}(\sigma) = \{(1, 3), (2, 3), (2, 4)\}$. It is clear by definition that $\sigma \lessdot_r \sigma'$ implies that σ' has exactly one more inversion than σ, namely the pair (σ_i, σ_{i+1}). Thus, $\sigma \leq_r \sigma'$ implies that $\mathrm{inv}(\sigma) \subset \mathrm{inv}(\sigma')$. It is in fact an equivalent condition, which gives a criterion to compare two given permutations.

Theorem 2.2.23. *Let $\sigma, \sigma' \in \mathfrak{S}_n$. We have $\sigma \leq_r \sigma'$ if and only if $\mathrm{inv}(\sigma) \subset \mathrm{inv}(\sigma')$.*

Covering relations are exactly all relations $\sigma \lessdot_r \sigma'$.

Theorem 2.2.24. *The (right) weak order on $\mathfrak{S}_n$ is a lattice, graded by the length (or number of inversions), and it is self-dual.*

It is worth noting that, from an enumerative perspective, the total number of intervals in the weak order on $\mathfrak{S}_n$ does not seem to admit such a nice formula as the three previous examples.

2.2.5 The permutree lattices

The Tamari lattice can also be defined as a lattice quotient of the weak order on $\mathfrak{S}_n$, and this can be generalized as the "Cambrian lattices of type A". This can be further generalized in two directions: as the Cambrian (and m-Cambrian) lattices in any (finite) Coxeter type which we will discuss in the next chapter, and as the permutree lattices, which are a wide family of lattice quotients of the symmetric group introduced by V. Pilaud and V. Pons [PP18], and we present these posets here. A joint generalization in all types does not exist yet, to our knowledge, though a partial answer was recently given in type B in [PPR22], and an idea is evoked in [PPT23].

To see the Tamari lattice as a quotient of the weak order, one can associate to each permutation its binary search tree.

Definition 2.2.25. Let $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathfrak{S}_n$. One can create a binary tree $T(\sigma)$ by reading each letter of σ and inserting it as a new child of a previously built node, such that the in-order reading of the nodes is always increasing. More precisely, when inserting the node σ_i in the tree built at the step $i-1$, one recursively compares σ_i to the label of the root, then inserts the node σ_i in its left subtree if σ_i is smaller, and in its right subtree otherwise.

The resulting binary tree (add any missing leaf) is called the **binary search tree** $T(\sigma)$ of σ.

Theorem 2.2.26. *The Tamari lattice is a lattice quotient of* $\mathrm{Weak}(\mathfrak{S}_n)$. *More precisely, the map that sends a permutation σ to its binary search tree is a surjective lattice morphism from the weak order on $\mathfrak{S}_n$ to the Tamari lattice.*

As we will see, this can be interpreted in the Coxeter group framework, and this inspired N. Reading to define the Cambrian lattices as a family of lattice quotients of the weak order [Rea06]. These constructions then were generalized for the symmetric group in a wider family of posets that we present quickly here. We will focus in greater detail on them in Section 7.3.4.

Permutree lattices are defined as the transitive closure of (increasing) rotations of some decorated trees called permutrees. These are special directed (unrooted) planar trees, whose nodes are labelled with integers from 1 to n. Each node has a type according to its number of incoming and outgoing arrows, and this gives a decoration n-tuple noted δ. One can define a rotation operation of permutrees that does not affect the decoration δ, and their transitive closure defines in fact lattices, naturally endowed with a surjective lattice morphism from the weak order on $\mathfrak{S}_n$, which makes them lattice quotients of $\mathrm{Weak}(\mathfrak{S}_n)$.

This family of posets contains in particular the weak order on $\mathfrak{S}_n$, the Tamari lattice, the type A Cambrian lattices, and the boolean lattice, which makes the family very rich and interesting to study.

Definition 2.2.27. A **permutree** of size n is a directed planar tree T with n internal vertices (called nodes) having each one or two incoming arrows and one or two outgoing arrows, together with a bijective labeling of its nodes $p\colon V \to [n]$ such that:

1. if a node v has two incoming arrows, then $p(v) > p(w)$ for all nodes w in the left subtree coming into v (called its left descendant), and $p(v) < p(w)$ for all nodes w in the right descendant of v.

2. if a node v has two outgoing arrows, then $p(v) > p(w)$ for all nodes w in the left subtree going out of v (called its left ancestor), and $p(v) < p(w)$ for all nodes w in the right ancestor of v.

For each node v of a permutree T, denote $\delta(T)_{p(v)} \in \{0,1\}^2$, where the first (resp. second) coordinate is the number of incoming arrows (resp. outgoing arrows) of v minus one. The

decoration of a permutree T is the n-tuple $\delta(T) = (\delta(T)_1, \dots, \delta(T)_n) \in \{\{0,1\}^2\}^n$, and we say that T is a δ-permutree. We denote $\mathcal{PT}(\delta)$ the set of δ-permutrees.

The permutrees are drawn in the plane, the labeling p of the nodes of a permutree can be seen as the order in which we see its nodes, from left to right. The first condition of the definition of permutrees then implies that when a node v has two incoming arrows, its left descendant lives entirely to the left of v, as all its nodes are labelled by numbers smaller than $p(v)$, and its right descendant lives entirely to the right of v, as all its nodes are labelled by numbers larger than $p(v)$. Note that The second condition is the same but for outgoing arrows.

Remark 2.2.28.

- The decoration of the first and the last nodes of a permutree do not matter, since conditions 1 and 2 are vacuous if $p(v) \in \{1, n\}$.

- Permutrees with all nodes decorated $(0,1)$ are in bijection with rooted binary trees with n nodes, the root node being the only one with an incoming arrow from a leaf, and each node is of degree 3. Nodes are labelled by the inorder labeling. The same is true if all nodes are decorated $(1,0)$, by up-down symmetry.

- Permutrees with all nodes decorated $(0,0)$ are in bijection with permutations of $[n]$, as each tree is then a directed path with n labelled nodes that can be read as the one-line notation of a permutation.

- Permutrees with all nodes decorated $(1,1)$ are in bijection with binary sequences of size $n-1$, i.e. elements of $\{0,1\}^{n-1}$. Indeed, in this case, each internal edge of the permutree connects two nodes with consecutive labels i and $i+1$ and is directed either from the smaller to the larger if $p(i) > p(i+1)$, or the other way around otherwise.

- Permutrees with all nodes decorated either $(0,1)$ or $(1,0)$ correspond to the "Cambrian trees" described in [CP17].

On these sets of trees, V. Pilaud and V. Pons defined a rotation operation of an edge, exchanging its orientation with a small rearrangement of the tree. This operation coincides with the rotation of the edges of a binary tree, but also with covering relations in the weak order and the boolean lattice in the cases where all nodes are decorated $(0,1)$, $(0,0)$ and $(1,1)$ respectively.

Definition 2.2.29. Let T be a permutree with an edge $i \to j$, from the node labelled i to the node labelled j, with $i < j$. If i has only one incoming arrow, let D be its descendant subtree, otherwise let D be its right descendant subtree. If j has only one outgoing arrow, let U be its ancestor subtree, otherwise let U be its left ancestor subtree.

Let T' be the permutree obtained by reversing the orientation of the edge $i \to j$, connecting the subtree D to j as its (left or only) descendant and the subtree U to i as its (right or only) ancestor. The transformation from T to T' is called the (increasing) **rotation** of the edge $i \to j$. Its reverse is called a decreasing rotation.

Remark that the decoration of T' is the same as the decoration of T. Thus, the set of permutrees $\mathcal{PT}(\delta)$ is closed under rotations.

Theorem 2.2.30. *For any decoration δ, the transitive closure of (increasing) rotations on the set $\mathcal{PT}(\delta)$ of δ-permutrees is a lattice, called the* **δ-permutree lattice**.

In fact, each such lattice is naturally endowed with a surjective lattice morphism from the weak order on $\mathfrak{S}_n$ and is thus a lattice quotient of $\mathrm{Weak}(\mathfrak{S}_n)$.

Two examples of permutree lattices can be seen in Figure 2.6.

Remark 2.2.31. The δ-permutree lattice coincides with:

- The Tamari lattice Tam_n when $\delta = (0,1)^n$,
- The weak order $\mathrm{Weak}(\mathfrak{S}_n)$ when $\delta = (0,0)^n$,
- The boolean lattice $\mathcal{B}_n$ when $\delta = (1,1)^n$,
- The Cambrian lattices $\mathrm{Camb}(\mathfrak{S}_n, c)$ when $\delta \in \{(0,1),(1,0)\}^n$.

V. Pilaud and V. Pons called the number of δ-permutrees the "factorial-Catalan number" $C(\delta)$ for it is a product of formulas interpolating between the Catalan and the factorial numbers. They also gave two recursive formulas to compute them, which we do not recall here. The number of their intervals, however, is not known in general, but since it already does not seem to behave well for the weak order on $\mathfrak{S}_n$, one does not expect a nice closed product formula.

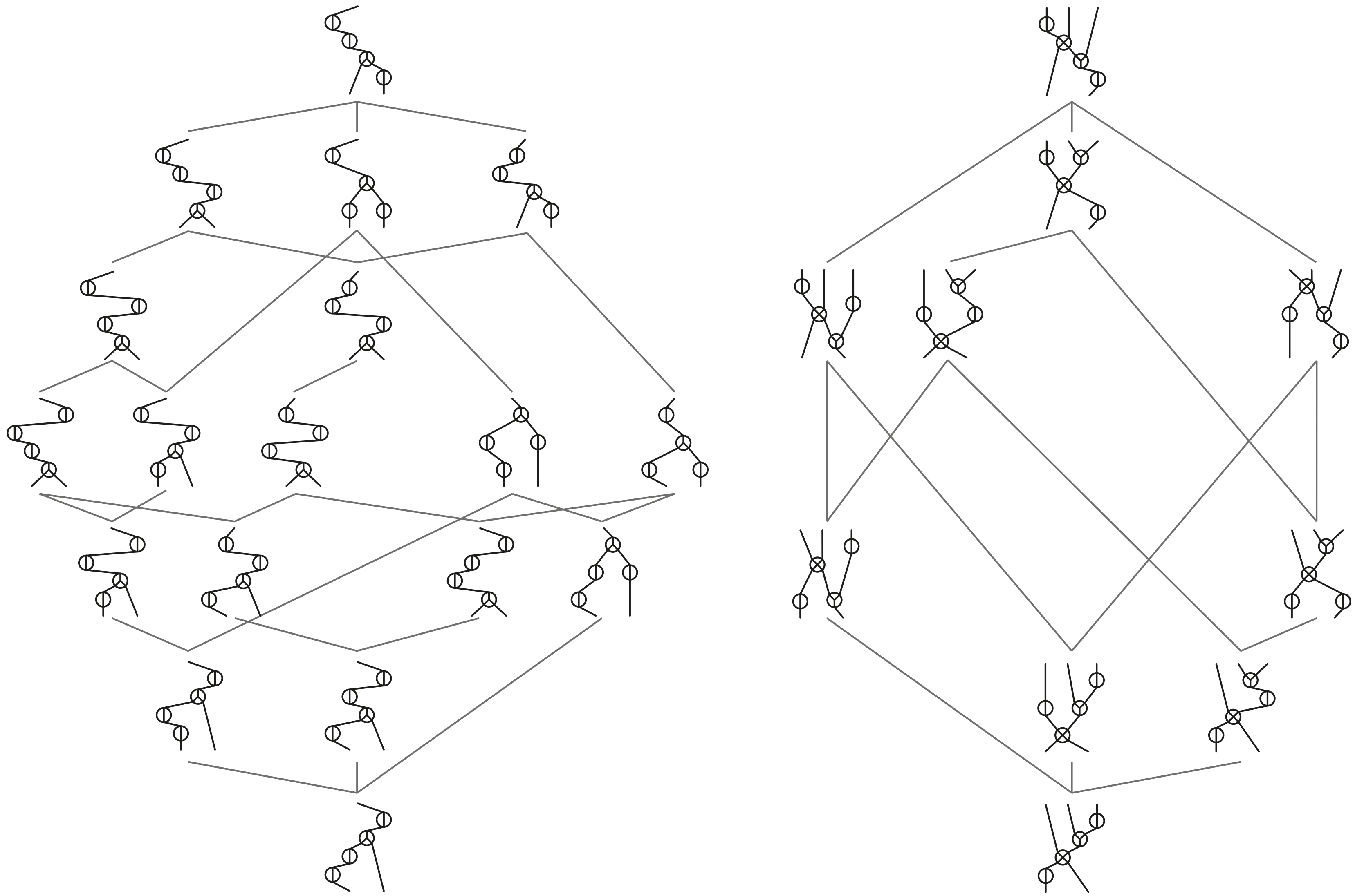


Figure 2.6: Two examples of permutree lattices for $\delta = ((0,0),(0,0),(1,0),(0,0))$ (left) and $\delta = ((0,0),(1,1),(0,1),(0,0))$ (right). The labeling p of the nodes correspond to the column in which the node appears, from left to right. The figures are borrowed from [PP18], with the agreement of the authors.

Chapter 3

Coxeter groups

The realm of Coxeter group theory is remarkably expansive, intertwining with diverse mathematical disciplines, thereby presenting a multitude of avenues for exploration — be they geometric, algebraic, or combinatorial in nature. This extensive framework was first introduced by H. S. M. Coxeter in [Cox34] as an abstract generalization of groups generated by orthogonal reflections in Euclidean spaces, and extends most of the core notions and ideas of the symmetric group presented earlier to this vast landscape. A general motto is that results proven on the symmetric group often have a counterpart in the realm of Coxeter groups, at least in the finite ones, which have been entirely classified in [Cox35], among which the type A groups corresponding to $\mathfrak{S}_n$. Saying it otherwise, the usual way to prove a result on Coxeter groups is to prove it for the symmetric group and then to wonder: "What about type B? D? (and E_8?)" Such proofs are however not really giving insights of *why* the result is *universally* true.

In this study, we will be mostly interested in combinatorial aspects of these groups, and especially of the finite ones, sometimes referred to as the spherical Coxeter groups. Many names in fact come from the more geometric aspects of Coxeter groups, that may be seen as reflection groups, as for instance the denomination of "spherical, affine and hyperbolic" Coxeter groups. In this chapter, we will recall the main results and notions needed later in the study, and present in particular the group presentation, root systems, the different poset structures, and the framework of subword complexes. For a more comprehensive study, we direct the reader's attention to [Hum90, BB05].

3.1 Definitions and tools

A Coxeter group can be seen as a group generated by involutions, called **(simple) reflections**, which satisfy some additional relations called **braid relations**. In the case of the symmetric group, the generators are the simple transpositions $(i, i+1)$. In what follows, we will use the notation $[s|t]^{m_{s,t}}$ as the product composed of $m_{s,t}$ alternating factors s and t and starting with the element s. For instance, the identity $[s|t]^3 = [t|s]^3$ reads $sts = tst$.

Definition 3.1.1. Let $\mathcal{S}$ be a finite set. For each $s, t \in \mathcal{S}$, choose $m_{s,t} \in \mathbb{N}_{>0} \cup \{\infty\}$, such that $m_{s,t} = 1$ if $s = t$ and $m_{s,t} = m_{t,s} \geq 2$ otherwise.

The associated **Coxeter group** W is the group generated by the elements of $\mathcal{S}$, subject to the relations $(st)^{m_{s,t}} = e$ for all $s, t \in \mathcal{S}$, where $(st)^\infty = e$ means no relation imposed between the generators s and t. In other words, $W = \langle \mathcal{S} \mid s^2 = e, [s|t]^{m_{s,t}} = [t|s]^{m_{s,t}} \rangle$.

The pair $(W, \mathcal{S})$ is called a **Coxeter system** and its **rank** is the cardinality of $\mathcal{S}$. The matrix $m = (m_{s,t})_{s,t\in\mathcal{S}}$ is called the **Coxeter matrix**. The elements of $\mathcal{S}$ are called **simple reflections** and the relations $[s|t]^{m_{s,t}} = [t|s]^{m_{s,t}}$ with $s \neq t$ are called **braid relations** and also **commutation relations** when $m_{s,t} = 2$.

The **Coxeter graph** associated to the pair $(W, \mathcal{S})$ is the graph on vertex set $\mathcal{S}$ with an edge between nodes s and t if $m_{s,t} \geq 3$, labelled with $m_{s,t}$ if $m_{s,t} \geq 4$. A Coxeter system is called **irreducible** if its Coxeter graph is connected.

The set of **reflections**, generally denoted $\mathcal{R}$, is the set of conjugates of simple reflections, i.e. $\mathcal{R} = \{s^w \mid s \in \mathcal{S}, w \in W\}$, where $s^w = w^{-1}sw$.

Remark that the product of two Coxeter groups (along with their Coxeter systems) corresponds to the (disjoint) union of their Coxeter graphs.

Theorem 3.1.2. *Every Coxeter system is isomorphic to the product of irreducible Coxeter systems (each corresponding to connected components of the Coxeter graph).*

Irreducible finite Coxeter systems are classified into types A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, and $I_2(m)$.

Example 3.1.3. The symmetric group $\mathfrak{S}_n$ is generated by the simple reflections $s_i = (i, i+1)$ for $i \in [n-1]$, and the braid relations are $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i \in [n-2]$ and $s_i s_j = s_j s_i$ for $|i-j| > 1$. Reflections correspond to all transpositions (i, j).

The Coxeter matrix of $A_4 = \mathfrak{S}_5$ is:

$$m = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{pmatrix}.$$

The presentation with generators and relations allows us to represent elements of a Coxeter group as words in the simple reflections, that we will call $\mathcal{S}$-words, or simply words. The empty word will be denoted ε. Following conventions of [STW18], we will use sans-serif font for letters and words, to distinguish with the equivalent "normal-fonted" generators and elements of the group. We will also later use **bold font** for elements in the Artin monoid. Each element admits many $\mathcal{S}$-words, but the shortest ones are those of interest and will be called reduced words.

Definition 3.1.4. Let $w \in W$. A **reduced ($\mathcal{S}$-)word** of w is a word $\mathsf{s}_1 \dots \mathsf{s}_k$ such that $w = s_1 \dots s_k$ as a product of elements of $\mathcal{S}$ and k is minimal in such expressions. The (Coxeter) **length** of w is the size $\ell_{\mathcal{S}}(w)$ of any reduced word of w.

Two words Q and Q' are called **commutation equivalent** if they can be obtained from each other by using a sequence of commutation relations, i.e. by a sequence of exchanges of consecutive commuting letters. We then write $\mathsf{Q} \equiv \mathsf{Q}'$.

A word u is **initial** in Q if it is a prefix of a word $\mathsf{Q}' \equiv \mathsf{Q}$, and **final** in Q if it is a suffix of some $\mathsf{Q}' \equiv \mathsf{Q}$. If u is a prefix (resp. a suffix) in Q, we write $\bar{\mathsf{u}}\mathsf{Q}$ (resp. $\mathsf{Q}\bar{\mathsf{u}}$) for the word obtained by removing the initial (resp. final) u from Q. We may extend this notation to initial (resp. final) u in Q, but $\bar{\mathsf{u}}\mathsf{Q}$ (resp. $\mathsf{Q}\bar{\mathsf{u}}$) is defined up to commutation equivalence.

Proposition 3.1.5. *For $s \in \mathcal{S}$ and $w \in W$, either $\ell_{\mathcal{S}}(sw) = \ell_{\mathcal{S}}(w) + 1$ or $\ell_{\mathcal{S}}(sw) = \ell_{\mathcal{S}}(w) - 1$. Similarly, $|\,\ell_{\mathcal{S}}(ws) - \ell_{\mathcal{S}}(w)| = 1$. We also have $\ell_{\mathcal{S}}(w) = \ell_{\mathcal{S}}(w^{-1})$.*

The following theorem, stated in [BB05, Theorem 3.3.1] is attributed to H. Matsumoto and J. Tits [Mat64, Tit69].

Theorem 3.1.6. *The set of reduced words of an element $w \in W$ is connected upon braid moves, i.e. using braid relations to rewrite reduced words.*

The use of words enables to define descent sets, and the notion of parabolic subgroups or quotients.

Definition 3.1.7. Let $w \in W$.

- The **left descent set** $\mathrm{des}_L(w)$ of w is the set $\mathrm{des}_L(w) = \{s \in \mathcal{S} \mid \ell_{\mathcal{S}}(sw) = \ell_{\mathcal{S}}(w) - 1\}$,

- The **right descent set** $\mathrm{des}_R(w)$ of w is the set $\mathrm{des}_R(w) = \{s \in \mathcal{S} \mid \ell_{\mathcal{S}}(ws) = \ell_{\mathcal{S}}(w) - 1\}$,
- The set of **covered reflections** of w is defined by $\mathrm{cov}_{\downarrow}(w) = \{wsw^{-1} \mid s \in \mathrm{des}_R(w)\}$,
- The set of **covering reflections** of w is defined by $\mathrm{cov}^{\uparrow}(w) = \{wsw^{-1} \mid s \in \mathcal{S} \setminus \mathrm{des}_R(w)\}$.

A simple reflection s is a left descent of an element w if and only if the corresponding letter s is initial in *some* reduced word of w, and similarly for right descents and final letters. On the symmetric group, these descent sets have a nice interpretation. The covered and covering reflections can also interpreted as inversions, as we shall see in the next section.

Example 3.1.8. In type A, for the symmetric group, when written with the one-line notation $(\sigma_1, \ldots, \sigma_n)$, multiplication by s_i on the right corresponds to exchange σ_i and σ_{i+1} whereas multiplying by s_i on the left corresponds to exchange the positions of i and $i+1$ in the one-line notation.

Thus, the right descent set of an element σ is the set of indices i such that $\sigma_i > \sigma_{i+1}$, and the left descent set of σ is the set of integers i such that $i+1$ appears before i in the one-line notation of σ.

The covered and covering reflections correspond to the inversions that we remove or add to the inversion set of an element σ when multiplying on the right by simple reflections. Covered (resp. covering) reflections can be seen as the set of pairs of consecutive entries $\sigma_i < \sigma_{i+1}$ (resp. $\sigma_{i+1} < \sigma_i$) in the one-line notation of σ.

Since a braid move does not affect the set of used letters, Theorem 3.1.6 allows defining the support $\mathrm{supp}(w)$ of an element $w \in W$, as well as parabolic subgroups and quotients.

Definition 3.1.9. The **support** of an element $w \in W$ is the set of letters used in some (and in fact each) reduced $\mathcal{S}$-word for w.

For $J \subset \mathcal{S}$, the **standard parabolic subgroup** W_J is the subgroup of W generated by J, i.e. the subgroup of elements whose support is a subset of J. Following [Rea07b], for $s \in \mathcal{S}$, we will write $W_{\langle s \rangle}$ for the maximal standard parabolic subgroup generated by $\langle s \rangle = \mathcal{S} \setminus \{s\}$.

The **parabolic quotient** W^J is the quotient set $W_J \backslash W = \{w \in W \mid \mathrm{des}_L(w) \cap J = \emptyset\}$.

A parabolic subgroup in general is defined as the conjugate of some standard parabolic subgroup.

Example 3.1.10. In type A again, the standard parabolic subgroup $\mathfrak{S}_{n\langle s_i \rangle}$ corresponds to the subset of permutations such that $\sigma_k \leq i$ if (and only if) $k \leq i$. Multiplying on the left by an element of $\mathfrak{S}_{n\langle s_i \rangle}$ will shuffle the entries $k \leq i$ together and the entries $k \geq i+1$ together in the one-line notation of σ. Thus, cosets of the corresponding parabolic quotient correspond to a fixed subset of positions for the integer of $[i]$ in the one-line notation, that is to say, two elements σ and τ are in the same coset in $\mathfrak{S}_{n\langle s_i \rangle} \backslash \mathfrak{S}_n$ if and only if the sets $\sigma^{-1}([i])$ and $\tau^{-1}([i])$ are equal.

3.2 Root systems and inversions

A crucial tool for what will follow is the notion of "inversions". It was first defined for the symmetric group, but was generalized to all Coxeter groups. It is strongly related to the Coxeter length, since the size of the inversion set is precisely equal to the length. Inversions of an element can be formulated using only the framework of $\mathcal{S}$-words and inversion sequences, and could have been defined from the very beginning, following for instance [BB05]. It is however useful to first introduce the notion of root systems, as a very similar definition of inversion sets can be stated using this more geometric perspective.

In all generality, it is possible to associate to every Coxeter system a "geometric representation", that is to say to embed it faithfully as a subgroup of $\mathrm{GL}(V)$ for some real vector space V. Better, we may ask that V has a basis indexed by $\mathcal{S}$ and that every generator of $\mathcal{S}$ is sent to an "orthogonal" reflection. By reflection, we mean a linear transformation of V that fixes some hyperplane and sends some nonzero vector to its negative, and by orthogonal, we mean that it preserves some well-chosen bilinear form on V. All of this is discussed in [Hum90, Section 5.3] in great detail. However, for our purpose, we will only need the case of finite Coxeter groups, for which everything is much simpler: V is a Euclidean space and each generator is sent to an orthogonal reflection. We will thus present only this case, and benefit from the fact that finite Coxeter groups are exactly the same as finite real reflection groups.

Definition 3.2.1. Let V be a Euclidean vector space.

- For $\alpha \in V \setminus \{0\}$, the **reflection** associated to α is the orthogonal reflection along the hyperplane orthogonal to α, and is denoted r_α.
- A **finite reflection group** is a finite group generated by a set of orthogonal reflections.
- A **root system** in V is a finite set $\Phi \subset V \setminus \{0\}$ such that:
    1. $\forall \alpha \in \Phi, \mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$,
    2. $\forall \alpha \in \Phi, r_\alpha(\Phi) = \Phi$.
- One can associate to a root system Φ the finite reflection group generated by the reflections r_α, for $\alpha \in \Phi$.

Theorem 3.2.2 ([Cox34, Cox35])**.** *Finite Euclidean reflection groups correspond exactly to finite Coxeter groups.*

In other words, to each finite Coxeter system $(W, \mathcal{S})$, one can associate an orthogonal reflection to each generator $s \in \mathcal{S}$, such that the group generated by these reflections is isomorphic to W, and conversely, any finite group that is generated by orthogonal reflections in a Euclidean space can be seen as a Coxeter group.

In fact, if Φ is a root system, then any hyperplane that does not contain any element of Φ cuts the set Φ into two parts (of equal cardinality, namely the size of the set of reflections $\mathcal{R}$). Moreover, when choosing one half as the set of "positive" roots, and taking the cone that they generate, then the extremal elements of the cone (those that are not positive linear combinations of others) form a linearly independent family Δ of "simple roots", and furthermore verifies that every element of Φ is a linear combination of simple roots with all nonnegative or all nonpositive coefficients.

Proposition 3.2.3. *Let W be the finite Coxeter group associated to the root system Φ, and $\mathcal{S}$ and $\mathcal{R}$ the sets of its simple reflections and reflections respectively.*

- *The set Φ can be written as $\Phi = \Phi^+ \sqcup \Phi^-$, where $\Phi^- = -\Phi^+$, and such that there exists a hyperplane H which separates Φ^+ and Φ^-.*
- *Once fixed such a set Φ^+, its elements are called* **positive roots** *(and their opposite are negative roots). They are in bijection with the set of reflections $\mathcal{R}$, where each reflection r sends exactly one root α_r to its negative, or reciprocally each positive root α is sent to the reflection r_α. The set of simple reflections $\mathcal{S}$ is in bijection with the set of* **simple roots** *Δ, which is linearly independent.*
- *Each root of Φ^+ is a linear combination of simple roots with nonnegative coefficients.*
- *The action of an element $w \in W$ on the set of roots corresponds, up to a sign, to the conjugation by w on the set of reflections $\mathcal{R}$, that is to say $w(\alpha_r) = \pm\alpha_{wrw^{-1}}$.*

Example 3.2.4. The most classical example for type A is to take $V = \mathrm{Vect}(\{e_i \mid 1 \le i \le n\})$, $\Delta = \{e_i - e_{i+1} \mid 1 \le i < n\}$, $\Phi^+ = \{e_i - e_j \mid 1 \le i < j \le n\}$, and $\Phi = \{e_i - e_j \mid 1 \le i \ne j \le n\}$.

Then the associate (finite) reflection group is precisely the symmetric group $\mathfrak{S}_n$, whose action can be set as $\sigma(e_i - e_j) = e_{\sigma(i)} - e_{\sigma(j)}$.

In that case indeed, reflections are in bijection with transpositions, under the transformation $r_{(i,j)} = e_i - e_j$ for $i < j$. The action of a permutation σ on a root $r_{(i,j)}$ rewrites as $\sigma(r_{(i,j)}) = \pm r_{(\sigma(i),\sigma(j))} = \pm r_{\sigma(i,j)\sigma^{-1}}$, that is to say the conjugation on reflections (with an additional sign).

From now on, we consider a Coxeter group with a root system attached to it. This additional sign that is not captured by the action on reflections will in fact be really important, and this leads to the notion of *inversions*.

Definition 3.2.5. Let $\mathsf{w} = \mathsf{s}_1\mathsf{s}_2\dots\mathsf{s}_p \in \mathcal{S}^*$. The **inversion sequence** of the word w is the sequence $\mathsf{inv}(\mathsf{w}) = (\alpha_{s_1}, s_1(\alpha_{s_2}), \dots, s_1\cdots s_{p-1}(\alpha_{s_p}))$ of roots.

The **inversion set** of an element $w \in W$ is the set $w(\Phi^-) \cap \Phi^+$.

Using this notion, we have a few useful results.

Proposition 3.2.6.

- *The inversion sequence of a word* w *contains no negative roots if and only if* w *is a reduced word. In this case, this set of roots is exactly the inversion set* $\mathrm{inv}(w)$ *of the corresponding element* w*, and we have* $\ell_{\mathcal{S}}(w) = |\,\mathrm{inv}(w)|$.
- *Any* $w \in W$ *is uniquely determined by its set of inversions* $\mathrm{inv}(w)$*. Such sets of inversions are exactly all "biclosed" subsets of* Φ^+*. These are subsets* $A \subseteq \Phi^+$ *such that if* $\alpha, \beta, \gamma \in \Phi^+$ *verify* $gg = a\alpha + b\beta$ *with* $a, b \ge 0$*, then* $\alpha, \beta \in A \Rightarrow \gamma \in A$ *and* $\gamma \in A \Rightarrow (\alpha \in A$ *or* $\beta \in A)$.
- *For* $w \in W$*, we have the equalities* $\mathrm{des}_R(w) = \mathrm{inv}(w) \cap \Delta$, $\mathrm{des}_L(w) = \mathrm{inv}(w^{-1}) \cap \Delta$*, and* $\mathrm{cov}_\downarrow(w) = -w(\Delta) \cap \Phi^+ = \mathrm{inv}(w) \cap w^{-1}(-\Delta)$*, that is to say that covered reflections of* w *are inversions of* w *that are mapped to simple roots under* w^{-1}.

For the purpose of this study, we define a **colored** version of roots, by taking the set of positive roots together with a nonnegative integer called the color. The idea is that negative roots will correspond to color 1 roots, and we have a conjugation action of the set of words $\mathcal{S}^*$ on colored roots that extends the action of the group W, where the color keeps track of the number of times a root has been sent to its negative.

Definition 3.2.7. Let W be a Coxeter group with positive roots Φ^+. The set of **colored roots** is the set $\Phi^{(\infty)} = \Phi^+ \times \mathbb{N} = \{\beta^{(k)} \mid \beta \in \Phi^+, k \ge 0\}$. We write $|\beta^{(k)}| = \beta$ the uncolored corresponding positive root.

The set of simple reflections $\mathcal{S}$ acts on $\Phi^{(\infty)}$ by:

$$s(\beta^{(k)}) = \begin{cases} \left[s(\beta)\right]^{(k)} & \text{if } \beta \ne \alpha_s \\ \beta^{(k+1)} & \text{if } \beta = \alpha_s \end{cases}. \tag{3.1}$$

Then, the set of words $\mathcal{S}^*$ acts on $\Phi^{(\infty)}$ by extending the action of $\mathcal{S}$. The **colored inversion sequence** of a word $\mathsf{w} = \mathsf{s}_1\mathsf{s}_2\dots\mathsf{s}_p$, still denoted $\mathsf{inv}(\mathsf{w})$ is defined by

$$\mathsf{inv}(\mathsf{w}) = \left(\alpha_{\mathsf{s}_1}^{(0)}, s_1(\alpha_{\mathsf{s}_2}^{(0)}), \dots, s_1\cdots s_{p-1}(\alpha_{\mathsf{s}_p}^{(0)})\right). \tag{3.2}$$

We also define the **colored reflection sequence** $\mathsf{inv}_{\mathcal{R}}(\mathsf{w})$ of w through the bijection between Φ^+ and $\mathcal{R}$

$$\mathsf{inv}_{\mathcal{R}}(\mathsf{w}) = \left(r_1^{(m_1)}, \dots, r_p^{(m_p)}\right), \tag{3.3}$$

where $r_i = r_{\beta_i}$ if $\mathsf{inv}(\mathsf{w}) = \left(\beta_1^{(m_1)}, \dots, \beta_p^{(m_p)}\right)$.

Example 3.2.8. In type A_2, with generators $\mathcal{S} = \{s, t\}$ and $u = sts = tst$, let us consider the word $\mathsf{w} = \mathsf{sttts}$.

- Its (uncolored) inversion sequence is $(\alpha_s, \alpha_u, -\alpha_u, \alpha_u, \alpha_t)$.
- Its colored inversion sequence is $(\alpha_s^{(0)}, \alpha_u^{(0)}, \alpha_u^{(1)}, \alpha_u^{(2)}, \alpha_t^{(0)})$.
- Its colored reflection sequence is $(s^{(0)}, u^{(0)}, u^{(1)}, u^{(2)}, t^{(0)})$.

3.3 Partial orders on Coxeter groups

In the scope of this study, our attention turns to partial order structures in relation with the Tamari lattice. Notably, this lattice is closely related to the weak order on the symmetric group, which in turn falls under the umbrella of Coxeter groups. We will see in fact in the upcoming section that the Tamari lattice is a type A Cambrian lattice. For this purpose, we now introduce the main poset structures on Coxeter groups that will be used in the study.

3.3.1 The Bruhat order

The Bruhat order (sometimes referred to as "strong" Bruhat order) is a partial order on the elements of a Coxeter group, that can be defined as the transitive closure of some relation. It is a graded poset but not a lattice. It will not be central in this study, but it still appears naturally and is worth mentioning.

Definition 3.3.1. Let $w \in W$ and $t \in \mathcal{R}$, if $\ell_{\mathcal{S}}(wt) > \ell_{\mathcal{S}}(w)$, we write $w \xrightarrow{t} wt$ and $u \to v$ for $u, v \in W$ if there exists such $t \in \mathcal{R}$ such that $u \xrightarrow{t} v$.

The **Bruhat order** $(W, \leq_B)$ is the transitive closure of the relation $\to$.

Not all relations $w \xrightarrow{t} wt$ will be covering relations, but it is useful to consider the directed graph on vertices W with all arrows $w \xrightarrow{t} wt$, referred to as the **Bruhat graph**. This can be noticed on the example on the symmetric group $\mathfrak{S}_3$ in Figure 3.1.

Proposition 3.3.2. *The covering relations in the Bruhat order are exactly the $w \xrightarrow{t} wt$ where $\ell_{\mathcal{S}}(wt) = \ell_{\mathcal{S}}(w) + 1$. The resulting poset is graded by the length $\ell_{\mathcal{S}}$. It is not a lattice.*

When W is finite, the group contains a unique longest element $w_\circ$, and the poset is bounded.

In fact, as the notation suggests, the Bruhat order is defined as "u is smaller that v if u is a subword of v". More precisely, if $\mathsf{w} = \mathsf{s}_1 \dots \mathsf{s}_p \in \mathcal{S}^*$, a **subword** u of w is a word of the form $\mathsf{u} = \mathsf{s}_{i_1} \dots \mathsf{s}_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq p$, and we denote this by $\mathsf{u} \sqsubseteq \mathsf{w}$. Note that this notion of subword does *not* require the letters to be consecutive, contrary to *factors* in the context of ν-paths, hence a different notation. This notion can be extended to elements by taking reduced words, and this is made precise in the following theorem.

Theorem 3.3.3 ([BB05, Corollary 2.2.3])**.** *For $u, w \in W$, the three statements are equivalent:*

1. *$u \leq_B w$ in the Bruhat order,*
2. *There exist reduced words u and w for u and w respectively, such that $\mathsf{u} \sqsubseteq \mathsf{w}$,*
3. *For any reduced word w of w, there exists a reduced word u of u such that $\mathsf{u} \sqsubseteq \mathsf{w}$.*

Proposition 3.3.4.

- *The inverse map $w \mapsto w^{-1}$ is a poset automorphism of the Bruhat order.*

- *When W is finite, the maps $w \mapsto w_\circ w$ and $w \mapsto w w_\circ$ are anti-automorphisms of the Bruhat order, and the map $\psi\colon w \mapsto w_\circ w w_\circ$ is an automorphism of the Bruhat order.*

Remark 3.3.5. The longest element in this case satisfies in particular that ${w_\circ}^2 = e$, where e is the neutral element of W, and $\mathrm{inv}(w) = \mathcal{R}$, so that $\ell_{\mathcal{S}}(w) = |\mathcal{R}|$.

Moreover, the set of simple reflections $\mathcal{S}$ is stable under the conjugation by $w_\circ$, which defines a bijection ψ on the group and on $\mathcal{S}$-words.

Lastly, a handy notion for (upcoming) subword complexes is the Demazure product of a word Q, which is the longest element w that admits a reduced expression as a subword of Q.

Definition 3.3.6. Let $\mathsf{Q} \in \mathcal{S}^*$, its **Demazure product** $\mathrm{Dem}(\mathsf{Q}) \in W$ is recursively defined as follows:

- $\mathrm{Dem}(\varepsilon) = e$,
- $\mathrm{Dem}(\mathsf{Qs}) = \mathrm{Dem}(\mathsf{Q})s$ if $\mathrm{Dem}(\mathsf{Q})(\alpha_s) \in \Phi^+$,
- $\mathrm{Dem}(\mathsf{Qs}) = \mathrm{Dem}(\mathsf{Q})$ if $\mathrm{Dem}(\mathsf{Q})(\alpha_s) \in \Phi^-$.

In other words, the Demazure product of a word Q is obtained by scanning the word from left to right and adding letters that increase the length of the resulting element and ignoring the others. This kind of "greedy" procedure turns out to appear often in our work.

3.3.2 The weak order

The (right) weak order is also a partial order on the elements of a Coxeter group, which can be defined in several ways. The easiest to formulate is to use inclusion of inversion sets, but it is possible to build it as the transitive closure of some relation, which makes it naturally extended by the Bruhat order, and also to state it in terms of words as "being a prefix". It is a graded lattice and it generalizes the weak order on the symmetric group defined in Definition 2.2.21. There exists a left weak order, which we will not mention much in this study, which is strongly related to the right one, can be defined in a very similar way, but it is in fact isomorphic to the right weak order via the inverse map. The names "left" and "right" come from the fact that we multiply by elements of the group on the left and on the right, respectively.

Recalling from Proposition 3.1.5 that multiplying an element $w \in W$ by a simple reflection s either increases or decreases the length by one, one can thus build an order on W as the transitive closure of relations of the form $w \xrightarrow{s} ws$, such that the length $\ell_{\mathcal{S}}$ is increasing along arrows. Since $\mathcal{S} \subset \mathcal{R}$, the Bruhat order naturally extends the resulting poset since it is the closure of *more* relations.

Definition 3.3.7. The (right) **weak order** $\mathrm{Weak}(W)$ is the transitive closure of all relations $w \xrightarrow{s} ws$, where $w \in W$, $s \in \mathcal{S}$, and $\ell_{\mathcal{S}}(w) < \ell_{\mathcal{S}}(ws)$.

An example for $W = \mathfrak{S}_3$ can be found in Figure 3.1.

Remark 3.3.8.

- The (right) weak order is extended by the Bruhat order.
- The weak order is a poset graded by the length $\ell_{\mathcal{S}}$, moreover bounded when W is finite.

Theorem 3.3.9. *For $u, w \in W$, the four statements are equivalent:*

1. *$u \leq w$ in the (right) weak order,*
2. *There exist reduced words u and w for u and w respectively, such that u is a prefix of w,*

3. $\mathrm{inv}(u) \subset \mathrm{inv}(w)$*,*

4. There exists $v \in W$ *such that* $uv = w$ *and* $\ell_{\mathcal{S}}(u) + \ell_{\mathcal{S}}(v) = \ell_{\mathcal{S}}(w)$*.*

Example 3.3.10. The type A Coxeter group $\mathfrak{S}_n$ produces precisely the weak order on permutations, as defined in Definition 2.2.21, and the inversion set of a permutation also corresponds to the one defined in Definition 2.2.22. Indeed, for a permutation $\sigma = (\sigma_1, \dots, \sigma_n)$, the condition $\sigma_i > \sigma_{i+1}$ is equivalent to have $\sigma(r_{(i,i+1)}) = \sigma(e_i - e_{i+1}) = e_{\sigma_i} - e_{\sigma_{i+1}} = -r_{(\sigma_{i+1},\sigma_i)} \in \Phi^-$.

Remark 3.3.11. The left weak order can be defined symmetrically by $u \leq_L w$ if and only if one of these equivalent conditions holds:

- $u^{-1} \leq w^{-1}$ in the right weak order,
- There exist reduced words u and w for u and w respectively, such that u is a suffix of w,
- $\mathrm{inv}(u^{-1}) \subset \mathrm{inv}(w^{-1})$,
- There exists $v \in W$ such that $vu = w$ and $\ell_{\mathcal{S}}(u) + \ell_{\mathcal{S}}(v) = \ell_{\mathcal{S}}(w)$.

The left weak order also is extended by the Bruhat order.

We can also inspect the behavior of the weak orders under the involutions of W as in Proposition 3.3.4 for the Bruhat order.

Proposition 3.3.12.

- *The inverse map* $w \mapsto w^{-1}$ *is a poset isomorphism between the left and the right weak orders.*
- *When* W *is finite, the maps* $w \mapsto w_\circ w$ *and* $w \mapsto w w_\circ$ *are anti-automorphisms of the weak orders, and the map* $\psi\colon w \mapsto w_\circ w w_\circ$ *is an automorphism of them.*

Theorem 3.3.13 ([BB05, Theorem 3.2.1])**.** *The weak order on* W *is a meet-semilattice. When* W *is finite, it is a lattice.*

Remark 3.3.14. Though the weak order is given by inclusion of inversion sets, the inversion set of $u \wedge v$ is *not* in general equal to $\mathrm{inv}\,(u) \cap \mathrm{inv}\,(v)$, nor does their join (when W is finite) have $\mathrm{inv}(u) \cup \mathrm{inv}(v)$ as inversion set. In fact, the correct inversion set of $u \wedge v$ is the biggest subset included in $\mathrm{inv}\,(u) \cap \mathrm{inv}\,(v)$ that is an inversion set, and the inversion set of $u \vee v$ is the smallest subset containing $\mathrm{inv}\,(u) \cup \mathrm{inv}\,(v)$ that is the inversion set of some element.

Remark 3.3.15. For $J \subset \mathcal{S}$, W_J inherits a structure of Coxeter group, and the weak order on W_J corresponds to the restriction of the weak order on W to W_J. In particular, for W finite, W_J contains a unique longest element that we note $w_\circ(J)$.

Proposition 3.3.16. *Every element* $w \in W$ *admits a unique* parabolic decomposition $w = w_J w^J$*, where* $w_J \in W_J$ *and* $w^J \in W^J$*. It is given by* $w_J = w \wedge w_\circ(J)$*.*

This w_J can also be constructed by writing $w = uv$ starting with $u = e$ and $v = w$, then looking for any simple root $s \in J$ that is a left descent of v, and replacing (u, v) by (us, sv). This process stops when v has no left descent in J (which means $v \in W^J$), and the resulting u is by construction in W_J. It does not in fact depend on the potential choices made in the construction, and this is again, in a way, a greedy procedure as for constructing the Demazure product of an element.

3.3.3 The absolute order

A third crucial order on Coxeter groups is in a way similar to both the Bruhat order and the weak order. The idea is to mimic the weak order, but using words in the alphabet $\mathcal{R}$ instead of $\mathcal{S}$, using a new notion of length called absolute length. Using $\mathcal{R}$ as a generating set instead of $\mathcal{S}$ is sometimes referred to as the "dual" point of view, and has been explored for instance to give a "dual" presentation for the Coxeter group (or the Artin group and monoid), see for instance [Bes03]. It is worth mentioning that it is absolutely not a duality in the sense of a "back-and-forth" process that would give a "bidual" kind of objects. In this new point of view, on finite Coxeter groups, the role of the longest element is now played by several elements called Coxeter elements appearing as maximal elements of the absolute order.

The resulting poset also resembles the Bruhat order since it does not privilege the right or the left side over the other, and can be rephrased as being an $\mathcal{R}$-subword. Moreover, its Hasse diagram is the Bruhat graph with some arrows flipped, so that all arrows are increasing according to the absolute length instead of the Coxeter length. It is a graded meet-semilattice.

Definition 3.3.17. Let $w \in W$, a **reduced $\mathcal{R}$-word** of w is a word $\mathsf{t}_1 \dots \mathsf{s}_k$ such that $w = t_1 \dots t_k$ as a product of elements of $\mathcal{R}$ and k is minimal in such expressions. The **absolute length** of w is the size $\ell_{\mathcal{R}}(w)$ of any reduced $\mathcal{R}$-word of w.

The **absolute order** $\mathrm{Abs}(W) = (W, \leq_{\mathcal{R}})$ is the poset on W defined by $u \leq_{\mathcal{R}} w$ if there exists $v \in W$ such that $uv = w$ and $\ell_{\mathcal{R}}(u) + \ell_{\mathcal{R}}(v) = \ell_{\mathcal{R}}(w)$.

Proposition 3.3.18.

- *Since the set $\mathcal{R}$ is closed under conjugation, $\ell_{\mathcal{R}}(wuw^{-1}) = \ell_{\mathcal{R}}(u)$ for all $u, w \in W$. Thus, if $uv = w$ with $\ell_{\mathcal{R}}(u)+\ell_{\mathcal{R}}(v) = \ell_{\mathcal{R}}(w)$, then we also have $v'u = w$ with $\ell_{\mathcal{R}}(v')+\ell_{\mathcal{R}}(u) = \ell_{\mathcal{R}}(w)$, for $v' = uvu^{-1}$.*

- *For any $w \in W$ and $t \in \mathcal{R}$, we have $\ell_{\mathcal{R}}(wt) = \ell_{\mathcal{R}}(w) \pm 1$. Thus, either $w \lessdot_{\mathcal{R}} wt$ or $w \lessdot_{\mathcal{R}} wt$, and the absolute order is graded by the absolute length.*

- *The Hasse diagram of the absolute order is the Bruhat graph with some flipped arrows (see Figure 3.1).*

- *The absolute order is a meet-semilattice.*

Since the conjugation by any element w permutes the set $\mathcal{R}$, this induces a poset automorphism of the absolute order. However, the poset does not behave well under multiplication by w_0 (on either side).

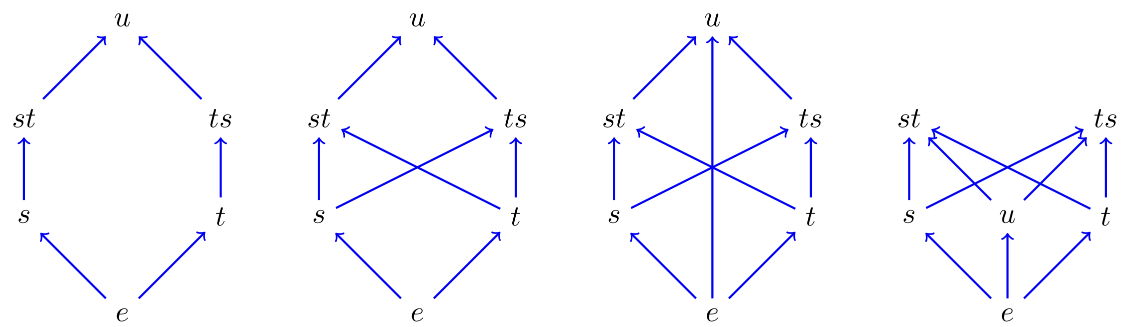
Proposition 3.3.19. *The inverse map and the conjugation by any element w are poset automorphisms of the absolute order.*

Example 3.3.20. For type A, the absolute length of a permutation $\sigma \in \mathfrak{S}_n$ is equal to n minus the number of cycles of σ. A reduced $\mathcal{R}$-word of σ is obtained by writing a cycle decomposition of σ and then splitting the cycles into transpositions.

Maximal elements in $\mathrm{Abs}(\mathfrak{S}_n)$ are long cycles, as for instance $c = (1, 2, \dots, n)$ (written as a cycle and not in the one-line notation). Since long cycles are conjugate to each other, all intervals $[e, c]$ are isomorphic to each other.

Proposition 3.3.21 ([Bra01, Theorem 3.7])**.** *The restriction of* $\mathrm{Abs}(\mathfrak{S}_n)$ *to any interval $[e, c]$ with c a long cycle as above is isomorphic to the noncrossing partition lattice* NCL_n *defined in Definition 2.2.15.*

Remark 3.3.22. In fact, Proposition 3.3.21 is a more general fact: in any finite Coxeter group, we can define Coxeter elements, usually denoted c, all of which are conjugated to each other, and thus all intervals $[e, c]$ in the absolute order are isomorphic to each other. The resulting poset can be called the **noncrossing partition lattice** $\mathrm{NCL}(W)$ (or $\mathrm{NCL}(W, c)$ if we want to specify the Coxeter element c) of type W.



Weak order Weak($\mathfrak{S}_3$) Bruhat order $(\mathfrak{S}_3, \leq_B)$ Bruhat graph of $\mathfrak{S}_3$ Absolute order Abs($\mathfrak{S}_3$)

Figure 3.1: The weak order, the Bruhat order, and the absolute order on $\mathfrak{S}_3$. The Bruhat graph contains all arrows of the Hasse diagram of the Bruhat order, with an extra edge $e \to u = sts = tst$. The Hasse diagram of the absolute order is the Bruhat graph with two arrows flipped.

3.4 The Cambrian lattices

The Cambrian lattices have already been mentioned as being a generalization of the Tamari lattice in the context of Coxeter groups. They were defined by N. Reading as subposets of the weak order on a Coxeter group. More precisely, each orientation of the edges of the Coxeter graph, or equivalently each choice of a standard Coxeter element, gives rise to a lattice congruence on the weak order, and thus to a lattice quotient called a Cambrian lattice. They can be defined in many equivalent ways, and we will present here the most useful ones for our purpose.

3.4.1 Cambrian lattices as subposets of the weak order

Given a standard Coxeter element c, the first definition of Cambrian lattices is to restrict the weak order to a certain subset of elements, that we call c-sortable. In what follows, not only will we choose a Coxeter element, but we will also choose a reduced expression for it, or equivalently a total order on the set of simple reflections. All the constructions that we will present will in fact *not* depend on the choice of such a reduced $\mathcal{S}$-word.

Definition 3.4.1. Let $(W, \mathcal{S})$ be a Coxeter system. A (standard) **Coxeter element** c is a product of all simple reflections, in any order. A **Coxeter word** c is a reduced $\mathcal{S}$-word for c, or equivalently a total order on $\mathcal{S}$, given by the order in which the elements of $\mathcal{S}$ appear in c.

Remark 3.4.2. All Coxeter elements are conjugated to each other, and their common multiplicative order is called the **Coxeter number** h. The term Coxeter elements is sometimes used to encompass all conjugates of standard Coxeter elements.

The choice of a Coxeter element $c = s_1 \dots s_n$ is equivalent to the choice of an orientation of the Coxeter graph, where an edge between s_i and s_j is oriented from s_i to s_j if the former appears before the latter in any reduced word for c.

This also gives a Coxeter word $c\big|_J$ (or element) in any standard parabolic subgroup W_J, by taking only the letters in J.

Indeed, as each simple reflection appears only once in any reduced word for c, only commutations moves can be used to rewrite a reduced word for c into another one, and thanks to Theorem 3.1.6, we obtain all such Coxeter words associated to c.

This choice of a Coxeter word c gives a privileged reduced word for each element $w \in W$, as the "first" subword w of the "infinite word" $\mathsf{c}^\infty = \mathsf{s}_1\mathsf{s}_2 \dots \mathsf{s}_n|\mathsf{s}_1\mathsf{s}_2 \dots \mathsf{s}_n|\mathsf{s}_1 \dots$ which is a reduced word for w. The vertical bar $|$, called a divider, is here used to separate copies of c (but is ignored when doing the product of elements).

	s	t	s	t	s	t	$\mathcal{C}_c(w)$
e	-	-					$\alpha_s^{(0)}, \alpha_t^{(0)}$
s	x	-	-				$\alpha_u^{(0)}, \alpha_s^{(1)}$
st	x	x	-	-			$\alpha_t^{(0)}, \alpha_u^{(1)}$
t	-	x		-			$\alpha_s^{(0)}, \alpha_t^{(1)}$
sts	x	x	x	-	-		$\alpha_s^{(1)}, \alpha_t^{(1)}$

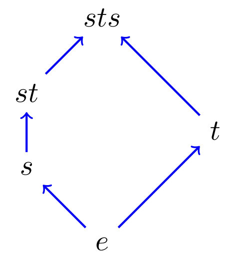


Figure 3.2: The sortable version of the Cambrian lattice $\mathrm{Camb}_{\mathrm{Sort}}(\mathfrak{S}_3, \mathsf{st})$. Each element is represented by its c-sorting word. The selected letters correspond to symbols x, and the first time a given letter is missing is denoted by -. The skip set—corresponding to skipped letters (-) conjugated by selected letters (x) preceding them—is also given.

Definition 3.4.3. Let c be a Coxeter word in W. Given an element $w \in W$, its **c-sorting word** $\mathsf{w}(\mathsf{c})$ is the lexicographically smallest (as a subset of indices) subword of c^∞ that is a reduced word for w. In other words, this is the leftmost subword of c^∞ that is a reduced word for w.

The c-sorting word can be interpreted as a sequence $(I_1, \dots, I_k)$ of subsets of $\mathcal{S}$, separated by the divider signs, where $\mathsf{c}\big|_{I_1}$ is the subword before the first divider, and so on.

An element $w \in W$ whose c-sorting word is $\mathsf{w}(\mathsf{c}) = \mathsf{c}\big|_{I_1} \dots \mathsf{c}\big|_{I_k}$ is said to be **c-sortable** when $I_1 \supseteq I_2 \supseteq \dots \supseteq I_k$. The set of c-sortable elements is denoted by $\mathrm{Sort}(W, \mathsf{c})$.

The (sortable version of the) **Cambrian lattice** $\mathrm{Camb}_{\mathrm{Sort}}(W, \mathsf{c})$ is the restriction of the (right) weak order to the set of c-sortable elements.

Example 3.4.4. In type A_2, with generators $\mathcal{S} = \{s, t\}$ and Coxeter word $\mathsf{c} = \mathsf{st}$, the element ts is not c-sortable since its c-sorting word is $\mathsf{t}|\mathsf{s}$, and so $I_1 = \{\mathsf{t}\} \supsetneq \{\mathsf{s}\} = I_2$. All five other elements are c-sortable. The poset is drawn corresponding Cambrian lattice $\mathrm{Camb}(\mathfrak{S}_3, \mathsf{st})$ is drawn.

Remark 3.4.5. The definition of the c-sorting word of an element $w \in W$ gives a greedy procedure to compute it. One starts with $\mathsf{u} = \varepsilon$ and $v = w$ and scans the word c^∞ from left to right. For each letter s, if it is a left descent of v, then (u, v) is replaced by (us, sv). This process terminates at $(\mathsf{w}(\mathsf{c}), e)$.

Furthermore, for $w \in W$, if c starts with the letter s, then we have the equivalence:

$$w \in \mathrm{Sort}(W, \mathsf{c}) \Leftrightarrow \begin{cases} sw \in \mathrm{Sort}(W, \bar{\mathsf{s}}\mathsf{cs}) & \text{if } s \in \mathrm{des}_L(w) \\ w \in \mathrm{Sort}(W_{\langle s \rangle}, \bar{\mathsf{s}}\mathsf{c}) & \text{if } s \notin \mathrm{des}_L(w) \end{cases}. \tag{3.4}$$

This kind of defining recurrence occurs repeatedly throughout the study of Cambrian lattices, as a fundamental aspect of their structure. We will refer to them as **c-Cambrian recurrences**, following [STW18].

Remark 3.4.6. Two different Coxeter words c and c' for the same Coxeter element c give rise to isomorphic lattices, for the c- and c'-sorting words for an element w are also commutation equivalent, and the c- and c'- sortable conditions are then equivalent, as the Cambrian recurrences coincide. Hence, we may write $\mathrm{Camb}_{\mathrm{Sort}}(W, c)$ with the Coxeter element rather than the Coxeter word.

Theorem 3.4.7 ([Rea07b, Theorem 1.1])**.** *The Cambrian lattice* $\mathrm{Camb}_{Sort}(W, c)$ *is a sublattice of the weak order on* W *(and hence, it is indeed a lattice). In fact, c-sortable elements are bottom elements of equivalence classes of a lattice congruence on* $\mathrm{Weak}(W)$*.*

Example 3.4.8. The type A Coxeter group $\mathfrak{S}_n$ gives lattices that can be interpreted in terms of posets on triangulations of a given convex polygons, as defined in [Rea06, Section 6]. The choice of the Coxeter element changes the shape of the polygon, deciding which labels appear "at the top" or "at the bottom", then the covering relations are given by increasing flips. A flip consists in removing one of the diagonals of the triangulation, which lived in some quadrilateral, and inserting the other diagonal of the quadrilateral instead. Each diagonal is a portion of a line in the plan, and thus has a slope. A flip is then increasing if the slope of the new diagonal is greater than the slope of the old one.

The case when all arrows of the Coxeter graph of $\mathfrak{S}_n$ are oriented in the same way corresponds to the choice of the long cycle $c^{\mathsf{lin}} = (1, 2, \dots, n)$ and will be referred to as the **linear type A case**. This gives a poset on triangulations of the $(n+2)$-gon that is isomorphic to the Tamari lattice defined in Definition 2.2.7 under the bijection from trees to triangulations described in the proof of Proposition 1.3.2 consisting of taking the dual of the triangulation. Indeed, increasing flips are in this case translated to left rotations of trees.

Remark 3.4.9 ([STW18, Proposition 6.2.5])**.** In the weak order, the inversion set of the meet of two elements was not necessarily the intersection of their inversion sets. For instance, in type A_2, with generators $\mathcal{S} = \{s, t\}$, the meet of st and ts is e, whose inversion set is empty, while the inversion sets of st and ts are $\{s, sts\}$ and $\{t, sts\}$ respectively, whose intersection is $\{sts\}$.

This property is however true when we only consider c-sortable elements: the meet (in the week order) of two c-sortable elements is c-sortable (so it is also their meet in the Cambrian order) and the inversion set of the meet is the intersection of the two inversion sets!

The join of two c-sortable elements is also c-sortable, however its inversion set is usually not the union of the two inversion sets.

3.4.2 Cambrian lattices as subword complexes

A second definition of Cambrian lattices involves the notion of **subword complexes**, introduced by A. Knutson and E. Miller in [KM04, KM05]. The authors prove that subword complexes are simplicial complexes with very good properties in general. For instance they are vertex-decomposable, hence shellable, and moreover homeomorphic to a ball or a sphere. In this context, another definition of the Cambrian lattice can be given, with a special choice of word and element for the simplicial complex. The equivalence of the two definitions of the Cambrian lattice will be stated in the next section, with yet a third definition. In view of later sections, we give a more general definition that can be found in [STW18, Section 3.2], as well as a subclass of "initial" subword complexes that behave best. For them, we still have the shellability result but they may have the homotopy type of a wedge of spheres.

Definition 3.4.10. Let $\mathsf{Q} = \mathsf{s}_1 \dots \mathsf{s}_p \in \mathcal{S}^*$, $w \in W$, and $a = \ell_{\mathcal{S}}(w) + 2g$ for some $g \geq 0$. The **subword complex** $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ is a simplicial complex whose vertex set is included in $[p]$, where each element of $[p]$ represents the position of the corresponding letter of the word Q. The facets of $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ are subsets of positions of letters in Q whose complement form an $\mathcal{S}$-word of length a for the element w.

The word Q is called the **search word**, and we may omit a when it is equal to $\ell_{\mathcal{S}}(w)$. Elements of a facet correspond by definition to the position of a letter of Q, and we may thus call letters the elements of a facet.

Remark 3.4.11. A "classical" subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, \ell_{\mathcal{S}}(w))$ is nonempty if and only if w is smaller than the Demazure product of Q in the Bruhat order, i.e. $w \sqsubseteq \mathrm{Dem}(\mathsf{Q})$, as defined in Definition 3.3.6. More generally, a subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ may be empty if there is no word of length a for w that is a subword of Q, for instance if $a > n$ or if $w \not\sqsubseteq \mathrm{Dem}(\mathsf{Q})$.

If Q and Q' are two commutation equivalent search words, then $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ and $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}', w, a)$ are isomorphic simplicial complexes, by a natural identification of the letters of Q and Q'. More precisely, if $\mathsf{Q} = \mathsf{s}_1 \dots \mathsf{s}_i \mathsf{s}_{i+1} \dots \mathsf{s}_p$ and $\mathsf{Q}' = \mathsf{s}_1 \dots \mathsf{s}_{i+1} \mathsf{s}_i \dots \mathsf{s}_p$, with $s_i s_{i+1} = s_{i+1} s_i$, then exchanging i and $i+1$ gives an isomorphism of simplicial complexes between $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ and $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}', w, a)$, for any $w \in W$ and $a \geq 0$.

Definition 3.4.12. Given a Coxeter word c, a subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ is **c-initial** (or simply initial when the context is clear) if

1. c is a prefix of Q,
2. Q is initial in c^{∞},
3. and $\bar{\mathsf{c}}\mathsf{Q}$ is a word for w of length a.

Equivalently, if the Coxeter word is $\mathsf{c} = \mathsf{s}_1 \dots \mathsf{s}_n$, a c-initial subword complex is of the form $\mathrm{SC}_{\mathcal{S}}(\mathsf{c}\, \mathsf{s}_{n+1} \dots \mathsf{s}_p, s_{n+1} \dots s_p, p-n)$, and thus, $[n]$ is a facet.

Theorem 3.4.13 ([STW18, Theorem 3.6.2, Corollary 3.6.3])**.** *If $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ is an initial subword complex, then it is vertex-decomposable, and the lexicographic order on the facets is a shelling order.*

Moreover, if Q is of length p and $\hat{\mathsf{Q}}$ is the word of length q obtained from Q by removing its longest initial subword that is also initial in $\mathsf{w}_\circ(\mathsf{c})$, then $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ is homotopy equivalent to a wedge of k spheres where k is the number of facets of $\mathrm{SC}_{\mathcal{S}}(\hat{\mathsf{Q}}, w, b)$, for $b = a - (p - q)$.

Before giving the definition of the Cambrian lattice in this context, we recall a few useful results on reduced words.

Lemma 3.4.14 ([STW18, Lemmas 2.6.3 and 2.6.5])**.** *Let c be a Coxeter element with two associated reduced words c and c', such that c starts with the letter s. Let $w \in W$ with reduced c-sorting and c'-sorting words $\mathsf{w}(\mathsf{c})$ and $\mathsf{w}(\mathsf{c}')$. Recall that ψ was defined in Proposition 3.3.4 as the conjugation by $w_\circ$. Then, we have the following properties:*

1. $\mathsf{w}(\mathsf{c}) \equiv \mathsf{w}(\mathsf{c}')$,
2. c *is initial in* $\mathsf{w}_\circ(\mathsf{c})$,
3. $\psi(\mathsf{c})$ *is final in* $\mathsf{w}_\circ(\mathsf{c})$,
4. $\psi(\mathsf{w}_\circ(\mathsf{c})) \equiv \mathsf{w}_\circ(\psi(\mathsf{c}))$,
5. $\mathsf{w}_\circ(\bar{\mathsf{s}}\mathsf{c}\mathsf{s}) \equiv \bar{\mathsf{s}} w_\circ(\mathsf{c})\psi(\mathsf{s})$,
6. $\mathsf{w}_\circ(\mathrm{rev}(\psi(\mathsf{c}))) \equiv \mathrm{rev}(\mathsf{w}_\circ(\mathsf{c}))$
7. $\mathsf{w}_\circ(\mathsf{c})$ *is initial in* c^{∞},
8. $\mathsf{w}_\circ(\mathsf{c})\psi(\mathsf{w}_\circ(\mathsf{c})) \equiv \mathsf{c}^h$.

Given a pure simplicial complex, one can define a flip graph on its facets, where a flip consists in changing exactly one vertex in a facet. When we have a total order on the vertices, and in particular for subword complexes, we can put an orientation on the flips, where a flip is increasing if the new vertex is greater than the old one. This defines a poset as the transitive closure of flips. The definition of Cambrian lattices as the flip poset of some subword complexes, that we will call "c-cluster subword complex", is due to C. Ceballos, J.-P. Labbé, and C. Stump [CLS14, Definition 2.5]. It is of use in general to define a generalization of the roots of a Coxeter group, called the colored roots, and to attach to a subword complex a root vector and a root configuration.

The translation of the c-cluster subword complex then coincides with the notion of c-cluster complex which was already considered in [RS11, Section 5], generalizing the constructions of [FZ03, IS10]. Cambrian lattices can be interpreted as partial orders on noncrossing partitions, however very different from the noncrossing partition lattice defined in Remark 3.3.22. These objects are deeply related to the associahedra, which are polytopes that realize the Cambrian lattices, that is to say that for any Coxeter element $c \in W$, some orientation of the 1-skeleton of the type W-associahedron coincides with the Hasse diagram of the Cambrian lattice $\mathrm{Camb}_{\mathrm{Clus}}(W, c)$ that we will define.

Definition 3.4.15. Let $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ be a subword complex with $\mathsf{Q} = \mathsf{s}_1 \dots \mathsf{s}_p$ with a facet I. The **root vector** r_I of I is the sequence of colored roots obtained by conjugating each letter of Q by letters not in I in the corresponding prefix. More precisely, it is the tuple $(r_I(1), \dots, r_I(p))$ where $r_I(i) = \mathsf{s}_1 \dots \widehat{\mathsf{s}}_{i_1} \dots \widehat{\mathsf{s}}_{i_k} \dots \mathsf{s}_{i-1} \left(\alpha_{s_i}^{(0)}\right)$, where $1 \leq i_1 < \cdots < i_k < i$ are the indices of letters of I appearing before s_i and are omitted.

The **root configuration** of I is the set $\mathrm{R}(I)$ of roots attached to elements of the facets, that is to say $\mathrm{R}(I) = \{r_I(i) \mid i \in I\}$.

Definition 3.4.16. Two facets I and J of a pure simplicial complex are said to be **adjacent** if $I \setminus \{i\} = J \setminus \{j\}$ for some indices $1 \leq i \neq j \leq p$. The **flip** from I to J is said to be **increasing** if $i < j$ (and decreasing otherwise). For a subword complex, define the **direction** of such a flip as the root $|r_I(i)|$.

The **flip graph** of $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ (or any pure simplicial complex) is the graph of increasing flips on the set of facets, and the **flip poset** is the transitive closure of increasing flips.

Remark 3.4.17. It may be possible (when $a > \ell_{\mathcal{S}}(w)$) that there are three or more facets mutually adjacent, i.e. facets I, J, K such that $I \setminus \{i\} = J \setminus \{j\} = K \setminus \{k\}$. In later cases, we may refer to (small) increasing flips as covering relations in the flip poset only, that is to say the flip from I to J is a small increasing flip if $I \setminus \{i\} = J \setminus \{j\}$ for $i < j$ and there is no $i < k < j$ such that $(I \setminus \{i\}) \cup \{k\}$ is a facet.

Proposition 3.4.18 ([STW18, Lemmas 3.3.1 to 3.3.5])**.**

1. *Let $r_I = (\beta_1^{(m_1)}, \dots, \beta_p^{(m_p)})$ be a root vector. Then if $\widetilde{\mathsf{Q}} = \mathsf{s}_1 \dots \widehat{\mathsf{s}}_{i_1} \dots \widehat{\mathsf{s}}_{i_{p-a}} \dots \mathsf{s}_p$ is the word obtained from Q by removing the letters of I, then $\mathsf{inv}(\widetilde{\mathsf{Q}})$ is equal to the sequence $(\beta_j^{(m_j)})_{j \notin I}$ of positive roots attached to letters not in I.*

2. *For $i \in I$, the color m_i is equal to the number of indices $j < i$ with $j \notin I$ such that $\beta_j = \beta_i$.*

3. *If I and J are adjacent facets, writing $I \setminus \{i\} = J \setminus \{j\}$, we have $\beta = |r_I(i)| = |r_J(j)|$ and the flip is increasing in the direction of increasing color of these roots. If $i < j$, then for all $i < k < j$ we have $|r_J(k)| = |s_\beta(r_I(k))|$ and $r_J(k) = r_I(k)$ for all other indices k.*

4. *Conversely, if I is a facet and $i \in I$ and $j \notin I$ are such that $|r_I(i)| = |r_I(j)|$, then $(I \setminus \{i\}) \cup \{j\}$ is also a facet.*

5. *A facet I of a subword complex $\mathrm{SC}_{\mathcal{S}}(Q, w, a)$ is uniquely determined by its root configuration $\mathrm{R}(I)$.*

Definition 3.4.19. Let c be a Coxeter word. The **c-cluster subword complex** is the subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{c}\mathsf{w}_\circ(\mathsf{c}), w_\circ)$. The (cluster version of the) **Cambrian lattice** $\mathrm{Camb}_{\mathrm{Clus}}(W, \mathsf{c})$ is the flip poset of the c-cluster subword complex.

An example of Cambrian lattice in type A_2 is given in Figure 3.3, and the root configuration of the facets is also computed.

We also define the notion of "dual subword complexes" which has to do with the use of $\mathcal{R}$-words instead of $\mathcal{S}$-words. This time, facets will be chosen as a fixed length subword such that the product of the letters is a given element.

Definition 3.4.20. Let Q be an $\mathcal{R}$-word, $w \in W$ and $a = \ell_{\mathcal{R}}(w) + 2g$ for some $g \geq 0$. The **dual subword complex** $\mathrm{SC}_{\mathcal{R}}(\mathsf{Q}, w, a)$ is a simplicial complex whose vertices are the positions of letters of the word Q and whose facets are subsets of positions of letters in Q whose product form an $\mathcal{R}$-word of length a for the element w. We may omit a when it is equal to $\ell_{\mathcal{R}}(w)$.

	s (12)	t (13)	s (23)	t (12)	s (13)	R(I)
12	x	x				$\alpha_s^{(0)}, \alpha_t^{(0)}$
23		x	x			$\alpha_u^{(0)}, \alpha_s^{(1)}$
34			x	x		$\alpha_t^{(0)}, \alpha_u^{(1)}$
45				x	x	$\alpha_s^{(1)}, \alpha_t^{(1)}$
15	x				x	$\alpha_s^{(0)}, \alpha_t^{(1)}$

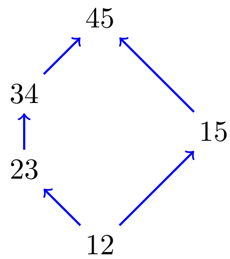


Figure 3.3: The subword complex version of the Cambrian lattice $\mathrm{Camb}_{\mathrm{Clus}}(\mathfrak{S}_3, \mathsf{st})$. Each element is represented by the corresponding facet of the subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{ststs}, w_\circ, 3)$ (or equivalently its dual subword complex). The root configuration—corresponding to the selected letters (x) conjugated by unselected letters preceding them—is also given.

Proposition 3.4.21 ([STW18, Proposition 3.4.2])**.** *Let Q be an $\mathcal{S}$-word of size p, $w \in W$ and $a = \ell_{\mathcal{S}}(w) + 2g$. Let $Q \in W$ be the product of letters of Q.*

Then we can consider the $\mathcal{R}$-word $\mathsf{Q}' = \mathsf{inv}_{\mathcal{R}}(\mathsf{Q})$ (by forgetting the colors), and then choosing $p - a$ letters in Q such that the complementary is equal to w is equivalent to choose $p - a$ letters in Q' whose product is equal to $w' = wQ^{-1}$. In other words, $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a) = \mathrm{SC}_{\mathcal{R}}(\mathsf{Q}', w', p - a)$.

Proof. The proof is quite easy, as multiplying a word Q by one of its inversions correspond to suppress the letter in the word, and suppressing $p - a$ letters in Q from the rightmost to the leftmost, such that the rest is a word for w gives an $\mathcal{R}$-word w' such that $w'Q = w$. $\square$

Corollary 3.4.22. *The* c*-cluster subword complex* $\mathrm{SC}_{\mathcal{S}}(\mathsf{cw}_\circ(\mathsf{c}), w_\circ)$ *is isomorphic to the dual subword complex* $\mathrm{SC}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{cw}_\circ(\mathsf{c})), c^{-1})$.

The reason for introducing this dual subword complex is that it translates directly to the world of cluster complexes, introduced by S. Fomin and A. Zelevinsky [FZ03], then extended by N. Reading into c-cluster algebras in all finite Coxeter groups [Rea07b, Rea07a]. These were defined using families of relations on the set $-\Delta \cup \Phi^+$ of "almost positive roots", called c-compatibility relations. The cluster complex is then the simplicial complex whose faces are the sets of almost positive roots that are pairwise c-compatible. Then, this simplicial complex is isomorphic to the two subword complexes defined in this section, as explained in [CLS14].

The equivalence of the (so far) two definitions of Cambrian lattices will be discussed in the next subsection, which focuses on the root configuration of facets to give a third definition of Cambrian lattices, this time interpreted as a new poset on noncrossing partitions.

3.4.3 Cambrian lattices as an order on noncrossing partitions

Noncrossing partitions are the third context in which we present an incarnation of the Cambrian lattices. They were introduced by G. Kreweras in [Kre72] and the lattice of noncrossing partitions was then understood as a subposet of the type A intersection lattice of a Coxeter hyperplane arrangement [Rei97]. This was further extended to all Coxeter types by T. Brady and C. Watt on the one hand and D. Bessis on the other hand [BW02, Bes03]. We give here the definitions and results that are relevant to our purpose, and describe the equivalence of the three definitions of Cambrian lattices.

Definition 3.4.23. Let $c \in W$ be a Coxeter element. An element $w \in W$ is called **c-noncrossing** if it satisfies $w \leq_{\mathcal{R}} c$ in the absolute order. The set of c-noncrossing elements is denoted by $\mathrm{NC}(W, c)$. The **noncrossing partition lattice** $\mathrm{NCL}(W, c)$ is the restriction to the interval $[e, c]$ of the absolute lattice $\mathrm{Abs}(W)$.

The **Kreweras complement** of a c-noncrossing partition w is defined by $\mathrm{Krew}_c(w) = cw^{-1}$.

Proposition 3.4.24. *If $u \leq_{\mathcal{R}} v \leq_{\mathcal{R}} w$, then $vu^{-1} \leq_{\mathcal{R}} v$ and $wv^{-1} \leq_{\mathcal{R}} wu^{-1}$.*

If particular, the Kreweras complement is a bijection on c-noncrossing partitions, and it is an anti-automorphism of $\mathrm{NCL}(W, c)$.

The proofs are immediate with Proposition 3.3.18. We also have that applying twice the Kreweras complement Krew_c is the same as conjugating by c. This agrees with the case of the classical noncrossing partition lattice and Kreweras complement of Definition 1.3.8.

Recall that the choice of a Coxeter word c results in a choice of a privileged reduced $\mathcal{S}$-word for all elements of the group in the form of their c-sorting word, and in particular for $w_\circ$. As the inversion set of $w_\circ$ is the entire set $\mathcal{R}$ of reflections, the inversion sequence of the word $\mathsf{w}_\circ(\mathsf{c})$ defines a special $\mathcal{R}$-word $\mathsf{R}(\mathsf{c}) = \mathsf{inv}_{\mathcal{R}}\, \mathsf{w}_\circ(\mathsf{c})$, which can be considered as a total order $\leq_{\mathsf{c}}$ on the set $\mathcal{R}$. This order will be called the **reflection order** (associated to c).

Proposition 3.4.25 ([ABW07, Theorem 3.5])**.** *Let c be a Coxeter word of a standard Coxeter element c. Each element $w \in \mathrm{NC}(W, c)$ has a unique reduced $\mathcal{R}$-word $\mathsf{w}_{\mathcal{R}}(\mathsf{c}) = \mathsf{r}_1\mathsf{r}_2 \dots \mathsf{r}_p$ such that $\mathsf{r}_1 <_{\mathsf{c}} \mathsf{r}_2 <_{\mathsf{c}} \cdots <_{\mathsf{c}} \mathsf{r}_p$. We will refer to $\mathsf{w}_{\mathcal{R}}(\mathsf{c})$ as the* **c-increasing word** *of w.*

Moreover, this writing of w as a reduced $\mathcal{R}$-word is the lexicographically smallest one.

In fact, the authors of [ABW07] prove that for any total order on $\mathcal{R}$, the lexicographically smallest reduced $\mathcal{R}$-word for w is increasing with respect to that total ordering, and that when the order is compatible with c, in some sense, then this is its unique increasing reduced $\mathcal{R}$-word. This enabled them to prove in particular that the noncrossing partition lattice is EL-shellable [ABW07, Theorem 1.1].

Remark 3.4.26. The property of being the lexicographically smallest reduced $\mathcal{R}$-word gives an algorithmic greedy procedure to find this unique increasing factorization of $w \in \mathrm{NC}(W, c)$. The algorithm is in flavor very similar to the one in Remark 3.4.5 for finding the c-sorting word of w:

One starts with $\mathsf{u} = \varepsilon$ and $v = w$, then scans the word $\mathsf{R}(\mathsf{c})$ from left to right. For each letter r, if we have $\ell_{\mathcal{R}}(rv) < \ell_{\mathcal{R}}(v)$ for the corresponding reflection r, then (u, v) is replaced by (ur, rv). This process terminates at $(\mathsf{w}_{\mathcal{R}}(\mathsf{c}), e)$.

Note that for two Coxeter words c and c' for the same Coxeter element c, the c-increasing and c'-increasing words of an element w are also commutation equivalent.

Proposition 3.4.27. *For each Coxeter word c, there is a canonical bijection between* $\mathrm{NC}(W, c)$ *and facets of the dual subword complex* $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^2, c)$.

Proof. Let I be a facet of $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^2, c)$. Let w_0 be the product of reflections corresponding to letters of I in the first copy of $\mathsf{R}(\mathsf{c})$ and w_1 be the product of reflections corresponding to letters of I in the second copy of $\mathsf{R}(\mathsf{c})$.

By definition, we have $w_0 w_1 = c$ and $\ell_{\mathcal{R}}(w_0) + \ell_{\mathcal{R}}(w_1) = \ell_{\mathcal{R}}(c)$. Hence w_0 and w_1 are noncrossing partitions, and moreover, $w_0 = \mathrm{Krew}_c(w_1)$. ☐

If I is a facet of $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^2, c)$ we write $I = \mathsf{r}_1^{(0)} \dots \mathsf{r}_k^{(0)} \mathsf{r}_{k+1}^{(1)} \dots \mathsf{r}_n^{(1)}$ if the first k letters are in the first copy of $\mathsf{R}(\mathsf{c})$ and the last $n - k$ letters are in the second copy of $\mathsf{R}(\mathsf{c})$. In other words, we have written c as a product of colored reflections with two colors, the letters being in increasing order within each color. Each facet is now a collection of colored roots, which recalls the root configuration of facets of subword complexes. [STW18, Proposition 3.4.2] We are now ready to define the third incarnation of Cambrian lattices, as a poset on noncrossing partitions. This definition is due to N. Williams, C. Stump and H. Thomas in [STW18, Section 4.4]. The set of elements will be the facets of the subword complex $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^2, c)$, *however*, the order will *not* be the flip poset of this simplicial complex! Instead, it will be the transitive closure of some rotations inspired by the transformation of the root configurations of facets under flips in the c-cluster subword complex, as described in Proposition 3.4.18. The idea is to take a root of color

	s (12)	u (13)	t (23)	s (12)	u (13)	t (23)
s^0t^0	x		x			
s^0t^1	x					x
s^1t^1				x		x
u^0s^1		x		x		
t^0u^1			x		x	

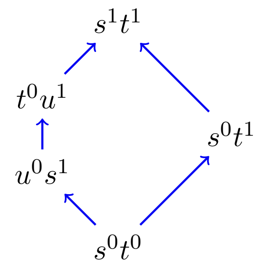


Figure 3.4: The noncrossing version of the Cambrian lattice $\mathrm{Camb}_{\mathrm{NC}}(\mathfrak{S}_3, \mathsf{st})$. Each element is represented by the corresponding factorization of c in $\mathsf{R}(\mathsf{c})^2$.

0 and change it into a root of color 1, conjugating all roots appearing in between—such that they remain in between after conjugation—and not affecting the others. This is made more precise in the following definition.

Definition 3.4.28. Let c be a Coxeter word for a Coxeter element c and $I = \mathsf{r}_1^{(m_1)} \dots \mathsf{r}_p^{(m_p)}$ be a facet of $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^2, c)$.

Let $1 \le i \le p$ be such that $m_i = 0$. Set $r'_i = r_i$ and $m'_i = 1$. For j such that $r_i^{(0)} < r_j^{(m_j)} < r_i^{(1)}$, set $r'_j = r_i r_j r_i = |r_i(\alpha_{r_j})|$, where the last equality between a reflection and a root is taken via the canonical bijection. If $r_i(\alpha_{r_j}) \in \Phi^+$, set $m'_j = m_j$ and otherwise, if $r_i(\alpha_{r_j}) \in \Phi^-$, set $m'_j = m_j - 1$. For all other j, set $r'_j = r_j$ and $m'_j = m_j$.

Then the **rotation** of I at index i is the facet I' whose reflection sequence is the sorted tuple of $((\mathsf{r}'_1)^{(m'_1)} \dots (\mathsf{r}'_n)^{(m'_n)})$.

The (noncrossing version of the) **Cambrian lattice** $\mathrm{Camb}_{\mathrm{NC}}(W, \mathsf{c})$ is the transitive closure of rotations of facets of $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^2, c)$.

Before stating the equivalence of the three definitions of Cambrian lattices, we need to introduce the notion of **skip set** of c-sortable elements.

Definition 3.4.29. Let $w \in \mathrm{Sort}(W, \mathsf{c})$ and such that $\mathsf{w}(\mathsf{c}) = \mathsf{c}\big|_{I_1} \dots \mathsf{c}\big|_{I_k}$.

For each simple reflection s, let j be the first index such that $\mathsf{s} \notin I_j$, this is the first "skipped" s in c^∞. The definition of c-sortable elements implies that $\mathsf{s} \notin I_i$ for all $i \ge j$. The idea will be to conjugate this first skipped s by all letters of $\mathsf{w}(\mathsf{c})$ appearing before it. More precisely, let $\beta_s^{(m_s)} = \mathsf{s}_1 \dots \mathsf{s}_i(\alpha_{\mathsf{s}}^{(0)})$, where $\mathsf{s}_1 \dots \mathsf{s}_i$ is the prefix of $\mathsf{w}(\mathsf{c})$ of all letters of c^∞ appearing before the first skipped s.

The **skip set** $\mathcal{C}_c(w)$ of w is defined as $\{\beta_s^{(m_s)} \mid s \in \mathcal{S}\}$. The skip set is given in Figure 3.2 for st-sortable elements in type A_2.

Theorem 3.4.30 ([STW18, Theorems 5.7.3 and 6.8.6])**.** *Let c be a Coxeter word in W.*

1. *$\mathrm{Camb}_{Clus}(W, \mathsf{c})$ and $\mathrm{Camb}_{NC}(W, \mathsf{c})$ are isomorphic by identifying the root configuration of a c-cluster with a facet of $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^2, c)$.*

2. *$\mathrm{Camb}_{Sort}(W, \mathsf{c})$ and $\mathrm{Camb}_{NC}(W, \mathsf{c})$ are isomorphic by identifying the skip set of a c-sortable element with a facet of $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^2, c)$.*

The translation between the three definitions of the Cambrian lattices can be noticed by comparing the examples given in type A_2 for the Coxeter word st in Figures 3.2 to 3.4.

These three constructions can in fact all be generalized in the broader context of Artin monoids and groups which are to Coxeter groups what the braid monoid is to the symmetric group. This will be addressed in Section 9.2.

Chapter 4

Generalizations of the Tamari lattice

The Tamari lattice appears in various guises and these different incarnations gave rise to different families of generalizations, among which:

- the m-Tamari lattices, further generalized as the ν-Tamari lattices,
- the alt-Tamari lattices, also containing the Dyck lattice, further generalized as the alt ν-Tamari lattices, containing the ν-Tamari lattices,
- the Cambrian lattices, further generalized as the m-Cambrian lattices,
- the posets of tilting modules,
- the permutree lattices, also containing the type A Cambrian lattices and weak order.

In this section, we present briefly the definitions of these families, along with relevant results or conjectures about them. While some results are already established, the proofs will be the object of forthcoming chapters. All these families and how they relate to each other are represented on Figures 1 and 2 at the beginning of the manuscript.

4.1 The m-Tamari lattices and the ν-Tamari lattices

The number of intervals in the Tamari lattice is surprisingly counted by the same numbers as the conjectured dimension of some representation of the symmetric group. More precisely, M. Haiman studied in [Hai94] the polynomial ring in two sets of n variables, together with the diagonal action of the symmetric group, and the resulting representation of $\mathfrak{S}_n$ when quotienting out by the ideal generated by non-constant invariant polynomials, which can be seen as the subspace of "diagonal harmonics". He conjectured that the dimension of this representation was equal to the number of so-called parking functions of size n, and that the multiplicity of the sign representation was equal to the Catalan number C_n. He also conjectured similar formulas for the case of diagonal harmonics with three sets of variables, and in particular gave a conjectural formula for the dimension of the alternating component, which was later proven by Chapoton to enumerate the intervals in the Tamari lattice of size n [Cha06].

Motivated by this intriguing connection, F. Bergeron and L.-F. Préville-Ratelle defined in [BPR12] the family of m-Tamari lattices, and conjectured that the number of intervals in these posets was equal to the dimension of the alternating component of the $\mathfrak{S}_n$-module of higher trivariate diagonal harmonics. A formula for their enumeration and connections to representation theory can be found in [BMFPR11, BMCPR13].

A further generalization of the m-Tamari lattices was introduced by L.-F. Préville-Ratelle and X. Viennot in [PRV17]. These posets are indexed by a lattice path ν and called the ν-Tamari lattices. Their total number of intervals is connected to the enumeration of nonseparable planar maps, as shown in [FPR17].

4.1.1 The m-Tamari lattices

The definition of the Tamari lattice on Dyck paths stated in Definition 2.2.11 can naturally be extended to m-Dyck paths. Indeed, the set of m-Dyck paths, viewed as Dyck paths whose rises' lengths are divisible by m, is stable under Tamari rotations.

Definition 4.1.1. Let $m, n \geq 1$. The **m-Tamari lattice** $\mathrm{Tam}_n^{(m)}$ of size n is the restriction of the Tamari lattice Tam_{mn} to the subset of m-Dyck paths of size n.

As the set of m-Dyck paths is stable under rotations, it is an upper ideal of Tam_{mn}, but it is in fact easy to see that it is an interval, and thus a lattice. Obviously, the case $m = 1$ is the usual Tamari lattice. M. Bousquet-Mélou, É. Fusy and L.-F. Préville-Ratelle counted intervals in the m-Tamari lattices, generalizing the enumeration of intervals in the Tamari lattice [Cha06].

Theorem 4.1.2 ([BMFPR11, Corollary 11])**.** *The number of intervals in the m-Tamari lattice $\mathrm{Tam}_n^{(m)}$ is equal to:*

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}. \tag{4.1}$$

4.1.2 The ν-Tamari lattices on ν-paths

In fact, one can view the m-Dyck paths as m-ballot paths, which are north-east lattice paths remaining above the path $(NE^m)^n$. One can then rephrase the definition of the m-Tamari lattice in terms of m-ballot paths using a relevant notion of altitude to define the equivalent notion of excursions and rotations.

This generalizes well to the context of ν-paths into the ν-Tamari lattices as defined by L.-F. Préville-Ratelle and X. Viennot in [PRV17].

Definition 4.1.3. For a lattice point p on a ν-path μ, define its **ν-altitude** $\mathrm{alt}_\nu(p)$ to be the maximum number of horizontal steps that can be added to the right of p without crossing ν. Given a valley EN of μ, let p be the lattice point between the east and north steps. Let q be the next lattice point of μ such that $\mathrm{alt}_\nu(q) = \mathrm{alt}_\nu(p)$, and $\mu_{[p,q]}$ be the subpath of μ that starts at p and ends at q.

Let μ' be the path obtained from μ by switching $\mu_{[p,q]}$ with the east step E that precedes it. The ν-rotation of μ at the valley p is defined to be $\mu \lessdot_\nu \mu'$.

The **ν-Tamari poset** Tam_ν is the transitive closure of ν-rotations on ν-paths.

An example is illustrated in Figure 4.1.

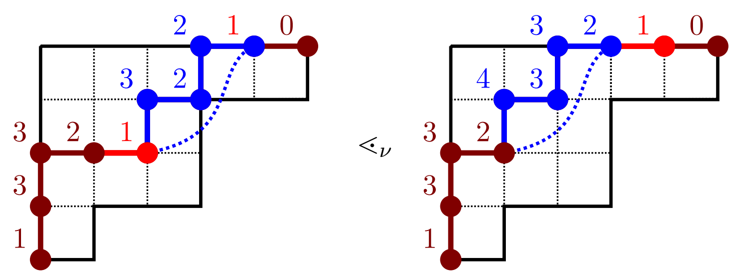


Figure 4.1: The rotation operation of a ν-path. Each node is labelled with its ν-altitude. The ν-excursion that is exchanged with an east step (in red) is highlighted in blue. The ν-elevation of the total path is $1 - 0 = 1$.

An example of the ν-Tamari poset for $\nu = ENEEN$ is illustrated in Figure 4.2.

Theorem 4.1.4 ([PRV17, Theorem 1.1])**.** *The ν-Tamari poset is a lattice. The ν-rotations are exactly its covering relations.*

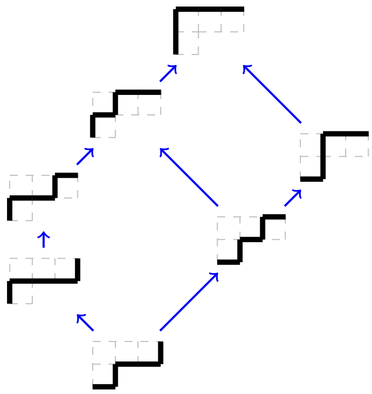


Figure 4.2: The ν-Tamari lattice for $\nu = ENEEN = (1, 2, 0)$.

Another approach to define the ν-Tamari lattice is to introduce the notion of *ν-elevation* of a subpath as the difference of ν-altitude between its ending point and its starting point. Recall that we can encode such a path ν as the sequence $(\nu_0, \ldots, \nu_n)$, where ν_i is the number of east steps of ν between its i-th and $i+1$-st north steps.

Definition 4.1.5. We write $\mathrm{elev}_\nu(E) = -1$ for an east step E and $\mathrm{elev}_\nu(N_i) = \nu_i$ if N_i is the i-th north step of a ν-path μ. For any subpath A of μ, we define its *ν-elevation* $\mathrm{elev}_\nu(A) = \sum_{a \in A} \mathrm{elev}_\nu(a)$ as the sum of the ν-elevations of the steps of A.

The *ν-excursion* of a north step N of a ν-path μ is defined as the shortest subpath A of μ that starts with this N and such that $\mathrm{elev}_\nu(A) = 0$.

Remark that the ν-altitude of a lattice point p of a ν-path μ is equal to the ν-elevation of the subpath of μ that starts at p and ends at the end of μ. It then follows from the definition of the ν-excursion that exchanging the east step E of a valley with the ν-excursion that follows it is exactly a covering relation in Tam_ν.

4.1.3 The ν-Tamari lattices on ν-trees

An alternative description of the ν-Tamari lattices in terms of ν-trees was presented in [CPS20].

Recall that the Ferrers diagram F_ν associated to the path ν is the collection of boxes that lie weakly above ν in the smallest rectangle containing ν. Let L_ν denote the set of lattice points inside F_ν.

Definition 4.1.6. Two points $p, q \in L_\nu$ are said **ν-incompatible** if p is strictly southwest or strictly northeast of q, and the smallest rectangle containing p and q lies entirely in F_ν. Otherwise, p and q are said to be **ν-compatible**.

A *ν-tree* is a maximal collection of pairwise ν-compatible elements in L_ν.

In particular, the vertex in the top-left corner of F_ν is ν-compatible with every other vertex, and belongs to every ν-tree. Connecting two consecutive elements (not necessarily at distance 1) in the same row or column allows us to visualize ν-trees as classical rooted binary trees [CPS20]. The vertex at top-left corner of F_ν is always the *root*. An example of a ν-tree and the rotation operation which we now describe is shown in Figure 4.3.

Let T be a ν-tree and $p, r \in T$ be two elements that do not lie in the same row or same column. We denote by $p \square r$ the smallest rectangle containing p and r, and write $p \llcorner r$ (resp. $p \urcorner r$) for the lower left corner (resp. upper right corner) of $p \square r$.

Definition 4.1.7. Let $p, q, r \in T$ be such that $q = p \llcorner r$ and no other element of T besides p, q, r lies in $p \square r$. The *ν-rotation* of T at q is defined as the set $T' = \big(T \setminus \{q\}\big) \cup \{q'\}$, where $q' = p \urcorner r$, and we write $T \lessdot_\nu T'$.

As proven in [CPS20, Lemma 2.10], any ν-rotation of a ν-tree is also a ν-tree.

Definition 4.1.8. The *rotation poset of ν-trees* Tam_ν^{tr} is the transitive closure of ν-rotations.

An example of the rotation poset of ν-trees for $\nu = ENEEN$ is illustrated on the right of Figure 4.4.

Theorem 4.1.9 ([CPS20])**.** *The ν-Tamari lattice is isomorphic to the rotation poset of ν-trees:*

$$\mathrm{Tam}_\nu \cong \mathrm{Tam}_\nu^{tr}\,.$$

In particular, the rotation poset of ν-trees is a lattice.

A bijection between these two posets is given by the **right flushing bijection** flush_ν introduced in [CPS20]. This bijection maps a ν-path $\mu = (\mu_0, \dots, \mu_n)$ to the unique ν-tree with $\mu_i + 1$ nodes at height i—the height of a node being its ordinate, considering that the path ν starts at the point $(0, 0)$. This tree can be recursively obtained by adding $\mu_i + 1$ nodes at height i from bottom to top, from right to left, avoiding forbidden positions. The forbidden positions are those above a node that is not the leftmost node in a row (these come from the initial points of the east steps in the path μ). In Figure 4.5, the forbidden positions are the ones that belong to the wiggly lines. Note that each lattice point of a ν-path with ordinate i corresponds by construction to a node at height i in the resulting ν-tree, but the order on each row is reversed.

The inverse flush_ν^{-1} of the right flushing bijection is called the **left flushing bijection**, and can be described similarly, adding points from left to right, from bottom to top, avoiding the forbidden position given by the wiggly lines. In other words, the left flushing bijection of a ν-tree is the ν-path that has as many nodes per row as the tree.

The Tamari lattice Tam_n corresponds to the ν-Tamari lattice where $\nu = (NE)^n$. In this sense, it is contained in the family of ν-Tamari lattices. Very interestingly, the ν-Tamari lattices are also contained as intervals in some Tamari lattice, as shown in [FPR17].

More precisely, L.-F. Préville-Ratelle and X. Viennot defined in [PRV17] a notion of **canopy** and proved that the set of binary trees with a fixed canopy is an interval that is isomorphic to a ν-Tamari lattice. This thus partitions the Tamari lattice Tam_n into intervals, in correspondence with paths ν of length $n - 1$. Then, defining a **synchronized interval** as an interval whose top and bottom have the same canopy, W. Fang and L.-F. Préville-Ratelle proved in [FPR17] that synchronized intervals were in bijection with nonseparable planar maps, these being enumerated by the following formula:

$$\frac{2(3n+3)!}{(n+2)!\,(2n+3)!} = \frac{2}{(n+2)(2n+3)}\binom{3n+3}{n+1}. \tag{4.2}$$

Note how this resembles Equation (2.5) for the number of intervals in the Tamari lattice, themselves in bijection with some family of maps.

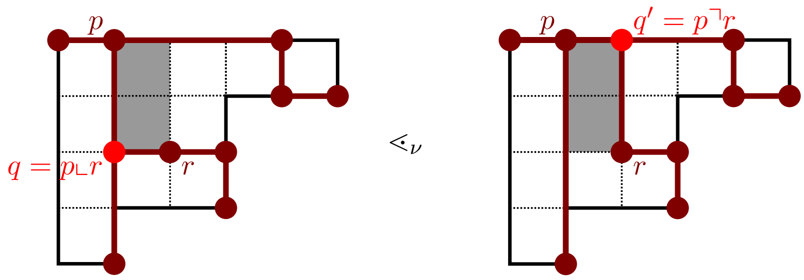


Figure 4.3: The rotation operation of a ν-tree.

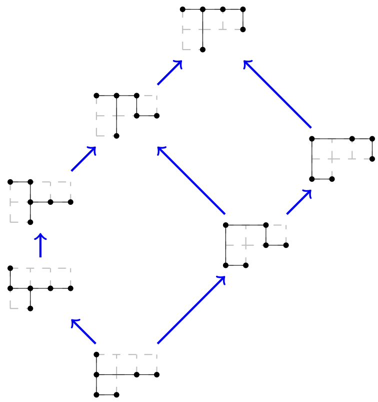


Figure 4.4: The rotation lattice of ν-trees for $\nu = ENEEN$.

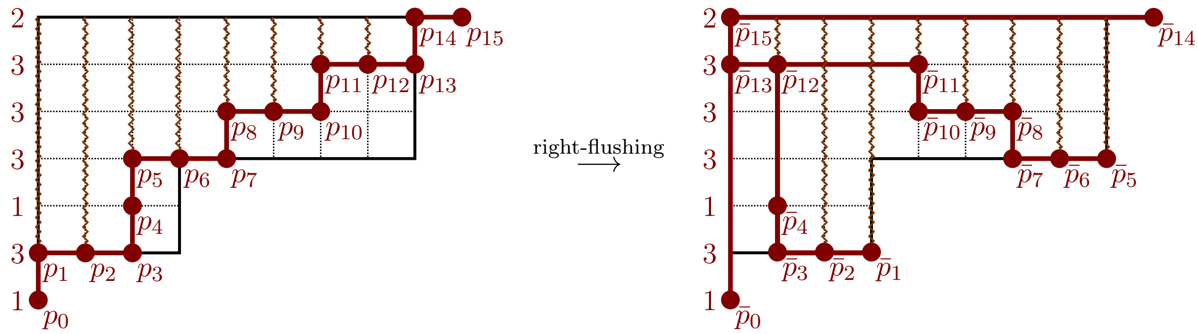


Figure 4.5: Right flushing bijection from ν-paths to ν-trees.

4.2 The alt-Tamari and alt ν-Tamari lattices

The Tamari and the Dyck lattices are two posets defined on Dyck paths that share many properties. In particular, both are lattices and the latter is an extension of the former. Moreover, their covering relations are in fact very similar: they consist of exchanging the down step of a valley with a prefix of the excursion that follows it, namely the entire excursion for the Tamari lattice and only the first up step for the Dyck lattice. More surprisingly, they share the same distribution of linear intervals with respect to their length, as first experimented by F. Chapoton and proven in Chapter 5.

Inspired by these similarities and coincidences, one can define a new family of partial orders on Dyck paths which includes these two lattices and such that these properties are satisfied. We call them **alt-Tamari lattices**. The term "alt" stands for "altitude", a notion that we use in order to define them.

In a broader scope, extending these structures on ballot paths to the world of ν-paths, we could generalize the Dyck and the Tamari lattices to the ν-Dyck and the ν-Tamari lattices, respectively, in Definitions 2.2.5 and 4.1.3. In this wider framework, these two posets still share all these properties described above, namely lattice and extension properties, similarity of covering relations and distribution of linear intervals. In fact, the whole family of alt-Tamari lattices can be generalized as the new family of **alt ν-Tamari lattices**, in which all these results still hold. This will be precisely the object of Chapter 6.

We outline here the main definitions of the objects without all proofs, but these will be detailed in Sections 5.4.1 and 6.3.

4.2.1 The alt-Tamari lattices

The alt-Tamari lattice $\mathrm{Tam}_n(\delta)$ is a poset on the set $\mathcal{Z}_n$ of Dyck paths of size n which depends on the choice of an **increment vector** $\delta = (\delta_1, \dots, \delta_n) \in \{0,1\}^n$. Given a Dyck path of size n, we will number its up steps with integers $\{1, \dots, n\}$ increasingly from left to right. For example, the path *ududdud* will be numbered $u_1u_2du_3ddu_4d$. We introduce the notions of δ-*altitude* of integer points of a Dyck path, of δ-*elevation* of its subpaths, and of δ-*excursions*.

Definition 4.2.1. Let $n \geq 1$ and $\delta = (\delta_1, \dots, \delta_n) \in \{0,1\}^n$ be an increment vector of size n and P a Dyck path of size n.

- We set the **δ-altitude** of the initial integer point of P to be equal to 0. Then, we declare that the up step u_i increases the δ-altitude by δ_i, while all down steps make it decrease by 1.
- The **δ-elevation** of a subword A of P is the change of δ-altitude between the starting and ending points of A. Thus we have $\mathrm{elev}_\delta(d) = -1$ and $\mathrm{elev}_\delta(u_i) = \delta_i$ for a down step d and the i-th up step u_i, and $\mathrm{elev}_\delta(A) = \sum_{s \in A} \delta(s)$.
- The **δ-excursion** of an up step u_i of P is the smallest subword C_i of P starting with u_i such that $\mathrm{elev}_\delta(C_i) = 0$.

For instance, if $\delta_i = 0$, the δ-excursion of u_i is reduced to u_i. If $\delta_i = 1$ for all i, then the δ-excursion of an up step u_i is exactly the excursion starting at u_i and the notion of δ-altitude is precisely the altitude, as defined in Definition 1.3.5.

This allows us to define δ-rotations and the alt-Tamari lattices.

Definition 4.2.2. Let $n \geq 1$ and $\delta = (\delta_1, \dots, \delta_n) \in \{0,1\}^n$ be an increment vector of size n and P a Dyck path of size n. Suppose that the i-th up step of P, u_i, is preceded by a down step d. Let C_i be the δ-excursion of u_i.

The **δ-rotation** of P at the up step u_i is the Dyck path Q obtained from P by exchanging the down step d that precedes u_i with the δ-excursion C_i. In other words, we can write $P = AdC_iB$ and $Q = AC_idB$. We denote δ-rotations by $P \lessdot_\delta Q$.

The **alt-Tamari lattice** $\mathrm{Tam}_n(\delta)$ is the transitive closure of δ-rotations on the set of Dyck paths.

Remark 4.2.3. There are two extreme choices for the increment vector δ.
If $\delta_i = 1$ for all $1 \leq i \leq n$, the alt-Tamari lattice $\mathrm{Tam}_n(\delta)$ coincides with the Tamari lattice.
If $\delta_i = 0$ for all $1 \leq i \leq n$, the alt-Tamari lattice $\mathrm{Tam}_n(\delta)$ coincides with the Dyck lattice.

The fact that the alt-Tamari posets are well-defined and the proof of the following motivating results for defining them will be the object of Chapters 5 and 6.

Theorem 4.2.4. *Let $n \geq 1$ and $\delta = (\delta_1, \dots, \delta_n) \in \{0,1\}^n$ be an increment vector of size n.*

The alt-Tamari poset $\mathrm{Tam}_n(\delta)$ is well-defined; it is a lattice and its distribution of linear intervals with respect to their length is the same as in the Tamari lattice Tam_n.

4.2.2 The alt ν-Tamari lattices on ν-paths

Let $\nu = (\nu_0, \dots, \nu_n)$ be a fixed lattice path using north and east unit steps. The alt-Tamari lattices can be generalized in the context of ν-paths to the alt ν-Tamari lattices.

We say that $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{N}^n$ is an **increment vector** with respect to ν if $\delta_i \leq \nu_i$ for all $1 \leq i \leq n$. Note that the first index of δ is 1 while the first index of ν (as a sequence recording east steps) is 0. We also define notions of δ-altitude, δ-elevation, δ-excursions and δ-rotations straightforwardly adapted from the case of Dyck paths.

Definition 4.2.5. Let $\nu = (\nu_0, \dots, \nu_n)$ be a fixed path and $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{N}^n$ an increment vector with respect to ν. Let μ be a ν-path.

- We set the **δ-altitude** $\mathrm{alt}_\delta(p)$ of the initial lattice point of μ to be equal to zero, and declare that the i-th north step of μ increases the δ-altitude by δ_i and an east step decreases the δ-altitude by 1.
- We define the **δ-elevation** of a subpath of μ as the difference of the δ-altitude between its ending point and its starting point.
- The **δ-excursion** of a north step N of a ν-path μ is defined as the shortest subpath A of μ that starts with this N and such that $\mathrm{elev}_\delta(A) = 0$.
- If an east step E of μ is followed by a north step N and its δ-excursion C, let μ' be the path obtained from μ by switching C with the east step E that precedes it. We say that $\mu \lessdot_\delta \mu'$ is a **δ-rotation**.

An example of δ-rotation is illustrated in Figure 4.6.

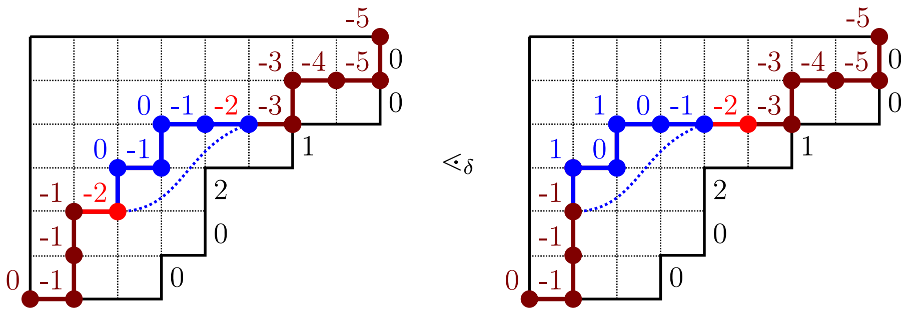


Figure 4.6: The δ-rotation operation of a ν-path for $\delta = (0, 0, 2, 1, 0, 0)$. Each node is labelled with its δ-altitude.

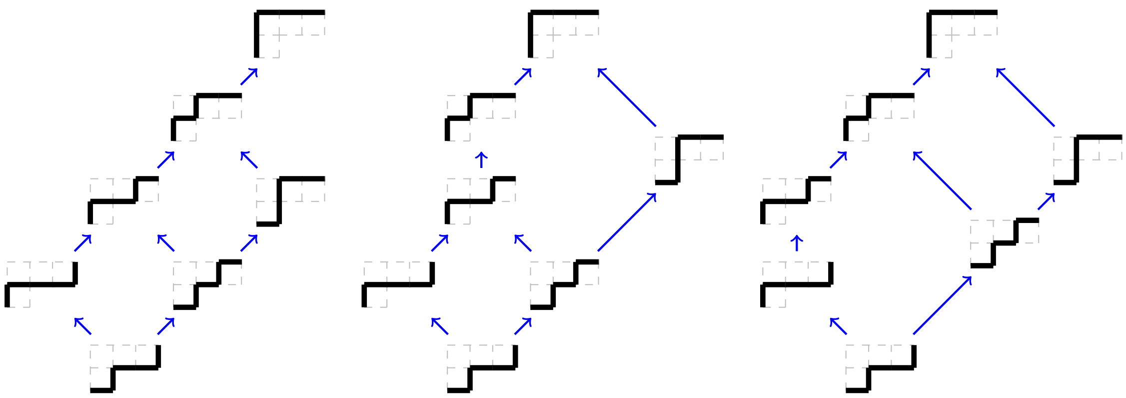


Figure 4.7: Examples of alt ν-Tamari lattices $\mathrm{Tam}_\nu(\delta)$ for $\nu = ENEEN = (1, 2, 0)$. Left: the ν-Dyck lattice, for $\delta^{\min} = (0, 0)$. Middle: the lattice for $\delta = (1, 0)$. Right: the ν-Tamari lattice, for $\delta^{\max} = (2, 0)$.
In each case, the number of linear intervals of length k is given by ℓ_k where $\ell = (\ell_0, \ell_1, \ell_2, \ell_3) = (7, 8, 4, 1)$. For instance, 7 represents the trivial intervals of length 0, which are just the elements of each poset; there are 8 linear intervals of length 1, which correspond to the edges; 4 linear interval of length 2, and 1 linear interval of length 3.

Definition 4.2.6. The **alt ν-Tamari poset** $\mathrm{Tam}_\nu(\delta)$ is the transitive closure of δ-rotations on the set of ν-paths.

Remark 4.2.7. For a fixed path ν, there are two extreme choices for the increment vector δ. If $\delta_i = \nu_i$ for all $1 \leq i \leq n$, the alt ν-Tamari lattice coincides with the ν-Tamari lattice. If $\delta_i = 0$ for all $1 \leq i \leq n$, the alt ν-Tamari lattice coincides with the ν-Dyck lattice. We denote these two cases by $\delta^{\max}$ and $\delta^{\min}$, respectively.

The three different alt ν-Tamari lattices for $\nu = ENEEN$ are illustrated in Figure 4.7. The following results generalizing those obtained on the alt-Tamari posets will be proved in Chapter 6.

Theorem 4.2.8. *The alt ν-Tamari poset $\mathrm{Tam}_\nu(\delta)$ is well-defined, it is a lattice, and its distribution of linear intervals with respect to their length does not depend on δ.*

4.2.3 The alt ν-Tamari lattices on (δ, ν)-trees

We give, again without proofs, another description of the alt ν-Tamari lattices inspired by Definition 4.1.8, using some trees instead of ν-paths. This description will be very useful when studying their (linear) intervals and the details will be presented in Section 6.3.2. One can prove in particular that the alt ν-Tamari lattice $\mathrm{Tam}_\nu(\delta)$ is isomorphic to the interval $[\nu, 1^\nu]$ in the $\check{\nu}$-Tamari lattice $\mathrm{Tam}_{\check{\nu}}$, for a well-chosen path $\check{\nu}$—hence it is a lattice.

Proposition 4.2.9. *Let ν be a path and δ an increment vector with respect to ν. Let $\check{\nu}_0 = \sum_{i=0}^n \nu_i - \sum_{i=1}^n \delta_i$ with $\delta_i \leq \nu_i$.*

Then $\check{\nu} = (\check{\nu}_0, \check{\nu}_1, \ldots, \check{\nu}_n) = (\check{\nu}_0, \delta_1, \ldots, \delta_n)$ is a path below ν whose endpoints are the same as ν, and the $\check{\nu}$-altitude and the δ-altitude coincide on ν-paths, up to a constant shift. Thus, the $\check{\nu}$-elevation and the δ-elevation coincide on ν-paths.

The alt ν-Tamari lattice $\mathrm{Tam}_\nu(\delta)$ is isomorphic to the restriction of the $\check{\nu}$-Tamari lattice to the subset of ν-paths.

Then, using the right flushing with respect to $\check{\nu}$ on the set of ν-paths, we obtain a set of $\check{\nu}$-trees that are in fact contained in a smaller shape than $F_{\check{\nu}}$. We call them (δ, ν)-trees and define them slightly differently below—we will prove in Chapter 6 that the two definitions are equivalent.

Definition 4.2.10. Let ν be a path, δ an increment vector with respect to ν and $\check{\nu}$ defined as above.

Let $\hat{\nu}$ be the path that starts at the point of coordinates $(\check{\nu}_0, 0)$, i.e. the lowest right corner of $F_{\check{\nu}}$, and which consists of the sequence of west and north steps

$$W^{\nu_0} N W^{\gamma_1} N W^{\gamma_2} \ldots N W^{\gamma_n}, \qquad \text{for } \gamma_i = \nu_i - \delta_i. \tag{4.3}$$

The shape $F_{\delta,\nu}$ is the subset of $F_{\check{\nu}}$ consisting of the boxes that are not below $\hat{\nu}$, and we denote by $L_{\delta,\nu}$ its set of lattice points. A **(δ, ν)-tree** is a maximal collection of pairwise $\check{\nu}$-compatible elements in $L_{\delta,\nu}$.

An example is illustrated on Figure 4.8.

Theorem 4.2.11. *The (δ, ν)-trees are exactly the images of ν-paths under the right flushing bijection with respect to $\check{\nu}$.*

The rotation poset of (δ, ν)-trees, denoted $\mathrm{Tam}_\nu^{tr}(\delta)$, is isomorphic to the alt ν-Tamari lattice $\mathrm{Tam}_\nu(\delta)$.

The three different rotation lattices of (δ, ν)-trees for $\nu = ENEEN$ are illustrated in Figure 4.9.

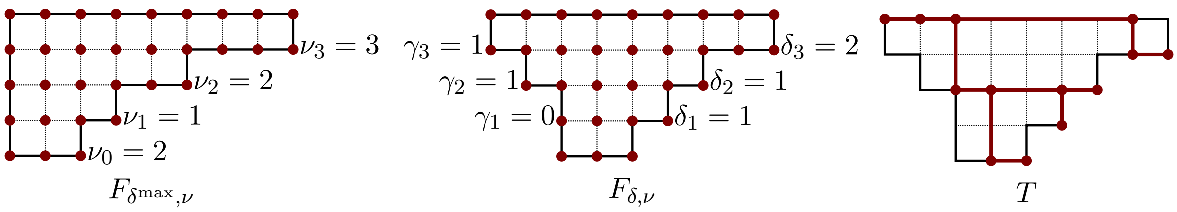


Figure 4.8: Left: The Ferrers diagram $F_{\delta^{\max},\nu}$ and its corresponding lattice points $L_{\delta^{\max},\nu}$ for $\nu = EENENENEEENEEEN = (2,1,2,3,0)$ and $\delta^{\max} = (1,2,3,0)$. Middle: The Ferrers diagram $F_{\delta,\nu}$ and its corresponding lattice points $L_{\delta,\nu}$ for the same ν and $\delta = (1,1,2,0)$; the paths $\check{\nu} = EEEENENENENEEN$ and $\hat{\nu} = WWNNWNWN$. Right: a (δ,ν)-tree.

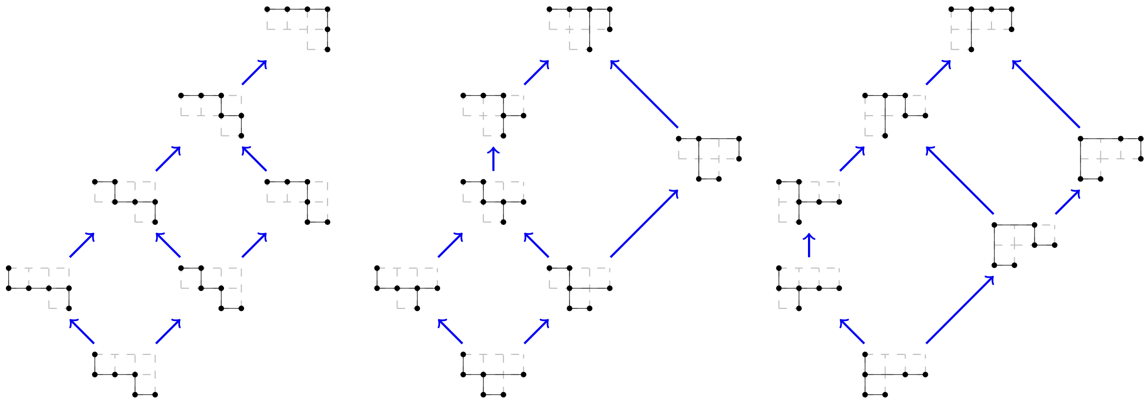


Figure 4.9: Examples of rotation lattices of (δ,ν)-trees for $\nu = ENEEN$. Left: the ν-Dyck lattice, for $\delta^{\min} = (0,0)$. Middle: the lattice for $\delta = (1,0)$. Right: the ν-Tamari lattice, for $\delta^{\max} = (2,0)$.

4.3 The Cambrian and m-Cambrian lattices

The Cambrian lattices introduced in Section 3.4 are a family of lattices defined by N. Reading, and the Tamari lattice appears as the linear type A Cambrian lattice. In Section 9.2, we define and study the m-Cambrian lattices, which are a generalization of them, introduced by C. Stump, H. Thomas and N. Williams in [STW18]. As for the m-Tamari lattices, m is a positive integer and the case $m = 1$ corresponds to the classical Cambrian lattices.

The idea to go from the Cambrian to the m-Cambrian lattices is to replace the Coxeter group W with the (positive) Artin monoid, which is somehow the space of words on the simple generators of W, quotiented by braid moves. In this framework, the group W embeds as a subset of the monoid, sending an element to its reduced word. Most notions defined in the Coxeter group have an equivalent in the Artin monoid, and in particular, the weak order and the c-sortability can be naturally extended. Then, the Cambrian lattice identifies as the restriction of the weak order to the c-sortable elements of the interval $[\boldsymbol{e}, \boldsymbol{w}_\circ]$ or equivalently to the c-sortable elements of the Artin monoid with Garside degree at most 1. Then the m-Cambrian lattice can naturally be defined as the restriction of the weak order (on the Artin monoid) to the c-sortable elements of the interval $[\boldsymbol{e}, \boldsymbol{w}_\circ^m]$ or equivalently to the c-sortable elements of the Artin monoid with Garside degree at most m.

In fact, the other two definitions of the Cambrian lattice can be extended in this direction, using subword complexes and m-chains in the noncrossing partition lattices, and all three constructions provide isomorphic lattices, as will be discussed in Section 9.2. A new construction of the m-Cambrian lattice is also presented in Section 9.4.

4.4 The posets of tilting modules

To complete the picture, the posets of tilting modules form another family of posets that generalize the Tamari lattice, though not central in this study. As for the Cambrian lattices, each choice of an orientation of a Coxeter graph (or Coxeter element) produces such a poset, and the Tamari lattice also appears as the linear type A poset of tilting modules. These posets can be defined in terms of representation theory of quivers, starting with an orientation of a Dynkin diagram, which is equivalent to choose a Coxeter element c in the corresponding (crystallographic) Coxeter group W.

They are in fact deeply connected to the Cambrian lattices, as they can also be defined as the **positive part** of the corresponding Cambrian lattices $\mathrm{Camb}(W, c)$, that is to say the restriction of the Cambrian lattices to the elements with full support, and this can be taken as a definition for all finite Coxeter groups. In terms of simplicial complexes, this is equivalent to taking the subcomplex of the Cambrian complex whose vertices are the elements that share no vertex with the minimal element.

Definition 4.4.1. Let c be a standard Coxeter element in the Coxeter group W. The **poset of tilting modules** $\mathrm{Tilt}(W, c)$ is the restriction of $\mathrm{Camb}(W, c)$ to the set of elements with full support. Equivalently, it is the flip poset of the subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{w}_\circ(\mathsf{c}), c^{-1}w_\circ)$.

Note that the Cambrian lattice $\mathrm{Camb}(W, c)$ was defined in Definition 3.4.19 as the flip poset of the dual subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{cw}_\circ(\mathsf{c}), w_\circ)$, and this definition is equivalent to removing the initial c of the search word.

In the broader context of quiver representation theory, which however lies out of the scope of this memoir, a **tilting module** in the category of modules $\mathsf{mod}Q$ over a quiver Q with n vertices is an object T which is a direct sum of n pairwise non-isomorphic indecomposable modules and satisfies $\mathrm{Ext}^1(T, T) = 0$. Then a natural partial order or the set of tilting modules gives rise to the so-called poset of tilting modules $\mathrm{Tilt}(Q)$, as introduced and studied by D. Happel and L. Unger, and by C. Riedtmann and A. Schofield in [RS91, HU05].

To pursue in this quiver representation framework, when studying a poset P, one can consider its incidence algebra $I(P)$ and look at the finite dimensional modules over this algebra, which form the **category of modules** $\mathsf{mod}P$. This is the same as taking the representations of the Hasse diagram of P viewed as a quiver, together with commuting relations, namely two paths with the same two endpoints are equal.

For both families of Cambrian lattices and posets of tilting modules, the choice of two different Coxeter elements c and c' in the same group W usually gives in general two non-isomorphic lattices, whose categories of modules are not equivalent. However, using the so-called flip-flop technique, Ladkani proved that their derived categories are always triangle-equivalent, and we say that the two categories of modules (or the posets, for short) are then **derived equivalent** [Lad07b, Lad07a]. The idea behind the flip-flop technique is that modifying slightly the orientation of the Dynkin diagram by transforming a sink into a source changed the posets in a quite understandable way. Such a transformation was then shown to preserve the derived category of modules, and any two posets of the family are connected through a chain of such modifications, viewed as small steps. In relation to this derived-equivalence result, the **Coxeter polynomial** is a discrete invariant of derived equivalence. That is to say, it can be computed directly from the poset, and if two posets are derived-equivalent, they will share the same Coxeter polynomial.

Theorem 4.4.2 ([Lad07b, Lad07a])**.** *Let c and c' be two Coxeter elements in the Coxeter group W. Then the posets of tilting modules* $\mathrm{Tilt}(W, c)$ *and* $\mathrm{Tilt}(W, c')$ *are derived equivalent and the same holds for the corresponding Cambrian lattices* $\mathrm{Camb}(W, c)$ *and* $\mathrm{Camb}(W, c')$.

In Part II, we study the distribution of linear intervals, and in this framework, we conjecture that, similarly to the results of Ladkani on derived equivalence, the distribution of linear intervals of two posets of tilting modules for two different Coxeter elements c and c' have the same distribution of linear intervals. We conjecture similar results for the Cambrian lattices, and even for the m-Cambrian lattices. Some evidence and conjectured formulas are given in Section 7.3.3.

Conjecture 4.4.3. Let c and c' be two Coxeter elements in the Coxeter group W. Then the posets of tilting modules $\mathrm{Tilt}(W, c)$ and $\mathrm{Tilt}(W, c')$ have the same distribution of linear intervals and the same holds for the corresponding Cambrian lattices $\mathrm{Camb}(W, c)$ and $\mathrm{Camb}(W, c')$.

Moreover, for all $m \geq 1$, this also holds for the m-Cambrian lattices $\mathrm{Camb}^{(m)}(W, c)$ and $\mathrm{Camb}^{(m)}(W, c')$.

4.5 The permutree lattices

Lastly, the permutree lattices are another family of lattices that contains the Tamari lattice and thus generalizes it, as presented in Section 2.2.5. They were introduced by V. Pilaud and V. Pons in [PP18], as lattice quotients of the weak order. Each choice of a decoration $\delta \in \{0, 1\}^{2n}$ produces such a permutree lattice $\mathcal{PT}(\delta)$.

In Section 7.3.4, we will particularly focus on their linear intervals. We also conjecture an equidistribution result of their linear intervals and a derived equivalence result on these posets akin to the alt-Tamari lattices, the Cambrian lattices and the posets of tilting modules.

Part II

Linear intervals in the Tamari world

Chapter 5

In the alt-Tamari lattices

This chapter is based on the prepublished article [Che22].

In Section 2.2, we defined two classical partial orders on the set of Dyck paths, namely the Dyck and the Tamari lattices. The former is an extension of the latter, which means that every pair of paths that defines an interval in the Tamari lattice also defines an interval in the Dyck lattice. This implies in particular that the Dyck lattice has more intervals than the Tamari lattice. In fact, the number of intervals of these two posets are known and have nice closed product formulas (see [BB09]).

Using the Lindström-Gessel-Viennot lemma, M. de Sainte-Catherine and G. Viennot proved in [dSCV86] that the number of intervals in the Dyck lattice is equal to

$$\frac{6(2n)!\,(2n+2)!}{n!\,(n+1)!\,(n+2)!\,(n+3)!}. \tag{5.1}$$

Using generating functions and solving a functional equation, as detailed in Chapter 8, Chapoton [Cha06] gave a formula for the number of intervals in the Tamari lattice, namely

$$\frac{2(4n+1)!}{(n+1)!\,(3n+2)!}. \tag{5.2}$$

Guided by computer experimentation, Chapoton proposed to study the enumeration of the subset of **linear intervals**, that is to say intervals which are totally ordered, as he observed that the numbers seemed to be the same in both lattices, even when distinguished according to their length, and furthermore that they were counted by a nice closed formula. However, when a pair of comparable paths (P, Q) defines a linear interval in the Tamari lattice, the corresponding interval in the Dyck lattice is not necessarily linear as well, or may have a different length, which suggests that there is a deeper explanation for this coincidence between these two very interesting posets.

In this chapter, we prove this result and we enumerate the intervals. We also define a new family of partial orders on Dyck paths called alt-Tamari posets, which we introduced quickly in Section 4.2.1. We prove the equidistribution of linear intervals in the entire family, which appears to explain the connection between the Dyck and the Tamari lattices.

5.1 Linear intervals in the Dyck lattice

We start with the case of the Dyck lattice, which is perhaps the most natural poset structure on Dyck paths, where P is smaller than Q if Q lies weakly above P. It is also the one where linear intervals have the simplest description.

5.1.1 Structure of linear intervals

We first define a family of "left" and "right" intervals in the Dyck lattice that we prove to be linear. Then we show that all linear intervals are either trivial, left or right intervals.

Definition 5.1.1. A left interval in the Dyck lattice is an interval $[P, Q]$ where the Dyck word Q is obtained from P by changing a factor $d^\ell u$ into ud^ℓ for some $\ell \geq 1$.

A right interval in the Dyck lattice is an interval $[P, Q]$ where the Dyck word Q is obtained from P by changing a factor du^ℓ into $u^\ell d$ for some $\ell \geq 1$.

Saying it otherwise, a left interval is obtained from a path P where an up step u is moved ℓ times to the left, and a right interval is obtained from a path P where a down step d is moved ℓ times to the right, hence the name. Such an interval is indeed linear of length ℓ since at every step, there is only one valley that can be changed into a peak such that the path remains under Q. Note that left and right intervals are exchanged by the mirror involution on Dyck paths mentioned in Remark 2.2.3.

Remark 5.1.2. Covering relations are exactly all intervals that are both left and right intervals, as can be seen in Figure 5.1.

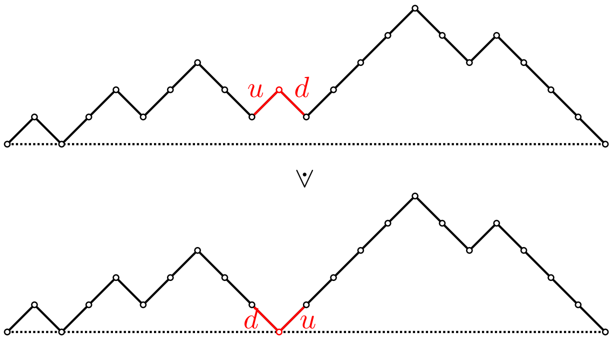


Figure 5.1: A covering relation in the Dyck lattice.

Proposition 5.1.3. *Linear intervals of length $\ell = 2$ in the Dyck lattice are either left or right intervals.*

Proof. Suppose we have a linear interval of length 2, that is of the form $P \lessdot Q \lessdot R$. Then the Dyck word Q is obtained from P by transforming a valley du into a peak ud.

If the next covering relation uses the last step d of the peak of Q we just produced, then the valley of P was followed by an up step and R is obtained from P by changing a factor duu into uud. Thus, $[P, R]$ is a right interval of length 2.

Similarly, if the next covering relation uses the first step u of the peak of Q we just produced, then the valley of P was preceded by a down step and R is obtained from P by changing a factor ddu into udd. Thus, $[P, R]$ is a left interval of length 2.

If the next covering relation happens at a valley somewhere else in Q, then this valley exists already in P and the two covering relations can be performed independently, thus $[P, R]$ would be a square as shown in Figure 5.2 and not a linear interval.

□

Proposition 5.1.4. *All linear intervals of length $\ell \geq 2$ in the Dyck lattice are either left or right intervals.*

Proof. The case for $\ell = 2$ is true thanks to Proposition 5.1.3, and we now proceed by induction.

Let $[P, Q]$ be a linear interval of length $\ell + 1 \geq 3$, and Q' be the lower cover of Q in $[P, Q]$. Then $[P, Q']$ is a linear interval of length $\ell \geq 2$. By induction it is either a left or right interval.

If $[P, Q']$ is a left interval, then Q' is obtained from P by changing a factor $d^\ell u$ into ud^ℓ. Furthermore, in the chain from P to Q', the peak created in Q' is formed by the first two steps

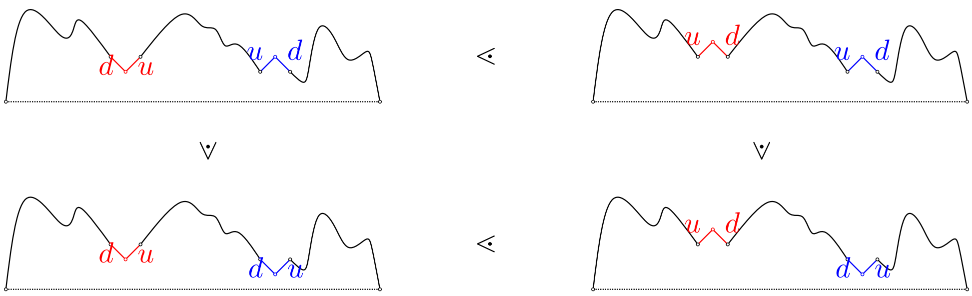


Figure 5.2: A square in the Dyck lattice.

of this factor ud^ℓ, and is not followed by an up step in Q'. Thus, the covering relation $Q' \lessdot Q$ has to use this first up step u of this factor ud^ℓ, as otherwise the interval would not be linear. Then Q is obtained from P by changing a factor $d^{\ell+1}u$ into $ud^{\ell+1}$ and $[P, Q]$ is a left interval of length $\ell + 1$.

If $[P, Q']$ is a right interval, then Q' is obtained from P by changing a factor du^ℓ into $u^\ell d$. Symmetrically, the covering relation $Q' \lessdot Q$ has to use this last down step d, and we get that $[P, Q]$ is a right interval of length $\ell + 1$. □

Remark 5.1.5. The paths in a right interval of length $\ell \geq 1$ have at least $\ell + 1$ up steps since the bottom element of the interval has du^ℓ as a factor. Similarly, paths in a left interval of length ℓ have at least $\ell + 1$ down steps since the bottom element has $d^\ell u$ as a factor.

Thus, for $n \geq 1$ there are no linear intervals of length $\ell \geq n$ in Dyck_n.

5.1.2 Combinatorial description of linear intervals

We have described the structure of all linear intervals in the Dyck lattice according to their length. We now deduce a combinatorial description of them in order to count them. More precisely, we will produce a functional equation on the generating function of the linear intervals of a fixed length in the Dyck lattice, that we will solve later in Section 5.3.

For $\ell \geq 0$, let $S_\ell(t)$ be the generating function of linear intervals of length ℓ in the Dyck lattices. For this section, $A(t) = \sum_{n \in \mathbb{N}} C_n t^n$ will denote the generating function of Dyck paths. Furthermore, as every Dyck path of size n has n down (resp. up) steps, the generating function of Dyck paths marked at a down (resp. up) step is equal to $tA'(t)$.

Firstly, a linear interval of length 0 is of the form $[P, P]$ where P is a nonempty Dyck path. We can thus write:

$$S_0 = A - 1. \tag{5.3}$$

Let $[P, Q]$ be a linear interval of length 1, *i.e.* a covering relation. Then Q is obtained from P by changing a valley du into a peak ud. Let uD_1d be the excursion whose first step is the up step of this valley of P. Then D_1 as a word is any Dyck word, maybe empty.

In fact, as we can see in Figure 5.3, any covering relation $P \lessdot Q$ can be understood as a path $D_\bullet$ with a marked down step $d_\bullet$, before which we insert duD_1 for the bottom element P and udD_1 for the top element Q. Hence, covering relations $P \lessdot Q$ are in bijection with pairs $(D_\bullet, D_1)$ consisting of a Dyck path $D_\bullet$ marked at a down step and a Dyck path D_1. The total number of up steps of P (and of Q) is one plus the sizes of $D_\bullet$ and Q since we need to insert an extra du (or ud). We have thus:

$$S_1 = t(tA')A = t^2 A' A. \tag{5.4}$$

Similarly, as in Figure 5.4, let $[P, Q]$ be a right interval of length $\ell \geq 2$. Then, Q is obtained from P by changing a factor du^ℓ to $u^\ell d$. Let us denote $d_1, \ldots, d_\ell$ the down steps of P matching

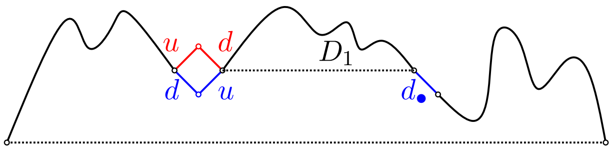


Figure 5.3: Decomposition of a covering relation in the Dyck lattice.

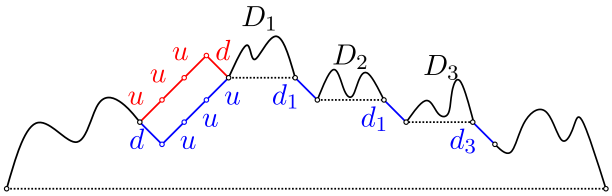


Figure 5.4: Decomposition of a right interval of length 3 in the Dyck lattice.

with these ℓ up steps. Then, in P we have a factor of the form $du^{\ell}D_1d_1\dots D_\ell d_\ell$, where $D_1,\dots,D_\ell$ are ℓ Dyck words, and in Q, we have instead the factor $u^{\ell}dD_1d_1\dots D_\ell d_\ell$.

Thus, any right interval of length $\ell \geq 2$ can be understood as a Dyck path $D_\bullet$ with a marked down step $d_\bullet(= d_\ell)$, before which we insert $du^{\ell}D_1d\dots dD_\ell$ for the bottom element and $u^{\ell}dD_1d\dots dD_\ell$ for the top element, with any ℓ Dyck words $D_1,\dots,D_\ell$. Right intervals of length ℓ in the Dyck lattice of size n are then in bijection with tuples $(D_\bullet, D_1,\dots,D_\ell)$ of $\ell+1$ Dyck paths, the first being marked at a down step, and whose total sizes add up to $n-\ell$.

Symmetrically, any left interval of length $\ell \geq 2$ can be understood as a Dyck path with a marked up step $u_\bullet$, after which we insert $D_1u\dots uD_\ell d^{\ell}u$ for the bottom element and $D_1u\dots uD_\ell ud^{\ell}$ for the top element, with ℓ Dyck words $D_1,\dots,D_\ell$. We have a similar bijection with tuples of paths $(D_\bullet, D_1,\dots,D_\ell)$ as previously.

Recall from Remark 5.1.2 that linear intervals of length at least 2 can not be both left and right. Thanks to this, for any $\ell \geq 2$, we finally have:

$$S_\ell = tA'(tA)^{\ell} + tA'(tA)^{\ell} = 2t^{\ell+1}A'A^{\ell}. \tag{5.5}$$

We have now expressed every generating function S_ℓ of linear intervals in the Dyck lattices as a function of the series A and A' of Catalan and marked Catalan objects. We gather the results in the following proposition.

Proposition 5.1.6. *Let $\ell \geq 0$, the generating function of linear intervals in the Dyck lattices satisfies the following equations:*

$$S_\ell = \begin{cases} A-1 & \text{if } \ell = 0,\\ t^2A'A & \text{if } \ell = 1,\\ 2t^{\ell+1}A'A^{\ell} & \text{if } \ell \geq 2.\end{cases}$$

In Section 5.3, we will rewrite these equations before solving them, but we first show in the next section that the generating functions of linear intervals in the Tamari lattices satisfy the same equations as above, proving the equidistribution result.

5.2 Linear intervals in the Tamari lattice

We now consider the case of the Tamari lattice. As explained in Remark 2.2.13, the Tamari lattice can be described on Dyck paths, in such a way that it is extended by the Dyck lattice. Thus, every pair of comparable elements (P, Q) in the Tamari lattice is still comparable in the

Dyck lattice. We will prove that both lattices have the same number of linear intervals of any length, however when $[P, Q]$ is linear interval in Tam_n, the pair (P, Q) will usually not define a linear interval in Dyck_n, or not necessarily of the same length.

In this section, we are using the description on binary trees. The result will be redundant with the more general case of the alt-Tamari posets in Section 5.4, but the study is inspiring for the next chapter, where the use of trees is very fruitful.

5.2.1 Structure of intervals

In [Cha06], the author introduces an operadic structure on intervals in the Tamari lattices with the operation of grafting of intervals. The author also defines new intervals and proves that every interval as a poset can be uniquely written as a product of new intervals. See this article for more details on the results of this section.

Recall that a (planar rooted) binary tree of size n has n nodes and $n+1$ leaves. We number the leaves from left to right, starting at 0.

Definition 5.2.1. Let $I = [P, Q]$ be an interval in Tam_n and $I' = [P', Q']$ an interval in Tam_m. The **grafting** of I' on I at the k-th leaf, for some $0 \leq k \leq n$, is the interval $I'' = [P'', Q''] \in \mathrm{Tam}_{n+m}$, where the bottom element P'' is obtained by grafting P' on the k-th leaf of P and the top element Q'' is obtained by grafting Q' on the k-th leaf of Q.

As a poset, I'' is isomorphic to the product of the intervals I and I'.

The grating operation is represented in Figure 5.5.

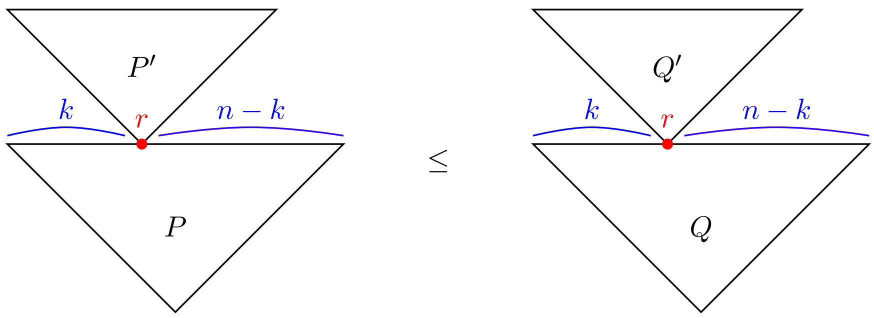


Figure 5.5: Grafting an interval $[P', Q']$ on the k-th leaf of the interval $[P, Q]$.

Definition 5.2.2. An interval in the Tamari lattice is **new** if it can not be written as the grafting of two intervals.

Note that we excluded the Tamari lattice of size 0, and thus, new intervals are well-defined. This gives a unique decomposition of intervals into new intervals. Any interval then has the structure of the product of the new intervals in its decomposition.

Remark 5.2.3. If an interval $I'' = [P'', Q'']$ is not new, then it decomposes into an interval $I' = [P', Q']$ grafted on the k-th leaf of some interval $I = [P, Q] \in \mathrm{Tam}_n$ for some $0 \leq k \leq n$.

Thus, there is a common node r in P'' and Q'' such that, excluding the leaves in the respective subtrees starting at r, there are k leaves on the left of r and $n - k$ leaves on its right, as shown in Figure 5.5. The node r is precisely the root node of P' and Q' that has been identified with the selected leaf of P and Q.

In fact, such common nodes between the bottom and top elements of an interval are precisely the root nodes of the new intervals in its decomposition.

Theorem 5.2.4 ([Cha06, Proposition 7.2])**.** *Every interval in the Tamari lattice has a unique decomposition into new intervals, and as a poset, is the product of these intervals.*

Corollary 5.2.5. *An interval in the Tamari lattice is trivial if and only if all the new intervals of its decomposition are trivial, and it is linear of length $\ell \geq 1$ if and only if one interval of its decomposition is linear of length ℓ and all the others are trivial.*

5.2.2 Structure of linear intervals

As in the Dyck lattice, we will define "left" and "right" intervals, and prove that they are exactly all nontrivial linear intervals. For this, let us first define two particular intervals L_n and R_n in Tam_{n+1}, for $n \geq 1$. We will see that they are actually the only linear new intervals in Tam_{n+1} and also its linear intervals of greatest length, namely n.

Definition 5.2.6. Let $n \geq 1$. The interval R_n has the right comb r_{n+1} as bottom element and its top element is obtained from r_{n+1} by performing one rotation at each node of the right side, from top to bottom. In other words, the top element of R_n is the tree whose root node has a leaf as right child and a right comb r_n as left child.

The interval L_n is the mirrored version of R_n. Its top element is the left comb ℓ_{n+1} and its bottom element is the tree whose root node has a leaf as left child and a left comb ℓ_n as right child.

Note that $L_1 = R_1$ is the interval whose bottom element is r_2 and whose top element is ℓ_2, it is linear of length 1 and new. The example of R_3 with all four elements is given in Figure 5.6, as well as the bottom and top elements of L_4.

Remark 5.2.7. For $n \geq 1$, both intervals L_n and R_n are linear of length n since there is only one possible way to perform a right rotation to go down in R_n at every step (or a left rotation to go up in L_n).

Moreover, they are new since their top and bottom elements do not share any common node as explained in Remark 5.2.3. Indeed, all nodes in the bottom element of L_n have exactly one leaf strictly on their left whereas all nodes in the top element of L_n have no leaves strictly on their left.

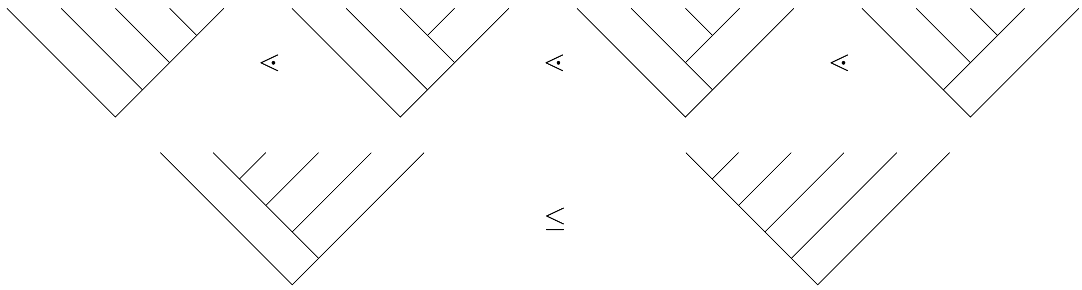


Figure 5.6: The intervals R_3 with all 4 elements (top) and L_4 (bottom).

We now prove that there are three kinds of linear intervals in the Tamari lattice, namely trivial intervals, "left" or "right" intervals.

Definition 5.2.8. A left (resp. right) interval is an interval obtained defined by first grafting trivial intervals (or nothing) on the leaves of an interval L_n (resp. R_n) for some $n \geq 1$ and then grafting or not the resulting interval on a trivial interval.

As a poset, such a left or right interval is the product of trivial intervals and one linear interval of length n, and thus, it is linear of length n. Note that covering relations are once again exactly all intervals that are both left and right interval since $L_n \neq R_n$ for $n \geq 2$. It remains to prove

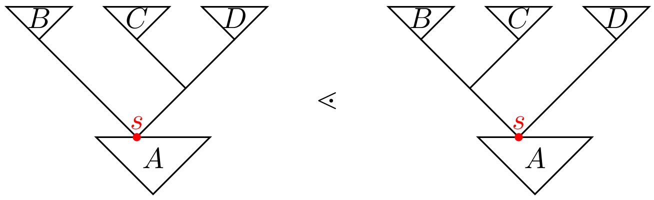


Figure 5.7: A covering relation in the Tamari lattice. It is both a left and a right interval.

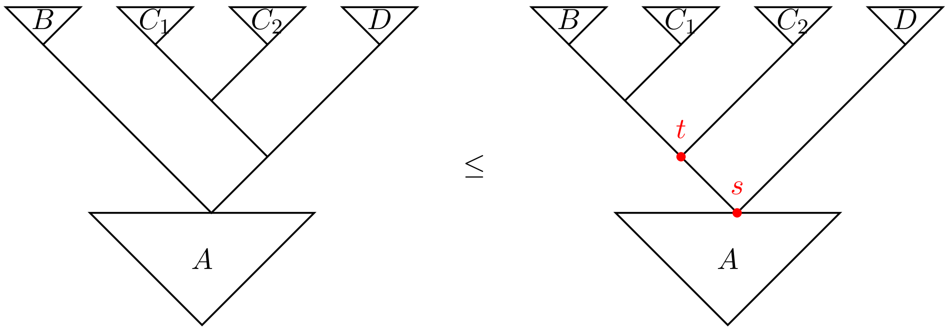


Figure 5.8: A left interval of length 2 in the Tamari lattice.

that all linear intervals are of this form. The structure of the proof is exactly the same as for the case of the Dyck lattice. However, it requires being more careful as more cases exist.

We already noticed that linear intervals of length 1 being precisely covering relations, they are indeed left and right intervals. We now address the case of linear intervals of greater length.

Proposition 5.2.9. *All linear intervals of length $\ell \geq 2$ in the Tamari lattice are either left or right intervals.*

Proof. We prove the result for $\ell = 2$ and then by induction.

Suppose we have a linear interval of length 2, that is of the form $P \lessdot Q \lessdot R$. Then we know that $P \lessdot Q$ is a covering relation at some node s as in Figure 5.7, for some trees A, B, C, D. More precisely, P (resp. Q) can be described as a comb r_2 (resp. ℓ_2) grafted on the k-th leaf of a tree A, with the trees B, C, D grafted from left to right on the leaves of r_2 (resp. ℓ_2).

Let us study the next possible rotations for the covering relation $Q \lessdot R$. As we can see in Figure 5.10, if we perform a rotation within A, B, C or D, then we get a square, thus, not a linear interval. More precisely, if the next rotation $Q \lessdot R$ is such that R can be described as a comb ℓ_2 grafted on the k-th leaf of a tree A', with the trees B', C', D' grafted from left to right on the leaves of ℓ_2, then we can define a tree P' as R where the left comb ℓ_2 is replaced by a right comb r_2. Then we have $P \lessdot P' \lessdot R$ and $[P, R]$ is not linear.

This being excluded, there remains at most three possible nodes for a rotation as in Figure 5.11, namely the same node s, its left successor t (if any), and its predecessor u (if s is a right child).

If we perform another rotation at the node s, we get a pentagon as in Figure 5.12, thus not a linear interval.

Otherwise, a rotation at the node t produces a left interval of length 2 as in Figure 5.8 and a rotation at the node u produces a right interval of length 2 similar to the one in Figure 5.9, both being linear intervals.

Let now $[P, Q]$ be a linear interval of length $\ell \geq 3$. Let Q' be the lower cover of Q in $[P, Q]$, such that $[P, Q']$ is a linear interval of length $\ell - 1 \geq 2$. By induction, it is either a left of a right interval.

Assume $[P, Q']$ is a left interval. Then, the last rotation in $[P, Q']$ happens at a node s as in

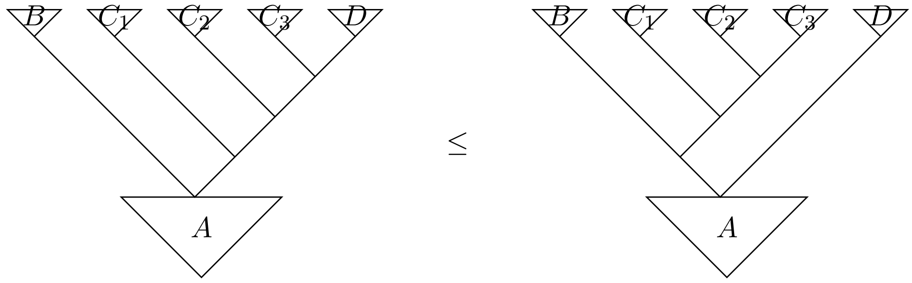


Figure 5.9: A right interval of length 3.

Figure 5.8. The node s is not a right child, so the only possible rotation to get a linear interval is at its left successor t. Thus, $[P, Q]$ is a left interval of length ℓ.

Assume $[P, Q']$ is a right interval, and the last rotation happens at a node s, then a rotation to its left child (if possible) would produce a nonlinear interval as in Figure 5.12. The only possible rotation is thus at the predecessor of s and $[P, Q]$ is a right interval of length ℓ. $\square$

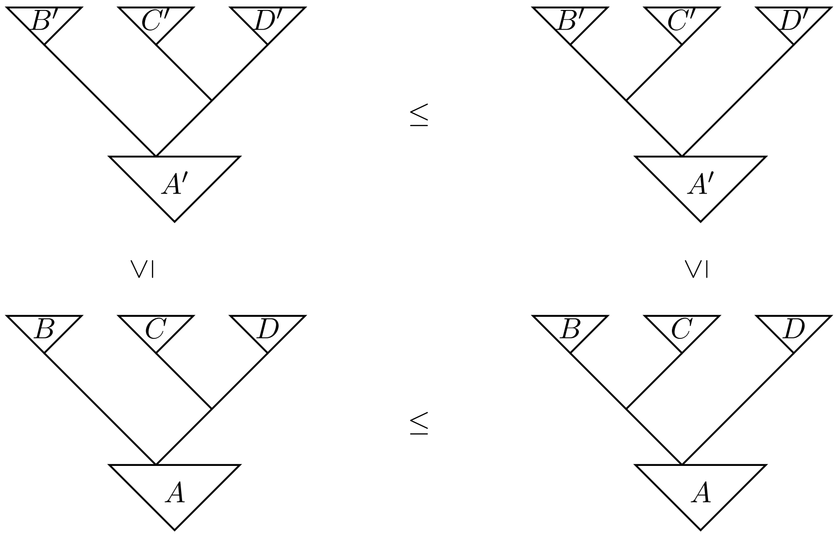


Figure 5.10: Nonlinear interval.

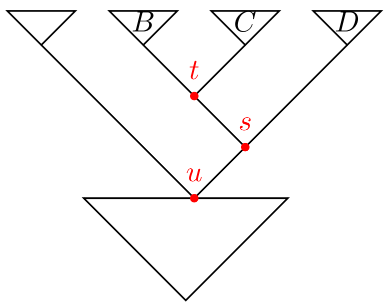


Figure 5.11: The tree Q after rotation at s.

We have proven that all linear intervals of length $\ell \geq 1$ decompose as an interval L_ℓ or R_ℓ, grafted on some tree A (possibly of size 0, which is equivalent to no grafting) with some (possibly trivial) trees grafted on the leaves of L_ℓ or R_ℓ. Thus, as long as the tree A or one of the trees we

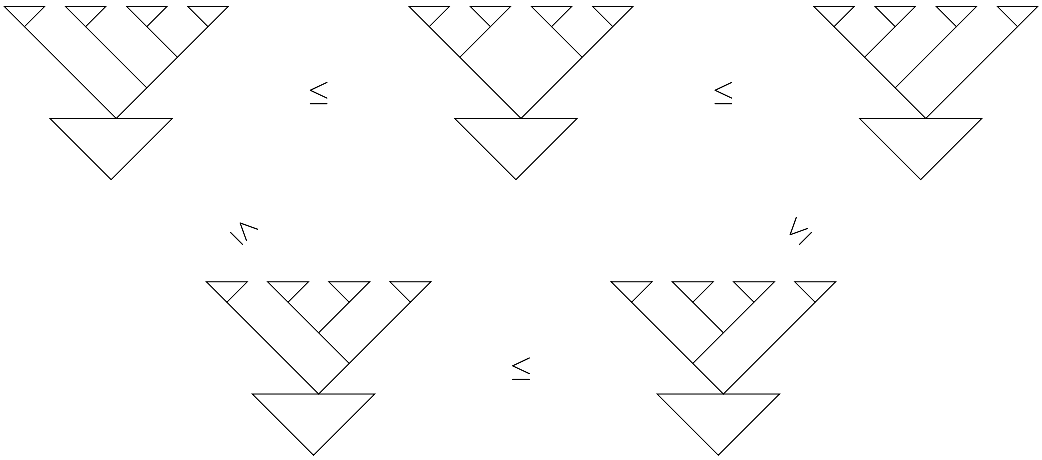


Figure 5.12: Two consecutive rotations at the same node produce a nonlinear interval.

grafted on L_ℓ or R_ℓ is a nontrivial tree, we get an interval which is not new. Consequently, L_ℓ and R_ℓ are the only linear intervals that are new in $\mathrm{Tam}_{\ell+1}$.

Remark 5.2.10. A linear interval of length $\ell \geq 1$ has at least $\ell + 1$ nodes (those of L_ℓ or R_ℓ). Thus, for $n \geq 1$ there are no linear intervals of length $\ell \geq n$ in Tam_n.

5.2.3 Combinatorial description of linear intervals

As for the Dyck lattice, we now give a combinatorial description of the linear intervals in the Tamari lattice in order to write an expression of their generating functions. In fact, we will produce the very same equations as for the Dyck lattice in Proposition 5.1.6, and this will prove that the two lattices have the same number of linear intervals of any fixed length.

For $\ell \geq 0$, let $T_\ell(t)$ be the generating function of linear intervals of length ℓ in the Tamari lattices. Recall that the binary trees are in bijection with Dyck paths, and that $A(t) = \sum_{n\in\mathbb{N}} C_n t^n$ is thus also their generating function. As for the Dyck paths, the generating function of binary trees marked at a node is equal to $tA'(t)$.

Proposition 5.2.11. *For any $\ell \geq 0$, the generating function $T_\ell(t)$ of linear intervals of length ℓ in the Tamari lattices is equal to the generating function $S_\ell(t)$ of linear intervals of length ℓ in the Dyck lattices.*

Proof. Again, a linear interval of length 0 is of the form $[P, P]$ where P is a tree not reduced to a leaf. We can write

$$T_0 = A - 1, \tag{5.6}$$

which proves that T_0 is equal to S_0 thanks to Equation (5.3).

A linear interval of length 1 is a covering relation. As we can see in Figure 5.7, the part with A, B and D can be understood as a tree with a marked node s, together with another tree C that we plug into the right edge out of s for the bottom element and into the left edge out of s for the top element. Note that A, B, C and D might be just leaves. All in all, if we call $T_\bullet$ the tree formed of A, B and D, we have a binary tree marked at the node s. Thus, covering relations are in bijection with pairs $(T_\bullet, C)$ consisting of a binary tree $T_\bullet$ marked at a node and a binary tree C. As the plugging of C creates an extra node, we get the same equation as Equation (5.4) for T_1:

$$T_1 = tA'tA = t^2 A'A. \tag{5.7}$$

Similarly, as in Figures 5.8 and 5.9, a linear interval of length $\ell \geq 2$ can be understood as a tree formed of A, B and D, with a marked node s and a direction (left or right), together with a sequence of ℓ trees $C_1, \ldots, C_\ell$. Those ℓ trees are then plugged into the selected edge going out of s or on a common branch, which is then plugged into the other edge out of s. Each plugging creates an extra node, and we have this equation on T_ℓ:

$$T_\ell = 2tA'(tA)^\ell = 2t^{\ell+1}A'A^\ell, \tag{5.8}$$

which proves thanks to Equation (5.5) the equality with S_ℓ.

□

As a corollary we have the desired result:

Theorem 5.2.12. *For any $n \geq 1$ and $\ell \geq 0$, the Tamari lattice* Tam_n *and the Dyck lattice* Dyck_n *have the same number of linear intervals of length ℓ.*

5.3 Enumeration of linear intervals

We have written equations on the generating functions of linear intervals of a fixed length in the Dyck and the Tamari lattices. In this section, we use Lagrange inversion [Sta24, ch. 5] as in Section 1.2 to get the coefficients of S_ℓ. Let us introduce the series $B = A - 1$ of nontrivial trees. We write $B = t(B+1)^2 = tF(B)$, where $F(x) = (x+1)^2$. Then, it follows that

$$\begin{aligned}(B+1)B' &= (B+1)^3 + 2B't(B+1)^2\\ B' &= \frac{(B+1)^3}{1-B}.\end{aligned}$$

For $\ell \geq 2$, we can write:

$$S_1 = t^2B'(B+1) = t^2\phi_1(B) \quad \text{and} \quad S_\ell = 2t^{\ell+1}B'(B+1)^\ell = 2t^{\ell+1}\phi_\ell(B),$$

where $\phi_\ell(x) = \dfrac{(1+x)^{\ell+3}}{1-x}$ for $\ell \geq 1$.

As $\phi_\ell(x)$ is a series in x with a constant term equal to 1 and $B(t)$ is a series in t with no constant term, we have $[t^0]\phi_\ell(B) = 1$.

We can write $\dfrac{1}{1-x} = \displaystyle\sum_{k\geq 0} x^k$ and $\dfrac{1}{(1-x)^2} = \displaystyle\sum_{k\geq 0}(k+1)x^k$. Let us compute $[t^n]\phi_\ell(B)$, using Lagrange inversion, for $n \geq 1$:

$$\begin{aligned}
[t^n]\phi_\ell(B) &= \frac{1}{n}[x^{n-1}]\phi_\ell'(x)F(x)^n\\
&= \frac{1}{n}[x^{n-1}]\left((\ell+3)\frac{(1+x)^{\ell+2}}{1-x} + \frac{(1+x)^{\ell+3}}{(1-x)^2}\right)((1+x)^2)^n\\
&= \frac{\ell+3}{n}[x^{n-1}]\frac{(1+x)^{\ell+2+2n}}{1-x} + \frac{1}{n}[x^{n-1}]\frac{(1+x)^{\ell+3+2n}}{(1-x)^2}\\
&= \frac{\ell+3}{n}\sum_{j=0}^{n-1}\binom{\ell+2+2n}{j} + \frac{1}{n}\sum_{j=0}^{n-1}\binom{\ell+3+2n}{j}(n-j-1+1)\\
&= \frac{\ell+3}{n}\sum_{j=0}^{n-1}\binom{\ell+2+2n}{j} + \sum_{j=0}^{n-1}\binom{\ell+3+2n}{j} - \frac{1}{n}\sum_{j=0}^{n-1}j\binom{\ell+3+2n}{j}\\
&= \frac{\ell+3}{n}\sum_{j=0}^{n-1}\binom{\ell+2+2n}{j} + \sum_{j=0}^{n-1}\binom{\ell+3+2n}{j} - \frac{1}{n}\sum_{j=0}^{n-2}(\ell+3+2n)\binom{\ell+2+2n}{j}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\ell+3}{n}\binom{\ell+2+2n}{n-1} + \sum_{j=0}^{n-1}\binom{\ell+3+2n}{j} - 2\sum_{j=0}^{n-2}\binom{\ell+2+2n}{j} \\
&= \frac{\ell+3}{n}\binom{\ell+2+2n}{n-1} + \sum_{j=0}^{n-1}\binom{\ell+2+2n}{j} + \sum_{j=1}^{n-1}\binom{\ell+2+2n}{j-1} - 2\sum_{j=0}^{n-2}\binom{\ell+2+2n}{j} \\
&= \frac{\ell+3}{n}\binom{\ell+2+2n}{n-1} + \binom{\ell+2+2n}{n-1} \\
[t^n]\phi_\ell(B) &= \binom{\ell+2+2n}{n}.
\end{aligned}
$$

We can notice that this formula is still true for $n = 0$. We now extract the coefficients of S_ℓ, and we have the enumeration of linear intervals in the Dyck and the Tamari lattices.

Theorem 5.3.1. *In the Tamari lattice* Tam_n *of size* n *and the Dyck lattice* Dyck_n *of size* n*, there are:*

- $\frac{1}{n+1}\binom{2n}{n}$ *linear intervals of length* 0*,*
- $\binom{2n-1}{n+1}$ *linear intervals of length* 1*,*
- $2\binom{2n-\ell}{n+1}$ *linear intervals of length* ℓ*, for* $2 \le \ell < n$*.*

Furthermore, there are no linear intervals of length $\ell \ge n$*.*

This adds up to $\frac{1}{n+1}\binom{2n}{n} + \binom{2n-1}{n-2} + 2\binom{2n-1}{n+2}$ *linear intervals in* Tam_n *and* Dyck_n*.*

Proof. For the intervals of length 0, this is the number of elements in Tam_n and Dyck_n.

As stated in Remarks 5.1.5 and 5.2.10, we already know that any linear interval in Dyck_n or Tam_n is of length at most $n-1$. Thus, we can fix $n > \ell > 0$ and use the previous results to get the number of intervals of length ℓ.

We have $[t^n]\phi_\ell(B) = \binom{\ell+2+2n}{n}$. Thus, we get:

$$[t^n]t^{\ell+1}\phi_\ell(B) = \binom{2n-\ell}{n-\ell-1}.$$

Now, for $\ell = 1$, we have $S_1 = t^2\phi_1(B)$ thus there are $\binom{2n-1}{n-2} = \binom{2n-1}{n+1}$ intervals of length 1.

For $2 \le \ell < n$, we have $S_\ell = 2t^{\ell+1}\phi_\ell(B)$, and there are $2\binom{2n-\ell}{n-\ell-1} = 2\binom{2n-\ell}{n+1}$ intervals of length ℓ.

Finally, we have $\sum_{\ell=2}^{n-1}\binom{2n-\ell}{n+1} = \sum_{\ell=0}^{n-3}\binom{n+1+\ell}{n+1} = \binom{2n-1}{n+2}$. This can be proven combinatorially, as a particular case of the identity $\sum_{k=0}^{b}\binom{a+k}{a} = \binom{a+b+1}{a+1}$. □

5.4 Linear intervals in the alt-Tamari lattices

In Section 4.2.1, we stated that the Dyck and the Tamari lattices can be described in a very similar manner, through covering relations, and we defined without any proof a new family of posets on Dyck paths, called the alt-Tamari lattices, which are described as the transitive reflexive closure of what we called δ-rotations. The term “alt” stands for “altitude”, a notion that we use in order to define them. In this section, we detail the construction of these posets, and we prove that the formulas in Theorem 5.3.1 also hold for the distribution of linear intervals in these new posets. These partial orders turn out to be lattices, hence the name of the section and chapter, but we will prove this in the next chapter only.

More precisely, the alt-Tamari poset $\mathrm{Tam}_n(\delta)$ depends on the choice of an increment vector $\delta = (\delta_1, \ldots, \delta_n) \in \{0,1\}^n$. We first prove that the alt-Tamari posets are well-defined, and that we have a result of refinement of posets whenever two functions δ and δ' are comparable. Moreover, in all these posets, we can again define trivial, left and right intervals and prove that all linear intervals are of this kind. We then give a combinatorial description of these intervals, from which it follows that we have a bijection between linear intervals of any two alt-Tamari posets of the same size, and this bijection preserves the length. This proves the main result of this section, namely Theorem 5.4.25 which states that the number of linear intervals in the alt-Tamari posets $\mathrm{Tam}_n(\delta)$, even when distinguished according to their length, does not depend on the choice of the increment vector δ.

5.4.1 Definition of the alt-Tamari posets

Given a Dyck path P of size $n \geq 1$, we number its up steps with integers $\{1, \ldots, n\}$ increasingly from left to right. For example, the path $uududdud$ will be numbered $u_1u_2du_3ddu_4d$.

If $\delta = (\delta_1, \ldots, \delta_n) \in \{0,1\}^n$ is an increment vector, we defined recursively the **δ-altitude** of integer points of the path P, starting at 0 for the initial point of P, and the vector δ encodes that the up step $u_i, 1 \leq i \leq n$ increases the δ-altitude by δ_i, while all down steps make the δ-altitude decrease by 1. We also defined the **δ-elevation** of a factor A of P as the change of δ-altitude between the starting and ending points of A. We have thus $\mathrm{elev}_\delta(d) = -1$ and $\mathrm{elev}_\delta(u_i) = \delta_i$ for a down step d and the i-th up step u_i, and $\mathrm{elev}_\delta(A) = \sum_{s \in A} \mathrm{elev}_\delta(s)$, where the sum is on the steps of A.

Finally, the **δ-excursion** of an up step u_i of P is the smallest factor C_i of P starting with u_i such that $\mathrm{elev}_\delta(C_i) = 0$. An example is provided in Figure 5.13.

Remark 5.4.1. The δ-excursion is well-defined since the excursion E_i as defined in Section 1.3.2 starting at the up step u_i satisfies $\mathrm{elev}_\delta(E_i) \leq 0$. Moreover, C_i is always a prefix of E_i.

For instance, if $\delta_i = 0$, the δ-excursion of u_i is reduced to u_i. If $\delta_i = 1$ for all i, then the δ-excursion of an up step u_i is exactly the excursion starting at u_i, as defined in Definition 1.3.5. In this case, remark that the notion of δ-altitude is precisely the altitude of Definition 1.3.5 as well.

Given an increment vector δ and a Dyck path P with a valley du_i, the **δ-rotation** of P at the up step u_i as the Dyck path Q obtained from P by exchanging the down step that precedes u_i with the δ-excursion of u_i. In other words, if C_i is the δ-excursion of u_i, we can write $P = AdC_iB$ and $Q = AC_idB$. We denote δ-rotations by $P \lessdot_\delta Q$. The notation will be justified by Proposition 5.4.9 as they will turn out to be covering relations in the alt-Tamari poset that we define later.

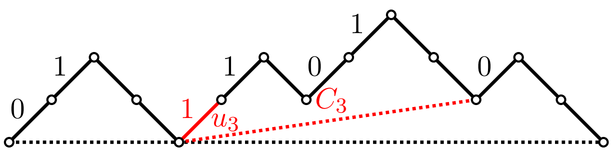


Figure 5.13: The δ-excursion C_3 of u_3 on a Dyck path for $\delta = (0, 1, 1, 1, 0, 1, 0)$.

Lemma 5.4.2. *If Q is the δ-rotation of P at the up step u_i, then Q is strictly greater than P in the Dyck lattice.*

Proof. Let C_i be the δ-excursion of u_i in P. We can write $P = AdC_iB$ as a word, and we have $Q = AC_idB$. The Dyck path Q is obtained from P by moving the down step d that precedes u_i to the right.

This can be achieved as a sequence of covering relations in the Dyck lattice. Indeed, when moving this down step d letter by letter to the right, either it is exchanged with an up step u_k, and it is a covering relation in the Dyck lattice, or it is exchanged with another down step and the Dyck path is unchanged. □

This proves that there are no cycles of δ-rotations. We can thus define the alt-Tamari poset $\mathrm{Tam}_n(\delta) = (\mathcal{Z}_n, \leq_\delta)$ as the transitive closure of δ-rotations on the set of Dyck paths of length n.

Remark 5.4.3.

- The choice of the first entry δ_1 does not change the poset since u_1 is never in a valley.
- The choice of the last entry δ_n of $dd \in \{0,1\}^n$ does not change the poset either. Indeed, the δ-excursion of the last step u_n can be either u_n or u_nd and in both cases, a δ-rotation at the last up step will always change du_nd into u_ndd.
- The case when the increment vector δ is such that $\delta_i = 1$ for all i coincides with the *Tamari lattice*, since the δ-excursion of any up step is always its full excursion.
- The case when the increment vector δ is such that $\delta_i = 0$ for all i coincides with the *Dyck lattice*, since the δ-excursion of any up step is reduced to the up step itself.

Lemma 5.4.4. *Let C_i be the δ-excursion of u_i in a Dyck path P. Let C_j be the δ-excursion of u_j in P with $i \neq j$.*

Either C_i and C_j are disjoint as factors, and we write $C_i \cap C_j = \emptyset$ or one is included in the other. Moreover, they do not end at the same step.

Proof. Suppose $i < j$. Suppose that C_i and C_j are not disjoint. Then u_j is a step of C_i.

Write $C_i = Au_jB$. Then as C_i is the δ-excursion of u_i, it implies that $\mathrm{elev}_\delta(C_i) = 0$ and all its strict prefixes w satisfy $\mathrm{elev}_\delta(w) > 0$, and so we have $\mathrm{elev}_\delta(A) > 0$.

Moreover, as C_j is the δ-excursion of u_j, we have $\mathrm{elev}_\delta(W) \geq 0$ for any prefix of C_j. Hence, $\mathrm{elev}_\delta(AW) = \mathrm{elev}_\delta(A) + \mathrm{elev}_\delta(W) > 0$.

It follows that AC_j is a prefix of C_i, hence C_j is included in C_i.

We can finally notice that C_i and C_j do not end at the same step, for the same reason. □

We define two useful integer vectors associated to a Dyck path P and increment vector δ, namely the "horizontal" vector $h(P)$ and the "length" vector $\ell(P;\delta)$. The former keeps tracks of the positions of the up steps and the latter keeps tracks of the lengths of δ-excursions. We will study how these vectors are modified along δ-rotations, and use this to prove that they are exactly covering relations in the alt-Tamari posets, and later to study linear intervals.

Definition 5.4.5. Let P be a Dyck path of size n and δ an increment vector.

For $1 \leq i \leq n$ we define $h_i(P)$ as the position of u_i in P and $h(P) = (h_1(P), \dots, h_n(P))$ its **horizontal vector**.

For $1 \leq i \leq n$, we define $\ell_i(P;\delta)$ as the length of the δ-excursion of u_i in P and the **length vector** $\ell(P;\delta)$ of P is $(\ell_1(P;\delta), \dots, \ell_n(P;\delta))$.

For instance, if $P = u_1u_2du_3ddu_4d$ and $\delta_i = 1$ for all i, then its horizontal vector is $h(P) = (1, 2, 4, 7)$ and its length vector is $\ell(P;\delta) = (6, 2, 2, 2)$. On the Dyck path P of Figure 5.13 we have $h(P) = (1, 2, 5, 6, 8, 9, 12)$ and $\ell(P;\delta) = (1, 2, 7, 2, 1, 2, 1)$.

Lemma 5.4.6. *Let P be a Dyck path with a valley du_i and C_i be the δ-excursion of u_i. Let Q be the result of the δ-rotation of P at u_i.*

For all j such that $u_j \in C_i$, we have $h_j(Q) = h_j(P) - 1$ and for all other j, $h_j(Q) = h_j(P)$.

Furthermore, if there exists some j such that the δ-excursion of u_j in P ends with the down step d of the valley du_i, then $\ell_j(Q;\delta) = \ell_j(P;\delta) + \ell_i(P;\delta)$. For all other j, we have $\ell_j(Q;\delta) = \ell_j(P;\delta)$.

Proof. The statement about the $h_j(Q)$ is immediate by definition of δ-rotations.

Suppose that there exists j such that the δ-excursion C_j of the up step u_j in P ends with the down step of the valley du_i. Write $C_j = Bd$, and $P = ABdC_iD$. Then $Q = ABC_idD$.

The equality $C_j = Bd$ implies $\mathrm{elev}_\delta(B) = 1$ and for all nonempty prefixes w of B, we have $\mathrm{elev}_\delta(w) > 0$. Similarly, for all prefixes w' of C_i, we have $\mathrm{elev}_\delta(w') \geq 0$. So for all prefixes w'' of BC_i, we have $\mathrm{elev}_\delta(w'') > 0$ and $\mathrm{elev}_\delta(BC_id) = \mathrm{elev}_\delta(BdC_i) = 0$.

Thus, BC_id is the δ-excursion of u_j in Q, and we have $\ell_j(Q;\delta) = \ell_j(P;\delta) + \ell_i(P;\delta)$.

Suppose that j is such that the δ-excursion C_j of the up step u_j in P does not end with the down step of the valley du_i.

If $C_j \cap dC_i = \emptyset$, then it is clear that the δ-excursion of u_j in Q is still C_j.

If $C_j \subseteq C_i$ then it is immediate as well that the δ-excursion of u_j in Q is still C_j, but shifted to the left.

If $C_i \subseteq C_j$ then we write $C_j = AdC_iB$ in P. The δ-excursion of u_j in Q will then be AC_idB and its length does not change either. □

Remark 5.4.7. This proves in particular that for all $Q \geq_\delta P$ in $\mathrm{Tam}_n(\delta)$, $h(Q) \leq h(P)$ and $\ell(Q;\delta) \geq \ell(P;\delta)$ component-wise.

One can wonder if we have the converse implication, which would give a characterization of the alt-Tamari order. This converse implication holds for the Dyck and the Tamari lattices (see Proposition 8.1.2). In what follows we only use the direct implication, and in the next chapter, we will embed these posets as intervals in a ν-Tamari lattice. In this framework, we can define bracket vectors, which are integer vectors attached to the objects in such a manner that the comparison in the lattice corresponds to the comparison of bracket vectors componentwise [CPS20].

Lemma 5.4.8. *Let P a Dyck path such that u_i is in the δ-excursion of u_j.*

For all $Q \geq P$ in $\mathrm{Tam}_n(\delta)$, u_i is in the δ-excursion of u_j.

Proof. It is sufficient to prove this for all covering relations of P. By definition, u_i is in the δ-excursion of u_j in P, if and only if $h_i(P) - h_j(P) \leq \ell_j(P;\delta)$.

Suppose Q is an upper cover of P. Because of Lemma 5.4.6, we have either $h_j(Q) = h_j(P) - 1$ or $h_j(Q) = h_j(P)$.

If $h_j(Q) = h_j(P) - 1$ then, the full δ-excursion of u_j is moved in the δ-rotation. Because u_i is in the δ-excursion of u_j in P, we have also $h_i(Q) = h_i(P) - 1$. In this case, $h_i(Q) - h_j(Q) = h_i(P) - h_j(P)$.

If $h_j(Q) = h_j(P)$ then either we have $h_i(Q) = h_i(P) - 1$ or $h_i(Q) = h_i(P)$. Thus, $h_i(Q) - h_j(Q)$ is either equal to $h_i(P) - h_j(P)$ or to $h_i(P) - h_j(P) - 1$.

In all cases, we have $h_i(Q) - h_j(Q) \leq h_i(P) - h_j(P) \leq \ell_j(P;\delta) \leq \ell_j(Q;\delta)$. The last inequality is again guaranteed by Lemma 5.4.6. Finally, the inequality $h_i(Q) - h_j(Q) \leq \ell_j(Q;\delta)$ implies that u_i is still in the δ-excursion of u_j in Q. □

Proposition 5.4.9. *The covering relations in the poset $\mathrm{Tam}_n(\delta)$ are exactly all the δ-rotations.*

Proof. As the poset is defined as the transitive closure of δ-rotations, all covering relations are δ-rotations. We have to prove the converse, namely that no δ-rotation can be achieved as a nontrivial sequence of δ-rotations.

Suppose Q is the δ-rotation of P at the up step u_i, and C_i is the δ-excursion of u_i in P. We write $P = AdC_iB$ and $Q = AC_idB$.

Then, since A and B are unchanged by the δ-rotation, no δ-rotation is possible at an up step in A or in B.

Suppose $P \lessdot_\delta Q_1 \lessdot_\delta \ldots \lessdot_\delta Q_k = Q$ is a sequence of δ-rotations from P to Q. Then all δ-rotations of the sequence have to happen at steps in C_i.

Because $h_i(Q) = h_i(P) - 1$ and u_i is the only up step of C_i that changes h_i, then one of those δ-rotations has to happen at u_i.

If the first δ-rotation happens at $u_j \neq u_i$, then $h_j(Q_1) = h_j(P) - 1 = h_j(Q)$. Thus, h_j can not decrease anymore. But then, as Lemma 5.4.8 ensures that u_j will always remain in the δ-excursion of u_i in all the sequence, the δ-rotation at u_i would make h_j decrease by 1 and this is not possible. This proves that the first δ-rotation has to happen at u_i and $Q_1 = Q$.

Therefore, $P \leq Q$ is the only chain from P to Q, and it is thus a covering relation. □

Let us now prove that there is a boolean structure of refinement in the family of the alt-Tamari posets. More precisely, given two increment vectors δ and δ' of size n, whenever we have $\delta \leq \delta'$ component-wise, then $\mathrm{Tam}_n(\delta)$ is an extension of $\mathrm{Tam}_n(\delta')$. This means that whenever we have two Dyck paths P and Q such that $P \leq_{\delta'} Q$ in $\mathrm{Tam}_n(\delta')$, then $P \leq_\delta Q$ in $\mathrm{Tam}_n(\delta)$.

Lemma 5.4.10. *Let P be a Dyck path of size n and $\delta \leq \delta'$ two increment vectors. The δ-excursion E_i of u_i in P is a prefix of the δ'-excursion E'_i of u_i.*

Proof. Let w be any strict prefix of E_i. As E_i is a δ-excursion, we have $\mathrm{elev}_\delta(w) > 0$. Then, as $\delta' \geq \delta$, we also have $\mathrm{elev}'_\delta(w) > 0$. Similarly, $\mathrm{elev}'_\delta(E_i) \geq \mathrm{elev}_\delta(E_i) = 0$. □

As the alt-Tamari lattice $\mathrm{Tam}_n(\delta')$ is defined as the transitive reflexive closure of δ'-rotations, it is sufficient to prove that any δ'-rotation $P \lessdot_{\delta'} Q$ defines an interval $[P, Q]$ in $\mathrm{Tam}_n(\delta)$.

Proposition 5.4.11. *Let $\delta \leq \delta'$ be two increment vectors of size n. Let $P \lessdot_{\delta'} Q$ be a covering relation in $\mathrm{Tam}_n(\delta')$. We have $P <_\delta Q$ in $\mathrm{Tam}_n(\delta)$.*

Proof. Using Lemmas 5.4.4 and 5.4.10, we will prove that we can build a chain from P to Q in $\mathrm{Tam}_n(\delta)$, just as we did to prove Lemma 5.4.2, namely that the Dyck lattice refines all alt-Tamari posets.

We write $P = AdC'_iB$ and $Q = AC'_idB$ with C'_i the δ'-excursion of u_i. We will exchange this down step d with down steps of δ-excursions until we reach Q and this will prove the result.

The δ-excursion C_i of u_i in P is a prefix of $C'_i = C_iD$. Letting $P_1 = AC_idDB$, we have $P \lessdot_\delta P_1$ in $\mathrm{Tam}_n(\delta)$.

Now, if D is empty, we have built a chain from P to Q. Otherwise, either D starts with a down step and exchanging the two down steps does not change the path or D starts with an up step u_j. In this second case, let C_j be the δ-excursion of u_j in P. Then C_j is a prefix of the δ'-excursion C'_j of u_j in P. Now, C'_j and C'_i are two δ'-excursions whose intersection is not empty. Lemma 5.4.4 ensures that C'_j is thus a prefix of D and the same holds for C_j. Then exchanging d and C_j is a δ-rotation.

By induction on the length of D, we build a chain from P to Q in $\mathrm{Tam}_n(\delta)$. □

5.4.2 Structure of linear intervals

Now we study the linear intervals in the alt-Tamari posets. As in the Tamari lattice and the Dyck lattice, we define left and right intervals, which we prove to be linear. Then we prove that all linear intervals of length $\ell \geq 1$ are either left or right intervals. Lastly, we give a combinatorial description and deduce similar bijections as previously.

Lemma 5.4.12. *Let δ be an increment vector. Let $P \lessdot_\delta Q$ be a covering relation. We can write $P = AdC_iB$ and $Q = AC_idB$ with C_i the δ-excursion of u_i.*

There are at most two covering relations $Q \lessdot_\delta Q'$ such that $[P, Q']$ is a linear interval. More precisely, a covering relation $Q \lessdot_\delta Q'$ that is not occurring at the valley du_i (assuming A ends with *a down step d) nor at the valley du_k (assuming B starts with the up step u_k) gives an interval $[P, Q']$ that is not linear.*

Proof. Let Q' be the δ-rotation of Q at the up step u_j with $u_j \neq u_i$ and $u_j \neq u_k$ if B starts with the up step u_k.

We want to prove that $[P, Q']$ is not linear. We have one saturated chain $P \lessdot_\delta Q \lessdot_\delta Q'$ from P to Q'. We will show that there is a different saturated chain from P to Q'.

Case 1: Suppose that $u_j \in C_i$. Then, we can write $Q' = AC_i'dB$ and if P' is the δ-rotation of P at u_j, we have $P' = AdC_i'B$, and we have another saturated chain $P \lessdot_\delta P' \lessdot_\delta Q'$ from P to Q'. Thus, $[P, Q]$ is a nonlinear interval as shown in Figure 5.14.

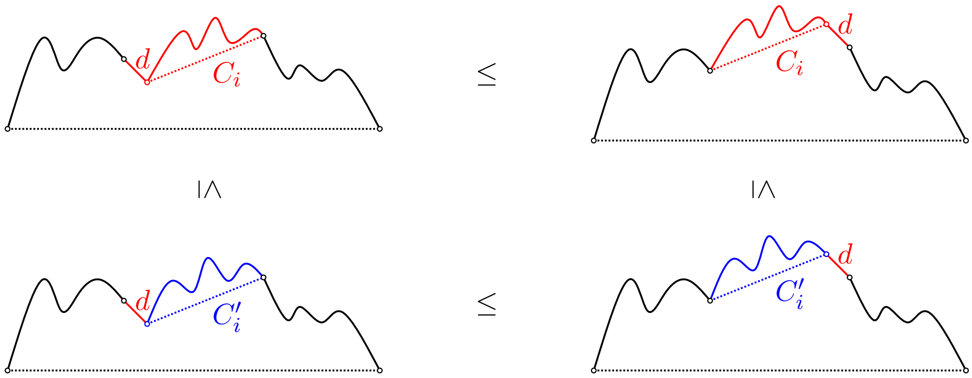


Figure 5.14: A square in an alt-Tamari poset.

Case 2: Suppose now that $u_j \notin C_i$. Let C_j be the δ-excursion of u_j in P. Then, two more cases occur. Either C_j is directly followed by C_i in P or not.

• Case 2.1: Suppose that $C_j = Ed$ and $P = A'dEdC_iB$ as in the first case of Lemma 5.4.6. Then, we can write $Q = A'dEC_idB$ and $Q' = A'EC_iddB$. Set $P' = A'EddC_iB$ and $P'' = A'EdC_idB$, and we have another saturated chain $P \lessdot_\delta P' \lessdot_\delta P'' \lessdot_\delta Q'$ from P to Q' as we can see in Figure 5.15.

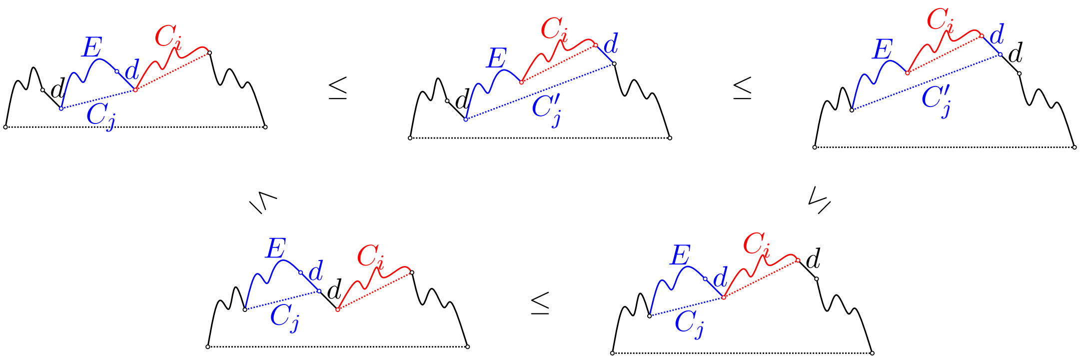


Figure 5.15: A pentagon in an alt-Tamari poset.

• Case 2.2: Suppose that C_j is not directly followed by C_i in P. Then the δ-excursion of u_j in Q is still C_j, and we can write $Q' = A'C_idB'$ after the δ-rotation at u_j. Let $P' = A'dC_iB'$ be the δ-rotation of P at u_j. We have again another saturated chain $P \lessdot_\delta P' \lessdot_\delta Q'$ from P to Q' and thus, a square similar to the one in Figure 5.14.

In all these cases, the interval $[P, Q']$ is not linear. □

Definition 5.4.13. We say that an interval $[P, Q]$ in the alt-Tamari poset is a left interval if we can write $P = Ad^\ell C_iB$ and $Q = AC_id^\ell B$ for some $\ell \geq 1$ with C_i a δ-excursion.

We say that $[P, Q]$ is a **right interval** if we have $P = AdC_1 \dots C_\ell B$ and $Q = AC_1 \dots C_\ell dB$ with ℓ δ-excursions $C_1, \dots, C_\ell$ for some $\ell \geq 1$.

Remark the cases $\ell = 1$ give δ-rotations, that is to say covering relations in $\mathrm{Tam}_n(\delta)$. Again, covering relations are exactly all intervals that are both left and right. Saying it otherwise, a left interval consists of moving ℓ times the same δ-excursion to the left whereas a right interval consists of moving ℓ times the same down step to the right. In particular, they are indeed intervals in the poset $\mathrm{Tam}_n(\delta)$.

Proposition 5.4.14. *A left interval is linear and the ℓ that appears in Definition 5.4.13 is its length.*

Proof. Let $[P, Q]$ be a left interval. We can write $P = Ad^\ell C_i B$ and $Q = AC_i d^\ell B$ with C_i the δ-excursion of u_i for some $\ell \geq 1$.

First, we clearly have a saturated chain of length ℓ from P to Q. One is clearly given by $P = P_0 \lessdot_\delta P_1 \lessdot_\delta \dots \lessdot_\delta P_\ell = Q$, where $P_j = Ad^{\ell-j} C_i d^j B$. We want to prove that it is the unique saturated chain from P to Q.

As in the proof of Proposition 5.4.9, for all $u_j \notin C_i$, $h_j(Q) = h_j(P)$ so in any chain from P to Q, only δ-rotations at up steps of C_i are possible. Moreover, for all $u_j \in C_i$, $h_j(Q) = h_j(P) - \ell$. Remark than only δ-rotations at u_i can make $h_i(Q')$ decrease by 1 for any $Q' \in [P, Q]$, so that any saturated chain from P to Q contains ℓ δ-rotations at u_i. Furthermore, because of Lemma 5.4.8, for all $u_j \in C_i$, each of these δ-rotation at u_i will make $h_j(Q')$ decrease by 1 for all $Q' \geq P$. Thus, any saturated chain from P to Q must contain exactly ℓ δ-rotations at u_i and no other δ-rotations.

Thus, any left interval is linear. $\square$

Proposition 5.4.15. *A right interval is linear and the ℓ that appears in Definition 5.4.13 is its length.*

Proof. Let $[P, Q]$ be a right interval. We can write $P = AdC_1 \dots C_\ell B$ and $Q = AC_1 \dots C_\ell dB$, where $C_1, \dots, C_\ell$ are ℓ δ-excursions for some $\ell \geq 1$. Recall that for $\ell = 1$, $[P, Q]$ is a covering relation and thus a linear interval of length 1.

Again, we can write a saturated chain of length ℓ from P to Q, namely $P = P_0 \lessdot_\delta P_1 \lessdot_\delta \dots \lessdot_\delta P_\ell = Q$, where $P_j = AC_1 \dots C_j dC_{j+1} \dots C_\ell B$. We prove by induction on ℓ that it is the unique saturated chain from P to Q. The case $\ell = 1$ is proven in Proposition 5.4.9.

For all up steps u_j in one of these ℓ δ-excursions, $h_j(Q) = h_j(P) - 1$ and we also have $\ell_j(Q; \delta) = \ell_j(P; \delta)$. For all other u_j, $h_j(Q) = h_j(P)$.

Let u_{i_j} be the first step of C_j. Because of Lemma 5.4.8, for all $Q' \geq P$, any δ-rotation that moves u_{i_j} will also move all up steps of C_j. Hence, in any saturated chain from P to Q, the only possible δ-rotations happen at the steps u_{i_j}.

Moreover, for all $1 \leq j \leq \ell - 1$, C_j and C_{j+1} are two consecutive δ-excursions in P so a δ-rotation at $u_{i_{j+1}}$ would change $\ell_{i_j}(P; \delta)$ into $\ell_{i_j}(P; \delta) + \ell_{i_{j+1}}(P; \delta)$, as stated in Lemma 5.4.6. Thus, in any saturated chain from P to Q, the first δ-rotation must happen at u_{i_1}.

Then, $[P_1, Q]$ is a right interval with $\ell - 1$ δ-excursions. By induction, $[P_1, Q]$ is a linear interval of length $\ell - 1$ and thus, $[P, Q]$ is an interval of length ℓ since it contains a unique saturated chain of length ℓ.

It follows that any right interval is linear. $\square$

Lemma 5.4.16. *Let $P = AdC_1C_2B$ with $C_1 = Ed$ the δ-excursion of u_i and C_2 the δ-excursion of u_j.*

If $Q = AEC_2ddB$, then the interval $[P, Q]$ is not linear.

Proof. We have two chains $P \lessdot_\delta AdEC_2dB \lessdot_\delta Q$ and $P \lessdot_\delta AC_1dC_2dB \lessdot_\delta AC_1C_2ddB \lessdot_\delta Q$ from P to Q. This is exactly the situation of Figure 5.15. $\square$

Proposition 5.4.17. *In the alt-Tamari poset* $\mathrm{Tam}_n(\delta)$*, all linear intervals of length* $\ell \geq 2$ *are either right or left intervals.*

Proof. We already know thanks to Lemma 5.4.12 that there are only two kinds of linear intervals of length 2, and they are precisely left and right intervals. We will prove the result by induction on length.

Let $[P, Q]$ be a linear interval of length $\ell + 1 \geq 3$. Let Q' be the lower cover of Q in $[P, Q]$, such that $[P, Q']$ is a linear interval of length $\ell \geq 2$. By induction, it is either a left of a right interval.

Suppose $[P, Q']$ is a left interval. Then we can write $P = Ad^{\ell}C_iB$ and $Q' = AC_id^{\ell}B$ with C_i the δ-excursion of some up step u_i.

As the last rotation in $[P, Q']$ occurs at u_i and C_i is followed by at least two down steps in Q', Lemma 5.4.12 ensures that the only possible δ-rotation $Q' \lessdot_\delta Q$ such that $[P, Q]$ is a linear interval is again at the up step u_i. This produces a left interval of length $\ell + 1$.

Suppose $[P, Q']$ is a right interval. Then we can write $P = AdC_1 \dots C_\ell B$ and $Q' = AC_1 \dots C_\ell dB$ with ℓ δ-excursions $C_1, \dots, C_\ell$.

Now, Lemma 5.4.12 ensures that there are only two up steps of Q' where a δ-rotation might produce an interval $[P, Q]$ that is still linear, namely the first step of C_ℓ and the first step of B.

Lemma 5.4.16 shows that a δ-rotation at the first step of C_ℓ would produce a nonlinear interval. Hence, the only possible δ-rotation $Q' \lessdot_\delta Q$ happens at the first step of B and this produces a right interval of length $\ell + 1$. □

5.4.3 Combinatorial description and counting

We have described the structure of all linear intervals of the alt-Tamari posets, according to their length. We can adapt the combinatorial description that we gave for the Dyck lattice, and produce again a decomposition which generalizes the cases of the Dyck and the Tamari lattices. This proves the main result of this section, namely that all the alt-Tamari posets on Dyck paths of size n share the same number of linear intervals of any length.

Remark 5.4.18. As all the alt-Tamari posets are defined on Dyck paths, it is immediate that they all have the same number C_n of trivial intervals, or in other words, (linear) intervals of length 0.

Furthermore, as a covering relation can be described as a Dyck path with a marked valley, it follows that every Dyck path has the same number of upper covers in all alt-Tamari posets. Thus, the total number of covering relations of all alt-Tamari posets is the same, namely the number of linear intervals of length 1.

In this section, we will prove that for any $\ell \geq 1$, all these posets have the same number of right (resp. left) intervals of length ℓ, namely $\binom{2n-\ell}{n+1}$. This will be proven through a bijection between a right (resp. left) interval of length ℓ and a pair consisting of a Dyck path marked at a down (resp. up) step and a sequence of ℓ Dyck paths (possibly trivial). The same decompositions work also for intervals of length $\ell = 1$, that is to say, they are in bijection with a pair of Dyck paths, the first being marked at a down (resp. up) step. We start with this case.

In all what follows, we will use the notion of excursions as defined in Section 1.3.2, not to be confused with δ-excursions of an up step as defined in Section 5.4.1, which is always a prefix of the excursion of this up step (see Remark 5.4.1).

Proposition 5.4.19. *Let* δ *be an increment vector. There is a bijection between covering relations in* $\mathrm{Tam}_n(\delta)$ *and pairs* (P_0, P_1) *of Dyck paths, where* P_0 *is marked at a down step, and the lengths of* P_0 *and* P_1 *add up to* $n - 1$*.*

Proof. Let $[P, Q]$ be a covering relation in $\mathrm{Tam}_n(\delta)$. We can write $P = AdC_iB$ and $Q = AC_idB$, where C_i is the δ-excursion of the up step u_i.

Let E be the excursion of u_i as defined in Section 1.3.2. We can write $E = C_iD$ with D a prefix of $B = DB'$. Remark that E starts with u_i and ends with the down step matching with u_i. Thus, we can write $E = u_iE'd$ with E' a possibly empty Dyck path.

Now, we can set $P_0 = AdB'$ and $P_1 = E'$, so that P_0 is a Dyck path marked at the down step between A and B' and P_1 is a possibly empty Dyck path. The lengths of P_0 and P_1 add up to $n - 1$.

Let us prove that it is a bijection. Let P_0 be a Dyck path of length $n_0 \geq 1$, marked at a down step, so that we can write $P_0 = AdB'$ with d the marked down step of P_0. Let P_1 be a Dyck path of length $n_1 \geq 0$. Let $n = n_0 + n_1 + 1$ and δ be an increment vector of size n.

Suppose that there are j up steps in A and let $i = j + 1$. We will construct P (resp. Q) by inserting a Dyck path into P_0 between d and B' (resp. between A and B'), so that its first up step will become the i-th up step of P and Q.

Let $E = u_iP_1d$, where the up steps of P_1 are relabelled starting with u_{i+1}. Obviously, E is the excursion of u_i.

Let then C_i be the δ-excursion of u_i in E, where u_i is considered to be the i-th up step. This is possible because a δ-excursion is always a prefix of the excursion. We can write $E = C_iD$ and $B = DB'$.

Then, setting $P = AdC_iB$ and $Q = AC_idB$, we have constructed a covering relation $P \lessdot_\delta Q$ in $\mathrm{Tam}_n(\delta)$. It is clear that this is the reciprocal of the decomposition described above and thus, this is a bijection. Moreover, we have $P = AdEB'$ and this writing does not depend on δ. □

Corollary 5.4.20. *Let δ and δ' be two increment vectors of the same size n. There is a bijection between covering relations in $\mathrm{Tam}_n(\delta)$ and $\mathrm{Tam}_n(\delta')$. Moreover, this bijection preserves the bottom elements of the covering relations.*

In particular, there are $\binom{2n-1}{n-2}$ covering relations in $\mathrm{Tam}_n(\delta)$ for any δ.

This decomposition is more or less directly adapted from the case of the Dyck lattice in Section 5.1.2. It is a special case of the next proposition on right intervals. The proof is in disguise an induction on ℓ and the previous proposition was the base case. We remove recursively from the bottom path P of the interval the last excursion corresponding to the rightmost δ-excursion, which gives a bijection between right intervals of length ℓ and a pair consisting of a right interval of length $\ell - 1$ and a Dyck path P_ℓ. We write directly the full decomposition.

Proposition 5.4.21. *Let δ be an increment vector and $\ell \geq 1$.*

There is a bijection between right intervals of length ℓ in $\mathrm{Tam}_n(\delta)$ and sequences $(P_0, P_1, \dots, P_\ell)$ of Dyck paths, where P_0 is marked at a down step, and the sizes of $P_0, \dots, P_\ell$ add up to $n - \ell$.

Proof. Let $[P, Q]$ be a right interval of length $\ell \geq 2$ in $\mathrm{Tam}_n(\delta)$. By definition of right intervals, we can write $P = AdC_1 \dots C_\ell B$ and $Q = AC_1 \dots C_\ell dB$ with ℓ δ-excursions $C_1, \dots, C_\ell$.

Let $u_{i_1}, \dots, u_{i_\ell}$ be the steps with which $C_1, \dots, C_\ell$ start respectively. Let $d_1, \dots, d_\ell$ be the down steps matching $u_{i_1}, \dots, u_{i_\ell}$.

Let D_ℓ be the part of the path P starting just after C_ℓ and ending just after d_ℓ. Then, $E_\ell = C_\ell D_\ell$ starts with u_{i_ℓ} and ends with its matching down step d_ℓ, so E_ℓ is the excursion of u_{i_ℓ}. In particular, if $E_\ell = C_\ell$ then D_ℓ is empty.

For $\ell > j \geq 1$, let D_j be the part of P starting just after D_{j+1} (which may be empty) and ending just after d_j. As previously, the excursion of u_{i_j} in P is either $E_j = C_j$ and D_j is empty or it is $E_j = C_jE_{j-1}D_j$ and D_j is not empty.

Now, we can write $P = AdC_1 \dots C_\ell D_\ell \dots D_1B'$. Set $P_0 = AdB'$, marked at the down step between A and B' and $C_jD_j = u_{i_j}P_jd_j$ for $1 \leq j \leq \ell$, and we have the decomposition as stated.

We now prove that this decomposition is bijective by constructing its reciprocal.

Let P_0 be a Dyck path of length $n_0 \geq 1$, marked at a down step, so that we can write $P_0 = AdB'$ with d the marked up step of P_0. Let $P_1, \ldots, P_\ell$ be ℓ Dyck paths of respective lengths $n_1, \ldots, n_\ell \geq 0$. Let $n = \sum_{i=0}^{\ell} n_i + \ell$ and δ be an increment vector of size n.

We will build P and Q by inserting these ℓ paths one by one and ℓ additional pairs of up and down steps between A and B'.

Let $i_1 - 1$ be the number of up steps in A. Let C_1 be the δ-excursion of the first step of uP_1d where the steps are relabelled starting with i_1. For $2 \leq j \leq \ell$, let $i_j - 1$ be the number of up steps in $AC_1 \ldots C_{j-1}$ and C_j be the δ-excursion of uP_jd where the steps are relabelled starting with i_j. Thus, for all $1 \leq j \leq \ell$, we can write $uP_jd = C_jD_j$.

Now, let $P = AdC_1 \ldots C_\ell D_\ell \ldots D_1 B'$. Writing $B = D_\ell \ldots D_1 B'$, we can set $Q = AC_1 \ldots C_\ell dB$. By construction, $[P, Q]$ is a right interval of length ℓ and again, this is clearly the reciprocal of the decomposition defined above. □

Corollary 5.4.22. *Let δ and δ' be two increment vectors of the same size n and $\ell \geq 2$. There is a bijection between right intervals of length ℓ in $\mathrm{Tam}_n(\delta)$ and $\mathrm{Tam}_n(\delta')$.*

In particular, there are $\binom{2n-\ell}{n+1}$ right intervals of length ℓ in $\mathrm{Tam}_n(\delta)$ for any δ.

Let us now focus on the case of the left intervals. Note that the case $\ell = 1$ in the next proposition does not correspond to the same decomposition as the case $\ell = 1$ for the right intervals. However, this decomposition is more directly adapted from the case of the left intervals of the Dyck paths, which it specializes to. Furthermore, in this case, the decomposition of the bottom path does not depend on the increment vector! Once again, the decomposition could be described by recursively removing excursions, but we write it directly for a general ℓ.

Proposition 5.4.23. *Let δ be an increment vector and $\ell \geq 1$.*

There is a bijection between left intervals of length ℓ in $\mathrm{Tam}_n(\delta)$ and sequences $(P_0, P_1, \ldots, P_\ell)$ of Dyck paths, where P_0 is marked at an up step, and the sizes of $P_0, \ldots, P_\ell$ add up to $n - \ell$.

Proof. Let $[P, Q]$ be a left interval of length $\ell \geq 2$ in $\mathrm{Tam}_n(\delta)$. By definition of left intervals, we can write $P = Ad^\ell C_i B$ and $Q = AC_i d^\ell B$, where C_i is the δ-excursion of the up step u_i.

Let $u_{i_1}, \ldots, u_{i_\ell}$ be the up steps of A matching the ℓ down steps of P between A and C_i. Then we can write $P = A' u_{i_1} P_1 u_{i_2} \ldots u_{i_\ell} P_\ell d^\ell C_i B$, where $P_1, \ldots, P_\ell$ are ℓ possibly trivial Dyck paths. We then set $P_0 = A' C_i B$, and it is a Dyck path marked at the first step of C_i. It is clear that the total number of up steps in $P_0, \ldots, P_\ell$ is $n - \ell$ since all up steps of P except $u_{i_1}, \ldots, u_{i_\ell}$ are in exactly one of these Dyck paths $P_0, \ldots, P_\ell$.

Let us prove that this decomposition is bijective by constructing its inverse.

Let P_0 be a Dyck path of length $n_0 \geq 1$, marked at an up step, so that we can write $P_0 = A'uB'$ with u the marked up step of P_0. Let $P_1, \ldots, P_\ell$ be ℓ Dyck paths of respective lengths $n_1, \ldots, n_\ell \geq 0$. Let $n = \sum_{i=0}^{\ell} n_i + \ell$ and δ be an increment vector of size n.

We will build P and Q by inserting these ℓ paths and ℓ additional pairs of up and down steps between A' and B'. Let $i_1 - 1$ be the number of up steps in A' and for $2 \leq j \leq \ell$ let $i_j = i_{j-1} + n_{j-1} + 1$.

Then, we can set $A = A' u_{i_1} P_1 u_{i_2} \ldots u_{i_\ell} P_\ell$. We set as well $P = Ad^\ell uB'$ and this is by construction a Dyck path of length n. Remark that the construction of P does not depend on δ.

Let now C be the δ-excursion of the up step u preceding B' in P. We can write $uB' = CB$ so that $P = Ad^\ell CB$. Setting finally $Q = ACd^\ell B$, we obtain a left interval $[P, Q]$ of length ℓ. Again, this is clearly the reciprocal of the decomposition of left intervals defined above. □

Corollary 5.4.24. *Let δ and δ' be two increment vectors of the same size n and $\ell \geq 2$. There is a bijection between left intervals of length ℓ in $\mathrm{Tam}_n(\delta)$ and $\mathrm{Tam}_n(\delta')$. Moreover, this bijection preserves the bottom element of the intervals.*

In particular, there are $\binom{2n-\ell}{n+1}$ left intervals of length ℓ in $\mathrm{Tam}_n(\delta)$ for any δ.

In fact, a more direct argument gives this bijection on left intervals between any two alt-Tamari lattices. Left intervals $[P, Q]$ of length ℓ in $\mathrm{Tam}_n(\delta)$ are in bijection with Dyck paths P marked at an up step u_i preceded by ℓ down steps. The reconstruction of the entire intervals then depends on the increment vector δ, which determines where the δ-excursion of u_i stops.

All in all, we proved that the linear intervals are equidistributed with respect to their length among the alt-Tamari posets.

Theorem 5.4.25. *For all $n \geq 1$ and $\ell \geq 0$, all the alt-Tamari posets have the same number of linear intervals of length ℓ.*

5.5 Interpretation of the numbers

The results of this section were obtained thanks to discussions with Vincent Pilaud. Note that they do not appear in the preprint [Che22].

The number $\binom{2n-\ell}{n+1} = \binom{2n-\ell}{n-\ell-1}$ counts lattice paths in a rectangular grid of size $(n+1) \times (n-\ell-1)$. In this section, we describe a bijection Φ (resp. Ψ) between a right (resp. left) interval of length ℓ and such a path. To make the bijection easier, we will use ballot paths and east steps E and north steps N instead of down and up steps.

5.5.1 Right intervals bijection

Let $[P, P']$ be a right interval of length $\ell \geq 1$ in $\mathrm{Tam}_n(\delta)$. We can write $P = AEC_1 \dots C_\ell B$ and $P' = AC_1 \dots C_\ell EB$ with ℓ consecutive δ-excursions $C_1, \dots, C_\ell$.

First, remark that we can recover the interval $[P, P']$ from $P_\bullet$, that is P marked at the east step E that follows A. We want to produce a path with $n-\ell-1$ north steps and $n+1$ east steps.

Let C_0 be the excursion (as in Definition 1.3.5) ending at the marked east step E in P. Writing $AE = A'C_0$, we have $P = A'C_0 \dots C_\ell B$. For $0 \leq i \leq \ell$, each subpath C_i starts with a north step, and we write $C_i = NC'_i$. Now we add an east step at the end of P, we suppress these $\ell+1$ north steps, and we finally move the prefix A' at the end of the result. We obtain a path $\Phi(P_\bullet) = C'_0 \dots C'_\ell BEA'$ which has $n-\ell-1$ north steps and $n+1$ east steps.

Lemma 5.5.1. *If $P_\bullet$ and $\Phi(P_\bullet)$ are as above, then the vertex v at the beginning of A' in $\Phi(P_\bullet)$ is the first vertex of $\Phi(P_\bullet)$ with the smallest altitude.*

Proof. On the one hand, since A' is a prefix of P, it is clear that the altitude of all vertices after v in $\Phi(P_\bullet)$ have an altitude weakly greater than the altitude of v. On the other hand, $C'_0 \dots C'_\ell BE$ is obtained from a suffix of P where one east step was added at the end and $\ell+1$ north steps were erased. Thus, there are strictly more east steps than north steps in any suffix of $C'_0 \dots C'_\ell BE$ and then the altitude of any vertex strictly on the left of v in $\Phi(P_\bullet)$ is strictly greater than the altitude of v. □

Proposition 5.5.2. *The map Φ is a bijection between right intervals of length $\ell \geq 1$ in $\mathrm{Tam}_n(\delta)$ and lattice paths with $n-\ell-1$ north steps and $n+1$ east steps.*

Proof. We prove it by describing the inverse bijection of Φ.

Let Q be a path with $n-\ell-1$ north steps and $n+1$ east steps. We want to build its preimage.

Let v be the first vertex of Q with the smallest altitude. It is preceded by an east step E and we can write $Q = Q'EA'$. As every vertex of Q after v has an altitude at least equal to the one of v, A' is the prefix of a Dyck path and similarly Q' is the suffix of a Dyck path.

Consider the path $P' = A'Q'$, which has now n east steps and $n-\ell-1$ north steps. As A' is a prefix of a Dyck path, it contains at least as many north steps as east steps, and thus, Q' contains at least $\ell+1$ more east steps than north steps.

We will add $\ell+1$ north steps in Q' and produce a Dyck path $P_\bullet$ marked at an east step E followed by ℓ consecutive δ-excursions, or in other words a right interval of length ℓ.

Add a north step after A', and then jump at the end of the excursion C_0 starting at this north step, and mark its last east step. We can write $AE = A'C_0$.

For $1 \leq i \leq \ell$, we recursively:

- Add a north step N_i after C_{i-1}. By construction, there are at least $\ell - i$ more east steps than north steps after N_i so the excursion of N_i is well-defined.

- Jump at the end of the δ-excursion C_i that starts with this north step N_i.

We have constructed a path $P = AEC_1 \dots C_\ell B$, and we mark it at E.

By construction, AE and each C_i are prefixes of a Dyck path so the same holds for $AEC_1 \dots C_\ell$. Furthermore, B is a suffix of Q and thus, it is also the suffix of a Dyck path. Finally, P contains n east steps and n north steps. Thus, it is a Dyck path of length n. It is marked at an east step and by construction, it is followed by ℓ δ-excursions.

Finally, it is clear that applying Φ to the marked Dyck path $P_\bullet$ will give exactly the path Q we started with. □

5.5.2 Left intervals bijection

The left bijection is very similar and actually easier because the bottom path of a left interval does not depend on δ.

Let $[P, P']$ be a left interval of length $\ell \geq 1$ in $\mathrm{Tam}_n(\delta)$. We can write $P = AE^\ell CB$ and $P' = ACE^\ell B$ for some $\ell \geq 1$ with C a δ-excursion. Again, note that knowing $P_\bullet$—that is P marked at the first (north) step of C—is sufficient to recover the entire interval $[P, P']$.

Let $\widetilde{C} = C'E$ be the excursion starting at the first step of C in P, and write $CB = C'EB'$. Now we suppress the ℓ consecutive east steps along with the last step of $\widetilde{C}$, we add a north step at the beginning of our path, and finally move B' at the beginning of the result. We obtain a path $\Psi(P_\bullet) = B'NAC'$, with $n+1$ north steps and $n-\ell-1$ east steps.

Lemma 5.5.3. *If $P_\bullet$ and $\Psi(P_\bullet)$ are as above, then the vertex v at the end of B' in $\Psi(P_\bullet)$ is the last vertex of $\Psi(P_\bullet)$ with the smallest altitude.*

Proof. On the one hand, since B' is a suffix of P, it is clear that the altitude of all vertices before v in $\Psi(P_\bullet)$ have an altitude greater than or equal to the altitude of v. On the other hand, NAC' is obtained from a prefix of P where one north step was added at the beginning and $\ell+1$ east steps were erased. Thus, there is strictly more north steps than east steps in any suffix of NAC' and then the altitude of any vertex strictly on the right of v in $\Psi(P_\bullet)$ is greater than the altitude of v. □

Proposition 5.5.4. *The map Ψ is a bijection between left intervals of length $\ell \geq 1$ in $\mathrm{Tam}_n(\delta)$ and lattice paths with $n+1$ north steps and $n-\ell-1$ east steps.*

Proof. We describe the inverse bijection of Ψ. Let Q be a path with $n+1$ north steps and $n-\ell-1$ east steps.

Let v be the last vertex of Q with the smallest altitude. It is followed by a north step N and we can write $Q = B'NQ'$. Since v is the last minimum of Q, then B' contains at most as many north steps as east steps and is also the suffix of a Dyck path. Thus, Q' contains at least $\ell+1$ more north steps as east steps, and furthermore, it is the prefix of a Dyck path.

We add an east step E at the end of Q. Let $\widetilde{C} = C'E$ be the excursion of this last east step. We can write $Q' = A\widetilde{C}$.

Now we define $P = AE^\ell \widetilde{C} B'$, and we mark it at the first step of $\widetilde{C}$ to get our marked path $P_\bullet$. Now, P is a path that contains n north steps and n east steps. Furthermore, B' is the suffix of a Dyck path, and thus, so is $E^\ell \widetilde{C} B$. Finally, as A is the prefix of a Dyck path, then P is a Dyck path.

By construction, $\Psi(P_\bullet)$ is exactly Q and thus, Ψ is indeed a bijection. $\square$

Note that again, Ψ and Φ are two different bijections on covering relations in the case $\ell = 1$.

Chapter 6

In the alt ν-Tamari lattices

The results in this chapter are based on the prepublished article [CC23], in a joint work with Cesar Ceballos.

In the previous chapter, we built a family of posets that contains the Tamari and the Dyck lattices. They are all described similarly on Dyck paths, and they behave very nicely regarding linear intervals, whose distribution with respect to their length is the same for all of them. We even proved that for each Dyck path P, the number of left intervals of a given length ℓ whose bottom element is P is the same in every alt-Tamari poset: it is equal to the number of times the word $d^\ell u$ is a factor of P, or say it otherwise, the number of (non-final) falls of size at least ℓ. Right intervals do not satisfy such a result: for a path Q, the number of right intervals whose top (or bottom) element is Q may differ among the family. However, one of the results of this chapter will imply that the distribution of right intervals with respect to length and grouped by having the same top element is the same in all alt-Tamari posets. Some other questions on the alt-Tamari posets remained open in the previous chapter, as for instance the fact that they are lattices and this will be addressed as well in this chapter.

Seeing the Dyck paths as ballot paths, it is very natural to generalize them to the sets of lattice paths using north and east steps and staying above a fixed path ν, called ν-paths. The Dyck order can straightforwardly be generalized to the set of ν-paths, and we obtain the ν-Dyck lattice Dyck_ν. As presented in Definition 4.1.3, it is also possible to generalize the Tamari lattice to the set of ν-paths, introducing a notion of horizontal distance, that we call here ν-altitude, in the so-called ν-Tamari lattice Tam_ν. Again, the ν-Dyck lattice extends the ν-Tamari lattice. Remarkably, we observe that the distribution of linear intervals with respect to their length is the same again in both lattices. The object of this chapter is to introduce a new family of posets that further generalizes the family of alt-Tamari posets in the context of ν-paths. They turn out to be lattices and we call them the alt ν-Tamari lattices. We generalize the results of the previous chapter, proving that the distribution of linear intervals only depends on the path ν. We additionally prove that all these posets are lattices and that an extension structure also holds among the family for a fixed path ν.

6.1 Left and right intervals in the ν-Dyck lattice

As for the Dyck lattice, the nontrivial linear intervals of the ν-Dyck lattice can be easily characterized into two different classes.

Definition 6.1.1. An interval $[P,Q]$ in Dyck_ν is a left interval if Q is obtained from P by transforming a subpath $E^\ell N$ into NE^ℓ for some $\ell \geq 1$. It is a right interval if Q is obtained from P by transforming a subpath EN^ℓ into $N^\ell E$ for some $\ell \geq 1$.

Lemma 6.1.2. *The linear intervals of length* 2 *are either left or right intervals.*

The proof is a direct adaptation of the proof of Proposition 5.1.3.

Proof. Let $P \lessdot Q \lessdot R$ be a linear interval of length 2. The covering relations $P \lessdot Q$ transforms a valley EN of P into a peak NE. If the next covering relation $Q \lessdot R$ happens at a valley of Q that is also a valley of P, then the interval $[P, R]$ is a square. Thus, this second covering relation must use either of the two steps of the peak NE that was created in Q. □

Proposition 6.1.3. *The left and right intervals in the previous definition are linear intervals of length ℓ. Moreover, all nontrivial linear intervals in* Dyck_ν *are either left or right intervals.*

Proof. If $[P, Q]$ is an interval of this form with $\ell \geq 1$, then there exists only one saturated chain from P to Q. Indeed, there is only one valley of P that is not a valley of Q and thus, any saturated chain from P to Q starts at this valley. We then obtain an interval of the same form, but ℓ has decreased by 1, and we conclude by induction.

Lemma 6.1.2 proves that all linear intervals of height $k = 2$ are either left or right intervals. Suppose that $[P, Q]$ is a linear interval of height $k + 1 \geq 3$. It is linear so Q has only one lower cover Q' in $[P, Q]$. Then $[P, Q']$ is linear of height k and thus by induction, it is of the prescribed form.

Suppose that Q' is obtained from P by transforming a subpath E^kN into NE^k, which creates this peak NE in Q' followed by $k - 1$ east steps. Then, Lemma 6.1.2 ensures that the covering relation $Q' \lessdot Q$ has to use the north step N of this peak and thus, Q is obtained from P by changing a subpath $E^{k+1}N$ into NE^{k+1}.

Suppose now that Q' is obtained from P by transforming a subpath EN^k into N^kE. Similarly, Lemma 6.1.2 ensures that $Q' \lessdot Q$ has to use the east step E of this peak and Q is obtained from P by transforming a subpath EN^{k+1} into $N^{k+1}E$. □

Corollary 6.1.4. *Left intervals of length ℓ in* Dyck_ν *are in bijection with ν-paths marked at a north step preceded by ℓ east steps. Right intervals of length ℓ in* Dyck_ν *are in bijection with ν-paths marked at an east step followed by ℓ north steps.*

6.2 Left and right intervals in the ν-Tamari lattice

The description of the ν-Tamari lattice as the rotation poset of ν-trees as explained in Section 4.1.3 gives an easy description of its linear intervals. We introduce once more two classes of "left" and "right" intervals that will describe all linear intervals.

Definition 6.2.1. An interval $[T, T']$ in Tam_ν^{tr} is a **left interval** if T' is obtained from T by applying $\ell > 0$ rotations at the first ℓ nodes of a consecutive sequence $q_0, \dots, q_{\ell-1}, q_\ell$ in the same row, from left to right. For example, applying two rotations at the first two nodes of the sequence $\bar{p}_{13}, \bar{p}_{12}, \bar{p}_{11}$ in Figure 4.5 (right). It is a **right interval** if T' is obtained from T by applying ℓ rotations at the first ℓ nodes of a consecutive sequence $q_0, \dots, q_{\ell-1}, q_\ell$ in the same column, from bottom to top. For example, applying two rotations at the first two nodes of the sequence $\bar{p}_3, \bar{p}_4, \bar{p}_{12}$ in Figure 4.5 (right).

Proposition 6.2.2. *The left and right intervals in the previous definition are linear intervals of length ℓ. Moreover, all nontrivial linear intervals in the rotation lattice on ν-trees are either left or right intervals.*

Proof. A left (resp. right) interval $[T, T']$ is indeed linear because there is a unique saturated chain from T to T' and ℓ is its length. Indeed, each element different from T' in the interval has only one upper cover that is below T'.

The converse is proven by induction, similarly as the simpler case of the Tamari lattice in Proposition 5.2.9.

Intervals of length 1 are all covering relations and thus both left and right intervals.

An interval which contains a saturated chain $T_0 \lessdot T_1 \lessdot T_2$ can be:

- a left interval when $T_1 \lessdot T_2$ is the rotation in the same row immediately on the right of $T_0 \lessdot T_1$,
- a right interval when $T_1 \lessdot T_2$ is the rotation in the same column immediately above $T_0 \lessdot T_1$,
- a pentagon when $T_1 \lessdot T_2$ is the rotation in the same column immediately under $T_0 \lessdot T_1$,
- a square otherwise.

In particular, this proves that all linear intervals of length 2 are exactly left or right intervals.

A linear interval $[T, T']$ of length $k \geq 3$ must contain a linear interval $[T, T'']$ of length $k-1$. By induction, $[T, T'']$ must be either a left interval and in this case $[T, T']$ is a left interval as well or a right interval and in this case $[T, T']$ is also a right interval. $\square$

Remark 6.2.3. The left flushing (see Section 4.1.3) of a left interval on the rotation lattice of ν-trees produces a left interval $[P, Q]$ of ν-paths in Tam_ν, where P is of the form AE^kBC with B some ν-excursion and Q is of the form ABE^kC. In other words, P is a ν-path with a valley preceded by k east steps.

The left flushing of a right interval on the rotation lattice of ν-trees produces a right interval $[P, Q]$ of ν-paths in Tam_ν, where P is of the form $AEB_1 \ldots B_kC$ with $B_1, \ldots, B_k$ being k consecutive ν-excursions, and Q is of the form $AB_1 \ldots B_kEC$.

6.3 The alt ν-Tamari lattice

Given a fixed path ν, the ν-Dyck lattice and the ν-Tamari lattice are two posets defined on the set of ν-paths with quite similar covering relations. In both cases, a covering relation consists of swapping the east step of a valley with a subpath that follows it. We can in fact define a whole family of posets that are described similarly, and we call them the alt ν-Tamari posets. We prove that the resulting posets are lattices and study their linear intervals.

6.3.1 On ν-paths

Let $\nu = (\nu_0, \ldots, \nu_n)$ be a fixed path. In Section 4.2.2, we gave the definitions of the alt ν-tamari lattices without proofs. We first recall the different notions introduced there.

We say that $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n$ is an **increment vector** with respect to ν if $\delta_i \leq \nu_i$ for all $1 \leq i \leq n$. We then introduce a notion of **δ-altitude** similar to the ν-altitude by setting to zero the δ-altitude of the initial lattice point of a ν-path μ, and declaring that the i-th north step of μ increases the δ-altitude by δ_i and an east step decreases the δ-altitude by 1. From here, we define **δ-rotations** as follows:

Definition 6.3.1. Let μ be a ν-path. Given a valley EN of μ, let p be the lattice point between the east and north steps. Let q be the next lattice point of μ such that $\mathrm{alt}_\delta(q) = \mathrm{alt}_\delta(p)$, and $\mu_{[p,q]}$ be the subpath of μ that starts at p and ends at q. Let μ' be the path obtained from μ by switching $\mu_{[p,q]}$ with the east step E that precedes it.

The **δ-rotation** of μ at the valley p is defined to be $\mu \lessdot_\delta \mu'$.

An example is illustrated in Figure 6.1. Note that a δ-rotation increases the number of boxes below the path, and therefore its transitive closure induces a poset structure on the set of ν-paths.

Definition 6.3.2. Let δ be an increment vector with respect to ν. The *alt ν-Tamari poset* $\mathrm{Tam}_\nu(\delta)$ is the transitive closure of δ-rotations on the set of ν-paths.

The three examples of the ν-Tamari poset for $\nu = ENEEN = (1, 2, 0)$ are illustrated on Figure 4.7.

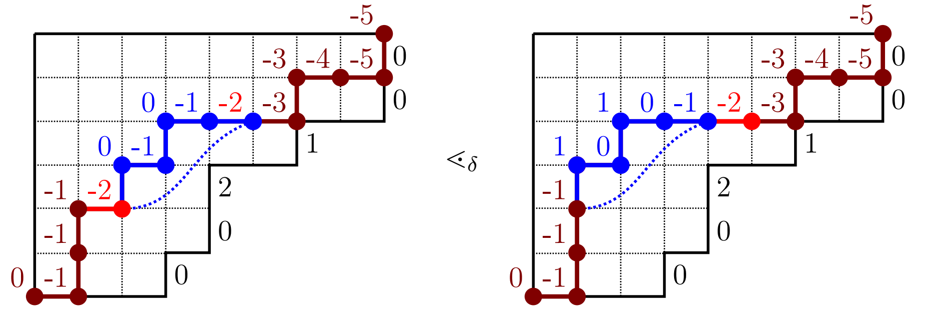


Figure 6.1: The δ-rotation operation of a ν-path for $\delta = (0, 0, 2, 1, 0, 0)$. Each node is labelled with its δ-altitude.

Remark 6.3.3. For a fixed path ν, there are two extreme choices for the increment vector δ. If $\delta_i = \nu_i$ for all $1 \leq i \leq n$, the alt ν-Tamari lattice coincides with the ν-Tamari lattice. If $\delta_i = 0$ for all $1 \leq i \leq n$, the alt ν-Tamari lattice coincides with the ν-Dyck lattice. We denote these two cases by $\delta^{\max}$ and $\delta^{\min}$, respectively.

Another approach to define the alt ν-Tamari poset is to introduce the notion of **δ-elevation** of a subpath as the difference of the δ-altitude between its ending point and its starting point. We thus write $\mathrm{elev}_\delta(E) = -1$ for an east step E and $\mathrm{elev}_\delta(N_i) = \delta_i$ if N_i is the i-th north step of a ν-path μ. For any subpath A of μ, we then have $\mathrm{elev}_\delta(A) = \sum_{a \in A} \mathrm{elev}_\delta(a)$ as the sum of the δ-elevation of the steps of A.

The **δ-excursion** of a north step N of a ν-path μ is defined as the shortest subpath A of μ that starts with this N and such that $\mathrm{elev}_\delta(A) = 0$. It follows from the definition of the δ-excursion that exchanging the east step E of a valley with the δ-excursion that follows it is exactly a covering relation in $\mathrm{Tam}_\nu(\delta)$.

Remark 6.3.4. Note that for $\delta = \delta^{\max}$, the δ-altitude is the ν-altitude shifted by $-\nu_0$, but the δ-elevation and the ν-elevation are equal.

For a general increment vector δ with respect to ν, it is not a priori clear that $\mathrm{Tam}_\nu(\delta)$ is a lattice. This is a consequence of the following proposition.

Proposition 6.3.5. *Let $\check{\nu}_0 = \sum_{i=0}^{n} \nu_i - \sum_{i=1}^{n} \delta_i$ with $\delta_i \leq \nu_i$. Then $\check{\nu} = (\check{\nu}_0, \check{\nu}_1, \ldots, \check{\nu}_n) = (\check{\nu}_0, \delta_1, \ldots, \delta_n)$ is a path below ν whose endpoints are the same as ν. Moreover, the following properties hold:*

1. *δ-rotations of a ν-path μ coincide with $\check{\nu}$-rotations of μ.*
2. *The alt ν-Tamari poset $\mathrm{Tam}_\nu(\delta)$ is the restriction of $\mathrm{Tam}_{\check{\nu}}$ to the subset of paths weakly above ν.*
3. *The covering relations of $\mathrm{Tam}_\nu(\delta)$ are exactly the δ-rotations.*
4. *The alt ν-Tamari poset $\mathrm{Tam}_\nu(\delta)$ is the interval $[\nu, 1^\nu]$ in $\mathrm{Tam}_{\check{\nu}}$.*

Here, $1^\nu = N^n E^m$ denotes the top path above ν and $\check{\nu}$, where $m = \nu_0 + \cdots + \nu_n = \check{\nu}_0 + \cdots + \check{\nu}_n$.

Proof. The fact that $\check{\nu}$ is weakly below ν follows from $\sum_{i=0}^{j} \check{\nu}_i - \sum_{i=0}^{j} \nu_i = \sum_{i=j+1}^{n} (\nu_i - \delta_i) \geq 0$, where equality holds for $j = n$.

Note that a ν-path μ is also a $\check{\nu}$-path, and for a subpath A of μ we have $\mathrm{elev}_\delta(A) = \mathrm{elev}_{\check{\nu}}(A)$. Since the δ-rotations (resp. $\check{\nu}$-rotations) are determined by the δ-elevation (resp. $\check{\nu}$-elevation), then Item (1) follows. Items (2) and (3) follow from Item (1).

The first three Items imply that the set of ν-paths is stable under $\check{\nu}$-rotations, so it is an upper ideal (or filter) in $\mathrm{Tam}_{\check{\nu}}$. For Item (4) we need to show that the path ν is the unique minimal element of this upper ideal. We will obtain that the restriction of $\mathrm{Tam}_{\check{\nu}}$ to the subset of paths weakly above ν is the interval $[\nu, 1^\nu]$ in $\mathrm{Tam}_{\check{\nu}}$. In other words, we need to show that every ν-path μ satisfies $\nu \leq_{\mathrm{Tam}_{\check{\nu}}} \mu$. Note that this property does not hold for an arbitrary path $\check{\nu}$ below ν, but for our particular choice this is equivalent to show that $\nu \leq_{\mathrm{Tam}_\nu(\delta)} \mu$ (by Item (1)). This holds because we can reach any ν-path μ by applying a sequence of δ-rotations: add the boxes between ν and μ one at a time from bottom to top, from right to left. Each of these steps corresponds to a δ-rotation because $\delta_i \leq \nu_i$. □

Corollary 6.3.6. *The alt ν-Tamari poset is a lattice.*

Proof. By Proposition 6.3.5 (4), the alt ν-Tamari poset $\mathrm{Tam}_\nu(\delta)$ is isomorphic to the interval $[\nu, 1^\nu]$ in $\mathrm{Tam}_{\check{\nu}}$. Since an interval in a lattice is also a lattice, we deduce that $\mathrm{Tam}_\nu(\delta)$ is a lattice. □

Remark 6.3.7. If we chose any other path $\check{\nu}$ weakly below ν that does not satisfy $\check{\nu}_i \leq \nu_i$, for all $i > 0$, then the restriction of $\mathrm{Tam}_{\check{\nu}}$ to the subset of ν-paths is not a lattice. It is an upper ideal but it has several minimal elements. The results on the number of linear intervals that are presented in the rest of the chapter do not hold either with this weaker condition (see Remark 6.5.2).

The boolean structure of refinement of the alt-Tamari posets can also be generalized to the alt ν-Tamari lattices. More precisely, the following proposition is a generalization of Proposition 5.4.11.

Proposition 6.3.8. *Let δ and δ' be two increment vectors with respect to ν such that $\delta_i \leq \delta'_i$ for all i. If $P < Q$ in $\mathrm{Tam}_\nu(\delta')$, then $P < Q$ in $\mathrm{Tam}_\nu(\delta)$.*

In other words, whenever $\delta \leq \delta'$, the poset $\mathrm{Tam}_\nu(\delta)$ is an *extension* of the poset $\mathrm{Tam}_\nu(\delta')$, meaning that it can be obtained from $\mathrm{Tam}_\nu(\delta')$ by adding some relations.

Proof. It is sufficient to prove that the result is true for covering relations in $\mathrm{Tam}_\nu(\delta')$, namely that if $P \lessdot_{\delta'} Q$, then $P < Q$ in $\mathrm{Tam}_\nu(\delta)$.

We can write $P = AEBC$ and $Q = ABEC$ for some δ'-excursion B. Note that for any north step N in B, the δ'-excursion of N is a factor of B and that the δ-excursion of any north step is a prefix of its δ'-excursion since we have $\delta \leq \delta'$. Thus, we can build a chain of δ-rotations from P to Q by exchanging the east step E between A and B with either the next δ-excursions if B starts with a north step, or with the first east step of B otherwise, which does not change the path. □

6.3.2 On (δ, ν)-trees

In Section 4.2.3, we gave an alternative definition of the alt ν-Tamari lattices, on objects that we called (δ, ν)-trees. We recall here the definitions, this time with proofs.

The alt ν-Tamari lattice $\mathrm{Tam}_\nu(\delta)$ is the interval $[\nu, 1^\nu]$ in $\mathrm{Tam}_{\check{\nu}}$. So, it can be described as the rotation lattice of $\check{\nu}$-trees that are above the $\check{\nu}$-tree T_ν corresponding to ν in $\mathrm{Tam}_{\check{\nu}}$. These trees can be described as maximal collections of pairwise compatible elements in a shape $F_{\delta,\nu}$ which we will now describe. This point of view is useful to show that all alt ν-Tamari lattices have the same number of linear intervals of any length.

Let δ, ν and $\check{\nu}$ as in Proposition 6.3.5. Let $F_{\check{\nu}}$ be the Ferrers diagram that lies weakly above $\check{\nu}$. We consider the lattice path $\hat{\nu}$ that starts at the lowest right corner of $F_{\check{\nu}}$ (the point with coordinates $(\check{\nu}_0, 0)$), and consists of the sequence of west and north steps

$$W^{\nu_0} N W^{\gamma_1} N W^{\gamma_2} \dots N W^{\gamma_n}, \qquad \text{for } \gamma_i = \nu_i - \delta_i. \tag{6.1}$$

We define $F_{\delta,\nu}$ to be the subset of $F_{\check{\nu}}$ consisting of the boxes that are not below $\hat{\nu}$, and denote by $L_{\delta,\nu}$ its set of lattice points. A (δ, ν)-*tree* is a maximal collection of pairwise $\check{\nu}$-compatible elements in $L_{\delta,\nu}$. An example is illustrated on the right of Figure 6.2.

Lemma 6.3.9. *The (δ, ν)-trees are exactly the $\check{\nu}$-trees that are contained in $L_{\delta,\nu}$.*

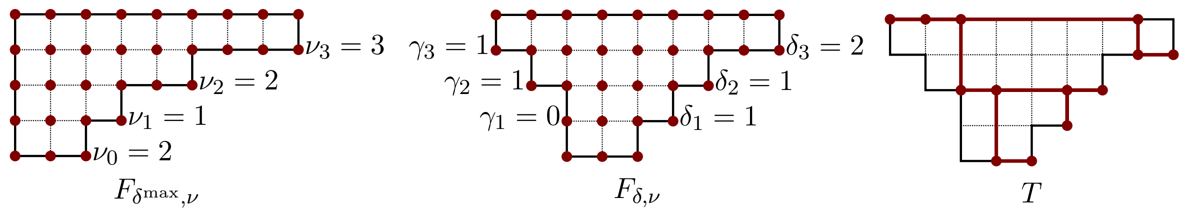


Figure 6.2: Left: The Ferrers diagram $F_{\delta^{\max},\nu}$ and its corresponding lattice points $L_{\delta^{\max},\nu}$ for $\nu = EENENEENEEEN = (2,1,2,3,0)$ and $\delta^{\max} = (1,2,3,0)$. Middle: The Ferrers diagram $F_{\delta,\nu}$ and its corresponding lattice points $L_{\delta,\nu}$ for the same ν and $\delta = (1,1,2,0)$; the path $\check{\nu} = EEEENENENEEN$ and $\hat{\nu} = WWNNWNWN$. Right: a (δ,ν)-tree.

Proof. A $\check{\nu}$-tree that is contained in $L_{\delta,\nu}$ is automatically a (δ,ν)-tree by definition. So, we just need to check that (δ,ν)-trees are $\check{\nu}$-trees. By definition, a (δ,ν)-tree is a maximal collection of pairwise $\check{\nu}$-compatible elements in $L_{\delta,\nu}$. As $L_{\delta,\nu} \subseteq L_{\check{\nu}}$, we want to prove that the maximality in $L_{\delta,\nu}$ implies the maximality in $L_{\check{\nu}}$.

Recall that the paths ν and $\check{\nu}$ have the same starting point $(0,0)$ and the same ending point (m,n), and that every ν-tree and every $\check{\nu}$-tree has exactly $m+n+1$ nodes (equal to the number of lattice points in ν and $\check{\nu}$). Furthermore, the shape $F_{\delta,\nu}$ fits in the $m \times n$ box with the top corners being $(0,n)$ and (m,n). In our example in Figure 6.15, $m = 11$ and $n = 7$. The ν-tree and the (δ,ν)-tree shown in this figure both have $m+n+1 = 19$ nodes. We want to show that every (δ,ν)-tree has exactly $m+n+1$ elements.

Let T be a (δ,ν)-tree. Label its elements $p_0, p_1, \ldots, p_r$ from bottom to top, from right to left. We will show that $r = m+n$, which implies that T has $m+n+1$ elements as desired.

Let us reconstruct T recursively, by adding the elements $p_0, p_1, \ldots, p_r$ one at a time in order. Note that if p_i is not the leftmost element in its row, then all the lattice points above p_i are forbidden in the next steps, because they are incompatible with an element $p_j \in T$ that is to the left of p_i in the same row.

Now, when we add an element p_j in the process of reconstructing T, then p_j is necessarily located at the rightmost position of its row that is not forbidden by any element before. Otherwise, let $p_j \in T$ be the node with the smallest index that does not satisfy that property, and let q be the rightmost lattice point in the same row that is not forbidden by any element p_i with $i < j$. In particular, q is on the right of p_j by assumption (and thus compatible with p_j), and q is compatible with every p_i with $i < j$. Moreover, for $k > j$, q is also compatible with $p_k \in T$, otherwise p_k, p_j would be incompatible. So, we can add the element q to T, creating a new compatible set, contradicting the maximality of T.

Furthermore, the following relation holds,

$$j = \mathrm{forb}(p_j) + \mathrm{height}(p_j)$$

where $\mathrm{height}(p_j)$ is the height of $p_j \in T$ and $\mathrm{forb}(p_j)$ is the number of $p_i \in T$ with $i < j$ such that p_i is not the leftmost node of T in its row. That is, $\mathrm{forb}(p_j)$ is the number of nodes before p_j that forbid the positions above them. This formula is clear because the j nodes $p_0, \ldots, p_{j-1}$ appearing before p_j either forbid positions above them (not the leftmost node of their row) or increase the height by one (the leftmost node of their row).

If we apply this formula to the last node p_r and assume that $r < m+n$, then

$$\mathrm{forb}(p_r) + \mathrm{height}(p_r) < m+n.$$

We can assume that $\mathrm{height}(p_r) = n$ (maximum possible height), otherwise we could add the top left corner of $F_{\delta,\nu}$ to T, creating a bigger compatible set and contradicting the maximality T. This implies that $\mathrm{forb}(p_r) < m$, which means that on the top row there are still some lattice points that are not forbidden. Adding one of these points contradicts the maximality of T. As a consequence, we have proven that $r = m + n$ as desired. $\square$

We define the **(δ, ν)-right flushing** $\mathrm{flush}_{\delta,\nu}$ as the restriction of the right flushing bijection $\mathrm{flush}_{\check{\nu}}$ (with respect to $\check{\nu}$) to set of ν-paths (thought as the subset of $\check{\nu}$-paths that are above ν).

Proposition 6.3.10. *The map $\mathrm{flush}_{\delta,\nu}$ is a bijection between the set of ν-paths and the set of (δ, ν)-trees. Moreover, two ν-paths are related by a δ-rotation $\mu \lessdot_\delta \mu'$ if and only if the corresponding trees are related by a $\check{\nu}$-rotation $T \lessdot_{\check{\nu}} T'$.*

Proof. By Proposition 6.3.5, δ-rotations of a ν-path μ coincide with $\check{\nu}$-rotations of μ, and the right flushing bijection $\mathrm{flush}_{\check{\nu}}$ transforms $\check{\nu}$-rotations on paths to $\check{\nu}$-rotations on the corresponding trees. Therefore, the second part of the proposition is clear. It remains to show that μ is a ν-path if and only if $\mathrm{flush}_{\check{\nu}}(\mu)$ is a (δ, ν)-tree, or equivalently a $\check{\nu}$-tree that is contained in $L_{\delta,\nu}$.

We start by proving the forward direction. First, note that the image of the bottom path, namely $T_\nu = \mathrm{flush}_{\check{\nu}}(\nu)$, is contained in $L_{\delta,\nu}$. More precisely, the shape $F_{\delta,\nu}$ has ν_k east steps on its boundary at height k. For $k > 0$, some of these east steps (exactly $\nu_k - \delta_k$) are on the left boundary, and some (exactly δ_k) are on the right boundary. The $\nu_k + 1$ nodes of T_ν at height k consist of the $\nu_k - \delta_k + 1$ points on the left boundary, and the δ_k end points of the east steps of the right boundary. At height $k = 0$, the $\nu_0 + 1$ nodes of T are all the lattice points at the bottom of $F_{\delta,\nu}$. This shows that T_ν is contained in $L_{\delta,\nu}$.

Now, every ν-path μ can be obtained by applying a sequence of $\check{\nu}$-rotations to the bottom path ν. Its image $\mathrm{flush}_{\check{\nu}}(\mu)$ is a $\check{\nu}$-tree that can be obtained by applying the corresponding sequence of $\check{\nu}$-rotations to the tree T_ν. Since T_ν is contained in $L_{\delta,\nu}$ and such rotations preserve this property, then $\mathrm{flush}_{\check{\nu}}(\mu)$ is also contained in $L_{\delta,\nu}$. This finishes the proof of the forward direction.

The backward direction is equivalent to the following statement: if T is a $\check{\nu}$-tree contained in $L_{\delta,\nu}$ then $\mu = \mathrm{flush}_{\check{\nu}}^{-1}(T)$ is weakly above ν. This is equivalent to show that the number of nodes in T at heights less than or equal to k is at most $\nu_0 + \cdots + \nu_k + k + 1$.

For $0 \leq k \leq n$, let L_k (resp. F_k) be the restriction of $L_{\delta,\nu}$ (resp. $F_{\delta,\nu}$) to the points with height less than or equal to k. The width of F_k is equal to $\nu_0 + \cdots + \nu_k$. The maximal number of compatible lattice points inside L_k is equal to $\nu_0 + \cdots + \nu_k + k + 1$. The restriction of T to L_k is a compatible set (not necessarily maximal). The result follows. $\square$

The (δ, ν)-right flushing bijection from ν-paths to (δ, ν)-trees is described in exactly the same way as in Section 4.1.3: we recursively add $\mu_i + 1$ nodes to the tree inside the shape $F_{\delta,\nu}$ from right to left, from bottom to top, while avoiding the forbidden positions above a node which is not the leftmost node in a row. Figure 6.15 shows the image of the path $\mu = (1, 0, 1, 1, 3, 2, 1, 2)$ for $\delta^{max} = (1, 0, 2, 2, 0, 3, 0)$ (left) and for $\delta = (0, 0, 1, 2, 0, 1, 0)$ (right), where the base path is $\nu = (3, 1, 0, 2, 2, 0, 3, 0)$.

Definition 6.3.11. The **rotation poset of (δ, ν)-trees** $\mathrm{Tam}^{tr}_\nu(\delta)$ is the transitive closure of $\check{\nu}$-rotations on (δ, ν)-trees.

The three examples of the rotation poset of (δ, ν)-trees for $\nu = ENEEN = (1, 2, 0)$ are illustrated on Figure 6.3.

Theorem 6.3.12. *The alt ν-Tamari lattice is isomorphic to the rotation poset of (δ, ν)-trees:*

$$\mathrm{Tam}_\nu(\delta) \cong \mathrm{Tam}^{tr}_\nu(\delta).$$

In particular, the rotation poset of (δ, ν)-trees is a lattice.

Proof. The alt ν-Tamari lattice is the poset on ν-paths whose covering relations are given by δ-rotations. The rotation poset of (δ, ν)-trees is poset on (δ, ν)-trees whose covering relations are $\check{\nu}$-rotations. The result is then a consequence of Proposition 6.3.10. The lattice property was proven for the alt ν-Tamari lattice in Corollary 6.3.6. $\square$

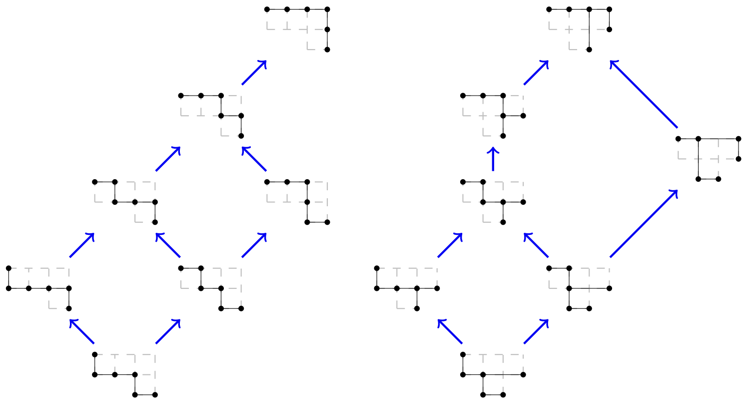


Figure 6.3: Examples of alt ν-Tamari lattices $\mathrm{Tam}_\nu(\delta)$ for $\nu = ENEEN = (1,2,0)$. Left: the ν-Dyck lattice, for $\delta = (0,0)$. Middle: the lattice for $\delta = (1,0)$. Right: the ν-Tamari lattice, for $\delta = (2,0)$.
In each case, the number of linear intervals of length k is given by ℓ_k where $\ell = (\ell_0, \ell_1, \ell_2, \ell_3) = (7,8,4,1)$. For instance, 7 represents the trivial intervals of length 0, which are just the elements of each poset; there are 8 linear intervals of length 1, which correspond to the edges; 4 linear interval of length 2, and 1 linear interval of length 3.

6.4 Left and right intervals in the alt ν-Tamari lattice

Since $\mathrm{Tam}_\nu(\delta)$ is an interval in $\mathrm{Tam}_{\tilde{\nu}}$, its linear intervals are linear intervals in $\mathrm{Tam}_{\tilde{\nu}}$. In terms of trees, this gives the following simple characterization.

Definition 6.4.1. An interval $[T, T']$ in $\mathrm{Tam}_\nu^{tr}(\delta)$ is a **left interval** (resp. **right interval**) if $[T, T']$ is a left interval (resp. right interval) in $\mathrm{Tam}_{\tilde{\nu}}^{tr}$, as defined in Definition 6.2.1.

Proposition 6.4.2. *The nontrivial linear intervals in* $\mathrm{Tam}_\nu^{tr}(\delta)$ *are either left or right intervals.*

Proof. This is a direct consequence of Proposition 6.2.2. □

In this section, we aim at characterizing the left and right intervals in terms of certain row and (reduced) column vectors associated to the (δ,ν)-trees. This will be used in Section 6.5, to show that the number of linear intervals in the alt ν-Tamari lattice $\mathrm{Tam}_\nu(\delta)$ is independent of the choice of δ.

6.4.1 Row vectors and left intervals

The **row vector** of a (δ,ν)-tree T is the vector

$$r(T) = (r_0, \dots, r_n),$$

where $r_i + 1$ is the number of nodes of T at height i.

Proposition 6.4.3. *A (δ,ν)-tree T is completely characterized by its row vector. Moreover, $(r_0, \dots, r_n)$ is the row vector of some (δ,ν)-tree if and only if*

1. *$r_i \geq 0$ for all i,*
2. *$\sum_{i=0}^{j} r_i \leq \sum_{i=0}^{j} \nu_i$ for all j, and*
3. *$\sum_{i=0}^{n} r_i = \sum_{i=0}^{n} \nu_i$.*

Proof. By Proposition 6.3.10, the map $\text{flush}_{\delta,\nu}$ is a bijection between the set of ν-paths and the set of (δ,ν)-trees. Moreover, this map preserves the number of points at each height, and therefore the row vector. Since ν-paths are characterized by their row vectors, then (δ,ν)-trees are characterized by their row vectors as well.

Furthermore, via the map $\text{flush}_{\delta,\nu}$, characterizing the row vectors of (δ,ν)-trees is equivalent to characterizing the row vectors of ν-paths. Condition (1) just says that every ν-path μ has at least one lattice point at each height. Condition (2) says that μ is weakly above ν, and Condition (3) says that μ and ν have the same ending points. □

Given a (δ,ν)-tree T, we say that an *ordered* set $L = \{p, q_0, q_1, \dots, q_\ell\} \subseteq T$ is a **horizontal $\mathbb{L}$** of T if L is the restriction of T to a rectangle R of the grid, such that p is the top-left corner of R, and $q_0, q_1, \dots, q_\ell$ appear in this order on the bottom side of R with q_0 being its left-bottom corner and q_ℓ its right-bottom corner. Note that no other element of T belongs to R. We say that the length of L is equal to ℓ. We denote by $T+L$ the (δ,ν)-tree obtained from T by rotating the nodes $q_0, q_1, \dots, q_{\ell-1}$ in T in this order. An example of these concepts is illustrated in Figure 6.4.

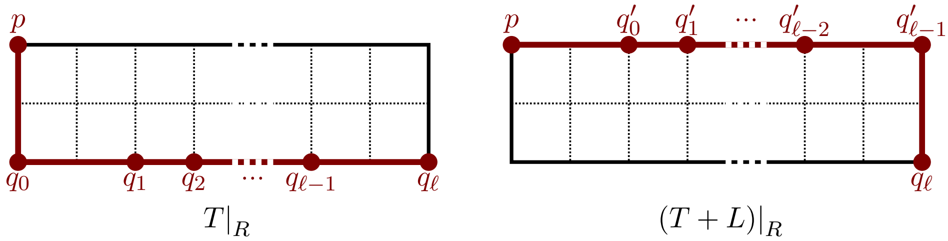


Figure 6.4: Schematic illustration of a horizontal $\mathbb{L}$ and the tree $T+L$.

Lemma 6.4.4. *Let L be a horizontal $\mathbb{L}$ of length ℓ of a (δ,ν)-tree T. Then, $[T, T+L]$ is a left interval of length ℓ in $\text{Tam}_\nu^{tr}(\delta)$. Moreover, every left interval of $\text{Tam}_\nu^{tr}(\delta)$ with bottom element T is of this form.*

Proof. This follows by the definition of left intervals. □

Proposition 6.4.5. *Let T be a (δ,ν)-tree with row vector $r(T) = (r_0, \dots, r_n)$. The number of left intervals of length ℓ with bottom element T in $\text{Tam}_\nu^{tr}(\delta)$ is equal to*

$$|\{0 \le i \le n-1 : \; r_i \ge \ell\}|.$$

Proof. By Lemma 6.4.4, the left intervals of length ℓ with bottom element T are of the form $[T, T+L]$ where L is a horizontal $\mathbb{L}$ of length ℓ of T. There is one such L for each $r_i \ge \ell$ with $0 \le i \le n-1$, where $q_0, \dots, q_\ell$ are the $\ell+1$ leftmost nodes of T at height i and p is the parent of q_0 in T. □

The previous two results, Lemma 6.4.4 and Proposition 6.4.5, characterize the left intervals in $\text{Tam}_\nu^{tr}(\delta)$ in terms of the row vectors of (δ,ν)-trees. Our next goal is to have a similar characterization for the right intervals with respect to certain column vectors. As we will see, column vectors are not enough for such a characterization, and we will need to consider a notion of reduced column vectors. Before going into that, we first introduce column vectors and present some of their properties.

6.4.2 Column vectors

Given a path ν from $(0,0)$ to (m,n), the *reversed path* $\overleftarrow{\nu}$ is the path from $(0,0)$ to (n,m) obtained by reading ν from right to left and replacing east steps by north steps and vice versa. Equivalently, $\overleftarrow{\nu} = (\overleftarrow{\nu}_0, \dots, \overleftarrow{\nu}_m)$ where $\overleftarrow{\nu}_i$ is the number of north steps of the path ν in column

$m-i$. For instance, if $\nu = ENEEN = (1,2,0)$ then $\overleftarrow{\nu} = ENNEN = (1,0,1,0)$. This notion is convenient to characterize column vectors.

In order to define the column vector of a (δ,ν)-tree, it is convenient to assign an order $j_0 \prec_\delta \cdots \prec_\delta j_m$ to the columns of $L_{\delta,\nu}$, obtained by reading the columns from shortest to longest, from right to left, as illustrated in Figure 6.5 (left). See also the three examples in Figure 6.6.

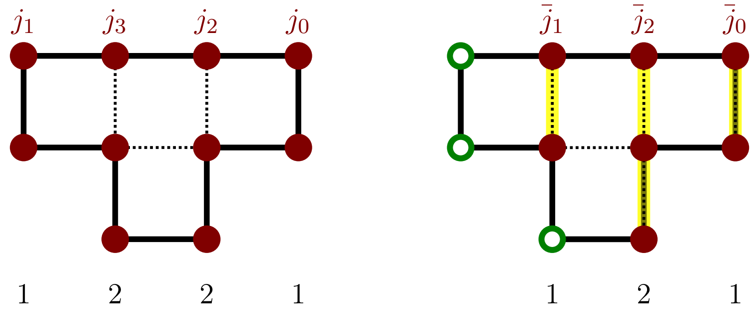


Figure 6.5: Left: the columns j_0, j_1, j_2, j_3 of $L_{\delta,\nu}$, their lengths are 1,1,2,2. Right: the reduced columns $\bar{j}_0, \bar{j}_1, \bar{j}_2$ of $L_{\delta,\nu}$ with lengths 1,1,2, where the relevant points are filled brown and the non-relevant points are unfilled green. In both cases, the columns are read from shortest to longest, from rigth to left.

The *column vector* of a (δ,ν)-tree T is the vector

$$c_\delta(T) = (c_0, \dots, c_m),$$

where $c_i + 1$ is the number of nodes of T in column j_i. For instance, the three (δ,ν)-trees (for the three choices of δ) in Figure 6.6, all have column vector $(0,1,0,1)$. This means, in each of the cases, there are $0+1$ nodes of the tree in column j_0, $1+1$ nodes in column j_1, $0+1$ nodes in column j_2, and $1+1$ nodes in column j_3. Equivalently, c_i counts the number of edges of T in column j_i.

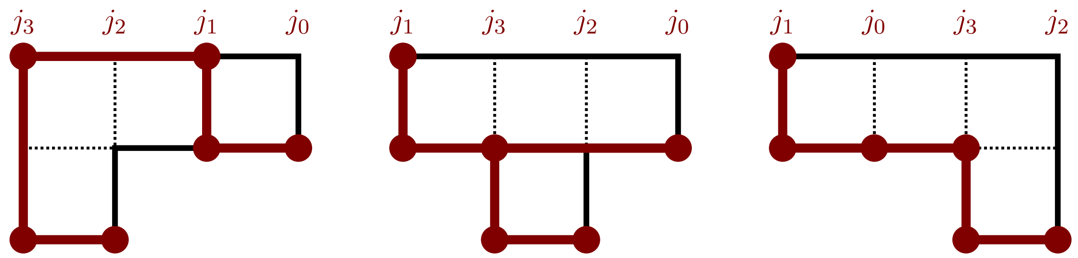


Figure 6.6: The columns j_0, j_1, j_2, j_3 of $L_{\delta,\nu}$ for $\nu = ENEEN$ and the three possible choices of $\delta = (2,0), (1,0)$ and $(0,0)$. The columns are read from shortest to longest, from right to left. The column vector of the shown trees is $c_\delta(T) = (0,1,0,1)$ in all three cases.

Proposition 6.4.6. *A (δ,ν)-tree T is completely characterized by its column vector. Moreover, $(c_0,\dots,c_m)$ is the column vector of some (δ,ν)-tree if and only if*

1. *$c_i \geq 0$ for all i,*
2. *$\sum_{i=0}^{j} c_i \leq \sum_{i=0}^{j} \overleftarrow{\nu}_i$ for all j, and*
3. *$\sum_{i=0}^{m} c_i = \sum_{i=0}^{m} \overleftarrow{\nu}_i$.*

We prove this proposition in several steps.

Lemma 6.4.7. *A (δ, ν)-tree T can be reconstructed from its column vector.*

Proof. Let T be a (δ, ν)-tree. Label the elements of the tree $p_0, p_1, ..., p_r$ from right to left, from bottom to top, as illustrated in Figure 6.7.

We reconstruct T recursively, by adding the elements $p_0, p_1, ..., p_r$ one at a time in order. Note that if p_i is not the top most element in its column, then all the lattice points on the left of p_i are forbidden in the next steps, because they are incompatible with an element $p_j \in T$ that is above p_i in the same column.

Now, when we add an element p_j in the process of reconstructing T, then p_j is necessarily located at the bottom most position of its column that is not forbidden by any element before. Otherwise, let $p_j \in T$ be the node with the smallest label that does not satisfy that property, and let q be the bottom most lattice point in the same column that is not forbidden by any element p_i with $i < j$. In particular, q is below p_j by assumption so it is compatible with p_j, and q is compatible with every p_i with $i < j$. Moreover, for $k > j$, q is also compatible with $p_k \in T$, otherwise p_k, p_j would be incompatible. So, we can add the element q to T, creating a new compatible set, contradicting the maximality of T.

The tree T can therefore be constructed by adding nodes from right to left, from bottom to top, avoiding forbidden positions. The forbidden positions are those to the left of a node that is not the top most node in a column. The number of points in each column is determined by the column vector. $\square$

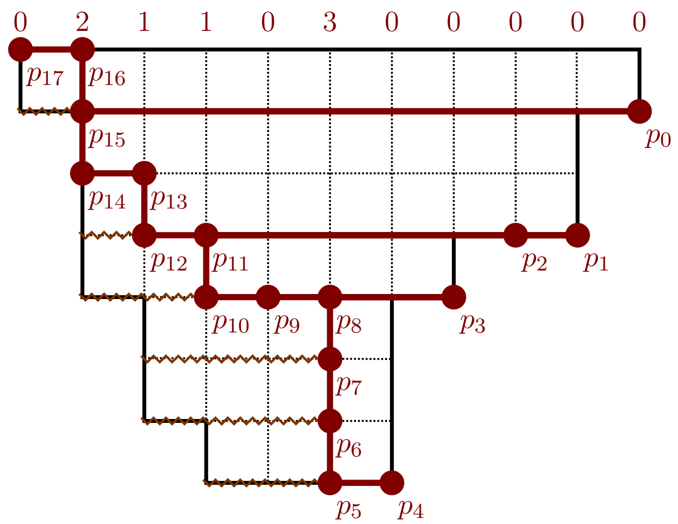


Figure 6.7: The down flushing algorithm.

We call the algorithm described in the previous proof the **down flushing algorithm**. Its input is a valid column vector $(c_0, \ldots, c_m)$ (or the number of nodes in each column), and its output is the unique (δ, ν)-tree such that $c_\delta(T) = (c_0, \ldots, c_m)$. Figure 6.7 illustrates an example, where the labels on top represent the number of nodes, minus 1, in each column, and the forbidden positions are the ones that belong to the wiggly lines.

Lemma 6.4.8. *Let δ, δ' be two increment vectors with respect to ν, such that δ' is obtained by either adding or subtracting 1 to one of the entries of δ. For every (δ, ν)-tree T, there is a unique (δ', ν)-tree T' such that*

$$c_\delta(T) = c_{\delta'}(T').$$

Proof. Uniqueness follows by Lemma 6.4.7, so we just need to prove existence.

Let T be a (δ,ν)-tree with column vector $c_\delta(T) = (c_0,\dots,c_m)$, and assume that δ' is obtained by subtracting 1 to a non-zero entry δ_a of δ. This operation produces a small transformation to the columns of $L_{\delta,\nu}$. All the columns of length larger than $n-a$ are moved one step to the right, while the subsequent column (of length $n-a$) is moved one step to their left. All other columns stay the same. The result is the new set $L_{\delta',\nu}$. An example is illustrated in Figure 6.9.

Consider the labeling $j_0,\dots,j_m$ of the columns of $L_{\delta,\nu}$ (and also of the columns of $L_{\delta',\nu}$) obtained by reading the columns from shortest to longest, from right to left, as before. Assume that j_{i_1} is the label of the column that was moved to the left under the small transformation that changes δ to δ'. We also consider the columns $j_{i_2},\dots,j_{i_k}$, consisting of the columns of $L_{\delta,\nu}$, from right to left, of length larger than $n-a$ that contain at least one node of T at height larger than or equal to a. The restriction of the tree T to the nodes at height larger than or equal to a in columns $j_{i_1},\dots,j_{i_k}$ is marked as a bold red path on the left of Figures 6.8 and 6.9. It is a subpath of the unique path of the tree from column j_{i_1} to the root of the tree. We will describe a small transformation to T that produces a (δ',ν)-tree T' with the same column vector as T. The result of this is illustrated on the right of Figures 6.8 and 6.9, and affects the tree at the red marked nodes. The brown points in the columns between $\bar{j}_{i_k}$ and $\bar{j}_{i_1}$ are also moved one step to the right, together with their column.

Note that the columns $j_{i_2},\dots,j_{i_k}$ of $L_{\delta',\nu}$ are positioned one step to the right of columns $j_{i_2},\dots,j_{i_k}$ of $L_{\delta,\nu}$, while column j_{i_1} was moved to some position to the left, see Figures 6.8 and 6.9.

Let A be the set of rows that contain at least one node of the marked bold red path of T. We apply the following transformation to T. For each node of T in a column j_{i_b}, for $2\le b\le k$, that belongs to A, we draw a node in T' in column j_{i_b} but shifted down $c_{j_{i_1}}$ positions within A. The $c_{j_{i_1}}+1$ nodes in column j_{i_1} are moved to the top rows of A. All other nodes of T remain intact in their columns. A schematic illustration of this transformation is shown in Figure 6.8, and an explicit example in Figure 6.9.

The result is a (δ',ν)-tree T' with the same column vector as T: $c_\delta(T) = c_{\delta'}(T')$. The reason why this procedure works is guaranteed by a direct analysis of the down flushing algorithm. Moreover, we can also recover T from T' by a similar transformation in the reverse direction. □

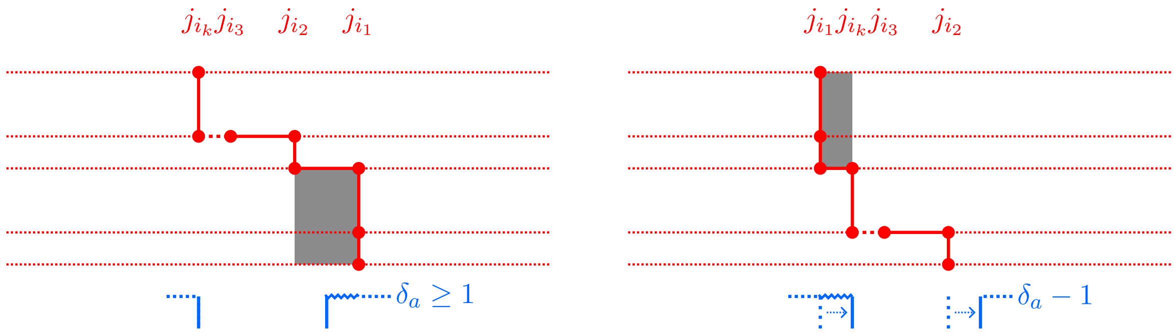


Figure 6.8: Schematic illustration of the transformation in the proof of Lemma 6.4.8.

Lemma 6.4.9. *Let δ,δ' be two increment vectors with respect to ν. For every (δ,ν)-tree T, there is a unique (δ',ν)-tree T' such that*

$$c_\delta(T) = c_{\delta'}(T').$$

Proof. Any two increment vectors with respect to ν can be connected by a sequence of increment vectors, such that each vector is obtained from the previous one by either adding or subtracting 1 to one of its entries. The result then follows by Lemma 6.4.8. □

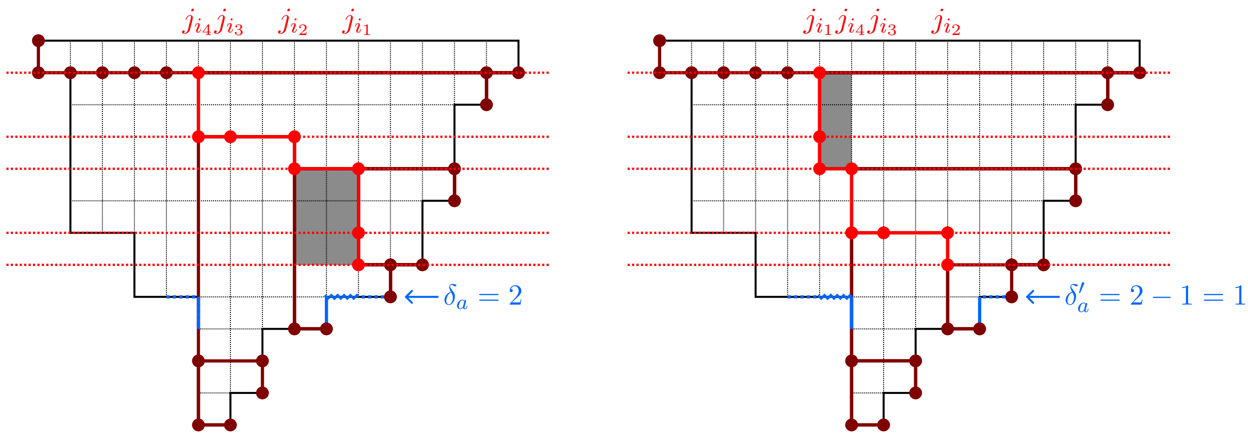


Figure 6.9: Example of the transformation in the proof of Lemma 6.4.8.

Proof of Proposition 6.4.6. A (δ, ν)-tree T is completely characterized by its column vector by Lemma 6.4.7. Furthermore, the characterization of column vectors of (δ, ν)-trees is independent of the choice of increment vector δ, by Lemma 6.4.9. So, we just need to prove the three conditions of the proposition for one particular choice of δ. We choose the extreme case $\delta = \delta^{\max}$, where $\delta_i = \nu_i$. In this case (δ, ν)-trees are just the classical ν-trees.

Classifying the column vectors of ν-trees is the same as classifying the row vectors of $\overleftarrow{\nu}$-trees, because reversing the path transforms column vectors to row vectors, and vice versa. The three conditions of the proposition are then equivalent to the three conditions of Proposition 6.4.3 (for the extreme maximal case δ). $\square$

6.4.3 Reduced column vectors

We say that a lattice point $p \in L_{\delta,\nu}$ is **non-relevant** if it is the leftmost point of a row of $L_{\delta,\nu}$. All other points in $L_{\delta,\nu}$ are called **relevant**. Figure 6.5 (right) illustrates an example where the relevant points are filled brown, and the non-relevant points are unfilled green.

The **reduced columns** are the columns of relevant points in $L_{\delta,\nu}$. These are shown in yellow in Figure 6.5 (right). The three examples for $\nu = ENEEN$ and all possible choices of δ are shown in Figure 6.10. The reduced columns are colored yellow here as well for easier visualization.

In order to define the reduced column vector of a (δ, ν)-tree, it is convenient to assign an order $\overline{j}_0 \prec_\delta \cdots \prec_\delta \overline{j}_{m-1}$ to the reduced columns of $F_{\delta,\nu}$, obtained by reading the reduced columns from shortest to longest, from right to left, as illustrated in Figure 6.5 (right). See also the three examples in Figure 6.10.

The **reduced column vector** of a (δ, ν)-tree T is the vector

$$\overline{c}_\delta(T) = (\overline{c}_0, \ldots, \overline{c}_{m-1}),$$

where $\overline{c}_i + 1$ is the number of nodes of T in reduced column $\overline{j}_i$. For instance, the three (δ, ν)-trees (for the three choices of δ) in Figure 6.10, all have reduced column vector $(0, 1, 0)$. This means, in each of the cases, that there are $0 + 1$ nodes of the tree in reduced column $\overline{j}_0$, $1 + 1$ nodes in reduced column $\overline{j}_1$, and $0 + 1$ nodes in reduced column $\overline{j}_2$. Note that the green nodes are non-relevant and do not belong to the reduced columns by definition, and so are not counted here.

Proposition 6.4.10. *A (δ, ν)-tree T is completely characterized by its reduced column vector. Moreover, $(\overline{c}_0, \ldots, \overline{c}_{m-1})$ is the reduced column vector of some (δ, ν)-tree if and only if*

1. *$\overline{c}_i \geq 0$ for all i,*
2. *$\sum_{i=0}^{j} \overline{c}_i \leq \sum_{i=0}^{j} \overleftarrow{\nu}_i$ for all j.*

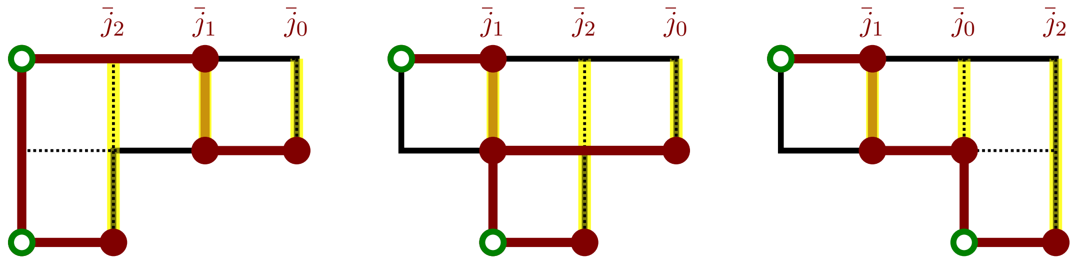


Figure 6.10: The ordering $\bar{j}_0 \prec_\delta \cdots \prec_\delta \bar{j}_2$ of the reduced columns of $L_{\delta,\nu}$ for $\nu = ENEEN$ and the three possible choices of $\delta = (2,0),(1,0)$ and $(0,0)$. The reduced columns (colored yellow) are read from shortest to longest, from right to left. The reduced column vector of the shown trees is $\bar{c}_\delta(T) = (0,1,0)$ in all three cases.

The proof of this proposition follows the same steps as the proof of Proposition 6.4.6 for column vectors. We write all the (somewhat repeated) details for self containment.

Lemma 6.4.11. *A (δ,ν)-tree T can be reconstructed from its reduced column vector.*

Proof. We proceed similarly as in the proof of Lemma 6.4.7, with the small difference that we need to be careful what to do with the non-relevant positions, which are not counted by the reduced column vector.

Let $\bar{c}_\delta(T) = (\bar{c}_0,\dots,\bar{c}_{m-1})$ be the reduced column vector of T. Similarly as before, the tree T can be reconstructed by adding nodes from right to left, from bottom to top, avoiding the forbidden positions that are to the left of a node that is not the top most node of its column. Here comes the tricky part. When we want to add the nodes in column $\bar{j}_i$, there are two possible scenarios:

(1) If there are non-relevant positions in column $\bar{j}_i$ that are not forbidden by any of the nodes added before in the process, then these non-relevant positions are automatically compatible with all the nodes of the tree T (the ones that were already added, and all the future ones). Therefore, all the non-relevant nodes in column $\bar{j}_i$ that are not forbidden by any previously added node should be added to T. After this we proceed adding $\bar{c}_i + 1$ nodes from bottom to top in the positions that are not forbidden in column $\bar{j}_i$.

(2) If all the non-relevant positions in column $\bar{j}_i$ are forbidden, then we just proceed adding $\bar{c}_i + 1$ nodes from bottom to top in the positions that are not forbidden in that column.

This procedure reconstructs the tree T and only depends on the reduced column vector.

An example is illustrated in Figure 6.11. Note that the unfilled green point p_{14} is non-relevant, and was forced to be added to T because it is not forbidden by any of the previously added nodes $p_1,\dots,p_{13}$. At this step of the process, one proceeds adding the 1+1 relevant points p_{15},p_{16} in that column, which are counted by the corresponding entry plus one of the reduced column vector. ☐

We call the algorithm described in the previous proof the *reduced down flushing algorithm*. Its input is a valid reduced column vector $(\bar{c}_0,\dots,\bar{c}_{m-1})$ (or the number of relevant nodes in each column), and its output is the unique (δ,ν)-tree such that $\bar{c}_\delta(T) = (\bar{c}_0,\dots,\bar{c}_{m-1})$.

Lemma 6.4.12. *Let δ,δ' be two increment vectors with respect to ν, such that δ' is obtained by either adding or subtracting 1 to one of the entries of δ. For every (δ,ν)-tree T, there is a unique (δ',ν)-tree T' such that*

$$\bar{c}_\delta(T) = \bar{c}_{\delta'}(T').$$

Moreover, the heights of the non-relevant nodes of T and T' coincide.

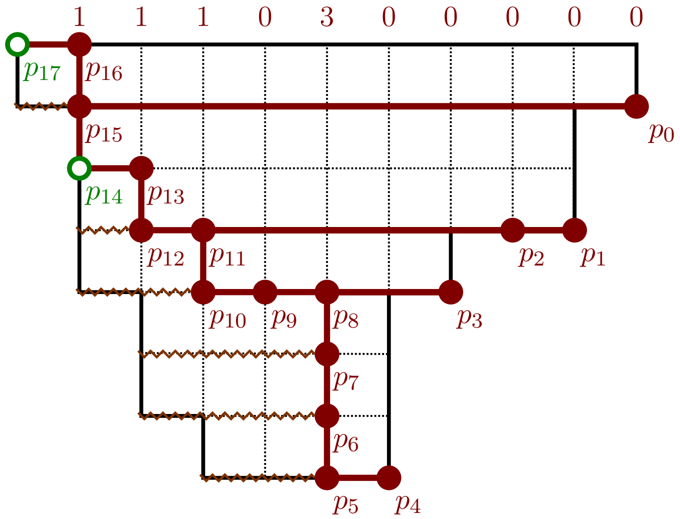


Figure 6.11: The reduced down flushing algorithm.

Proof. Uniqueness follows by Lemma 6.4.11, so we just need to prove existence.

Let T be a (δ, ν)-tree with reduced column vector $\bar{c}_\delta(T) = (\bar{c}_0, \dots, \bar{c}_{m-1})$, and assume that δ' is obtained by subtracting 1 to a non-zero entry δ_a of δ. This operation produces a small transformation to the reduced columns of $L_{\delta,\nu}$ (which is slightly different to the transformation in the proof on Lemma 6.4.8). All the reduced columns of length larger than $n-a$ are moved one step to the right, while the subsequent reduced column (of length $n-a$) is moved one step to their left. All other columns stay the same. An example is illustrated in Figure 6.13. Here, we have chosen the same example as in Figure 6.9, to highlight the differences with the transformation described in the proof of Lemma 6.4.8.

Consider the labeling $\bar{j}_0, \dots, \bar{j}_{m-1}$ of the reduced columns of $L_{\delta,\nu}$ (and also of the reduced columns of $L_{\delta',\nu}$) obtained by reading the reduced columns from shortest to longest, from right to left, as before. Assume that $\bar{j}_{i_1}$ is the label of the reduced column that was moved to the left under the small transformation that changes δ to δ'. We also consider the columns $\bar{j}_{i_2}, \dots, \bar{j}_{i_k}$, consisting of the reduced columns of $L_{\delta,\nu}$, from right to left, of length larger than $n-a$ that contain at least one node of T at height larger than or equal to a. The restriction of the tree T to the nodes at height larger than or equal to a in the reduced columns $\bar{j}_{i_1}, \dots, \bar{j}_{i_k}$ is marked as a bold red path on the left of Figures 6.12 and 6.13. It is a subpath of the unique path of the tree from column $\bar{j}_{i_1}$ to the root of the tree.

Note that column j_{i_4}, of length larger than $n-a$ in Figure 6.9, is now a reduced column of length $n-a$. That is why there is no $\bar{j}_{i_4}$ in our example in Figure 6.13.

We will describe a small transformation to T that produces a (δ', ν)-tree T' with the same reduced column vector as T. The result of this is illustrated on the right of Figures 6.12 and 6.13, and affects the red marked nodes of the tree. The brown and green points between the columns $\bar{j}_{i_k}$ and $\bar{j}_{i_1}$ are also moved one step to the right.

Note that the columns $\bar{j}_{i_2}, \dots, \bar{j}_{i_k}$ of $L_{\delta',\nu}$ are positioned one step to the right of columns $\bar{j}_{i_2}, \dots, \bar{j}_{i_k}$ of $L_{\delta,\nu}$, while column $\bar{j}_1$ was moved to some position to the left, see Figures 6.12 and 6.13.

Let A be the set of rows that contain at least one node of the marked bold red path of T. We apply the following transformation to T. For each node T in a reduced column $\bar{j}_{i_b}$, for $2 \le b \le k$, that belongs to A, we draw a node in T' in the reduced column $\bar{j}_{i_b}$ but shifted down $\bar{c}_{j_{i_1}}$ positions withing A. The $\bar{c}_{j_{i_1}} + 1$ nodes in reduced column $\bar{j}_{i_1}$ are moved to the top rows of A. All other relevant nodes of T remain intact in their reduced columns, and all non-relevant nodes remain intact in their "not reduced" columns. A schematic illustration of this transformation is shown in Figure 6.12, and an explicit example in Figure 6.13.

The result is a (δ', ν)-tree T' with the same reduced column vector as T: $\bar{c}_\delta(T) = \bar{c}_{\delta'}(T')$, and such that the heights of the non-relevant nodes are preserved. The reason why this procedure works is guaranteed by a direct analysis of the reduced down flushing algorithm. Moreover, we can also recover T from T' by a similar transformation in the reverse direction. □

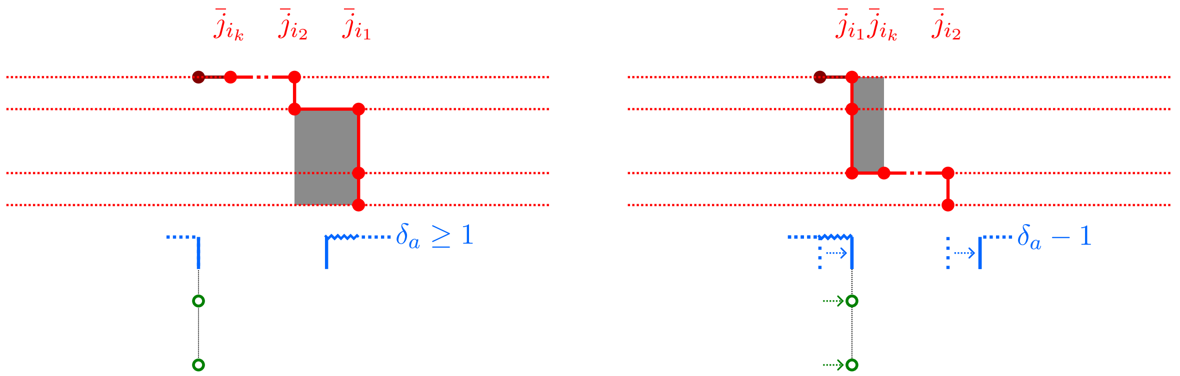


Figure 6.12: Schematic illustration of the transformation in the proof of Lemma 6.4.8.

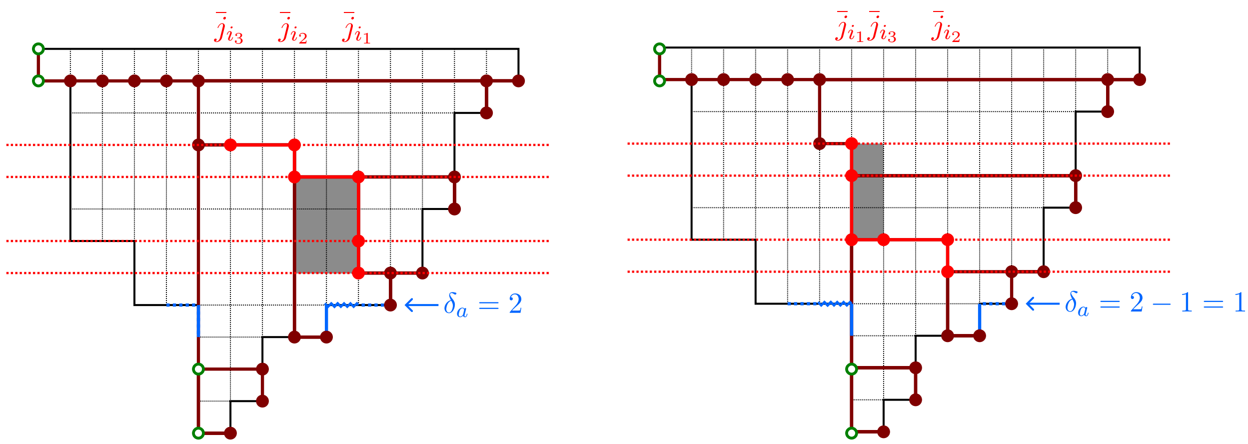


Figure 6.13: Example of the transformation in the proof of Lemma 6.4.8.

Lemma 6.4.13. *Let δ, δ' be two increment vectors with respect to ν. For every (δ, ν)-tree T, there is a unique (δ', ν)-tree T' such that*

$$\bar{c}_\delta(T) = \bar{c}_{\delta'}(T').$$

Moreover, the heights of the non-relevant nodes of T and T' coincide.

Proof. Any two increment vectors with respect to ν can be connected by a sequence of increment vectors, such that each vector is obtained from the previous one by either adding or subtracting 1 to one of its entries. The result then follows by Lemma 6.4.12. □

Proof of Proposition 6.4.10. A (δ, ν)-tree T is completely characterized by its reduced column vector by Lemma 6.4.11. Furthermore, the characterization of reduced column vectors of (δ, ν)-trees is independent of the choice of increment vector δ, by Lemma 6.4.13. So, we just need to prove the two conditions of the proposition for one particular choice of δ. We choose the extreme case $\delta = \delta^{\max}$, where $\delta_i = \nu_i$. In this case (δ, ν)-trees are just the classical ν-trees.

The reduced column vector $(\bar{c}_0, \dots, \bar{c}_{m-1})$ of a ν-tree T is obtained from the row vector $(r_0, \dots, r_m)$ of the corresponding $\overleftarrow{\nu}$-tree $\overleftarrow{T}$ by removing its last entry r_m. The two conditions of the proposition are then equivalent to the first two conditions of Proposition 6.4.3 (for the extreme maximal case δ). The third condition was about the number of points in the top row of $\overleftarrow{T}$, which correspond to the non-relevant points in T. □

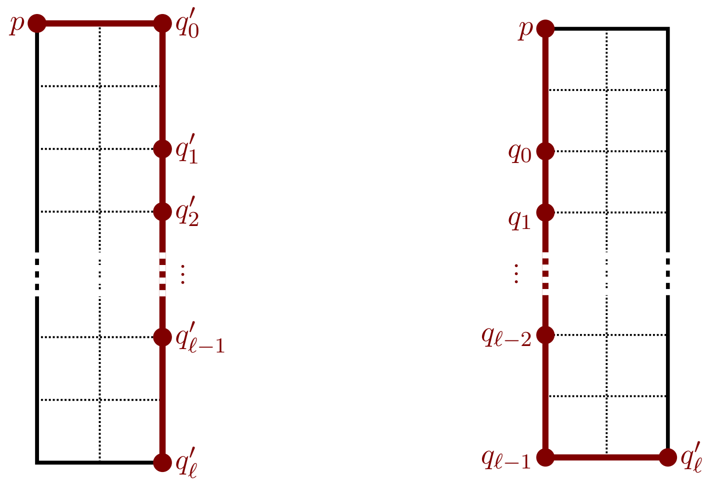
6.4.4 Reduced column vectors and right intervals

We are finally ready to provide our characterization of right intervals in $\mathrm{Tam}_\nu^{tr}(\delta)$ in terms of reduced column vectors.

Given a (δ, ν)-tree T, we say that an *ordered* set $L = \{p, q'_0, q'_1, \dots, q'_\ell\} \subseteq T$ is a **vertical $\mathbb{L}$** of T if L is the restriction of T to a rectangle $R \subseteq F_{\delta,\nu}$ of the grid, such that p is the top-left corner of R, and $q'_0, q'_1, \dots, q'_\ell$ appear in this order from top to bottom on the right side of R, with q'_0 being its top-right corner and q'_ℓ its bottom-right corner. Note that no other elements of T belong to R. We say that the length of L is equal to ℓ. We denote by $T - L$ the (δ, ν)-tree obtained from T by rotating down the nodes $q'_0, q'_1, \dots, q'_{\ell-1}$ in T in this order.

Note that the condition $R \subseteq F_{\delta,\nu}$ is crucial here, to guaranty that the result after applying these rotations is still contained in the Ferrers diagram $F_{\delta,\nu}$, otherwise $T - L$ would not be a (δ, ν)-tree. In particular, if $R \subseteq F_{\delta,\nu}$ then $q'_0, q'_1, \dots, q'_\ell$ are all relevant nodes in T, and contribute to the reduced column vector. Vice versa, if q'_ℓ is relevant then $p \llcorner q'_\ell \in F_{\delta,\nu}$ because of the reduced down flushing algorithm, and thus $R \subseteq F_{\delta,\nu}$.

An example of these concepts is illustrated in Figure 6.14.



$T|_R$ $(T - L)|_R$

Figure 6.14: Schematic illustration of a vertical $\mathbb{L}$ and the tree $T - L$.

Lemma 6.4.14. *Let L be a vertical $\mathbb{L}$ of length ℓ of a (δ, ν)-tree T. Then, $[T - L, T]$ is a right interval of length ℓ in $\mathrm{Tam}_\nu^{tr}(\delta)$. Moreover, every right interval of $\mathrm{Tam}_\nu^{tr}(\delta)$ with top element T is of this form.*

Proof. This follows by the definition of right intervals. □

Proposition 6.4.15. *Let T be a (δ, ν)-tree with reduced column vector $\bar{c}_\delta(T) = (\bar{c}_0, \dots, \bar{c}_{m-1})$. The number of right intervals of length ℓ with top element T in $\mathrm{Tam}_\nu^{tr}(\delta)$ is equal to*

$$|\{0 \leq i \leq m-1 : \ \bar{c}_i \geq \ell\}|.$$

Proof. By Lemma 6.4.14, the right intervals of length ℓ with top element T are of the form $[T - L, T]$ where L is a vertical $\mathbb{L}$ of length ℓ of T. There is one such L for each $\bar{c}_i \geq \ell$ with

$0 \leq i \leq m-1$, where $q'_0, \ldots, q'_\ell$ are the $\ell+1$ top most nodes of T at column $\bar{j}_i$ and p is the parent of q_0 in T. □

6.5 Bijections between linear intervals

Using the tools developed in the previous section, we are now ready to prove one of our main results.

Theorem 6.5.1. *For a fixed path ν, all alt ν-Tamari lattices* $\mathrm{Tam}_\nu(\delta)$ *have the same number of linear intervals of length ℓ.*

This is a direct consequence of Proposition 6.4.2 and Corollaries 6.5.5 and 6.5.7 below, which show that the number of left intervals and the number of right intervals of length ℓ are preserved for any choice of δ. Indeed, we prove more refined versions of these results in Propositions 6.5.4 and 6.5.6.

Remark 6.5.2. If we chose any other path $\check{\nu}$ weakly below ν that does not satisfy $\check{\nu}_i \leq \nu_i$, for all $i > 0$, then the restriction of $\mathrm{Tam}_{\check{\nu}}$ to the subset of ν-paths does not satisfy the enumerative result of Theorem 6.5.1.

More precisely, this poset still has the same number of left intervals (the left flushing argument presented afterwards still works) as all alt ν-Tamari lattices. But based on computational experiments, it seems to have fewer right intervals. For instance, for $\nu = (1, 1, 1)$ and $\check{\nu} = (1, 2, 0)$, the distribution of linear intervals in the resulting poset is $(5, 5, 1)$ but the distribution of linear intervals in Tam_ν is $(5, 5, 2)$.

Remark 6.5.3. Theorem 6.5.1 generalizes the results obtained in Theorem 5.4.25 for the staircase $\nu = (NE)^n$. However, in this more general case, we usually do not have a closed formula counting the linear intervals of length ℓ similar to the one presented in Theorem 5.3.1.

In the m-Tamari lattice, where $\nu = (NE^m)^n$, one can adapt the decomposition given in Section 5.1.2 in order to find a closed formula for the number of right intervals of length ℓ:

$$m\binom{mn+n-\ell}{n-\ell-1}.$$

We were not able to find a nice formula for the number of left intervals in this case. For $n = 5$ and $m = 2$, the distribution of left intervals in this lattice is $(728, 442, 222, 112, 47, 18, 5, 1)$. Since 47 is a prime number, such a nice product formula does not seem to exist.

6.5.1 The horizontal flushing and left intervals

We define the **horizontal flushing** $f^h_{\delta,\delta'}$ as the map between the set of (δ, ν)-trees and the set of (δ', ν)-trees characterized by the property

$$f^h_{\delta,\delta'}(T) = T' \quad \Longleftrightarrow \quad r(T) = r(T').$$

That is, the map that preserves the row vector of the tree. This map is uniquely determined by this property, and can be computed as the composition

$$f^h_{\delta,\delta'}(T) = \mathrm{flush}_{\delta',\nu} \circ \mathrm{flush}^{-1}_{\delta,\nu},$$

which sends a (δ, ν)-tree to the unique ν-path with the same row vector, and then to the corresponding (δ', ν)-tree. In particular, $f^h_{\delta,\delta'}$ is a bijection, and can be described using a **horizontal flushing algorithm**:

If $r(T) = (r_0, \ldots, r_n)$, then T' can be reconstructed by adding $r_i + 1$ nodes, from bottom to top, from right to left, avoiding the forbidden positions that are above the nodes that are not the leftmost nodes in their row.

This gives a natural correspondence between the horizontal $\mathbb{L}$'s of T and the horizontal $\mathbb{L}$'s of T': an L of length ℓ in row i (here we mean that the bottom part of the $\mathbb{L}$ is in row i) of T corresponds to the unique horizontal $\mathbb{L}$ of the same length in row i of T'. By abuse of notation, we denote by $f^h_{\delta,\delta'}(L) = L'$ the horizontal $\mathbb{L}$ of T' associated to L, a horizontal $\mathbb{L}$ of T.

Proposition 6.5.4. *Let T be a (δ,ν)-tree and $T' = f^h_{\delta,\delta'}(T)$ be its corresponding (δ',ν)-tree. We also denote by $L' = f^h_{\delta,\delta'}(L)$ the horizontal $\mathbb{L}$ of T' associated to L, a horizontal $\mathbb{L}$ of T.*

1. *The number of left intervals of length ℓ in $\mathrm{Tam}^{tr}_\nu(\delta)$ with bottom element T is equal to the number of left intervals of length ℓ in $\mathrm{Tam}^{tr}_\nu(\delta')$ with bottom element T'.*

2. *The map*
$$[T, T+L] \to [T', T'+L']$$
is a bijection between the left intervals of $\mathrm{Tam}^{tr}_\nu(\delta)$ and the left intervals of $\mathrm{Tam}^{tr}_\nu(\delta')$.

Proof. By Proposition 6.4.5, the number of left intervals with bottom element T depends only on the row vector $r(T)$. Since $r(T) = r(T')$, then Item (1) follows. Item (2) is straight forward from the characterization of left intervals in Lemma 6.4.4. □

An example of the bijection between left intervals is illustrated in Figure 6.15. The maximal horizontal $\mathbb{L}$'s are marked red for easier visualization.

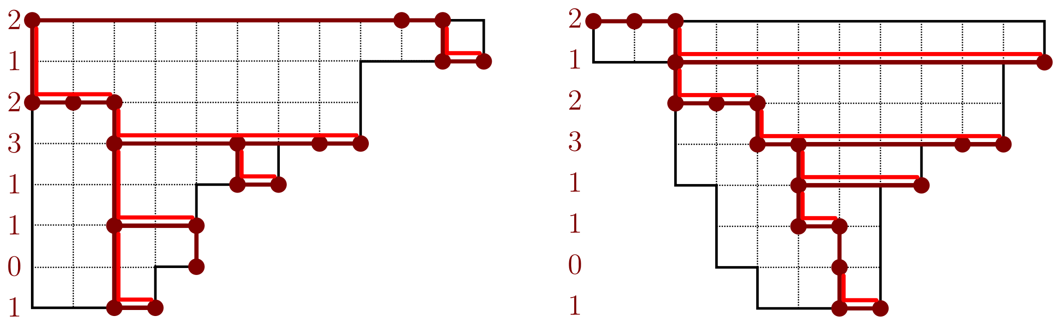


Figure 6.15: Bijection between left intervals for $\nu = (3,1,0,2,2,0,3,0)$, $\delta^{max} = (1,0,2,2,0,3,0)$, and $\delta = (0,0,1,2,0,1,0)$. Both trees have row vector $(1,0,1,1,3,2,1,2)$, whose entries plus one count the number of nodes in each of the rows.

Corollary 6.5.5. *The number of left intervals of length ℓ in $\mathrm{Tam}^{tr}_\nu(\delta)$ and $\mathrm{Tam}^{tr}_\nu(\delta')$ are the same.*

Proof. This is a direct consequence of Proposition 6.5.4. □

6.5.2 The reduced vertical flushing and right intervals

We define the **vertical flushing** $f^v_{\delta,\delta'}$ as the map between the set of (δ,ν)-trees and the set of (δ',ν)-trees characterized by the property

$$f^v_{\delta,\delta'}(T) = T' \quad \Longleftrightarrow \quad \bar{c}_\delta(T) = \bar{c}_{\delta'}(T').$$

That is, the map that preserves the reduced column vector of the tree.

This map is uniquely determined by this property by Lemma 6.4.13. In particular, $f^v_{\delta,\delta'}$ is a bijection, and can be described using a **vertical flushing algorithm**:

If $\bar{c}_\delta(T) = (\bar{c}_0,\ldots,\bar{c}_{m-1})$, then T' can be reconstructed by adding nodes, from right to left, from bottom to top, avoiding the forbidden positions that are to the left of the nodes that are not the top most nodes in their column. The difference here is that the number of nodes that we add to a column whose reduced column is labeled $\bar{j}_i$, is not necessarily equal to $\bar{c}_{j_i}+1$: we first add all the non-relevant nodes that are not forbidden by any of the previously added nodes; then we continue adding $\bar{c}_{j_i}+1$ relevant nodes from bottom to top in the non-forbidden available positions.

This also gives a natural correspondence between the vertical $\mathbb{L}$'s of T and the vertical $\mathbb{L}$'s of T': an L of length ℓ in reduced column $\bar{j}_i$ (here we mean that the right part of the L is in reduced column $\bar{j}_i$) of T corresponds to the unique vertical $\mathbb{L}$ of the same length in reduced column $\bar{j}_i$ of T'. By abuse of notation, we denote by $f^v_{\delta,\delta'}(L) = L'$ the vertical $\mathbb{L}$ of T' associated to L, a vertical $\mathbb{L}$ of T.

Proposition 6.5.6. *Let T be a (δ,ν)-tree and $T' = f^v_{\delta,\delta'}(T)$ be its corresponding (δ',ν)-tree. We also denote by $L' = f^v_{\delta,\delta'}(L)$ the vertical $\mathbb{L}$ of T' associated to L, a vertical $\mathbb{L}$ of T.*

1. *The number of right intervals of length ℓ in $\mathrm{Tam}^{tr}_\nu(\delta)$ with top element T is equal to the number of right intervals of length ℓ in $\mathrm{Tam}^{tr}_\nu(\delta')$ with top element T'.*

2. *The map*
$$[T, T-L] \to [T', T'-L']$$
is a bijection between the right intervals of $\mathrm{Tam}^{tr}_\nu(\delta)$ and the right intervals of $\mathrm{Tam}^{tr}_\nu(\delta')$.

Proof. By Proposition 6.4.15, the number of right intervals with top element T depends only on the reduced column vector $\bar{c}_\delta(T)$. Since $\bar{c}_\delta(T) = \bar{c}_{\delta'}(T')$, then Item (1) follows. Item (2) is straight forward from the characterization of right intervals in Lemma 6.4.14. □

Examples of the bijection between right intervals are illustrated in Figures 6.16 and 6.17. The maximal vertical $\mathbb{L}$'s are marked red for easier visualization. The green nodes are the non-relevant nodes.

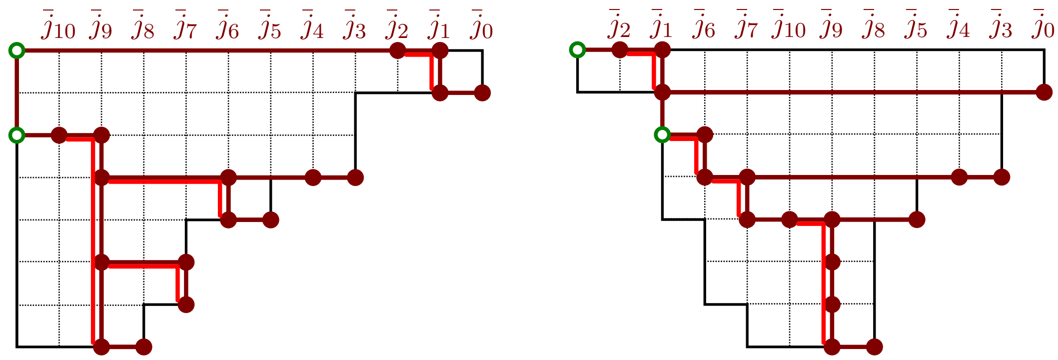


Figure 6.16: Bijection between right intervals for $\nu = (3,1,0,2,2,0,3,0)$, δ^{max}, and $\delta = (0,0,1,2,0,1,0)$. Both trees have reduced column vector $(0,1,0,0,0,0,1,1,0,3,0)$, whose entries plus one count the number of relevant nodes in the reduced columns. The green non-relevant nodes are not counted.

Corollary 6.5.7. *The number of right intervals of length ℓ in $\mathrm{Tam}^{tr}_\nu(\delta)$ and $\mathrm{Tam}^{tr}_\nu(\delta')$ are the same.*

Proof. This is a direct consequence of Proposition 6.5.6. □

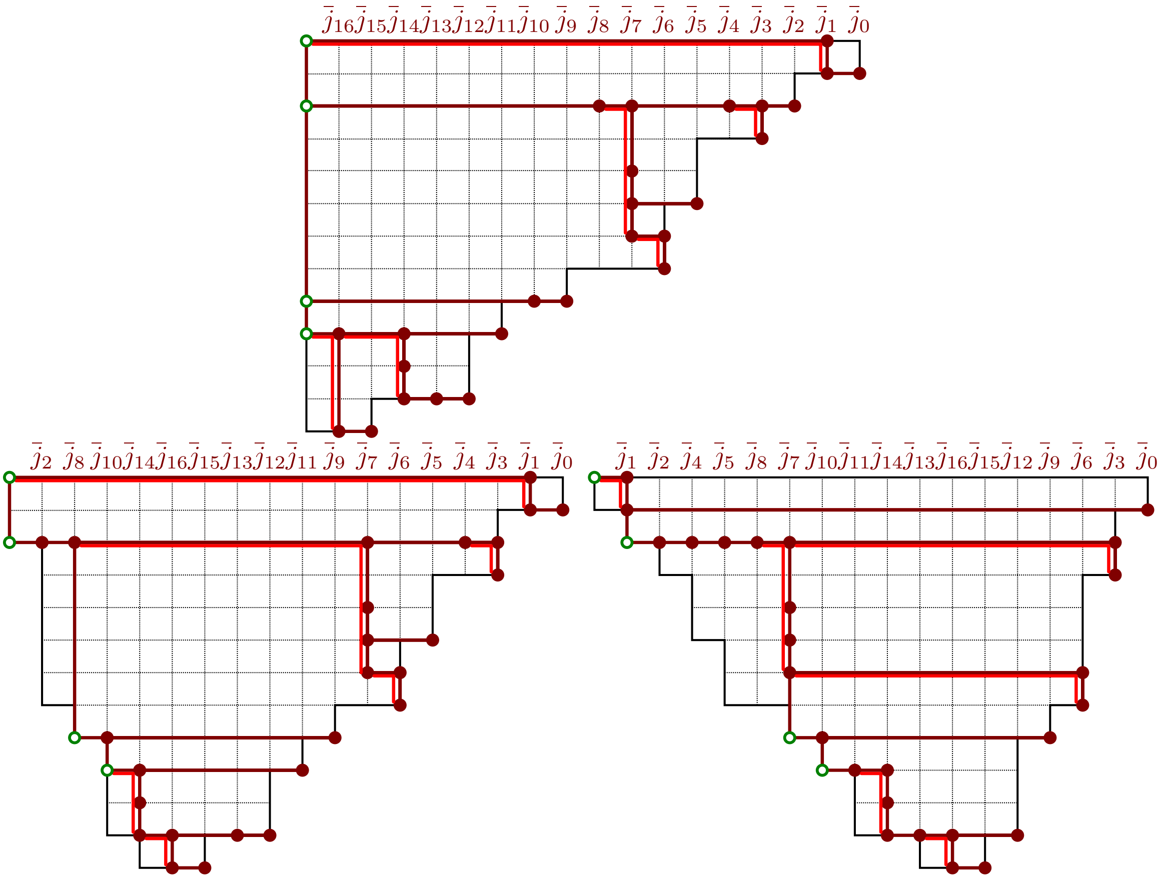


Figure 6.17: Bijection between right intervals for $\nu = (2,3,0,1,2,3,0,1,0,2,1,2,0)$, δ^{max}, $\delta = (2,0,1,1,2,0,1,0,2,0,2,0)$ and $\delta' = (1,0,0,1,1,0,0,0,2,0,1,0)$. The three trees have reduced column vector $(0,1,0,1,0,0,1,3,0,0,0,0,0,0,2,0,1)$.

Chapter 7

In other posets related to the Tamari lattice

This chapter is based on unpublished work. We compute the distribution of linear intervals in the weak order on the symmetric group. We also consider other partial orders related to the Tamari lattice. In particular, inspired by [Der23], we introduce a "greedy alt ν-Tamari lattice", and in a special case, we prove that its distribution of linear intervals is the same as the one of the Tamari lattice. We also provide several conjectures on enumeration formulas for linear intervals in several posets as well as equidistribution conjectures on several families of posets. Finally, we study in greater detail the case of the permutree lattices.

The study of linear intervals was very fruitful in the Tamari and the Dyck lattices, and this was then extended in the ν-Tamari direction. We could define a big new family of posets, namely the alt ν-Tamari lattices, which seemed to have good properties. They are all lattices and they possess a rich underlying geometric structure. More precisely, they seem to be realizable as a polytopal complex in some Euclidean space, as it was proved for the case of the ν-Tamari lattices in [CPS19], using arrangements of tropical hyperplanes. In particular, one can give integer coordinates to each (δ, ν)-tree (seen as a $\check{\nu}$-tree) to draw the Hasse diagram of the posets in this perspective. Moreover, the alt ν-Tamari lattices also behave very nicely with respect to linear intervals, the number of which we proved to be independent of the choice of the increment vector δ.

A more algebraic direction which is worth mentioning is to look at the categories of modules over the incidence algebras of a poset. In the case of the alt ν-Tamari lattices, these categories $\mathsf{mod}\,\mathrm{Tam}_\nu(\delta)$ and $\mathsf{mod}\,\mathrm{Tam}_\nu(\delta')$ seem to be derived-equivalent for a fixed path ν. A similar result has been proven in other families of generalizations of the Tamari lattice, namely the posets of tilting modules and the Cambrian lattices [Lad07b, Lad07a]. Moreover, these new posets could be useful as intermediate steps to prove the conjectured derived equivalence between the categories of modules of the ν-Tamari lattice and the ν-Dyck lattice.

Looking at linear intervals seems to be very promising in other posets as well, and in particular in posets related to the Tamari lattice. We conjecture for instance similar equidistribution of linear intervals results in the families of the Cambrian lattices and the posets of tilting modules, namely that the number of linear intervals would not depend on the choice of the Coxeter element or orientation of the Dynkin diagram, and this result seems in fact to hold in all types. This is also conjectured to remain true in the more general context of the m-Cambrian lattices.

It is also worth noting that the enumeration of linear intervals is also an invariant of posets that is additive under sum of posets and can be made multiplicative as well under Cartesian products, as are the cardinality and relation number (Remark 2.1.7). To do so, we have to distinguish trivial intervals and linear intervals which are not trivial. Indeed, all intervals in a Cartesian product of posets are the Cartesian product of intervals in each poset, and the result is trivial when all intervals are trivial and linear but nontrivial if and only if exactly one of the intervals in the product is linear and all others are trivial.

Definition 7.0.1. Let ϵ be a formal variable with $\epsilon^2 = 0$. Given a (finite) poset P, its **linear relation number** is the polynomial $L(P) = T(P) + U(P)\epsilon \in \mathbb{Z}[\epsilon]$ where $T(P)$ is the number of trivial intervals of P and $U(P)$ counts its nontrivial linear intervals.

Proposition 7.0.2. *Let P and Q be two posets. We have $L(P + Q) = L(P) + L(Q)$ and $L(P \times Q) = L(P) \cdot L(Q)$. In other words, the linear relation number is additive under sum of posets and multiplicative under Cartesian products.*

This observation could be helpful in generalizing some results or in counting linear intervals in posets where intervals are understood through a recursive decomposition.

Moreover, interesting enumeration formulas also seem to appear in other posets, which is another reason for studying linear intervals. For instance, we also have an easy closed formula for the number of linear intervals in the weak order on the symmetric group and in a "greedy" version of the Dyck lattice. We seem to also have such formulas for type B and D Cambrian lattices and posets of tilting modules, or in the more exotic Pallo's comb poset. One could also study other families containing the Tamari lattice, such as the permutrees posets, and see if similar formulas exist.

However, some posets do not have any linear interval of length 2 or more, as for instance the third classical poset on Catalan objects, namely the noncrossing partition lattice. The Boolean lattice B_n and the set partition lattice P_n are two other examples. In these cases, all intervals of height 2 are isomorphic either to the Boolean lattice B_2 or to the set partition lattice P_2, which are not totally ordered. The posets which do not have linear intervals of height 2 are said to be **2-thick**, and for lattices of finite length, this is equivalent to be *relatively complemented* [Bjö81].

7.1 In the weak order

Let us focus first on the (right) weak order in the symmetric group presented in Section 2.2.4. Computational experimentation led to conjecture that the number of linear intervals in the weak order on $\mathfrak{S}_n$ has a nice distribution, again with respect to the length. Discussions with Viviane Pons and Vincent Pilaud led to the enumerative result presented in this section.

Interestingly, as in the ν-Tamari lattices, we can classify nontrivial linear intervals into "left" and "right" intervals, which both correspond to covering relations for $\ell = 1$ and are distinct otherwise. Recall from Theorem 2.2.23 that the covering relations in the weak order on $\mathfrak{S}_n$ are of the form $\sigma \lessdot_r \sigma'$ where σ' is obtained from σ by exchanging two consecutive entries $\sigma_i < \sigma_{i+1}$ in the one-line notation, which is equivalent to $\sigma' = \sigma s_i$ and $\sigma(i) < \sigma(i+1)$.

Definition 7.1.1. A **right interval** in the weak order on $\mathfrak{S}_n$ is an interval $[\sigma, \sigma']$ where $\sigma' = \sigma s_i s_{i+1} \dots s_{i+\ell-1}$ for some $\ell \geq 1$ and $1 \leq i \leq n - \ell$, with $\sigma(i) < \sigma(i + k)$ for all $1 \leq k \leq \ell$.

A **left interval** in the weak order on $\mathfrak{S}_n$ is an interval $[\sigma, \sigma']$ where $\sigma' = \sigma s_{i+\ell-1} \dots s_i$ for some $\ell \geq 1$ and $1 \leq i \leq n - \ell$, with $\sigma(i + \ell) > \sigma(i + k)$ for all $0 \leq k \leq \ell - 1$.

Such a right (resp. left) interval is clearly linear of length ℓ, since we added ℓ inversions to σ, all of which being of the form $(\sigma(i), k)$ (resp. $(k, \sigma(i+\ell))$). Thus, there is only one saturated chain from σ to σ'. For example, the interval $18236574 \lessdot_r 18326574 \lessdot_r 18362574 \lessdot_r 18365274 \lessdot_r 18365724$ is a right interval of length 4 in the weak order on $\mathfrak{S}_8$.

Proposition 7.1.2. *All linear intervals of length $\ell \geq 2$ in the weak order are either left or right intervals.*

Proof. We prove the result by induction on $\ell \geq 2$.

Suppose that $\sigma \lessdot_r \sigma' \lessdot_r \sigma''$ is a saturated chain of length 2. Then, $\sigma' = \sigma s_i$ for some i such that $\sigma(i) < \sigma(i + 1)$, and $\sigma'' = \sigma' s_k$ for some k with $\sigma'(k) < \sigma'(k + 1)$. Obviously, $k \neq i$. We then have three cases.

- If $\sigma(i) < \sigma(i+2)$ and $\sigma'' = \sigma s_i s_{i+1}$, then $[\sigma, \sigma'']$ is a right interval.
- If $\sigma(i-1) > \sigma(i+1)$ and $\sigma'' = \sigma s_i s_{i-1}$, then $[\sigma, \sigma'']$ is a left interval.
- If $\sigma'' = \sigma' s_k$ with $k \notin \{i-1, i, i+1\}$, then $[\sigma, \sigma'']$ is a square with $\sigma \lessdot_r \sigma s_k \lessdot_r \sigma''$ as the other saturated chain from σ to σ'.

We quickly analyze the case $\ell = 3$ to conclude on the general case. Suppose that $[\sigma, \sigma']$ is linear of length $\ell = 3$, then it must contain a linear interval $[\sigma, \sigma'']$ of length 2, which is either a left or a right interval. If it is a left interval, then so is $[\sigma, \sigma']$, and if it is a right interval, then so is $[\sigma, \sigma']$. Indeed, if for instance $[\sigma, \sigma'']$ is a right interval, then $\sigma'' = \sigma s_i s_{i+1}$ for some i, and by the case $\ell = 2$, σ' can be either equal to $\sigma s_i s_{i+1} s_{i+2}$, which is a right interval, or to $\sigma s_i s_{i+1} s_i = \sigma s_{i+1} s_i s_{i+1}$, which is a hexagon, hence not linear. A symmetric argument holds for left intervals.

Now by induction, if a linear interval $[\sigma, \sigma']$ is linear of length $\ell \geq 3$, then it must contain a linear interval $[\sigma, \sigma'']$ of length $\ell - 1$, which is either a left or a right interval. Again, if it is a left interval, then so is $[\sigma, \sigma']$, and if it is a right interval, then so is $[\sigma, \sigma']$.

Remark that this also proves that there are no linear intervals of length n or greater in the weak order on $\mathfrak{S}_n$. □

This being proven, we can now enumerate the linear intervals in the weak order on $\mathfrak{S}_n$ according to their length.

Theorem 7.1.3. *In the weak order* $\mathrm{Weak}(\mathfrak{S}_n)$*, there are:*

- $n!$ *linear intervals of length* 0*,*
- $\dfrac{(n-1)\,n!}{2}$ *linear intervals of length* 1*,*
- $\dfrac{2\,(n-\ell)\,n!}{\ell+1}$ *linear intervals of length* ℓ *for* $2 \leq \ell \leq n-1$*,*
- *no intervals of length* $\ell \geq n$*.*

Proof. For $\ell = 0$, this is the number of elements of $\mathfrak{S}_n$.

For $\ell \geq 1$, we will prove that the number of right intervals of length ℓ in $\mathrm{Weak}(\mathfrak{S}_n)$ is equal to $\dfrac{(n-\ell)\,n!}{\ell+1}$, and the same will be true for left intervals by symmetry. We will conclude since covering relations are the only intervals that are both left and right.

Suppose that $[\sigma, \sigma']$ is a right interval of length ℓ in $\mathrm{Weak}(\mathfrak{S}_n)$. Then, $\sigma' = \sigma s_i s_{i+1} \dots s_{i+\ell-1}$ for some $1 \leq i \leq n-\ell$, with $\sigma(i) < \sigma(i+k)$ for all $1 \leq k \leq \ell$.

To build such an interval, one first needs to choose an integer $1 \leq i \leq n-\ell$, which accounts for a factor $n-\ell$, then a subset $\{\sigma(i), \dots, \sigma(i+\ell)\} \subset [n]$ of size $\ell+1$, which can be done in $\binom{n}{\ell+1}$ ways. The integer $\sigma(i)$ needs to be the smallest element of this subset. Finally, one can choose any permutation of the remaining ℓ elements of this subset and any permutation of the remaining $n-\ell-1$ elements, which contributes for a factor $\ell!\,(n-\ell-1)!$.

All in all, there are $(n-\ell)\,(n-\ell-1)!\,\ell!\dbinom{n}{\ell+1} = \dfrac{(n-\ell)\,n!}{(\ell+1)}$ right intervals of length ℓ in $\mathrm{Weak}(\mathfrak{S}_n)$. □

The total number of linear intervals does not seem to factorize very well, the first terms are $[1, 2, 2^4, 2^3 \cdot 13, 2^8 \cdot 3, 2^6 \cdot 3^2 \cdot 11, 2^5 \cdot 3^3 \cdot 67, 2^5 \cdot 3^3 \cdot 673, \dots]$ ([OEIS, A344216]). It is worth noting that the total number of intervals in the weak order on $\mathfrak{S}_n$ does not seem to admit a nice product either, the first terms being $[1, 3, 17, 151, \dots]$ ([OEIS A007767]), the last two being prime numbers.

7.2 In the greedy alt ν-Tamari posets

Recently, new posets have been introduced on the set of Dyck paths, and in particular the *"greedy Tamari" poset* and the *"dexter" poset* on the set of Dyck paths of size n [Cha20, Der23]. Both are extended by the Tamari lattice by construction, they are conjectured to be anti-isomorphic through a simple bijection and Chapoton proved that the latter was a meet semilattice.

The greedy Tamari poset GTam_n is defined similarly as the Tamari lattice Tam_n, with covering relations consisting of exchanging the down step of a valley with a part of the path immediately following it. However, instead of exchanging the down step of the valley with the smallest subpath that is a Dyck path starting with the up step of the valley—or in other word its excursion—one exchanges the largest subpath that is a Dyck path starting with this up step.

The dexter poset Dext_n is also defined similarly, with covering relations consisting of moving excursions to the left, but only if the excursion is not followed by a down step, and exchanging it with one or several down steps preceding it.

Definition 7.2.1. Let P be a Dyck path. Suppose that for some i, the i-th up step u_i of P is preceded by a down step d, and let D be the largest subpath of P starting with this up step and being a Dyck path itself. Equivalently, D is the union of the consecutive excursions following d. We then write $P = AdDB$, with B possibly empty. Let $Q = ADdB$ be the Dyck path obtained from P by exchanging d with this factor D. We say that Q is obtained from P by a **greedy Tamari rotation** and write $P \lessdot_G Q$.

We define the **greedy Tamari poset** as the transitive closure of greedy Tamari rotations.

Definition 7.2.2. Let P be a Dyck path. Suppose that for some i, the i-th up step u_i of P is preceded by a down step d and that the excursion E_i of u_i is not followed by a down step. Let $k \geq 1$ the number of down steps preceding u_i in P. We then write $P = Ad^kE_iB$ with B perhaps empty (or otherwise starting with an up step). Then for any $1 \leq j \leq k$, we say that $Q = Ad^{k-j}E_id^jB$ is obtained from P by a **dexter rotation** and write $P \lessdot_D Q$.

We define the **dexter poset** Dext_n as the transitive closure of dexter rotations.

Theorem 7.2.3 ([Cha20, Theorem 7.6])**.** *The dexter poset* Dext_n *is a meet semilattice.*

Conjecture 7.2.4 ([BMC23])**.** The greedy Tamari poset GTam_n and the dexter poset Dext_n are anti-isomorphic.

Similarly to the alt ν-Tamari lattices' case, nontrivial linear intervals in the greedy Tamari lattice can be classified into two classes, that we call left and right intervals. We prove that for this definition, left intervals of length ℓ are in bijection with paths marked at a valley preceded by ℓ down steps, as in the Tamari lattice. However, the distribution of right intervals is usually different from the one in the Tamari lattice, which has usually more right intervals.

Dermenjian extended in [Der23] this idea to define a *"greedy ν-Tamari" poset* (that he calls ν-Greedy order), where the covering relations consist of exchanging an east step with all the consecutive ν-excursions following it at once.

In the same spirit, one can define a *"greedy ν-Dyck" poset* GDyck_ν by considering the set of Dyck paths of size n and defining covering relations consisting of exchanging the east step of a valley with the entire rise following it, or more generally, a *"greedy alt ν-Tamari" poset* by exchanging an east step with all the consecutive δ-excursions following it at once.

Definition 7.2.5. Let ν be a fixed north-east lattice path and δ an increment vector with respect to ν. Let μ be a ν-path. Suppose that the i-th north step N_i of μ is preceded by an east step E, and let η be the union of all consecutive δ-excursions starting with N_i. Let μ' be the path obtained from μ by exchanging η with the east step E preceding it. We say that μ' is obtained from μ by a **greedy δ-rotation** and write $\mu \lessdot_{G\delta} \mu'$.

We define the **greedy alt ν-Tamari poset** $\mathrm{GTam}_\nu(\delta)$ as the transitive closure of greedy δ-rotations.

Remark 7.2.6.

- Remark that in the definition of greedy alt ν-rotations, η can not be followed by a north step, as it would be followed by a δ-excursion.
- The case with $\delta = \delta^{max}$, which we can call the greedy ν-Tamari poset GTam_ν, corresponds to the ν-Greedy order of Dermenjian.
- The case with $\delta = \delta^{min}$ corresponds with what we called earlier the greedy ν-Dyck poset GDyck_ν, which seems to have good properties.
- Since greedy alt ν-rotations can be achieved by a sequence of alt ν-Tamari rotations, $\mathrm{Tam}_\nu(\delta)$ is an extension of $\mathrm{GTam}_\nu(\delta)$. However, we do not have any extension relation between two greedy alt ν-Tamari posets $\mathrm{GTam}_\nu(\delta)$ and $\mathrm{GTam}_\nu(\delta')$.

Proposition 7.2.7. *The poset* $\mathrm{GTam}_\nu(\delta)$ *is well-defined and its covering relations are exactly all greedy* δ*-rotations.*

Proof. Since $\mathrm{GTam}_\nu(\delta)$ is extended by $\mathrm{Tam}_\nu(\delta)$, it is a well-defined poset as there can be no cyclic sequence of greedy δ-rotations.

It is sufficient to prove that if $\mu \lessdot_{G\delta} \mu'$ is a greedy δ-rotation, then $\mu \leq_{G\delta} \mu'$ is the only saturated chain in the interval $[\mu, \mu']$.

Remark that by definition, $[\mu, \mu']$ must be a right interval in $\mathrm{Tam}_\nu(\delta)$, and thus by Proposition 6.4.2, it is a linear interval and contains a unique saturated chain. Conclude by remarking that no other greedy δ-rotation of μ would produce an element in the interval $[\mu, \mu']$ in $\mathrm{Tam}_\nu(\delta)$. $\square$

These posets are new and we conjecture them all to be join-semilattices. The greedy ν-Dyck lattice GDyck_ν appears in recent work of P. Nadeau and V. Tewari [NT23, Section 5] as subposets of a lattice they define. Thanks to their result, one can show that the join of two ν-paths in their lattice is a ν-path itself and thus, all greedy ν-Dyck lattice are join-semilattices, and even lattices if ν has no two consecutive north steps (except maybe at the beginning of the path).

In all of these posets, one can again define "left" and "right" intervals, being again exactly all the nontrivial linear intervals. Left intervals consist of moving ℓ times the same subpath η to the left, whereas a right interval consists of moving an east step to the right after ℓ *almost consecutive* sequences of δ-excursions. When ν is the staircase path $(NE)^n$, in the case that we will call "Greedy Dyck lattice" and denote GDyck_n, we can moreover prove that the distribution of right intervals in GDyck_n is also the same as the one in the alt-Tamari lattices. We will prove this with a bijective argument.

Definition 7.2.8. An interval $[\mu, \mu']$ in $\mathrm{GTam}_\nu(\delta)$ is a **left interval** if for some i, the i-th north step N_i of μ is preceded by (at least) $\ell \geq 1$ east steps, and if η is the union of consecutive δ-excursions starting with N_i, then μ' is obtained from μ by exchanging η with the ℓ east steps preceding it.

An interval $[\mu, \mu']$ in $\mathrm{GTam}_\nu(\delta)$ is a **right interval** if for some $\ell \geq 1$, there exists a subpath $E\eta_1 E\eta_2 \ldots \eta_\ell$ in μ, where each η_i is the union of consecutive δ-excursions and η_ℓ is not followed by a north step, and μ' is obtained from μ by sending the first east step of this subpath immediately after η_ℓ.

Proposition 7.2.9. *Left and right intervals are linear and the* ℓ *appearing in their definition is their length. Furthermore, all nontrivial linear intervals in* $\mathrm{GTam}_\nu(\delta)$ *are either left or right intervals, and they are both left and right if and only if they are greedy* δ*-rotations.*

Proof. The proofs are really similar to the ones of Propositions 5.4.14, 5.4.15 and 5.4.17. One can define a "greedy length vector" which does no longer record the size of the δ-excursion of each north step, but records instead the total size of all consecutive δ-excursions starting from each north step.

It is then clear that for both left and right intervals, one can produce a saturated chain in $\mathrm{GTam}_\nu(\delta)$ whose length is the ℓ appearing in the definition. It remains to see that it is unique. For this, applying the same arguments on greedy length vectors is enough to conclude that no other saturated chain exists in a left or right interval.

For $\ell = 1$, the definition gives greedy δ-rotations, which are covering relations. Then, the proof for classifying linear intervals follows by induction on the length $\ell \geq 2$.

For $\ell = 2$, a case study shows that two consecutive greedy δ-rotations can produce a right or left interval, a square (in most cases) or a pentagon if the first greedy δ-rotation makes the greedy length vector change, and the second greedy δ-rotation happens at a north step whose entry in this vector was changed.

Finally, for $\ell \geq 3$, taking the lower cover of the top element produces a linear interval of length $\ell - 1$, which is either left or right, and forces the full interval to be of the same kind. □

Remark 7.2.10. Similarly as Remark 6.2.3, left intervals of length ℓ in $\mathrm{GTam}_\nu(\delta)$ are in bijection with ν-paths marked at a valley preceded by (at least) ℓ east steps.

Theorem 7.2.11. *The distribution of left intervals with respect to their length is the same in* $\mathrm{GTam}_\nu(\delta)$ *and in* Tam_ν. *The distribution of all linear intervals with respect to their length is the same in* GDyck_n *and in* Tam_n.

Proof. The statement for left intervals follows immediately from Remark 7.2.10.

For the second statement, we will give a decomposition of right intervals of length ℓ in GDyck_n, in the same flavor as in the alt-Tamari lattices cases in Proposition 5.4.21. More precisely, we will describe a bijection with sequences $(P_0, P_1, \dots, P_\ell)$ of Dyck paths whose total sizes add up to $n - \ell$, with P_0 marked at an east step.

Recall that in the greedy Dyck case, covering relations consist of exchanging the east step of a valley with the entire vertical run following it. Let $[P, Q]$ be a right interval of length ℓ in GDyck_n. One can write $P = AEN^{i_1}E \dots N^{i_\ell}EB$, and $Q = AN^{i_1}E \dots N^{i_\ell}EEB$, with $i_j > 0, 1 \leq j \leq \ell$. Then, right intervals of length ℓ are in bijection with paths $P_\bullet = P$ marked at an east step (the one following A) followed by a vertical run and whose $\ell - 1$ next east steps are followed by a vertical run as well.

Then one removes from $P_\bullet$ the excursion $E_\ell = NP_\ell E$ starting at the last of these vertical runs and gets a pair $(P'_\bullet, P_\ell)$, with $P'_\bullet$ corresponding to a right interval of length $\ell - 1$. Proceeding inductively, one finally gets a sequence $(P_0, P_1, \dots, P_\ell)$ as prescribed.

In particular, using Proposition 5.4.21, this gives a bijection with right intervals in any alt-Tamari lattice of the same size, and thus with right intervals in Tam_n. □

In most cases, the greedy alt ν-Tamari poset does not have a unique minimal element. The only cases where it does have a unique minimal element are GDyck_ν when the path ν does not have two consecutive north steps (except perhaps at the beginning). It seems that, except maybe when $\mathrm{GTam}_\nu(\delta)$ has a minimal element—and might be a lattice—then it also possesses fewer right intervals than the corresponding ν-Tamari lattice, similarly as in Remark 6.5.2 when the path $\check{\nu}$ does not satisfy $\check{\nu}_i \leq \nu_i$ for all $i > 0$.

7.3 Conjectures about other posets

7.3.1 In further generalized alt ν-Tamari lattices

In Section 6.3.2, we constructed a family of posets alt ν-Tamari $\mathrm{Tam}^{tr}_\nu(\delta)$ as the rotation poset on trees in a shape $F_{\delta,\nu}$, which was obtained from the Ferrers diagram F_ν living above ν by moving some columns to the left. More precisely, any permutation of the columns of F_ν such that the sizes of the columns is unimodal (increasing then decreasing) in the resulting shape corresponds to such a shape $F_{\delta,\nu}$, and the rotation poset of trees living in this shape produces an alt ν-Tamari lattice, whose distribution of linear intervals was proven to be the same as in the ν-Tamari lattice.

As discussed with Matias von Bell and Cesar Ceballos, computer experimentation led them to conjecture that this equidistribution result still holds if one permutes both the rows and columns of the shape F_ν in such a way that the resulting shape has unimodal sequences of rows and columns. Such a shape can be found in Figure 7.1, to be compared with Figure 4.8. More precisely, they prove[1] that the rotation poset of the trees living in such a shape is still a lattice, with an algorithmic computation of joins and meets. Its linear intervals would still be understood as trivial, left or right intervals, and the distribution of left and right intervals with respect to their length would be the same as in the ν-Tamari lattice.

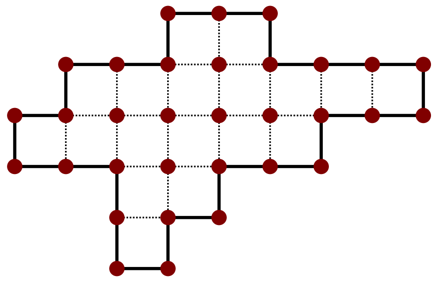


Figure 7.1: The diagram corresponding to a shape whose column and row sizes are the unimodal sequences $(1, 2, 4, 4, 3, 2, 1, 1)$ and $(2, 7, 6, 2, 1)$.

It is now known that for a fixed path ν, the Hasse diagrams of all resulting posets are the dual graphs of different triangulations of the same flow polytope, given by different framings. The possibility of obtaining such posets on ν-Catalan objects via flow polytopes was hinted in [vB22, Section 6.1] and in upcoming work, M. von Bell and C. Ceballos show that these posets are in fact lattices, that they call framing lattices. Moreover, computational evidence suggests that the distribution of linear intervals would not depend on the framing in all the lattices obtained for a fixed graph. This family of framing lattices remarkably contains a vast amount of known lattices and in particular some already mentioned in this manuscript, namely the alt ν-Tamari lattices, but also the type A Cambrian lattices, permutree lattices, or the s-weak order [Pil20, DMP+23]. We can expect that it would also contain the type A m-Cambrian lattices.

7.3.2 In Pallo's comb posets

Pallo's comb posets are less well-known posets on the set of Dyck paths which have been introduced by Pallo [Pal03] and studied later [CSS14] by Csar, Sengupta and Suksompong, showing some nice properties on them. In particular, they are graded meet-semilattices, all their intervals are distributive lattices (as are the ν-Tamari lattices and the alt ν-Tamari lattices), and they are extended by the dexter semilattices [Cha20].

They are defined as the transitive closure of some covering relations of the Tamari lattices, namely Tamari rotations of paths occurring at a valley which is also a contact. Since the excursion ends at height 0, it can not be followed by a down step, so any such relation is also a cover in the dexter poset. They were initially defined on binary trees, selecting Tamari rotations occurring at nodes on the rightmost branch of the bottom tree.

Definition 7.3.1. Pallo's comb poset Comb_n is the transitive closure of the subset of Tamari rotations $P \lessdot Q$ where $P = AdEB$, $Q = AEdB$ and the path Ad is itself a Dyck path (and so is B), namely $\mathrm{elev}(Ad) = 0$.

[1] Private communication.

Interestingly, the poset Comb_n has a nice conjectured distribution of linear intervals, as observed by Chapoton ([OEIS, A344191]).

Conjecture 7.3.2. Pallo's comb poset Comb_n possesses:

- $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ linear intervals of length 0.
- $\dfrac{2^{\ell-1}\,(\ell+3)}{n+2}\dbinom{2n-\ell}{n+1}$ linear intervals of length ℓ for $1 \leq \ell \leq n-1$.
- no interval of length $\ell \geq n$.

This adds up to $C_n\,\dfrac{n^2+2}{n+2}$.

Edelman has also considered in [Ede89] a poset which was built similarly as the transitive closure of a subset of covering relations of the weak order in the symmetric group. Csar, Sengupta and Suksompong showed that the map sending a permutation to its sorting tree, which is known to define a lattice congruence on the weak order and presents the Tamari lattice as a quotient lattice of the weak order, is also an order preserving surjection from Edelman's poset to Pallo's comb poset.

Note however that Edelman's poset does not seem to have a nice distribution of linear intervals as for instance on size 4, it has 13 intervals of length 3, which is prime, and it also has a linear interval of length 5.

As a quick remark, J.-C. Aval and F. Chapoton introduced in [AC18] a generalization of Pallo's comb poset. Using triangulations instead of Dyck paths, one can notice that the bottom element of the Tamari lattice has all its diagonals issued from the same vertex. Such diagonals exactly correspond to contacts of Dyck paths—this will be further explored in Section 9.3, where we will introduce the notion of *initial diagonals*. Then, a poset isomorphic to Pallo's comb poset can be defined by allowing only flips of such initial diagonals. J.-C. Aval and F. Chapoton generalized this idea by replacing triangulations with $(m+2)$-angulations, for some $m \geq 1$. The authors study in particular the intervals in these new posets, which have nice properties (see [AC18, Proposition 14]). The distribution of linear intervals in these posets also seems to admit nice formulas.

7.3.3 In the (m-)Cambrian lattices and the posets of tilting modules

The type A Cambrian lattices and posets of tilting modules are two families of posets that generalize the Tamari lattice, as well as the alt-Tamari lattices. We conjecture that these two families also have the same linear interval distribution as the Tamari lattice, as explained in Chapter 4. The three families seem to share other properties, such as the fact that they are all lattices, and their categories of modules seem to be derived equivalent, which is only conjectural for the alt-Tamari lattices, and has been proven by Ladkani for the two other families [Lad07a, Lad07b].

Recall that, once a ground field k is fixed, for each finite poset P, one can consider its category of modules $\mathsf{mod}P$, that consists of finite dimensional modules over its incidence algebra. One can then compare the posets, using this framework of categories. In particular, the categories of modules of two posets P and Q might be **derived-equivalent**, that is to say that there is a triangle-equivalence between their derived categories, in which case we say that P and Q are derived-equivalent.

Given a Coxeter group W, each standard Coxeter element c (or choice of orientation of the Coxeter graph) gives rise to a Cambrian lattice $\mathrm{Camb}(W, c)$ and a poset of tilting modules $\mathrm{Tilt}(W, c)$. For both families, the choice of two different Coxeter elements c and c' in the same group W usually gives in general two non-isomorphic lattices, whose categories of modules are not equivalent. However, using the so-called flip-flop technique, Ladkani proved that they were derived-equivalent [Lad07a, Lad07b]. The idea behind the flip-flop technique is that modifying slightly the orientation of the oriented Coxeter graph by transforming a sink into a source changed the posets in a quite understandable way. Such a transformation was then shown to preserve the derived category of modules, and any two posets of the family are connected through a chain of such modifications, viewed as small steps.

Chapoton surprisingly observed that the two very well-known posets on Dyck paths, namely the Dyck and the Tamari lattices seemed to be derived-equivalent as well. This result of derived equivalence is expected to hold in the entire family of alt-Tamari lattices, but it is only conjectural for now. In fact, these new posets could be used as intermediate steps to prove the result for the two extreme cases of the Dyck and the Tamari lattices, using for instance the boolean structure of extensions of posets within the family mentioned in Proposition 6.3.8.

Regarding linear intervals, Theorem 5.4.25 stated that the distribution of linear intervals was the same for all the alt-Tamari lattices of a given size. The same result seems to be true as well for the type A Cambrian lattices and posets of tilting modules, as conjectured by Chapoton ([OEIS, A344136]). It is not known whether the derived equivalence result and the linear interval distribution result are related, but if not, it is still possible that these results would arise from a common reason in these three families. In fact, the equidistribution result seems to hold in all Coxeter types for both families, namely that it does not depend on the choice of the Coxeter element, and even to generalize to the m-Cambrian lattices. Moreover, as for the Tamari lattice, some nice product formulas seem to appear for the infinite Coxeter families.

Conjecture 7.3.3. For any finite Coxeter group W, the distribution of linear intervals in the Cambrian lattices $\mathrm{Camb}(W, c)$ does not depend on the choice of the Coxeter element c.

Remark 7.3.4. This conjecture has been checked for:

1. type A until $A_7 = \mathfrak{S}_8$ where the distribution of linear intervals is given by the sequence $(\ell_0, \ldots, \ell_7) = (1430, 5005, 4004, 1430, 440, 110, 20, 2)$ for all 64 standard Coxeter elements, where ℓ_k is the number of linear intervals of length k. This is indeed the same distribution as all the alt-Tamari lattices of size 8,

2. type B until B_7 whose distribution is $(3432, 12012, 10296, 5544, 1980, 720, 216, 48, 6)$ for all 64 standard Coxeter elements,

3. type D until D_7 whose distribution is $(2508, 8778, 7560, 3276, 896, 210, 28)$ for all 64 standard Coxeter elements,

4. types E_6, E_7, F_4, H_3 and H_4, whose distribution sequences are respectively $(833, 2499, 2142, 952, 224, 14)$,$(4160, 14560, 13104, 6240, 1840, 300, 20)$,$(105, 210, 168, 112, 37)$, $(32, 48, 36, 26, 18, 2)$ and $(280, 560, 480, 352, 251, 68, 3)$ for all standard Coxeter elements.

Conjecture 7.3.5. For any finite Coxeter group W, the distribution of linear intervals in the posets of tilting modules $\mathrm{Tilt}(W, c)$ does not depend on choice of the Coxeter element c, or telling it otherwise, for all orientations of the Coxeter graph, viewed as a quiver.

Remark 7.3.6. This conjecture has been checked for:

1. type A until $A_8 = \mathfrak{S}_9$ where the distribution of linear intervals is given by the sequence $(\ell_0, \ldots, \ell_7) = (1430, 5005, 4004, 1430, 440, 110, 20, 2)$ for all 128 standard Coxeter elements. It is the same distribution as all the alt-Tamari lattices of size 8 (and not 9!),

2. type B until B_7,

3. type D until D_7,

4. exceptional types (partially for E_8).

For both cases, the conjecture seems to hold well in generality. One may explore whether the flip-flop technique would be a good way to prove the result, as it was for the derived equivalence. However, it might also be specific to these contexts, it is not clear that two posets obtained from each other by a flip-flop transformation would have the same distribution of linear intervals in general.

Note also that the numbers appearing in the distributions of the infinite families seem to factor very well, with a lot of small prime numbers. Type A is conjectured to be the one computed for the Tamari lattice, but types B and D led Chapoton to conjecture a formula for both types in each of the two families.

Conjecture 7.3.7. Let $F(W;t)$ and $G(W;t)$ be the generating functions of linear intervals in the Cambrian lattices and the posets of tilting modules of type W respectively (conjecturally independent of orientation of the Dynkin diagram after Conjectures 7.3.3 and 7.3.5), where t keeps track of the length of the intervals. Then we have:

$$F(B_n;t) = \binom{2n}{n} + \frac{n}{2}\binom{2n}{n}t + \binom{2n-2}{n}t^3 + \sum_{k=2}^{n+1}(n-1)\binom{2n-k+1}{n}t^k \quad \text{[OEIS, A344228]} \tag{7.1}$$

$$F(D_n;t) = \frac{3n-2}{n}\binom{2n-2}{n-1} + \frac{3n-2}{2}\binom{2n-2}{n-1}t + 2\binom{2n-4}{n-1}t^3 + \sum_{k=2}^{n-1}\frac{2(3n-k-1)(2n-k-2)}{n-1}\binom{2n-k-3}{n-2}t^k \quad \text{[OEIS, A344321]} \tag{7.2}$$

$$G(B_n;t) = \binom{2n-1}{n} + (n-1)\binom{2n-2}{n-1}t + \binom{2n-3}{n-3}t^3 + \sum_{k=2}^{n}\frac{2n^2-4n+k}{n}\binom{2n-k-1}{n-1}t^k \quad \text{[OEIS, A344717]} \tag{7.3}$$

$$G(D_n;t) = \frac{3n-4}{n}\binom{2n-3}{n-1} + \frac{(n-2)(3n-4)}{n-1}\binom{2n-4}{n-2}t + 2\binom{2n-5}{n-4}t^3 + \sum_{k=2}^{n-1}\frac{(3n-k-3)(2n-k-3)}{n-1}\binom{2n-k-3}{n-2}t^k. \quad \text{[OEIS, A344728]} \tag{7.4}$$

Remark 7.3.8. For type D, the formula seems to be slightly nicer by changing n into $n+1$.

Changing furthermore n into $n-1$ in the formula for type B Cambrian makes all four formulas very similar, with all terms of the last sum being of the form of a simple fraction times $\binom{2n-k-1}{n-1}t^k$.

This is to compare with the term $2\binom{2n-k}{n+1}t^k$ in type A.

Conjecture 7.3.9. For any finite Coxeter group W and any integer $m > 0$, the distribution of linear intervals in the m-Cambrian lattices $\mathrm{Camb}^{(m)}(W,c)$ does not depend on the choice of the Coxeter element c.

Remark 7.3.10. This conjecture has been checked for:

1. type A for $n+m \leq 8$. The distribution for $\mathrm{Camb}^{(3)}(A_5,c)$ is for instance always $(7084, 26565, 30030, 26355, 14520, 4634, 824, 66)$. The fifth coefficient factorizes as $1090 = 2 \cdot 5 \cdot 109$, which hints that there is no nice product formula, but maybe a summation formula.

2. type B for $n+m \leq 7$ (and some more cases). The distribution for $\mathrm{Camb}^{(2)}(B_5, c)$ is for instance always $(3003, 10010, 11297, 8316, 4070, 1391, 383, 37)$.

3. type D for $n+m \leq 7$ (and some more cases).

4. some exceptional types for $m=2$ and $m=3$.

Note that even in type A, B, and D, the numbers factor not so well. There might however still exist a summation formula.

7.3.4 In permutree lattices

Permutree lattices are a wide family of posets defined by V. Pilaud and V. Pons [PP18] as the transitive closure of (increasing) rotations of some decorated permutrees. Recall that permutrees defined in Definition 2.2.27 are directed (unrooted) planar trees whose nodes are labelled with integers from 1 to n, each one with one or two incoming and one or two outgoing arrows. Each node has a type according to its number of incoming and outgoing arrows, and this gives a decoration n-tuple denoted δ. Furthermore, the rotation of a tree as defined in Definition 2.2.29 consists of reversing an increasing edge $i \to j$ and changing very locally the tree, and this procedure does not affect the decoration δ. This then defines a poset on each set of permutrees with a fixed decoration δ, which turns out to be a lattice, and in fact a lattice quotient of the weak order on the symmetric group. This family of posets contains in particular the weak order, the Tamari lattice, the type A Cambrian lattices, and the boolean lattice, which makes it very interesting to study.

Regarding their linear intervals, we think that in all of these posets, (nontrivial) linear intervals would also be classified as left or right intervals, and this might be a way to understand the distribution of linear intervals. In particular, there could possibly be a way to prove the conjectured result that all type A Cambrian lattices have the same distribution of linear intervals, namely the one of the Tamari lattice. Furthermore, the existence of product formulas for the weak order and the Tamari lattice (conjecturally for all type A Cambrian lattices) gives hope for the existence of interpolating formulas for all lattices of the family, that could be computed from the fixed decoration δ.

Recall that thanks to Remark 2.2.28, the decoration of the first and the last nodes of a permutree do not matter in the definition of the objects and in fact of the poset.

Proposition 7.3.11. *If $\delta_i = (1,1)$, then the δ-permutree lattice is isomorphic to the product of the δ'-permutree lattice and the δ''-permutree lattices, where $\delta' = (\delta_1, \ldots, \delta_i)$ and $\delta'' = (\delta_i, \ldots, \delta_n)$.*

Similarly to the boolean structure of refinement in the alt-Tamari lattices whenever two increment vectors were comparable componentwise, there is a boolean structure of lattice quotients on the family of permutree lattices of size n. In particular, for every decoration δ of size n, the boolean lattice is a lattice quotient of δ-permutree lattice, which itself is a lattice quotient of the weak order on $\mathfrak{S}_n$.

Proposition 7.3.12. *Let δ and δ' be two decorations of size n. Then, $\mathcal{PT}(\delta')$ is a lattice quotient of $\mathcal{PT}(\delta)$ whenever $\delta \leq \delta'$ componentwise, i.e. each entry of δ_i is smaller that the corresponding entry in δ'_i for all $i \in [n]$. In other words, there is a surjective lattice morphism from the δ-permutree lattice to the δ'-permutree lattice.*

Proposition 7.3.13. *The number of δ-permutrees only depends on the positions of decorations $(0,0)$ and $(1,1)$ in δ.*

Telling it otherwise, if two decorations δ and δ' differ by "flipping" a $(1,0)$ to a $(0,1)$, then the numbers of δ-permutrees and of δ'-permutrees are the same. In fact, V. Pilaud and V. Pons called the number of δ-permutrees the "factorial-Catalan number" $C(\delta)$ and gave two recursive formulas to compute them. We do not give the formulas nor the details here, but from Propositions 7.3.11 and 7.3.13, we only have to compute the number of δ-permutrees with decorations $\delta \in \{(0,0),(0,1)\}^n$.

Proposition 7.3.14. *The number of covering relations in the δ-permutree lattice is equal to $\frac{n-1}{2} C(\delta)$.*

Proof. One only needs to notice that each δ-permutree possesses $n-1$ internal edges which can be rotated to obtain a covering relation. Every covering relation is then obtained twice. □

The distributions of linear intervals in the weak order and in the Tamari lattice have been previously discussed and gave rise to surprisingly nice product formulas, as shown in Theorems 5.3.1 and 7.1.3. It is worth noting as well that for both lattices, we could classify nontrivial linear intervals into "left" or "right" intervals, and in both cases, covering relations were exactly all intervals being both left and right. Moreover, the boolean lattice is 2-thick, meaning that it has no linear intervals of length 2 or more. This can be proven directly, or understood multiplicatively as the boolean lattice is isomorphic to a product of 2-chains.

Since the number of linear intervals of length 0 is precisely the cardinality of the poset, the distribution of linear intervals can be seen as a refinement of the cardinality. Both invariants (under poset isomorphism) are additive with respect to disjoint union of posets. In fact, as mentioned in Proposition 7.0.2, we can also compute the number of linear intervals in a product of posets in a multiplicative way, using a vanishing variable for nontrivial intervals.

Furthermore, Conjecture 7.3.3 predicts that the distribution of linear intervals in the type A Cambrian lattices would not depend on the Coxeter element, or telling it otherwise, that it is the same as in the Tamari lattice. Propositions 7.3.13 and 7.3.14 can be rephrased in terms of linear intervals, namely that the number of linear intervals of length 0 and 1 in the δ-permutree only depends on the positions of decorations $(0,0)$ and $(1,1)$ in δ. Moreover, the same seems to be true for the Coxeter polynomial, which also suggests that if two decorations differ only by changing some $(0,1)$ into $(1,0)$, the corresponding posets have derived-equivalent categories of modules. This leads to the following conjectures:

Conjecture 7.3.15. The distribution of linear intervals with respect to their length in the δ-permutree lattice only depends on the positions of decorations $(0,0)$ and $(1,1)$ in δ.

Conjecture 7.3.16. The derived category of the category of modules of the δ-permutree lattice only depends, up to triangle-equivalence, on the positions of decorations $(0,0)$ and $(1,1)$ in δ.

These conjectures are based on experimental evidence. The distribution of linear intervals and the Coxeter polynomials of all permutree lattices of size $n \leq 6$ has been computed thanks to data provided by D. Tamayo. Note that Conjectures 7.3.15 and 7.3.16 are respectively a generalization of Conjecture 7.3.3 and of the result of Ladkani [Lad07a] in the case of the type A Cambrian lattices.

Because of the multiplicative behavior of the distribution of linear intervals, one only needs to focus on δ-permutree lattices with decoration $\delta \in \{(0,0),(0,1),(1,0)\}^n$, and if Conjecture 7.3.15 holds, only to the case $\delta \in \{(0,0),(0,1)\}^n$. It seems quite promising that in all these lattices, linear intervals would still be classified as trivial, left and right intervals, as detailed hereafter, and a decomposition or a bijective proof could transform $(1,0)$ decorations into $(0,1)$ decorations. One could even hope for a product formula that would interpolate between the formulas enumerating the Tamari lattice's and the weak order's linear intervals, resembling possibly the one for $C(\delta)$.

Definition 7.3.17. We say that an interval $[T,T']$ in a permutree lattice is a **right interval** if T' is obtained from T by a sequence of rotations of edges $i \to j_1, \dots, i \to j_\ell$ with always the same source node i, for some $\ell \geq 1$. This implies in particular that for all $1 \leq k \leq \ell$, we have $i < j_k$ and that j_{k+1} is the leftmost parent of j_k in T.

We say that an interval $[T, T']$ in a permutree lattice is a left interval if T' is obtained from T by a sequence of rotations of edges $i_1 \to j, \ldots, i_\ell \to j$ with always the same target node j, for some $\ell \geq 1$. This implies in particular that for all $1 \leq k \leq \ell$, we have $i_k < j$ and that i_{k+1} is the rightmost child of i_k in T.

V. Pilaud and V. Pons defined two involutions on permutrees, that they call vertical and horizontal "symmetrees", which consist of respectively reversing all arrows and inverting the labeling, namely changing $p(v)$ into $n+1-p(v)$ for all nodes v. These involutions provide in fact anti-isomorphisms between two possibly different permutree lattices. Both exchange (and reverse) left and right intervals.

Remark 7.3.18.

- For the Tamari lattice Tam_n when $\delta = (0,1)^n$, the anti-involution corresponds to the horizontal symmetry of Remark 2.2.9. The definitions of left and right intervals as permutrees coincide with the definitions given in Definition 5.2.8, except that left and right are exchanged. Indeed, if $[T, T']$ is a right interval as permutrees, then the nodes $j_1, \ldots, j_\ell$ form a left comb, and they all undergo a rotation, from bottom to top.

- For the weak order $\mathrm{Weak}(\mathfrak{S}_n)$, when $\delta = (0,0)^n$, the two symmetrees correspond with left and right multiplication with $w_\circ$. Left and right intervals correspond exactly to the definitions given in Definition 7.1.1.

- For the boolean lattice $\mathcal{B}_n$ when $\delta = (1,1)^n$, symmetrees correspond to taking the complement of a subset (and additionally inverting the labeling for the horizontal one). It does not contain left or right intervals of length at least 2 since a node can not have a right parent and a right child at the same time in the boolean lattice, for instance.

Proposition 7.3.19. *nontrivial linear intervals in the permutree lattices must be left and right intervals.*

Proof. We first inspect the case of two consecutive rotations, then the case of three consecutive rotations, and we finally conclude. We will use the central symmetree in order to handle cases more efficiently since it changes an increasing arrow $i \to j$ into an increasing arrow $n+1-j \to n+1-i$, exchanging the roles of sources and targets of arrows.

Suppose that T is a permutree with an increasing edge $i \to j$ that we first rotate, producing a tree T'. Note that T and T' differ by exactly three edges. Precisely, the arrow $i \to j$ has been reversed, the rightmost incoming edge of i is now the leftmost incoming edge of j, and the leftmost outgoing edge of j is now the rightmost outgoing edge of i.

- Suppose that we rotate an edge $i' \to j'$ with $i', j' \notin \{i, j\}$, then the two rotations can be performed independently since the arrow already existed in T, and the resulting interval $[T, T'']$ is a square.

- Suppose that we rotate an edge $j \to k$. This implies that this edge already existed in T and k was not the leftmost parent of j. Then, the interval is either a square if j is of incoming degree 2 or a pentagon if j is of incoming degree 1.

- Suppose that we rotate an edge $i \to k$. This is then the definition of a right interval of length 2.

- Other cases are obtained by symmetree: rotating an edge $k \to i$ produces a nonlinear interval and rotating an edge $k \to j$ produces a left interval.

Now suppose that we have consecutively rotated arrows $i \to j$ and $i \to k$, producing the tree T''. This implies that $i < k$ and k was the leftmost parent of j in T. This also implies that j is now the leftmost child of k in T'', since it was the rightmost child of i in T'.

Because of the previous cases, the only candidate edges in T'' for producing a linear interval (of length 3) are an increasing arrow from i if any, which would produce a right interval, or the arrow $j \to k$, since j is the leftmost child of k. We must have $j < k$ for this edge to be increasing. However, the interval resulting of the successive rotations $i \to j$, $i \to k$ and $j \to k$ is never linear, it is either a pentagon if j is of incoming degree 1 or a hexagon if j is of incoming degree 2.

Using the central symmetree, this proves that the only candidates for linear intervals of length 3 are right and left intervals. By induction, the results naturally extends to all linear intervals. □

Remark that Proposition 7.3.19 also implies that there are no intervals of length n or more in the permutree lattices on trees with n nodes, since the definition of a left or a right interval of length n involves $n+1$ different nodes. It remains to prove that left and right intervals are indeed linear, and this would prove the following conjecture. It seems that very recent work of D. Tamayo [Tam23] provides a cubical representation of the permutree lattices, for which right intervals are indeed exactly lines, using a notion of inversion set. We expect that this should achieve the proof that nontrivial linear intervals are exactly left and right intervals.

Conjecture 7.3.20. Left and right intervals are exactly all nontrivial linear intervals in all δ-permutree lattices.

To conclude this section, let us emphasize that the permutree lattices and the alt-Tamari lattices are two families of posets that seem to be defined in a quite similar manner and to have similar behaviors. More precisely, both families contain the Tamari lattices and are defined as the transitive closure of some rotations, which are indeed exactly all covering relations in the resulting partial orders. All posets of the families are defined with a decoration of size n, namely an integer $\delta_i \in \{0,1\}$ for each up step of a Dyck path and a tuple $\delta(T)_i \in \{0,1\}^2$ for each node of a permutree. Moreover, the first and last entries are not really relevant, as remarked in Remarks 2.2.28 and 5.4.3. All members of the family are in fact lattices, and there are relations between two of these lattices whenever their decorations are comparable componentwise. For the alt-Tamari lattices, we proved in Proposition 5.4.11 that whenever $\delta \leq \delta'$, then $\mathrm{Tam}_n(\delta)$ is an extension of $\mathrm{Tam}_n(\delta')$, and for the permutree lattices, Proposition 7.3.12 states that whenever $\delta \leq \delta'$, then $\mathcal{PT}(\delta')$ is a lattice quotient of $\mathcal{PT}(\delta)$. Finally, it seems that linear intervals behave very well in both families, as we just mentioned. In particular, the fact that their distribution is independent of the increment vector in the alt-Tamari lattices is very similar to the conjectured statement that changing all decorations $(1,0)$ into $(0,1)$ in the permutree lattices would not affect the distribution either, and similarly for the conjectured derived equivalences.

From all of these surprising similarities, one could wonder whether there is a way to define a new huge family of "alt permutree" posets that would contain both the alt-Tamari and the permutree lattices and that would keep these similar properties. This family would possibly be defined on permutrees with a decoration vector $\delta \in \{\{0,1\}^3\}^n$ by adding another component to the existing decoration, which would *alter* the rotation relation of permutrees. If such a family can be defined, one also could dream of an extension in the ν-Tamari direction, or to other Coxeter types!

Part III

Intervals in the m-eralizations

Chapter 8

Enumerating the intervals in the m-Tamari lattices

The enumeration of intervals in the Tamari lattice is due to Chapoton [Cha06]. He used the description of the Tamari lattice on binary trees, as in Definition 2.2.7, to decompose the intervals into smaller "indecomposable" intervals, where the root of the bottom element had a leaf as a left child. He then transformed indecomposable intervals into smaller intervals, and used a catalytic parameter to count how many times each interval was reached. This enabled him to write an equation on the generating function of intervals in the Tamari lattices, that he solved.

Inspired by the work of Haiman on diagonal harmonics, as detailed in the introduction, F. Bergeron then gave in [Ber12] a new description of the Tamari lattice on Dyck paths (see Definition 2.2.11). He then generalized the poset as the m-Tamari lattice, as explained in Definition 4.1.1, and conjectured a formula for the number of their intervals. This question was resolved by M. Bousquet-Mélou, É. Fusy and L.F. Préville-Ratelle in [BMFPR11], using a decomposition of the intervals on (m-)Dyck paths, inspired by the decomposition of F. Chapoton.

In the next section, we present the details of their decomposition in the case of the Tamari lattice, as it is illustrative for the general case, where most arguments translate straightforwardly, and also because we made some small changes to their proof because of slightly different conventions. These small differences are motivated by what will follow, namely refinement of the equations obtained in [BMFPR11], and a conjecture of [STW18] in relation with Cambrian lattices.

8.1 In the Tamari lattice

Recall that the Tamari lattice can be described on Dyck paths as the transitive closure of rotations of Dyck paths, where a rotation consists in sending the down step of a valley after the excursion that follows it. A first important idea is to associate bijectively to each path P a vector of integers, in such a way that the order reads easily as the componentwise comparison. This idea was initially due to S. Huang and D. Tamari in [HT72] and is usually referred to as "bracket vectors". This is how they proved the lattice property of the Tamari lattices. The notion was even extended to the ν-Tamari lattices in [CPS20].

The most natural way to do this for Dyck paths is to associate to each path the sequence of the sizes of its excursions. This is what we called the length vector in Chapter 5 for the alt-Tamari lattices, and it is in fact almost equivalent to the original one of S. Huang and D. Tamari. Recall that we generally denote u_i the i-th up step of a path P.

Definition 8.1.1. Let P be a Dyck path of size n. Its length vector is $\ell(P) = (\ell_1(P), \dots, \ell_n(P))$, where $\ell_i(P)$ is the length of the excursion of P starting at the i-th up step u_i.

Proposition 8.1.2 ([BMFPR11, Proposition 5]). *Let P and Q be two Dyck paths of length n. Then $P \leq Q$ in Tam_n if and only if $\ell(P) \leq \ell(Q)$ componentwise.*

Thanks to Remark 5.4.7, we already have the direct implication, which is easy by examining how cover relations modify the length vector of a path. Recall in fact that thanks to Lemma 5.4.6, if P is a Dyck path with a valley du_k, then, noting u_i the matching up step of d in P, the excursions of u_i and u_k are consecutive. Hence, if P' is the rotation of P at the valley du_k, then the length vector of P' is obtained from the length vector of P by changing $\ell_k(P)$ into $\ell_i(P) + \ell_k(P) \geq \ell_k(P)$.

The other implication is slightly more involved but we describe the proof of [BMFPR11]. We prove a lemma first.

Lemma 8.1.3. *If $\ell(P) \leq \ell(Q)$ componentwise, then P is below Q in the Dyck lattice.*

Proof. We proceed by induction on the size n of the paths.

If $n = 1$ then the result is trivial. Otherwise, let P' and Q' be the paths obtained by suppressing the initial up steps of P and Q respectively, as well as their matching down steps, that is to say the first and last steps of their first excursion. It is clear that $\ell(P')$ and $\ell(Q')$ are obtained from $\ell(P)$ and $\ell(Q)$ by suppressing the first coordinate. Thus, we still have $\ell(P') \leq \ell(Q')$ componentwise, and by induction P' is below Q'.

Now adding back the initial up steps and their matching down steps, we see that P remains below Q in the Dyck lattice since the initial part of length $\ell_1(Q) - 2$ of Q' that is lifted is at least as long as the initial part of P' that is lifted. □

Remark in particular that this implies that the first non-initial contact of Q is also a contact of P, and in fact all contacts of Q are contacts of P. We now prove the proposition, following [BMFPR11].

Proof of Proposition 8.1.2. Now we want to prove that $\ell(P) \leq \ell(Q)$ implies indeed that $P \leq Q$ in Tam_n.

We know that P is below Q thanks to Lemma 8.1.3. We proceed by induction on $|\ell(Q) - \ell(P)| = \sum_{i=1}^{n} (\ell_i(Q) - \ell_i(P))$, where each term is nonnegative by assumption.

If $|\ell(Q) - \ell(P)| = 0$, then P and Q have the same length vector, and thus $P = Q$. Otherwise, let i be the smallest integer such that $\ell_i(P) < \ell_i(Q)$. We first prove that P and Q coincide at least up to their common i-th up step.

The paths P and Q are different and P is below Q, so they coincide until a certain point, after which P has a down step and Q has an up step. Let u_j be the matching up step of this down step of P, so that P and Q coincide up to at least u_j. Moreover, since u_j is a common step of P and Q, and its matching down step in P is not a step of Q, we have $\ell_j(P) < \ell_j(Q)$, hence $i \leq j$ and we proved that P and Q coincide at least up to their common i-th up step.

Now let d be the matching down step of u_i in P. Note that since u_i is also an up step of Q and that $\ell_i(Q) > \ell_i(P)$, then d is not the final step of P. However, this down step d can not be followed by a down step. Indeed, if there were a down step d' following d, then the matching up step $u_{i'}$ in P would be strictly before u_i, and thus an up step of Q, and we would have $\ell_{i'}(P) < \ell_{i'}(Q)$. This would contradict the minimality of i. Thus, this down step d of P is followed by an up step u_k.

Now, the last part of the proof consists in applying a rotation to P at the valley du_k to produce a path P' that still satisfies $\ell(P') \leq \ell(Q)$ but such that the quantity $|\ell(Q) - \ell(P')|$ decreases.

It is sufficient to remark that, since P is below Q, we have that $\ell_i(P) + \ell_k(P) \leq \ell_i(Q)$. Thus, if P' is obtained from P by a rotation at the valley du_k, by the discussion above, $\ell(P')$ is obtained from $\ell(P)$ by replacing $\ell_i(P)$ by $\ell_i(P) + \ell_k(P)$. Thus, we have $\ell(P') \leq \ell(Q)$ componentwise, and $|\ell(Q) - \ell(P')| = |\ell(Q) - \ell(P)| - \ell_k(P) < |\ell(Q) - \ell(P)|$.

By induction, we can conclude that $P \lessdot P' \leq Q$ in Tam_n. ☐

From these proofs, two important key points emerge.

1. If $[P, Q]$ is an interval in Tam_n, then any contact of Q is also a contact of P. We can then cut the interval into an indecomposable interval, up to the first contact of Q, and the rest, which may be empty. We call this the **splitting** of the interval $[P, Q]$.

2. Suppressing the initial and final steps of their respective first excursion produces two paths P' and Q' whose length vectors are obtained from $\ell(P)$ and $\ell(Q)$ by suppressing the first coordinate. Thus, as we had $\ell(P) \leq \ell(Q)$ componentwise, we still have $\ell(P') \leq \ell(Q')$ componentwise, and $[P', Q']$ is an interval in Tam_{n-1}. We call this the **reduction** of $[P, Q]$.

This is exactly the way we decompose intervals in the Tamari lattice into smaller intervals of the Tamari lattice, but in order to keep track of their number, we need to be able to count how many times each interval is reached under such transformations. This is where the **catalytic parameter** comes into play, that is to say an additional statistic on intervals that is used for this purpose. In our case, we use the number of contacts of the bottom path of the interval as catalytic parameter. We can also introduce the initial rise, and we observe interesting results.

Definition 8.1.4. Let $I = [P, Q]$ be an interval in Tam_n. A **contact** $c(I)$ of I is defined as a **non-initial contact** of its bottom path P.

The **initial rise** $r(I)$ of I is the size of the initial rise of Q, that is to say the number of up steps at the beginning of Q.

An interval $I = [P, Q]$ is said **indecomposable** if Q has no contact except its first and last vertices.

Proposition 8.1.5. *Let I be an interval in Tam_n. Then there are exactly $c(I)+1$ indecomposable intervals in Tam_{n+1} whose reduction is I, that is to say, that are mapped to I by the transformation that suppresses the initial and final steps of the first excursion of its bottom and top paths respectively. Moreover, the preimages of I under the reduction map have respectively 1, $2, \dots,$ and $c(I) + 1$ contacts, respectively.*

Proof. Let P be a Dyck path. Let uAd be its initial excursion, such that $P = uAdB$, with A and B possibly empty Dyck paths. Then if P' is obtained from P by reduction, then $P' = AB$, and there is a contact between A and B.

Conversely, each contact of P', including the initial one, gives rise to such a path P that is mapped to P'. Such a path P can be obtained by inserting a down step before any contact of P' and an up step at the beginning of the path. When doing so, this produces a path which has 1 initial contact and as many non-initial contacts as there was in P' at the right of the down step that is inserted. This "inverse" process will be called **expansion** of P' at its chosen contact.

Saying it otherwise, if P' has a non-initial contacts, the expansion of P' at its initial contact produces a path with $a + 1$ non-initial contacts, and each next contact of P' that is chosen produces a path with one less (non-initial) contact, until the last insertion of a down step that produces a path with only one initial and one final contacts.

Now if $I = [P', Q']$ is an interval with $c(I)$ contacts, then to produce an indecomposable interval $[P, Q]$ in Tam_{n+1} that is mapped to I under this transformation, there is no choice for Q but inserting an up step before the beginning of Q' and a down step after the end of Q'. For P the discussion above shows that there are exactly $c(I) + 1$ choices for expanding P'. We have to check that each such choice of P indeed gives an interval $[P, Q]$. It suffices to remark that $\ell_1(Q) = 2n + 2 \geq \ell_1(P)$ and that $\ell_i(P) = \ell_{i-1}(P') \leq \ell_{i-1}(Q') = \ell_i(Q)$ for all $i \geq 2$, and concluding via Proposition 8.1.2.

Remark that in this case, the initial rise of Q is equal to $r(I) + 1$. ☐

Remark 8.1.6. Now let us rephrase this in terms of generating functions. We count each interval $I \in \mathrm{Tam}_n$ with a weight $x^{c(I)}y^{r(I)}t^n$. Then there are $c(I)+1$ indecomposable intervals in Tam_{n+1} that are reduced to I, and their weights are respectively $x^1y^{r(I)+1}t^{n+1}, \dots, x^{c(I)+1}y^{r(I)+1}t^{n+1}$. This adds up to

$$\frac{x^{c(I)+2}-x}{x-1}y^{r(I)+1}t^{n+1} = xyt\,\frac{x\left(x^{c(I)}y^{r(I)}t^n\right)-\left(y^{r(I)}t^n\right)}{x-1}. \tag{8.1}$$

Definition 8.1.7. Let Δ be the (shifted) **divided difference operator** defined by

$$\Delta(S(x)) = x\frac{S(x)-S(1)}{x-1}, \tag{8.2}$$

and extended by linearity to any formal series.

Remark that Equation (8.1) rewrites as

$$\frac{x^{c(I)+2}-x}{x-1}y^{r(I)+1}t^{n+1} = yt\,\Delta\Big(x\left(x^{c(I)}y^{r(I)}t^n\right)\Big). \tag{8.3}$$

Now we are ready to prove the equation that is satisfied by the generating function of intervals in the Tamari lattices.

Proposition 8.1.8 ([BMFPR11, Proposition 7])**.** *Let $F(t;x,y)$ be the (decorated) generated series of Tamari intervals, where x keeps track of the number of contacts and y of the initial rise. Then F satisfies the equation:*

$$F(t;x,y) = yt\ (F(t;x,1)+1)\ \Delta(x(F(t;x,y)+1)). \tag{8.4}$$

Proof. We do not consider the Tamari lattice on Dyck paths of size 0. Then, to write this equation, we remark that any interval $[P,Q]$ can be split into an indecomposable interval $[P_1,Q_1]$ until the first contact of Q and the rest, which is another Tamari interval $[P_2,Q_2]$ (or nothing if $[P,Q]$ was already indecomposable). As P is below Q, this contact of Q is also a contact of P.

Remark that the sizes and number of non-initial contacts of $[P_1,Q_1]$ and $[P_2,Q_2]$ add up to the size and number of non-initial contacts of $[P,Q]$ respectively, and that the initial rise of $[P,Q]$ is exactly the initial rise of $[P_1,Q_1]$, or saying it otherwise, that the initial rise of $[P_2,Q_2]$ is ignored.

Writing $G(t;x,y)$ the generating function of indecomposable intervals, we therefore have the equation

$$F(t;x,y) = G(t;x,y)\,(1+F(t;x,1)). \tag{8.5}$$

Now, we study the reduction of intervals.

If $[P_1,Q_1]$ is the unique interval in Tam_1, then its weight is xyt and we have $xyt = yt\Delta(x)$. Otherwise, Proposition 8.1.5 shows that any other indecomposable interval $[P_1,Q_1]$ can be reconstructed uniquely as the expansion of some interval $[P',Q']$. Moreover by linearity of Δ and summing Equation (8.3) over all Tamari intervals, we obtain

$$G(t;x,y) = xyt + yt\,\Delta(xF(t;x,y)) = yt\,\Delta(x(F(t;x,y)+1)). \tag{8.6}$$

Combining Equations (8.5) and (8.6), we finally obtain the desired expression for $F(t;x,y)$. □

This equation was already solved in [Cha06] by F. Chapoton. It turns out that the unique solution of Equation (8.4) is in fact algebraic in x, y and t.

The equation was then generalized to the m-Tamari case and solved in [BMFPR11]. We draw the reader's attention to a few differences compared to this article:

- Our generating function does not contain any constant term since we do not consider the Tamari lattice on Dyck paths of size 0. The main reason for this is that in the Cambrian lattice, this would correspond to the group $\mathfrak{S}_0$ which would correspond to "type A_{-1}", but this does not make so much sense. Moreover, the equations behave better when dealing with more variables as we will add more statistics later.

- We do not count the initial contact of the bottom path, which results in a shifted version of the series, and we also use a shifted version of the divided difference operator. The main reason is to have the observed symmetry between the initial rise and the number of contacts, and also for a nicer writing of the equations.

Theorem 8.1.9 ([Cha06, Theorem 2.1])**.** *For $n \geq 1$, the number of intervals in the Tamari lattice* Tam_n *is equal to*

$$\frac{2(4n+1)!}{(n+1)!\,(3n+2)!} = \frac{2}{n(n+1)}\binom{4n+1}{n-1}. \tag{8.7}$$

8.2 In the m-Tamari lattice

The m-Tamari lattice $\mathrm{Tam}_n^{(m)}$ was introduced in Definition 4.1.1 as the restriction of the Tamari lattice Tam_{mn} to the set of m-Dyck paths, i.e. Dyck paths of size mn whose rises' lengths are all multiples of m. Noticing that Tamari rotations can only fuse two rises into one or move them around, this proves that the set of m-Dyck paths is an upper ideal (or filter) in Tam_{mn}. It is in fact the interval between the path $(u^m d^m)^n$ and the top element $u^{mn} d^{mn}$ in Tam_{mn}, and thus also a lattice itself. As an example, the 2-Tamari lattice of size 3 is represented in Section 8.2.

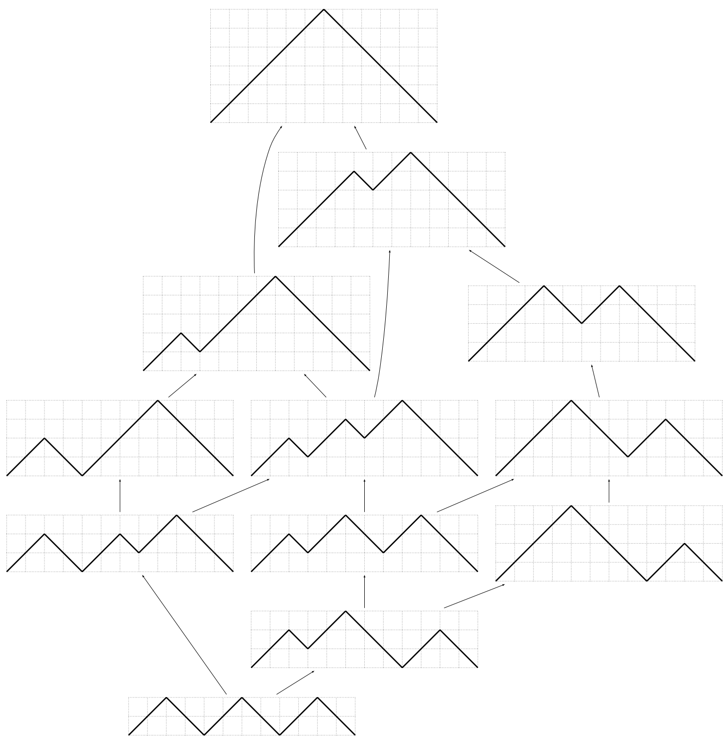


Figure 8.1: The 2-Tamari lattice of size 3 $\mathrm{Tam}_3^{(2)}$.

8.2.1 Adapting the decomposition

The generalization of the decomposition of intervals in the Tamari lattice to the case of the m-Tamari lattice needs small adaptations, but the main ideas are the same. It remains true that any contact of the top path Q of an interval $[P, Q]$ is also a contact of the bottom path P, however, the reduction of $[P, Q]$ produces an interval in Tam_{mn-1}. What we want instead is to produce an interval in a smaller m-Tamari lattice, and one way to achieve this is to apply m times this splitting-reduction process.

Definition 8.2.1. Let $m \geq 1$ and $I = [P, Q]$. Extending the notions of Definition 8.1.4, the number of **contacts** $c(I)$ of I is defined as the number of non-initial contacts of P, and the **initial rise** $r(I)$ of I is the size of the initial rise of Q **divided by** m. Similarly, the size of an interval is the size of its paths, namely their number of up steps divided by m.

Proposition 8.2.2 ([BMFPR11, Proposition 8])**.** *Let $m \geq 1$ and $F(x, y) \equiv F^{(m)}(t; x, y)$ be the (decorated) generated series of m-Tamari intervals, where x keeps track of the number of contacts and y of the initial rise. Then*

$$F(x, y) = yt\left((F(x, 1) + 1) \cdot \Delta\right)^{(m)} \Big(x\big(F(x, y) + 1\big)\Big), \tag{8.8}$$

where the power (m) means that the operator $G \mapsto (F(x, 1) + 1) \cdot \Delta(G)$ is applied m times.

Proof. If $[P, Q]$ is an m-Tamari interval, then we can apply m consecutive times the splitting and the reduction, each time splitting the interval into two at the first contact of the top element and reducing the first (or only) interval of the splitting.

At each of the m splittings, the second interval (if any) is an m-Tamari interval, since the bottom path is an m-Dyck path. This contributes a factor $F(x, 1) + 1$ to the generating function, since the initial rise of the second interval is ignored.

At each of the m reductions, one initial step is suppressed and its corresponding down step as well. In the end, the bottom path of the interval is a Dyck path, and since the only up steps that were suppressed were the first m initial ones, all its rises are indeed multiples of m, including its initial rise, whose length is one less than for the path P. As it is an interval in some Tamari lattice, it is indeed an m-Tamari interval.

Conversely, each m-Tamari interval $[P, Q]$ can be reconstructed uniquely as the result of m consecutive expansion-gluing processes. However, the up steps in the bottom path must be inserted consecutively, in order for P to be an m-Dyck path, and thus, the initial contact can not be chosen for expanding the bottom path, except at the very first expansion. The rest of the process is however the same as for the Tamari intervals, except it is repeated m times.

Starting with the $x(F(x, y)+1)$, where $F(x, y)$ is the generating function of m-Tamari intervals, and the factor x accounts for the initial contact which can be chosen for the first expansion, each expansion translates into applying the operator Δ to the generating function and each gluing translates into a multiplication by $F(x, y) + 1$.

The initial rise of the resulting interval is then equal to one plus the initial rise of the initial interval, and the size of the interval is equal to one plus the size of all intervals involved in the process. □

The proof is an abbreviated rephrasing of the proof of [BMFPR11, Proposition 8] with our slightly different conventions. They then solved the equations after finding a rational parametrization of the generating function, with the *kernel method* and heavy computation. We refer to the original article for the details of the resolution, as it is not the main focus of this thesis.

Theorem 8.2.3 ([BMFPR11, Corollary 11]). *For $m, n \geq 1$, the number of intervals in the m-Tamari lattice $\mathrm{Tam}_n^{(m)}$ is equal to*

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}. \tag{8.9}$$

In fact, they provide a formula for the number of intervals in the m-Tamari lattice with a given number of contacts. Very interestingly and surprisingly, a symmetry between x and y in some intermediate equation implied a symmetry in x and y in the solution, that is to say that the joint distribution of contacts and initial rise is symmetrical. This suggested the existence of a bijection between m-Tamari intervals that preserves the size but exchanges the number of contacts and the initial rise. The question was then answered by V. Pons in [Pon19], using objects called **Tamari Interval Posets**, which will be presented in Section 8.3.

8.2.2 More statistics on intervals

One can add more statistics on intervals, and write a *refined* version of Equation (8.8), by following how the statistics behave in the decomposition of intervals. In this section, we present a few more statistics that we decorate the intervals with, and we present the refined equations in Section 8.2.3. The main motivation for these additional statistics is the links with other objects or domains, most being only conjectural. In particular, as we will detail in Section 9.4.2, we can define statistics on m-Cambrian intervals in linear type A on the one hand, and on m-Tamari intervals on the other hands, for which we conjecture the distributions to coincide.

In [BMFPR11], the authors also consider in the final comments an additional statistics, namely the **height** of the intervals, which is the maximal length of a chain within the interval (see Definition 2.1.4). This statistic was considered in F. Bergeron and L.-F. Préville-Ratelle's paper [BPR12] in relation with a q-analogue of their conjectures.

Among general properties of intervals, one can also consider the Möbius invariant or even their homotopy type, as defined in Definition 2.1.20. However, these notions do not integrate well in this study comparing m-Tamari and linear type A m-Cambrian intervals, because the refined versions of the previous distributions no longer match up.

Two other interesting statistics on intervals are the numbers of ways to extend each interval, that is to say the number of elements covering the top element and the number of elements covered by the bottom element. In fact, in [Cha18], F. Chapoton introduced four statistics on intervals, namely these two as well as the number of ways of shrinking an interval, i.e. the number of elements of the interval covered by the top element and the number of elements of the interval covering the bottom element. These correspond to arrows that go into or out of the interval in the Hasse diagram of the poset.

It turns out that in the case of the Tamari lattice, as F. Chapoton shows in his article, the generating function with three of these variables shows a ternary symmetry, which no known bijection explained. The proof of this result was analogous to the initial-rise and contact symmetry, that is to say that the algebraic equation that he finds is totally symmetric in the three variables.

Definition 8.2.4. Let $I = [a, b]$ be an interval in a poset $(P, \leq)$.

- The **bottom-out degree** $\mathrm{bout}(I)$ of I is the number of elements a' such that $a' \lessdot a$.
- The **top-out degree** $\mathrm{tout}(I)$ of I is the number of elements b' such that $b \lessdot b'$.
- The **bottom-in degree** $\mathrm{bin}(I)$ of I is the number of elements a' such that $a \lessdot a' \leq b$.
- The **top-in degree** $\mathrm{tin}(I)$ of I is the number of elements b' such that $a \leq b' \lessdot b$.

Theorem 8.2.5 ([Cha18, Theorem 2.1])**.** *Let $F(t; u, v, \bar{u}, \bar{v})$ be the generating function of Tamari intervals where $u, v, \bar{u}, \bar{v}$ keep track respectively of the bottom-out, top-out, bottom-in and top-in degrees.*

The specialization $F(t; u, v, \bar{u}, 1)$ is totally symmetric in $u, v, \bar{u}$, and the same holds for $F(t; u, v, 1, \bar{v})$.

The second statement comes automatically with the first by the self-duality of the Tamari lattice, which exchanges the top-out and bottom-out degrees on the one hand and the top-in and bottom-in degrees on the other hand. This theorem implies in particular that the joint distributions of any two of these statistics (except for the pair $(\mathrm{bin}(I), \mathrm{tin}(I))$) is symmetric and is the same. In his article, F. Chapoton also noticed that the coefficients of these joint distributions however had big prime factors, which implied that there was no hope for a nice product formula as for the intervals. Nevertheless, A. Bostan, F. Chyzak and V. Pilaud recently proved in [BCP23, Theorem 1] that the joint distribution of the *sum* of these statistics does admit nice formulas, resembling and refining Equation (2.5) for the number of intervals in the Tamari lattice.

Theorem 8.2.6 ([BCP23, Theorem 1])**.** *For any $n \geq 1$, $k \geq 0$, the number of intervals in the Tamari lattice Tam_n with out-degree k (i.e. the sum of the bottom-out and top-out degrees) is equal to*

$$\frac{2}{n(n+1)}\binom{n+1}{k+2}\binom{3n}{k}. \tag{8.10}$$

We can play the same game with the intervals in the m-Tamari lattices, for a fixed integer $m \geq 1$. However, as we lose the self-duality, we do not have the automatic symmetry under the simultaneous exchange of bottom-out and top-out degrees and of bottom-in and top-in degrees. We conjecture that some ternary symmetry survives for m-Tamari intervals.

Conjecture 8.2.7. Let $F^{(m)}(t; u, v, \bar{u}, \bar{v})$ be the generating function of m-Tamari intervals where $u, v, \bar{u}, \bar{v}$ keep track respectively of the bottom-out, top-out, bottom-in and top-in degrees.

The specialization $F^{(m)}(t; u, v, \bar{u}, 1)$ is totally symmetric in $u, v, \bar{u}$.

The specialization $F(t; u, v, 1, \bar{v})$ is only symmetric in u and v.

The last statistics that we introduce here are much more detailed as they consist of integer partitions, instead of integers. They will be called rise partition and contact partition, as they contain the information of the initial rises and contacts, respectively, as a marked part. They also contain the information of the out degrees defined in Definition 8.2.4 as the number of unmarked parts. These partitions were already considered in [PR12, Conjecture 17]. Here, we view the m-Dyck paths as ballot paths, so that we do not have to divide the rises' size by m, and we are considering excursions of the paths within the scope of ν-excursions, for $\nu = (NE^m)^n$.

Definition 8.2.8. Let $m, n \geq 1$ and $I = [P, Q]$ be an interval in $\mathrm{Tam}_n^{(m)}$.

- The **rise partition** $\lambda^{\mathsf{top}}(I)$ is defined as the partition $\lambda \vdash n$ whose parts are the sizes of the rises of the top element Q, with a marked part (underlined) corresponding to the initial rise. The marked part can be erased to produce the partition $\widetilde{\lambda}^{\mathsf{top}}(I) \vdash n-k$, where $k \geq 1$ is the initial rise of I.

- The **contact partition** $\lambda^{\mathsf{bot}}(I)$ is defined as the partition $\lambda \vdash n$ whose parts are the sizes of sequences of consecutive excursions of the bottom element P, with a marked part (underlined) corresponding to the non-initial contacts (as each corresponds to the last step of an excursion at height 0, which are all consecutive). The marked part can be erased to produce the partition $\widetilde{\lambda}^{\mathsf{bot}}(I) \vdash n-k$, where $k \geq 1$ is the number of non-initial contacts of I.

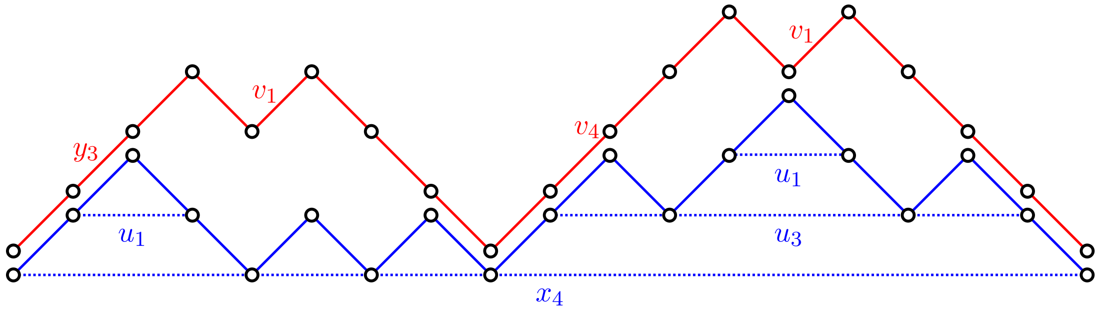


Figure 8.2: A decorated interval I in the Tamari lattice of size 9. The contact partition is $\lambda^{\mathsf{bot}}(I) = (\underline{4}, 3, 1, 1)$ and the rise partition is $\lambda^{\mathsf{top}}(I) = (4, \underline{3}, 1, 1)$.

An example of decorated interval in the Tamari lattice can be found in Figure 8.2. We use variables x_i and y_j to keep track of the marked parts and variables u_i and v_j for the unmarked parts, as we will see in Section 8.2.3.

Proposition 8.2.9. *Let $I = [P, Q]$ be an interval in $\mathrm{Tam}_n^{(m)}$. The number of parts of $\widetilde{\lambda}^{\mathsf{top}}(I)$ and $\widetilde{\lambda}^{\mathsf{bot}}(I)$ correspond respectively to the top-out and bottom-out degrees of I.*

Proof. Each non-initial rise of Q is preceded by a down step and thus corresponds to a valley of Q, hence to an upper cover, and vice-versa.

Similarly, each sequence of consecutive excursions of P ends with an excursion followed by a down step and hence a lower cover of P, and vice-versa. □

Remark 8.2.10. Let $I = [P, Q]$ be an interval in $\mathrm{Tam}_n^{(m)}$. The top-out degree of I corresponds to the number of valleys of Q, and the bottom-out degree of I corresponds to the number of excursions of P followed by a down step.

Remark 8.2.11. The partitions for m-Tamari intervals can be computed in the representation on m-Dyck paths seen as Dyck paths. However, this needs a few adjustments:

- The rise partition can directly be computed, but each part is a multiple of m. One shall then divide them by m to get the actual rise partition associated to the top element.
- The contact partition can also be computed on the Dyck paths of size mn, but one shall only consider "relevant" up steps, that is to say one up step out of m up steps, starting with the first one. Indeed, by definition, up steps come by sequences of m consecutive steps.

To end this section, we discuss how to compute effectively the height of an interval $[P, Q]$ in the m-Tamari lattice. Indeed, every other statistic (except the Möbius invariant, that we are not considering) is computed either from P or from Q. However, it is not the case of the height, which depends on both bounds of the interval. We present two very similar greedy algorithms to compute a longest chain between two elements P and Q, if any, and then we introduce a few tools to prove that they indeed are of the correct length. The first algorithm is useful to keep track of the length of the longest chain when gluing two intervals and when computing the height algorithmically, and the second one is useful to follow the length of the longest chain in the expansion of an interval.

Proposition 8.2.12. *Let $P, Q \in \mathrm{Tam}_n$. One starts with $P_0 = P$. For each $i \geq 0$, if $P_i \neq Q$, consider the first step where P_i and Q do not coincide.*

- *If it is an up step in P_i and a down step in Q, then we stop and conclude that $P \not\leq Q$.*
- *Otherwise, let u_j be the leftmost up step of P_i that is not an up step of Q and let P_{i+1} be the result of the rotation of P_i at the up step u_j.*

Then this procedure stops and produces a longest chain from P to Q if $P \leq Q$.

Proposition 8.2.13. *Let $P, Q \in \mathrm{Tam}_n$. One starts with $P_0 = P$. For each $i \geq 0$, if $P_i \neq Q$, consider the first step where P_i and Q do not coincide.*

- *If it is an up step in P_i and a down step in Q, then we stop and conclude that $P \not\leq Q$.*
- *Otherwise, among all up steps of P that are not up steps of Q and of minimal height between these, let $u_{j'}$ be the leftmost one. Then, let P_{i+1} be the result of the rotation of P_i at the up step $u_{j'}$.*

This greedy procedure also stops and produces a longest chain from P to Q if $P \leq Q$.

In order to prove this, we will reuse the horizontal vector defined in Definition 5.4.5, and we introduce an order on the integers $[n]$ corresponding to containment of excursions. These ideas coincide with some in Section 8.3, where we introduce Tamari interval-posets, as defined by G. Châtel and V. Pons in [CP15].

Definition 8.2.14. Let P and Q be two Dyck paths of size n.

- The **containment order** $\leq_P$ with respect to P is the poset on $[n]$ where $j \leq_P i$ if the excursion of u_j is contained in the excursion of u_i.
- The **horizontal vector** $h(P)$ of P is the vector of indices of up steps of P (in increasing order).
- The **horizontal difference vector** $h(P,Q)$ is the vector $h(P) - h(Q)$ where the difference is computed componentwise.
- The **height vector** $k(P,Q)$ is defined as the Möbius inversion of $h(P,Q)$ with respect to $\leq_P$, that is to say
$$h_j(P,Q) = \sum_{j \leq_P i} k_i(P,Q). \tag{8.11}$$

Remark 8.2.15. The bijection between Dyck paths and planar trees described in the proof of Proposition 1.3.4 consisted of walking around the tree (counterclockwise) and recording the height of the nodes along the circuit. If we take a planar forest and connect all nodes to a root to produce a planar tree, then we can produce a Dyck path by walking clockwise instead.

The order $\leq_P$ corresponds to the order induced by the forest corresponding to P under this bijection (labelled in the pre-order), where $j \leq_P i$ if j is a descendant of i.

Corollary 8.2.16. *If j is a maximal element in $\leq_P$, then $k_j(P,Q) = h_j(P,Q)$. Otherwise, j is covered by a unique i_j, and we have $k_j(P,Q) = h_j(P,Q) - h_{i_j}(P,Q)$.*

Proof. It is sufficient to remark that every interval for $\leq_P$ is a path and then the Möbius function as defined in Definition 2.1.20 satisfies $\mu_P(j,j) = 1$, $\mu_P(j,i) = -1$ if i covers j and $\mu_P(j,i) = 0$ otherwise. Then, the result follows with the (dual) Möbius inversion (see Remark 2.1.26). □

Proposition 8.2.17. *Let P and Q be two Dyck paths of size n. Then $P \leq Q$ if and only if all entries of $k(P,Q) = (k_1, \ldots, k_n)$ are nonnegative.*

Moreover, in this case, for each chain from P to Q of maximal length, k_i is the number of covering relations in the chain involving the excursion of the i-th up step. In particular, the height of the interval $[P,Q]$ is equal to $\sum_{i=1}^{n} k_i$.

Lemma 8.2.18. *Let $P \lessdot P'$ be a covering relation in Tam_n exchanging the excursion E_i of the i-th up step u_i with the down step preceding it. Let j be the index of the excursion preceding E_i in P, i.e. ending at the down step preceding E_i.*

Then $\leq_{P'}$ is a refinement of $\leq_P$, and more precisely, it is obtained by adding the upper cover $i \leq_{P'} j$ and all implied relations.

Let Q be any Dyck path of size n. Then, $k(P',Q)$ is obtained from $k(P,Q)$ by changing $k_i(P,Q)$ into $k_i(P,Q) - 1 - k_j(P,Q)$ (and only this entry).

Proof. The first part is immediate with the definitions of $\leq_{P'}$ and $\leq_P$. In fact it was already proven in Lemma 5.4.8.

For the second part, using Corollary 8.2.16, we know that $k_i(P',Q) = h_i(P',Q) - h_j(P',Q)$ and since u_i was moved one step to the left but not u_j, we have $k_i(P',Q) = h_i(P,Q) - 1 - h_j(P,Q)$. Then, either i and j were not covered by any element in $\leq_P$ or by the same element k, and in both cases, we have, again using Corollary 8.2.16, $k_i(P',Q) = k_i(P,Q) - 1 - k_j(P,Q)$.

Finally, for all $i' \neq i$:

- either $i' \leq_P i$ and then its upper cover j' is the same in $\leq_P$ and $\leq_{P'}$. Moreover $h_{i'}(P',Q) = h_{i'}(P,Q) - 1$ and $h_{j'}(P',Q) = h_{j'}(P,Q) - 1$, and thus $k_{i'}(P',Q) = k_{i'}(P,Q)$,
- or $i' \not\leq_P i$ and then $h_{i'}(P',Q) = h_{i'}(P,Q)$ and the same holds for the upper cover of i' in $\leq_P$ if any. Thus, we also have $k_{i'}(P',Q) = k_{i'}(P,Q)$.

□

Proof of Propositions 8.2.12, 8.2.13 and 8.2.17. Firstly, it is clear by Möbius inversion that $k(P,Q)$ has all entries equal to 0 if and only if P and Q are the same Dyck path.

Then, because of Lemma 8.2.18, if some entry $k_i(P,Q)$ is negative, then for all $P' \geq P$, $k_i(P',Q) \leq k_i(P,Q)$ is negative and in particular P' can not be equal to Q and Q is not greater than P in Tam_n.

Conversely, if all entries of $k(P,Q) = (k_1, \ldots, k_n)$ are nonnegative, again with Lemma 8.2.18, the length of any chain from P to Q is at most $\sum_{i=1}^n k_i$. Moreover, the greedy procedures described in Propositions 8.2.12 and 8.2.13 both produce such a chain of length $\sum_{i=1}^n k_i$.

Indeed, if $(P_i)_i$ is the sequence of paths produced by either algorithm, then the key point is that if u_j is the chosen up step of P_i to perform a rotation $P_i \lessdot P_{i+1}$, then $k_{j'}(P_i,Q) = 0$ for $u_{j'}$ the first up step of the excursion preceding u_j in P_i. This is because by definition of u_j, P_i and Q agree until at least $u_{j'}$. Hence, the entry $k_j(P_i,Q)$ decreases by exactly one, and by induction, both procedures produce a chain from P to Q of length $\sum_{i=1}^n k_i$. □

8.2.3 Generalizing the decomposition

Now that we have defined several additional statistics on the intervals and that we know how to compute them, we can produce functional equations on the generating functions of m-Tamari intervals by understanding how they evolve through the expansion and gluing processes. First, we will consider the set of statistics composed of contacts, initial rise, height, bottom-out and top-out degrees, with one variable for each extra statistic, and then we will consider the height together with the rise and contact partitions, with infinite sets of variables to keep track of the partitions, one for each possible size of a part.

We need a modified version of the operator Δ, in order to keep track of the modification of the height of the interval as well as the bottom-out degree.

Proposition 8.2.19. *Let $m \geq 1$ and $F \equiv F^{(m)}(t;x,y,u,v,q)$ be the (decorated) generated series of m-Tamari intervals, where x, y, u, v, and q keep track respectively of the number of contacts, the initial rise, the bottom-out degree, the top-out degree and the height. Then*

$$F = yt\left(\left(vF(y=1)+1\right)\cdot\underline{\Delta}\right)^{(m)}\Big(x\big(F+1\big)\Big), \tag{8.12}$$

where $\underline{\Delta}(S(x)) = ux\frac{S(qx)-S(1)}{qx-1} + \frac{1-u}{q}S(qx)$ *and* $F(y=1)$ *means that the variable* y *is evaluated to* 1 *while all other variables are kept unchanged.*

Proof. With the exact same decomposition of m-Tamari intervals as in Proposition 8.2.2, we have to keep track of the different parameters through gluing and expansion of intervals, using in particular Remark 8.2.10 and Propositions 8.2.12 and 8.2.13 for dealing with out-degrees and height.

Firstly, when gluing an interval $I_2 = [P_2, Q_2]$ at the right of an interval $I_1 = [P_1, Q_1]$:

- The number of non-initial contacts of I_1 and I_2 add up.
- The initial rise of I_2 is ignored (which means y is set to 1 for the corresponding factor).
- The height of the intervals adds up, thanks to Proposition 8.2.12. Indeed, the greedy chain that the greedy algorithm produces will have first all its covering relations in the part coming from I_1 and then all in the part coming from I_2, somehow as a "concatenation" of the longest chains.
- The bottom-out degreeS add up because all excursions followed by a down step must already live either within P_1 or within P_2.
- The top-out degree of the gluing is equal to one plus the sum of those of I_1 and I_2, since there is one valley that is created when gluing Q_1 and Q_2.

All of this explains the factor $(vF(y=1)+1)$ each time we decide to glue an interval or not to do it.

Secondly, when expanding an interval $I' = [P', Q']$ into $I = [P, Q]$:

- Only the first occurrence is authorized to use the initial contact, hence the initial factor of $x(F+1)$.
- As previously, each choice of a contact of P' produces an interval $[P, Q]$, having from 1 (when choosing the last contact) to $c(I')$ non-initial contacts.
- At the end of these m steps of expansion-gluing, m initial up steps have been created, which contributes to an increase of the size and the initial rise by one each, hence the factor yt.
- The interval obtained by the expansion at the last contact (resulting in 1 non-initial contact) is isomorphic as a poset to the interval $[P', Q']$. Hence its height is not modified. However, every next contact to the left (resulting in one more non-initial contact than the previous) increases the height by one more because of Proposition 8.2.13. Hence, if the number of non-initial contacts of $[P, Q]$ is k, then the height has increased by $k-1$ in comparison with I'.
- All choices of contacts but the leftmost one will produce an interval with one more relevant excursion of P' that was at height 0 (hence not followed by a down step) now at height 1 and followed by a down step. This gives a factor u to account for this increment of the bottom-out degree, except for the interval with the biggest number of contacts.
- The top path Q' is obtained from Q by adding an up step at the beginning and a down step at the end, which does not affect the number of valleys.

All of this gives that each expansion process of an interval with k counted contacts, i.e. whose weight is $x^k W$, where W is a monomial into the other variables, will produce k intervals of weight uxW, uqx^2W, $\ldots$, $uq^{k-2}x^{k-1}W$, and $q^{k-1}x^kW$.

All this adds up to

$$\left(ux(1+(qx)+\cdots+(qx)^{k-2})+q^{k-1}x^k\right)W=\left(ux\frac{(qx)^k-1}{qx-1}-uq^{k-1}x^k+q^{k-1}x^k\right)W$$
$$=\underline{\Delta}\left(x^k\right)W=\underline{\Delta}\left(x^kW\right).$$

□

We can in fact state an even more refined version, by keeping track of the rise and contact partitions, additionally to the height of intervals. One way to keep track of an integer partition is to use an infinite family of variables, one for each part size. For instance, given a set $\underline{u}=(u_i)_{i\in\mathbb{N}}$ of variables, the partition $\lambda=(6,4,4,3,1,1,1,1)$ would be recorded as $u_6u_4^2u_3u_1^4$ — or possibly with extra factors u_0 according to the situation, that we get rid of at the end by specializing this variable to 1.

In our case, we will use two infinite families of variables for each partition, one for the unmarked parts and one for the marked part and keep the variables q for the height and t for the size. For this purpose, let $\underline{x},\underline{y},\underline{u},\underline{v}$ be four infinite sets of formal variables, and let us introduce several operators before stating the theorem. We use the $\underline{u}$ variables for the contact partition, without the marked part for which one x_i variable is used, i.e. for the number of non-initial contacts. Similarly, the $\underline{v}$ variables keep track of the rise partition, with one y_j variable for the initial rise.

Definition 8.2.20.

- We define the **shift operators** S_x and S_y that increase the index of x_i and y_j by one, changing them into x_{i+1} and y_{j+1}, respectively—the monomials of our intervals will always have only one variable of the families $(x_i)_i$ and $(y_j)_j$.

$$S_x=\sum_{i\geq 0}x_{i+1}\frac{\partial}{\partial x_i},$$
$$S_y=\sum_{j\geq 0}y_{j+1}\frac{\partial}{\partial y_j}.$$

- The **split operator** Δ_x is the equivalent of $\underline{\Delta}$ in this context. It selects an x_i variable and splits into two parts of the contact partition, one marked part of size j, and one unmarked part of size $i-j$, for all possible $1\leq j\leq i$. It also multiplies each factor by a factor q^{j-1}, accordingly.

$$\Delta_x=\sum_{i\geq 1}\left(\sum_{j=1}^{i}q^{j-1}x_ju_{i-j}\right)\frac{\partial}{\partial x_i}.$$

- The **merge operator** Θ_x is used for gluing operation, by merging two marked parts x_i and x_j into one x_{i+j}. The $\frac{1}{2}$ factor compensates the double derivative when $i=j$ and each pair $i\neq j$ appearing twice.

$$\Theta_x=\sum_{i,j\geq 1}\frac{1}{2}x_{i+j}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}.$$

- The **unmark operator** Ψ_y is transforming a marked part y_j into an unmarked part v_j.

$$\Psi_y=\sum_{j\geq 0}v_j\frac{\partial}{\partial y_j}.$$

Theorem 8.2.21. *Let $F^{(m)}(t; \underline{x}, \underline{y}, \underline{u}, \underline{v}, q)$ be the generating function of m-Tamari intervals, with t and q keeping track of the size and height, and $\underline{x}, \underline{u}$ (resp. $\underline{y}, \underline{v}$) recording the marked and unmarked contact partition parts (resp. rise partition parts), then*

$$F^{(m)}(t; \underline{x}, \underline{y}, \underline{u}, \underline{v}, q) = \left[\Theta_x\left(\Psi_y\left(F^{(m)} + x_0 y_0\right) \cdot \Delta_x\right)\right]^{(m)}\left(tS_yS_x\left(F^{(m)} + x_0y_0\right)\right)\Bigg|_{u_0=1, v_0=1}. \tag{8.13}$$

Proof. To prove that the equation is correct, we will decompose the equation and see that it corresponds to what happens during the m rounds of expansion-gluing. We illustrated this process for 2-Tamari intervals in Figure 8.4. We need to prove in particular that each application of the operators S_x, S_y, Δ_x, and Ψ_y are to series with only one variable x_i or y_i (accordingly to their index) and operator Θ_x is applied to a series with exactly two factors in the $\underline{x}$ family of variables, so that their behavior corresponds to those described in Definition 8.2.20.

As we do not take into account the Tamari lattice of size 0, every interval will have exactly one marked part x_i and one marked part y_j for some $i, j \geq 1$.

- We start with an interval, or maybe an empty pair, hence the term $x_0 y_0$. The application of S_x accounts for the initial contact. The operator S_y and the multiplication by t accommodate for the increment of initial rise and size that occur at the end of the process. This can indeed be done at the very beginning since none of the subsequent operations interfere with that.
- Then follow m iterations of expansion, each handled by applying first the splitting Δ_x, and gluing, realized by first multiplying by a factor $\Psi_y\left(F^{(m)} + x_0 y_0\right)$ and then applying the merging operator Θ_x to the result.
  - Each time, we apply Δ_x to a series whose factors all contain exactly one marked contact variable x_i. The operator thus transforms the x_i into $q^{j-1} x_j u_{i-j}$ in all possibly manners for $1 \leq j \leq i$, and sums all the resulting monomials. This accounts for selecting one of the contacts and splitting the marked part into j non-initial contacts at the right of the inserted step and the remaining $i - j$ that are not contacts anymore. This also contributes to a q^{j-1} factor, as in Proposition 8.2.19. By construction, the factors of the resulting series all contain exactly one marked contact variable x_j for some j. The expansions of the interval Figure 8.2 at all contacts (including the initial contact) are depicted in Figure 8.3.
  - The term $x_0 y_0$ that we add to $F^{(m)}$ corresponds to choosing no interval to glue. Applying the unmark operator Ψ_y to this sum transforms the unique marked part y_j of each term into an unmarked rise part u_j. Multiplying the result with the series obtained at the previous step produces a series whose terms all have exactly one marked rise part and two marked contact parts, one from each interval glued together. All rises of the interval on the right are non-initial, all non-initial contacts of both intervals are non-initial contacts of the result. Applying finally the merge operator Θ_x gathers those two terms into one marked contact part. All terms of the resulting series have exactly one marked rise factor x_i and one marked contact factor y_j for some $i, j \geq 1$.
- The final specialization of u_0 and v_0 to 1 is to get rid of all empty unmarked parts.

□

Proposition 8.2.22. *One could rewrite the operators and equations to get rid of all unnecessary variables u_0 and v_0.*

Specializing u_0 and v_0 to 1 to forget empty parts, each u_i and v_j variables to u and v respectively, and each x_i and y_j to x^i and y^j respectively, (8.13) is transformed into (8.12).

Proof. Each unmarked contact part u_i for $i \geq 1$ correspond to a sequence of i consecutive excursions that are not at height 0. For each such sequence of consecutive excursions, the last one is indeed followed by a down step. Similarly, each unmarked rise part v_j corresponds to a non-initial rise, starting at a valley.

- The term $tS_yS_x\big(F^{(m)} + x_0y_0\big)$ is transformed into $txy(F^{(m)} + 1)$.
- The action of the operator Δ_x translates to the action of $\underline{\Delta}$.
- The unmark operator Ψ_y applied to $F^{(m)} + x_0y_0$ gives the evaluation $y = 1$ in $F^{(m)} + 1$, and the merge operator Θ_x is accounting for the transformation of x^ix^j into x^{i+j}.

□

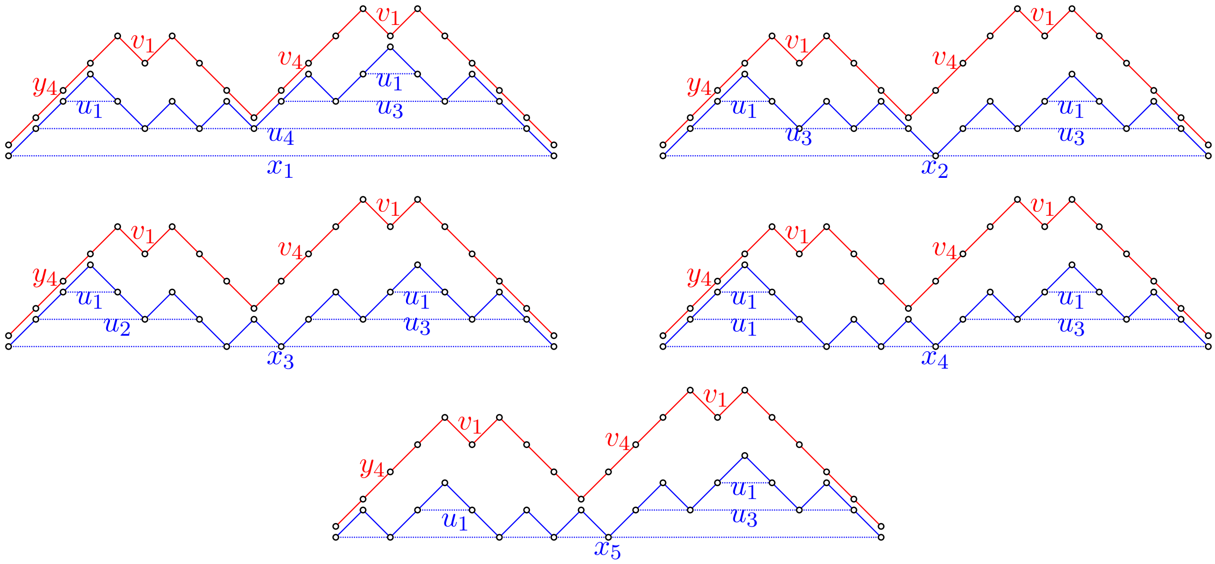


Figure 8.3: Possible expansions of the Tamari interval of Figure 8.2 with 4 non-initial contacts. Note that each bottom (blue) path is the result of the rotation of the first contact at height 0 of the bottom path of the next interval.

It would be interesting to investigate whether such functional equations appear in the world of combinatorial maps. Indeed, as mentioned earlier, it is remarkable that Tamari intervals are in bijection with simple triangulations, synchronized Tamari intervals are in bijection with simple quadrangulations, and new (or modern) intervals are in bijection with bipartite maps. Such an equation could be used as a guide to identify an interesting subset of maps which would be in bijection with m-Tamari intervals. Some resembling operators on series with infinite sets of variables indeed appeared in the work of G. Chapuy and M. Dołęga on the enumeration of some maps or constellations with a given degree sequence, see for instance [CD22], or in the work of W. Fang regarding combinatorial maps and constellations (see [Fan16, Chapter 4]). One could thus study the different operators appearing in (8.13) and the relations they satisfy.

8.3 Tamari and m-Tamari interval-posets

Tamari interval-posets are posets on integers satisfying some simple rules, namely that for each integer i, the set of elements weakly below i is an interval of consecutive integers (containing i). They were introduced by G. Châtel and V. Pons in [CP15], where the authors proved that they are in bijection with Tamari intervals. They also gave a description using initial and final forests,

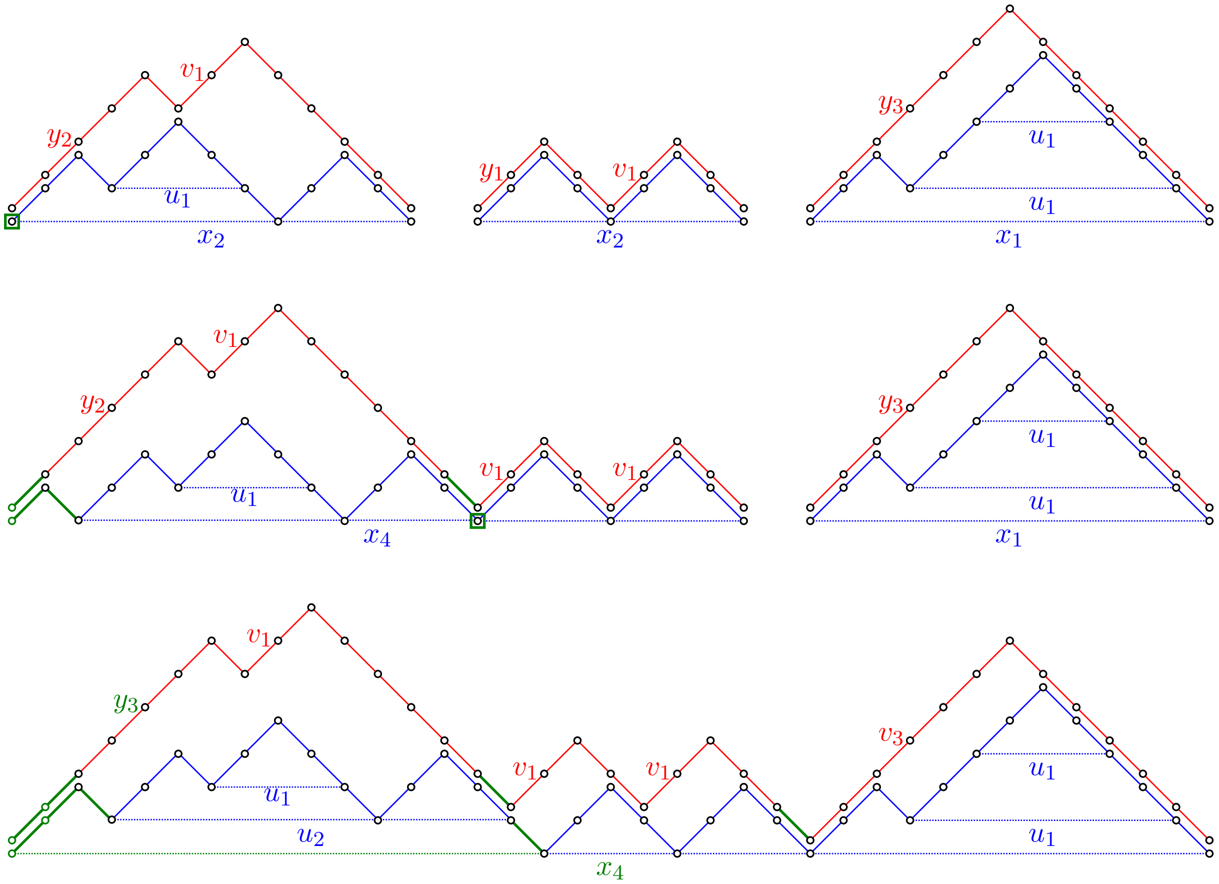


Figure 8.4: Example of expansion-gluing process of 2-Tamari intervals. The chosen contact for the expansion is framed in green. Note that at the second step, the choice of the initial contact for the expansion of the bottom path is forbidden since it would not have created a 2-Dyck path.

by splitting relations into increasing or decreasing relations, which grants a natural involution on these objects, by exchanging the two forests. They proved a refined version of the enumeration of Tamari and m-Tamari intervals by giving a recursive way to count the number of intervals with a fixed top element.

In [Pon19], V. Pons described statistics on Tamari interval-posets of size n, namely the number of Tamari inversions, and two partitions of n with a marked part. She proved that the distribution of these three decorations coincides with the distribution of height, rise and contact partitions on Tamari intervals, as described in Section 8.2.2. She also described a class of m-divisible Tamari interval-posets, which she showed to be in bijection with m-Tamari intervals. These m-divisible interval-posets are also equipped with a natural involution, and this implies the result conjectured in [PR12, Conjecture 17], generalizing the symmetry of initial rise and contacts established in [BMFPR11].

Definition 8.3.1. A **Tamari interval-poset** of size n is a poset $(I, \triangleleft)$ on the set $[n]$ such that for all triples of integers $a < b < c$:

- $a \triangleleft c$ implies $b \triangleleft c$,
- $c \triangleleft a$ implies $b \triangleleft a$.

A relation $a \triangleleft b$ is called **increasing** if $a \leq b$ in the natural order and **decreasing** if $a \geq b$.

Because of the first condition in the definition, the Hasse diagram of increasing relations is a (planar rooted) forest, and the same is true for decreasing relations. These two forests will be referred to as initial and final forests respectively. The two integer partitions decorating Tamari interval-posets are defined as the sorted sequence of the number of children of each node, and the number of roots of the forest will be an additional marked part.

Definition 8.3.2.

- Let $(I, \triangleleft)$ be a Tamari interval-poset of size n. Its **initial forest** (resp. **final forest**) is the Hasse diagram of the increasing relations (resp. decreasing relations), viewed as a poset.
- An element $a \in I$ is called an **increasing root** (resp. a decreasing root) if there is no increasing (resp. decreasing) relation $a \triangleleft b$ in I. Their number is denoted $\mathrm{ir}(I)$ (resp. $\mathrm{dr}(I)$).
- The **increasing children** (resp. decreasing children) of $b \in I$ are elements covered by b in the initial forest (resp. final forest) of I.
- The **initial forest partition** $\widetilde{\lambda}^{\mathsf{top}}(I)$ (resp. **final forest partition** $\widetilde{\lambda}^{\mathsf{bot}}(I)$) is defined as the sorted sequence of the number of increasing children (resp. decreasing children) of elements of I.

The number of roots can be added to the respective partitions so that they form partitions of n with one marked part. They will correspond to the contact and rise marked partitions of Tamari intervals, and the height will also translate as some statistic on interval-posets called *Tamari inversions*. Additionally, there is a natural involution on Tamari interval-posets that exchanges the forests, hence the two marked partitions, and preserves the number of Tamari inversions.

Definition 8.3.3. Let $(I, \triangleleft)$ be a Tamari interval-poset. A **Tamari inversion** of I is a pair $a < b$ such that:

- there is no $a \leq k < b$ with $b \triangleleft k$,
- and there is no $a < k \leq b$ with $a \triangleleft k$.

The number of Tamari inversions of I is denoted $k(I)$.

An example of interval-poset is given in Figure 8.5, as well as its initial and final forest partitions, and its Tamari inversions.

Proposition 8.3.4. *Let $(I, \triangleleft)$ be a Tamari interval-poset of size n. Let $(I', \triangleleft')$ be the poset defined by $a \triangleleft' b$ if $n+1-a \triangleleft n+1-b$. We write $I' = \psi(I)$ and call ψ the* **complement involution**.

Then $(I', \triangleleft')$ is a Tamari interval-poset of size n such that $\widetilde{\lambda}^{\mathsf{top}}(I') = \widetilde{\lambda}^{\mathsf{bot}}(I)$, $\widetilde{\lambda}^{\mathsf{bot}}(I') = \widetilde{\lambda}^{\mathsf{top}}(I)$, and $k(I) = k(I')$.

We can bijectively transform Tamari intervals into Tamari interval-posets, as described in [CP15] on binary trees or in [Pon19, Propositions 18 and 19] on Dyck paths. The transformation is illustrated in Figure 8.5. The contact partition can be identified with the final forest partition, but it does not work so well for the rise partition. However, in order to fix this problem, another involution on Tamari interval-posets called "left branch involution" is defined in [Pon19], which we refer to for the definition.

Definition 8.3.5. Let P be a Dyck path of size n.

The **final forest** $F_{\geq}(P)$ of P is the poset on $[n]$ where $j \triangleleft i$ if the j-th up step of P is in the excursion of the i-th up step of P.

The **initial forest** $F_{\leq}(P)$ of P is the poset on $[n]$ where $i \triangleleft j$ if the j-th up step of P is the first up step following the last step of the excursion of the i-th up step of P.

Remark 8.3.6. The final forest corresponds to the containment order $\leq_P$ defined in Definition 8.2.14, as illustrated in Figure 8.5.

As noted in Remark 8.2.15, we can transform Dyck paths into binary trees via the bijection corresponding to a clockwise tour of the binary tree. Then, the initial forest is obtained from this binary tree via the classical bijection from trees to forests described in the proof of Proposition 1.3.2 which transforms right children into right siblings. Note that the final forest can also be obtained from this same binary tree by transforming left children into left siblings instead.

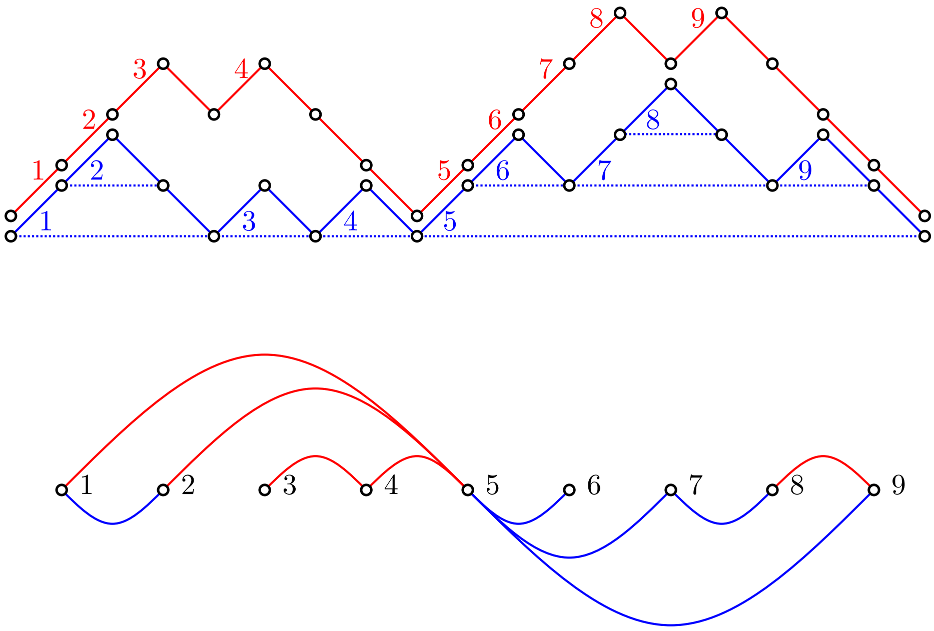


Figure 8.5: A transformation of a Tamari interval into an interval-poset I. The arcs on the top are increasing relations and the ones on the bottom are decreasing relations. The increasing roots are 5, 6, 7, and 9, and the vertices 4, 5, and 9 have respectively 1, 3, and 1 incoming increasing arcs. Hence, the initial forest partition is $\widetilde{\lambda}^{\mathsf{top}}(I) = (3,1,1)$. Similarly, the final forest partition is $\widetilde{\lambda}^{\mathsf{bot}}(I) = (3,1,1)$. The Tamari inversions are $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$, $(6,7)$, $(6,9)$, and $(7,9)$, so $k(I) = 7$.

Theorem 8.3.7 ([CP15, Theorem 2.8])**.** *Interval-posets are in bijection with Tamari intervals.*

More precisely, $[P,Q]$ is an interval in Tam_n if and only if there is a Tamari interval-poset whose final forest is $F_{\geq}(P)$ and initial forest is $F_{\leq}(Q)$.

Theorem 8.3.8 ([Pon19, Propositions 25, 28, and 50])**.** *Let I be the interval-poset corresponding to an interval $[P,Q]$ in Tam_n under the bijection of Theorem 8.3.7. Then $\widetilde{\lambda}^{\mathsf{bot}}(I) = \widetilde{\lambda}^{\mathsf{bot}}(P)$ and the number of Tamari inversions of I is equal to the height of $[P,Q]$.*

Moreover, there exists an involution ϕ called **left branch involution** *on Tamari interval-posets such that if $I = [P,Q]$ $\widetilde{\lambda}^{\mathsf{bot}}(\phi(I)) = \widetilde{\lambda}^{\mathsf{bot}}(I)$, $\widetilde{\lambda}^{\mathsf{top}}(\phi(I)) = \widetilde{\lambda}^{\mathsf{top}}(I)$ and the height $h(I)$ of I is equal to $k(\phi(I))$.*

Combining this result with the complement involution of Proposition 8.3.4, we get a combinatorial explanation of the symmetric distribution of non-initial contacts and initial rises in Tamari intervals.

We are in fact interested in m-Tamari intervals and their statistics. They can be defined on m-Dyck paths, as in Definition 4.1.1, and considered as a subset of bigger Tamari intervals, then transformed into Tamari interval-posets. However, this does not behave well with respect to the statistics defined in Section 8.2.2. Instead, using again some trees called m-grafting trees, V. Pons described in [Pon19] another subset of Tamari interval-posets in bijection with m-Tamari intervals, that we shall describe now. They are referred to as *rise-contact-m-divisible Tamari interval-posets* but we will call them m-Tamari interval-posets for short.

Definition 8.3.9. A Tamari interval-poset I of size mn is an **m-Tamari interval-poset** of size n if all parts of $\widetilde{\lambda}^{\mathsf{top}}(I)$ and $\widetilde{\lambda}^{\mathsf{bot}}(I)$ are divisible by m.

In this case, when considered as an m-Tamari interval-poset, we divide all parts sizes of the partitions by m to define the relevant initial and final forest partitions $\widetilde{\lambda}^{\mathsf{top}}_{(m)}(I)$, and $\widetilde{\lambda}^{\mathsf{bot}}_{(m)}(I)$.

Remark 8.3.10. The image under the complement involution ψ of an m-Tamari interval-poset is again an m-Tamari interval-poset.

Transforming the Tamari interval-posets arising from an m-Tamari interval into an m-grafting tree, then using a so-called "expansion" operation, V. Pons could produce an m-Tamari interval-poset, through this bijective process.

Theorem 8.3.11 ([Pon19, Propositions 67, 72, and 73])**.** *For all $m \geq 1$, there exists a bijection $\theta_{(m)}$ between the sets of m-Tamari intervals and m-Tamari interval-posets such that:*

$$\begin{aligned}
\widetilde{\lambda}^{\mathsf{bot}}_{(m)}(\theta_{(m)}(I)) &= \widetilde{\lambda}^{\mathsf{bot}}(I), \\
\widetilde{\lambda}^{\mathsf{top}}_{(m)}(\theta_{(m)}(I)) &= \widetilde{\lambda}^{\mathsf{top}}(I), \\
k(\theta_{(m)}(I)) &= mh(I) + \frac{nm(m-1)}{2}.
\end{aligned}$$

Again, combining the theorem with Remark 8.3.10, one gets the general result conjectured by L.-F. Préville-Ratelle in his thesis [PR12, Conjecture 17] about the involution on m-Tamari intervals which exchanges the rise and the contact vectors, while keeping unchanged the height of the intervals. In particular, this explains combinatorially and generalizes the symmetry of the joint union of the rise and the non-initial contact statistics on m-Tamari intervals proven in [BMFPR11].

Theorem 8.3.12 ([Pon19, Theorem 61])**.** *The joint distribution of height, rise and contact partitions on m-Tamari intervals is symmetric into the two marked partitions.*

The main takeout of this section is the introduction of m-Tamari interval-posets, which are simple combinatorial objects on which we can read all statistics on m-Tamari intervals we are interested in. Besides, they possess a natural symmetry through the complement involution ψ, as defined in Proposition 8.3.4. Our focus on such statistics and involution is motivated by the comparison to come with the linear type A m-Cambrian lattices, as we detail in Section 9.4.2.

Chapter 9

The m-Cambrian lattice

The Tamari lattice can be seen as the linear type A Cambrian lattice, as explained in Example 3.4.8. This context is another place where an "m-eralization" of the Tamari lattice can be defined, namely the m-Cambrian lattices and especially the linear type A case. These were introduced in [STW18] and are the main object of this chapter.

The general idea of this new object is to go from the Coxeter group W to the positive Artin monoid $\boldsymbol{B}^+$, and to extend naturally the structures defined in Section 3.4. The (positive) Artin monoid has a similar presentation as the Coxeter group, removing the relations $s_i^2 = 1$ and keeping only braid relations. In other words, elements can be seen as words in the generators up to braid relations, and this hence gives a notion of (right) weak order $\mathrm{Weak}(\boldsymbol{B}^+)$. The type A Artin monoid is usually referred to as the *braid monoid* $\mathfrak{S}_n$.

All three descriptions of the Cambrian lattice given in Section 3.4 can be generalized and give in fact equivalent definitions of the so-called m-Cambrian lattice. One is defined as the restriction of the weak order on the Artin monoid to the c-sortable elements of some interval, another as some initial subword complex, and the last one as a transitive closure of rotation relation on chains in the noncrossing partition lattice.

We first present the definitions and results needed for this chapter, then the three equivalent definitions of the m-Cambrian lattice and the translation from one to another. We also provide a tentative new definition of these posets. This definition seems to be very powerful as it is both simple to describe the elements and the comparison relations, which could be very useful to manipulate the m-Cambrian lattices. It also gives a poset structure on m-chains in the Cambrian lattices, which, to the best of our knowledge, has never been considered before, even in the linear type A case, i.e. the Tamari lattice. This case was in fact the starting point of this PhD project. Indeed, a conjecture of [STW18] states that the number of intervals in the linear type A m-Cambrian lattice would coincide with the number of m-Tamari intervals. We also formulate some refined conjecture, as well as some approaches towards the desired result, unfortunately still conjectural.

9.1 The Artin group and monoid

Given a Coxeter system $(W, \mathcal{S})$, one can define the Artin group $\boldsymbol{B}$ (sometimes called Artin-Tits group or generalized braid group) by removing the condition that generators are of order two. One can also define the (positive) Artin monoid $\boldsymbol{B}^+$ as the submonoid of $\boldsymbol{B}$ generated by the (positive) generators of $\boldsymbol{B}$, or in other words the subset of elements that can be expressed without using the inverse of the generators. The Artin group appears naturally as the fundamental group of the complement of the (complex) hyperplane arrangement associated to the Coxeter system.

It is clear that there is a morphism from the monoid generated by a set of generators subject to braid relations to the Artin group. However, it is not obvious that this is an injective morphism, and this is called the *embedding property* [Par02]. Natural questions such as the word problem (i.e. deciding whether two words represent the same element) are solved for some classes of Artin groups (as the spherical ones [BS72, Satz 6.6]) but not for general Artin groups. This is however easier to solve for Artin monoids [BS72, Satz 6.3], in which the (Coxeter) length can be easily defined, as well as the right weak order.

In what follows, we will only consider spherical positive Artin monoids, where the weak order is even a lattice. Recall that, following the [STW18] convention, we use normal font for elements w in the Coxeter group W, sans serif font for $\mathcal{S}$-words $\mathsf{Q} \in \mathcal{S}^*$ and bold font for elements $\boldsymbol{w}$ in the Artin monoid $\boldsymbol{B}^+$.

Definition 9.1.1. Let $(W, \mathcal{S})$ be a Coxeter system. Recall that the group W is given by the group presentation $\langle \mathcal{S} \mid s^2 = e, [s|t]^{m_{s,t}} = [t|s]^{m_{s,t}} \rangle$, with $m_{s,t} = 1$ if and only if $s = t$ and $m_{s,t} = m_{t,s} \geq 2$ otherwise.

Let $\boldsymbol{\mathcal{S}}$ be a copy of $\mathcal{S}$. The **Artin group** is the group

$$\boldsymbol{B} = \langle \boldsymbol{\mathcal{S}} \mid [\boldsymbol{s}|\boldsymbol{t}]^{m_{s,t}} = [\boldsymbol{t}|\boldsymbol{s}]^{m_{s,t}}, s \neq t \rangle. \tag{9.1}$$

The **positive Artin monoid** $\boldsymbol{B}^+$ is the submonoid of $\boldsymbol{B}$ generated by $\boldsymbol{\mathcal{S}}$.

There are obviously a canonical surjective group morphism from $\boldsymbol{B}$ to W and a canonical monoid morphism from $\boldsymbol{B}^+$ to W defined by sending each generator $\boldsymbol{s}$ to its corresponding generator s in W. We also have a canonical surjective monoid morphism from $\mathcal{S}^*$ to $\boldsymbol{B}^+$.

Thanks to Theorem 3.1.6, since all reduced words of an element $w \in W$ are connected through braid moves, there is also a natural injection $W \hookrightarrow \boldsymbol{B}^+$. Identifying elements of the monoid with $\mathcal{S}$-words subject to braid relations, the right weak order can naturally be defined on $\boldsymbol{B}^+$ as being initial, as in Definition 3.1.4. This naturally extends the right weak order on W under the canonical injection. For spherical Coxeter systems, if $\boldsymbol{w}_\circ$ is the element of $\boldsymbol{B}^+$ corresponding to the longest element $w_\circ$ of W, this identifies $\mathrm{Weak}(W)$ as the interval $[\boldsymbol{e}, \boldsymbol{w}_\circ]$ in $\mathrm{Weak}(\boldsymbol{B}^+)$.

Definition 9.1.2. The **(Coxeter) length** of an element $\boldsymbol{w} \in \boldsymbol{B}^+$ is the size $\ell_{\boldsymbol{\mathcal{S}}}(\boldsymbol{w})$ of any expression of $\boldsymbol{w}$ as a product of elements of $\boldsymbol{\mathcal{S}}$.

The **right weak order** $\mathrm{Weak}(\boldsymbol{B}^+)$ on $\boldsymbol{B}^+$ is the partial order defined by $\boldsymbol{u} \leq \boldsymbol{w}$ if there exists $\boldsymbol{v} \in \boldsymbol{B}^+$ such that $\boldsymbol{uv} = \boldsymbol{w}$ and $\ell_{\boldsymbol{\mathcal{S}}}(\boldsymbol{u}) + \ell_{\boldsymbol{\mathcal{S}}}(\boldsymbol{v}) = \ell_{\boldsymbol{\mathcal{S}}}(\boldsymbol{w})$.

Proposition 9.1.3. *The weak order* $\mathrm{Weak}(W)$ *on* W *is isomorphic to the interval* $[\boldsymbol{e}, \boldsymbol{w}_\circ]$ *in* $\mathrm{Weak}(\boldsymbol{B}^+)$.

One can use colored roots and inversion sequences of an $\mathcal{S}$-word as in Definition 3.2.7 to define inversion sets of elements of $\boldsymbol{B}^+$. Indeed, a braid move only permutes the colored inversion sequence of an $\mathcal{S}$-word.

Definition 9.1.4. Let $\boldsymbol{w} = \boldsymbol{s}_1\boldsymbol{s}_2 \dots \boldsymbol{s}_p \in \boldsymbol{B}^+$ and $\mathsf{w} = \mathsf{s}_1\mathsf{s}_2 \dots \mathsf{s}_p$ be an $\mathcal{S}$-word corresponding to $\boldsymbol{w}$. Let $\mathsf{inv}(\mathsf{w}) = \left(\beta_1^{(m_1)}, \dots, \beta_p^{(m_p)}\right)$ be the colored inversion sequence of w.

The **colored inversion set** of $\boldsymbol{w}$ is the set

$$\mathrm{inv}(\boldsymbol{w}) = \left\{\beta_1^{(m_1)}, \dots, \beta_p^{(m_p)}\right\}. \tag{9.2}$$

Remark 9.1.5. Contrary to the case of the Coxeter group, the colored inversion set of an element of the Artin monoid is not enough to determine the element. For example $\boldsymbol{sstt}$ would have the same colored inversion set as $\boldsymbol{ttss}$, for any $\boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{\mathcal{S}}$. But provided $\boldsymbol{s}$ and $\boldsymbol{t}$ do not commute, the elements are different.

Therefore, the weak order on $\boldsymbol{B}^+$ is no longer equivalent to the inclusion of inversion sets, as it was for the Coxeter group.

Recall that for a colored root $\beta^{(m)}$, we denoted $|\beta^{(m)}|$ the corresponding positive root β. It is useful to notice that the color of a root appearing in an inversion sequence is always the smallest possible color, i.e. the number of times the corresponding positive root appears before in the inversion sequence. Thus, the computation of the inversion set can firstly be done without taking the colors into account.

Lemma 9.1.6 ([STW18, Lemma 2.9.3])**.** *Let* $\mathsf{Q} = \mathsf{s}_1 \dots \mathsf{s}_p$ *be an* $\mathcal{S}$*-word with colored inversion sequence* $\mathsf{inv}(\mathsf{Q}) = \left(\beta_1^{(m_1)}, \dots, \beta_p^{(m_p)}\right)$*. The following two statements hold:*

1. *For any* $1 \le i \le p$*, we have* $\beta_i = |s_1 \dots s_{i-1}(\alpha_{s_i})|$*,*
2. *For any* $1 \le i \le p$*, the color* m_i *is equal to the number of* $j < i$ *such that* $\beta_j = \beta_i$*.*

In Section 3.4.2, we used subword complexes to define the Cambrian lattices. One of the constructions uses this framework, that can be extended to the Artin monoid. While the original notion of subword complexes can be generalized in several ways, in the case of Coxeter initial subword complexes, all three definitions coincide. The proof of [STW18] relies on a notion of dual braid monoid that we do not present here. This was developed by D. Bessis in [Bes03] for spherical Artin monoids. Though we do not use it extensively in this work, Garside theory is a very rich framework that encompasses in particular many properties of the Artin monoid and the "dual" Artin monoid. This can be used to prove the embedding property, gives normal form to each element, solve the word problem, as well as other interesting structural properties.

Definition 9.1.7. The **Garside factorization** of an element $\boldsymbol{w} \in \boldsymbol{B}^+$ is the factorization of $\boldsymbol{w}$ as a product of elements $(\boldsymbol{w}^{(i)})$ of $[\boldsymbol{e}, \boldsymbol{w}_\circ]$ defined recursively as follows: Starting with $\boldsymbol{w}_1 = \boldsymbol{w}$, as long as $\boldsymbol{w}_i \neq \boldsymbol{e}$, one sets $\boldsymbol{w}^{(i)} = \boldsymbol{w}_i \wedge \boldsymbol{w}_\circ$ and $\boldsymbol{w}_{i+1} = (\boldsymbol{w}^{(i)})^{-1}\boldsymbol{w}_i$.

The **Garside degree** $\deg(\boldsymbol{w})$ of $\boldsymbol{w}$ is the number of Garside factors of $\boldsymbol{w}$, i.e. the smallest k such that $\boldsymbol{w}_k = \boldsymbol{e}$.

Then we have $\boldsymbol{w} = \boldsymbol{w}^{(1)} \cdot \boldsymbol{w}^{(2)} \cdot \dots \cdot \boldsymbol{w}^{(k)}$, where the **Garside factors** $\boldsymbol{w}^{(i)}$ are all initial in $\boldsymbol{w}_\circ$ and can thus be considered as elements of W.

Proposition 9.1.8 ([Mic99, Corollary 4.2])**.** *A factorization* $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \cdot \dots \cdot \boldsymbol{v}_k$ *with* $\boldsymbol{v}_i \le \boldsymbol{w}_\circ$ *is the Garside factorization of* $\boldsymbol{w} = \boldsymbol{v}_1 \dots \boldsymbol{v}_k$ *if and only if* $\boldsymbol{v}_i\boldsymbol{v}_{i+1} \wedge \boldsymbol{w}_\circ = \boldsymbol{v}_i$ *for all* $1 \le i < k$*.*

This is equivalent to the inclusions $\mathrm{des}_R(\boldsymbol{v}_i) \supseteq \mathrm{des}_L(\boldsymbol{v}_{i+1})$ *for all* $1 \le i < k$*.*

Our main use of the Garside factorization is that the Garside degree of an element is also the smallest k such that the element is in $[\boldsymbol{e}, \boldsymbol{w}_\circ^k]$.

Definition 9.1.9. Let $m \ge 1$. The **m-eralized weak order** $\mathrm{Weak}^{(m)}(W)$ is the restriction of the weak order $\mathrm{Weak}(\boldsymbol{B}^+)$ to the interval $[\boldsymbol{e}, \boldsymbol{w}_\circ^m]$. Its set of elements is denoted $W^{(m)}$.

Theorem 9.1.10 ([STW18, Proposition 2.11.4 and Theorem 2.11.15])**.** *An element* $\boldsymbol{u} \in \boldsymbol{B}^+$ *is in* $W^{(m)}$ *if and only if its Garside degree is at most* m*.*

The map $\boldsymbol{w} \mapsto \mathrm{rev}(\boldsymbol{w}_\circ^m \overline{\boldsymbol{w}})$ *is an anti-automorphism of* $\mathrm{Weak}^{(m)}(W)$*, which is a lattice.*

Recall that given a search word Q, an element $w \in W$ and a length a, the subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ was defined in Definition 3.4.10 as the simplicial complex whose facets are complements of subwords of length a in Q that are words of w. Using the Artin monoid and inversion sets of its elements, two natural generalizations of subword complexes arise naturally, as we present here.

Definition 9.1.11. Let Q be a search word and $\boldsymbol{w} \in \boldsymbol{B}^+$. The **Artin subword complex** $\mathrm{SC}_{\mathcal{S}}^{\boldsymbol{B}}(\mathsf{Q}, \boldsymbol{w})$ is the simplicial complex whose vertices are positions of letters in Q and whose facets are subset of positions in Q whose complement form an $\mathcal{S}$-word for $\boldsymbol{w}$.

Let Q be a search word and X be a set of colored roots. The **inversion set subword complex** $\mathrm{SC}_{\mathcal{S}}^{\mathrm{col}}(\mathsf{Q}, X)$ is the simplicial complex whose vertices are positions of letters in Q and whose facets are subsets of positions in Q whose complement has X as inversion set.

Proposition 9.1.12 ([STW18, Proposition 3.2.4]). *Let $\mathsf{Q} = \mathsf{s}_1 \dots \mathsf{s}_p$ be a search word and let $\boldsymbol{w} \in \boldsymbol{B}^+$. Let $w \in W$ be the element corresponding to $\boldsymbol{w}$ under the canonical projection.*

Then we have injections

$$\mathrm{SC}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{Q}, \boldsymbol{w}) \hookrightarrow \mathrm{SC}^{\mathrm{col}}_{\mathcal{S}}(\mathsf{Q}, \mathrm{inv}(\boldsymbol{w})) \hookrightarrow \mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, \ell_{\boldsymbol{\mathcal{S}}}(\boldsymbol{w})). \tag{9.3}$$

Remark 9.1.13. Both injections can be strict. For example, for $\mathsf{Q} = \mathsf{ssttss}$ in $\mathfrak{S}_3$ generated by $\{s, t\}$, and $\boldsymbol{w} = \boldsymbol{sstt}$, we have:

- $\mathrm{SC}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{Q}, \boldsymbol{w})$ has only $\{5, 6\}$ as a facet;
- $\mathrm{SC}^{\mathrm{col}}_{\mathcal{S}}(\mathsf{Q}, \mathrm{inv}(\boldsymbol{w}))$ has $\{1, 2\}$ and $\{5, 6\}$ as facets;
- $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, e, \ell_{\boldsymbol{\mathcal{S}}}(\boldsymbol{w}))$ has facets $\{\{1, 2\}, \{1, 5\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{5, 6\}\}$.

These three definitions in fact agree in the case of Coxeter initial subword complexes. These behave particularly nicely since they additionally have very good properties and are for instance vertex decomposable simplicial complexes, as recalled in Theorem 3.4.13.

Theorem 9.1.14 ([STW18, Theorem 3.5.1]). *Let $\mathsf{c} = \mathsf{s}_1 \dots \mathsf{s}_n$ be a Coxeter word and $\mathsf{Q} = \mathsf{s}_1 \dots \mathsf{s}_n \mathsf{s}_{n+1} \dots \mathsf{s}_p$ be initial in c^∞. Set $\boldsymbol{w} = \boldsymbol{s}_{n+1} \dots \boldsymbol{s}_p \in \boldsymbol{B}^+$ and consider its corresponding element $w \in W$. Then the two injections are equalities, namely*

$$\mathrm{SC}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{Q}, \boldsymbol{w}) = \mathrm{SC}^{\mathrm{col}}_{\mathcal{S}}(\mathsf{Q}, \mathrm{inv}(\boldsymbol{w})) = \mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, p - n). \tag{9.4}$$

Note that the Cambrian lattices as defined on subword complexes were indeed Coxeter initial subword complexes.

9.2 Definitions of the m-Cambrian lattices

Each definition of the Cambrian lattices described in Section 3.4 can be generalized by introducing an integer parameter $m \geq 1$, and this gives three isomorphic posets, namely the m-Cambrian lattices. We present here the definitions and the bijections between them, based on the work of [STW18].

Again, each of them depends on the choice of a (standard) Coxeter element in the Coxeter group, and the first two constructions are presented with the choice of a Coxeter word but are equivalent when two words are commutation equivalent, i.e. represent the same Coxeter element.

9.2.1 m-Cambrian lattices as subposets of the weak order

In Section 3.4.1, we presented the Cambrian lattice as the restriction of the weak order on the set of c-sortable elements, for c a chosen Coxeter word. The weak order can be generalized to the Artin monoid (see Definition 9.1.2), and the notion of c-sortability also generalizes straightforwardly, as we shall see. From there, the Cambrian lattice can be recovered as the restriction of $\mathrm{Weak}(\boldsymbol{B}^+)$ to the set of c-sortable elements in the interval $[\boldsymbol{e}, \boldsymbol{w}_\circ] = W^{(1)}$. The m-Cambrian lattice then is obtained by changing $\boldsymbol{w}_\circ$ into $\boldsymbol{w}_\circ^m$.

Definition 9.2.1. Let c be a Coxeter word in W. The **c-sorting word** $\mathsf{w}(\mathsf{c})$ of an element $\boldsymbol{w} \in \boldsymbol{B}^+$ is the lexicographically smallest (as a sequence of positions) subword of c^∞ that is a word for $\boldsymbol{w}$. Again, it can be interpreted as a sequence $(I_1, \dots, I_k)$ of subsets of $\mathcal{S}$, separated by the divider signs, where $\mathsf{c}\big|_{I_1}$ is the subword before the first divider, and so on.

An element $\boldsymbol{w} \in \boldsymbol{B}^+$ whose c-sorting word is $\mathsf{w}(\mathsf{c}) = \mathsf{c}\big|_{I_1} \dots \mathsf{c}\big|_{I_k}$ is said to be **c-sortable** if $I_1 \supseteq I_2 \supseteq \dots \supseteq I_k$. The set of c-sortable elements of $W^{(m)}$ is denoted by $\mathrm{Sort}^{(m)}(W, \mathsf{c})$.

The (sortable version of the) **m-Cambrian lattice** $\mathrm{Camb}^{(m)}_{\mathrm{Sort}}(W, \mathsf{c})$ is the restriction of $\mathrm{Weak}^{(m)}(W)$ to $\mathrm{Sort}^{(m)}(W, \mathsf{c})$.

Remark 9.2.2. The greedy algorithm described in Remark 3.4.5 to compute the c-sorting word of an element $w \in W$ extends to the entire Artin monoid in the exact same way: starting with $\mathsf{u} = \varepsilon$ and $\boldsymbol{v} = \boldsymbol{w}$, one reads c^∞ letter by letter and replaces (u, v) by $(\mathsf{u}\mathsf{s}, \boldsymbol{s}^{-1}\boldsymbol{v})$ whenever $\boldsymbol{s}$ is initial in $\boldsymbol{v}$.

Theorem 9.2.3 ([STW18, Theorems 6.6.4])**.** *The m-Cambrian lattice* $\mathrm{Camb}^{(m)}_{Sort}(W, \mathsf{c})$ *is a sublattice of the m-eralized weak order* $\mathrm{Weak}^{(m)}(W)$.

Unlike the Cambrian lattices, the m-Cambrian lattice is however *not* a lattice quotient of $\mathrm{Weak}^{(m)}(W)$. However, and contrary to $\mathrm{Weak}^{(m)}(W)$, the order is characterized by inclusion of (colored) inversion sets and meets are easy to compute as they correspond to intersections of inversion sets.

Theorem 9.2.4 ([STW18, Theorem 6.6.2])**.** *For* $\boldsymbol{u}, \boldsymbol{w} \in \mathrm{Sort}^{(m)}(W, \mathsf{c})$, $\mathrm{inv}(\boldsymbol{u} \wedge \boldsymbol{w}) = \mathrm{inv}(\boldsymbol{u}) \cap \mathrm{inv}(\boldsymbol{w})$. *Moreover, we have $\boldsymbol{u} \le \boldsymbol{w}$ if and only if* $\mathrm{inv}(\boldsymbol{u}) \subseteq \mathrm{inv}(\boldsymbol{w})$.

As mentioned earlier, Garside factorization can be used to determine which elements are in $W^{(m)}$. Though we will not use it extensively in this work, it is worth noting that c-sortability can also be characterized with the Garside factorization. For this, we ask that each factor is sortable with the relevant Coxeter word of the relevant parabolic subgroup. Indeed, if $\boldsymbol{w} = \boldsymbol{w}^{(1)} \cdot \dots \cdot \boldsymbol{w}^{(k)}$ is the Garside factorization of $\boldsymbol{w}$, then each factor can be seen as an element of W, and we have $w_{i+1} \in W_{\mathrm{des}_R(w_i)}$ by Proposition 9.1.8. The new relevant Coxeter word then lives in $W_{\mathrm{des}_R(w_i)}$. Each generator of this parabolic subgroup is a right descent s_k of w_i and corresponds to a covered reflection $w_i s (w_i)^{-1}$ of w_i, as defined in Definition 3.1.7. The choice of a Coxeter word induces a total order on the set of reflections that we called the *reflection order* $\le_{\mathsf{c}}$. The relevant ordering on $\mathrm{des}_R(w_i)$ is then chosen as the one arising from the reflection ordering of covered reflections of w_i.

Definition 9.2.5. Let c be a Coxeter word and $w \in W$. Let $\mathrm{des}_R(w) = \{s_{i_1}, s_{i_2}, \dots, s_{i_k}\}$ be ordered so that $w s_{i_1} w^{-1} <_{\mathsf{c}} w s_{i_2} w^{-1} <_{\mathsf{c}} \dots <_{\mathsf{c}} w s_{i_k} w^{-1}$.

The **twisted restriction** of c with respect to w is the parabolic Coxeter word $\mathsf{c}\big|^{w} = \mathsf{s}_{i_1}\mathsf{s}_{i_2}\dots\mathsf{s}_{i_k}$.

Let $\boldsymbol{w} = w^{(1)} \cdot \dots \cdot w^{(k)}$ be the Garside factorization of $\boldsymbol{w} \in \boldsymbol{B}^+$. Then $\boldsymbol{w}$ is **factorwise c-sortable** if $w^{(i)}$ is $\mathsf{c}^{(i)}$-sortable for all $1 \le i \le k$, where we set $\mathsf{c}^{(1)} = \mathsf{c}$ and $\mathsf{c}^{(i+1)} = \mathsf{c}^{(i)}\big|^{w^{(i)}}$ for $1 \le i < k$. The set of factorwise c-sortable elements is denoted $\mathrm{Sort}^{(m)}_{\mathrm{fact}}(W, \mathsf{c})$.

Proposition 9.2.6 ([STW18, Corollary 6.4.4 and Proposition 6.4.6])**.** *The sets* $\mathrm{Sort}^{(m)}(W, \mathsf{c})$ *and* $\mathrm{Sort}^{(m)}_{fact}(W, \mathsf{c})$ *coincide. Moreover, if $w^{(1)} \cdot \dots \cdot w^{(k)}$ is the Garside factorization of $\boldsymbol{w} \in \boldsymbol{B}^+$, then the c-sorted word of $\boldsymbol{w}$ is commutation equivalent to the concatenation of $\mathsf{c}^{(i)}$-sorted words of its Garside factors, namely* $\mathsf{w}(\mathsf{c}) \equiv \big[\mathsf{w}^{(1)}(\mathsf{c}^{(1)})\big] \cdot \dots \cdot \big[\mathsf{w}^{(k)}(\mathsf{c}^{(k)})\big]$.

Finally, one can define *skip sets* for any sortable element, exactly as they were defined in Definition 3.4.29, in order to translate to the forthcoming definition of the m-Cambrian lattice on m-eralized noncrossing partitions in Section 9.2.3.

Definition 9.2.7. Let $\boldsymbol{w} \in \mathrm{Sort}^{(m)}_{\mathrm{fact}}(W, \mathsf{c})$ for some $m \ge 1$ and such that $\mathsf{w}(\mathsf{c}) = \mathsf{c}\big|_{I_1} \dots \mathsf{c}\big|_{I_k}$.

For each simple reflection s, let j be the first index such that $\mathsf{s} \notin I_j$, i.e. the index of the first skipped s of $\boldsymbol{w}$. Let $\beta_s^{(m_s)} = \mathsf{s}_1 \dots \mathsf{s}_i(\alpha_{\mathsf{s}}^{(0)})$, where $\mathsf{s}_1 \dots \mathsf{s}_i$ is the prefix of $\mathsf{w}(\mathsf{c})$ of all letters of c^∞ appearing before this first skipped s.

The **skip set** $\mathcal{C}_c(\boldsymbol{w})$ of $\boldsymbol{w}$ is defined as $\{\beta_s^{(m_s)} \mid s \in \mathcal{S}\}$.

9.2.2 m-Cambrian lattices as subword complexes

We used subword complexes to define the Cambrian lattices in Section 3.4.2. As we saw in Section 9.1, subword complexes can be generalized to the context of the Artin monoid, and all generalizations coincide in the case of Coxeter initial subword complexes. Viewing the Cambrian lattice as the restriction of the weak order to the c-sortable elements of the interval $[\boldsymbol{e}, \boldsymbol{w}_\circ]$, a first definition of the m-Cambrian lattice was given in Section 9.2.1, by replacing $\boldsymbol{w}_\circ$ by $\boldsymbol{w}_\circ^m$. We now present a second definition, again by replacing $w_\circ$ by ${w_\circ}^m$ in the definition as a subword complex. The subword complexes that we consider are c-initial, as were already those involved in the definition of the Cambrian lattices (see Definition 3.4.19).

This general definition is also due to C. Stump, H. Thomas, and N. Williams [STW18], but a first occurrence of this generalization in the context of cluster complexes appeared in the work of S. Fomin and N. Reading in [FR05] for so-called bipartite Coxeter element.

Definition 9.2.8. Let c be a Coxeter word. The **m-eralized c-cluster subword complex** is the c-initial subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{c}{\mathsf{w}_\circ}^m(\mathsf{c}), {w_\circ}^m, m\,\ell_{\mathcal{S}}(w_\circ))$, which thus coincides with the Artin subword complex $\mathrm{SC}^{\boldsymbol{B}}_{\boldsymbol{\mathcal{S}}}(\mathsf{c}{\mathsf{w}_\circ}^m(\mathsf{c}), \boldsymbol{w}_\circ^m)$.

The (cluster version of the) **m-Cambrian lattice** $\mathrm{Camb}^{(m)}_{\mathrm{Clus}}(W, \mathsf{c})$ is the flip poset of the m-eralized c-cluster subword complex.

Remark 9.2.9. Because of Proposition 3.4.21, the m-eralized c-cluster subword complex coincides with the dual subword complex $\mathrm{SC}_{\mathcal{R}}(\mathsf{inv}_{\mathcal{R}}(\mathsf{c}{\mathsf{w}_\circ}^m(\mathsf{c})), c^{-1})$.

Recall that for any subword complex, we defined in Definition 3.4.15 the **root configuration** of its facets as the set of colored roots attached to elements of the facets. If $\mathsf{Q} = \mathsf{s}_1 \dots \mathsf{s}_p$ is the search word and I is a facet, the root attached to $k \in I$ corresponds to α_{s_k} conjugated by the prefix of the complement of I before s_k.

These simplicial complexes can be seen as (generalized) cluster complexes, and can be defined as some m-eralized c-compatibility relations on the set of *m-colored almost positive roots* $(\Phi^+)^m \cup \Delta$. In the type A case, in particular, each m-colored almost positive root can be identified with some relevant diagonal in an $(mn+2)$-gon, and the compatibility relation corresponds to the noncrossing condition. Each facet then corresponds to an $(m+2)$-angulation of this $(mn+2)$-gon, as we shall detail in Section 9.2.3 for the linear Coxeter element.

9.2.3 m-Cambrian lattices as chains on noncrossing partitions

The third definition of the m-Cambrian lattice is based on noncrossing partitions. This time, the m-eralization does not consist of replacing $w_\circ$ with ${w_\circ}^m$. Instead, they will be defined on m-tuples of noncrossing partitions, in a similar vein to factorwise c-sortable elements.

More precisely, we can define m-eralized noncrossing partitions in several manners, namely as m-chains in the noncrossing partition lattice, as m-delta sequences, and even as m-divisible noncrossing partitions in the type A case. The last definition is due to P.H. Edelman in [Ede80] and was historically the first studied generalization of noncrossing partitions. The generalization to any finite Coxeter group is due to D. Armstrong [Arm09].

Definition 9.2.10. Let c be a Coxeter element in W.

The **m-eralized noncrossing partitions** are m-multichains in the noncrossing partition lattice $\mathrm{NCL}(W, c)$, i.e. sequences $w_1 \leq_{\mathcal{R}} w_2 \leq_{\mathcal{R}} \cdots \leq_{\mathcal{R}} w_m \leq_{\mathcal{R}} c$. The set of m-eralized noncrossing partitions is denoted $\mathrm{NC}^{(m)}(W, c)$. The **support** of an m-eralized noncrossing partition is defined as $\mathrm{supp}(w_m)$.

The **m-delta sequences** are factorizations of c as a product of $m+1$ c-noncrossing partitions, i.e. $(m+1)$-tuples of c-noncrossing partitions $(\delta_0, \delta_1, \dots, \delta_m)$ such that $\delta_0\delta_1 \dots \delta_m = c$ and $\sum_{i=0}^m \ell_{\mathcal{R}}(\delta_i) = \ell_{\mathcal{R}}(c)$. The set of m-delta sequences is denoted $\mathrm{NC}^{(m)}_{\delta}(W, c)$.

Proposition 9.2.11 ([Arm09, Lemma 3.2.4]). *The sets* $\mathrm{NC}^{(m)}(W,c)$ *and* $\mathrm{NC}_\delta^{(m)}(W,c)$ *are in bijection via:*

$$\begin{aligned} \mathrm{NC}^{(m)}(W,c) &\longleftrightarrow \mathrm{NC}_\delta^{(m)}(W,c) \\ (w_1 \leq_{\mathcal{R}} \cdots \leq_{\mathcal{R}} w_m) &\longmapsto (cw_m^{-1}, w_m w_{m-1}^{-1}, \ldots, w_2 w_1^{-1}, w_1) \\ (\delta_m \leq_{\mathcal{R}} \delta_{m-1}\delta_m \leq_{\mathcal{R}} \cdots \leq_{\mathcal{R}} \delta_1 \ldots \delta_m) &\longleftarrow (\delta_0, \delta_1, \ldots, \delta_m). \end{aligned}$$

Remark 9.2.12. Recall that the choice of a Coxeter word c is equivalent to a total ordering of reflections $\leq_{\mathsf{c}}$ in W, and that thanks to Proposition 3.4.25, each element of $\mathrm{NC}(W,c)$ has a unique reduced $\mathcal{R}$-word $\mathsf{w}_{\mathcal{R}}(\mathsf{c}) = \mathsf{r}_1 \ldots \mathsf{r}_p$ that is c-increasing.

Then, by definition, an m-delta sequence corresponds bijectively to a facet of the dual subword complex $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^{m+1}, c)$. Indeed, by numerating from 0 to m the $m+1$ copies of $\mathsf{R}(\mathsf{c})$ in $\mathsf{R}(\mathsf{c})^{m+1}$, one can identify δ_i with its unique c-increasing word in the i-th copy of $\mathsf{R}(\mathsf{c})$.

One could define an m-eralized noncrossing partition poset, but this is not the partial order that we focus on in this work. Instead, exactly as in Definition 3.4.28 for the Cambrian lattice, we can define rotations of facets of this dual subword complex, inspired by the modifications of the root configuration of a facet under a flip in the c-cluster subword complex.

Definition 9.2.13. Let c be a Coxeter word for a Coxeter element c and $I = \mathsf{r}_1^{(m_1)} \ldots \mathsf{r}_p^{(m_p)}$ be a facet of $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^{m+1}, c)$.

Let $1 \leq i \leq p$ be such that $m_i < m$. Set $r'_i = r_i$ and $m'_i = m_i + 1$. For j such that $r_i^{(m_i)} < r_j^{(m_j)} < r_i^{(m_i+1)}$, set $r'_j = r_i r_j r_i = |r_i(\alpha_{r_j})|$, where the last equality between a reflection and a root is taken via the canonical bijection. If $r_i(\alpha_{r_j}) \in \Phi^+$, set $m'_j = m_j$ and otherwise, if $r_i(\alpha_{r_j}) \in \Phi^-$, set $m'_j = m_j - 1$. For all other j, set $r'_j = r_j$ and $m'_j = m_j$.

Then the **rotation** of I at index i is the facet I' whose reflection sequence is the sorted tuple of $((\mathsf{r}'_1)^{(m'_1)}, \ldots, (\mathsf{r}'_n)^{(m'_n)})$.

The (noncrossing version of the) ***m*-Cambrian lattice** $\mathrm{Camb}_{\mathrm{NC}}^{(m)}(W,\mathsf{c})$ is the transitive closure of rotations of facets of $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^{m+1}, c)$.

In other words, a rotation consists in sending a reflection r_i which is not in the last copy of $\mathsf{R}(\mathsf{c})$ into the next copy of $\mathsf{R}(\mathsf{c})$, conjugating all reflections in between, and not affecting the others. As in the Cambrian case, the skip set of a c-sortable element and the root configuration of a facet are identifying the three versions of the m-Cambrian lattices.

Theorem 9.2.14 ([STW18, Theorems 5.7.3 and 6.8.6]). *Let* c *be a Coxeter word in* W*.*

1. $\mathrm{Camb}_{Clus}^{(m)}(W,\mathsf{c})$ *and* $\mathrm{Camb}_{NC}^{(m)}(W,\mathsf{c})$ *are isomorphic by identifying the root configuration of a facet of the m-eralized* c*-cluster with a facet of* $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^{m+1}, c)$*.*

2. $\mathrm{Camb}_{Sort}^{(m)}(W,\mathsf{c})$ *and* $\mathrm{Camb}_{NC}^{(m)}(W,\mathsf{c})$ *are isomorphic by identifying the skip set of a* c*-sortable element with a facet of* $\mathrm{SC}_{\mathcal{R}}(\mathsf{R}(\mathsf{c})^{m+1}, c)$*.*

Remark 9.2.15. All definitions of the m-Cambrian lattices were given for a choice of a Coxeter word c. However, they depend only on the Coxeter element c since the constructions for two commuting Coxeter words are easily proven to be isomorphic.

The definition of the m-Cambrian lattice on m-noncrossing partitions reveals that $\mathrm{Camb}_{\mathrm{NC}}^{(m)}(W,c)$ and $\mathrm{Camb}_{\mathrm{NC}}^{(m)}(W,\psi(c))$ are in fact dual posets thanks to Lemma 3.4.14(4,6). Indeed, by reading backwards the word $\mathsf{R}(\mathsf{c})$, one obtains the inversion sequence of $w_\circ(\psi(\mathsf{c}))$, and we can identify m-delta sequences for c with m-delta sequences for $\psi(\mathsf{c})$ by reversing the order of the factors and reading them backwards. Then, increasing rotations are naturally translated to decreasing rotations under this identification.

9.3 Linear type A intervals

In our study, we are particularly interested in the linear type A case, that is to say for the symmetric group with Coxeter element the long cycle $c = (1, 2, \dots, n) = s_1 s_2 \dots s_{n-1}$. As we have seen in Example 3.4.8, this case corresponds to the Tamari lattice for $m = 1$.

A nice combinatorial description of the linear type A m-Cambrian lattice as a poset on $(m+2)$-angulations of an $(mn+2)$-gon was made in [Fre16]. A total ordering on the diagonals can be given, and corresponds in fact more or less to the lexicographic order for a suitable numbering of the vertices of the $(mn+2)$-gon. Slightly more precisely, once chosen a vertex numbered 1, the vertex labelled 2 will be the one located m counterclockwise steps away, and so on and so forth. For an even value of m, a small accommodation must be done for all vertices to get a number. This corresponds to the natural numbering of vertices for $m = 1$.

Then, an $(m + 2)$-angulation of the $(mn + 2)$-gon is a maximal collection of noncrossing diagonals that cut the $(mn+2)$-gon into $(m+2)$-gons. A flip consists in removing some diagonal and replacing it with some other diagonal of this $(2m + 2)$-gon and is increasing if the new diagonal is strictly greater in the total order on diagonals. The resulting flip poset on the set of $(mn + 2)$-angulations is isomorphic to the linear type A m-Cambrian lattice. In fact, all of this can be quite straightforwardly translated to the description of the m-Cambrian lattice as a subword complex in Section 9.2.2. The compatibility corresponds to the noncrossing condition on diagonals. This is illustrated in Figure 9.1. One can notice that the linear type A 2-Cambrian lattice of size 3 is not isomorphic to the 2-Tamari lattice of size 3 showed in Section 8.2.

Another viewpoint on this is to use pseudoline arrangements on sorting networks (see [PP12, PS15]), which can be generalized by taking a specific sorting network and requiring that every pair of line crosses exactly m times. Both descriptions are in fact quite directly equivalent to the subword complex definition.

One of the reasons why we are interested in these objects, and the starting point of this work is a conjecture in [STW18] according to which, even though the linear type A m-Cambrian lattice and the m-Tamari lattice are not isomorphic, they seem to share the same number of intervals. They do not however share the number of 3-multichains.

Conjecture 9.3.1 ([STW18, Conjecture 6.10.2])**.** The m-Tamari lattice and the linear type A m-Cambrian lattice have the same number of intervals.

Moreover, if each Tamari interval $[P, Q]$ is labelled with a parking function attached to the top element Q, and each m-Cambrian interval whose top element is some $w_1 \leq_{\mathcal{R}} \cdots \leq_{\mathcal{R}} w_m$ is labelled with a coset of the parabolic subgroup $W_{\mathrm{Fix}(w_1)}$, then the number of labelled intervals is the same on each side.

The authors also state a refined version of this conjecture that would imply both, since the number of parking functions and of cosets of the corresponding parabolic subgroups depend only on the integer partition that is defined from the interval.

Conjecture 9.3.2 ([STW18, Conjecture 6.10.3])**.** For any $m, n \geq 1$, and $\lambda \vdash n$, there are as many intervals $[P, Q]$ in the m-Tamari lattice such that $\lambda^{\mathsf{top}}(Q) = \lambda$ as intervals $[I, J]$ in the linear type A m-Cambrian lattice such that the last component of the delta sequence J has cycle type λ.

Remark 9.3.3. For $n = 3$ and $m = 2$, the 2-Tamari lattice $\mathrm{Tam}_3^{(2)}$ has 12 elements, 58 intervals and 189 3-multichains, whereas the linear type A 2-Cambrian lattice $\mathrm{Camb}^{(2)}(\mathfrak{S}_3, c^{\mathsf{lin}})$ has 12 elements, 58 intervals but 188 3-multichains.

Note also that for $m = 2$ and $n = 4$, there are two different 2-Cambrian lattices of type A (up to isomorphism). The one corresponding to the linear Coxeter element has 703 intervals whereas the other has 714 intervals.

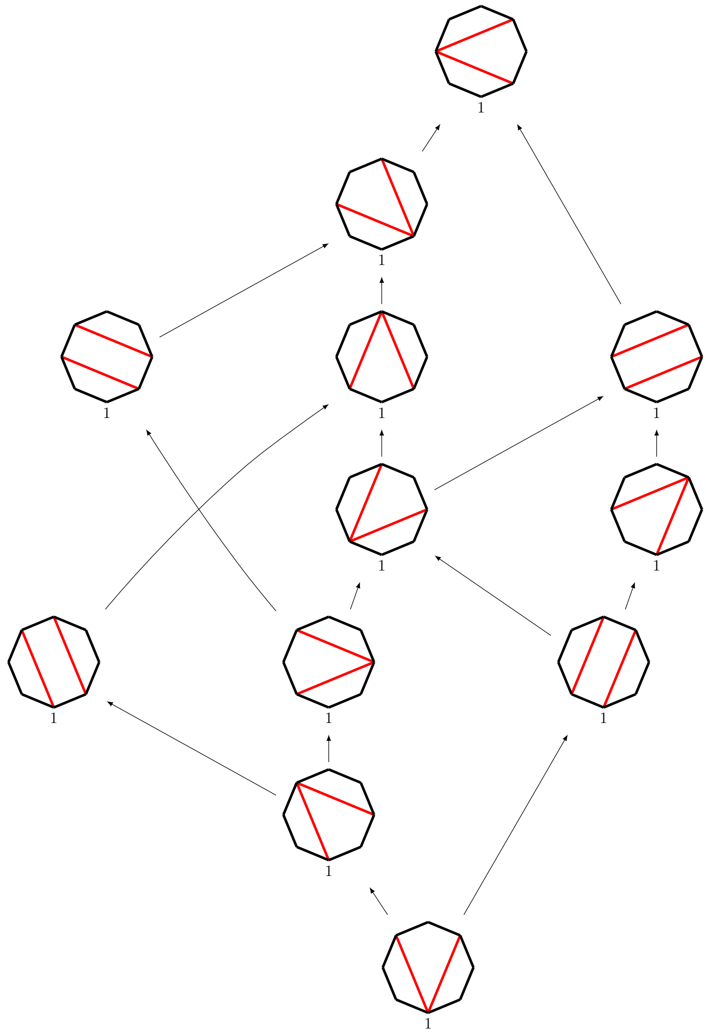


Figure 9.1: The 2-Cambrian lattice on $\mathfrak{S}_3$ for the linear Coxeter element $\mathrm{Camb}^{(2)}(\mathfrak{S}_3, (1,2,3))$.

In this section, we will define more precisely these different statistics on the linear type A m-Cambrian lattices, and we will state some refined conjectures that would imply the conjectures above. For each functional equation on the m-Tamari intervals that appears in Chapter 8, we will provide an adequate conjecture. The idea of decorating the intervals with more and more statistics is to try to find sets of statistics that are equidistributed and that would constrain a conjectural bijection between the intervals of these two lattices, since we will want them to preserve these statistics. The functional equation approach is also another way to attack the conjecture, as they possess only one solution. Finding a decomposition of m-Cambrian intervals proving that their generating function satisfies the same functional equation would also imply the conjecture.

A first set of statistics that we define on m-Cambrian intervals are initial and final diagonals that we conjecture to be equidistributed with contacts and initial rise, as in the original enumeration of m-Tamari intervals in [BMFPR11]. The names come from the description on $(m+2)$-angulations, where the minimal element of the poset has all diagonals touching the smallest vertex, and the maximal element has all diagonals touching the largest vertex.

Definition 9.3.4. Let $I = [J_0, J_1]$ be an interval in the linear type A m-Cambrian lattice $\mathrm{Camb}^{(m)}_{\mathrm{Clus}}(W, c)$. The number of **initial diagonals** of the interval I is equal to $J_0 \cap \widehat{0}$ and the number of **final diagonals** of I is equal to $J_1 \cap \widehat{1}$, where $\widehat{0}$ and $\widehat{1}$ are the minimal and maximal facets of the subword complex, respectively.

Remark 9.3.5. When represented on $(m+2)$-angulations, the element $\widehat{0}$ has all diagonals starting at the same "initial" vertex and the element $\widehat{1}$ has all diagonals starting at the same "final" vertex, hence the name of the statistics.

When represented on m-eralized noncrossing partitions, these statistics read very well. If $(\delta_0, \ldots, \delta_m)$ is an m-delta sequence, the number of initial diagonals is equal to the size of the block containing 1 in δ_0 minus one, and the number of final diagonals is equal to the size of the block containing n in δ_m minus one.

The next statistics that we are interested in are the bottom-out and top-out degrees, as well as the height of an interval. The out-degrees can also read well on delta sequences, where each element δ_i is seen as a noncrossing partition in $\mathrm{NC}(W, c)$. The height can be computed with a greedy algorithm as well, as we will see in Section 9.4.

Proposition 9.3.6. *If $I = [(\delta_i)_i, (\delta'_i)_i]$ is an interval in $\mathrm{Camb}^{(m)}_{NC}(W, c)$, the* **bottom-out degree** *of I as defined in Definition 8.2.4 is equal to the number of blocks of δ_0 minus one. Similarly, the* **top-out degree** *of I is equal to the number of blocks of δ_m minus one.*

Lastly, we can define a top and a bottom partition, marked at the parts that correspond to the initial and final diagonals.

Definition 9.3.7. Let $I = [(\delta_i)_i, (\delta'_i)_i]$ be an interval in $\mathrm{Camb}^{(m)}_{\mathrm{NC}}(W, c)$. The **bottom partition** $\widetilde{\lambda}^{\mathsf{bot}}(I)$ of I is defined as the integer partition of sizes of the blocks of δ_0 not containing 1. The **top partition** $\widetilde{\lambda}^{\mathsf{top}}(I)$ of I is defined as the integer partition of sizes of the blocks of δ'_m not containing n.

Remark 9.3.8. As for the m-Tamari lattice, the number of blocks of the bottom partition is equal to the bottom-out degree, and the number of blocks of the top partition is equal to the top-out degree. Moreover, the size of the "missing block" of the bottom partition is equal to the number of initial diagonals plus one, and the size of the "missing block" of the top partition is equal to the number of final diagonals plus one.

Remark also that thanks to Remark 9.2.15, the linear type A m-Cambrian lattice is self-dual since $\psi(\mathsf{c})$ is also a linear Coxeter element in W. Under this poset isomorphism, that we denote by slight abuse ψ, the bottom and top partitions are exchanged.

We can then state conjectures on equidistribution of these statistics between the m-Tamari lattice and the linear type A m-Cambrian lattice.

Conjecture 9.3.9. For $m, n \geq 1$, and $a, b \geq 0$, there are as many intervals I:

- in $\mathrm{Tam}^{(m)}_n$ with a non-initial contacts and b as initial rise,
- and in $\mathrm{Camb}^{(m)}(\mathfrak{S}_n, c^{\mathsf{lin}})$ with $a+1$ initial diagonals and $b+1$ final diagonals.

In particular the generating function of linear type A m-Cambrian intervals is solution to (8.8).

Conjecture 9.3.10. For $m, n \geq 1$, and $a, b, c, d, e \geq 0$, there are as many intervals I:

- in $\mathrm{Tam}^{(m)}_n$ with a non-initial contacts, b as initial rise, bottom-out degree c, top-out degree d, and height e,
- and in $\mathrm{Camb}^{(m)}(\mathfrak{S}_n, c^{\mathsf{lin}})$ with $a+1$ initial diagonals, $b+1$ final diagonals, bottom-out degree c, top-out degree d, and height e.

In particular the generating function of linear type A m-Cambrian intervals is solution to (8.12).

Conjecture 9.3.11. For $m, n \geq 1$, and $e \geq 0$, λ_0, and λ_1 two integer partitions, there are as many intervals I:

- in $\mathrm{Tam}_n^{(m)}$ of height e, with $\widetilde{\lambda}^{\mathsf{bot}}(I) = \lambda_0$ and $\widetilde{\lambda}^{\mathsf{top}}(I) = \lambda_1$,
- and in $\mathrm{Camb}^{(m)}(\mathfrak{S}_n, c^{\mathsf{lin}})$ of height e, with $\widetilde{\lambda}^{\mathsf{bot}}(I) = \lambda_0$ and $\widetilde{\lambda}^{\mathsf{top}}(I) = \lambda_1$.

In particular the generating function of linear type A m-Cambrian intervals is solution to (8.13).

Lastly, the m-Tamari interval-posets defined in Definition 8.3.9 possess the additional symmetry given by the complement involution, that is not so obvious in the m-Tamari intervals, but very similar to the one on type A m-Cambrian lattice.

One can expect in fact a bijection between m-Tamari interval-posets and linear type A m-Cambrian intervals that would preserve all these statistics and the two involutions, and is the most refined conjecture of this nature presented in this work.

Conjecture 9.3.12. There is a bijection ξ between m-Tamari interval-posets and intervals in $\mathrm{Camb}^{(m)}(W, c^{\mathsf{lin}})$ such that:

$$\begin{aligned}
\widetilde{\lambda}^{\mathsf{top}}_{(m)}(I) &= \widetilde{\lambda}^{\mathsf{top}}(\xi(I)), \\
\widetilde{\lambda}^{\mathsf{bot}}_{(m)}(I) &= \widetilde{\lambda}^{\mathsf{bot}}(\xi(I)), \\
k(I) &= mh(\xi(I)) + \frac{nm(m-1)}{2}, \\
\psi(I) &= \psi(\xi(I)).
\end{aligned}$$

Lastly, we can also observe a ternary symmetry when considering the bottom-out, top-out and one of the top-in or bottom-in degrees, as in the m-Tamari lattice.

Conjecture 9.3.13. Let $G^{(m)}(t; u, v, \bar{u}, \bar{v})$ be the generating function of linear type A m-Cambrian intervals where $u, v, \bar{u}, \bar{v}$ keep track respectively of the bottom-out, top-out, bottom-in and top-in degrees.

The specialization $G^{(m)}(t; u, v, \bar{u}, 1)$ is totally symmetric in $u, v, \bar{u}$, and similarly for $G^{(m)}(t; u, v, 1, \bar{v})$.

Note that the self-duality makes the two statements equivalent.

Remark 9.3.14.

- These conjectures have been checked for at least $n + m \leq 7$.
- The bijection conjectured in Conjecture 9.3.11 for $m = 1$ can *not* be the identity since the computation of both partitions is different in the two versions of the Tamari lattice, so there is already a nontrivial bijection to find between Tamari intervals.
- Such a bijection between m-Tamari and (linear type A) m-Cambrian intervals for $m \geq 2$ can not be of the form $(P, Q) \mapsto (f(P), g(P, Q))$ or $(f(P, Q), g(Q))$ since the distribution of the sizes of principal ideals and filters are different in the two lattices. Thus, it is necessarily of the form $(f(P, Q), g(P, Q))$.
- We could have considered other natural statistics, for instance the Möbius invariant of the intervals. However, this invariant does not refine all conjectures. In particular, it is not compatible with the height parameter. Nevertheless, it seems to be compatible with the 4 other parameters of Conjecture 9.3.10, as tested for $m = 2, n \leq 6$ and $m = 3, n \leq 5$.

9.4 A new definition

We have described the m-Cambrian lattice on different sets of objects. However, these definitions have the drawback that either the objects are complicated to describe, or it is difficult to decide whether two elements are comparable.

For instance, in the definition on sortable elements, if we represent objects with their c-sorting words, then it is easy to tell which elements are c-sortable, but not straightforward to tell whether they are comparable. If however we represent them as their sets of colored inversions, then the comparison criterion is nothing but the inclusion, but it is hard to tell whether such a set of colored roots is the inversion set of a c-sortable element.

In the definition with subword complexes, it is easy to describe the facets of the subword complex, but only covering relations are immediately readable from there. Similarly, for the definition on m-noncrossing partitions, whether with m-chains or m-delta sequences, objects are easy to describe but the comparison is not immediate either, as only covering relations are easy to understand.

In this section, we present a new definition of the m-Cambrian lattice, which however is still conjectural and is contingent on the yet unproven Assumption 1. The idea of this description was shaped by an idea introduced by Corentin Henriet for linear type A, and led to discussions with Wenjie Fang and Corentin Henriet to bring forward a conjectural definition in the general Coxeter framework and exploration of the linear type A. This new definition consists in giving a comparison criterion on m-chains of noncrossing partitions. It relies on a greedy algorithm to compute an *increasing chain* between two elements if comparable, which also produces a chain of maximal length, if any. This seems to be the more convenient definition of these lattices since both the objects and comparison relations are easy to describe.

Experimentally, one could expect a similar definition on m-chains in the so-called shard intersection order. It was indeed proven in [STW18, Theorem 6.9.4] that the set of m-chains in the shard intersection order contains all m-eralized noncrossing partition lattices, and in [STW18, Theorem 6.9.5] that the m-Cambrian definition on noncrossing partitions could also be transported to this wider setting.

9.4.1 The greedy algorithm

Recall from Definition 3.4.16 that the **direction** of a flip between two adjacent facets I and J is the root $|r_I(i)|$ attached to the vertex $i \in I$ that is not present in J. This time, we will attach a colored root to each covering relation $I \lessdot J$ as the colored root $r_I(i)$ where $I \setminus J = \{i\}$. Then we can define increasing chains by requiring that the sequence of colored roots attached to the flips is increasing in the reflection order.

Definition 9.4.1. Let $I \lessdot J$ be a covering relation in the flip poset of some subword complex. Let $i \in I$ be the unique element of I that is not in J. The **flip root** $r(I, J)$ attached to this increasing flip $I \lessdot J$ is the colored root $r_I(i)$.

A saturated chain $I_0 \lessdot I_1 \lessdot \ldots \lessdot I_k$ in a c-initial subword complex is **c-increasing** if $r(I_0, I_1) <_{\mathsf{c}} r(I_1, I_2) <_{\mathsf{c}} \cdots <_{\mathsf{c}} r(I_{k-1}, I_k)$ in the reflection order associated to c (extended as the lexicographic order on colored roots).

Proposition 9.4.2. *There exists at most one c-increasing (saturated) chain between two facets I and J of* $\mathrm{Camb}^{(m)}_{Clus}(W, \mathsf{c})$.

We need a few useful lemmas before proving this proposition, relying in particular on Proposition 3.4.18. For each colored root r, we attach to a facet I a vector space $V_r(I)$ spanned by all roots smaller than r. We study how this vector space is modified under some rotations to prove the uniqueness of the increasing chain. Recall that acting by a reflection r on a root β produces a root $\beta + a\alpha_r$ for some $a \in \mathbb{R}$.

Lemma 9.4.3. *Let I be a facet of $\mathrm{Camb}^{(m)}_{Clus}(W, \mathsf{c})$. If its root configuration is $\mathrm{R}(I) = \{r_i^{(m_i)}\}$, then the set $\{r_i\} \subset \Phi^+$ is a basis of the ambient space $\mathrm{Vect}\,(\Phi)$.*

Proof. Let $\mathsf{c} = \mathsf{s}_1 \dots \mathsf{s}_n$. Using that $\mathrm{Camb}^{(m)}_{\mathrm{Clus}}(W, \mathsf{c})$ possesses a minimal element $\widehat{0}$, and that every element is reachable from it by a sequence of flips, we can prove this lemma by induction.

First, as $\mathrm{Camb}^{(m)}_{\mathrm{Clus}}(W, \mathsf{c})$ is a c-initial subword complex, it possesses a facet $\widehat{0} = [n]$ consisting of the first n positions of the search word. Its root configuration is then a free family. Indeed, by induction on $i \leq n$, the i-th root $r_{\widehat{0}}(i)$ of the configuration is $(s_1 \dots s_{i-1}) \cdot \alpha_{s_i} \in \alpha_{s_i} + \mathrm{Vect}\left(\alpha_{s_1}, \dots, \alpha_{s_{i-1}}\right)$ and thus $r_{\widehat{0}}(i) \notin \mathrm{Vect}\left((r_{\widehat{0}}(j))_{j<i}\right) = \mathrm{Vect}\left(\alpha_{s_1}, \dots, \alpha_{s_{i-1}}\right)$.

Now suppose that I is a facet such that the set of (colorless) roots in $\mathrm{R}(I)$ is a basis of $\mathrm{Vect}\,(\Phi)$. We want to prove that for any facet J covering I in the flip graph, then the set of (colorless) roots in $\mathrm{R}(J)$ is also a basis of $\mathrm{Vect}\,(\Phi)$.

Let J be the upper cover obtained from I by flipping the vertex i, changing it into j. We have $|r_I(i)| = |r_J(j)|$. Moreover, for all other $k \in I$, $r_J(k) \in r_I(k) + \mathrm{Vect}\,(|r_I(i)|)$ by Proposition 3.4.18, and thus the family $\{|r_J(k)|\}$ is still a basis of $\mathrm{Vect}\,(\Phi)$, which achieves the proof. □

The following remark on how the root configuration changes under a flip is exactly the reason why the Cambrian rotations on noncrossing partitions and m-eralized noncrossing partitions are defined as they are (see Definitions 3.4.28 and 9.2.13). It is derived from Proposition 3.4.18 and the fact that for a c-initial subword complex, no two roots of the root configuration can correspond to the same positive root since they form a basis.

Remark 9.4.4. If $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ is a c-initial subword complex, then a covering relation $I \lessdot J$ in the flip poset changes the root configuration by

1. changing a root $r_i^{(m_i)}$ into $r_i^{(m_i+1)}$,
2. transforming every root $r_j^{(m_j)}$ such that $r_i^{(m_i)} <_{\mathsf{c}} r_j^{(m_j)} <_{\mathsf{c}} r_i^{(m_i+1)}$ by conjugating it by r_i and fixing the new color so that the result remains between $r_i^{(m_i)}$ and $r_i^{(m_i+1)}$,
3. not changing any other root of the root configuration.

This remark allows us to prove the following lemma immediately.

Lemma 9.4.5. *Let $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ be a c-initial subword complex and $r \in \Phi^{(\infty)}$ a colored root.*

Let I be a facet, and $V_r(I) = \mathrm{Vect}\,(r_I(j) \mid j \in I, r_I(j) \leq_{\mathsf{c}} r)$ be the vector space spanned by the roots of the root configuration of I that are smaller than r.

For J is an upper cover of I, if $r(I, J) = r$ then $V_r(J) \subsetneq V_r(I)$, and if $r(I, J) >_{\mathsf{c}} r$ then $V_r(J) = V_r(I)$.

The case where $r(I, J) <_{\mathsf{c}} r$ is not tackled in this lemma as it is not needed in the proof and is more complicated.

We can now prove Proposition 9.4.2.

Proof of Proposition 9.4.2. Let I and J be two facets of $\mathrm{Camb}^{(m)}_{\mathrm{Clus}}(W, \mathsf{c})$ such that there is a c-increasing chain $I = I_0 \lessdot I_1 \lessdot \dots \lessdot I_k = J$.

Let r be the smallest root of the root configuration $\mathrm{R}(I)$ of I that is not in $\mathrm{R}(J)$. By studying how the root configuration changes under a covering relation, using Remark 9.4.4, we will prove that the first flip from a c-increasing chain from I to J must be the flip whose flip root is r, by induction on r. Let I' be an upper cover of I in $\mathrm{Camb}^{(m)}_{\mathrm{Clus}}(W, \mathsf{c})$.

Suppose that $r(I, I') = r' <_{\mathsf{c}} r$. We will prove that $I' \not\leq J$ in $\mathrm{Camb}^{(m)}_{\mathrm{Clus}}(W, \mathsf{c})$, and thus that I' can not be the first step of a c-increasing chain from I to J. Let $I'_1 = I' \lessdot \dots \lessdot I'_j$ be any saturated chain starting at I'. Setting $I'_0 = I$, this gives a saturated chain starting with the flip $I \lessdot I'$.

Let r'' be the smallest colored root such that $r'' = r(I'_i, I'_{i+1})$ for some i. Then, using Lemma 9.4.5 with the root $r'' < r$, we have on the one hand $V_{r''}(I) = V_{r''}(J)$, and on the other hand $V_{r''}(I) = V_{r''}(I'_i) \supsetneq V_{r''}(I'_{i+1}) \supseteq V_{r''}(I'_j)$. Hence, $I' \not\leq J$ in $\mathrm{Camb}^{(m)}_{\mathrm{Clus}}(W, \mathsf{c})$.

Now consider a c-increasing chain $I \lessdot I'_1 \lessdot \ldots \lessdot I'_j$. Because of the previous case, we know that $r(I, I'_1) \geq r$.

Suppose that $r(I, I'_1) > r$. Then $r(I'_i, I'_{i+1}) > r$ for all i since the chain is c-increasing, and thus $V_r(I) = V_r(I'_i)$ for all i, using again Lemma 9.4.5. But on the other hand, Lemma 9.4.5 also gives $V_r(I) \supsetneq V_r(I_1) \supseteq V_r(J)$.

Thus, any c-increasing chain from I to J must start with the flip $I \lessdot I'$ where $r(I, I') = r$. Now either $I' = J$ or the smallest root in $\mathrm{R}(I')$ not in $\mathrm{R}(J)$ is strictly greater than r, and we can conclude by induction that there is a unique c-increasing chain from I to J. $\square$

Remark 9.4.6. The proof of Proposition 9.4.2 also gives a greedy algorithm to compute an increasing chain between two elements I and J, if any, this time considering them as delta sequences, or as subwords of $\mathsf{R}(\mathsf{c})^{m+1}$.

Starting with $I_0 = I$, one reads the word $\mathsf{R}(\mathsf{c})^{m+1}$ from left to right. For each letter r,

- if $r \in I_j$ and $r \in J$, or if $r \notin I_j$ and $r \notin J$, then continue to the next letter,
- if $r \in I_j$ and $r \notin J$, then replace I_j by its upper cover I_{j+1} such that $r(I_j, I_{j+1}) = r$,
- if $r \notin I_j$ and $r \in J$, then there is no increasing chain from I to J.

The result is the increasing chain from I to J, if any.

We have proven that if there is a c-increasing chain from I to J, then it is unique. We want to prove that if $I \leq J$ then there exists such a chain. The proof is not complete and relies on the following assumption.

Assumption 1. Let $I_0 \lessdot I_1 \lessdot I_2$ in $\mathrm{Camb}^{(m)}(W, \mathsf{c})$ be such that $r(I_0, I_1) = r >_{\mathsf{c}} r' = r(I_1, I_2)$.
Then $r' \in I_0$ and setting I'_1 the upper cover of I_0 such that $r(I_0, I'_1) = r'$, we have $I'_1 \leq I_2$.

Start of proof. As $r' <_{\mathsf{c}} r$ it is clear that $r' \in I_0$ since it was not affected by the first flip $I_0 \lessdot I_1$. It is flippable since it is not of color m as it was flippable in I_1.

Then, I'_1 is well-defined. We want to prove that $I'_1 \leq I_2$, which in fact implies $I'_1 < I_2$ since I_1 and I_0 can not be simultaneously the result of the lower cover of I_2 in the direction $|r'|$.

Firstly, by Remark 9.4.4, r' is the smallest root of I_0 that is not a root of I_2 since I_0 and I_2 agree before $min(r, r')$. Thus, if there exists a c-increasing chain from I_0 to I_2, it must start with $I_0 \lessdot I'_1$.

Here is a list of ideas to achieve the proof:

- As we proceeded to two rotations in two different root directions from I_0 to I_2, the interval $[I_0, I_2]$ is isomorphic to a Cambrian lattice of rank 2, using a stronger version of the non-leaving face property stated in Assumption 3. Moreover, all rank 2 Cambrian lattices $\mathrm{Camb}(I_2(k), \mathsf{c})$ are the union of a 2-chain that is not c-increasing and of a c-increasing chain of length $k \geq 2$.
- Given two elements I and J, one can use a modified version of the greedy algorithm, that reads the word $\mathsf{R}(\mathsf{c})^{m+1}$ from left to right, and for each letter r, if it is in one facet and not the other, then it proceeds to the rotation in the direction of r in the corresponding facet. Then we must prove that at the end of the algorithm, the two facets have been transformed into their join $I \vee J$.

- A stronger statement may be proven and used to imply this one: if $I \leq J$ and r is the smallest root of $I \setminus J$, then the upper cover of I in the direction of r remains under J. The proof could maybe be performed by inspiration of the proof of Proposition 4.4.1 in [STW18], namely by applying the so-called Cambrian rotation to the facets I and J by conjugating both by their common prefix before r. This process conjugates r to a simple reflection, which is initial for the new Coxeter word. The two facets I and J have been transformed into facets I^* and J^*, such that $I^* \leq J^*$.

$\square$

We capture the last idea in the following assumption, equivalent to the previous one. It is clear that it implies Assumption 1, and the converse holds by several applications of Assumption 1, as explained afterwards in the proof of Theorem 9.4.7.

Assumption 2. Let $[I, J]$ be a nontrivial interval in $\mathrm{Camb}^{(m)}_{\mathrm{NC}}(W, \mathsf{c})$. Let r be the smallest colored root of $I \setminus J$ and I' be the upper cover of I in the direction of r, i.e. such that $r(I, I') = r$. Then $I' \leq J$.

In other words, the first step of the greedy algorithm described in Remark 9.4.6 always remains under J.

Theorem 9.4.7. *Under Assumption 1, if $I \leq J$ in $\mathrm{Camb}^{(m)}(W, \mathsf{c})$, then there exists a c-increasing chain from I to J, and it is of maximal length among chains from I to J. In particular, the greedy algorithm described in Remark 9.4.6 produces a chain of length equal to the height of the interval.*

Proof. Suppose $I \leq J$, then there exists a saturated chain $I = I_0 \lessdot I_1 \lessdot \ldots \lessdot I_k = J$ which may not be c-increasing.

Let i be the greatest index such that the subchain from I_0 to I_i is c-increasing and composed of the i smallest flip roots of the chain. Let j be the index of the smallest flip root $r(I_j, I_{j+1})$ of the subchain from I_i to I_k. By induction on i and $j - i$, we will produce a c-increasing chain from I to J of length at least k by modifying locally the chain, using Assumption 1.

If $i = k$ then the chain is already c-increasing and we are done. Otherwise, by maximality of i, we must have $j - i > 0$. Then, $I_{j-1} \lessdot I_j \lessdot I_{j+1}$ respects the conditions of Assumption 1, and we can produce a c-increasing chain $I_{j-1} \lessdot I'_j \lessdot \ldots \lessdot I_{j+1}$ where $r(I_{j-1}, I'_j) = r(I_j, I_{j+1})$. Then we can replace $I_{j-1} \lessdot I_j \lessdot I_{j+1}$ by $I_{j-1} \lessdot I'_j \lessdot \ldots \lessdot I_{j+1}$ to produce a new chain from I to J of length at least k. Then, either $j - i - 1 > 0$ and we continue the induction as before, or $j - i - 1 = 0$ and now by construction the chain is c-increasing at the first $i + 1$ steps, with the $i + 1$ smallest flip roots of the chain, so i has increased by at least one.

By induction, as there is no infinite chain in the poset, the process terminates and produces an increasing chain from I to J.

The maximality of the length of the increasing chain comes from the uniqueness of the c-increasing chain, as proven in Proposition 9.4.2, and the fact that starting with any chain, on can produce this unique c-increasing chain by the process described above, and it is at least as long as the chain we started with. $\square$

For what follows, we will suppose that Assumption 1 holds. This allows us to define the m-Cambrian lattice as the poset where I is smaller than J if there is a c-increasing chain from I to J. However, the greedy procedure of Remark 9.4.6 can be rewritten slightly differently to produce a very nice interpretation on m-chains in the noncrossing partition lattice. The key idea is to mark a pause between two copies of $\mathsf{R}(\mathsf{c})$ while reading the word $\mathsf{R}(\mathsf{c})^{m+1}$ and to "take an instant picture" of the current state of the algorithm.

Lemma 9.4.8. *Let c be a Coxeter word in W and $m > 1$. Then the $(m-1)$-Cambrian lattice $\mathrm{Camb}^{(m-1)}_{NC}(W, \mathsf{c})$ is isomorphic to the interval of $\mathrm{Camb}^{(m)}_{NC}(W, \mathsf{c})$ consisting of m-delta sequences $(\delta_i)_{0 \leq i \leq m}$ whose first part δ_0 is equal to e.*

Proof. There is a straightforward bijection from $\mathrm{Camb}_{\mathrm{NC}}^{(m-1)}(W,\mathsf{c})$ to m-delta sequences with a trivial first part, namely the map $(\delta_0,\dots,\delta_{m-1}) \mapsto (e,\delta_0,\dots,\delta_{m-1})$. It is also clear that this map preserves the order, and this is a bijection since moreover the set of such m-delta sequences with a trivial first part is stable under m-Cambrian covering relations. $\square$

Lemma 9.4.9. *Let $\delta = (\delta_i)_{0\le i\le m}$ and $\delta' = (\delta'_i)_{0\le i\le m}$ be two m-delta sequences such that $\delta \le \delta'$ in $\mathrm{Camb}_{NC}^{(m)}(W,c)$.*

Then for all $1 \le i \le m$, the 1-delta sequences $(\delta_0\cdots\delta_{i-1}, \delta_i\cdots\delta_m)$ and $(\delta'_0\cdots\delta'_{i-1}, \delta'_i\cdots\delta'_m)$ are comparable in $\mathrm{Camb}(W,c)$.

Proof. We fix a Coxeter word c for c. The choice is not important since two different Coxeter words give two isomorphic posets. We prove the statement by induction on m.

We will prove that if $\delta = (\delta_i)_{0\le i\le m}$ and $\delta' = (\delta'_i)_{0\le i\le m}$ are two m-delta sequences such that $\delta \le \delta'$ in $\mathrm{Camb}_{\mathrm{NC}}^{(m)}(W,\mathsf{c})$, then the two $(m-1)$-delta sequences $\widetilde{\delta} = (\delta_0\delta_1, \delta_2,\dots,\delta_m)$ and $\widetilde{\delta}' = (\delta'_0\delta'_1, \delta'_2,\dots,\delta'_m)$ satisfy $\widetilde{\delta} \le \widetilde{\delta}'$ in $\mathrm{Camb}_{\mathrm{NC}}^{(m-1)}(W,\mathsf{c})$.

The same result will then hold for the $(m-1)$-delta sequences obtained by multiplying the last two parts thanks to the duality of $\mathrm{Camb}_{\mathrm{NC}}^{(m)}(W,\mathsf{c})$ and $\mathrm{Camb}_{\mathrm{NC}}^{(m)}(W,\psi(\mathsf{c}))$ explained in Remark 9.2.15, obtained by reading the word $\mathsf{R}(\mathsf{c})^{m+1}$ backwards. The result will then follow by induction on m.

Recall that thanks to Proposition 3.4.25, any noncrossing partition $\delta \in \mathrm{NC}(W,\mathsf{c})$ can be uniquely written as a (reduced) subword of $\mathsf{R}(\mathsf{c})$. Now consider the two m-delta sequences $\delta = (\delta_i)_{0\le i\le m}$ and $\delta' = (\delta'_i)_{0\le i\le m}$. We use a modified version of the greedy algorithm of Remark 9.4.6 on the first copy of $\mathsf{R}(\mathsf{c})$ to transform both δ_0 and δ'_0 into e.

We start with $\delta_* = \delta$ and $\delta'_* = \delta'$, and we read the first copy of $\mathsf{R}(\mathsf{c})$ in $\mathsf{R}(\mathsf{c})^{m+1}$ from left to right. For each letter r:

- If $r \notin \delta_*$ and $r \notin \delta'_*$, we continue with the next letter.
- If $r \in \delta_*$ and $r \notin \delta'_*$, then we replace δ_* by its upper cover in the direction of r. Thanks to Theorem 9.4.7, it remains under δ'_* since this is the first move of the c-increasing chain.
- If $r \in \delta_*$ and $r \in \delta'_*$, then we replace δ_* and δ'_* by their respective upper covers in the direction of r. Because of the previous case, the comparison $\delta_* \le \delta'_*$ is preserved, by changing first the top element and then the bottom one.
- The case $r \notin \delta_*$ and $r \in \delta'_*$ can not happen according to Theorem 9.4.7 because $\delta_* \le \delta'_*$.

This will transform δ_1 and δ'_1 into $\delta_0\delta_1$ and $\delta'_0\delta'_1$ respectively, and not affect the other parts. We conclude with Lemma 9.4.8 that $\widetilde{\delta} \le \widetilde{\delta}'$. $\square$

Definition 9.4.10. Let $w = (w_1 \le_{\mathcal{R}} \cdots \le_{\mathcal{R}} w_m)$ and $w' = (w'_1 \le_{\mathcal{R}} \cdots \le_{\mathcal{R}} w'_m)$ be two m-chains in the noncrossing partition lattice $\mathrm{NCL}(W,c)$. We define a binary relation on $\mathrm{NC}^{(m)}(W,c)$ by $w \le_{\mathrm{Gr}} w'$ if

1. $w_i \le w'_i$ in $\mathrm{Camb}(W,c)$ for all $1 \le i \le m$,
2. and $w_i \le_{\mathcal{R}} w'_{i+1}$ in $\mathrm{NCL}(W,c)$ for all $1 \le i < m$.

We define the (greedy version of the) **m-Cambrian lattice** $\mathrm{Camb}_{\mathrm{Gr}}^{(m)}(W,c)$ as the set $\mathrm{NC}^{(m)}(W,c)$ together with the relation $\le_{\mathrm{Gr}}$.

The relation $w \le_{\mathrm{Gr}} w'$ is illustrated in Figure 9.2.

Theorem 9.4.11. *Under Assumption 1, for any standard Coxeter element $c \in W$ and $m \ge 1$, $\mathrm{Camb}_{Gr}^{(m)}(W,c)$ is a poset isomorphic to the m-Cambrian lattice $\mathrm{Camb}_{NC}^{(m)}(W,c)$.*

$$\begin{tikzcd}
w'_1 \arrow[r, "\leq_{\mathcal{R}}"] & w'_2 \arrow[r, "\leq_{\mathcal{R}}"] & w'_3 \arrow[r, "\leq_{\mathcal{R}}"] & \cdots \arrow[r, "\leq_{\mathcal{R}}"] & w'_m \\
w_1 \arrow[u, "\leq"] \arrow[ur, "\leq_{\mathcal{R}}"] \arrow[r, "\leq_{\mathcal{R}}"] & w_2 \arrow[u, "\leq"] \arrow[ur, "\leq_{\mathcal{R}}"] \arrow[r, "\leq_{\mathcal{R}}"] & w_3 \arrow[u, "\leq"] \arrow[ur, "\leq_{\mathcal{R}}"] \arrow[r, "\leq_{\mathcal{R}}"] & \cdots \arrow[ur, "\leq_{\mathcal{R}}"] \arrow[r, "\leq_{\mathcal{R}}"] & w_m \arrow[u, "\leq"]
\end{tikzcd}$$

Figure 9.2: An illustration of the greedy comparison criterion between two m-chains w and w' in $\mathrm{NCL}(W, c)$. Arrows going up are comparisons in $\mathrm{Camb}(W, c)$ whereas arrows going right or up-right are comparisons in $\mathrm{NCL}(W, c)$. Reading the "ladder" from left to right gives an m-chain in some poset on intervals in $\mathrm{Camb}(W, c)$.

Proof. We fix a Coxeter word c in order to dispose of the total order on reflections. We will build an isomorphism between $\mathrm{Camb}^{(m)}_{\mathrm{Gr}}(W, c)$ and $\mathrm{Camb}^{(m)}_{\mathrm{NC}}(W, c)$.

Let $(\delta_i)_{0\leq i\leq m} \leq (\delta'_i)_{0\leq i\leq m}$ be an interval in $\mathrm{Camb}^{(m)}_{\mathrm{NC}}(W, c)$. Let $w_k = \delta_{m-k+1}\dots\delta_m$ and $w'_k = \delta'_{m-k+1}\dots\delta'_m$, for $1 \leq k \leq m$ be the corresponding m-chains in $\mathrm{NCL}(W, c)$. Using the greedy algorithm of Remark 9.4.6 and Theorem 9.4.7, we will prove that indeed $(w_i)_i = w \leq_{\mathrm{Gr}} w' = (w'_i)_i$, and vice-versa.

Firstly, starting with $I = (\delta_i)_i$ and $J = (\delta'_i)_i$, reading the word $\mathsf{R}(\mathsf{c})^{m+1}$ from left to right leads to a c-increasing chain from I to J, meaning that we never find a letter of J that is not also a letter of I. Thus, we read $\mathsf{R}(\mathsf{c})^{m+1}$ letter by letter and apply a covering relation to each letter that is in I but not in J, transforming I progressively. We mark a pause after each copy of $\mathsf{R}(\mathsf{c})$ at the end of which we have applied covering relations to I so that I and J agree on all letters up to this copy.

We start by reading the first copy of $\mathsf{R}(\mathsf{c})$. At the end of this copy, we have transformed δ_0 into δ'_0 using covering relations. By further examination, we can see that δ_1 has been changed into some δ^*_1 but for all $i \geq 2$, δ_i is unchanged.

Since we have used m-Cambrian flips, we have $(\delta_0, \dots, \delta_m) \leq (\delta'_0, \delta^*_1, \delta_2, \dots, \delta_m)$ in $\mathrm{Camb}^{(m)}_{\mathrm{NC}}(W, \mathsf{c})$. Thus, $(\delta'_0, \delta^*_1, \delta_2, \dots, \delta_m)$ is an m-delta sequence on the one hand, which corresponds to the m-noncrossing partition $w_1 \leq_{\mathcal{R}} \cdots \leq_{\mathcal{R}} w_{m-1} \leq_{\mathcal{R}} w'_m$. In particular, we have the last inequality $w_{m-1} \leq_{\mathcal{R}} w'_m$. On the other hand, with Lemma 9.4.9 for $i = 1$, we obtain $\delta_0 \geq \delta'_0$ and $w_m \leq w'_m$ in $\mathrm{Camb}(W, \mathsf{c})$.

We can now continue the greedy algorithm and stop after each copy of $\mathsf{R}(\mathsf{c})$. After the i-th copy of $\mathsf{R}(\mathsf{c})$, we have transformed I into the m-delta sequence $(\delta'_0, \dots, \delta'_{i-1}, \delta^*_i, \delta_{i+1}, \dots, \delta_m)$. In particular, we obtain on the one hand $\delta_{i+1}\dots\delta_m = w_{m-i} \leq_{\mathcal{R}} w'_{m-i+1} = (\delta^*_i\delta_{i+1}\dots\delta_m)$, and on the other hand, using Lemma 9.4.9 for $i+1$, we obtain $w_{m-i+1} \leq w'_{m-i+1}$ in $\mathrm{Camb}(W, \mathsf{c})$.

Before the last copy of $\mathsf{R}(\mathsf{c})$, we have transformed I into J, and a last use of Lemma 9.4.9 gives $\delta_m \geq \delta'_m$, i.e. $w_1 \leq w'_1$.

Reciprocally, starting with two m-noncrossing partitions $w = (w_i)_i$ and $w' = (w'_i)_i$ such that $w \leq_{\mathrm{Gr}} w'$, the same computation as above prove that the greedy algorithm allows transforming each noncrossing partition w_i into w'_i in decreasing order starting with $i = m$, and using only Cambrian rotations. Thus, $w \leq w'$ in the m-Cambrian lattice $\mathrm{Camb}^{(m)}_{\mathrm{NC}}(W, \mathsf{c})$. □

We finish this section by a last result that we mentioned, and whose proof is similar to the one of Proposition 9.4.2, namely the so-called non-leaving face property. This is formulated on the flip poset of Coxeter-initial subword complexes, as the fact that a position that is flipped out can never be reached again. Recall that a face of the subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ consists of a set of positions in the search word Q such that the complement contains at least one word of length a for w. Equivalently, the proposition states that if the bottom and the top elements are contained in some face, then the entire interval also is contained in this face.

A property of this name was first introduced by C. Ceballos and V. Pilaud in [CP16] and generalized by N. Williams for all type W associahedra in [Wil17]. It however focused on the shortest path between two vertices of some polytopes, or "geodesics", which they proved to be contained into any face containing both ends. Note that these type W associahedra are realized by the type W Cambrian lattices, i.e. their Hasse diagram is an orientation of their 1-skeleton. Here, we will prove that any monotone path between two vertices of the m-Cambrian lattices is contained into any face containing both ends, which is a different statement, but still similar in flavor.

Proposition 9.4.12. *Let I and J be two facets of a c-initial subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}, w, a)$ such that $I \leq J$ in the corresponding flip poset.*

Then for all $i \in I \cap J$, for all $K \in [I, J]$, we have $i \in K$.

Proof. Let $\mathsf{Q} = \mathsf{s}_1 \dots \mathsf{s}_p$ be initial in c^∞, with c is initial in Q.

Recall that for any facet I, we attached a root vector $r_I = (r_I(i))_{1\leq i\leq p}$ in Definition 3.4.15, and we defined in particular the root configuration as the set of entries of the root vector corresponding to elements in I.

For each $i \in [p]$, we can define the vector space $V_I(i) = \mathrm{Vect}\left(\{r_I(j) \mid j \in I, j \leq i\}\right)$ generated by all roots of I appearing before position i. Thanks to Lemma 9.4.3, we know that the dimension of $V_I(i)$ is the size of $I \cap [i]$. We study the evolution of $V_I(i)$ under any flip $I \lessdot I'$. In particular, we will prove that if we flip i out of I, then i can never be flipped back in, because $r_{I'}(i) \notin V_{I'}(i)$.

Recall that a flip $I \lessdot I'$ consists in removing an element $j \in I$ and replacing it with the next $j' > j$ such that $|r_I(j')| = |r_I(j)|$. The root of any $k \in I$ such that $j < k < j'$ is conjugated by $r_I(j)$, which corresponds to adding a multiple of $r_I(j)$ to $r_I(k)$.

If $j > i$, we have immediately $V_I(i) = V_{I'}(i)$.

If $j \leq i$, if $j' \leq i$, then the dimensions of $V_I(i)$ and of $V_{I'}(i)$ are equal, and if $j' > i$, then the dimension of $V_I(i)$ is strictly greater than the dimension of $V_{I'}(i)$. Moreover, since the roots attached to I are linearly independent, for any $k \in [i] \setminus \{j, j'\}$, we have $r_{I'}(k) = r_I(k) + \lambda r_I(j)$ for some λ.

Thus, if furthermore $k \in I$, since $r_I(k) \in V_I(i)$ by definition, we have $r_{I'}(k) \in V_I(i)$. In particular, this proves that for any increasing flip $I \lessdot I'$, we have

$$\forall i, \quad V_{I'}(i) \subseteq V_I(i). \tag{9.5}$$

Moreover, if $r_I(k) \notin V_I(i)$, then $r_{I'}(k) = r_I(k) + \lambda r_I(j) \notin V_I(i)$. In particular, using (9.5), we have

$$\forall i, k, \quad r_I(k) \notin V_I(i) \Rightarrow r_{I'}(k) \notin V_{I'}(i). \tag{9.6}$$

To conclude, if $j = i$, we have $V_I(i) = V_{I'}(i) \oplus r_{I'}(i)$ since $r_I(i) = r_{I'}(i)$ and $i \in I$ but $i \notin I'$. Thus, $r_{I'}(i) \notin V_{I'}(i)$. Then, by (9.6), for all $J' \geq I'$, we have $r_{J'}(i) \notin V_{J'}(i)$, and thus $i \notin J'$. In particular, $J \not\geq I'$.

This proves that for all $K \in [I, J]$, we have $i \in K$ since flipping the vertex i implies leaving the interval. ☐

We have conjecturally a stronger assumption, that seems to hold very well in type A ans in fact should hold in every type.

Assumption 3. Let c be a Coxeter word in a Coxeter group W of rank n. Let I and J be two facets of the c-initial subword complex $\mathrm{SC}_{\mathcal{S}}(\mathsf{c}w_\circ(\mathsf{c})^m, w, a)$ such that $I \leq J$ in the corresponding flip poset. Suppose that $i \in I \cap J$.

Then for all $K \in [I, J]$, we have $i \in K$. Moreover, this interval is isomorphic as a poset to an interval in the m-Cambrian lattice corresponding to some parabolic subgroup of W, of rank $n - 1$.

Start of a proof. Let $\mathsf{Q} = \mathsf{s}_1 \dots \mathsf{s}_i \dots \mathsf{s}_p = \mathsf{c}w_\circ{}^m(\mathsf{c})$.

Using the shift operator defined in [STW18, Section 5.4] $i-1$ times, we can produce the search word $\mathsf{Q}' = \mathsf{s}_i \dots \mathsf{s}_p \psi^m(\mathsf{s}_1) \dots \psi^m(\mathsf{s}_{i-1})$ and the corresponding Coxeter word c' in which s_i is initial.

Thanks to [STW18, Propositions 5.4.5 and 5.4.7], the set of facets of $\mathrm{SC}_{\mathcal{S}}(\mathsf{Q}', w_\circ{}^m, m\,\ell_{\mathcal{S}}(w_\circ{}^m))$ containing the first position of Q' is in bijection with the elements of $\mathrm{Camb}^{(m)}_{\mathrm{Clus}}(W_{\langle s_i \rangle}, \mathsf{c}')$. Applying the reverse of the shift operator to rotate back a certain number of times then should produce the Cambrian lattice in the desired parabolic subgroup, itself isomorphic to the restriction of $\mathrm{Camb}^{(m)}_{\mathrm{Clus}}(W, \mathsf{c})$ to the set of elements containing i. $\square$

9.4.2 The linear type A case

In this section, we will focus on the linear type A case, as it is of major interest with regard to their number of intervals (see Section 9.3). We will translate the new conjectural description of the m-Cambrian lattices, using in particular the Tamari interval-posets defined in Section 8.3.

Remark 9.4.13. If Assumption 1 holds, then Theorem 9.4.11 implies that an interval in the m-Cambrian lattice can be understood as a pair of m-tuples of noncrossing partitions $((w_i)_{1 \le i \le m}, (w'_i)_{1 \le i \le m})$ satisfying:

1. $w_i \le_{\mathcal{R}} w_{i+1}$ in $\mathrm{NCL}(W, c)$ for all $1 \le i < m$,
2. $w'_i \le_{\mathcal{R}} w'_{i+1}$ in $\mathrm{NCL}(W, c)$ for all $1 \le i < m$,
3. $w_i \le_{\mathcal{R}} w'_{i+1}$ in $\mathrm{NCL}(W, c)$ for all $1 \le i < m$,
4. $w_i \le w'_i$ in $\mathrm{Camb}_{\mathrm{NC}}(W, c)$ for all $1 \le i \le m$.

Instead of thinking of an m-Cambrian interval interval as some pairs of m-chains of noncrossing partitions, one can think of it as an m-tuple of intervals in $\mathrm{Camb}_{\mathrm{NC}}(W, c)$ satisfying some compatibility relations, relying on the comparison rules in $\mathrm{NCL}(W, c)$. In particular, even in linear type A, this gives a new poset structure on intervals in the Tamari lattice that, to the best of our knowledge, has never been studied before.

In this section, we describe one way to look at Tamari interval-posets as arch diagrams, and to describe the different compatibility relations that are implied by this new definition of the m-Cambrian lattice.

Firstly, when drawing a Tamari interval-poset $(I, \triangleleft)$, we can remember that if $i \triangleleft j$ is an increasing relation, then for all $k \in [i, j]$ in the natural order, we have $k \triangleleft j$. In particular, if there are several covering relations in the initial forest of the form $i \triangleleft j$ for same j, then only the one with the smallest i is necessary, since the others are implied by it. Similarly, if there are several covering relations in the final forest of the form $i \triangleleft j$ for same i, then only the one with the largest j is necessary. We can thus erase these "implied arches".

Then, it remains to study under which conditions an initial and a final forests are compatible, and to notice that this is already readable with the "non-implied arches".

Definition 9.4.14. For each $i \in [n]$ we consider the corresponding point $(i, 0)$ in the plane.

An **initial arch diagram** I^{ini} of size n is a collection of arches (i.e. pairs (i, j) with $i < j$) drawn in the upper half plane such that:

- no two arches cross, i.e. there is no two arches (i, k) and (j, l) with $i < j < k < l$,
- no two arches start at the same point,
- no two arches end at the same point.

A **final arch diagram** I_{fin} of size n is defined similarly, except that the arches are drawn in the lower half plane.

An initial and a final arch diagrams of the same size are **Tamari-compatible** if for each initial arch (i,k) there is no final arch (j,l) with $i \leq j < k \leq l$. Such a Tamari-compatible pair $I = (I^{\mathsf{ini}}, I_{\mathsf{fin}})$ is called a **Tamari arch diagram** of size n.

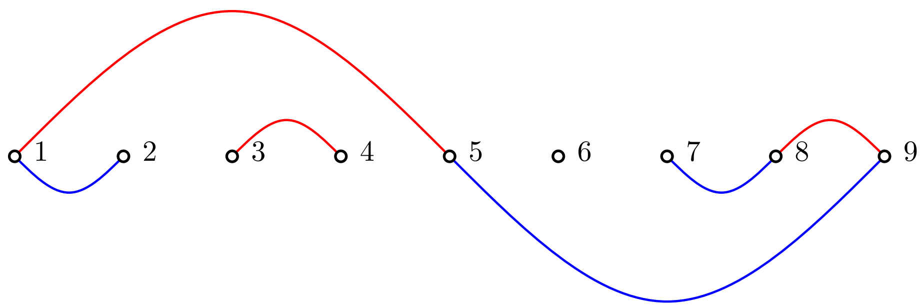


Figure 9.3: The Tamari arch diagram corresponding to the interval-poset of Figure 8.5. Increasing arches $(2,5)$ and $(4,5)$ have been erased and decreasing arches $(5,6)$ and $(5,7)$ have been erased.

Proposition 9.4.15. *Erasing implied arches of Tamari interval-posets gives a bijection with Tamari arch diagrams.*

The Tamari arch diagram corresponding to the Tamari interval-poset of Figure 8.5 is illustrated in Figure 9.3.

Proof. The arcs of the initial forest are always noncrossing. Indeed, if $i < j < k < l$, the existence of relations $i \lhd k$ and $j \lhd l$ imply that $j \lhd k$ and $k \lhd l$. Hence $j \lhd l$ is not a covering relation. The case of the final forest is symmetric.

The erasing process ensures furthermore that no two initial (resp. final) arches start or end at the same point.

Finally, the compatibility condition holds because the existence of an increasing arch (i,k) and a decreasing arch (j,l) for $i \leq j < k \leq l$ would imply respectively that $j \lhd k$ and $k \lhd l$, which is not possible since we have a poset.

Hence, the erasing process gives a map from Tamari interval-posets to Tamari arch diagrams.

We now prove that this is a one-to-one correspondence. Starting with a Tamari arch diagram, we can define a relation $\lhd$ on $[n]$ by adding all implied relations. Namely, we set $i \lhd^* j$ if there is an increasing arch (i',j) for some $i' < i$, or a decreasing arch (i,j') for some $j' > j$, and we take the transitive closure $\lhd$ of the relation $\lhd^*$. This gives a Tamari interval-poset, as long as the relation is antisymmetric.

To prove this, we first remark that if there is an increasing relation $i \lhd j$, then there exists a sequence of increasing arches $(i_1,i_2),(i_2,i_3),\dots,(i_d,j)$ for some $i_1 < \dots < i_d < j$ with $i_0 \leq i$.

As $\lhd$ is the transitive closure of $\lhd^*$, we must have a sequence of relations $i = j_0 \lhd^* j_1 \lhd^* \dots \lhd^* j_k = j$. We study the different cases for the first two relations, and then we conclude for all such sequences.

- If $i < j_1$ and $j_2 < j_1$, then there exists an increasing arch (i',j_1) with $i' \leq i$, and a decreasing arch (j_2,j') with $j' \geq j_1$. Compatibility relations imply that $j_2 < i' \leq i$. This implies that the sequence above from i to j must end with an increasing relation.
- Symmetrically, if $i > j_1$ and $j_2 > j_1$ then we have a decreasing arch (j_1,i') with $i \leq i'$ and an increasing arch (j',j_2) with $j' \leq j_1$. Compatibility relations imply that $j' < i < j_2$. This implies that already $i \lhd^* j_2$. This implies that the sequence above from i to j can be chosen so that any decreasing relation followed by an increasing relation can be replaced by

only one increasing relation. Combining with the previous point, we can assume that the sequence above from i to j is constituted of only increasing relations.

- If $i < j_1 < j_2$, then either there exists an increasing arch (i', j_2) or two increasing arches (i', j_1) and (j_1, j_2), for some $i' \leq i$. Combining with the previous points, we can assume that there is a sequence of consecutive increasing arches from i_1 to j, for some $i_1 \leq i$.

The symmetric statement holds for any decreasing relation $j \triangleleft i$.

Now if the relation $\triangleleft$ was not antisymmetric, there would be $i < j$ such that $i \triangleleft j$ and $j \triangleleft i$. This would imply the existence of some consecutive increasing arches $(i_1, i_2), (i_2, i_3), \ldots, (i_d, j)$ with $i_1 \leq i$ and some decreasing arch (j_1, j_2) with $j_1 < j \leq j_2$. This is not possible because there would be one of those increasing arches that would be incompatible with the decreasing arch (j_1, j_2).

This proves that the map is indeed bijective. □

We can even describe a nice correspondence between linear type A Cambrian intervals and such arch diagrams. We will not prove that it is bijective since we will not use it. Unfortunately, it does not seem to generalize to a direct bijection between the linear type A m-Cambrian lattice and the m-Tamari interval-posets.

Recall that the linear type A Cambrian lattice can be described on 1-delta sequences, i.e. on the factorizations of the long cycle $c^{\mathsf{lin}} = (1, 2, \ldots, n)$ as a subword of $\mathsf{R}(\mathsf{c})^2$, or equivalently as a product of two noncrossing partitions (δ_0, δ_1) such that $\ell_{\mathcal{R}}(\delta_0) + \ell_{\mathcal{R}}(\delta_1) = n - 1$. The total order induced by c^{lin} on the set of reflections is the lexicographic order on the pairs (i, j) with $i < j$.

Recall that a noncrossing partition $\delta_i \in \mathrm{NC}(W, c^{\mathsf{lin}})$ can be uniquely written as a c^{lin}-increasing word, thanks to Proposition 3.4.25.

Remark 9.4.16. Given an interval $\delta \leq \delta'$ in $\mathrm{Camb}_{\mathrm{NC}}(\mathfrak{S}_n, c^{\mathsf{lin}})$, one can define an arch diagram by putting a decreasing arch (i, j) for every letter $(i, j)^{(1)}$ in δ and an increasing arch (i, j) for every letter $(i, j)^{(0)}$ in δ'.

Saying it otherwise, if $\delta = (\delta_0, \delta_1)$, one takes the c^{lin}-increasing word of δ_1 and puts a decreasing arch for each corresponding reflection, producing a final arch diagram δ_{fin}. Symmetrically, an initial arch diagram δ'^{ini} can be produced from δ'_0 and increasing arches.

Note that one can attach to an initial arch diagram a classical noncrossing partition bijectively by setting that $i < j$ are in the same part if there is a sequence of consecutive increasing arches from i to j. This gives back exactly δ'_0 as a noncrossing partition in $\mathrm{NC}(W, c^{\mathsf{lin}})$. Obviously, the same holds for δ_1 from a final arch diagram.

Each Tamari arch diagram corresponds to a Tamari interval. It remains to describe the compatibility conditions between intervals. For this, it is in fact easier to shift the final arch diagram so that $i \in [n]$ corresponds to the point $i' = (i + \frac{1}{2}, 0)$.

We will thus represent Tamari arch diagrams on a set of $2n$ points $(1 < 1' < 2 < 2' < \cdots < n < n')$ of alternating color, with an initial arch diagram on the unprimed integers and a final arch diagram on the primed integers. This is practical for it is not necessary to tell apart the initial and the final forests, and the Tamari-compatibility condition translates well. This gives a new definition of Tamari arch diagrams as a bicolored version.

Definition 9.4.17. A bicolored arch diagram of size n is a collection of arches on the set $(1 < 1' < 2 < 2' < \cdots < n < n')$ such that:

- both ends of an arch are simultaneously primed or unprimed,
- no two unprimed arches cross, no two primed arches cross,
- no two arches share the same beginning, nor the same end,

- there is no two arches (i,k) and (j',l') with $i < j' < k < l'$.

We now need to describe the noncrossing compatibility conditions of Remark 9.4.13, in order to complete the picture. It is easy to translate from Remark 9.4.16 the compatibility condition between two initial or two final arch diagrams, the third condition in fact also reads well.

Definition 9.4.18. A Tamari arch diagram $I = (I^{\mathsf{ini}}, I_{\mathsf{fin}})$, or equivalently a bicolored arch diagram is **Kreweras-compatible** if there is no two arches (i',k') and (j,l) with $i' < j < k' < l$.

Saying it otherwise, a bicolored arch diagram is Kreweras-compatible if and only if no two arches are crossing even when all are drawn in the same half plane (and no two arches share a beginning nor an end).

Definition 9.4.19. Let $I = (I^{\mathsf{ini}}, I_{\mathsf{fin}})$ and $J = (J^{\mathsf{ini}}, J_{\mathsf{fin}})$ be two bicolored arch diagrams of size n. Let I_{fin} and I^{ini} be the corresponding final and initial arch diagrams of I, and similarly J_{fin} and J^{ini}.

We say that the bicolored arch diagrams I and J are **Cambrian-compatible** if

- J^{ini} refines I^{ini}, that is to say that for every arc $(i,j) \in J^{\mathsf{ini}}$, there is a sequence of consecutive increasing arcs from i to j in I^{ini},
- I_{fin} refines J_{fin},
- I_{fin} and J^{ini} are Kreweras-compatible.

An **m-sequence** of arch diagrams of size n is a sequence $(I_1, \dots, I_m)$ of bicolored arch diagrams of size n such that for all $1 \le i < m$, I_i and I_{i+1} are Cambrian-compatible.

These m-sequences of arch diagrams are very similar to the m-Tamari interval-posets defined in Definition 8.3.9. The former correspond to sequences of m Tamari interval-posets of size n with some compatibility relations, the latter to Tamari interval-posets of size mn with some restrictions.

Moreover, there is a natural way to attach to an m-sequence of arch diagrams a top and a bottom partitions. Indeed, the top partition as defined in Definition 9.3.7 is as the partition whose parts are the sizes of blocks of δ'_m, the one containing n being marked. This translates with Remark 9.4.16 and Theorem 9.4.11 to the initial forest partition of the first Tamari arch diagram of the m-sequence. The bottom partition is defined symmetrically as the final forest partition of the last Tamari arch diagram of the m-sequence.

Additionally, thanks to Theorem 9.4.7, the height of the corresponding m-Cambrian interval corresponds to the sum of the heights of the Tamari intervals, i.e. the sum of the numbers of Tamari inversions of the arch diagrams of the m-sequence.

It is worth noting that the natural involution on linear type A m-Cambrian lattices still translates on the arch diagrams as the involution that returns the sequence as well as the arch diagrams of the sequence (with the complement involution).

All in all, we can formulate a conjecture equivalent to Conjecture 9.3.12.

Conjecture 9.4.20. There is a bijection ξ between m-sequences of arch diagrams of size n and m-Tamari interval-posets of size n, that behaves well with respect to the top and bottom partitions, the height and the involutions.

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Le treillis de Tamari est un ordre partiel sur les objets comptés par les nombres de Catalan. Plusieurs descriptions de ce treillis existent et donnent lieu à différentes familles de généralisations. Dans cette thèse, on étudie ces différents ordres partiels et notamment leurs intervalles, en particulier d'un point de vue énumératif.
Après une première partie préliminaire, une seconde partie concerne concerne l'étude de la sous-famille des intervalles linéaires dans le treillis de Tamari et ses différentes généralisations. On définit en particulier les familles des ordres alt-Tamari et alt ν-Tamari. On prouve bijectivement des résultats d'équidistribution de ces intervalles linéaires, que l'on énumère dans le cas des treillis alt-Tamari.
Une troisième partie se penche sur une conjecture de Stump, Thomas et Williams selon laquelle les treillis m-Cambriens en type A linéaire et m-Tamari auraient le même nombre d'intervalles. On présente et généralise l'étude dans le cas m-Tamari, puis on étudie les treillis m-Cambriens, dont on propose une nouvelle description conjecturale.

The Tamari lattice is a partial order on objects counted by the Catalan numbers. There are several descriptions of this lattice, which lead to different families of generalizations. In this manuscript, we study these different partial orders and their intervals, especially from an enumerative perspective.
After a first preliminary part, a second part focuses on the study of the subfamily of linear intervals in the Tamari lattice and its generalizations. We define in particular the new families of alt-Tamari and alt ν-Tamari orders. We prove bijectively some equidistributy result of these linear intervals, that we enumerate in the case of the alt-Tamari lattices.
A third part is motivated by a conjecture of Stump, Thomas and Williams, according to which the m-Cambrian lattices in linear type A and the m-Tamari lattices would have the same number of intervals. We present and generalize the study in the m-Tamari case, then we study the m-Cambrian lattices, for which we propose a new conjectural description.

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Enumerative study of intervals in lattices of Tamari type

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Résumé

Le treillis de Tamari est un ordre partiel sur les objets comptés par les nombres de Catalan. Plusieurs descriptions de ce treillis existent et donnent lieu à différentes familles de généralisations. Dans cette thèse, on étudie ces différents ordres partiels et notamment leurs intervalles, en particulier d'un point de vue énumératif.

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<u>Mots clés :</u> Treillis de Tamari, ordre partiel, intervalles dans un poset, combinatoire énumérative, chemin de Dyck, arbre binaire, intervalles linéaires, treillis alt-Tamari, treillis alt ν-Tamari, groupe de Coxeter, groupe symétrique, treillis cambrien, treillis m-cambrien, permutarbre, nombres de Catalan, ordre faible, treillis m-Tamari, intervalle-poset, treillis ν-Tamari

Résumé en anglais

The Tamari lattice is a partial order on objects counted by the Catalan numbers. There are several descriptions of this lattice, which lead to different families of generalizations. In this manuscript, we study these different partial orders and their intervals, especially from an enumerative perspective.

After a first preliminary part, a second part focuses on the study of the subfamily of linear intervals in the Tamari lattice and its generalizations. We define in particular the new families of alt-Tamari and alt ν-Tamari orders. We prove bijectively some equidistributivity results of these linear intervals, that we enumerate in the case of the alt-Tamari lattices.

A third part is motivated by a conjecture of Stump, Thomas and Williams, according to which the m-Cambrian lattices in linear type A and the m-Tamari lattices would have the same number of intervals. We present and generalize the study in the m-Tamari case, then we study the m-Cambrian lattices, for which we propose a new conjectural description.

<u>Keywords :</u> Tamari lattice, poset, intervals in a poset, enumerative combinatorics, Dyck paths, binary tree, linear intervals, alt-Tamari lattice, alt ν-Tamari lattice, Coxeter group, symmetric group, Cambrian lattice, m-Cambrian lattice, permutree, Catalan numbers, weak order, m-Tamari lattice, interval-poset, ν-Tamari lattice