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Basile CORON

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Algebraic invariants of matroids and generalized operads

THÈSE dirigée par :

Mr DOTSENKO Vladimir	Professeur, université de Strasbourg

RAPPORTEURS :

Mr ADIPRASITO Karim	Professeur, université de Copenhage
Mme FEICHTNER Eva Maria	Professeure, université de Bremen

AUTRES MEMBRES DU JURY :

Mr DUPONT Clément	Maître de conférence, université de Montpellier
Mr ENRIQUEZ Benjamin	Professeur, université de Strasbourg
Mme GALVEZ Maria	Professeure, université polytechnique de Catalogne

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Basile Coron

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Vladimir Dotsenko, directeur de thèse
Karim Alexander Adiprasito, rapporteur
Eva Maria Feichtner, rapporteuse
Clément Dupont, examinateur
Benjamin Enriquez, examinateur
Maria Immaculada Gálvez Carillo, examinatrice

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Algebraic invariants of matroids and generalized operads

Basile Coron

Contents

Introduction (English)

The present document aims at defining and studying new structures of operadic type on algebraic invariants of matroids.

Part 1: hyperplane arrangements and operadic structures

A hyperplane arrangement is a finite collection of hyperplanes in a finite dimensional vector space. For any hyperplane arrangement $\mathcal{H}$, the intersection lattice $\mathcal{L}_{\mathcal{H}}$ of $\mathcal{H}$ is the set of all possible intersections $\{\bigcap_{H \in S} H, S \subset \mathcal{H}\}$ ordered by reverse inclusion. As its name indicates, it is a lattice, with supremum given by the intersection. This lattice encapsulates all the combinatorial information of the hyperplane arrangement.

Example 0.0.1. Consider the arrangement Braid_3 given by the hyperplanes $\{z_1 = z_2\}, \{z_2 = z_3\}$, and $\{z_1 = z_3\}$ in $\mathbb{C}^3$. The Hasse diagram of the intersection lattice of this hyperplane arrangement is drawn in Figure 1. It is the set of partitions of $\{1, 2, 3\}$ ordered by refinement.

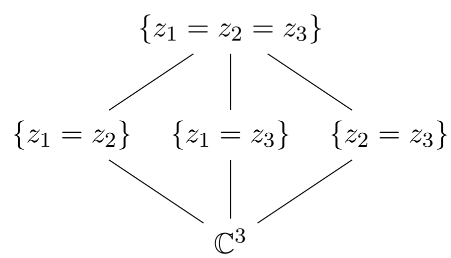

Figure 1: Hasse diagram of $\mathcal{L}_{\mathsf{Braid}_3}$

An important leitmotiv in the study of hyperplane arrangements is sorting the information which is determined by the intersection lattice, i.e. the information of combinatorial nature, and the information which is not. A classical theorem in this spirit is the following.

Proposition 0.0.2 (Orlik-Solomon, [27]). *Let $\mathcal{H}$ be a complex hyperplane arrangement and let $\mathcal{A}_{\mathcal{H}}$ be the complement of the union of the hyperplanes of $\mathcal{H}$. The cohomology algebra of $\mathcal{A}_{\mathcal{H}}$ is isomorphic to the quotient of the exterior algebra $\Lambda[e_H,\ H \in \mathcal{H}]$ (with each e_H in degree 1) by the elements $\delta(e_{H_1}...e_{H_k})$ for all $\{H_1, ..., H_k\}$ such that*

$$\dim H_1 \cap ... \cap H_k > n - k$$

(where δ denotes the unique derivation sending all the generators to 1). The cohomology algebra of the projectivization $\mathbb{P}\mathcal{A}_{\mathcal{H}}$ is isomorphic to the subalgebra generated by the elements $e_H - e_{H'}$ for any two hyperplanes H and H'.

The dimensions of the intersections of hyperplanes are given by the rank function of $\mathcal{L}_{\mathcal{H}}$, meaning the above algebras are determined by $\mathcal{L}_{\mathcal{H}}$. In other words the cohomology algebra of a hyperplane arrangement complement and its projectivization are "combinatorial". We will denote those algebras by $\mathrm{OS}(\mathcal{L}_H)$ and $\overline{\mathrm{OS}}(\mathcal{L}_{\mathcal{H}})$ respectively.

To go further in the study of the projective arrangement complement one may be interested in finding a model for $H^\bullet(\mathbb{P}\mathcal{A}_H)$. A classical way of achieving this goal is to find a "good" compactification of the projective complement

$$\mathbb{P}\mathcal{A}_{\mathcal{H}} \hookrightarrow \overline{Y}_{\mathcal{H}}$$

and then look at the Leray spectral sequence of this inclusion. In order for this spectral sequence to be tractable, one would like the complement of $\mathbb{P}\mathcal{A}_{\mathcal{H}}$ in the compactification to be a divisor with normal crossings, in which case the compactification is called "wonderful". De Concini and Procesi showed in [8] that there exist several such compactifications which can be obtained by successively blowing up some of the intersections of hyperplanes in $\mathbb{P}(\mathbb{C}^n)$. If $\mathcal{G}$ is some subset of $\mathcal{L}_{\mathcal{H}}$, let us denote by $\overline{Y}_{\mathcal{H},\mathcal{G}}$ the result of successively blowing up $\mathbb{P}(\mathbb{C}^n)$ along the elements of $\mathcal{G}$ (by increasing size). In order for $\overline{Y}_{\mathcal{H},\mathcal{G}}$ to be a wonderful compactification, the subset $\mathcal{G}$ needs to contain all the non transversal intersections. This can be axiomatized by the notion of building set, introduced in a purely combinatorial setting by Feichtner-Kozlov [13]. We refer to Subsection 1.1 for the definition of the symbols used in the following definition.

Definition 0.0.3 (Building set). Let $\mathcal{L}$ be a lattice. A *building set* $\mathcal{G}$ of $\mathcal{L}$ is a subset of $\mathcal{L} \setminus \{\hat{0}\}$ such that for every element X of $\mathcal{L}$ the morphism of posets

$$\prod_{G \in \max \mathcal{G}_{\leq X}} [\hat{0}, G] \xrightarrow{\vee} [\hat{0}, X]$$

is an isomorphism (where $\max \mathcal{G}_{\leq X}$ is the set of maximal elements of $\mathcal{G} \cap [\hat{0}, X]$).

In conclusion, we have a wonderful compactification $\overline{Y}_{\mathcal{H},\mathcal{G}}$ of $\mathbb{P}\mathcal{A}_{\mathcal{H}}$ for each building set $\mathcal{G}$ of $\mathcal{L}_{\mathcal{H}}$. The projective smooth varieties $\overline{Y}_{\mathcal{H},\mathcal{G}}$ are stratified by the exceptional divisors $\mathcal{D}_G, G \in \mathcal{G}$ obtained after each blow up. For any subset $\mathcal{S} \subset \mathcal{G}$ the intersection $\bigcap_{G \in \mathcal{S}} \mathcal{D}_G$ is non-empty if and only if $\mathcal{S}$ forms a nested set of $(\mathcal{L}, \mathcal{G})$.

Definition 0.0.4 (Nested set)**.** Let $\mathcal{L}$ be a lattice and $\mathcal{G}$ a building set of $\mathcal{L}$. A subset $\mathcal{S}$ of $\mathcal{G}$ is called a *nested set* if for every subset $\mathcal{A} \subset \mathcal{S}$ of pairwise incomparable elements, the join of the elements of $\mathcal{A}$ does not belong to $\mathcal{G}$ whenever $\mathcal{A}$ contains at least two elements.

De Concini and Procesi showed that for any nested set $\mathcal{S}$ of $(\mathcal{L}_{\mathcal{H}}, \mathcal{G})$, the stratum

$$\overline{Y}_{\mathcal{S}} \coloneqq \bigcap_{G \in \mathcal{S}} \mathcal{D}_G$$

is isomorphic to a product of wonderful compactifications of "smaller" hyperplane arrangements $\mathcal{H}_{\mathcal{S}}^G$ and building sets $\mathcal{G}_{\mathcal{S}}^G$ indexed by the elements of $\mathcal{S}$:

$$\overline{Y}_{\mathcal{S}} \simeq \prod_{G \in \mathcal{S}} \overline{Y}_{\mathcal{H}_{\mathcal{S}}^G, \mathcal{G}_{\mathcal{S}}^G}.$$

The inclusion of the stratum induces a morphism in cohomology

$$H^\bullet(\overline{Y}_{\mathcal{H},\mathcal{G}}) \to \bigotimes_{G \in \mathcal{S}} H^\bullet(\overline{Y}_{\mathcal{H}_{\mathcal{S}}^G, \mathcal{G}_{\mathcal{S}}^G}). \tag{1}$$

To be precise, the hyperplane arrangements $\mathcal{H}_{\mathcal{S}}^G$ are obtained by what is called restriction/contraction of the hyperplane arrangement $\mathcal{H}$ and $\mathcal{G}_{\mathcal{S}}^G$ is a building set "induced" by $\mathcal{G}$ on $\mathcal{H}_{\mathcal{S}}^G$ (see Subsection 1.1).

Definition 0.0.5 (Restriction/Contraction)**.** Let $\mathcal{H}$ be a hyperplane arrangement in some vector space V and F some element in $\mathcal{L}_{\mathcal{H}}$. The *restriction* of $\mathcal{H}$ along F is the arrangement in V/F given by the elements of $\mathcal{L}_{\mathcal{H}}$ containing F. The *contraction* of $\mathcal{H}$ along F is the hyperplane arrangement in F given by all the hyperplanes of F that can be obtained as the intersection of some element in $\mathcal{L}_{\mathcal{H}}$ with F.

The hyperplane arrangement $\mathcal{H}_{\mathcal{S}}^G$ is the contraction along $\bigvee \mathcal{S}_{<G}$ of the restriction along G of $\mathcal{H}$.

A second important leitmotiv in the study of hyperplane arrangements is trying to relate the properties of a hyperplane arrangement with the properties of its restrictions/contractions. More precisely, we are interested in "hereditary" properties, that subsist when taking any restriction/contraction, and in "inductive" properties, that pass to the hyperplane arrangement if they are satisfied for every restriction/contraction. In this light, morphisms (4) seem of utmost importance. For some specific hyperplane arrangements and building sets the datum of the algebras $H^\bullet(\overline{Y}_{\mathcal{H},\mathcal{G}})$ together with morphisms (4) form types of structures which have already been extensively studied.

Example 0.0.6. Consider the braid arrangement Braid_n given by the diagonal hyperplanes $\{z_i = z_j\} \subset \mathbb{C}^n$. Its intersection lattice is the partition lattice over $\{1, ..., n\}$. It admits a building set $\mathcal{G}_n$ given by the partitions with only one equivalence class containing more than two elements. In this case it turns out that the hyperplane arrangements $\mathcal{H}_{\mathcal{S}}^{G}$ appearing in (4) are isomorphic to smaller braid arrangements for any nested set $\mathcal{S}$. The algebras $\{H^\bullet(\overline{Y}_{\mathsf{Braid}_n,\mathcal{G}_n}), n \in \mathbb{N}^*\}$ together with morphisms (4) form an object called a (co)operad.

Operads were introduced in the early seventies by Boardman, Vogt and May, and have played an increasingly prominent role in algebraic topology as well as general algebra. In a few words, operads are a formal abstraction of the collection of multi-variable operations on an object, much like monoids form an abstraction of the collection of endomorphisms of an object. In an operad $\mathcal{P}$, elements (which should be thought of as formal operations) are classified by the number of inputs they take, i.e. $\mathcal{P} = \{\mathcal{P}(0), \mathcal{P}(1), \mathcal{P}(2), ...\}$, and can be composed as one would compose multi-variable operations. For instance, if we have a multivariable operation μ_1 having say 2 inputs, and a second multivariable operation μ_2 having say 3 inputs, then one can compose those operations to get a new operation taking 4 inputs, for instance:

$$x_1, x_2, x_3, x_4 \to \mu_1(\mu_2(x_1, x_2, x_3), x_4).$$

This means that we should have a corresponding map: $\mathcal{P}(2) \times \mathcal{P}(3) \to \mathcal{P}(4)$, which shall be part of the datum of the operad. Of course all the different composition maps should satisfy the same associativity axioms as the usual composition. One can also consider the additional datum of a symmetric action on the operations, mimicking the symmetric action given by the permutation of variables. Naturally, this action is required to satisfy the same compatibility with composition as the usual permutation of variables.

There are many equivalent ways to formally define an operad. The one of interest to us is to view an operad as a monoidal functor defined on a particular monoidal category which can be constructed as follow. Start with the groupoid FinSet with finite sets as objects and all bijections as morphisms. Consider then the monoidal category $\mathcal{T}$ obtained by adding to the free symmetric monoidal category generated by FinSet some morphisms

$$\bigotimes_{v \text{ vertex of } t} \mathsf{Inputs}(v) \xrightarrow{t} \mathsf{Leaves}(t),$$

for each rooted tree t, with $\mathsf{Inputs}(v)$ the set of ingoing edges (the "inputs") of the vertex v and $\mathsf{Leaves}(t)$ the set of leaves of t. The composition of those morphisms is defined by substitution of trees, i.e. if t is some rooted tree and for each vertex v of t we have a rooted tree t_v with leaves $\mathsf{Inputs}(v)$, then $t \circ (\otimes t_v)$ is the tree constructed by substituting t_v at each vertex v in t. Finally, one needs to quotient by some relations expressing the compatibility between the symmetries and the composition morphisms. This construction is also called the "monad of trees". With this construction at hand, an operad can be very simply defined as a (strong) monoidal functor from $\mathcal{T}$ to some arbitrary symmetric monoidal category. The spaces of operations are the images of the finite sets by the functor, the composition morphisms are the images of the tree morphisms by the functor, the associativity axiom comes from the associativity of the composition in $\mathcal{T}$ and the compatibility of the functor with the composition, the symmetric action comes from the images of the bijections by the functor.

Over the last two decades many variants of operads appeared in the literature, such as props (Mac Lane [23]), properads (Vallette [32]), cyclic operads (Getzler-Kapranov [16]), modular operads (Getzler-Kapranov [17]) and so on. All those objects can be defined as monoidal functors from a suitable symmetric monoidal category constructed in a similar way to the one introduced in the previous paragraph. A general framework for such categories was laid out by Ralph Kaufmann and Benjamin Ward, who introduced the notion of a Feynman category [20]. Before presenting the definition let us start by setting some notations. If $\mathcal{C}$ is some category we denote by $\mathrm{Sym}(\mathcal{C})$ the free symmetric monoidal category generated by $\mathcal{C}$. For any functor $F : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ a symmetric monoidal category, there is a unique (strong) monoidal functor $\mathrm{Sym}(F) : \mathrm{Sym}(\mathcal{C}) \to \mathcal{D}$ which restricts to F on $\mathcal{C}$. If $\mathcal{C}$ is any category we denote by $\mathcal{C}^{\mathrm{iso}}$ the category with the same objects as $\mathcal{C}$ and morphisms the isomorphisms of $\mathcal{C}$. If we are given a diagram of categories $\mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{G} \mathcal{E}$, the comma category $(F \downarrow G)$ is the category having for objects triples $(c \in \mathcal{C}, e \in \mathcal{E}, \phi : F(c) \to G(e))$ and for morphisms commutative diagrams. If the functors F and G are clear from the context we will write instead $(\mathcal{C} \downarrow \mathcal{E})$.

Definition 0.0.7 (Feynman category)**.** A triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ is a *Feynman category* if $\mathcal{V}$ is a groupoid, $\mathcal{F}$ is a symmetric monoidal category and $\imath : \mathcal{V} \to \mathcal{F}$ is a functor such that:

1. The functor $\imath$ induces an equivalence of categories $\mathrm{Sym}(\imath) : \mathrm{Sym}(\mathcal{V}) \to \mathcal{F}^{\mathrm{iso}}$.
2. The functor $\imath$ induces an equivalence of categories $\mathrm{Sym}((\mathcal{F} \downarrow \mathcal{V})^{\mathrm{iso}}) \to (\mathcal{F} \downarrow \mathcal{F})^{\mathrm{iso}}$.
3. For every object $\star \in \mathcal{V}$, the comma category $(\mathcal{F} \downarrow \star)$ is essentially small (i.e. is equivalent to a small category).

Example 0.0.8.

- The triple $(\mathsf{FinSet}, \mathcal{T}, \imath)$ with FinSet and $\mathcal{T}$ the categories introduced previously and $\imath$ the obvious inclusion is a Feynman category.
- Consider the groupoid $\star$ with only one object and just the identity morphism. Denote $\mathcal{A}$ the monoidal category with objects the finite sets, morphisms the surjections and monoidal structure given by disjoint union. The triple $(\star, \mathcal{A}, \imath)$ with $\imath$ the obvious inclusion is a Feynman category.

An object of $\mathcal{V}$ will be called an arity and a morphism of $\mathcal{F}$ will be called a structural morphism. In plain English, Definition 0.0.7 means that a Feynman category can be constructed by adding morphisms of the form

$$\bigotimes \text{Objects} \longrightarrow \text{Object}$$

to the free symmetric monoidal category generated by some groupoid, as we did to construct $\mathcal{T}$ out of FinSet.

Definition 0.0.9 (Operad over a Feynman category)**.** Let $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ be a Feynman category and $\mathcal{C}$ a symmetric monoidal category. An *operad over* $\mathfrak{F}$ in $\mathcal{C}$ is a strong monoidal functor from $\mathcal{F}$ to $\mathcal{C}$, and a cooperad over $\mathfrak{F}$ in $\mathcal{C}$ is a strong monoidal functor from $\mathcal{F}$ to $\mathcal{C}^{op}$. A module over $\mathfrak{F}$ in $\mathcal{C}$ is a functor from $\mathcal{V}$ to $\mathcal{C}$.

(Co)operads (resp. modules) over $\mathfrak{F}$ will also be called $\mathfrak{F}$-(co)operads (resp. $\mathfrak{F}$-modules).

Example 0.0.10.

- An operad over $(\mathsf{FinSet}, \mathcal{T}, \imath)$ is a classical operad.
- An operad over $(\star, \mathcal{A}, \imath)$ is a monoid.

The first objective of this thesis is to find a Feynman category over which the collection of algebras $H^\bullet(\overline{Y}_{\mathcal{H},\mathcal{G}})$ for all pairs $(\mathcal{H}, \mathcal{G})$ together with morphisms (4) for all nested sets form a (co)operad. Example 0.0.6 shows that this Feynman category should "contain" in some sense the Feynman category encoding classical operads.

Instead of dealing with hyperplane arrangements we shall be working at the combinatorial level, meaning with lattices. This means that we will lose all the information which is not combinatorial, but we acquire more generality because we can work with lattices which are not necessarily intersection lattices of some hyperplane arrangement. The class of lattices we will focus on in this thesis is that of geometric lattices.

Definition 0.0.11 (Geometric lattice)**.** A finite lattice $(\mathcal{L}, \leq)$ is said to be *geometric* if it satisfies the following properties:

- For every pair of elements $G_1 \leq G_2$, all the maximal chains of elements between G_1 and G_2 have the same cardinal. *(Jordan–Hölder property)*
- The rank function $\rho : \mathcal{L} \to \mathbb{N}$ which assigns to any element G of $\mathcal{L}$ the cardinal of any maximal chain of elements from $\hat{0}$ to G (not counting $\hat{0}$) satisfies the inequality
$$\rho(G_1 \wedge G_2) + \rho(G_1 \vee G_2) \leq \rho(G_1) + \rho(G_2)$$
for every G_1, G_2 in $\mathcal{L}$. *(Sub-modularity)*
- Every element in $\mathcal{L}$ can be obtained as the supremum of some set of atoms (i.e. elements of rank 1). *(Atomicity)*

This class strictly contains every lattice arising as the intersection lattice of some hyperplane arrangement (the rank function being given by the codimension). See [28] Example 2.1.22 for a geometric lattice not representable over any field. One could think of geometric lattices as a combinatorial abstraction of hyperplane arrangements. We have drawn in Figure 2 the Hasse diagrams of some posets. Let us quickly sort which of those posets are geometric lattices. Poset (a) is not even a lattice because the two top elements do not have an infimum. Poset (b) is a lattice but is not geometric because it is not atomic. Poset (c) is not a geometric lattice because it is not atomic and it does not satisfy the Jordan–Hölder property. Poset (d) is an atomic lattice satisfying the Jordan–Hölder property but is not geometric because its rank function is not submodular. Finally, posets (e) and (f) are geometric lattices. The first one is the intersection lattice of the arrangement given in Example 0.0.1 , and the second one is the intersection lattice of the coordinate arrangement $\{z_1 = 0\}, \{z_2 = 0\}, \{z_3 = 0\}$.

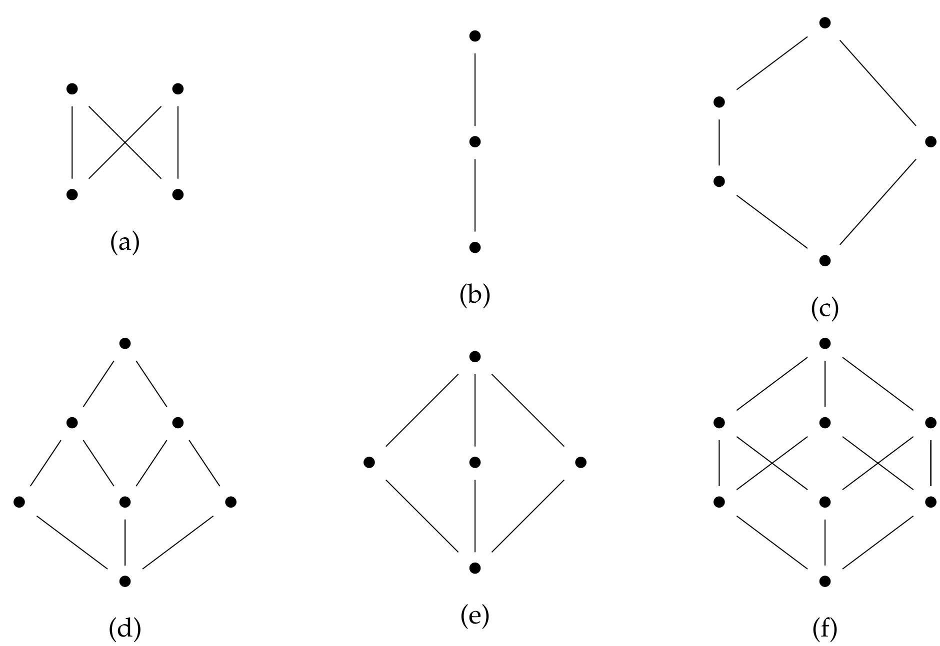

Figure 2: Hasse diagrams of some posets

In this document every lattice will be assumed to be geometric. A geometric lattice is equivalent to the datum of what is called a loopless simple matroid, also known as a combinatorial geometry. There are many different definitions of a matroid and we list some of them here, referring to [35] for more details.

Definition 0.0.12 (Matroids via independent subsets)**.** A *matroid* is a pair of a finite set E and a set $\mathcal{I}$ of subsets of E (the "independent" subsets) satisfying the axioms

- For any I in $\mathcal{I}$, every subset of I belongs to $\mathcal{I}$.
- For any I, J in $\mathcal{I}$, if $\#J > \#I$ there exists an element a in J and not in I such that $I \cup \{a\}$ is independent.

Definition 0.0.13 (Matroids via closure operator)**.** A *matroid* is a pair of a finite set E and an application (the "closure operator")

$$\sigma : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$$

satisfying the axioms

- For any $X \in \mathcal{P}(E)$ we have $X \subseteq \sigma(X)$.
- For any $X \subseteq Y \in \mathcal{P}(E)$ we have $\sigma(X) \subseteq \sigma(Y)$.
- For any $X \in \mathcal{P}(E)$ we have $\sigma(\sigma(X)) = \sigma(X)$.
- For any $X \in \mathcal{P}(E)$ and $a, b \in E$, if $a \in \sigma(X \cup \{b\}) \setminus \sigma(X)$ then $b \in \sigma(X \cup \{a\}) \setminus \sigma(X)$.

Definition 0.0.14 (Matroids via circuits)**.** A *matroid* is a pair of a finite set E and a set $\mathcal{C}$ of subsets of E (the "circuits") satisfying the axioms

- The empty set is not a circuit.
- If $C_1 \subseteq C_2 \in \mathcal{C}$ then $C_1 = C_2$.
- If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there exists a circuit $C \subseteq C_1 \cup C_2 \setminus \{e\}$.

One passes from the independent subset definition to the circuit definition by defining the circuits as the minimal non-independent subsets. One passes from the circuit definition to the closure definition by putting

$$\sigma(X) := X \cup \{x \,|\, \exists C \in \mathcal{C} \text{ with } C \subseteq X \cup \{x\} \text{ and } x \in C\}.$$

Finally, one passes from the closure definition to the independent subset definition by defining an independent subset as a subset S such that for every $s \in S$, the element s does not belong to $\sigma(S \setminus \{s\})$. Matroids form a combinatorial abstraction of the general notion of independence across mathematics, encompassing linear independence, affine independence, algebraic independence and so on. Matroids are also very strongly related to graph theory via the observation that for any graph G with set of edges E, the subsets of E given by the minimal cycles of G satisfy the circuit axioms (0.0.14), and thus they define a matroid. The independent subsets of this matroid are given by the subforests of G. We will denote this matroid by M_G.

Example 0.0.15. Consider the graph G depicted in Figure 3. The circuits of M_G are $\{1, 2\}$, $\{3\}$, $\{1, 4, 5\}$ and $\{2, 4, 5\}$. The independent subsets of M_G are $\emptyset$, $\{1\}$, $\{2\}$, $\{4\}$, $\{5\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 4\}$, $\{2, 5\}$, $\{4, 5\}$ together with the subsets obtained by adding 6 to the latter subsets.

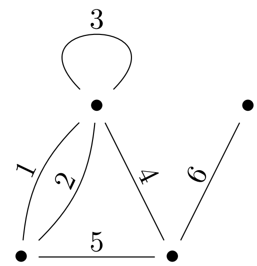

Figure 3: A graph with labelled edges

A matroid $(E, \mathcal{I})$ is said to be loopless if every singleton is independent. A matroid is said to be simple if for every pair of elements a and b, if $\{a\}$ and $\{b\}$ are independent then $\{a, b\}$ is also independent. A flat of a matroid $M = (E, \sigma)$ is a subset $F \subseteq E$ such that $\sigma(F)$ is equal to F. The set of flats of M denoted by $\mathcal{L}_M$ ordered by inclusion is a geometric lattice with meet given by the intersection. Conversely if $\mathcal{L}$ is a geometric lattice then the datum (E, σ) where E is the set of atoms of $\mathcal{L}$ and σ is the map defined by

$$\sigma(X) = \bigcap_{\substack{F \in \mathcal{L} \\ X \subset \mathrm{At}_{\leq}(F)}} \mathrm{At}_{\leq}(F)$$

is a simple loopless matroid. Those two constructions are inverse to each other on simple loopless matroids. In this document we will mainly use the axiomatization by geometric lattices for convenience (it is the axiomatization that makes restriction/contraction the most transparent).

In [8], De Concini and Procesi gave an explicit presentation by generators and relations of the cohomology rings $H^\bullet(\overline{Y}_{\mathcal{H},\mathcal{G}})$, which only depends on $\mathcal{L}_\mathcal{H}$ and $\mathcal{G}$ (in other words, $H^\bullet(\overline{Y}_{\mathcal{H},\mathcal{G}})$ is "combinatorial"). In [14], Feichtner and Yuzvinsky generalized those rings to any pair $(\mathcal{L}, \mathcal{G})$ where $\mathcal{G}$ is a building set of some atomic lattice $\mathcal{L}$. A pair $(\mathcal{L}, \mathcal{G})$ with $\mathcal{G}$ the building set of some lattice $\mathcal{L}$ will be called a "built lattice". In this document we will restrict to geometric lattices. We will call those rings the Feichtner–Yuzvinsky rings and denote them $\mathrm{FY}(\mathcal{L}, \mathcal{G})$. If $\mathcal{G} = \mathcal{L} \setminus \{\hat{0}\}$ we get the so-called Chow ring of the geometric lattice/combinatorial geometry $\mathcal{L}$. One could think of Feichtner–Yuzvinsky algebras as invariants which help us study matroids, the same way cohomology algebras help us study spaces. In the realizable case, meaning when the geometric lattice $\mathcal{L}$ is the intersection lattice of a hyperplane arrangement, the Feichtner–Yuzvinsky ring is the cohomology algebra of a complex smooth projective manifold and therefore it satisfies Poincaré duality and the Kähler package.

A third important leitmotiv in the study of hyperplane arrangements/matroids is that properties of realizable matroids that assuredly seem deeply geometrical, may often magically subsist in the non-realizable setting. For instance, a celebrated result of Adiprasito-Huh-Katz [1] states that every combinatorial Chow ring satisfies (a purely algebraic version of) Poincaré duality and the Kähler package.

Definition 0.0.16 (Poincaré duality)**.** A graded commutative algebra A over a field $\mathbb{K}$ is said to satisfy *Poincaré duality* if there exists a top degree n such that we have $A^p = 0$ for all $p \geq n+1$, and an isomorphism of vector spaces $A^n \simeq \mathbb{K}$ (the "degree map") such that the multiplication

$$A^p \otimes A^{n-p} \to A^n \simeq \mathbb{K}$$

is a non-degenerate pairing for all $p \leq n$.

Definition 0.0.17 (Kähler package)**.** A graded commutative algebra A over $\mathbb{Q}$ is said to satisfy the *Kähler package* if it has a top degree n and an isomorphism

$$\deg : A^n \xrightarrow{\sim} \mathbb{Q}$$

such that there exists a non-empty cone $\Sigma \subset A^1$ of elements ℓ satisfying the following properties.

- The multiplication map
$$\cdot \ell^{n-2k} : A^k \to A^{n-k}$$
is an isomorphism for all $k \leq n/2$. (*Hard Lefschetz property*)
- The bilinear form
$$Q^k_\ell : A^k \times A^k \to \mathbb{Q}$$
defined by $Q^k_\ell(a,b) = (-1)^k \deg(a\ell^{2n-k}b)$ is positive defined on $\ker(\cdot \, \ell^{2n-k+1})$. (*Hodge-Riemann relations*)

The result of Adiprasito-Huh-Katz was later generalized to all Feichtner–Yuzvinsky rings by Pezzoli-Pagaria [29].

In [5], Bibby, Denham and Feichtner showed that the morphisms (4), which are geometrical in nature (they are induced by the inclusion of strata) also exist in the purely combinatorial setting. More precisely, for any nested $\mathcal{S}$ in some building set $\mathcal{G}$ of some geometric lattice $\mathcal{L}$ we have a morphism of algebras

$$\mathrm{FY}(\mathcal{L}, \mathcal{G}) \to \bigotimes_{G \in \mathcal{S}} \mathrm{FY}(\mathcal{I}^G_{\mathcal{S}}, \mathrm{Ind}_{\mathcal{I}^G_{\mathcal{S}}}(\mathcal{G})), \tag{2}$$

with

$$\mathcal{I}^G_{\mathcal{S}} := [\bigvee_{\substack{G' \in \mathcal{S} \\ G' < G}} G', G] \subset \mathcal{L}$$

(those intervals being the intersection lattices of the hyperplane arrangements $\mathcal{H}_S^G$ in the realizable case) and $\mathrm{Ind}_{\mathcal{I}_S^G}(\mathcal{G})$ the building set "induced" by $\mathcal{G}$ on $\mathcal{I}_S^G$ (see Definition 1.1.13). In addition to those structural morphisms, each isomorphism of poset $\mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ sending some building set $\mathcal{G}$ of $\mathcal{L}$ to some building set $\mathcal{G}'$ of $\mathcal{L}'$ (called an isomorphism of built lattices) induces an isomorphism of algebras

$$\mathrm{FY}(\mathcal{L}, \mathcal{G}) \xrightarrow{\sim} \mathrm{FY}(\mathcal{L}', \mathcal{G}'). \tag{3}$$

In Section 2 we will construct a Feynman category $\mathfrak{LBS} = (\mathbf{LBS}_{\mathrm{irr}}, \mathbf{LBS}, \imath)$ such that the datum of the algebras $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ together with the structural morphisms (5) and the symmetries (6) forms a cooperad over $\mathfrak{LBS}$ (in the category of graded commutative algebras). To put it in a nutshell the objects of **LBS** are going to be the built lattices, the symmetries are going to be the isomorphisms of built lattices defined above and the nested sets will form the structural morphisms. The combinatorial core of the construction is to define a suitable "composition" of nested sets.

Having completed this first step, our new objectives are three-fold. Firstly, we would be interested in studying the Feynman category $\mathfrak{LBS}$ itself. We shall mainly be concerned with finding a (graded) presentation of $\mathfrak{LBS}$, i.e. find a collection of morphisms of $\mathfrak{LBS}$ which generate every other morphism via tensorization and composition, and work out which relations those generators satisfy. A presentation of $\mathfrak{LBS}$ is given in Proposition 2.3.1 and Proposition 2.3.2. From a practical point of view, presentations of Feynman categories are quite useful to define operads. For instance in the case of classical operads, we know that the rooted trees with only two inner vertices together with the isomorphisms generate every morphism in $\mathcal{T}$ via tensorization and composition, and we know which relations those generators satisfy. This implies that classical operads can be defined by specifying only their partial compositions (composition of just two operations), and the symmetric group action in each arity.

Our second objective will be to introduce several new $\mathfrak{LBS}$-operads, other than the operad of Feichtner–Yuzvinsky algebras. In Section 3 and Section 5, we show that both the families

$$\{\mathrm{OS}(\mathcal{L}), (\mathcal{L}, \mathcal{G})\} \quad \text{and} \quad \{\overline{\mathrm{OS}}(\mathcal{L}), (\mathcal{L}, \mathcal{G})\}$$

admit an $\mathfrak{LBS}$-(co)operadic structure. The structural morphisms of the second one are given by a combinatorial generalization of the residue morphisms in the realizable case. When restricted to the partition lattices with building set of partitions with only one non-trivial equivalence class, this gives the linear dual of a well-known operad called Grav introduced by Getzler in [15]. The restriction of the operad of Orlik–Solomon algebras is the linear dual of a classical operad callled Ger which encodes Gerstenhaber algebras.

Finally, we would also like to study in more details the $\mathfrak{LBS}$-cooperad of Feichtner–Yuzvinsky algebras, which will be denoted by $\mathbb{FY}$. We also denote by $\mathbb{FY}^\vee$ the operad over $\mathfrak{LBS}$ obtained by dualizing the objects, the structural morphisms and the symmetries of $\mathbb{FY}$. One of our underlying ambitions is to be able to relate the important properties of the algebras $\mathrm{FY}(\mathcal{L},\mathcal{G})$ highlighted previously, with the properties of the global object $\mathbb{FY}$. Strikingly, even though operads over Feynman categories are in general much more complex than associative algebras, one can study them the same way one would study an associative algebra. For instance, one can first try to look for a "presentation" of $\mathbb{FY}^\vee$. In our operadic context this means finding a set of elements in each object $\mathbb{FY}^\vee(\mathcal{L},\mathcal{G})$ which generate every other element in every arity by sum and product over structural morphisms, and then examining which relations those products satisfy. In Section 3, Proposition 3.1.3, we show that the degree maps generate $\mathbb{FY}^\vee$ in the above sense, and we describe the relations between those generators. In Remark 3.1.4 we explain how the fact that the degree maps operadically generate every other element in $\mathbb{FY}^\vee$ is essentially equivalent to the fact that the Feichtner–Yuzvinsky algebras satisfy Poincaré duality.

For associative algebras, a computational tool to deal with presentations of algebras is the theory of Gröbner bases. The general idea of Gröbner bases is to start by choosing an order on the generators of the presentation. This order is then used to derive an order on all monomials, which is compatible in some sense with the multiplication of monomials (we call such orders "admissible"). We then use this order to rewrite monomials in the quotient algebra:

$$\text{greatest term } \longrightarrow \sum \text{lesser terms},$$

for every relation $R = \text{greatest term } - \sum \text{lower terms}$ in some subset $\mathcal{B}$ of the ideal generated by the relations of the presentation. The greatest term is usually called the "leading term". The subset $\mathcal{B}$ is called a Gröbner basis when it contains "enough" elements. More precisely, we want that every leading term of some relation in the quotient algebra is divisible by the leading term of some element of $\mathcal{B}$. In general the goal is to find a Gröbner basis as little as possible so that the rewriting is as easy as possible. At the end of the rewriting process (which stops if the monomials are well-ordered) we are left with all the monomials which are not rewritable i.e. which are not divisible by a leading term of some element of $\mathcal{B}$. Those monomials are called "normal" and they form a linear basis of our algebra exactly when $\mathcal{B}$ is a Gröbner basis. This basis comes with a multiplication table given by the rewriting process.

It turns out that this general strategy can be applied to structures which are much more general and complex than associative algebras, such as operads. Loosely speaking, all we need in order to implement this machinery is to be able to make (reasonable) sense of the key words used above, such as "monomials", "admissible orders" and "divisibility between monomials". For operads over a Feynman category, the only non-trivial part is to construct admissible orders on monomials out of orders on generators. The main issue comes from the symmetries, because usually the compatibility with symmetries is too strong and prevents us from finding any admissible order. In order to circumvent this problem, drawing inspiration from the case of classical operads which was sorted out by Dotsenko and Khoroshkin in [11], in Section 4 we introduce a notion of a "shuffle" operad over $\mathfrak{LBS}$, and we develop a theory of Gröbner bases for shuffle operads over $\mathfrak{LBS}$.

Another important notion in the theory of associative algebras is Koszul duality. In English we say that a (graded) associative algebra is Koszul when it is generated by elements of grading 1, the relations between those generators are generated by elements of grading 2, relations between relations are generated in grading 3 and so on. The formal definition revolves around the so-called "bar" and "cobar" operators (denoted B and Ω respectively)

$$\mathrm{B} : \{ \text{ dg graded coassociative coalgebras } \} \leftrightarrows \{ \text{ dg graded associative algebras } \} : \Omega.$$

A graded associative algebra is said to be Koszul when there is a quasi-isomorphism between $\Omega(H(\mathrm{B}A))$ and A. In this case $\Omega(H(\mathrm{B}A))$ is the minimal model of A.

Kaufmann and Ward have shown in [20] that for certain Feynman categories called "cubical", one has a similar construction of bar/cobar operators for operads over those Feynman categories, which allows us to define Koszulness of those operads exactly as we did for associative algebras. In Section 5 Proposition 5.1.2 we show that the Feynman category $\mathfrak{LBS}$ is cubical. We then show the following theorem.

Theorem 0.0.18 (Corollary 5.3.3)**.** *The linear dual of the cooperad of Feichtner–Yuzvinsky algebras is Koszul with Koszul dual the cooperad of projective Orlik–Solomon algebras.*

This duality restricts to the duality between Hypercom and Grav on partition lattices, proved by Getzler in [15]. The proof of Getzler relies on the observation that the second page of the Leray spectral sequence of the inclusion

$$\mathbb{P}\mathcal{A}_{\mathsf{Braid}_n} \hookrightarrow \overline{Y}_{\mathsf{Braid}_n, \mathcal{G}_n}$$

is the bar construction of Hypercom, together with the fact that this spectral sequence degenerates at the second page (by a mixed Hodge theory argument). In the possibly non realizable case those geometric constructions do not exist but we still have a (purely combinatorial) Leray model for Orlik–Solomon algebras, a result proved by Bibby, Denham and Feichtner in [5]. This combinatorial Leray model is the bar construction of the operad of Feichtner–Yuzvinsky algebras, which implies that the latter operad is Koszul. Alternatively, we also prove the two results

Proposition 0.0.19 (Proposition 5.6.1)**.** *An $\mathfrak{LBS}$-operad admitting a quadratic Gröbner basis is Koszul.*

Theorem 0.0.20 (Corollary 4.5.7)**.** *The operad of Feichtner–Yuzvinsky algebras admits a quadratic Gröbner basis.*

This leads to the same conclusion.

In Section 6 we conclude the first part of this document by giving some further directions toward possible generalizations or modifications of $\mathfrak{LBS}$ which may lead to other applications.

Part 2: Koszulness of Feichtner–Yuzvinsky algebras

The second part of this manuscript is devoted to the study of the Koszulness of some Feichtner–Yuzvinsky algebras, using the operadic structure developed in the first part. We would like to stress here that this is a priori a problem unrelated to that of Koszulness of the operad $\mathbb{FY}^\vee$. However, we shall see that one can use the operadic structure to derive Koszulness of some of the Feichtner–Yuzvinsky algebras.

Koszulness is a particularly interesting property to ask of the cohomology ring of a formal space because it allows a direct computation of other rational homotopy invariants such as the rational homotopy Lie algebra (see Berglund [4]). Since the wonderful compactifications of hyperplane arrangements are known to be formal, it is natural to ask which Feichtner–Yuzvinsky algebras are Koszul, a question raised by Dotsenko in [10]. This question is largely open.

A classical way to prove the Koszulness of a given algebra is to find a quadratic Gröbner basis for this algebra. Feichtner and Yuzvinsky computed explicit Gröbner bases for the Feichtner–Yuzvinsky rings, but those bases are almost never quadratic. In fact, the Feichtner–Yuzvinsky rings themselves are not necessarily quadratic. One of the first results proving the Koszulness of some Feichtner–Yuzvinsky algebras was given by Dotsenko who proved that the Feichtner–Yuzvinsky algebras associated to the braid arrangements with minimal building sets (see Example 0.0.6) are Koszul. In a nutshell, Dotsenko introduced an explicit order on the generators of the Feichtner–Yuzvinsky rings and then used the (classical) operadic structure on this collection of rings to construct a bijection between the algebraic normal monomials associated to the latter order and relations of degree 2, and the operadic normal monomials obtained in a previous work via Gröbner bases for operads (Dotsenko-Khoroshkin [11]). By a dimension argument this implies that the relations of weight 2 form a quadratic Gröbner basis of the Feichtner–Yuzvinsky rings in question. More recently, Mastroeni-McCullough [26] proved that the combinatorial Chow rings are all Koszul, using the notion of Koszul filtrations.

In this document we will generalize the strategy of Dotsenko, using the extended operadic structure introduced in the first part of this document. However, we shall see that in order to extend the argument one needs to restrict the class of lattices we are considering. Recall that an element a in a lattice $\mathcal{L}$ is called modular if for any $b \leq a$ and c in $\mathcal{L}$ we have the identity

$$a \wedge (b \vee c) = b \vee (a \wedge c).$$

For instance a normal subgroup of a finite group G is modular in the lattice of subgroups of G. For an example of a non-modular element let us go back to poset (c) in Figure 2.

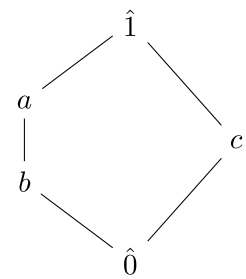

Figure 4: A poset with a non-modular element

We have

$$a \wedge (b \vee c) = a \wedge \hat{1} = a \neq b = b \vee \hat{0} = b \vee (a \wedge c),$$

which shows that a is not modular in this lattice. In [30] Stanley introduced the following definition.

Definition 0.0.21 (Stanley)**.** A lattice $\mathcal{L}$ is called *supersolvable* if it admits a maximal chain of modular elements.

The denomination comes from the fact that the lattice of subgroups of a finite supersolvable group is supersolvable. Supersolvable lattices have very nice properties in general. For instance, we have the following classical result.

Theorem 0.0.22 (Yuzvinsky, [36])**.** *The Orlik–Solomon algebra of a supersolvable lattice admits a quadratic Gröbner basis.*

In this document we will prove a similar result for Feichtner–Yuzvinsky algebras. We first introduce the following notion of supersolvability for built lattices.

Definition 0.0.23. A built lattice $(\mathcal{L}, \mathcal{G})$ is *supersolvable* if $\mathcal{L}$ admits a maximal chain

$$\hat{0} = G_1 < ... < G_n = \hat{1}$$

of modular elements in $\mathcal{G}$ such that for all G' in $\mathcal{G}$ and all $i \leq n$ we have $G' \wedge G_i \in \mathcal{G} \cup \hat{0}$.

The main result of the second part of this PhD is the following theorem.

Theorem 0.0.24 (Theorem 8.3.1)**.** *Let $(\mathcal{L}, \mathcal{G})$ be a supersolvable built lattice. The algebra $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ admits a quadratic Gröbner basis and is therefore Koszul.*

Theorem 0.0.22 immediately implies that combinatorial Chow rings of supersolvable lattices have quadratic Gröbner bases, strengthening the result of Mastroeni-McCullough for supersolvable lattices. Turning our attention towards minimal building sets, Theorem 0.0.22 also gives us the following result.

Theorem 0.0.25 (Theorem 8.4.5)**.** *Let $\mathcal{L}$ be a supersolvable lattice and $\mathcal{G}_{\min}$ the building set of irreducible elements of $\mathcal{L}$. The algebra $\mathrm{FY}(\mathcal{L}, \mathcal{G}_{\min})$ admits a quadratic Gröbner basis and is therefore Koszul.*

Stanley [30] proved that the geometric lattices associated to chordal graphs (i.e. graphs such that every cycle has a chord) are supersolvable. Alternatively, one can associate to a graph G a built lattice $(\mathcal{L}_G, \mathcal{G}_G)$ where $\mathcal{L}_G$ is the lattice associated to G and $\mathcal{G}_G$ is the building set of connected closed subgraphs of G. Stanley's original argument also shows that $(\mathcal{L}_G, \mathcal{G}_G)$ is a supersolvable built lattice. This implies by Theorem 0.0.22 that its Feichtner–Yuzvinsky algebra admits a quadratic Gröbner basis. Since the complete graphs are chordal we recover the result of Dotsenko.

In [21], Losev and Manin introduced moduli spaces of stable curves with marked points of two types, where the points of the first type are not allowed to coincide with any other points, and those of second type are allowed to coincide between them. Those moduli spaces form the components of an object called the "extended modular operad", introduced by Losev and Manin in the sequel [22]. In [25], Manin asked if the cohomology algebras of those moduli spaces are Koszul. By considering the family of chordal graphs $G_{m,n}$, where $G_{m,n}$ has $m+n$ vertices, the first m vertices are neighbors of every vertices and the last n vertices are neighbors only of the first m vertices, one obtains the following result.

Theorem 0.0.26 (Theorem 9.2.1)**.** *The cohomology algebras of the components of the extended modular operad in genus 0 have quadratic Gröbner bases and are therefore Koszul.*

Layout of the document

In Section 1 we define the main combinatorial characters of the story.

In Section 2 we define the Feynman category $\mathfrak{LBS}$ and give a presentation of this Feynman category.

In Section 3 we define the cooperad of Feichtner–Yuzvinsky algebras and the cooperad of Orlik–Solomon algebras.

In Section 4 we develop a theory of Gröbner bases for operads over $\mathfrak{LBS}$.

In Section 5 we show that $\mathfrak{LBS}$ is cubical and we unpack the Koszul duality theory for operads over $\mathfrak{LBS}$. We show that the (co)operad of Feichtner–Yuzvinsky algebras is Koszul.

In Section 6 we give some general comments towards possible generalizations and modifications of $\mathfrak{LBS}$.

In Section 7 we make more explicit the relation between geometric lattices and matroids, which will be used in the proof of the main result of the second part.

In Section 8 we prove Theorem 0.0.22 and we deduce Theorem 0.0.23.

In Section 9 we concentrate our attention toward supersolvable built lattices associated to chordal graphs, which leads to Theorem 0.0.24.

Finally, in Section 10 we take a step back and give some general comments for further research on Koszulness of Feichtner–Yuzvinsky algebras.

Introduction (Français)

Le document qui suit vise à introduire et étudier de nouvelles structures de type opéradique sur certains invariants algébriques de matroïdes.

Partie 1: Arrangements d'hyperplans et structures opéradiques

Un arrangement d'hyperplans est une collection finie d'hyperplans dans un espace vectoriel de dimension finie. Pour tout arrangement d'hyperplans $\mathcal{H}$, le treillis d'intersection $\mathcal{L}_\mathcal{H}$ de $\mathcal{H}$ est l'ensemble $\{\bigcap_{H \in S} H, S \subset \mathcal{H}\}$ des intersections possibles ordonné par l'inclusion renversée. Comme son nom l'indique il s'agit d'un treillis, dont le supremum est donné par l'intersection. Ce treillis contient toute l'information combinatoire de l'arrangement d'hyperplans.

Exemple 0.0.1. Considérons l'arrangement Braid_3 donné par les hyperplans $\{z_1 = z_2\}$,-$\{z_2 = z_3\}$, et $\{z_1 = z_3\}$ dans $\mathbb{C}^3$. Le diagramme de Hasse du treillis d'intersection de cet arrangement d'hyperplans est représenté sur la Figure 5. Il s'agit de l'ensemble des partitions de $\{1, 2, 3\}$ ordonnées par raffinement.

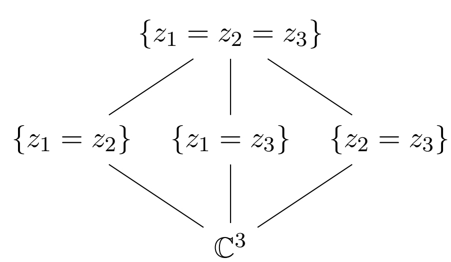

Figure 5: Diagramme de Hasse de $\mathcal{L}_{\mathsf{Braid}_3}$

Un leitmotif important dans l'étude des arrangements d'hyperplans est de comprendre quelle information est déterminée par le treillis d'intersection (l'information "combinatoire") et quelle information ne l'est pas. Un théorème classique dans cet esprit est le suivant.

Proposition 0.0.2 (Orlik-Solomon, [27])**.** *Soit $\mathcal{H}$ un arrangement d'hyperplans et $\mathcal{A}_\mathcal{H}$ le complémentaire de la réunion des hyperplans de $\mathcal{H}$. L'algèbre de cohomologie de $\mathcal{A}_\mathcal{H}$ est isomorphe au quotient de l'algèbre extérieure $\Lambda[e_H,\, H \in \mathcal{H}]$ (avec chaque e_H en degré 1) par les éléments $\delta(e_{H_1}...e_{H_k})$ pour tout $\{H_1, ..., H_k\}$ tel que*

$$\dim H_1 \cap ... \cap H_k > n - k$$

(où δ dénote l'unique dérivation qui envoie tous les générateurs sur 1). L'algèbre de cohomologie de la projectivisation $\mathbb{P}\mathcal{A}_\mathcal{H}$ est isomorphe à la sous-algèbre engendrée par les éléments $e_H - e_{H'}$, $H, H' \in \mathcal{H}$.

La dimension des intersections d'hyperplans est donnée par la fonction de rang du treillis $\mathcal{L}_\mathcal{H}$, ce qui implique que l'algèbre décrite ci-dessus est déterminée par $\mathcal{L}_\mathcal{H}$. En d'autres termes, l'algèbre de cohomologie du complémentaire d'un arrangement d'hyperplans ainsi que celle de sa projectivisation sont de nature "combinatoire". Nous noterons ces algèbres $\mathrm{OS}(\mathcal{L}_H)$ et $\overline{\mathrm{OS}}(\mathcal{L}_\mathcal{H})$ respectivement.

Pour aller plus loin dans l'étude des complémentaires projectivisés d'arrangements, il est intéressant de trouver un modèle pour $H^\bullet(\mathbb{P}\mathcal{A}_H)$. Un moyen classique d'atteindre ce but est de trouver une "bonne" compactification du complémentaire projectif

$$\mathbb{P}\mathcal{A}_\mathcal{H} \hookrightarrow \overline{Y}_\mathcal{H}$$

pour ensuite examiner la suite spectrale de Leray de cette inclusion. Pour que cette suite spectrale soit calculable, on aimerait que le complémentaire de $\mathbb{P}\mathcal{A}_\mathcal{H}$ dans la compactification soit un diviseur à croisements normaux, auquel cas cette compactification est dite "merveilleuse". De Concini et Procesi ont montré dans [8] qu'il existe plusieurs compactifications merveilleuses du complémentaire projectivisé d'un arrangement d'hyperplans, qui peuvent être obtenues par éclatements successifs le long de certaines intersections d'hyperplans bien choisies. Si $\mathcal{G}$ est un sous-ensemble de $\mathcal{L}_\mathcal{H}$ on notera $\overline{Y}_{\mathcal{H},\mathcal{G}}$ le résultat obtenu après éclatement successif de $\mathbb{P}(\mathbb{C}^n)$ le long des éléments de $\mathcal{G}$ (par taille croissante). Pour que $\overline{Y}_{\mathcal{H},\mathcal{G}}$ soit une compactification merveilleuse l'ensemble $\mathcal{G}$ doit contenir toutes les intersections non-transverses. Ceci peut être axiomatisé par la notion d'ensemble de construction, introduite dans un cadre combinatoire par Feichtner-Kozlov [13]. On réfère à la section 1.1 pour la définition des symboles utilisés dans la définition suivante.

Définition 0.0.3 (Ensemble de construction)**.** Soit $\mathcal{L}$ un treillis. Un *ensemble de construction* $\mathcal{G}$ de $\mathcal{L}$ est un sous-ensemble de $\mathcal{L} \setminus \{\hat{0}\}$ tel que pour tout élément X de $\mathcal{L}$ le morphisme

$$\prod_{G \in \max \mathcal{G}_{\leq X}} [\hat{0}, G] \xrightarrow{\vee} [\hat{0}, X]$$

est un isomorphisme.

En conclusion, nous avons une compactification merveilleuse $\overline{Y}_{\mathcal{H},\mathcal{G}}$ de $\mathbb{P}\mathcal{A}_{\mathcal{H}}$ pour tout ensemble de construction $\mathcal{G}$ de $\mathcal{L}_{\mathcal{H}}$. Les variétés projectives lisses $\overline{Y}_{\mathcal{H},\mathcal{G}}$ sont stratifiées par les diviseurs exceptionnels $\mathcal{D}_G, G \in \mathcal{G}$ obtenus après chaque éclatement. Pour tout sous-ensemble $\mathcal{S} \subset \mathcal{G}$ l'intersection $\bigcap_{G\in\mathcal{S}} \mathcal{D}_G$ est non-vide si et seulement si $\mathcal{S}$ forme un ensemble niché de $(\mathcal{L}, \mathcal{G})$.

Définition 0.0.4 (Ensemble niché)**.** Soit $\mathcal{L}$ un treillis et $\mathcal{G}$ un ensemble de construction de $\mathcal{L}$. Un sous-ensemble $\mathcal{S}$ de $\mathcal{G}$ est dit *niché* si pour toute partie $\mathcal{A} \subset \mathcal{S}$ d'éléments deux à deux incomparables, le supremum des éléments de $\mathcal{A}$ n'appartient pas à $\mathcal{G}$ dès que $\mathcal{A}$ est de cardinal supérieur ou égal à deux.

De Concini et Procesi ont montré que pour tout ensemble niché $\mathcal{S}$ de $(\mathcal{L}_{\mathcal{H}}, \mathcal{G})$, la strate

$$\overline{Y}_{\mathcal{S}} := \bigcap_{G\in\mathcal{S}} \mathcal{D}_G$$

est isomorphe à un produit de compactifications merveilleuses d'arrangements d'hyperplans plus "petits" $\mathcal{H}^G_{\mathcal{S}}$ le long d'ensembles de construction $\mathcal{G}^G_{\mathcal{S}}$, indexés par les éléments de $\mathcal{S}$:

$$\overline{Y}_{\mathcal{S}} \simeq \prod_{G\in\mathcal{S}} \overline{Y}_{\mathcal{H}^G_{\mathcal{S}},\mathcal{G}^G_{\mathcal{S}}}.$$

L'inclusion de la strate induit un morphisme en cohomologie

$$H^\bullet(\overline{Y}_{\mathcal{H},\mathcal{G}}) \to \bigotimes_{G\in\mathcal{S}} H^\bullet(\overline{Y}_{\mathcal{H}^G_{\mathcal{S}},\mathcal{G}^G_{\mathcal{S}}}). \tag{4}$$

Pour être tout à fait précis, les arrangements d'hyperplans $\mathcal{H}^G_{\mathcal{S}}$ sont obtenus par ce qu'on appelle restriction/contraction de l'arrangement d'hyperplans initial $\mathcal{H}$, et l'ensemble de construction $\mathcal{G}^G_{\mathcal{S}}$ est "induit" par $\mathcal{G}$ sur $\mathcal{H}^G_{\mathcal{S}}$ (voir Section 1.1).

Définition 0.0.5 (Restriction/Contraction)**.** Soit $\mathcal{H}$ un arrangement d'hyperplans dans un espace vectoriel V, et F un élément de $\mathcal{L}_{\mathcal{H}}$. La *restriction* de $\mathcal{H}$ le long de F est l'arrangement d'hyperplans dans V/F donné par les éléments de $\mathcal{L}_{\mathcal{H}}$ contenant F. La *contraction* de $\mathcal{H}$ le long de F est l'arrangement d'hyperplans donné par tous les hyperplans de F qui peuvent être obtenus par l'intersection d'un élément de $\mathcal{L}_{\mathcal{H}}$ avec F.

L'arrangement d'hyperplans $\mathcal{H}^G_{\mathcal{S}}$ est la contraction le long de $\bigvee \mathcal{S}_{<G}$ de la restriction le long de G de $\mathcal{H}$.

Un second leitmotif important dans l'étude des arrangements d'hyperplans est d'essayer de relier les propriétés d'un arrangement d'hyperplans avec les propriétés de ses restrictions/contractions. Plus précisément, nous sommes intéressés par les propriétés "héréditaires" des arrangements, qui subsistent après restriction/contraction, et les propriétés "inductives" qui passent à tout l'arrangement d'hyperplans si elles sont vérifiées sur chaque restriction/contraction. Sous cet angle, les morphismes (4) semblent revêtir une importance particulière. Pour certains arrangements d'hyperplans et ensembles de constructions spécifiques, la donnée des algèbres $H^\bullet(\overline{Y}_{\mathcal{H},\mathcal{G}})$ avec les morphismes (4) forme des types de structures qui ont déjà été intensivement étudiés.

Exemple 0.0.6. Considérons l'arrangement de tresse Braid_n donné par les hyperplans diagonaux $\{z_i = z_j\} \subset \mathbb{C}^n$. Son treillis d'intersection est l'ensemble des partitions de $\{1, ..., n\}$ ordonné par raffinement. Ce treillis admet un ensemble de construction donné par l'ensemble des partitions qui contiennent une seule classe d'équivalence avec strictement plus d'un élément. Dans ce cas on vérifie que les arrangements $\mathcal{H}_\mathcal{S}^G$ apparaissant dans (4) sont isomorphes à des arrangements de tresses plus petits, pour n'importe quel ensemble niché $\mathcal{S}$. Les algèbres $\{H^\bullet(\overline{Y}_{\mathsf{Braid}_n,\mathcal{G}_n}), n \in \mathbb{N}^*\}$ avec les morphismes (4) forment un objet appelé une (co)opérade. Il s'agit du dual (linéaire) de l'opérade Hypercom qui encode les algèbres hypercommutatives.

Les opérades ont été introduites vers le début des années soixante-dix par Boardmann, Vogt et May, et ont joué un rôle de plus en plus important en topologie algébrique et en algèbre générale. En quelques mots, les opérades forment une abstraction formelle de la collection des opérations à plusieurs variables d'un objet, de la même manière que les monoïdes forment une abstraction formelle de la collection des endomorphismes d'un objet. Dans une opérade $\mathcal{P}$, les éléments (qui devraient être vus comme des opérations formelles) sont classifiés par le nombre d'entrées qu'ils prennent, c'est-à-dire

$$\mathcal{P} = \{\mathcal{P}(0), \mathcal{P}(1), \mathcal{P}(2), ...\},$$

et peuvent être composés comme l'on composerait des opérations à plusieurs variables. Par exemple, si l'on a une opération à deux variables μ_1, et une opération à trois variables μ_2, alors on peut composer ces deux opérations pour en obtenir une troisième, qui prendra cette fois-ci quatre entrées. Par exemple:

$$x_1, x_2, x_3, x_4 \to \mu_1(\mu_2(x_1, x_2, x_3), x_4).$$

Par conséquent, nous devrions avoir une application correspondante: $\mathcal{P}(2)\times\mathcal{P}(3) \to \mathcal{P}(4)$, qui fera partie de la donnée de l'opérade $\mathcal{P}$. Bien sûr toutes les applications de compositions doivent satisfaire les même axiomes d'associativité que la composition usuelle. On pourrait aussi considérer la donnée supplémentaire d'une action du groupe symétrique sur les opérations, qui devrait "mimer" l'action par permutation des variables que l'on a pour les opérations à plusieurs variables. Naturellement, cette action devra bien sûr satisfaire la même compatibilité avec les morphismes de composition que la permutation des variables.

Il existe plusieurs manières équivalentes de définir formellement une opérade. Celle qui va nous intéresser dans ce document est de voir une opérade comme un foncteur monoïdal défini sur une catégorie monoïdale construite de la manière suivante. Considérons le groupoïde FinSet ayant pour objets les ensembles finis et pour morphismes les bijections. On construit la catégorie monoïdale $\mathcal{T}$ en ajoutant à la catégorie symétrique monoïdale libre engendrée par FinSet des morphismes

$$\bigotimes_{v \text{ sommet de } t} \mathsf{Inputs}(v) \xrightarrow{t} \mathsf{Leaves}(t),$$

pour tout arbre enraciné t, avec $\mathsf{Inputs}(v)$ l'ensemble des arêtes rentrantes du sommet v et $\mathsf{Leaves}(t)$ l'ensemble des feuilles de t. La composition de ces morphismes est définie par substitution des arbres, c'est-à-dire si t est un arbre enraciné et pour tout sommet v de t on a un arbre enraciné t_v ayant pour feuilles $\mathsf{Inputs}(v)$, alors $t \circ (\otimes t_v)$ est l'arbre construit en substituant tout sommet v par l'arbre t_v dans t. Finalement, il faut encore quotienter la catégorie obtenue par certaines relations qui expriment la compatibilité des symétries avec les morphismes t. Cette construction est aussi appelée la "monade des arbres". Avec cette nouvelle construction on peut définir une opérade simplement comme un foncteur monoïdal (fort) depuis $\mathcal{T}$ dans une catégorie symétrique monoïdale de notre choix. Les ensembles d'opérations $\mathcal{P}(0), \mathcal{P}(1), ..., \mathcal{P}(n)$ sont les images par le foncteur $\mathcal{P}$ des ensembles finis, les morphismes de compositions sont les images par le foncteur $\mathcal{P}$ des morphismes représentés par les arbres enracinés, l'associativité des morphismes de composition vient de l'associativité de la composition des morphismes dans la catégorie $\mathcal{T}$, et enfin l'action du groupe symétrique vient de l'image par le foncteur des bijections entre ensembles finis.

Au cours des deux dernières décennies de nombreuses variantes des opérades sont apparues dans la littérature, comme par exemple les props (Mac Lane [23]), les propérades (Vallette [32]), les opérades cycliques (Getzler-Kapranov [16]), les opérades modulaires (Getzler-Kapranov [17]) et ainsi de suite. Tous ces objets peuvent être définis comme des foncteurs monoïdaux sur certaines catégories monoïdales symétriques bien choisies, définies d'une manière similaire à celle utilisée pour construire $\mathcal{T}$ à partir de FinSet dans le paragraphe précédent. Un cadre général pour de telles catégories a été développé par Ralph Kaufmann et Benjamin Ward, qui ont introduit la notion de catégorie de Feynman [20]. Avant de présenter la définition formelle, commençons par introduire quelques notations. Si $\mathcal{C}$ est une catégorie, on note $\mathrm{Sym}(\mathcal{C})$ la catégorie symétrique monoïdale libre engendrée par $\mathcal{C}$. Pour tout foncteur $F : \mathcal{C} \to \mathcal{D}$ avec $\mathcal{D}$ une catégorie symétrique monoïdale, il existe un unique foncteur monoïdal fort $\mathrm{Sym}(F) : \mathrm{Sym}(\mathcal{C}) \to \mathcal{D}$ qui se restreint à F sur $\mathcal{C}$. On note $\mathcal{C}^{\mathrm{iso}}$ la catégorie ayant les même objets que $\mathcal{C}$ et dont les morphismes sont les isomorphismes de $\mathcal{C}$. Si on se donne un diagramme de catégories $\mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{G} \mathcal{E}$, la catégorie virgule $(F \downarrow G)$ est la catégorie ayant pour objets les triplets

$(c \in \mathcal{C}, e \in \mathcal{E}, \phi : F(c) \to G(e))$ et pour morphismes les diagrammes commutatifs. Si les foncteurs F et G peuvent être déduits du contexte on notera plus simplement $(\mathcal{C} \downarrow \mathcal{E})$.

Définition 0.0.7 (Catégorie de Feynman)**.** Un triplet $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ est une *catégorie de Feynman* si $\mathcal{V}$ est un groupoïde, $\mathcal{F}$ est une catégorie monoïdale symétrique et $\imath : \mathcal{V} \to \mathcal{F}$ est un foncteur tel que:

1. Le foncteur $\imath$ induit une équivalence de catégories $\mathrm{Sym}(\imath) : \mathrm{Sym}(\mathcal{V}) \to \mathcal{F}^{\mathrm{iso}}$.
2. Le foncteur $\imath$ induit une équivalence de catégories $\mathrm{Sym}((\mathcal{F} \downarrow \mathcal{V})^{\mathrm{iso}}) \to (\mathcal{F} \downarrow \mathcal{F})^{\mathrm{iso}}$.
3. Pour tout objet $\star \in \mathcal{V}$, la catégorie virgule $(\mathcal{F} \downarrow \star)$ est essentiellement petite (i.e. est équivalente à une petite catégorie).

Exemple 0.0.8.

- Le triplet $(\mathsf{FinSet}, \mathcal{T}, \imath)$ avec FinSet et $\mathcal{T}$ les catégories introduites précédemment et $\imath$ l'inclusion évidente est une catégorie de Feynman.
- Soit $\star$ le groupoïde avec un seul objet et un seul morphisme. On pose $\mathcal{A}$ la catégorie symétrique monoïdale ayant pour objets les ensembles finis et pour morphismes les surjections, et dont la structure monoïdale est donnée par l'union disjointe. Le triplet $(\star, \mathcal{A}, \imath)$ avec $\imath$ l'inclusion évidente est une catégorie de Feynman.

Un objet de $\mathcal{V}$ sera appelé une arité et un morphisme de $\mathcal{F}$ sera appelé un morphisme structurel. De manière plus informelle, on peut résumer la Définition 0.0.7 par le fait qu'une catégorie de Feynman peut être construite en ajoutant des morphismes de la forme

$$\bigotimes \text{Objets} \longrightarrow \text{Objet}$$

à une catégorie symétrique monoïdale libre engendrée par un groupoïde.

Définition 0.0.9 (Opérade sur une catégorie de Feynman)**.** Soit $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ une catégorie de Feynman et $\mathcal{C}$ une catégorie symétrique monoïdale. Une *opérade sur* $\mathfrak{F}$ dans $\mathcal{C}$ est un foncteur monoïdal fort de $\mathcal{F}$ dans $\mathcal{C}$, et une coopérade sur $\mathfrak{F}$ dans $\mathcal{C}$ est un foncteur monoïdal fort de $\mathcal{F}$ dans $\mathcal{C}^{op}$. Un module sur $\mathfrak{F}$ dans $\mathcal{C}$ est un foncteur de $\mathcal{V}$ dans $\mathcal{C}$.

Les (co)opérades (resp. modules) sur $\mathfrak{F}$ seront aussi appelés des $\mathfrak{F}$-(co)opérades (resp. $\mathfrak{F}$-modules).

Exemple 0.0.10.

- Une opérade sur $(\mathsf{FinSet}, \mathcal{T}, \imath)$ est une opérade au sens classique.
- Une opérade sur $(\star, \mathcal{A}, \imath)$ est un monoïde.

Le premier objectif de cette thèse est de construire une catégorie de Feynman sur laquelle la collection des algèbres $H^\bullet(\overline{Y}_{\mathcal{H},\mathcal{G}})$ pour toute paire $(\mathcal{H}, \mathcal{G})$, avec les morphismes (4) pour tout ensemble niché, forme une (co)opérade. L'exemple 0.0.6 montre que cette catégorie de Feynman devrait "contenir" en un certain sens la catégorie de Feynman encodant les opérades classiques.

Au lieu de travailler avec les arrangements d'hyperplans nous nous placerons à un niveau combinatoire, c'est-à-dire nous travaillerons avec les treillis. Cela implique de perdre toute l'information qui n'est pas combinatoire, mais cela permet aussi d'accéder à un plus haut niveau de généralité car nous pourrons potentiellement travailler avec des treillis qui ne sont pas des treillis d'intersection. La classe de treillis sur laquelle nous nous concentrerons dans ce document est celle des treillis géométriques.

Définition 0.0.11 (Treillis géométrique)**.** Un treillis fini $(\mathcal{L}, \leq)$ est dit *géométrique* si il satisfait les propriétés suivantes:

- Pour toute paire d'éléments $G_1 \leq G_2$, toutes les chaînes maximales d'éléments entre G_1 et G_2 ont le même cardinal. *(Propriété de Jordan–Hölder)*
- La fonction de rang $\rho : \mathcal{L} \to \mathbb{N}$ qui assigne à un élément G de $\mathcal{L}$ le cardinal d'une chaîne maximale d'éléments entre $\hat{0}$ et G (sans compter $\hat{0}$) satisfait l'inégalité

  $$\rho(G_1 \wedge G_2) + \rho(G_1 \vee G_2) \leq \rho(G_1) + \rho(G_2)$$

  pour tout G_1, G_2 dans $\mathcal{L}$. *(Sous-modularité)*
- Tout élément de $\mathcal{L}$ peut être obtenu comme le suprémum d'un ensemble d'atomes (éléments de rang 1). *(Atomicité)*

Cette classe contient strictement tous les treillis d'intersections d'arrangements d'hyperplans (la fonction de rang étant donnée par la codimension). Voir [28] Exemple 2.1.22 pour un treillis géométrique qui n'est pas le treillis d'intersection d'un arrangement d'hyperplans dans aucun corps. Les treillis géométriques peuvent être vus comme une abstraction combinatoire des arrangements d'hyperplans. Sont représentés sur la figure 6 quelques diagrammes de Hasse de certains ensembles partiellement ordonnés (posets). Déterminons lesquels de ces posets sont des treillis géométriques. Le poset (a) n'est même pas un treillis car les deux éléments du haut n'ont pas d'infimum (entre autres). Le poset (b) est un treillis mais n'est pas géométrique car il n'est pas atomique. Le poset (c) n'est pas un treillis géométrique car il n'est pas atomique et ne satisfait pas la propriété de Jordan-Hölder. Le poset (d) est un treillis atomique satisfaisant la propriété de Jordan-Hölder mais n'est pas géométrique car sa fonction de rang n'est pas sous-modulaire. Enfin, les posets (e) et (f) sont des treillis géométriques. Le premier est le treillis d'intersection de l'arrangement donné dans l'exemple 0.0.1, et le second est le treillis d'intersection de l'arrangement donné par les hyperplans $\{z_1 = 0\}, \{z_2 = 0\}, \{z_3 = 0\} \subset \mathbb{C}^3$.

Dans ce document tous les treillis seront supposés être géométriques. La donnée d'un treillis géométrique est équivalente à la donnée de ce qu'on appelle un matroïde simple

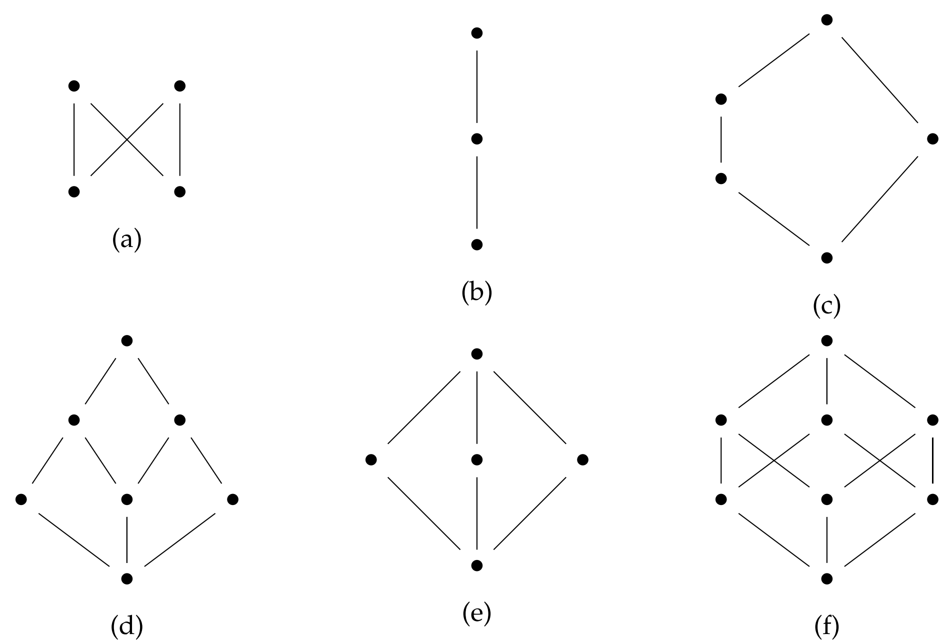

Figure 6: Diagramme de Hasse de quelques posets

sans boucle, aussi connu sous le nom de géométrie combinatoire. Il existe plusieurs définitions de matroïdes, partiellement listées ci-dessous. Nous référons à [35] pour plus de détails.

Définition 0.0.12 (Matroïdes via ensembles indépendants)**.** Un *matroïde* est une paire d'un ensemble fini E et d'un ensemble $\mathcal{I}$ de parties de E (appelées les sous-ensembles "indépendants") satisfaisant les axiomes

- Pour tout I dans $\mathcal{I}$, toute partie de I appartient à $\mathcal{I}$.
- Pour tout I, J dans $\mathcal{I}$, si $\#J > \#I$ il existe un élément a dans J et pas dans I tel que $I \cup \{a\}$ soit indépendant.

Définition 0.0.13 (Matroïdes via opérateur de cloture)**.** Un *matroïde* est une paire d'un ensemble fini E et d'une application (l'opérateur de "cloture")

$$\sigma : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$$

satisfaisant les axiomes

- Pour tout $X \in \mathcal{P}(E)$ on a $X \subseteq \sigma(X)$.
- Pour tout $X \subseteq Y \in \mathcal{P}(E)$ on a $\sigma(X) \subseteq \sigma(Y)$.

- Pour tout $X \in \mathcal{P}(E)$ on a $\sigma(\sigma(X)) = \sigma(X)$.
- Pour tout $X \in \mathcal{P}(E)$ et $a, b \in E$, si $a \in \sigma(X \cup \{b\}) \setminus \sigma(X)$ alors $b \in \sigma(X \cup \{a\}) \setminus \sigma(X)$.

Définition 0.0.14 (Matroïdes via les circuits)**.** Un *matroïde* est une paire d'un ensemble fini E et d'un ensemble $\mathcal{C}$ de parties de E (les "circuits") satisfaisant les axiomes

- L'ensemble vide n'est pas un circuit.
- Si $C_1 \subseteq C_2 \in \mathcal{C}$ alors $C_1 = C_2$.
- Si $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ et $e \in C_1 \cap C_2$ alors il existe un circuit $C \subseteq C_1 \cup C_2 \setminus \{e\}$.

Toutes ces définitions sont équivalentes. On passe de la définition par ensembles indépendants à la définition par circuits en définissant les circuits comme étant les ensembles non indépendants minimaux. On passe de la définition par circuits à la définition par opérateur de cloture en posant

$$\sigma(X) \coloneqq X \cup \{x \,|\, \exists C \in \mathcal{C} \text{ avec } C \subseteq X \cup \{x\} \text{ and } x \in C\}.$$

Enfin, on passe de la définition par opérateur de cloture à la définition par ensembles indépendants en définissant un ensemble indépendant comme étant un sous-ensemble S tel que pour tout $s \in S$, l'élément s n'appartient pas à la cloture de $S \setminus \{s\}$. Les matroïdes forment une abstraction combinatoire de la notion générale d'indépendance en mathématiques, qui comprend par exemple l'indépendance linéaire, l'indépendance affine, l'indépendance algébrique et ainsi de suite. Les matroïdes sont aussi fortement reliés à la théorie des graphes par l'observation que pour tout graphe G avec ensemble d'arêtes E, les parties de E formant des cycles minimaux de G satisfont les axiomes des circuits d'un matroïdes (0.0.14), et définissent donc un matroïde. Les sous-ensembles indépendants de ce matroïde sont donnés par les sous-forêts de G. On notera ce matroïde M_G.

Exemple 0.0.15. Soit G le graphe représenté sur la figure 7. Les circuits de M_G sont $\{1, 2\}$, $\{3\}$, $\{1, 4, 5\}$ et $\{2, 4, 5\}$. Les sous-ensembles indépendants de M_G sont $\emptyset$, $\{1\}$, $\{2\}$, $\{4\}$, $\{5\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 4\}$, $\{2, 5\}$, $\{4, 5\}$ avec tous les sous-ensembles obtenus en ajoutant 6 à ces sous-ensembles.

Une boucle dans un matroïde $(E, \mathcal{I})$ est un élément x de E tel que $\{x\}$ soit dépendant. Un matroïde est dit simple si tout sous-ensemble de cardinal 2 sans boucle est indépendant. Un fermé d'un matroïde $M = (E, \sigma)$ est un sous-ensemble $F \subseteq E$ tel que $\sigma(F)$ soit égal à F. L'ensemble des fermés de M dénoté $\mathcal{L}_M$ ordonné par l'inclusion est un treillis géométrique, dont l'infimum est donné par l'intersection. Dans l'autre sens si $\mathcal{L}$ est un treillis géométrique, alors la paire (E, σ) avec E l'ensemble des atomes de $\mathcal{L}$ et σ l'application définie par

$$\sigma(X) = \bigcap_{\substack{F \in \mathcal{L} \\ X \subset \mathrm{At}_{\leq}(F)}} \mathrm{At}_{\leq}(F)$$

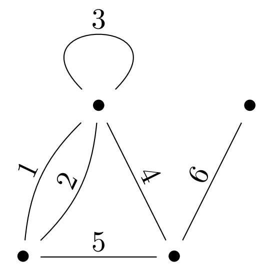

Figure 7: Un graphe

est un matroïde simple sans boucle. Ces deux constructions sont inverses l'une de l'autre sur les matroïdes simples sans boucle. Dans ce document on utilisera principalement l'axiomatisation par les treillis géométriques, qui convient mieux à nos objectifs (c'est l'axiomatisation qui rend les restrictions/contractions les plus transparentes).

Dans [8], De Concini et Procesi ont donné une présentation explicite par générateurs et relations des algèbres de cohomologie $H^\bullet(\overline{Y}_{\mathcal{H},\mathcal{G}})$, qui dépend uniquement de $\mathcal{L}_\mathcal{H}$ et de $\mathcal{G}$ (en d'autres termes, $H^\bullet(\overline{Y}_{\mathcal{H},\mathcal{G}})$ est "combinatoire"). Dans [14], Feichtner et Yuzvinsky ont généralisé ces anneaux pour toute paire $(\mathcal{L}, \mathcal{G})$ avec $\mathcal{G}$ un ensemble de construction de $\mathcal{L}$ un treillis. Une telle paire $(\mathcal{L}, \mathcal{G})$ sera appelée un "treillis construit". Dans ce document nous nous restreindrons aux treillis géométriques. Nous appellerons ces anneaux les anneaux de Feichtner–Yuzvinsky et on les notera $\mathrm{FY}(\mathcal{L}, \mathcal{G})$. Si $\mathcal{G} = \mathcal{L} \setminus \{\hat{0}\}$ on obtient un anneau appelé l'anneau de Chow du treillis géométrique $\mathcal{L}$. On peut voir les anneaux de Feichtner–Yuzvinsky comme des invariants qui nous aident à étudier les matroïdes, de la même manière que les anneaux de cohomologie nous aident à étudier les espaces. Dans le cas réalisable, c'est-à-dire quand le treillis géométrique $\mathcal{L}$ est le treillis d'intersection d'un arrangement d'hyperplans, l'anneau de Feichtner–Yuzvinsky est l'anneau de cohomologie d'une variété complexe projective lisse ce qui implique qu'il satisfait la dualité de Poincaré et le Kähler package.

Un troisième leitmotif important dans l'étude des arrangements d'hyperplans/matroïdes est que certaines propriétés des matroïdes réalisables qui semblent pourtant profondément géométriques, ont tendance à subsister dans le cas non-réalisable (c'est-à-dire purement combinatoire). Par exemple un résultat célèbre de Adiprasito-Huh-Katz [1] montre que tous les anneaux de Chow combinatoires satisfont une forme purement algébrique de la dualité de Poincaré et du Kähler package.

Définition 0.0.16 (Dualité de Poincaré)**.** Une algèbre commutative graduée A sur un corps $\mathbb{K}$ satisfait la *dualité de Poincaré* si il existe un degré maximal n tel que l'on ait $A^p = 0$ pour tout $p \geq n+1$, et un isomorphisme d'espace vectoriel $A^n \simeq \mathbb{K}$ (l'application de degré) tel que la multiplication

$$A^p \otimes A^{n-p} \to A^n \simeq \mathbb{K}$$

soit un appariement non dégénéré pour tout $p \leq n$.

Définition 0.0.17 (Kähler package)**.** Une algèbre commutative graduée A sur $\mathbb{Q}$ satisfait le *Kähler package* si il existe un degré maximal n et un isomorphisme

$$\deg : A^n \xrightarrow{\sim} \mathbb{Q}$$

tel qu'il existe un cone non-vide $\Sigma \subset A^1$ d'éléments ℓ satisfaisant les propriétés suivantes.

- La multiplication
$$\cdot\, \ell^{n-2k} : A^k \to A^{n-k}$$
est un isomorphisme pour tout $k \leq n/2$. (*Propriété de Lefschetz forte*)

- La forme bilinéaire
$$Q_\ell^k : A^k \times A^k \to \mathbb{Q}$$
définie par $Q_\ell^k(a,b) = (-1)^k \deg(a\ell^{2n-k}b)$ est définie positive sur $\ker(\cdot\, \ell^{2n-k+1})$. (*Relations de Hodge-Riemann*)

Le résultat de Adiprasito-Huh-Katz a ensuite été généralisé à tous les anneaux de Feichtner–Yuzvinsky par Pezzoli-Pagaria [29].

Dans [5], Bibby, Denham et Feichtner ont montré que les morphismes (4), qui sont de nature géométrique (ils sont induits par l'inclusion de strates) existent dans un cadre purement combinatoire. Plus précisément pour tout ensemble niché $\mathcal{S}$ dans un ensemble de construction $\mathcal{G}$ d'un treillis géométrique $\mathcal{L}$ on a un morphisme d'algèbre

$$\mathrm{FY}(\mathcal{L},\mathcal{G}) \to \bigotimes_{G \in \mathcal{S}} \mathrm{FY}(\mathcal{I}_\mathcal{S}^G, \mathrm{Ind}_{\mathcal{I}_\mathcal{S}^G}(\mathcal{G})), \tag{5}$$

avec

$$\mathcal{I}_\mathcal{S}^G := [\bigvee_{\substack{G' \in \mathcal{S} \\ G' < G}} G', G] \subset \mathcal{L}$$

(ces intervalles sont les treillis d'intersection des arrangements d'hyperplans $\mathcal{H}_\mathcal{S}^G$ dans le cas réalisable) et $\mathrm{Ind}_{\mathcal{I}_\mathcal{S}^G}(\mathcal{G})$ l'ensemble de construction "induit" par $\mathcal{G}$ sur $\mathcal{I}_\mathcal{S}^G$ (voir Définition 1.1.13). En plus de ces morphismes structurels, tout isomorphisme de poset $\mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ qui envoie un ensemble de construction $\mathcal{G}$ de $\mathcal{L}$ sur un ensemble de construction $\mathcal{G}'$ de $\mathcal{L}'$ (qu'on appellera un isomorphisme de treillis construit) induit un isomorphisme d'algèbre

$$\mathrm{FY}(\mathcal{L},\mathcal{G}) \xrightarrow{\sim} \mathrm{FY}(\mathcal{L}',\mathcal{G}'). \tag{6}$$

Dans la Section 2 on construit une catégorie de Feynman $\mathfrak{LBS} = (\mathbf{LBS}_{\mathrm{irr}}, \mathbf{LBS}, \imath)$ telle que la donnée des algèbres $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ avec les morphismes structurels (5) et les isomorphismes (6) forme une coopérade sur $\mathfrak{LBS}$ (dans la catégorie des algèbres commutatives graduées). Pour résumer la construction, les objets de **LBS** seront les treillis (géométriques) construits, les isomorphismes seront les isomorphismes de treillis construits définis au-dessus, et les morphismes structurels seront donnés par les ensembles nichés. Le coeur de la construction est d'arriver à définir une bonne "composition" des ensembles nichés.

Après avoir atteint cette première étape, nos nouveaux objectifs se diviseront en trois catégories. Pour commencer, il serait intéressant d'étudier la catégorie de Feynman $\mathfrak{LBS}$ en elle-même. On sera principalement préoccupé par la question de trouver une bonne présentation de $\mathfrak{LBS}$, c'est-à-dire un ensemble de morphismes structurels de $\mathfrak{LBS}$ qui génère tous les autres morphismes par tensorisation et composition, ainsi que les relations que ces générateurs satisfont. Une présentation de $\mathfrak{LBS}$ est donnée par les Propositions 2.3.1 et 2.3.2. D'un point de vue pratique, les présentations de catégories de Feynman sont utiles pour définir des opérades sur ces catégories de Feynman. Par exemple dans le cas des opérades classiques, on sait que les arbres avec deux sommets internes avec les isomorphismes génèrent tous les autres morphismes de $\mathcal{T}$ par composition et tensorisation, et on sait quelles relations ces générateurs satisfont. Ceci implique que les opérades classiques peuvent être définies en spécifiant simplement les compositions partielles (composition de deux opérations), ainsi que l'action du groupe symétrique en chaque arité.

Notre deuxième objectif sera d'introduire plusieurs nouvelles $\mathfrak{LBS}$-opérades, autres que l'opérade des anneaux de Feichtner–Yuzvinsky. Dans les sections 3 et 5, on montre que les familles d'algèbres

$$\{\mathrm{OS}(\mathcal{L}), (\mathcal{L}, \mathcal{G})\} \quad \text{et} \quad \{\overline{\mathrm{OS}}(\mathcal{L}), (\mathcal{L}, \mathcal{G})\}$$

admettent une structure $\mathfrak{LBS}$-opéradique. Les morphismes structurels de la deuxième sont donnés par une généralisation combinatoire des morphismes de résidus dans le cas réalisable. Lorsque l'on se restreint aux treillis de partitions avec l'ensemble de construction des partitions avec une seule classe d'équivalence non triviale, on obtient le dual linéaire d'une opérade bien connue nommée Grav, introduite par Getzler dans [15]. La restriction de l'opérade des algèbres d'Orlik–Solomon est le dual linéaire de l'opérade classique nommée Ger qui encode les algèbres de Gerstenhaber.

Enfin, dans un troisième temps on s'intéressera de plus près à la $\mathfrak{LBS}$-coopérade des anneaux de Feichtner–Yuzvinsky, que l'on notera $\mathbb{FY}$. On notera aussi $\mathbb{FY}^\vee$ l'opérade sur $\mathfrak{LBS}$ obtenue en dualisant (au sens linéaire) les objets, les morphismes structurels et les isomorphismes de $\mathbb{FY}$. Un de nos espoirs est de pouvoir relier les propriétés importantes des algèbres de Feichtner–Yuzvinsky mises en lumières précédemment, et les propriétés de l'objet global $\mathbb{FY}$. De manière particulièrement frappante, alors que les opérades sur des catégories de Feynman sont en général bien plus complexes que de simples algèbres associatives, on peut les étudier de la même manière. Par exemple on peut d'abord s'intéresser à trouver une "présentation" de $\mathbb{FY}^\vee$. Dans notre contexte opéradique cela veut dire trouver un ensemble d'éléments dans chaque objet $\mathbb{FY}^\vee(\mathcal{L}, \mathcal{G})$ qui génère tous les autres éléments en chaque arité par somme et produit le long de morphismes structurels, et ensuite examiner quelles relations ces générateurs satisfont. Dans la section 3, Proposition 3.1.3, on montre que les applications de degré génèrent $\mathbb{FY}^\vee$ au sens expliqué ci-dessus, et on décrit les relations entre ces générateurs. En remarque 3.1.4 on explique comment le fait que les applications de degré génèrent $\mathbb{FY}^\vee$ est essentiellement équivalent au fait que les algèbres de Feichtner–Yuzvinsky satisfont la dualité de Poincaré.

Pour les algèbres associatives, un outil calculatoire pour gérer les présentations d'algèbres est la théorie des bases de Gröbner. L'idée générale des bases de Gröbner est de commencer par choisir un ordre sur les générateurs de la présentation. Cet ordre est ensuite utilisé pour obtenir un ordre sur tous les monômes, qui est compatible en un certain sens avec la multiplication (ces ordres sont dits "admissibles"). On utilise alors cet ordre pour réécrire les monômes dans l'algèbre quotient:

$$\text{terme dominant } \longrightarrow \sum \text{reste des termes},$$

pour toute relation $R = \text{terme dominant } - \sum \text{reste des termes}$ dans un sous-ensemble $\mathcal{B}$ de l'idéal généré par les relations de la présentation. Le sous-ensemble $\mathcal{B}$ est appelé une base de Gröbner quand il contient "assez" d'éléments. Plus précisément, on demande que tout terme dominant d'une relation dans l'algèbre quotient soit divisible par le terme dominant d'un élément de $\mathcal{B}$. En général le but est de trouver une base de Gröbner aussi petite que possible pour que la réécriture soit aussi facile que possible. A la fin du processus de réécriture il ne reste que les monômes qui ne sont pas réécrivables, c'est-à-dire qui ne sont pas divisibles par le terme dominant d'un élément de $\mathcal{B}$. Ces monômes sont appelés les monômes "normaux" et forment une base linéaire de l'algèbre quotient si et seulement si $\mathcal{B}$ est une base de Gröbner. La table de multiplication de cette base est donnée par le processus de réécriture.

Il se trouve que cette stratégie générale peut être appliquée à des structures beaucoup plus générales et complexes que les algèbres associatives, telles que les opérades. De manière informelle, tout ce dont nous avons besoin pour implémenter cette stratégie est de pouvoir donner un sens aux mots clefs utilisés ci-dessus, tels que "monômes", "ordres admissibles" et "divisibilité entre monômes". Pour les opérades sur une catégorie de Feynman, la seule partie non-triviale est de construire des ordres admissibles sur les monômes à partir d'un ordre sur les générateurs. Le problème principal provient des isomorphismes, car la compatibilité avec les isomorphismes est en générale trop forte et nous empêche de trouver des ordres admissibles. Pour éviter ce problème, en s'inspirant du cas des opérades classiques traité par Dotsenko et Khoroshkin [11], dans la section 4 on introduit la notion d'opérade "shuffle" sur $\mathfrak{LBS}$ ce qui permet de développer une théorie de bases de Gröbner pour les $\mathfrak{LBS}$-opérades.

Une autre notion importante dans la théorie des algèbres associatives (graduées) est la dualité de Koszul. Informellement, on dit qu'une algèbre associative graduée est Koszul si elle est générée par ses éléments de degré 1, les relations entre éléments de degré 1 sont générés par des éléments de degré 2, les relations entre relations de degré 2 sont générées en degré 3 et ainsi de suite. Une définition formelle possible utilise les opérateurs "bar" et "cobar" (que l'on notera B et Ω respectivement):

$$\mathrm{B} : \{ \text{ dg graded coassociative coalgebras } \} \leftrightarrows \{ \text{ dg graded associative algebras } \} : \Omega.$$

Une algèbre associative graduée est Koszul si il existe un quasi-isomorphisme entre l'algèbre différentielle graduée $\Omega(H(\mathrm{B}A))$ et A. Dans ce cas $\Omega(H(\mathrm{B}A))$ est le modèle minimal de A.

Kaufmann et Ward ont montré dans [20] que pour certaines catégories de Feynman dites "cubiques", on a un opérateur bar et un opérateur cobar entre (co)opérades sur ces catégories de Feynman, ce qui permet de définir une notion de Koszulité de la même manière que pour les algèbres associatives. Dans la section 5 Proposition 5.1.2 on montre que la catégorie de Feynman $\mathfrak{LBS}$ est cubique. On montre ensuite

Théorème 0.0.18 (Corollary 5.3.3)**.** *Le dual linéaire de la coopérade des anneaux de Feichtner–Yuzvinsky est Koszul, avec pour dual de Koszul la coopérade des algèbres d'Orlik–Solomon projectives.*

Cette dualité se restreint à la dualité entre Hypercom et Grav sur les treillis de partition, qui a été prouvée par Getzler dans [15]. La preuve de Getzler repose sur l'observation que la deuxième page de la suite spectrale de Leray de l'inclusion

$$\mathbb{P}\mathcal{A}_{\mathsf{Braid}_n} \hookrightarrow \overline{Y}_{\mathsf{Braid}_n,\mathcal{G}_n}$$

est la construction bar de Hypercom, et le fait que cette suite spectrale converge à la deuxième page, par un argument de théorie de Hodge mixte. Dans un cadre purement combinatoire ces outils géométriques n'existent pas mais on a quand même un modèle de Leray (purement combinatoire) pour les algèbres d'Orlik–Solomon, un résultat prouvé par Bibby, Denham et Feichtner dans [5]. Ce modèle de Leray combinatoire est la construction bar de l'opérade des anneaux de Feichtner–Yuzvinsky, ce qui implique que cette opérade est Koszul. De manière alternative, on montre aussi les deux résultats suivant.

Proposition 0.0.27 (Proposition 5.6.1)**.** *Une $\mathfrak{LBS}$-opérade qui admet une base de Gröbner quadratique est Koszul.*

Théorème 0.0.19 (Corollaire 4.5.7)**.** *L'opérade des anneaux de Feichtner–Yuzvinsky admet une base de Gröbner quadratique.*

Ceci mène à la même conclusion.

Dans la section 6 on conclut la première partie de ce document en donnant de nouvelles directions de recherches possibles pour potentiellement généraliser $\mathfrak{LBS}$, ce qui pourrait mener à d'autres applications.

Part 2: Koszulité des algèbres de Feichtner–Yuzvinsky

La deuxième partie de ce manuscrit est dévouée à l'étude de la Koszulité de certaines algèbres de Feichtner–Yuzvinsky, à l'aide de la structure opéradique mise en lumière en première partie. Il est important de mentionner ici qu'il s'agit d'un problème a priori non relié à la Koszulité de l'opérade $\mathbb{FY}^\vee$. Cependant, nous verrons que l'on peut quand même utiliser la structure opéradique sur les anneaux de Feichtner–Yuzvinsky pour déduire des résultats sur les anneaux eux-même.

La Koszulité est une propriété particulièrement intéressante à vérifier sur les anneaux de cohomologie des espaces formels car elle permet de calculer directement d'autres invariants d'homotopie rationnelle tels que l'algèbre de Lie homotopique (voir Berglund [4]). Puisque nous savons que les compactifications merveilleuses sont formelles il est naturel de se demander quelles algèbres de Feichtner–Yuzvinsky sont Koszul, une question posée par Dotsenko dans [10]. Cette question est encore largement ouverte.

Un moyen classique de prouver la Koszulité d'une algèbre donnée est de trouver une base de Gröbner quadratique de cette algèbre. Feichtner et Yuzvinsky ont calculé une base de Gröbner explicite des anneaux de Feichtner–Yuzvinsky, mais ces bases de Gröbner ne sont presque jamais quadratiques. En fait, les anneaux de Feichtner–Yuzvinsky eux-même ne sont pas nécessairement quadratiques. Un des premiers résultats montrant la Koszulité de certaines algèbres de Feichtner–Yuzvinsky a été donné par Dotsenko qui a montré que les algèbres de Feichtner–Yuzvinsky associées aux arrangements de tresses avec ensemble de construction minimal (Exemple 0.0.6) sont toutes Koszul. En quelques mots, Dotsenko a introduit un ordre explicite sur les générateurs de ces anneaux de Feichtner–Yuzvinsky et a ensuite utilisé la structure opéradique (au sens classique) pour construire une bijection entre monômes normaux algébriques associés aux relations de degré 2, et les monômes normaux opéradiques associés à une base de Gröbner opéradique connue par des travaux précédents (voir Dotsenko-Khoroshkin [11]). Par un argument de dimension cela implique que les relations de degré 2 forment une base de Gröbner des anneaux de Feichtner–Yuzvinsky en question. Plus récemment, Mastroeni-McCullough [26] ont montré que les anneaux de Chow combinatoires sont tous Koszul, en utilisant la notion de filtration de Koszul.

Dans ce document nous allons généraliser la stratégie de Dotsenko, en utilisant la structure opéradique étendue introduite en première partie. Cependant, nous allons voir que l'argument de Dotsenko ne s'étend pas à tous les treillis géométriques et il nous faudra restreindre la classe des treillis que l'on considère.

Rappelons qu'un élément a dans un treillis $\mathcal{L}$ est dit modulaire si pour tout $b \leq a$ et c dans $\mathcal{L}$ on a l'identité

$$a \wedge (b \vee c) = b \vee (a \wedge c).$$

Par exemple un sous-groupe normal d'un groupe fini G est modulaire dans le treillis des sous-groupes de G. Pour un exemple d'élément non-modulaire revenons au poset (c) de la figure 6.

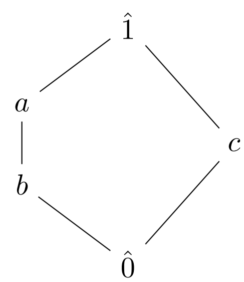

Figure 8: Un poset avec un élément non modulaire

On a

$$a \wedge (b \vee c) = a \wedge \hat{1} = a \neq b = b \vee \hat{0} = b \vee (a \wedge c),$$

ce qui montre que a n'est pas modulaire dans ce treillis. Dans [30] Stanley a introduit la définition suivante

Définition 0.0.20 (Stanley)**.** Un treillis $\mathcal{L}$ est dit *supersolvable* si il admet une chaîne maximale d'éléments modulaires.

La dénomination vient du fait que le treillis des sous-groupes d'un groupe fini modulaire supersolvable est supersolvable. Les treillis supersolvables ont de bonnes propriétés en général. Par exemple on a le résultat classique suivant.

Théorème 0.0.21 (Yuzvinsky, [36])**.** *L'algèbre d'Orlik–Solomon d'un treillis supersolvable admet une base de Gröbner quadratique et est donc Koszul.*

Dans ce document nous allons prouver un résultat similaire pour les algèbres de Feichtner–Yuzvinsky. On commence par introduire la notion de supersolvabilité pour les treillis construits.

Definition 0.0.28. Un treillis construit $(\mathcal{L}, \mathcal{G})$ est *supersolvable* si $\mathcal{L}$ admet une chaîne maximale

$$\hat{0} = G_1 < ... < G_n = \hat{1}$$

d'éléments de $\mathcal{G}$ modulaires tels que pour tout G' dans $\mathcal{G}$ et tout $i \leq n$ on a $G' \wedge G_i \in \mathcal{G} \cup \hat{0}$.

Le résultat principal de la deuxième partie de cette thèse est le théorème suivant.

Théorème 0.0.22 (Théorème 8.3.1)**.** *Soit $(\mathcal{L}, \mathcal{G})$ un treillis construit supersolvable. L'algèbre $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ admet une base de Gröbner quadratique et est donc Koszul.*

Théorème 0.0.22 implique immédiatement que les anneaux de Chow combinatoires de treillis supersolvables admettent une base de Gröbner quadratique, ce qui renforce le résultat de Mastroeni-McCullough pour les treillis supersolvables. Du côté des ensembles de constructions minimaux, Théorème 0.0.22 implique également le résultat suivant.

Théorème 0.0.23 (Théorème 8.4.5)**.** *Soit $\mathcal{L}$ un treillis supersolvable et $\mathcal{G}_{\min}$ l'ensemble de construction des éléments irreductibles de $\mathcal{L}$. L'algèbre $\mathrm{FY}(\mathcal{L}, \mathcal{G}_{\min})$ admet une base de Gröbner quadratique et est donc Koszul.*

Stanley [30] a montré que les treillis géométriques associés aux graphes cordaux (c'est-à-dire les graphes tels que tout cycle de longueur supérieure ou égale à quatre admet une corde) sont supersolvables. Tout treillis graphique $\mathcal{L}_G$ admet un ensemble de construction particulier noté $\mathcal{G}_G$ qui est constitué des sous-graphes fermés connexes de G. L'argument original de Stanley montre aussi que si G est un graphe cordal alors $(\mathcal{L}_G, \mathcal{G}_G)$ est un treillis construit supersolvable. Ceci implique par Théorème 0.0.22 que l'algèbre de Feichtner–Yuzvinsky $\mathrm{FY}(\mathcal{L}_G, \mathcal{G}_G)$ admet une base de Gröbner quadratique. Puisque les graphes complets sont cordaux on retrouve le résultat de Dotsenko (dans le cas des graphes complets les ensembles de construction $\mathcal{G}_G$ et $\mathcal{G}_{\min}$ coincident).

Dans [21], Losev et Manin ont introduit des espaces de module pour les courbes stables avec points marqués de deux types, ou les points du premier type ne sont pas autorisés à coïncider mais les points du second peuvent coïncider entre eux. Ces espaces de module forment les composantes d'un objet appelé "l'opérade modulaire étendue", introduite par Losev et Manin dans la suite [22]. Dans [25], Manin a posé la question de la Koszulité des algèbres de cohomologie de ces espaces de module. En considérant la famille de graphes cordaux $G_{m,n}$, où $G_{m,n}$ a $m+n$ sommets, les m premiers sommets sont voisins de tous les autres sommets et les n derniers sommets sont voisins seulement des m premiers sommets, on obtient le résultat suivant.

Théorème 0.0.24 (Théorème 9.2.1)**.** *L'algèbre de cohomologie des composantes de l'opérade modulaire étendue en genre 0 possède une base de Gröbner quadratique et est donc Koszul.*

Structure du document

En Section 1 on définit les personnages combinatoires principaux de cette histoire.

En Section 2 on définit la catégorie de Feynman $\mathfrak{LBS}$ et on en donne une présentation.

En Section 3 on définit la coopérade des algèbres de Feichtner–Yuzvinsky et la coopérade des algèbres d'Orlik–Solomon.

En Section 4 on développe une théorie de bases de Gröbner pour les opérades sur $\mathfrak{LBS}$.

En Section 5 on montre que $\mathfrak{LBS}$ est cubique et on explicite la théorie de dualité de Koszul pour les opérades sur $\mathfrak{LBS}$. On montre que la (co)opérade des algèbres de Feichtner–Yuzvinsky est Koszul.

En Section 6 on donne des commentaires généraux pouvant servir pour des modifications ou des généralisations de $\mathfrak{LBS}$.

En Section 7 on explicite la relation entre treillis géométriques et matroïdes et on introduit la notion de treillis construit supersolvable.

En Section 8 on prouve Théorème 0.0.22 et on déduit Théorème 0.0.23.

En Section 9 on concentre notre attention sur les treillis construits graphiques ce qui mène au Théorème 0.0.24.

Enfin, en Section 10 on donne des remarques d'ordre général pouvant servir pour une recherche ultérieure sur la Koszulité des algèbres de Feichtner–Yuzvinsky.

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Chapter 1

Combinatorial preliminaries

In this section we introduce the main combinatorial objects which will be used throughout this document.

1.1 Lattices, building sets and nested sets

Definition 1.1.1 (Lattice)**.** A finite poset $\mathcal{L}$ is called a *lattice* if every pair of elements in $\mathcal{L}$ admits a supremum and an infimum.

The supremum of two elements G_1, G_2 is denoted by $G_1 \vee G_2$ and called their *join*, while their infimum is denoted by $G_1 \wedge G_2$ and called their *meet*.

Remark 1.1.2. Since $\mathcal{L}$ is supposed to be finite, having supremums and infimums for pairs of elements implies having supremums and infimums for any subset S of $\mathcal{L}$, which will be denoted by $\bigvee S$ and $\bigwedge S$ respectively. As a consequence, every lattice admits an upper bound (the supremum of $S = \mathcal{L}$) and a lower bound (the infimum of $S = \mathcal{L}$) which will be denoted by $\hat{1}$ and $\hat{0}$ respectively.

Definition 1.1.3 (Geometric lattice)**.** A finite lattice $(\mathcal{L}, \leq)$ is said to be *geometric* if it satisfies the following properties:

- For every pair of elements $G_1 \leq G_2$, all the maximal chains of elements between G_1 and G_2 have the same cardinal. *(Jordan-Hölder property)*
- The rank function $\rho : \mathcal{L} \to \mathbb{N}$ which assigns to any element G of $\mathcal{L}$ the cardinal of any maximal chain of elements from $\hat{0}$ to G (not counting $\hat{0}$) satisfies the inequality

  $$\rho(G_1 \wedge G_2) + \rho(G_1 \vee G_2) \leq \rho(G_1) + \rho(G_2)$$

  for every G_1, G_2 in $\mathcal{L}$. *(Sub-modularity)*

- Every element in $\mathcal{L}$ can be obtained as the supremum of some set of atoms (i.e. elements of rank 1). *(Atomicity)*

One of the reasons to study this particular class of lattices is that the intersection poset of any hyperplane arrangement is a geometric lattice. In fact, one may think of geometric lattices as a combinatorial abstraction of hyperplane arrangements. In addition, this object is equivalent to the datum of a loopless simple matroid via the lattice of flats construction (see [35] for a reference on matroid theory) and therefore it has connections to many other areas in mathematics (graph theory for instance).

Here is a list of some important well-known geometric lattices.

Example 1.1.4.

- If X is any finite set, the set $\mathcal{P}(X)$ of subsets of X ordered by inclusion is a geometric lattice with join the union and meet the intersection. It is the intersection lattice of the hyperplane arrangement of coordinate hyperplanes in $\mathbb{C}^X$. Those geometric lattices are called boolean lattices and denoted by $\mathcal{B}_X$.
- If X is any finite set, the set Π_X of partitions of X ordered by refinement is a geometric lattice. It is the intersection lattice of the so-called *braid* arrangement which consists of the diagonal hyperplanes $\{z_i = z_j\}$ in $\mathbb{C}^X$. Those geometric lattices are called partition lattices.
- If $G = (V, E)$ is any graph one can construct the graphical matroid M_G associated to G and then consider $\mathcal{L}_G$ the lattice of flats associated to M_G (see [35] for the details of this construction). Those lattices are said to be *graphical*. This family of geometric lattices contains the two previous ones because $\mathcal{B}_X$ is the lattice associated to any tree with edges X and Π_X is the lattice associated to the complete graph with vertices X. For any graph $G = (V, E)$ the geometric lattice $\mathcal{L}_G$ is the intersection lattice of the hyperplane arrangement $\{\{z_u = z_v\}, (u, v) \in E\}$ in $\mathbb{C}^V$.

We have the following important fact about geometric lattices.

Proposition 1.1.5 ([35])**.** *Let $(\mathcal{L}, \leq)$ be a geometric lattice. For every $G_1 \leq G_2 \in \mathcal{L}$, the interval $[G_1, G_2] = \{G \in \mathcal{L} \mid G_1 \leq G \leq G_2\}$ ordered by the restriction of $\leq$ is a geometric lattice.*

In the rest of this article every lattice will be assumed to be geometric unless stated otherwise.

Definition 1.1.6 (Building set)**.** Let $\mathcal{L}$ be a geometric lattice. A *building set* $\mathcal{G}$ of $\mathcal{L}$ is a subset of $\mathcal{L} \setminus \{\hat{0}\}$ such that for every element X of $\mathcal{L}$ the morphism of posets

$$\prod_{G \in \max \mathcal{G}_{\leq X}} [\hat{0}, G] \xrightarrow{\vee} [\hat{0}, X] \tag{1.1}$$

is an isomorphism (where $\max \mathcal{G}_{\leq X}$ is the set of maximal elements of $\mathcal{G} \cap [\hat{0}, X]$).

The elements of $\max \mathcal{G}_{\leq X}$ will be called the *factors* of X in $\mathcal{G}$. In the rest of the paper we will prefer the more suggestive notation $\mathrm{Fact}_{\mathcal{G}}(X)$ to refer to the set of those elements.

Definition 1.1.7 (Built lattice)**.** The datum of a lattice $\mathcal{L}$ and a building set $\mathcal{G}$ of $\mathcal{L}$ will be called a *built lattice*. If $\mathcal{G}$ contains $\hat{1}$ we say that $(\mathcal{L}, \mathcal{G})$ is *irreducible*.

The definition of a building set makes sense for a larger class of posets, as shown in [14], but in this paper we will restrict ourselves to the case of geometric lattices. In this particular context, building sets are geometrically motivated by the construction of wonderful compactifications for hyperplane arrangement complements. In a nutshell, building sets are sets of intersections of a hyperplane arrangement that one can successively blow up in order to obtain a wonderful compactification of its complement (see [8] for more details). Each blowup creates a new exceptional divisor, so the wonderful compactification is equipped with a family of irreducible divisors indexed by $\mathcal{G}$. This family of divisors forms a normal crossing divisor when $\mathcal{G}$ is a building set.

There are a few key examples to keep in mind throughout this story.

Example 1.1.8.

- Every lattice $\mathcal{L}$ admits $\mathcal{L} \setminus \{\hat{0}\}$ as a building set.
- Every lattice $\mathcal{L}$ also admits a unique minimal building set which consists of all the elements G of $\mathcal{L}$ such that $[\hat{0}, G]$ is not a product of proper subposets.
- From the definition one can see that a building set of some lattice $\mathcal{L}$ must contain all the atoms of $\mathcal{L}$. If $\mathcal{L}$ is a boolean lattice (see Example 1.1.4) then its set of atoms is in fact a building set (the minimal one). This fact characterizes boolean lattices.
- If $\mathcal{L}$ is the lattice of partitions of some finite set (see Example 1.1.4) then the subset of partitions with only one block having more than two elements is a building set of $\mathcal{L}$. This is the minimal building set of $\mathcal{L}$.
- If $\mathcal{L}$ is a graphical lattice (see Example 1.1.4) then the set of elements of $\mathcal{L}$ corresponding to sets of edges which are connected forms a building set of $\mathcal{L}$. This family of examples contains the two previous ones (by considering graphs with disconnected edges for the former and complete graphs for the latter).
- Alternatively, if $G = (V, E)$ is a graph one can consider the boolean lattice $\mathcal{B}_V$. This lattice has a building set made up of the "tubes" of G, that is sets of vertices of G such that the induced subgraph on those vertices is connected. This leads to the notion of graph associahedra introduced in [6].

The additional choice of the building set adds a lot of information. For instance for graphs we have the following result (which to the best of our knowledge is new).

Proposition 1.1.9. *Let G_1 and G_2 be two graphs with no isolated vertices, no loops and no parallel edges. If there exists an isomorphism of posets f between $\mathcal{L}_{G_1}$ and $\mathcal{L}_{G_2}$ such that we have $f(\mathcal{G}_{G_1}) = \mathcal{G}_{G_2}$, then G_1 and G_2 are isomorphic.*

In other words the datum of the built lattice $(\mathcal{L}_G, \mathcal{G}_G)$ is enough to recover G. As is well known the datum $\mathcal{L}_G$ alone is not enough because for instance two trees having the same number of edges have the same associated geometric lattice (a boolean lattice). The datum of the connected subgraphs $\mathcal{G}_G$ alone is not enough either because the three-cycle and the star-shaped graph with 3 edges both have the same connected subgraphs (every subgraph is connected).

Proof. We start with the following lemma.

Lemma 1.1.10. *The only graphs with associated built lattice of the form $(\mathcal{B}_n, \mathcal{G}_{\max})$ for some n are the star-shaped graphs.*

Proof. If $\mathcal{L}_G$ is a boolean lattice then G is a tree. One can see that the only trees with all subgraphs connected are the star-shaped trees. □

We can assume that G_1 and G_2 are both connected. If G_1 has no vertex with strictly more than one neighbor then G_1 has just a single edge and so does G_2. Otherwise denote by x_0 some vertex of G_1 having strictly more than one neighbor. We will construct an isomorphism of graphs $\hat{f}_{x_0}$ between G_1 and G_2, induced by f and the choice of the vertex x_0. Denote by $\mathrm{Star}(x_0)$ the set of edges of G_1 attached to x_0. By Lemma 1.1.10 the image of $\mathrm{Star}(x_0)$ by f is a star-shaped subgraph of G_2 with more than two edges. We define $\hat{f}_{x_0}(x_0)$ as the center of this star-shaped subgraph. Next we have the following lemma.

Lemma 1.1.11. *Let γ be a path in G_1 starting from x_0 and which does not contain any cycle. The set of edges $f(\gamma)$ is a path in G_2 starting from $\hat{f}_{x_0}(x_0)$ which does not contain any cycle*

Proof. We prove the result by induction on the length of γ. If γ contains one edge the result is obvious. If γ contains two edges, say e_1, e_2 with e_1 attached to x_0, then $\{f(e_1), f(e_2)\}$ is connected in G_2 and $f(e_1)$ is attached to $\hat{f}_{x_0}(x_0)$. If $f(e_2)$ is attached to $\hat{f}_{x_0}(x_0)$ then let e_0 be an edge attached to x_0 different from e_1. The map f sends $\{e_0, e_1, e_2\}$ to a star-shaped graph which contradicts Lemma 1.1.10. This proves that $f(\gamma)$ is a path starting from $\hat{f}_{x_0}(x_0)$ without any cycle. Let γ be a path starting from x_0 without any cycle and with at least 3 edges. Denote by e_1, e_2, e_3 the last three edges of γ in this order when coming from x_0. By induction $f(\gamma \setminus \{e_3\})$ is a path starting from $\hat{f}_{x_0}(x_0)$ without any cycle. Denote by y_1 the end vertex of this path. By the fact that f must send connected subsets to connected subsets we get that $f(e_3)$ is either attached to y_1 or to the other end of e_2. If it is attached to the other end of e_2 then f sends $\{e_0, e_1, e_2\}$ to a star-shaped graph which contradicts Lemma 1.1.10. As a consequence we see that $f(\gamma)$ is a path starting from $\hat{f}_{x_0}(x_0)$. By the fact that f preserves the rank on both sides we see that this path cannot contain any cycle. □

Finally, for any vertex x in G_1 we define $\hat{f}_{x_0}(x)$ as the end vertex of $f(\gamma)$ with γ any path from x_0 to x without any cycle. Let us prove that this definition does not depend on the choice of γ, by induction on γ. Let γ' be another path from x_0 to x without any cycle. Let x_1 be the first vertex at which γ and γ' meet (after x_0). Denote by γ^{x_1} (resp. γ'^{x_1}) the part of γ (resp. γ') which goes from x_0 to x_1. The set of edges $\gamma^{x_1} \cup \gamma'^{x_1}$ forms a circuit of G_1 and therefore $f(\gamma^{x_1} \cup \gamma'^{x_1})$ must be a circuit of G_2. By Lemma 1.1.11 this implies that $f(\gamma^{x_1})$ and $f(\gamma'^{x_1})$ have the same endpoint and we can conclude by induction. By construction $\hat{f}_{x_0}$ is a morphism a graph. i.e. sends neighbors to neighbors. Also by construction $\hat{f}_{x_0}$ has an inverse given by $\widehat{f^{-1}}_{\hat{f}_{x_0}(x_0)}$ which finishes the proof. □

Additionally, one can check that $\hat{f}_{x_0}$ does not depend on x_0. We have almost proved that the functor $G \to (\mathcal{L}_G, \mathcal{G}_G)$ is fully faithful on isomorphisms, but this is not quite true because of the one edge graph, which has a non-trivial automorphism (swapping the extremities of the edge), contrary to its associated built lattice.

One could get a slightly more general version of Proposition 1.1.9 allowing parallel edges, by just working at the level of the graphical matroid and not the associated geometric lattice. In order to allow loops one would need a notion of building set at the level of the matroid. We investigate this question in Subsection 6.1.

A key fact about building sets is that any interval $[G_1, G_2]$ in some built lattice $(\mathcal{L}, \mathcal{G})$ admits an "induced" building set which we describe now. We start by introducing a useful notation.

Notation 1.1.12. For any element G of some lattice $\mathcal{L}$ and a subset X of $\mathcal{L}$, we denote by $G \vee X$ the set of elements of $\mathcal{L}$ which can be obtained as the join of G and some element of X.

Definition 1.1.13 (Induced building set)**.** Let $G_1 < G_2$ be two elements in some built lattice $(\mathcal{L}, \mathcal{G})$. We denote by $\mathrm{Ind}_{[G_1,G_2]}(\mathcal{G})$ the set $(G_1 \vee \mathcal{G}) \cap [G_1, G_2] \setminus \{G_1\}$ and we call it the *induced building set* on $[G_1, G_2]$.

Lemma 1.1.14 ([5] Lemma 2.8.5)**.** *The subset* $\mathrm{Ind}_{[G_1,G_2]}(\mathcal{G}) \subset [G_1, G_2]$ *is a building set of* $[G_1, G_2]$.

We will often write $\mathrm{Ind}(\mathcal{G})$ instead of $\mathrm{Ind}_{[G_1,G_2]}(\mathcal{G})$ if the interval can be deduced from the context. We have the obvious lemma.

Lemma 1.1.15. *For any elements* $X_1 \leq X_2 \leq X_3 \leq X_4$ *in some lattice* $\mathcal{L}$ *with building set* $\mathcal{G}$*, we have the equality of building set*

$$\mathrm{Ind}_{[X_2,X_3]}(\mathrm{Ind}_{[X_1,X_4]}(\mathcal{G})) = \mathrm{Ind}_{[X_2,X_3]}(\mathcal{G}).$$

A subset of pairwise incomparable elements in a poset will be called an *antichain*.

Definition 1.1.16 (Nested set)**.** Let $(\mathcal{L}, \mathcal{G})$ be a built lattice. A subset $\mathcal{S}$ of $\mathcal{G}$ is called a *nested set* if for every antichain $\mathcal{A}$ in $\mathcal{S}$ which contains at least two elements, the join of the elements of $\mathcal{A}$ does not belong to $\mathcal{G}$. A nested set $\mathcal{S} \subset \mathcal{G}$ is said to be *irreducible* if it contains $\max \mathcal{G}$.

Example 1.1.17. A chain of elements in some building set $\mathcal{G}$ is always nested and those are the only nested sets of the maximal building set ($\mathcal{G} = \mathcal{L} \setminus \{\hat{0}\}$).

Geometrically, nested sets correspond to sets of divisors in the wonderful compactification which have a nontrivial intersection. There are two crucial lemmas regarding nested sets.

Lemma 1.1.18 ([13] Proposition 2.8)**.** *Let $\mathcal{G}$ be a building set of a geometric lattice $\mathcal{L}$ and let X be any element of $\mathcal{L}$. The subset $\mathrm{Fact}_{\mathcal{G}}(X)$ is a nested antichain in $\mathcal{G}$ and furthermore it is the only nested antichain in $\mathcal{G}$ having join X.*

Lemma 1.1.19 ([29] Proposition 2.4)**.** *A nested set of $\mathcal{L}$ is a forest in the Hasse diagram of $\mathcal{L}$. More precisely for every K in $\mathcal{L}$, $\mathcal{S}_{>K}$ is either empty or has a unique minimal element.*

We next introduce a map $\mathrm{Comp}_G : \mathrm{Ind}_{[G,G']}(\mathcal{G}) \to \mathcal{G}$ which will help us define our composition of nested sets in the next section. Let G'' be an element of $\mathrm{Ind}_{[G,G']}(\mathcal{G})$ and F an element such that $G'' = G \vee F$ and which is maximal amongst elements satisfying this equality (such an F exists by definition of $\mathrm{Ind}(\mathcal{G})$ and by finiteness of $\mathcal{L}$). Let us denote by $\{G_i, i \leq n\}$ the factors of G in $\mathcal{G}$. We have equalities

$$G'' = G \vee F = \bigvee_{i \leq n} G_i \vee F = \bigvee_{\substack{i \leq n \\ G_i \not\leq F}} G_i \vee F$$

but the elements in the join on the right form an antichain which is nested by maximality of F and therefore by Lemma 1.1.18 those elements are exactly the factors of G'' in $\mathcal{G}$. From this quick analysis it appears that such a maximal element F is in fact unique and we can make the following definition.

Definition 1.1.20. For any element G'' in some induced building set $\mathrm{Ind}_{[G,G']}$, we define $\mathrm{Comp}_G(G'')$ to be the unique maximal element of $\mathcal{G}$ satisfying

$$\mathrm{Comp}_G(G'') \vee G = G''.$$

If G can be deduced from the context we will omit it. We have the simple lemma.

Lemma 1.1.21. *The map Comp_G is injective.*

Proof. The map Comp_G has a left inverse given by taking the join with G. □

1.2 Combinatorial invariants

To the objects introduced in the previous section one can associate various rings that generalize cohomology rings in the realizable case.

1.2.1 The Feichtner–Yuzvinsky rings

Definition 1.2.1. For every built lattice $(\mathcal{L}, \mathcal{G})$ we define the Feichtner–Yuzvinsky graded commutative ring $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ by

$$\mathrm{FY}(\mathcal{L}, \mathcal{G}) = \mathbb{Z}[x_G,\, G \in \mathcal{G}]/\mathcal{I}_{\mathrm{aff}},$$

with all the generators in degree 2, and $\mathcal{I}_{\mathrm{aff}}$ the ideal generated by elements

$$\sum_{G \geq H} x_G$$

for every atom H, and elements

$$\prod_{G \in X} x_G$$

for every set $X \subset \mathcal{G}$ which is not nested.

The subscript "aff" stands for affine. In the realizable case, the ring $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ is the cohomology ring of the wonderful compactification associated to the building set $\mathcal{G}$ (see [8] for the computation of the cohomology ring). Those rings were generalized to arbitrary built lattices by Feichtner and Yuzvinsky in [14].

The Feichtner–Yuzvinsky rings admit two other useful presentations.

Proposition 1.2.2. *For every built lattice $(\mathcal{L}, \mathcal{G})$ we have the other classical presentation*

$$\mathrm{FY}(\mathcal{L}, \mathcal{G}) \simeq \mathbb{Z}[x_G,\, G \in \mathcal{G} \setminus \{\hat{1}\}]/\mathcal{I}_{\mathrm{proj}}$$

where $\mathcal{I}_{\mathrm{proj}}$ is the ideal generated by elements

$$\sum_{\hat{1} > G \geq H_1} x_G - \sum_{\hat{1} > G \geq H_2} x_G$$

for every pair of atoms H_1 and H_2, and elements

$$\prod_{G \in X} x_G$$

for every set $X \subset \mathcal{G} \setminus \{\hat{1}\}$ which is not nested.

Additionally, we have the presentation

$$\mathrm{FY}(\mathcal{L},\mathcal{G}) \simeq \mathbb{Z}[h_G,\, G \in \mathcal{G}]/\mathcal{I}_{\mathrm{wond}}$$

where $\mathcal{I}_{\mathrm{wond}}$ is the ideal generated by relations

$$h_H$$

for every atom H and

$$\prod_{G' \in \mathcal{A}} (h_G - h_{G'})$$

for every $G \in \mathcal{G}$ and $\mathcal{A}$ an antichain in $\mathcal{G}$ such that $\bigvee \mathcal{A}$ is equal to G. The change of variable between the last presentation and the defining presentation is given by

$$h_G = \sum_{G' \geq G} x_{G'}.$$

The first (defining) presentation will be called the *affine* presentation, the second the *projective* presentation and the last one the *wonderful* presentation. The first two presentations appear in [14] (as a definition) while the second appeared first in [12] for the braid arrangement and in [2] for general maximal building sets. It is widely used in [29].

Proof. The proof can be found in [29] (Theorem 2.9). □

In [14], the authors address the issue of finding a Gröbner basis for $\mathrm{FY}_{\mathrm{aff}}(\mathcal{L},\mathcal{G})$ (see [3] for a reference on Gröbner bases) and they show that when considering any linear order on generators refining the reverse order on $\mathcal{G}$, although the elements defining $\mathcal{I}_{\mathrm{aff}}$ do not form a Gröbner basis in general, one can still describe a fairly manageable Gröbner basis.

Theorem 1.2.3 ([14] Theorem 2)**.** *Elements of the form $(\prod_{G \in \mathcal{S}} x_G) h_{G'}^{\rho(G')-\rho(\bigvee S)}$ with $\mathcal{S}$ any nested set and G' any element of $\mathcal{G}$ satisfying $G' > \bigvee \mathcal{S}$, together with the usual $\prod_{G \in X} x_G$ for every non-nested set X, form a Gröbner basis of $\mathrm{FY}_{\mathrm{aff}}(\mathcal{L},\mathcal{G})$ for any linear order on generators refining the reversed order of $\mathcal{L}$. The normal monomials with respect to this Gröbner basis are monomials of the form*

$$x_{G_1}^{\alpha_1} ... x_{G_n}^{\alpha_n}$$

where the G_i's form a nested set $\mathcal{S}$ and for every $i \leq n$ we have $\alpha_i < \mathrm{rk}[\bigvee \mathcal{S}_{<G_i}, G_i]$.

Notice that $\mathcal{S}$ can be empty in which case we get the already known relations $h_H = 0$ for any atom H. Using that the above monomials form a linear basis of $\mathrm{FY}(\mathcal{L},\mathcal{G})$ one can see that every Feichtner–Yuzvinsky algebra is in fact of finite dimension and that the part of maximal grading, which is $2(\mathrm{rk}(\mathcal{L})-1)$, has dimension one (generated by $x_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-1}$).

One can also find a Gröbner basis for the wonderful presentation.

Corollary 1.2.4. *Elements of the form* $(\prod_{G'\in\mathcal{A}}(h_G - h_{G'}))h_G^{\rho(G)-\rho(\mathcal{A})}$ *for every antichain* $\mathcal{A}\subset\mathcal{G}$ *without atoms and every* $G \geq \bigvee\mathcal{A}$ *form a Gröbner basis of* $\mathrm{FY}_{\mathrm{wond}}(\mathcal{L},\mathcal{G})$ *for any linear order on generators refining the reversed order of* $\mathcal{L}$.

Proof. Indeed one can see that the leading terms of those elements are terms of the form $(\prod_{G'\in\mathcal{A}} h_{G'})h_G^{\rho(G)-\rho(\bigvee\mathcal{A})}$ and therefore the normal monomials with respect to those relations are elements of the form $\prod_{G\in\mathcal{S}} h_G^{\alpha_G}$ for any nested set $\mathcal{S}$ and any positive integers α_G satisfying the relations

$$\alpha_G \ < \rho(G) - \rho(\bigvee \mathcal{S}_{<G})$$

for all G in $\mathcal{S}$, which are in obvious bijection with the normal monomials for the affine Gröbner basis. This proves that those monomials form a linear basis of $\mathrm{FY}_{\mathrm{wond}}(\mathcal{L},\mathcal{G})$ (linearly independent with the right cardinality), which implies that the elements above form a Gröbner basis of $\mathcal{I}_{\mathrm{wond}}$. □

This Gröbner basis will be of use in subsequent sections. In the next and last preliminary subsection we introduce another important combinatorial invariant.

1.2.2 The Orlik–Solomon algebras

In this document "graded commutative" means with Koszul signs.

Definition 1.2.5 (Orlik–Solomon algebra)**.** Let $\mathcal{L}$ be a geometric lattice. We define the Orlik–Solomon graded commutative algebra $\mathrm{OS}(\mathcal{L})$ by

$$\mathrm{OS}(\mathcal{L}) = \Lambda[e_H,\ H \text{ atom of } \mathcal{L}]/\mathcal{I}$$

where $\mathcal{I}$ is the ideal generated by elements of the form $\delta(e_{H_1}\wedge ... \wedge e_{H_n})$ for any circuit $\{H_1, ..., H_n\}$ and δ is the unique derivation of degree -1 satisfying $\delta(e_H) = 1$. All the generators e_H have degree 1.

A circuit is a notion coming from matroid theory. In the language of geometric lattices it is a set of atoms $C = \{H_i,\ i \leq n\}$ such that $\rho(\bigvee C)$ is $n-1$ and $\rho(\bigvee X') = |X'|$ for all proper subsets $X' \subset X$.

In the complex realizable case this algebra is the cohomology ring of the complement of the hyperplane arrangement (see Orlik-Solomon [27]).

We denote by $\overline{\mathrm{OS}}(\mathcal{L})$ the subalgebra of $\mathrm{OS}(\mathcal{L})$ generated by elements of the form $e_H - e_{H'}$ for every pair of atoms H, H'. In the complex realizable case the algebra $\overline{\mathrm{OS}}(\mathcal{L})$ is the cohomology ring of the projective complement. We have the following important lemma.

Lemma 1.2.6 ([36] Section 2.4)**.** *For every geometric lattice* $\mathcal{L}$ *we have the equality*

$$\overline{\mathrm{OS}}(\mathcal{L}) = \ker\delta = \mathrm{Im}\,\delta.$$

Chapter 2

The Feynman category

In this section we show that the combinatorial objects introduced in the previous section (geometric lattices, building sets and nested sets) can be bundled up into a Feynman category.

2.1 A short introduction to Feynman categories

The notion of Feynman categories was introduced by R. Kaufmann and B. Ward in [20]. Loosely speaking, Feynman categories encode types of operadic structures.

Notation. Let $\mathcal{C}$ be a category. We denote by $\mathcal{C}^{\text{iso}}$ the subcategory of $\mathcal{C}$ having the same objects as $\mathcal{C}$ but only its isomorphisms as morphisms. We denote by $\mathrm{Sym}(\mathcal{C})$ the free symmetric monoidal category generated by $\mathcal{C}$. For any functor $F : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ a symmetric monoidal category, there is a unique induced strong monoidal functor $\mathrm{Sym}(F) : \mathrm{Sym}(\mathcal{C}) \to \mathcal{D}$. If we are given a diagram of categories $\mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{G} \mathcal{E}$, the comma category $(F \downarrow G)$ is the category having for objects triples $(c \in \mathcal{C}, e \in \mathcal{E}, \phi : F(c) \to G(e))$ and for morphisms suitable commutative diagrams. If the functors F and G are clear from the context we will write instead $(\mathcal{C} \downarrow \mathcal{E})$.

Definition 2.1.1 (Feynman category)**.** A triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ is a *Feynman category* if $\mathcal{V}$ is a groupoid, $\mathcal{F}$ is a symmetric monoidal category and $\imath : \mathcal{V} \to \mathcal{F}$ is a functor such that:

1. The functor $\imath$ induces an equivalence of categories $\mathrm{Sym}(\imath) : \mathrm{Sym}(\mathcal{V}) \to \mathcal{F}^{\text{iso}}$.

2. The functor $\imath$ induces an equivalence of categories $\mathrm{Sym}((\mathcal{F} \downarrow \mathcal{V})^{\text{iso}}) \to (\mathcal{F} \downarrow \mathcal{F})^{\text{iso}}$.

3. For every object $\star \in \mathcal{V}$, the comma category $(\mathcal{F} \downarrow \star)$ is essentially small (i.e. is equivalent to a small category).

Definition 2.1.2 (Operad over a Feynman category)**.** Let $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ be a Feynman category and $\mathcal{C}$ a symmetric monoidal category. An *operad over* $\mathfrak{F}$ in $\mathcal{C}$ is a strong monoidal functor from $\mathcal{F}$ to $\mathcal{C}$, and a *cooperad over* $\mathfrak{F}$ in $\mathcal{C}$ is a strong monoidal functor from $\mathcal{F}$ to $\mathcal{C}^{op}$. A *module over* $\mathfrak{F}$ in $\mathcal{C}$ is a functor from $\mathcal{V}$ to $\mathcal{C}$.

(Co)operads (resp. modules) over $\mathfrak{F}$ will also be called $\mathfrak{F}$-(co)operads (resp. $\mathfrak{F}$-modules).

Example 2.1.3. As described in the introduction, there exist a Feynman category $\mathfrak{Op}$ encoding classical operads i.e. such that operads over $\mathfrak{Op}$ are classical operads and modules over $\mathfrak{Op}$ are $\mathbb{S}$-modules.

2.2 Construction of the Feynman category

Let us start by defining the underlying groupoid of our Feynman category.

Definition 2.2.1. We define LBS to be the groupoid having as objects the built lattices and morphisms

$$\mathrm{Mor}_{LBS}((\mathcal{L}, \mathcal{G}), (\mathcal{L}', \mathcal{G}')) = \{f : \mathcal{L}' \xrightarrow{\sim} \mathcal{L} \text{ isomorphism of poset satisfying } f(\mathcal{G}') = \mathcal{G}\}.$$

We denote by $\mathbf{LBS}_{\mathrm{irr}}$ the full subcategory of LBS having as objects the irreducible built lattices.

The groupoid $\mathbf{LBS}_{\mathrm{irr}}$ will play the role of $\mathcal{V}$ in Definition 2.1.1. We will add morphisms to LBS in order to get the right category $\mathcal{F}$.

Proposition 2.2.2. *The category LBS admits a symmetric monoidal structure $\otimes$ given by*

$$(\mathcal{L}, \mathcal{G}) \otimes (\mathcal{L}', \mathcal{G}') = (\mathcal{L} \times \mathcal{L}', \mathcal{G} \times \{\hat{0}\} \cup \{\hat{0}\} \times \mathcal{G}').$$

Furthermore the inclusion $\imath : \boldsymbol{LBS}_{\mathrm{irr}} \to LBS$ induces an equivalence of categories

$$\mathrm{Sym}(\imath) : \mathrm{Sym}(\boldsymbol{LBS}_{\mathrm{irr}}) \to LBS.$$

Proof. The fact that a product of two geometric lattices is again a geometric lattice is classical and the proof can be found in [35]. Additionally $\mathcal{G} \times \{\hat{0}\} \cup \{\hat{0}\} \times \mathcal{G}'$ is indeed a building set of $\mathcal{L} \times \mathcal{L}'$ because for any $(X, X') \in \mathcal{L} \times \mathcal{L}'$ we have the isomorphisms

$$\begin{aligned}
[(\hat{0}, \hat{0}), (X, X')] &\simeq [\hat{0}, X] \times [\hat{0}, X'] \\
&\simeq \prod_{G \in \mathrm{Fact}_{\mathcal{G}}(X)} [\hat{0}, G] \times \prod_{G' \in \mathrm{Fact}_{\mathcal{G}'}(X')} [\hat{0}, G'] \\
&\simeq \prod_{G'' \in \mathrm{Fact}_{\mathcal{G} \times \{\hat{0}\} \cup \{\hat{0}\} \times \mathcal{G}'}((X, X'))} [(\hat{0}, \hat{0}), G''].
\end{aligned}$$

Besides, one can see that $\otimes$ is functorial and satisfies the associativity/symmetry axioms of a symmetric monoidal product, the unit being $(\{\hat{0}\}, \emptyset)$.

For the last claim we show that $\mathrm{Sym}(\imath)$ is essentially surjective and fully faithful. Let $(\mathcal{L}, \mathcal{G})$ be an object of LBS. If we denote by $\{G_i, i \leq n\}$ the factors of $\hat{1}$ in $\mathcal{G}$ then we have an isomorphism $[\hat{0}, \hat{1}] \simeq \prod_i [\hat{0}, G_i]$ and $\mathcal{G}$ is sent to $\mathcal{G} \cap [\hat{0}, G_1] \cup ... \cup \mathcal{G} \cap [\hat{0}, G_n]$. In other words $(\mathcal{L}, \mathcal{G})$ is isomorphic to $([\hat{0}, G_1], \mathcal{G} \cap [\hat{0}, G_1]) \otimes ... \otimes ([\hat{0}, G_n], \mathcal{G} \cap [\hat{0}, G_n])$ and $\mathrm{Sym}(\imath)$ is essentially surjective.

Finally, let $\bigotimes_{i \leq n} (\mathcal{L}_i, \mathcal{G}_i)$ and $\bigotimes_{j \leq n'} (\mathcal{L}'_j, \mathcal{G}'_j)$ be two elements of $\mathrm{Sym}(\mathbf{LBS}_{\mathrm{irr}})$ (here $\otimes$ denotes the free symmetric monoidal product in $\mathrm{Sym}(\mathbf{LBS}_{\mathrm{irr}})$) and let ϕ be an isomorphism in LBS between $\bigotimes_{i \leq n} (\mathcal{L}_i, \mathcal{G}_i)$ and $\bigotimes_{j \leq n'} (\mathcal{L}'_j, \mathcal{G}'_j)$. Such an isomorphism is given by a bijection between the factors of both sides (the isomorphism induces a bijection between the maximal elements of the building set of the domain and the maximal elements of the building set of the target) together with isomorphisms between corresponding summands. This datum is exactly equivalent to an isomorphism in $\mathrm{Sym}(\mathbf{LBS}_{\mathrm{irr}})$ between $\bigotimes_{i \leq n} (\mathcal{L}_i, \mathcal{G}_i)$ and $\bigotimes_{j \leq n'} (\mathcal{L}'_j, \mathcal{G}'_j)$, which proves that $\mathrm{Sym}(\imath)$ is fully faithful.

□

We now add structural morphisms to LBS to get our Feynman category. Let $(\mathcal{L}, \mathcal{G})$ be an irreducible built lattice and $\mathcal{S} = \{G_i, i \leq n\}$ an irreducible linearly ordered nested set of $\mathcal{G}$. For any $G \in \mathcal{S}$ we define $\tau_{\mathcal{S}}(G) \coloneqq \bigvee \mathcal{S}_{<G}$ and we set

$$(\mathcal{L}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}}) \coloneqq \bigotimes_i ([\tau_{\mathcal{S}}(G_i), G_i], \mathrm{Ind}_{[\tau_{\mathcal{S}}(G_i), G_i]}(\mathcal{G}))$$

which is an object of LBS. We will view $\mathcal{S}$ as a new formal morphism $(\mathcal{L}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}}) \xrightarrow{\mathcal{S}} (\mathcal{L}, \mathcal{G})$ that we will add by hand to LBS. However, to this end one must specify how those new morphisms compose with each other and with the isomorphisms. This is the object of the next definition/lemma.

From now on, every nested set is assumed to be irreducible unless stated otherwise. If $\mathcal{S}$ is a nested set, the intervals $[\tau_{\mathcal{S}}(G), G]$ for G any element of $\mathcal{S}$ will be called the "local intervals of $\mathcal{S}$".

Let $\mathcal{S} = \{G_i, i \leq n\}$ be a nested set in $(\mathcal{L}, \mathcal{G})$ and let there be given additional linearly ordered nested sets $\mathcal{S}_i$'s in each irreducible built lattice $([\tau_{\mathcal{S}}(G_i), G_i], \mathrm{Ind}_{[\tau_{\mathcal{S}}(G_i), G_i]}(\mathcal{G}))$. We define

$$\mathcal{S} \circ (\mathcal{S}_i)_i \coloneqq \mathcal{S} \cup \bigcup_i \{\mathrm{Comp}_{\tau_{\mathcal{S}}(G_i)}(K) \, , \, K \in \mathcal{S}_i\} \tag{2.1}$$

which comes naturally equipped with a linear order (by concatenating the linear orders of the $\mathcal{S}_i$'s).

Remark 2.2.3. For $\mathcal{G} = \mathcal{G}_{\max}$ the operation $\circ$ is just the concatenation of chains.

We have the key lemma.

Lemma 2.2.4. *$\mathcal{S} \circ (\mathcal{S}_i)_i$ is a nested set of $(\mathcal{L}, \mathcal{G})$.*

Proof. By Lemma 1.1.21 and Lemma 1.1.19 we have that all the elements of the form $\mathrm{Comp}(K)$ with K in some $\mathcal{S}_i$ are distinct and not in $\mathcal{S}$, and they are partitioned according to the unique minimal element of $\mathcal{S}$ above them.

Let $\mathcal{A} = \{G_i | i \in I\} \sqcup \bigsqcup_{j \in J} \mathcal{A}_j$ be an antichain in $\mathcal{S} \circ (\mathcal{S}_i)_i$ partitioned according to the previous remark, such that the join of $\mathcal{A}$ belongs to $\mathcal{G}$. Let us prove that $\mathcal{A}$ is a singleton. Since $\mathcal{A}$ is an antichain, I and J are disjoint. Let M be the set of maximal elements of $\{G_i, i \in I\} \cup \{G_j, j \in J\}$. Since $\bigvee \mathcal{A}$ belongs to $\mathcal{G}$ and $\mathcal{S}$ is a nested set, M is a singleton. If this singleton belongs to $\{G_i, i \in I\}$, then by the fact that $\mathcal{A}$ is an antichain we must have $\mathcal{A} = M$. If this singleton belongs to $\{G_j, j \in J\}$, let us denote it $\{G_j\}$. By nestedness of $\mathcal{S}_j$ we see that $\mathcal{A}_j$ is a singleton, which we denote by $\{\mathrm{Comp}(K)\}$ with K some element in $\mathcal{S}_j$. We have

$$\mathrm{Comp}(K) \vee \tau_{\mathcal{S}}(G_j) = \bigvee \mathcal{A} \vee \tau_{\mathcal{S}}(G_j) = K.$$

By maximality of $\mathrm{Comp}(K)$ (see Definition 1.1.20), this means that $\bigvee \mathcal{A}$ is equal to $\mathrm{Comp}(K)$ and by the fact that $\mathcal{A}$ is an antichain, $\mathcal{A}$ must be equal to the singleton $\{\mathrm{Comp}(K)\}$. □

Lemma 2.2.5. *We have an isomorphism of built lattices*

$$\bigotimes_i (\mathcal{L}_{\mathcal{S}_i}, \mathcal{G}_{\mathcal{S}_i}) \xrightarrow{\Phi} (\mathcal{L}_{\mathcal{S}\circ(\mathcal{S}_i)_i}, \mathcal{G}_{\mathcal{S}\circ(\mathcal{S}_i)_i}).$$

Proof. We just need to show that the irreducible built lattices appearing on the left are canonically isomorphic to the irreducible built lattices appearing on the right. Let K be some element in some $\mathcal{S}_i$ and let $K_1, \dots, K_p$ be the maximal elements of $(\mathcal{S}_i)_{<K}$. We need to find an isomorphism of built lattice

$$([\tau_{\mathcal{S}_i}(K), K], \mathrm{Ind}_{[\tau_{\mathcal{S}_i}(K), K]}(\mathrm{Ind}_{[\tau_{\mathcal{S}}(G_i), G_i]}(\mathcal{G}))) \xrightarrow{\Phi} ([\bigvee_i \mathrm{Comp}(K_i) \vee \bigvee_{G_j < \mathrm{Comp}(K)} G_j, \mathrm{Comp}(K)], \mathrm{Ind}(\mathcal{G})).$$

Such an isomorphism is given by taking the join with $\tau_{\mathcal{S}}(G_i)$. The fact that this is indeed an isomorphism of poset comes from the building set isomorphism

$$[\hat{0}, K] \simeq [\hat{0}, \mathrm{Comp}(K)] \times \prod_{\substack{G_j \in \max \mathcal{S}_{<G_i} \\ G_j \nleq \mathrm{Comp}(K)}} [\hat{0}, G_j].$$

The fact that it sends building set to building set comes from Lemma 1.1.15. □

Finally, we have to prove that the operation $\circ$ on nested sets is "associative".

Lemma 2.2.6. *Let $\mathcal{S}$ be a nested set in $(\mathcal{L},\mathcal{G})$. Let $(\mathcal{S}_i)_i$ be nested sets in every local interval of $\mathcal{S}$ and for every i let $(\mathcal{S}^i_j)_j$ be nested sets in every local interval of $\mathcal{S}_i$. The nested sets $\mathcal{S}\circ(\mathcal{S}_i\circ(\mathcal{S}^i_j)_j)_j$ and $(\mathcal{S}\circ(\mathcal{S}_i)_i)\circ(\mathcal{S}^i_j)_{i,j}$ are equal (the last composition being performed via the isomorphism Φ of Lemma 2.2.5).*

Proof. This is a statement about the Comp operation. When necessary we put the lattice in which we are doing the Comp operation in superscript. We denote $\mathcal{S}=\{G_i\,,\,i\leq n\}$ and choose some $i_0\leq n$. We then denote $\mathcal{S}_{i_0}=\{K_j\,,\,j\leq m\}$ and choose some $j_0\leq m$. We also define $I_0 \coloneqq \{i\,|\,G_i\in\max\mathcal{S}_{<G_{i_0}}\}$ and $J_0\coloneqq\{j\,|\,K_j\in\max((\mathcal{S}_{i_0})_{<K_{j_0}})\}$.

We partition I_0 into the following subsets:

$$\begin{aligned}
I_0^{\text{ext}} &\coloneqq \{i\in I_0\,|\,G_i\not\leq \mathrm{Comp}(K_{j_0})\},\\
I_0^{\text{int}} &\coloneqq \{i\in I_0\,|\,G_i\leq \mathrm{Comp}(K_{j_0}) \text{ and } \forall j\in J_0,\, G_i\not\leq\mathrm{Comp}(K_j)\},\\
I_0^{j} &\coloneqq \{i\in I_0\,|\,G_i\leq \mathrm{Comp}(K_j)\}.
\end{aligned}$$

It is a partition because the set $\{K_j,\, j\in J_0\}$ is nested. The map Φ is taking the join with $\bigvee_{i\in I_0^{\text{ext}}} G_i$.

The lemma amounts to showing that for any L in $\mathrm{Ind}_{[\tau_{\mathcal{S}_{i_0}}(K_{j_0}),K_{j_0}]}(\mathcal{G})$ we have the equality

$$\mathrm{Comp}^{\mathcal{L}}_{\tau_{\mathcal{S}}(G_{i_0})}(\mathrm{Comp}^{[\tau_{\mathcal{S}}(G_{i_0}),G_{i_0}]}_{\tau_{\mathcal{S}_{i_0}}(K_{j_0})}(L)) \quad=\quad \mathrm{Comp}^{\mathcal{L}}_{\bigvee\{\mathrm{Comp}(K_j),\,j\in J_0\}\,\vee\bigvee\{G_i,\,i\in I_0^{\text{int}}\}}(\Phi^{-1}(L)).$$

We have

$$\begin{aligned}
\mathrm{Comp}^{\mathcal{L}}_{\tau_{\mathcal{S}}(G_{i_0})}(\mathrm{Comp}^{[\tau_{\mathcal{S}}(G_{i_0}),G_{i_0}]}_{\tau_{\mathcal{S}_{i_0}}(K_{j_0})}(L))\vee\bigvee_{j\in J_0}\mathrm{Comp}(K_j)\vee\bigvee_{i\in I_0^{\text{int}}}G_i\vee\bigvee_{i\in I_0^{\text{ext}}}G_i = \\
\mathrm{Comp}^{[\tau_{\mathcal{S}}(G_{i_0}),G_{i_0}]}_{\tau_{\mathcal{S}_{i_0}}(K_{j_0})}(L)\vee\bigvee_{j\in J_0}\mathrm{Comp}(K_j)\vee\bigvee_{i\in I_0^{\text{int}}}G_i\vee\bigvee_{i\in I_0^{\text{ext}}}G_i = L.
\end{aligned}$$

Applying Φ^{-1} to both sides we get

$$\mathrm{Comp}^{\mathcal{L}}_{\tau_{\mathcal{S}}(G_{i_0})}(\mathrm{Comp}^{[\tau_{\mathcal{S}}(G_{i_0}),G_{i_0}]}_{\tau_{\mathcal{S}_{i_0}}(K_{j_0})}(L))\vee\bigvee_{j\in J_0}\mathrm{Comp}(K_j)\vee\bigvee_{i\in I_0^{\text{int}}}G_i = \Phi^{-1}(L).$$

We need to prove that $\mathrm{Comp}^{\mathcal{L}}_{\tau_{\mathcal{S}}(G_{i_0})}(\mathrm{Comp}^{[\tau_{\mathcal{S}}(G_{i_0}),G_{i_0}]}_{\tau_{\mathcal{S}_{i_0}}(K_{j_0})}(L))$ is the biggest element in $\mathcal{G}$ which satisfies this equation.

Let $G' \in \mathcal{G}$ such that we have

$$G' \vee \bigvee_{j \in J_0} \mathrm{Comp}(K_j) \vee \bigvee_{i \in I_0^{\mathrm{int}}} G_i = \Phi^{-1}(L).$$

Applying Φ on both sides we get

$$G' \vee \bigvee_{j \in J_0} \mathrm{Comp}(K_j) \vee \bigvee_{i \in I_0^{\mathrm{int}}} G_i \vee \bigvee_{i \in I_0^{\mathrm{ext}}} G_i = L. \tag{2.2}$$

The element $G' \vee \bigvee_{i \in I} G_i$ is below L and belongs to $\mathrm{Ind}_{[\tau_{\mathcal{S}}(G_{i_0}), G_{i_0}]}(\mathcal{G})$ so it is below one of the factors of L in $\mathrm{Ind}_{[\tau_{\mathcal{S}}(G_{i_0}), G_{i_0}]}(\mathcal{G})$. Those factors are $\mathrm{Comp}^{[\tau_{\mathcal{S}}(G_{i_0}), G_{i_0}]}_{\tau_{\mathcal{S}_{i_0}}(K_{j_0})}(L)$ or K_j for some j in J. By equation (2.2), $G' \vee \bigvee_{i \in I} G_i$ cannot be below any K_j so we have

$$G' \vee \bigvee_{i \in I} G_i \leq \mathrm{Comp}^{[\tau_{\mathcal{S}}(G_{i_0}), G_{i_0}]}_{\tau_{\mathcal{S}_{i_0}}(K_{j_0})}(L),$$

which implies

$$G' \leq \mathrm{Comp}^{\mathcal{L}}_{\tau_{\mathcal{S}}(G_{i_0})}(\mathrm{Comp}^{[\tau_{\mathcal{S}}(G_{i_0}), G_{i_0}]}_{\tau_{\mathcal{S}_{i_0}}(K_{j_0})}(L)).$$

□

We are now in position to make the following definition.

Definition 2.2.7. **LBS** is the monoidal category defined as follow.

- The objects of **LBS** are built lattices.
- The morphisms of **LBS** are generated (via composition and tensoring) by
  1. Structural morphisms
  $$(\mathcal{L}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}}) \xrightarrow{\mathcal{S}} (\mathcal{L}, \mathcal{G})$$
  for every totally ordered nested set in some irreducible built lattice $(\mathcal{L}, \mathcal{G})$. The composition of those morphisms is given by 2.1.
  2. Isomorphisms between built lattices
  $$(\mathcal{L}, \mathcal{G}) \xrightarrow{\sim} (\mathcal{L}', \mathcal{G}')$$
  for each isomorphism of poset $f : \mathcal{L}' \to \mathcal{L}$ such that $f(\mathcal{G}') = \mathcal{G}$,

quotiented by relations

$$(\mathcal{L}_{f(\mathcal{S})},\mathcal{G}_{f(\mathcal{S})}) \overset{f(\mathcal{S})}{\to} (\mathcal{L}',\mathcal{G}') \overset{f}{\to} (\mathcal{L},\mathcal{G}) \sim (\mathcal{L}_{f(\mathcal{S})},\mathcal{G}_{f(\mathcal{S})}) \overset{\otimes f}{\to} (\mathcal{L}_{\mathcal{S}},\mathcal{G}_{\mathcal{S}}) \overset{\mathcal{S}}{\to} (\mathcal{L},\mathcal{G}) \tag{2.3}$$

for any isomorphism of irreducible built lattice $f : (\mathcal{L}',\mathcal{G}') \to (\mathcal{L},\mathcal{G})$, and for every permutation σ of $\mathcal{S}$:

$$(\mathcal{L}_{\mathcal{S}},\mathcal{G}_{\mathcal{S}}) \overset{\sigma}{\to} (\mathcal{L}_{\mathcal{S}_\sigma},\mathcal{G}_{\mathcal{S}_\sigma}) \overset{\mathcal{S}_\sigma}{\to} (\mathcal{L},\mathcal{G}) \sim (\mathcal{L}_{\mathcal{S}},\mathcal{G}_{\mathcal{S}}) \overset{\mathcal{S}}{\to} (\mathcal{L},\mathcal{G}) \tag{2.4}$$

where $\mathcal{S}_\sigma$ denotes the nested set equipped with the new linear order given by σ. Finally we also impose the relation

$$\{\hat{1}\} \sim \mathrm{Id}_{(\mathcal{L},\mathcal{G})}.$$

- The monoidal structure is the same as the one on LBS, which in addition acts on nested sets (which are now considered as morphisms) by disjoint union.

Proposition 2.2.8. *The triple $\mathfrak{LBS} = (\boldsymbol{LBS}_{\mathrm{irr}}, \boldsymbol{LBS}, \imath)$ with $\imath$ the obvious inclusion is a Feynman category.*

Proof. This is a consequence of Proposition 2.2.2 and of the construction itself. □

We conclude this subsection by proving a general lemma on the composition of nested sets which will be important later on.

Lemma 2.2.9. *Let $\mathcal{S}$ be some irreducible nested set in some irreducible built lattice and $\mathcal{S}' \subset \mathcal{S}$ a subset containing $\hat{1}$. For any G' in $\mathcal{S}'$ we put $\mathcal{S}'_{G'} := (\mathcal{S} \vee \tau_{\mathcal{S}'}(G')) \cap (\tau_{\mathcal{S}'}(G'), G']$. For any G' in $\mathcal{S}'$, $\mathcal{S}'_{G'}$ is a nested set in $([\tau_{\mathcal{S}'}(G'), G'], \mathrm{Ind}(\mathcal{G}))$ and we have the equality between nested sets:*

$$\mathcal{S} = \mathcal{S}' \circ (\mathcal{S}'_{G'})_{G' \in \mathcal{S}'}. \tag{2.5}$$

Proof. Let G' be any element of $\mathcal{S}'$ and $G'_1, ..., G'_n$ the maximal elements of $\mathcal{S}'_{<G'}$. Let G be some element in $\mathcal{S}$ which is below G' and not below any of the G'_i's. We denote $K := \tau_{\mathcal{S}'}(G') \vee G$. By the fact that $\mathcal{S}$ is nested, G is the maximal element of $\mathcal{G}$ satisfying the equality

$$G \vee \tau_{\mathcal{S}'}(G') = K.$$

This proves the equality

$$G = \mathrm{Comp}_{\tau_{\mathcal{S}'}(G')}(\tau_{\mathcal{S}'}(G') \vee G),$$

which implies equality (2.5).

For the nestedness, assume we have $G_1, ..., G_k$ some elements of $\mathcal{S}$ such that $\tau_{\mathcal{S}'}(G') \vee G_1, ..., \tau_{\mathcal{S}'}(G') \vee G_k$ are elements of $(\tau_{\mathcal{S}'}(G'), G']$ and such that $\bigvee_i \tau_{\mathcal{S}'}(G') \vee G_i$ belongs to

$\mathrm{Ind}_{[\tau_{\mathcal{S}'}(G'),\hat{1}]}(\mathcal{G})$. The factors of $\bigvee_i \tau_{\mathcal{S}'}(G') \vee G_i$ in $\mathcal{G}$ are some of the $(G_i')'s$ and the element $\mathrm{Comp}_{\tau_{\mathcal{S}'}(G')}(\bigvee_i \tau_{\mathcal{S}'}(G') \vee G_i)$. Since $\mathrm{Comp}_{\tau_{\mathcal{S}'}(G')}$ is increasing (it has a left inverse given by taking the join with $\tau_{\mathcal{S}'}(G')$), by the first part of this proof we have

$$G_i = \mathrm{Comp}_{\tau_{\mathcal{S}'}(G')}(\tau_{\mathcal{S}'}(G') \vee G_i) \leq \mathrm{Comp}_{\tau_{\mathcal{S}'}(G')}(\bigvee_i \tau_{\mathcal{S}'}(G') \vee G_i)$$

for all $i \leq k$ and by the building set isomorphism we must have

$$\bigvee_i G_i = \mathrm{Comp}_{\tau_{\mathcal{S}'}(G')}(\bigvee_i \tau_{\mathcal{S}'}(G') \vee G_i).$$

By nestedness of $\mathcal{S}$ the G_i's do not form an antichain and therefore the $\tau_{\mathcal{S}'}(G') \vee G_i$'s do not either. This proves that $(\tau_{\mathcal{S}'}(G') \vee \mathcal{S}) \cap (\tau_{\mathcal{S}'}(G'), \hat{1}]$ is a nested set. □

2.3 Presentation of $\mathfrak{LBS}$

In this section we give a presentation of the category **LBS**, that is a set of morphisms that generate every other morphisms in $\mathfrak{LBS}$ via composition and tensoring, together with the relations they satisfy. As mentioned in the introduction, from a practical point of view having a presentation of $\mathfrak{LBS}$ will make defining operads over $\mathfrak{LBS}$ a lot easier (in the next section).

Proposition 2.3.1. *Every morphism in **LBS** can be obtained as a composition and tensoring of morphisms of the form $\{G, \hat{1}\}$ and isomorphisms.*

From now on since all our nested sets must contain $\hat{1}$ we will omit it (for instance the above generators will be written $\{G\}$).

Proof. Iterate Lemma 2.2.9 with $\mathcal{S}'$ of the form $\{G\}$. □

Proposition 2.3.2. *Relations between compositions of generators $\{G\}$ are all generated by the relations*

$$\{G_1\} \circ (\{G_2\} \otimes \mathrm{Id}) = \{G_2\} \circ (\mathrm{Id} \otimes \{G_1\}) \tag{2.6}$$

for every pair $G_1 < G_2$, relations

$$\{G_1\} \circ (\{G_1 \vee G_2\} \otimes \mathrm{Id}) = \{G_2\} \circ (\{G_1 \vee G_2\} \otimes \mathrm{Id}) \circ \sigma^{2,3} \tag{2.7}$$

for every pair G_1, G_2 of non comparable elements forming a nested set (with $\sigma^{2,3}$ the transposition swapping the two last summands) and relations

$$f \circ f(\{G\}) = \{G\} \circ (f_{[G,\hat{1}]} \otimes f_{[\hat{0},G]}) \tag{2.8}$$

for every isomorphism $f : (\mathcal{L}, \mathcal{G}) \xrightarrow{\sim} (\mathcal{L}', \mathcal{G}')$ in LBS and every element $G \in \mathcal{G} \setminus \{\hat{1}\}$.

Proof. One can check that those relations are indeed satisfied in $\mathfrak{LBS}$. In order to prove that they generate all relations we start with the following lemma.

Lemma 2.3.3. *Let $(\mathcal{L}, \mathcal{G})$ be a built lattice and $\lhd$ a linear order on $\mathcal{L}$. Any morphism in $\mathfrak{LBS}$*

$$F : \bigotimes_i (\mathcal{L}_i, \mathcal{G}_i) \to (\mathcal{L}, \mathcal{G})$$

can be uniquely written as a composition

$$\bigotimes_i (\mathcal{L}_i, \mathcal{G}_i) \xrightarrow{\otimes f_i} (\mathcal{L}'_i, \mathcal{G}'_i) \xrightarrow{\sigma} (\mathcal{L}'_{\sigma(i)}, \mathcal{G}'_{\sigma(i)}) \xrightarrow{\mathcal{S}} (\mathcal{L}, \mathcal{G}), \tag{2.9}$$

where

- *The f_i's are isomorphisms in $\boldsymbol{LBS}_{\mathrm{irr}}$.*
- *σ is a permutation of the summands.*
- *$\mathcal{S}$ is an irreducible nested set of $(\mathcal{L}, \mathcal{G})$ with total order given by restriction of $\lhd$.*

Proof. By iteration of relations (2.3) and (2.4) one can see that every morphism can be written in the form (2.9). For the unicity we define an invariant $\iota(F) : \bigsqcup_i (\mathcal{L}_i \backslash \{\hat{0}\}) \to \mathcal{L} \backslash \{\hat{0}\}$ for every morphism $F : \bigotimes_i (\mathcal{L}_i, \mathcal{G}_i) \to (\mathcal{L}, \mathcal{G})$ in **LBS** by setting

- For any nested set in some built lattice $(\mathcal{L}, \mathcal{G})$:

  $$\iota(\mathcal{S}) : \bigsqcup_G ([\tau_{\mathcal{S}}(G), G] \setminus \{\tau_{\mathcal{S}}(G)\}) \hookrightarrow \mathcal{L} \setminus \{\hat{0}\}$$

  is the obvious inclusion.
- For any isomorphism $(\mathcal{L}', \mathcal{G}') \xrightarrow{f} (\mathcal{L}, \mathcal{G})$ in **LBS**, $\iota(f)$ is equal to f^{-1}.

and then extending to all morphisms by composition. One can see that ι preserves the relations (2.3) and (2.4) and therefore it passes to the quotient and gives a well-defined invariant for every morphism in $\mathfrak{LBS}$.

Now if $F : \bigotimes_i (\mathcal{L}_i, \mathcal{G}_i) \to (\mathcal{L}, \mathcal{G})$ can be written as a composition

$$\bigotimes_i (\mathcal{L}_i, \mathcal{G}_i) \xrightarrow{\otimes f_i} (\mathcal{L}'_i, \mathcal{G}'_i) \xrightarrow{\sigma} (\mathcal{L}'_{\sigma(i)}, \mathcal{G}'_{\sigma(i)}) \xrightarrow{\mathcal{S}} (\mathcal{L}, \mathcal{G}),$$

then we see that $\mathcal{S}$ and the $f'_i s$ can be extracted from $\iota(F)$ ($\mathcal{S}$ is just the image by $\iota(F)$ of $\bigsqcup_i \{\hat{1}_{\mathcal{L}_i}\}$ in $\mathcal{L} \setminus \{\hat{0}\}$ and the $f'_i s$ are just restrictions of $\iota(F)$ to the suitable subsets) and σ is the only permutation that permutes the summands in the right order when $\mathcal{S}$ is given the order $\lhd$. This proves the unicity. $\square$

Assume now that we have two sequences of morphisms in **LBS**

$$\varphi = A \to X_1 \to \dots \to X_n \to (\mathcal{L}, \mathcal{G}) \text{ and } \psi = A \to X'_1 \to \dots \to X'_{n'} \to (\mathcal{L}, \mathcal{G}),$$

such that every morphism in ϕ or ψ is either a generator of the form $\mathrm{Id} \otimes \dots \otimes \mathrm{Id} \otimes \{G\} \otimes \mathrm{Id} \otimes \dots \otimes \mathrm{Id}$ or an isomorphism, and $(\mathcal{L}, \mathcal{G})$ is some irreducible built lattice. We want to prove that if the composition of the morphisms of φ is equal to the composition of the morphisms of ψ then there is a chain of equivalences of the form (2.6), (2.7) or (2.8) (possibly tensored and composed with other common morphisms) between φ and ψ.

First, by iteration of relation (2.8) one can assume that the only morphisms in φ and ψ which are not isomorphisms are the first morphisms of φ and ψ respectively. By Lemma 2.3.3 we can assume that those two isomorphisms are only permutations of summands and that the nested obtained by composition of the other morphisms of φ is the same as the one obtained by composition of the other morphisms of ψ. We will denote this nested set by $\mathcal{S}$. We also denote by $\mathcal{S}_i$ (resp. $\mathcal{S}'_i$) the nested set obtained by composing the last i morphisms of φ (resp. ψ). By construction of the composition of nested sets (see Section 2.2) we have

$$\mathcal{S}_1 \subsetneq \dots \subsetneq \mathcal{S}_n = \mathcal{S}$$

and

$$\mathcal{S}'_1 \subsetneq \dots \subsetneq \mathcal{S}'_{n'} = \mathcal{S},$$

and the cardinal of the nested sets increases exactly by one at each step. Let us denote $\mathcal{S}_1 = \{G_\varphi\}$. By the equations above there exist some $j \leq n'$ such that we have

$$\mathcal{S}'_j \setminus \mathcal{S}'_{j-1} = \{G_\varphi\}.$$

One can find a chain of relations of the form (2.6) and (2.7) between ψ and some ψ' such that the first morphism of ψ' is $\{G_\varphi\}$ (applying relations (2.6) and (2.7) allows one to swap successively the morphism corresponding to G_φ in ψ with the morphism after, until it reaches the end). This means that we can assume that the last morphism of ψ is $\{G_\varphi\}$. We denote by $\mathcal{S}^{<G_\varphi}$ (resp. $\mathcal{S}^{>G_\varphi}$) the nested set obtained by composing the morphisms of φ which correspond to generators in $[\hat{0}, G]$ (resp. $[G, \hat{1}]$), and we denote similarly $\mathcal{S}'^{<G_\psi}$, $\mathcal{S}'^{>G_\psi}$ the same constructions but with ψ. We have

$$\{G_\varphi\} \circ (\mathcal{S}^{<G_\varphi}, \mathcal{S}^{>G_\varphi}) = \mathcal{S} = \{G_\varphi\} \circ (\mathcal{S}'^{<G_\varphi}, \mathcal{S}'^{>G_\varphi})$$

which implies that we have

$$\mathcal{S}^{<G_\varphi} = \mathcal{S}'^{<G_\varphi}$$
$$\mathcal{S}^{>G_\varphi} = \mathcal{S}'^{>G_\varphi}$$

by Lemma 1.1.21 and we conclude by induction. □

2.4 $\mathfrak{LBS}$ is a graded Feynman category

An important feature of $\mathfrak{LBS}$ is that morphisms of **LBS** can be graded in the following sense.

Definition 2.4.1. A *degree function* on a Feynman category $(\mathcal{V}, \mathcal{F}, \imath)$ is a map

$$\deg : \mathrm{Mor}(\mathcal{F}) \to \mathbb{N}$$

such that

- $\deg(\phi \circ \psi) = \deg(\phi) + \deg(\psi)$
- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
- Morphisms of degree 0 and 1 generate $\mathrm{Mor}(\mathcal{F})$ by compositions and tensor products.

Furthermore the degree function is said to be *proper* if the degree 0 morphisms are exactly the isomorphisms. A graded Feynman category is a Feynman category equipped with a degree function.

Example 2.4.2. The Feynman category $\mathfrak{Op}$ encoding classical operads has a proper grading given by defining the degree of a tree t as the number of inner vertices of t minus one.

For $\mathfrak{LBS}$ we define a proper degree function by putting

$$\deg(f) = 0 \text{ for all isomorphisms } f,$$
$$\deg(\{G\}) = 1 \text{ for all } G \text{ different from } \hat{1}.$$

One can see that the relations introduced in Proposition 2.3.2 are homogeneous with respect to this grading and therefore we can extend this grading to every morphism in $\mathfrak{LBS}$, which makes $\mathfrak{LBS}$ a properly graded Feynman category.

Chapter 3

(Co)operads over $\mathfrak{LBS}$

In this section we show that the algebraic invariants introduced in Section 1 (Feichtner–Yuzvinsky algebras, Orlik–Solomon algebras) can be bundled up to form various (co)operads over $\mathfrak{LBS}$ (see Definition 2.1.2).

3.1 The Feichtner–Yuzvinsky cooperad

In this section we define an $\mathfrak{LBS}$-cooperad structure on the family of Feichtner–Yuzvinsky rings.

3.1.1 Definition of the cooperad

Lemma 3.1.1. *The map $(\mathcal{L}, \mathcal{G}) \to \mathrm{FY}(\mathcal{L}, \mathcal{G})$ can be upgraded to a (strong) monoidal functor from LBS to **grComRing**^{op} where **grComRing** is the symmetric monoidal category of graded commutative rings.*

Proof. Let $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{L}', \mathcal{G}')$ be two built lattices and let $(\mathcal{L}'', \mathcal{G}'') = (\mathcal{L}, \mathcal{G}) \otimes (\mathcal{L}', \mathcal{G}')$. We have an isomorphism of algebras

$$\begin{array}{rcccc} \mathrm{FY}(\mathcal{L}, \mathcal{G}) & \otimes & \mathrm{FY}(\mathcal{L}', \mathcal{G}') & \xrightarrow{\sim} & \mathrm{FY}(\mathcal{L}'', \mathcal{G}'') \\ x_G & \otimes & 1 & \to & x_G \\ 1 & \otimes & x_{G'} & \to & x_{G'} \end{array}$$

with inverse

$$\begin{array}{c} \mathrm{FY}(\mathcal{L}'', \mathcal{G}'') \to \mathrm{FY}(\mathcal{L}, \mathcal{G}) \otimes \mathrm{FY}(\mathcal{L}', \mathcal{G}') \\ x_G \to \begin{cases} x_G \otimes 1 & \text{if } G \in \mathcal{G} \\ 1 \otimes x_G & \text{otherwise.} \end{cases} \end{array}$$

If ϕ is an isomorphism $(\mathcal{L}', \mathcal{G}') \xrightarrow{\sim} (\mathcal{L}, \mathcal{G})$ in LBS, it induces the isomorphism of algebras

$$\begin{aligned} \mathrm{FY}(\mathcal{L}, \mathcal{G}) &\to \mathrm{FY}(\mathcal{L}', \mathcal{G}') \\ x_G &\to x_{\phi(G)} \end{aligned}$$

which is compatible with composition. □

Next we upgrade FY into a monoidal functor from **LBS** to **grComRing**^{op}. Thanks to the presentation of **LBS** given in Subsection 2.3 we only need to specify the action of FY on nested sets of cardinal one and then check that it satisfies the right relations. For every $G \in \mathcal{G} \setminus \{\hat{1}\}$ we set (using this time the wonderful presentation)

$$\begin{array}{rccl} \mathrm{FY}(\{G\}): & \mathrm{FY}(\mathcal{L}, \mathcal{G}) & \longrightarrow & \mathrm{FY}([G, \hat{1}], \mathrm{Ind}(\mathcal{G})) \otimes \mathrm{FY}([\hat{0}, G], \mathrm{Ind}(\mathcal{G})) \\ & h_{G'} & \longrightarrow & \begin{cases} 1 \otimes h_{G'} & \text{if } G' \leq G \\ h_{G \vee G'} \otimes 1 & \text{otherwise.} \end{cases} \end{array}$$

In the realizable case this morphism of algebra is induced by the inclusion of the stratum $\overline{Y}_{\{G, \hat{1}\}}$ in the wonderful compactification $\overline{Y}_{\mathcal{L}, \mathcal{G}}$.

Let us quickly justify why this map passes to the quotient. If H is an atom of $\mathcal{L}$ which is below G then h_H is sent to $h_H \otimes 1$ which is zero in the target algebra. On the contrary if H is an atom of $\mathcal{L}$ which is not below G then this means by sub-modularity of $\mathcal{L}$ that $H \vee G$ is an atom of $[G, \hat{1}]$ and thus h_H is sent to zero again. Notice that this is the first time we have actually used the geometricity of our lattices. Now let $X = \{G_1, ..., G_n\}$ be some elements in $\mathcal{G}$ having join $G' \in \mathcal{G}$. Let us assume that the first k G_i's are the elements of X below G. If $G' \leq G$ then $\prod_i (h_G - h_{G_i})$ is sent to

$$1 \otimes \prod_i (h_G - h_{G_i}),$$

which is zero in the target algebra. Otherwise if $G' > G$ then $\prod_{i>k} (h_{G'} - h_{G_i})$ is sent to

$$\prod_{i>k} (h_{G \vee G'} - h_{G \vee G_i}) \otimes 1,$$

which is also zero in the target algebra.

Proposition 3.1.2. *The maps* $\mathrm{FY}(\{G\})$ *satisfy the relations given in Proposition 2.3.2.*

Proof. Let $G_1 < G_2$ be two comparable elements in $\mathcal{G} \setminus \{\hat{1}\}$. We have to check the equality of algebra morphisms

$$(\mathrm{FY}(\{G_2\}) \otimes \mathrm{Id}) \circ \mathrm{FY}(\{G_1\}) = (\mathrm{Id} \otimes \mathrm{FY}(\{G_1\}) \circ \mathrm{FY}(\{G_2\}),$$

and it is enough to check it on generators. Let G be any element in $\mathcal{G}$. If $G \leq G_1$ one can check that both morphisms send h_G to $1 \otimes 1 \otimes h_G$. If $G \leq G_2$ and $G \nleq G_1$ one can see that both morphisms send h_G to $1 \otimes h_{G_1 \vee G} \otimes 1$. Lastly, if $G \nleq G_2$ one can check that both morphisms send h_G to $h_{G_2 \vee G} \otimes 1 \otimes 1$.

Let G_1, G_2 be two non-comparable elements of $\mathcal{G} \setminus \{\hat{1}\}$ forming a nested set. We have to check the equality of algebra morphisms

$$(\mathrm{FY}(\{G_1 \vee G_2\}) \otimes \mathrm{Id}) \circ \mathrm{FY}(\{G_1\}) = \sigma^{2,3} \circ (\mathrm{FY}(\{G_1 \vee G_2\}) \otimes \mathrm{Id}) \circ \mathrm{FY}(\{G_2\}).$$

It amounts again to a simple verification on generators with a small dichotomy. Let G be any element in $\mathcal{G}$. If $G \leq G_1$ this means that G cannot be below G_2 and we see that both morphisms send h_G to $1 \otimes 1 \otimes h_G$, if $G \leq G_2$ by a similar argument both morphisms send h_G to $1 \otimes h_G \otimes 1$. Finally if G is neither below G_1 nor below G_2 then both morphisms send h_G to $h_{G \vee G_1 \vee G_2} \otimes 1 \otimes 1$.

We can give an explicit formula for general morphisms $\mathrm{FY}(\mathcal{S})$. If G is any element in $\mathcal{G}$ let G' be the unique minimal element of $\mathcal{S}_{>G}$. We then have

$$\mathrm{FY}(\mathcal{S})(h_G) = 1^{\otimes} \otimes h_{\tau_{\mathcal{S}}(G') \vee G} \otimes 1^{\otimes},$$

where $1^{\otimes}$ means that we put a 1 in every interval which is not $[\tau_{\mathcal{S}}(G_{i_0}), G_{i_0}]$.

Finally, we need to check that the morphisms $\mathrm{FY}(\{G\})$ satisfy relation (2.8). Let $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{L}', \mathcal{G}')$ be two built lattices and $f : (\mathcal{L}', \mathcal{G}') \xrightarrow{\sim} (\mathcal{L}, \mathcal{G})$ an isomorphism in $\mathfrak{LBS}$, i.e. an isomorphism of poset $f : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that $f(\mathcal{G})$ is equal to $\mathcal{G}'$. Let G be an element in $\mathcal{G} \setminus \{\hat{1}\}$. We have to check the equality between algebra morphisms

$$\mathrm{FY}(\{f(G)\}) \circ \mathrm{FY}(f) = (\mathrm{FY}(f_{|[G,\hat{1}]}) \otimes \mathrm{FY}(f_{|[\hat{0},G]})) \circ \mathrm{FY}(\{G\}).$$

Let h_K be some generator in $\mathrm{FY}(\mathcal{L}, \mathcal{G})$. If $K \leq G$ one can check that both morphisms send h_K to $1 \otimes h_{f(K)}$. Otherwise if $K \nleq G$ one can check that both morphisms send h_K to $h_{f(G) \vee f(K)} \otimes 1$. □

In the sequel we will write $\mathbb{FY}$ when referring to the $\mathfrak{LBS}$-cooperad and not just the algebras. We also write $\mathbb{FY}^{\vee}$ for the (linear) dual operad (apply linear duality to all objects and morphisms). This is an $\mathfrak{LBS}$-operad in the category of graded coalgebras.

3.1.2 A quadratic presentation for $\mathbb{FY}^{\vee}$

In this section we exhibit a quadratic presentation for the operad $\mathbb{FY}^{\vee}$. Let us first quickly recall what this means in the context of operads over general Feynman categories. This is all part of the theory developed by R. Kaufmann and B. Ward in [20].

Let $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, i)$ be a Feynman category and M an $\mathfrak{F}$-module in some monoidal category $\mathcal{C}$ (see Definition 2.1.2). If $\mathcal{C}$ is cocomplete there exists a "free" $\mathfrak{F}$-operad in $\mathcal{C}$ generated by M denoted by $\mathfrak{F}(M)$. The $\mathfrak{F}$-operad $\mathfrak{F}(M)$ satisfies the universal property that for any morphism of $\mathfrak{F}$-module between M and some $\mathfrak{F}$-operad $\mathbb{P}$ (viewed as an $\mathfrak{F}$-module), there exists a unique morphism of $\mathfrak{F}$-operad between $\mathfrak{F}(M)$ and $\mathbb{P}$ which extends the first morphism. Concretely, $\mathfrak{F}(M)$ is given by the left Kan extension of $M : \mathrm{Sym}(\mathcal{V}) \to \mathcal{C}$ along $\mathrm{Sym}(\imath)$. The left Kan extension universal property is exactly the freeness universal property.

If furthermore $\mathfrak{F}$ is assumed to be a graded Feynman category and $\mathcal{C}$ is a category of modules for instance, then free operads are naturally graded (i.e. components in all arity are graded and structural morphisms preserve this grading).

In addition, if we are given an $\mathfrak{F}$-operad $\mathbb{P}$ and M a sub $\mathfrak{F}$-module of $\mathbb{P}$ then one can define the ideal $\langle M \rangle$ generated by M in $\mathbb{P}$ by considering all possible elements in $\mathbb{P}$ which can be obtained as the composition (along some structural morphism in $\mathfrak{F}$) of an element of M with elements of $\mathbb{P}$. Finally, we can define the quotient of an operad by an ideal in an obvious way (just take the quotient in each arity and notice that the morphisms pass to the quotient).

With all those notions at hand we can define a quadratic operad over a graded Feynman category to be the quotient of a free operad by an ideal generated by degree 1 elements.

Since we have proved that $\mathfrak{LBS}$ is a graded Feynman category in Section 2.4, this vocabulary applies to $\mathfrak{LBS}$-operads.

Let Gen be the $\mathfrak{LBS}$-module in the category of graded $\mathbb{Z}$-modules defined by

$$\mathrm{Gen}(\mathcal{L}, \mathcal{G}) = \mathbb{Z}[2(\mathrm{rk}(\mathcal{L}) - 1)]$$

and

$$\mathrm{Gen}(f) = \mathrm{Id}$$

for every irreducible built lattice $(\mathcal{L}, \mathcal{G})$ and all isomorphism of built lattice f.

If $(\mathcal{L}, \mathcal{G})$ is an irreducible built lattice we denote by $\Psi_{(\mathcal{L},\mathcal{G})}$ the canonical generator of $\mathrm{Gen}(\mathcal{L}, \mathcal{G})$. We also denote for all irreducible nested sets $\mathcal{S}$ of $(\mathcal{L}, \mathcal{G})$:

$$\Psi_{\mathcal{S}} := \mathfrak{LBS}(\mathrm{Gen})(\mathcal{S})((\Psi_{([\tau_{\mathcal{S}}(G), G], \mathrm{Ind}(\mathcal{G}))})_{G \in \mathcal{S}}),$$

which is an element of $\mathfrak{LBS}(\mathrm{Gen})(\mathcal{L}, \mathcal{G})$. We are now able to state the main result of this section.

Proposition 3.1.3. *The $\mathfrak{LBS}$-operad $\mathbb{FY}^\vee$ is isomorphic to the quotient of $\mathfrak{LBS}(\mathrm{Gen})$ by the ideal generated by the elements*

$$\sum_{\hat{1}>G\geq H_1} \Psi_{\{G\}} - \sum_{\hat{1}>G\geq H_2} \Psi_{\{G\}}, \tag{3.1}$$

for all atoms H_1, H_2 in some irreducible built lattice $(\mathcal{L}, \mathcal{G})$.

Proof. We have a map of $\mathfrak{LBS}$-modules $\mathrm{Gen} \xrightarrow{\pi} \mathbb{FY}^\vee$ sending $\Psi_{(\mathcal{L},\mathcal{G})}$ to the linear form which is zero in all degrees except degree $2(\mathrm{rk}(\mathcal{L}) - 1)$ where it takes value 1 on $x_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-1}$ (which linearly generates $\mathrm{FY}^{2(\mathrm{rk}(\mathcal{L})-1)}(\mathcal{L}, \mathcal{G})$).

This map is a natural transformation since for any isomorphism $f : (\mathcal{L}, \mathcal{G}) \to (\mathcal{L}', \mathcal{G}')$ between built lattices, f preserves the top element and therefore $\mathbb{FY}(f)$ preserves $h_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-1} = x_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-1}$ which implies that $\mathbb{FY}(f)^\vee$ sends $\pi(\Psi_{(\mathcal{L},\mathcal{G})})$ to $\pi(\Psi_{(\mathcal{L}',\mathcal{G}')})$.

This map extends to an $\mathfrak{LBS}$-operadic map $\mathfrak{LBS}(\mathrm{Gen}) \xrightarrow{\hat{\pi}} \mathbb{FY}^\vee$ (by universal property of free operads). Our goal is to prove that this map passes to the quotient by the elements (3.1) and that the induced morphism is an isomorphism. The proof splits into three steps.

Step 1: The map $\hat{\pi}$ is surjective.

Going back to explicit formulas for general left Kan extensions in cocomplete categories yields

$$\mathfrak{LBS}(\mathrm{Gen})(\mathcal{L}, \mathcal{G}) = \bigoplus_{\otimes_i(\mathcal{L}_i,\mathcal{G}_i)\to(\mathcal{L},\mathcal{G})} \bigotimes \mathrm{Gen}((\mathcal{L}_i, \mathcal{G}_i))/\sim\ ,$$

where the sum is over all possible maps $\otimes_i(\mathcal{L}_i, \mathcal{G}_i) \to (\mathcal{L}, \mathcal{G})$ in $\mathfrak{LBS}$ and the equivalence relation $\sim$ identifies components corresponding to equivalent maps (two maps that can be obtained from one another by precomposing with isomorphisms).

More explicitly if we have two maps $\otimes_i(\mathcal{L}_i, \mathcal{G}_i) \xrightarrow{\psi} (\mathcal{L}, \mathcal{G})$ and $\otimes_i(\mathcal{L}'_i, \mathcal{G}'_i) \xrightarrow{\phi} (\mathcal{L}, \mathcal{G})$ such that there exist isomorphisms $f_i : (\mathcal{L}_i, \mathcal{G}_i) \xrightarrow{\sim} (\mathcal{L}'_i, \mathcal{G}'_i)$ satisfying

$$\psi = \phi \circ (\otimes_i f_i)$$

then we have

$$\otimes_i \alpha_i \sim \otimes_i \mathrm{Gen}(f_i)(\alpha_i)$$

for every element $\otimes\alpha_i$ in $\otimes_i \mathrm{Gen}((\mathcal{L}_i, \mathcal{G}_i))$ (this formula holds for any $\mathfrak{LBS}$-module). Likewise if we have an equality of the form

$$\psi = \phi \circ \sigma$$

with sigma some permutation of the $(\mathcal{L}_i, \mathcal{G}_i)$'s then we have

$$\otimes_i \alpha_i \sim \otimes_i \alpha_{\sigma(i)}.$$

Finally, replacing $\mathrm{Gen}(\mathcal{L}, \mathcal{G})$ by its definition we get (for every irreducible built lattice $(\mathcal{L}, \mathcal{G})$):

$$\mathfrak{LBS}(\mathrm{Gen})((\mathcal{L}, \mathcal{G})) = \mathbb{Z}\langle \{\otimes_i (\mathcal{L}_i, \mathcal{G}_i) \to (\mathcal{L}, \mathcal{G})\}/\sim \rangle$$

(generators are equivalence classes of maps in $\mathfrak{LBS}$ having target $(\mathcal{L}, \mathcal{G})$, with the equivalence relation being the precomposition with isomorphisms). With this identification the map $\hat{\pi}$ becomes

$$\begin{aligned} \mathfrak{LBS}(\mathrm{Gen})(\mathcal{L}, \mathcal{G}) &\longrightarrow \mathbb{FY}^\vee(\mathcal{L}, \mathcal{G}) \\ [\mu : \otimes_i (\mathcal{L}_i, \mathcal{G}_i) \to (\mathcal{L}, \mathcal{G})] &\longrightarrow (\alpha \to \otimes_i \pi(\Psi_{(\mathcal{L}_i, \mathcal{G}_i)})(\mathbb{FY}(\mu)(\alpha))). \end{aligned}$$

Let us fix an irreducible built lattice $(\mathcal{L}, \mathcal{G})$ and some linear order extending the order on $\mathcal{L}$. Amongst all maps of the form $\otimes_i (\mathcal{L}_i, \mathcal{G}_i) \to (\mathcal{L}, \mathcal{G})$ we have the maps given by linearly ordered nested sets

$$(\mathcal{L}_\mathcal{S}, \mathcal{G}_\mathcal{S}) \xrightarrow{\mathcal{S}} (\mathcal{L}, \mathcal{G})$$

whose linear order is given by our chosen global linear order on $\mathcal{L}$. It is enough to prove the surjectivity of $\hat{\pi}$ restricted to equivalence classes of those morphims. Passing to the dual we must prove the injectivity of the map

$$\begin{array}{ccc} \mathbb{FY}(\mathcal{L}, \mathcal{G}) & \longrightarrow & \mathbb{Z}\langle \{\text{irreducible nested sets of } (\mathcal{L}, \mathcal{G})\}\rangle \\ \alpha & \longrightarrow & (\pi(\Psi_\mathcal{S})(\mathbb{FY}(\mathcal{S})(\alpha)))_\mathcal{S} \end{array} \tag{3.2}$$

where $\pi(\Psi_\mathcal{S})$ denotes the tensor product of maps $\otimes_{G \in \mathcal{S}} \pi(\Psi_{([\tau_\mathcal{S}(G), G], \mathrm{Ind}(\mathcal{G}))})$.

Let α be an element of $\mathbb{FY}(\mathcal{L}, \mathcal{G})$ which is sent to zero by the above map, meaning that for every nested set $\mathcal{S}$ in $(\mathcal{L}, \mathcal{G})$ we have $\pi(\Psi_\mathcal{S})(\mathbb{FY}(\mathcal{S})(\alpha)) = 0$. We can assume that α is homogeneous and by Corollary 1.2.4 we can write it uniquely as a sum of normal monomials

$$\alpha = \sum_{\substack{\mathcal{S} \text{ nested} \\ \mu\, \mathcal{S}-\text{admissible}}} \lambda_{\mathcal{S},\mu} h^\mu_\mathcal{S}$$

where $h^\mu_\mathcal{S}$ denotes the monomial $\prod_{G \in \mathcal{S}} h_G^{\mu(G)}$. We have to prove that all the $\lambda_{\mathcal{S},\mu}$'s are zero. Arguing by contradiction, let G_0 some element of $\mathcal{G}$ which is minimal amongst elements belonging to some nested set $\mathcal{S}$ such that there exist some $\mathcal{S}$-admissible index μ satisfying $\lambda_{\mathcal{S},\mu} \neq 0$. We denote by $\mathcal{S}_0$ and μ_0 the corresponding nested set and index for G_0.

For any irreducible built lattice $(\mathcal{L}, \mathcal{G})$ and any $k < \mathrm{rk}$, one can construct a nested set $\mathcal{S}(\mathcal{L}, \mathcal{G}, k)$ having only local intervals with rank 1 except the top interval having rank k, by picking any maximal chain $\hat{0} \prec X_1 \prec .. \prec X_n \prec \hat{1}$ in $\mathcal{L}$ and putting

$$\mathcal{S}(\mathcal{L}, \mathcal{G}, k) = \{X_1\} \circ \{X_2\} \circ ... \circ \{X_{n-k+1}\}.$$

The formula makes sense because each X_i is an atom in $[X_{i-1}, \hat{1}]$ and must therefore belong to $\mathrm{Ind}_{[X_{i-1},\hat{1}]}(\mathcal{G})$. Notice that we have

$$\hat{\pi}(\Psi_{\mathcal{S}(\mathcal{L},\mathcal{G},k)})(h_{\hat{1}}^{k-1}) = 1.$$

For any nested set $\mathcal{S}'$ in $([G_0, \hat{1}], \mathrm{Ind}(\mathcal{G}))$ and any nested set $\mathcal{S}$ in $\mathcal{G}$ with admissible index μ we have

$$\hat{\pi}(\Psi_{\{G\}\circ(\mathcal{S}',\mathcal{S}([\hat{0},G],\mathrm{Ind}(\mathcal{G}),\mu_0(G_0)+1)})(h_{\mathcal{S}}^{\mu}) = 0$$

if $\mathcal{S}$ does not contain some element with strictly positive index and which is below G_0. By minimality of G_0 this means that we have

$$\begin{aligned}\hat{\pi}(\Psi_{\{G\}\circ(\mathcal{S}',\mathcal{S}([\hat{0},G],\mathrm{Ind}(\mathcal{G}),\mu_0(G_0)+1)})(\alpha) &= \sum_{\substack{\mathcal{S} \text{ nested} \\ \mu\, \mathcal{S}-\text{admissible} \\ \mu(G_0)=\mu_0(G_0)}} \lambda_{\mathcal{S},\mu}\hat{\pi}(\Psi_{\{G\}\circ(\mathcal{S}',\mathcal{S}([\hat{0},G],\mathrm{Ind}(\mathcal{G}),\mu_0(G_0)+1)})(h_{\mathcal{S}}^{\mu}) \\ &= \sum_{\substack{\mathcal{S} \text{ nested} \\ \mu\, \mathcal{S}-\text{admissible} \\ \mu(G_0)=\mu_0(G_0)}} \lambda_{\mathcal{S},\mu}\hat{\pi}(\Psi_{\mathcal{S}'})(h_{G_0\vee\mathcal{S}}^{\mu}/h_{G_0}^{\mu(G_0)}) \\ &= 0.\end{aligned}$$

One can check that the monomials $h_{G_0\vee\mathcal{S}}^{\mu}/h_{G_0}^{\mu(G_0)}$ are normal monomials in the irreducible built lattice $([G, \hat{1}], \mathrm{Ind}(\mathcal{G}))$. By induction we get that $\lambda_{\mathcal{S}_0,\mu_0}$ is equal to zero which is a contradiction.

Remark 3.1.4. Let us remark that for any irreducible nested set $\mathcal{S}$ the linear form

$$\hat{\pi}(\Psi_{\mathcal{S}}) \circ \mathbb{FY}(\mathcal{S}) : \mathrm{FY}(\mathcal{L}, \mathcal{G}) \to \mathbb{Z}$$

is zero on $\mathrm{FY}^k(\mathcal{L}, \mathcal{G})$ for k different from $2(\mathrm{rk}(\mathcal{L}) - \#\mathcal{S})$, and for $k = 2(\mathrm{rk}(\mathcal{L}) - \#\mathcal{S})$ one can check that it is equal to the multiplication by $x_{\mathcal{S}\setminus\{\hat{1}\}}$, via the identification

$$\mathrm{FY}(\mathcal{L}, \mathcal{G})^{2(\mathrm{rk}(\mathcal{L})-1)} \simeq \mathbb{Z}$$

(given by choosing the generator $x^{\mathrm{rk}(\mathcal{L})-1}$). The injectivity of the map (3.2) can be rewritten as the condition that for all element $\alpha \in \mathbb{FY}^{2k}(\mathcal{L}, \mathcal{G})$, if $\alpha x_{\mathcal{S}} = 0$ for all nested set $\mathcal{S} \subset \mathcal{G}\backslash\{\hat{1}\}$ of cardinal $\mathrm{rk}(\mathcal{L}) - 1 - k$, then $\alpha = 0$. In other words the Feichtner–Yuzvinsky algebras satisfy Poincaré duality.

Step 2: The map $\hat{\pi}$ sends the elements (3.1) to zero.

We must check that the linear forms

$$\phi_H = \hat{\pi}\left(\sum_{\hat{1} > G \geq H} \mathfrak{LBS}(\mathrm{Gen})(\{G\}) \left(\Psi_{([G,\hat{1}],\mathrm{Ind}(\mathcal{G}))}, \Psi_{([\hat{0},G],\mathrm{Ind}(\mathcal{G}))}\right)\right)$$

indexed by atoms H are in fact all equal and it is enough to check it on normal monomials. Those linear forms are zero in degree other than $2(\mathrm{rk}(\mathcal{L}) - 2)$. Fortunately, normal monomials of degree $2(\mathrm{rk}(\mathcal{L}) - 2)$ are rather simple.

Lemma 3.1.5. *For any irreducible built lattice $(\mathcal{L}, \mathcal{G})$, the only degree $2(\mathrm{rk}(\mathcal{L}) - 2)$ normal monomials are the monomials $h_G^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1}$ with G any element of $\mathcal{G}$ (when G is an atom we get the monomial $h_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-2}$).*

Proof. The proof hinges on the following result.

Lemma 3.1.6. *Let $\mathcal{S}$ be a nested set containing $\hat{1}$ in some irreducible built lattice $(\mathcal{L}, \mathcal{G})$. We have the equality*

$$\sum_{G \in \mathcal{S}} \mathrm{rk}([\tau_{\mathcal{S}}(G), G]) = \mathrm{rk}(\mathcal{L}).$$

Proof. The proof goes by induction on the rank of $\mathcal{L}$. Let G_0 be a minimal element in $\mathcal{S}$. $G_0 \vee (\mathcal{S} \setminus \{G_0\})$ is a nested set in $([G_0, \hat{1}], \mathrm{Ind}(\mathcal{G}))$ so by induction hypothesis the sum of the rank of its intervals is the rank of $[G_0, \hat{1}]$ but by nestedness of $\mathcal{S}$ taking the join with G_0 establishes a bijection between intervals of $\mathcal{S}$ which are not the interval $[\hat{0}, G_0]$ and intervals of $G_0 \vee (\mathcal{S} \setminus \{G_0\})$, and this bijection preserves the rank of the intervals. Consequently, by induction we have

$$\begin{aligned}\sum_{G \in \mathcal{S}} \mathrm{rk}([\tau_{\mathcal{S}}(G), G]) &= \mathrm{rk}([\hat{0}, G_0]) + \sum_{G \in \mathcal{S}\setminus\{G_0\}} \mathrm{rk}([\tau_{\mathcal{S}}(G), G]) \\ &= \mathrm{rk}([\hat{0}, G_0]) + \mathrm{rk}([G_0, \hat{1}]) \\ &= \mathrm{rk}(\mathcal{L}).\end{aligned}$$

□

Now if $\mathcal{S}$ is any irreducible nested set then the degree of any normal monomial of the form $h_{\mathcal{S}}^{\mu(\mathcal{S})}$ for some $\mathcal{S}$-admissible index is at most $\sum_{G\in\mathcal{S}} \mathrm{rk}([\tau_{\mathcal{S}}(G), G]) - 1$ which is equal by the previous lemma to $\mathrm{rk}(\mathcal{L}) - |\mathcal{S}|$. This means that to have degree $2(\mathrm{rk}(\mathcal{L}) - 2)$ the cardinality of $\mathcal{S}$ must be at most two (counting $\hat{1}$) which proves the result. For normal monomials with underlying nested set not containing $\hat{1}$ we just add $\hat{1}$ to the nested set and follow the same line of argument. □

Let $h_G^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)-1}$ be any degree $2(\mathrm{rk}(\mathcal{L}) - 2)$ normal monomial and H some atom. Going back to the definition of ϕ_H we get

$$\phi_H(h_G^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)-1}) = \sum_{G' \geq H} (\pi(\Psi_{[G',\hat{1}]}) \otimes \pi(\Psi_{[\hat{0},G']})) \left(\mathbb{FY}(\{G\})(h_G^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)-1})\right).$$

If $H \leq G$ then the only term which is not zero in this sum is the term $G' = G$ which gives

$$\begin{aligned}
\phi_H(h_G^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1}) &= (\pi(\Psi_{[G,\hat{1}]}) \otimes \pi(\Psi_{[\hat{0},G]})) \left(\mathbb{FY}(\{G\})(h_G^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1})\right) \\
&= (\pi(\Psi_{[G,\hat{1}]}) \otimes \pi(\Psi_{[\hat{0},G]})) \left(h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1} \otimes h_G^{\mathrm{rk}(G)-1}\right) \\
&= 1.
\end{aligned}$$

If $H \nleq G$ the only term which is not zero in this sum is the term $G' = H$ which gives

$$\begin{aligned}
\phi_H(h_G^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1}) &= (\pi(\Psi_{[H,\hat{1}]}) \otimes \pi(\Psi_{[\hat{0},H]})) \left(\mathbb{FY}(\{H\})(h_G^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1}\right) \\
&= (\pi(\Psi_{[H,\hat{1}]}) \otimes \pi(\Psi_{[\hat{0},H]})) \left(h_{G\vee H}^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}\mathcal{L}-\mathrm{rk}(G)-1} \otimes 1\right) \qquad (3.3) \\
&= \pi(\Psi_{[H,\hat{1}]}) \left(h_{G\vee H}^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}\mathcal{L}-\mathrm{rk}(G)-1}\right).
\end{aligned}$$

The monomial in the last equation is not a normal monomial in the irreducible built lattice $([H, \hat{1}], \mathrm{Ind}(\mathcal{G}))$ and therefore we must rewrite it. By geometricity of $\mathcal{L}$ there exist atoms $H_1, ..., H_{rk(\mathcal{L})-\mathrm{rk}(G)-1}$ such that $\hat{1}$ is equal to the join of $G \vee H$ with those atoms. This means that we have the relation

$$(h_{\hat{1}} - h_{G\vee H})(h_{\hat{1}} - h_{H_1})...(h_{\hat{1}} - h_{H_{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)-1}}) = 0$$

in the algebra $\mathrm{FY}([H, \hat{1}], \mathrm{Ind}(\mathcal{G}))$, and replacing the h_{H_i}'s by zero we get

$$h_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)} = h_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)-1} h_{G\vee H}$$

which implies

$$h_{G\vee H}^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-\mathrm{rk}(G)-1} = h_{\hat{1}}^{\mathrm{rk}(\mathcal{L})-2}$$

This equality together with equation (3.3) leads directly to

$$\phi_H(h_G^{\mathrm{rk}(G)-1} h_{\hat{1}}^{\mathrm{rk}(\hat{1})-\mathrm{rk}(G)-1}) = 1$$

as in the case $H \leq G$, which proves that ϕ_H does not depend on H.

Step 3: The kernel of $\hat{\pi}$ is generated by the relations (3.1).

We postpone the proof of this last step to Subsection 4.5, where it will be an application of our theory of Gröbner bases for $\mathfrak{LBS}$-operads. □

3.2 The Feichtner–Yuzvinsky operad

One can define another operad out of the Feichtner–Yuzvinsky algebras as follow. Let $(\mathcal{L}, \mathcal{G})$ be any irreducible built lattice. We put

$$\mathbb{FY}^{\mathrm{PD}}(\mathcal{L}, \mathcal{G}) \coloneqq \mathrm{FY}(\mathcal{L}, \mathcal{G}).$$

For any isomorphism of built lattice $f : (\mathcal{L}, \mathcal{G}) \xrightarrow{\sim} (\mathcal{L}', \mathcal{G}')$ we define

$$\begin{array}{rccc} \mathbb{FY}^{\mathrm{PD}}(f): & \mathrm{FY}(\mathcal{L}, \mathcal{G}) & \longrightarrow & \mathrm{FY}(\mathcal{L}', \mathcal{G}') \\ & x_G & \longrightarrow & x_{f^{-1}(G)} \end{array}$$

and for any $G \in \mathcal{G} \setminus \{\hat{1}\}$ we put

$$\begin{array}{rccc} \mathbb{FY}^{\mathrm{PD}}(\{G\}): & \mathbb{FY}^{\mathrm{PD}}([G, \hat{1}], \mathrm{Ind}(\mathcal{G})) \otimes \mathbb{FY}^{\mathrm{PD}}([\hat{0}, G], \mathrm{Ind}(\mathcal{G})) & \to & \mathbb{FY}^{\mathrm{PD}}(\mathcal{L}, \mathcal{G}) \\ & \prod_i x_{G'_i}^{\alpha_i} \otimes \prod_j x_{G_j}^{\alpha_j} & \to & x_G \prod_i x_{\mathrm{Comp}_G(G'_i)}^{\alpha_i} \prod_j x_{G_j}^{\alpha_j} \end{array}$$

if all the G_j's are different from G.

Let us show that this morphism is well-defined. For any antichain $\{G_i\}$ below G and such that $\bigvee_i G_i$ belongs to $\mathcal{G}$ we have

$$\mathbb{FY}^{\mathrm{PD}}(1 \otimes \prod_i x_{G_i}) = x_G \prod_i x_{G_i} = 0.$$

For any antichain $\{G \vee G_i\}$ in $\mathrm{Ind}_{[G,\hat{1}]}(\mathcal{G})$ having join G' in $\mathrm{Ind}_{[G,\hat{1}]}(\mathcal{G})$ we have

$$\mathbb{FY}^{\mathrm{PD}}(\prod_i x_{G \vee G_i} \otimes 1) = x_G \prod_i x_{\mathrm{Comp}_G(G \vee G_i)} = 0$$

because either G' belongs to $\mathcal{G}$ and the elements $\{G, \mathrm{Comp}_G(G \vee G_i)\}$ have join G' in $\mathcal{G}$, either G' does not belong to $\mathcal{G}$ and the elements $\mathrm{Comp}_G(G \vee G_i)$ have join $\mathrm{Comp}_G(G')$.

For any atom H below G we have

$$\begin{aligned}\mathbb{FY}^{\mathrm{PD}}(1 \otimes \sum_{\substack{G' \geq H \\ G' < G}} x_{G'}) = \sum_{\substack{G' \geq H \\ G' < G}} x_G x_{G'} \\ &= x_G(h_H - \sum_{G' \geq G} x_{G'} - \sum_{G', G \text{ incomparables}} x_{G'}) \\ &= x_G(- \sum_{G' \geq G} x_{G'} - \sum_{G', G \text{ incomparables}} x_{G'}),\end{aligned}$$

which does not depend on H.

Finally, if $G \vee H$ is an atom in $[G, \hat{1}]$, either $G \vee H$ belongs to $\mathcal{G}$ in which case we have

$$\mathbb{FY}^{\mathrm{PD}}(\{G\})(h_H \otimes \hat{1}) = x_G h_{G \vee H} = 0,$$

either $G \vee H$ does not belong to $\mathcal{G}$ in which case we have

$$\mathbb{FY}^{\mathrm{PD}}(\{G\})(h_H \otimes \hat{1}) = x_G h_H = 0.$$

Our next order of business is to prove that those morphisms satisfy the relations in Proposition 2.3.2. Let $(\mathcal{L}, \mathcal{G})$ be some irreducible built lattice. If G_1 and G_2 are two non comparable elements forming a nested set in $\mathcal{G} \setminus \{\hat{1}\}$, one can check that both

$$\mathbb{FY}^{\mathrm{PD}}(\{G_1\}) \circ (\mathbb{FY}^{\mathrm{PD}}(\{G_1 \vee G_2\}) \otimes \mathrm{Id})$$

and

$$\mathbb{FY}^{\mathrm{PD}}(\{G_2\}) \circ (\mathbb{FY}^{\mathrm{PD}}(\{G_1 \vee G_2\}) \otimes \mathrm{Id}) \circ \sigma_{2,3}$$

send $\prod_i x_{G_i} \otimes \prod_j x_{G'_j} \otimes \prod_k x_{G''_k}$ to $x_{G_1} x_{G_2} \prod_i x_{\mathrm{Comp}_{G_1 \vee G_2}(G_i)} \prod_j x_{G'_j} \prod_k x_{G''_k}$.

If $G_1 < G_2 < \hat{1}$ one can check that both

$$\mathbb{FY}^{\mathrm{PD}}(\{G_1\}) \circ (\mathbb{FY}^{\mathrm{PD}}(\{G_2\}) \otimes \mathrm{Id})$$

and

$$\mathbb{FY}^{\mathrm{PD}}(\{G_2\}) \circ (\mathrm{Id} \otimes \mathbb{FY}^{\mathrm{PD}}(\{G_1\}))$$

send $\prod_i x_{G_i} \otimes \prod_j x_{G'_j} \otimes \prod_k x_{G''_k}$ to $x_{G_1} x_{G_2} \prod_i x_{\mathrm{Comp}_{G_2}(G_i)} \prod_j x_{\mathrm{Comp}_{G_1}(G'_j)} \prod_k x_{G''_k}$.

Finally, if $f : (\mathcal{L}, \mathcal{G}) \xrightarrow{\sim} (\mathcal{L}', \mathcal{G}')$ is an isomorphism of irreducible built lattice and G' is some element in $\mathcal{G}'$, one can check that both

$$\mathbb{FY}^{\mathrm{PD}}(f) \circ \mathbb{FY}^{\mathrm{PD}}(\{f(G')\})$$

and

$$\mathbb{FY}^{\mathrm{PD}}(\{G'\}) \circ (\mathbb{FY}^{\mathrm{PD}}(f_{|[G',\hat{1}]}) \otimes \mathbb{FY}^{\mathrm{PD}}(f_{[\hat{0},G']}))$$

send $\prod_i x_{G_i} \otimes \prod_j x_{G'_j}$ to $x_{G'} \prod_i x_{f^{-1}(\mathrm{Comp}_{f(G')}(G_i))} \prod_j x_{f^{-1}(G'_j)}$.

The operads $\mathbb{FY}$ and $\mathbb{FY}^{\mathrm{PD}}$ are strongly related via Poincaré duality. If we denote by PD the isomorphism $\mathrm{FY}(\mathcal{L},\mathcal{G}) \xrightarrow{\sim} \mathrm{FY}(\mathcal{L},\mathcal{G})^\vee$ given by Poincaré duality we have the equality between morphisms

$$\mathbb{FY}^{\mathrm{PD}}(\{G\}) = \mathrm{PD}^{-1} \circ \mathbb{FY}^\vee(\{G\}) \circ (\mathrm{PD} \otimes \mathrm{PD}).$$

3.3 The affine Orlik–Solomon cooperad

In this section we introduce an $\mathfrak{LBS}$-cooperadic structure on the Orlik–Solomon algebras, which extends the linear dual of the Gerstenhaber operad (defined on partition lattices with minimal building set).

3.3.1 Definition

Let $(\mathcal{L},\mathcal{G})$ be an irreducible built lattice. For all $G \in \mathcal{G} \setminus \{\hat{1}\}$ we define a morphism of algebras $\mathbb{OS}(\{G\}) : \mathrm{OS}(\mathcal{L}) \to \mathrm{OS}([G,\hat{1}]) \otimes \mathrm{OS}([\hat{0},G])$ by

$$\mathbb{OS}(\{G\})(e_H) = \begin{cases} 1 \otimes e_H & \text{if } H \leq G \\ e_{G\vee H} \otimes 1 & \text{otherwise,} \end{cases}$$

for every generator e_H.

Lemma 3.3.1. *The morphism $\mathbb{OS}(\{G\})$ is well defined.*

Proof. We have to check that $\mathbb{OS}(\{G\})$ vanishes on elements of the form δcircuit. Let $C = \{H_i\} \sqcup \{H'_j\}$ be a circuit with the H_i's below G and the H'_j's not below G. We have

$$\begin{aligned} \mathbb{OS}(\{G\})(\delta(\prod C)) &= \mathbb{OS}(\{G\})(\delta(\prod e_{H_i} \wedge \prod e_{H'_j})) \\ &= \mathbb{OS}(\{G\})(\delta(\prod e_{H_i}) \wedge \prod e_{H'_j} \pm \prod e_{H_j} \wedge \delta(\prod e_{H'_j})) \\ &= \delta(\prod e_{H_i}) \otimes \prod e_{G\vee H'_j} \pm \prod e_{H_j} \otimes \delta(\prod e_{G\vee H'_j}). \end{aligned} \tag{3.4}$$

If the H_i's form a set of dependent atoms of $\mathcal{L}$ then we have $\delta(\prod e_{H_i}) = \prod e_{H_i} = 0$ (these identities holding in both $\mathrm{OS}(\mathcal{L})$ and $\mathrm{OS}([\hat{0},G])$). If on the contrary the H_i's form a set of independent atoms, by the fact that C is dependent there exists an atom H'_{j_0} in C which is below $\bigvee H_i \vee \bigvee_{j<j_0} H'_j$. By taking the join with G in this relation we obtain $G \vee H'_{j_0} \leq G \vee \bigvee_{j<j_0} H'_j$ which implies that the $G \vee H'_j$'s form a set of dependent atoms of $[G,\hat{1}]$ which shows that we have $\delta(\prod e_{G\vee H'_j}) = \prod e_{G\vee H'_j} = 0$ in the algebra $\mathrm{OS}([G,\hat{1}])$. □

Lemma 3.3.2. $\mathbb{OS}$ *extends to an* $\mathfrak{LBG}$*-cooperad of graded commutative algebras.*

Proof. On objects we set $\mathbb{OS}(\mathcal{L},\mathcal{G}) = \mathrm{OS}(\mathcal{L})$ for every built lattice $(\mathcal{L},\mathcal{G})$. For structural morphisms we use the morphisms $\mathbb{OS}(\{G\})$ introduced above on each generator of $\mathfrak{LBG}$ (one-element nested sets). On isomorphisms the action is

$$\mathbb{OS}(\mathcal{L},\mathcal{G}) \to \mathbb{OS}(\mathcal{L}',\mathcal{G}')$$
$$e_H \to e_{f(H)}$$

for any isomorphism $f : (\mathcal{L}',\mathcal{G}') \xrightarrow{\sim} (\mathcal{L},\mathcal{G})$ in $\mathfrak{LBG}$ (an isomorphism of posets must preserve the atoms).

We have to check that those morphisms satisfy the relations given in Proposition 2.3.2. Let $G_1 \leq G_2 < \hat{1}$ be two comparable elements in the building set $\mathcal{G}$ of a lattice $\mathcal{L}$. Let us prove the equality of morphisms

$$(\mathrm{Id} \otimes \mathbb{OS}(\{G_1\})) \circ \mathbb{OS}(\{G_2\}) = (\mathbb{OS}(\{G_2\}) \otimes \mathrm{Id}) \circ \mathbb{OS}(\{G_1\}).$$

Since we are dealing with morphisms of algebras it is enough to prove the equality on generators which amounts to a simple verification. For any atom H in $\mathcal{L}$, if $H \leq G_1$ then both morphisms send e_H to $1 \otimes 1 \otimes e_H$. If $H \leq G_2$ and $H \nleq G_1$ then both morphisms send e_H to $1 \otimes e_{G_1 \vee H} \otimes 1$. Lastly, if $H \nleq G_2$ then both morphisms send e_H to $e_{G_2 \vee H} \otimes 1 \otimes 1$.

Let G_1 and G_2 be two non-comparable elements of $\mathcal{G} \setminus \{\hat{1}\}$ forming a nested set. We have to show the equality of morphisms

$$(\mathbb{OS}(\{G_1 \vee G_2\}) \otimes \mathrm{Id}) \circ \mathbb{OS}(\{G_2\}) = \sigma^{2,3} \circ (\mathbb{OS}(\{G_1 \vee G_2\}) \otimes \mathrm{Id}) \circ \mathbb{OS}(\{G_1\}).$$

For any atom H in $\mathcal{L}$, if $H \leq G_1$ then by nestedness $H \nleq G_2$ and consequently both morphisms send e_H to $1 \otimes e_H \otimes 1$. If $H \leq G_2$ then by the same argument both morphisms send e_H to $1 \otimes 1 \otimes e_H$. Finally if $H \nleq G_1$ and $H \nleq G_2$ then both morphisms send e_H to $e_{G_1 \vee G_2 \vee H} \otimes 1 \otimes 1$ which finishes the proof.

Lastly, we need to prove the equality

$$\mathbb{OS}(\{f(G)\}) \circ \mathbb{OS}(f) = (\mathbb{OS}(f_{|[G,\hat{1}]}) \otimes \mathbb{OS}(f_{|[\hat{0},G]})) \circ \mathbb{OS}(\{G\})$$

for every isomorphism $f : (\mathcal{L}',\mathcal{G}') \xrightarrow{\sim} (\mathcal{L},\mathcal{G})$ in **LBS** and $G \in \mathcal{G} \setminus \{\hat{1}\}$. For any atom H of $\mathcal{L}$, if $H \leq G$ then both morphisms send e_H to $1 \otimes e_{f(H)}$. If on the contrary $H \nleq G$ then both morphisms send e_H to $e_{f(G) \vee f(H)} \otimes 1$.

Finally, one can check that $\mathbb{OS}$ is strong monoidal, which finishes the proof. □

Chapter 4

Gröbner bases for operads over $\mathfrak{LBS}$

In this section we develop a theory of Gröbner bases for operads over $\mathfrak{LBS}$.

Classical Gröbner bases [3] are a computational tool which is used to work out quotients of free associative algebras. The general idea is to start by introducing an order on generators of the free algebra. This order is then used to derive an order on all monomials, which is compatible in some sense with the multiplication of monomials (we call such orders “admissible”).

We then use this order to rewrite monomials in the quotient algebra:

$$\text{greatest term} \longrightarrow \sum \text{lower terms},$$

for every relation $R = \text{greatest term} - \sum \text{lower terms}$ in some subset $\mathcal{B}$ of the quotient ideal (usually the greatest term is called the “leading term” and we will use this denomination).

The subset $\mathcal{B}$ is called a Gröbner basis when it contains “enough” elements. More precisely we want that every leading term of some relation in the quotient ideal is divisible by the leading term of some element of $\mathcal{B}$.

The goal is to find a Gröbner basis as little as possible so that the rewriting is as easy as possible. At the end of the rewriting process (which stops if the monomials are well-ordered) we are left with all the monomials which are not rewritable i.e. which are not divisible by a leading term of some element of $\mathcal{B}$. Those monomials are called “normal” and they form a linear basis of our algebra exactly when $\mathcal{B}$ is a Gröbner basis. This basis comes with multiplication tables given by the rewriting process.

It turns out that this general strategy can be applied to structures which are much more general and complex than associative algebras, such as operads for instance. Loosely speaking, all we need in order to implement this strategy is to be able to make (reasonable) sense of the key words used above, such as "monomials", "admissible orders" and "divisibility between monomials".

For operads over a Feynman category, the only non-trivial part is to construct admissible orders on monomials out of orders on generators. The main issue comes from the symmetries, because usually the compatibility with symmetries is too strong and prevents us from finding any admissible order. In order to circumvent this problem, drawing inspiration from the case of classical operads which was sorted out by Dotsenko and Khoroshkin in [11], we introduce a notion of a "shuffle" operad over $\mathfrak{LBS}$.

4.1 Shuffle operads over $\mathfrak{LBS}$

We could directly define a shuffle $\mathfrak{LBS}$-operad to be an $\mathfrak{LBS}$-operad without symmetries but this definition would not be completely satisfactory, in particular when trying to find an admissible order on nested sets. Essentially, we would like to have a bit more data than just built lattices to construct those orders in a functorial way (for instance in the case of shuffle operads the new important data is the linear ordering of the entries, see [11]).

In view of what has just been said we make the following definition.

Definition 4.1.1. A *directed built lattice* is a triple $(\mathcal{L}, \mathcal{G}, \lhd)$ where $(\mathcal{L}, \mathcal{G})$ is a built lattice and $\lhd$ is a linear order on atoms of $\mathcal{L}$. A directed built lattice $(\mathcal{L}, \mathcal{G}, \lhd)$ is said to be *irreducible* if $(\mathcal{L}, \mathcal{G})$ is irreducible.

We are going to do the same construction all over again but with directed built lattices instead of built lattices.

Definition 4.1.2. Let $(\mathcal{L}, \mathcal{G}, \lhd)$ be a directed built lattice and $[G_1, G_2]$ be an interval of $\mathcal{L}$. The interval $[G_1, G_2]$ admits an induced directed built lattice structure given by the building set $\mathrm{Ind}_{[G_1,G_2]}(\mathcal{G})$ and the linear order $\lhd_{\mathrm{ind}}$ defined by

$$K_1 \lhd_{\mathrm{ind}} K_2 \Leftrightarrow \min\{H \,|\, G_1 \vee H = K_1, H \text{ atom}\} \;\lhd \min\{H \,|\, G_1 \vee H = K_2, H \text{ atom}\} \quad (4.1)$$

for any pair of elements K_1, K_2 covering G_1 (i.e. atoms of $[G_1, G_2]$).

Both minima in (4.1) are well defined by geometricity of $\mathcal{L}$. As in the case of built lattices, doing a double induction is the same as doing a single induction directly on the smallest interval (Lemma 1.1.15).

Definition 4.1.3. The monoidal product $\otimes$ on directed built lattices is defined by

$$(\mathcal{L}_1, \mathcal{G}_1, \lhd_1) \otimes (\mathcal{L}_2, \mathcal{G}_2, \lhd_2) = (\mathcal{L}_1 \times \mathcal{L}_2, \mathcal{G}_1 \times \{0\} \sqcup \{0\} \times \mathcal{G}_2, \lhd)$$

where $\lhd$ is defined by putting all the atoms of $\mathcal{L}_1$ before the atoms of $\mathcal{L}_2$.

As a quick reminder, for any nested set $\mathcal{S}$ and G an element of $\mathcal{S}$ we have defined in Subsection 2.2 the notation $\tau_{\mathcal{S}}(G) \coloneqq \bigvee \mathcal{S}_{<G}$. If $\mathcal{S}$ is an (ordered) nested set in a directed built lattice $(\mathcal{L}, \mathcal{G}, \lhd)$ we denote

$$(\mathcal{L}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}}, \lhd_{\mathcal{S}}) \coloneqq \bigotimes_{G \in \mathcal{S}} ([\tau_{\mathcal{S}}(G), G], \mathrm{Ind}(\mathcal{G}), \lhd_{\mathrm{ind}}).$$

We are now able to make the following definition.

Definition 4.1.4. The Feynman category $\mathfrak{LBS}_{\mathrm{III}}$ is the triple $(\mathcal{V}, \mathcal{F}, \imath)$ where

- The objects of the groupoid $\mathcal{V}$ are the directed irreducible lattices and its morphisms are defined by

$$\begin{aligned}\mathrm{Mor}_{\mathcal{V}}((\mathcal{L}, \mathcal{G}, \lhd), (\mathcal{L}', \mathcal{G}', \lhd')) = \{ f : \mathcal{L}' \xrightarrow{\sim} \mathcal{L} \mid f \text{ poset isomorphism,} \\ \text{increasing with respect to } \lhd \text{ and } \lhd' \text{ and such that } f(\mathcal{G}') = \mathcal{G} \}.\end{aligned}$$

- The objects of the category $\mathcal{F}$ are monoidal products of directed built lattices and its morphisms are given by structural morphisms $(\mathcal{L}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}}, \lhd_{\mathcal{S}}) \xrightarrow{\mathcal{S}} (\mathcal{L}, \mathcal{G}, \lhd)$ together with the tensored isomorphisms of $\mathcal{V}$ and the permutations of monoidal summands, quotiented by the relations
$$\mathcal{S} \circ \sigma \sim \sigma \cdot \mathcal{S}$$
for every ordered nested set $\mathcal{S}$ in some directed irreducible nested set and σ some permutation of the summands of $(\mathcal{L}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}}, \lhd_{\mathcal{S}})$ acting on $\mathcal{S}$ by changing the order of the elements.

- The functor $\imath$ is the obvious inclusion.

We define the composition of nested sets in this context exactly as it was defined in Section 2.2. In subsequent sections a shuffle $\mathfrak{LBS}$-operad will mean an operad over the Feynman category $\mathfrak{LBS}_{\mathrm{III}}$.

4.2 Monomials and divisibility

In order to be able to talk about Gröbner bases one needs a suitable notion of "monomials" in a free $\mathfrak{LBS}_{\mathrm{III}}$-operad as well as a notion of divisibility between those monomials. Let

M be a module over $\mathfrak{LBS}_{\mathrm{III}}$ in some category of vector spaces over some field. We have defined in Section 3.1.2 the free $\mathfrak{LBS}_{\mathrm{III}}$-operad $\mathfrak{LBS}_{\mathrm{III}}(M)$ generated by the module M. By the same analysis conducted in the latter section we have the explicit formula for each directed irreducible built lattice $(\mathcal{L}, \mathcal{G}, \lhd)$:

$$\mathfrak{LBS}_{\mathrm{III}}(M)(\mathcal{L}, \mathcal{G}, \lhd) = \bigoplus_{\substack{\mathcal{S} \subset \mathcal{G} \\ \mathcal{S} \text{ nested set}}} \bigotimes_{G \in \mathcal{S}} M([\tau_{\mathcal{S}}(G), G], \mathrm{Ind}(\mathcal{G}), \lhd_{\mathrm{ind}}). \tag{4.2}$$

If furthermore we are given a basis $\mathrm{B}(\mathcal{L}, \mathcal{G}, \lhd)$ of every vector space $M(\mathcal{L}, \mathcal{G}, \lhd)$ then we can make the following definition.

Definition 4.2.1 (Monomial)**.** A *monomial* in $\mathfrak{LBS}_{\mathrm{III}}(M)$ is an element which is a tensor of elements of the basis $\bigsqcup_{(\mathcal{L}', \mathcal{G}', \lhd')} \mathrm{B}(\mathcal{L}', \mathcal{G}', \lhd')$.

In other words a monomial corresponds to the datum $(\mathcal{S}, (e_G)_{G \in \mathcal{S}})$ of a nested set $\mathcal{S}$ and elements of the basis $e_G \in \mathrm{B}([\tau_{\mathcal{S}}(G), G], \mathrm{Ind}(\mathcal{G}), \lhd_{\mathrm{ind}})$ for each G in $\mathcal{S}$. Monomials are stable under composition in $\mathfrak{LBS}_{\mathrm{III}}(M)$ and by (4.2) they form a basis of every vector space $\mathfrak{LBS}_{\mathrm{III}}(M)(\mathcal{L}, \mathcal{G}, \lhd)$. Additionally, we have a notion of divisibility between monomials.

Definition 4.2.2 (Division between monomials)**.** Let m_1 and m_2 be two monomials in some arity $\mathfrak{LBS}_{\mathrm{III}}(M)(\mathcal{L}, \mathcal{G}, \lhd)$. We say that m_1 *divides* m_2 if m_2 can be expressed as a composition:

$$m_2 = \mathfrak{LBS}_{\mathrm{III}}(M)(\mathcal{S})((\alpha_G)_{G \in \mathcal{S}})$$

for some nested set $\mathcal{S}$, where one of the α_G's is m_1 and the rest are elements of the basis $\bigsqcup_{(\mathcal{L}, \mathcal{G}, \lhd)} B(\mathcal{L}, \mathcal{G}, \lhd)$.

4.3 An admissible order on monomials

Let M be an $\mathfrak{LBS}_{\mathrm{III}}$-module, with a basis $\mathrm{B}(\mathcal{L}, \mathcal{G}, \lhd)$ of $M(\mathcal{L}, \mathcal{G}, \lhd)$ for each $(\mathcal{L}, \mathcal{G}, \lhd)$. Assume that we have a total order $\dashv$ of those bases in each arity. In this section we will construct an order on monomials induced by $\dashv$, which is compatible with the composition of monomials in the sense of Proposition 4.3.6. We start by defining a total order $\lhd^*$ on $\mathcal{L}$, induced by the direction $\lhd$. For any element G in $\mathcal{L}$ we denote by $w(G)$ the word in the alphabet $\mathrm{At}(\mathcal{L})$ (the set of atoms of $\mathcal{L}$) given by the list of atoms below G in increasing order.

Definition 4.3.1. Given two elements G_1, G_2 in $\mathcal{L}$, we say $G_1 \lhd^* G_2$ if $w(G_2)$ is an initial subword of $w(G_1)$, or if $w(G_1)$ is less than $w(G_2)$ for the lexicographic order.

There are a few important lemmas/remarks to be made about this order. First we prove that $\lhd^*$ is compatible with the already existing order on $\mathcal{L}$.

Lemma 4.3.2. *The total order $\lhd^*$ extends the reversed lattice order on $\mathcal{L}$.*

Proof. Let $G_1 < G_2$ be two comparable elements in $\mathcal{G}$, such that $w(G_1)$ is not an initial subword of $w(G_2)$. One can write

$$w(G_1) = uH_1w_1$$
$$w(G_2) = uH_2w_2$$

with u, w_1, w_2 some words in the alphabet $\mathrm{At}(\mathcal{L})$ and H_1, H_2 two different atoms. Since we have $\mathrm{At}_{\leq}(G_1) \subset \mathrm{At}_{\leq}(G_2)$ we immediately get $H_2 \lhd H_1$ which shows that $w(G_2)$ is smaller than $w(G_1)$ for the lexicographic order. □

Next we prove that $\lhd^*$ behaves well with respect to restriction to intervals.

Lemma 4.3.3. *Assume we are given an interval $[G_1, G_2]$ in some irreducible directed built lattice $(\mathcal{L}, \mathcal{G}, \lhd)$. The two total orders $\lhd^*_{\mathrm{ind}}$ and $\lhd^*_{|[G_1,G_2]}$ on $[G_1, G_2]$ are the same.*

Proof. Since we are comparing two total orders we only need to prove one implication, for instance $K \lhd^*_{|[G_1,G_2]} K' \Rightarrow K \lhd^*_{\mathrm{ind}} K'$. If $w(K')$ is included in $w(K)$ then this means that we have $K' \leq K$ which proves that we have $K \lhd^*_{\mathrm{ind}} K'$ by the previous lemma. If not then one can write $w(K) = uHw$ and $w(K') = uH'w'$ with u, w and w' some words and $H \lhd H'$ two atoms of $\mathcal{L}$. The atom H cannot be below G_1 otherwise H would also be a letter in $w(K')$. If H' is below G_1 then let H'' be the first letter bigger than H' in $w(K')$ which is not below G_1 and such that $G_1 \vee H''$ does not belong to $G_1 \vee u$ (such a letter exists because we have assumed that $w(K')$ is not included in $w(K)$). In this case one can write

$$w_{[G_1,G_2]}(K) = v(G \vee H)t$$
$$w_{[G_1,G_2]}(K') = v(G \vee H'')t'$$

with v, t, t' some words in $\mathrm{At}([G_1, G_2])$. This implies that we have $K \lhd^*_{\mathrm{ind}} K'$.

□

Finally, we prove that $\lhd^*$ is compatible with the join in $\mathcal{L}$.

Lemma 4.3.4. *Let G, G_1 and G_2 be three elements in $\mathcal{L}$ such that $\mathrm{Fact}_{\mathcal{G}}(G_1)$ and $\mathrm{Fact}_{\mathcal{G}}(G_2)$ are both disjoint from $\mathrm{Fact}_{\mathcal{G}}(G)$ and such that $\mathrm{Fact}_{\mathcal{G}}(G_1) \cup \mathrm{Fact}_{\mathcal{G}}(G)$ and $\mathrm{Fact}_{\mathcal{G}}(G_2) \cup \mathrm{Fact}_{\mathcal{G}}(G)$ are both nested antichains. Then we have the equivalence*

$$G_1 \lhd^* G_2 \Leftrightarrow G \vee G_1 \ \lhd^* \ G \vee G_2.$$

Proof. Let us start by proving the direct implication. If $G_1 > G_2$ then we have $G \vee G_1 > G \vee G_2$ (the strictness coming from the nestedness condition) which proves the result by Lemma 4.3.2. Otherwise write $w(G_1) = uH_1w_1$ and $w(G_2) = uH_2w_2$ where u, w_1 and w_2

are some words and H_1 is strictly smaller than H_2. By the nestedness condition in the proposition we have

$$w(G \vee G_1) = \mathsf{sh}(w(G), w(G_1)),$$
$$w(G \vee G_2) = \mathsf{sh}(w(G), w(G_2)),$$

where $\mathsf{sh}(\cdot,\cdot)$ is the operation which merges two given words with increasing letters into a word with increasing letters. From this we see that one can write

$$w(G \vee G_1) = u' H_1 w_1',$$
$$w(G \vee G_2) = u' H_2' w_2',$$

with u', w_1', w_2' some words and H_2' an atom of $\mathcal{L}$. If H_2' is in $w(G)$ then H_1 is strictly smaller than H_2' (otherwise H_2' would belong to u'). If on the contrary H_2' belongs to $w(G_2)$ then H_2' is equal to H_2 and H_1 is strictly smaller than H_2'.

For the converse, assume we have $G \vee G_1 \lhd^* G \vee G_2$. If $G \vee G_2 < G \vee G_1$ then $G_2 < G_1$ by nestedness which implies $G_1 \lhd^* G_2$ by Lemma 4.3.2. Otherwise, write $w(G \vee G_1) = uH_1w_1$ and $w(G \vee G_2) = uH_2w_2$ for u, w_1, w_2 some words and H_1 strictly smaller than H_2. One can check that H_1 necessarily belongs to $w(G_1)$ which means that we can write $w(G_1) = u'H_1w_1'$ and $w(G_2) = u'H_2'w_2'$ where H_2' is the first letter of $w(G \vee G_2)$ which belongs to $w(G_2)$ and which comes after H_2. We immediately get $H_1 \lhd H_2 \unlhd H_2'$ which finishes the proof. □

For any monomial $m = (\mathcal{S}, (e_G)_G)$ and G_0 some element in $\min_\leq \mathcal{S}$, we denote

$$G_0 \vee m := (G_0 \vee (\mathcal{S} \setminus \{G_0\}), (e_{\mathrm{Comp}_{G_0}(G)})_G),$$

which is a well-defined monomial in $\mathfrak{LBS}_{\mathrm{III}}(M)([G_0, \hat{1}], \mathrm{Ind}(\mathcal{G}), \lhd_{\mathrm{ind}})$, by Lemma 2.2.9. For any nested set $\mathcal{S}$ we denote $\mathsf{MM}(\mathcal{S}) := \min_{\lhd^*_\dashv} \min_\leq \mathcal{S}$.

Definition 4.3.5 (An admissible order on monomials)**.** We define a total order $\lhd^*_\dashv$ on monomials in the following inductive way. For $m_1 = (\mathcal{S}_1, (e^1_G)_{G\in\mathcal{S}_1})$ and $m_2 = (\mathcal{S}_2, (e^2_G)_{G\in\mathcal{S}_2})$ we put $m_1 \lhd^*_\dashv m_2$ if there exists some G in $\min_\leq S_1 \cap \min_\leq S_2$ such that $e^1_G = e^2_G$ and $G \vee m_1 \lhd^*_\dashv G \vee m_2$, or if there is no such G and $\mathsf{MM}(\mathcal{S}_1) \lhd^* \mathsf{MM}(\mathcal{S}_2)$ or $\mathsf{MM}(\mathcal{S}_1) = \mathsf{MM}(\mathcal{S}_2)$ and $e^1_{\mathsf{MM}(\mathcal{S}_1)} \dashv e^2_{\mathsf{MM}(\mathcal{S}_2)}$.

One can check that this definition does not depend on the choice of the element G because if we have two different elements G and G' in $\min_\leq \mathcal{S}_1 \cap \min_\leq \mathcal{S}_2$ such that $e^1_G = e^2_G$ and $e^1_{G'} = e^2_{G'}$ then we have the equality of monomials

$$(G \vee G') \vee (G \vee m_i) = (G \vee G') \vee (G' \vee m_i)$$

for $i = 1, 2$. If there is no ambiguity on the order $\dashv$ we write $\lhd^*$ instead of $\lhd^*_\dashv$.

Proposition 4.3.6. *The order on monomials $\lhd^*$ is compatible with the composition of monomials. More precisely, if $\mathcal{S}$ is any nested set in some directed irreducible built lattice $(\mathcal{L}, \mathcal{G}, \lhd)$ and we have some generators $e_G \in B([\tau_{\mathcal{S}}(G), G], \mathrm{Ind}(\mathcal{G}), \lhd_{\mathrm{ind}})$ for all G in $\mathcal{S}$ except for G_0 in $\mathcal{S}$ where we have monomials $m_1 = (\mathcal{S}_1, (e^1_G)_G), m_2 = (\mathcal{S}_2, (e^2_G)_G) \in \mathfrak{LBG}_{\mathrm{III}}([\tau_{\mathcal{S}}(G_0), G_0], \mathrm{Ind}(\mathcal{G}), \lhd_{\mathrm{ind}})$, with $\#\mathcal{S}_1 = \#\mathcal{S}_2$, then we have*

$$m_1 \lhd^* m_2 \Rightarrow \mathfrak{LBG}_{\mathrm{III}}(M)(\mathcal{S})((e_G)_G, m_1) \lhd^* \mathfrak{LBG}_{\mathrm{III}}(M)(\mathcal{S})((e_G)_G, m_2).$$

Proof. The proof goes by induction on $\#\mathcal{S} + \#\mathcal{S}_1$. The initialization at $\#\mathcal{S} = 0$ or $\#\mathcal{S}_1 = 0$ is obvious. The induction step is a consequence of Lemma 4.3.4. □

Since every element in a free $\mathfrak{LBG}_{\mathrm{III}}$-operad can be uniquely written as a sum of monomials, we can make the following definition.

Definition 4.3.7 (Leading term)**.** If f is an element in some free $\mathfrak{LBG}_{\mathrm{III}}$-operad with total order on generators $\dashv$ then the *leading term* of f, denoted by $\mathsf{Lt}(f)$, is the biggest monomial with respect to $\lhd^*_{\dashv}$ which has a non-zero coefficient in f.

At last, everything has been leading to the following definition.

Definition 4.3.8 (Gröbner basis)**.** Let $\mathcal{I}$ be an operadic ideal in some free $\mathfrak{LBG}_{\mathrm{III}}$-operad $\mathfrak{LBG}_{\mathrm{III}}(M)$, where M is some $\mathfrak{LBG}_{\mathrm{III}}$-module in some category of vector spaces which is endowed with a basis and a well-order $\dashv$ of this basis in each arity. A *Gröbner basis* of $\mathcal{I}$ relative to $\dashv$ is a subset $\mathcal{B}$ of $\mathcal{I}$ such that every leading term relative to $\lhd^*_{\dashv}$ of some element of $\mathcal{I}$ is divisible by the leading term of some element of $\mathcal{B}$.

A Gröbner basis is said to be quadratic if it contains only degree 1 elements.

Definition 4.3.9 (Normal monomial)**.** A *normal monomial* with respect to some set of elements $\mathcal{B}$ is a monomial which is not divisible by the leading term of some element in $\mathcal{B}$.

Proposition 4.3.10. *The set of normal monomials with respect to some set of elements $\mathcal{B}$ in an ideal $\mathcal{I} \subset \mathfrak{LBG}_{\mathrm{III}}(M)$ linearly generates $\mathfrak{LBG}_{\mathrm{III}}(M)/\mathcal{I}$ in every arity. This set of monomials is linearly independant if and only if $\mathcal{B}$ is a Gröbner basis of $\mathcal{I}$.*

Proof. The proof is the same as in every other context where we have a notion of Gröbner basis. Let us just point out that since the basis is well ordered one can see that the monomials are well ordered as well. □

4.4 Relating $\mathfrak{LBG}$-operads and shuffle $\mathfrak{LBG}$-operads

There is an obvious functor between Feynman categories:

$$\begin{aligned} \mathrm{III} : \mathfrak{LBG}_{\mathrm{III}} &\to \mathfrak{LBG} \\ (\mathcal{L}, \mathcal{G}, \lhd) &\to (\mathcal{L}, \mathcal{G}) \\ \mathcal{S} &\to \mathcal{S}. \end{aligned}$$

This allows us to define a forgetful functor from $\mathfrak{LBS}$-operads/modules to shuffle $\mathfrak{LBS}$-operads/modules by precomposition.

Definition 4.4.1. For any $\mathfrak{LBS}$-operad (resp. module) $\mathbb{P}$ we define the shuffle $\mathfrak{LBS}$-operad (resp. module) $\mathbb{P}_{\text{Ш}}$ by

$$\mathbb{P}_{\text{Ш}} := \mathbb{P} \circ \text{Ш}.$$

As in the classical operadic case this functor enjoys very nice properties which are listed and proved below.

Proposition 4.4.2. *1. For any $\mathfrak{LBS}$-module M we have an isomorphism of shuffle operads:*

$$\mathfrak{LBS}_{\text{Ш}}(M_{\text{Ш}}) \simeq \mathfrak{LBS}(M)_{\text{Ш}}. \tag{4.3}$$

2. *Let R be a sub-$\mathfrak{LBS}$-module in some free $\mathfrak{LBS}$-operad $\mathfrak{LBS}(M)$ with M some $\mathfrak{LBS}$-module. The $\mathfrak{LBS}$-module $R_{\text{Ш}}$ can be identified with a sub-$\mathfrak{LBS}_{\text{Ш}}$-module of the free $\mathfrak{LBS}_{\text{Ш}}$-operad $\mathfrak{LBS}_{\text{Ш}}(M_{\text{Ш}})$. The $\mathfrak{LBS}_{\text{Ш}}$-module $\langle R \rangle_{\text{Ш}}$ can be identified with an ideal of $\mathfrak{LBS}(\mathrm{Gen})_{\text{Ш}}$. The isomorphism* (4.3) *sends (via identifications) the (shuffle) ideal $\langle R_{\text{Ш}} \rangle$ to the shuffle ideal $\langle R \rangle_{\text{Ш}}$. As a consequence we have the isomorphism of $\mathfrak{LBS}_{\text{Ш}}$-operads*

$$\mathfrak{LBS}_{\text{Ш}}(M_{\text{Ш}})/\langle R_{\text{Ш}} \rangle \xrightarrow{\sim} \mathfrak{LBS}(M)_{\text{Ш}}/\langle R \rangle_{\text{Ш}}.$$

3. *Lastly, we have an isomorphism of shuffle operads*

$$\mathfrak{LBS}(M)_{\text{Ш}}/\langle R \rangle_{\text{Ш}} \simeq (\mathfrak{LBS}(M)/\langle R \rangle)_{\text{Ш}} \ .$$

Proof. 1. By the definition of left Kan extensions we have a morphism of $\mathfrak{LBS}$-modules

$$M \to \mathfrak{LBS}(M).$$

Applying the forgetful functor we get a morphism of $\mathfrak{LBS}_{\text{Ш}}$-modules

$$M_{\text{Ш}} \to \mathfrak{LBS}(M)_{\text{Ш}}.$$

By universal property of free $\mathfrak{LBS}_{\text{Ш}}$-operads (which comes from the universal property of left Kan extensions) we get a morphism of $\mathfrak{LBS}_{\text{Ш}}$-operads

$$\mathfrak{LBS}_{\text{Ш}}(M_{\text{Ш}}) \to \mathfrak{LBS}(M)_{\text{Ш}}.$$

Let us take a closer look at this morphism.

By unpacking the construction of free operads (left Kan extensions) we get the following formula for every irreducible directed built lattice $(\mathcal{L}, \mathcal{G}, \lhd)$:

$$\mathfrak{LBS}_{\text{Ш}}(M_{\text{Ш}})(\mathcal{L}, \mathcal{G}, \lhd) \simeq \bigoplus_{\substack{\mathcal{S} \subset \mathcal{G} \\ \mathcal{S} \text{ irreducible} \\ \text{nested set}}} M_{\mathcal{S}}.$$

We also have

$$\mathfrak{LBS}(M)_{Ш}(\mathcal{L},\mathcal{G},\lhd) = \mathfrak{LBS}(M)(\mathcal{L},\mathcal{G}) = \left(\bigoplus_{\otimes_i(\mathcal{L}_i,\mathcal{G}_i)\to(\mathcal{L},\mathcal{G})} \bigotimes_i M(\mathcal{L}_i,\mathcal{G}_i) \right) / \sim$$

where $\sim$ identifies components corresponding to equivalent maps via isomorphisms, as explained at the beginning of the proof of Theorem 3.1.3. The above morphism sends the component $M(\mathcal{S})$ to the equivalence class of the component $M(\mathcal{S})$.

However by Lemma 2.3.3 the nested sets in $\mathcal{S}$ with linear order $\lhd^*$ form a system of representatives for the equivalence classes of morphisms which means that the above morphism is indeed a linear isomorphism in each arity.

2. For the first identification if we start with the injective morphism $R \hookrightarrow \mathfrak{LBS}(M)$, then apply the forgetful functor (which preserves injective morphisms since it is the right adjoint to left Kan extension), and then compose with isomorphism (4.3) we get an injective morphism $R_{Ш} \hookrightarrow \mathfrak{LBS}_{Ш}(M_{Ш})$. For the second identification we have an injective morphism $\langle R\rangle \hookrightarrow \mathfrak{LBS}(M)$ and applying the forgetful functor gives us an injective morphism $\langle R\rangle_{Ш} \hookrightarrow \mathfrak{LBS}(M)_{Ш}$. By unraveling again the explicit description of isomorphism (4.3) we see that it sends the shuffle ideal $\langle R_{Ш}\rangle$ to the shuffle ideal $\langle R\rangle_{Ш}$.

3. We have an obvious identification of components in each arity between the two shuffle operads and one can check that this identification is operadic.

□

4.5 Application: the example of $\mathbb{FY}^\vee$

We have proved in Subsection 3.1.2 that if Gen is the $\mathfrak{LBS}$-module with one generator $\Psi_{(\mathcal{L},\mathcal{G})}$ of degree $2(\mathrm{rk}(\mathcal{L})-1)$ in each arity $(\mathcal{L},\ \mathcal{G})$, and $\mathcal{I}$ is the ideal generated by the elements

$$\sum_{G.\geq H_1} \mathfrak{LBS}(\mathrm{Gen})(\{G\})(\Psi_{([G,\hat{1}],\mathrm{Ind}(\mathcal{G}))}, \Psi_{([\hat{0},G],\mathrm{Ind}(\mathcal{G}))}) - \sum_{G\geq H_2} \mathfrak{LBS}(\mathrm{Gen})(\{G\})(\Psi_{([G,\hat{1}],\mathrm{Ind}(\mathcal{G}))}, \Psi_{([\hat{0},G],\mathrm{Ind}(\mathcal{G}))}) \quad (4.4)$$

for each pair of atoms H_1 and H_2, then we have a surjective morphism of $\mathfrak{LBS}$-operads:

$$\mathfrak{LBS}(\mathrm{Gen})/\mathcal{I} \xrightarrow{\pi} \mathbb{FY}^\vee.$$

Let us denote by R the linear span of the elements (4.4), which is a sub $\mathfrak{LBS}$-module of $\mathfrak{LBS}(\mathrm{Gen})$. By Proposition 4.4.2 we have an isomorphism of shuffle $\mathfrak{LBS}$-operads

$$\mathfrak{LBS}_{\text{Ш}}(\mathrm{Gen}_{\text{Ш}})/\langle R_{\text{Ш}}\rangle \xrightarrow{\sim} \mathfrak{LBS}(\mathrm{Gen})_{\text{Ш}}/\langle R\rangle_{\text{Ш}}.$$

Let us use our theory of Gröbner bases for shuffle $\mathfrak{LBS}$-operads to study the operad $\mathfrak{LBS}_{\text{Ш}}(\mathrm{Gen}_{\text{Ш}})/\langle R_{\text{Ш}}\rangle$. Notice that $R_{\text{Ш}}$ is just the linear span of elements of the form

$$\sum_{G\geq H_1}\mathfrak{LBS}_{\text{Ш}}(\mathrm{Gen}_{\text{Ш}})(\{G\})(\Psi_{([G,\hat{1}],\mathrm{Ind}(\mathcal{G}))},\Psi_{([\hat{0},G],\mathrm{Ind}(\mathcal{G}))})- \sum_{G\geq H_2}\mathfrak{LBS}_{\text{Ш}}(\mathrm{Gen}_{\text{Ш}})(\{G\})(\Psi_{([G,\hat{1}],\mathrm{Ind}(\mathcal{G}))},\Psi_{([\hat{0},G],\mathrm{Ind}(\mathcal{G}))}).$$

We denote by $\mathcal{B}$ the set of those elements, and we put

$$\Psi_{\mathcal{S}} := \mathfrak{LBS}_{\text{Ш}}(\mathrm{Gen}_{\text{Ш}})(\mathcal{S})((\Psi_{([\tau_{\mathcal{S}}(G),G],\mathrm{Ind}(\mathcal{G}))})_{G\in\mathcal{S}}).$$

By applying the same arguments as in the proof of Theorem 3.1.3 we get a surjective morphism of $\mathfrak{LBS}_{\text{Ш}}$-operads.

$$\mathfrak{LBS}_{\text{Ш}}(\mathrm{Gen}_{\text{Ш}})/\langle R_{\text{Ш}}\rangle \longrightarrow \mathbb{FY}^{\vee}_{\text{Ш}}. \tag{4.5}$$

We will compute the normal monomials associated to $\mathcal{B}$ and find that they have the desired cardinality, which will prove that $\mathcal{B}$ forms a Gröbner basis of $\langle R_{\text{Ш}}\rangle$, and that morphism (4.5) is an isomorphism.

To describe those monomials in a natural way we introduce a classical tool in poset combinatorics called "EL-labeling".

Definition 4.5.1 (EL-labeling)**.** Let P be a finite poset with set of covering relations $\mathrm{Cov}(P)$. An *EL-labeling* of P is a map $\lambda : \mathrm{Cov}(P) \to \mathbb{N}$ such that for any two comparable elements $X < Y$ in P there exists a unique maximal chain going from X to Y which has increasing λ labels (when reading the covering relations from bottom to top) and this unique maximal chain is minimal for the lexicographic order on maximal chains (comparing the words given by the successive λ labels from bottom to top).

We refer to [33] for more details on this notion. The main result we will use about EL-labelings is the following.

Proposition 4.5.2. *Let $\mathcal{L}$ be a geometric lattice. Any linear ordering $H_1 \lhd \ldots \lhd H_n$ of the atoms of $\mathcal{L}$ induces an* EL*-labeling $\lambda_\lhd$ of $\mathcal{L}$ defined by*

$$\lambda_\lhd(X \prec Y) = \min\{i \mid X \vee H_i = Y\}$$

for any covering relation $X \prec Y$ in $\mathcal{L}$.

Proof. The proof of this result can be found in [33]. □

If λ is an EL-labelling of some poset P and $X < Y$ are two comparable elements we denote by $\omega_{X,Y,\lambda}$ the unique maximal chain from X to Y which has increasing λ labels. If the EL-labelling can be deduced from the context we will drop it from the notation.

We also define $\omega^k_{X,Y,\lambda}$ to be the the chain $\omega_{X,Y,\lambda}$ truncated at height k for any positive integer k which is less than the length of $\omega_{X,Y,\lambda}$. More precisely if $\omega_{X,Y,\lambda} = \{X_0 = X \prec X_1 \prec ... \prec X_n = Y\}$ then $\omega^k_{X,Y,\lambda} := \{X_0 \prec ... \prec X_k\}$.

We will also need a new definition in nested set combinatorics.

Definition 4.5.3 (Cluster)**.** An irreducible nested set $\mathcal{S}$ is called a *cluster* if all its local intervals except possibly the top one have rank 1. A cluster is said to be *proper* if its top interval has rank strictly greater than 1.

Clusters can be constructed out of truncated maximal chains in the following way. If ω is a chain $\omega = \{X_0 = \hat{0} \prec X_1 \prec ... \prec X_n\}$ in some built geometric lattice $\mathcal{L}$ then we put:

$$\mathcal{S}(\omega) := \{X_1\} \circ \{X_2\} \circ ... \circ \{X_n\},$$

which is a cluster. This formula makes sense even if the X_i's do not belong to $\mathcal{G}$ because each X_i covers X_{i-1} and therefore is an atom in $[X_{i-1}, \hat{1}]$ which must belong to the induced building set.

We can finally state the main result of this section.

Proposition 4.5.4. *The normal monomials with respect to $\mathcal{B}$ in arity $(\mathcal{L}, \mathcal{G}, \lhd)$ are the monomials of the form*

$$\mathfrak{LBS}_{\mathrm{III}}(\mathrm{Gen}_{\mathrm{III}})(\mathcal{S}) \left(\Psi_{\mathcal{S}(\omega^{k_G}_{\tau_{\mathcal{S}}(G),G,\lambda_\lhd})} \right)$$

where $\mathcal{S}$ is some nested set without any rank 1 intervals and the k_G's are integers strictly less than $\mathrm{rk}([\tau_{\mathcal{S}}(G), G]) - 1$, except $k_{\hat{1}}$ which can be equal to $\mathrm{rk}([\tau_{\mathcal{S}}(\hat{1}), \hat{1}]) - 1$. Furthermore this decomposition is unique.

Proof. We start with the following lemma.

Lemma 4.5.5. *Any irreducible nested set $\mathcal{S}$ in some irreducible built lattice can be written as*

$$\mathcal{S} = \mathcal{S}' \circ (\mathcal{S}'_{G'})_{G' \in \mathcal{S}'}$$

where S' is an irreducible nested set with no rank 1 intervals and the $\mathcal{S}'_{G'}s$ are proper clusters except $\mathcal{S}_{\hat{1}}$ which is any cluster.

The nested set $\mathcal{S}'$ will be called the frame of $\mathcal{S}$ and denoted by $\mathrm{fr}(\mathcal{S})$.

Proof. We put $\mathrm{fr}(\mathcal{S}) = \{G \in \mathcal{S} \text{ s.t. } \mathrm{rk}([\tau_{\mathcal{S}}(G), G]) > 1\} \cup \{\hat{1}\}$ and conclude by Lemma 2.2.9. □

Now let us proceed with the proof of the statement. We denote by $\mathfrak{M}(\mathcal{L}, \mathcal{G}, \lhd)$ the normal monomials with respect to $\mathcal{B}$ in arity $(\mathcal{L}, \mathcal{G}, \lhd)$ and $\mathfrak{M}'(\mathcal{L}, \mathcal{G}, \lhd)$ the monomials of the form

$$\mathfrak{LBS}_{\mathrm{III}}(\mathrm{Gen})(\mathcal{S})\left(\Psi_{\omega^{k_G}_{\tau_{\mathcal{S}}(G),G,\lambda_\lhd}}\right).$$

Our goal is to show the equality $\mathfrak{M}(\mathcal{L}, \mathcal{G}, \lhd) = \mathfrak{M}'(\mathcal{L}, \mathcal{G}, \lhd)$ for all irreducible directed built lattice $(\mathcal{L}, \mathcal{G}, \lhd)$. However we have a bijection between $\mathfrak{M}'(\mathcal{L}, \mathcal{G}, \lhd)$ and normal monomials of $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ with respect to the Gröbner basis introduced in Theorem 1.2.3, given by

$$\mathfrak{LBS}_{\mathrm{III}}(\mathrm{Gen})(\mathcal{S})\left(\Psi_{\omega^{k_G}_{\tau_{\mathcal{S}}(G),G,\lambda_\lhd}}\right) \to \prod_{G\in\mathcal{S}} x_G^{\mathrm{rk}([\tau_{\mathcal{S}}(G),G])-k_G-1},$$

and we have the surjective morphism of $\mathfrak{LBS}_{\mathrm{III}}$-operads (4.5). This means that it is enough to prove the inclusion $\mathfrak{M}(\mathcal{L}, \mathcal{G}, \lhd) \subset \mathfrak{M}'(\mathcal{L}, \mathcal{G}, \lhd)$ (see Proposition 4.3.10). By Lemma 4.5.5 it is enough to prove that for any cluster $\mathcal{S}$, if $\Psi_{\mathcal{S}}$ is a normal monomial then $\mathcal{S}$ is of the form $\omega^k_{\hat{0},\hat{1},\lambda_\lhd}$.

Leading terms of elements of $\mathcal{B}$ are monomials of the form $\Psi_{\{H\}}$ where H is not the minimal atom. Let $\mathcal{S}$ be any cluster such that $\Psi_{\mathcal{S}}$ is a normal monomial, i.e. is not divisible by any Ψ_H with H not minimal. For any $G \in \mathcal{S} \setminus \hat{1}$, let us denote by H_G the smallest atom such that $\tau_{\mathcal{S}}(G) \vee H_G = G$.

Lemma 4.5.6. *The map $G \to H_G$ is increasing, with respect to the order $\leq$ on the domain and the order $\lhd$ on the codomain.*

Proof. Let $G_1 < G_2$ be two elements in $\mathcal{S} \setminus \{\hat{1}\}$ such that there is no element in $\mathcal{S}$ strictly between G_1 and G_2. By Lemma 2.2.9 we can write $\mathcal{S} = (\mathcal{S} \setminus \{G_1\}) \circ \{\tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H_{G_1}\}$. Since $\Psi_{\mathcal{S}}$ is not divisible by any monomial Ψ_H where H is not minimal this means that $\tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H_{G_1}$ is the minimal atom in $[\tau_{\mathcal{S}\setminus\{G_1\}}(G_2), G_2]$, but this interval contains the atom $\tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H_{G_2}$ so we have $\tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H_{G_1} \lhd_{\mathrm{ind}} \tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H_{G_2}$.

Technically this means

$$\min\{H \mid \tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H = \tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H_{G_1}\} < \min\{H \mid \tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H = \tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H_{G_2}\}. \quad (4.6)$$

By nestedness of $\mathcal{S}$ we have

$$\{H \mid \tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H = \tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H_{G_1}\} = \{H \mid \tau_{\mathcal{S}}(G_1) \vee H = G_1\},$$

which has minimal element H_{G_1}, and

$$\{H \mid \tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H = \tau_{\mathcal{S}\setminus\{G_1\}}(G_2) \vee H_{G_2}\} = \{H \mid \tau_{\mathcal{S}}(G_2) \vee H = G_2\},$$

which has minimal element H_{G_2}. Inequality (4.6) concludes the proof. □

Let us denote $\mathcal{S} = \{G_1, ..., G_n\} \cup \{\hat{1}\}$ with $H_{G_1} \lhd ... \lhd H_{G_n}$. By the previous lemma and successive applications of Lemma 2.2.9 we get

$$\mathcal{S} = \{H_{G_1}\} \circ \{H_{G_1} \vee H_{G_2}\} \circ ... \circ \{H_{G_1} \vee ... \vee H_{G_n}\}.$$

What is left to prove is that the chain $H_{G_1} \prec ... \prec H_{G_1} \vee ... \vee H_{G_n}$ is exactly the chain $\omega^n_{\hat{0},\hat{1},\lambda_\lhd}$.

We consider the concatenation of chains

$$H_{G_1} \prec ... \prec H_{G_1} \vee ... \vee H_{G_n} \prec \omega_{H_{G_1} \vee ... \vee H_{G_n} \hat{1} \lambda_{\lhd_{\mathrm{ind}}}}.$$

This chain has increasing labels everywhere except possibly at $H_{G_1} \vee ... \vee H_{G_n}$.

By Lemma 2.2.9 we have $\mathcal{S} = (\mathcal{S} \setminus \{G_n\}) \circ \{\tau_{\mathcal{S}\setminus\{G_n\}}(\hat{1}) \vee H_{G_n}\}$ so if $\Psi_{\mathcal{S}}$ is a normal monomial this means that $\tau_{\mathcal{S}\setminus\{G_n\}}(\hat{1}) \vee H_{G_n}$ is the minimal atom in $[\tau_{\mathcal{S}\setminus\{G_n\}}(\hat{1}), \hat{1}]$. Therefore, H_{G_n} is smaller than all the atoms which are not below $\tau_{\mathcal{S}\setminus\{G_n\}}(\hat{1})$ and consequently the maximal chain introduced previously also has increasing labels at $H_{G_1} \vee ... \vee H_{G_n}$.

By Proposition 4.5.2 the chain $H_{G_1} \prec ... \prec H_{G_1} \vee ... \vee H_{G_n} \prec \omega_{H_{G_1} \vee ... \vee H_{G_n}, \hat{1}, \lambda_{\lhd_{\mathrm{ind}}}}$ must be the chain $\omega_{\hat{0},\hat{1},\lambda_\lhd}$ and therefore $H_{G_1} \prec ... \prec H_{G_1} \vee ... \vee H_{G_n}$ is the chain $\omega^n_{\hat{0},\hat{1},\lambda_\lhd}$ which concludes the proof. □

Corollary 4.5.7. *The morphism*

$$\mathfrak{LBS}(\mathrm{Gen})/\mathcal{I} \xrightarrow{\pi} \mathbb{FY}^\vee$$

is an isomorphism and the shuffle $\mathfrak{LBS}$-operad $\mathbb{FY}^\vee_{\mathrm{III}}$ admits a quadratic Gröbner basis.

Chapter 5

Koszulness of $\mathfrak{LBS}$-operads

In [20], R. Kaufmann and B. Ward constructed a Koszul duality theory for operads over certain well-behaved Feynman categories called "cubical". This cubicality condition is what allows us to define odd operads and a cobar construction on odd operads, which is central in Koszul duality theory.

In the first subsection we prove that $\mathfrak{LBS}$ is cubical. Then we unpack Koszulness for operads over $\mathfrak{LBS}$, following the definitions in [20]. Finally, we prove that having a quadratic Gröbner basis implies being Koszul and we apply this result to $\mathbb{FY}^{\vee}$.

5.1 $\mathfrak{LBS}$ is cubical

Let us start with some reminders on the notion of "cubicality" (we refer to [20] for more details).

Given $(\mathcal{V}, \mathcal{F}, \imath)$ a graded Feynman category (see Section 2.4) and A, B two objects in $\mathcal{F}$ we denote by $C_n^+(A, B)$ the set of composable chains of morphisms of degree less than 1 having exactly n morphisms of degree 1, quotiented by relations:

$$A \to \ldots \to X_{i-1} \xrightarrow{f} X_i \xrightarrow{g} X_{i+1} \to \ldots \to B \sim A \to \ldots \to X_{i-1} \xrightarrow{g \circ f} X_{i+1} \to \ldots \to B \quad (5.1)$$

provided f or g has degree 0. There is a composition map going from $C_n^+(A, B)$ to $\mathrm{Hom}_{\mathcal{F}}(A, B)$ given by composing all the morphisms of the chain (the equivalence relation preserves this composition). This map will be denoted by $c_{A,B}$.

Definition 5.1.1 (Cubical Feynman category)**.** A graded Feynman category $(\mathcal{V}, \mathcal{F}, \imath)$ is called *cubical* if the degree function is proper and if for every A, B objects of $\mathcal{F}$ there is a free $\mathbb{S}_n$ action on $C_n^+(A, B)$ such that

- The composition map is invariant over the action of $\mathbb{S}_n$.
- The composition map defines a bijection $c_{A,B} : C_n^+(A,B)_{\mathbb{S}_n} \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{F}}(A,B)$.
- The $\mathbb{S}_n$ action is compatible with concatenation of sequences (considering the inclusion $\mathbb{S}_p \times \mathbb{S}_q \subset \mathbb{S}_{p+q}$).

Proposition 5.1.2. *The Feynman categories $\mathfrak{LBS}$ and $\mathfrak{LBS}_{\mathrm{III}}$ are cubical.*

Proof. We only prove the result for $\mathfrak{LBS}$, the same arguments also work for $\mathfrak{LBS}_{\mathrm{III}}$. By Section 2.3, degree 0 and degree 1 morphisms generate every morphism in $\mathfrak{LBS}$. Let us now define an explicit faithful symmetric action on $C_n^+(A,B)$ for every A, B in $\mathfrak{LBS}$. It is enough to define it for B an irreducible built lattice.

Using relation (2.3) and relation (5.1) we can see that every chain ψ in $C_n^+(A,B)$ has a representative of the form

$$A \xrightarrow{(\bigotimes_i f_i \otimes (\{G\}\circ(g_1\otimes g_2))\otimes \bigotimes_j f_j)\circ\nu} A' \xrightarrow{\phi} B, \tag{5.2}$$

where the f_i's, $f_j's$ and g_1, g_2 are isomorphisms in $\mathbf{LBS}_{\mathrm{irr}}$, ν is a permutation of the summands of A and ϕ is an element of $C_{n-1}^+(A',B)$ which contains only degree 1 morphisms which are nested sets of cardinality one. By Lemma 2.3.3 this representative is in fact unique. We denote $\psi' = [A'' \xrightarrow{\mathrm{Id}\otimes\{G\}\otimes\mathrm{Id}} A' \xrightarrow{\phi} B]$. The composition of the chain ϕ is a nested set $\mathcal{S}$, and we have a linear ordering $G_1, G_2, ..., G_n$ of $\mathcal{S}$ given by reading the chain ϕ say from right to left. Let σ be any element of $\mathbb{S}_n$. We define

$$\sigma\cdot\psi' := [A'' \xrightarrow{\sigma'} A''' \xrightarrow{\mathrm{Id}^{\otimes}\otimes\{G_{\sigma(1)}\vee\ldots\vee G_{\sigma(n)}\}\otimes\mathrm{Id}^{\otimes}} \ldots \xrightarrow{\{G_{\sigma(1)}\}} B]$$

where

- The formula $\mathrm{Id}^{\otimes}\otimes\{G_{\sigma(1)}\vee\ldots\vee G_{\sigma(i)}\}\otimes\mathrm{Id}^{\otimes}$ means that we tensor the morphism $\{G_{\sigma(1)}\vee\ldots\vee G_{\sigma(i)}\}$ by the identities of the summands of the codomain not containing $G_{\sigma(1)}\vee\ldots\vee G_{\sigma(i)}$.
- σ' is the only permutation of the summands of A'' which gives us A'''.

Finally, we put

$$\sigma\cdot\psi := [A \xrightarrow{\bigotimes_i f_i\otimes g_1\otimes g_2\otimes\bigotimes_j f_j} A'' \xrightarrow{\sigma\cdot\psi'} B].$$

First, let us prove that this defines an action of $\mathbb{S}_n$. Assume σ is a product $\sigma_1\sigma_2$ with $\sigma_1, \sigma_2 \in \mathbb{S}_n$. We have

$$\begin{aligned}
&\sigma_1 \cdot (\sigma_2 \cdot \psi) = \sigma_1 \cdot [A \xrightarrow{\bigotimes_i f_i \otimes g_1 \otimes g_2 \otimes \bigotimes_j f_j} A'' \xrightarrow{\sigma_2 \cdot \psi'} B] \\
&= \sigma_1 \cdot [A \xrightarrow{\bigotimes_i f_i \otimes g_1 \otimes g_2 \otimes \bigotimes_j f_j} A'' \xrightarrow{\sigma_2'} A_2''' \xrightarrow{\mathrm{Id}^{\otimes} \otimes \{G_{\sigma_2(1)} \vee \ldots \vee G_{\sigma_2(n)}\} \otimes \mathrm{Id}^{\otimes}} \ldots \xrightarrow{\{G_{\sigma_2(1)}\}} B] \\
&= [A \xrightarrow{\bigotimes_i f_i \otimes g_1 \otimes g_2 \otimes \bigotimes_j f_j} A'' \xrightarrow{\sigma_2'} A_2''' \xrightarrow{\sigma_1'} A_1''' \xrightarrow{\mathrm{Id}^{\otimes} \otimes \{G_{\sigma_1(\sigma_2(n))}\} \otimes \mathrm{Id}^{\otimes}} \ldots \xrightarrow{\{G_{\sigma_1(\sigma_2(1))}\}} B] \\
&= [A \xrightarrow{\bigotimes_i f_i \otimes g_1 \otimes g_2 \otimes \bigotimes_j f_j} A'' \xrightarrow{(\sigma_1\sigma_2)'} A_1''' \xrightarrow{\mathrm{Id}^{\otimes} \otimes \{G_{\sigma_1\sigma_2(1)} \vee \ldots \vee G_{\sigma_1\sigma_2(n)}\} \otimes \mathrm{Id}^{\otimes}} \ldots \xrightarrow{\{G_{\sigma_1\sigma_2(1)}\}} B] \\
&= (\sigma_1\sigma_2) \cdot \psi.
\end{aligned}$$

Second, let us remark that by relation (2.4), we have $c_{A'',B}(\sigma \cdot \psi') = c_{A'',B}(\psi')$ which implies $c_{A,B}(\sigma \cdot \psi) = c_{A,B}(\psi)$ i.e. the composition map is invariant by the action of $\mathbb{S}_n$.

Third, we see that the action is free because of the unicity of decomposition (5.2).

Fourth and lastly, the composition map is bijective after passing to the quotient by the action of $\mathbb{S}_n$. The surjectivity immediately comes from the fact that morphisms of degree 0 and degree 1 generate every morphism in $\mathfrak{LBS}$. The injectivity is a consequence of the unicity of decomposition (5.2). □

5.2 Definition of Koszulness for $\mathfrak{LBS}$-operads

5.2.1 Odd operads over cubical Feynman categories

Let $\mathfrak{F} = (\mathcal{F}, \mathcal{V}, \imath)$ be cubical category. An odd operad over $\mathfrak{F}$ is an operad over the Feynman category $\mathfrak{F}^{\mathrm{odd}} = (\mathcal{F}^{\mathrm{odd}}, \mathcal{V}, \imath^{\mathrm{odd}})$ where $\mathcal{F}^{\mathrm{odd}}$ is the category enriched in abelian groups having the same objects as $\mathcal{F}$ and morphisms

$$\mathcal{F}^{\mathrm{odd}}(X, Y) = \mathbb{Z} < C_n^+(X, Y) > /\sigma.\phi - \epsilon(\sigma)\phi.$$

with composition given by concatenating chains of morphisms. Since $\mathcal{V}$ only has isomorphisms it is clear that $\mathcal{V}$ is also embedded in $\mathcal{F}^{\mathrm{odd}}$ and we call this embedding $\imath^{\mathrm{odd}}$.

In the case of $\mathfrak{LBS}$, the category $\mathbf{LBS}^{\mathrm{odd}}$ is generated by isomorphisms and generators $\{G\}^{\mathrm{odd}}$ for each element G which is not the maximal element in some building set of some lattice, quotiented by relations

$$\{G_1\}^{\mathrm{odd}} \circ (\{G_1 \vee G_2\}^{\mathrm{odd}} \otimes \mathrm{Id}) = -\{G_2\}^{\mathrm{odd}} \circ (\{G_1 \vee G_2\}^{\mathrm{odd}} \otimes \mathrm{Id}) \circ \sigma_{2,3} \qquad (5.3)$$

for every nested antichain $\{G_1 \neq \hat{1}, G_2 \neq \hat{1}\}$ in some building set, relations

$$\{G_1\}^{\mathrm{odd}} \circ (\{G_2\}^{\mathrm{odd}} \otimes \mathrm{Id}) = -\{G_2\}^{\mathrm{odd}} \circ (\mathrm{Id} \otimes \{G_1\}^{\mathrm{odd}}) \qquad (5.4)$$

for every chain $G_1 < G_2 < \hat{1}$ in some building set, as well as relations

$$f \circ f(\{G\}^{\mathrm{odd}}) = \{G\}^{\mathrm{odd}} \circ (f_{[G,\hat{1}]} \otimes f_{[\hat{0},G]}) \tag{5.5}$$

for every isomorphism f between built lattices.

5.2.2 An example of an odd $\mathfrak{LBS}$ cooperad

The family of projective Orlik–Solomon algebras $\{\overline{\mathrm{OS}}(\mathcal{L})\}_{(\mathcal{L},\mathcal{G})}$ has an odd cooperadic structure over $\mathfrak{LBS}$ which extends the dual of the odd operad Grav. It will be denoted by $\overline{\mathbb{OS}}$ and defined as follow.

- For any built lattice $(\mathcal{L},\mathcal{G})$ we define
$$\overline{\mathbb{OS}}(\mathcal{L},\mathcal{G}) \coloneqq \overline{\mathrm{OS}}(\mathcal{L}).$$
- For any element $G \in \mathcal{G} \setminus \{\hat{1}\}$ in some irreducible built lattice $(\mathcal{L},\mathcal{G})$, we define
$$\begin{array}{cccc} \overline{\mathbb{OS}}(\{G\}^{\mathrm{odd}}): & \overline{\mathrm{OS}}(\mathcal{L}) & \longrightarrow & \overline{\mathrm{OS}}([G,\hat{1}]) \otimes \overline{\mathrm{OS}}([\hat{0},G]) \\ & \prod_i e_{H_i} \prod_j e_{H'_j} & \longrightarrow & \delta(\prod_i e_{G\vee H_i}) \otimes \prod_j e_{H'_j} \end{array}$$
where the H_i's are atoms not below G and the H'_j's are atoms below G.

Let us check that the image belongs to the tensored projective Orlik–Solomon algebras. By Lemma 1.2.6 it is enough to show that $\overline{\mathbb{OS}}(\{G\}^{\mathrm{odd}})(\delta(\prod_{\mathcal{H}} e_H))$ belongs to $\overline{\mathrm{OS}}([G,\hat{1}]) \otimes \overline{\mathrm{OS}}([\hat{0},G])$ for all sets of atoms $\mathcal{H}$. We partition $\mathcal{H}$ into $\{H_i\} \sqcup \{H'_j\}$ where the H_i's are atoms not below G and the H'_j's are atoms below G. We then have

$$\begin{aligned} \overline{\mathbb{OS}}(\{G\}^{\mathrm{odd}})(\delta(\prod_{\mathcal{H}} e_H)) &= \overline{\mathbb{OS}}(\{G\}^{\mathrm{odd}})(\delta(\prod_i e_{H_i})\prod_j e_{H'_j} \pm \prod_i e_{H_i}\delta(\prod_j e_{H'_j})) \\ &= \delta(\delta(\prod_i e_{G\vee H_i})) \otimes \prod_j e_{H'_j} \pm \delta(\prod_i e_{G\vee H_i}) \otimes \delta(\prod_j e_{H'_j}) \\ &= \pm\delta(\prod_i e_{G\vee H_i}) \otimes \delta(\prod_j e_{H'_j}) \in \overline{\mathrm{OS}}([G,\hat{1}]) \otimes \overline{\mathrm{OS}}([\hat{0},G]). \end{aligned}$$

- For any isomorphism of built lattice $f : (\mathcal{L}',\mathcal{G}') \xrightarrow{\sim} (\mathcal{L},\mathcal{G})$ we define $\overline{\mathbb{OS}}(f)$ as the restriction of $\mathbb{OS}(f)$ to the projective subalgebra.

We must check that the morphisms $\overline{\mathbb{OS}}(\{G\}^{\mathrm{odd}})$ and $\overline{\mathbb{OS}}(f)$ satisfy relations (5.3), (5.4) and (5.5) above. Let $\{G_1 \neq \hat{1}, G_2 \neq \hat{1}\}$ be a nested antichain in some irreducible built lattice $(\mathcal{L},\mathcal{G})$ and let $\alpha = \delta(\prod_{i\leq n} e_{H_i} \prod_{j\leq n'} e_{H'_j} \prod_{k\leq n''} e_{H''_k})$ be an element in $\overline{\mathrm{OS}}(\mathcal{L})$ where the H_i's are atoms below neither G_1 nor G_2, the H'_j's atoms below G_2 and the H''_k's below G_1

(by nestedness of $\{G_1, G_2\}$ there can be no atom below both G_1 and G_2). In this case one can check that the morphism

$$(\overline{\mathbb{OS}}(\{G_1 \vee G_2\}^{\mathrm{odd}}) \otimes \mathrm{Id}) \circ \overline{\mathbb{OS}}(\{G_1\}^{\mathrm{odd}})$$

sends α to $\delta(\prod_i e_{G_1 \vee G_2 \vee H_i}) \otimes \delta(\prod_j e_{H'_j}) \otimes \delta(\prod_k e_{H''_k})$ whereas the morphism

$$\sigma_{2,3} \circ (\overline{\mathbb{OS}}(\{G_1 \vee G_2\}^{\mathrm{odd}}) \otimes \mathrm{Id}) \circ \overline{\mathbb{OS}}(\{G_2\}^{\mathrm{odd}})$$

sends α to the opposite. Let $G_1 < G_2 < \hat{1}$ be a chain in some irreducible built lattice $(\mathcal{L}, \mathcal{G})$ and let $\alpha = \delta(\prod_{i \leq n} e_{H_i} \prod_{j \leq n'} e_{H'_j} \prod_{k \leq n''} e_{H''_k})$ be an element in $\overline{\mathrm{OS}}(\mathcal{L})$ where the H_i's are atoms not below G_2, the H'_j's are atoms below G_2 and not below G_1 and the H'''_k's are atoms below G_1. In this case one can check that the morphism

$$(\overline{\mathbb{OS}}(\{G_2\}^{\mathrm{odd}}) \otimes \mathrm{Id}) \circ \overline{\mathbb{OS}}(\{G_1\}^{\mathrm{odd}})$$

sends α to $\delta(\prod_i e_{G_2 \vee H_i}) \otimes \delta(\prod_j G_1 \vee e_{H'_j}) \otimes \delta(\prod_k e_{H''_k})$ whereas the morphism

$$(\mathrm{Id} \otimes \overline{\mathbb{OS}}(\{G_2\}^{\mathrm{odd}})) \circ \overline{\mathbb{OS}}(\{G_2\}^{\mathrm{odd}})$$

sends α to the opposite. At last, equation (5.5) is also easily verified.

To conlude, we have shown that $\overline{\mathbb{OS}}$ is an odd $\mathfrak{LBS}$-cooperad (in graded abelian groups).

5.2.3 The bar/cobar construction

Let $\mathcal{C}$ be some complete cocomplete symmetric monoidal abelian category. We denote by $\mathrm{Ch}\,\mathcal{C}$ the category of chain complexes over $\mathcal{C}$. Let $\mathfrak{F}$ be a cubical Feynman category.

In [20], R. Kaufmann and B. Ward define a bar operator

$$\mathrm{B} : \mathfrak{F} - Ops_{\mathrm{Ch}\,\mathcal{C}} \to \mathfrak{F}^{\mathrm{odd}} - Ops_{\mathrm{Ch}\,\mathcal{C}^{op}}$$

and a cobar operator

$$\Omega : \mathfrak{F}^{\mathrm{odd}} - Ops_{\mathrm{Ch}\,\mathcal{C}^{op}} \to \mathfrak{F} - Ops_{\mathrm{Ch}\,\mathcal{C}}$$

which form an adjunction $\Omega \rightleftharpoons \mathrm{B}$ and such that the counit

$$\Omega\mathrm{B} \Longrightarrow \mathrm{Id}$$

is a level-wise quasi-isomorphism. Informally, those functors are defined by taking free constructions together with a differential constructed using the degree 1 generators. Let us describe Ω explicitely in our case. Let $\mathbb{P}$ be an $\mathfrak{LBS}^{\mathrm{odd}}$ cooperad in $\mathrm{Ch}\,\mathcal{C}^{op}$. We have

$$\Omega(\mathbb{P}) := (\mathfrak{LBS}(\mathbb{P}), \mathrm{d}_\Omega + \mathrm{d}_\mathbb{P}),$$

where $\mathrm{d}_{\mathbb{P}}$ is the obvious differential coming from $\mathbb{P}$ and d_Ω is defined as follow. Recall the explicit formula

$$\mathfrak{LBS}(\mathbb{P})(\mathcal{L},\mathcal{G}) = \bigoplus_{\otimes(\mathcal{L}_i,\mathcal{G}_i)\xrightarrow{f}(\mathcal{L},\mathcal{G})} \bigotimes_i \mathbb{P}(\mathcal{L}_i,\mathcal{G}_i)/\sim$$

where $\sim$ identifies components corresponding to equivalent maps (maps that can be obtained from each other by precomposition of isomorphisms). For any $\otimes_i p_i \in \bigotimes_{i\leq n}\mathbb{P}(\mathcal{L}_i,\mathcal{G}_i)$ we put

$$\mathrm{d}_\Omega([(\otimes p_i, f : \bigotimes_i(\mathcal{L}_i,\mathcal{G}_i) \to (\mathcal{L},\mathcal{G}))]) := \sum_{\substack{j\leq n\\ G\in\mathcal{G}_j\setminus\{\hat{1}\}}} [(\mathrm{Id}\otimes\mathbb{P}(\{G\})\otimes\mathrm{Id})(\otimes p_i), f\circ(\mathrm{Id}\otimes\{G\}\otimes\mathrm{Id}))].$$

The cubicality condition and the fact that $\mathbb{P}$ is odd ensures that $\mathrm{d}_\Omega + \mathrm{d}_{\mathbb{P}}$ is a differential. Let us now describe B explicitely for $\mathfrak{LBS}$-operads. Let $\mathbb{P}$ be an $\mathfrak{LBS}$-operad in $\mathrm{Ch}\,\mathcal{C}$. We have

$$\mathrm{B}(\mathbb{P}) = (\mathfrak{LBS}^{\mathrm{odd}}(\mathbb{P}), \mathrm{d}_{\mathrm{B}} + \mathrm{d}_{\mathbb{P}}),$$

where $\mathrm{d}_{\mathbb{P}}$ is the obvious differential coming from $\mathbb{P}$ and d_{B} is defined as follow. We have the formula

$$\mathfrak{LBS}^{\mathrm{odd}}(\mathbb{P})(\mathcal{L},\mathcal{G}) = \bigoplus_{\substack{n\in\mathbb{N}\\ \psi\in C_n^+((\mathcal{L}',\mathcal{G}'),(\mathcal{L},\mathcal{G}))}} \mathbb{P}(\mathcal{L}',\mathcal{G}')/\sim,$$

where the equivalence relation $\sim$ is given by

$$(\mathbb{P}(f)(\alpha),\psi) \sim (\alpha, \psi\circ f)$$

for every isomorphism f and

$$(\alpha,\psi) \sim \epsilon(\sigma)(\alpha,\sigma.\psi)$$

for every permutation σ. Let ψ be an element in $C_n^+((\mathcal{L}',\mathcal{G}'),(\mathcal{L},\mathcal{G}))$, and let α be an element of $\mathbb{P}(\mathcal{L}',\mathcal{G}')$. We have

$$\mathrm{d}_{\mathrm{B}}((\alpha,[\psi])) := \sum_{\substack{\phi:(\mathcal{L}'',\mathcal{G}'')\to(\mathcal{L},\mathcal{G})\\ G \text{ s.t. } c(\phi\circ\{G\})=c(\psi)}} \epsilon(\phi)((\mathrm{Id}\otimes\mathbb{P}(\{G\})\otimes\mathrm{Id})(\alpha),\phi),$$

where c is the composition map and $\epsilon(\phi)$ is the signature of the permutation sending ψ to $\phi\circ\{G\}$. One can check that this descends to the quotient by the equivalence relation $\sim$.

5.2.4 Quadratic duality and Koszul duality

In this subsection we assume that $\mathcal{C}$ is a category of vector spaces over some field. For any graded Feynman category $\mathfrak{F}$, an $\mathfrak{F}$-quadratic data is a pair (M, R) with M an $\mathfrak{F}$-module and R a submodule of $\mathfrak{F}_1(M)$, where $\mathfrak{F}_1(M)$ denotes the part of weight 1 in the free $\mathfrak{F}$-operad $\mathfrak{LBS}(M)$. Notice that an $\mathfrak{F}$-quadratic data can also be seen as an $\mathfrak{F}^{\mathrm{odd}}$-quadratic data since $\mathfrak{F}$ and $\mathfrak{F}^{\mathrm{odd}}$ have the same modules and we have

$$\mathfrak{F}_1^{\mathrm{odd}}(M) = \mathfrak{F}_1(M).$$

If $\mathbb{P}$ is an $\mathfrak{F}$-operad which is a quotient $\mathfrak{F}(M)/\langle R\rangle$ for some $\mathfrak{F}$-quadratic data (M, R), we define

$$\mathbb{P}^! := \mathfrak{F}^{\mathrm{odd}}(M^\vee)/\langle R^\perp\rangle,$$

which is an $\mathfrak{F}^{\mathrm{odd}}$-operad. We have a morphism of differential graded $\mathfrak{F}^{\mathrm{odd}}$-operads

$$(\mathbb{P}^!)^\vee \to \mathrm{B}\mathbb{P}, \tag{5.6}$$

which is induced by the morphism of $\mathfrak{F}$-modules given by the composition

$$(\mathbb{P}^!)^\vee \twoheadrightarrow M^\vee \hookrightarrow \mathbb{P}.$$

We say that $\mathbb{P}$ is Koszul with Koszul dual $(\mathbb{P}^!)^\vee$ if morphism (5.6) is a quasi-isomorphism. We refer to [20] and [34] for more details. This coincides with the classical Koszul duality theories (Koszul duality for operads for instance).

In addition to having a homological degree (given by the grading of $\mathfrak{LBS}_{\mathrm{III}}$), the odd cooperad $\mathrm{B}\mathbb{P}$ has a weight grading coming from the grading of $\mathbb{P}$. The differential preserves this weight grading.

One can check that the map

$$(\mathbb{P}^!)^\vee \to \mathrm{B}\mathbb{P}$$

is injective and its image is exactly the kernel of $\mathrm{d_B}$ in the diagonal {weight grading = degree}, which is also the homology of the diagonal since the elements on the diagonal are the highest degree elements in their respective weight component. As a consequence $\mathbb{P}$ is Koszul if and only if the homology of $\mathrm{B}\mathbb{P}$ is concentrated on the diagonal.

5.3 Koszulness of $\mathbb{FY}^{\mathrm{PD}}$ using the projective combinatorial Leray model

In [5] the authors define a differential bigraded algebra $B(\mathcal{L}, \mathcal{G})$ as follow.

Definition 5.3.1 (Combinatorial Leray model [5])**.** Let $(\mathcal{L},\mathcal{G})$ be an irreducible built lattice. The differential bigraded algebra $B(\mathcal{L},\mathcal{G})$ is defined as the quotient of the free commutative algebra $\mathbb{Q}[e_G, x_G, G \in \mathcal{G}]$ by the ideal $\mathcal{I}$ generated by

1. The elements $e_{\mathcal{S}}x_{\mathcal{T}}$ with $\mathcal{S}\cup\mathcal{T}$ not nested.
2. The elements $\sum_{G\geq H} x_G$ for all atoms H of $\mathcal{L}$.
3. The element $e_{\hat{1}}$.

The generators e_G have bidegree $(0,1)$ and the generators x_G have bidegree $(2,0)$. The differential d of this algebra has bidegree $(2,-1)$ and is defined by

$$\mathrm{d}(e_G) = x_G$$
$$\mathrm{d}(x_G) = 0.$$

The authors of [5] have shown that we have an isomorphism of graded vector spaces

$$B^{\bullet,d}(\mathcal{L},\mathcal{G}) \simeq \bigoplus_{\substack{\mathcal{S}\text{ irreducible}\\ \text{nested set of }(\mathcal{L},\mathcal{G})\\ \#\mathcal{S}=d+1}} \mathbb{FY}^{\mathrm{PD}}(\mathcal{S})$$

for every integer d ([5] Proposition 5.1.4). Those isomorphisms give an isomorphism of complexes between $B(\mathcal{L},\mathcal{G})$ and $\mathrm{B}\mathbb{FY}^{\mathrm{PD}}(\mathcal{L},\mathcal{G})$. For each irreducible built lattice $(\mathcal{L},\mathcal{G})$ we have a morphism of differential graded algebras

$$\overline{\mathrm{OS}}(\mathcal{L}) \to B^{\bullet}(\mathcal{L},\mathcal{G}),$$

induced by the map $e_H \to \sum_{G\geq H} e_G$. One can check that this is a morphism of $\mathfrak{LBS}^{\mathrm{odd}}$-cooperads. The main result of [5] is the following.

Theorem 5.3.2 ([5], Theorem 5.5.1)**.** *The morphism $\overline{\mathrm{OS}}(\mathcal{L}) \xrightarrow{\sim} B^{\bullet}(\mathcal{L},\mathcal{G})$ is a quasi-isomorphism for every pair $(\mathcal{L},\mathcal{G})$.*

This immediately implies:

Corollary 5.3.3. *The operad $\mathbb{FY}^{\mathrm{PD}}$ is Koszul with Koszul dual $\overline{\mathbb{OS}}$.*

Remark 5.3.4. The algebra structure on $\mathrm{B}\mathbb{FY}^{\mathrm{PD}}(\mathcal{L},\mathcal{G})$ coming from the isomorphism

$$\mathrm{B}\mathbb{FY}^{\mathrm{PD}}(\mathcal{L},\mathcal{G}) \simeq B(\mathcal{L},\mathcal{G})$$

can be defined purely operadically as follow. Let α be some element in $\mathbb{FY}^{\mathrm{PD}}(\mathcal{S})$ for some nested set $\mathcal{S}$ in some built lattice $(\mathcal{L},\mathcal{G})$ and β some element of $\mathbb{FY}^{\mathrm{PD}}(\mathcal{S}')$ for some nested set $\mathcal{S}'$ in the same built lattice. The product of α and β in $\mathrm{B}\mathbb{FY}^{\mathrm{PD}}(\mathcal{L},\mathcal{G})$ is given by

$$\alpha\cdot\beta = \begin{cases} \mathbb{FY}(\mathcal{S}')(\alpha)\mathbb{FY}(\mathcal{S})(\beta) & \text{if } \mathcal{S}\cap\mathcal{S}' = \{\hat{1}\} \text{ and } \mathcal{S}\cup\mathcal{S}' \text{ is a nested set.} \\ 0 & \text{otherwise.} \end{cases}$$

In the first row, $\mathcal{S}'$ is viewed as a nested set of $(\mathcal{L}_\mathcal{S}, \mathcal{G}_\mathcal{S})$ and $\mathcal{S}$ is viewed as a nested set of $(\mathcal{L}_{\mathcal{S}'}, \mathcal{G}_{\mathcal{S}'})$ via Lemma 2.2.9. The product takes place in the algebra $\mathrm{FY}(\mathcal{L}_{\mathcal{S}\cup\mathcal{S}'}, \mathcal{G}_{\mathcal{S}\cup\mathcal{S}'})$. It is interesting to note that we have used the operadic structure of $\mathbb{FY}$ and not that of $\mathbb{FY}^{\mathrm{PD}}$. This shows that Poincaré duality plays an important role when trying to relate the properties of $\mathbb{FY}$ and the properties of the Feichtner–Yuzvinsky algebras.

5.4 Koszulness and the affine combinatorial Leray model

In [5] the authors also define a Leray model $\hat{B}(\mathcal{L}, \mathcal{G})$ for the (affine) Orlik-Solomon algebras, just by taking out the relation $e_{\hat{1}} = 0$ in $B(\mathcal{L}, \mathcal{G})$. One can also interpret this affine Leray model as a bar construction in a larger Feynman category $\mathfrak{LBSmod}$ defined as follow. The set of objects of the underlying groupoid of $\mathfrak{LBSmod}$ is

$$\mathrm{Obj}(\mathbf{LBS}_{\mathrm{irr}}) \sqcup \mathrm{Obj}(\mathbf{LBS}_{\mathrm{irr}}).$$

For each irreducible built lattice $(\mathcal{L}, \mathcal{G})$ we will denote the two copies of $(\mathcal{L}, \mathcal{G})$ in $\mathfrak{LBSmod}$ by $(\mathcal{L}, \mathcal{G})^{\mathrm{proj}}$ and $(\mathcal{L}, \mathcal{G})^{\mathrm{aff}}$, for reasons which will be clear later. If $(\mathcal{L}, \mathcal{G})$ is an irreducible built lattice, the structural morphisms of $\mathfrak{LBSmod}$ with target $(\mathcal{L}, \mathcal{G})^{\mathrm{proj}}$ are labelled by irreducible nested sets

$$\bigotimes_{G\in\mathcal{S}}([\tau_\mathcal{S}(G), G], \mathrm{Ind}(\mathcal{G}))^{\mathrm{proj}} \xrightarrow{\mathcal{S}} (\mathcal{L}, \mathcal{G})^{\mathrm{proj}},$$

with composition as in $\mathfrak{LBS}$. In other words when restricting $\mathfrak{LBSmod}$ to the "projective" arities we get the Feynman category $\mathfrak{LBS}$. The structural morphisms of $\mathfrak{LBSmod}$ with target $(\mathcal{L}, \mathcal{G})^{\mathrm{aff}}$ are labelled by nested sets which can either contain $\hat{1}$ or not. If $\mathcal{S}$ contains $\hat{1}$ then we have the morphism

$$\bigotimes_{G\in\mathcal{S}}([\tau_\mathcal{S}(G), G], \mathrm{Ind}(\mathcal{G}))^{\mathrm{proj}} \xrightarrow{\mathcal{S}^{\mathrm{aff}}} (\mathcal{L}, \mathcal{G})^{\mathrm{aff}}$$

and if $\mathcal{S}$ does not contain $\hat{1}$ we have the morphism

$$\bigotimes_{G\in\mathcal{S}}([\tau_\mathcal{S}(G), G], \mathrm{Ind}(\mathcal{G}))^{\mathrm{proj}} \otimes ([\tau_\mathcal{S}(\hat{1}), \hat{1}], \mathrm{Ind}(\mathcal{G}))^{\mathrm{aff}} \xrightarrow{\mathcal{S}^{\mathrm{aff}}} (\mathcal{L}, \mathcal{G})^{\mathrm{aff}}.$$

The composition of those morphisms is defined as in $\mathfrak{LBS}$. This Feynman category encodes pairs $(\mathbb{P}, \mathbb{M})$ with $\mathbb{P}$ an $\mathfrak{LBS}$-operad and $\mathbb{M}$ a "$\mathbb{P}$-module" ($\mathbb{P}$ is the restriction to the projective part and $\mathbb{M}$ the restriction to the affine part). A set of generating morphisms of $\mathfrak{LBSmod}$ is given by

$$\{G, \hat{1}\}^{\mathrm{proj}}\ (G \neq \hat{1}) \text{ and } \{G\}^{\mathrm{aff}}.$$

One can define an odd $\mathfrak{LBSmod}$-cooperad $\mathbb{OS}_{\mathrm{tot}}$ by setting

$$\mathbb{OS}_{\mathrm{tot}}((\mathcal{L},\mathcal{G})^{\mathrm{proj}}) = \overline{\mathrm{OS}}(\mathcal{L}), \quad \mathbb{OS}_{\mathrm{tot}}((\mathcal{L},\mathcal{G})^{\mathrm{aff}}) = \mathrm{OS}(\mathcal{L}),$$

for each irreducible built lattice $(\mathcal{L},\mathcal{G})$, and

$$\mathbb{OS}_{\mathrm{tot}}(\{G,\hat{1}\}^{\mathrm{proj}}) = \overline{\mathbb{OS}}(\{G,\hat{1}\})$$

together with

$$\mathbb{OS}_{\mathrm{tot}}(\{G\}^{\mathrm{aff}}) = (\delta \otimes \mathrm{Id}) \circ \mathbb{OS}(\{G\}).$$

In this case the odd $\overline{\mathbb{OS}}$-comodule structure on $\mathbb{OS}$ comes from the morphism of $\mathfrak{LBS}$-operads

$$\mathbb{OS} \xrightarrow{\delta} \overline{\mathbb{OS}}.$$

On the other hand one can define an $\mathfrak{LBSmod}$-operad $\mathbb{FY}^{\mathrm{PD}}_{\mathrm{tot}}$ by setting

$$\mathbb{FY}^{\mathrm{PD}}_{\mathrm{tot}}((\mathcal{L},\mathcal{G})^{\mathrm{proj}}) = \mathrm{FY}(\mathcal{L},\mathcal{G}), \quad \mathbb{FY}^{\mathrm{PD}}_{\mathrm{tot}}((\mathcal{L},\mathcal{G})^{\mathrm{aff}}) = \mathrm{FY}(\mathcal{L},\mathcal{G}),$$

and

$$\mathbb{FY}_{\mathrm{tot}}(\{G,\hat{1}\}^{\mathrm{proj}}) = \mathbb{FY}^{\mathrm{PD}}(\{G,\hat{1}\}^{\mathrm{proj}}),$$

together with

$$\mathbb{FY}_{\mathrm{tot}}(\{G\}^{\mathrm{aff}}) = \mathbb{FY}(\{G,\hat{1}\})$$

for $G \neq \hat{1}$, and finally $\mathbb{FY}_{\mathrm{tot}}(\{\hat{1}\}^{\mathrm{aff}})$ is set to be the multiplication by $x_{\hat{1}}$. Exactly as for the projective part, one can use the results of [5] to see that $\hat{B}(\mathcal{L},\mathcal{G})$ is isomorphic to $\mathrm{B}\mathbb{FY}^{\mathrm{PD}}_{\mathrm{tot}}$ and the morphism $e_H \to \sum_{G \geq H} e_G$ induces a quasi-isomorphism of odd $\mathfrak{LBSmod}$-cooperads

$$\mathbb{OS}_{\mathrm{tot}} \xrightarrow{\simeq} \mathrm{B}\mathbb{FY}^{\mathrm{PD}}_{\mathrm{tot}},$$

which implies Koszulness of the $\mathfrak{LBSmod}$-operad $\mathbb{FY}^{\mathrm{PD}}_{\mathrm{tot}}$.

5.5 Koszulness via shuffle operads

As in the case of classical operads and their shuffle counterpart, we have the key proposition.

Proposition 5.5.1. *Let $\mathbb{P}$ be an $\mathfrak{LBS}$-operad. $\mathbb{P}$ is Koszul if and only if $\mathbb{P}_{\text{Ш}}$ is Koszul.*

Proof. By Proposition 4.4.2 we have isomorphisms of shuffle $\mathfrak{LBS}^{\mathrm{odd}}$-operads

$$(\mathbb{P}_{\text{Ш}})^{!} \simeq (\mathbb{P}^{!})_{\text{Ш}}$$

which gives an isomorphism of shuffle $\mathfrak{LBS}^{\mathrm{odd}}$-cooperad (in $\mathrm{Ch}\,\mathcal{C}$):

$$((\mathbb{P}_{\text{Ш}})^{!})^{\vee} \simeq ((\mathbb{P}^{!})^{\vee})_{\text{Ш}}.$$

On the other hand, we have

$$(\mathfrak{LBS}^{\mathrm{odd}}(\mathbb{P}))_{\mathrm{III}} \simeq \mathfrak{LBS}^{\mathrm{odd}}_{\mathrm{III}}(\mathbb{P}_{\mathrm{III}}).$$

By going back to explicit formulas one can check that those isomorphisms are compatible with the bar differential and that we have a commutative diagram

$$\begin{array}{ccc} (\mathbb{P}^!)^\vee_{\mathrm{III}} & \longrightarrow & (\mathrm{B}\mathbb{P})_{\mathrm{III}} \\ \downarrow \simeq & & \downarrow \simeq \\ ((\mathbb{P}_{\mathrm{III}})^!)^\vee & \longrightarrow & \mathrm{B}(\mathbb{P}_{\mathrm{III}}), \end{array}$$

but of course in every arity $(\mathcal{L}, \mathcal{G}, \lhd)$ we have the commutative diagram of complexes

$$\begin{array}{ccc} (\mathbb{P}^!)^\vee(\mathcal{L}, \mathcal{G}) & \longrightarrow & \mathrm{B}\mathbb{P}(\mathcal{L}, \mathcal{G}) \\ \downarrow \simeq & & \downarrow \simeq \\ ((\mathbb{P}^!)^\vee)_{\mathrm{III}}(\mathcal{L}, \mathcal{G}, \lhd) & \longrightarrow & (\mathrm{B}\mathbb{P})_{\mathrm{III}}(\mathcal{L}, \mathcal{G}, \lhd). \end{array}$$

Combining the two diagrams in every arity finishes the proof. □

5.6 Koszulness and Gröbner bases

As in the case of classical shuffle operads, we have the key proposition.

Proposition 5.6.1. *Let $\mathbb{P}$ be a shuffle $\mathfrak{LBS}$-operad. If $\mathbb{P}$ admits a quadratic Gröbner basis then $\mathbb{P}$ is Koszul.*

Proof. This is just an adaptation of the proof given in [19] to our setting, and translating Gröbner basis language in "PBW basis" language. We denote $\mathbb{P} = \mathfrak{LBS}_{\mathrm{III}}(M)/\langle R \rangle$. Let us use our total well-order on monomials to construct a filtration on $\mathrm{B}\mathbb{P}$. We will denote this total order by "$<$". Let $m = (\mathcal{S}, (e_G)_{G \in \mathcal{S}})$ be some monomial. We define

$$\mathrm{F}_m\mathrm{B}\mathbb{P} = < \{m_1 \otimes ... \otimes m_n \in \mathbb{P}(\mathcal{S}') \,|\, m_1, \dots, m_n \text{ monomials s.t. } \mathfrak{LBS}_{\mathrm{III}}(\mathcal{S}')((m_i)_i) \leq m\} >$$

where the brackets $<, >$ denote the linear span. The bar differential preserves this filtration. We will now show that the associated spectral sequence collapses at the first page and its homology is concentrated on the diagonal. The complex $E^0_m\mathrm{B}\mathbb{P}$ is spanned by elements of the form

$$\mathbb{P}(\mathcal{S}_1)((e_G)_{G \in \mathcal{S}_1}) \otimes ... \otimes \mathbb{P}(\mathcal{S}_n)((e_G)_{G \in \mathcal{S}_n}),$$

where the nested sets $\mathcal{S}_i$ are such that there exist some nested set $\mathcal{S}'$ satisfying $\mathcal{S} = \mathcal{S}' \circ (\mathcal{S}_i)_i$, and such that the monomials $\mathbb{P}(\mathcal{S}_i)((e_G)_{G \in \mathcal{S}_i})$ are all normal.

For any G in $\mathcal{S} \setminus \{\hat{1}\}$ we denote by $\mathrm{n}(G)$ the unique minimum of $\mathcal{S}_{>G}$. We also denote by $\mathrm{Adm}(m)$ the set of elements of $\mathcal{S} \setminus \{\hat{1}\}$ such that $\mathbb{P}(\{G\})(e_G, e_{\mathrm{n}(G)})$ is a normal monomial. By the fact that our Gröbner basis is quadratic we see that $E^0_m \mathrm{B}\mathbb{P}$ is isomorphic to the augmented dual of the combinatorial complex $C_\bullet(\Delta_{\mathrm{Adm}(m)})$, which has trivial homology except when $\mathrm{Adm}(m) = \emptyset$, in which case the complex is reduced to $\mathbb{K}$ on the diagonal (with generator given by $\otimes_G e_G$). By a standard spectral sequence argument this concludes the proof. $\square$

As a corollary of this proposition and 4.5.7 we get the following result.

Corollary 5.6.2. *The operad $\mathbb{FY}^\vee$ is Koszul.*

Chapter 6

Further directions for the operadic structure

In this section we highlight some possible ways to extend/refine $\mathfrak{LBS}$ which seem natural to us and may lead to further applications.

6.1 Working with matroids instead of geometric lattices

One possible refinement of $\mathfrak{LBS}$ would be to do everything with matroids instead of geometric lattices, which would allow us to take loops and parallel elements into account. For now this refinement is useless because all the operads we know (Feichtner–Yuzvinsky rings, Orlik–Solomon algebras) do not "see" the loops and parallel elements (i.e. factor through the lattice of flat construction). However, it may happen that some finer invariants of matroids which detect loops and parallel elements may also have an operadic structure. In order to implement this refinement it will be beneficial to have a purely matroidal axiomatization of building sets. Let us describe a possible way to obtain that. Recall that a matroid can be defined by its rank function as follow.

Definition 6.1.1 (Matroid, via rank function)**.** Let E be a finite set. A *matroid* structure on E is the datum of a map

$$\mathrm{rk} : \mathcal{P}(E) \to \mathbb{N}$$

called the rank function, satisfying the following properties.

1. The rank function takes value 0 on the empty set.
2. For every $A, B \in \mathcal{P}(E)$ we have

$$\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) \leq \mathrm{rk}(A) + \mathrm{rk}(B).$$

3. For every $A \in \mathcal{P}(E)$ and $x \in E$ we have

$$\mathrm{rk}(A \cup \{x\}) \leq \mathrm{rk}(A) + 1.$$

Here is a possible way of axiomatizing a building set in terms of the rank function.

Definition 6.1.2 (Building decomposition)**.** A *building decomposition* of a matroid (E, rk) is a function ν which assigns to every subset $X \subset E$ a partition of X and which satisfies the following axioms.

1. If $X \subset Y$ are two subsets of E, then $\nu(X)$ refines the restriction of $\nu(Y)$ to X.

2. If $\nu(X)$ is the partition with blocks $P_1|...|P_n$ then for all $i \leq n$ the partition $\nu(P_i)$ is the partition with only one block.

3. For all $X \subset E$, if $\nu(X)$ is the partition $P_1|...|P_n$ then $\mathrm{rk}(X) = \mathrm{rk}(P_1) + ... + \mathrm{rk}(P_n)$.

On simple loopless matroids the datum of a building decomposition is equivalent to the datum of a building set on the lattice of flats. One can construct a building set out of a building decomposition by considering the flats which have a partition with only one block. On the other hand, one can construct a building decomposition out of a building set by setting $\nu(X)$ to be the partition induced by the factor decomposition of $\sigma(X)$.

Example 6.1.3. Let $G = (V, E)$ be a graph and M_G its cycle matroid. M_G admits a building decomposition given by the partitions into connected components for each subset of E. Naturally if we look at the induced building set on the lattice of flats this gives the graphical building set introduced in Example 1.1.8.

We also have a natural notion of induced building decomposition on restrictions and contractions of matroids.

Definition 6.1.4. Let $M = (E, \mathrm{rk})$ be a matroid with building decomposition ν and S a subset of the ground set E. The contraction M^S admits a building decomposition $\mathrm{Ind}^S(\nu)$ given by $\mathrm{Ind}^S(\nu)(X) = \nu(X \cup S)_{|X}$ for every $X \subset E \setminus S$. The restriction M_S admits a building decomposition $\mathrm{Ind}_S(\nu)$ given by $\mathrm{Ind}_S(\nu)(X) = \nu(X)$ for every $X \subset S$.

Working with those definitions, we are fairly certain everything should work in the same fashion as in Section 2, by just replacing the built lattices $([\hat{0}, G], \mathrm{Ind}(\mathcal{G}))$, $([G, \hat{1}], \mathrm{Ind}(\mathcal{G}))$ by the matroidal restrictions/contractions $(M_G, \mathrm{Ind}(\nu))$, $(M^G, \mathrm{Ind}(\nu))$, for G such that $\nu(G)$ is the trivial partition (a priori we would not even need G to be closed, which would give additional structural morphisms).

6.2 The polymatroidal generalization

One can also naturally consider an exension of $\mathfrak{LBS}$ to polymatroids, which form a combinatorial abstraction of subspace arrangements. This is justified by the fact that the wonderful compactification story also works for subspace arrangements and the cohomology algebras give us a natural candidate for an operad over this bigger Feynman category. It has been shown by Pagaria and Pezzoli [29] that those cohomology rings also admit natural generalizations to arbitrary polymatroids and that they also have a Hodge theory. Here are some reminders on polymatroids.

Definition 6.2.1 (Polymatroid)**.** Let E be a finite set. A *polymatroid* structure on E is the datum of a function

$$\mathrm{cd} : \mathcal{P}(E) \to \mathbb{N}$$

satisfying

1. $\mathrm{cd}(\emptyset) = 0$.
2. For any subsets $A \subset B$ of E we have $\mathrm{cd}(A) \leq \mathrm{cd}(B)$.
3. For any subsets A, B of E we have

$$\mathrm{cd}(A \cap B) + \mathrm{cd}(A \cup B) \leq \mathrm{cd}(A) + \mathrm{cd}(B).$$

The letters "cd" stand for codimension. If we ask that cd take value 1 on singletons we get a classical matroid. We can define the lattice of flats of a polymatroid by considering the subsets F of E such that $\mathrm{cd}(F \cup \{x\}) > \mathrm{cd}(F)$ for all $x \notin F$. However, for general polymatroids the lattice of flats does not contain enough information and needs to be considered together with cd to recover the polymatroid (for matroids "cd" is just the rank function of the lattice of flats and does not add any information). In [29] the authors introduced a notion of building set for polymatroids.

Definition 6.2.2. Let $P = (E, \mathrm{cd})$ be a polymatroid with lattice of flats $\mathcal{L}$. A *building set* of P is a subset $\mathcal{G}$ of $\mathcal{L} \setminus \{\hat{0}\}$ such that for any X in $\mathcal{L}$ the join gives an isomorphism of posets

$$\prod_{G \in \mathrm{Fact}_{\mathcal{G}}(X)} [\hat{0}, G] \xrightarrow{\sim} [\hat{0}, X]$$

and we additionally have

$$\mathrm{cd}(X) = \sum_{G \in \mathrm{Fact}_{\mathcal{G}}(X)} \mathrm{cd}(G).$$

Notice that the last condition is automatically verified for matroids ($\mathrm{cd} = \mathrm{rk}$). The authors also give suitable generalizations of nested sets, and they show that for any G in $\mathcal{L}$, the (polymatroidal) contraction $([G, \hat{1}], \mathrm{Ind}(\mathrm{cd}))$ has an induced (polymatroidal) building set given (as in the matroidal case) by

$$\mathrm{Ind}_{[G,\hat{1}]}(\mathcal{G}) = (\mathcal{G} \vee G) \cap (G, \hat{1}],$$

and the same goes for (polymatroidal) restrictions. With those definitions we are fairly certain one can readily extend $\mathfrak{LBS}$ to polymatroids.

In [29], the authors also introduce a generalization of the Feichtner–Yuzvinsky algebras to the polymatroidal setting as follows.

Definition 6.2.3. Let $(\mathcal{L}, \mathrm{cd})$ be a polymatroid with some building set $\mathcal{G}$. The algebra $\mathrm{FY}(\mathcal{L}, \mathcal{G}, \mathrm{cd})$ is defined by

$$\mathrm{FY}(\mathcal{L}, \mathcal{G}, \mathrm{cd}) = \mathbb{Q}[x_G,\, G \in \mathcal{G}]/\mathcal{I},$$

with all the generators in degree 2 and $\mathcal{I}$ the ideal generated by elements

$$\prod_{i \leq n} x_{G_i}$$

where $\{G_1, ..., G_n\}$ is not nested and elements

$$\left(\sum_{G \geq H} x_G \right)^{\mathrm{cd}(H)}$$

for any atom H.

As in the matroidal case, one can get another presentation by considering the change of variable $h_G := \sum_{G' \geq G} x_{G'}$. The algebra morphisms

$$\begin{array}{ccc} \mathrm{FY}(\mathcal{L}, \mathrm{cd}, \mathcal{G}) & \longrightarrow & \mathrm{FY}([G,\hat{1}], \mathrm{Ind}(\mathrm{cd}), \mathrm{Ind}_{[G,\hat{1}]}(\mathcal{G})) \otimes \mathrm{FY}([\hat{0}, G], \mathrm{Ind}(\mathrm{cd}), \mathrm{Ind}_{[\hat{0},G]}(\mathcal{G})) \\ h_{G'} & \longrightarrow & \begin{cases} h_{G \vee G'} \otimes 1 & \text{if } G' \not\leq G \\ 1 \otimes h_G & \text{otherwise.} \end{cases} \end{array}$$

are well-defined and give us an operadic structure on the family of generalized Feichtner–Yuzvinsky algebras.

6.3 Adding morphisms to $\mathfrak{LBS}$

One could also consider adding more morphisms of degree 0 in $\mathfrak{LBS}$. This is justified by the fact that the Feichtner–Yuzvinsky algebras have a lot more functoriality than what we have in $\mathfrak{LBS}$. More precisely let $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{L}', \mathcal{G}')$ be two built lattices and let $f : \mathcal{L} \longrightarrow \mathcal{L}'$ be a poset morphism which sends $\mathcal{G}$ to $\mathcal{G}'$, atoms of $\mathcal{L}$ to atoms of $\mathcal{L}'$ and which is compatible with the join on both sides, i.e.

$$f(G_1 \vee G_2) = f(G_1) \vee f(G_2)$$

for all G_1, G_2 in $\mathcal{L}$. With those hypotheses the map induced by

$$\begin{array}{ccc} \mathrm{FY}(\mathcal{L}, \mathcal{G}) & \xrightarrow{\mathrm{FY}(f)} & \mathrm{FY}(\mathcal{L}', \mathcal{G}') \\ h_G & \longrightarrow & h_{f(G)} \end{array}$$

is a well-defined map of algebras. This incentivizes us to formally add such morphisms in $\mathfrak{LBS}$. Some of those morphisms are very natural to add in their own right. For instance if $\mathcal{G} \subset \mathcal{G}'$ are two building sets of some lattice $\mathcal{L}$ then the identity of $\mathcal{L}$ satifies the above conditions. In the realizable case the corresponding map $\mathrm{FY}(f)$ is induced by the blow down

$$\overline{Y}_{\mathcal{L},\mathcal{G}'} \to \overline{Y}_{\mathcal{L},\mathcal{G}}.$$

If f is the inclusion of some interval $[\hat{0}, G] \hookrightarrow \mathcal{L}$ then f satisfies the above conditions when taking the induced building set on $[\hat{0}, G]$. For instance this includes the various inclusions $\Pi_n \simeq [\hat{0}, [\![1, n+1]\!] \setminus \{i\}|i] \hookrightarrow \Pi_{n+1}$. The corresponding morphisms

$$\mathrm{FY}^{\vee}(\Pi_{n+1}, \mathcal{G}_{\min}) \to \mathrm{FY}^{\vee}(\Pi_n, \mathcal{G}_{\min})$$

are induced by forgetting some marked point on the genus 0 curve.

Chapter 7

Additional reminders in combinatorics

In this section we explicit the relation between geometric lattices and matroids and we introduce the notion of supersolvable built lattices.

7.1 Geometric lattices and matroids

Let us describe in more details the correspondence between simple loopless matroids and geometric lattices. There are several equivalent definitions of matroids. We refer to [35] for more details.

Definition 7.1.1 (Matroids via independent subsets)**.** A *matroid* is a pair of a finite set E and a set $\mathcal{I}$ of subsets of E (the "independent" subsets) satisfying the axioms

- For any I in $\mathcal{I}$, every subset of I belongs to $\mathcal{I}$.
- For any I, J in $\mathcal{I}$, if $\#J > \#I$ there exists an element a in J and not in I such that $I \cup \{a\}$ is independent.

Definition 7.1.2 (Matroids via closure operator)**.** A matroid is a pair of a finite set E and an application (the "closure operator")

$$\sigma : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$$

satisfying the axioms

- For any $X \in \mathcal{P}(E)$ we have $X \subseteq \sigma(X)$.
- For any $X \subseteq Y \in \mathcal{P}(E)$ we have $\sigma(X) \subseteq \sigma(Y)$.

- For any $X \in \mathcal{P}(E)$ we have $\sigma(\sigma(X)) = \sigma(X)$.
- For any $X \in \mathcal{P}(E)$ and $a, b \in E$, if $a \in \sigma(X \cup \{b\}) \setminus \sigma(X)$ then $b \in \sigma(X \cup \{a\}) \setminus \sigma(X)$.

Definition 7.1.3 (Matroids via circuits)**.** A *matroid* is a pair of a finite set E and a set $\mathcal{C}$ of subsets of E (the "circuits") satisfying the axioms

- The empty set is not a circuit.
- If $C_1 \subseteq C_2 \in \mathcal{C}$ then $C_1 = C_2$.
- If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there exists a circuit $C \subseteq C_1 \cup C_2 \setminus \{e\}$.

One can replace the last axiom by a stronger version which we will use later in this document.

$$\text{If } C_1, C_2 \in \mathcal{C},\ e \in C_1 \cap C_2,\ f \in C_1 \setminus C_2, \\ \text{then there exists a circuit } C \subseteq C_1 \cup C_2 \setminus \{e\} \text{ containing } f. \quad (7.1)$$

One passes from the circuit definition to the closure definition by putting

$$\sigma(X) := X \cup \{x \mid \exists C \in \mathcal{C} \text{ with } C \subseteq X \cup \{x\} \text{ and } x \in C\}. \quad (7.2)$$

A matroid $(E, \mathcal{I})$ is said to be simple loopless if every subset of E of cardinal less than two is independent. A flat of a matroid $M = (E, \sigma)$ is a subset $F \subseteq E$ such that $\sigma(F)$ is equal to F. The set of flats of M denoted by $\mathcal{L}_M$ ordered by inclusion is a geometric lattice with meet given by the intersection. Conversely if $\mathcal{L}$ is a geometric lattice then the datum (E, σ) where E is the set of atoms of $\mathcal{L}$ and σ is the map defined by

$$\sigma(X) = \bigcap_{\substack{F \in \mathcal{L} \\ X \subset \mathrm{At}_{\leq}(F)}} \mathrm{At}_{\leq}(F)$$

is a simple loopless matroid. Those two constructions are inverse to each other on simple loopless matroids. In the sequel we will freely identify an element of some geometric lattice with the set of atoms below this element. For instance if G_1 and G_2 are two elements of some geometric lattice $\mathcal{L}$ then $G_1 \cup G_2$ will mean $\mathrm{At}_{\leq}(G_1) \cup \mathrm{At}_{\leq}(G_2)$. Finally, notice that by definition, for any subset $S \subseteq \mathcal{L}$ with $\mathcal{L}$ some geometric lattice we have

$$\bigvee S = \sigma(\bigcup_{X \in S} X) \quad (7.3)$$

where σ is the closure operator of the associated matroid.

7.2 Supersolvable built lattices

Recall that a lattice $\mathcal{L}$ is said to be distributive if for every triple $X, Y, Z \in \mathcal{L}$ we have the equality

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z).$$

An element X in a lattice $\mathcal{L}$ is said to be modular if for every pair Y, Z of elements of $\mathcal{L}$ with $Y \leq X$ we have the equality

$$X \wedge (Y \vee Z) = Y \vee (X \wedge Z).$$

The following definition is due to Stanley [30].

Definition 7.2.1 (Supersovable lattice (1))**.** A lattice is said to be *supersolvable* if there exists a maximal chain M of elements of $\mathcal{L}$ such that for every pair $X < Y$ of elements in $\mathcal{L}$ the sublattice generated by M, X and Y is distributive.

Stanley [30] proved that this is equivalent to a (seemingly) weaker assumption.

Definition 7.2.2 (Supersolvable lattice (2))**.** A lattice is said to be *supersolvable* if it admits a maximal chain of modular elements.

Fact 7.2.3. Supersolvability is a hereditary condition (meaning it is stable by taking intervals) because if G is some element in some supersolvable lattice $\mathcal{L}$ with maximal chain of modular elements

$$\hat{0} = M_1 < ... < M_n = \hat{1},$$

the maximal chains

$$M_1 \wedge G \leq ... \leq M_n \wedge G$$

and

$$M_1 \vee G \leq ... \leq M_n \vee G$$

are maximal chains (with possibly multiple occurencies) of modular elements of $[\hat{0}, G]$ and $[G, \hat{1}]$ respectively (see [30]).

We introduce the following variant for built lattices.

Definition 7.2.4 (Supersolvable built lattices)**.** A built lattice $(\mathcal{L}, \mathcal{G})$ is said to be *supersolvable* if it admits a maximal chain $\hat{0} = G_1 < ... < G_n = \hat{1}$ of modular elements in $\mathcal{G}$ such that for any element G in $\mathcal{G}$, the element $G_i \wedge G$ belongs to $\mathcal{G} \cup \{\hat{0}\}$ for all $i \leq n$.

By Fact 7.2.3, supersolvability for built lattices is a hereditary condition (meaning it is stable by taking intervals and induced building set).

Example 7.2.5. Let $\mathcal{B}_4$ be the boolean lattice of $\{1,2,3,4\}$. If we put

$$\mathcal{G} := \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,2,3\},\{1,2,3,4\},\{2,3,4\},\{2,3\}\}$$

then the built lattice $(\mathcal{B}_4,\mathcal{G})$ is supersolvable. Indeed the chain

$$\{1\} \subset \{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\}$$

is a maximal chain of modular elements in $\mathcal{G}$ (all the elements of $\mathcal{B}_4$ are modular), and we have

$$\{2,3,4\} \wedge \{1,2,3\} = \{2,3\} \in \mathcal{G}.$$

On the contrary if one puts $\mathcal{G}' := \mathcal{G} \setminus \{2,3\} \cup \{3,4\}$ then $(\mathcal{B}_4,\mathcal{G}')$ is not a supersolvable built lattice because we have

$$\{1,2,3\} \wedge \{2,3,4\} = \{2,3\} \notin \mathcal{G}.$$

and any maximal chain of elements in $\mathcal{G}$ must contain either $\{1,2,3\}$ or $\{2,3,4\}$.

One can immediately see that if $\mathcal{L}$ is a supersolvable lattice then $(\mathcal{L},\mathcal{G}_{\max})$ is a supersolvable built lattice. In Section 8 and Section 9 we will introduce other large classes of supersolvable built lattices.

Let us prove here a small general lemma which will be useful later on.

Lemma 7.2.6. *Let $\mathcal{L}$ be a geometric lattice, G a modular element of $\mathcal{L}$ and C a circuit in $\mathcal{L}$. At least one of the following propositions is true:*

- $C \subset G$.
- $C \cap G = \emptyset$.
- *There exists a circuit C' such that $C' \cap G$ is a singleton H and $C' \setminus \{H\}$ is included in $C \setminus (G \cap C)$.*

Proof. Assume the first two propositions are not true. Let us denote

$$I := C \cap G,$$
$$J := C \setminus I.$$

Let H be any element of I, which is not empty by assumption. By modularity of G we have

$$G \wedge (\sigma(I \setminus \{H\}) \vee \sigma(J)) = \sigma(I \setminus \{H\}) \vee (G \wedge \sigma(J)).$$

Since J is not empty I is independent and the element on the left has rank at least $\#I$. Since $\sigma(I \setminus \{H\})$ only has rank $\#I - 1$ we must have

$$G \wedge \sigma(J) \neq \hat{0}.$$

By Formula (7.2) this implies that we have the desired circuit C'. □

Let $(\mathcal{L}, \mathcal{G})$ be a supersolvable built lattice with some chosen maximal chain of modular elements $\omega = \{\hat{0} = G_1 < ... < G_n = \hat{1}\}$. For any G in $\mathcal{G}$ and any G' in $\mathrm{Ind}_{[G,\hat{1}]}(\mathcal{G})$ we denote by $\mathrm{d}_{\omega,G}(G')$ the coatom in the maximal chain of modular elements induced by ω on $[G, G']$ (see Fact 7.2.3). An element of the form $\mathrm{d}_{\omega,G}(G')$ will be called an *initial segment* of G' relative to G. In practice we will drop ω from the notation. If G is equal to $\hat{0}$ we also drop it from the notation. In the sequel whenever we introduce a supersolvable built lattice we implicitly choose a particular maximal chain of modular elements of this built lattice.

The rest of this document will be devoted to the proof of the following theorem.

Theorem 7.2.7. *Let $(\mathcal{L}, \mathcal{G})$ be a supersolvable built lattice. The algebra $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ admits a quadratic Gröbner basis and is therefore Koszul.*

Let us include here a small step toward the above result which we will need later on.

Proposition 7.2.8. *Let $(\mathcal{L}, \mathcal{G})$ be a supersolvable built lattice. The Feichtner–Yuzvinsky ring $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ is quadratic.*

Proof. It is enough to prove that the nested set complex of $(\mathcal{L}, \mathcal{G})$ is flag, meaning that for any anti-chain $G_1, ..., G_n$ in $\mathcal{G}$ with $n \geq 2$, if $G_1 \vee ... \vee G_n$ belongs to $\mathcal{G}$ then there exists $i \neq j \leq n$ such that $G_i \vee G_j$ belongs to $\mathcal{G}$. Assume the contrary is true and there exist an anti-chain $G_1, ..., G_n$ such that we have

$$G_1 \vee ... \vee G_n \in \mathcal{G}$$

and

$$G_i \vee G_j \notin \mathcal{G} \ \forall i \neq j \leq n.$$

By restricting to a smaller interval we can assume $G_1 \vee ... \vee G_n = \hat{1}$. By Fact 7.2.3, the element $\mathrm{d}(\hat{1}) \wedge (G_i \vee G_j)$ is either equal to $G_i \vee G_j$ or is covered by $G_i \vee G_j$. In the latter case, using the building set isomorphism (1.1) we see that either G_i or G_j is below $\mathrm{d}(\hat{1})$. As a consequence we see that there is at most one integer $i \leq n$ such that G_i is not below $\mathrm{d}(\hat{1})$ (in fact there is exactly one such i). By reordering let us assume that this integer is n. By atomicity there exist an element $X < G_n$ such that we have $G_1 \vee ... \vee G_{n-1} \vee X = \mathrm{d}(\hat{1})$. If $G'_1, ...G'_k$ are the factors of X in $\mathcal{G}$, the anti-chain $G_1, ..., G_k, G'_1, ..., G'_k$ is a new counter-example to the flagness of the nested set complex of $(\mathcal{L}, \mathcal{G})$. We get a contradiction by reiterating this process. □

Chapter 8

Quadratic Gröbner bases for Feichtner–Yuzvinsky algebras of supersolvable built lattices

In the first subsection we define an order on the generators of the Feichtner–Yuzvinsky algebras of supersolvable lattices and we compute the normal monomials of weight 2 associated to this order. In the next subsection we define a bijection between the algebraic normal monomials associated to the latter order and the operadic normal monomials introduced in Proposition 4.5.4. The construction of this bijection will be done by induction and using the operadic structure. By a dimension argument this bijection will show that the algebraic normal monomials form a basis of the Feichtner–Yuzvinksky algebras, which will show that the set of weight 2 relations forms a Gröbner basis of the Feichtner–Yuzvinsky algebras.

8.1 The order on generators and the normal monomials of weight 2

Definition/Proposition 8.1.1. Let $(\mathcal{L}, \mathcal{G})$ be a supersolvable built lattice. The transitive closure of the relations

$$\mathrm{d}^k(G) \dashv G \dashv G' \tag{8.1}$$

for all k and all $G, G' \in \mathcal{G}$ with $G' \leq G$ and G' not an initial segment of G, is anti-symmetric and thus defines a partial order.

Proof. One can define an explicit total order containing the relations (8.1) as follow. Let $\lhd$ be a total order on the atoms of $\mathcal{L}$ extending the relations $H \dashv H'$ for all pairs of atoms H, H' such that there exists an integer k satisfying

$$H \leq \mathrm{d}^k(G) \text{ and } H \nleq \mathrm{d}^k(G).$$

For any element G in $\mathcal{G}$ let us denote by $\mathsf{w}(G)$ the word with letters $\mathrm{At}_{\leq}(G)$ written in increasing order. We define a total order on $\mathcal{G}$, also denoted $\lhd$, by putting

$$G \lhd G' \Leftrightarrow \mathsf{w}(G) \text{ is less than } \mathsf{w}(G') \text{ for the lexicographic order.}$$

This order contains relations (8.1). □

In the sequel whenever we introduce a supersolvable built lattice we implicitly choose an associated total order on $\mathcal{G}$ as in the above proof.

Proposition 8.1.2. *Let $(\mathcal{L}, \mathcal{G})$ be a supersolvable built lattice and let $\alpha = h_{G_1} h_{G_2}$ be a monomial in $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ with $G_1 \lhd G_2$. Let us denote by G the join $G_1 \vee G_2$. The monomial α is normal if and only if one of the three following conditions is verified.*

- *The element G does not belong to $\mathcal{G}$.*
- *The element G belongs to $\mathcal{G}$ and G_1 is an initial segment of $G_2 = G$.*
- *The element G belongs to $\mathcal{G}$, $G_2 \ngeq G_1$, G_2 is not covered by G and we have $G_1 = \mathrm{d}^k(G)$ where k is the maximal integer satisfying*

$$\mathrm{d}^k(G) \vee G_2 = G.$$

Proof. We have an obvious bijection between the monomials described in the above proposition and the normal monomials given by 1.2.3, sending $h_{\mathrm{d}^i{}_{G'}(G)} h_{G'}$ to $x_{G'} x_G$ if G' is not covered by G, sending $h_{\mathrm{d}(G)} h_G$ to x_G^2 and sending $h_{G_1} h_{G_2}$ to $x_{G_1} x_{G_2}$ when $\{G_1, G_2\}$ is a nested set. By a dimension argument it is enough to prove that the normal monomials of weight 2 with respect to $\lhd$ are included in the monomials described in the proposition.

Assume $G = G_1 \vee G_2$ belongs to $\mathcal{G}$. If $G \lhd G_1$ and $G \lhd G_2$ then $h_{G_1} h_{G_2}$ is the leading term of the relation

$$(h_G - h_{G_1})(h_G - h_{G_2}).$$

If $G_1 \unlhd G \lhd G_2$ and G_2 is covered by G then $h_{G_1} h_{G_2}$ is the leading term of the relation

$$(h_G - h_{G_1})(h_G - h_{G_2}) - h_G(h_G - h_{G_2}).$$

Finally, if $G_1 \unlhd G \lhd G_2$ then G_1 is some initial segment $\mathrm{d}^k(G)$. If k is not the maximal integer such that we have

$$\mathrm{d}^k(G) \vee G_2 = G,$$

we see that $h_{G_1} h_{G_2}$ is the leading term of the relation

$$(h_G - h_{G_1})(h_G - h_{G_2}) - (h_G - h_{\mathrm{d}^{k+1}(G)})(h_G - h_{G_2}).$$

□

Notice that the normal monomials do not really depend on $\lhd$ but only on the chosen maximal chain of modular elements. We next come to an important lemma which highlights a first connection between our normal monomials and supersolvability.

Lemma 8.1.3. *Let $(\mathcal{L}, \mathcal{G})$ be a supersolvable built lattice. Let G_1 and G_2 be two non-comparable elements of $\mathcal{G}$ such that we have $G_1 \lhd G_2$, $G_1 \vee G_2 \in \mathcal{G}$, G_2 is not covered by $G_1 \vee G_2$ and G_1 is an initial segment of $G_1 \vee G_2$. Then $h_{G_1} h_{G_2}$ is a normal monomial if and only if $G_1 \wedge G_2 \leq d(G_1)$.*

The forward statement is always true but for the converse we need the supersolvability hypothesis. For instance consider $\mathcal{L}$ the graphical lattice associated to a 5-cycle and number the edges (i.e. the atoms) from 1 to 5. If we pick $G_1 = \{1, 2, 3\}$ and $G_2 = \{4, 5\}$ in the maximal building set, then we have $G_1 \wedge G_2 = \hat{0}$ but $h_{G_1} h_{G_2}$ is not normal because we have $\mathrm{d}(G_1) \vee G_2 = G_1 \vee G_2 = \hat{1}$.

Proof. With the hypothesis on G_1 and G_2 we have

$$\begin{aligned} h_{G_1} h_{G_2} \text{ is normal } &\Leftrightarrow \mathrm{d}(G_1) \vee G_2 < G_1 \vee G_2 \\ &\Leftrightarrow G_1 \nleq \mathrm{d}(G_1) \vee G_2 \\ &\Leftrightarrow G_1 \wedge (\mathrm{d}(G_1) \vee G_2) < G_1 \\ &\Leftrightarrow \mathrm{d}(G_1) \vee (G_1 \wedge G_2) < G_1 \\ &\Leftrightarrow G_1 \wedge G_2 \leq \mathrm{d}(G_1). \end{aligned}$$

The fourth equivalence comes from the fact that by supersolvability G_1 is modular in the interval $[\hat{0}, G_1 \vee G_2]$. □

8.2 A bijection between algebraic normal monomials and operadic normal monomials

Let $(\mathcal{L}, \mathcal{G})$ be a supersolvable built lattice and let $\lhd$ be a total order on the atoms of $\mathcal{L}$ obtained as in the proof of Definition/Proposition 8.1.1. Let us denote $\mathsf{ANM}(\mathcal{L}, \mathcal{G}, \lhd)$ the algebraic normal monomials with respect to the order $\lhd$ and the relations of weight 2, i.e. is the set of monomials in $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ which are not divisible by the leading term (with respect to $\lhd$) of some relation of weight 2. Let us denote by $\mathsf{ONM}(\mathcal{L}, \mathcal{G}, \lhd)$ the set of operadic normal monomials with respect to the order on atoms $\lhd$ (see Proposition 4.5.4), viewed as nested sets. Notice that in the supersolvable case the nested set $\mathcal{S}(\omega^{k_G}_{\tau_{\mathcal{S}}(G), G})$ is simply some truncation of the maximal chain $\{G > \mathrm{d}_{\tau_{\mathcal{S}}(G)}(G) > \mathrm{d}^2_{\tau_{\mathcal{S}}(G)}(G) > \ldots > \tau_{\mathcal{S}}(G)\}$. This means that the operadic normal monomials do not depend on the particular choice of $\lhd$ but only on the choice of the maximal chain of modular elements of $(\mathcal{L}, \mathcal{G})$. Such choice being implicit we will drop $\lhd$ from all notations.

8.2.1 From operadic normal monomials to algebraic normal monomials

We first define maps

$$\mathsf{ONM}(\mathcal{L},\mathcal{G}) \xrightarrow{\Phi_{\mathcal{L},\mathcal{G}}} \mathsf{ANM}(\mathcal{L},\mathcal{G}).$$

by induction on the rank of $\mathcal{L}$. Let $(\mathcal{L},\mathcal{G})$ be a supersolvable built lattice. For any element $G \neq \hat{1} \in \mathcal{G}$ and any algebraic monomial $\alpha = \prod_{G' \in I} h_{G'}$ in $\mathrm{FY}([G,\hat{1}],\mathrm{Ind}(\mathcal{G}))$ we define the algebraic monomial in $\mathrm{FY}(\mathcal{L},\mathcal{G})$:

$$\mathsf{Supp}_G(\alpha) = \prod_{\substack{G' \in I \\ G' \in \mathcal{G}}} h_{\mathrm{d}^{i_{G',G}}(G')} \prod_{\substack{G' \in I \\ G' \notin \mathcal{G}}} h_{G'^{\perp}}$$

where for any $G' \in \mathcal{G}$, $i_{G',G}$ is the biggest integer such that we have $\mathrm{d}^{i_{G',G}}(G') \vee G = G'$, and for any $G' \notin \mathcal{G}$, the element $G'^{\perp}$ is the factor of G' in $\mathcal{G}$ different from G. Finally, for any operadic normal monomial $\mathcal{S}$ in some supersolvable built lattice $(\mathcal{L},\mathcal{G},\lhd)$ with G the maximal element of $\mathcal{S}'$ for the order $\lhd$ we define by induction (on both the cardinal of $\mathcal{S}$ and the rank of $\mathcal{L}$) the map

$$\Phi_{(\mathcal{L},\mathcal{G})}(\mathcal{S}) := \mathsf{Supp}_G(\Phi_{[\mathcal{G},\hat{1}],\mathrm{Ind}(\mathcal{G})}(G \vee \mathcal{S}_{\nleq G}))\Phi_{[\hat{0},G],\mathrm{Ind}(\mathcal{G})}(\mathcal{S}_{\leq G}).$$

initialized on empty nested sets by

$$\Phi(\emptyset) = h_{\hat{1}} h_{\mathrm{d}(\hat{1})} ... h_{\mathrm{d}^{\mathrm{rk}\mathcal{L}-2}(\hat{1})}.$$

One can check that $G \vee \mathcal{S}_{\nleq G}$ is an operadic monomial so our map is well-defined. This map sends an operadic normal monomial to some algebraic monomial, which will turn out to be normal but we will not need this fact.

8.2.2 From algebraic normal monomials to operadic normal monomials

We are concerned with finding an inverse for Φ. Let us define a candidate

$$\mathsf{ANM}(\mathcal{L},\mathcal{G}) \xrightarrow{\Psi_{\mathcal{L},\mathcal{G}}} \mathsf{ONM}(\mathcal{L},\mathcal{G})$$

by induction on the rank of $\mathcal{L}$ and the weight of the monomial. We will drop the built lattice from the notation if it can be deduced from the context. We initialize with

$$\Psi(1) = \omega_{\hat{0},\hat{1}} = \{\hat{1} > \mathrm{d}(\hat{1}) > ... > \hat{0}\}.$$

Let $\alpha = \prod_i h_{G_i}$ be some algebraic normal monomial in $\mathrm{FY}(\mathcal{L},\mathcal{G})$ and let us denote by G the maximum of the $G_i's$ with respect to $\lhd$. If $G = \hat{1}$ then by Proposition 8.1.2 we see that all the G_i's except G are below $\mathrm{d}(\hat{1})$ so we put

$$\Psi_{\mathcal{L},\mathcal{G}}(\alpha) = \Psi_{[\hat{0},\mathrm{d}(\hat{1})],\mathrm{Ind}(\mathcal{G})}(\alpha/h_{\hat{1}}),$$

where $\Psi_{[\hat{0},\mathrm{d}(\hat{1})],\mathrm{Ind}(\mathcal{G})}(\alpha/h_{\hat{1}})$ is viewed as a normal monomial in $(\mathcal{L},\mathcal{G})$. If G is different from $\hat{1}$, let us denote respectively

$$\alpha_{\leq G} = \prod_{G_i \leq G} h_{G_i},$$
$$\alpha_{\nleq G} = \prod_{G_i \nleq G} h_{G_i}.$$

We put

$$\Psi_{\mathcal{L},\mathcal{G}}(\alpha) := \{G\} \circ (\Psi_{[\hat{0},G],\mathrm{Ind}(\mathcal{G})}(\alpha_{\leq G}), \Psi_{[G,\hat{1}],\mathrm{Ind}(\mathcal{G})}(G \vee \alpha_{\nleq G})), \tag{8.2}$$

with $G \vee \alpha_{\nleq G}$ a notation for $\prod_{G_i \nleq G} h_{G \vee G_i}$. One can check that this defines an operadic normal monomial.

As a side remark let us remind the reader that we have

$$\mathrm{FY}(\{G\})(\alpha) := \alpha_{\leq G} \otimes G \vee \alpha_{\nleq G},$$

so we are in fact using again the (co)operadic structure on the Feichtner–Yuzvinsky rings.

We must prove that the monomial $G \vee \alpha_{\nleq G}$ is normal in $\mathrm{FY}([G,\hat{1}],\mathrm{Ind}(\mathcal{G}))$. This is implied by the following lemma.

Lemma 8.2.1. *Let $(\mathcal{L},\mathcal{G})$ be a supersolvable built lattice and let $G_1 \lhd G_2 \lhd G_3$ be elements in $\mathcal{G}$. If $h_{G_1}h_{G_2}$, $h_{G_1}h_{G_3}$ and $h_{G_2}h_{G_3}$ are normal then $h_{G_1 \vee G_3}h_{G_2 \vee G_3}$ is normal in $\mathrm{FY}([G_3,\hat{1}],\mathrm{Ind}(\mathcal{G}))$.*

This is the technical core of the article. The statement is not true in general without the supersolvability condition, as shown by the following example. Let $\mathcal{L}$ be the graphical lattice associated to a 6-cycle with edges $\{1,...,6\}$. Consider the elements $G_1 = \{1,2\}, G_2 = \{3,4\}$ and $G_3 = \{5,6\}$. One can quickly check that the monomials $h_{G_1}h_{G_2}$, $h_{G_1}h_{G_3}$ and $h_{G_2}h_{G_3}$ are normal in $\mathrm{FY}(\mathcal{L},\mathcal{G}_{\max})$. However, $h_{G_1 \vee G_3}h_{G_2 \vee G_3}$ is not normal in $\mathrm{FY}([G_3,\hat{1}],\mathcal{G}_{\max})$ for two reasons: $G_2 \vee G_3$ is covered by $G_1 \vee G_2 \vee G_3 = \hat{1}$ and we have

$$\{1\} \vee G_2 \vee G_3 = G_1 \vee G_2 \vee G_3.$$

If a built lattice has a small rank it can happen that it satisfies Lemma 8.2.1 without being supersolvable (see Subsection 10.1).

Proof. The statement is obvious when two of the G_i's are comparable so we can assume that the elements G_1, G_2, G_3 are not comparable. We make a disjunction on whether $G_i \vee G_j$ belongs to $\mathcal{G}$ for $i, j \leq 3$.

Case 1. $G_i \vee G_j \notin \mathcal{G}$ for all $i \neq j \leq 3$.

In this case we have $(G_1 \vee G_3) \vee (G_2 \vee G_3) \notin \mathrm{Ind}(\mathcal{G})$. Indeed, by the proof of Proposition 7.2.8 the element $G_1 \vee G_2 \vee G_3$ does not belong to $\mathcal{G}$, and if $G_1 \vee G_2 \vee G_3$ is equal to $G_1 \vee G$ with $G \in \mathcal{G}$ and G_1, G nested we immediately get $G_1 \vee G_2 = G \in \mathcal{G}$ contradicting the initial hypothesis.

Case 2. $G_1 \vee G_2 \notin \mathcal{G}, G_1 \vee G_3 \notin \mathcal{G}, G_2 \vee G_3 \in \mathcal{G}$.

Let us show that $\{G_1 \vee G_3, G_2 \vee G_3\}$ is a nested anti-chain in $\mathrm{Ind}(\mathcal{G})$ as in the previous case. By contradiction assume that $G_1 \vee G_2 \vee G_3$ belongs to $\mathcal{G}$. By restriction we can assume $G_1 \vee G_2 \vee G_3 = \hat{1}$. Since $h_{G_2} h_{G_3}$ is normal and $G_2 \vee G_3 \in \mathcal{G}$ there exists some integer k such that we have $\mathrm{d}^k(\hat{1}) \wedge (G_2 \vee G_3) = G_2$. We have

$$\begin{aligned}
\mathrm{d}^k(\hat{1}) &= \mathrm{d}^k(\hat{1}) \wedge (G_2 \vee G_1 \vee G_3) \\
&= G_2 \vee (\mathrm{d}^k(\hat{1}) \wedge (G_1 \vee G_3)) \\
&= G_2 \vee (\mathrm{d}^k(\hat{1}) \wedge G_1) \wedge (\mathrm{d}^k(\hat{1}) \wedge G_3) \\
&\leq G_2 \vee G_1 \vee G_2 \\
&= G_1 \vee G_2.
\end{aligned}$$

By nested-ness this implies $\mathrm{d}^k(\hat{1}) = G_2$ which contradicts $G_1 \lhd G_2$. If $G_1 \vee G_2 \vee G_3$ belongs to $\mathrm{Ind}(\mathcal{G})$ but not to $\mathcal{G}$ we immediately get a contradiction as in the previous case.

Case 3. $G_1 \vee G_2 \notin \mathcal{G}, G_1 \vee G_3 \in \mathcal{G}, G_2 \vee G_3 \notin \mathcal{G}$.

This is similar to the previous case.

Case 4. $G_1 \vee G_2 \notin \mathcal{G}, G_1 \vee G_3 \in \mathcal{G}, G_2 \vee G_3 \in \mathcal{G}$.

Once again let us show that $\{G_1 \vee G_3, G_2 \vee G_3\}$ is a nested anti-chain in $\mathrm{Ind}(\mathcal{G})$. By contradiction assume that $G_1 \vee G_2 \vee G_3$ belongs to $\mathcal{G}$. By restriction we can assume that we have $G_1 \vee G_2 \vee G_3 = \hat{1}$. By assumption there exists an integer k_1 such that we have $\mathrm{d}^{k_1}(\hat{1}) \wedge (G_1 \vee G_3) = G_1$ and an integer k_2 such that we have $\mathrm{d}^{k_2}(\hat{1}) \wedge (G_2 \vee G_3) = G_2$. Let us denote $k := \min(k_1, k_2)$. Let us assume $k_1 \geq k_2$, the other case being symmetric. By modularity of $\mathrm{d}^k(\hat{1})$ and definition of k we have

$$\begin{aligned}
\mathrm{d}^k(\hat{1}) &= \mathrm{d}^k(\hat{1}) \wedge (G_1 \vee G_2 \vee G_3) \\
&= G_1 \vee (\mathrm{d}^k(\hat{1}) \wedge (G_2 \vee G_3)) \\
&= G_1 \vee G_2.
\end{aligned}$$

This contradicts the fact that $G_1 \vee G_2$ does not belong to $\mathcal{G}$. If $G_1 \vee G_2 \vee G_3$ belongs to $\mathrm{Ind}(\mathcal{G})$ and not to $\mathcal{G}$ we immediately get a contradiction as in the previous cases.

Case 5. $G_1 \vee G_2 \in \mathcal{G}$, $G_1 \vee G_3 \notin \mathcal{G}$ and $G_2 \vee G_3 \notin \mathcal{G}$.

We can either have $G_1 \vee G_2 \vee G_3 \notin \mathcal{G}$ or the contrary. In the first case the building set isomorphism

$$[\hat{0}, G_1 \vee G_2 \vee G_3] \simeq [\hat{0}, G_1 \vee G_2] \times [\hat{0}, G_3]$$

immediately gives the result. In the second case we can assume $G_1 \vee G_2 \vee G_3 = \hat{1}$. Let us prove that $G_1 \vee G_3$ is an initial segment of $G_1 \vee G_2 \vee G_3$ in $[G_3, \hat{1}]$. By assumption there exists an integer k such that we have $\mathrm{d}^k(\hat{1}) \wedge (G_1 \vee G_2) = G_1$. We will prove the equality

$$\mathrm{d}^k(\hat{1}) \vee G_3 = G_1 \vee G_3.$$

We have

$$\begin{aligned}
\mathrm{d}^k(\hat{1}) &= \mathrm{d}^k(\hat{1}) \wedge (G_1 \vee G_2 \vee G_3) \\
&= G_1 \vee (\mathrm{d}^k(\hat{1}) \wedge (G_2 \vee G_3)) \text{ (by modularity of } \mathrm{d}^k(\hat{1})) \\
&\leq G_1 \vee (\mathrm{d}^k(\hat{1}) \wedge G_2) \vee (\mathrm{d}^k(\hat{1}) \wedge G_3) \text{ (by nested-ness of} \{G_2, G_3\}) \\
&\leq G_1 \vee G_1 \vee G_3 \text{ (by definition of } k) \\
&= G_1 \vee G_3.
\end{aligned}$$

The other inequality is obvious. Let us now show the inequality

$$\mathrm{d}(G_1) \vee G_2 \vee G_3 < G_1 \vee G_2 \vee G_3. \tag{8.3}$$

According to Lemma 8.1.3 it is enough to prove the inequality

$$(G_1 \vee G_3) \wedge (G_2 \vee G_3) \leq \mathrm{d}(G_1) \vee G_3. \tag{8.4}$$

An atom H below $G_1 \vee G_3$ is either below G_1 or G_3 by nested-ness, and similarly for $G_2 \vee G_3$. As a consequence, an atom below $G_1 \vee G_3$ and below $G_2 \vee G_3$ is either below G_3 or is below $G_1 \wedge G_2$, which is below $\mathrm{d}(G_1)$ by Lemma 8.1.3.

One must also prove that $G_2 \vee G_3$ is not covered by $G_1 \vee G_2 \vee G_3$. By inequality (8.3) if $G_2 \vee G_3$ is covered by $G_1 \vee G_2 \vee G_3$ then we have $\mathrm{d}(G_1) \vee G_3 \leq G_2 \vee G_3$. In this case by nested-ness $\mathrm{d}(G_1)$ is either below G_2 or G_3. In the first case we immediately obtain that G_2 is covered by $G_1 \vee G_2$ which is a contradiction. In the second case we get $G_1 \wedge G_3 \neq \hat{0}$ which contradicts the fact that $G_1 \vee G_3$ does not belong to $\mathcal{G}$.

Case 6. $G_1 \vee G_2 \in \mathcal{G}$, $G_2 \vee G_3 \in \mathcal{G}$.

In this case we necessarily have $G_1 \vee G_2 \vee G_3 \in \mathcal{G}$. Let us prove that $G_1 \vee G_3$ is an initial segment of $G_1 \vee G_2 \vee G_3$ in $[G_3, G_1 \vee G_2 \vee G_3]$. It is enough to prove that G_1 is an initial segment of $G_1 \vee G_2 \vee G_3$. By restriction we can assume $G_1 \vee G_2 \vee G_3 = \hat{1}$. Since G_1 is an initial segment of $G_1 \vee G_2$ there exists an integer k_1 such that we have $\mathrm{d}^{k_1}(\hat{1}) \wedge (G_1 \vee G_2) = G_1$. Since G_2 is an initial segment of $G_2 \vee G_3$ there exists an integer k_2 such that we have $\mathrm{d}^{k_2}(\hat{1}) \wedge (G_2 \vee G_3) = G_2$. The integer k_1 is greater or equal than k_2 because the opposite inequality would imply $G_2 \leq G_1$. Let us prove the equality

$$\mathrm{d}^{k_1}(\hat{1}) = G_1.$$

We have

$$\begin{aligned}
\mathrm{d}^{k_1}(\hat{1}) &= \mathrm{d}^{k_1}(\hat{1}) \wedge (G_1 \vee G_2 \vee G_3) \\
&= G_1 \vee (\mathrm{d}^{k_1}(\hat{1}) \wedge (G_2 \vee G_3)) \text{ (by modularity of } \mathrm{d}^{k_1}(\hat{1})) \\
&\leq G_1 \vee (\mathrm{d}^{k_1}(\hat{1}) \wedge G_2) \text{ (by } k_1 \geq k_2) \\
&\leq G_1 \vee G_1 \text{ (by definition of } k_1) \\
&= G_1.
\end{aligned}$$

The opposite inequality is obvious. Let us now prove inequality 8.4. By supersolvability the lattice generated by $G_1 = \mathrm{d}^{k_1}(\hat{1}), G_3$ and $G_2 \vee G_3$ is distributive. This implies

$$\begin{aligned}
(G_1 \vee G_3) \wedge (G_2 \vee G_3) &= (G_1 \wedge (G_2 \vee G_3)) \vee (G_3 \wedge (G_2 \vee G_3)) \\
&= (G_1 \wedge (G_2 \vee G_3)) \vee G_3 \\
&\leq (G_1 \wedge G_2) \vee G_3 \text{ (by } k_1 \geq k_2) \\
&\leq \mathrm{d}(G_1) \vee G_3.
\end{aligned}$$

As in the other cases one can check that $G_2 \vee G_3$ is not covered by $G_1 \vee G_2 \vee G_3$.

Case 7. $G_1 \vee G_2 \in \mathcal{G}$, $G_1 \vee G_3 \in \mathcal{G}$, $G_2 \vee G_3 \notin \mathcal{G}$.

In this case we necessarily have $G_1 \vee G_2 \vee G_3 \in \mathcal{G}$. As always by restriction we can assume $G_1 \vee G_2 \vee G_3 = \hat{1}$. By assumption there exists an integer k_2 such that we have $\mathrm{d}^{k_2}(\hat{1}) \wedge (G_1 \vee G_2) = G_1$ and an integer k_3 such that we have $\mathrm{d}^{k_3}(\hat{1}) \wedge (G_1 \vee G_3) = G_1$. Let us denote $k := \max(k_2, k_3)$. We will prove the equality

$$\mathrm{d}^{k}(\hat{1}) = G_1.$$

We have

$$\begin{aligned}
\mathrm{d}^k(\hat{1}) &= \mathrm{d}^k(\hat{1}) \wedge (G_1 \vee G_2 \vee G_3) \\
&= G_1 \vee (\mathrm{d}^k(\hat{1}) \wedge (G_2 \vee G_3)) \text{ (by modularity of } \mathrm{d}^k(\hat{1})) \\
&= G_1 \vee (\mathrm{d}^k(\hat{1}) \wedge G_2) \vee (\mathrm{d}^k(\hat{1}) \wedge G_3) \text{ (by nested-ness of } \{G_2, G_3\}) \\
&\leq G_1 \vee G_1 \text{ (by definition of } k) \\
&= G_1.
\end{aligned}$$

The opposite inequality is obvious. This implies that $G_1 \vee G_3$ is an initial segment of $G_1 \vee G_2 \vee G_3$ in $[G_3, \hat{1}]$. Let us now prove inequality 8.4. By supersolvability the lattice generated by $G_1 = \mathrm{d}^{k_1}(\hat{1}), G_3$ and $G_2 \vee G_3$ is distributive. This implies

$$\begin{aligned}
(G_1 \vee G_3) \wedge (G_2 \vee G_3) &= (G_1 \wedge (G_2 \vee G_3)) \vee (G_3 \wedge (G_2 \vee G_3)) \\
&= (G_1 \wedge (G_2 \vee G_3)) \vee G_3 \\
&= (G_1 \wedge G_2) \vee (G_1 \wedge G_3) \vee G_3 \text{ (by nested-ness of } \{G_2, G_3\}) \\
&\leq \mathrm{d}(G_1) \vee G_3 \text{ (by Lemma 8.1.3).}
\end{aligned}$$

As in the other cases one can check that $G_2 \vee G_3$ is not covered by $G_1 \vee G_2 \vee G_3$.

□

8.3 Proof of Theorem 7.2.7

In this subsection we give the proof of the main theorem of the second part of this manuscript and some immediate corollaries.

Theorem 8.3.1. *Let $(\mathcal{L}, \mathcal{G})$ be a supersolvable built lattice. The algebra $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ admits a quadratic Gröbner basis and is therefore Koszul.*

Proof. By dimension it is enough to prove that the map Φ is a left inverse of Ψ, which we will do by induction. The base cases are obvious.

Let $\alpha = \prod_{G' \in I} h_{G'}$ be a normal algebraic monomial with maximal element G with respect to $\lhd$. If $G = \hat{1}$ then every element $G' \in I$ is of the form $\mathrm{d}^k(\hat{1})$ for some k. In this case we can explicitly compute

$$\Psi(\prod_{i \leq n} h_{\mathrm{d}^{k_i}(\hat{1})}) = \{\mathrm{d}^k(\hat{1}) \,|\, k+1 \notin \{k_i, i \leq n\}\},$$

and

$$\Phi(\{\mathrm{d}^{k_i}(\hat{1}),\, i \leq n\}) = \prod_{k-1 \notin \{k_i, i \leq n\}} h_{\mathrm{d}^k(\hat{1})},$$

which proves reciprocity. If $G \neq \hat{1}$ by induction one can prove that the maximal element (for $\lhd$) of $\Psi(\alpha)$ is G. We then have by definition

$$\begin{aligned}\Phi \circ \Psi(\alpha) &= \Phi(\{G\} \circ (\Psi(G \vee \alpha_{\not\leq G}), \Psi(\alpha_{\leq G}))) \\ &= \mathsf{Supp}_G(\Phi(\Psi(G \vee \alpha)))\Phi(\Psi(\alpha_{\leq G})).\end{aligned}$$

However, by normality of α and maximality of G we get $\mathsf{Supp}_G(G \vee \alpha) = \alpha_{\not\leq G}$ which concludes the proof by induction. □

When restricting our attention to the maximal building set we get the immediate corollary.

Corollary 8.3.2. *Let $\mathcal{L}$ be a supersolvable lattice. The combinatorial Chow ring $\mathrm{FY}(\mathcal{L}, \mathcal{G}_{\max})$ admits a quadratic Gröbner basis and is therefore Koszul.*

In the next subsection we shall see that this is also true for the minimal building set. As mentioned in the introduction this strengthens the result of Mastroeni and McCullough [26] proving Koszulness of the combinatorial Chow rings, in the particular case of supersolvable lattices.

8.4 Minimal building sets of supersolvable lattices

A lattice is said to be irreducible if it is not a product of proper subposets. The minimal building set $\mathcal{G}_{\min}$ of a lattice $\mathcal{L}$ is the set of elements G of $\mathcal{L}$ such that $[\hat{0}, G]$ is irreducible. We have the key proposition.

Proposition 8.4.1. *Let $\mathcal{L}$ be an irreducible supersolvable lattice. The built lattice $(\mathcal{L}, \mathcal{G}_{\min})$ is supersolvable.*

Proof. It is enough to prove that if $\mathcal{L}$ is a supersolvable irreducible lattice then $\mathrm{d}(\hat{1})$ is irreducible. In fact here $\mathrm{d}(\hat{1})$ can be any modular coatom (it does not need to be part of a maximal chain of modular elements). We will prove the contraposition of this statement. Assume that $\mathrm{d}(\hat{1})$ decomposes as a product $[\hat{0}, G_1] \times ... \times [\hat{0}, G_n]$. Denote by R the set of atoms which are not below $\mathrm{d}(\hat{1})$. We have the following lemma.

Lemma 8.4.2. *There exists an integer $i \leq n$ such that we have the inclusion $(\bigvee R) \setminus R \subset G_i$.*

Proof. Recall from Subsection 7.1 that we have

$$\bigvee R = \sigma(R) = R \cup \{i \in \mathrm{At}(\mathcal{L}) \mid \exists C \text{ circuit s.t. } i \in C \subset R \cup \{i\}\}.$$

Let us prove that there exists some integer i such that for any atom $H \notin R$ and any circuit $C \subset R \cup \{H\}$ containing H we have $H \leq G_i$, by induction on the cardinal of the circuits. The base case is when the circuits have length 3. Let H_1, H_2 be two different atoms in R. Since $\mathrm{d}(\hat{1})$ is a coatom we have $H_2 \leq \hat{1} = \mathrm{d}(\hat{1}) \vee H_1$ so there exists a circuit C containing H_1 and H_2 and such that we have $C \setminus \{H_1, H_2\} \subset \mathrm{d}(\hat{1})$. Since $\mathrm{d}(\hat{1})$ is modular by Lemma 7.2.6 there exists a circuit C' containing some element H' in $\mathrm{d}(\hat{1})$ and such that $C' \setminus \{H'\}$ is equal to $\{H_1, H_2\}$. Such a circuit is in fact unique, the atom H' being necessarily equal to $\mathrm{d}(\hat{1}) \wedge (H_1 \vee H_2)$. Consider now three atoms H_1, H_2, H_3 in R. The element $\mathrm{d}(\hat{1}) \wedge (H_1 \vee H_2 \vee H_3)$ has rank at most 2 and it contains the three atoms $\mathrm{d}(\hat{1}) \wedge (H_i \vee H_j)$ for $i \neq j \leq 3$. If two of those atoms are equal, say

$$\mathrm{d}(\hat{1}) \wedge (H_1 \vee H_2) = \mathrm{d}(\hat{1}) \wedge (H_2 \vee H_3) =: H,$$

then by sub-modularity of $\mathcal{L}$ we have

$$\rho(H_1 \vee H_3 \vee H) \leq \rho(H_1 \vee H) + \rho(H_3 \vee H) - \rho((H_1 \vee H) \wedge (H_3 \vee H)) \leq 2 + 2 - 2 = 2$$

(since we have $H \vee H_2 \leq (H_1 \vee H) \wedge (H_3 \vee H)$). This implies $H \leq H_1 \vee H_3$ and therefore $\mathrm{d}(\hat{1}) \wedge (H_1 \vee H_3)$ is also equal to H. If the three atoms are different then they must form a circuit, and thus they must all belong to some same G_i. This concludes the initialization. Let us now assume that all the atoms $\mathrm{d}(\hat{1}) \wedge (H_1 \vee H_2)$ are below G_1 for instance.

Let C be a circuit of arbitrary length, containing a unique element H not in R. Let H_1, H_2 be two atoms in C different from H. By the initialization part there exists a circuit $\{H_1, H_2, H'\}$ with H' in G_1. If H' is equal to H then we have $H \in G_1$. If not, by Axiom (7.1) one can construct a circuit C', containing H and not containing H_1, such that C' is included in $C \cup \{H'\}$. If C' does not contain H', then we are done by induction. If C' contains H' then since $\mathrm{d}(\hat{1})$ is modular by Lemma 7.2.6 there exists a circuit C'' containing some element H'' in $\mathrm{d}(\hat{1})$ and such that $C'' \setminus \{H''\}$ is contained in $C' \cap R$. By induction the atom H'' belongs to G_1. If $H'' = H$ we are done. Otherwise by Axiom (7.1) there exists a circuit C''' containing H, contained in $C' \cup \{H''\}$ and not containing some element in $C' \cap R$. Reiterating this process we get a circuit containing H and some elements in G_1 which proves that H belongs to G_1. □

From this we deduce the second lemma.

Lemma 8.4.3. *The set of atoms $G_1 \cup R$ is closed.*

Proof. Let H be some element contained in a circuit C contained in $G_1 \cup R \cup \{H\}$. If H is in R we are done. If H is the unique element of C under $\mathrm{d}(\hat{1})$ then by the previous lemma we are also done. Otherwise using Lemma 7.2.6 together with Axiom (7.1) (as we did in the previous lemma) gives us a circuit contained in $\mathrm{d}(\hat{1})$, containing H with every other element in G_1. This proves that H belongs to G_1. □

Finally, we get the concluding lemma.

Lemma 8.4.4. *$\mathcal{L}$ is isomorphic to* $[\hat{0}, G_1 \cup R] \times [\hat{0}, G_2] \times ... \times [\hat{0}, G_n]$.

Proof. We will prove that every circuit is either contained in $G_1 \cup R$ or in some G_i with $i \geq 2$. Let C be a circuit in $\mathcal{L}$. If C is contained in $\mathrm{d}(\hat{1})$ then the result comes from the isomorphism $[\hat{0}, \mathrm{d}(\hat{1})] \simeq [\hat{0}, G_1] \times ... \times [\hat{0}, G_n]$. If $C \nsubseteq \mathrm{d}(\hat{1})$ and $C \cap \mathrm{d}(\hat{1})$ is a singleton then by the previous lemma we have $C \subset G_1 \cup R$. If $C \nsubseteq \mathrm{d}(\hat{1})$ and $C \cap \mathrm{d}(\hat{1})$ is not a singleton, pick H any atom in $C \cap \mathrm{d}(\hat{1})$. By iterating Lemma 7.2.6 as in the previous proof we obtain a circuit C' containing H, contained in $\mathrm{d}(\hat{1})$ and containing some elements in G_1. The isomorphism

$$[\hat{0}, \mathrm{d}(\hat{1})] \simeq [\hat{0}, G_1] \times [\hat{0}, G_n]$$

implies that this circuit should be contained in G_1 which proves the result. □

□

The above proposition and Theorem 7.2.7 imply the following theorem.

Theorem 8.4.5. *Let $\mathcal{L}$ be a supersolvable lattice. The algebra* $\mathrm{FY}(\mathcal{L}, \mathcal{G}_{\min})$ *has a quadratic Gröbner basis and is therefore Koszul.*

Proof. Any supersolvable lattice $\mathcal{L}$ decomposes as a product of irreducible supersolvable lattices

$$\mathcal{L} \simeq \mathcal{L}_1 \times ... \times \mathcal{L}_n.$$

We then have

$$\mathrm{FY}(\mathcal{L}, \mathcal{G}_{\min}) \simeq \mathrm{FY}(\mathcal{L}_1, \mathcal{G}_{\min}) \otimes ... \otimes \mathrm{FY}(\mathcal{L}_n, \mathcal{G}_{\min})$$

and we can conclude by Proposition 8.4.1 and Theorem 7.2.7. □

Chapter 9

Application to the extended modular operad

9.1 Chordal graphs

In [30] Stanley proved that if G is a chordal graph (meaning every cycle in G has a chord), the geometric lattice associated to G is supersolvable. This result is based on the following lemma by Dirac [9].

Lemma 9.1.1 (Dirac, [9])**.** *Every chordal graph admits a vertex v such that the graph induced by the neighboors of v is a complete graph.*

Such vertices are called "simplicial". If we remove a simplicial vertex from a chordal graph, the graph we obtain is chordal and this graph is a coatom in the original graph. This means we can reiterate the process and get a maximal chain in the lattice associated with a chordal graph. One can then check that this maximal chain contains only modular elements. We have a "built" variant of this result. Let us remind the reader that in Example 1.1.8 we have defined a built lattice $(\mathcal{L}_G, \mathcal{G}_G)$ for every simple graph G, with $\mathcal{L}_G$ the usual graphical matroid associated to G and $\mathcal{G}_G$ the building set of connected subgraphs of G.

Lemma 9.1.2. *Let G be a connected chordal graph. The built lattice $(\mathcal{L}_G, \mathcal{G}_G)$ associated to G is supersolvable.*

Proof. Let us choose a maximal chain of modular elements as in the last paragraph. By construction those elements are connected subgraphs of G. Let G' be a closed connected subgraph of G. For any integer k less than the rank of G, the element $\mathrm{d}^k(\hat{1}) \wedge G'$ can be obtained from G' by successively removing simplicial vertices of G' and therefore it is connected. $\square$

Corollary 9.1.3. *For all chordal graph G, the Feichtner–Yuzvinsky algebra* $\mathrm{FY}(\mathcal{L}_G, \mathcal{G}_G)$ *admits a quadratic Gröbner basis.*

9.2 The components of the extended modular operad

In [21], Losev and Manin introduced new moduli stacks $\overline{L}_{g,S}$ for stable curves of genus g with painted marked points indexed by S of two types (say "black" and "white") where the points of type black are allowed to coincide and the points of type white are not. Those stacks are the components of the so-called "extended modular operad" (see [22]). We will deduce from Corollary 9.1.3 the following result.

Theorem 9.2.1. *The cohomology algebras of the components of the extended modular operads in genus 0 are Koszul.*

Proof. It is part of the folklore that if S is a (colored) set with m white points and n black points and $*$ is some chosen white point, then the moduli space $\overline{L}_{0,S}$ is isomorphic to the wonderful compactification of the graphical arrangement

$$\{z_i = z_j \,|\, i \neq * \text{ white }, \, j \neq * \text{ white or black}\},$$

with respect to the building set of connected components (see 1.1.8). The corresponding graph denoted $G_{m-1,n}$ has $m+n-1$ vertices, with the first $m-1$ vertices connected to every other vertices and the last n vertices connected only to the first $m-1$ vertices. We notice that $G_{m,n}$ is a chordal graph for every m and n and therefore we can conclude by Corollary 9.1.3.

Let us summarize here the main line of arguments giving the stated isomorphism. In the sequel [25], Manin remarked that the moduli stacks $\overline{L}_{g,S}$ are part of the formalism of Hassett spaces introduced by Hassett in [18]. In the latter article, the author introduces the moduli problem of curves with weighted points, where one fixes a "weight data" consisting of a vector $(g, \mathsf{w}) = (g, w_1, ..., w_n) \in \mathbb{N} \times (]0,1] \cap \mathbb{Q})^n$ and one then seeks to parametrize the nodal curves of genus g with n marked points $(s_i)_{i \leq n}$ which are allowed to coincide "up to their weights", meaning that if the points $s_i, i \in I$ coincide then we require

$$\sum_{i \in I} w_i \leq 1,$$

and satisfying a (weighted) stability condition (see [18]). If the first p weights are 1 and the last $n-p$ weights are small enough (precisely $\sum_{i>p} w_i < 1$) then the above condition means exactly that the first p points cannot coincide with any other point and the last $n-p$ points can coincide only between them, and we recover the painted moduli problem of

Losev and Manin. Hassett proved that there exists a Deligne–Mumford stack $\overline{\mathcal{M}}_{g,\mathsf{w}}$ representing the above weighted moduli problem.

In genus 0 the stability condition can be simply stated: for any irreducible component T of the nodal curve, we require

$$\sum_{i\,\text{s.t.} s_i \in T} w_i + \#\text{nodes of } T > 2.$$

In addition, in genus 0 the moduli stack $\overline{\mathcal{M}}_{0,\mathsf{w}}$ is a smooth projective scheme (called a Hassett space). If the weights are either 1 or very small then we call those Hassett spaces "heavy/light". Indexes with weight 1 are called heavy and the other indexes are called light. Work of Cavalieri-Hampe-Markwig-Ranganathan [7] shows that a "heavy/light" Hassett space is a tropical compactification of the projective complement $\mathcal{M}_{0,\mathsf{w}}$ of the same graphical arrangement $\{z_i = z_j \,|\, i \neq *\,\text{heavy},\ j \neq *\,\text{heavy or light}\}$ (with $*$ some chosen heavy index). To put it in a nutshell this means that there exists an embedding of $\mathcal{M}_{0,\mathsf{w}}$ in a torus $\mathbb{T}^n$ together with a fan Σ in $\mathbb{R}^n$ having support the tropicalization of $\mathcal{M}_{0,\mathsf{w}}$ and such that the closure of $\mathcal{M}_{0,\mathsf{w}}$ in the toric variety $X(\Sigma)$ is the Hassett space $\overline{\mathcal{M}}_{0,\mathsf{w}}$ (we refer to [24] for an introduction to tropical geometry). The fan Σ introduced in [7] is none other than the Bergman fan associated to the buit lattice $(\mathcal{L}_{G_{m,n}}, \mathcal{G}_{G_{m,n}})$ (see [14] for the definition of the Bergman fan of a built lattice).

In [31], Tevelev has shown that the tropical compactification of a projective hyperplane arrangement complement along the bergman fan of some building set $\mathcal{G}$ of the correponding lattice can in fact be identified with the wonderful compactification of De Concini and Procesi along the same building set $\mathcal{G}$, which is the stated isomorphism. $\square$

Chapter 10

Further considerations

10.1 Towards a classification of Koszul Feichtner–Yuzvinsky algebras

We would like to emphasize the fact that we know plenty of Feichtner–Yuzvinsky algebras $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ which admit quadratic Gröbner bases and such that the built lattice $(\mathcal{L}, \mathcal{G})$ is not supersolvable, especially in low rank. For instance, if C_4 and C_5 are respectively the 4 and 5-cycles then the lattices $\mathcal{L}_{C_4}$ and $\mathcal{L}_{C_5}$ are not supersolvable but the built lattices $(\mathcal{L}_{C_4}, \mathcal{G}_{C_4})$ and $(\mathcal{L}_{C_5}, \mathcal{G}_{C_5})$ are so small that they still satisfy the key Lemma 8.2.1 and therefore their Feichtner–Yuzvinsky algebras will be Koszul. However, for the wonderful presentation and for the order used in this article, based on a few examples it feels to the author that the supersolvability condition should be close to necessary, in high enough rank. For instance if C_n is the n-cycle then one can easily check that the relations of weight 2 do not form a Gröbner basis of the algebra $\mathrm{FY}(\mathcal{L}_{C_n}, \mathcal{G}_{C_n})$ with respect to the order considered in this article for $n \geq 6$. We still do not know if $\mathrm{FY}(\mathcal{L}_{C_6}, \mathcal{G}_{C_6})$ is Koszul or not. In order to produce a quadratic Gröbner basis of this algebra one would either need to consider a different order, or even a different presentation (which should also be different from the classical presentation since one can show that no order on monomials induces a quadratic Gröbner basis for the classical presentation; the argument is completely analogous to that of Dotsenko [10] for the case of the building set of connected subgraphs of the complete graphs).

Let us also highlight the fact that even the question of quadraticity of Feichtner–Yuzvinsky algebras is not completely clear. We know that the building sets having a flag nested set complex give quadratic Feichtner–Yuzvinsky algebras but this condition is not necessary, as shown by the following example. Consider C_4 the 4-cycle with edges numbered from 1 to 4. The set of flats

$$\mathcal{G} = \{\hat{1}, \{1,2\}, \{1\}, \{2\}, \{3\}, \{4\}\}$$

is a building set of $\mathcal{L}_{\mathcal{C}_4}$ which has a non-flag nested set complex since $\{2, 3, 4\}$ is not nested and does not contain any proper subset which is not nested. However, the Feichtner–Yuzvinsky algebra of this built lattice is the algebra generated by $h_{\hat{1}}$ and $h_{\{1,2\}}$ with relations

$$h_{\hat{1}}^3 = 0,$$
$$h_{\hat{1}} h_{\{1,2\}} = h_{\hat{1}}^2,$$
$$h_{\{1,2\}}^2 = 0,$$

which is quadratic since the first relation is a consequence of the last two which are quadratic. This "pathology" has to do with the fact that the minimal building set of $\mathcal{L}_{C_4}$ (which is just the atoms together with the maximal element) does not have a flag nested set complex.

Proposition 10.1.1. *Let $\mathcal{L}$ be a lattice and $\mathcal{G}'$ a building set of $\mathcal{L}$ such that $(\mathcal{L}, \mathcal{G}')$ has a flag nested set complex. If $\mathcal{G}$ is a building set of $\mathcal{L}$ containing $\mathcal{G}'$ and such that* $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ *is quadratic, then the nested set complex of $(\mathcal{L}, \mathcal{G})$ is flag.*

Proof. Assume that we have non-comparable elements $G_1, ..., G_n$, with $n \geq 3$, such that we have $G := \bigvee_i G_i \in \mathcal{G}$ and $G_i \vee G_j \notin \mathcal{G}$ for all $i \neq j$. If G belongs to $\mathcal{G}'$ then decomposing the elements $G_1, ..., G_n$ in $\mathcal{G}'$ immediately yields a contradiction to the flag-ness of the nested set complex of $(\mathcal{L}, \mathcal{G}_{\min})$. If G does not belongs to $\mathcal{G}'$, to lighten the notation let us assume $G = \hat{1}$ (just restrict to the interval $[\hat{0}, G]$). We have some decomposition

$$\mathcal{L} \simeq [\hat{0}, F_1] \times ... \times [\hat{0}, F_p] \tag{10.1}$$

with $\{F_i, i \leq p\}$ the factors of G in $\mathcal{G}'$ and $p \geq 2$. Let j be some index less than p. By isomorphism (10.1) we have $F_j = \bigvee_i (F_j \wedge G_i)$. If we decompose the elements $F_j \wedge G_i$ as the join of their factors in $\mathcal{G}$ we can see that there is at most two indexes i such that we have $F_j \wedge G_i \neq \hat{0}$ (otherwise we get a new family of non comparable elements contradicting the flag-ness of $\mathcal{N}(\mathcal{L}, \mathcal{G})$, but this time with join F_j which is irreducible). In addition, there cannot be two such indexes, because if say $F_j = (G_{i_1} \wedge F_j) \vee (G_{i_2} \wedge F_j)$ with $i_1 \neq i_2$ then F_j is an element of $\mathcal{G}$ below $G_1 \vee G_2$ which is neither below G_{i_1} nor below G_{i_2} which contradicts the fact that we have $G_{i_1} \vee G_{i_2} \notin \mathcal{G}$. In conclusion for each j there is exactly one i such that we have $G_i \wedge F_j \neq \hat{0}$, and this implies that we in fact have $F_j \leq G_i$. By using isomorphism (10.1) one more time we get that each G_i is a join of some F_j's, and this forms a partition of the F_j's. Finally, if $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ is a quadratic algebra then the relation $(h_{\hat{1}} - h_{G_1})...(h_{\hat{1}} - h_{G_n})$ can be written as a sum of relations of weight 2, multiplied by monomials. One of the terms of this sum shall be of the form $h_{\hat{1}}^{n-2}(h_{\hat{1}} - h_{G'_1})(h_{\hat{1}} - h_{G'_2})$ with G'_1 and G'_2 two elements in $\mathcal{G}$ with join $\hat{1}$. By isomorphism (10.1) we have $G'_1 = (G_1 \wedge G'_1) \vee ... \vee (G_n \wedge G'_1)$, and similarly for G'_2. If there are more than three indexes i such that we have $G_i \wedge G'_1 \neq \hat{0}$, then decomposing the elements $G_i \wedge G'_1$ in $\mathcal{G}$ yields a new obstruction to the flag-ness of

$\mathcal{N}(\mathcal{L}, \mathcal{G})$, and we can conclude by some induction. If there are two indexes $i_1 \neq i_2$ such that we have $G'_1 \wedge G_{i_1} \neq \hat{0}$ and $G'_1 \wedge G_{i_2} \neq \hat{0}$ then G'_1 contradicts the fact that we have $G_{i_1} \vee G_{i_2} \notin \mathcal{G}$. Finally, we get $G'_1 = G_{i_1}$ and $G'_2 = G_{i_2}$ for some i_1, i_2, which contradicts $G_{i_1} \vee G_{i_2} \notin \mathcal{G}$. □

As we know from Proposition 8.4.1, if a lattice $\mathcal{L}$ is supersolvable the nested set complex associated to its minimal building set is flag. By the previous proposition this implies that for any building set $\mathcal{G}$ of $\mathcal{L}$, if $\mathrm{FY}(\mathcal{L}, \mathcal{G})$ is quadratic then the nested set complex of $(\mathcal{L}, \mathcal{G})$ is flag.

10.2 Conceptualizing the proofs of Koszulness

It would be very beneficial if one could explain in a more conceptual way the strategy for proving the Koszul property introduced by Dotsenko and extended in this document. In this direction, it could be of interest to check if an analogous strategy could work to reprove the following classical theorem of Yuzvinsky.

Theorem 10.2.1 (Yuzvinsky, [36])**.** *If $\mathcal{L}$ is a supersolvable geometric lattice then the algebra $OS(\mathcal{L})$ admits a quadratic Gröbner basis.*

The corresponding (co)operad would be the cooperad of Orlik–Solomon algebras introduced in Subsection 3.3. Having this other example may lead to a better understanding of the phenomena at play and perhaps give new applications.

It would also be interesting to find an operadic characterization of supersolvable lattices, which would explain why they behave so well with respect to the operadic structure.

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In the first part of this PhD thesis we introduce a global operadic structure on some algebraic invariants of matroids such as the generalized combinatorial Chow rings and the Orlik–Solomon algebras. We develop a theory of Gröbner bases for this new operadic structure, which we use in order to prove Koszulness of the operad of generalized combinatorial Chow rings.

In the second part we use this new operadic structure to prove the Koszulness of some generalized combinatorial Chow rings.

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE
UMR 7501
Université de Strasbourg et CNRS
7 Rue René Descartes
67 084 STRASBOURG CEDEX

Tél. 03 68 85 01 29
Fax 03 68 85 03 28

https://irma.math.unistra.fr
irma@math.unistra.fr

IRMA 2023/003
https://tel.archives-ouvertes.fr/tel-04088498

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Université de Strasbourg

Basile CORON

Algebraic invariants of matroids and generalized operads

École doctorale Mathématiques, sciences de l'information et de l'ingénieur | ED 269 Université de Strasbourg

Résumé

Dans la première partie de cette thèse on introduit une structure opéradique globale sur certains invariants algébriques de matroïdes tels que les anneaux de Chow combinatoires généralisés et les algèbres d'Orlik-Solomon. On développe une théorie de bases de Gröbner pour cette nouvelle structure opéradique, que l'on utilise pour montrer la Koszulité de l'opérade des anneaux de Chow combinatoires.

Dans la deuxième partie on utilise cette nouvelle structure opéradique pour prouver la Koszulité de certains anneaux de Chow combinatoires généralisés.

Mots-clefs : Matroïdes, Opérades, Koszulité

Résumé en anglais

In the first part of this PhD thesis we introduce a global operadic structure on some algebraic invariants of matroids such as the generalized combinatorial Chow rings and the Orlik-Solomon algebras. We develop a theory of Gröbner bases for this new operadic structure, which we use in order to prove Koszulness of the operad of generalized combinatorial Chow rings.

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Keywords : Matroids, Operads, Koszulness