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**La diminution de niveau pour les formes automorphes sur les surfaces modulaires de Picard**

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**Hao Fu**

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Discipline/ Spécialité : MATHÉMATIQUES

# LA DIMINUTION DE NIVEAU POUR LES FORMES AUTOMORPHES SUR LES SURFACES MODULAIRES DE PICARD

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# Résumé

Le principe de Mazur donne un critère selon lequel une représentation galoisienne irréductible mod $\ell$ provenant d'une forme modulaire de niveau $Np$ (avec $p$ premier par rapport à $N$) peut également provenir d'une forme modulaire de niveau $N$. Dans cette thèse nous démontrons un résultat analogue montrant que une représentation galoisienne mod $\ell$ provenant d'une représentation automorphe cuspidale stable du groupe de similitude unitaire $G = \mathrm{GU}(1,2)$ qui est Steinberg en un nombre premier inerte $p$ peut également provenir d'une représentation automorphe de $G$ qui est non ramifiée en $p$.

# Abstract

Mazur's principle gives a criterion under which an irreducible mod $\ell$ Galois representation arising from a modular form of level $Np$ (with $p$ prime to $N$) can also arise from a modular form of level $N$. We prove an analogous result showing that a mod $\ell$ Galois representation arising from a stable cuspidal automorphic representation of the unitary similitude group $G = \mathrm{GU}(1, 2)$ which is Steinberg at an inert prime $p$ can also arise from an automorphic representation of $G$ that is unramified at $p$.

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# Chapitre 0

# Introduction

Le théorème de diminution du niveau est un résultat profond dans le domaine de la théorie des nombres, et plus précisément dans la théorie des représentations galoisiennes et les formes modulaires. Ce théorème, établi par Kenneth Ribet en 1986, a joué un rôle central dans la démonstration du dernier théorème de Fermat par Andrew Wiles en 1994.

Le dernier théorème de Fermat stipule qu'il n'existe pas de solutions entières non triviales pour l'équation

$$a^n + b^n = c^n$$

pour $n > 2$. Ce théorème est resté sans preuve pendant plus de trois siècles après sa proposition par Pierre de Fermat en 1637. Il est facile de voir que le problème se ramène au cas où $n = \ell$ est un nombre premier plus grand que 3. En 1984, Gerhard Frey a démontré que si une telle solution existait, il serait possible de construire une courbe elliptique semi-stable sur $\mathbb{Q}$

$$y^2 = x(x - a^\ell)(x + b^\ell)$$

appelée la *courbe de Frey* telle qu'il ait une mauvaise réduction exactment en les nombres premiers $p$ qui divisent le produit $abc$ et sa représentation galoisienne mod $\ell$ satisfasse des propriétés de ramification très particulières. C'est à ce moment-là que le théorème de diminution de niveau entre en jeu. Ce théorème, s'appuyant sur les travaux de Jean-Pierre Serre ([Ser87b, Ser87a]), est établi par Ribet. Il est également connu dans la littérature sous le nom du *principe de Mazur*.

**Theorem 0.0.1.** [Rib90, Theorem 1.1] *Soit $N$ un entier positif et $p, \ell$ des nombres premiers distincts tels que $\ell$ soit impair et $(p, N) = 1$. Soit $f$ une nouvelle forme de poids 2 et de niveau $Np$ et $\overline{\rho}_{f,\ell}$ la représentation galoisienne résiduelle mod $\ell$ attachée à $f$. Supposons que*

1. *$\overline{\rho}_{f,\ell}$ soit absolument irréductible ;*
2. *$\overline{\rho}_{f,\ell}$ soit non ramifiée en $p$ ;*
3. *$p \not\equiv 1 \bmod \ell$.*

*Alors il existe une nouvelle forme $g$ de poids 2 et de niveau $N$ telle que $\overline{\rho}_{f,\ell} \cong \overline{\rho}_{g,\ell}$.*

L'importance de ce théorème réside dans le fait qu'il a permis de réduire la preuve du dernier théorème de Fermat à la conjecture de Shimura-Taniyama-Weil, qui prétend que la représentation galoisienne mod $\ell$ de toute courbe elliptique sur

$\mathbb{Q}$ provient d'une forme modulaire de poids 2. En effet, si la conjecture de Shimura-Taniyama-Weil pour les courbes elliptiques est vraie, alors le théorème 0.0.1 nous permettra de montrer que la représentation galoisienne mod $\ell$ attachée à la courbe de Frey provient d'une forme modulaire de poids 2 et de niveau 2. Cependant, une telle forme modulaire n'existe pas, ce qui conduit à une contradiction. Cette découverte a conduit Andrew Wiles à démontrer le dernier théorème de Fermat en prouvant la conjecture de Shimura-Taniyama-Weil pour les courbes elliptiques semi-stables.

Nous revenons à la démonstration du théorème 0.0.1. Ribet a incorporé la représentation galoisienne donnée dans un module de torsion de la jacobienne d'une courbe modulaire. Une étape clé est d'analyser l'action de Frobenius sur la partie torique de la jacobienne. L'hypothèse $p \not\equiv 1 \bmod \ell$ a été éliminée par Ribet dans un travail ultérieur ([Rib91]), où il a choisi un autre nombre premier $q$ tel que $q \not\equiv 1 \bmod \ell$ et a transféré la forme modulaire donnée à celle attachée à l'algèbre de quaternions indéfinie ramifiée en $pq$ par la correspondance de Jacquet-Langlands. Ensuite, le soi-disant "astuce de commutation $(p, q)$" lui permet de diminuer le niveau à $p$ tout en utilisant le principe de Mazur pour diminuer le niveau à $q$. Pour une explication plus détaillée de la méthode de Ribet, voir [Wan22].

Plus tard, Jarvis ([Jar99]) et Rajaei ([Raj01]) ont démontré des résultats similaires sur la diminution du niveau des représentations galoisiennes attachées à des courbes de Shimura sur des corps totalement réels, après une avancée majeure de Carayol dans [Car86]. La géométrie de la mauvaise réduction de la courbe de Shimura combinée à un calcul explicite des cycles évanescents révèle que le groupe des composantes de la jacobienne de la courbe de Shimura est Eisensteinien. Dans la même veine, van Hoften ([vH21]) et Wang ([Wan22]) ont étudié la diminution de niveau pour les variétés modulaires de Siegel de niveau paramodulaire sous différentes hypothèses techniques. Pour le groupe de similitude unitaire de signature (1,2), Helm a prouvé la diminution du niveau en une place scindée dans l'extension quadratique imaginaire sur un corps totalement réel dans [Hel06]. Boyer a traité le cas des variétés de Shimura unitaires de type Kottwitz-Harris-Taylor dans [Boy19].

Cette thèse est divisée en trois parties. Dans la première partie, nous présentons une preuve concise du théorème 0.0.1 afin d'illustrer notre méthode. Nous commençons par rappeler les notions de base, notamment les formes modulaires, les algèbres de Hecke, la représentation galoisienne et la suite spectrale de poids. Ensuite, nous présentons la géométrie de la fibre spéciale mod $p$ des courbes modulaires de niveau $Np$, qui consiste en deux copies de la fibre spéciale de niveau $N$ se coupant transversalement au lieu supersingulier. Nous réalisons $\overline{\rho}_{f,\ell}$ dans le premier groupe de cohomologie étale de la fibre générique de la courbe modulaire de niveau $Np$, sur lequel existe une filtration donnée par la suite spectrale de poids. Finalement, nous adaptons l'argument classique et concluons en utilisant l'involution d'Atkin-Lehner qui a la même action que le Frobenius sur le lieu supersingulier.

Dans la deuxième partie, nous traitons de la diminution du niveau pour le groupe de similitude unitaire de signature (1,2). Soit $F$ une extension quadratique imaginaire sur $\mathbb{Q}$ et $G := \mathrm{GU}(1,2)$ le groupe de similitude unitaire quasi-déployé correspondant de signature (1,2). Fixons un nombre premier $p$ inerte dans $F$ et un sous-groupe compact ouvert $K^p$ de $G(\mathbb{A}^{\infty,p})$, où $\mathbb{A}^{\infty,p}$ est l'anneau des adèles finis en dehors de $p$. Soit $K_p \subset G(\mathbb{Q}_p)$ un sous-groupe hyperspécial, et $\mathrm{Iw}_p \subset K_p$ un sous-groupe Iwahori. Soit $S$ (resp. $S_0(p)$) le modèle intégral de la variété de Shimura attachée à $G$ de niveau $K^pK_p$ (resp. $K^p\mathrm{Iw}_p$). Le théorème principal est le suivant :

**Theorem 0.0.2.** *Soit $\pi$ une représentation cuspidale automorphe stable de $G(\mathbb{A})$ cohomologique à coefficient trivial. Fixons un nombre premier $\ell \neq p$. Soit $\mathfrak{m}$ l'idéal maximal mod $\ell$ de l'algèbre de Hecke sphérique attachée à $\pi$. Soit $\overline{\rho}_{\pi,\ell}$ la représentation galoisienne mod $\ell$ attachée à $\pi$. Supposons que*

1. $(\pi^{\infty,p})^{K^p} \neq 0$;
2. *$\pi_p$ est la représentation de Steinberg de $G_p$ tordue par un caractère non ramifié ;*
3. *si $i \neq 2$ alors* $\mathrm{H}^i(S \otimes F^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} = 0$;
4. *$\overline{\rho}_{\pi,\ell}$ est absolument irréductible ;*
5. *$\overline{\rho}_{\pi,\ell}$ est non ramifiée en $p$ ;*
6. $\ell \nmid (p-1)(p^3+1)$.

*Alors il existe une représentation automorphe cuspidale $\tilde{\pi}$ de $G(\mathbb{A})$ telle que $(\tilde{\pi}^\infty)^{K^pK_p} \neq 0$ et $\overline{\rho}_{\tilde{\pi},\ell} \cong \overline{\rho}_{\pi,\ell}$.*

Nous adaptons la stratégie de Ribet. Étant donné que la jacobienne n'est pas disponible pour les surfaces de Shimura, inspirés par Helm, nous utilisons la suite spectrale de poids-monodromie pour analyser les analogues du groupe de composantes de la jacobienne de $S$ et $S_0(p)$. Pour ce faire, nous avons besoin d'une étude détaillée de la géométrie des fibres spéciales. La surface $S \otimes \mathbb{F}_{p^2}$ a été étudiée par Wedhorn dans [Wed01] et Vollaard dans [Vol10]. Ils ont montré que le lieu supersingulier est constitué de composantes géométriques irréductibles qui sont des courbes de Fermat de degré $p+1$ s'intersectant transversalement en des points superspéciaux. Le complément du lieu supersingulier est le lieu $\mu$-ordinaire qui est dense.

La géométrie de $S_0(p)$ est plus compliquée. L'étude des modèles locaux dans [Bel02] implique que $S_0(p)$ a une réduction semi-stable en $p$. Nous définissons trois strates fermées $Y_0, Y_1, Y_2$ dans $S_0(p) \otimes \mathbb{F}_{p^2}$. Nous montrons qu'elles sont toutes lisses et que leur union est $S_0(p) \otimes \mathbb{F}_{p^2}$. Nous étudions en outre les relations entre ces strates et $S \otimes \mathbb{F}_{p^2}$. En particulier, $Y_0$ est isomorphe au éclatement de $S \otimes \mathbb{F}_{p^2}$ le long des points superspéciaux ; $Y_1$ admet un morphisme purement inséparable vers $Y_0$ ; et $Y_2$ est un fibré en $\mathbb{P}^1$ sur la normalisation du lieu supersingulier de $S \otimes \mathbb{F}_{p^2}$ qui est géométriquement une union disjointe de courbes de Fermat. Les intersections deux à deux $Y_i \cap Y_j$ sont transversales et paramétrées par des variétés de Shimura discrètes attachées à $G'$, où $G'$ est la forme intérieure unique de $G$ qui coïncide avec $G$ à tous les places finies et est compacte modulo le centre à l'infini. Cela peut être vu comme une incarnation géométrique du transfert de Jacquet-Langlands. De plus, nous montrons que les points géométriques de $Y_0 \cap Y_1 \cap Y_2$ sont en bijection avec la variété de Shimura discrète attachée à $G'$ de niveau $K^p\mathrm{Iw}_p$. Tous les morphismes sont équivariants sous la correspondance de Hecke première à $p$, et définis sur $\mathbb{F}_{p^2}$, donc compatibles avec l'action de Frobenius en prenant la fibre géométrique. Le résultat ressemble à ceux de [dSG18] et [Vol10], mais est adapté aux applications arithmétiques en préservant l'équivariance de Hecke et la structure schématique.

Par la formule de Matsushima, la représentation automorphe $\pi$ donnée contribue à la cohomologie d'intersection de la compactification de Baily-Borel de $S_0(p)$. Heureusement, nous pouvons ignorer la compactification car la cohomologie du bord de la compactification de Baily-Borel s'annule lorsque on a localise à $\mathfrak{m}$ par l'irréductibilité de la représentation résiduelle galoisienne. Nous écrivons ensuite la suite spectrale monodromie-poids pour la surface $S_0(p)$.

Nous sommes prêts à prouver le théorème 0.0.2 par l'absurde. Si le niveau ne peut pas être abaissé, l'hypothèse (3) dans le théorème 0.0.2 éliminerait la possibilité que $\pi$ apparaisse dans la cohomologie étale de $S \otimes \mathbb{F}_p^{\mathrm{ac}}$. La suite spectrale monodromie-poids se dégénèrerait à la première page et donnerait lieu à une filtration de $\mathrm{H}^2(S_0(p) \otimes F^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}}$ dont les pièces graduées sont données par les groupes de cohomologie de $Y_0 \cap Y_1 \cap Y_2$. La condition non ramifiée sur la représentation galoisienne résiduelle forcerait $\overline{\rho}_{\pi,\ell}$ à vivre dans la cohomologie étale de $(Y_0 \cap Y_1 \cap Y_2) \otimes \mathbb{F}_p^{\mathrm{ac}}$. Nous trouvons alors une contradiction en étudiant les valeurs propres généralisées de l'action de Frobenius.

Dans la troisième partie de ma thèse, je présente un résultat selon lequel la cohomologie étale de la fibre générique de la surface modulaire de Picard, localisée en un idéal maximal approprié, s'annule en dehors du degré intermédiaire 2. La preuve repose sur le choix d'un nombre premier $q$ inert tel que la surface modulaire de Picard ait une bonne réduction en $q$, ce qui nous permet de relier la question à l'annulation de la cohomologie de la fibre spéciale en $q$. Nous utilisons le fait que le lieu ordinaire de la compactification minimale est affine, ainsi que la séquence spectrale de Deligne sur le diviseur à croisements normaux. Gardons la notation ci-dessus du théorème 0.0.2. Le résultat principal est le suivant :

**Theorem 0.0.3.** *Soit $\pi$ une représentation automorphe cuspidale stable de $G(\mathbb{A})$ cohomologique à coefficient trivial. Soit $\ell$ un nombre premier. Soit $\mathfrak{m}$ l'idéal maximal mod $\ell$ de l'algèbre de Hecke sphérique attachée à $\pi$ et $\overline{\rho}_{\mathfrak{m}}$ la représentation galoisienne résiduelle attachée à $\mathfrak{m}$. Supposons que $\overline{\rho}_{\mathfrak{m}}$ soit absolument irréductible et qu'il existe un nombre premier $q \neq \ell$ tel que*

1. *$q$ est inerte dans $F$;*
2. *$\ell \nmid (q-1)(q^3+1)$;*
3. *$K_q$ est hyperspécial ;*
4. *$\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_q)$ n'est pas conjugué à une matrice de la forme $\mathrm{diag}(-\nu q, \nu, -\nu q^{-1})$ ou $\mathrm{diag}(\nu q^2, \nu, \nu q^{-2})$ pour un certain $\nu \in \mathbb{F}_\ell^{\mathrm{ac},\times}$.*

*Alors*

$$\mathrm{H}^i(S \otimes \mathbb{Q}^{\mathrm{ac}}, \mathbb{F}_\ell^{\mathrm{ac}})_{\mathfrak{m}} = 0$$

*pour $i \neq 2$.*

En conséquence, le théorème 0.0.3 nous permet de supprimer la condition (3) dans le théorème 0.0.2, en admettant un condition faible sur la représentation galoisienne.

# Chapitre 1

# Level lowering of modular forms

## 1.1 Preliminaries

Level lowering was proposed by Serre [Ser87b, Ser87a] and proven by Mazur and Ribet [Rib90] in the setting of modular forms, which is a key step in deducing Fermat's Last theorem from the Shimura-Taniyama-Weil conjecture. To demonstrate our method, we will prove a level lowering theorem of modular forms, using the same strategy that will be employed later on. For a positive integer $N$, define principal congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ :

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod N \right\},$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod N \right\}$$

and $\Gamma_1(M, N) := \Gamma_1(M) \cap \Gamma_0(N)$. Define

$$\mathfrak{H}^+ := \{\tau \in \mathbb{C} : \mathrm{im}(\tau) > 0\}, \quad \mathfrak{H}^- := \{\tau \in \mathbb{C} : \mathrm{im}(\tau) < 0\}, \quad \mathfrak{H}^\pm = \mathfrak{H}^+ \sqcup \mathfrak{H}^+.$$

For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\tau \in \mathfrak{H}^\pm$, define

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \quad j(\gamma, \tau) = c\tau + d.$$

Moreover, for any integer $k$, define the weight-$k$ operator $[\gamma]_k$ on functions $f : \mathfrak{H}^\pm \to \mathbb{C}$ by

$$(f[\gamma]_k)(\tau) = j(\gamma, \tau)^{-k} f(\gamma(\tau)), \quad \tau \in \mathfrak{H}^\pm.$$

**Definition 1.1.1.** *Let $k, r, N$ be integers. Suppose that $(r, N) = 1$. A function $f : \mathfrak{H}^+ \to \mathbb{C}$ is a modular form of weight $k$ and level $\Gamma_1(r, N)$ if*

1. *$f$ is holomorphic on $\mathfrak{H}^+$,*
2. *$f$ is weight-$k$ invariant under $\Gamma_1(r, N)$, that is*

$$f[\gamma]_k = f, \quad \gamma \in \Gamma_1(r, N).$$

*In particular, for* $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, *we have* $f(\tau+1) = f(\tau)$, *thus* $f$ *admits a Fourier expansion*

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}.$$

3. $f[\gamma]_k$ *is holomorphic at* $\infty$ *for all* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, *that is,* $a_n$ *of* $f[\gamma]_k$ *vanishes for* $n < 0$.

*If in addition,* $a_0$ *of* $f[\gamma]_k$ *vanishes for all* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, *then* $f$ *is a* cusp *form of weight* $k$ *and level* $\Gamma_1(r,N)$. *The* $\mathbb{C}$*-vector space of modular forms (resp. cusp forms) of weight* $k$ *and level* $\Gamma_1(r,N)$ *is finite-dimensional, denoted by* $M_k(\Gamma_1(r,N))$ *(resp.* $S_k(\Gamma_1(r,N))$*).*

*There is a well-defined Petersson inner product*

$$\langle\, , \rangle_{\Gamma_1(r,N)} : S_k(\Gamma_1(r,N)) \times S_k(\Gamma_1(r,N)) \longrightarrow \mathbb{C}$$

*given by*

$$\langle f, g \rangle_{\Gamma_1(r,N)} = \frac{1}{V_{\Gamma_1(r,N)}} \int_{\Gamma_1(r,N)\backslash\mathfrak{H}^+} f(\tau)\overline{g(\tau)}(\mathrm{Im}(\tau))^k d\mu(\tau)$$

*where* $d\mu$ *is the hyperbolic measure and* $V_{\Gamma_1(r,N)}$ *is the volume of* $\Gamma_1(r,N)\backslash\mathfrak{H}^+$.

### 1.1.1 Hecke operators on the modular forms

For $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ we have a decomposition of the double coset

$$\Gamma_1(r,N)\alpha\Gamma_1(r,N) = \coprod_j \Gamma_1(r,N)\beta_j$$

where $j$ runs over a finite set. For an integer $k$, we can define the weight-$k$ Hecke operator

$$\begin{aligned}[\Gamma_1(r,N)\alpha\Gamma_1(r,N)]_k : S_k(\Gamma_1(r,N)) &\to S_k(\Gamma_1(r,N)) \\ f &\mapsto \sum_j f[\beta_j]_k.\end{aligned}$$

The definition does not depend on the choice of $\beta_j$'s. In particular, for $d \in (\mathbb{Z}/r\mathbb{Z})^*$, we have a well-defined *diamond operator*

$$\langle d \rangle := [\Gamma_1(r,N) \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \Gamma_1(r,N)]_k$$

where $\begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \in \Gamma_0(r) \cap \Gamma_0(N) \cong \Gamma_0(rN)$ such that $d \equiv \delta \mod r$. For a prime number $p$, we can also define

$$T_p := [\Gamma_1(r,N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(r,N)]_k.$$

All these operators commute with each other. We say $f \in S_k(\Gamma_1(r,N))$ is an *eigenform* of weight $k$ and level $\Gamma_1(r,N)$ if $f$ is an eigenvector for all diamond operators and $T_p$ with $(p,N) = 1$ and $a_1 = 1$.

### 1.1.2 Hecke algebra

Let $\mathbb{T}^N_{k,r,N}$ be the $\mathbb{Z}$-subalgebra of $\mathrm{End}(S_k(\Gamma_1(r,N))$generated by all $T_p$ for all prime numbers $p$ prime to $N$ and $\langle d\rangle$ for $d \in (\mathbb{Z}/r\mathbb{Z})^*$. Then $\mathbb{T}^N_{k,r,N}$ is a commutative $\mathbb{Z}$-algebra of finite type.

Let $A$ be a unitary commutative ring. An eigenform $f = \sum_{n\geq 0}^{\infty} a_n q^n$ of weight $k$ and level $\Gamma_1(r,N)$ with Fourier coefficients in $A$ gives rise to a homomorphism $\theta_f : \mathbb{T}^{rN}_{k,r,N} \to A$ sending $T_p$ to $a_p$ for all prime numbers $p$ not dividing $N$.

### 1.1.3 Two subspaces of $S_k(\Gamma_1(r,pN))$

Let $p$ be a prime number not dividing $N$. There are two ways to embed $S_k(\Gamma_1(r,N))$ into $S_k(\Gamma_1(r,pN))$ :

$$\begin{aligned} S_k(\Gamma_1(r,N)) &\to S_k(\Gamma_1(r,pN)) \\ i_p : f &\mapsto f \\ \alpha_p : f &\mapsto f\left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k : \tau \mapsto p^{k-1} f(p\tau). \end{aligned}$$

Define the subspaces

$$S_k(\Gamma_1(r,pN))^{p-\mathrm{old}} := i_p S_k(\Gamma_1(r,N)) + \alpha_p S_k(\Gamma_1(r,N)).$$

We then can define the subspace $S_k(\Gamma_1(r,pN))^{p-\mathrm{new}}$ to be the orthogonal complement of $S_k(\Gamma_1(r,pN))^{p-\mathrm{old}}$ under the Petersson inner product.

**Remark 1.1.2.** $\mathbb{T}^{pN}_{k,r,pN}$ *preserves* $S_k(\Gamma_1(r,pN))^{p-old}$ *and* $S_k(\Gamma_1(r,pN))^{p-new}$. *Denote by* $\mathbb{T}^{N,p-old}_{k,r,pN}$ *and* $\mathbb{T}^{pN,p-new}_{k,r,pN}$ *the restriction of* $\mathbb{T}^{pN}_{k,r,pN}$ *to* $\mathrm{End}_{\mathbb{C}}(S_k(\Gamma_1(r,pN))^{p-old})$ *and* $\mathrm{End}_{\mathbb{C}}(S_k(\Gamma_1(r,pN))^{p-new})$. *Then* $\mathbb{T}^{pN,p-old}_{k,r,pN}$ *and* $\mathbb{T}^{pN,p-new}_{k,r,pN}$ *are quotients of* $\mathbb{T}^{pN}_{k,r,pN}$. *Moreover,* $\mathbb{T}^{pN,p-old}_{k,r,pN}$ *is isomorphic to* $\mathbb{T}^{N}_{k,r,N}$.

## 1.2 Modular curve

Let $r \geq 4$ be a positive integer and $N$ be a prime number coprime to $r$. Let $\overline{\mathfrak{H}}^+ := \mathfrak{H}^+ \cup \mathbb{Q} \cup \{\infty\}$ be the compactification of $\mathfrak{H}$. We can then define the complex modular curve

$$Y_1(r,N)_{\mathbb{C}} := \Gamma_1(r,N)\backslash\mathfrak{H}^+, X_1(r,N)_{\mathbb{C}} := \Gamma_1(r,N)\backslash\overline{\mathfrak{H}}^+.$$

One gets $X_1(r,N)_{\mathbb{C}}$ by adding a finite number of points called *cusps* to $Y_1(r,N)_{\mathbb{C}}$. Since $X_1(r,N)_{\mathbb{C}}$ is naturally a compact Riemann surface, it is a projective variety over $\mathbb{C}$. Let $\Omega^1$ be the sheaf of differential 1-forms on $X_1(r,N)_{\mathbb{C}}$. We have a canonical isomorphism

$$\begin{aligned} S_2(\Gamma_1(r,N)) &\to \mathrm{H}^0(X_1(r,N)_{\mathbb{C}}, \Omega^1) \\ f &\to f dz. \end{aligned}$$

### 1.2.1 Integral model

To facilitate arithmetic applications, we shall use adèlic language to express $Y_1(r,N)_{\mathbb{C}}$. Let $\mathbb{A}^\infty$ be the finite adèle over $\mathbb{Q}$. Let $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$. Define open compact subgroups of $\mathrm{GL}_2(\mathbb{A}^\infty)$

$$K_1(r) := \left\{ g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \mid g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod r \right\},$$

$$K_0(N) := \left\{ g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}$$

and $K_1(r,N) := K_1(r) \cap K_0(N)$. By [Mil03, Lemma 2.3], $\mathbb{Q}^\times \backslash \{\pm\} \times \mathbb{A}^{\infty,\times} / \det(K_1(r,N))$ is finite and discrete where $\{\pm\} = \{+,-\}$ is a discrete two-element set, $\mathbb{Q}^\times$ acts on both sets on the left, and $\det(K_1(r,N))$ acts on $\mathbb{A}^{\infty,\times}$ on the right.

**Proposition 1.2.1.** [Mil03, proposition 2.7] *We have* $\mathrm{SL}_2(\mathbb{Q}) \cap K_1(r,N) = \Gamma_1(r,N)$ *and a bijection*

$$Y_1(r,N)_{\mathbb{C}} \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathfrak{H}^\pm \times \mathrm{GL}_2(\mathbb{A}^\infty)/K_1(r,N).$$

Define the moduli problem

**Definition 1.2.2.** *For a* $\mathbb{Z}[1/rN]$*-algebra* $R$, $\mathbf{Y}_1(r,N)(R)$ *is the isomorphism classes of* $(E, C_N)$, *where* $E$ *is an elliptic curve over* $R$, $P_r$ *is a point of* exact order $r$ *of* $E$, *and* $C_N$ *is a cyclic subgroups of* $E$ *of order* $N$.

Deligne and Rapoport in [DR73], Katz and Mazur in [KM85] showed that the moduli problem $\mathbf{Y}_1(r,N)$ is represented by a smooth affine curve over $\mathbb{Z}[1/rN]$, still denoted by $\mathbf{Y}_1(r,N)$, such that we have an isomorphism of complex varieties

$$\mathbf{Y}_1(r,N)(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathfrak{H}^\pm \times \mathrm{GL}_2(\mathbb{A}^\infty)/K_1(r,N) \cong Y_1(r,N)_{\mathbb{C}}.$$

**Remark 1.2.3.** *For a prime number* $p$ *not dividing* $rN$, *we can replace* $C_{pN}$ *by a pair* $(C_p, C_N)$ *of cyclic subgroups of order* $p$ *and* $N$, *respectively.*

There are two natural maps

$$\mathbf{Y}_1(r,N) \xleftarrow{\pi_1} \mathbf{Y}_1(r,pN) \xrightarrow{\pi_2} \mathbf{Y}_1(tr,N)$$

defined by the following : for a $\mathbb{Z}[1/rpN]$-algebra $R$ and $(E, P_r, C_N, C_p) \in \mathbf{Y}_1(r,pN)(R)$, we have

$$\pi_1 : (E, P_r, C_N, C_p) \mapsto (E, P_r, C_N),$$
$$\pi_2 : (E, P_r, C_N, C_p) \mapsto (E/C_p, P_r \bmod C_p, C_N \bmod C_p).$$

These morphisms extend in a unique way to morphisms from $\mathbf{X}_1(r,pN)$ to $\mathbf{X}_1(r,N)$. The Hecke action $T_p$ on the cohomology is defined as the composition

$$T_p : \mathrm{H}^0(X_1(r,N)_{\mathbb{C}}, \Omega^1_{X_1(r,N)_{\mathbb{C}}}) \xrightarrow{\pi_2^*} \mathrm{H}^0(X_1(r,pN)_{\mathbb{C}}, \Omega^1_{X_1(r,pN)_{\mathbb{C}}}) \xrightarrow{\pi_{1,!}} \mathrm{H}^0(X_1(r,N)_{\mathbb{C}}, \Omega^1_{X_1(r,N)_{\mathbb{C}}})$$

induced by

$$\pi_2^*\Omega_{X_1(r,N)_{\mathbb{C}}} \to \Omega_{X_1(r,pN)_{\mathbb{C}}} \cong \pi_1^!\Omega^1_{X_1(r,N)_{\mathbb{C}}}.$$

We can then verify that the two definitions of $T_p$ coincide, in the sense that the following diagram commutes :

$$\begin{array}{ccc} \mathrm{H}^0(X_1(r,N)_{\mathbb{C}}, \Omega^1_{X_1(r,N)_{\mathbb{C}}}) & \xrightarrow{T_p} & \mathrm{H}^0(X_1(r,N)_{\mathbb{C}}, \Omega^1_{X_1(r,N)_{\mathbb{C}}}) \\ \downarrow \cong & & \downarrow \cong \\ S_2(\Gamma_1(r,N)) & \xrightarrow{T_p} & S_2(\Gamma_1(r,N)) \end{array}$$

**Remark 1.2.4.** *In general, for $g \in \mathrm{GL}_2(\mathbb{A}^\infty)$ and $K$ a neat open compact subgroup of $\mathrm{GL}_2(\mathbb{A}^\infty)$, define*

$$\mathrm{Sh}_K := \mathrm{GL}_2(\mathbb{Q})\backslash\mathfrak{H}^\pm \times \mathrm{GL}_2(\mathbb{A}^\infty)/K.$$

*We have a correspondence*

$$\begin{array}{ccccc} & & \mathrm{Sh}_{K\cap gKg^{-1}} & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \mathrm{Sh}_K & & & & \mathrm{Sh}_K \end{array}$$

*where $\pi_1$ is induced by the natural injection $K \cap gKg^{-1} \to K$ and $\pi_2$ is given by $[x,a] \mapsto [x,ag]$ for $(x,a) \in \mathfrak{H}^\pm \times \mathrm{GL}_2(\mathbb{A}^\infty)$. The correspondence induces an action on the cohomology*

$$g : \mathrm{H}^i(\mathrm{Sh}_K, \mathbb{F}_\ell) \xrightarrow{\pi_2^*} \mathrm{H}^i(\mathrm{Sh}_{K\cap gKg^{-1}}, \mathbb{F}_\ell) \xrightarrow{\pi_{1,!}} \mathrm{H}^i(\mathrm{Sh}_K, \mathbb{F}_\ell)$$

*In particular, recall that*

$$\mathbf{Y}_1(r,pN)(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathfrak{H}^\pm \times \mathrm{GL}_2(\mathbb{A}^\infty)/K_1(r,pN),$$
$$\mathbf{Y}_1(r,N)(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathfrak{H}^\pm \times \mathrm{GL}_2(\mathbb{A}^\infty)/K_1(r,N),$$

*and notice that $K_1(r,pN) = K_1(r,N) \cap gK_1(r,N)g^{-1}$ with $g \in \mathrm{GL}_2(\mathbb{A}^\infty)$ such that $g_p = \mathrm{diag}(p^{-1},1)$ and $g_{p'} = \mathrm{id}$ for $p' \neq p$. Then we see that $g$ induces a correspondence over $\mathbb{C}$ and an action on $\mathrm{H}^i(Y_1(r,N)_{\mathbb{C}}, \mathbb{F}_\ell)$ which coincides with the action of $T_p$.*

For $d \in (\mathbb{Z}/r\mathbb{Z})^*$, the diamond operator $\langle d \rangle$ is induced by the action on $\mathbf{Y}_1(r,pN)$, still denoted by $\langle d \rangle$ :

$$\langle d \rangle : \mathbf{Y}_1(r,pN) \to \mathbf{Y}_1(r,pN)$$
$$(E, P_r, C_N, C_p) \mapsto (E, dP_r, C_N, C_p).$$

We also have the Atkin-Lehner involution $w_p$ on $\mathbf{Y}_1(r,pN)$ :

$$w_p : \mathbf{Y}_1(r,pN) \to \mathbf{Y}_1(r,pN) \tag{1.1}$$
$$(E, P_r, C_N, C_p) \mapsto (E/C_p, P_r \bmod C_p, C_N \bmod C_p, E[p]/C_p) \tag{1.2}$$

such that $\pi_2 = \pi_1 \circ w_p$ and $w_p^2 = \langle p \rangle$. In the adèlic language, $w_p$ corresponds to $g \in \mathrm{GL}_2(\mathbb{A}^\infty)$ where $g_{p'} = \mathrm{id}$ for $p' \neq p$ and $g_p = \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$.

### 1.2.2 Geometry of the special fiber

The integral model $\mathbf{Y}_1(r,N)$ has good reduction modulo $p$ and $\mathbf{Y}_1(r,pN)$ has semistable reduction modulo $p$. Consider the special fiber $\mathbf{Y}_1(r,N)_{\mathbb{F}_p}$ and $\mathbf{Y}_1(r,pN)_{\mathbb{F}_p}$. We have two closed immersions $\Phi_1, \Phi_2 : \mathbf{Y}_1(r,N)_{\mathbb{F}_p} \to \mathbf{Y}_1(r,pN)_{\mathbb{F}_p}$ defined as follows : for an $\mathbb{F}_p$-algebra $R$ and $(E, C_N) \in Y_1(r,N)_{\mathbb{F}_p}(R)$,

$$\Phi_1(E, P_r, C_N) = (E, P_r, C_N, \ker F), \quad \Phi_2(E, P_r, C_N) = (E^{(p)}, FP_r, FC_N, \ker V),$$

where $F : E \to E^{(p)}$ is the relative Frobenius and $V : E^{(p)} \to E$ is the Verschiebung satisfying $FV = p, VF = p$.

To summarize we have the following diagram

$$\begin{array}{ccccc}
\mathbf{Y}_1(r,N)_{\mathbb{F}_p} & & & & \mathbf{Y}_1(r,N)_{\mathbb{F}_p} \\
& \searrow^{\Phi_1} & & \swarrow^{\Phi_2} & \\
& & \mathbf{Y}_1(r,pN)_{\mathbb{F}_p} & & \\
& \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\
\mathbf{Y}_1(r,N)_{\mathbb{F}_p} & & & & \mathbf{Y}_1(r,N)_{\mathbb{F}_p}.
\end{array}$$

such that

$$\Phi_2 = w_p\Phi_1, \quad \pi_2 = \pi_1 w_p, \quad \pi_1\Phi_1 = \mathrm{id}, \quad \pi_2\Phi_2 = \langle p\rangle, \quad \pi_1\Phi_2 = \pi_2\Phi_1 = F.$$

Then we can show that

$$\mathbf{Y}_1(r,pN)_{\mathbb{F}_p} = \mathrm{im}\Phi_1 \cup \mathrm{im}\Phi_2.$$

Moreover, the intersection $\mathrm{im}\Phi_1 \cap \mathrm{im}\Phi_2$ is the *supersingular locus* $Y_1(r,pN)^{\mathrm{ss}}$ where the elliptic curve $E$ is supersingular (i.e., $E(\mathbb{F}_p^{\mathrm{ac}})[p] = 0$).

## 1.3 Galois representation

Keep the notation of Section 1.1.2. Let $N$ be a positive integer, $p, \ell, r$ be distinct prime numbers such that $(p, rN\ell) = 1$. Let $f = \sum_{n\geq 1} a_n q^n$ be an eigenform of weight $k$ and level $\Gamma_1(r,N)$. Let $\mathbb{T} := \mathbb{T}^N_{2,r,N}$ to be the prime-to-$N$ Hecke algebra. Let $K_f = \mathbb{Q}(\{a_n\})$ be the field generated by $a_n$'s with $(n,N) = 1$. One can show that $K_f$ is a finite extension over $\mathbb{Q}$, and $a_n \in O_f$ for $(n,N) = 1$ where $O_f$ is the ring of integers of $K_f$. Let $\lambda$ be a place of $K_f$ over $\ell$ and $O_{f,\lambda}$ be the completion of $O_f$ with respect to $\lambda$. By Section 1.1.2 we then have a homomorphism $\theta_f : \mathbb{T} \to K_f$. Then $\theta_f$ factors through $O_f$. Define

$$\mathfrak{m} := \ker(\mathbb{T} \xrightarrow{\theta_f} O_{f,\lambda} \longrightarrow O_{f,\lambda}/\lambda).$$

Eichler-Shimura and Deligne showed the existence of a semisimple mod $\ell$ Galois representation attached to $f$

$$\overline{\rho}_{\mathfrak{m}} = \overline{\rho}_{f,\ell} : \mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}/\mathfrak{m})$$

such that for prime numbers $p$ not dividing $rN\ell$, $\overline{\rho}_{\mathfrak{m}}$ is unramified at $p$ and

$$\operatorname{Tr}\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_p) = T_p \mod \mathfrak{m}, \quad \det\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_p) = \langle p\rangle p^{k-1} \mod \mathfrak{m}.$$

Now we discuss Galois representations with values in a complete local ring. The Hecke algebra $\mathbb{T}\otimes\mathbb{Z}_\ell$ is a semi-local complete $\mathbb{Z}_\ell$-algebra thus admits a decomposition

$$\mathbb{T}\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \cong \prod_i \mathbb{T}_{\mathfrak{m}_i} \tag{1.3}$$

where the product is over a finite number of maximal ideals $\mathfrak{m}_i$ of $\mathbb{T}$ such that $\mathbb{T}/\mathfrak{m}_i$ is of characteristique $\ell$, and $\mathbb{T}_{\mathfrak{m}_i}$ is the localization of $\mathbb{T}$ with respect to $\mathfrak{m}_i$. Carayol showed the existence of Galois representations taking value in a complete local ring :

**Proposition 1.3.1.** [Car94, Theorem 3] *Suppose $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible. Then there exists a continuous representation*

$$\rho_{\mathfrak{m}} : \mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{m}})$$

*such that for all prime numbers $p$ not dividing $rN\ell$, $\rho_{\mathfrak{m}}$ is unramified and satisfies the relations*

$$\operatorname{Tr}\rho_{\mathfrak{m}}(\mathrm{Frob}_p) = \theta_f(T_p), \quad \det\rho_{\mathfrak{m}}(\mathrm{Frob}_p) = \langle p\rangle p^{k-1}.$$

By reduction over $\mathfrak{m}$ we recover the residual Galois representation defined above

$$\overline{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{m}}) \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{m}}/\mathfrak{m}\mathbb{T}_{\mathfrak{m}}).$$

## 1.4 Level lowering

We can state a variant of level lowering theorem proved by Mazur and written down by Ribet. We set the weight $k = 2$ so that we only need constant sheaves on the modular curves. Keep the notation of Section 1.3 replacing $N$ by $pN$.

**Theorem 1.4.1.** [Rib90, Theorem 1.1] *Let $N$ be a positive integer and $r, p, \ell$ be distinct prime numbers such that $\ell$ is odd and $(p, rN\ell) = 1$. Let $f \in S_2(\Gamma_1(r, pN))^{p-new}$ be an eigenform of weight 2 and level $\Gamma_1(r, pN)$. Let $\mathfrak{m} \subset \mathbb{T} := \mathbb{T}^{pN}_{2,r,pN}$ be the prime-to-$pN$ maximal ideal attached to $f$ and $\overline{\rho}_{\mathfrak{m}}$ be the mod $\ell$ residual Galois representation. Suppose that*

1. *$\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible ;*
2. *$\overline{\rho}_{\mathfrak{m}}$ is unramified at $p$ ;*
3. *$p \not\equiv \pm 1 \bmod \ell$.*

*Then there exists an eigenform $g$ of weight 2 and level $\Gamma_1(r, N)$ such that $\overline{\rho}_{f,\ell} \cong \overline{\rho}_{g,\ell}$.*

To prove Theorem 1.4.1, by [DS74, Lemme 6.11], it suffices to show the following theorem

**Theorem 1.4.2.** *Suppose that $(p, N) = 1, (\ell, pN) = 1$ and $p \not\equiv 1 \mod \ell$. Suppose also that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible and unramified at $p$. Then $\theta_f : \mathbb{T} \to O_{L,\lambda}/\lambda$ factors through $\mathbb{T}^{p-old}$, which is the image of $\mathbb{T}$ in $\mathrm{End}(S_2(\Gamma_1(r, pN))^{p-old}$.*

In other words, we look for a morphism $\mathbb{T}^{p-\text{old}} \to O_{f,\lambda}/\lambda O_{f,\lambda}$ such that the following diagram commute

$$\begin{array}{ccccc}
\mathbb{T} & \twoheadrightarrow & \mathbb{T}^{p-\text{new}} & \longrightarrow & O_{f,\lambda} \\
\downarrow & & & & \downarrow \\
\mathbb{T}^{p-\text{old}} & & \dashrightarrow & & O_{f,\lambda}/\lambda O_{f,\lambda}
\end{array}$$

where $\mathbb{T}^{p-\text{new}} := \mathbb{T}_{2,r,pN}^{pN,p-\text{new}}$. We need to realize the given Galois representation $\overline{\rho}_{\mathfrak{m}}$ in the cohomology of modular curves. Define the parabolic cohomology

$$\mathrm{H}^1_p(\mathbf{Y}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{Z}_\ell) = \mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, j_*\mathbb{Z}_\ell)$$

where $j$ is the inclusion of $\mathbf{Y}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}$ in $\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}$. In his report [Car94] Carayol explained that $\overline{\rho}_{\mathfrak{m}}$ lives in the $\mathfrak{m}$-torsion of the first cohomology of the modular curve.

Let $M := \mathrm{H}^1_p(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{Z}_\ell)$. Deonte by $M_{\mathfrak{m}}$ the localization of $M$ at $\mathfrak{m}$. The action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/\mathbb{Q})$ on $M$ commutes with $\mathbb{T}$ so makes $M_{\mathfrak{m}}$ a $\mathbb{T}_{\mathfrak{m}}[\mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/\mathbb{Q})]$-module.

**Proposition 1.4.3.** *We have an isomorphism of* $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/\mathbb{Q})$*-modules*

$$\mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)[\mathfrak{m}] \cong \overline{\rho}_{\mathfrak{m}}^{\oplus n}$$

*for a positive integer $n$.*

*Proof.* It suffices to show that

$$\mathrm{H}^1_p(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_{\mathfrak{m}}[\mathfrak{m}] \cong \overline{\rho}_{\mathfrak{m}}^{\oplus n}.$$

Indeed, we have

$$\mathrm{H}^1_p(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_{\mathfrak{m}} \cong \mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_{\mathfrak{m}}$$

since the difference comes from Eisenstein series and vanishes after the localization provided that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible. We claim that the $\mathbb{F}_\ell$-cohomology is the reduction of the $\mathbb{Z}_\ell$-cohomology :

$$M_{\mathfrak{m}} \otimes \mathbb{F}_\ell \cong M_{\mathfrak{m}}/\ell M_{\mathfrak{m}} \cong \mathrm{H}^1_p(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_{\mathfrak{m}}.$$

Indeed, the cokernel of the injection

$$\mathrm{H}^1_p(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{Z}_\ell)_{\mathfrak{m}} \otimes \mathbb{F}_\ell \to \mathrm{H}^1_p(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_{\mathfrak{m}}$$

vanishes since $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/\mathbb{Q})$ acts via its abelian quotient.

It remains to show that $M_{\mathfrak{m}} \otimes \mathbb{F}_\ell[\mathfrak{m}]$ is the direct sum of several copies of $\overline{\rho}_{\mathfrak{m}}$. By Matsushima formula,

$$M_{\mathfrak{m}} \otimes \mathbb{Q}_\ell^{\mathrm{ac}}$$

is $\rho_{\mathfrak{m}}$-typic in the sense of [Sch18, Definition 5.2]. By [Sch18, Proposition 5.4] we see that $M_{\mathfrak{m}}$ is also $\rho_{\mathfrak{m}}$-typic, i.e.,

$$M_{\mathfrak{m}} \cong \rho_{\mathfrak{m}} \otimes_{\mathbb{T}_{\mathfrak{m}}} J$$

for some finitely-generated $\mathbb{T}_{\mathfrak{m}}$-module $J$. Thus

$$(M_{\mathfrak{m}} \otimes \mathbb{F}_\ell)[\mathfrak{m}] \cong (J/\ell J)[\mathfrak{m}] \otimes \overline{\rho}_{\mathfrak{m}}$$

from which the proposition follows. □

We recall the weight spectral sequence for semistable schemes :

**Theorem 1.4.4.** [Sai03, 1.1 ; Corollary 2.2.4] *Let $X$ be a proper scheme over $\mathbb{Z}_p$ with strictly semistable reduction at $p$ ; that is, a regular scheme with smooth generic fiber whose special fiber $Y$ is a divisor with normal crossings. For each $i$, let $Y^{(i)}$ denote the disjoint union of all $i+1$-fold intersections of distinct irreducible components of $Y$. Then there is a spectral sequence*

$$\mathrm{E}_1^{r,s} = \bigoplus_{i \geq \max(0,-p)} \mathrm{H}^{s-2i}(Y^{(r+2i)}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell(-i)) \Rightarrow \mathrm{H}^{r+s}(X_{\mathbb{Q}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)$$

*where the maps $d_1^{r,s} : \mathrm{E}_1^{r,s} \to \mathrm{E}_1^{r+1,s}$ are alternative sums of Gysin or restriction maps.*

*Moreover, let $I_p$ be the inertia group of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ac}}/\mathbb{Q}_p)$, define the (p-adic) tame quotient homomorphism $t_\ell : I_p \to \mathbb{F}_\ell(1)$ by sending $\sigma \in I_K$ to $(\sigma(p^{1/\ell^n})/p^{1/\ell^n})_n$. Choose $T \in I_p$ such that $t_\ell(T)$ is a generator of $\mathbb{F}_\ell(1)$. Then, the endomorphism $\nu := T - 1$ of $R\psi\mathbb{F}_\ell$ induces a map*

$$\begin{array}{ccc} \mathrm{E}_1^{p,q} = \bigoplus_{i\geq \max(0,-p)} \mathrm{H}^{q-2i}(Y^{(p+2i)}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell(-i)) & \longrightarrow & \mathrm{H}^{p+q}(X_{\mathbb{Q}_p^{\mathrm{ac}}}, \mathbb{F}_\ell) \\ \downarrow 1\otimes t_\ell(T) & & \downarrow T-1=\nu \\ \mathrm{E}_1^{p+2,q-2} = \bigoplus_{i\geq \max(0,-p)} \mathrm{H}^{q-2i}(Y^{(p+2i)}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell(-i+1)) & \longrightarrow & \mathrm{H}^{p+q}(X_{\mathbb{Q}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)(1) \end{array}$$

*of the weight spectral sequence.*

We apply the weight spectral sequence to the integral model $\mathbf{X}_1(r,pN)_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ with semistable reduction at $p$ : the first page $\mathrm{E}_{1,\mathfrak{m}}$ of the weight spectral sequence localized at $\mathfrak{m}$ reads

$$\begin{array}{ll} \mathrm{H}^0(\mathbf{X}_1(r,N)^{\mathrm{ss}}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m}(-1) \xrightarrow{\mathrm{d}_\mathfrak{m}^{-1,2}} & \mathrm{H}^2(\mathbf{X}_1(r,N)_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m} \oplus \mathrm{H}^2(\mathbf{X}_1(r,N)_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m} \\ & \mathrm{H}^1(\mathbf{X}_1(r,N)_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m} \oplus \mathrm{H}^1(\mathbf{X}_1(r,N)_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m} \\ & \mathrm{H}^0(\mathbf{X}_1(r,N)_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m} \oplus \mathrm{H}^0(\mathbf{X}_1(r,N)_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m} \xrightarrow{\mathrm{d}_\mathfrak{m}^{0,0}} \mathrm{H}^0(\mathbf{X}_1(r,N)^{\mathrm{ss}}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m} \end{array}$$

which converges to $\mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m}$. The spectral sequence degenerates at page 2 and gives rise to a filtration $\mathrm{Fil}^* \mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m}$ of $\mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m}$. Put $\mathrm{Gr}_i := \mathrm{Fil}^i / \mathrm{Fil}^{i+1}$, then

$$\begin{aligned} \mathrm{Gr}_{-1} &\cong \ker \mathrm{d}_\mathfrak{m}^{-1,2} \subset \mathrm{H}^0(\mathbf{X}_1(r,N)^{\mathrm{ss}}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m}(-1), \\ \mathrm{Gr}_0 &\cong \mathrm{H}^0(\mathbf{X}_1(r,N)_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell))_\mathfrak{m} \oplus \mathrm{H}^0(\mathbf{X}_1(r,N)_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell))_\mathfrak{m} \\ \mathrm{Gr}_1 &\cong \operatorname{coker} \mathrm{d}_\mathfrak{m}^{0,0} \end{aligned}$$

such that the action of the nilpotent operator $\nu$ on $\mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m}$ is given by

$$\mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m} \twoheadrightarrow \mathrm{Gr}_{-1} \longrightarrow \mathrm{Gr}_1(-1) \hookrightarrow \mathrm{H}^0(\mathbf{X}_1(r,N)^{\mathrm{ss}}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m}(-1).$$

In particular, $\ker \nu = \mathrm{Fil}^0 \mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m}$.

We study the $\mathrm{Gal}(\mathbb{F}_p^{\mathrm{ac}}/\mathbb{F}_{p^2})$-action on $\mathrm{H}^0(\mathbf{X}_1(r,pN)^{\mathrm{ss}}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_\mathfrak{m}$.

**Lemma 1.4.5.** *Let* $\mathrm{Frob}_p \in \mathrm{Gal}(\mathbb{F}_p^{\mathrm{ac}}/\mathbb{F}_p)$ *be a geometric Frobenius. We have* $\mathrm{Frob}_p = w_p$ *on* $\mathrm{H}^0(\mathbf{X}_1(r,N)^{\mathrm{ss}}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_{\mathfrak{m}}$.

*Proof.* The action of $\mathrm{Frob}_p$ on $\mathrm{H}^0(\mathbf{X}_1(r,pN)^{\mathrm{ss}}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_{\mathfrak{m}}$ coincides with that induced by the relative Frobenius $F$ on $\mathbf{X}_1(r,pN)_{\mathbb{F}_p^{\mathrm{ac}}}$. By [DR73, V.1], for a point $(E, P_r, C_N, C_p)$ in $X^{\mathrm{ss}}(\mathbb{F}_p^{\mathrm{ac}})$, we have

$$\begin{aligned} w_p(E, P_r, C_N, C_p) &= (E^{(p)}, FP_r, FC_N, \ker V) \\ &= (E^{(p)}, FP_r, FC_N, FC_p) = F(E, P_r, C_N, C_p) \end{aligned} \quad (1.4)$$

since the $E[p]$ has the unique rank-$p$ subgroup $\ker F = \ker V$. □

**Lemma 1.4.6.** *Suppose* $\overline{\rho}_{\mathfrak{m}}$ *is unramified at* $p$. *Then there exists a rank 1* $O_{f,\lambda}/\lambda$*-submodule* $W \subset \mathrm{H}^0(\mathbf{X}_1(r,N)_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell))_{\mathfrak{m}} \oplus \mathrm{H}^0(\mathbf{X}_1(r,N)_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell))_{\mathfrak{m}}$ *on which* $\mathbb{T}$ *acts via* $\theta_f$.

*Proof.* Proposition 1.4.3 and the unramifiedness of $\overline{\rho}_{\mathfrak{m}}$ at $p$ implies that

$$\mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)[\mathfrak{m}] \subset \ker \nu.$$

as a $\mathbb{T}$-submodule of $\mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)_{\mathfrak{m}}$. Suppose that the lemma is false. Then $\mathrm{H}^1(\mathbf{X}_1(r,pN)_{\mathbb{Q}^{\mathrm{ac}}}, \mathbb{F}_\ell)[\mathfrak{m}] \subset \mathrm{H}^0(\mathbf{X}_1(r,N)^{\mathrm{ss}}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_{\mathfrak{m}}$. However, by Lemma 1.4.5 we have $\mathrm{Frob}_p = w_p$ on $\mathrm{H}^0(\mathbf{X}_1(r,N)^{\mathrm{ss}}_{\mathbb{F}_p^{\mathrm{ac}}}, \mathbb{F}_\ell)_{\mathfrak{m}}$. Since $w_p^2 = \langle p \rangle \in \mathbb{T}$ (see (1.1)), we conclude that $\rho_{\mathfrak{m}}(\mathrm{Frob}_p)^2 \equiv \mathrm{diag}(\lambda, \lambda) \mod \ell$ for some $\lambda \in \mathbb{F}_\ell^{\mathrm{ac},\times}$. However, since $f$ is new at $p$, if $\pi$ is the automorphic representation corresponding to $f$, then $\pi_p$ is an unramified twist of the Steinberg representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. By the local-global compatibility, $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_p)$ is conjugate to a matrix of the form

$$\nu \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

for some $\nu \in \mathbb{F}_\ell^{\mathrm{ac},\times}$. Thus we deduce that $p \equiv \pm 1 \bmod \ell$, a contradiction. □

*Proof of Theorem 1.4.2.* In Lemma 1.4.6 the action of $\mathbb{T}$ on $W$ factors through $\mathbb{T} \to \mathbb{T}^{p\text{-old}}$. We finish the proof. □

# Chapitre 2

# Level lowering of automorphic representations on the Picard modular surface

## 2.1 Introduction

The level lowering problem was proposed by Serre[Ser87b, Ser87a] in the name of epsilon conjecture and served as a key step in deducing Fermat last theorem from Shimura-Taniyama-Weil conjecture. Ribet proved the following theorem, which he called also Mazur's principle.

**Theorem 2.1.1.** [Rib90, Theorem 1.1] *Let $N$ be a positive integer and let $p, \ell$ be distinct prime numbers such that $\ell$ is odd and $(p, N) = 1$. Let $f$ be a newform of weight 2 and level $Np$ and $\overline{\rho}_{f,\ell}$ be the mod $\ell$ residual Galois representation attached to $f$. Suppose*

1. *$\overline{\rho}_{f,\ell}$ is absolutely irreducible ;*
2. *$\overline{\rho}_{f,\ell}$ is unramified at $p$ ;*
3. *$p \not\equiv 1 \bmod \ell$.*

*Then there exists a newform $g$ of weight 2 and level $N$ such that $\overline{\rho}_{f,\ell} \cong \overline{\rho}_{g,\ell}$.*

In his original proof, Ribet embedded the given Galois representation into some torsion module of the Jacobian of a modular curve. A key step is to analyze the Frobenius action on the toric part of Jacobians. The assumption $p \not\equiv 1 \bmod \ell$ was removed by Ribet later in [Rib91], where he took another prime number $q$ such that $q \not\equiv 1 \bmod \ell$ and transferred the given modular form to the one attached to the indefinite quaternion algebra ramified at $pq$ by Jacquet-Langlands correspondence. Then the so-called $(p, q)$ switch trick allows him to lower the level at $p$ while by Mazur's principle he can further lower the level at $q$. For a more precise explanation of Ribet's method, see [Wan22].

Later Jarvis ([Jar99]) and Rajaei ([Raj01]) proved similar results on level lowering of Galois representations attached to Shimura curves over totally real fields after a major breakthrough by Carayol in [Car86]. The geometry of bad reduction of Shimura curve combined with an explicit calculation of nearby cycles shows the component group of the Jacobian of the Shimura curve is Eisenstein. Along the same line van Hoften ([vH21]) and Wang ([Wan22]) studied level lowering for Siegel modular threefold of paramodular level under different technical assumptions. For unitary similitude group of signature(1,2), Helm proved level lowering at a place split in the quadratic imaginary extension over a totally real field in [Hel06]. Boyer treated the case for unitary Shimura varieties of Kottwitz-Harris-Taylor type in [Boy19].

In this article we deal with level lowering at a rational prime inert in a quadratic imaginary extension for the unitary similitude group of signature (1,2).

Let $F$ be a quadratic imaginary extension over $\mathbb{Q}$ and $G := \mathrm{GU}(1,2)$ be the corresponding quasi-split unitary similitude group of signature (1,2). Fix a prime number $p$ inert in $F$ and an open compact subgroup $K^p$ of $G(\mathbb{A}^{\infty,p})$ where $\mathbb{A}^{\infty,p}$ is the finite adèle over $\mathbb{Q}$ outside $p$. Let $K_p \subset G(\mathbb{Q}_p)$ be a hyperspecial subgroup, and $\mathrm{Iw}_p \subset K_p$ be an Iwahoric subgroup. Let $S$ (resp. $S_0(p)$) be the integral model of Shimura variety attached to $G$ of level $K^pK_p$ (resp. $K^p\mathrm{Iw}_p$). The main theorem is

**Theorem 2.1.2** (Theorem 2.4.1)**.** *Let $\pi$ be a stable automorphic cuspidal representation of $G(\mathbb{A})$ cohomological with trivial coefficient. Fix a prime number $\ell \neq p$. Let $\mathfrak{m}$ be the mod $\ell$ maximal ideal of the spherical Hecke algebra attached to $\pi$. Let $\overline{\rho}_{\pi,\ell}$ be the mod $\ell$ Galois representation attached to $\pi$. Suppose*

1. $(\pi^{\infty,p})^{K^p} \neq 0$;
2. *$\pi_p$ is the Steinberg representation of $G_p$ twisted by an unramified character ;*
3. *if $i \neq 2$ then* $\mathrm{H}^i(S \otimes F^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} = 0$;
4. *$\overline{\rho}_{\pi,\ell}$ is absolutely irreducible ;*
5. *$\overline{\rho}_{\pi,\ell}$ is unramified at $p$ ;*
6. $\ell \nmid (p-1)(p^3+1)$.

*Then there exists a cuspidal automorphic representation $\tilde{\pi}$ of $G(\mathbb{A})$ such that $(\tilde{\pi}^\infty)^{K^pK_p} \neq 0$ and*

$$\overline{\rho}_{\pi,\ell} \cong \overline{\rho}_{\tilde{\pi},\ell}.$$

We adapt Ribet's strategy. As Jacobian is unavailable for Shimura surfaces, inspired by Helm we use weight-monodromy spectral sequence to analyze analogues of the component group of Jacobians of $S$ and $S_0(p)$. In order to do so, we need a detailed study on the geometry of special fibers. The surface $S \otimes \mathbb{F}_{p^2}$ was studied by Wedhorn in [Wed01] and Vollaard in [Vol10]. They showed that the supersingular locus consists of geometric irreducible components which are Fermat curves of degree $p+1$ intersecting transversally at superspecial points. The complement of supersingular locus is $\mu$-ordinary locus which is dense.

The geometry of $S_0(p)$ is more complicated. The study of local models in [Bel02] implies that $S_0(p)$ has semistable reduction at $p$. We define three closed strata $Y_0, Y_1, Y_2$ in $S_0(p) \otimes \mathbb{F}_{p^2}$. We show they are all smooth and their union is $S_0(p) \otimes \mathbb{F}_{p^2}$. We further study relations between these strata and $S \otimes \mathbb{F}_{p^2}$. In particular, $Y_0$ is isomorphic to the blowup of $S \otimes \mathbb{F}_{p^2}$ at superspecial points ; $Y_1$ admits a purely inseparable morphism to the latter ; and $Y_2$ is a $\mathbb{P}^1$-bundle over the normalization of the supersingular locus of $S \otimes \mathbb{F}_{p^2}$ which is geometrically a disjoint union of Fermat curves. The pairwise intersections $Y_i \cap Y_j$ are transversal and parameterized by discrete Shimura varieties attached to $G'$, where $G'$ is the unique inner form of $G$ which coincides with $G$ at all finite places and is compact modulo center at infinity.

This can be viewed as a geometric incarnation of Jacquet-Langlands transfer. Moreover, we show the geometric points of $Y_0 \cap Y_1 \cap Y_2$ are in bijection with the discrete Shimura variety attached to $G'$ of level $K^p\mathrm{Iw}_p$. All the morphisms are equivariant under prime-to-$p$ Hecke correspondence, and defined over $\mathbb{F}_{p^2}$ thus compatible with the Frobenius action when taking the geometric fiber. The result bears a resemblance to those of [dSG18] and [Vol10], but is tailored for arithmetic applications by preserving Hecke equivariance and schematic structure.

By Matsushima's formula, the given automorphic representation $\pi$ contributes to the intersection cohomology of Baily-Borel compactification of $S_0(p)$. Fortunately, we can ignore the compactification since the cohomology of the boundary of Borel-Serre compactification vanishes when localized at $\mathfrak{m}$ by the irreducibility of the residual Galois representation. We then write down the weight-monodromy spectral sequence for the surface $S_0(p)$.

We are ready to prove the main theorem by contradiction. If there were no level lowering, the torsion-free assumption would eliminate the possiblity that $\pi$ appears in the localized cohomology of $S \otimes \mathbb{F}_p^{\mathrm{ac}}$. The weight-monodromy spectral sequence would degenerate at the first page and give rise to a filtration of $\mathrm{H}^2(S_0(p) \otimes F^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}$ with the graded pieces given by the cohomology groups of $Y_0 \cap Y_1 \cap Y_2$. The unramified condition on the residual Galois representation would force $\overline{\rho}_{\pi,\ell}$ to live in the localized cohomology of $(Y_0 \cap Y_1 \cap Y_2) \otimes \mathbb{F}_p^{\mathrm{ac}}$. We then find a contradiction by studying the generalized eigenvalues of the Frobenius action.

The article is organized as follows : after introducing the relevant Shimura varieties in Section 2.2, we study the geometry of special fiber of Shimura varieties in Section 2.3. Finally we carry out the proof of the main theorem in Section 2.4 .

### 2.1.1 Notation and conventions

The following list contains basic notation and conventions we fix throughout the article, we will use them without further comments.

— We denote by $\mathbb{A}$ the ring of adèles over $\mathbb{Q}$. For a set $\square$ of places of $\mathbb{Q}$, we denote by $\mathbb{A}^\square$ the ring of adèles away from $\square$. For a number field $F$, we put $\mathbb{A}_F^\square := \mathbb{A}^\square \otimes_\mathbb{Q} F$. If $\square = \{v_1, \ldots, v_n\}$ is a finite set, we will also write $\mathbb{A}^{v_1,\ldots,v_n}$ for $\mathbb{A}^\square$.

— For a field $K$, denote by $K^{\mathrm{ac}}$ the algebraic closure of $K$ and put $G_K := \mathrm{Gal}(K^{\mathrm{ac}}/K)$. Denote by $\mathbb{Q}^{\mathrm{ac}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. When $K$ is a subfield of $\mathbb{Q}^{\mathrm{ac}}$, we take $G_K$ to be $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/K)$ hence a subgroup of $G_\mathbb{Q}$.

— For every rational prime $p$, we fix an algebraic closure $\mathbb{Q}_p^{\mathrm{ac}}$ of $\mathbb{Q}_p$with the residue field $\mathbb{F}_p^{\mathrm{ac}}$, and an isomorphism $\iota_p : \mathbb{Q}_p^{\mathrm{ac}} \cong \mathbb{C}$.

— For an algebraic group $G$ over $\mathbb{Q}$, set $G_p := G(\mathbb{Q}_p)$ for a rational prime $p$ and $G_\infty := G(\mathbb{R})$.

— Let $X$ be a scheme. The cohomology group $\mathrm{H}^\bullet(X, -)$ will always be computed on the small étale site of $X$. If $X$ is of finite type over a subfield of $\mathbb{C}$, then $\mathrm{H}^\bullet(X(\mathbb{C}), -)$ will be understood as the Betti cohomology of the associated complex analytic space $X(\mathbb{C})$.

— Let $R$ be a ring. Given two $R$-modules $M_1 \subset M_2$, and $s \in \mathbb{N}$ an integer. denote by $M_1 \overset{s}{\subset} M_2$ if the length of the $R$-module $M_2/M_1$ is $s$ (hence finite).

— Let $R$ be a ring and $M$ be a set. Denote by $R[M]$ the set of functions on $M$

with compact support with values in $R$.
- If a base ring is not specified in the tensor operation $\otimes$, then it is $\mathbb{Z}$.
- For a scheme $S$ (resp. Noetherian scheme $S$ ), we denote by $\mathsf{Sch}_{/S}$ (resp. $\mathsf{Sch}'_{/S}$ ) the category of $S$-schemes (resp. locally Noetherian $S$-schemes). If $S = \operatorname{Spec} R$ is affine, we also write $\mathsf{Sch}_{/R}$ ( resp. $\mathsf{Sch}'_{/R}$ ) for $\mathsf{Sch}_{/S}$ ( resp. $\mathsf{Sch}'_{/S}$).
- The structure sheaf of a scheme $X$ is denoted by $\mathcal{O}_X$.
- For a scheme $X$ over an affine scheme Spec $R$ and an $R$-algebra $S$, we write $X \otimes_R S$ or even $X_S$ for $X \times_{\operatorname{Spec} R} \operatorname{Spec} S$.
- For a scheme $S$ in characteristic $p$ for some rational prime $p$, we denote by $\sigma : S \to S$ the absolute $p$-power Frobenius morphism. For a perfect field $\kappa$ of characteristic $p$, we denote by $W(\kappa)$ its Witt ring, and by abuse of notation, $\sigma : W(\kappa) \to W(\kappa)$ the canonical lifting of the $p$-power Frobenius map.
- Denote by $\mathbb{P}^1$ the projective line scheme over $\mathbb{Z}$, and $\mathbb{G}_{m,R} = \operatorname{Spec} R[T, T^{-1}]$ the multiplicative group scheme over a ring $R$. Let $\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ be the Weil restriction of $\mathbb{G}_{m,\mathbb{C}}$ to $\mathbb{R}$.

## 2.2 Shimura varieties, integral models and moduli interpretations

In this section we introduce some Shimura varieties associated with the group of unitary similitudes.

Let $F = \mathbb{Q}(\sqrt{\Delta})$ be a quadratic imaginary extension of $\mathbb{Q}$ with $\Delta \in \mathbb{Z}$ a negative square-free element. Let $\mathfrak{c}$ be the nontrivial element in $\mathrm{Gal}(F/\mathbb{Q})$, and write $a^{\mathfrak{c}}$ or $\mathfrak{c}(a)$ for the action of $\mathfrak{c}$ on $a$ for $a \in F$. Fix an embedding $\tau_0 : F \to \mathbb{C}$ such that $\tau_0(\sqrt{\Delta}) \in \mathbb{R}_{>0} \cdot \sqrt{-1}$. Then $\Sigma_\infty := \{\tau_0, \tau_1 = \tau_0 \circ \mathfrak{c}\}$ is the set of all complex embeddings of $F$. Let $O_F$ be the ring of integers of $F$, $F^{ac}$ be an algebraic closure of $F$. Let $(\Lambda = O_F^3, \psi)$ be the free $O_F$-module of rank 3 equipped with the hermitian form

$$\psi(u, v) = {}^t u \Phi \bar{v}$$

where

$$\Phi = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}.$$

Then $\psi$ is of signature (1,2) over $\mathbb{R}$. Denote by $e_0, e_1, e_2 \in \Lambda$ the standard basis vectors. We put also

$$\langle u, v \rangle_\psi := \mathrm{Tr}_{F/\mathbb{Q}}(\frac{1}{\sqrt{\Delta}} \psi(u, v))$$

which is a non-degenerate alternating form $V \times V \to \mathbb{Q}$. Let $G = \mathrm{GU}(\Lambda, \psi)$ be the group of unitary similitudes defined over $\mathbb{Z}$ by

$$G(R) = \{(g, \nu(g)) \in \mathrm{GL}_{O_F \otimes_{\mathbb{Z}} R}(\Lambda \otimes_{\mathbb{Z}} R) \times R^\times : \psi(gx, gy) = \nu(g)\psi(x, y), \forall x, y \in \Lambda \otimes_{\mathbb{Z}} R\} \tag{2.1}$$

for any $\mathbb{Z}$-algebra $R$. Note that $G$ can be also defined as the similitude group of $\langle \_, \_ \rangle_\psi$.

Let $p$ be a prime number inert in $F$.

### 2.2.1 Bruhat-Tits tree and open compact subgroups of $G_p$

#### Bruhat-Tits tree

[BG06, 3.1] Let $\mathcal{T}$ be the Bruhat-Tits building of $G_p$. According to [Tit79] or [Cho94, 1.4], it is a tree, and its vertices decompose into two parts $\mathcal{V} \coprod \tilde{\mathcal{V}}$. Every vertex of $\mathcal{V}$ (resp. of $\tilde{\mathcal{V}}$) has $p^3+1$ (resp. $p+1$) neighbours which are all in $\tilde{\mathcal{V}}$) (resp. in $\mathcal{V}$). The points of $\mathcal{V}$ are *hyperspecial* points in the sense of [Tit79], those of $\tilde{\mathcal{V}}$ are special points which are not hyperspecial. We denote by $\mathcal{E}$ the set of (non-oriented) edges of $\mathcal{T}$.

The tree $\mathcal{T}$ is endowed with an action of $G_p$. The center $Z_p \subset G_p$ acts on $\mathcal{T}$ trivially. The action of $G_p$ on $\mathcal{V}$ (resp. $\tilde{\mathcal{V}}$) is transitive, and the stabilizer of a vertex $v$ acts transitively on the set of vertices of $\mathcal{V}$ with distance $n$ from $v$ [Cho94, 1.4, 1.5], and therefore on the set of elements of $\mathcal{E}$ of origin $v$.

#### Maximal compact subgroup

[BG06, 3.2] According to [BT72], a maximal compact subgroup of $G_p$ fixes one and only one vertex of $\mathcal{T}$, which defines a bijection between the set of maximal compact subgroups of $G_p$ and $\mathcal{V} \coprod \tilde{\mathcal{V}}$. There are therefore two conjugacy classes of maximal compact subgroups of $G_p$, those who fix a vertex of $\mathcal{V}$, which we call *hyperspecial*, and those who fix a vertex of $\tilde{\mathcal{V}}$, which we call *special*.

Let $v \in \mathcal{V}$ and $v' \in \tilde{\mathcal{V}}$. We denote by $K_v$ and $K_{v'}$ the maximal compact subgroup which fixes $v$ and $v'$. Then $K_v$ is conjugate to $K_p := G(\mathbb{Z}_p)$, which is the stabilizer of the standard self-dual lattice

$$\Lambda_0 = \Lambda \otimes \mathbb{Z}_p = \langle e_0, e_1, e_2 \rangle_{O_{F_p}}.$$

In the meanwhile, $K_{v'}$ is conjugate to $\tilde{K}_p$ which is the stablizer of the lattice

$$\Lambda_1 = \langle pe_0, e_1, e_2 \rangle_{O_{F_p}}.$$

Assume that $v$ and $v'$ are neighbors. The stabilizer $K_v \cap K_{v'}$ of the edge $(v, v')$ is an *Iwahoric subgroup* of $G_p$.

### 2.2.2 Picard modular surface over $\mathbb{C}$

Define the bounded symmetric domain associated with $G$ as

$$\mathcal{B} = \{(z_0 : z_1 : z_2) \in \mathbb{P}^2(\mathbb{C}) \mid \bar{z}_0 z_2 + \bar{z}_1 z_1 + \bar{z}_2 z_0 < 0\}$$

which is biholomorphic to the unit ball in $\mathbb{C}^2$. The group $G(\mathbb{R})$ acts by projective linear transformations on $\mathbb{P}^2(\mathbb{C})$, the action of $G(\mathbb{R})$ preserves $\mathcal{B}$ and induces a transitive action on $\mathcal{B}$. Denote by $K_\infty$ the stabilizer of the "center" $(-1 : 0 : 1)$. Then we have an homeomorphism

$$G(\mathbb{R})/K_\infty \cong \mathcal{B}.$$

### 2.2.3 Shimura varieties for unitary groups

Consider the Deligne homomorphism

$$\begin{aligned} h_0 \colon \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times &\longrightarrow G(\mathbb{R}) \\ z = x + \sqrt{-1}y &\longmapsto (\mathrm{diag}(\overline{z}, z, \overline{z}), z\overline{z}) \end{aligned}$$

where $\overline{z}$ is the complex conjugate of $z$, and $G(\mathbb{R})$ acts on $\mathrm{Hom}_{\mathbb{R}\text{-group scheme}}(\mathbb{S}, G)$ by conjugation. The stabilizer of $h_0$ of $G(\mathbb{R})$ is $K_\infty$, and there exists a bijection between $\mathcal{B}$ and the $G(\mathbb{R})$-conjugacy class $X$ of $h_0$.

For an compact open subgroup $K \subset G(\mathbb{A}^\infty)$, the Shimura variety $\mathrm{Sh}(G, K)$ of level $K$ is a quasi-projective algebraic variety defined over $F$ whose complex points are identified with

$$\mathrm{Sh}(G, K)(\mathbb{C}) := G(\mathbb{Q})\backslash G(\mathbb{A})/KK_\infty \simeq G(\mathbb{Q})\backslash[X \times G(\mathbb{A}^\infty)/K].$$

In this article, we will consider the Shimura varieties $\mathrm{Sh}(G, K)$ with $K$ of the form $K = K^pK_p$, $K^p\tilde{K}_p$ or $K^p\mathrm{Iw}_p$, where $K^p$ is a fixed open compact subgroup of $G(\mathbb{A}^\infty)$, as well as their canonical integral models over $O_{F,(p)}$.

### 2.2.4 Dieudonné theory on abelian schemes

We first introduce some general notations on abelian schemes.

**Definition 2.2.1.** [LTX+22, Definition 3.4.5] *Let $A$ and $B$ be two abelian schemes over a scheme $S \in \mathtt{Sch}_{/\mathbb{Z}_{(p)}}$. We say that a morphism of $S$-abelian schemes $\varphi : A \to B$ is a* quasi-isogeny *if there is an integer $n$ such that $n\varphi$ is an isogeny. We say that a morphism of $S$-abelian schemes $\varphi : A \to B$ is a* quasi-$p$-isogeny *if there exists some $c \in \mathbb{Z}_{(p)}^\times$ such that $c\varphi$ is a isogeny. A quasi-isogeny $\varphi$ is* prime-to-$p$ *if there exist two integers $n, n'$ both coprime to $p$ such that $n\varphi$ and $n'\varphi^{-1}$ are both isogenies.*

*We denote by $A^\vee$ the* dual abelian scheme *of $A$ over $S$. A* quasi-polarization *of $A$ is a quasi-isogeny $\lambda : A \to A^\vee$ such that $n\lambda$ is a polarization of $A$ for some $n \in \mathbb{Z}$. A quasi-polarization $\lambda : A \to A^\vee$ is called* $p$-principal *if $\lambda$ is a prime-to-$p$ quasi-isogeny.*

**Notation 2.2.2.** *Let $A$ be an abelian variety over a scheme $S$. We denote by $\mathrm{H}_1^{\mathrm{dR}}(A/S)$ (resp. $\mathrm{Lie}_{A/S}$, resp. $\omega_{A/S}$ ) the* relative de Rham homology *(resp.* Lie algebra*, resp.* dual Lie algebra*) of $A/S$. They are all locally free $\mathcal{O}_S$-modules of finite rank. We have* Hodge exact sequence

$$0 \to \omega_{A^\vee/S} \to \mathrm{H}_1^{\mathrm{dR}}(A/S) \to \mathrm{Lie}_{A/S} \to 0. \tag{2.2}$$

*When the base $S$ is clear from the context, we sometimes suppress it from the notation.*

**Definition 2.2.3.** *Let $S \in \mathsf{Sch}_{/\mathbb{Z}_{(p)}}$.*

1. *An $O_F$-*abelian scheme *over $S$ is a pair $(A, i_A)$ in which $A$ is an abelian scheme over $S$ and $i_A : O_F \to \mathrm{End}_S(A) \otimes \mathbb{Z}_{(p)}$ is a ring homomorphism of algebras.*

2. *An* unitary $O_F$-abelian scheme *over $S$ is a triple $(A, i_A, \lambda_A)$ in which $(A, i_A)$ is an $O_F$-abelian scheme over $S$, and $\lambda_A : A \to A^\vee$ is a quasi-polarization such that $i_A(a^{\mathfrak{c}})^\vee \circ \lambda_A = \lambda_A \circ i(a)$ for every $a \in O_F$.*
3. *For two $O_F$-abelian schemes $(A, i_A)$ and $(A', i'_A)$ over $S$, a (quasi-)homomorphism from $(A, i_A)$ to $(A', i'_A)$ is a (quasi-)homomorphism $\varphi : A \to A'$ such that $\varphi \circ i_A(a) = i'_A(a) \circ \varphi$ for every $a \in O_F$. We will usually refer to such $\varphi$ as an $O_F$-linear (quasi-)homomorphism.*

*Moreover, we will usually suppress the notion $i_A$ if the argument is insensitive to it.*

**Definition 2.2.4** (Signature type)**.** *Let $(A, i_A)$ be an $O_F$-abelian scheme of dimension 3 over a scheme $S \in \mathsf{Sch}_{/O_F \otimes \mathbb{P}}$. Let $r, s$ be two nonnegative integers with $r + s = 3$. We say that $(A, i_A)$ has signature type $(r, s)$ if for every $a \in O_F$, the characteristic polynomial of $i_A(a)$ on $\mathrm{Lie}_{A/S}$ is given by*

$$(T - \tau_0(a))^r (T - \tau_1(a))^s \in \mathcal{O}_S[T].$$

**Remark 2.2.5.** *Let $A$ be an $O_F$-abelian scheme of dimension 3 of signature type $(r, s)$ over a scheme $S \in \mathsf{Sch}_{/k}$. Consider the decomposition*

$$\begin{aligned} O_F \otimes_{\mathbb{Z}} k &\xrightarrow{\simeq} k \times k \\ a \otimes x &\longmapsto (\overline{\tau_0(a)}x, \overline{\tau_1(a)}x) \end{aligned}$$

*where the bar denotes the mod $p$ quotient map. Then for any $O_F \otimes k$-module $N$ we have a canonical decomposition*

$$N = N_0 \oplus N_1 \tag{2.3}$$

*where $a \in O_F$ acts on $N_i$ through $\tau_i$. Then (2.2) induces two short exact sequences*

$$0 \to \omega_{A^\vee/S,i} \to \mathrm{H}_1^{\mathrm{dR}}(A/S)_i \to \mathrm{Lie}_{A/S,i} \to 0,\ i = 0, 1$$

*of locally free $\mathcal{O}_S$-modules of ranks $s, 3, r$ and $r, 3, s$.*

**Notation 2.2.6.** *Let $(A, \lambda_A)$ be a unitary $O_F$-abelian scheme of signature type $(r, s)$ over a scheme $S \in \mathsf{Sch}_{/O_{F,(p)}}$. We denote*

$$\langle\, , \rangle_{\lambda_A, i} : \mathrm{H}_1^{\mathrm{dR}}(A/S)_i \times \mathrm{H}_1^{\mathrm{dR}}(A/S)_{i+1} \to \mathcal{O}_S,\ \ i = 0, 1$$

*the $\mathcal{O}_S$-bilinear alternating pairing induced by the quasi-polarization $\lambda_A$, which is perfect if and only if $\lambda_A$ is $p$-principal. Moreover, for an $\mathcal{O}_S$-submodule $\mathcal{F} \subseteq \mathrm{H}_1^{\mathrm{dR}}(A/S)_i$, we denote by $\mathcal{F}^\perp \subseteq \mathrm{H}_1^{\mathrm{dR}}(A/S)_{i+1}$ (where $i \in \mathbb{Z}/2\mathbb{Z}$) its (right) orthogonal complement under the above pairing, if $\lambda$ is clear from the context.*

**Notation 2.2.7.** *In notation 2.2.6, put*

$$A^{(p)} := A \times_{S,\sigma} S,$$

*where $\sigma$ is the absolute Frobenius morphism of $S$. Then we have*

1. *a canonical isomorphism $\mathrm{H}_1^{\mathrm{dR}}(A^{(p)}/S) \simeq \sigma^* \mathrm{H}_1^{\mathrm{dR}}(A/S)$ of $\mathcal{O}_S$-modules ;*

2. *the Frobenius homomorphism* $\mathrm{Fr}_A : A \to A^{(p)}$ *which induces the* Verschiebung map
$$\mathtt{V}_A := (\mathrm{Fr}_A)_* : \mathrm{H}_1^{\mathrm{dR}}(A/S) \to \mathrm{H}_1^{\mathrm{dR}}(A^{(p)}/S)$$
*of* $\mathcal{O}_S$*-modules ;*

3. *the Verschiebung homomorphism* $\mathrm{Ver}_A : A^{(p)} \to A$ *which induces the* Frobenius map
$$\mathtt{F}_A := (\mathrm{Ver}_A)_* : \mathrm{H}_1^{\mathrm{dR}}(A^{(p)}/S) \to \mathrm{H}_1^{\mathrm{dR}}(A/S)$$
*of* $\mathcal{O}_S$*-modules.*

*In what follows, we will suppress* $A$ *in the notations* $\mathtt{F}_A$ *and* $\mathtt{V}_A$ *if the reference to* $A$ *is clear.*

In Notation 2.2.7, we have $\ker \mathtt{F} = \operatorname{im} \mathtt{V} = \omega_{A^{(p)}/S}$ and $\ker \mathtt{V} = \operatorname{im} \mathtt{F}$.

**Notation 2.2.8.** *Suppose that* $S = \operatorname{Spec} \kappa$ *for a perfect field* $\kappa$ *of characteristic* $p$ *containing* $\mathbb{F}_{p^2}$*. Then we have a canonical isomorphism* $\mathrm{H}_1^{\mathrm{dR}}(A^{(p)}/\kappa) \simeq \mathrm{H}_1^{\mathrm{dR}}(A/\kappa) \otimes_{\kappa,\sigma} \kappa$*.*

1. *By abuse of notation, we have*
   - *the* $(\kappa, \sigma)$*-linear Frobenius map* $\mathtt{F} : \mathrm{H}_1^{\mathrm{dR}}(A/\kappa) \to \mathrm{H}_1^{\mathrm{dR}}(A/\kappa)$ *and*
   - *the* $(\kappa, \sigma^{-1})$*-linear Verschiebung map* $\mathtt{V} : \mathrm{H}_1^{\mathrm{dR}}(A/\kappa) \to \mathrm{H}_1^{\mathrm{dR}}(A/\kappa)$*.*
2. *We have the* covariant *Dieudonné module* $\mathcal{D}(A)$ *associated to the* $p$*-divisible group* $A[p^\infty]$*, which is a free* $W(\kappa)$*-module, such that* $\mathcal{D}(A)/p\mathcal{D}(A)$ *is canonically isomorphic to* $\mathrm{H}_1^{\mathrm{dR}}(A/\kappa)$*. Again by abuse of notation, we have*
   - *the* $(W(\kappa), \sigma)$*-linear Frobenius map* $\mathtt{F} : \mathcal{D}(A) \to \mathcal{D}(A)$ *lifting the one above, and*
   - *the* $(W(\kappa), \sigma^{-1})$*-linear Verschiebung map* $\mathtt{V} : \mathcal{D}(A) \to \mathcal{D}(A)$ *lifting the one above, respectively, satisfying* $\mathtt{F} \circ \mathtt{V} = \mathtt{V} \circ \mathtt{F} = p$*.*

**Remark 2.2.9.** *Similar to 2.3 we also have a decomposition*
$$\mathcal{D}(A) = \mathcal{D}(A)_0 \oplus \mathcal{D}(A)_1.$$

Let $(A, \lambda_A)$ be a unitary $O_F$-abelian scheme of signature type $(r, s)$ over Spec $\kappa$. We have a pairing
$$\langle\, ,\, \rangle_{\lambda_A} : \mathcal{D}(A) \times \mathcal{D}(A) \to W(\kappa)$$
lifting the one in Notation 2.2.6. We denote by $\mathcal{D}(A)^{\perp_A}$ the $W(\kappa)$-dual of $\mathcal{D}(A)$
$$\mathcal{D}(A)^{\perp_A} := \{x \in \mathcal{D}(A)[1/p] \mid \langle x, y\rangle_{\lambda_A} \in W(\kappa), \forall y \in \mathcal{D}(A).\}$$
as a submodule of $\mathcal{D}(A)[1/p]$. We have the following properties :

1. The direct summands in (2.3) are totally isotropic with respect to $\langle , \rangle_{\lambda_A}$.
2. we have
$$\langle \mathtt{F}x, y\rangle_{\lambda_A} = \langle x, \mathtt{V}y\rangle_{\lambda_A}^{\sigma},\ \langle i_A(a)x, y\rangle_{\lambda_A} = \langle x, i_A(a^{\mathtt{c}})y\rangle_{\lambda_A}$$
   for $a \in O_F$.

Next we review some facts from the Serre-Tate theory [Kat81] and the Grothendieck-Messing theory [Mes72], tailored to our application. Consider a closed immersion $S \hookrightarrow \hat{S}$ in $\mathtt{Sch}_{/\mathbb{Z}_{p^2}}$ on which $p$ is locally nilpotent, with its ideal sheaf equipped with a PD structure, and a unitary $O_F$-abelian scheme $(A, \lambda)$ of signature type $(r, s)$ over $S$. We let $\mathrm{H}_1^{\mathrm{cris}}(A/\hat{S})$ be the evaluation of the first relative crystalline homology of $A/S$ at the PD-thickening $S \hookrightarrow \hat{S}$, which is a locally free $\mathcal{O}_{\hat{S}} \otimes O_F$-module. The polarization $\lambda$ induces a pairing

$$\langle , \rangle_{\lambda,i}^{\mathrm{cris}} : \mathrm{H}_1^{\mathrm{cris}}(A/\hat{S})_i \times \mathrm{H}_1^{\mathrm{cris}}(A/\hat{S})_{i^c} \to \mathcal{O}_{\hat{S}},\ i = 0, 1. \tag{2.4}$$

We define two groupoids

— $\mathrm{Def}(S, \hat{S}; A, \lambda)$, whose objects are unitary $O_F$-abelian schemes $(\hat{A}, \hat{\lambda})$ of signature type $(r, s)$ over $\hat{S}$ that lift $(A, \lambda)$ ;

— $\mathrm{Def}'(S, \hat{S}; A, \lambda)$, whose objects are pairs $(\hat{\omega}_0, \hat{\omega}_1)$ where $\hat{\omega}_i \subseteq \mathrm{H}_1^{\mathrm{cris}}(A/\hat{S})_i$ is a subbundle that lifts $\omega_{A^\vee/S,i} \subseteq \mathrm{H}_1^{\mathrm{dR}}(A/S)_i$ for $i = 0, 1$, such that $\langle \hat{\omega}_0, \hat{\omega}_1 \rangle_{\lambda,1}^{\mathrm{cris}} = 0$.

**Proposition 2.2.10.** *The functor from* $\mathrm{Def}(S, \hat{S}; A, \lambda)$ *to* $\mathrm{Def}'(S, \hat{S}; A, \lambda)$ *sending* $(\hat{A}, \hat{\lambda})$ *to* $(\omega_{\hat{A}^\vee,0}, \omega_{\hat{A}^\vee,1})$ *is a natural equivalence.*

### 2.2.5 Moduli problems

Fix an open compact subgroup $K^p \subset G(\mathbb{A}^{\infty,p})$.

**Definition 2.2.11.** *Let* $S$ *be the moduli problem that associates with every* $O_{F,(p)}$*-algebra* $R$ *the set* $S(R)$ *of equivalence classes of triples* $(A, \lambda_A, \eta_A)$*, where*

— $(A, \lambda_A)$ *is a unitary* $O_F$*-abelian scheme of signature type (1,2) over* $R$ *such that* $\lambda_A$ *is p-principal ;*

— $\eta_A$ *is a* $K^p$*-level structure, that is, for a chosen geometric point* $s$ *on every connected component of* $\operatorname{Spec} R$*, a* $\pi_1(\operatorname{Spec} R, s)$*-invariant* $K^p$*-orbit of isomorphisms*

$$\eta_A : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} \mathrm{H}_1^{\mathrm{et}}(A, \mathbb{A}^{\infty,p})$$

*such that the skew hermitian pairing* $\langle \_, \_ \rangle_\psi$ *on* $V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ *corresponds to the* $\lambda_A$*-Weil pairing on* $\mathrm{H}_1^{\mathrm{et}}(A, \mathbb{A}^{\infty,p})$ *up to scalar.*

*Two triples* $(A, \lambda_A, \eta_A)$ *and* $(A', \lambda_{A'}, \eta_{A'})$ *are equivalent if there is a prime-to-p* $O_F$*-linear isogeny* $\varphi : A \to A'$ *such that*

— *there exists* $c \in \mathbb{Z}_{(p)}^\times$ *such that* $\varphi^\vee \circ \lambda_{A'} \circ \varphi = c\lambda_A$*; and*

— *the* $K^p$*-orbit of maps* $v \mapsto \varphi_* \circ \eta_A(v)$ *for* $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ *coincides with* $\eta_{A'}$*.*

*Given* $g \in K^p \backslash G(\mathbb{A}^{\infty,p})/K'^p$ *such that* $g^{-1}K^p g \subset K'^p$*, we have a map* $S(K^p)(R)$ *to* $S(K'^p)(R)$ *by changing* $\eta_A$ *to* $\eta_A \circ g$*.*

**Definition 2.2.12.** *Let* $\widetilde{S}$ *be the moduli problem that associates with every* $O_{F,(p)}$*-algebra* $R$ *the set* $\widetilde{S}(R)$ *of equivalence classes of triples* $(\tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}})$*, where*

1. $(\tilde{A}, \lambda_{\tilde{A}})$ *is a unitary* $O_F$*-abelian scheme of signature type (1,2) over* $R$ *such that* $\ker \lambda_{\tilde{A}}[p^\infty]$ *is contained in* $\tilde{A}[p]$ *of rank* $p^2$*;*
2. $\eta_{\tilde{A}}$ *is a* $K^p$*-level structure.*

*The equivalence relation and the action of* $G(\mathbb{A}^{\infty,p})$ *are defined similarly as in Definition 2.2.15.*

**Definition 2.2.13.** *The moduli problem $S_0(p)$ associates with every $O_F \otimes \mathbb{Z}_{(p)}$-algebra $R$ the set $S_0(p)(R)$ of equivalence classes of sextuples $(A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha)$ where*

1. $(A, \lambda_A, \eta_A)$ *is an element in* $S(R)$.
2. $(\tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}})$ *is an element in* $\tilde{S}(R)$.
3. $\alpha : A \to \tilde{A}$ *is an $O_F$-linear quasi-p-isogeny such that*
$$p\lambda_A = \alpha^\vee \circ \lambda_{\tilde{A}} \circ \alpha.$$
4. $\ker \alpha \subset A[p]$ *is a Raynaud $O_F$-subgroup scheme of rank $p^2$, which is isotropic for the $\lambda_A$-Weil pairing*
$$e_p : A[p] \times A[p] \to \mu_p.$$

*For the definition of Raynaud subgroup, see* [dSG18, 1.2.1].

*Two septuplets $(A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha)$ and $(A', \lambda'_A, \eta'_A, \tilde{A}', \lambda_{\tilde{A}'}, \eta_{\tilde{A}'}, \alpha')$ are equivalent if there are $O_F$-linear prime-to-p quasi-isogenies $\varphi : A \to A'$ and $\varphi' : \tilde{A} \to \tilde{A}'$ such that*

— *there exists $c \in \mathbb{Z}_{(p)}^\times$ such that $\varphi^\vee \circ \lambda_{A'} \circ \varphi = c\lambda_A$ and $\varphi^\vee \circ \lambda_{\tilde{A}'} \circ \varphi = c\lambda_{\tilde{A}}$.*
— *the $K^p$-orbit of maps $v \mapsto \varphi_* \circ \eta_A(v)$ for $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ coincides with $\eta_{A'}$.*
— *the $K^p$-orbit of maps $v \mapsto \varphi'_* \circ \eta_{A'}(v)$ for $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ coincides with $\eta_{\tilde{A}'}$.*

It is well known that, for sufficiently small $K^p$, the three moduli problems $S$, $\tilde{S}$ and $S_0(p)$ are all representable by quasi-projective schemes over $O_{F,(p)}$, still denoted by $S, \tilde{S}, S_0(p)$ by abuse of notation, and give integral models of $\mathrm{Sh}(G, K^pK_p)$, $\mathrm{Sh}(G, K^p\tilde{K}_p)$ and $\mathrm{Sh}(G, K^p\mathrm{Iw}_p)$ respectively. We have natural forgetful maps $\pi : S_0(p) \to S$ sending $(A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha)$ to $(A, \lambda_A, \eta_A)$, and $\tilde{\pi} : S_0(p) \to \tilde{S}$ sending $(A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha)$ to $(\tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}})$. This gives rise to the diagram

$$\begin{array}{ccc} & S_0(p) & \\ {}^{\pi}\swarrow & & \searrow^{\tilde{\pi}} \\ S & & \tilde{S} \end{array}$$

**Remark 2.2.14.** *For the convenience of readers, we recall why $S$ is an integral model of $\mathrm{Sh}(G, K^pK_p)$. We shall content ourselves with describing a canonical bijection $S(\mathbb{C}) \simeq \mathrm{Sh}(G, K^pK_p)(\mathbb{C})$, which determines uniquely an isomorphism $S \otimes_{O_{F,(p)}} F \cong \mathrm{Sh}(G, K^pK_p)$. It suffices to assign to each point*
$$s = (A, \lambda_A, \eta_A) \in S(\mathbb{C})$$
*a point in*
$$\mathrm{Sh}(G, K^pK_p)(\mathbb{C}) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}^\infty)/K^pK_p)$$
*Let $H := \mathrm{H}_1(A, \mathbb{Q})$, which is an $F$-vector space by the action of $O_F$ on $A$. The polarization $\lambda_A$ induces a structure of skew hermitian space on $H$. By Hodge theory, the composed map*
$$H \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{H}_1^{\mathrm{dR}}(A, \mathbb{R}) \to \mathrm{H}_1^{\mathrm{dR}}(A, \mathbb{C}) \to \mathrm{Lie}_A$$

*is an isomorphism, which gives a complex structure on $H\otimes_{\mathbb{Q}}\mathbb{R}$. The signature condition on $A$ ensures an isomorphism of (skew) hermitian spaces $H\otimes_{\mathbb{Q}}\mathbb{R}\cong V\otimes_{\mathbb{Q}}\mathbb{R}$. Now look at the place $p$. Since $A$ is an abelian variety up to prime-to-$p$ isogeny, the $\mathbb{Z}_p$-module $\Lambda_H := \mathrm{H}_1^{\text{ét}}(A,\mathbb{Z}_p)$ is well-defined and gives a self-dual lattice in $H\otimes_{\mathbb{Q}}\mathbb{Q}_p \cong \mathrm{H}_1^{\text{ét}}(A,\mathbb{Q}_p)$. Hence there exists an isomorphism of hermitian spaces $H\otimes_{\mathbb{Q}}\mathbb{Q}_p\cong V\otimes_{\mathbb{Q}}\mathbb{Q}_p$. In addition to the prime-to-p level structure $\eta_A$, the Hasse principle implies that there exists globally an isomorphism of hermitian spaces $\xi : H \xrightarrow{\sim} V$ over $F$ up to similitude. Fix such a $\xi$. First, the complex structure on $H\otimes_{\mathbb{Q}}\mathbb{R}$ transfers via $\xi$ to a homomorphism of $\mathbb{R}$-algebras $\mathbb{C}\to \mathrm{End}_F(V)\otimes_{\mathbb{Q}}\mathbb{R}$, which leads to an element $x\in X = G(\mathbb{R})/K_\infty$ because of the signature condition. Secondly, postcomposing with $\xi$, the level structure $\eta_A$ gives a coset $g^pK^p := \xi\circ\eta_A \in G(\mathbb{A}^{\infty,p})/K^p$. At last, there exists a coset $g_pK_p\in G(\mathbb{Q}_p)/K_p$ such that $\xi(\Lambda_H) = g_p(\Lambda_0)$ as lattices of $V\otimes_{\mathbb{Q}}\mathbb{Q}_p$ for any representative $g_p$ of $g_pK_p$. Note that a different choice of $\xi$ differs by the left action of an element of $G(\mathbb{Q})$. It follows that the class*

$$[x, g^pK^p, g_pK_p] \in G(\mathbb{Q})\backslash(X\times G(\mathbb{A}^{\infty,p})/K^p \times G(\mathbb{Q}_p)/K_p)$$

*does not depend on the choice of $\xi$, and gives the point of $\mathrm{Sh}(G, K^pK_p)$ corresponding to $s\in S(\mathbb{C})$.*

### 2.2.6 An inner form of $G$

Let $(W,\psi_W)$ be a hermitian space over $F$ of dimension 3 such that it is isomorphic to $(V\otimes_{\mathbb{Q}}\mathbb{A}^\infty,\psi)$ as hermitian spaces over $\mathbb{A}^\infty$, and $(W\otimes_{\mathbb{Q}}\mathbb{R},\psi_W)$ has signature $(0,3)$. Such a $(W,\psi_W)$ exists and is unique up to isomorphism. Let $G'$ be the unitary similitude group over $\mathbb{Q}$ attached to $(W,\psi_W)$. Then $G'$ is an inner form of $G$ such that $G'(\mathbb{A}^\infty)\cong G(\mathbb{A}^\infty)$. In the sequel, we fix such an isomorphism so that $K^p$ and $K_p$ are also viewed respectively as subgroups of $G'(\mathbb{A}^{\infty,p})$ and $G'(\mathbb{Q}_p)$. As $G'(\mathbb{R})$ is compact modulo center, for an open compact subgroup $K'\subseteq G'(\mathbb{A}^\infty)$, we have a finite set

$$\mathrm{Sh}(G',K') := G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/K'.$$

We will give moduli interpretations for $\mathrm{Sh}(G',K^pK_p)$, $\mathrm{Sh}(G',K^p\tilde{K}_p)$ and $\mathrm{Sh}(G',K^p\mathrm{Iw}_p)$.

**Definition 2.2.15.** *The moduli problem $T$ is to associate with every $O_{F,(p)}$-algebra $R$ the set $T(R)$ of equivalence classes of triples $(B,\lambda_B,\eta_B)$, where*

- *$(B,\lambda_B)$ is a unitary $O_F$-abelian scheme of signature type (0,3) over $R$ such that $\lambda_B$ is $p$-principal;*
- *$\eta_B$ is a $K^p$-level structure, that is, for a chosen geometric point $s$ on every connected component of $\mathrm{Spec}\,R$, $\eta_B$ is a $\pi_1(\mathrm{Spec}\,R, s)$-invariant $K^p$-orbit of isomorphisms*

$$\eta_B : W\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p} \xrightarrow{\sim} \mathrm{H}_1(B,\mathbb{A}^{\infty,p})$$

*of hermitian spaces over $F\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p}$.*

*Two triples $(B,\lambda_B,\eta_B)$ and $(B',\lambda_{B'},\eta_{B'})$ are equivalent if there is a prime-to-p $O_F$-linear isogeny $\varphi : B\to B'$ such that*

- *there exists $c\in\mathbb{Z}_{(p)}^\times$ such that $\varphi^\vee\circ\lambda_{B'}\circ\varphi = c\lambda_B$; and*
- *the $K^p$-orbit of maps $v\mapsto \varphi_*\circ\eta_B(v)$ for $v\in W\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p}$ coincides with $\eta_{B'}$.*

*Given $g\in K^p\backslash G'(\mathbb{A}^{\infty,p})/K'^p$ such that $g^{-1}K^pg\subset K'^p$, we have a map $T(K^p)(U)$ to $T(K^{p\prime})(U)$ by changing $\eta_A$ to $\eta_A\circ g$.*

**Definition 2.2.16.** *The moduli problem* $\widetilde{T}$ *is to associate with every* $O_F \otimes \mathbb{Z}_{(p)}$*-algebra* $R$ *the set* $\widetilde{T}(R)$ *of equivalence classes of triples* $(\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}})$*, where*

1. $(\tilde{B}, \lambda_{\tilde{B}})$ *is a unitary* $O_F$*-abelian scheme of signature type (0,3) over* $R$ *such that* $\ker \lambda_{\tilde{B}}[p^\infty]$ *is contained in* $\tilde{B}[p]$ *of rank* $p^2$*;*
2. $\eta_{\tilde{B}}$ *is a* $K^p$*-level structure.*

*The equivalence relation and the action of* $G(\mathbb{A}^{\infty,p})$ *are defined similarly as in Definition 2.2.12.*

**Definition 2.2.17.** *The moduli problem* $T_0(p)$ *is to associate with every* $O_F \otimes \mathbb{Z}_{(p)}$*-algebra* $R$ *the set* $T_0(p)(R)$ *of equivalence classes of sextuples* $(B, \lambda_B, \eta_B, \tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, \beta)$ *where*

1. $(B, \lambda_B, \eta_B) \in T(R)$*;*
2. $(\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}) \in \tilde{T}(R)$*;*
3. $\beta : \tilde{B} \to B$ *is an isogeny such that*
$$p\lambda_{\tilde{B}} = \beta^\vee \circ \lambda_B \circ \beta;$$
4. $\ker \beta$ *is a* $O_F$*-subgroup scheme of* $\tilde{B}[p]$ *of rank* $p^4$*, which is isotropic for the* $\lambda_{\tilde{B}}$*-Weil pairing*
$$e_p : \tilde{B}[p] \times \tilde{B}[p] \to \mu_p.$$

*The equivalence relation and the action of* $G(\mathbb{A}^{\infty,p})$ *are defined similarly as in Definition 2.2.13.*

For sufficiently small $K^p$, three moduli problems defined above are representable by quasi-projective schemes over $O_{F,(p)}$. By abuse of notation we still denote them by $T, \tilde{T}, T_0(p)$.

**Proposition 2.2.18.** *We have the uniformization maps*

$$\upsilon : T(\mathbb{C}) \cong \mathrm{Sh}(G', K^p K_p)$$
$$\tilde{\upsilon} : \tilde{T}(\mathbb{C}) \cong \mathrm{Sh}(G', K^p \tilde{K}_p)$$
$$\upsilon_0 : T_0(p)(\mathbb{C}) \cong \mathrm{Sh}(G', K^p \mathrm{Iw}_p)$$

*which is equivariant under prime-to-p Hecke correspondence. That is, given* $g \in K^p \backslash G(\mathbb{A}^{\infty,p})/K'^p$*, we have the commutative diagram*

$$\begin{array}{ccc} T(K^p)(\mathbb{C}) & \xrightarrow{\upsilon} & \mathrm{Sh}(G', K^p K_p) \\ \downarrow g & & \downarrow g \\ T(K'^p)(\mathbb{C}) & \xrightarrow{\upsilon} & \mathrm{Sh}(G', K'^p K_p) \end{array}$$

*for* $g \in K^p \backslash G(\mathbb{A}^{\infty,p})/K'^p$ *such that* $g^{-1}K^p g \subset K'^p$*. Here we use* $T(K^p)$ *to emphasize the dependence of* $T$ *on* $K^p$*. Similar diagrams hold for* $\tilde{T}$ *and* $T_0(p)$*.*

*Proof.* Similar to Remark 2.2.14. It is worthwhile noting the signature type condition forces the image of $\mathbb{C} \to \mathrm{End}_F(W) \otimes \mathbb{R}$ lies in the center $F \otimes \mathbb{R}$. $\square$

## 2.3 The geometry of geometric special fiber

Let $k$ be a prefect field. Denote by $S_k$ or $S \otimes k$ the base change of $S$ to $k$. If $k = \mathbb{F}_{p^2}$ we denote still by $S$ the special fiber $S \otimes \mathbb{F}_{p^2}$. Same notation holds for other integral models.

### 2.3.1 The geometry of $S$

We recall the Ekedahl-Oort stratification on $S$, which has been studied extensively in [Wed01, BW06, VW11]. Given $(A, i_A, \lambda_A, \eta_A) \in S(k)$. Define two standard Dieudonné modules as "building blocks" of $\mathcal{D}(A[p])$ :

**Definition 2.3.1.** [BW06, 3.2],[VW11, 2.4, 3.1]

1. *Define a superspecial unitary Dieudonné module $\mathcal{S}$ over $k$ as follows. It is a free $W(k)$-module of rank 2 with a base $\{g, h\}$. Set*

$$\mathcal{S}_0 = W(k)g,\ \mathcal{S}_1 = W(k)h,\ \mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1.$$

*$\mathcal{S}$ is equipped by the natural $O_F \otimes W(k)$ action.*

*Define an alternating form on $\mathcal{S}$ by $\langle g, h\rangle = -1$. Define a $(W(k), \sigma)$-linear map $\mathtt{F}$ on $\mathcal{S}$ by $\mathtt{F}g = ph$ and $\mathtt{F}h = -g$. Define a $(W(k), \sigma^{-1})$-linear map $\mathtt{V}$ by $\mathtt{V}h = g$ and $\mathtt{V}g = -ph$. This makes $\mathcal{S}$ is a unitary Dieudonné module of signature $(0, 1)$. Write by $\overline{\mathcal{S}}$ its reduction mod $p$.*

2. *For an integer $r \geqslant 1$ define a unitary Dieudonné module $\mathcal{B}(r)$ over $k$ as follows. It is a free $W(k)$-module of rank $2r$ with a base $(e_1, \ldots, e_r, f_1, \ldots, f_r)$. Set*

$$\mathcal{B}(r)_0 = W(k)e_1 \oplus \cdots \oplus W(k)e_r,$$
$$\mathcal{B}(r)_1 = W(k)f_1 \oplus \cdots \oplus W(k)f_r,\ \mathcal{B}(r) = \mathcal{B}(r)_0 \oplus \mathcal{B}(r)_1.$$

*The alternating form is defined by*

$$\langle e_i, f_j\rangle = (-1)^i \delta_{ij}.$$

*Finally, define a $\sigma$-linear map $\mathtt{F}$ and a $\sigma^{-1}$-linear map $\mathtt{V}$ by*

$$\begin{aligned}
\mathtt{V}e_i &= pf_{i+1}, && \text{for } i = 1, \ldots, r-1,\\
\mathtt{V}e_r &= f_1, &&\\
\mathtt{V}f_i &= e_{i+1}, && \text{for } i = 1, \ldots, r-1,\\
\mathtt{V}f_n &= pe_1, &&\\
\mathtt{F}e_1 &= (-1)^r f_r, &&\\
\mathtt{F}e_i &= pf_{i-1}, && \text{for } i = 2, \ldots, r,\\
\mathtt{F}f_1 &= pe_r, &&\\
\mathtt{F}f_i &= e_{i-1}, && \text{for } i = 2, \ldots, r,
\end{aligned}$$

*This is a unitary Dieudonné module of signature $(1, r-1)$. Write by $\overline{\mathcal{B}}(r)$ its reduction mod $p$.*

**Proposition 2.3.2.** *Let $x = (A, i_A, \lambda_A, \eta_A) \in S(k)$. Then $\mathcal{D}(A[p]) \cong \mathcal{D}(A)/p$ is isomorphic to*

$$\overline{\mathcal{B}}(r) \oplus \overline{\mathcal{S}}^{\oplus 3-r}$$

*for some integer $r$ with $1 \leqslant r \leqslant 3$.*

The *Ekedahl-Oort stratification* in our case is given by

$$S = S_1 \bigsqcup S_2 \bigsqcup S_3,$$

where each $S_i$ is a reduced locally closed subscheme, and a geometric point $(A, i_A, \lambda_A, \eta_A) \in S(k)$ lies in $S_i(k)$ if and only if

$$\mathrm{H}_1^{\mathrm{dR}}(A/k)_0 \cong \overline{\mathcal{B}}(i) \oplus \overline{\mathcal{S}}^{\oplus 3-i}.$$

All $S_i$ are equidimensional [Wed01, Section 6], and we have $\dim S_2 = 2, \dim S_1 = 0$ and $\dim S_3 = 1$.

The open stratum $S_2$ is usually called the *$\mu$-ordinary* locus, and denoted by $S_\mu$. Its complement $S_{\mathrm{ss}} := S_1 \cup S_3 = S \backslash S_2$ is the *supersingular* locus, i.e., the associated $F$-isocrystal $(\mathcal{D}(A)[1/p], \mathtt{F})$ of $A$ has Newton slope $1/2$. Furthermore, the stratum $S_1$ is exactly the locus where $\mathtt{F}\mathcal{D}(A) = \mathtt{V}\mathcal{D}(A)$ holds. It is called the *superspecial* locus and denoted by $S_{\mathrm{sp}}$. The stratum $S_3$ is called *general supersingular* locus, denoted by $S_{\mathrm{gss}}$. We will study the irreducible components of supersingular locus $S_{\mathrm{ss}}$.

### 2.3.2 Unitary Deligne-Lusztig variety

Let $\kappa$ be a field containing $\mathbb{F}_{p^2}$ and denote by $\overline{\kappa}$ one of its algebraic closure. Recall $\sigma : S \to S$ denotes the absolute $p$-power Frobenius morphism for schemes $S$ in characteristic $p$.

**Definition 2.3.3.** *Consider a pair $(\mathscr{V}, \{\,,\,\})$ in which $\mathscr{V}$ is a $\kappa$-linear space of dimension 3, and $\{\,,\,\} : \mathscr{V} \times \mathscr{V} \to \kappa$, is a non-degenerate pairing that is $\kappa$-linear in the first variable and $(\kappa, \sigma)$-linear in the second variable. For every $\kappa$-scheme $S$, put $\mathscr{V}_S := \mathscr{V} \otimes_\kappa \mathcal{O}_S$. Then there is a unique pairing $\{\,,\,\}_S : \mathscr{V}_S \times \mathscr{V}_S \to \mathcal{O}_S$ extending $\{\,,\,\}$ that is $\mathcal{O}_S$-linear in the first variable and $(\mathcal{O}_S, \sigma)$-linear in the second variable. For a subbundle $H \subseteq \mathscr{V}_S$, we denote by $H^\perp \subseteq \mathscr{V}_S$ its orthogonal complement under $\{\,,\,\}_S$ defined by*

$$H^\perp = \{x \in \mathscr{V}_S \mid \{x, H\}_S = 0\}.$$

*When the pairing is induced by a (quasi-)polarization $\lambda_A$ of an abelian variety $A$, we write $\perp_{\bar{A}}$ instead of $\perp$ to specify.*

**Definition 2.3.4.** *We say that a pair $(\mathscr{V}, \{\,,\,\})$ is* admissible *if there exists an $\mathbb{F}_{p^2}$-linear subspace $\mathscr{V}_0 \subseteq \mathscr{V}_{\overline{\kappa}}$ such that the induced map $\mathscr{V}_0 \otimes_{\mathbb{F}_{p^2}} \overline{\kappa} \to \mathscr{V}_{\overline{\kappa}}$ is an isomorphism, and $\{x, y\} = -\{y, x\}^\sigma = \{x, y\}^{\sigma^2}$ for every $x, y \in \mathscr{V}_0$.*

**Definition 2.3.5.** *Let $\mathrm{DL}(\mathscr{V}, \{\,,\,\})$ be the moduli problem associating with every $\kappa$-algebra $R$ the set $\mathrm{DL}(\mathscr{V}, \{\,,\,\})(R)$ of subbundles $H$ of $\mathscr{V}_R$ of rank 2 such that $H^\perp \subseteq H$. We call $\mathrm{DL}(\mathscr{V}, \{\,,\,\}, h)$ the* (unitary) Deligne-Lusztig variety *attached to $(\mathscr{V}, \{\,,\,\})$ of rank 2.*

**Proposition 2.3.6.** *Consider an admissible pair* $(\mathscr{V},\{\ ,\ \})$. *Then* $\mathrm{DL}(\mathscr{V},\{\ ,\ \})$ *is represented by a projective smooth scheme over* $\kappa$ *of dimension* 1 *with a canonical isomorphism for its tangent sheaf*

$$\mathcal{T}_{\mathrm{DL}(\mathscr{V},\{,\})/\kappa} \simeq \mathcal{H}om(\mathcal{H}/\mathcal{H}^{\perp}, \mathscr{V}_{\mathrm{DL}(\mathscr{V},\{,\})}/\mathcal{H})$$

*where* $\mathcal{H} \subseteq \mathscr{V}_{\mathrm{DL}(\mathscr{V},\{,\})}$ *is the universal subbundle. Moreover,* $\mathrm{DL}(\mathscr{V},\{,\}) \otimes_{\kappa} \bar{\kappa}$ *is isomorphic to the Fermat curve* $\mathscr{C} \subset \mathbb{P}^2_{\bar{\kappa}}$ :

$$\mathscr{C} : \{(x:y:z) \in \mathbb{P}^2_{\bar{\kappa}} | x^{p+1} + y^{p+1} + z^{p+1} = 0\}.$$

*Proof.* For the first part, see [LTX+22, Proposition A.1.3]. For the second part, by admissibility there exists an $\mathbb{F}_{p^2}$-linear space $\mathscr{V}_0$ such that $\mathscr{V}_0 \otimes \bar{\kappa} \to \mathscr{V}_{\bar{\kappa}}$ is an isomorphism. Fix an element $\delta \in \mathbb{F}_{p^2}^{\times}$ such that $\delta^{\sigma} = -\delta$. Then we can find a basis $\{e_1, e_2, e_3\}$ of $\mathscr{V}_0$ which can be regarded as a basis of $\mathscr{V}_{\bar{\kappa}}$ such that $\{e_i, e_j\} = \delta\delta_{ij}$. Take a rank 2 $\bar{\kappa}$-subspace $H$ of $\mathscr{V}_{\bar{\kappa}}$. If $e_3 \notin H$, we can assume $H = \{ze_1 + xe_3, ze_2 + ye_3\}$ where $x, y, z \in \bar{\kappa}$ and $z \neq 0$. Then $H^{\perp} = \{-x^p e_1 - y^p e_2 + z^p e_3\}$. The condition $H^{\perp} \subset H$ is equivalent to $H^{\perp} \cap H \neq \{0\}$, i.e.,

$$\begin{vmatrix} z & 0 & x \\ 0 & z & y \\ -x^p & -y^p & z^p \end{vmatrix} = z(x^{p+1} + y^{p+1} + z^{p+1}) = 0.$$

Thus $x^{p+1} + y^{p+1} + z^{p+1} = 0$. It is easy to see the map $\{ze_1 + xe_3, ze_2 + ye_3\} \mapsto (x : y : z)$ extends to an isomorphism $\mathrm{DL}(\mathscr{V},\{,\}) \otimes_{\kappa} \bar{\kappa} \cong \mathscr{C}$. □

**Notation 2.3.7.** *Take a point* $t = (B, \lambda_B, \eta_B) \in T(\kappa)$. *Then* $B[p^{\infty}]$ *is a supersingular* $p$*-divisible group by the signature condition and the fact that* $p$ *is inert in* $F$. *From Notation 2.2.7, we have the* $(\kappa, \sigma)$*-linear Frobenius map*

$$\mathtt{F} : \mathrm{H}_1^{\mathrm{dR}}(B/\kappa)_i \to \mathrm{H}_1^{\mathrm{dR}}(B/\kappa)_{i+1}, \quad i \in \mathbb{Z}/2\mathbb{Z}.$$

*which can be lifted to*

$$\mathtt{F} : \mathcal{D}(B)_i \to \mathcal{D}(B)_{i+1}.$$

*We define a pairing*

$$\{\ ,\ \}_t : \mathrm{H}_1^{\mathrm{dR}}(B/\kappa)_i \times \mathrm{H}_1^{\mathrm{dR}}(B/\kappa)_i \to \kappa$$

*by the formula* $\{x, y\}_t := \langle x, \mathtt{F}y\rangle_{\lambda_B}$. *This pairing can also be lifted to*

$$\{\ ,\ \}_t : \mathcal{D}(B)_i \times \mathcal{D}(B)_i \to W(\kappa)$$

*To ease notation, we put*

$$\mathscr{V}_t := \mathrm{H}_1^{\mathrm{dR}}(B/\kappa)_1.$$

**Lemma 2.3.8.** *The pair* $(\mathscr{V}_t, \{\ ,\ \}_t)$ *is admissible of rank 3. In particular, the Deligne-Lusztig variety* $\mathrm{DL}_t := \mathrm{DL}(\mathscr{V}_t, \{\ ,\ \}_t)$ *is a geometrically irreducible projective smooth scheme in* $\mathsf{Sch}_{/\kappa}$ *of dimension 1 with a canonical isomorphism for its tangent sheaf*

$$\mathcal{T}_{\mathrm{DL}_t/\kappa} \simeq \mathcal{H}om(\mathcal{H}/\mathcal{H}^{\perp}, (\mathscr{V}_t)_{\mathrm{DL}_t}/\mathcal{H})$$

*where* $\mathcal{H} \subseteq (\mathscr{V}_t)_{D_t}$ *is the universal subbundle.*

*Proof.* It follows from the construction that $\{\ ,\ \}_t$ is $\kappa$-linear in the first variable and $(\kappa, \sigma)$-linear in the second variable. Thus by Proposition 2.3.6 it suffices to show that $(\mathscr{V}_t, \{,\}_t)$ is admissible.

Note that we have a canonical isomorphism $(\mathscr{V}_t)_{\bar{\kappa}} = \mathrm{H}_1^{\mathrm{dR}}(B/\kappa)_i \otimes_\kappa \bar{\kappa} \simeq \mathrm{H}_1^{\mathrm{dR}}(B_{\bar{\kappa}}/\bar{\kappa})_i$, and that the $(\bar{\kappa}, \sigma)$-linear Frobenius map $\mathtt{F} : \mathrm{H}_1^{\mathrm{dR}}(B_{\bar{\kappa}}/\bar{\kappa})_i \to \mathrm{H}_1^{\mathrm{dR}}(B_{\bar{\kappa}}/\bar{\kappa})_{i+1}$ and the $(\bar{\kappa}, \sigma^{-1})$-linear Verschiebung map $\mathtt{V} : \mathrm{H}_1^{\mathrm{dR}}(B_{\bar{\kappa}}/\bar{\kappa})_{i+1} \to \mathrm{H}_1^{\mathrm{dR}}(B_{\bar{\kappa}}/\bar{\kappa})_i$ are both bijective. Thus, we obtain a $(\bar{\kappa}, \sigma^2)$-linear bijective map $\mathtt{V}^{-1}\mathtt{F} : \mathrm{H}_1^{\mathrm{dR}}(B_{\bar{\kappa}}/\bar{\kappa})_i \to \mathrm{H}_1^{\mathrm{dR}}(B_{\bar{\kappa}}/\bar{\kappa})_i$. Denote by $\mathscr{V}_0$ the invariant subspace of $\mathrm{H}_1^{\mathrm{dR}}(B_{\bar{\kappa}}/\bar{\kappa})_i$ under $\mathtt{V}^{-1}\mathtt{F}$. Then the canonical map $\mathscr{V}_0 \otimes_{\mathbb{F}_{p^2}} \bar{\kappa} \to \mathrm{H}_1^{\mathrm{dR}}(B/\bar{\kappa})_i = (\mathscr{V}_t)_{\bar{\kappa}}$ is an isomorphism. For $x, y \in \mathscr{V}_0$, we have

$$\{x, y\}_t = \langle x, \mathtt{F}y\rangle_{\lambda_B} = \langle \mathtt{V}x, y\rangle_{\lambda_B}^\sigma = \langle \mathtt{F}x, y\rangle_{\lambda_B}^\sigma = -\langle y, \mathtt{F}x\rangle_{\lambda_B}^\sigma = -\{y, x\}_t^\sigma$$

Thus, $(\mathscr{V}_t, \{\ ,\ \}_t)$ is admissible. The lemma follows. □

### 2.3.3 Basic correspondence

We define a new moduli problem which gives the normalization of the supersingular locus $S_{\mathrm{ss}}$.

**Definition 2.3.9.** *The moduli problem $N$ associates with every $\mathbb{F}_{p^2}$-algebra $R$ the set $N(R)$ of equivalence classes of sextuples $(B, \lambda_B, \eta_B, A, \lambda_A, \eta_A, \gamma)$ where*

1. *$(B, \lambda_B, \eta_B) \in T(R)$;*
2. *$(A, \lambda_A, \eta_A) \in S(R)$;*
3. *$\gamma : A \to B$ is an $O_F$-linear isogeny such that*

$$p\lambda_A = \gamma^\vee \circ \lambda_B \circ \gamma;$$

*Note that condition (3) implies that $\ker(\gamma)$ is a subgroup scheme of $A[p]$ stable under $O_F$. Two septuplets $(B, \lambda_B, \eta_B, A, \lambda_A, \eta_A, \gamma)$ and $(B', \lambda'_B, \eta'_B, A', \lambda_{A'}, \eta_{A'}, \gamma')$ are equivalent if there are $O_F$-linear prime-to-p quasi-isogenies $\varphi : B \to B'$ and $\psi : A \to A'$ such that*

- *there exists $c \in \mathbb{Z}_{(p)}^\times$ such that $\varphi^\vee \circ \lambda_{B'} \circ \varphi = c\lambda_B$ and $\psi^\vee \circ \lambda_{A'} \circ \psi = c\lambda_A$.*
- *the $K^p$-orbit of maps $v \mapsto \varphi_* \circ \eta_B(v)$ for $v \in V' \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ coincides with $\eta_{B'}$.*
- *the $K^p$-orbit of maps $v \mapsto \varphi'_* \circ \eta_A(v)$ for $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ coincides with $\eta_{A'}$.*

We obtain in an obvious way a correspondence

$$\begin{array}{ccccc} & & N & & \\ & \swarrow^{\theta} & & \searrow^{\nu} & \\ T & & & & S_{\mathrm{ss}} \end{array} \qquad (2.5)$$

**Theorem 2.3.10.** *In diagram (2.5), take a point*

$$t = (B, \lambda_B, \eta_B) \in T(\kappa)$$

*where $\kappa$ is a field containing $\mathbb{F}_{p^2}$. Put $N_t := \theta^{-1}(t)$, and denote by $(B, \lambda_B, \eta_B, \mathcal{A}, \lambda_{\mathcal{A}}, \eta_{\mathcal{A}}, \gamma)$ the universal object over the fiber $N_t$.*

1. *The fiber $N_t$ is a smooth scheme over $\kappa$, with a canonical isomorphism for its tangent bundle*

$$\mathcal{T}_{N_t/\mathbb{F}_{p^2}} \simeq (\omega_{\mathcal{A}^\vee,1}, \ker \alpha_{*,1}/\omega_{\mathcal{A}^\vee,1})$$

2. *The assignment sending $(B, \lambda_B, \eta_B, A, \lambda_A, \eta_A; \gamma) \in N_t(R)$ for every $R \in \mathtt{Sch}'_{/\kappa}$ to the subbundle*

$$U := \delta_{*,0}^{-1}(\omega_{A^\vee/R,0}) \subseteq \mathrm{H}_1^{\mathrm{dR}}(B/R)_0 \cong \mathrm{H}_1^{\mathrm{dR}}(B/\kappa)_0 \otimes_\kappa \mathcal{O}_R = \mathscr{V}_t \otimes_\kappa R.$$

*induces an isomorphism*

$$\zeta_t : N_t \cong \mathrm{DL}(\mathscr{V}_t, \{\ ,\ \})$$

*where $\delta : B \to A$ is the unique quasi-$p$-isogeny such that $\gamma \circ \delta = p\,\mathrm{id}_B$ and $\delta \circ \gamma = p\,\mathrm{id}_A$. In particular, $N_t$ is isomorphic to the Fermat curve $\mathscr{C}$.*

*Proof.* See [LTX+22, Theorem 4.2.5]. □

We can define a moduli problem for $S_{\mathrm{ssp}}$.

**Definition 2.3.11.** *Let $S_{\mathrm{ssp}}(R)$ be the set of points $(A, \lambda_A, \eta_A) \in S(R)$ for $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$, where*

$$\mathtt{V}\omega_{A^\vee/R,0} = 0.$$

**Remark 2.3.12.** *The definition is equivalent to $\mathtt{V}\omega_{A^\vee/R,1} = 0$. Indeed, by comparing the rank we have $(\ker \mathtt{V})_0 = \omega_{A^\vee/R,0}$, which is equivalent to $(\ker \mathtt{V})_1 = \omega_{A^\vee/R,1}$ by duality.*

**Remark 2.3.13.** *The conditions $\omega_{A^\vee/R,0} = (\ker \mathtt{V})_0$ and $\omega_{A^\vee/R,1} = (\ker \mathtt{V})_1$ imply $S_{\mathrm{ssp}}$ is smooth of dimension 0.*

**Definition 2.3.14.** *Let $M$ be the moduli problem associating with every $\mathbb{F}_{p^2}$-algebra $R$ the set $M(R)$ of equivalence classes of septuplets $(\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, A, \lambda_A, \eta_A, \delta')$ where*

1. $(\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}) \in \tilde{T}(R)$;
2. $(A, \lambda_A, \eta_A) \in S(R)$;
3. *$\delta' : \tilde{B} \to A$ is a $O_F$-linear quasi-$p$-isogeny such that*
   (a) $\ker \delta'[p^\infty] \subseteq \tilde{B}[p]$;
   (b) $\lambda_{\tilde{B}} = \delta'^\vee \circ \lambda_A \circ \delta'$;
   (c) *the $K^p$-orbit of maps $v \mapsto \delta'_* \circ \eta_{\tilde{B}}(v)$ for $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ coincides with $\eta_A$.*

*The equivalence relations are defined in a similar way.*

There is a natural correspondence

$$\begin{array}{ccc} & M & \\ {}^{\rho'}\swarrow & & \searrow^{\rho} \\ \tilde{T} & & S \end{array}$$

**Lemma 2.3.15.** *The morphism $\rho$ factors through $S_{\mathrm{ssp}}$. Moreover, $M$ is smooth of dimension 0.*

*Proof.* Take a point $(\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, A, \lambda_A, \eta_A, \delta') \in M(R)$ for $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$. By Remark 2.3.12 it suffices to show $\mathtt{V}_A \omega_{A^\vee/R,0} = 0$. By condition (3) and the proof of Lemma 3.4.12(1)(4) of [LTX+22], we have

$$\mathrm{rank}_{\mathcal{O}_R} \ker \delta'_{*,0} + \mathrm{rank}_{\mathcal{O}_R} \ker \delta'_{*,1} = 1.$$

We claim $\delta'_{*,1}$ is an isomorphism, since otherwise $\mathrm{rank}_{\mathcal{O}_R} \ker \delta'_{*,0} = 0$ and $\mathrm{rank}_{\mathcal{O}_R} \mathrm{im}\, \delta'_{*,0} = 3$, which imply $\mathrm{rank}_{\mathcal{O}_R} \omega_{A^\vee/R,0} = 3$ by $\delta'_{*,0}\omega_{\tilde{B}/R,0} \subset \omega_{A^\vee/R,0}$, contradicting the signature condition on $A$. We conclude that $\mathrm{im}\, \delta'_{*,1} = \mathrm{H}_1^{\mathrm{dR}}(A/R)_1$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{H}_1^{\mathrm{dR}}(\tilde{B}/R)_0 & \xrightarrow{\delta'_{*,0}} & \mathrm{H}_1^{\mathrm{dR}}(A/R)_0 \\ \downarrow{\scriptstyle \mathtt{V}_{\tilde{B}}} & & \downarrow{\scriptstyle \mathtt{V}_A} \\ \mathrm{H}_1^{\mathrm{dR}}(\tilde{B}^{(p)}/R)_0 & \xrightarrow{\delta'^{(p)}_{*,1}} & \mathrm{H}_1^{\mathrm{dR}}(A^{(p)}/R)_0 \end{array} \quad . \tag{2.6}$$

Thus we have

$$\mathtt{V}_A \omega_{A^\vee/R,0} = \mathtt{V}_A \,\mathrm{im}\, \delta'_{*,0} = \delta'^{(p)}_{*,1}(\mathrm{im}\, \mathtt{V}_{\tilde{B}})_0 = \delta'^{(p)}_{*,1}\omega_{\tilde{B}^{(p)\vee}/R,0} = (\delta'_{*,1}\omega_{\tilde{B}^\vee/R,1})^{(p)} = 0$$

where we have used $\omega_{\tilde{B}^\vee/R,1} = 0$. We have proved $\rho$ factors through $S_{\mathrm{ssp}}$. The signature condition and Remark 2.3.13 imply $\tilde{B}$ and $A$ have trivial deformation. Thus $M$ is smooth of dimension 0. □

**Lemma 2.3.16.** *The morphism $\rho$ induces isomorphisms of $\mathbb{F}_{p^2}$-schemes*

$$\rho : M \cong S_{\mathrm{ssp}}, \quad \rho' : M \cong \tilde{T}$$

*which are both equivariant under the prime-to-p Hecke correspondence. That is, given $g \in K^p \backslash G(\mathbb{A}^{\infty,p})/K'^p$ such that $g^{-1}K^p g \subset K'^p$, we have a commutative diagram*

$$\begin{array}{ccc} S_{\mathrm{ssp}}(K^p) & \xrightarrow{g} & S_{\mathrm{ssp}}(K'^p) \\ {\scriptstyle \varphi(K_p)}\downarrow & & \downarrow{\scriptstyle \varphi(K'^p)} \\ \tilde{T}(K^p) & \xrightarrow{g} & \tilde{T}(K'^p) \end{array}$$

*Proof.* We show that $\rho$ is an isomorphism. Since $M$ and $S_{\mathrm{ssp}}$ are smooth of dimension 0, it suffices to check that for every algebraically closed field $\kappa$ containing $\mathbb{F}_{p^2}$, $\rho$ induces a bijection on $\kappa$-points. We will construct an inverse map $\theta$ of $\rho$. Given a point $s' = (A, \lambda_A, \eta_A) \in S_{\mathrm{ssp}}(\kappa)$. We list properties of $\mathcal{D}(A)$ :

1. $\mathtt{V}\mathcal{D}(A) = \mathtt{F}\mathcal{D}(A)$. This follows from lifting the definition $\mathtt{V}\omega_{A^\vee/\kappa} = 0$ of $S_{\mathrm{ssp}}$.
2. $\mathcal{D}(A)_0^{\perp_A} = \mathcal{D}(A)_1, \mathcal{D}(A)_1^{\perp_A} = \mathcal{D}(A)_0$. This is because $\lambda$ is self-dual, or equivalently $\mathcal{D}(A)^{\perp_A} = \mathcal{D}(A)$.
3. We have a chain of $W(\kappa)$-modules

$$p\mathcal{D}(A)_0 \overset{2}{\subset} \mathtt{F}\mathcal{D}(A)_1 \overset{1}{\subset} \mathcal{D}(A)_0, \quad p\mathcal{D}(A)_1 \overset{1}{\subset} \mathtt{F}\mathcal{D}(A)_0 \overset{2}{\subset} \mathcal{D}(A)_1.$$

   This follows from [Vol10, Lemma 1.4] and in particulier $A$ is of signature type (1,2).

Set

$$\mathcal{D}_{\tilde{B},0} = \mathtt{V}\mathcal{D}(A)_1, \quad \mathcal{D}_{\tilde{B},1} = \mathcal{D}(A)_1, \quad \mathcal{D}_{\tilde{B}} = \mathcal{D}_{\tilde{B},0} \oplus \mathcal{D}_{\tilde{B},1}.$$

We verify that $\mathcal{D}_{\tilde{B}}$ is $\mathtt{F}, \mathtt{V}$-stable. Indeed, since $\mathcal{D}(A)$ is $\mathtt{V}^{-1}\mathtt{F}$-invariant, it suffices to verify the condition for $\mathtt{V}$ : we have $\mathtt{V}\mathcal{D}_{\tilde{B}} = \mathtt{V}^2\mathcal{D}(A)_1 + \mathtt{V}\mathcal{D}(A)_1 = \mathtt{V}\mathcal{D}(A)_1 + p\mathcal{D}(A)_1 \subset \mathcal{D}_{\tilde{B}}$ since $\mathtt{V}^2 = \mathtt{FV} = p$.

The chain (3) implies $\mathcal{D}_{\tilde{B}} \subset \mathcal{D}(A)$ as $W(\kappa)$-lattices in $\mathcal{D}(A)[1/p]$.

By Dieudonné theory there exists an abelian 3-fold $\tilde{B}$ such that $\mathcal{D}(\tilde{B}) = \mathcal{D}_{\tilde{B}}$, and the injection $\mathcal{D}(\tilde{B}) \to \mathcal{D}(A)$ is induced by a prime-to-$p$ isogeny $\delta' : \tilde{B} \to A$. Define the endormorphism structure $i_{\tilde{B}}$ on $\tilde{B}$ by $i_{\tilde{B}}(a) = \delta'^{-1} \circ i(a) \circ \delta'$ for $a \in O_F$. Then $(\tilde{B}, i_{\tilde{B}})$ is an $O_F$-abelian scheme. Let $\lambda_{\tilde{B}}$ be the unique polarization such that

$$\lambda_{\tilde{B}} = \delta'^{\vee} \circ \lambda_A \circ \delta'.$$

The pairings induced by $\lambda_A$ and $\lambda_{\tilde{B}}$ have the relation

$$\langle x, y \rangle_{\lambda_A} = \langle x, y \rangle_{\lambda_{\tilde{B}}},\ x, y \in \mathcal{D}(A).$$

Define the level structure $\eta_{\tilde{B}}$ by $\eta_{\tilde{B}} = {\delta'_*}^{-1} \circ \eta_A$. We verify

1. $\mathcal{D}(\tilde{B})$ is of signature type (0,3). Indeed, this follows from
$$\mathrm{Lie}(\tilde{B}) \cong \mathcal{D}_{\tilde{B}}/\mathtt{V}\mathcal{D}_{\tilde{B}} \cong \mathcal{D}(A)_1/p\mathcal{D}(A)_1.$$
2. $\ker \lambda_{\tilde{B}}$ is a finite group scheme of rank $p^2$. Indeed, from covariant Dieudonne theory it is equivalent to show $\mathcal{D}(\tilde{B}) \overset{2}{\subset} \mathcal{D}(\tilde{B})^{\perp_{\tilde{B}}}$. Thus it suffices to show $\mathcal{D}(\tilde{B})_0 \overset{1}{\subset} \mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}}$. From (2) it is equivalent to show $\mathtt{F}\mathcal{D}(A)_1 \overset{1}{\subset} \mathcal{D}(A)_0$ which comes from (3).
3. $\ker \delta'[p^\infty] \subset \tilde{B}[p]$. It suffices to show $p\mathcal{D}(A) \subset \mathcal{D}(\tilde{B})$, which is by definition.

Finally we set $\theta(s') = (\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, A, \lambda_A, \eta_A, \delta')$. To verify $\theta$ is equivariant under prime-to-$p$ Hecke correspondence, it suffices to consider the associativity of the following diagram

$$V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{g} V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\eta_A} \mathrm{H}_1(A, \mathbb{A}^{\infty,p}) \xrightarrow{\delta'^{-1}_*} \mathrm{H}_1(\tilde{B}, \mathbb{A}^{\infty,p})$$

for $g \in K^p \backslash G(\mathbb{A}^{\infty,p})/K'^p$. It is easy to verify $\theta$ and $\rho$ are the inverse of each other.

We show that $\rho'$ is an isomorphism. Since $M$ and $\tilde{T}$ are smooth and have dimension 0, it suffices to check that for every algebraically closed field $\kappa$ containing $\mathbb{F}_{p^2}$, $\rho'$ induces a bijection on $\kappa$-points. We will construct an inverse map $\theta'$ of $\rho'$. Given $t = (\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}) \in \tilde{T}(\kappa)$, we list properties of $\mathcal{D}(\tilde{B})$ :

1. $\mathtt{V}\mathcal{D}(\tilde{B})_0 = \mathtt{F}\mathcal{D}(\tilde{B})_0$. In fact, since $\mathcal{D}(\tilde{B})$ is of signature (0,3), [Vol10, Lemma 1.4] gives
$$\mathcal{D}(\tilde{B})_0 = \mathtt{V}\mathcal{D}(\tilde{B})_1 = \mathtt{F}\mathcal{D}(\tilde{B})_1.$$
2. $\mathcal{D}(\tilde{B})_1 \overset{1}{\subset} \mathcal{D}(\tilde{B})_0^{\perp_{\tilde{B}}}$ and $\mathcal{D}(\tilde{B})_0 \overset{1}{\subset} \mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}}$. Indeed, since $\ker \lambda_{\tilde{B}}[p^\infty]$ is a $\tilde{B}[p]$-subgroup scheme of rank $p^2$, by covariant Dieudonné theory we have $\mathcal{D}(\tilde{B}) \overset{2}{\subset} \mathcal{D}(\tilde{B})^{\perp_{\tilde{B}}}$, and the claim follows.

3. We have the chain of $W(\kappa)$-lattice

$$\mathcal{D}(\tilde{B})_1 \overset{1}{\subset} \mathtt{V}^{-1}\mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}} \overset{2}{\subset} \frac{1}{p}\mathcal{D}(\tilde{B})_1.$$

Indeed, $\ker \lambda_{\tilde{B}} \subset \tilde{B}[p]$ gives $\mathcal{D}(\tilde{B})_0^{\perp_{\tilde{B}}} \subset (1/p)\mathcal{D}(\tilde{B})_1$. The claim comes from (2b) and the fact that $\mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}} = (\mathtt{V}^{-1}\mathcal{D}(\tilde{B})_0)^{\perp_{\tilde{B}}} = \mathtt{F}(\mathcal{D}(\tilde{B})_0)^{\perp_{\tilde{B}}}$.

We set

$$\mathcal{D}_{A,0} = \mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}}, \mathcal{D}_{A,1} = \mathcal{D}(\tilde{B})_1, \mathcal{D}_A = \mathcal{D}_{A,0} \oplus \mathcal{D}_{A,1}.$$

That $\mathcal{D}_A$ is $\mathtt{F}, \mathtt{V}$-stable follows from (2c). By covariant Dieudonné theory there exists an abelian 3-fold $A$ such that $\mathcal{D}(A) = \mathcal{D}_A$, and the inclusion $\mathcal{D}(\tilde{B}) \to \mathcal{D}(A)$ is induced by a prime-to-$p$ isogeny $\delta' : \tilde{B} \to A$. Define the endormorphism structure $i_A$ on $A$ by $i_A(a) = \delta' \circ i_{\tilde{B}}(a) \circ \delta'^{-1}$ for $a \in O_F$. Then $(A, i_A)$ is an $O_F$-abelian scheme. Let $\lambda_A$ be the unique polarization such that

$$\lambda_{\tilde{B}} = \delta'^{\vee} \circ \lambda_A \circ \delta'.$$

The pairings induced by $\lambda_A$ and $\lambda_{\tilde{B}}$ have the relation

$$\langle x, y \rangle_{\lambda_A} = \langle x, y \rangle_{\lambda_{\tilde{B}}}, \ \ x, y \in \mathcal{D}(A).$$

Define the level structure $\eta_A$ by $\eta_A = \delta'_* \circ \eta_{\tilde{B}}$. We verify

1. $\mathcal{D}(A)$ is of signature (1,2) : calculate the Lie algebra

$$\frac{\mathcal{D}(A)}{\mathtt{V}\mathcal{D}(A)} = \frac{\mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}} + \mathcal{D}(\tilde{B})_1}{\mathtt{V}\mathcal{D}(\tilde{B})_1 + \mathtt{V}\mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}}}.$$

The claim follows from (2c).

2. $\mathcal{D}(A)$ is self-dual with respect to $\langle , \rangle_{\lambda_A}$. Indeed, it suffices to show $\mathcal{D}(A)_0^{\perp_A} = \mathcal{D}(A)_1$. Since $\mathcal{D}(A)_0^{\perp_A} = \mathcal{D}(A)_0^{\perp_{\tilde{B}}}$, it is enough to verify $\mathcal{D}(A)_0^{\perp_{\tilde{B}}} = \mathcal{D}(A)_1$, which is exactly our construction.

Finally we set $\theta(t') = (\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, A, \lambda_A, \eta_A, \delta')$. The equivariance under prime-to-$p$ Hecke correspondence is clear. □

### 2.3.4 The geometry of $S_0(p)$

We define three closed subschemes $Y_i, i = 0, 1, 2$ of $S_0(p)$ over $\mathbb{F}_{p^2}$ as follows : for an $\mathbb{F}_{p^2}$-algebra $R$, a point $s = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in S_0(p)(R)$ belongs to

— $Y_0(R)$ if and only if $\omega_{\tilde{A}^\vee/R,0} = \mathrm{im}\alpha_{*,0}$;
— $Y_1(R)$ if and only if $\omega_{A^\vee/R,1} = \ker \alpha_{*,1}$;
— $Y_2(R)$ if and only if $\omega_{\tilde{A}^\vee/R,1} = \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0^{\perp_{\tilde{A}}}$.

**Remark 2.3.17.** *In [dSG18], the authors define two strata $\overline{Y}_m, \overline{Y}_{et}$. We will see that $Y_0$ coincides with their $\overline{Y}_m$ and $Y_1$ coincides with their $\overline{Y}_{et}$.*

We are going to show these three strata are all smooth of dimension 2.

**Lemma 2.3.18.** *Take $s = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in S_0(p)(R)$ for a scheme $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$.*

1. *If* $s \in Y_0(R)$ *then*
   (a) $\omega_{\tilde{A}^\vee/R,1} \subseteq \mathrm{im}\alpha_{*,1}$;
   (b) $(\ker \mathtt{V}_A)_1 = \ker \alpha_{*,1}$.
2. *If* $s \in Y_1(R)$ *then*
   (a) $\ker \alpha_{*,0} \subseteq \omega_{A^\vee/R,0}$;
   (b) $\alpha_{*,0}(\omega_{A^\vee/R,0}) = \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_1^{\perp_{\tilde{A}}}$.
3. *If* $s \in Y_2(R)$ *then*
   (a) $\ker \alpha_{*,0} \subseteq \omega_{A^\vee/R,0}$.

*Proof.* Denote by $\breve{\alpha} : \tilde{A} \to A$ the unique isogeny such that $\breve{\alpha} \circ \alpha = p\,\mathrm{id}_A$ and $\alpha \circ \breve{\alpha} = p\,\mathrm{id}_{\tilde{A}}$.

1. (a) The condition $\omega_{\tilde{A}^\vee/R,0} = \mathrm{im}\alpha_{*,0}$ implies $\omega_{\tilde{A}^\vee/R,0}^{\perp_{\tilde{A}}} = (\mathrm{im}\alpha_{*,0})^{\perp_{\tilde{A}}}$. On the other hand, we have $\langle \mathrm{im}\,\alpha_{*,0}, \mathrm{im}\,\alpha_{*,1}\rangle_{\lambda_{\tilde{A}}} = \langle \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0, \breve{\alpha}_{*,1}\,\mathrm{im}\,\alpha_{*,1}\rangle_{\lambda_{\tilde{A}}} = 0$, which implies $\mathrm{im}\alpha_{*,1} = (\mathrm{im}\alpha_{*,0})^{\perp_{\tilde{A}}}$ by comparing the rank. We also have $\langle \omega_{\tilde{A}^\vee/R,0}, \omega_{\tilde{A}^\vee/R,1}\rangle_{\lambda_{\tilde{A}}} = 0$, thus (1a) follows.

   (b) It suffices to show $\ker \alpha_{*,1} \subseteq (\ker \mathtt{V}_A)_1$. The condition (1a) implies $\omega_{\tilde{A}^\vee/R,1} \subseteq \mathrm{im}\alpha_{*,1} = \ker \breve{\alpha}_{*,1}$. We also have $(\ker \mathtt{V}_A)_1 = (\mathrm{im}\,\mathtt{F}_A)_1$. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_1 & \xrightarrow{\breve{\alpha}_{*,1}} & \mathrm{H}_1^{\mathrm{dR}}(A/R)_1 \\ \downarrow \mathtt{V}_{\tilde{A}} & & \downarrow \mathtt{V}_A \\ \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}^{(p)}/R)_0 & \xrightarrow{\breve{\alpha}^{(p)}_{*,0}} & \mathrm{H}_1^{\mathrm{dR}}(A^{(p)}/R)_0 \end{array} . \tag{2.7}$$

   Thus we have

$$\begin{aligned} \mathtt{V}_A \ker \alpha_{*,1} = \mathtt{V}_A\,\mathrm{im}\,\breve{\alpha}_{*,1} &= \breve{\alpha}^{(p)}_{*,0}(\mathrm{im}\,\mathtt{V}_{\tilde{A}})_0 \\ &= \breve{\alpha}^{(p)}_{*,0}\omega_{(\tilde{A}^{(p)})^\vee/R,0} = (\breve{\alpha}_{*,1}\omega_{\tilde{A}^\vee/R,1})^{(p)} = 0, \end{aligned}$$

   and (1b) follows.

2. (a) The condition $\omega_{A^\vee/R,1} = \ker \alpha_{*,1}$ implies $\omega_{A^\vee/R,1}^{\perp_A} = (\ker \alpha_{*,1})^{\perp_A}$. On the other hand, we have $\omega_{A^\vee/R,0} = \omega_{A^\vee/R,1}^{\perp_A}$ and

$$\begin{aligned} \langle \ker \alpha_{*,0}, \ker \alpha_{*,1}\rangle_{\lambda_A} &= \langle \mathrm{im}\,\breve{\alpha}_{*,0}, \ker \alpha_{*,1}\rangle_{\lambda_A} \\ &= \langle \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0, \alpha_{*,1}\ker \alpha_{*,1}\rangle_{\lambda_A} = 0. \end{aligned}$$

   Thus (2a) follows.

   (b) (2a) implies $\mathrm{rank}_{\mathcal{O}_R}\, \alpha_{*,0}\omega_{A^\vee/R,0} = 1$. On the other hand, we have

$$\begin{aligned} \langle \alpha_{*,0}\omega_{A^\vee/R,0}, \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_1\rangle_{\lambda_{\tilde{A}}} &= \langle \omega_{A^\vee/R,0}, \breve{\alpha}_{*,1}\,\mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_1\rangle_{\lambda_{\tilde{A}}} \\ &= \langle \omega_{A^\vee/R,0}, \ker \alpha_{*,1}\rangle_{\lambda_A} = \langle \omega_{A^\vee/R,0}, \omega_{A^\vee/R,1}\rangle_{\lambda_A} = 0. \end{aligned}$$

   Thus $\alpha_{*,0}\omega_{A^\vee/R,0} \subseteq \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_1^{\perp_{\tilde{A}}}$. By comparing the rank (2b) follows.

3. (a) Since $\omega_{A^\vee/R,0}^{\perp_A} = \omega_{A^\vee/R,1}$, by taking the dual it suffices to show $\omega_{A^\vee/R,1} \subset (\ker \alpha_{*,0})^{\perp_A}$. Since $\ker \alpha_{*,0} = \operatorname{im} \breve{\alpha}_{*,0}$, it suffices to show that $\langle \omega_{A^\vee/R,1}, \operatorname{im} \breve{\alpha}_{*,0} \rangle_{\lambda_A} = 0$. By the equality $\langle \alpha_* x, y\rangle_{\lambda_{\tilde{A}}} = \langle x, \breve{\alpha}_* y\rangle_{\lambda_A}$ it suffices to show that $\langle \alpha_{*,1}\omega_{A^\vee/R,1}, \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0\rangle_{\lambda_{\tilde{A}}} = 0$, which follows from the conditions $\omega_{\tilde{A}^\vee/R,1} = \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}^\vee/R)_0^{\perp_{\tilde{A}}}$ and $\alpha_{*,1}\omega_{A^\vee/R,1} \subseteq \omega_{\tilde{A}^\vee/R,1}$.

□

**Proposition 2.3.19.** *1. $Y_0$ is smooth of dimension 2 over $\mathbb{F}_{p^2}$. Moreover, let $(\mathcal{A}, \tilde{\mathcal{A}}, \alpha)$ denote the universal object on $Y_0$. Then the tangent bundle $\mathcal{T}_{Y_0/\mathbb{F}_{p^2}}$ of $Y_0$ fits into an exact sequence*

$$0 \to \mathcal{H}om(\omega_{\mathcal{A}^\vee,1}, \alpha_{*,1}^{-1}\omega_{\tilde{\mathcal{A}}^\vee,1}/\omega_{\mathcal{A}^\vee,1}) \to \mathcal{T}_{Y_0/\mathbb{F}_{p^2}} \to \mathcal{H}om(\alpha_{*,1}^{-1}\omega_{\tilde{\mathcal{A}}^\vee,1}/\ker \alpha_{*,1}, \mathrm{H}_1^{\mathrm{dR}}(\mathcal{A})_1/\alpha_{*,1}^{-1}\omega_{\tilde{\mathcal{A}}^\vee,1}) \to 0 \quad (2.8)$$

*2. $Y_1$ is smooth of dimension 2 over $\mathbb{F}_{p^2}$. Moreover, let $(\mathcal{A}, \tilde{\mathcal{A}}, \alpha)$ denote the universal object on $Y_1$. Then the tangent bundle $\mathcal{T}_{Y_1/\mathbb{F}_{p^2}}$ of $Y_1$ fits into an exact sequence*

$$0 \to \mathcal{H}om(\omega_{\tilde{\mathcal{A}}^\vee,1}, \omega_{\tilde{\mathcal{A}}^\vee,0}^{\perp_{\tilde{\mathcal{A}}}}/\omega_{\tilde{\mathcal{A}}^\vee,1}) \to \mathcal{T}_{Y_1/\mathbb{F}_{p^2}} \to \mathcal{H}om(\omega_{\tilde{\mathcal{A}}^\vee,0}/\mathrm{H}_1^{\mathrm{dR}}(\tilde{\mathcal{A}})_1^{\perp_{\tilde{\mathcal{A}}}}, \mathrm{H}_1^{\mathrm{dR}}(\tilde{\mathcal{A}})_1/\omega_{\tilde{\mathcal{A}}^\vee,0}) \to 0 \quad (2.9)$$

*3. $Y_2$ is smooth of dimension 2 over $\mathbb{F}_{p^2}$. Moreover, let $(\mathcal{A}, \tilde{\mathcal{A}}, \alpha)$ denote the universal object on $Y_2$. Then the tangent bundle $\mathcal{T}_{Y_2/\mathbb{F}_{p^2}}$ of $Y_2$ fits into an exact sequence*

$$0 \to \mathcal{H}om(\omega_{\tilde{\mathcal{A}}^\vee,0}/\alpha_{*,0}\omega_{\mathcal{A}^\vee,0}, \mathrm{H}_1^{\mathrm{dR}}(\tilde{\mathcal{A}})_0/\omega_{\tilde{\mathcal{A}}^\vee,0}) \to \mathcal{T}_{Y_2/\mathbb{F}_{p^2}} \to \mathcal{H}om(\omega_{\mathcal{A}^\vee,0}/\ker \alpha_{*,0}, \mathrm{H}_1^{\mathrm{dR}}(\mathcal{A})_0/\omega_{\mathcal{A}^\vee,0}) \to 0. \quad (2.10)$$

*Proof.* 1. We show $Y_0$ is formally smooth using deformation theory. Consider a closed immersion $R \hookrightarrow \hat{R}$ in $\mathtt{Sch}'_{/\mathbb{F}_{p^2}}$ defined by an ideal sheaf $\mathcal{I}$ with $\mathcal{I}^2 = 0$. Take a point $y = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_0(R)$. By Proposition 2.2.10 lifting $y$ to an $\hat{R}$-point is equivalent to lifting

— $\omega_{A^\vee/R,0}$ (resp. $\omega_{\tilde{A}^\vee/R,0}$) to a rank 2 subbundle $\hat{\omega}_{A^\vee,0}$ (resp. $\hat{\omega}_{\tilde{A}^\vee,0}$) of $\mathrm{H}_1^{\mathrm{cris}}(A/\hat{R})_0$ (resp. $\mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0$),

— $\omega_{A^\vee/R,1}$ (resp. $\omega_{\tilde{A}^\vee/R,1}$) to a rank 1 subbundle $\hat{\omega}_{A^\vee,1}$ (resp. $\hat{\omega}_{\tilde{A}^\vee,1}$) of $\mathrm{H}_1^{\mathrm{cris}}(A/\hat{R})_1$ (resp. $\mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_1$),

subject to the following requirements

(a) $\hat{\omega}_{A^\vee,0}$ and $\hat{\omega}_{A^\vee,1}$ are orthogonal complement of each other under $\langle\ ,\ \rangle_{\lambda_A}^{\mathrm{cris}}$ (2.4);

(b) $\hat{\omega}_{\tilde{A}^\vee,0}$ and $\hat{\omega}_{\tilde{A}^\vee,1}$ are orthogonal under $\langle\ ,\ \rangle_{\lambda_{\tilde{A}}}^{\mathrm{cris}}$ (2.4);

(c) $\hat{\omega}_{A^\vee,1} \subseteq \alpha_{*,1}^{-1}\hat{\omega}_{\tilde{A}^\vee,1}$;

(d) $\hat{\omega}_{\tilde{A}^\vee,0} = \alpha_{*,0}\mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0$;

Since $\langle\ ,\ \rangle^{\mathrm{cris}}_{\lambda_A,0}$ is a perfect pairing, $\hat{\omega}_{A^\vee,0}$ is uniquely determined by $\hat{\omega}_{A^\vee,1}$ by (3a). Moreover, $\hat{\omega}_{\tilde{A}^\vee,0}$ is uniquely determined by $\mathrm{H}^{\mathrm{cris}}_1(\tilde{A}/\hat{R})_0$. Therefore, it suffices to give the lifts $\hat{\omega}_{A^\vee,1}$ and $\hat{\omega}_{\tilde{A}^\vee,1}$ subject to condition (1c) above. But lifting $\omega_{\tilde{A}^\vee/R,1}$ is the same as lifting its preimage $\alpha^{-1}_{*,1}\omega_{\tilde{A}^\vee/R,1}$ to a rank 2 subbundle $\hat{\omega}'_{A^\vee,1}$ of $\mathrm{H}^{\mathrm{cris}}_1(\tilde{A}/\hat{R})_0$ containing $\ker\alpha_{*,1}$. Thus the tangent space $T_{Y_0/\mathbb{F}_{p^2},y}$ at $y$ fits canonically into an exact sequence

$$0 \to \mathcal{H}om(\omega_{A^\vee/R,1}, \alpha^{-1}_{*,1}\omega_{\tilde{A}^\vee/R,1}/\omega_{A^\vee/R,1}) \to T_{Y_0/\mathbb{F}_{p^2},y}$$

$$\to \mathcal{H}om(\alpha^{-1}_{*,1}\omega_{\tilde{A}^\vee/R,1}/\ker\alpha_{*,1}, \mathrm{H}^{\mathrm{dR}}_1(A/R)_1/\alpha^{-1}_{*,1}\omega_{\tilde{A}^\vee/R,1}) \to 0 \quad (2.11)$$

Thus, $Y_0$ is formally smooth over $\mathbb{F}_{p^2}$ of dimension 2.

2. Now we show $Y_1$ is formally smooth. Consider a closed immersion $R \hookrightarrow \hat{R}$ in $\mathtt{Sch}'_{/\mathbb{F}_{p^2}}$ defined by an ideal sheaf $\mathcal{I}$ with $\mathcal{I}^2 = 0$. Take a point $y = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_1(R)$. By proposition 2.2.10 to lift $y$ to an $\hat{R}$-point is equivalent to lift
   — $\omega_{A^\vee/R,0}$ (resp. $\omega_{\tilde{A}^\vee/R,0}$) to a rank 2 subbundle $\hat{\omega}_{A^\vee,0}$ (resp. $\hat{\omega}_{\tilde{A}^\vee,0}$) of $\mathrm{H}^{\mathrm{cris}}_1(A/\hat{R})_0$ (resp. $\mathrm{H}^{\mathrm{cris}}_1(\tilde{A}/\hat{R})_0$),
   — $\omega_{A^\vee/R,1}$ (resp. $\omega_{\tilde{A}^\vee/R,1}$) to a rank 1 subbundle $\hat{\omega}_{A^\vee,1}$ (resp. $\hat{\omega}_{\tilde{A}^\vee,1}$) of $\mathrm{H}^{\mathrm{cris}}_1(A/\hat{R})_1$ (resp. $\mathrm{H}^{\mathrm{cris}}_1(\tilde{A}/\hat{R})_1$),
   subject to the following requirements
   (a) $\hat{\omega}_{A^\vee,0}$ and $\hat{\omega}_{A^\vee,1}$ are orthogonal complement of each other under $\langle\ ,\ \rangle^{\mathrm{cris}}_{\lambda_A,0}$ (2.4) ;
   (b) $\hat{\omega}_{\tilde{A}^\vee,0}$ and $\hat{\omega}_{\tilde{A}^\vee,1}$ are orthogonal under $\langle\ ,\ \rangle^{\mathrm{cris}}_{\lambda_{\tilde{A}},0}$;
   (c) $\alpha_{*,0}\hat{\omega}_{A^\vee,0} \subseteq \hat{\omega}_{\tilde{A}^\vee,0}$;
   (d) $\hat{\omega}_{A^\vee,1} = \ker\alpha_{*,1}$.

   Since $\langle\ ,\ \rangle^{\mathrm{cris}}_{\lambda_A,0}$ is a perfect pairing, $\hat{\omega}_{A^\vee,0}$ is uniquely determined by $\hat{\omega}_{A^\vee,1} = \ker\alpha_{*,1}$ by (3a) and (2d). On the other hand, we have $\alpha_{*,0}\omega_{A^\vee/R,0} = \mathrm{H}^{\mathrm{dR}}_1(\tilde{A}/R)^{\perp_{\tilde{A}}}_1$ by Lemma 2.3.18(2b). To summarize, lifting $y$ to an $\hat{R}$-point is equivalent to lifting $\omega_{\tilde{A}^\vee/R,0}$ to a subbundle $\hat{\omega}_{\tilde{A}^\vee,0}$ containing $\mathrm{H}^{\mathrm{cris}}_1(\tilde{A}/\hat{R})^{\perp_{\tilde{A}}}_1$, and lifting $\omega_{\tilde{A}^\vee/R,1}$ to a subbundle $\hat{\omega}_{\tilde{A}^\vee,1}$ of $\hat{\omega}^{\perp_{\tilde{A}}}_{\tilde{A}^\vee,0}$ where the latter has $\mathcal{O}_{\hat{R}}$-rank 2. Thus the tangent space $\mathcal{T}_{Y_1/\mathbb{F}_{p^2},y}$ at $y$ fits canonically into an exact sequence

$$0 \to \mathcal{H}om(\omega_{\tilde{A}^\vee/R,1}, \omega^{\perp_{\tilde{A}}}_{\tilde{A}^\vee/R,0}/\omega_{\tilde{A}^\vee/R,1}) \to \mathcal{T}_{Y_1/\mathbb{F}_{p^2},y}$$

$$\to \mathcal{H}om(\omega_{\tilde{A}^\vee/R,0}/\,\mathrm{H}^{\mathrm{dR}}_1(\tilde{A}/R)^{\perp_{\tilde{A}}}_1, \mathrm{H}^{\mathrm{dR}}_1(\tilde{A}/R)_1/\omega_{\tilde{A}^\vee/R,0}) \to 0 \quad (2.12)$$

   Thus, $Y_1$ is formally smooth over $\mathbb{F}_{p^2}$ of dimension 2.

3. We show $Y_2$ is formally smooth using deformation theory. Consider a closed immersion $R \hookrightarrow \hat{R}$ in $\mathtt{Sch}'_{/\mathbb{F}_{p^2}}$ defined by an ideal sheaf $\mathcal{I}$ with $\mathcal{I}^2 = 0$. Take a point $y = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_2(R)$. We return to the proof of Proposition 3. By proposition 2.2.10 to lift $y$ to an $\hat{R}$-point is equivalent to lifting

— $\omega_{A^\vee/R,0}$ (resp. $\omega_{\tilde{A}^\vee/R,0}$) to a rank 2 subbundle $\hat{\omega}_{A^\vee,0}$ (resp. $\omega_{\tilde{A}^\vee,0}$) of $\mathrm{H}_1^{\mathrm{cris}}(A/\hat{R})_0$ ( resp. $\mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0$),
— $\omega_{A^\vee/R,1}$ (resp. $\omega_{\tilde{A}^\vee/R,1}$) to a rank 1 subbundle $\hat{\omega}_{A^\vee,1}$ (resp. $\omega_{\tilde{A}^\vee,1}$) of $\mathrm{H}_1^{\mathrm{cris}}(A/\hat{R})_1$ ( resp. $\mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_1$),

subject to the following requirements

(a) $\hat{\omega}_{A^\vee,0}$ and $\hat{\omega}_{A^\vee,1}$ are orthogonal complement of each other under $\langle\ ,\ \rangle^{\mathrm{cris}}_{\lambda_A,0}$ (2.4) ;

(b) $\hat{\omega}_{\tilde{A}^\vee,1}$ is the orthogonal complement of $\mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0$ under $\langle\ ,\ \rangle^{\mathrm{cris}}_{\lambda_{\tilde{A}},0}$ ;

(c) $\alpha_{*,i}\hat{\omega}_{A^\vee,i} \subseteq \hat{\omega}_{\tilde{A}^\vee,i}$ for $i = 0, 1$.

(d) $\ker \alpha_{*,0} \subseteq \hat{\omega}_{A^\vee,0}$(Lemma 2.3.18(3a)).

Since $\langle\ ,\ \rangle^{\mathrm{cris}}_{\lambda_A,0}$ is a perfect pairing, $\hat{\omega}_{A^\vee,1}$ is uniquely determined by $\hat{\omega}_{A^\vee,0}$ by (3a). Moreover, $\hat{\omega}_{\tilde{A}^\vee,1}$ is uniquely determined by $\mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0$ by (1b). Given a lift $\hat{\omega}_{A^\vee,0}$ with condition (3d) and define $\hat{\omega}_{A^\vee,1} := \hat{\omega}^{\perp_A}_{A^\vee,0}$. We claim $\alpha_{*,1}\hat{\omega}_{A^\vee,1} \subset \hat{\omega}_{\tilde{A}^\vee,1}$. Indeed, aince $\omega_{\tilde{A}^\vee,1} = \mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0^{\perp_{\tilde{A}}}$, it suffices to check $\langle \alpha_{*,1}\hat{\omega}_{A^\vee,1}, \mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0\rangle_{\lambda_{\tilde{A}}} = 0$. However, we have

$$\begin{aligned}\langle \alpha_{*,1}\hat{\omega}_{A^\vee,1}, \mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0\rangle_{\lambda_{\tilde{A}}} &= \langle \hat{\omega}_{A^\vee,1}, \breve{\alpha}_{*,0}\mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0\rangle_{\lambda_{\tilde{A}}} \\ &= \langle \hat{\omega}_{A^\vee,1}, \ker\alpha_{*,0}\rangle_{\lambda_{\tilde{A}}} \subset \langle \hat{\omega}_{A^\vee,1}, \hat{\omega}_{A^\vee,0}\rangle_{\lambda_{\tilde{A}}} = 0. \quad (2.13)\end{aligned}$$

The claim follows. To summarize, lifting $y$ to an $\hat{R}$-point is equivalent to lifting $\omega_{A^\vee/R,0}$ to a subbundle $\hat{\omega}_{A^\vee,0}$ of $\mathrm{H}_1^{\mathrm{cris}}(A/\hat{R})_0$ containing $\ker\alpha_{*,0}$ and lifting $\omega_{\tilde{A}^\vee/R,0}$ to a subbundle $\hat{\omega}_{\tilde{A}^\vee,0}$ of $\mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0$ containing $\alpha_{*,0}\hat{\omega}_{A^\vee,0}$. Thus the tangent space $T_{Y_2/\mathbb{F}_{p^2},y}$ at $y$ fits canonically into an exact sequence

$$\begin{aligned}0 \to \mathcal{H}om(\omega_{\tilde{A}^\vee/R,0}/\alpha_{*,0}\omega_{A^\vee/R,0}, \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0/\omega_{\tilde{A}^\vee/R,0}) \to T_{Y_2/\mathbb{F}_{p^2},y} \\ \to \mathcal{H}om(\omega_{A^\vee/R,0}/\ker\alpha_{*,0}, \mathrm{H}_1^{\mathrm{dR}}(A/R)_0/\omega_{A^\vee/R,0}) \to 0. \quad (2.14)\end{aligned}$$

Thus $Y_2$ is smooth over $\mathbb{F}_{p^2}$ of dimension 2.

□

**Lemma 2.3.20.** *$S_0(p)$ is the union of three strata defined over $\mathbb{F}_{p^2}$*

$$S_0(p) = Y_0 \cup Y_1 \cup Y_2.$$

*Proof.* By Hilbert's Nullstellensatz, it suffices to show that

$$S_0(p)(\kappa) = Y_0(\kappa) \cup Y_1(\kappa) \cup Y_2(\kappa)$$

for an algebraically closed field $\kappa$ of characteristic $p$. Take $s = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in S_0(p)(\kappa)$. Suppose $s \notin Y_0(\kappa) \cup Y_1(\kappa)$, that is, $\omega_{\tilde{A}^\vee/R,0} \neq \mathrm{im}\alpha_{*,0}$ and $\omega_{A^\vee/R,1} \neq \ker\alpha_{*,1}$. It follows that $\omega_{A^\vee/R,1} \cap \ker\alpha_{*,1} = \{0\}$ by the rank condition and therefore $\alpha_{*,1}$ induces an isomorphism $\omega_{\tilde{A}^\vee/R,1} = \alpha_{*,1}\omega_{A^\vee,1}$. Thus $\langle \mathrm{im}\,\alpha_{*,0}, \omega_{\tilde{A}^\vee/R,1}\rangle_{\lambda_{\tilde{A}}} = \langle \mathrm{im}\,\alpha_{*,0}, \alpha_{*,1}\omega_{A^\vee,1}\rangle_{\lambda_{\tilde{A}}} = 0$. On the other hand, we have $\langle \omega_{\tilde{A}^\vee/R,0}, \omega_{\tilde{A}^\vee/R,1}\rangle_{\lambda_{\tilde{A}}} = 0$. Since $\omega_{\tilde{A}^\vee/R,0} \neq \mathrm{im}\alpha_{*,0}$ , we conclude $\langle \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0, \omega_{\tilde{A}^\vee/R,1}\rangle_{\lambda_{\tilde{A}}} = 0$. Thus $s \in Y_2(\kappa)$ and the lemma follows. □

### 2.3.5 Relation between strata of $S_0(p)$ and $S$

**Definition 2.3.21.** *Let $S^{\#}$ be the moduli scheme that associates with every scheme $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$, the isomorphism classes of pairs $(A, \lambda_A, \eta_A, \mathcal{P}_0)$ where*

1. $(A, \lambda_A, \eta_A) \in S(R)$;
2. $\mathcal{P}_0$ *is a line subbundle of* $\ker(\mathtt{V} : \omega_{A^\vee/R,0} \to \omega_{A^{\vee(p)}/R,0})$.

Given a point $(A, \lambda_A, \eta_A) \in S(R)$ for a scheme $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$, recall (Notation (2.2.7)) that we have the locally free $\mathcal{O}_R$-module $\mathrm{H}_1^{\mathrm{dR}}(A/R)$, the Frobenius map $\mathtt{V}_A : \mathrm{H}_1^{\mathrm{dR}}(A/R)_i \to \mathrm{H}_1^{\mathrm{dR}}(A^{(p)}/R)_{i+1}$ and the Verschiebung map $\mathtt{F}_A : \mathrm{H}_1^{\mathrm{dR}}(A^{(p)}/R)_{i+1} \to \mathrm{H}_1^{\mathrm{dR}}(A/R)_i$ for $i = 0, 1$ satisfying $\ker \mathtt{F}_A = \operatorname{im} \mathtt{V}_A = \omega_{A^{(p)}/R}, \ker \mathtt{V}_A = \operatorname{im} \mathtt{F}_A$. If no confusion arises we denote them by $\mathtt{F}$ and $\mathtt{V}$. The $p$-principal polarization $\lambda_A$ induces a perfect pairing $\langle\ ,\ \rangle$ on $\mathrm{H}_1^{\mathrm{dR}}(A/R)$. Denote by $H^\perp$ the orthogonal complement of a subbundle $H$ of $\mathrm{H}_1^{\mathrm{dR}}(A/R)$ under the pairing $\langle\ ,\ \rangle$.

**Proposition 2.3.22.** *$S^{\#}$ is smooth of dimension 2 over $\mathbb{F}_{p^2}$. Moreover, let $(\mathcal{A}, \mathcal{P}_0)$ denote the universal object on $S^{\#}$. Then the tangent bundle $\mathcal{T}_{S^{\#}/\mathbb{F}_{p^2}}$ of $S^{\#}$ fits into an exact sequence*

$$0 \to \mathcal{H}om(\omega_{\mathcal{A}^\vee/S^{\#},1}, \mathcal{P}_0^\perp/\omega_{\mathcal{A}^\vee/S^{\#},1}) \to \mathcal{T}_{S^{\#}/\mathbb{F}_{p^2}} \to \mathcal{H}om(\mathcal{P}_0^\perp/(\ker \mathtt{V})_1, \mathrm{H}_1^{\mathrm{dR}}(\mathcal{A}/S^{\#})_1/\mathcal{P}_0^\perp) \to 0 \quad (2.15)$$

*Proof.* We show $S^{\#}$ is formally smooth using deformation theory. Consider a closed immersion $R \hookrightarrow \hat{R}$ in $\mathtt{Sch}'_{/\mathbb{F}_{p^2}}$ defined by an ideal sheaf $\mathcal{I}$ with $\mathcal{I}^2 = 0$. Take a point $s = (A, \lambda_A, \eta_A, \mathcal{P}_0) \in S^{\#}(R)$. By proposition 2.2.10 lifting $s$ to an $\hat{R}$-point is equivalent to lifting

— $\omega_{A^\vee/R,0}$ (resp. $\omega_{A^\vee/R,1}$) to a rank 2 (resp. rank 1) subbundle $\hat{\omega}_{A^\vee,0}$ (resp. $\hat{\omega}_{A^\vee,1}$) of $\mathrm{H}_1^{\mathrm{cris}}(A/\hat{R})_0$ ( resp. $\mathrm{H}_1^{\mathrm{cris}}(A/\hat{R})_1$),

— $\mathcal{P}_0$ to a rank 1 subbundle $\hat{\mathcal{P}}_0$ of $(\ker \mathtt{V})_0$.

subject to the following requirements

1. $\hat{\omega}_{A^\vee,0}$ and $\hat{\omega}_{A^\vee,1}$ are orthogonal complement of each other under $\langle\ ,\ \rangle^{\mathrm{cris}}_{\lambda_A}$ (2.4) ;
2. $\hat{\mathcal{P}}_0 \subseteq \hat{\omega}_{A^\vee,0}$;

Since $\langle\ ,\ \rangle^{\mathrm{cris}}_{\lambda_A,0}$ is a perfect pairing, $\hat{\omega}_{A^\vee,0}$ is uniquely determined by $\hat{\omega}_{A^\vee,1}$ by (1). In the meanwhile, lifting $\mathcal{P}_0$ is equivalent to lifting $\mathcal{P}_0^\perp$ to a rank 2 subbundle $\hat{\mathcal{P}}_1$ of $\mathrm{H}_1^{\mathrm{cris}}(A/\hat{R})_1$ subject to the conditions

1. $(\ker \mathtt{V})_0^\perp = (\ker \mathtt{V})_1 \subseteq \hat{\mathcal{P}}_1$ ;
2. $\hat{\omega}_{A^\vee,0}^\perp = \hat{\omega}_{A^\vee,1} \subseteq \hat{\mathcal{P}}_1$.

Therefore, it suffices to give the lifts $\hat{\omega}_{A^\vee,1}$ and $\hat{\mathcal{P}}_1$ subject to the conditions (1). Thus the tangent space $T_{S^{\#}/\mathbb{F}_{p^2},s}$ at $s$ fits canonically into an exact sequence

$$0 \to \mathcal{H}om(\omega_{\mathcal{A}^\vee/S^{\#},1}, \mathcal{P}_0^\perp/\omega_{\mathcal{A}^\vee/S^{\#},1}) \to \mathcal{T}_{S^{\#}/\mathbb{F}_{p^2},s} \to \mathcal{H}om(\mathcal{P}_0^\perp/(\ker \mathtt{V})_1, \mathrm{H}_1^{\mathrm{dR}}(\mathcal{A}/S^{\#})_1/\mathcal{P}_0^\perp) \to 0 \quad (2.16)$$

Thus, $S^{\#}$ is formally smooth over $\mathbb{F}_{p^2}$ of dimension 2.

□

**Remark 2.3.23.** *By [dSG18, 2.3], $S^{\#}$ is the moduli space represented by the blow up of $S$ at the superspecial points. Indeed, for $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$ and $(A, \lambda_A, \eta_A, \mathcal{P}_0) \in S^{\#}(R)$, if $A$ is not superspecial then $\mathcal{P}_0 = \ker(\mathtt{V}\,|_{\omega_{A^\vee/R,0}})$ is unique. At superspecial points, since $\mathtt{V}\,|_{\omega_{A^\vee/R,0}}$ vanishes, the additional datum $\mathcal{P}_0$ amounts to a choice of a subline bundle $\omega_{A^\vee/R,0}$.*

**Proposition 2.3.24.** *1. There is an isomorphism of $\mathbb{F}_{p^2}$-schemes*

$$\pi_0^{\#} : Y_0 \xrightarrow{\sim} S^{\#}$$

*defined as follows : given a point $y = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_0(R)$ for a scheme $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$, define*

$$\pi_0^{\#}(y) = (A, \lambda_A, \eta_A, (\alpha_{*,1}^{-1}\omega_{\tilde{A}^\vee/R,1})^{\perp}) \in S^{\#}(R).$$

*2. There is a purely inseparable morphism of $\mathbb{F}_{p^2}$-schemes*

$$\pi_1^{\#} : Y_1 \to S^{\#}$$

*defined as follows : given a point $y = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_1(R)$ for a scheme $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$, define*

$$\pi_1^{\#}(y) = (A, \lambda_A, \eta_A, (\alpha_{*,1}^{-1}(\ker \mathtt{V}_{\tilde{A}})_1)^{\perp}) \in S^{\#}(R).$$

*Proof.* 1. We check $\pi_0^{\#}$ is well-defined. Given a point $y = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_0(R)$ for a scheme $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$, we need to show $(\alpha_{*,1}^{-1}\omega_{\tilde{A}^\vee/R,1})^{\perp} \subseteq (\ker \mathtt{V}_A)_0 \cap \omega_{A^\vee/R,0}$. Firstly we show $(\alpha_{*,1}^{-1}\omega_{\tilde{A}^\vee/R,1})^{\perp} \subseteq \omega_{A^\vee/R,0}$. By duality it suffices to show $\omega_{A^\vee/R,1} \subset \alpha_{*,1}^{-1}\omega_{\tilde{A}^\vee/R,1}$, which follows from functoriality. Secondly we show $(\alpha_{*,1}^{-1}\omega_{\tilde{A}^\vee/R,1})^{\perp} \subseteq (\ker \mathtt{V}_A)_0$. By duality it suffices to show $(\ker \mathtt{V}_A)_1 \subseteq \alpha_{*,1}^{-1}\omega_{\tilde{A}^\vee/R,1}$. The condition $\operatorname{im} \alpha_{*,0} = \omega_{\tilde{A}^\vee/R,0}$ implies $\operatorname{im} \alpha_{*,0}^{(p)} = \omega_{(\tilde{A}^{(p)})^\vee/R,0}$. The commutative diagram (2.7) then implies $\alpha_{*,1}(\ker \mathtt{V}_A)_1 = \alpha_{*,1}(\operatorname{im} \mathtt{F}_A)_1 = \mathtt{F}_{\tilde{A}} \operatorname{im} \alpha_{*,0}^{(p)} = \mathtt{F}_{\tilde{A}}\omega_{(\tilde{A}^{(p)})^\vee/R,1} = 0$. Thus $\pi_0^{\#}$ is well-defined.

Since $S^{\#}$ is smooth over $\mathbb{F}_{p^2}$, to show that $\pi_0^{\#}$ is an isomorphism, it suffices to check that for every algebraically closed field $\kappa$ containing $\mathbb{F}_{p^2}$, we have

(a) $\pi_0^{\#}$ induces a bijection on $\kappa$-points ;

(b) $\pi_0^{\#}$ induces an isomorphism on the tangent spaces at every $\kappa$-point.

For (1a), it suffices to construct a map $\theta : S^{\#}(\kappa) \to Y_0(\kappa)$ inverse to $\pi_0^{\#}$. Take a point $s = (A, \lambda_A, \eta_A, \mathcal{P}_0) \in S^{\#}(\kappa)$. We will construct a point $y = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_0(\kappa)$. Recall that there is a perfect pairing $\langle\ ,\ \rangle$ on $\mathcal{D}(A)$ lifting that on $\mathrm{H}_1^{\mathrm{dR}}(A/\kappa)$. Given a $W(\kappa)$-submodule $M$ of $\mathcal{D}(A)$ denote by $M^\vee$ the dual lattice

$$M^\vee := \{x \in \mathcal{D}(A) \mid \langle x, M\rangle \in W(\kappa)\}.$$

We list miscellaneous properties of $\mathcal{D}(A)$ and $\mathcal{P}_0$ :

(a) We have two chains of $W(\kappa)$-modules

$$p\mathcal{D}(A)_0 \overset{2}{\subset} \mathtt{F}\mathcal{D}(A)_1 \overset{1}{\subset} \mathcal{D}(A)_0,\ p\mathcal{D}(A)_1 \overset{1}{\subset} \mathtt{F}\mathcal{D}(A)_0 \overset{2}{\subset} \mathcal{D}(A)_1.$$

Here, for an inclusion of $W(\kappa)$-modules $N \overset{i}{\subset} M$, the number $i$ above $\subset$ means $\dim_\kappa(M/N) = i$.

(b) $\mathcal{D}(A)$ is self dual : $\mathcal{D}(A)_0^\perp = \mathcal{D}(A)_1,\ \mathcal{D}(A)_1^\perp = \mathcal{D}(A)_0$.

(c) The preimage of $(\ker \mathtt{V}_A)_0 \cap \omega_{A^\vee/S,0}$ under the reduction map $\mathcal{D}(A)_0 \to \mathcal{D}(A)_0/p\mathcal{D}(A)_0 \cong \mathrm{H}_1^{\mathrm{dR}}(A/R)_0$ is $\mathtt{F}\mathtt{D}(A)_1 \cap \mathtt{V}\mathcal{D}(A)_1$.

(d) $\mathcal{P}_0$ is a $\kappa$-vector subspace of $\ker \mathtt{V} \cap \omega_{A^\vee/R,0}$ of dimension 1.

(e) Denote by $\tilde{\mathcal{P}}_0$ the preimage of $\mathcal{P}_0$ under the reduction map $\mathcal{D}(A)_0 \to \mathrm{H}_1^{\mathrm{dR}}(A/\kappa)_0$. Then we have chains of $W(\kappa)$-modules

$$p\mathtt{V}\mathcal{D}(A)_1 \overset{1}{\subset} p\mathcal{D}(A)_0 \overset{1}{\subset} \tilde{\mathcal{P}}_0 \subset \mathtt{F}\mathcal{D}(A)_1 \cap \mathtt{V}\mathcal{D}(A)_1,\ \tilde{\mathcal{P}}_0 \overset{2}{\subset} \mathtt{V}\mathcal{D}(A)_0.$$

$$p\mathtt{F}\mathcal{D}(A)_1 \overset{1}{\subset} p\mathcal{D}(A)_0 \overset{1}{\subset} \tilde{\mathcal{P}}_0 \subset \mathtt{F}\mathcal{D}(A)_1 \cap \mathtt{V}\mathcal{D}(A)_1,\ \tilde{\mathcal{P}}_0 \overset{2}{\subset} \mathtt{F}\mathcal{D}(A)_0.$$

We set

$$\mathcal{D}_{\tilde{A},0} = \mathtt{F}(\tilde{\mathcal{P}}_0)^\vee,\ \mathcal{D}_{\tilde{A},1} = \mathtt{V}^{-1}\mathcal{D}(A)_0,\ \mathcal{D}_{\tilde{A}} = \mathcal{D}_{\tilde{A},0} + \mathcal{D}_{\tilde{A},1}.$$

We verify that $\mathcal{D}_{\tilde{A}}$ is $\mathtt{F}, \mathtt{V}$-stable and satisfies the following chain conditions :

(a) $\mathtt{V}\mathcal{D}_{\tilde{A},0} \overset{2}{\subset} \mathcal{D}_{\tilde{A},1}$. It suffices to check $(\tilde{\mathcal{P}}_0)^\vee \overset{2}{\subset} p^{-1}\mathtt{V}^{-1}\mathcal{D}(A)_0$. By taking duals, this is equivalent to $p\mathtt{F}\mathcal{D}(A)_1 \overset{2}{\subset} \tilde{\mathcal{P}}_0$, which follows from (1e).

(b) $\mathtt{F}\mathcal{D}_{\tilde{A},0} \overset{2}{\subset} \mathcal{D}_{\tilde{A},1}$. It suffices to check $(\tilde{\mathcal{P}}_0)^\vee \overset{2}{\subset} p^{-1}\mathtt{F}^{-1}\mathcal{D}(A)_0$. By taking duals, this is equivalent to $p\mathtt{V}\mathcal{D}(A)_1 \overset{2}{\subset} \tilde{\mathcal{P}}_0$, which follows from (1e).

(c) $\mathtt{V}\mathcal{D}_{\tilde{A},1} \overset{1}{\subset} \mathcal{D}_{\tilde{A},0}$. It suffices to check $\mathtt{F}^{-1}\mathcal{D}(A)_0 \overset{1}{\subset} (\tilde{\mathcal{P}}_0)^\vee$. By taking duals, this is equivalent to $\tilde{\mathcal{P}}_0 \overset{1}{\subset} \mathtt{V}\mathcal{D}(A)_1$, which follows from (1e).

(d) $\mathtt{F}\mathcal{D}_{\tilde{A},1} \overset{1}{\subset} \mathcal{D}_{\tilde{A},0}$. It suffices to check $\mathtt{V}^{-1}\mathcal{D}(A)_0 \overset{1}{\subset} (\tilde{\mathcal{P}}_0)^\vee$. By taking duals, this is equivalent to $\tilde{\mathcal{P}}_0 \overset{1}{\subset} \mathtt{F}\mathcal{D}(A)_0$, which follows from (1e).

(e) $\mathcal{D}(A)_0 \overset{1}{\subset} \mathcal{D}_{\tilde{A},0},\ \mathcal{D}(A)_1 \overset{1}{\subseteq} \mathcal{D}_{\tilde{A},1}$. Same as (1c) and (1a).

Thus we have an inclusion $\mathcal{D}(A) \subseteq \mathcal{D}_{\tilde{A}}$. By covariant Dieudonné theory there exists an abelian 3-fold $\tilde{A}$ such that $\mathcal{D}(\tilde{A}) = \mathcal{D}_{\tilde{A}}$, and the inclusion $\mathcal{D}(A) \subseteq \mathcal{D}_{\tilde{A}}$ is induced by a prime-to-$p$ isogeny $\alpha : A \to \tilde{A}$. Define the endormorphism structure $i_{\tilde{A}}$ on $\tilde{A}$ by $i_{\tilde{A}}(a) = \alpha \circ i_A(a) \circ \alpha^{-1}$ for $a \in O_F$. Then $(\tilde{A}, i_{\tilde{A}})$ is an $O_F$-abelian scheme. Let $\lambda_{\tilde{A}}$ be the unique polarization such that

$$p\lambda_A = \alpha^\vee \circ \lambda_{\tilde{A}} \circ \alpha.$$

The pairings induced by $\lambda_{\tilde{A}}$ and $\lambda_B$ have the relations

$$\langle x, y\rangle_{\lambda_A} = p^{-1}\langle x, y\rangle_{\lambda_{\tilde{A}}},\ x, y \in \mathcal{D}(A).$$

For a $W(\kappa)$-submodule $M$ of $\mathcal{D}(A)$, we have

$$M^{\vee_A} = pM^{\vee_{\tilde{A}}}.$$

Define the level structure $\eta_{\tilde{A}}$ on $\tilde{A}$ by $\eta_{\tilde{A}} = \alpha_* \circ \eta_A$. We verify

(a) $\mathcal{D}(\tilde{A})$ is of signature (1,2). This is by definition.

(b) $\ker\alpha$ is a Raynaud subgroup of $A[p]$. It suffices to show $\mathcal{D}(A)_0 \overset{1}{\subseteq} \mathcal{D}(\tilde{A})_0$ and $\mathcal{D}(A)_1 \overset{1}{\subseteq} \mathcal{D}(\tilde{A})_1$, which follows from (1e).

(c) $\mathcal{D}(\tilde{A})_1 \overset{1}{\subset} \mathcal{D}(\tilde{A})_0^{\perp_{\tilde{A}}}$. It suffices to show $\mathtt{V}^{-1}\mathcal{D}(A)_0 \overset{1}{\subset} p^{-1}\mathtt{F}\tilde{\mathcal{P}}_0$, or equivalently $p\mathcal{D}(A)_0 \overset{1}{\subset} \tilde{\mathcal{P}}_0$, which follows from (1e).

(d) $\mathcal{D}(\tilde{A})_0 \overset{1}{\subset} \mathcal{D}(\tilde{A})_1^{\perp_{\tilde{A}}}$. This is the dual version of (1c).

(e) $\ker\lambda_{\tilde{A}}[p^\infty]$ is a $\tilde{A}[p]$-subgroup scheme of rank $p^2$. Indeed, from covariant Dieudonné theory it is equivalent to show $\mathcal{D}(\tilde{A}) \overset{2}{\subset} \mathcal{D}(\tilde{A})^{\perp_{\tilde{A}}}$. Thus it suffices to show $\mathcal{D}(\tilde{A})_0 \overset{1}{\subset} \mathcal{D}(\tilde{A})_1^{\perp_{\tilde{A}}}$ and $\mathcal{D}(\tilde{A})_1 \overset{1}{\subset} \mathcal{D}(\tilde{A})_0^{\perp_{\tilde{A}}}$ which follows from (1c) and (1d).

(f) $\omega_{\tilde{A}^\vee/R,0} = \mathrm{im}\alpha_{*,0}$, $\omega_{\tilde{A}^\vee/R,1} \subset \mathrm{im}\alpha_{*,1}$. It suffices to check $\mathtt{V}\mathcal{D}(\tilde{A})_0 \subseteq \mathcal{D}(A)_1$, $\mathtt{V}\mathcal{D}(\tilde{A})_1 \subseteq \mathcal{D}(A)_0$, which follows from (1e).

Finally we set $\theta(s) = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha)$. By (1) we see $\theta(s) \in Y_0(\kappa)$. It is easy to verify $\theta$ is the inverse of $\pi_0^\#$.

For (1b), the morphism $\pi_0^\#$ induces the identification $\alpha_{*,1}^{-1}\omega_{\tilde{A}^\vee/\kappa,1} = \mathcal{P}_0^\perp$. Combined with Lemma 2.3.18 (1b), we see two exact sequences of tangent bundle (2.8) and (2.15) coincide. The proposition follows.

2. We check $\pi_1^\#$ is well-defined. Given a point $y = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_1(R)$ for a scheme $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$. We need to show $(\alpha_{*,1}^{-1}(\ker \mathtt{V}_{\tilde{A}})_1)^\perp \subseteq (\ker \mathtt{V}_A)_0 \cap \omega_{A^\vee/R,0}$.

Firstly we show $(\alpha_{*,1}^{-1}(\ker \mathtt{V}_{\tilde{A}})_1)^\perp \subseteq \omega_{A^\vee/R,0}$. By duality it suffices to show $\omega_{A^\vee/R,1} \subset \alpha_{*,1}^{-1}(\ker \mathtt{V}_{\tilde{A}})_1$, which follows from the condition $\omega_{A^\vee/R,1} = \ker\alpha_{*,1}$. Secondly we show $(\alpha_{*,1}^{-1}(\ker \mathtt{V}_{\tilde{A}})_1)^\perp \subseteq (\ker \mathtt{V}_A)_0$. By duality it suffices to show $(\ker \mathtt{V}_A)_1 \subseteq \alpha_{*,1}^{-1}(\ker \mathtt{V}_{\tilde{A}})_1$, which is again from the commutative diagram (2.7). Thus $\pi_1^\#$ is well-defined.

To show that $\pi_1^\#$ is a purely inseparable morphism, it suffices to check that for every algebraically closed field $\kappa$ containing $\mathbb{F}_{p^2}$, $\pi$ induces a bijection on $\kappa$-points. We construct an inverse map $\theta$ of $\pi_1^\#$. Take a point $s = (A, \lambda_A, \eta_A, \mathcal{P}_0) \in S^\#(\kappa)$.

We set

$$\mathcal{D}_{\tilde{A},0} = \mathtt{V}(\tilde{\mathcal{P}}_0)^\vee,\ \mathcal{D}_{\tilde{A},1} = \mathtt{F}^{-1}\mathcal{D}(A)_0,\ \mathcal{D}_{\tilde{A}} = \mathcal{D}_{\tilde{A},0} + \mathcal{D}_{\tilde{A},1}.$$

In an entirely similar manner we can construct a point $\theta(s) = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_1(\kappa)$. It is easy to verify that $\theta$ is the inverse of $\pi_1^\#$.

□

We now introduce a new moduli problem to show $Y_2$ is a $\mathbb{P}^1$-bundle over $N$.

**Definition 2.3.25.** *Let $P$ be the moduli problem associating with every $\mathbb{F}_{p^2}$-algebra $R$ the set $P(R)$ of equivalence classes of undecuples $(A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, B, \lambda_B, \eta_B, \alpha, \delta)$ where*

*1.* $(A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in S_0(p)(R)$;

2. $(A, \lambda_A, \eta_A, B, \lambda_B, \eta_B, \delta \circ \alpha) \in N(R)$;
3. $\delta : \tilde{A} \to B$ *is a* $O_F$*-linear quasi-p-isogeny such that*
   (a) $\ker \delta[p^\infty] \subseteq \tilde{A}[p]$;
   (b) $\lambda_{\tilde{A}} = \delta^\vee \circ \lambda_B \circ \delta$;
   (c) *the* $K^p$*-orbit of maps* $v \mapsto \delta_* \circ \eta_{\tilde{A}}(v)$ *for* $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ *coincides with* $\eta_B$.

*Two undecuples* $(B, \lambda_B, \eta_B, A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha, \delta)$ *and* $(B', \lambda'_B, \eta'_B, A', \lambda_{A'}, \eta_{A'}, \tilde{A}', \lambda_{\tilde{A}'}, \eta_{\tilde{A}'}, \alpha', \delta')$ *are equivalent if there are* $O_F$*-linear prime-to-p quasi-isogenies* $\varphi : B \to B'$, $\psi : A \to A'$ *and* $\phi : \tilde{A} \to \tilde{A}'$ *such that*

— *there exists* $c \in \mathbb{Z}_{(p)}^\times$ *such that* $\varphi^\vee \circ \lambda_{B'} \circ \varphi = c\lambda_B, \psi^\vee \circ \lambda_{A'} \circ \psi = c\lambda_A$ *and* $\phi^\vee \circ \lambda_{\tilde{A}'} \circ \phi = c\lambda_{\tilde{A}}$;
— *the* $K^p$*-orbit of maps* $v \mapsto \varphi_* \circ \eta_B(v)$ *for* $v \in W \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ *coincides with* $\eta_{B'}$;
— *the* $K^p$*-orbit of maps* $v \mapsto \psi_* \circ \eta_A(v)$ *for* $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ *coincides with* $\eta_{A'}$;
— *the* $K^p$*-orbit of maps* $v \mapsto \phi_* \circ \eta_{\tilde{A}}(v)$ *for* $v \in V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ *coincides with* $\eta_{\tilde{A}'}$.

**Lemma 2.3.26.** *Take a point* $s = (B, \lambda_B, \eta_B, A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha, \delta) \in P(R)$ *for a scheme* $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$. *Then*

1. $\delta_{*,0} : \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0 \to \mathrm{H}_1^{\mathrm{dR}}(B/R)_0$ *is an isomorphism and* $\mathrm{rank}_{\mathcal{O}_R} \ker \delta_{*,1} = 1$.
2. $\omega_{\tilde{A}^\vee/R,1} = \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0^{\perp_{\lambda_{\tilde{A}}}}$.

*Proof.* 1. Denote by $\gamma$ the quasi-$p$-isogeny $\gamma := \delta \circ \alpha : A \to B$. The relation $p\lambda_A = \alpha^\vee \circ \lambda_{\tilde{A}} \circ \alpha$ and $\lambda_{\tilde{A}} = \delta^\vee \circ \lambda_B \circ \delta$ implies

$$p\lambda_A = \gamma^\vee \circ \lambda_B \circ \gamma.$$

By [LTX+22, Lemma 3.4.12(2),(3a),(3b),(4)], we have

$$\mathrm{rank}_{\mathcal{O}_R}(\ker \alpha_{*,0}) - \mathrm{rank}_{\mathcal{O}_R}(\ker \alpha_{*,1}) = 0,$$
$$\mathrm{rank}_{\mathcal{O}_R}(\ker \alpha_{*,0}) + \mathrm{rank}_{\mathcal{O}_R}(\ker \alpha_{*,1}) = 2,$$
$$\mathrm{rank}_{\mathcal{O}_R}(\ker \gamma_{*,0}) - \mathrm{rank}_{\mathcal{O}_R}(\ker \gamma_{*,1}) = -1,$$
$$\mathrm{rank}_{\mathcal{O}_R}(\ker \gamma_{*,0}) + \mathrm{rank}_{\mathcal{O}_R}(\ker \gamma_{*,1}) = 3,$$
$$\mathrm{rank}_{\mathcal{O}_R}(\ker \delta_{*,0}) + \mathrm{rank}_{\mathcal{O}_R}(\ker \delta_{*,1}) = 1.$$

The solution is

$$\mathrm{rank}_{\mathcal{O}_R} \ker \alpha_{*,0} = 1,\ \mathrm{rank}_{\mathcal{O}_R} \ker \gamma_{*,0} = 1,\quad \mathrm{rank}_{\mathcal{O}_R} \ker \alpha_{*,1} = 1,\ \mathrm{rank}_{\mathcal{O}_R} \ker \gamma_{*,1} = 2.$$

We claim $\mathrm{rank}_{\mathcal{O}_R} \ker \delta_{*,0} = 0$ since otherwise $\delta_{*,1}$ is an isomorphism and therefore $\mathrm{rank}_{\mathcal{O}_R} \ker \alpha_{*,1} = \mathrm{rank}_{\mathcal{O}_R} \ker \gamma_{*,1}$ which is absurd. Then by comparing the ranks we conclude $\delta_{*,0}$ is an isomorphism. (1) follows.

2. By comparing the rank it suffices to show $\langle \omega_{\tilde{A}^\vee/R,1}, \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0 \rangle_{\lambda_{\tilde{A}}} = 0$. We claim $\omega_{\tilde{A}^\vee/R,1} = \ker \delta_{*,1}$. Indeed, by (1) it suffices to show $\omega_{\tilde{A}^\vee/R,1} \subseteq \ker \delta_{*,1}$. The signature condition of $B$ implies $\omega_{B^\vee/R,1} = 0$. Thus $\omega_{\tilde{A}^\vee/R,1} \subseteq \ker \delta_{*,1}$ and the claim follows. On the other hand, from $\lambda_{\tilde{A}} = \delta^\vee \circ \lambda_B \circ \delta$ we have $\langle x, y \rangle_{\lambda_{\tilde{A}}} = \langle \delta_* x, \delta_* y \rangle_{\lambda_B}$ for $x, y \in \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)$. Therefore $\langle \ker \delta_{*,1}, \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0 \rangle_{\lambda_{\tilde{A}}} = 0$. We can then conclude $\langle \omega_{\tilde{A}^\vee/R,1}, \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0 \rangle_{\lambda_{\tilde{A}}} = 0$ and (2) follows.

□

**Proposition 2.3.27.** *$P$ is smooth of dimension 2 over $\mathbb{F}_{p^2}$. Moreover, let $(\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{B}, \alpha, \delta)$ denote the universal object over $P$. Then the tangent bundle $\mathcal{T}_{\mathcal{P}/\mathbb{F}_{p^2}}$ of $\mathcal{P}$ fits into an exact sequence*

$$0 \to \mathcal{H}om(\omega_{\tilde{\mathcal{A}}^\vee/P,0}/\alpha_{*,0}\omega_{\mathcal{A}^\vee/P,0}, \mathrm{H}_1^{\mathrm{dR}}(\tilde{\mathcal{A}}/P)_0/\omega_{\tilde{\mathcal{A}}^\vee/P,0}) \to \mathcal{T}_{P/\mathbb{F}_{p^2}} \to \mathcal{H}om(\omega_{\mathcal{A}^\vee/P,0}/\ker\alpha_{*,0}, \mathrm{H}_1^{\mathrm{dR}}(\mathcal{A}/P)_0/\omega_{\mathcal{A}^\vee/P,0}) \to 0. \quad (2.17)$$

*Proof.* The proof resembles that of Proposition 2.3.19(3). We show $P$ is formally smooth using deformation theory. Consider a closed immersion $R \hookrightarrow \hat{R}$ in $\mathtt{Sch}'_{/\mathbb{F}_{p^2}}$ defined by an ideal sheaf $\mathcal{I}$ with $\mathcal{I}^2 = 0$. Take a point $s = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, B, \lambda_B, \eta_B, \alpha, \delta) \in P(R)$. Denote by $\breve{\delta} : B \to \tilde{A}$ the unique quasi-$p$-isogeny such that $\breve{\delta} \circ \delta = \mathrm{id}_{\tilde{A}}$ and $\delta \circ \breve{\delta} = \mathrm{id}_B$ . By proposition 2.2.10 lifting $s$ to an $\hat{R}$-point is equivalent to lifting

- $\omega_{A^\vee/R,0}$ (resp. $\omega_{\tilde{A}^\vee/R,0}$) to a rank 2 subbundle $\hat{\omega}_{A^\vee,0}$ (resp. $\omega_{\tilde{A}^\vee,0}$) of $\mathrm{H}_1^{\mathrm{cris}}(A/\hat{R})_0$ ( resp. $\mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0$),
- $\omega_{A^\vee/R,1}$ (resp. $\omega_{\tilde{A}^\vee/R,1}$) to a rank 1 subbundle $\hat{\omega}_{A^\vee,1}$ (resp. $\omega_{\tilde{A}^\vee,1}$) of $\mathrm{H}_1^{\mathrm{cris}}(A/\hat{R})_1$ ( resp. $\mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_1$),
- $\omega_{B^\vee/R,0}$ (resp. $\omega_{B^\vee/R,1}$) to a rank 3 (resp. rank 0) subbundle $\hat{\omega}_{B^\vee,0}$ (resp. $\hat{\omega}_{B^\vee,1}$) of $\mathrm{H}_1^{\mathrm{cris}}(B/\hat{R})_0$ ( resp. $\mathrm{H}_1^{\mathrm{cris}}(B/\hat{R})_1$),

subject to the requirements in the proof of Proposition 2.3.19(3) and

1. $\delta_{*,1}\hat{\omega}_{\tilde{A}^\vee,1} \subseteq \hat{\omega}_{B^\vee,1}$.

We verify (1) holds. Indeed, since $\lambda_B$ is $p$-principal, it suffices to show that $\langle \delta_{*,1}\hat{\omega}_{\tilde{A}^\vee,1}, \mathrm{H}_1^{\mathrm{cris}}(B/\hat{R})_0\rangle_{\lambda_B} = 0$. However, the same argument as Lemma 2.3.26(1) shows $\delta_{*,0} : \mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0 \to \mathrm{H}_1^{\mathrm{cris}}(B/\hat{R})_0$ is an isomorphism. Thus we have $\langle \delta_{*,1}\hat{\omega}_{\tilde{A}^\vee,1}, \mathrm{H}_1^{\mathrm{cris}}(B/\hat{R})_0\rangle_{\lambda_B} = \langle \hat{\omega}_{\tilde{A}^\vee,1}, \mathrm{H}_1^{\mathrm{cris}}(\tilde{A}/\hat{R})_0\rangle_{\lambda_{\tilde{A}}} = 0$ by Proposition 2.3.19(3b) and therefore (1) holds. We conclude the requirements are the same as those in Proposition 3. Thus the tangent space $\mathcal{T}_{P/\mathbb{F}_{p^2},s}$ at $s$ fits into an exact sequence

$$0 \to \mathcal{H}om(\omega_{\tilde{A}^\vee/R,0}/\alpha_{*,0}\omega_{A^\vee/R,0}, \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/R)_0/\omega_{\tilde{A}^\vee/R,0}) \to \mathcal{T}_{P/\mathbb{F}_{p^2},s} \to \mathcal{H}om(\omega_{A^\vee/R,0}/\ker\alpha_{*,0}, \mathrm{H}_1^{\mathrm{dR}}(A/R)_0/\omega_{A^\vee/R,0}) \to 0. \quad (2.18)$$

We have shown $P$ is smooth over $\mathbb{F}_{p^2}$ of dimension 2. □

**Lemma 2.3.28.** *The natural forgetful map $\tilde{\nu}$ induces an isomorphism of $\mathbb{F}_{p^2}$-schemes*

$$\tilde{\nu} : P \cong Y_2.$$

*Proof.* Since $Y_2$ is smooth over $\mathbb{F}_{p^2}$ by Proposition 2.3.19(3), to show that $\tilde{\nu}$ is an isomorphism, it suffices to check that for every algebraically closed field $\kappa$ containing $\mathbb{F}_{p^2}$, we have

1. $\tilde{\nu}$ induces a bijection on $\kappa$-points ; and
2. $\tilde{\nu}$ induces an isomorphism on the tangent spaces at every $\kappa$-point.

For (1), we construct an inverse map $\theta(\kappa)$ of $\tilde{\nu}$. Take a point $y = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_2(\kappa)$. We have the following facts :

1. We have two chains
$$\mathcal{D}(A)_0 \overset{1}{\subset} \mathcal{D}(\tilde{A})_0, \quad \mathcal{D}(A)_1 \overset{1}{\subset} \mathcal{D}(\tilde{A})_1$$
since $\ker \alpha$ is a Raynaud subgroup of $A[p]$.
2. $\mathcal{D}(\tilde{A})_0^{\perp_{\tilde{A}}} = p^{-1}\mathtt{V}\mathcal{D}(\tilde{A})_0$. Indeed, this is by taking the preimage of the condition $\omega_{\tilde{A}^\vee/R,1} = \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/\kappa)_0^{\perp_{\tilde{A}}}$ under the reduction map $\mathcal{D}(\tilde{A})_0 \to \mathcal{D}(\tilde{A})_0/p\mathcal{D}(\tilde{A})_0 \cong \mathrm{H}_1^{\mathrm{dR}}(\tilde{A}/\kappa)_0$.
3. $\mathtt{V}\mathcal{D}(\tilde{A})_0 = \mathtt{F}\mathcal{D}(\tilde{A})_0$. Rewrite (2) as $\mathcal{D}(\tilde{A})_0^{\perp_A} = \mathtt{V}\mathcal{D}(\tilde{A})_0$ by identifying $\mathcal{D}(\tilde{A})$ as a lattice in $\mathcal{D}(A)[1/p]$ and taking account of the relation $\mathcal{D}(\tilde{A})_0^{\perp_A} = p\mathcal{D}(\tilde{A})_0^{\perp_{\tilde{A}}}$. By taking the $\lambda_A$-dual we get $\mathcal{D}(\tilde{A})_0 = (\mathtt{V}\mathcal{D}(\tilde{A})_0)^{\perp_A} = \mathtt{F}^{-1}\mathcal{D}(\tilde{A})_0^{\perp_A} = \mathtt{F}^{-1}\mathtt{V}\mathcal{D}(\tilde{A})_0$. Thus (3) follows.
4. There is a chain of $W(\kappa)$-lattice in $\mathcal{D}(A)_0[1/p]$ :
$$p\mathcal{D}(A)_1 \overset{1}{\subseteq} \mathtt{V}\mathcal{D}(A)_0 \overset{1}{\subseteq} \mathtt{V}\mathcal{D}(\tilde{A})_0 = \mathcal{D}(\tilde{A})_0^{\perp_A} \overset{1}{\subseteq} \mathcal{D}(A)_1.$$

Indeed, the first inclusion follows from (3) and the second follows from (1). Now we define
$$\mathcal{D}_{B,0} = \mathcal{D}(\tilde{A})_0,\ \mathcal{D}_{B,1} = p^{-1}\mathtt{V}\mathcal{D}_{B,0},\ \mathcal{D}_B = \mathcal{D}_{B,0} + \mathcal{D}_{B,1}.$$
We can easily verify $\mathcal{D}_B$ is $\mathtt{F},\mathtt{V}$-stable from the fact that $\mathcal{D}_{B,0}$ is $\mathtt{V}^{-1}\mathtt{F}$-invariant. Moreover, we have an injection $\mathcal{D}(\tilde{A}) \to \mathcal{D}_B$. By covariant Dieudonné theory there exists an abelian 3-fold $B$ such that $\mathcal{D}(B) = \mathcal{D}_B$, and the inclusion $\mathcal{D}(\tilde{A}) \to \mathcal{D}(B)$ is induced by an isogeny $\delta : \tilde{A} \to B$. Let $\lambda_B$ be the unique polarization such that
$$\lambda_{\tilde{A}} = \delta^\vee \circ \lambda_B \circ \delta.$$

We have the relation
$$\langle x, y\rangle_{\lambda_{\tilde{A}}} = \langle x, y\rangle_{\lambda_B},\ x, y \in \mathcal{D}(\tilde{A}).$$
Define the level structure $\eta_B$ by $\eta_B = \delta_* \circ \eta_{\tilde{A}}$. We verify
1. $\mathcal{D}(B)$ is of signature type (0,3) : this follows from the definition.
2. $\mathcal{D}(B)$ is self-dual with respect to $\langle\ ,\ \rangle_{\lambda_B}$. Indeed, as above it suffices to show $\mathcal{D}(B)_1 = \mathcal{D}(B)_0^{\perp_B}$, which is equivalent to $\mathtt{V}\mathcal{D}_{B,0} = \mathcal{D}_{B,0}^{\perp_A}$, which follows from (4).

Finally we set $\theta(y) = (B, \lambda_B, \eta_B, A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha, \delta)$.

For (2), take $s \in P(\kappa)$ and thus $y = \tilde{\nu}(s) \in Y_2(\kappa)$. Under the morphism $\tilde{\nu}$ the exact sequences (2.17) and (2.10) coincide. Thus (2) follows and $\tilde{\nu}$ is an isomorphism. □

**Proposition 2.3.29.** *1. Define $\mathscr{V}$ by*
$$\mathscr{V}(R) := \mathrm{H}_1^{\mathrm{dR}}(B/R)_0/\gamma_{*,0}\omega_{A^\vee/R,0}$$
*where $(A, \lambda_A, \eta_A, B, \lambda_B, \eta_B, \gamma) \in N(R)$ for every $\mathbb{F}_{p^2}$-algebra $R$. Then $\mathscr{V}$ is a locally free sheaf of rank 2 over $N$.*

2. *The assignment sending a point* $(A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, B, \lambda_B, \eta_B, \alpha, \delta) \in P(R)$ *for every* $\mathbb{F}_{p^2}$*-algebra* $R$ *to the subbundle*

$$I := \delta_{*,0}\omega_{\tilde{A}^\vee/R,0}/\delta_{*,0}\alpha_{*,0}\omega_{A^\vee/R,0} \subseteq \mathscr{V}(R)$$

*induces an isomorphism of* $\mathbb{F}_{p^2}$*-schemes*

$$\mu : P \simeq \mathbb{P}(\mathscr{V}).$$

*The relations of morphisms are summarized in the following diagram*

$$\begin{array}{ccccc} \mathbb{P}(\mathscr{V}) & \xleftarrow{\cong} & Y_2 & \hookrightarrow & S_0(p) \\ \downarrow & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\nu} & S_{\mathrm{ss}} & \hookrightarrow & S \end{array} \quad (2.19)$$

*Proof.* 1. It suffices to show $\delta_{*,0}\alpha_{*,0}\omega_{A^\vee/R,0}$ is locally free $\mathcal{O}_R$-module of rank 1. Since $\delta_{*,0}$ is an isomorphism by Lemma 2.3.26(1), it suffices to show $\alpha_{*,0}\omega_{A^\vee/R,0}$ is locally free of rank 1, which follows from Lemma 2.3.18(3a).

2. To show $I$ is locally free of rank 1, the argument is the same as (1). Now we show $\mu$ is an isomorphism. Since $Y_2$ is smooth, it suffices to check that for every algebraically closed field $\kappa$ containing $\mathbb{F}_{p^2}$, we have
   (a) $\mu$ induces a bijection on $\kappa$-points ;
   (b) $\mu$ induces an isomorphism on the tangent spaces at every $\kappa$-point.

   To show (2a), it suffices to construct an inverse map $\theta$. Take $p' = (A, \lambda_A, \eta_A, B, \lambda_B, \eta_B, \gamma, I) \in \mathbb{P}(\mathscr{V})(\kappa)$ where $I$ is a locally free rank 1 $\mathcal{O}_\kappa$-submodule of $\mathscr{V}(\kappa)$. We list miscellaneous properties of $\mathcal{D}(A)$ and $\mathcal{D}(B)$ :
   (a) $\mathtt{V}\mathcal{D}(B) = \mathtt{F}\mathcal{D}(B)$. In fact, since $\mathcal{D}(B)$ is of signature (0,3), [Vol10, Lemma 1.4] gives

   $$\mathcal{D}(B)_0 = \mathtt{V}\mathcal{D}(B)_1 = \mathtt{F}\mathcal{D}(B)_1,$$

   which implies $\mathcal{D}(B)_0$ and $\mathcal{D}(B)_1$ are both $\mathtt{V}^{-1}\mathtt{F}$-invariant.
   (b) $\mathcal{D}(B)_0^{\perp_B} = \mathcal{D}(B)_1$ and $\mathcal{D}(B)_1^{\perp_B} = \mathcal{D}(B)_0$. This follows from the self-dual condition of $\lambda_B$.
   (c) We have chains of $W(\kappa)$-module

   $$p\mathcal{D}(B)_0 \overset{1}{\subseteq} \mathcal{D}(A)_0 \overset{2}{\subseteq} \mathcal{D}(B)_0,\ p\mathcal{D}(B)_1 \overset{1}{\subseteq} \mathcal{D}(A)_1 \overset{2}{\subseteq} \mathcal{D}(B)_1.$$

   (d) Denote by $\tilde{I}$ the preimage of $I$ under the composition of the reduction map $\mathcal{D}(B)_0 \to \mathcal{D}(B)_0/p\mathcal{D}(B)_0 \cong \mathrm{H}_1^{\mathrm{dR}}(B/R)_0$ and the quotient map $\mathrm{H}_1^{\mathrm{dR}}(B/\kappa)_0 \to \mathscr{V}(\kappa)$. Then we have a chain of $W(\kappa)$-module

   $$p\mathcal{D}(B)_0 \overset{2}{\subset} \tilde{I} \overset{1}{\subset} \mathcal{D}(B)_0.$$

   Now define

   $$\mathcal{D}_{\tilde{A},0} = D(B)_0, \mathcal{D}_{\tilde{A},1} = \mathtt{V}^{-1}\tilde{I}, \mathcal{D}_{\tilde{A}} = \mathcal{D}_{\tilde{A},0} + \mathcal{D}_{\tilde{A},1}$$

   We verify that $\mathcal{D}_{\tilde{A}}$ is $\mathtt{F}, \mathtt{V}$-stable and has the following chain conditions :

(a) $\mathtt{V}\mathcal{D}_{\tilde{A},0} \cong \mathtt{F}\mathcal{D}_{\tilde{A},0}$. This follows from (2a).

(b) $\mathtt{V}\mathcal{D}_{\tilde{A},0} \overset{2}{\subset} \mathcal{D}_{\tilde{A},1}$. The rank condition of $I$ gives $p\mathcal{D}(B)_0 \overset{2}{\subset} \tilde{I}$, thus we have $\mathtt{F}\mathcal{D}(B)_0 \overset{2}{\subset} \mathtt{V}^{-1}\tilde{I}$.

(c) $\mathtt{V}\mathcal{D}_{\tilde{A},1} \overset{1}{\subset} \mathcal{D}_{\tilde{A},0}$. This is by definition.

(d) $\mathtt{F}\mathcal{D}_{\tilde{A},1} \overset{1}{\subset} \mathcal{D}_{\tilde{A},0}$. This is equivalent to $\mathcal{D}_{\tilde{A},1} \overset{1}{\subset} \mathtt{F}^{-1}\mathcal{D}_{\tilde{A},0}$. The claim follows from the fact that $\mathtt{F}^{-1}\mathcal{D}_{\tilde{A},0} = \mathtt{V}^{-1}\mathcal{D}_{\tilde{A},0}$.

We also have an inclusion $\delta : D_{\tilde{A}} \subset \mathcal{D}(B)$ by definition. By covariant Dieudonné theory there exists an abelian 3-fold $A$ such that $\mathcal{D}(\tilde{A}) = \mathcal{D}_{\tilde{A}}$, and the inclusion $\mathcal{D}(\tilde{A}) \to \mathcal{D}(B)$ is induced by a prime-to-$p$ isogeny $\delta : \tilde{A} \to B$. Define the endormorphism structure $i_{\tilde{A}}$ on $\tilde{A}$ by $i_{\tilde{A}}(a) = \delta^{-1} \circ i_B(a) \circ \delta$ for $a \in O_F$. Then $(\tilde{A}, i_{\tilde{A}})$ is an $O_F$-abelian scheme. Let $\lambda_{\tilde{A}}$ be the unique polarization such that

$$\lambda_{\tilde{A}} = \delta^{\vee} \circ \lambda_B \circ \delta.$$

The pairings induced by $\lambda_{\tilde{A}}$ and $\lambda_B$ have the relation

$$\langle x, y\rangle_{\lambda_{\tilde{A}}} = \langle x, y\rangle_{\lambda_B}, \ \ x, y \in \mathcal{D}(A).$$

Define the level structure $\eta_{\tilde{A}}$ on $\tilde{A}$ by $\eta_{\tilde{A}} = \delta_*^{-1} \circ \eta_B$. We verify

(a) $\mathcal{D}(\tilde{A})$ is of signature (1,2). This follows from (2b) and (2c).

(b) $\mathcal{D}(\tilde{A})_1 \overset{1}{\subset} \mathcal{D}(\tilde{A})_0^{\perp_{\tilde{A}}}$. Consider $\mathcal{D}(\tilde{A})_0^{\perp_{\tilde{A}}} = \mathcal{D}(B)_0^{\perp_B} = p^{-1}\mathtt{V}\mathcal{D}(B)_0$. The claim follows from the definition and (2d).

(c) $\mathcal{D}(\tilde{A})_0 \overset{1}{\subset} \mathcal{D}(\tilde{A})_1^{\perp_{\tilde{A}}}$. This is the dual version of (2b).

(d) $\ker \lambda_{\tilde{A}}[p^{\infty}]$ is a $\tilde{A}[p]$-subgroup scheme of rank $p^2$. Indeed, from covariant Dieudonné theory it is equivalent to show $\mathcal{D}(\tilde{A}) \overset{2}{\subset} \mathcal{D}(\tilde{A})^{\perp_{\tilde{A}}}$. Thus it suffices to show $\mathcal{D}(\tilde{A})_0 \overset{1}{\subset} \mathcal{D}(\tilde{A})_1^{\perp_{\tilde{A}}}$ and $\mathcal{D}(\tilde{A})_1 \overset{1}{\subset} \mathcal{D}(\tilde{A})_0^{\perp_{\tilde{A}}}$ which follows from (2b) and (2c).

Now we prove (2b). Indeed, a deformation argument shows that the tangent space $\mathcal{T}_{\mathbb{P}(\mathscr{V})/\mathbb{F}_{p^2}, p'}$ at $p'$ fits into an exact sequence

$$\begin{aligned} 0 \to \mathcal{H}om(I, \mathscr{V}(R)/I) &\to \mathcal{T}_{\mathbb{P}(\mathscr{V})/\mathbb{F}_{p^2}, p} \\ &\to \mathcal{H}om(\omega_{A^{\vee}/R,0}/\ker\gamma_{*,0}, \mathrm{H}_1^{\mathrm{dR}}(A/R)_0/\omega_{A^{\vee}/R,0}) \to 0 \end{aligned} \quad (2.20)$$

which coincides with (2.17) under $\mu$. Thus (2b) follows.

□

### 2.3.6 Intersection of irreducible components of $S_0(p)$

Define $Y_{i,j} := Y_i \times_{S_0(p)} Y_j$ and $Y_{i,j,k} := Y_i \times_{S_0(p)} Y_j \times_{S_0(p)} Y_k$. The intersection of irreducible components are parametrized by some discrete Shimura varieties :

**Proposition 2.3.30.** *1. Denote by $\pi_{0,1}$ the restriction of the morphism $\pi$ on $Y_{0,1}$. Then $\pi_{0,1}$ factors through $S_{\mathrm{ssp}}$. Moreover, denote by $(\mathcal{A}, \lambda_{\mathcal{A}}, \eta_{\mathcal{A}})$ the universal object on $S_{\mathrm{ssp}}$. Let $\mathbb{P} := \mathbb{P}(\omega_{\mathcal{A}^{\vee},0})$ be the projective bundle associated with*

$\omega_{\mathcal{A}^\vee,0}$. *Then the assignment sending a point* $(A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_{0,1}(R)$ *for every* $\mathbb{F}_{p^2}$*-algebra* $R$ *to the subbundle*

$$I := (\alpha_{*,1}^{-1}\omega_{\tilde{A}^\vee/R,1})^\perp \subseteq \omega_{A^\vee/R,0}$$

*induces an isomorphism of* $\mathbb{F}_{p^2}$*-schemes*

$$\varphi_{0,1} : Y_{0,1} \cong \mathbb{P}$$

*The morphism* $\varphi_{0,1}$ *is equivariant under the prime-to-p Hecke correspondence. That is, given* $g \in K^p\backslash G(\mathbb{A}^{\infty,p})/K'^p$ *such that* $g^{-1}K^p g \subset K'^p$*, we have a commutative diagram*

$$\begin{CD} Y_{0,1}(K_p) @>{\varphi_{0,1}(K_p)}>> Y_{0,1}(K'^p) \\ @V{g}VV @VV{g}V \\ \mathbb{P}(K_p) @>{\varphi_{0,1}(K'^p)}>> \mathbb{P}(K'^p) \end{CD} .$$

*To summarize, we have the commutative diagram*

$$\begin{array}{ccccc} \mathbb{P} & \xleftarrow{\cong} & Y_{0,1} & \hookrightarrow & S_0(p) \\ & \searrow & \downarrow{\scriptstyle \pi_{0,1}} & & \downarrow{\scriptstyle \pi} \\ & & S_{\mathrm{ssp}} & \hookrightarrow & S \end{array} . \tag{2.21}$$

2. *The restriction of the morphism* $\tilde{\pi} := \pi \circ \tilde{\nu}^{-1}$ *on* $Y_{0,2}$ *in the diagram* (2.21) *is an isomorphism of* $\mathbb{F}_{p^2}$*-schemes which is equivariant under the prime-to-p Hecke correspondence.*

$$\tilde{\pi}_{0,2} := \tilde{\pi}\mid_{Y_{0,2}} : Y_{0,2} \cong N.$$

$$Y_{0,2} \hookrightarrow Y_2 \xrightarrow{\tilde{\nu}^{-1}} P \xrightarrow{\pi} N .$$

3. *The morphism* $\tilde{\pi}$ *induces a finite flat purely inseparable map*

$$\tilde{\pi}_{1,2} : Y_{1,2} \to N.$$

*which is equivariant under the prime-to-p Hecke correspondence.*

*Proof.* 1. We show $\pi_{0,1}$ factors through $S_{\mathrm{ssp}}$. Take $y = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_{0,1}(R)$ for a scheme $R \in \mathtt{Sch}'_{/\mathbb{F}_{p^2}}$, we need to show $\mathtt{V}\omega_{A^\vee/R,1} = 0$. By definition we have $\omega_{A^\vee/R,1} = \ker \alpha_{*,1}$; By Lemma 2.3.18(1b) we have $\mathtt{V} \ker \alpha_{*,1} = 0$. Thus $(A, \lambda_A, \eta_A) \in S_{\mathrm{ssp}}(R)$.

It is easy to see $\varphi_{0,1}$ is well-defined. Now we show it is an isomorphism. A deformation argument shows $Y_{0,1}$ is smooth with tangent bundle

$$\mathcal{T}_{Y_{0,1}} \cong \mathcal{H}om(\alpha_{*,1}^{-1}\omega_{\tilde{A}^\vee,1}/\ker \alpha_{*,1}, \mathrm{H}_1^{\mathrm{dR}}(\mathcal{A})_1/\alpha_{*,1}^{-1}\omega_{\tilde{A}^\vee,1}). \tag{2.22}$$

Thus it suffices to check that for every algebraically closed field $\kappa$ containing $\mathbb{F}_{p^2}$,

(a) $\varphi_{0,1}$ induces a bijection on $\kappa$-points ;

(b) $\varphi_{0,1}$ induces an isomorphism on the tangent spaces at every $\kappa$-point.

To show (1b), it suffices to construct an inverse map $\theta$. Take $p = (A, \lambda_A, \eta_A, I) \in \mathbb{P}(\kappa)$ where $I$ is a locally free rank 1 sub $\kappa$-module of $\omega_{A^\vee/R,0}$. We list miscellaneous properties of $\mathcal{D}(A)$ :

(a) $\mathtt{F}\mathcal{D}(A)_0 = \mathtt{V}\mathcal{D}(A)_0$. This is by $\mathtt{V}\omega_{A^\vee/R,1} = 0$.

(b) Denote by $\widetilde{I^\perp}$ the preimage of $I^\perp$ under the reduction map $\mathcal{D}(A)_1 \to \mathcal{D}(A)_1/p\mathcal{D}(A)_1 \cong \mathrm{H}_1^{\mathrm{dR}}(A/R)_1$. Then the condition $\omega_{A^\vee/\kappa,1} \subset I^\perp$ lifts as a chain of $W(\kappa)$-module

$$\mathtt{V}\mathcal{D}(A)_0 \overset{1}{\subset} \widetilde{I^\perp} \overset{1}{\subset} \mathcal{D}(A)_1 \overset{1}{\subset} \mathtt{F}^{-1}\mathcal{D}(A)_0.$$

Now define

$$\mathcal{D}_{\tilde{A},0} = \mathtt{V}^{-1}\widetilde{I^\perp}, \mathcal{D}_{\tilde{A},1} = \mathtt{V}^{-1}\mathcal{D}(A)_0, \mathcal{D}_{\tilde{A}} = \mathcal{D}_{\tilde{A},0} + \mathcal{D}_{\tilde{A},1}$$

We verify that $\mathcal{D}_{\tilde{A}}$ is $\mathtt{F}, \mathtt{V}$-stable and has the following chain conditions :

(a) $\mathtt{V}\mathcal{D}_{\tilde{A},0} \overset{2}{\subset} \mathcal{D}_{\tilde{A},1}$ and $\mathtt{F}\mathcal{D}_{\tilde{A},0} \overset{2}{\subset} \mathcal{D}_{\tilde{A},1}$. By (1a) it suffices to show that $\widetilde{I^\perp} \overset{2}{\subset} \mathtt{F}^{-1}\mathcal{D}(A)_0$, which is by (1b).

(b) $\mathtt{V}\mathcal{D}_{\tilde{A},1} \overset{1}{\subset} \mathcal{D}_{\tilde{A},0}$ and $\mathtt{F}\mathcal{D}_{\tilde{A},1} \overset{1}{\subset} \mathcal{D}_{\tilde{A},0}$. By (1a) it suffices to show $\mathtt{V}\mathcal{D}(A)_0 \overset{1}{\subset} \widetilde{I^\perp}$, which is by (1b).

We also have an inclusion $\alpha_* : \mathcal{D}(A) \subset \mathcal{D}_{\tilde{A}}$ by definition. By covariant Dieudonné theory there exists an abelian 3-fold $\tilde{A}$ such that $\mathcal{D}(\tilde{A}) = \mathcal{D}_{\tilde{A}}$, and $\alpha_*$ is induced by a prime-to-$p$ isogeny $\alpha : A \to \tilde{A}$. Define the endormorphism structure $i_{\tilde{A}}$ on $\tilde{A}$ by $i_A(a) = \alpha^{-1} \circ i_{\tilde{A}}(a) \circ \alpha$ for $a \in O_F$. Then $(\tilde{A}, i_{\tilde{A}})$ is an $O_F$-abelian scheme. Let $\lambda_{\tilde{A}}$ be the unique polarization such that

$$p\lambda_A = \alpha^\vee \circ \lambda_{\tilde{A}} \circ \alpha.$$

The pairings induced by $\lambda_A$ and $\lambda_{\tilde{A}}$ are related by

$$\langle x, y\rangle_{\lambda_A} = p^{-1}\langle \alpha_* x, \alpha_* y\rangle_{\lambda_{\tilde{A}}},\ x, y \in \mathcal{D}(A).$$

Define the level structure $\eta_{\tilde{A}}$ on $\tilde{A}$ by $\eta_{\tilde{A}} = \alpha_* \circ \eta_A$. We verify

(a) $\mathcal{D}(\tilde{A})$ is of signature (1,2). This follows from (1)

(b) $\mathcal{D}(\tilde{A})_1 \overset{1}{\subset} \mathcal{D}(\tilde{A})_0^{\vee_{\tilde{A}}}$. Consider $\mathcal{D}(\tilde{A})_0^{\vee_{\tilde{A}}} = (\mathtt{V}^{-1}\widetilde{I^\perp})^{\vee_{\tilde{A}}} = p^{-1}\mathtt{F}(\widetilde{I^\perp})^{\vee_A}$. The claim follows from the definition and (1b).

(c) $\mathcal{D}(\tilde{A})_0 \overset{1}{\subset} \mathcal{D}(\tilde{A})_1^{\vee_{\tilde{A}}}$. This is the dual version of (2b).

(d) $\ker \lambda_{\tilde{A}}[p^\infty]$ is a $\tilde{A}[p]$-subgroup scheme of rank $p^2$. Indeed, from covariant Dieudonné theory it is equivalent to show $\mathcal{D}(\tilde{A}) \overset{2}{\subset} \mathcal{D}(\tilde{A})^{\perp_{\tilde{A}}}$. Thus it suffices to show $\mathcal{D}(\tilde{A})_0 \overset{1}{\subset} \mathcal{D}(\tilde{A})_1^{\perp_{\tilde{A}}}$ and $\mathcal{D}(\tilde{A})_1 \overset{1}{\subset} \mathcal{D}(\tilde{A})_0^{\perp_{\tilde{A}}}$ which follows from (2b) and (2c).

(e) $\omega_{\tilde{A}^\vee/\kappa,0} = \operatorname{im} \alpha_{*,0}$ and $\omega_{A^\vee/\kappa,1} = \ker \alpha_{*,1}$. These are from the definition of $\mathcal{D}(\tilde{A})_1$ and (1a).

Finally we set $\theta(p) = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha)$. The equivariance under prime-to-$p$ Hecke correspondence is clear.

To show (1b), denote by $\mathcal{I} \subseteq \omega_{\mathcal{A}^\vee,0}$ the universal subbundle (of rank 1). Then we have an isomorphism

$$\mathcal{T}_{\mathbb{P}/S_{\mathrm{ssp}}} \simeq \mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\mathcal{I}^\perp/\omega_{\mathcal{A}^\vee,1}, \mathrm{H}_1^{\mathrm{dR}}(\mathcal{A})_1/\mathcal{I}^\perp). \tag{2.23}$$

Under the morphism $\varphi_{0,1}$ we have

$$I^\perp = \alpha_{*,1}^{-1}\omega_{A/\kappa,1}, \ \ker \alpha_{*,1} = \omega_{A/\kappa,1}.$$

Thus the expression of tangent space (2.22) and (2.23) coincide. Thus $\varphi_{0,1}$ is an isomorphism.

2. Since $N$ is smooth over $\mathbb{F}_{p^2}$ by Proposition 2.3.19(3), to show that $\varphi_{0,2}$ is an isomorphism, it suffices to check that for every algebraically closed field $\kappa$ containing $\mathbb{F}_{p^2}$, we have
   (a) $\varphi_{0,2}$ induces a bijection on $\kappa$-points ; and
   (b) $\varphi_{0,2}$ induces an isomorphism on the tangent spaces at every $\kappa$-point.

   For (2a), we construct an inverse map $\theta$ of $\varphi_{0,2}$. Take a point

$$n = (A, \lambda_A, \eta_A, B, \lambda_B, \eta_B, \gamma) \in N(\kappa).$$

   We define

$$\mathcal{D}_{\tilde{A},0} = \mathcal{D}(B)_0,\ \mathcal{D}_{\tilde{A},1} = \mathtt{V}^{-1}\mathcal{D}(A)_0,\ \mathcal{D}_{\tilde{A}} = \mathcal{D}_{\tilde{A},0} \oplus \mathcal{D}_{\tilde{A},1}.$$

   We can easily verify $\mathcal{D}_B$ is $\mathtt{F},\mathtt{V}$-stable from the fact that $\mathcal{D}_{B,0}$ is $\mathtt{V}^{-1}\mathtt{F}$-invariant. We also have an inclusion $\alpha_* : \mathcal{D}(A) \subset \mathcal{D}_{\tilde{A}}$ by definition. By covariant Dieudonné theory there exists an abelian 3-fold $\tilde{A}$ such that $\mathcal{D}(\tilde{A}) = \mathcal{D}_{\tilde{A}}$, and $\alpha_*$ is induced by a prime-to-$p$ isogeny $\alpha : A \to \tilde{A}$. Define the endormorphism structure $i_{\tilde{A}}$, polarization $\lambda_{\tilde{A}}$ and prime-to-$p$ level structure $\eta_{\tilde{A}}$ in a similar way. We verify
   (a) $\mathcal{D}(\tilde{A})$ is of signature (1,2). This is by definition.
   (b) $\mathcal{D}(\tilde{A})_1 \overset{1}{\subset} \mathcal{D}(\tilde{A})_0^{\perp_{\tilde{A}}}$. Consider $\mathcal{D}(\tilde{A})_0^{\perp_{\tilde{A}}} = p\tilde{I}^{\perp_{\tilde{A}}} = \tilde{I}^{\perp_A}$. The claim follows from the definition and (1b).
   (c) $\mathcal{D}(\tilde{A})_0 \overset{1}{\subset} \mathcal{D}(\tilde{A})_1^{\perp_{\tilde{A}}}$. This is the dual version of (2b).
   (d) $\ker \lambda_{\tilde{A}}[p^\infty]$ is a $\tilde{A}[p]$-subgroup scheme of rank $p^2$. Indeed, from covariant Dieudonné theory it is equivalent to show $\mathcal{D}(\tilde{A}) \overset{2}{\subset} \mathcal{D}(\tilde{A})^{\perp_{\tilde{A}}}$. Thus it suffices to show $\mathcal{D}(\tilde{A})_0 \overset{1}{\subset} \mathcal{D}(\tilde{A})_1^{\perp_{\tilde{A}}}$ and $\mathcal{D}(\tilde{A})_1 \overset{1}{\subset} \mathcal{D}(\tilde{A})_0^{\perp_{\tilde{A}}}$ which follows from (2b) and (2c).
   (e) $\omega_{\tilde{A}^\vee/\kappa,0} = \operatorname{im} \alpha_{*,0}$ and $\omega_{A^\vee/\kappa,1} = \ker \alpha_{*,1}$. These are from the definition of $\mathcal{D}(\tilde{A})_1$ and (1a).

   Finally we set $\theta(n) = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha)$. The equivariance under prime-to-$p$ Hecke correspondence is clear.

   For (2b), take $p \in P(\kappa)$ and thus $y = \tilde{\nu}(p) \in Y_2(\kappa)$. By the proof of Proposition 3 and Proposition 2.3.27, the canonical morphism of tangent space

$$\mathcal{T}_{Y_2,y} \to \tilde{\nu}_*\mathcal{T}_{P,p}$$

   is an isomorphism.

3. To show that $\varphi_{1,2}$ is a purely inseparable morphism, we need check that for every algebraically closed field $\kappa$ containing $\mathbb{F}_{p^2}$, $\varphi_{1,2}$ induces a bijection on $\kappa$-points. We construct an inverse map $\theta$ of $\varphi_{0,2}$. Take a point $n = (A, \lambda_A, \eta_A, B, \lambda_B, \eta_B, \gamma) \in N(\kappa)$. We define

$$\mathcal{D}_{\tilde{A},0} = \mathcal{D}(B)_0,\ \mathcal{D}_{\tilde{A},1} = p^{-1}\mathtt{V}\mathcal{D}(A)_0,\ \mathcal{D}_{\tilde{A}} = \mathcal{D}_{\tilde{A},0} + \mathcal{D}_{\tilde{A},1}.$$

We can easily verify $\mathcal{D}_B$ is $\mathtt{F}, \mathtt{V}$-stable from the fact that $\mathcal{D}_{B,0}$ is $\mathtt{V}^{-1}\mathtt{F}$-invariant. In a entirely similar way we can construct a point $\theta(n) = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha)$.

□

**Definition 2.3.31.** *Let $\tilde{M}$ be the moduli problem associating with every $\mathbb{F}_{p^2}$-algebra $R$ the set $\tilde{M}(R)$ of equivalence classes of tuples*

$$(\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, B, \lambda_B, \eta_B, \delta', \alpha, \delta)$$

*where*

1. $(\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, A, \lambda_A, \eta_A, \delta') \in M(R)$;
2. $(A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_{0,2}(R)$;
3. $(A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, B, \lambda_B, \eta_B, \alpha, \delta) \in P(R)$;
4. $(B, \lambda_B, \eta_B, \tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, \delta' \circ \alpha \circ \delta) \in T_0(p)(R)$.

*The equivalence relations are defined in a similar way.*

There is a natural correspondence

$$\begin{array}{ccc} & \tilde{M} & \\ {}^{\tilde{\rho}'}\swarrow & & \searrow^{\tilde{\rho}} \\ T_0(p) & & Y_{0,2} \end{array}$$

**Lemma 2.3.32.** *The morphism $\tilde{\rho}$ factors through $Y_{0,1,2}$. Moreover, $\tilde{M}$ is smooth of dimension 0.*

*Proof.* Take a point $(\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, B, \lambda_B, \eta_B, \delta', \alpha, \delta) \in \tilde{M}(R)$ for an $\mathbb{F}_{p^2}$-algebra $R$. By Lemma 2.3.18(1b) we have $(\ker \mathtt{V})_1 = \ker \alpha_{*,1}$. By Remark 2.3.13 we have $(\ker \mathtt{V})_1 = \omega_{A^\vee/R,1}$. Thus $\omega_{A^\vee/R,1} = \ker \alpha_{*,1}$ and $\tilde{\rho}$ factors through $Y_{0,1,2}$. It is easy to see $\tilde{B}, A, \tilde{A}, B$ have trivial deformation. Thus $\tilde{M}$ is smooth of dimension 0. □

**Lemma 2.3.33.** *1. The morphism $\tilde{\rho}$ induces an isomorphism of $\mathbb{F}_{p^2}$-schemes*

$$\tilde{\rho} : \tilde{M} \cong Y_{0,1,2}$$

*which is equivariant under the prime-to-p Hecke correspondence.*

*2. The morphism $\tilde{\rho}'$ is an isomorphism of $\mathbb{F}_{p^2}$-schemes*

$$\tilde{\rho}' : \tilde{M} \cong T_0(p)$$

*which is equivariant under the prime-to-p Hecke correspondence.*

*Proof.* 1. Since $\tilde{M}$ and $Y_{0,1,2}$ are smooth of dimension 0, to show that $\tilde{\rho}$ is an isomorphism, it suffices to check that for every algebraically closed field $\kappa$ containing $\mathbb{F}_{p^2}$, $\tilde{\rho}$ induces a bijection on $\kappa$-points. Take a point $y = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y_{0,1,2}(\kappa)$. We set

$$\tilde{\theta}(y) = (\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, B, \lambda_B, \eta_B, \delta', \alpha, \delta),$$

where $\tilde{B}$ with $\delta'$ are constructed in Lemma 2.3.16 and $B$ with $\delta$ are constructed in Lemma 2.3.28. It is easy to verify $\tilde{\theta}(y) \in \tilde{M}(R)$ and $\tilde{\theta}$ is the inverse of $\tilde{\rho}$. The equivariance under prime-to-$p$ Hecke correspondence is clear.

2. Since $\tilde{M}$ and $T_0(p)$ are smooth of dimension 0, to show that $\tilde{\rho}'$ is an isomorphism, it suffices to check that for every algebraically closed field $\kappa$ containing $\mathbb{F}_{p^2}$, $\tilde{\rho}'$ induces a bijection on $\kappa$-points. Take a point $t = (\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, B, \lambda_B, \eta_B, \beta) \in \tilde{T}(\kappa)$. We list properties of $\mathcal{D}(B)$ and $\mathcal{D}(\tilde{B})$ :

   (a) $\mathcal{D}(B)$ and $\mathcal{D}(\tilde{B})$ is $\mathtt{V}^{-1}\mathtt{F}$-invariant. In fact, since $\mathcal{D}(B)$ is of signature (0,3), [Vol10, Lemma 1.4] gives

   $$\mathcal{D}(B)_0 = \mathtt{V}\mathcal{D}(B)_1 = \mathtt{F}\mathcal{D}(B)_1,$$

   which implies $\mathcal{D}(B)_0$ and $\mathcal{D}(B)_1$ are both $\mathtt{V}^{-1}\mathtt{F}$-invariant. The argument is identical for $\mathcal{D}(\tilde{B})$.

   (b) $\mathcal{D}(B)_0^{\perp_B} = \mathcal{D}(B)_1$ and $\mathcal{D}(B)_1^{\perp_B} = \mathcal{D}(B)_0$. This follows from the self-dual condition of $\lambda_B$.

   (c) We have a chain of $W(\kappa)$-lattice

   $$\mathcal{D}(\tilde{B})_1 \overset{1}{\subset} \mathtt{V}^{-1}\mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}} \overset{2}{\subset} \frac{1}{p}\mathcal{D}(\tilde{B})_1.$$

   Indeed, $\ker \lambda_{\tilde{B}} \subset \tilde{B}[p]$ gives $\mathcal{D}(\tilde{B})_0^{\perp_{\tilde{B}}} \subset (1/p)\mathcal{D}(\tilde{B})_1$. The claim comes from (2b) and the fact that $\mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}} = (\mathtt{V}^{-1}\mathcal{D}(\tilde{B})_0)^{\perp_{\tilde{B}}} = \mathtt{F}(\mathcal{D}(\tilde{B})_0)^{\perp_{\tilde{B}}}$.

   (d) We have a relation

   $$p\mathcal{D}(B)_0 \overset{1}{\subset} \mathcal{D}(\tilde{B})_0 \overset{2}{\subset} \mathcal{D}(B)_0,\ p\mathcal{D}(B)_1 \overset{1}{\subset} \mathcal{D}(\tilde{B})_1 \overset{2}{\subset} \mathcal{D}(B)_1.$$

   Indeed, we have $p\mathcal{D}(B) \subset \mathcal{D}(\tilde{B})$ since $\ker \beta \in \tilde{B}[p]$ and there is an exact sequence

   $$0 \to \mathcal{D}(\tilde{B}) \to \mathcal{D}(B) \to \mathcal{D}(\ker \beta) \to 0$$

   by covariant Dieudonné theory.

   We set

   $$\mathcal{D}_{\tilde{A},0} = \mathtt{V}\mathcal{D}(B)_1,\ \mathcal{D}_{\tilde{A},1} = \mathtt{V}^{-1}\mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}},\ \mathcal{D}_{\tilde{A}} = \mathcal{D}_{\tilde{A},0} + \mathcal{D}_{\tilde{A},1}.$$

   We verify that $\mathcal{D}_{\tilde{A}}$ is $\mathtt{F}, \mathtt{V}$-stable. Indeed, since $\mathcal{D}(B)$ and $\mathcal{D}(\tilde{B})$ are $\mathtt{V}^{-1}\mathtt{F}$-invariant, it suffices to verify the condition for $\mathtt{V}$ : we have $\mathtt{V}\mathcal{D}_{\tilde{A}} = \mathtt{V}^2\mathcal{D}(B)_1 + \mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}}$. Then it suffices to show $p\mathcal{D}(B)_0 \subset p\mathcal{D}(\tilde{B})_1^{\perp_B}$ and $p\mathcal{D}(\tilde{B})_1^{\perp_B} \subset D(B)_0$ since $\mathtt{V}^2 = \mathtt{F}\mathtt{V} = p$. Then it suffices to show $\mathcal{D}(\tilde{B})_1 \subset D(B)_0^{\perp_B}$ and $p\mathcal{D}(B)_0^{\perp_B} \subset \mathcal{D}(\tilde{B})_1$, which are from (2d). By covariant Dieudonné theory there exists an abelian 3-fold $A$ such that $\mathcal{D}(\tilde{A}) = \mathcal{D}_{\tilde{A}}$, and the inclusion $\mathcal{D}(\tilde{A}) \to \mathcal{D}(B)$ is induced by a prime-to-$p$ isogeny $\delta : \tilde{A} \to B$. Define the endormorphism structure $i_{\tilde{A}}$ on $\tilde{A}$ by $i_{\tilde{A}}(a) = \delta^{-1} \circ i_B(a) \circ \delta$ for $a \in O_F$. Then $(A, i_A)$ is an $O_F$-abelian scheme. Let $\lambda_{\tilde{A}}$ be the unique polarization such that

$$\lambda_{\tilde{A}} = \delta^\vee \circ \lambda_B \circ \delta.$$

The pairings induced by $\lambda_{\tilde{A}}$ and $\lambda_B$ have the relation

$$\langle x, y\rangle_{\lambda_{\tilde{A}}} = \langle x, y\rangle_{\lambda_B},\ x, y \in \mathcal{D}(A).$$

Define the level structure $\eta_{\tilde{A}}$ on $\tilde{A}$ by $\eta_{\tilde{A}} = \delta_*^{-1} \circ \eta_B$. We verify

(a) $\mathcal{D}(\tilde{A})$ is of signature (1,2) : calculate the Lie algebra

$$\frac{\mathcal{D}(\tilde{A})}{\mathtt{V}\mathcal{D}(\tilde{A})} = \frac{\mathtt{V}\mathcal{D}(B)_1 + \mathtt{V}^{-1}\mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}}}{\mathcal{D}(\tilde{B})_1^{\perp_{\tilde{B}}} + p\mathcal{D}(B)_1}.$$

The argument is the same as that in verifying $\mathcal{D}_{\tilde{A}}$ is $\mathtt{F}, \mathtt{V}$-stable.

(b) $\ker \lambda_{\tilde{A}}[p^\infty]$ is a $\tilde{A}[p]$-subgroup scheme of rank $p^2$. Indeed, from covariant Dieudonné theory it is equivalent to show $\mathcal{D}(\tilde{A}) \overset{2}{\subset} \mathcal{D}(\tilde{A})^{\perp_{\tilde{A}}}$. Thus it suffices to show $\mathcal{D}(\tilde{A})_0 \overset{1}{\subset} \mathcal{D}(\tilde{A})_1^{\perp_{\tilde{A}}}$, which is equivalent to $p\mathcal{D}(\tilde{B})_0^{\perp_B} \overset{1}{\subset} \mathcal{D}(B)_1$, which is equivalent to $p\mathcal{D}(B)_0 \overset{1}{\subset} D(\tilde{B})_0$, which comes from (2d).

We have constructed $\tilde{A}$ and $\delta$, while $A$ and $\delta'$ are constructed in Lemma 2.3.16. The inclusion $\mathcal{D}(A) \subset \mathcal{D}(\tilde{A})$ is then induced by a prime-to-$p$ isogeny $\alpha : A \to \tilde{A}$.

Finally we set $\tilde{\theta}'(t) = (\tilde{B}, \lambda_{\tilde{B}}, \eta_{\tilde{B}}, A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, B, \lambda_B, \eta_B, \delta', \alpha, \delta)$. It is easy to verify $\tilde{\theta}'$ is the inverse of $\tilde{\rho}'$. The equivariance under prime-to-$p$ Hecke correspondence is clear.

□

## 2.4 Level lowering

### 2.4.1 Langlands group of $G$

Denote by $Z$ the center of $G$ and $G_0$ the unitary group associated with $G$. By [Kni01, p. 378] we have $Z(\mathbb{A}) = \mathbb{A}_F^\times$ and

$$G(\mathbb{A}) = Z(\mathbb{A})G_0(\mathbb{A}).$$

Let $P$ be the parabolic of $G$ and $M \subset P$ be the Levi factor of $G$ such that $P(\mathbb{Q})$ consists of matrices under the standard basis of $(\Lambda, \psi)$ of the form

$$P(\mathbb{Q}) = \left\{ \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \middle| a, b, c \in F^\times, ac^\mathfrak{c} = bb^\mathfrak{c} \right\}$$

and $M \subset P$ be the subgroup of diagonal matrices. The Langlands dual group of $G$ and $G_0$ are

$$\widehat{G_0} = \mathrm{GL}_3(\mathbb{C}), \quad \hat{G} = \mathrm{GL}_3(\mathbb{C}) \times \mathbb{C}^\times,$$

$$^L G_0 = \widehat{G_0} \rtimes \mathrm{Gal}(F/\mathbb{Q}), \quad ^L G = \widehat{G} \rtimes \mathrm{Gal}(F/\mathbb{Q}).$$

Let $c$ be the nontrivial element in $\mathrm{Gal}(F/\mathbb{Q})$. The action of $c$ on $\widehat{G}$ is given by

$$c(g, \lambda) = (\Phi({}^t g)^{-1}\Phi, \lambda \det g).$$

The embedding $G_0 \hookrightarrow G$ corresponds to the natural projection $\widehat{G} \to \widehat{G_0}$.

Let $p$ be a rational prime unramified in $F$. By Satake's classification, each unramified principal series $\sigma_p$ of $G_p$ corresponds to a $\widehat{G}$-conjugacy class of semisimple elements in $\widehat{G} \rtimes \mathrm{Frob}_p$ where $\mathrm{Frob}_p$ is the image of an Frobenius element at $p$, called the Langlands/Satake (semisimple) parameter of $\sigma_p$.

### 2.4.2 Classification of unramified principal series at an inert place

Keep the notation of Section 2.4.1. Suppose $p$ is inert in $F$. Let $\mathrm{LC}(P_p \backslash G_p)$ be the space of locally constant functions on $P_p \backslash G_p$, equipped with the natural action by $G_p$ via right multiplication. Let $\mathrm{St}_p$ be the quotient space of $\mathrm{LC}(P_p \backslash G_p)$ by the constant function. Then $\mathrm{St}_p$ is an irreducible admissible representation of $G_p$, called the Steinberg representation of $G_p$.

Moreover, let $\nu : G \to \mathbb{G}_m$ be the similitude homomorphism. For any $\beta \in \mathbb{C}^\times$, let $\mu_\beta : G_p \to \mathbb{C}^\times$ be the composite

$$\mu_\beta : G_p \xrightarrow{g \mapsto \frac{\det g}{\nu(g)}} \mathbb{Q}_p^\times \xrightarrow{x \mapsto \beta^{\mathrm{val}_p(x)}} \mathbb{C}^\times$$

Any unramified character of $M_p$ has the form

$$\begin{aligned} \chi_{\alpha,\beta} : & M_p \to \mathbb{C}^\times \\ & \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \mapsto \alpha^{\mathrm{val}_p(a) - \mathrm{val}_p(b)} \beta^{\mathrm{val}_p(b)} \end{aligned}$$

where $\alpha, \beta \in \mathbb{C}^*$ and $\mathrm{val}_p$ is the $p$-adic valuation on $F_p$.

Denote by $I_{\alpha,\beta} := \mathrm{Ind}_{P_p}^{G_p}(\chi_{\alpha,\beta})$ be the normalized unitary induction of $\chi_{\alpha,\beta}$, viewed as a character on $P_p$ trivial on its unipotent radical. Then $I_{\alpha,\beta}|_{G_{0,p}}$ coincides with $I(\alpha)$ in the notation of [BG06, 3.6.5, 3.6.6]. We list the properties of $I_{\alpha,\beta}$ :

1. If $\alpha \neq p^{\pm 2}, -p^{\pm 1}$, then $I_{\alpha,\beta}$ is irreducible.
2. If $\alpha = p^{\pm 2}$, $I_{\alpha,\beta}$ has two Jorden-Holder foctors : $\mathrm{St}_p \otimes \mu_\beta$ and $\mu_\beta$.
3. If $\alpha = -p^{\pm 1}$, then $I_{\alpha,\beta}$ has two Jordan-Hölder factors, $\pi_\beta^n$ which is unramified and non-tempered, and $\pi_\beta^2$ which is ramified and square-integrable.
4. The central character of $I_{\alpha,\beta}$ is
$$\begin{aligned} Z_p \cong F_p^\times \to & \mathbb{C}^* \\ b \mapsto & \beta^{\mathrm{val}_p(b)}. \end{aligned}$$
5. For all $\alpha, \beta \in \mathbb{C}^*, \dim I_{\alpha,\beta}^{K_p} = \dim I_{\alpha,\beta}^{\tilde{K}_p} = 1, \dim I_{\alpha,\beta}^{\mathrm{Iw}_p} = 2$.
6. $\mathrm{St}_p^{K_p} = \mathrm{St}_p^{\tilde{K}_p} = 0, \dim(\pi_\beta^n)^{K_p} = \dim(\pi_\beta^2)^{\tilde{K}_p} = 1, (\pi_\beta^n)^{\tilde{K}_p} = (\pi_\beta^2)^{K_p} = 0.$

7. Let $\pi_p$ be an admissible irreducible representation of $G_p$. Then $\pi_p^{\mathrm{Iw}_p} \neq 0$ if and only if it is a Jordan-Hölder factor of $I_{\alpha,\beta}$ for $\alpha, \beta \in \mathbb{C}^*$[Car79, Theorem 3.8].
8. The Langlands parameter of $I_{\alpha,\beta}$ is the $\widehat{G}$-conjugacy class of

$$t_{\alpha,\beta} = \left( \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta/\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, 1 \right) \rtimes c.$$

Note that $t_{\alpha,\beta}^2 = \left( \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}, \beta \right) \in \widehat{G}$.

### 2.4.3 Automorphic and Galois representation

Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$. Let $\pi_0$ be the restriction of $\pi$ to $G_0(\mathbb{A})$ and $\chi_\pi$ be the central character of $\pi$. Recall that Rogawski defined, a base change map from automorphic representations of $G_0(\mathbb{A})$(resp. $G(\mathbb{A})$) to $G_0(\mathbb{A}_F) \cong \mathrm{GL}_3(\mathbb{A}_F)$(resp. $\mathrm{GL}_3(\mathbb{A}_F) \times \mathrm{GL}_1(\mathbb{A}_F)$). Denote by $\pi_{0F}$(resp. $\pi_F$) the base change of $\pi_0$(resp. $\pi$). By [Rog92, Lemma 4.1.1], we have

$$\pi_F = \pi_{0F} \otimes \overline{\chi}_\pi$$

as a representation of $\mathrm{GL}_3(\mathbb{A}_F) \times \mathrm{GL}_1(\mathbb{A}_F)$, where $\overline{\chi}_\pi$ is the character $z \mapsto \chi_\pi(\bar{z})$. We say $\pi$ is *stable* [Rog90, Theorem 13.3] if $\pi_{0F}$ is a cuspidal representation.

Let $\Box$ be a finite set of places of $\mathbb{Q}$ containing the archimedean place such that $\pi$ is unramified outside $\Box$, $\ell$ be a rational prime and fix an isomorphism $\iota_\ell : \mathbb{Q}_\ell^{\mathrm{ac}} \to \mathbb{C}$. Let $p \nmid \Box$ be a finite place of $\mathbb{Q}$ unramified in $F$, $t_{\pi,p} \in {}^L G$ be the Satake parameter of $\pi_p$, well defined up to $\widehat{G}$-conjugacy, and $t_{\pi_0,p} \in^L G_0$ be the image of $t_{\pi,p}$ via the canonical projection ${}^L G \to^L G_0$.

1. If $pO_F = ww^{\mathfrak{c}}$ splits, then $t_{\pi_0,p} \in \widehat{G}_0 = \mathrm{GL}_3(\mathbb{C})$ and

$$\{t_{\pi_{0F},w}, t_{\pi_{0F},w^{\mathfrak{c}}}\} = \{t_{\pi_0,p}, {}^t t_{\pi_0,p}^{-1}\};$$

2. If $p$ is inert in $F$, then $t_{\pi_0,p} \in \widehat{G}_0 \rtimes \mathrm{Frob}_p$ and $t_{\pi_{0F},p} = t_{\pi_0,p}^2 \in \mathrm{GL}_3(\mathbb{C})$. If $\pi_p = I_{\alpha,\beta}$ for $\alpha, \beta \in \mathbb{C}^\times$, then $t_{\pi_{0F},p} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}$.

Assume now $\pi$ is stable and cohomological with trivial coefficient, i.e.,

$$\mathrm{H}^*(\mathfrak{g}, K_\infty; \pi_\infty) \neq 0$$

where $K_\infty$ is defined in Section 2.2.2 and $\mathfrak{g} = \mathrm{Lie}\, G(\mathbb{R}) \otimes \mathbb{C}$. Blasius and Rogawski [BR92, 1.9] defined a semisimple 3-dimensional $\ell$-adic representation

$$\rho_{\pi_0,\ell} : \mathrm{Gal}(F^{\mathrm{ac}}/F) \to \mathrm{GL}_3(\mathbb{Q}_\ell^{\mathrm{ac}})$$

attacted to $\pi_0$(or $\pi_{0,F}$) that is characterized as follows :

1. $\rho_{\pi_0,\ell}$ is unramified outside $\Box \cup \{\ell\}$.

2. Let $w$ be a non-archimedean place of $F$ with $w \nmid \Box\ell$ and $\mathrm{Frob}_w \in \mathrm{Gal}(F_w^{\mathrm{ac}}/F_w) \hookrightarrow \mathrm{Gal}(F^{\mathrm{ac}}/F)$ be a geometric Frobenius of $w$. Then the characteristic polynomial of $\rho_{\pi_0,\ell}(\mathrm{Frob}_w)$ coincides with that of $\iota_\ell(t_{\pi_{0F},w})q_w$, where $t_{\pi_{0F},w} \in \mathrm{GL}_3(\mathbb{C})$ is the Satake parameter of $\pi_{0F}$ at $w$, which is well defined up to conjugation.

Since $\pi$ is cohomological with trivial coefficient, $\chi_{\pi,\infty} : \mathbb{C}^\times \to \mathbb{C}^\times$ is trivial. By class field theory, $\iota_\ell \circ \chi_\pi$ can be viewed as a character of $\mathrm{Gal}(F^{\mathrm{ac}}/F)$. We put

$$\rho_{\pi,\ell} := \rho_{\pi_0,\ell} \otimes (\iota_\ell \circ \overline{\chi}_\pi). \tag{2.24}$$

Let $L/\mathbb{Q}_\ell$ be a sufficiently large finite extension such that $\mathrm{Im}(\rho_{\pi,\ell}) \subseteq \mathrm{GL}_3(L)$. Let $M^\circ$ be a $\mathrm{Gal}(F^{\mathrm{ac}}/F)$-stable $O_L$-lattice in the representation space of $\rho_{\pi,\ell}$. We denote by $\overline{\rho}_{\pi,\ell}$ the semi-simplification of $M^\circ/\varpi_L M^\circ$ as $\mathrm{Gal}(F^{\mathrm{ac}}/F)$-representation. By Brauer-Nesbitt theorem, $\overline{\rho}_{\pi,\ell}$ is independent of the choice of $M^\circ$.

By the local-global compatibility, if $p$ is inert in $F$ and $\pi_p \cong \mathrm{St}_p \otimes \mu_\beta$ for some $\beta \in \mathbb{C}^\times$, then the multiset of eigenvalues of $\overline{\rho}_{\pi,\ell}(\mathrm{Frob}_p)$ is $\{\iota_\ell^{-1}(\beta)p^4, \iota_\ell^{-1}(\beta)p^2, \iota_\ell^{-1}(\beta)\}$ mod $\ell$.

### 2.4.4 Spherical Hecke algebra

(cf. [BG06, 3.3.1]) For a finite place $p$ of $\mathbb{Q}$ at which $G$ is unramified, let $K_p$ denote a hyperspecial subgroup of $G_p$. Denote by $\mathbb{T}(G_p, K_p) := \mathbb{Z}[K_p\backslash G_p/K_p]$ the Hecke algebra of all $\mathbb{Z}$-valued locally constant, compactly supported bi-$K_p$-invariant functions on $G_p$. It is known that $\mathbb{T}(G_p, K_p)$ is a *commutative* algebra with unit element given by the characteristic function of $K_p$. We put $K^\Box := \prod_{p\notin\Box} K_p$. Denote by $\mathbb{T}(G^\Box, K^\Box)$ the prime-to-$\Box$ spherical Hecke algebra

$$\mathbb{T}(G^\Box, K^\Box) := \bigotimes_{p\notin\Box} \mathbb{T}(G_p, K_p).$$

Suppose $(\pi^\Box)^{K^\Box} \neq 0$. Then $\dim(\pi^\Box)^{K^\Box} = 1$ and there exists a homomorphism $\phi_\pi : \mathbb{T}^\Box \to O_L$ such that $T \in \mathbb{T}^\Box$ acts on $(\pi^\Box)^{K^\Box}$ via $\iota_\ell(\phi_\pi(T))$. Let $\lambda$ be the place in $L$ over $\mathbb{Q}_\ell$. Define

$$\overline{\phi}_{\pi,\ell} : \mathbb{T}^\Box \xrightarrow{\phi_\pi} O_L \to O_L/\lambda, \quad \mathfrak{m} := \ker \overline{\phi}_{\pi,\ell}. \tag{2.25}$$

The residual Galois representation $\overline{\rho}_{\pi,\ell}$ depends only on $\mathfrak{m}$ thus is also denoted by $\overline{\rho}_{\mathfrak{m}}$.

With the above preparations we can state the main theorem :

**Theorem 2.4.1.** *Let $\pi$ be a stable cuspidal automorphic representation of $G(\mathbb{A})$ cohomological with trivial coefficient. Let $p$ be a prime number inert in $F$. Suppose that*

1. *$\pi_p \cong \mathrm{St}_p \otimes \mu_\beta$ for some $\beta \in \mathbb{C}^\times$ as defined in Section 2.4.2 ;*
2. *if $i \neq 2$ then $\mathrm{H}^i(S \otimes F^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} = 0$;*
3. *$\overline{\rho}_{\pi,\ell}$ is absolutely irreducible ;*
4. *$\overline{\rho}_{\pi,\ell}$ is unramified at $p$ ;*
5. *$\ell \nmid (p-1)(p^3+1)$.*

*Then there exists a cuspidal automorphic representation* $\tilde{\pi}$ *of* $G(\mathbb{A})$ *such that* $\tilde{\pi}^{K^pK_p} \neq 0$ *and* $\overline{\rho}_{\tilde{\pi},\ell} \cong \overline{\rho}_{\pi,\ell}$.

To prove the theorem, we will firstly use the Rapoport-Zink weight-monodromy spectral sequence to study the cohomology of Picard modular surface, then we argue by contradiction. We need some preliminaries on the compactification of Shimura varieties.

### 2.4.5 Borel-Serre compactification of $S_0(p)$

Let $S_0(p)^{\mathrm{BS}}$ be the Borel-Serre compactification of $S_0(p)(\mathbb{C})$ and $\partial S_0(p)^{\mathrm{BS}}$ the boundary. By [NT16, Lemma 3.10] we have a $G(\mathbb{A}^\infty)$-equivariant isomorphism

$$\begin{aligned}\partial S_0(p)^{\mathrm{BS}}(\mathbb{C}) &\cong P(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/K^p\mathrm{Iw}_p \times e(P)) \\ &\cong \mathrm{Ind}_{P(\mathbb{A}^\infty)}^{G(\mathbb{A}^\infty)} P(\mathbb{Q})\backslash(P(\mathbb{A}^\infty)/K_P^p\mathrm{Iw}_p \times e(P)) \quad (2.26)\end{aligned}$$

where $e(P)$ is the smooth manifold with corners described in [BS73, §7.1] and $K_P^p = K^p \cap P(\mathbb{A}^{\infty,p})$.

**Lemma 2.4.2.** *Keep the notations and assumptions of Theorem 2.4.1. We have*

$$\mathrm{H}^*(\partial S_0(p)^{\mathrm{BS}}, \mathbb{F}_\ell)_{\mathfrak{m}} = 0.$$

*Proof.* Suppose on the contrary that $\mathrm{H}^*(\partial S_0(p)^{\mathrm{BS}}, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$. We will show that $\overline{\rho}_{\pi,\ell}$ is reducible, which contradicts the condition (3) in Theorem 2.4.1. Since $\overline{\rho}_{\pi,\ell} \cong \overline{\rho}_{\pi_0,\ell} \otimes (\iota_\ell \circ \overline{\chi}_\pi)$ by (2.24), it suffices to show that $\overline{\rho}_{\pi_0,\ell}$ is reducible. Put $K_P^\square = K^\square \cap P(\mathbb{A}^\square)$, $K_M^\square = K^\square \cap M(\mathbb{A}^\square)$, etc. We have a Satake map

$$\mathcal{N} : \mathbb{T}(G^\square, K_G^\square) \to \mathbb{T}(M^\square, K_M^\square).$$

Following the argument of [ACC+22, p. 36] or [NT16, Theorem 4.2], since $\mathfrak{m}$ is in the support of $\mathrm{H}^*(\partial S_0(p)^{\mathrm{BS}}, \mathbb{F}_\ell)$, there exists a subgroup $K'_M \subset K_M$ with $(K'_M)^\square = K_M^\square$ and a maximal ideal $\mathfrak{m}'$ of $\mathbb{T}(M^\square, K_M{}^\square)$ in the support of $\mathrm{H}^0(M(\mathbb{Q})\backslash M(\mathbb{A}^\infty)/K'_M, \mathbb{F}_\ell)$ such that $\mathfrak{m} = \mathcal{N}^{-1}(\mathfrak{m}')$. In other words, there exists a homomorphism $\overline{\theta}_{\pi,\ell} : \mathbb{T}(M^\square, K_M^\square) \to \overline{L}$ for a finite extension $\overline{L}$ of $\mathbb{F}_\ell$ such that $\overline{\phi}_{\pi,\ell} = \overline{\theta}_{\pi,\ell} \circ \mathcal{N}$.

Put $H := \mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m$. The standard Levi $M$ is a torus

$$\begin{aligned} M &\cong H \times H \\ \mathrm{diag}(a,b,c) &\mapsto (a,b).\end{aligned}$$

We can now assume $K'_M = K'_H \times K'_H$ which implies

$$\mathbb{T}(M^\square, K'^\square_M) \cong \mathbb{T}(H^\square, K'^\square_H) \otimes \mathbb{T}(H^\square, K'^\square_H).$$

Since $H(\mathbb{A}) = \mathbb{A}_F^\times$, $\overline{\theta}_{\pi,\ell}$ is equivalent to two Hecke characters $\overline{\psi}_1, \overline{\psi}_2 : \mathbb{A}_F^\times/F^\times K'_H \to \overline{L}$.

By class field theory, $\overline{\psi}_1$ and $\overline{\psi}_2$ correspond to two Galois characters $\overline{\sigma}_1, \overline{\sigma}_2 : \mathrm{Gal}(F^{\mathrm{ac}}/F) \to \overline{L}^\times$ such that for a place $w$ in $F$ and a uniformizer $\varpi_w$ in $F_w \subset \mathbb{A}_F^\times$, we have

$$\overline{\sigma}_i(\mathrm{Frob}_w) = \overline{\psi}_i(\varpi_w).$$

We claim that

$$\overline{\rho}_{\pi_0,\ell} \cong (\overline{\sigma}_1 \oplus \overline{\sigma}_2 \cdot \overline{\sigma}_2^{\mathfrak{c},\vee} \oplus \overline{\sigma}_1^{\mathfrak{c},\vee}) \otimes \epsilon_\ell \tag{2.27}$$

where $\epsilon_\ell$ is the $\ell$-adic cyclotomic character and $\overline{\sigma}_i^{\mathfrak{c},\vee}(g) := \overline{\sigma}_i((g^{\mathfrak{c}})^{-1})$. Indeed, by Chebatorev density and Brauer-Nesbitt theorem, it suffices to verify that for every place $q = ww^{\mathfrak{c}}$ split in $F$, the eigenvalues of $\mathrm{Frob}_w$ for $\overline{\rho}_{\pi_0,\ell}$ and $(\overline{\sigma}_1 \oplus \overline{\sigma}_2 \cdot \overline{\sigma}_2^{\mathfrak{c},\vee} \oplus \overline{\sigma}_1^{\mathfrak{c},\vee}) \otimes \epsilon_\ell$ coincide.

To show this, recall that

$$\begin{aligned} G(\mathbb{Q}_q) &= \left\{ g \in \mathrm{GL}_3(F \otimes_{\mathbb{Q}} \mathbb{Q}_q) \mid {}^t g^{\mathfrak{c}} \Phi g = \nu(g)\Phi \text{ for some } \nu(g) \in \mathbb{Q}_q^\times \right\} \\ &= \left\{ g = (g_w, g_{w^{\mathfrak{c}}}) \in \mathrm{GL}_3(F_w) \times \mathrm{GL}_3(F_{w^{\mathfrak{c}}}) \mid g_{w^{\mathfrak{c}}} = \nu(g)\Phi({}^t g_w^{-1})\Phi \right\}. \end{aligned}$$

Therefore, we have an isomorphism

$$\begin{aligned} G_q &\cong \mathrm{GL}_3(F_w) \times \mathbb{Q}_q^\times \\ g &\mapsto (g_w, \nu(g)) \end{aligned}$$

under which $g = \mathrm{diag}(a, b, c) \in M_q$ is identified with $(\mathrm{diag}(a_w, b_w, c_w), b_w b_{w^{\mathfrak{c}}})$. If $T \subset \mathrm{GL}_3$ denotes the diagonal torus, we have an isomorphism

$$\begin{aligned} T_q \times \mathbb{Q}_q^\times &\cong M_q \\ (\mathrm{diag}(a, b, c), \nu) &\mapsto \mathrm{diag}((a, \nu/c), (b, \nu/b), (c, \nu/a)). \end{aligned}$$

Since $H_q \cong F_w^\times \times F_{w^{\mathfrak{c}}}^\times$, we have

$$\begin{aligned} T_q \times \mathbb{Q}_q^\times &\cong H_q \times H_q \\ (\mathrm{diag}(a, b, c), \nu) &\mapsto ((a, \nu/c), (b, \nu/b)). \end{aligned}$$

The local component at $q$ of $\overline{\psi}_1\overline{\psi}_2$ is given by

$$\begin{aligned} (\overline{\psi}_1\overline{\psi}_2)_q : (\mathrm{diag}(a, b, c), \nu) &\mapsto \overline{\psi}_{1,w}(a)\overline{\psi}_{1,w^{\mathfrak{c}}}(\nu/c)\overline{\psi}_{2,w}(b)\overline{\psi}_{2,w^{\mathfrak{c}}}(\nu/b) \\ &= (\overline{\psi}_{1,w^{\mathfrak{c}}}\overline{\psi}_{2,w^{\mathfrak{c}}})(\nu)\overline{\psi}_{1,w}(a)(\overline{\psi}_{2,w}\overline{\psi}_{2,w^{\mathfrak{c}}}^{-1})(b)\overline{\psi}_{1,w^{\mathfrak{c}}}^{-1}(c). \end{aligned}$$

Let $\widehat{M} = \widehat{T} \times \mathbb{G}_m$ be the torus over $\mathbb{Z}_\ell$ dual to $M_{\mathbb{Q}_q}$. By duality, the group of the unramified characters of $M_{\mathbb{Q}_q}$ with values in $\overline{L}^\times$ is isomorphic to

$$X^*(M_{\mathbb{Q}_q}) \otimes L^\times = X_*(\widehat{M}) \otimes L^\times \cong \widehat{M}(\overline{L}),$$

where $X^*(M_{\mathbb{Q}_q})$ (resp. $X_*(\widehat{M})$) denotes the character group of $M_{\mathbb{Q}_q}$ (resp. the cocharacter group of $\widehat{M}$). With this identification $(\overline{\psi}_1\overline{\psi}_2)_q$ corresponds to the semisimple element

$$\left( \begin{pmatrix} \overline{\psi}_{1,w}(q) & 0 & 0 \\ 0 & (\overline{\psi}_{2,w}/\overline{\psi}_{2,w^{\mathfrak{c}}})(q) & 0 \\ 0 & 0 & \overline{\psi}_{1,w^{\mathfrak{c}}}^{-1}(q) \end{pmatrix}, \nu \right) \in \widehat{M}(\overline{L}).$$

By Section 2.4.3 and Satake isomorphism the eigenvalues of $\overline{\rho}_{\pi_0,\ell}(\mathrm{Frob}_w)$ are given by

$$q\{\overline{\psi}_{1,w}(q), (\overline{\psi}_{2,w}\overline{\psi}_{2,w^{\mathfrak{c}}}^{-1})(q), \overline{\psi}_{1,w^{\mathfrak{c}}}^{-1}(q)\}.$$

By Chebotarev density, the equality (2.27) holds. This finishes the proof of the lemma. □

**Corollary 2.4.3.** *Denote by $S_0(p)^{\mathrm{BB}}$ the Baily-Borel compactification of $S_0(p)$. Then we have canonical isomorphisms*

$$\mathrm{H}^2_c(S_0(p)\otimes F^{\mathrm{ac}},\mathbb{F}_\ell)_{\mathfrak{m}} \cong \mathrm{IH}^2(S_0(p)^{\mathrm{BB}}\otimes F^{\mathrm{ac}},\mathbb{F}_\ell)_{\mathfrak{m}} \cong \mathrm{H}^2(S_0(p)\otimes F^{\mathrm{ac}},\mathbb{F}_\ell)_{\mathfrak{m}}. \quad (2.28)$$

*Proof.* One has an exact sequence of Betti cohomology [CS19, Remark 1.5]

$$0\to \mathrm{H}^1(\partial S_0(p)^{\mathrm{BS}},\mathbb{F}_\ell)\to \mathrm{H}^2_c(S_0(p),\mathbb{F}_\ell)\to \mathrm{H}^2(S_0(p),\mathbb{F}_\ell)\to \mathrm{H}^2(\partial S_0(p)^{\mathrm{BS}},\mathbb{F}_\ell)\to 0 \quad (2.29)$$

which is equivariant under $\mathbb{T}(G^{\square},K^{\square})$-action. By [HLR86, 1.8] the intersection cohomology group $\mathrm{IH}^2(S_0(p)^{\mathrm{BB}}\otimes F^{\mathrm{ac}},\mathbb{F}_\ell)_{\mathfrak{m}}$ is the image of the map $\mathrm{H}^2_c(S_0(p)\otimes F^{\mathrm{ac}},\mathbb{F}_\ell)_{\mathfrak{m}}\to \mathrm{H}^2(S_0(p)\otimes F^{\mathrm{ac}},\mathbb{F}_\ell)_{\mathfrak{m}}$. The corollary then follows from Lemma 2.4.2. □

### 2.4.6 Generalities on the weight-monodromy spectral sequence

[Sai03, Corollary 2.2.4], [Liu19, 2.1]. Let $K$ be a henselian discrete valuation field with residue field $\kappa$ and a separable closure $\bar{K}$. We fix a prime $p$ that is different from the characteristic of $\kappa$. Throughout this section, the coefficient ring $\Lambda$ will be $\mathbb{F}_\ell$. We first recall the following definition.

**Definition 2.4.4** (Strictly semistable scheme)**.** *Let $X$ be a scheme locally of finite presentation over Spec $O_K$. We say that $X$ is strictly semistable if it is Zariski locally étale over*

$$\mathit{Spec}\ O_K[t_1,\dots,t_n]/(t_1\cdots t_s-\varpi)$$

*for some integers $0\le s\le n$ (which may vary) and a uniformizer $\varpi$ of $K$.*

Let $X$ be a proper strictly semistable scheme over $O_K$. The special fiber $X_\kappa := X\otimes_{O_K}\kappa$ is a normal crossing divisor of $X$. Suppose that $\{X_1,\dots,X_m\}$ is the set of irreducible components of $X_\kappa$. For $r\ge 0$, put

$$X_\kappa^{(r)} = \coprod_{I\subset\{1,\dots,m\},|I|=r+1}\ \bigcap_{i\in I} X_i.$$

Then $X_\kappa^{(r)}$ is a finite disjoint union of smooth proper $\kappa$-schemes of codimension $r$. From [Sai03, page 610], we have the pullback map

$$\delta_r^*:\mathrm{H}^s(X_{\bar\kappa}^{(r)},\Lambda(j))\to \mathrm{H}^s(X_{\bar\kappa}^{(r+1)},\Lambda(j))$$

and the pushforward (Gysin) map

$$\delta_{r*}:\mathrm{H}^s(X_{\bar\kappa}^{(r)},\Lambda(j))\to \mathrm{H}^{s+2}(X_{\bar\kappa}^{(r-1)},\Lambda(j+1))$$

for every integer $j$. These maps satisfy the formula

$$\delta_{r-1}^*\circ\delta_{r*}+\delta_{r+1*}\circ\delta_r^*=0$$

for $r\ge 1$. For reader's convenience, we recall the definition here. For subsets $J\subset I\subset\{1,\dots,m\}$ such that $|I|=|J|+1$, let $i_{JI}:\bigcap_{i\in I}X_i\to\bigcap_{i\in J}X_i$ denote the closed immersion. If $I=\{i_0<\cdots<i_r\}$ and $J=I\backslash\{i_j\}$, then we put $\epsilon(J,I)=(-1)^j$. We define $\delta_r^*$ to be the alternating sum $\sum_{I\subset J,|I|=|J|-1=r+1}\epsilon(I,J)i_{IJ}^*$ of the pullback maps, and $\delta_{r*}$ to be the alternating sum $\sum_{I\supset J,|I|=|J|+1=r+1}\epsilon(J,I)i_{JI*}$ of the Gysin maps.

**Remark 2.4.5.** *In general, the maps $\delta_r^*$ and $\delta_{r*}$ depend on the ordering of the irreducible components of $X_\kappa$. However, it is easy to see that the composite map $\delta_{1*} \circ \delta_0^*$ does not depend on such ordering.*

Let us recall the weight spectral sequence attached to $X$. Denote by $K^{\mathrm{ur}} \subset K^{\mathrm{ac}}$ the maximal unramified extension, with the residue field $\bar{\kappa}$ which is a separable closure of $\kappa$. Then we have $G_K/I_K \simeq G_\kappa$. Denote by $t_0 : I_K \to \Lambda_0(1)$ the ($p$-adic) tame quotient homomorphism, that is, the one sending $\sigma \in I_K$ to $(\sigma(\varpi^{1/p^n})/\varpi^{1/p^n})_n$ for a uniformizer $\varpi$ of $K$. We fix an element $T \in I_K$ such that $t_0(T)$ is a topological generator of $\Lambda_0(1)$.

We have the weight spectral sequence $\mathrm{E}_X$ attached to the (proper strictly semistable) scheme $X$, where

$$(\mathrm{E}_X)_1^{r,s} = \bigoplus_{i \geq \max(0,-r)} \mathrm{H}^{s-2i}(X_{\bar{\kappa}}^{(r+2i)}, \Lambda(-i)) \Rightarrow \mathrm{H}^{r+s}(X_{\bar{K}}, \Lambda) \tag{2.30}$$

This is also known as the Rapoport-Zink spectral sequence, first studied in [RZ82] ; here we will follow the convention and discussion in [Sai03]. For $t \in \mathbb{Z}$, put ${}^t\mathrm{E}_X = \mathrm{E}_X(t)$ and we will suppress the subscript $X$ in the notation of the spectral sequence if it causes no confusion. By [Sai03, Corollary 2.8(2)], we have a map $\mu : \mathrm{E}_\bullet^{\bullet-1,\bullet+1} \to \mathrm{E}_\bullet^{\bullet+1,\bullet-1}$ of spectral sequences (depending on $T$ ) and its version for ${}^r\mathrm{E}$. The map $\mu^{r,s} := \mu_1^{r,s} : {}^t\mathrm{E}_1^{p-1,q+1} \to {}^t\mathrm{E}_1^{r+1,s-1}$ is the sum of its restrictions to each direct summand $\mathrm{H}^{s+1-2i}(X_{\bar{\kappa}}^{(2i+1)}, \Lambda(r-i))$, and such restriction is the tensor product by $t_0(T)$ (resp. the zero map) if $\mathrm{H}^{s+1-2i}(X_{\bar{\kappa}}^{(2i+1)}, \Lambda(t-i+1))$ does (resp. does not) appear in the target. The map $\mu^{r,s}$ induces a map, known as the monodromy operator,

$$\tilde{\mu}^{r,s} : {}^t\mathrm{E}_2^{r-1,s+1} \to {}^t\mathrm{E}_2^{r+1,s-1}(-1)$$

of $\Lambda[G_\kappa]$-modules.

### 2.4.7 Weight-monodromy spectral sequence for $S_0(p)$

We will try to apply the weight-monodromy spectral sequence to the surface $f : S_0(p) \to \mathrm{Spec}(O_F \otimes \mathbb{Z}_{(p)})$. In the derivation of weight-monodromy spectral sequence $f$ is required to be proper to get $\mathrm{H}^i(S_0(p) \otimes \mathbb{F}_p^{\mathrm{ac}}, R\Psi\mathbb{Z}_\ell) \cong \mathrm{H}^i(S_0(p) \otimes F^{\mathrm{ac}}, \mathbb{Z}_\ell)$. However, in our case $f$ is not proper. Fortunately, according to [LS18, Corollary 4.6], $\mathrm{H}^i(S_0(p) \otimes \mathbb{F}_p^{\mathrm{ac}}, R\Psi\mathbb{Z}_\ell) \cong \mathrm{H}^i(S_0(p) \otimes F^{\mathrm{ac}}, \mathbb{Z}_\ell)$ still holds. Put

$$Y^{(2)} = Y_{0,1,2} \otimes \mathbb{F}_p^{\mathrm{ac}},\ Y^{(1)} = (Y_{0,1} \sqcup Y_{0,2} \sqcup Y_{1,2}) \otimes \mathbb{F}_p^{\mathrm{ac}},\ Y^{(0)} = (Y_0 \sqcup Y_1 \sqcup Y_2) \otimes \mathbb{F}_p^{\mathrm{ac}}.$$

The spectral sequence (2.30) with $\Lambda = \mathbb{F}_\ell$ reads

$$\begin{array}{lll}
\mathrm{H}^0(Y^{(2)})_{\mathfrak{m}}(-2) \to \mathrm{H}^2(Y^{(1)})_{\mathfrak{m}}(-1) & \to & \mathrm{H}^4(Y^{(0)})_{\mathfrak{m}} \\
\mathrm{H}^1(Y^{(1)})_{\mathfrak{m}}(-1) & \to & \mathrm{H}^3(Y^{(0)})_{\mathfrak{m}} \\
\mathrm{H}^0(Y^{(1)})_{\mathfrak{m}}(-1) & \to & \mathrm{H}^2(Y^{(0)})_{\mathfrak{m}} \oplus \mathrm{H}^0(Y^{(2)})_{\mathfrak{m}}(-1) \to \mathrm{H}^2(Y^{(1)})_{\mathfrak{m}} \\
 & & \mathrm{H}^1(Y^{(0)})_{\mathfrak{m}} \to \mathrm{H}^1(Y^{(1)})_{\mathfrak{m}} \\
 & & \mathrm{H}^0(Y^{(0)})_{\mathfrak{m}} \to \mathrm{H}^0(Y^{(1)})_{\mathfrak{m}} \to \mathrm{H}^0(Y^{(2)})_{\mathfrak{m}}.
\end{array}$$

Here we omit the coefficient $\mathbb{F}_\ell$ in the cohomology group.

**Lemma 2.4.6.** *Let $G_0$ (resp. $G'_0$) be the unitary group attached to $G$ (resp. $G'$) as in Section 2.4.1. Recall the inner form $G'$ defined in Section 2.2.6. Put $G_{0,p} := G_0(\mathbb{Q}_p), K_{0,p} := K_p \cap G_{0,p}, K_0^p := K^p \cap G_0^p$. Let $K^1_{0,p}$ be the kernel of the reduction map $G_0(O_p) \to G_0(\mathbb{F}_{p^2})$. Then we have an isomorphism*

$$\iota_\ell \mathrm{H}^1(N \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{Q}_\ell^{\mathrm{ac}}) \mid_{G_0(\mathbb{A})} \cong \mathrm{Map}_{K_{0,p}}(G'_0(\mathbb{Q})\backslash G'_0(\mathbb{A}^\infty)/K_0^p, \Omega_3) \tag{2.31}$$

*of $\mathbb{C}[K_0^pK^1_{p,0}\backslash G'_0(\mathbb{A}^\infty)/K_0^pK^1_{p,0}]$-modules, where $(\rho_{\Omega_3}, \Omega_3)$ is the Tate-Thompson representation of $K_{0,p}$ in [LTX+22, C.2] and the right hand side of the isomorphism denotes the locally constant maps $f : G'_0(\mathbb{Q})\backslash G'_0(\mathbb{A}^\infty)/K_0^p \to \Omega_3$ such that $f(gk) = \rho_{\Omega_3}(k^{-1})f(g)$ for $k \in K_{0,p}$ and $g \in G'_0(\mathbb{A}^\infty)$. Moreover, let $\pi_0^\square$ be an irreducible admissible representation of $G_0(\mathbb{A}^\square)$ such that $(\pi_0^\square)^{K_0^\square}$ is a constituent of $\iota_\ell \mathrm{H}^1(N \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{Q}_\ell^{\mathrm{ac}})$. Then one can complete $\pi_0^\square$ to an automorphic representation $\pi'_0 = \pi_0^\square \otimes \prod_{q\in\square} \pi'_{0,q}$ of $G'_0(\mathbb{A})$ such that $\mathrm{BC}(\pi'_{0,p})$ is a constituent of an unramified principal series of $\mathrm{GL}_3(F_p)$ with Satake parameter $\{-p, 1, -p^{-1}\}$, where $\mathrm{BC}$ denotes the local base change from $G_{0,p}$ to $\mathrm{GL}_3(F_p)$.*

*Proof.* Recall the fiber of the morphism $N \to T$ is geometrically a Fermat curve of degree $p + 1$ where $T(\mathbb{C}) \cong G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/K^pK_p$ by Theorem 2.3.10(2). Take $t \in T(\mathbb{F}_p^{\mathrm{ac}})$, then $\mathrm{H}^1(N \otimes \mathbb{F}_p^{\mathrm{ac}} \cap \theta^{-1}(t), \mathbb{Q}_\ell^{\mathrm{ac}}) \mid_{G_0(\mathbb{A})}$ is a representation of $G_0(\mathbb{F}_p^{\mathrm{ac}}) = K_{0,p}/K^1_{0,p}$, isomorphic to $\Omega_3$. For the remaining part, note that the right-hand side of (2.31) is a $\mathbb{C}[K_0^pK^1_{p,0}\backslash G_0(\mathbb{A}^\infty)/K_0^pK^1_{p,0}]$-submodule of $\mathrm{Map}(G'_0(\mathbb{Q})\backslash G'_0(\mathbb{A}^\infty)/K_0^pK^1_{0,p}, \mathbb{C})$. In particular, we can complete $\pi_0^\square$ to an automorphic representation $\pi'_0 = \pi_0^\square \otimes \prod_{q\in\square} \pi'_{0,q}$ of $G'_0(\mathbb{A})$ such that $\pi'_{0,p}|_{K_{0,p}}$ contains $\Omega_3$. The same argument as [LTX+22, Theorem 5.6.4(ii)] then implies $\pi'_{0,p} \cong c\text{-Ind}_{K_{p,0}}^{G_0}(\Omega_3) \cong \pi^s(1)$ where $\pi^s(1)$ appears in [Rog90, Proposition 13.1.3(d)]. The base change $\mathrm{BC}(\pi^s(1))$ has the Satake parameter $\{-p, 1, -p^{-1}\}$ by [Rog90, Proposition 13.2.2(c)]. The lemma follows. □

**Lemma 2.4.7.** *Keep the notations and assumptions of Theorem 2.4.1. Suppose there is no level-lowering, i.e., there is no automorphic representation $\pi'$ of $G(\mathbb{A})$ such that $\pi'^{K^pK_p} \neq 0$ and $\overline{\rho}_{\pi',\ell} \cong \overline{\rho}_{\pi,\ell}$. Then one has*

1. $\mathrm{H}^2(S \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} = 0$;
2. $\mathrm{H}^2(T \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} = 0$;
3. $\mathrm{H}^0(\tilde{T} \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} = 0$;
4. $\mathrm{H}^*(S^\# \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} = 0$;
5. $\mathrm{H}^*(N \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} = 0$;

*Proof.* 1. Suppose $\mathrm{H}^2(S \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$. By [LS18, Corollary 4.6], we have $\mathrm{H}^2(S \otimes F^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} \cong \mathrm{H}^2(S \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$. The universal coefficient theorem gives the exact sequence

$$0 \to \mathrm{H}^i(S \otimes F^{\mathrm{ac}}, \mathbb{Z}_\ell)_{\mathfrak{m}} \otimes \mathbb{F}_\ell \to \mathrm{H}^i(S \otimes F^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} \to \mathrm{H}^{i+1}(S \otimes F^{\mathrm{ac}}, \mathbb{Z}_\ell)_{\mathfrak{m}}[\ell] \to 0, \quad i \in \mathbb{Z}$$

which implies that $\mathrm{H}^2(S \otimes F^{\mathrm{ac}}, \mathbb{Z}_\ell)_{\mathfrak{m}}$ is torsion-free and non-zero. Thus there exists a cuspidal automorphic representation $\tilde{\pi}$ of $G(\mathbb{A})$ such that the $\tilde{\pi}$-isotypic component $\mathrm{H}^2(S \otimes F^{\mathrm{ac}}, \mathbb{Z}_\ell)_{\mathfrak{m}}[\tilde{\pi}] \otimes \mathbb{Q}_\ell^{\mathrm{ac}} \neq 0$ and $\tilde{\pi}^{K^pK_p} \neq 0$ since $S$ is of level $K^pK_p$. Moreover, by Section 2.4.3 the prime-to-$\Box$ Hecke equivariance implies $\overline{\rho}_{\tilde{\pi},\ell}(\mathrm{Frob}_q) = \overline{\rho}_{\pi,\ell}(\mathrm{Frob}_q)$ for $q \notin \Box$. Finally, Chebotarev density ensures $\overline{\rho}_{\tilde{\pi},\ell} \cong \overline{\rho}_{\pi,\ell}$. This contradicts the no-level-lowering assumption.

2. Suppose $\mathrm{H}^0(T \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^0(T \otimes F^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \neq 0$. Since $\mathrm{H}^0(T \otimes F^{\mathrm{ac}}, \mathbb{Z}_\ell)_\mathfrak{m}$ is torsion-free, there exists an irreducible automorphic representation $\pi'$ of $G'(\mathbb{A})$ such that $\pi'^{K^pK_p} \neq 0$. By [Clo00, Theorem 2.4] we can transfer $\pi'$ to an automorphic representation $\tilde{\pi}$ of $G(\mathbb{A})$ such that the finite components $\tilde{\pi}^\infty$ and $\pi'^\infty$ coincide. In particular $\tilde{\pi}^{K^pK_p} \neq 0$. The prime-to-$\Box$ Hecke equivariance and Chebatrov density then imply that $\overline{\rho}_{\tilde{\pi},\ell} \cong \overline{\rho}_{\pi,\ell}$, contradicting the no-level-lowering assumption.

3. Suppose $\mathrm{H}^0(\tilde{T} \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \neq 0$. By the same argument as (2), there is an irreducible automorphic representation $\pi'$ of $G'(\mathbb{A})$ such that $(\pi')^{K^p\tilde{K}_p} \neq 0$ and we can again transfer $\pi'$ to an automorphic representation $\tilde{\pi}$ of $G(\mathbb{A})$ such that the finite components $\tilde{\pi}^\infty$ and $\pi'^\infty$ coincide. In particular $\tilde{\pi}^{K^p\tilde{K}_p} \neq 0$. The prime-to-$\Box$ Hecke equivariance and Chebotarev density then imply that $\overline{\rho}_{\tilde{\pi},\ell} \cong \overline{\rho}_{\pi,\ell}$.
On the other hand, by Section 2.4.2(7), $\tilde{\pi}_p$ is a Jordan-Hölder factor of $I_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{C}^\times$.
If $\alpha \neq p^{\pm 2}, -p^{\pm 1}$, then $\tilde{\pi}_p \cong I_{\alpha,\beta}$ thus $\tilde{\pi}_p^{K_p} \neq 0$ by Section 2.4.2(5), contradicting the no-level-lowering assumption.
If $\alpha = p^{\pm 2}$, then $\tilde{\pi}_p \cong \mathrm{St}_p \otimes \mu_\beta$ or $\mu_\beta$. The first case is excluded since it has no non-trivial $\tilde{K}_p$-fixed vector by Section 2.4.2(6). The second case is excluded as $\tilde{\pi}$ is tempered.
If $\alpha = -p^{\pm 1}$, then $\tilde{\pi}_p \cong \pi_\beta^n$ or $\pi_\beta^2$. The former is excluded since it has no non-trivial $\tilde{K}_p$-fixed vector by Section 2.4.2(6). For the latter the multiset of eigenvalues of $\overline{\rho}_{\tilde{\pi},\ell}(\mathrm{Frob}_p)$ would be $\{-p, 1, -p^{-1}\}$ up to a scalar, leaving two possibilities : if $p^2 \equiv -p \bmod \ell$ then $p \equiv -1 \mod \ell$ thus $p^2 \equiv 1 \mod \ell$, if $p^2 \equiv -p^{-1} \bmod \ell$ then $p^3 \equiv -1 \mod \ell$, both contradicting our assumption.

4. Let $E$ be the exceptional divisor of the blowup $S^\#$ of $S$ along the superspecial locus $S_{\mathrm{ssp}}$. Consider the corresponding blow up square

$$\begin{array}{ccc} E & \xrightarrow{j} & S^\# \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle b} \\ S_{\mathrm{ssp}} & \xrightarrow{i} & S \end{array} .$$

We have a distinguished triangle [The18, Tag 0EW5]

$$\mathbb{F}_\ell \to Ri_*(\mathbb{F}_\ell\,|_{S_{\mathrm{ssp}}}) \oplus Rb_*(\mathbb{F}_\ell\,|_{S^\#}) \to Rc_*(\mathbb{F}_\ell\,|_E) \to \mathbb{F}_\ell[1]$$

where $c = i \circ \pi = b \circ j$. This induces an exact sequence of localized étale cohomology

$$\mathrm{H}^i(S \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \to \mathrm{H}^i(S^\# \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \oplus \mathrm{H}^i(S_{\mathrm{ssp}} \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \\ \to \mathrm{H}^i(E \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \to \mathrm{H}^{i+1}(S \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}$$

compatible with the $\mathbb{T}(G^\square, K^\square)_\mathfrak{m}$-action. Since $\mathrm{H}^*(S \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} = 0$ by (1) and $\mathrm{H}^0(S_{\mathrm{ssp}} \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^0(\tilde{T} \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} = 0$ by Lemma 2.3.16 and (3), we have an isomorphism of $\mathbb{T}(G^\square, K^\square)_\mathfrak{m}$-modules

$$\mathrm{H}^i(S^\# \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^i(E \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}.$$

Therefore, it suffices to show $\mathrm{H}^*(E \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} = 0$. Since $E$ is a $\mathbb{P}^1$-bundle over $S_{\mathrm{ssp}}$ by the proof of Proposition 2.3.24(1), we have $\mathrm{H}^*(E \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} = \mathrm{H}^*(S_{\mathrm{ssp}} \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}[X]/X^2 = 0$ and finish the proof.

5. Firstly, we have $\mathrm{H}^i(N \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^i(T \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} = 0$ for $i = 0, 2$ by (2). If $\mathrm{H}^1(N \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \neq 0$, then $\pi^\square$ appears in $\mathrm{H}^1(N \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{Z}_\ell)_\mathfrak{m} \otimes \mathbb{Q}_\ell^{\mathrm{ac}}$ since $\mathrm{H}^1(N \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{Z}_\ell)_\mathfrak{m}$ is torsion-free. By Lemma 2.4.6 we can complete $\pi^\square$ to an automorphic representation $\pi' = \pi^\square \otimes \prod_{q \in \square} \pi'_q$ of $G'(\mathbb{A})$ such that the Satake parameter of $\mathrm{BC}(\pi'_{p,0})$ is $\{p, 1, p^{-1}\}$. We can again transfer $\pi'$ to an automorphic representation $\tilde{\pi}$ of $G(\mathbb{A})$ such that the finite components $\tilde{\pi}^\infty$ and $\pi'^\infty$ coincide. Then the multiset of eigenvalues of $\overline{\rho}_{\tilde{\pi},\ell}(\mathrm{Frob}_p)$ would be $\{-p, 1, -p^{-1}\}$ up to a scalar. Comparing the eigenvalues of $\overline{\rho}_{\tilde{\pi},\ell}$ and $\overline{\rho}_{\pi,\ell}$ as in Lemma 2.4.7(3) leads to a contradiction.

□

**Corollary 2.4.8.** *1.* $\mathrm{H}^*(Y^{(0)}, \mathbb{F}_\ell)_\mathfrak{m} = 0$;

*2.* $\mathrm{H}^*(Y^{(1)}, \mathbb{F}_\ell)_\mathfrak{m} = 0$.

*Proof.* 1. By Proposition 2.3.24 we have isomorphisms of $\mathbb{T}(G^\square, K^\square)_\mathfrak{m}$-module $\mathrm{H}^i(Y_0 \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^i(Y_1 \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^i(S^\# \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} = 0$ for $i = 0, 1, 2$. Now we show $\mathrm{H}^*(Y_2 \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} = 0$. By Lemma 2.3.28, Proposition 2.3.29 and Lemma 2.4.7(5), $Y_2$ is a $\mathbb{P}^1$-bundle over $N$ and thus

$$\mathrm{H}^*(Y_2 \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^*(N \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}[X]/X^2 = 0.$$

2. By Proposition 2.3.30(1), $Y_{0,1}$ is a $\mathbb{P}^1$-bundle over $\tilde{T}$. Thus we have an isomorphism of $\mathbb{T}(G^\square, K^\square)_\mathfrak{m}$-modules

$$\mathrm{H}^*(Y_{0,1} \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^*(\tilde{T} \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}[X]/X^2 = 0$$

by [Mil80, Proposition 10.1] and Lemma 2.4.7(3). By Proposition 2.3.30(2)(3), $Y_{0,2}$ is isomorphic to $N$, $Y_{1,2} \to N$ is a purely inseparable map, thus we have isomorphisms of $\mathbb{T}(G^\square, K^\square)_\mathfrak{m}$-modules

$$\mathrm{H}^i(Y_{0,2} \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^i(Y_{1,2} \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^i(N \otimes \mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}.$$

By Lemma 2.4.7(5) they all vanish.

□

**Corollary 2.4.9.** *The spectral sequence* (3.16) *localized at* $\mathfrak{m}$ *degenerates at* $E_1$.

*Proof.* By Poincaré duality it suffices to show $E_\mathfrak{m}^{-1,4} = \mathrm{H}^2(Y^{(1)}, \mathbb{F}_\ell)_\mathfrak{m} = 0$ and $E_\mathfrak{m}^{0,3} = \mathrm{H}^3(Y^{(0)}, \mathbb{F}_\ell)_\mathfrak{m} = 0$, which follow from Lemma 2.4.7. □

We study the $\mathrm{Gal}(\mathbb{F}_p^{\mathrm{ac}}/\mathbb{F}_{p^2})$-action on $\mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^0(T_0(p)\otimes\mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}$. Consider the Iwahoric Hecke algebra $\mathbb{T}(G_p, \mathrm{Iw}_p) := \mathbb{Z}[\mathrm{Iw}_p\backslash G_p/\mathrm{Iw}_p]$. The $\mathrm{Gal}(\mathbb{F}_p^{\mathrm{ac}}/\mathbb{F}_{p^2})$-action and the $\mathbb{T}(G_p, \mathrm{Iw}_p)$-action on $\mathrm{H}^0(T_0(p)\otimes\mathbb{F}_p^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}$ commute. Let $\phi_{\mathrm{Iw}_p}$ denote the action of $\mathbb{T}(G_p, \mathrm{Iw}_p)$ on $\mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell)_\mathfrak{m}$. For $a \in Z(\mathbb{Q}_p) = F_p^\times$, denote by $\langle a\rangle \in \mathbb{T}(G_p, \mathrm{Iw}_p)$ the characteristic function of $a\mathrm{Iw}_p$.

**Lemma 2.4.10.** *The action of* $\mathrm{Frob}_{p^2}$ *and* $\langle p^{-1}\rangle$ *on* $\mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell)_\mathfrak{m}$ *coincide.*

*Proof.* Take $s = (A, \lambda_A, \eta_A, \tilde{A}, \lambda_{\tilde{A}}, \eta_{\tilde{A}}, \alpha) \in Y^{(2)}(\mathbb{F}_p^{\mathrm{ac}})$. $\tilde{A}$ is superspecial by Lemma 2.3.28(3) and Lemma 2.3.33(2a).

Since $A$ and $\tilde{A}$ are superspecial, there are supersingular elliptic curves $E$ and $\tilde{E}$ defined over $\mathbb{F}_{p^2}$ such that $A = (E^{\oplus 3})\otimes\mathbb{F}_p^{\mathrm{ac}}$ and $\tilde{A} = (\tilde{E}^{\oplus 3})\otimes\mathbb{F}_p^{\mathrm{ac}}$. It is well known that the relative Frobenius $\mathrm{Fr}_E : E \to E^{(p^2)} \cong E$ coincides with the isogeny $-p : E \to E$, and $\mathrm{Fr}_{\tilde{E}} : \tilde{E} \to \tilde{E}^{(p^2)} \cong E$ coincides with the isogeny $-p : \tilde{E} \to \tilde{E}$. It turns out that the action of $\mathrm{Frob}_{p^2}$ and $\langle -p^{-1}\rangle$ on $\mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell)_\mathfrak{m}$ coincide. We conclude by remarking that $\langle -p^{-1}\rangle = \langle p^{-1}\rangle$. □

**Lemma 2.4.11.** $\phi_{\mathrm{Iw}_p}(\langle p^{-1}\rangle)$ *lies in the image of* $Z(\mathbb{A}^\Box)/K^\Box \cap Z(\mathbb{A}^\Box)$ *in* $\mathrm{End}_{\mathbb{F}_\ell}(\mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell)_\mathfrak{m})$.

*Proof.* Let $\underline{p} \in Z(\mathbb{A}^\infty) \cong (\mathbb{A}_F^\infty)^\times$ be the element whose $p$-component is $p$ and other components are 1. By definition the action of $\underline{p}$ and $\langle p\rangle$ coincide. Since the action of $\underline{p}^{-1}$ on $\mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell)_\mathfrak{m}$ factors through $Z(\mathbb{A}^\infty)/Z(\mathbb{Q})(K^p\mathrm{Iw}_p \cap Z(\mathbb{A}^\infty))$, it suffices to show that there exist $g^\Box \in Z(\mathbb{A}^\Box)$ and $f \in Z(\mathbb{Q})$, such that $g^\Box f\underline{p}^{-1} \in K^\Box \cap Z(\mathbb{A}^\Box)$, which follows from the weak approximation. □

### 2.4.8 Proof of the main theorem

*Proof.* [Proof of Theorem 2.4.1] Suppose there is no level-lowering, i.e., there is no automorphic representation $\tilde{\pi}$ of $G(\mathbb{A})$ such that $\tilde{\pi}^{K^pK_p} \neq 0$ and $\overline{\rho}_{\tilde{\pi},\ell} \cong \overline{\rho}_{\pi,\ell}$. By Zucker's conjecture and the Matsushima formula we have the decomposition [BR92, 1.9]

$$\mathrm{IH}^2(S_0(p)\otimes F^{\mathrm{ac}}, \mathbb{Z}_\ell)_\mathfrak{m}\otimes\mathbb{Q}_\ell^{\mathrm{ac}} = \bigoplus_{\tilde{\pi}} \iota_\ell^{-1}\tilde{\pi}^{K^p\mathrm{Iw}_p}\otimes\rho_{\tilde{\pi},\ell} \tag{2.32}$$

where $\tilde{\pi}$ runs over irreducible automorphic representations of $G(\mathbb{A})$ such that $\tilde{\pi}_\infty$ is cohomological with trivial coefficient and $\overline{\rho}_{\tilde{\pi},\ell} \cong \overline{\rho}_{\pi,\ell}$. By Corollary 2.4.3 and the absolute irreducibility of $\overline{\rho}_{\pi,\ell}$, every irreducible Jordan-Hölder factor of $\mathrm{H}^2(S_0(p)\otimes F^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}$ is isomorphic to $\overline{\rho}_{\pi,\ell}$. The weight-monodromy spectral sequence, which degenerates at $E_1$ by Lemma 2.4.9, gives a filtration $\mathrm{Fil}^*\,\mathrm{H}^2(S_0(p)\otimes F^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}$ on $\mathrm{H}^2(S_0(p)\otimes F^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}$ of $\mathbb{T}(G^\Box, K^\Box)_\mathfrak{m}$-modules. Put $\mathrm{Gr}_p := \mathrm{Fil}^p/\mathrm{Fil}^{p+1}$. Then by Lemma 2.4.7 the non-zero terms are

$$\begin{aligned}\mathrm{Gr}_{-2} &= \mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell(-2))_\mathfrak{m},\\ \mathrm{Gr}_0 &= \mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell(-1))_\mathfrak{m},\\ \mathrm{Gr}_2 &= \mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell)_\mathfrak{m}.\end{aligned}$$

The monodromy operator $\tilde{\mu}$ in Section 2.4.6 boils down to identity maps $\mathrm{Gr}_{-2} \to \mathrm{Gr}_0(-1)$ and $\mathrm{Gr}_0 \to \mathrm{Gr}_2(-1)$. In particular, $\ker\tilde{\mu} = \mathrm{Fil}^2 \cong \mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell)_\mathfrak{m}$. The unramifiedness of $\overline{\rho}_{\pi,\ell}$ at $p$ implies that $\mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell)[\mathfrak{m}] \subset \mathrm{H}^0(Y^{(2)}, \mathbb{F}_\ell)_\mathfrak{m}$ contains a copy of $\overline{\rho}_{\pi,\ell}\,|_{\mathrm{Gal}(\mathbb{F}_p^{\mathrm{ac}}/\mathbb{F}_{p^2})}$. However, by Lemma 2.4.10 and Lemma 2.4.11, $\mathrm{Frob}_{p^2}$ acts as the scalar $\chi_{\pi,\ell}(\underline{p})^{-1}$ on $\mathrm{H}^0(Y^{(2)},\mathbb{F}_\ell)_{\mathfrak{m}}[\mathfrak{m}]$ where $\chi_{\pi,\ell} := \iota_\ell^{-1}\circ\chi_\pi$ and $\chi_\pi$ is the central character. On the other hand, the multiset of eigenvalues of $\overline{\rho}_{\pi,\ell}(\mathrm{Frob}_p)$ is $\{p^2, 1, p^{-2}\} \mod \ell$ up to multiplication by a common scalar. We then deduce that $p^2 \equiv 1 \mod l$, contradicting the assumption. □

# Chapitre 3

# Vanishing theorems for Picard modular surfaces

## 3.1 Introduction

In Chapter 2, we proved an analogue of the level lowering theorem for Picard modular surfaces. An essential assumption is that the cohomology of the generic fiber of Picard modular surface localized at a suitable Hecke maximal ideal concentrates in the middle degree. In this chapter, we aim at giving a criterion for this to be true.

Let $F$ be an imaginary quadratic extension of $\mathbb{Q}$, $O_F$ be its ring of integers and $G$ be the unitary similitude group over $\mathbb{Z}$ of signature (1,2) defined in (2.1).

Fix an open compact subgroup $K^q \subset G(\mathbb{A}^{\infty,q})$. Let $\mathbf{S}$ be the Picard modular surface attached to $G$ of level $K^qK_q$ defined over $O_F \otimes \mathbb{Z}_{(q)}$ (cf. Section 2.2.11 where we use the notation $p$ instead of $q$ and $S$ instead of $\mathbf{S}$).

We state the main theorem :

**Theorem 3.1.1.** *Let $\pi$ be a stable cuspidal automorphic representation of $G(\mathbb{A})$ cohomological with trivial coefficient. Let $\ell$ be a prime number and fix an isomorphism $\iota_\ell : \mathbb{Q}_\ell^{\mathrm{ac}} \to \mathbb{C}$. Denote by $\lambda$ the place in the field of definition of $\pi^\infty$ over $\ell$ induced by $\iota_\ell$. Choose a finite set $\Box$ of places of $\mathbb{Q}$ outside which $\pi$ is unramified. Let $\mathfrak{m} \subset \mathbb{T}(G^\Box, K^\Box)$ be the $\mod \lambda$ Hecke maximal ideal attached to $\pi$. Denote by $\overline{\rho}_\mathfrak{m}$ the residual Galois representation attached to $\pi$. Suppose that $\overline{\rho}_\mathfrak{m}$ is absolutely irreducible and there is a prime number $q$ such that*

1. *$q$ is inert in $F$ ;*
2. *$\ell \nmid (q-1)(q^3+1)$ ;*
3. *$K_q$ is hyperspecial ;*
4. *$\overline{\rho}_\mathfrak{m}(\mathrm{Frob}_q)$ is not conjugate to a matrix of the form $\mathrm{diag}(-\nu q, \nu, -\nu q^{-1})$ or $\mathrm{diag}(\nu q^2, \nu, \nu q^{-2})$ for some $\nu \in \mathbb{F}_\ell^{\mathrm{ac},\times}$.*

*Then*

$$\mathrm{H}^i(\mathbf{S} \otimes \mathbb{Q}^{\mathrm{ac}}, \mathbb{F}_\ell^{\mathrm{ac}})_\mathfrak{m} = 0$$

*for $i \neq 2$.*

**Remark 3.1.2.** *We expect the existence of such a prime number $q$ to be implied by the assumption that the image of $\overline{\rho}_\mathfrak{m}$ is big enough, similar to* [LTX+19, Lemma 2.6.1].

## 3.2 Hecke operators

Recall that in Section 2.2.6 we define an inner form $G'$ of $G$ over $\mathbb{Q}$ such that $G'_q \cong G_q$ for all finite places $q$ and $G'$ is of signature $(0,3)$ at infinity. We also define moduli problems $\mathbf{T}, \widetilde{\mathbf{T}}, \mathbf{T}_0(q)$ over $O_F \otimes \mathbb{Z}_{(q)}$ attached to $G'$ with level $K_q, \widetilde{K}_q, \mathrm{Iw}_q$ at $q$. We have natural maps

$$\begin{array}{ccc} & \mathbf{T}_0(q) & \\ \swarrow^{\psi} & & \searrow^{\widetilde{\psi}} \\ \mathbf{T} & & \widetilde{\mathbf{T}} \end{array} \tag{3.1}$$

equivariant under prime-to-$q$ Hecke correspondence.

**Definition 3.2.1.** *Let $R$ be a commutative ring. We define morphisms*

$$\begin{aligned}
T := K_q \operatorname{diag}(q,1,q^{-1})K_q \in \mathbb{Z}[K_q\backslash G_q/K_q] &: \mathrm{H}^0(\mathbf{T}\otimes \mathbb{F}_q^{\mathrm{ac}}, R) \to \mathrm{H}^0(\mathbf{T}\otimes \mathbb{F}_q^{\mathrm{ac}}, R),\\
J := \widetilde{K}_q K_q \in \mathbb{Z}[\widetilde{K}_q\backslash G_q/K_q] &: \mathrm{H}^0(\mathbf{T}\otimes \mathbb{F}_q^{\mathrm{ac}}, R) \to \mathrm{H}^0(\widetilde{\mathbf{T}}\otimes \mathbb{F}_q^{\mathrm{ac}}, R),\\
\widetilde{J} := K_q \widetilde{K}_q \in \mathbb{Z}[K_q\backslash G_q/\widetilde{K}_q] &: \mathrm{H}^0(\widetilde{\mathbf{T}}\otimes \mathbb{F}_q^{\mathrm{ac}}, R) \to \mathrm{H}^0(\mathbf{T}\otimes \mathbb{F}_q^{\mathrm{ac}}, R),\\
I_1 := K_q \mathrm{Iw}_q \in \mathbb{Z}[K_q\backslash G_q/\mathrm{Iw}_q] &: \mathrm{H}^0(\mathbf{T}_0(q)\otimes \mathbb{F}_q^{\mathrm{ac}}, R) \to \mathrm{H}^0(\mathbf{T}\otimes \mathbb{F}_q^{\mathrm{ac}}, R),\\
\widetilde{I}_1 := \widetilde{K}_q \mathrm{Iw}_q \in \mathbb{Z}[\widetilde{K}_q\backslash G_q/\mathrm{Iw}_q] &: \mathrm{H}^0(\mathbf{T}_0(q)\otimes \mathbb{F}_q^{\mathrm{ac}}, R) \to \mathrm{H}^0(\widetilde{\mathbf{T}}\otimes \mathbb{F}_q^{\mathrm{ac}}, R),\\
{}^t I_1 := \mathrm{Iw}_q K_q \in \mathbb{Z}[\mathrm{Iw}_q\backslash G_q/K_q] &: \mathrm{H}^0(\mathbf{T}\otimes \mathbb{F}_q^{\mathrm{ac}}, R) \to \mathrm{H}^0(\mathbf{T}_0(q)\otimes \mathbb{F}_q^{\mathrm{ac}}, R),\\
{}^t \widetilde{I}_1 := \mathrm{Iw}_q \widetilde{K}_q \in \mathbb{Z}[\mathrm{Iw}_q\backslash G_q/\widetilde{K}_q] &: \mathrm{H}^0(\widetilde{\mathbf{T}}\otimes \mathbb{F}_q^{\mathrm{ac}}, R) \to \mathrm{H}^0(\mathbf{T}_0(q)\otimes \mathbb{F}_q^{\mathrm{ac}}, R).
\end{aligned}$$

*We also define* $I_2 := \widetilde{J} \circ \widetilde{I}_1, {}^t I_2 := {}^t \widetilde{I}_1 \circ J$.

The following lemma describes the relations between these morphisms

**Lemma 3.2.2.** *We have*

1. $\widetilde{J} \circ J = T + (q^3+1)\,\mathrm{id}$;
2. $I_1 \circ {}^t I_1 = (q^3+1)\,\mathrm{id}$;
3. $I_1 \circ {}^t I_2 = (q^3+1)\,\mathrm{id} + T$;
4. $I_2 \circ {}^t I_1 = (q^3+1)\,\mathrm{id} + T$;
5. $I_2 \circ {}^t I_2 = (q^3+1)(q+1)\,\mathrm{id} + (q+1)T$.

*Proof.* Recall that for an open compact subgroup $K'^q K'_q$ of $G'(\mathbb{A}^\infty)$ and a commutative ring $R$, $\mathrm{H}^0(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/K'^q K'_q), R)$ is the $R$-module of locally constant, compactly supported functions $f : G'(\mathbb{A}^\infty) \to R$ such that $f(qgk) = f(g)$ for all $q \in G'(\mathbb{Q}), g \in G'(\mathbb{A}^\infty)$ and $k \in K'^q K'_q$.

Let $R[G'_q/K'_q]$ be the $R$-module of locally constant, compactly supported functions $f : G'_q \to R$ such that $f(gk) = f(g)$ for all $g \in G'_q, k \in K'_q$.

Since $G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/K'^q G'_q$ is a quotient of the finite set $G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/K'^q K'_q,$, we have a decomposition

$$G'(\mathbb{A}^\infty) = \coprod_{i=1}^{h} G'(\mathbb{Q}) x_i K'^q G'_q$$

for some $x_1, \cdots, x_h \in G'(\mathbb{A}^\infty)$. Therefore we have

$$G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/K'^q K'_q \cong \sqcup_i G'(\mathbb{Q})\backslash G'(\mathbb{Q}) x_i K'^q G'_q/K'^q K_q = \sqcup_i \Gamma_i\backslash G'_q/K'_q$$

where $\Gamma_i := G'(\mathbb{Q}) \cap x_i K^q x_i^{-1}$. For $f \in \mathrm{H}^0(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/K'^q K'_q), R)$, define functions $f_i \in R[\Gamma_i\backslash G'_q/K'_q]$ by sending $g_q \in G'_q$ to $f(x_i g_q)$. Then the morphism

$$\begin{aligned} \mathrm{H}^0(G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/K'^q K'_q), R) &\to \bigoplus_{i=1}^h R[\Gamma_i\backslash G'_q/K'_q] \\ f &\mapsto (f_i)_{1\le i\le h} \end{aligned}$$

is an isomorphism of $R$-modules.

Keep the notations of the Bruhat-Tits tree of $G_q$ in Section 2.2.1. Since $G_q \cong G'_q$, we have the bijections

$$G'_q/K_q \cong \mathcal{V}, \quad G'_q/\widetilde{K}_q \cong \widetilde{\mathcal{V}}, \quad G'_q/\mathrm{Iw}_q \cong \mathcal{E}.$$

To summarize, we have isomorphisms of $R$-modules

$$\mathrm{H}^0(\mathbf{T}\otimes \mathbb{F}_q^{\mathrm{ac}}, R) \cong \bigoplus_{i=1}^h R[\Gamma_i\backslash \mathcal{V}], \quad \mathrm{H}^0(\Gamma_i\backslash \widetilde{\mathbf{T}}\otimes \mathbb{F}_q^{\mathrm{ac}}, R) \cong \bigoplus_{i=1}^h R[\Gamma_i\backslash \widetilde{\mathcal{V}}],$$

and

$$\mathrm{H}^0(\mathbf{T}_0(q)\otimes \mathbb{F}_q^{\mathrm{ac}}, R) \cong \bigoplus_{i=1}^h R[\Gamma_i\backslash \mathcal{E}].$$

Thus the morphisms defined in Definition 3.2.1 are given by the corresponding morphisms with the same notation on the Bruhat-Tits tree :

**Definition 3.2.3.** [BG06, Lemme 3.5.1] *Let $d$ be the distance function between vertices of $\mathcal{X}$. For $x \in \mathcal{V}, \widetilde{\mathcal{V}}, \mathcal{E}$, denote by $\delta_x$ be the characteristic function of $x$. Define $G'_q$-equivariant morphisms*

$$\begin{aligned} T : R[\mathcal{V}] &\to R[\mathcal{V}] \\ \delta_x &\mapsto \sum_{d(y,x)=2} \delta_y \end{aligned} \qquad \begin{aligned} J : R[\mathcal{V}] &\to R[\widetilde{\mathcal{V}}] \\ \delta_x &\mapsto \sum_{d(x',x)=1} \delta_{x'} \end{aligned} \qquad \begin{aligned} \widetilde{J} : R[\widetilde{\mathcal{V}}] &\to R[\mathcal{V}] \\ \delta_{x'} &\mapsto \sum_{d(x',x)=1} \delta_x \end{aligned}$$

*and*

$$\begin{aligned} I_1 : R[\mathcal{E}] &\to R[\mathcal{V}] \\ \delta_{(x,x')} &\mapsto \delta_x \end{aligned} \qquad \begin{aligned} I_2 : R[\mathcal{E}] &\to R[\mathcal{V}] \\ \delta_{(x,x')} &\mapsto \sum_{d(y,x')=1} \delta_y \end{aligned}$$

$$\begin{aligned} {}^tI_1 : R[\mathcal{V}] &\to R[\mathcal{E}] \\ \delta_x &\mapsto \sum_{d(x,x')=1} \delta_{(x,x')} \end{aligned} \qquad \begin{aligned} {}^tI_2 : R[\mathcal{V}] &\to R[\mathcal{E}] \\ \delta_x &\mapsto \sum_{d(x',x)=1, d(x',y)=1} \delta_{(x',y)}. \end{aligned}$$

To prove Lemma 3.2.2, it suffices to verify the relations between morphisms on the tree, which is an easy combinatoric calculation similar to [BG06, Lemme 3.5.3]. □

The following lemma calculate the eigenvalue of Hecke operators on unramified principle series :

**Lemma 3.2.4.** *Let $\pi'$ be an irreducible admissble automorphic representation of $G'(\mathbb{A}^\infty)$ appearing in (3.6) such that $\pi'_q \cong I_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{C}^*$. Let $\phi_{\pi'}$ be the Hecke homomorphism defined in Section 2.4.4. Then we have*

$$\phi_{\pi'}(T_{\mathfrak{m}}) = q^2(\alpha + \alpha^{-1}) + (q-1). \tag{3.2}$$

*Proof.* Recall that in Section 2.4.1 we have the dual group $\widehat{G} \cong (\mathrm{GL}_3 \times \mathbb{G}_m)(\mathbb{C})$ of $G$ over $\mathbb{Q}_q$ and the Langlands dual group ${}^LG = (\mathrm{GL}_3 \times \mathbb{G}_m)(\mathbb{C}) \rtimes \{1, \sigma\}$ where the involution $\sigma$ sends $g \in \mathrm{GL}_3(\mathbb{C})$ to $\Phi({}^tg^{-1})\Phi$.

Let $M_q := M(\mathbb{Q}_q)$ be the split torus of $G_q$ defined in Section 2.4.1. Let $\widehat{M}_q \cong D \times \mathbb{G}_m$ be the dual group of $M_q$ where $D \subset \mathrm{GL}_3(\mathbb{C})$ is the diagnonal matrix. Let $X^*(\widehat{M}_q)$ be the character group of $\widehat{M}_q$ and $\mathbb{Z}[X^*(\widehat{M}_q)] \cong \mathbb{Z}^3 \oplus \mathbb{Z}$ be the group algebra of $X^*(\widehat{M}_q)$ with coefficient in $\mathbb{Z}$. Let $W$ be the Weyl group of $G_q$. For a character $\nu \in X^*(\widehat{M}_q)$, denote by $[\nu] \in \mathbb{Z}[X^*(\widehat{M}_q)]$ the characteristic function of $\nu$. Define characters $\mu : \widehat{M}_q \to \mathbb{G}_m$ by $(\mathrm{diag}(a,b,c), \nu) \mapsto a/c$ and $\mu_c : \widehat{M}_q \to \mathbb{G}_m$ by $(\mathrm{diag}(a,b,c), \nu) \mapsto \nu$.

The Satake isomorphism reads

$$\mathrm{Sat} : \mathbb{Z}[K_q \backslash G_q / K_q] \cong \mathbb{Z}[X^*(\widehat{M}_q)]^W \cong \mathbb{Z}[[\mu_c], [\mu_c^{-1}], [\mu] + [\mu^{-1}]].$$

By [Car79, (35),(39)], the value of $\phi_{\pi'}(T_{\mathfrak{m}})$ is given by

$$\phi_{\pi'}(T_{\mathfrak{m}}) = \int_{\mathbb{Z}[X^*(\widehat{M}_q)]} \mathrm{Sat}(T_{\mathfrak{m}})(m)\chi_{\alpha,\beta}(m)dm.$$

It then boils down to calculate the Satake transform of $T_{\mathfrak{m}}$.

For an algebraic representation $\rho$ of ${}^LG$, denote by $\chi(\rho)$ the character of $\rho$ restricted on $\widehat{M}_q\sigma$. Let $\rho_{\mathrm{std}}$ be the standard representation $\widehat{G} \cong \mathrm{GL}_3 \times \mathbb{G}_m \to \mathrm{GL}_3$ given by $(g, \nu) \mapsto g\nu$. Define $\rho := \rho_{\mathrm{std}} \otimes \rho_{\mathrm{std}}^\vee$ where $\rho_{\mathrm{std}}^\vee$ denotes the dual representation. Since the restriction of $\rho$ on the similitude factor $\mathbb{G}_{\mathrm{m}} \subset \widehat{G}$ is trivial, we can apply the calculation in [LTX+22, Lemma B.1.2] to get

$$[\chi(\rho)] = [\mu] + [\mu^{-1}] + 1$$

in $\mathbb{Z}[X^*(\widehat{M}_q)]$. By [XZ18, Lemma 9.2.4] and [Zhu18, (5.2.1)] we have

$$q^2[\chi(\rho)] = q^2 - q + 1 + \mathrm{Sat}(T_{\mathfrak{m}}),$$

from which we derive that

$$\mathrm{Sat}(T_{\mathfrak{m}}) = q^2([\mu] + [\mu^{-1}] + 1) - (q^2 - q + 1) = q^2([\mu] + [\mu^{-1}]) + q - 1,$$

Since $\chi_{\alpha,\beta}([\mu]) = \alpha$, the lemma follows. □

**Lemma 3.2.5.** *Suppose that $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_q)$ is not conjugate to a matrix of the form $\mathrm{diag}(-\nu q, \nu, -\nu q^{-1})$ or $\mathrm{diag}(\nu q^2, \nu, \nu q^{-2})$ for some $\nu \in \mathbb{F}_\ell^{\mathrm{ac},\times}$. Then the morphisms localized at $\mathfrak{m}$*

$$J_{\mathfrak{m}} : \mathrm{H}^0(\mathbf{T} \otimes \mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} \to \mathrm{H}^0(\widetilde{\mathbf{T}} \otimes \mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} \tag{3.3}$$

$$\widetilde{J}_{\mathfrak{m}} : \mathrm{H}^0(\widetilde{\mathbf{T}} \otimes \mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} \to \mathrm{H}^0(\mathbf{T} \otimes \mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} \tag{3.4}$$

$$(I_{1,\mathfrak{m}}, \widetilde{I}_{1,\mathfrak{m}}) : \mathrm{H}^0(\mathbf{T}_0(q) \otimes \mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} \to \mathrm{H}^0(\mathbf{T} \otimes \mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} \oplus \mathrm{H}^0(\widetilde{\mathbf{T}} \otimes \mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell)_{\mathfrak{m}} \tag{3.5}$$

*are isomorphisms.*

*Proof.* We have decompositions

$$\mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{Z})_{\mathfrak{m}}\otimes\mathbb{C}\simeq\bigoplus_{\pi'}m(\pi')\cdot\pi'^{K^qK_q},\tag{3.6}$$

$$\mathrm{H}^0(\widetilde{\mathbf{T}}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{Z})_{\mathfrak{m}}\otimes\mathbb{C}\simeq\bigoplus_{\pi'}m(\pi')\cdot\pi'^{K^q\widetilde{K}_q},\tag{3.7}$$

$$\mathrm{H}^0(\mathbf{T}_0(q)\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{Z})_{\mathfrak{m}}\otimes\mathbb{C}\simeq\bigoplus_{\pi'}m(\pi')\cdot\pi'^{K^q\mathrm{Iw}q}\tag{3.8}$$

where $\pi'$ runs over all irreducible admissible representations of $G'(\mathbb{A})$ with coefficients in $\mathbb{C}$ such that $\pi'_\infty$ is trivial and the mod $\ell$ Hecke homomorphism $\overline{\phi}_{\pi',\ell}$ defined in (2.25) satisfies $\overline{\phi}_{\pi',\ell}=\overline{\phi}_{\pi,\ell}$, and $m(\pi')$ denotes the automorphic multiplicity of $\pi'$.

We then claim the following equalities :

$$\mathrm{rank}_{\mathbb{Z}_\ell}\mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{Z}_\ell)_{\mathfrak{m}}=\mathrm{rank}_{\mathbb{Z}_\ell}\mathrm{H}^0(\widetilde{\mathbf{T}}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{Z}_\ell)_{\mathfrak{m}},\tag{3.9}$$

$$\mathrm{rank}_{\mathbb{Z}_\ell}\mathrm{H}^0(\mathbf{T}_0(q)\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{Z}_\ell)_{\mathfrak{m}}=2\mathrm{rank}_{\mathbb{Z}_\ell}\mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{Z}_\ell)_{\mathfrak{m}},\tag{3.10}$$

It is equivalent to show that the dimension of $\mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{Z})_{\mathfrak{m}}\otimes\overline{\mathbb{Q}}_\ell$ and $\mathrm{H}^0(\widetilde{\mathbf{T}}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{Z})_{\mathfrak{m}}\otimes\overline{\mathbb{Q}}_\ell$ are both half the dimension of $\mathrm{H}^0(\mathbf{T}_0(q)\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{Z})_{\mathfrak{m}}\otimes\overline{\mathbb{Q}}_\ell$.

By (7) in Section 2.4.2, for any $\pi'$ in the direct sums (3.6), (3.7) or (3.8), the $q$-component $\pi'_q$ has to be a Jordan-Hölder factor of the unramified principle series $I_{\alpha,\beta}$ for some $\alpha,\beta\in\mathbb{C}^\times$. The assumption (4) in Theorem 3.1.1 then implies that $\alpha\neq\{-q,-q^{-1},q^2,q^{-2}\}$, from which we deduce that $I_{\alpha,\beta}$ is irreductible and therefore $\pi'_q\cong I_{\alpha,\beta}$ by (1) in Section 2.4.2. On the other hand, we have $\dim I_{\alpha,\beta}^{K_q}=\dim I_{\alpha,\beta}^{\widetilde{K}_q}=1,\dim I_{\alpha,\beta}^{\mathrm{Iw}_q}=2$ by (5) in Section 2.4.2, thus the claim follows.

Now we head back to show that $J_{\mathfrak{m}}$ and $\widetilde{J}_{\mathfrak{m}}$ are isomorphisms. By (3.9) it suffices to show that the composition $\widetilde{J}_{\mathfrak{m}}\circ J_{\mathfrak{m}}:\mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{F}_\ell)_{\mathfrak{m}}\to\mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{F}_\ell)_{\mathfrak{m}}$ is an isomorphism. Indeed, if so then $J_{\mathfrak{m}}$ (resp. $\widetilde{J}_{\mathfrak{m}}$) is an injective (resp. surjective) morphism between free $\mathbb{F}_\ell$-vector spaces of the same rank. Consequently, $J_{\mathfrak{m}}$ and $\widetilde{J}_{\mathfrak{m}}$ are isomorphisms.

By Lemma 3.2.2(1), we have

$$\widetilde{J}_{\mathfrak{m}}\circ J_{\mathfrak{m}}=T_{\mathfrak{m}}+(q^3+1)\,\mathrm{id}_{\mathfrak{m}}\,.$$

Thus it suffices to show that for every $\pi'$ appearing in (3.6),

$$\phi_{\pi'}(T_{\mathfrak{m}})+(q^3+1)\not\equiv 0\mod\mathfrak{m}$$

where $\phi_{\pi'}$ is defined in Section 2.4.4. By Lemma 3.2.4 we have

$$\phi_{\pi'}(T_{\mathfrak{m}})=q^2(\alpha+\alpha^{-1})+(q-1).\tag{3.11}$$

Since $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_q)$ is not conjugate to a matrix of the form $\mathrm{diag}(-\nu q,\nu,-\nu q^{-1})$ or $\mathrm{diag}(\nu q^2,\nu,\nu q^{-2})$ for some $\nu\in\mathbb{F}_\ell^{\mathrm{ac},\times}$, the relation (2) between Satake parameter and the Langlands parameter in Section 2.4.3 then implies that $\alpha\not\equiv-q,-q^{-1},q^2,q^{-2}$ mod $\mathfrak{m}$. Thus

$$\phi_{\pi'}(T_{\mathfrak{m}})+q^3+1\equiv q^2(\alpha+\alpha^{-1}+q+q^{-1})\not\equiv 0\bmod\mathfrak{m},$$

and $\widetilde{J}_{\mathfrak{m}}\circ J_{\mathfrak{m}}$ is an isomorphism. Therefore $J_{\mathfrak{m}}$ and $\widetilde{J}_{\mathfrak{m}}$ are both isomorphisms.

To show that $(I_{1,\mathfrak{m}}, \widetilde{I}_{1,\mathfrak{m}})$ is an isomorphism, by (3.4) it suffices to show that the composition of the morphisms

$$(I_{1,\mathfrak{m}}, I_{2,\mathfrak{m}}) : \mathrm{H}^0(\mathbf{T}_0(q)\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{F}_\ell)_\mathfrak{m} \to \mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{F}_\ell)_\mathfrak{m} \oplus \mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{F}_\ell)_\mathfrak{m} \quad (3.12)$$

is injective. Consider the composition $I_\mathfrak{m} : (I_{1,\mathfrak{m}} \oplus I_{2,\mathfrak{m}}) \circ ({}^tI_{1,\mathfrak{m}} \oplus^t I_{2,\mathfrak{m}}) :$

$$\begin{array}{ccccc}
\mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{F}_\ell)_\mathfrak{m} & & & & \mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{F}_\ell)_\mathfrak{m} \\
 & \searrow^{{}^tI_{1,\mathfrak{m}}} & & \swarrow^{{}^tI_{2,\mathfrak{m}}} & \\
\oplus & & \mathrm{H}^0(\mathbf{T}_0(q)\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{F}_\ell)_\mathfrak{m} & & \oplus \\
 & \swarrow_{I_{1,\mathfrak{m}}} & & \searrow_{I_{2,\mathfrak{m}}} & \\
\mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{F}_\ell)_\mathfrak{m} & & & & \mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}},\mathbb{F}_\ell)_\mathfrak{m}.
\end{array}$$

Then by (3.10) it suffices to show that $I_\mathfrak{m}$ is an isomorphism. Since the source and target of $I_\mathfrak{m}$ are $\mathbb{F}_\ell$ vector space of the same dimension, it suffices to show that $I_\mathfrak{m}$ is injective. By Lemma 3.2.2, the intersection matrix is calculated as

$$\begin{pmatrix} I_{1,\mathfrak{m}} \circ^t I_{1,\mathfrak{m}} & I_{1,\mathfrak{m}} \circ^t I_{2,\mathfrak{m}} \\ I_{2,\mathfrak{m}} \circ^t I_{1,\mathfrak{m}} & I_{2,\mathfrak{m}} \circ^t I_{2,\mathfrak{m}} \end{pmatrix} = \begin{pmatrix} (q^3+1)\,\mathrm{id}_\mathfrak{m} & (q^3+1)\,\mathrm{id}_\mathfrak{m} + T_\mathfrak{m} \\ (q^3+1)\,\mathrm{id}_\mathfrak{m} + T_\mathfrak{m} & (q^3+1)(q+1)\,\mathrm{id}_\mathfrak{m} + (q+1)T_\mathfrak{m} \end{pmatrix}$$

whose determinant is $-(T_\mathfrak{m}+(q^3+1)\,\mathrm{id}_\mathfrak{m})(T_\mathfrak{m}-q(q^3+1)\,\mathrm{id}_\mathfrak{m})$. For every $\pi'$ appearing in (3.6) such that $\pi'_q \cong I_{\alpha,\beta,q}$ for some $\alpha,\beta \in \mathbb{C}^*$, the determinant acts on $\pi_q'^{K_q}$ as the scalar

$$\begin{aligned} &-(\phi_{\pi'}(T_\mathfrak{m}) + (q^3+1))(\phi_{\pi'}(T_\mathfrak{m}) - q(q^3+1)) \mod \mathfrak{m} \\ &\qquad = -(\alpha+\alpha^{-1}+q+q^{-1})(\alpha+\alpha^{-1}-q^2-q^{-2}) \mod \mathfrak{m}. \end{aligned} \quad (3.13)$$

which is nonzero since $\alpha \not\equiv -q, -q^{-1}, q^2, q^{-2} \mod \mathfrak{m}$ by the same argument below (3.11). Thus $I_\mathfrak{m}$ is injective and $(I_{1,\mathfrak{m}}, \widetilde{I}_{1,\mathfrak{m}})$ is an isomorphism. □

## 3.3 Proof of the main theorem

Denote by $\mathbf{S}^*$ the minimal compactification of the moduli problem $\mathbf{S}$. As a topological space, it is obtained by adding a finite set of points to $\mathbf{S}$, corresponding to CM elliptic curves. It is well known that the ordinary locus $\mathbf{S}^{*,\mathrm{ord}}$ of $\mathbf{S}^*$ is affine. The natural map $\mathbf{S} \to \mathbf{S}^*$ is an open immersion.

Let $(\mathbf{S}\otimes\mathbb{F}_q^{\mathrm{ac}})^{\mathrm{ss}}$ be the supersingular locus of $\mathbf{S}\otimes\mathbb{F}_q^{\mathrm{ac}}$ which coincides with that of $\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}}$. Let $(\mathbf{S}\otimes\mathbb{F}_q^{\mathrm{ac}})^{\mathrm{ssp}}$ be the superspecial locus of $\mathbf{S}\otimes\mathbb{F}_q^{\mathrm{ac}}$ which coincides with that of $\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}}$.

Let $(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^\#$ be the blowup of $\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}}$ along $(\mathbf{S}\otimes\mathbb{F}_q^{\mathrm{ac}})^{\mathrm{ssp}}$ with the canonical morphism $\pi : (\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^\# \to \mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}}$.

Let $(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\#,\mathrm{ss}} := \pi^{-1}(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\mathrm{ss}}$ be the supersingular locus of $(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\#,\mathrm{ss}}$. Let $\widetilde{(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})}^{\mathrm{ss}} \subset (\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^\#$ be the strict transformation of $(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\mathrm{ss}}$. Let $(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\#,\mathrm{ssp}} := \pi^{-1}(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\mathrm{ssp}}$ be the exceptional divisor of $(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^\#$. Then we have

$$(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\#,\mathrm{ss}} = \widetilde{(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})}^{\mathrm{ss}} \cup (\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\#,\mathrm{ssp}}$$

There are canonical morphisms

$$\theta : \widetilde{(\mathbf{S}^* \otimes \mathbb{F}_q^{ac})}^{ss} \to \mathbf{T} \otimes \mathbb{F}_q^{ac}$$

with each geometric fiber isomorphic to the Fermat curve $\mathscr{C}$ (cf. Proposition 2.3.6) and

$$\widetilde{\theta} : (\mathbf{S} \otimes \mathbb{F}_q^{ac})^{\#,ssp} \to (\mathbf{S} \otimes \mathbb{F}_q^{ac})^{ssp} \cong \widetilde{\mathbf{T}} \otimes \mathbb{F}_q^{ac},$$

with each geometric fiber isomorphic to $\mathbb{P}^1$. Moreover, we have an isomorphism

$$\theta_0(q) : \widetilde{(\mathbf{S}^* \otimes \mathbb{F}_q^{ac})}^{ss} \cap (\mathbf{S}^* \otimes \mathbb{F}_q^{ac})^{\#,ssp} \cong \mathbf{T}_0(q) \otimes \mathbb{F}_q^{ac}.$$

In the following we replace $\mathbb{F}_q^{ac}$ by $\mathbb{F}_q^{ac}$ if needed.

We are going to use the following spectral sequence by Deligne in the $\ell$-adic setting, see [Pet17, Theorem 3.3(ii), Example 3.5].

**Proposition 3.3.1.** *Let $X$ be a smooth algebraic variety of dimension $n$ over an algebraic closed field $k$ such that* $\mathrm{char}(k) \neq \ell$. *Suppose $D = D_1 \cup \ldots \cup D_k$ a strict normal crossing divisor. Consider the stratification of $X$ by the various intersections of the components of $D$. For $I \subset \{1, \ldots, k\}$, let $D_I = \bigcap_{i \in I} D_i$, including $D_\emptyset = X$. Let $D_i := \sqcup_{|I|=i} D_I$. There is a spectral sequence*

$$\mathrm{E}_1^{i,j} = \mathrm{H}^{2n+2i+j}(D_{-i}, \mathbb{F}_\ell) \Longrightarrow \mathrm{H}^{2n+i+j}(X \backslash D, \mathbb{F}_\ell)$$

*where the morphisms $\mathrm{d}_1^{i,j} : \mathrm{E}_1^{i,j} \to \mathrm{E}_1^{i+1,j}$ are alternating sums of Gysin maps.*

*Proof of Theorem 3.1.1.* Since $K_q$ is hyperspecial, $\mathbf{S}$ has smooth reduction at $q$. Since

$$\mathrm{H}^i(\mathbf{S}^* \otimes \mathbb{Q}^{ac}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^i(\mathbf{S}^* \otimes \mathbb{Q}_q^{ac}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^i(\mathbf{S}^* \otimes \mathbb{F}_q^{ac}, \mathbb{F}_\ell)_\mathfrak{m},$$

it suffices to show that $\mathrm{H}^i(\mathbf{S}^* \otimes \mathbb{F}_q^{ac}, \mathbb{F}_\ell)_\mathfrak{m} \neq 0$ for $i \neq 2$. Since $\overline{\rho}_\mathfrak{m}$ is absolutely irreducible, the same argument as Lemma 2.4.2 implies that the compact support cohomology is isomorphic to the ordinary cohomology

$$\mathrm{H}_c^i(\mathbf{S} \otimes \mathbb{F}_q^{ac}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^i(\mathbf{S} \otimes \mathbb{F}_q^{ac}, \mathbb{F}_\ell)_\mathfrak{m}. \tag{3.14}$$

By the Poincaré duality, it suffices to show that $\mathrm{H}^i(\mathbf{S}^* \otimes \mathbb{F}_q^{ac}, \mathbb{F}_\ell)_\mathfrak{m} = 0$ for $i > 2$. The same argument as Lemma 2.4.7(4) implies that

$$\mathrm{H}^i((\mathbf{S}^* \otimes \mathbb{F}_q^{ac})^{\#}, \mathbb{F}_\ell)_\mathfrak{m} \cong \mathrm{H}^i(\mathbf{S}^* \otimes \mathbb{F}_q^{ac}, \mathbb{F}_\ell)_\mathfrak{m}.$$

Thus it suffices to show that $\mathrm{H}^i((\mathbf{S}^* \otimes \mathbb{F}_q^{ac})^{\#}, \mathbb{F}_\ell)_\mathfrak{m} = 0$ for $i > 2$. We now apply the weight spectral sequence in Proposition 3.3.1 to $(\mathbf{S}^* \otimes \mathbb{F}_q^{ac})^{\#}$. Define

$$D_0 = (\mathbf{S}^* \otimes \mathbb{F}_q^{ac})^{\#}, \quad D_1 = \widetilde{(\mathbf{S}^* \otimes \mathbb{F}_q^{ac})}^{ss} \sqcup (\mathbf{S}^* \otimes \mathbb{F}_q^{ac})^{\#,ssp},$$
$$D_2 = \widetilde{(\mathbf{S}^* \otimes \mathbb{F}_q^{ac})}^{ss} \cap (\mathbf{S}^* \otimes \mathbb{F}_q^{ac})^{\#,ssp} \cong \mathbf{T}_0(q) \otimes \mathbb{F}_q^{ac}. \tag{3.15}$$

The first page $\mathrm{E}_1$ of the spectral sequence writes as

$$\begin{array}{ccccc}
\mathrm{H}^0(D_2, \mathbb{F}_\ell)_\mathfrak{m} & \xrightarrow{\mathrm{d}_{1,\mathfrak{m}}^{-2,4}} & \mathrm{H}^2(D_1, \mathbb{F}_\ell)_\mathfrak{m} & \xrightarrow{\mathrm{d}_{1,\mathfrak{m}}^{-1,4}} & \mathrm{H}^4(D_0, \mathbb{F}_\ell)_\mathfrak{m} \\
 & & \mathrm{H}^1(D_1, \mathbb{F}_\ell)_\mathfrak{m} & \xrightarrow{\mathrm{d}_{1,\mathfrak{m}}^{-1,3}} & \mathrm{H}^3(D_0, \mathbb{F}_\ell)_\mathfrak{m} \\
 & & \mathrm{H}^0(D_1, \mathbb{F}_\ell)_\mathfrak{m} & \longrightarrow & \mathrm{H}^2(D_0, \mathbb{F}_\ell)_\mathfrak{m} \\
 & & & & \mathrm{H}^1(D_0, \mathbb{F}_\ell)_\mathfrak{m} \\
 & & & & \mathrm{H}^0(D_0, \mathbb{F}_\ell)_\mathfrak{m}
\end{array} \tag{3.16}$$

where $\mathrm{d}_{1,\mathfrak{m}}^{-2,4}$ is defined by

$$\mathrm{H}^0(D_2, \mathbb{F}_\ell) \xrightarrow{(\mathrm{Gys}_1, -\mathrm{Gys}_2)} \mathrm{H}^2((\widetilde{\mathbf{S}^* \otimes \mathbb{F}_q^{\mathrm{ac}}})^{\mathrm{ss}} \otimes \mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell) \oplus \mathrm{H}^2((\mathbf{S}^* \otimes \mathbb{F}_q^{\mathrm{ac}})^{\#,\mathrm{ssp}} \otimes \mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell)$$

where $\mathrm{Gys}_1$ and $\mathrm{Gys}_2$ are the Gysin maps for the closed immersions of $D_2$ into $(\widetilde{\mathbf{S}^* \otimes \mathbb{F}_q^{\mathrm{ac}}})^{\mathrm{ss}}$ and $(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\#,\mathrm{ssp}}$. The spectral sequence converges to $\mathrm{H}^*((\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\mathrm{ord}}, \mathbb{F}_\ell)_\mathfrak{m}$ where $(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\mathrm{ord}}$ is the ordinary locus $(\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\#} - (\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\#,\mathrm{ss}} = (\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\#} - D_1$. By [GN17, Corollary 1.3], $(\mathbf{S}^* \otimes \mathbb{F}_q^{\mathrm{ac}})^{\mathrm{ord}}$ is affine, thus its cohomology vanishes for degree $> 2$ by Artin-Grothendieck vanishing theorem (cf. [Han20, Theorem 1.1]).

We claim that $\mathrm{d}_{1,\mathfrak{m}}^{-2,4}$ is an isomorphism. Indeed, by Section 3.3 and (3.15) the morphism

$$\mathrm{d}_{1,\mathfrak{m}}^{-2,4} : \mathrm{H}^0(D_2, \mathbb{F}_\ell)_\mathfrak{m} \to \mathrm{H}^2(D_1, \mathbb{F}_\ell)_\mathfrak{m}$$

is identified with

$$(\psi_{!,\mathfrak{m}}, \widetilde{\psi}_{!,\mathfrak{m}}) = (I_{1,\mathfrak{m}}, \widetilde{I}_{1,\mathfrak{m}}) : \mathrm{H}^0(\mathbf{T}_0(q)\otimes\mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \to \mathrm{H}^0(\mathbf{T}\otimes\mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m} \oplus \mathrm{H}^0(\widetilde{\mathbf{T}}\otimes\mathbb{F}_q^{\mathrm{ac}}, \mathbb{F}_\ell)_\mathfrak{m}$$

where $\psi$ and $\widetilde{\psi}$ are the natural projections in (3.1). The isomorphism (3.5) then implies that $\mathrm{d}_{1,\mathfrak{m}}^{-2,4}$ is an isomorphism.

We now show that $\mathrm{H}^4((\mathbf{S}^* \otimes \mathbb{F}_q^{\mathrm{ac}})^{\#}, \mathbb{F}_\ell)_\mathfrak{m} = 0$. It is easy to see $\mathrm{E}_{2,\mathfrak{m}}^{0,4} = \mathrm{E}_{\infty,\mathfrak{m}}^{0,4} = 0$ which is a graded piece of $\mathrm{H}^4((\mathbf{S}^* \otimes \mathbb{F}_q^{\mathrm{ac}})^{\mathrm{ord}}, \mathbb{F}_\ell)_\mathfrak{m} = 0$. On the other hand, $\mathrm{E}_{2,\mathfrak{m}}^{0,4} = \mathrm{coker}(\mathrm{d}_{1,\mathfrak{m}}^{-1,4})$. Thus it suffice to show that $\mathrm{d}_{1,\mathfrak{m}}^{-2,4}$ is surjective which is deduced from the claim.

We now show that $\mathrm{H}^3((\mathbf{S}^* \otimes \mathbb{F}_q^{\mathrm{ac}})^{\#}, \mathbb{F}_\ell)_\mathfrak{m} = 0$. Calculate

$$\mathrm{E}_2^{-1,4} = \ker \mathrm{d}_{1,\mathfrak{m}}^{-2,4},$$
$$\mathrm{E}_2^{0,3} = \mathrm{coker}(\mathrm{H}^1(D_1, \mathbb{F}_\ell)_\mathfrak{m} \to \mathrm{H}^3((\mathbf{S}^* \otimes \mathbb{F}_q^{\mathrm{ac}})^{\#}, \mathbb{F}_\ell)_\mathfrak{m} = \mathrm{H}^3((\mathbf{S}^* \otimes \mathbb{F}_q^{\mathrm{ac}})^{\#}, \mathbb{F}_\ell)_\mathfrak{m}.$$

By Lemma 2.4.6 we have $\mathrm{H}^1(D_1, \mathbb{F}_\ell)_\mathfrak{m} = 0$. Thus we have $\mathrm{E}_{3,\mathfrak{m}}^{0,3} = \mathrm{E}_{\infty,\mathfrak{m}}^{0,3} = 0$ which is a graded piece of $\mathrm{H}^3((\mathbf{S}^* \otimes \mathbb{F}_q^{\mathrm{ac}})^{\mathrm{ord}}, \mathbb{F}_\ell)_\mathfrak{m} = 0$. On the other hand, $\mathrm{E}_{3,\mathfrak{m}}^{0,3} = \mathrm{coker}(\ker \mathrm{d}_{1,\mathfrak{m}}^{-2,4} \to \mathrm{H}^3((\mathbf{S}^*\otimes\mathbb{F}_q^{\mathrm{ac}})^{\#}, \mathbb{F}_\ell)_\mathfrak{m})$. Thus it suffices to show $\ker \mathrm{d}_{1,\mathfrak{m}}^{-2,4} = 0$, which is deduced from the claim. We finish the proof of Theorem 3.1.1. □

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## LA DIMINUTION DE NIVEAU POUR LES FORMES AUTOMORPHES SUR LES SURFACES MODULAIRES DE PICARD


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## Résumé

Le principe de Mazur donne un critère selon lequel une représentation galoisienne irréductible mod $\ell$ provenant d'une forme modulaire de niveau $Np$ (avec $p$ premier par rapport à $N$) peut également provenir d'une forme modulaire de niveau $N$. Dans cette thèse nous démontrons un résultat analogue montrant que une représentation galoisienne mod $\ell$ provenant d'une représentation automorphe cuspidale stable du groupe de similitude unitaire $G=\mathrm{GU}(1,2)$ qui est Steinberg en un nombre premier inerte $p$ peut également provenir d'une représentation automorphe de $G$ qui est non ramifiée en $p$.



**Mots clés :** Surface modulaire de Picard, diminution de niveau, principe de Mazur, représentation galoisienne



## Résumé en anglais

Mazur's principle gives a criterion under which an irreducible mod $\ell$ Galois representation arising from a modular form of level $Np$(with $p$ prime to $N$) can also arise from a modular form of level $N.$ We prove an analogous result showing that a mod $\ell$ Galois representation arising from a stable cuspidal automorphic representation of the unitary similitude group $G=\mathrm{GU}(1,2)$ which is Steinberg at an inert prime $p$ can also arise from an automorphic representation of $G$ that is unramified at $p$.



**Keywords :** Picard modular surface, level lowering, Mazur's principle, Galois representation

Le principe de Mazur donne un critère selon lequel une représentation galoisienne irréductible mod $\ell$ provenant d'une forme modulaire de niveau $Np$ (avec $p$ premier par rapport à $N$) peut également provenir d'une forme modulaire de niveau $N$. Dans cette thèse nous démontrons un résultat analogue montrant que une représentation galoisienne mod $\ell$ provenant d'une représentation automorphe cuspidale stable du groupe de similitude unitaire $G = \mathrm{GU}(1,2)$ qui est Steinberg en un nombre premier inerte $p$ peut également provenir d'une représentation automorphe de $G$ qui est non ramifiée en $p$.

Mazur's principle gives a criterion under which an irreducible mod $\ell$ Galois representation arising from a modular form of level $Np$ (with $p$ prime to $N$) can also arise from a modular form of level $N$. We prove an analogous result showing that a mod $\ell$ Galois representation arising from a stable cuspidal automorphic representation of the unitary similitude group $G = \mathrm{GU}(1,2)$ which is Steinberg at an inert prime $p$ can also arise from an automorphic representation of $G$ that is unramified at $p$.

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