Université de Strasbourg

UNIVERSITÉ DE STRASBOURG

École doctorale Mathématiques, sciences de l'information et de l'ingénieur | ED 269 Université de Strasbourg

ÉCOLE DOCTORALE MATHEMATIQUES, SCIENCES DE L'INFORMATION ET DE L'INGENIEUR – ED269

Mathématiques

THÈSE présentée par :

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Soutenue le : **24 Octobre 2023**

Pour obtenir le grade de : **Docteur de l'université de Strasbourg**

Discipline/ Spécialité : Mathématiques

Fibré vectoriel fini sur une courbe algébrique

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Thèse

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE

UMR 7501

Strasbourg

présentée pour obtenir le grade de docteur de
l'Université de Strasbourg
Spécialité MATHÉMATIQUES

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G-Torseurs Essentiellement Finis

Soutenue le 24 Octobre 2023
devant la commission d'examen

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Introduction

Soit X une courbe connexe projective lisse sur un corps algébriquement clos k. Soit $g = g(X)$ le genre de X. En 1938, Weil a introduit la notion de fibré vectoriel fini. Un fibré vectoriel E est dit fini s'il existe deux polynômes distincts, $f, g \in \mathbb{N}[x]$, tels que le fibré vectoriel $f(E)$ soit isomorphe à $g(E)$ (voir [Wei38]). Pour $k = \mathbb{C}$, il a prouvé qu'un fibré vectoriel est fini si et seulement s'il provient d'une représentation de $\pi_1(X)$ qui se factorise par un groupe fini. Près de 40 ans plus tard, dans [Nor76], Nori a introduit la notion de fibré vectoriel essentiellement fini comme un sous-quotient d'un fibré vectoriel fini. La catégorie des fibrés vectoriels essentiellement finis est une catégorie tannakienne, et le groupe correspondant est connu sous le nom de groupe fondamental de Nori qui est un schéma en pro-groupes sur k dont les k points sont isomorphes au groupe fondamental étale, $\pi_1^{\text{et}}(X)$, lorsque k est de caractéristique 0 (voir [Sza09, Corollaire 6.7.20] et aussi, [EHS08]).

En considérant un fibré vectoriel comme un GL_n-torseur, nous sommes amenés à nous poser la question suivante : Peut-on généraliser la notion de fibré vectoriel essentiellement fini à une notion de G-torseur essentiellement fini, pour G un groupe algébrique affine ? Nori a prouvé qu'un fibré vectoriel E est essentiellement fini si et seulement s'il existe un schéma en groupe fini Γ, un Γ-torseur F_Γ et une représentation V de Γ telle que $E \cong F_\Gamma \times^\Gamma V$. On est donc conduit à la définition suivante

Definition 0.1. Un G-torseur sur X est dit essentiellement fini si il admet une réduction à un groupe fini.

Vu à travers la correspondance entre les fibrés vectoriels et les GL_n-torseurs, cela concorde avec la définition usuelle de fibré vectoriel essentiellement fini. On démontre le théorème suivant :

Theorem 0.2. *Soit G un groupe réductif, connexe et soit F_G un G-torseur. Les assertions suivantes sont équivalentes :*

1. *Le G-torseur F_G est essentiellement fini.*
2. *Il existe une représentation fidèle $\rho : G \longrightarrow GL_V$ tel que $\rho_* F_G$ est un fibré vectoriel essentiellement fini.*
3. *Pour toute représentation $\rho\colon G \to GL_V$, $\rho_* F_G$ est un fibré vectoriel essentiellement fini.*
4. *Il existe un morphisme surjectif et propre $f\colon Y \to X$ tel que $f^* F_G$ est trivial.*

Notons également que, puisque la semistabilité peut être vérifiée sur le fibré adjoint, tout G-torseur essentiellement fini est semistable. Nous donnons une autre preuve de ce fait, n'utilisant pas la représentation adjointe.

Considérons M_G^{ss} l'espace des modules des G-torseurs semistables sur X, pour G un groupe réductif connexe. On rappelle que les composantes connexes de M_G^{ss} sont indexées par le groupe fondamental algébrique de G, $\pi_1(G)$. Si un G-torseur, F_G est dans la composante connexe correspondant à $d \in \pi_1(G)$, on dit que F_G est de degré d. Les fibrés vectoriels essentiellement fini sont de degré zéro. On prouve le théorème suivant :

Theorem 0.3. *Pour tout groupe réductif connexe G, tous les G-torseurs essentiellement finis sur X ont un degré qui est de torsion.*

Encore une fois, cela généralise le cas de $G = \mathrm{GL}_n$, puisque dans ce cas $\pi_1(G) = \mathbb{Z}$, est sans torsion. Nous montrons également que si X est une courbe elliptique, alors tous les G-torseurs essentiellement finis sont de degré 0.

Notons $M_G^{\mathrm{ef},0}$ les k-points correspondant aux G-torseurs essentiellement finis dans $M_G^{\mathrm{ss},0}$. Si $G = \mathrm{GL}_n$ et $n = 1$ les G-torseurs essentiellement finis correspondent aux fibrés en droites essentiellement finis c'est à dire aux fibrés en droites de torsions (Voir le Lemme 3.1 [Nor76]). Ainsi $M_{\mathrm{GL}_1}^{\mathrm{ef},0}$ est dense dans $M_{\mathrm{GL}_1}^{\mathrm{ss},0} = \mathrm{Jac}^0(X)$ puisque les points de torsion sont denses dans toute variété abélienne. En caractéristique positive, Ducrohet et Mehta on montré que $M_{\mathrm{GL}_n}^{\mathrm{ef},0} \subset M_{\mathrm{GL}_n}^{\mathrm{ss},0}$ est dense pour tout n, lorsque $g \geq 2$ et de même pour les fibrés vectoriels de déterminant trivial i.e SL_n-torseur. (Ils montrent en fait qu'un plus petit ensemble d'objets appelés fibrés vectoriels Frobenius perodique, sont denses [DM10]). Cependant en caractéristique zéro, on en sait beaucoup moins sur la densité des fibrés vectoriels essentiellement finis. Nous pouvons donc nous demander si $M_{\mathrm{GL}_n}^{\mathrm{ef},0}$ est dense dans $M_{\mathrm{GL}_n}^{\mathrm{ss},0}$ pour $n > 1$. Plus généralement, nous nous intéressons à la question de savoir si $M_G^{\mathrm{ef},0}$ est dense dans $M_G^{\mathrm{ss},0}$ pour G un groupe réductif connexe sur un corps algébriquement clos k.

Si $g = 0$, c'est-à-dire si $X \cong \mathbb{P}^1$, il est bien connu que $M_G^{\mathrm{ss},0}(k)$ est un singleton. Il est donc clair que tout G-torseur essentiellement fini sur $\mathbb{P}^1$ est trivial. Nous donnons une preuve différente de ce résultat en utilisant une interprétation tannakienne de la classification des G-torseurs sur $\mathbb{P}^1$ (voir [Ans18]) et la définition des torseurs essentiellement finis. Si $g = 1$, c'est-à-dire si X est une courbe elliptique, nous prouvons que $M_G^{\mathrm{ef},0}$ est dense dans $M_G^{\mathrm{ss},0}(k)$ pour tous les groupes réductifs connexes. Ceci découle des travaux de Frăţilă [Fră21] et de Laszlo [Las98]. Au contraire, si $g \geq 2$ et que la caractéristique de k est zéro, nous montrons le théorème suivant :

Theorem 0.4. *Pour tout groupe réductif connexe, semisimple de rang 1, $M_G^{ef,0} \subset$*

$M_G^{ss,0}$ *n'est pas dense.*

Le travail principal consiste à prouver le théorème pour les PGL_2-torseurs. Notons également que cela montre que $M^{\mathrm{ef},0}_{\mathrm{GL}_2} \subset M^{\mathrm{ss},0}_{\mathrm{GL}_2}$ n'est pas dense. En caractéristique 0, Weissman [Wei22] a obtenu indépendamment ce résultat de non-densité pour $M^{\mathrm{ef},0}_{\mathrm{GL}_n}$ avec $n > 1$.

Par le théorème de Narasimhan et Seshadri, les points de $M^{\mathrm{ss},0}_{\mathrm{GL}_n}(\mathbb{C})$ sont aussi les classes d'isomorphisme de représentations $\pi_1(X) \longrightarrow \mathrm{U}_n(\mathbb{C})$, c'est-à-dire qu'il existe un homeomorphisme analytique entre $M^{\mathrm{ss},0}_{\mathrm{GL}_n}(\mathbb{C})$ et la variété de caractères

$$\mathrm{Hom}(\pi_1(X), \mathrm{U}_n(\mathbb{C}))/\sim .$$

En particulier, les fibrés vectoriels finis correspondent à des représentations unitaires du $\pi_1(X)$ qui se factorisent par des groupes finis. Comme la topologie de Zariski est plus grossière que la topologie analytique, nous voyons comme corollaire de la non-densité des fibrés vectoriels finis que l'ensemble des représentations unitaires de rang n du $\pi_1(X)$ qui se factorisent par des groupes finis n'est pas dense à l'intérieur de $\mathrm{Hom}(\pi_1(X), \mathrm{U}_n(\mathbb{C}))/\sim$.

Let X be a smooth projective connected curve over an algebraically closed field k. Let $g = g(X)$ be the genus of X. In 1938 Weil introduced the notion of a finite vector bundle; a vector bundle E is called finite if there are two distinct polynomials, $f, g \in \mathbb{N}[x]$, such that the vector bundle $f(E)$ is isomorphic to $g(E)$ (see [Wei38]). For $k = \mathbb{C}$, he proved that a vector bundle is finite if and only if it arises from a representation of $\pi_1(X)$ which factors through a finite group. Almost 40 years later, in [Nor76], Nori introduced the notion of an essentially finite vector bundle as a subquotient of a finite one. The category of essentially finite vector bundles forms a Tannakian category, and the corresponding group is known as the Nori fundamental group, a pro-group scheme over k whose k points are isomorphic to the étale fundamental group, $\pi_1^{\mathrm{et}}(X)$, when k is of characteristic 0 (see [Sza09, Corollary 6.7.20] and also e.g., [EHS08]).

Viewing a vector bundle as a GL_n-torsor, we are led to the question: can we generalise the notion of an essentially finite vector bundle, to a notion of an essentially finite G-torsor, for G an affine algebraic group? Nori proved that a vector bundle E is essentially finite if and only if there exists a finite group scheme Γ, a Γ-torsor F_Γ and a representation V of Γ such that $E \cong F_\Gamma \times^\Gamma V$. Hence, we are led to the following definition

Definition 0.5. An **essentially finite** G-torsor is a G-torsor over X which admits a reduction to a finite group.

Under the correspondence between vector bundles and GL_n-torsors, this agrees with the known definition of essentially finite vector bundles. We prove the following.

Theorem 0.6. *Let G be a connected, reductive group, and let F_G be a G-torsor. Then the following are equivalent.*

1. *The G-torsor F_G is essentially finite.*
2. *There exists a faithful representation $\rho\colon G \to GL_V$ such that $\rho_* F_G$ is an essentally finite vector bundle.*
3. *For every representation $\rho\colon G \to GL_V$, $\rho_* F_G$ is an essentally finite vector bundle.*
4. *There exists a proper surjective morphism $f : Y \to X$ such that $f^* F_G$ is trivial.*

Note also that since semistability can be checked on the adjoint bundle, every essentially finite G-torsor is semistable. We give a self-contained proof of this fact, not using the adjoint representation.

Let now M_G^{ss} denote the moduli space of semistable G-bundles over X, for G a connected reductive group. Recall that the connected components of M_G^{ss} are indexed by the algebraic fundamental group of G, $\pi_1(G)$. If a G-bundle, F_G, lies in a component corresponding to $d \in \pi_1(G)$, then it is said to have degree d. Essentially finite vector bundles always have degree 0. We prove the following.

Theorem 0.7. *For any connected reductive group G, every essentially finite G-torsor over X is of torsion degree.*

Again this generalises the case for $G = \mathrm{GL}_n$, since in this case $\pi_1(G) = \mathbb{Z}$, which is torsion-free. We also show that if X is an elliptic curve then all essentially finite G-bundles have degree 0.

Let now $M_G^{\mathrm{ef},0}$ denote the k-points of the essentially finite G-torsors of degree 0, inside $M_G^{\mathrm{ss},0}$, and let $G = \mathrm{GL}_n$. If $n = 1$, then essentially finite G-bundles correspond to essentially finite line bundles, which correspond to torsion line bundles (see Lemma 3.1 [Nor76]). Hence, $M_{\mathrm{GL}_1}^{\mathrm{ef}}$ is dense inside $M_{\mathrm{GL}_1}^{\mathrm{ss},0} = \mathrm{Jac}^0(X)$ since torsion points are dense in any abelian variety. In positive characteristic Ducrohet and Mehta have shown that $M_{\mathrm{GL}_n}^{\mathrm{ef},0} \subset M_{\mathrm{GL}_n}^{\mathrm{ss},0}$ is dense for all n when $g \geq 2$, and similarly for vector bundles with trivial determinant (they show in fact that a smaller set of objects, called Frobenius periodic vector bundles, are dense; see [DM10]). However, in characteristic zero much less seems to be known about the density of essentially finite bundles when the rank is greater than 1. Hence, we may ask whether $M_{\mathrm{GL}_n}^{\mathrm{ef},0}$ is dense in $M^{\mathrm{ss},0}_{\mathrm{GL}_n}$ for $n > 1$, when $\mathrm{char}(k) = 0$. More generally, we are interested in the question of whether $M^{\mathrm{ef},0}_G$ is dense in $M^{\mathrm{ss},0}_G$ for arbitrary connected reductive groups G over an arbitrary, algebraically closed field k.

If $g = 0$, that is if $X \cong \mathbb{P}^1$, then it is well-known that $M^{\mathrm{ss},0}_G(k)$ is a singleton. Hence it is clear that every essentially finite G-torsor over $\mathbb{P}^1$ is trivial. We give a self-contained proof of this result using a Tannakian interpretation of both the classification of G-torsors over $\mathbb{P}^1$ (see [Ans18]) and the definition of essentially finite torsors. If $g = 1$, that is if X is an elliptic curve, then we prove that $M^{\mathrm{ef},0}_G$ is dense in $M^{\mathrm{ss},0}_G$ for all connected, reductive groups. This follows from work of Frăţilă [Fră21] and Laszlo [Las98]. On the contrary, if $g \geq 2$ and $\mathrm{char}(k) = 0$, then we show the following.

Theorem 0.8. *Let* $\mathrm{char}(k) = 0$. *For all connected, reductive groups of semisimple rank 1,* $M^{ef,0}_G \subset M^{ss,0}_G$ *is not dense.*

The main work lies in proving the theorem for PGL_2-torsors. Note also that this shows that $M^{\mathrm{ef}}_{\mathrm{GL}_2}$ is not dense in $M^{\mathrm{ss},0}_{\mathrm{GL}_2}$. In characteristic 0, Weissman [Wei22] has independently obtained this non-density result for $M^{\mathrm{ef}}_{\mathrm{GL}_n}$ for all $n \geq 1$.

By the theorem of Narasimhan and Seshadri, the points of $M^{\mathrm{ss},0}_{\mathrm{GL}_n}(\mathbb{C})$ are also the isomorphism classes of representations $\pi_1(X) \longrightarrow \mathrm{U}_n(\mathbb{C})$, i.e., there is an analytic homeomorphism between $M^{\mathrm{ss},0}_{\mathrm{GL}_n}(\mathbb{C})$ and the character variety $\mathrm{Hom}(\pi_1(X), \mathrm{U}_n(\mathbb{C}))/\sim$. In particular finite vector bundles correspond to unitary representations of $\pi_1(X)$ which factor through finite groups. As the Zariski topology is coarser than the analytic topology we see as a corollary to non-density for rank n vector bundles that the set of rank n unitary representations of $\pi_1(X)$ which factor through finite groups is not dense inside $\mathrm{Hom}(\pi_1(X), \mathrm{U}_n(\mathbb{C}))/\sim$.

1 Essentially finite vector bundle

Ce premier chapitre introduit les bases nécessaires pour la suite. Nous débutons par une revue détaillée des fondamentaux des fibrés en droite, mettant en avant leurs propriétés caractéristiques, notamment le célèbre théorème de Riemann-Roch.

Ensuite, nous introduisons les notions essentielles de degré d'un fibré vectoriel et de pente, en soulignant leurs propriétés élémentaires qui joueront un rôle central dans notre compréhension des fibrés vectoriels et de leur classification.

La semi-stabilité est un concept fondamental que nous abordons ensuite, montrant que la catégorie des fibrés vectoriels semi-stables à pente fixée constitue une catégorie abélienne.

L'un des points clefs de ce chapitre est la notion de fibré vectoriel fini, dont l'origine remonte à A. Weil en 1939. Nous montrons comment, dans le cas complexe, ces fibrés sont en correspondance avec les représentations du groupe fondamental dont les images sont finies, établissant ainsi un lien profond entre la topologie et la théorie des fibrés vectoriels.

Enfin, nous explorons la notion de fibré vectoriel essentiellement fini, telle que formulée par M. Nori, des décennies après les travaux de Weil. Ils sont définis comme des sous-quotients de fibrés vectoriels finis.

1.1 Line bundles and divisors

Let k be an algebraically closed field and let $X \to \mathrm{Spec}(k)$ be a smooth projective and connected curve. By a vector bundle we mean a locally free coherent sheaf.

For $x \in X$, we let $\mathcal{O}_X(-x)$ denote the sheaf of functions vanishing at x; that is, for $U \subseteq X$, we have $\mathcal{O}_X(-x)(U) = \{f \in \mathcal{O}_X(U) \mid f(x) = 0\}$.
By construction, this is a subsheaf of $\mathcal{O}_X$ and, in fact, $\mathcal{O}_X(-x)$ is a line bundle on X
For $x \in X$, we let k_x denote the skyscraper sheaf of x whose sections over $U \subseteq X$ are given by $k_x(U) = \begin{cases} k & \text{if } x \in U \\ 0 & \text{else} \end{cases}$.
The skyscraper sheaf is not a locally free sheaf; it is a torsion sheaf which is supported on the point x.

Since $H^0(X, k_x) = k_x(X) = k$ and $H^1(X; k_x) = 0$ we have $\chi(X, k_x) = 1$.
There is a short exact sequence of sheaves

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to k_x \to 0$$

where the morphism $\mathcal{O}_X \to k_x$ is given by evaluating a function at x. We can tensor this exact sequence by a line bundle $\mathcal{L}$ to obtain

$$0 \to \mathcal{L}(-x) \to \mathcal{L} \to k_x \to 0$$

where $\mathcal{L}(-x)$ is a line bundle, whose sections are the sections of $\mathcal{L}$ which vanish at x. Hence by additivity of Euler characteristics [Gro61, Proposition 2.5.2], we have the following formula

$$\chi(X, \mathcal{L}) = \chi(X, \mathcal{L}(-x)) + 1$$

Recall that a **Weil divisor** over X is a finite formal sum of points $D = \sum_{x \in X} m_x[x]$, we define is **degree** by $\deg(D) = \sum_{x \in X} m_x$. We say that a divisor is **effective**, denoted by $D \geq 0$, if $m_x \geq 0$ for all x.
For a rational function $f \in k(X)$, we define the associated principal divisor

$$\operatorname{div}(f) = \sum_{x \in X(k)} \operatorname{ord}_x(f)[x]$$

where $\operatorname{ord}_x(f)$ is the order of vanishing of f at x. We say that two divisors are **linearly equivalent** if their difference is a principal divisor.
For a Weil divisor D, we define a line bundle $\mathcal{O}_X(D)$ by

$$\mathcal{O}_X(D)(U) = \{f \in k(X)^* \mid (\operatorname{div}(f) + D)_{|U} \geq 0\}$$

As X is smooth, the above construction $D \mapsto \mathcal{O}_X(D)$ determines an isomorphism from the group of Weil divisor modulo linear equivalence to the Picard group of isomorphism classes of line bundles. In particular any line bundle $\mathcal{L}$ over X is isomorphic to $\mathcal{O}_X(D)$ for some divisor D. For proofs, see [Har77, Chapter 6].
For an effective divisor D, the dual line bundle $\mathcal{O}_X(-D)$ is isomorphic to the ideal sheaf of the subscheme $D \subseteq X$ given by this effective divisor and we have a short exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to k_D \to 0$$

where k_D denotes the skyscraper sheaf supported on D, thus k_D is a torsion sheaf. In particular, any effective divisor admits a non-zero section $\mathcal{O}_X \to \mathcal{O}_X(D)$. In fact, a line bundle $\mathcal{O}_X(D)$ admits a non-zero section if and only if D is linearly equivalent to an effective divisor by [Har77, Proposition 7.7 page 157]

Theorem 1.1 (Riemann-Roch)**.** *Let $\mathcal{L} = \mathcal{O}_X(D)$ be a line bundle over X. then*

$$\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X) + deg(D)$$

Proof. We can write $D = [x_1] + ... + [x_n] - [y_1] - ... - [y_m]$ and then proceed by induction on $n + m \in \mathbb{Z}$. The base case $D = 0$ is obvious. Now assume that the equality has been proved for D, then we have to show it for $D + [x]$ and $D - [x]$ for $x \in X$. For $D - [x]$ we use the following short exact sequence

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to k_x \to 0$$

and we tensor it by $\mathcal{O}_X(D)$ to obtain

$$0 \to \mathcal{O}_X(D - [x]) \to \mathcal{O}_X(D) \to k_x \to 0$$

By the additivity of of Euler characteristics we have

$$\begin{aligned} \chi(X, \mathcal{O}_X(D)) &= \chi(X, \mathcal{O}_X(D - [x])) + 1 \\ \chi(X, \mathcal{O}_X(D - [x])) &= \chi(X, \mathcal{O}_X(D)) - 1 \\ \chi(X, \mathcal{O}_X(D - [x])) &= \chi(X, \mathcal{O}_X) + \deg(D) - 1 \\ &= \chi(X, \mathcal{O}_X) + \deg(D - [x]) \end{aligned} \tag{1.1}$$

For $D + [x]$ we use the following short exact sequence

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to k_x \to 0$$

and we tensor it by $\mathcal{O}_X(D + [x])$ to obtain

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + [x]) \to k_x \to 0$$

By the additivity of of Euler characteristics we have

$$\begin{aligned} \chi(X, \mathcal{O}_X(D + [x])) &= \chi(X, \mathcal{O}_X(D)) + 1 \\ &= \chi(X, \mathcal{O}_X) + \deg(D) + 1 \\ &= \chi(X, \mathcal{O}_X) + \deg(D + [x]) \end{aligned} \tag{1.2}$$

□

1.2 Notion of degree

Definition 1.2. Let $\mathcal{E}$ be a vector bundle of rank r on X, we define the **degree** of $\mathcal{E}$ by $\deg(\mathcal{E}) = \chi(X, \mathcal{E}) - r\chi(X, \mathcal{O}_X)$

Lemma 1.3. Let $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ be a short exact sequence of nonzero vector bundles on X. Then

$$\deg(\mathcal{E}_2) = \deg(\mathcal{E}_1) + \deg(\mathcal{E}_3)$$

Proof. Follows immediately from additivity of Euler characteristics [Gro61, Proposition 2.5.2] and the additivity of ranks. □

Lemma 1.4. Let $\mathcal{E}$ be a vector bundle on X, then $T(\mathcal{E}) = \sup\{\deg(\mathcal{F}) \mid \mathcal{F} \subseteq \mathcal{E}\}$ is finite.

Proof. Since the global sections functor is left exact, we get

$$H^0(X, \mathcal{F}) \subseteq H^0(X, \mathcal{E})$$

This implies that $h^0(X, \mathcal{F}) \leq h^0(X, \mathcal{E})$. Now by definition we have

$$h^0(X, \mathcal{F}) - h^1(X, \mathcal{F}) = \deg(\mathcal{F}) + \mathrm{rk}(\mathcal{F})(1 - g)$$

where g is the genus of X.
From this we get

$$\deg(\mathcal{F}) + \mathrm{rk}(\mathcal{F})(1 - g) + h^1(X, \mathcal{F}) = h^0(X, \mathcal{F}) \leq h^0(X, \mathcal{E})$$

Now,

- If $g = 0$ then $\deg(\mathcal{F}) \leq h^0(X, \mathcal{E}) - \mathrm{rk}(\mathcal{F}) - h^1(X, \mathcal{F})$ and since $\mathrm{rk}(\mathcal{F}) \geq 0$ and $h^1(X, \mathcal{F}) \geq 0$ we get $\deg(\mathcal{F}) \leq h^0(X, \mathcal{E})$.
- If $g = 1$ we see that $\deg(\mathcal{F}) \leq h^0(X, \mathcal{E}) - h^1(X, \mathcal{F})$ and since $h^1(X, \mathcal{F}) \geq 0$ we see that $\deg(\mathcal{F}) \leq h^0(X, \mathcal{E})$.
- If $g \geq 2$, $\deg(\mathcal{F}) \leq h^0(X, \mathcal{E}) - h^1(X, \mathcal{F}) + \mathrm{rk}(\mathcal{F})(g - 1)$ again since $h^1(X, \mathcal{F}) \geq 0$ and $\mathrm{rk}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{E})$ then $\deg(\mathcal{F}) \leq h^0(X, \mathcal{E}) + \mathrm{rk}(\mathcal{E})(g - 1)$

□

Lemma 1.5. Let $\mathcal{E}$ be a vector bundle of rank r on X, then

$$\deg(\mathcal{E}) = \deg(\bigwedge^r \mathcal{E})$$

Proof. See [Le 97, Theorem 2.6.9 page 33]. □

Lemma 1.6. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be vector bundle on X, then

$$\deg(\mathcal{E}_1 \otimes \mathcal{E}_2) = \mathrm{rk}(\mathcal{E}_1)\deg(\mathcal{E}_2) + \mathrm{rk}(\mathcal{E}_2)\deg(\mathcal{E}_1)$$

Proof. Using the previous Lemma it follow from the fact that $\det(\mathcal{E}_1 \otimes \mathcal{E}_2) = \det(\mathcal{E}_1)^{\mathrm{rk}(\mathcal{E}_2)} \otimes \det(\mathcal{E}_2)^{\mathrm{rk}(\mathcal{E}_1)}$ □

1.3 Notion of slope

Definition 1.7. Let $\mathcal{E}$ be a vector bundle on X we define the **slope** of $\mathcal{E}$ by $\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$.

Lemma 1.8. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be vector bundle on X, then $\mu(\mathcal{E}_1 \otimes \mathcal{E}_2) = \mu(\mathcal{E}_1) + \mu(\mathcal{E}_2)$

Proof. We have

$$\begin{aligned} \mu(\mathcal{E}_1 \otimes \mathcal{E}_2) &= \frac{\deg(\mathcal{E}_1 \otimes \mathcal{E}_2)}{\mathrm{rk}(\mathcal{E}_1 \otimes \mathcal{E}_2)} \\ &= \frac{\mathrm{rk}(\mathcal{E}_1)\deg(\mathcal{E}_2) + \mathrm{rk}(\mathcal{E}_2)\deg(\mathcal{E}_1)}{\mathrm{rk}(\mathcal{E}_1)\mathrm{rk}(\mathcal{E}_2)} \\ &= \mu(\mathcal{E}_1) + \mu(\mathcal{E}_2) \end{aligned} \tag{1.1}$$

□

Proposition 1.9. Let $\mathcal{E}$ be a vector bundle on X, then $\mu_{\max}(\mathcal{E}) = \sup\{\mu(\mathcal{F}) \mid \mathcal{F} \subseteq \mathcal{E}, \mathcal{F} \neq 0\}$ is finite.

Proof. Let $\mathcal{F}$ be a non trivial sub-bundle of $\mathcal{E}$, we get $\mu(\mathcal{F}) \leq \frac{T(\mathcal{E})}{\mathrm{rk}(\mathcal{F})}$, then $\mu_{\max}(\mathcal{E}) \leq T(\mathcal{E})$ we conclude by the Lemma 1.4. □

Lemma 1.10. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be vector bundle on X, then we have

$$\mu_{\max}(\mathcal{E}_1 \oplus \mathcal{E}_2) \leq \max(\mu_{\max}(\mathcal{E}_1), \mu_{\max}(\mathcal{E}_2))$$

Proof. Consider a subbundle $\mathcal{F} \subseteq \mathcal{E}_1 \oplus \mathcal{E}_2$, and denote by $j : \mathrm{Spec}(K(X)) \to X$ the inclusion of the generic point of X. The pullback $j^*\mathcal{F}$ decomposes as a direct sum $j^*\mathcal{F} \cong \mathcal{F}_1 \oplus \mathcal{F}_2$, where the $\mathcal{F}_i$ are vector bundles over $\mathrm{Spec}(K(X))$ corresponding to sub-spaces of the vector spaces $j^*\mathcal{E}_i$. The $j_*\mathcal{F}_i$ are locally free sub-sheaves of the $\mathcal{E}_i$, hence there exist sub-bundles $\mathcal{G}_i$ such that $j_*\mathcal{F}_i \subseteq \mathcal{G}_i \subseteq \mathcal{E}_i$ that have the same rank as the $j_*\mathcal{F}_i$. Then it suffices to prove $\mu(\mathcal{G}_1 \oplus \mathcal{G}_2) \leq \max(\mu(\mathcal{G}_1), \mu(\mathcal{G}_2))$, in fact we have

$$\begin{aligned}\mu(\mathcal{G}_1 \oplus \mathcal{G}_2) &= \frac{\deg(\mathcal{G}_1 \oplus \mathcal{G}_2)}{\mathrm{rk}(\mathcal{G}_1 \oplus \mathcal{G}_2)} \\ &= \frac{\deg(\mathcal{G}_1) + \deg(\mathcal{G}_2)}{\mathrm{rk}(\mathcal{G}_1) + \mathrm{rk}(\mathcal{G}_2)} \\ &\leq \max(\frac{\deg(\mathcal{G}_1)}{\mathrm{rk}(\mathcal{G}_1)}, \frac{\deg(\mathcal{G}_2)}{\mathrm{rk}(\mathcal{G}_2)}) \\ &= \max(\mu(\mathcal{G}_1), \mu(\mathcal{G}_2))\end{aligned} \tag{1.2}$$

□

Lemma 1.11. Let $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ be a short exact sequence of nonzero vector bundles on X.

1. If $\mu(\mathcal{E}_1) \leq \mu(\mathcal{E}_2)$, then $\mu(\mathcal{E}_1) \leq \mu(\mathcal{E}) \leq \mu(\mathcal{E}_2)$.
2. If $\mu(\mathcal{E}_1) \geq \mu(\mathcal{E})$, then $\mu(\mathcal{E}_1) \geq \mu(\mathcal{E}) \geq \mu(\mathcal{E}_2)$.
3. If $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$, then $\mu(\mathcal{E}_1) = \mu(\mathcal{E}) = \mu(\mathcal{E}_2)$.

Proof. 1. By Lemma 1.3 we have $\mu(\mathcal{E}) = \frac{d_1+d_2}{r_1+r_2}$ and by assumption we have $\mu(\mathcal{E}_1) = \frac{d_1}{r_1} \leq \mu(\mathcal{E}_2) = \frac{d_2}{r_2}$ i.e $d_1 \leq \frac{r_1 d_2}{r_2}$. Then

$$\begin{aligned}\mu(\mathcal{E}) &= \frac{d_1 + d_2}{r_1 + r_2} \\ &\leq \frac{\frac{r_1 d_2}{r_2} + d_2}{r_1 + r_2} \\ &\leq \frac{d_2(r_1 + r_2)}{r_2(r_1 + r_2)} \\ &= \mu(\mathcal{E}_2)\end{aligned} \tag{1.3}$$

Using $d_2 \geq \frac{r_2 d_1}{r_1}$ we get $\mu(\mathcal{E}_1) \leq \mu(\mathcal{E})$

2. Same calculation as 1. give the result.

3. Obvious by 1. and 2.

□

1.4 Notion of stability

Definition 1.12. Let $\mathcal{E}$ be a vector bundle on X, $\mathcal{E}$ is called **semistable** (resp **stable**) if for every non trivial sub-bundle $\mathcal{F}$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ (resp $\mu(\mathcal{F}) < \mu(\mathcal{E})$).

Lemma 1.13. Let $\mathcal{E}$ be a vector bundle on X, $\mathcal{E}$ is semistable (resp stable) if and only if for every surjections $\mathcal{E} \to \mathcal{F}$ we have $\mu(\mathcal{F}) \geq \mu(\mathcal{E})$ (resp $\mu(\mathcal{F}) > \mu(\mathcal{E})$).

Proof. Suppose that $\mathcal{E}$ is semistable and let $\mathcal{E} \to \mathcal{F}$ be a surjection, then we have a short exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

Where $\mathcal{G}$ is the kernel of the surjection. By semistability of $\mathcal{E}$ we have $\mu(\mathcal{G}) \leq \mu(\mathcal{E})$ then $\deg(\mathcal{G})\mathrm{rk}(\mathcal{E}) \leq \deg(\mathcal{E})\mathrm{rk}(\mathcal{G})$ hence $(\deg(\mathcal{E}) - \deg(\mathcal{F}))\mathrm{rk}(\mathcal{E}) \leq \deg(\mathcal{E})(\mathrm{rk}(\mathcal{E}) - \mathrm{rk}(\mathcal{F}))$ i.e $\mu(\mathcal{F}) \geq \mu(\mathcal{E})$.
Let $\mathcal{F}$ be a sub-bundle of $\mathcal{E}$ so we have the following short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{E}/\mathcal{F} \to 0$$

Then the same calculus in reverse direction shows that $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$.
And likewise with strict inequalities □

Remark 1.14.

1. All line bundles are stable since they do not even have non-trivial sub-bundles.

2. Let $\mathcal{E}$ be a vector bundle on X. If $\deg(\mathcal{E})$ and $\mathrm{rank}(\mathcal{E})$ are coprime then $\mathcal{E}$ is semistable if and only if $\mathcal{E}$ is stable.

3. Let $\mathcal{E}$ be a vector bundle and $\mathcal{L}$ be a line bundle on X. Using the formula 1.8 we see that $\mathcal{E}$ is semistable (resp stable) if and only if $\mathcal{E} \otimes \mathcal{L}$ is semistable (resp stable).

4. Let $\mathcal{E}$ be a vector bundle on X. Using 1.13 we see that $\mathcal{E}$ is semistable (resp stable) if and only if $\mathcal{E}^*$ is semistable (resp stable).

Proposition 1.15. Let $\mathcal{E}$ and $\mathcal{F}$ be two semistable vector bundles on X. If $\mu(\mathcal{F}) < \mu(\mathcal{E})$ then $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = \{0\}$.

Proof. Let $\alpha : \mathcal{E} \to \mathcal{F}$ be a non trivial morphism, then $\mathrm{Im}(\alpha)$ is a proper sub-bundle of $\mathcal{F}$. As $\mathcal{E}$ is semi-stable we have, by the Lemma 1.13 $\mu(\mathcal{E}) \leq \mu(\mathrm{Im}(\alpha))$, then $\mu(\mathcal{F}) < \mu(\mathrm{Im}(\alpha))$ but $\mathcal{F}$ is semi-stable, contradiction. □

Proposition 1.16. Let $\mathcal{E}$ be a semistable vector bundle and let $\mathcal{F}$ be any vector bundle on X. If $\mu(\mathcal{E}) > \mu_{\max}(\mathcal{F})$, then any morphism $f : \mathcal{E} \to \mathcal{F}$ is zero.

Proof. Assume $\mathrm{Im}(f) \neq 0$. Since $\mathcal{E}$ is semistable $\mu(\mathrm{Im}(f)) \geq \mu(\mathcal{E}) > \mu_{\max}(\mathcal{F})$ by Lemma 1.13. So $\mu(\mathrm{Im}(f)) > \mu_{\max}(\mathcal{F})$, a contradiction. □

Proposition 1.17. Let $\mathcal{E}$ and $\mathcal{F}$ be two stable vector bundles on X with $\mu(\mathcal{E}) = \mu(\mathcal{F})$ then $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = \{0\}$ or $\mathcal{E} \cong \mathcal{F}$

Proof. Suppose that $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \neq \{0\}$ and let $\alpha : \mathcal{E} \to \mathcal{F}$ be a non trivial morphism, then $\mathrm{Im}(\alpha)$ is a non trivial sub-bundle of $\mathcal{F}$. We have a surjection $\mathcal{E} \to \mathrm{Im}(\alpha)$ so by semistability of $\mathcal{E}$ we have $\mu(\mathcal{E}) \leq \mu(\mathrm{Im}(\alpha))$ and by semistability of $\mathcal{F}$ we have $\mu(\mathrm{Im}(\alpha)) \leq \mu(\mathcal{F})$ then $\mu(\mathcal{E}) = \mu(\mathrm{Im}(\alpha)) = \mu(\mathcal{F})$. As $\mathcal{E}$ and $\mathcal{F}$ are stable the only possiblity is $\mathcal{F} \cong \mathrm{Im}(\alpha) \cong \mathcal{E}$ □

Proposition 1.18. Let $\mathcal{E}$ be a stable vector bundle over X. Then $\mathcal{E}$ is simple i.e $\mathrm{End}(\mathcal{E}) = k$.

Proof. Suppose that there exists an endomorphism α which is not a scalar multiple of the identity. Then for some $x \in X(k)$ we have the restricted morphism $\alpha_x \in \mathrm{End}(E \otimes k(x))$ wich is not a scalar multiple of the identity. Take any eigenvalue $\lambda \in k$ of α_x, then $\alpha - \lambda Id_E$ is not surjective. We deduce by the Proposition 1.17 that $\alpha - \lambda Id_E = 0$ wich is a contradiction. □

Corollary 1.19. Let $\mathcal{E}$ be a stable vector bundle over X. Then $\mathcal{E}$ is indecomposable.

Proof. If $\mathcal{E}$ is decomposable then it has non-trivial endomorphisms, given by different homotheties on each factors. □

Remark 1.20. 1. Let $\mathcal{E}$ be a vector bundle on X and let $\mathcal{E}_1 \subseteq \mathcal{E}$ be a proper subsheaf which is a vector bundle of the same rank. Then $\deg(\mathcal{E}_1) < \deg(\mathcal{E})$ and therefore $\mu(\mathcal{E}_1) < \mu(\mathcal{E})$. To prove this, we can replace $\mathcal{E}$ and $\mathcal{E}_1$ by their top exterior powers by Lemma 1.5 and thereby reduce to the case where $\mathcal{E}$ and $\mathcal{E}_1$ are line bundles, in which case the result is obvious.

2. Let $\mathcal{E}$ be a vector bundle on X which is semistable of slope μ and let $\mathcal{E}_1 \subseteq \mathcal{E}$ be a coherent subsheaf. Then $\mathcal{E}_1$ is also a vector bundle, but not necessarily a vector sub-bundle since the quotient $\mathcal{E}/\mathcal{E}_1$ might not be a vector bundle. However, $\mathcal{E}_1$ is always contained in a vector sub-bundle $\tilde{\mathcal{E}_1} \subseteq \mathcal{E}$ of the same rank. Using the previous Remark we obtain

$$\mu(\mathcal{E}_1) \leq \mu(\tilde{\mathcal{E}_1}) \leq \mu$$

Moreover, the first inequality is strict if $\mathcal{E}_1$ is not a subbundle of $\mathcal{E}$.

Proposition 1.21. Let $\mathcal{E}$ be a vector bundle on X. Then there is a unique semistable sub-bundle $\mathcal{E}_1 \subseteq \mathcal{E}$ such that for all sub-bundles $\mathcal{F} \subseteq \mathcal{E}$

1. $\mu(\mathcal{F}) \leq \mu(\mathcal{E}_1)$
2. $\mu_{\max}(\mathcal{E}) > \mu_{\max}(\mathcal{E}/\mathcal{E}_1)$

This sub-bundle is called the **destabilising bundle** of $\mathcal{E}$.

Proof. By Lemma 1.9 there existe $n_0 \in \mathbb{N}$ such that for all sub-bundle $\mathcal{F}$ of $\mathcal{E}$ we have $\mu(\mathcal{F}) \leq n_0$ i.e $\{\mu(\mathcal{F}) \mid \mathcal{F} \subseteq \mathcal{E}, \mathcal{F} \neq 0\}$ is bounded. Since the possible slopes of sub-bundles of $\mathcal{E}$ lie in the set

$$\{\frac{d}{r} : d \in \mathbb{Z}, 1 \leq r \leq \mathrm{rk}(\mathcal{E})\}$$

Then $\{\mu(\mathcal{F}) \mid \mathcal{F} \subseteq \mathcal{E}, \mathcal{F} \neq 0\}$ is a discrete subset of $\mathbb{R}$ so the sup is attained i.e there exist a sub-bundle $\mathcal{E}_1$ such that $\mu_{\max}(\mathcal{E}) = \mu(\mathcal{E}_1)$. Then it's immediate that for sub-bundle $\mathcal{F} \subseteq \mathcal{E}$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E}_1)$. We can choose $\mathcal{E}_1$ to be a sub-bundle of maximal rank among sub-bundles of slope $\mu_{\max}(\mathcal{E})$.
If $\mathcal{F}$ is a sub-bundle of $\mathcal{E}_1$ it's a sub-bundle of $\mathcal{E}$ then by 1. $\mu(\mathcal{F}) \leq \mu(\mathcal{E}_1)$ and $\mathcal{E}_1$ is semistable. Consider now a sub-bundle $\mathcal{E}_2$ of $\mathcal{E}$, strictly containing $\mathcal{E}_1$ and such that $\mu(\mathcal{E}_2/\mathcal{E}_1) = \mu_{\max}(\mathcal{E}/\mathcal{E}_1)$. Since the sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_2/\mathcal{E}_1 \to 0$$

is exact, $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1)$ if and only if $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2)$ by Lemma 1.11. But $\mathcal{E}_2$ is a sub-bundle of $\mathcal{E}$, so $\mu(\mathcal{E}_2) \leq \mu_{\max}(\mathcal{E}) = \mu(\mathcal{E}_1)$. Since $\mu(\mathcal{E}_2) = \mu(\mathcal{E}_1)$ would contradict the maximality of $\mathrm{rk}(\mathcal{E}_1)$ for sub-bundles of $\mathcal{E}$ having slope $\mu_{\max}(\mathcal{E})$, one has $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2)$, which implies that $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1)$, i.e. $\mu_{\max}(\mathcal{E}) > \mu_{\max}(\mathcal{E}/\mathcal{E}_1)$.

Let $\mathcal{F}_1$ be another sub-bundle of maximal rank among sub-bundles of maximal slope and consider the following sequence

$$0 \to \mathcal{F}_1 \to \mathcal{E} \to \mathcal{E}/\mathcal{E}_1$$

We know that $\mathcal{F}_1$ is semistable and $\mu(\mathcal{F}_1) = \mu_{\max}(\mathcal{E}) > \mu_{\max}(\mathcal{E}/\mathcal{E}_1)$, Lemma 1.16 shows that the composed map $\mathcal{F}_1 \to \mathcal{E} \to \mathcal{E}/\mathcal{E}_1$ is zero. So $\mathcal{F}_1 \subseteq \mathcal{E}_1$, therefore $\mathcal{F}_1 = \mathcal{E}_1$ □

Proposition 1.22. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two semistable vector bundles over X with $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2) = \mu$, and $\mathcal{E}$ an extension of $\mathcal{E}_2$ by $\mathcal{E}_1$. Then $\mathcal{E}$ is semistable of slope μ.

Proof. We have the following short exact sequence.

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

By Lemma 1.11 we have that $\mu(\mathcal{E}) = \mu$. Let $\mathcal{F}$ be a proper sub-bundle of $\mathcal{E}$ and denote by $\mathcal{F}_1$ the destabilising bundle of $\mathcal{F}$. Suppose that $\mu(\mathcal{F}) > \mu$ then $\mu(\mathcal{F}_1) > \mu$, by Proposition 1.15 we have that $\mathrm{Hom}(\mathcal{F}_1, \mathcal{E}_2) = \{0\}$. We deduce that $\mathcal{F}_1$ is a sub-bundle of $\mathcal{E}_1$ wich is a contradiction since $\mathcal{E}_1$ is semistable. □

Consider a rational number μ and let $C(\mu)$ denote the category of semistable vector bundles of slope μ on X.

Proposition 1.23. The category $C(\mu)$ is abelian

Proof. This category is an additive sub-category of the category of vector bundles because it has direct sums by Proposition 1.22.
Consider a non-zero map $f : \mathcal{E} \to \mathcal{F}$ of semistable bundles of slope μ. By semistablity of $\mathcal{E}$ and $\mathcal{F}$ we have that $\mu(\mathrm{Im}(f)) = \mu$, by Remark 1.20 it is forced to be a sub-bundle of $\mathcal{F}$, so f has constant rank and the kernel and cokernel are vector bundles. We have two exact sequences

$$0 \to \ker(f) \to \mathcal{E} \to \mathrm{Im}(f) \to 0$$

$$0 \to \mathrm{Im}(f) \to \mathcal{F} \to \mathrm{coker}(f) \to 0$$

Applying Lemma 1.11 we conclude that $\ker(f)$ and $\mathrm{coker}(f)$ also have slope μ. Every sub-bundle of $\ker(f)$ can also be regarded as a sub-bundle of $\mathcal{E}$, and therefore has slope smaller than μ by virtue of our assumption that $\mathcal{E}$ is semistable. This proves that $\ker(f)$ is semistable of slope μ. We claim that $\mathrm{coker}(f)$ is also semistable of slope μ. Assume otherwise: then there exists a sub-bundle $\mathcal{F}_1 \subseteq \mathrm{coker}(f)$ of slope $> \mu$. Let $\mathcal{F}_0$ be the inverse image of $\mathcal{F}_1$ in $\mathcal{F}$, so that we have an exact sequence

$$0 \to \mathrm{Im}(f) \to \mathcal{F}_0 \to \mathcal{F}_1 \to 0$$

Applying Lemma 1.11, we deduce that $\mu(\mathcal{F}_0) > \mu$, contradicting the semistability of $\mathcal{F}$.

□

1.5 Finite bundle

Let $\mathbb{K}(X)$ be the Grothendieck ring associated to the additive monoid $\mathbb{VB}(X)$ corresponding to the isomorphism classes of vector bundles over X. The Krull-Schmidt-Remak theorem holds [Ati56], in particular, $[W]$, where W runs through all indecomposable vector bundles on X, form a free basis for $\mathbb{K}(X)$.

Definition 1.24. For a vector bundle $\mathcal{E}$, $S(\mathcal{E})$ is the collection of all the indecomposable components of $\mathcal{E}^{\otimes n}$, for all non-negative integers n.

Lemma 1.25. *Let $\mathcal{E}$ be a vector bundle on X. The following are equivalent :*

1. *$[\mathcal{E}]$is integral over $\mathbb{Z}$ in $\mathbb{K}(X)$.*
2. *There are polynomials P and Q with non-negative integer coefficients, such that $P(\mathcal{E}) \cong Q(\mathcal{E})$, and $P \neq Q$.*
3. *$S(\mathcal{E})$ is finite.*

Proof.

1. (1) $\Rightarrow$ (2) : Let $R \in \mathbb{Z}[T]$ unitary such that $R([\mathcal{E}]) = 0$ and $R \neq 0$. Choose $P, Q \in \mathbb{Z}[T]$ such that P and Q have non-negative coefficient, and $R = P - Q$. Then $P([\mathcal{E}]) = Q([\mathcal{E}])$ in $\mathbb{K}(X)$, but $\mathbb{VB}(X)$ as a monoid, has the cancellation property so it follows that $P(\mathcal{E}) \cong Q(\mathcal{E})$.
2. (2) $\Rightarrow$ (1) : We write $R = P - Q$, then $R([\mathcal{E}]) = 0$.
3. (1) $\Rightarrow$ (3) : Let $R \in \mathbb{Z}[T]$ unitary such that $R([\mathcal{E}]) = 0$ and $R \neq 0$. Let d be the degree of R, then any member of $S(\mathcal{E})$ is an indecomposable componant of $\mathcal{E}^{\otimes r}$ for $0 \leq r \leq d-1$, then $S(\mathcal{E})$ is finite.
4. (3) $\Rightarrow$ (1) : Suppose that $S(\mathcal{E})$ is finite and consider the free additive subgroup $G \subseteq \mathbb{K}(X)$ with basis $S(\mathcal{E})$. Consider the $\mathbb{Z}$-linear map $m_{\mathcal{E}} : \mathbb{K}(X) \to \mathbb{K}(X)$

define by $[\mathcal{F}] \mapsto [\mathcal{F} \otimes \mathcal{E}]$, then we have $m_{\mathcal{E}}(G) \subseteq G$. The characteristic polynomial χ is a monic polynomial in $\mathbb{Z}[T]$ with $\chi(m_{\mathcal{E}}) = 0$ as an endomorpism of $\mathbb{K}(X)$. We obtain $\chi(m_{\mathcal{E}})([\mathcal{O}_X]) = \chi([\mathcal{E}]) = 0$

□

Definition 1.26. A vector bundle $\mathcal{E}$ on X is said to be **finite** if it satisfies any of the equivalent hypothesis of previous lemma

Lemma 1.27. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two finite vector bundles on X then $\mathcal{E}_1 \oplus \mathcal{E}_2$ and $\mathcal{E}_1 \otimes \mathcal{E}_2$ are finite.*

Proof. This is obvious because integral elements form a ring □

Lemma 1.28. *Let $\mathcal{E}$ be a vector bundle on X, then $\mathcal{E}$ is finite if and only if $\mathcal{E}^*$ is finite.*

Proof. If P and Q are polynomials with non-negative integer we have $P(\mathcal{E}) \cong Q(\mathcal{E})$ if and only if $P(\mathcal{E}^*) \cong Q(\mathcal{E}^*)$. □

Lemma 1.29. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two vector bundles on X such that $\mathcal{E}_1 \oplus \mathcal{E}_2$ is finite then $\mathcal{E}_1$ and $\mathcal{E}_2$ are finite.*

Proof. The result follow from the inclusion $S(\mathcal{E}_i) \subseteq S(\mathcal{E}_1 \oplus \mathcal{E}_2)$. □

Proposition 1.30. Let $\mathcal{L}$ be a line bundle on X, then $\mathcal{L}$ is finite if and only if $\mathcal{L}^{\otimes m} \cong \mathcal{O}_X$ for some $m > 0$.

Proof. Suppose that $\mathcal{L}^{\otimes m} \cong \mathcal{O}_X$ for some $m > 0$ then $\mathcal{L}$ is finite if we consider $P = T^m$ and $Q = 1$. Conversely if we suppose $\mathcal{L}$ finite then $S(\mathcal{L})$ is a finite set and $S(\mathcal{L})$ consists of $\mathcal{L}^{\otimes r}$ for $r \geq 0$. The result follows because all tensor powers $\mathcal{L}^{\otimes r}$ are invertible and invertible sheaves are indecomposable. □

Theorem 1.31. *Let $\mathcal{E}$ be a finite vector bundle on X, then $\mathcal{E}$ is semistable of slope 0.*

Proof. By Lemma 1.10 and the defintion of $S(\mathcal{E})$ we have $\mu(\mathcal{F}) \leq \sup\{\mu_{max}(\mathcal{F}) \mid \mathcal{F} \in S(\mathcal{E})\} =: R(\mathcal{E})$, for every locally free sheaf $\mathcal{F}$ that is a subbundle of some $\mathcal{E}^{\otimes i}$. For every $j > 0$ we have $\mu(\mathcal{F}^{\otimes j}) = j\mu(\mathcal{F})$ by the Formula 1.8. On the other hand, it is a subbundle of $\mathcal{E}^{\otimes ij}$, so $\mu(\mathcal{F}^{\otimes j}) \leq R(\mathcal{E})$, this is only possible if $\mu(\mathcal{F}) \leq 0$ for all $\mathcal{F}$. In particular we have $\mu(\mathcal{E}) \leq 0$, but the dual $\mathcal{E}^*$ is again finite so $\mu(\mathcal{E}^*) \leq 0$ must hold as well. Since $\mu(\mathcal{E}^*) = -\mu(\mathcal{E})$, this shows $\mu(\mathcal{E}) = 0$, and therefore also $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for all subbundles $\mathcal{F} \subseteq \mathcal{E}$. □

Definition 1.32. A vector bundle $\mathcal{E}$ of rank r on X is said **isotrivial** if there exists an étale cover $f : Y \to X$ such that $f^*\mathcal{E} \cong \mathcal{O}_Y^{\oplus r}$

Theorem 1.33. *Suppose that $char(k) = 0$ and let $\mathcal{E}$ be a vector bundle of rank r on X which is isotrivial, then $\mathcal{E}$ is finite.*

Proof. By definition there exist an étale cover $f : Y \to X$ such that $f^*\mathcal{E} \cong \mathcal{O}_Y^{\oplus r}$. We consider $\mathcal{A} = f_*(\mathcal{O}_Y)$, this is an $\mathcal{O}_X$-algebra and we have the existence of trace morphism $Tr_{\mathcal{A}/\mathcal{O}_X} : \mathcal{A} \to \mathcal{O}_X$. Such an $\mathcal{O}_X$-linear morphism allows us to find a section of $\mathcal{O}_X$-algebra $\mathcal{O}_X \hookrightarrow \mathcal{A}$ as the caracteristic is zero.
So each $\mathcal{E}^{\otimes n}$ is a direct summand of $\mathcal{A}^{\oplus l}$. Hence, the indecomposable coherent $\mathcal{O}_X$-modules appearing in $\mathcal{E}^{\otimes n}$ are isomorphic to certain indecomposable components of $\mathcal{A}$ by the uniqueness of the Krull-Schmidt-Remak theorem. Then $S(\mathcal{E})$ is finite since $\mathcal{A}$ has finitely many indecomposable components, we conclude that $\mathcal{E}$ is finite. □

1.6 Monodromy of finite bundle in the complex case

Let $X \to \operatorname{Spec}(\mathbb{C})$ be a smooth projective and connected curve over complex numbers, we will identify X with $X(\mathbb{C})$ the compact Riemann surface associated to X.

Let $\pi : \tilde{X} \longrightarrow X$ be the universal cover of X , the group $\pi_1(X)$ acts freely over $\tilde{X}$ and $X \cong \tilde{X}/\pi_1(X)$.
Let $\rho : \pi_1(X) \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-linear representation, this representation induces a linear action of $\pi_1(X)$ over the vector bundle $\tilde{X} \times V$ define by

$$\gamma.(\tilde{x}, v) = (\gamma.\tilde{x}, \rho(\gamma)(v))$$

We write E_ρ the holomorphic vector bundle on X, quotient of $\tilde{X} \times V$ by the previous action and we write $\mathcal{E}_\rho$ the associated locally free $\mathcal{O}_X$-module.

We have a lot of compatibility with algebraic operations :

Proposition 1.34. Let ρ_1 and ρ_2 two representations of $\pi_1(X)$, then we get :

- $E_{\rho_1 \oplus \rho_2} \cong E_{\rho_1} \oplus E_{\rho_2}$
- $E_{\rho_1 \otimes \rho_2} \cong E_{\rho_1} \otimes E_{\rho_2}$
- $\Lambda^n E_\rho \cong E_{\Lambda^n \rho}$ for every $n > 0$.

- $Sym^n E_\rho \cong E_{Sym^n \rho}$ for every $n > 0$.
- $E_\rho^* \cong E_{{}^t\rho^{-1}}$
- If $\rho_1 \cong \rho_2$ then $E_{\rho_1} \cong E_{\rho_2}$

Proof. See [Ses82, Proposition 34 page 39] □

By the last and first items of the last proposition we get a morphism of monoids

$$\mathrm{Rep}_{\mathbb{C}}(\pi_1(X)) \longrightarrow \mathbb{VB}(X) \longrightarrow \mathbb{K}(X)$$

define by $[(\rho, V)] \mapsto [\mathcal{E}_\rho] \mapsto [(\mathcal{E}_\rho, 0)]$ and by the universal property of the Grothendieck group we get a morphism of groups

$$\varphi : \mathbb{K}(\mathrm{Rep}_{\mathbb{C}}(\pi_1(X))) \longrightarrow \mathbb{K}(X)$$

This morphism of groups is in fact a morphism of rings by the second item of the previous proposition.

Proposition 1.35. Let $\rho : \pi_1(X) \longrightarrow \mathrm{GL}(V)$ be a representation such that the image is finite. Then $\mathcal{E}_\rho$ is a finite vector bundle on X.

Proof. Let $G = \mathrm{Im}(\rho)$, since G is a finite group, any complex G-module is completely reducible, and furthermore, there are only finitely many isomorphism classes of irreducible G-modules, then by a similar argument as for finite bundle, $[(\rho, V)]$ is integral over $\mathbb{Z}$ in $\mathbb{K}(\mathrm{Rep}_{\mathbb{C}}(\pi_1(X)))$, we deduce that $[\mathcal{E}_\rho]$ is integral over $\mathbb{Z}$ in $\mathbb{K}(X)$ because image of integral elements by a morphism of rings is integral, so $\mathcal{E}_\rho$ is a finite vector bundle over X. □

Theorem 1.36. *A holomorphic vector bundle $\mathcal{E}$ which is indecomposable on X come from an irreducible representation ρ of $\pi_1(X)$ if and only if $deg(\mathcal{E}) = 0$.*

Proof. See Atiyah [Ati57] or Weil [Wei38]. □

Theorem 1.37. *Let $\mathcal{E}$ be a finite vector bundle on X then there exists a representation $\rho : \pi_1(X) \longrightarrow GL(V)$ such that $G = Im(\rho)$ is finite and $\mathcal{E}_\rho \cong \mathcal{E}$.*

Proof. Let $\mathcal{E}$ be a finite vector bundle over Y, by Lemma 1.29 we can suppose that $\mathcal{E}$ is indecomposable, we know that $\mu(\mathcal{E}) = 0$ so $\deg(\mathcal{E}) = 0$, then by Theorem 1.36 there exist an irreducible representation $\rho : \pi_1(X) \longrightarrow \mathrm{GL}(V)$ such that $\mathcal{E}_\rho \cong \mathcal{E}$.

By the Selberg's Lemma ([Alp87]) it suffices to show that every element of $G = \mathrm{Im}(\rho)$ is torsion.
Let $g \in G$, we know that there are polynomials P and Q with non-negative integer coefficients, such that $P(\mathcal{E}) \cong Q(\mathcal{E})$, and $P \neq Q$, or for every $x \in X$ the fiber $\mathcal{E}_x \cong V$ so $P(V) \cong Q(V)$ as G-modules. We deduce that $S(V)$ is finite and there are finitely many G-modules, say $V_1, ..., V_k$, such that every $V^{\otimes j}$ admits a decomposition as G-modules

$$V^{\otimes j} = \sum_{i=1}^{k} a_{i,j} V_i$$

where $a_{i,j}$ are non-negative integers.
So there are finitely many complex numbers, say $(\lambda_i)_{1 \leq i \leq N}$, such that all the eigenvalues for the action of g on any $V^{\otimes j}$ are contained in $(\lambda_i)_{1 \leq i \leq N}$. Since the i-th power of an eigenvalue for the action of g on V becomes an eigenvalue for the action of g on $V^{\otimes i}$, it is that all the eigenvalues for the action of g on V must be roots of unity. So for some $N > 0$ we have that g^N as 1 for only eigenvalue, we claim that $g^N = Id$.
Let A_n denote the $(n+1) \times (n+1)$ matrix whose (i, j)-th entry is 1 if $i = j$ or $i+1 = j$ and 0 otherwise. Then A_n has a standard action on $\mathbb{C}^{n+1}$ which is indecomposable. We will denote $\mathbb{C}^{n+1}$ equipped with the action of A_n by V_n. If $n \geq m$ then the Jordan canonical form of $A_n \otimes A_m$ has the form

$$A_{n+m} \oplus A_{n+m-2} \oplus ... \oplus A_{n-m}$$

Indeed, let W denote the standard two dimensional representation of $\mathrm{SL}(2, \mathbb{C})$, and let $\mathrm{Sym}^n(W)$ be its n-th symmetric power. Denote by A the element

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of $\mathrm{SL}(2, \mathbb{C})$. For the representation $\mathrm{Sym}^n(W)$ of $\mathrm{SL}(2, \mathbb{C})$, there exists a basis with respect to which A acts by the matrix A_n. The assertion now follows from the fact that for $n \geq m$, the tensor product $\mathrm{Sym}^n(W) \otimes \mathrm{Sym}^m(W)$ is equivalent to

$$\mathrm{Sym}^{n+m}(W) \oplus \mathrm{Sym}^{n+m-2}(W) \oplus ... \oplus \mathrm{Sym}^{n-m+2}(W) \oplus \mathrm{Sym}^{n-m}(W)$$

as $\mathrm{SL}(2, \mathbb{C})$-modules, see [FH91, page 151]
The indecomposable components which occur in V for the action of g^N are of the form V_n for some n. If $g^N \neq Id$, then not all such n are zero. Since there are only finitely many indecomposable components occurring in all tensor powers of V , we conclude that there will only be finitely many indecomposable components occurring

in all tensor powers of V_n for the action of A_n. On the other hand, the assertion about the Jordan canonical form of $A_n \otimes A_m$ implies that V_{kn} occurs as an indecomposable component for the action of $A_n^{\otimes k}$ on $V_n^{\otimes k}$. Hence V_{kn} occurs as a indecomposable component of $V_n^{\otimes k}$ for all k, which is a contradiction. $\square$

Proposition 1.38. Let $\mathcal{E}$ be a finite vector bundle of rank r on X then $\mathcal{E}$ is isotrivial.

Proof. By the previous theorem there exist a representation $\rho : \pi_1(X) \longrightarrow \mathrm{GL}(V)$ such that $\mathrm{Im}(\rho)$ is finite and $\mathcal{E}_\rho \cong \mathcal{E}$. Let $G = \mathrm{Ker}(\rho)$, we know that $\pi_1(X)/G \cong \mathrm{Im}(\rho)$ is finite so G correspond to a Galois étale cover $f : Z \longrightarrow X$, it is obvious by the definition of $\mathcal{E}_\rho$ that $f^*\mathcal{E} \cong \mathcal{O}_Z^{\oplus r}$. $\square$

Corollary 1.39. There are a bijection between :

1. Equivalence classes of finite vector bundle on X.
2. Isotrivial vector bundle on X.
3. Vector bundle on X coming from a representation of $\pi_1(X)$ with finite image.

Example 1.40. Let X be a smooth projective curve of genus 2 and D_3 be the dihedral group of order 6, if α is 3-root of unity, the elements of the group can be represented in $\mathrm{GL}_2(\mathbb{C})$ by $\begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^{-k} \end{pmatrix}$ and $\begin{pmatrix} 0 & \alpha^{-k} \\ \alpha^k & 0 \end{pmatrix}$ with $0 \leq k \leq 2$.
Let $A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ $A_2 \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^{-2} \end{pmatrix}$ $B_1 = \begin{pmatrix} 0 & \alpha^{-1} \\ \alpha^1 & 0 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & \alpha^{-2} \\ \alpha^2 & 0 \end{pmatrix}$.
We check that $A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1} = I_2$ then we get a morphism

$$\rho : \pi_1(X) \to D_3 \hookrightarrow \mathrm{GL}_2(\mathbb{C})$$

Finally $\mathcal{E}_\rho$ is a finite bundle on X.

1.7 Torsors

Let G be an affine group scheme define over k.

Definition 1.41. A **G-torsor** over X is a scheme P over X with an action of G such that there exists an fppf cover, $(U_i \to X)_{i\in I}$ such that for each $i \in I$ there is a $G|_{U_i}$-equivariant isomorphism $P|_{U_i} \cong G|_{U_i}$

Proposition 1.42. Let G acts on the right on a X-scheme $j : P \to X$. The following are equivalent :

1. P is a G-torsor.
2. The scheme P is faithfully flat and locally of finite-type over X, and $(p_1, a) : P \times_X G \to P \times_X P$ given by $(p, g) \mapsto (p, p.g)$ is an isomorphism.

Proof. This is [Mil80, page 120] □

When G is a finite group, a G-torsor naturally gives a finite vector bundle.

Lemma 1.43. Let $j : P \to X$ be a G-torsor over X with G a finite group scheme. The vector bundle $\mathcal{E}_P = j_*\mathcal{O}_P$ is a finite vector bundle on X.

Proof. Consider the isomorphism $P \times_X G_X \to P \times_X P$ given by the previous Proposition. The $\mathcal{O}_X$-algebra corresponding to the left hand side is isomorphic as an $\mathcal{O}_X$-module to $\mathcal{E}_P^{\oplus n}$ where n is the order of the group $G(k)$. The right hand side corresponds to $\mathcal{E}_P^{\otimes 2}$, whence an isomorphism $\mathcal{E}_P^{\oplus n} \cong \mathcal{E}_P^{\otimes 2}$ and $\mathcal{E}_P$ is finite. □

Let Y be any quasi projective G-variety and let P be a G-torsor over X. For example Y could be a G-module. Then we denote by $P(Y)$ the associated bundle with fibre type Y which is the following object: $P(Y) = (P \times Y)/G$ for the twisted action of G on $P \times Y$ given by $g.(p, y) = (p.g, g^{-1}.y)$.
Any G-equivariant map $f : Y \to Z$ will induce a morphism $P(f) : P(Y) \to P(Z)$
A section $s : X \to P(Y)$ is given by a morphism $\tilde{s} : P \to Y$ such that $\tilde{s}(p.g) = g^{-1}\tilde{s}(p)$ and $s(x) = (p, \tilde{s}(p))$ where $p \in P$ is such that $j(p) = x$, where $j : P \to X$

Definition 1.44. Let H be an affine group scheme. If $\varphi : H \to G$ is a morphism of groups and Q is a H-torsor over X, the associated bundle $Q(G)$, for the action of H on G by left multiplication through φ, is naturally a G–torsor. We denote this G–torsor often by φ_*Q and we say this torsor is obtained from Q by extension of structure group.

Remark 1.45. In the special case when $\varphi : G \to \mathrm{GL}_V$ is a representation of G, we denote φ_*P by V_P

Definition 1.46. Let H be an affine group scheme and $j : P \to X$ a G-torsor. We say that P **admit a reduction to H** if there exist a group morphism $\varphi : H \to G$ and an H-torsor $f : Q \to X$ such that $P \cong \varphi_*Q$

Remark 1.47. It is easy to see that the pullback of P along f is the trivial G-torsor.

M.V.Nori (see [Nor76]) gives an alternative description of G-torsors, which I briefly recall. Let $\mathrm{Rep}_k(G)$ denote the category of all finite dimensional representations of an affine group scheme G, or equivalently, left G-modules. By a G-module (or representation) we shall always mean a left G-module (or a left representation). Given a G-torsor P over X and a left G-module V, the associated vector bundle is denoted by $P(V)$. Consider the functor

$$F(P) : \mathrm{Rep}_k(G) \to \mathrm{Bun}_X$$

wich sends any V to the vector bundle $P(V)$ and sends any morphism between two G-modules to the naturally induced morphism between the two corresponding vector bundles. The functor $F(P)$ enjoys several natural abstract properties. For example, it is compatible with the algebra structures of $\mathrm{Rep}_k(G)$ and Bun_X defined using direct sum and tensor product operations. Furthermore, $F(P)$ takes an exact sequence of G-modules to an exact sequence of vector bundles, it also takes the trivial G-module to the trivial line bundle on X, and the dimension of V also coincides with the rank of the vector bundle $F(P)(V)$. Nori proves that the collection of G-torsors over X is in one-to-one correspondence with the collection of functors F from $\mathrm{Rep}_k(G)$ to Bun_X satisfying the following properties:

1. Strict : a morphism of vector bundles is said to be strict if its cokernel is also locally free. Let $u : V \to W$ G-module map. Then we need the induced morphism $F(u) : F(V) \to F(W)$ to be strict. In particular, $\mathrm{Ker}(F(u))$ and $\mathrm{Im}(F(u))$ are also locally free.
2. Exact : $\mathrm{Ker}(F(u)) = F(\mathrm{Ker}(u)), \mathrm{Coker}(F(u)) = F(\mathrm{Coker}(u))$.
3. Faithfull: $F(\mathrm{Hom}(V, W)) \hookrightarrow \mathrm{Hom}(F(V), F(W))$
4. Tensor functor : $F(V \otimes W) = F(V) \otimes F(W)$ and $F(trivial) = \mathcal{O}_X$
5. The functor F_x (defined by $F_x(V) = F(V)_x$) is a fibre functor on the category $\mathrm{Rep}_k(G)$

1.8 Essentially finite bundle

Definition 1.48. A vector bundle $\mathcal{E}$ is essentially finite if it is semistable of slope 0, and moreover there is a finite bundle $\mathcal{G}$ and semistable degree 0 subbundles $\mathcal{F}_2 \subseteq \mathcal{F}_1 \subseteq \mathcal{G}$ such that $\mathcal{E} = \mathcal{F}_1/\mathcal{F}_2$.

We will denote by $\mathrm{EF}(X)$ the full subcategory of $C(0)$ consisting of essentially finite vector bundles.

Remark 1.49. 1. Let $\mathcal{G}$ be a finite vector bundle and $\mathcal{E}$ be a sub-bundle of $\mathcal{G}$ with slope 0. Then $\mathcal{E}$ is essentially finite.

2. Let $\mathcal{G}$ be a finite vector bundle and $\mathcal{E}$ be a sub-bundle of $\mathcal{G}$ with slope 0. Then $\mathcal{G}/\mathcal{E}$ is essentially finite.

3. By definition the category $\mathrm{EF}(X)$ is abelian.

Lemma 1.50. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two essentially finite vector bundles over X then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is essentially finite.*

Proof. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two essentially finite vector bundles over X which are subquotients of the finite bundle $\mathcal{G}_1$ and $\mathcal{G}_2$, so we have $\mathcal{E}_1 = \mathcal{F}_1^{'}/\mathcal{F}_1^{''}$ with $\mathcal{F}_1^{''} \overset{\iota_1}{\hookrightarrow} \mathcal{F}_1^{'} \hookrightarrow \mathcal{G}_1$ and $\mathcal{E}_2 = \mathcal{F}_2^{'}/\mathcal{F}_2^{''}$ with $\mathcal{F}_2^{''} \overset{\iota_2}{\hookrightarrow} \mathcal{F}_2^{'} \hookrightarrow \mathcal{G}_2$.
We see that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is isomorphic to a quotient of $\mathcal{F}_1^{'} \otimes \mathcal{F}_2^{'}$ by $I = \mathrm{Im}(\iota_1) + \mathrm{Im}(\iota_2)$, so it is a subquotient of $\mathcal{G}_1 \otimes \mathcal{G}_2$ wich is finite by Lemma 1.27.
We have to show that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is semistable of slope 0, by 1.8 the slope is 0. We have that $\mathcal{F}_1^{'} \otimes \mathcal{F}_2^{'}$ is a sub-bundle of $\mathcal{G}_1 \otimes \mathcal{G}_2$ with slope 0 by formula 1.8, then it's semistable. We have the following exact sequence

$$0 \to I \to \mathcal{F}_1^{'} \otimes \mathcal{F}_2^{'} \to \mathcal{E}_1 \otimes \mathcal{E}_2 \to 0$$

Then using 1.3 we have that $\mu(I) = 0$ then I it's semistable and using the fact that $C(0)$ is an abelian category we deduce that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is semistable. □

Lemma 1.51. Let $\mathcal{E}$ be an essentially finite vector bundle on X. Then $\mathcal{E}^*$ is an essentially finite vector bundle.

Proof. By definition $\mathcal{E}$ is a subquotient of a finite bundle $\mathcal{G}$, so we have $\mathcal{E} = \mathcal{F}_1/\mathcal{F}_2$ with $\mathcal{F}_2 \overset{\iota}{\hookrightarrow} \mathcal{F}_1 \hookrightarrow \mathcal{G}$. We have the following exact sequence,

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{E} \to 0$$

Then we get the following exact sequence

$$0 \to \mathcal{E}^* \to \mathcal{F}_1^* \to \mathcal{F}_2^* \to 0$$

By the Remark 1.49, $\mathcal{F}_1^*$ is essentially finite then $\mathcal{E}^*$ is essentially finite. □

Theorem 1.52. *Let $x : Spec(k) \to X$ be a k-rationnal point. Then the category $EF(X)$ is a neutral tannakian category over k.*

Proof. We know that $\mathrm{EF}(X)$ is abelian, using the Lemma 1.50 and 1.51 we have that $\mathrm{EF}(X)$ is tensor and rigid. Finally, consider the functor given by $\mathcal{E} \mapsto x^*\mathcal{E}$. It is a faithful exact tensor functor with values in the category of finite dimensional k-vector spaces, i.e. a fibre functor on $\mathrm{EF}(X)$. □

Definition 1.53. Let $x : Spec(k) \to X$ be a k-rationnal point. The **Nori fundamental group scheme** of X with base point x is the affine k-group scheme corresponding via Tannaka Duality to the neutral Tannakian category $\mathrm{EF}(X)$ and the fibre functor x^*. We denote it by $\pi_1^N(X, x)$.

For a subset S of $\mathrm{Ob}(\mathrm{EF}(X))$, let $S^* = \{\mathcal{E}^* \mid \mathcal{E} \in S\}$. Let $\tilde{S} = S \cup S^*$ and $\bar{S} = \{\mathcal{E}_1 \otimes ... \otimes \mathcal{E}_m \mid \mathcal{E}_i \in \tilde{S}\}$. Let $\mathrm{EF}(X, S)$ be the full sub tannakian category of $\mathrm{EF}(X)$, this determines an affine group scheme wich we call $\pi_1^N(X, S, x)$, such that

$$\alpha_S : \mathrm{EF}(X, S) \to \mathrm{Rep}_k(\pi_1^N(X, S, x))$$

is an equivalence of categories. Let F_S be the inverse of α_S; then F_S can be regarded as a functor from $\mathrm{Rep}_k(\pi_1^N(X, S, x))$ to Bun_X such that the composite $x^* \circ F_S$ is the forgetful functor. In particular there is a $\pi_1^N(X, S, x)$-torsor $\tilde{X}_S$ such that $F_S = F(\tilde{X}_S)$. The functor $x^* \circ F_S$ and $F(\tilde{X}_S|_x)$ coincide and there is a natural isomorphism of $\tilde{X}_S|_x$ with G, wich is equvalent to specifying a k-point $\tilde{x}_S$.
Now, if S is a subset of Q, there is a natural transformation of Tannaka categories from $\mathrm{EF}(X, S)$ to $\mathrm{EF}(X, Q)$ wich determines a natural morphism $\rho_S^Q : \pi_1^N(X, Q, x) \to \pi_1^N(X, S, x)$ by [DM82, Corollary 2.9] and it follows that $\tilde{X}_S$ is induced from $\tilde{X}_Q$ by the morphism ρ_S^Q.

Lemma 1.54. Let S be a finite collection of finite vector bundles. Then $\pi_1^N(X, S, x)$ is a finite group scheme.

Proof. Let $\mathcal{E}$ be the direct sum of all the members of S and their duals. Then $\mathcal{E}$ is a finite vector bundle by Lemma 1.27 and 1.28, then $S(\mathcal{E})$ is a finite set. We see that $S(\mathcal{E})$ generates the abelian category $\mathrm{EF}(X, S)$ in the sense of [Saa72, page 135] and therefore $\pi_1^N(X, S, x)$ is finite by [Saa72, page 156 4.3.2] □

Remark 1.55. In particular when $S = \{\mathcal{E}\}$ is just one finite vector bundle we will denote $\pi_1^N(X, S, x)$ by $\pi_1^N(X, \langle\mathcal{E}\rangle, x)$.

Proposition 1.56. Let S be any finite collection of essentially finite vector bundles. Then there is a G-torsor P on X with G a finite group scheme, such that the image of $F(P)$ contains the given collection S.

Proof. For each $\mathcal{E} \in S$, choose $\mathcal{G}$ such that $\mathcal{E}$ is a quotient of semistable bundle of $\mathcal{G}$, and $\mathcal{G}$ a finite vector bundle. Let Q be the collection of $\mathcal{G}$ as constructed. Note that S is a subset of $\mathrm{Ob}(\mathrm{EF}(X, Q))$.
Put $G = \pi_1^N(X, Q, x)$ and $P = \tilde{X}_Q$. By the previous Lemma, G is a finite group scheme and let α_Q be the equivalence of categories from $\mathrm{EF}(X, Q)$ to $\mathrm{Rep}_k(\pi_1^N(X, S, x))$. We know that $F(P) \circ \alpha_Q(\mathcal{G}) = \mathcal{G}$ for all object $\mathcal{G}$ of $\mathrm{EF}(X, Q)$. Thus proving the Proposition. □

Consider now the category $\mathrm{FTors}_{X,x}$ where object are triples (P, G, p), where G is a finite group scheme over k, P is a left G-torsor over X and p is a k-rational point in the fibre of P above x. A morphism $(P, G, p) \to (Q, G', q)$ in this category is given by a pair of morphisms $\varphi : G \to G'$, $\psi : P \to Q$ such that the G-action on P is compatible with the G'-action on Q , and moreover $\psi(p) = q$.

Definition 1.57. The $\pi_1^N(X, x)$-torsor $\tilde{X}$ is the **universal covering** of X.

The universal property possessed by $\pi_1^N(X, x)$ and $\tilde{X}$ is given by the following :

Theorem 1.58. *There is an equivalence of categories between $FTors_{X,x}$ and the category of finite group schemes G over k equipped with a k-group scheme morphism $\pi_1^N(X, x) \to G$.*

Proof. This is [Nor76, Proposition 3.11] □

As a direct consequence of Nori's work, we have a nice description of the essentially finite vector bundles:

Proposition 1.59. A vector bundle $\mathcal{E}$ on X is essentially finite if and only if there exists a finite k-group scheme G and a G-torsor $j : P \to X$ such that $j^*\mathcal{E} \cong \mathcal{O}_P^{rk(\mathcal{E})}$

Proof. This is Proposition 1.56 and Theorem 1.58. □

Corollary 1.60. In $\mathrm{char}(k) = 0$. A vector bundle is essentially finite if and only if he is finite.

Proof. It suffices to show that an essentially finite vector bundle is finite.
Let $\mathcal{E}$ an essentially finite vector bundle, then there exists a finite k-group scheme G and a G-torsor $j : P \to X$ such that $j^*\mathcal{E} \cong \mathcal{O}_P^{rk(\mathcal{E})}$. In $\mathrm{char}(k) = 0$ by a Theorem of Cartier G is étale then j is étale by [Mil80, Proposition 4.2], so $\mathcal{E}$ is finite. □

1.9 Frobenius Periodic Bundle

Suppose first that $\mathrm{char}(k) = p > 0$ and let σ_X denote the absolute Frobenius of X.

Definition 1.61. A vector bundle $\mathcal{E}$ is **Frobenius periodic** if $\mathcal{E} \cong (\sigma_X^n)^*\mathcal{E}$ for some $n \geq 1$.

The following Proposition is due to Lange and Stuhler:

Proposition 1.62. If vector bundle $\mathcal{E}$ is Frobenius periodic then $\mathcal{E}$ is isotrivial. If $\mathcal{E}$ is an stable isotrivial vector bundle then $\mathcal{E}$ is Frobienius periodic.

Proof. The first part is [LS77, Theorem 1.4] and the second part is [BD07, Theorem 1.5] □

Corollary 1.63. A Frobenius periodic vector bundle is essentially finite.

Proof. Let $\mathcal{E}$ be a Frobenius periodic vector bundle, then $\mathcal{E}$ is isotrivial, so there is a Galois cover trivializing $\mathcal{E}$, then by Proposition 1.59, $\mathcal{E}$ is essentially finite. □

2 Essentially Finite Torsors

Dans ce chapitre, nous abordons la stabilité et le degré dans le contexte des G-torseurs, notamment dans le cas particulier où G est le groupe linéaire GL_n, qui correspond au cas des fibrés vectoriels.

Nous introduisons la notion de G-torseurs essentiellement finis, qui se révèle être analogue à celle des fibrés vectoriels essentiellement finis. Nous démontrons qu'ils sont semi-stables et que leur degré est de torsion. Enfin nous en donnons une caractérisations Tannakienne.

This chapter presents results achieved through collaboration with Stefan Reppen in [GR23].

2.1 Stability and degree

Let k be an algebraically closed field and let $X \to \mathrm{Spec}(k)$ be a smooth projective and connected curve.
Let G be a connected, reductive group over k. Given a maximal torus $T \subset G$ let $X^*(T)$ denote the group of characters of T and let $X_*(T)$ denote the group of cocharacters. Let further $\Phi \subset X^*(T)$ denote the corresponding roots and let $\Phi^\vee \subset X_*(T)$ denote the corresponding coroots. We let $\pi_1(G)$ denote the algebraic fundamental group of G, namely,

$$\pi_1(G) = X_*(T)/\operatorname{span}\{\Phi^\vee\}. \tag{2.1}$$

Given a parabolic $P \subset G$ with Levi quotient L, let $\Phi_L^\vee \subset \Phi^\vee$ denote the coroots of L. We write $\pi_1(P) \coloneqq \pi_1(L)$.
Let $\mathcal{M}_G$ denote the stack of G-torsors over X, let $\mathcal{M}_G^{\mathrm{ss}}$ denote the substack of semistable G-torsors and let M_G^{ss} denote the moduli space of semistable G-torsors (see [Ram96a], [Ram96b] and [GLS$^+$08]). If we consider another curve, Y, then for clarity we may also write $\mathcal{M}_{G,Y}$ to denote the stack of G-torsors over Y. We define $\mathcal{M}_{G,Y}^{\mathrm{ss}}$ and $M_{G,Y}^{\mathrm{ss}}$ analogously.

Recall that the connected components of $\mathcal{M}_G$ are labeled by $\pi_1(G)$, that is,

$$\pi_0(\mathcal{M}_G) = \pi_1(G). \tag{2.2}$$

If $\check\lambda \in \pi_1(G)$, let $\mathcal{M}_G^{\check\lambda} \subset \mathcal{M}_G$ denote the corresponding component. Define similarly $\mathcal{M}_G^{\mathrm{ss},\check\lambda}$ and $M_G^{\mathrm{ss},\check\lambda}$ to be the components in $\mathcal{M}_G^{\mathrm{ss}}$ respectively M_G^{ss} corresponding to $\check\lambda$.

Definition 2.1. If F_G is an object of $\mathcal{M}_G^{\check{\lambda}}$, then F_G is said to be of **degree** $\check{\lambda}$.

We also have that $\pi_0(\mathcal{M}_P) \cong \pi_0(\mathcal{M}_L) = \pi_1(P)$ and we similarly say that a P-torsor is of degree $\check{\lambda}_P$ if it lies in the component corresponding to $\check{\lambda}_P$.

Lemma 2.2. Suppose $\varphi : G \to H$ is a morphism of smooth connected algebraic groups and let F_G be a G-torsor of degree 0. Then $\varphi_* F_G$ has degree 0.

Proof. By [Hof10] we have a commutative diagram of pointed sets

$$\begin{array}{ccc} \pi_1(G) & \longrightarrow & \pi_0(\mathcal{M}_G) \\ \downarrow & & \downarrow \\ \pi_1(H) & \longrightarrow & \pi_0(\mathcal{M}_H), \end{array} \tag{2.3}$$

where all morphisms are the natural ones induced by φ and where the left vertical map is a group morphism. The statement follows. □

Remark 2.3. In particular, if F_G is a G-bundle of degree 0 then $\deg V_{F_G} = 0$ for all representations V of G.

The center of G can be described as

$$Z(G) = \bigcap_{\alpha \in \Phi} \ker(\alpha) \subset T. \tag{2.4}$$

By composition via the inclusion $Z(G) \to T$ we have a natural map

$$X_*(Z(G)) \to X_*(T) \to \pi_1(G). \tag{2.5}$$

Upon tensoring with $\mathbb{Q}$ this induces an isomorphism $X_*(Z(G))_{\mathbb{Q}} \cong \pi_1(G)_{\mathbb{Q}}$. Following [Sch15] the definition of the slope map and subsequently the definition of a semistable G-torsor is as follows.

Definition 2.4. For a parabolic subgroup, P, such that $B \subset P \subset G$, with corresponding Levi L, the **slope map** $\phi_P : \pi_1(P) \to X_*(T)_{\mathbb{Q}}$ is the map given by

$$\phi_P : \pi_1(P) \to \pi_1(P)_{\mathbb{Q}} \cong X_*(Z(L))_{\mathbb{Q}} \to X_*(T)_{\mathbb{Q}}. \tag{2.6}$$

Example 2.5. For $G = \mathrm{GL}_n$, we will describe the slope map ϕ_G. We have that $L = G$ so $Z(L) = \mathrm{Diag}_n$, the scalar matrices of rank n. We also have the standard identifications $X_*(\mathrm{Diag}_n) \cong \mathbb{Z}$ and $X_*(T) \cong \mathbb{Z}^n$. Further, we may write $\pi_1(G) = \mathbb{Z}{\cdot}\bar{e}_1$, where $e_i : t \mapsto \mathrm{diag}(1, ..., 1, t, 1, ..., 1)$ with t in the i^{th} position, and $\overline{(-)}$ represents the image in $\pi_1(G)$. Then we have that $\overline{(a, ..., a)} = na\bar{e}_1$, hence the morphism $X_*(\mathrm{Diag}_n) \to \pi_1(G)$ is simply

$$\begin{aligned} X_*(\mathrm{Diag}_n) \to X_*(T) \to \pi_1(G) = \mathbb{Z}\bar{e}_1 \\ a \mapsto (a, ..., a) \mapsto \overline{(a, ..., a)} = na\bar{e}_1, \end{aligned} \tag{2.7}$$

i.e., multiplication by n. Thus, upon tensoring with $\mathbb{Q}$ the morphism ϕ_G from (2.6) is given by

$$\begin{aligned} \pi_1(G) \to \pi_1(G)_{\mathbb{Q}} \xrightarrow{\cong} X_*(\mathrm{Diag}_n)_{\mathbb{Q}} \to X_*(T)_{\mathbb{Q}} \\ a \mapsto \frac{a}{1} \mapsto \frac{a}{n} \mapsto (\frac{a}{n}, ..., \frac{a}{n}). \end{aligned} \tag{2.8}$$

Now let P be an arbitrary parabolic of $G = \mathrm{GL}_n$, with Levi factor $L = \prod_{i=1}^m \mathrm{GL}_{n_i}$. Then $Z(L) = \prod_{i=1}^m \mathrm{Diag}_{n_i} \cong \mathbb{Z}^m$. The isomorphism $\pi_1(P)_{\mathbb{Q}} \to X_*(Z(L))_{\mathbb{Q}}$ is the inverse to the morphism

$$\begin{aligned} X_*(Z(L)) \cong \mathbb{Z}^m \to \mathbb{Z}^n \to \mathbb{Z}^m \cong \pi_1(P) \\ (a_1, ..., a_m) \mapsto (a_1, ..., a_1, a_2, ..., a_2, ..., a_m, ..., a_m) \mapsto (n_1 a_1, n_2 a_2, ..., n_m a_2), \end{aligned} \tag{2.9}$$

where a_i occurs n_i times in the tuple in the middle. Thus, the slope map ϕ_P is given by

$$\begin{aligned} \pi_1(P) \to \pi_1(P)_{\mathbb{Q}} \cong X_*(Z(L))_{\mathbb{Q}} \to X_*(T)_{\mathbb{Q}} \\ (a_1, ..., a_m) \mapsto (\frac{a_1}{1}, ..., \frac{a_m}{1}) \mapsto (\frac{a_1}{n_1}, ..., \frac{a_m}{n_m}) \mapsto (\frac{a_1}{n_1}, ..., \frac{a_1}{n_1}, ..., \frac{a_m}{n_m}, ..., \frac{a_m}{n_m}). \end{aligned} \tag{2.10}$$

Definition 2.6. Let F_G be a G–torsor of degree $\check{\lambda}$. We say that F_G is **semistable** if for each parabolic $P \subset G$ and each reduction F_P of F_G to P, of degree $\check{\lambda}_P$, we have that

$$\phi_P(\check{\lambda}_P) \le \phi_G(\check{\lambda}). \tag{2.11}$$

Remark 2.7. If $\phi_P(\check{\lambda}_P) < \phi_G(\check{\lambda})$ then F_G is called **stable**.

Example 2.8. Again let $G = \mathrm{GL}_n$, we show why this definition gives back the usual slope semi-stability for vector bundles.
Let now $\mathcal{E}$ be a vector bundle, let $P \subset G$ be a parabolic with Levi factor $L = \prod_{i=1}^m \mathrm{GL}_{n_i}$ and let F_P be a reduction of $\mathcal{E}$ to P. This amounts to giving a filtration $0 \subset \mathcal{E}_1 \subset ... \subset \mathcal{E}_m = \mathcal{E}$, where $\mathrm{rk}\,\mathcal{E}_i - \mathrm{rk}\,\mathcal{E}_{i-1} = n_i$. Then $\deg(F_P) =$

$(\deg(\pi_{1,*}F_P), ..., \deg(\pi_{m,*}F_P))$ where $\pi_i : P \to L \to \mathrm{GL}_{n_i}$ is the composition of the projections $P \to L$ and $L \to \mathrm{GL}_{n_i}$. Then we see that

$$\begin{aligned}\phi_P(\deg(F_P)) &= (\frac{\deg(\pi_{1,*}F_P)}{n_1}, ..., \frac{\deg(\pi_{1,*}F_P)}{n_1}, ..., \frac{\deg(\pi_{m,*}F_P)}{n_m}, ..., \frac{\deg(\pi_{m,*}F_P)}{n_m}) \\ &= (\mu(\mathcal{E}_1), ..., \mu(\mathcal{E}_1), ..., \mu(\mathcal{E}_m/\mathcal{E}_{m-1}), ..., \mu(\mathcal{E}_m/\mathcal{E}_{m-1})).\end{aligned} \tag{2.12}$$

Since $\phi_G(\deg(\mathcal{E})) = (\mu(\mathcal{E}), ..., \mu(\mathcal{E}))$ we see that Definition 2.6 agrees with the usual slope semi-stability definition.

Now we recall some results of [Sch15] regarding the slope map. To this end, let $\lambda \in X^*(T)$ be a dominant character and let V be a finite-dimensional G-representation of highest weight λ. If P is a parabolic with Levi factor L, and if $V = \bigoplus_{\nu \in X^*(T)} V[\nu]$ is the weight space-decomposition of V, then let

$$V[\lambda + \mathbb{Z}\Phi_L] := \bigoplus_{\nu \in \lambda + \mathbb{Z}\Phi_L} V[\nu], \tag{2.13}$$

where Φ_L are the roots of the Levi L. Then we have the following result.

Proposition 2.9. Keep the notation as above. Let F_G be a G-torsor of degree $\check{\lambda}_G$. Then the slope of the vector bundle V_{F_G} is given by

$$\mu(V_{F_G}) = \langle \phi_G(\check{\lambda}_G), \lambda \rangle. \tag{2.14}$$

Furthermore, if F_P is a P-torsor of degree $\check{\lambda}_P$ with corresponding Levi bundle F_L, then the vector bundle $V[\lambda + \mathbb{Z}\Phi_L]_{F_L}$ has slope

$$\mu(V[\lambda + \mathbb{Z}\Phi_L]_{F_L}) = \langle \phi_P(\check{\lambda}_P), \lambda \rangle. \tag{2.15}$$

Proof. This is [Sch15, Proposition 3.2.5(b), (c)]. □

2.2 Essentially finite Torsors

Definition 2.10. An **essentially finite** G-torsor is a G-torsor over X which admits a reduction to a finite group.

Remark 2.11. Although we have fixed a smooth, projective, connected curve X over k for simplicity of the exposition, this definition makes sense over an arbitrary scheme. Similarly, we may use the same definition for arbitrary affine groups, not necessarily connected reductive.

Remark 2.12. Note that if $\varphi : \Gamma \to G$ is a map from a finite group Γ, then we obtain an injection $\tilde{\varphi} : \Gamma/\ker(\varphi) \hookrightarrow G$. If F_Γ is a Γ-torsor, then $\varphi_* F_\Gamma = \tilde{\varphi}_*(\pi_* F_\Gamma)$ as G–torsors, so we can always assume Γ to be a subgroup of G.

Example 2.13.

1. The trivial G-torsor $G \times X$ is essentially finite since it admits a reduction to the trivial group.
2. If Γ is finite then every Γ-torsor F_Γ is essentially finite since $F_\Gamma \cong \mathrm{id}_* F_\Gamma$.
3. Note that if $\alpha : G \to G'$ is a morphism of algebraic groups and F_G is an essentially finite G-torsor, then $\alpha_* F_G$ is an essentially finite G'-torsor.

We will now give two equivalent conditions for a G-bundle to be essentially finite; one in terms of the Nori fundamental group, and one Tannakian interpretation. Since k is algebraically closed, there is a rational point x of X.

Proposition 2.14. A G-bundle F_G is essentially finite if and only if there exists a morphism $\rho \colon \pi_1^N(X, x) \to G$ such that $\rho_* \widetilde{X} \cong F_G$.

Proof. Let F_G be an essentially finite G-torsor, let $\iota \colon \Gamma \hookrightarrow G$ be a finite subgroup of G and let $j \colon F_\Gamma \to X$ be a Γ-torsor such that $\iota_* F_\Gamma \cong F_G$. Let y be a rational point of F_Γ such that $j(y) = x$. Then j defines a pointed finite torsor $(F_\Gamma, y) \to (X, x)$. By [Nor76, Proposition 3.11], there is a morphism $\pi_1^N(X, x) \to \Gamma$, which we compose with ι to get a morphism $\rho \colon \pi_1^N(X, x) \to G$ such that $F_G \cong \rho_* \widetilde{X}$.

Conversely, suppose that we have a morphism $\rho \colon \pi_1^N(X, x) \to G$ such that $\rho_* \widetilde{X} \cong F_G$. Since $\pi_1^N(X, x) = \varprojlim_i A_i$ is the inverse limit of its finite quotients A_i (see [Nor82]), there is some i and a morphism $\rho_i \colon A_i \to G$ such that ρ factors

$$\rho \colon \pi_1^N(X, x) \xrightarrow{\pi_i} A_i \xrightarrow{\rho_i} G \tag{2.1}$$

where π_i is the projection. Since $\rho_* \widetilde{X} \cong \rho_{i,*}(\pi_{i,*} \widetilde{X})$ we see that F_G is essentially finite.

□

Proposition 2.15. A G-torsor F_G is essentially finite if and only if there exists a finite group Γ, a Γ-torsor F_Γ, and a tensor functor $\alpha : \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(\Gamma)$ such that :

1. we have that $\omega_\Gamma \circ \alpha = \omega_G$, where $\omega_G : \mathrm{Rep}_k(G) \to \mathrm{Vec}_k$ and $\omega_\Gamma : \mathrm{Rep}_k(\Gamma) \to \mathrm{Vec}_k$ are the forgetful functors; and

2. we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_k(G) & \xrightarrow{F_G} & \mathrm{Vec}_X \\ {\scriptstyle\alpha}\downarrow & \nearrow_{F_\Gamma} & \\ \mathrm{Rep}_k(\Gamma) & & \end{array} \tag{2.2}$$

Proof. If F_G is essentially finite, coming from a finite group Γ, a group morphism $\varphi : \Gamma \to G$ and a Γ-torsor F_Γ, then we take α to be the induced functor from φ. Conversely, every such α, by [DM82, Corollary 2.9], comes from a group morphism $\varphi : \Gamma \to G$. □

Remark 2.16. If a G-torsor F_G is essentially finite then there exists a finite group Γ and a Γ-torsor $j_\Gamma : F_\Gamma \to X$ such that $j_\Gamma^* F_G$ is trivial.

Proposition 2.17. Under the correspondence between vector bundles of rank n and GL_n-torsors, a GL_n-torsor is essentially finite if and only if the corresponding vector bundle is essentially finite.

Proof. Let F_{GL_n} be a GL_n-torsor, and let Γ be a finite subgroup of GL_n, $\alpha : \Gamma \to \mathrm{GL}_n$ and let $j : F_\Gamma \to X$ be a Γ-torsor such that $F_{\mathrm{GL}_n} = \alpha_* F_\Gamma$. Then F_{GL_n} is trivialised by $j : F_\Gamma \to X$ so the corresponding vector bundle E is also trivialised by $j : F_\Gamma \to X$. Thus, E is essentially finite.

Conversely suppose E is an essentially finite vector bundle. Then there is a finite group $\iota : \Gamma \to \mathrm{GL}_n$ and a Γ-torsor $F_\Gamma \to X$ such that $E = F_\Gamma \times^\Gamma \mathbb{A}^n$. Then we have that

$$E = F_\Gamma \times^\Gamma \mathbb{A}^n \cong F_\Gamma \times^\Gamma \mathrm{GL}_n \times^{\mathrm{GL}_n} \mathbb{A}^n \cong \iota_* F_\Gamma \times^{\mathrm{GL}_n} \mathbb{A}^n, \tag{2.3}$$

whence the vector bundle associated to $\iota_* F_\Gamma$ is E. Hence, the bundle corresponding to E is isomorphic to $\iota_* F_\Gamma$, hence essentially finite. □

Lemma 2.18. Let Y be a proper and connected scheme over k. A G-bundle F_G over Y is trivial if and only if for any faithful representation $\rho\colon G \to GL_V$, $\rho_* F_G$ is trivial.

Proof. The idea of this can be found in [BD13, Lemma 4.5], but we spell out the details since their assumptions on the base scheme are different from ours. Suppose that $\rho\colon G \to GL_V$ is any faithful representation. Consider the long exact sequence of pointed sets (see [DG70, III, §4, 4.6])

$$1 \to G(Y) \xrightarrow{\rho} GL_V(Y) \xrightarrow{\pi} (GL_V/G)(Y) \xrightarrow{\delta} H^1(Y,G) \xrightarrow{\rho_*} H^1(Y,GL_V), \tag{2.4}$$

where $\pi\colon GL_V \to GL_V/G$ is the canonical projection. The morphism δ takes a Y-point $y\colon Y \to GL_V/G$ to the G-bundle $\delta(y) := Y \times_{GL_V/G,y,\pi} GL_V$. Since G is reductive, GL_V/G is affine and hence, using that Y is proper and connected, y is constant. That is, we have a factorisation $y\colon Y \to \operatorname{Spec} k \to GL_V/G$. Since k is algebraically closed, $(GL_V/G)(k) = GL_V(k)/G(k)$, and hence y being constant implies that there is a lift $\tilde{y}\colon Y \to GL_V$ of y. By the universal propery of fiber products we thus see that $\delta(y)$ admits a section, whence $\delta(y)$ is trivial. Hence, by exactness of the sequence a G-bundle F_G is trivial if and only if ρ_*F_G is trivial. □

Theorem 2.19. *Let G be a connected, reductive group, and let F_G be a G-bundle. Then the following are equivalent.*

1. *The G-bundle F_G is essentially finite.*
2. *There exists a faithful representation $\rho\colon G \to GL_V$ such that ρ_*F_G is an essentally finite vector bundle.*
3. *For every representation $\rho\colon G \to GL_V$, ρ_*F_G is an essentally finite vector bundle.*
4. *There exists a proper surjective morphism $f\colon Y \to X$ such that f^*F_G is trivial.*

Proof. By above we see that 1. implies 3., and it is clear that 3. implies 2. By [BdS11] 4. is equivalent to 3. Hence we prove that 2. implies 3. and that 3. implies 1.

First suppose that 2. holds, let $\varphi\colon G \to GL_W$ be a faithful representation such that φ_*F_G is essentially finite and let $\rho\colon G \to GL_V$ be an arbitrary representation. Since φ_*F_G is essentially finite there is a proper surjective morphism $f\colon Y \to X$ such that $f^*\varphi_*F_G$ is trivial. Since any restriction of $f^*\varphi_*F_G$ to a connected component of Y is trivial, we may assume that Y is connected. Thus, since $f^*\varphi_*F_G \cong \varphi_*f^*F_G$, we see from Lemma 2.18 that f^*F_G is trivial. Hence, $f^*\rho_*F_G \cong \rho_*f^*F_G$ is trivial, which implies that ρ_*F_G is essentially finite (again by [BdS11]). This proves that 2. implies 3.

Now assume that 3. holds. Then the functor $F_G\colon \mathrm{Rep}_k(G) \to Bun_X$ factors through the category of essentially finite vector bundles, hence induces a group morphism

$\rho\colon \pi_1^N(X,x) \to G$ such that $\rho_* \widetilde{X} \cong F_G$. Thus, by Proposition 2.14 F_G is essentially finite.

□

Proposition 2.20. Every essentially finite G-torsor is semistable.

Proof. Let F_G be such a torsor. Let further $P \subset G$ be a parabolic of G, let λ be a dominant character and let V be a representation of highest weight λ. Since F_G is essentially finite, the associated vector bundle V_{F_G} is essentially finite, hence semistable. Hence, using Proposition 2.9, we have that

$$\langle \psi_G(\check{\lambda}_G), \lambda \rangle = \mu(V_{F_G}) \geq \mu(V[\lambda + \mathbb{Z}\Phi_L]_{F_L}) = \langle \psi_P(\check{\lambda}_P), \lambda \rangle. \tag{2.5}$$

That is, for every dominant character $\lambda \in X^*(T)_{\mathbb{Q}}$ we have that

$$\langle \psi_G(\check{\lambda}_G) - \psi_P(\check{\lambda}_P)), \lambda \rangle \geq 0. \tag{2.6}$$

Since the cone of cocharacters with non-negative pairing with all dominant characters is double-dual to the cone of simple coroots, we see that

$$\psi_G(\check{\lambda}_G) - \psi_P(\check{\lambda}_P) \geq 0. \tag{2.7}$$

□

Proposition 2.21. Let F_G be an essentially finite G-torsor. Then its degree is torsion as an element of $\pi_1(G)$.

Proof. Let F_G be such a bundle. Let $j : F_\Gamma \to X$ be a finite bundle such that $F_G \cong F_\Gamma \times^\Gamma G$. Let T be a maximal torus and $B \supset T$ a Borel containing T, and choose a reduction F_B of F_G to a Borel. We know that $j^* F_G$ is trivial. Since

$$j^* F_B \times^B G = j^*(F_B \times^B G) = j^* F_G, \tag{2.8}$$

we see that $j^* F_B \times^B G$ is trivial. We have that $\pi_0(\mathcal{M}_{B,F_\Gamma}) = \pi_0(\mathcal{M}_{T,F_\Gamma}) = X_*(T)$ and this maps surjectively onto $\pi_0(\mathcal{M}_{G,F_\Gamma})$. The fact that $j^* F_B$ maps to the trivial torsor means that it corresponds to 0 in $\pi_1(G) = X_*(T)/\Phi^\vee = \pi_0(\mathcal{M}_{G,F_\Gamma})$. This implies that the degree of $j^* F_B$, seen as an element in $X_*(T)$, is a sum of coroots. The equality $\pi_0(\mathcal{M}_B) = \pi_0(\mathcal{M}_T)$ is induced by the morphism $\pi_T : B \to T$, so $\pi_{T,*} j^* F_B$

also corresponds to a sum of coroots. Since $\pi_{T,*}j^*F_B = j^*\pi_{T,*}F_B$, the conclusion follows if we can show that the morphism

$$j^* : \mathcal{M}_{T,X} \to \mathcal{M}_{T,F_\Gamma} \tag{2.9}$$

has the property that, if j^*F_T has degree in $\Phi^\vee$, then the same holds for a multiple of $\deg(F_T)$.

If F_T corresponds to the cocharacter μ_{F_T}, then j^*F_T corresponds to the cocharacter $\mu_{j^*F_T} = \deg(j)\mu_{F_T}$. Thus if $\mu_{F_T} = \sum_{i=1}^n a_i\alpha_i^\vee + \mu$, where α_i are the simple roots and $\mu \in X_* \setminus \Phi^\vee$ then

$$\mu_{j^*F_T} = \sum_{i=1}^n \deg(j)a_i\alpha_i^\vee + \deg(j)\mu = \sum_{i=1}^n a_i'\alpha_i^\vee$$

Hence, $\deg(j)\mu \in \Phi^\vee$.

We now apply this to our situation above, i.e., with $F_T \coloneqq \pi_{T,*}F_B$, and since $\pi_1(G) = X_*(T)/\Phi^\vee$ we can conclude that $\deg(F_G)$ is torsion. □

Proposition 2.22. Let G be a connected, reductive group. If X is an elliptic curve, then every essentially finite G-bundle over X has degree 0.

Proof. We argue by induction on the dimension of G. If $\dim(G) = 1$ then $G \cong \mathbb{G}_m$ and the result follows since it is true for all vector bundles. Suppose now that $\dim(G) = n > 1$. Let F_G be an essentially finite G-torsors of degree d. By [Fră21] there is a proper Levi L and a degree $d' \in \pi_1(L)$ such that the inclusion $\iota\colon L \to G$ induces a surjection $\mathcal{M}_{L,X}^{d'} \to \mathcal{M}_{G,X}^{d}$. Let F_L be a reduction of structure group of F_G to L. Since F_G is essentially finite there is a faithful representation $\rho\colon G \to \mathrm{GL}_V$ such that $\rho_*F_G \cong (\rho\circ\iota)_*F_L$ is essentially finite. By Theorem 2.19 this implies that F_L is essentially finite. Since L is a proper Levi, by induction $d' = 0$, whence $d = 0$. □

If the characteristic of k is positive, there is a stronger notion of semistability, defined as follows. Let $\sigma_X : X \to X$ denote the absolute Frobenius of X.

Definition 2.23. A G-torsor F_G is said to be **strongly semistable** if for all $n \geq 0$, $(\sigma_X^n)^*F_G$ is semistable.

Proposition 2.24. Every essentially finite G-torsor is strongly semistable.

Proof. For any algebraic group H, and any H-torsor, if $\sigma_H : H \to H$ denotes the absolute Frobenius of H, then we have that (see , [Las01, page 655]

$$(\sigma_H)_* F_H \cong \sigma_X^* F_H. \tag{2.10}$$

Let now F_G be an essentially finite G-torsor. Let $j : F_\Gamma \to X$ be a finite bundle such that $F_G \cong F_\Gamma \times^\Gamma G$. Since the push-forward along group morphisms commutes with pullbacks, we have that

$$j_*(\sigma_\Gamma)_* F_\Gamma \cong j_* \sigma_X^* F_\Gamma \cong (\sigma_X)^* j_* F_\Gamma \cong (\sigma_X)^* F_G. \tag{2.11}$$

Hence $(\sigma_X)^* F_G$ is essentially finite and thus semistable. The statement follows similarly via induction. □

Definition 2.25. A G-torsor F_G is **Frobenius periodic** if $F_G \cong (\sigma_X^n)^* F_G$ for some $n \geq 1$.

Proposition 2.26. A Frobenius periodic G-torsor is isotrivial.

Proof. See [BD07, page 496]. □

Corollary 2.27. A Frobenius periodic G-torsor is essentially finite.

2.3 The prestack of essentially finite torsors

Let $\mathcal{M}_G^{\text{ef}}$ denote the functor

$$\begin{aligned} \mathcal{M}_G^{\text{ef}} : \mathbf{Aff}_k^{\text{op}} &\to \mathbf{Grpds} \\ U &\mapsto \Big\{\text{essentially finite } G\text{-torsors over } U \times X\Big\} + \Big\{\text{isomorphism of } G\text{-torsors}\Big\}. \end{aligned} \tag{2.1}$$

It is immediate that $\mathcal{M}_G^{\text{ef}}$ is a subfunctor of $\mathcal{M}_G^{\text{ss}}$.

Proposition 2.28. The functor $\mathcal{M}_G^{\text{ef}}$ is a k-prestack.

Proof. First suppose that $f : U' \to U$ is a morphism in $\mathbf{Aff}_k^{\text{op}}$ and suppose F_G is an essentially finite G-torsor over $U \times X$. Let $(U_i \to U)$ be a cover and $(g_{ij} : g_{ij} \in G(U_{ij}))$ a cocycle for F_G. Then $(f^* U_i \to U')$ is a cover of U' and $(f^* g_{ij})_{ij}$ is a cocycle for $f^* F_G$. Indeed, since $g_{ij} g_{jk} = g_{ik}$ on U_{ijk} we see that

$$f^* g_{ij} f^* g_{jk}(x) = g_{ij}(f(x)) g_{jk}(f(x)) = g_{ik}(f(x)) = f^* g_{ik}(x). \tag{2.2}$$

The torsor f^*F_G is also essentially finite since if $g_{ij} \in \Gamma(U_{ij}) \subset G(U_{ij})$ for some finite group Γ, then $f^*g_{ij} = g_{ij} \circ f$ also takes values in Γ. Since $\mathcal{M}_G^{\mathrm{ss}}$ is a lax functor we see that $\mathcal{M}_G^{\mathrm{ef}}$ is one as well.

Next it is clear that if $F_G, F'_G \in \mathcal{M}_G^{\mathrm{ef}}(U)$, then $\underline{\mathrm{Isom}}(F_G, F'_G) : \mathbf{Aff}_{/U} \to \mathbf{Set}$ is a sheaf since homomorphisms of finite G-torsors are simply homomorphisms of G-torsors and $\mathcal{M}_G^{\mathrm{ss}}$ is a stack. □

Remark 2.29. Note however that $\mathcal{M}_G^{\mathrm{ef}}$ is not a stack since the descent data is not necessarily effective. Indeed, let $G = \mathrm{GL}_n$ and let E be a vector bundle which is not essentially finite. Let further $(U_i \to X)$ be a trivialising cover of E, with trivilising morphisms $\phi_i : E|_{U_i} \to \mathcal{O}_{U_i}^n$. Then $E|_{X\times U_i}$ with the morphisms $(\mathrm{id} \times \phi_j^{-1}) \circ (\mathrm{id} \times \phi_i)$ form a descent data for $E|_{X\times X} \in \mathcal{M}_G(X \times X)$. Now, if $E|_{X\times X}$ is essentially finite, then so is E. Indeed, by [BdS11, Theorem 1] we have a proper surjective morphism $f : Y \to X \times X$ such that $f^*E_{X\times X}$ is trivial, and by composing with the projection $X \times X \to X$ we have a proper surjective morphism $g : Y \to X$ such that g^*E is trivial. Since E was assumed not to be essentially finite, we conclude that $E|_{X\times X}$ is not essentially finite and the descent data constructed is not effective.

The following statement is immediate, but will be important for us in the final section.

Proposition 2.30. Let G and G' be reductive groups. The isomorphism $\mathcal{M}_{G\times G'}^{\mathrm{ss}} \xrightarrow{\cong} \mathcal{M}_G^{\mathrm{ss}} \times \mathcal{M}_{G'}^{\mathrm{ss}}$ restricts to an isomorphism

$$\mathcal{M}_{G\times G'}^{\mathrm{ef}} \cong \mathcal{M}_G^{\mathrm{ef}} \times \mathcal{M}_{G'}^{\mathrm{ef}}. \tag{2.3}$$

Proof. The isomorphism on objects is given by

$$\begin{aligned} F_{G\times G'} &\mapsto (\pi_* F_{G\times G'}, \pi'_* F_{G\times G'}), \\ (F_G, F_{G'}) &\mapsto F_G \times F_{G'}, \end{aligned} \tag{2.4}$$

where $\pi : G \times G' \to G$ and $\pi' : G \times G' \to G'$ are the projections. If $\Gamma \subset G \times G'$ is a finite structure group of $F_{G\times G'}$, then $\pi(\Gamma)$ and $\pi'(\Gamma')$ are evidently finite structure groups of F_G and $F_{G'}$ respectively. Similarly, finite structure groups Γ and Γ' of F_G, respectively $F_{G'}$, give a finite structure group, $\Gamma \times \Gamma'$ of $F_G \times F_{G'}$. □

3 Density of essentially finite torsors

Dans ce chapitre on s'intéresse à la densité des G-torseurs essentiellement finis de degré zéro dans l'espace des modules des G-torseurs semi-stables de degré zéro.

Après avoir donné quelques propriétés dynamiques de l'espace des modules, on sépare l'étude en trois parties, en fonction du genre de la courbe.

Lorsque le genre est nul, on montre que tout les G-torseurs essentiellement finis sont triviaux.

Lorsque le genre est égal à 1, on démontre que les G-torseurs essentiellement finis forment un sous ensemble dense de l'espace des modules G-torseurs semi-stables.

Lorsque le genre est supérieur ou égal à 2, on montre que lorsque G est un groupe semi simple de rang 1, ce qui inclus donc les groupes GL_2, SL_2 et PGL_2 ne forme pas un sous-ensemble dense. Dans ce dernier cas la preuve se ramène à létude du cas $G = \mathrm{PGL}_2$.

This chapter presents results achieved through collaboration with Stefan Reppen in [GR23].

3.1 Preliminaries

Proposition 3.1. Suppose $\pi : G \to H$ is a morphism of reductive algebraic groups such that $\pi(Z(G)^0) \subset Z(H)^0$. If π admits a section $s : H \to G$ such that $s(Z(H)^0) \subset Z(G)^0$, then density of $M_G^{\mathrm{ef},0}$ in $M_G^{\mathrm{ss},0}$ implies density of $M_H^{\mathrm{ef},0}$ in $M_H^{\mathrm{ss},0}$.

Proof. Suppose that $M_H^{\mathrm{ef},0}$ is not dense in $M_H^{\mathrm{ss},0}$. Since π_* takes essentially finite G-torsors to essentially finite H-torsors, by Lemma 2.2 we have a commutative diagram as follows

$$\begin{array}{ccc} M_G^{\mathrm{ef},0} & \underset{s_*}{\overset{\pi_*}{\rightleftarrows}} & M_H^{\mathrm{ef},0} \\ \big\downarrow & & \big\downarrow \\ M_G^{\mathrm{ss},0} & \underset{s_*}{\overset{\pi_*}{\rightleftarrows}} & M_H^{\mathrm{ss},0} \end{array} \tag{3.1}$$

Since π_* is a morphism, $\pi_*\left(\overline{M_G^{ef,0}}\right) \subset \overline{M_H^{ef,0}}$. Suppose now on the contrary that $M_G^{\mathrm{ef},0}$ is dense in $M_G^{\mathrm{ss},0}$. Pick any $F \in M_H^{\mathrm{ss},0}$. Then $s_*F \in M_G^{\mathrm{ss},0} = \overline{M_G^{\mathrm{ef},0}}$. But since

$\pi_* s_* = \mathrm{id}$ we see that

$$F = \pi_* s_* F \in \pi_*\left(\overline{M_G^{\mathrm{ef},0}}\right) \subset \overline{M_H^{\mathrm{ef},0}}, \tag{3.2}$$

which implies that $\overline{M_H^{\mathrm{ef},0}} = M_H^{\mathrm{ss},0}$. Contradiction. □

Remark 3.2. The condition on the centers is to make sure that the pushforward of a semistable bundle is semistable.

Corollary 3.3. Let G be a direct product of reductive groups G_1 and G_2. If $M_{G_i}^{\mathrm{ef},0}$ is not dense in $M_{G_i}^{\mathrm{ss},0}$ for some $i = 1, 2$, then $M_G^{\mathrm{ef},0}$ is not dense in $M_G^{\mathrm{ss},0}$.

Proof. We use the projection $\pi_i : G \to G_i$ and apply the previous proposition. □

Proposition 3.4. Let $G = T$ be a torus. Then M_T^{ef} is dense in $M_T^{\mathrm{ss},0}$.

Proof. First suppose $T = \mathbb{G}_m$. Then $M_T^{\mathrm{ss},0} = \mathrm{Jac}^0(X)$ is the Jacobian of X and essentially finite $\mathbb{G}_m$-torsors corresponds to finite line bundles which corresponds to torsion points on $\mathrm{Jac}(X)$, which are dense. If $T \cong \mathbb{G}_m^r$ for $r > 1$, then we apply Proposition 2.30 and the statement follows. □

3.2 Genus 0

Let now $X = \mathbb{P}_k^1$, where k is an arbitrary algebraically closed field. By Proposition 2.14 we immediately have the following statement.

Proposition 3.5. Every essentially finite G-torsors over X is trivial.

Proof. Since $\pi_1^N(X, x)$ is trival, the statement follows from Proposition 2.14. □

It is also well-known that $M_G^{\mathrm{ss},0}(k)$ is a singleton so the density statement is immediate.

For the remainder of this section, we give a different proof of Proposition 3.5, which might be interesting in its own right. We do this by using the Tannakian interpretation of essentially finite G-torsors and the classification of G-torsors on X.

The classification of G-torsors on X was initially done by Grothendieck [Gro57] and by Harder [Har68] for characterstic p. In [Ans18] Anschütz gives a Tannakian interpretation of this classification. We thus begin by introducing the relevant notions from [Ans18].

Over X there is a canonical $\mathbb{G}_m$-torsor

$$\begin{aligned} \eta : \mathbb{A}^2 \setminus \{0\} &\to X \\ (x_0, x_1) &\mapsto [x_0 : x_1], \end{aligned} \tag{3.1}$$

often called the Hopf bundle. Pushforward along this bundle defines an exact, faithful tensor functor

$$\begin{aligned} \mathcal{E} : \mathrm{Rep}_k(\mathbb{G}_m) &\to \mathrm{Bun}_X \\ V &\mapsto \mathbb{A}^2 \setminus \{0\} \times^{\mathbb{G}_m} V, \end{aligned} \tag{3.2}$$

Taking the Harder-Narashiman filtration of a vector bundle over X defines a fully faithful tensor functor

$$HN : \mathrm{Bun}_X \to \mathrm{FilBun}_X \tag{3.3}$$

from Bun_X to the category of filtered vector bundles. Finally we can take the graded pieces of a filtered vector bundle and this defines an exact tensor functor

$$\mathrm{Gr} : \mathrm{FilBun}_X \to \mathrm{GrBun}_X, \tag{3.4}$$

where GrBun_X is the category of graded vector bundles.

Proposition 3.6. The composition

$$\mathcal{E}_{\mathrm{Gr}} : \mathrm{Rep}_k(\mathbb{G}_m) \xrightarrow{\mathcal{E}} \mathrm{Bun}_X \xrightarrow{HN} \mathrm{FilBun}_X \xrightarrow{\mathrm{Gr}} \mathrm{GrBun}_X \tag{3.5}$$

is an equivalence of tensor categories onto its essential image, which consists of graded bundles $E = \bigoplus_{n \in \mathbb{Z}} E_i$ such that each E_i is semistable of slope i.

Proof. This is [Ans18, Lemma 2.3]. □

The main Theorem of Grothendieck, restated in the Tannaka language by Anschütz is now given by

Proposition 3.7. Let G be a reductive group over k. The composition with $\mathcal{E}$ defines faithful functor

$$\Phi : \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \to \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Bun}_X), \tag{3.6}$$

which induces a bijection

$$\mathrm{Hom}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Rep}_k(\mathbb{G}_m)) \cong H^1_{\text{ét}}(X, G) \tag{3.7}$$

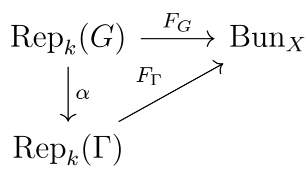
on isomorphism classes.

Proof. This is [Ans18, Theorem 3.3]. □

The inverse of this is given by composition with $\mathcal{E}_{\mathrm{Gr}}^{-1} \circ \mathrm{Gr} \circ \mathrm{HN}$. Using this we can now describe all essentially finite G-torsors on X.

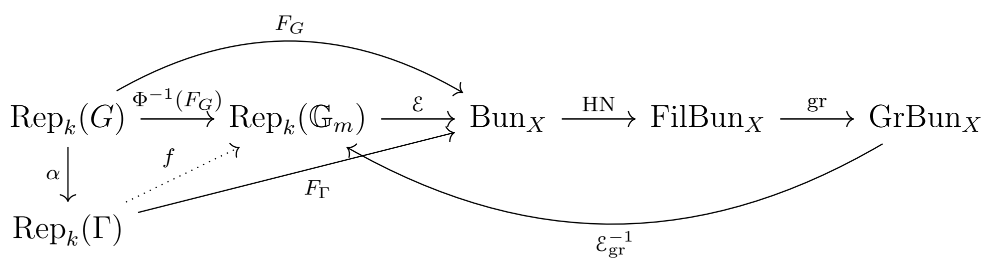
Proposition 3.8. Every essentially finite G-torsors over X is trivial.

Proof. Let $F_G : \mathrm{Rep}_k(G) \to \mathrm{Bun}_X$ be an essentially finite torsor. By Proposition 2.15 there exists a commutative diagram of tensor functors



(3.8)

for some finite group Γ. By [Ans18] this sits inside the following larger diagram



(3.9)

where f is defined to be the composition

$$f := \mathcal{E}_{\mathrm{gr}}^{-1} \circ \mathrm{gr} \circ \mathrm{HN} \circ F_\Gamma. \tag{3.10}$$

Since all functors are tensor functors, so is f. By [DM82, Corollary 2.9] f is induced by a morphism

$$\tilde{f} : \mathbb{G}_m \to \Gamma. \tag{3.11}$$

Since $\mathbb{G}_m$ is connected and Γ is discrete we see that $\tilde{f}$ and thus f is the trivial map. But this implies that

$$F_G \cong \mathcal{E} \circ \Phi^{-1}(F_G) \cong \mathcal{E} \circ \mathcal{E}_{\mathrm{gr}}^{-1} \circ \mathrm{gr} \circ \mathrm{HN} \circ F_G \cong \mathcal{E} \circ \mathcal{E}_{\mathrm{gr}}^{-1} \circ \mathrm{gr} \circ \mathrm{HN} \circ F_\Gamma \circ \alpha \cong \mathcal{E} \circ f \circ \alpha \tag{3.12}$$

is the trivial torsor. □

3.3 Genus 1

In the case when X is an elliptic curve, the density result follows almost immediately from known properties of M_G^{ss}, studied by Laszlo [Las98] in characteristic 0 and Frăţilă in charactierstic p [Fră21].

Proposition 3.9. Suppose X is an elliptic curve. Then M_G^{ef} is dense in $M_G^{\mathrm{ss},0}$ for any reductive group G.

Proof. Let T be a maximal torus of G and let W be the corresponding Weyl group. Then, by [Las98, Theorem 4.16] and [Fră21, Theorem 1.1], we have an isomorphism

$$\varphi : M_T^{\mathrm{ss},0}/W \to M_G^{\mathrm{ss},0} \tag{3.1}$$

induced by the inclusion $\iota : T \hookrightarrow G$. Since $\iota_*(M_T^{\mathrm{ef}}) \subset M_G^{\mathrm{ef}}$, the result follows from Proposition 3.4. □

3.4 Genus>1

Let now X be of genus $g \geq 2$. Suppose first that $\mathrm{char}(k) = p > 0$. In [DM10, Proposition 4.1 and corollary 5.1] the authors proved that, for any $n > 0$, the set of k-points in $M_{\mathrm{GL}_n}^{\mathrm{ss},0}$ (resp $M_{\mathrm{SL}_n}^{\mathrm{ss}}$) Frobenius periodic is dense. Hence, the set of k-points corresponding to essentially finite vector bundles is also dense by Corollary 1.63. Hence, we may state the following.

Proposition 3.10. Let k be of characteristic $p > 0$. For any $n > 1$, $M_{\mathrm{PGL}_n}^{\mathrm{ef},0}$ is dense in $M_{\mathrm{PGL}_n}^{\mathrm{ss},0}$.

Proof. This follows from the previous discussion and the fact that the projection $\mathrm{GL}_n \to \mathrm{PGL}_n$ induces a surjection $M_{\mathrm{GL}_n}^{\mathrm{ss},0} \to M_{\mathrm{PGL}_n}^{\mathrm{ss},0}$ by [Ser58, Proposition 18] which takes essentially finite GL_n-torsors to essentially finite PGL_n-torsors. □

Let now k be of characteristic zero. We restrict ourselves to split reductive groups of semisimple rank 1. By classical results (see e.g., [Mil17, Chapter 21]) these are all given by the following list.

Proposition 3.11. Let G be a split reductive group of semisimple rank 1. Then, up to isomorphism, G is one of the following groups:

$$\mathrm{GL}_2 \times \mathbb{G}_m^r, \qquad \mathrm{SL}_2 \times \mathbb{G}_m^r, \qquad \mathrm{PGL}_2 \times \mathbb{G}_m^r, \qquad r \in \mathbb{N}. \tag{3.1}$$

Hence, by Proposition 3.1 applied to the projection map, if we show non-density for SL_2, GL_2, and PGL_2, we show it for all split reductive groups of semisimple rank 1. Now, by known results [BLS98, I.3 page 7], the quotient maps on the respective groups induce dominant morphisms

$$\begin{aligned} M^{\mathrm{ss}}_{\mathrm{SL}_2} &\to M^{\mathrm{ss},0}_{\mathrm{PGL}_2} \\ M^{\mathrm{ss},0}_{\mathrm{GL}_2} &\to M^{\mathrm{ss},0}_{\mathrm{PGL}_2}. \end{aligned} \tag{3.2}$$

Thus, to show non-density for split reductive groups of semisimple rank 1 it suffices to show it for PGL_2, which we do now.

To do this we need a bound on the dimension of $M^{\mathrm{ss}}_{\mathrm{O}(2)}$. For a connected reductive group G it is well-known that $\dim \mathcal{M}_G = \dim(G)(g-1)$ (see e.g. [Sor00]). Since $\mathrm{O}(2)$ is not connected, we compute $\dim \mathcal{M}_{\mathrm{O}(2)}$ following the approach for connected reductive groups.

Lemma 3.12. We have that $\dim \mathcal{M}_{\mathrm{O}(2)} = g-1$.

Proof. Let $F_{\mathrm{O}(2)}$ be an $\mathrm{O}(2)$ bundle and let $\mathfrak{o}_2$ denote the Lie algebra of $\mathrm{O}(2)$. Let further $\mathrm{Ad} : \mathrm{O}(2) \to \mathrm{GL}(\mathfrak{o}_2)$ denote the adjoint representation and let $E \coloneqq \mathrm{Ad}_* F_{\mathrm{O}(2)}$. By definition we know that the dimension of $\mathcal{M}_{\mathrm{O}(2)}$ at the point $F_{\mathrm{O}(2)}$ is the rank of the cotangent complex at $F_{\mathrm{O}(2)}$, which is equal to $-\chi(X, E)$. By Riemann-Roch we thus have that

$$\begin{aligned} \dim \mathcal{M}_{\mathrm{O}(2)} &= -\deg(E) - \mathrm{rk}(E)\chi(X, \mathcal{O}_X) \\ &= -\deg(E) + g - 1. \end{aligned} \tag{3.3}$$

By identifying $\mathrm{O}(2)$ as the matrices

$$\mathrm{O}(2) = T' \coprod T' \{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \}, \;\; T' = \{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} : t \in \mathbb{G}_m \}, \tag{3.4}$$

one sees immediately that the adjoint representation is self dual. Then we see that $\mathfrak{o}_2 \cong \{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} : a \in k \}$ and for $M \in \mathrm{O}(2)$ we have that

$$\mathrm{Ad}(M) \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} = M \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} M^{-1} = \begin{cases} \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} & M \in T' \\ \begin{bmatrix} -a & 0 \\ 0 & a \end{bmatrix} & M \in T' \{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \} \end{cases}. \tag{3.5}$$

If $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ denotes the standard basis of $\mathfrak{o}_2$, and $e_1^\vee : \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \mapsto a$, the corresponding basis for $\mathfrak{o}_2^\vee$, then the action of $O(2)$ on $\mathfrak{o}_2^\vee$ is given by

$$M \cdot ae_1^\vee = \begin{cases} ae_1^\vee & M \in T' \\ -ae_1^\vee & M \in T'\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\} \end{cases}. \tag{3.6}$$

We thus see that $\mathfrak{o}_2$ and $\mathfrak{o}_2^\vee$ are isomorphic as $\mathrm{O}(2)$-modules.

Hence, $E \cong E^\vee$ and thus $\deg(E) = -\deg(E)$ whence $\deg(E) = 0$. We conclude that $\dim \mathcal{M}_{\mathrm{O}(2)} = g - 1$. □

Lemma 3.13. Let ι denote an inclusion $\iota \colon \mathrm{O}(2) \hookrightarrow \mathrm{PGL}_2$. If $F_{O(2)}$ is a semistable $\mathrm{O}(2)$-bundle then $\iota_* F_{O(2)}$ is a semistable PGL_2-bundle.

Proof. The proof of [BS02, Proposition 2.6] applies verbatim, since an $\mathrm{O}(2)$-bundle $F_{\mathrm{O}(2)}$ is semistable if and only if $\iota'_* F_{\mathrm{O}(2)}$ is semistable, where $\iota' \colon \mathrm{O}(2) \hookrightarrow \mathrm{GL}_2$ is the standard representation. □

Proposition 3.14. The subset of essentially finite PGL_2-torsors is not dense inside $M^{\mathrm{ss},0}_{\mathrm{PGL}_2}$.

Proof. By [NvdPT08] the finite subgroups of PGL_2 are given by S_4, A_5, A_4 and for all $n \in \mathbb{N}$, μ_n and D_n. Furthermore, for each finite subgroup there is only one conjugacy class by [Bea10, Proposition 4.1]. Hence, for a given finite subgroup Γ, we may choose any embedding $\iota : \Gamma \hookrightarrow \mathrm{PGL}_2$ and unambiguously consider $\iota_* \mathcal{M}_\Gamma \subset \mathcal{M}^{\mathrm{ss},0}_{\mathrm{PGL}_2}$.

Now, for any such group Γ, $\iota_* M_\Gamma \subset M^{\mathrm{ss},0}_{\mathrm{PGL}_2}$ is a finite number of points. Indeed, we have that

$$H^1_{\mathrm{et}}(X, \Gamma) = \mathrm{Hom}(\pi_1(X), \Gamma) \tag{3.7}$$

and since $\pi_1(X)$ is (pro)finitely generated, we see that $H^1_{\mathrm{et}}(X, \Gamma)$ is a finite set. Hence, to prove the proposition it is enough to show that the essentially finite torsors whose finite group is isomorphic to D_n or μ_n for some $n > 0$, is not dense. By abuse of notation, we still denote this subset by $M^{\mathrm{ef}}_{\mathrm{PGL}_2}$.

Let $\pi : \mathrm{GL}_2 \to \mathrm{PGL}_2$ denote the quotient morphism. From [NvdPT08, Section 2] we thus see that we may choose the embedding such that for every such Γ, we have a commutative diagram

$$\Gamma \hookrightarrow \pi(\mathrm{O}(2)) \hookrightarrow \mathrm{PGL}_2 \ , \quad \text{with } \iota\colon \Gamma \to \mathrm{PGL}_2 \tag{3.8}$$

where $\mathrm{O}(2) \subset \mathrm{GL}_2$ is realized as the matrices

$$\mathrm{O}(2) = T' \coprod T'\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}, \ \ T' = \{\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{G}_m\}. \tag{3.9}$$

Since $\pi(\mathrm{O}(2)) \cong \mathrm{O}(2)$, and since $\iota'\colon O(2) \cong \pi(\mathrm{O}(2)) \hookrightarrow \mathrm{PGL}_2$ is a closed embedding, the induced morphism $\iota'_* : \mathcal{M}_{\mathrm{O}(2)} \to \mathcal{M}_{\mathrm{PGL}_2}$ is locally of finite type by [Hof10, Fact 2.3]. By Lemma 3.13 this induces a map $\iota'_* : \mathcal{M}^{\mathrm{ss}}_{\mathrm{O}(2)} \to \mathcal{M}^{\mathrm{ss}}_{\mathrm{PGL}_2}$, which induces by the universal property of the coarse moduli space a morphism of finite type schemes $M^{\mathrm{ss}}_{\mathrm{O}(2)} \to M^{\mathrm{ss}}_{\mathrm{PGL}_2}$. By taking base change along $M^{\mathrm{ss},0}_{\mathrm{PGL}_2}$ we obtain an open subscheme $U \subset M^{\mathrm{ss}}_{\mathrm{O}(2)}$ and a morphism of finite type $f\colon U \to M^{\mathrm{ss},0}_{\mathrm{PGL}_2}$. We thus obtain a Cartesian diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & M^{\mathrm{ss},0}_{\mathrm{PGL}_2} \\ \big\downarrow & & \big\downarrow \\ M^{\mathrm{ss}}_{\mathrm{O}(2)} & \xrightarrow{\iota'_*} & M^{\mathrm{ss}}_{\mathrm{PGL}_2} \end{array} \ , \tag{3.10}$$

Now, for any essentially finite PGL_2-torsor, F_{PGL_2}, we may assume that $F_{\mathrm{PGL}_2} = \iota'_* F_{\mathrm{O}(2)}$ where $F_{\mathrm{O}(2)}$ is an essentially finite $\mathrm{O}(2)$-torsor.

Hence, we have a finite type morphism $f : U \to M^{\mathrm{ss},0}_{\mathrm{PGL}_2}$ of projective varieties such that

$$M^{\mathrm{ef}}_{\mathrm{PGL}_2} \subset f(U). \tag{3.11}$$

Thus, it suffices to show that f is not dominant. Suppose it was. Then we obtain an inclusion of functions fields

$$k(M^{\mathrm{ss},0}_{\mathrm{PGL}_2}) \hookrightarrow k(U). \tag{3.12}$$

This implies that

$$3g-3 = \dim M^{\mathrm{ss},0}_{\mathrm{PGL}_2} = \mathrm{tr.deg}_k k(M^{\mathrm{ss},0}_{\mathrm{PGL}_2}) \leq \mathrm{tr.deg}_k k(U) = \dim U = \dim M^{\mathrm{ss}}_{\mathrm{O}(2)} \leq g-1, \tag{3.13}$$

where the last inequality follows from Lemma 3.12. □

From the statement for PGL_2 we obtain the same statement for SL_2.

Corollary 3.15. The subset of essentially finite SL_2-torsors is not dense inside $M^{\mathrm{ss},0}_{\mathrm{SL}_2}$.

Proof. Since the map $M^{\mathrm{ss}}_{\mathrm{SL}_2} \to M^{\mathrm{ss},0}_{\mathrm{PGL}_2}$ is dominant this follows from Proposition 3.14. □

From this we obtain the same statement for GL_2.

Corollary 3.16. The subset of essentially finite GL_2-torsors is not dense inside $M^{\mathrm{ss},0}_{\mathrm{GL}_2}$.

Proof. The same proof as above applies, or we have the following. Consider the map

$$\det : M^{\mathrm{ss},0}_{\mathrm{GL}_2} \to \mathrm{Jac}^0(X). \tag{3.14}$$

Since $\det^{-1}(\mathcal{O}_X) = M^{\mathrm{ss}}_{\mathrm{SL}_2}$ by Corollary 3.15 we obtain the desired result. □

Finally, the complete statement is the following.

Corollary 3.17. For any split reductive group G, of semi-simple rank 1, the essentially finite G-torsors are not dense in $M^{\mathrm{ss},0}_G$.

Proof. This follows from the classification of split reductive groups and Proposition 3.14. □

References

[Alp87] Roger C. Alperin. An elementary account of Selberg's lemma. English. *Enseign. Math. (2)*, 33:269–273, 1987.

[Ans18] Johannes Anschütz. A Tannakian classification of torsors on the projective line. English. *C. R., Math., Acad. Sci. Paris*, 356(11-12):1203–1214, 2018. ISSN: 1631-073X. DOI: 10.1016/j.crma.2018.10.006.

[Ati56] Michael F. Atiyah. On the Krull-Schmidt theorem with application to sheaves. English. *Bull. Soc. Math. Fr.*, 84:307–317, 1956.

[Ati57] M. F. Atiyah. Complex analytic connections in fibre bundles. *Transactions of the American Mathematical Society*, 85(1):181–207, 1957.

[BD07] Indranil Biswas and Laurent Ducrohet. An analog of a theorem of Lange and Stuhler for principal bundles. English. *C. R., Math., Acad. Sci. Paris*, 345(9):495–497, 2007. ISSN: 1631-073X. DOI: 10.1016/j.crma.2007.10.010.

[BD13] Indranil Biswas and João Pedro Dos Santos. Triviality criteria for bundles over rationally connected varieties. English. *J. Ramanujan Math. Soc.*, 28(4):423–442, 2013. ISSN: 0970-1249.

[BdS11] Indranil Biswas and João Pedro Pinto dos Santos. Vector bundles trivialized by proper morphisms and the fundamental group scheme. English. *J. Inst. Math. Jussieu*, 10(2):225–234, 2011. ISSN: 1474-7480. DOI: 10.1017/S1474748010000071.

[Bea10] Arnaud Beauville. Finite subgroups of $\mathrm{PGL}_2(K)$. English. In *Vector bundles and complex geometry. Conference on vector bundles in honor of S. Ramanan on the occasion of his 70th birthday, Madrid, Spain, June 16–20, 2008.* Pages 23–29. Providence, RI: American Mathematical Society (AMS), 2010. ISBN: 978-0-8218-4750-3.

[BLS98] Arnaud Beauville, Yves Laszlo, and Christoph Sorger. The Picard group of the moduli of G-bundles on a curve. English. *Compos. Math.*, 112(2):183–216, 1998. ISSN: 0010-437X. DOI: 10.1023/A:1000477122220.

[BS02] V. Balaji and C. S. Seshadri. Semistable principal bundles. I: Characteristic zero. English. *J. Algebra*, 258(1):321–347, 2002. ISSN: 0021-8693. DOI: 10.1016/S0021-8693(02)00502-1.

[DG70] Michel Demazure and Pierre Gabriel. Groupes algébriques. Tome I: Géométrie algébrique. Généralités. Groupes commutatifs. Avec un appendice 'Corps de classes local' par Michiel Hazewinkel. French. Paris: Masson et Cie, Éditeur; Amsterdam: North-Holland Publishing Company. xxvi, 700 p. (1970). 1970.

[DM10] Laurent Ducrohet and Vikram B. Mehta. Density of vector bundles periodic under the action of Frobenius. English. *Bull. Sci. Math.*, 134(5):454–460, 2010. ISSN: 0007-4497. DOI: 10.1016/j.bulsci.2009.11.001.

[DM82] Pierre Deligne and J. S. Milne. Tannakian categories. English. Hodge cycles, motives, and Shimura varieties, Lect. Notes Math. 900, 101-228 (1982). 1982.

[EHS08] Hélène Esnault, Phùng Hô Hai, and Xiaotao Sun. On Nori's fundamental group scheme. English. In *Geometry and dynamics of groups and spaces. In memory of Alexander Reznikov. Partly based on the international conference on geometry and dynamics of groups and spaces in memory of Alexander Reznikov, Bonn, Germany, September 22–29, 2006*, pages 377–398. Basel: Birkhäuser, 2008. ISBN: 978-3-7643-8607-8.

[FH91] William Fulton and Joe Harris. *Representation theory. A first course.* English, volume 129 of *Grad. Texts Math.* New York : Springer-Verlag, 1991.

[Fră21] Dragoş Frăţilă. Revisiting the moduli space of semistable G-bundles over elliptic curves. English. *Ann. Inst. Fourier*, 71(2):615–641, 2021. ISSN: 0373-0956. DOI: 10.5802/aif.3405.

[GLS+08] Tomás L. Gómez, Adrian Langer, Alexander H. W. Schmitt, and Ignacio Sols. Moduli spaces for principal bundles in arbitrary characteristic. English. *Adv. Math.*, 219(4):1177–1245, 2008. ISSN: 0001-8708. DOI: 10.1016/j.aim.2008.05.015.

[GR23] Archia Ghiasabadi and Stefan Reppen. Essentially finite G-torsors, 2023. arXiv: 2211.02909 [math.AG].

[Gro57] A. Grothendieck. Sur la classification des fibres holomorphes sur la sphère de Riemann. French. *Am. J. Math.*, 79:121–138, 1957. ISSN: 0002-9327. DOI: 10.2307/2372388.

[Gro61] Alexander Grothendieck. Éléments de géométrie algébrique : III. Étude cohomologique des faisceaux cohérents, Première partie. fr. *Publications Mathématiques de l'IHÉS*, 11:5–167, 1961.

[Har68] Günter Harder. Semisimple group schemes over complete curves. German. *Invent. Math.*, 6:107–149, 1968. ISSN: 0020-9910. DOI: 10.1007/BF01425451.

[Har77] Robin Hartshorne. *Algebraic geometry*. English, volume 52 of *Grad. Texts Math.* Springer, Cham, 1977.

[Hof10] Norbert Hoffmann. On moduli stacks of G-bundles over a curve. English. In *Affine flag manifolds and principal bundles*, pages 155–163. Basel: Birkhäuser, 2010. ISBN: 978-3-0346-0287-7. DOI: 10.1007/978-3-0346-0288-4_5.

[Las01] Yves Laszlo. A non-trivial family of bundles fixed by the square of Frobenius. English. *C. R. Acad. Sci., Paris, Sér. I, Math.*, 333(7):651–656, 2001. ISSN: 0764-4442. DOI: 10.1016/S0764-4442(01)02109-7.

[Las98] Yves Laszlo. About G-bundles over elliptic curves. English. *Ann. Inst. Fourier*, 48(2):413–424, 1998. ISSN: 0373-0956. DOI: 10.5802/aif.1623.

[Le 97] Joseph Le Potier. *Lectures on vector bundles*. English, volume 54 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 1997. ISBN: 0-521-48182-1.

[LS77] Herbert Lange and Ulrich Stuhler. Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe. German. *Math. Z.*, 156:73–83, 1977. ISSN: 0025-5874. DOI: 10.1007/BF01215129.

[Mil17] J. S. Milne. *Algebraic groups. The theory of group schemes of finite type over a field*. English, volume 170 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2017. DOI: 10.1017/9781316711736.

[Mil80] J. S. Milne. *Étale cohomology*. English, volume 33 of *Princeton Math. Ser.* Princeton University Press, Princeton, NJ, 1980.

[Nor76] Madhav V. Nori. On the representations of the fundamental group. English. *Compos. Math.*, 33:29–41, 1976. ISSN: 0010-437X.

[Nor82] Madhav V. Nori. The fundamental group-scheme. English. *Proc. Indian Acad. Sci., Math. Sci.*, 91:73–122, 1982. ISSN: 0253-4142. DOI: 10.1007/BF02967978.

[NvdPT08] K. A. Nguyen, M. van der Put, and J. Top. Algebraic subgroups of $\mathrm{GL}_2(\mathbb{C})$. English. *Indag. Math., New Ser.*, 19(2):287–297, 2008. ISSN: 0019-3577. DOI: 10.1016/S0019-3577(08)80004-3.

[Ram96a] A. Ramanathan. Moduli for principal bundles over algebraic curves. I. English. *Proc. Indian Acad. Sci., Math. Sci.*, 106(3):301–328, 1996. ISSN: 0253-4142. DOI: 10.1007/BF02867438.

[Ram96b] A. Ramanathan. Moduli for principal bundles over algebraic curves. II. English. *Proc. Indian Acad. Sci., Math. Sci.*, 106(4):421–449, 1996. ISSN: 0253-4142. DOI: 10.1007/BF02837697.

[Saa72] Neantro Saavedra Rivano. *Categories tannakiennes*. French, volume 265 of *Lect. Notes Math.* Springer, Cham, 1972.

[Sch15] Simon Schieder. The Harder-Narasimhan stratification of the moduli stack of G-bundles via Drinfeld's compactifications. English. *Sel. Math., New Ser.*, 21(3):763–831, 2015. ISSN: 1022-1824. DOI: 10.1007/s00029-014-0161-y.

[Ser58] Jean-Pierre Serre. Espaces fibrés algébriques. *Séminaire Claude Chevalley*, 3, 1958. URL: http://www.numdam.org/item/SCC_1958__3__A1_0/.

[Ses82] C. S. Seshadri. *Fibrés vectoriels sur les courbes algébriques*. fr, number 96 in Astérisque. Société mathématique de France, 1982.

[Sor00] Christoph Sorger. Lectures on moduli of principal G-bundles over algebraic curves. English. In *Moduli spaces in algebraic geometry. Lecture notes of the school on algebraic geometry, Trieste, Italy, July 26–August 13, 1999*, pages 1–57. Trieste: The Abdus Salam International Centre for Theoretical Physics (ICTP), 2000. ISBN: 92-95003-00-4. URL: www.ictp.trieste.it/~pub_off/lectures/.

[Sza09] Tamás Szamuely. *Galois groups and fundamental groups*. English, volume 117 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2009. ISBN: 978-0-521-88850-9.

[Wei22] Dario Weissmann. A functorial approach to the stability of vector bundles, 2022. DOI: 10.48550/ARXIV.2211.08260. URL: https://arxiv.org/abs/2211.08260.

[Wei38] André Weil. Généralisation des fonctions abéliennes. French. *J. Math. Pures Appl. (9)*, 17:47–87, 1938.

Soit X une courbe projective lisse de genre g, définie sur un corps algébriquement clos k, et soit G un groupe réductif connexe sur k. Nous disons qu'un G-torseur est essentiellement fini s'il admet une réduction à un groupe fini, généralisant la notion de fibrés vectoriels essentiellement finis à des groupes G arbitraires.
Nous donnons une interprétation tannakienne de tels torseurs, et nous prouvons que tous les G-torseurs essentiellement finis ont un degré de torsion, et que ce degré est égal à 0 si X est une courbe elliptique.
Nous étudions ensuite la densité de l'ensemble des k-points des G-torseurs essentiellement finis de degré 0, noté $M_G^{ef,0}$, à l'intérieur de $M_G^{ss,0}$, les k-points de tous les G-torseurs semi-stables de degré 0. Nous montrons que lorsque $g = 1$, $M_G^{ef,0}$ est dense dans $M_G^{ss,0}$. Quand $g > 1$ et quand $car(k) = 0$, nous montrons que pour tout groupe réductif semi-simple de rang 1, $M_G^{ef,0}$ n'est pas dense dans $M_G^{ss,0}$.

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Résumé

Soit X une courbe projective lisse de genre g, définie sur un corps algébriquement clos k, et soit G un groupe réductif connexe sur k. Nous disons qu'un G-torseur est essentiellement fini s'il admet une réduction à un groupe fini, généralisant la notion de fibrés vectoriels essentiellement finis à des groupes G arbitraires.
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Résumé en anglais

Let X be a smooth projective curve of genus g, defined over an algebraically closed field k, and let G be a connected reductive group over k. We say that a G-torsor is essentially finite if it admits a reduction to a finite group, generalizing the notion of essentially finite vector bundles to arbitrary groups G. We give a Tannakian interpretation of such torsors, and we prove that all essentially finite G-torsors have torsion degree, and that the degree is 0 if X is an elliptic curve. We then study the density of the set of k-points of essentially finite G-torsors of degree 0, denoted M_{G}^{ef,0}, inside M_{G}^{ss,0}, the k-points of all semistable degree 0 G-torsors. We show that when g=1, M_{G}^{ef,0} inside M_{G}^{ss,0} is dense. When g>1 and when \Char(k)=0, we show that for any reductive group of semisimple rank 1, M_{G}^{\ef,0} inside M_{G}^{\ss,0} is not dense.