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Structures algébriques associées aux relations de double mélange entre valeurs polylogarithmes multiples aux racines de l'unité

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Thèse

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>
> Al-Khawarizmi

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Contents

Introduction en français

On appelle *valeur polylogarithme multiple* ou *valeur L-multiple* (abrégé en MLV) tout nombre complexe défini par la série

$$L_{(k_1,\ldots,k_r)}(z_1,\ldots,z_r) := \sum_{0<m_1<\cdots<m_r} \frac{z_1^{m_1}\cdots z_r^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}, \tag{1}$$

où $r, k_1, \ldots, k_r$ sont des entiers naturels non nuls et $z_1, \ldots, z_r$ appartiennent à μ_N le groupe des racines N^{th} de l'unité dans $\mathbb{C}$ avec N un entier naturel non nul. La série (1) converge si et seulement si $(k_r, z_r) \neq (1,1)$. Ces valeurs ont été définies par Goncharov dans [Gon98] et [Gon01a] et ont été étudiées par d'autres comme Arakawa et Kaneko dans [ArKa] et apparaissent comme une généralisation de valeurs zêta multiples qui, à leur tour, généralisent les valeurs spéciales de la fonction zêta de Riemann. Parmi les relations satisfaites par les MLV, notre propos ici se porte sur les relations de *double mélange et régularisation*. En effet, on sait que les MLV peuvent s'exprimer en termes d'intégrales itérées[1] comme suit (voir, par exemple, [Gon98, Theorem 2.1])

$$L_{(k_1,\ldots,k_r)}(z_1,\ldots,z_r) = \int_0^1 \Omega_0^{k_r-1}\Omega_{z_r}\Omega_0^{k_{r-1}-1}\Omega_{z_{r-1}z_r}\cdots\Omega_0^{k_1-1}\Omega_{z_1\cdots z_r} \tag{2}$$

où $\Omega_0 = \frac{ds}{s}$ et $\Omega_z = \frac{ds}{z^{-1}-s}$ pour $z \in \mu_N$. Ensuite, le produit de MLV peut s'écrire comme combinaison $\mathbb{Q}$-linéaire d'autres MLV (voir [Gon98, §2]).

Exemple. En utilisant la somme itérée (1) on obtient pour $(k_1, z_1), (k_2, z_2) \in \mathbb{N}_{>0} \times \mu_N \diagdown \{(1,1)\}$

$$\begin{aligned} L_{k_1}(z_1)L_{k_2}(z_2) &= \sum_{m_1,m_2\in\mathbb{N}^*} \frac{z_1^{m_1}z_2^{m_2}}{k_1^{m_1}k_2^{m_2}} = \left(\sum_{0<m_1<m_2} + \sum_{0<m_2<m_1} + \sum_{0<m_1=m_2}\right)\frac{z_1^{m_1}z_2^{m_2}}{k_1^{m_1}k_2^{m_2}} \\ &= L_{(k_1,k_2)}(z_1,z_2) + L_{(k_2,k_1)}(z_2,z_1) + L_{k_1+k_2}(z_1z_2). \end{aligned}$$

[1] Ici, nous utilisons les notations de [Kas, Chapitre XIX, §11].

En utilisant l'intégrale itérée (2) on obtient en prenant $k_1 = 2$ et $k_2 = 3$

$$\begin{aligned}
L_2(z_1)L_3(z_2) &= \int_0^1 \Omega_0\Omega_{z_1} \int_0^1 \Omega_0^2\Omega_{z_2} \\
&= \int_0^1 \Omega_0\Omega_{z_1}\Omega_0^2\Omega_{z_2} + 2\int_0^1 \Omega_0^2\Omega_{z_1}\Omega_0\Omega_{z_2} + 3\int_0^1 \Omega_0^3\Omega_{z_1}\Omega_{z_2} \\
&\quad + 3\int_0^1 \Omega_0^3\Omega_{z_2}\Omega_{z_1} + \int_0^1 \Omega_0^2\Omega_{z_2}\Omega_0\Omega_{z_1} \\
&= L_{(3,2)}(z_1^{-1}z_2, z_1) + 2L_{(2,3)}(z_1^{-1}z_2, z_1) + 3L_{(1,4)}(z_1^{-1}z_2, z_1) \\
&\quad + 3L_{(1,4)}(z_1z_2^{-1}, z_2) + L_{(2,3)}(z_1z_2^{-1}, z_2)
\end{aligned}$$

où la deuxième égalité provient du fait que pour tout $(n,m) \in \mathbb{N}^2$ et toutes 1-formes différentielles à valeurs complexes $\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_{n+m}$ sur un intervalle $[a,b]$ on a (voir, par exemple, [Kas, Chapter XIX, §11])

$$\int_a^b \omega_1 \cdots \omega_n \int_a^b \omega_{n+1} \cdots \omega_{n+m} = \sum_{\sigma} \int_a^b \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(n)}\, \omega_{\sigma^{-1}(n+1)} \cdots \omega_{\sigma^{-1}(n+m)},$$

où les éléments σ sont des (n,m)-battages du groupe symétrique $\mathfrak{S}_{n+m}$.

On obtient deux systèmes de relations associés à la somme itérée (1) et à l'intégrale itérée (2) respectivement. Les travaux de Ihara, Kaneko et Zagier dans [IKZ], basés sur [Hof97], permettent de décrire ces relations de manière formelle comme suit :

- On considère $\mathfrak{H} = \mathbb{Q}\langle X\rangle$ la $\mathbb{Q}$-algèbre non commutative de polynômes en des variables x_0, x_z $(z \in \mu_N)$, $\mathfrak{H}^1 = \mathbb{Q} \oplus \bigoplus_{z\in\mu_N} \mathfrak{H}x_z$ et $\mathfrak{H}^0 = \mathbb{Q} \oplus \bigoplus_{z\in\mu_N} x_0\mathfrak{H}x_z$. Dans ce cas, on a (inclusion d'algèbres)
$$\mathfrak{H}^0 \subset \mathfrak{H}^1 \subset \mathfrak{H}.$$
- On définit l'application $\mathbb{Q}$-linéaire $Z : \mathfrak{H}^0 \to \mathbb{R}$ donnée par $1 \mapsto 1$ et pour tout $r \in \mathbb{N}_{>0}$,
$$u_1 \cdots u_r \longmapsto \int_0^1 \Omega_{u_1} \cdots \Omega_{u_r},$$
où $\Omega_{x_0} = \Omega_0$ et $\Omega_{x_z} = \Omega_z$. On a donc, d'après (2),
$$Z(x_0^{k_r-1}x_{z_r} \cdots x_0^{k_1-1}x_{z_1\cdots z_r}) = L_{(k_1,\ldots,k_r)}(z_1, \ldots z_r). \tag{3}$$
- Pour $(k,z) \in \mathbb{N}_{>0} \times \mu_N$, posons $y_{k,z} = x_0^{k-1}x_z$. L'algèbre $\mathfrak{H}^1$ est engendrée par $(y_{k,z})_{(k,z)\in\mathbb{N}_{>0}\times\mu_N}$. De plus pour $(k,z) \in \mathbb{N}_{>0} \times \mu_N \diagdown (1,1)$ on a $Z(y_{k,z}) = L_k(z)$.
- On définit un autre produit d'algèbre $*$ sur $\mathfrak{H}^1$ par récurrence par
$$\begin{cases} 1 * w = w * 1 = w \\ y_{k,z_1}w_1 * y_{l,z_2}w_2 = \\ y_{k,z_1}(w_1 * y_{l,z_1^{-1}z_2}w_2) + y_{l,z_2}(y_{k,z_1z_2^{-1}}w_1 * w_2) + y_{k+l,z_1z_2}(w_1 * w_2), \end{cases}$$
où $k,l \in \mathbb{N}_{>0}$, $z_1, z_2 \in \mu_N$ et w, w_1, w_1 des mots dans $\mathfrak{H}^1$. Ce produit est appelé *produit harmonique* (ou *produit stuffle*). Le couple $(\mathfrak{H}^0, *)$ est une sous-algèbre de $(\mathfrak{H}^1, *)$.

- Le système de relations associé à la somme itérée (1) est alors équivalent au fait que l'application $Z : (\mathfrak{H}^0, *) \to \mathbb{R}$ soit un morphisme d'algèbres.

- On définit un autre produit d'algèbre ш sur $\mathfrak{H}$ par récurrence par

$$\begin{cases} 1 ш w = w ш 1 = w \\ x_{z_1} w_1 ш x_{z_2} w_2 = x_{z_1}(w_1 ш x_{z_2} w_2) + x_{z_2}(x_{z_1} w_1 ш w_2), \end{cases}$$

  où $z_1, z_2 \in \mu_N \sqcup \{0\}$ et w, w_1, w_2 des mots dans $\mathfrak{H}$. Ce produit est appelé *produit de battage* (ou *produit shuffle*). Les couples $(\mathfrak{H}^1, ш)$ et $(\mathfrak{H}^0, ш)$ sont des sous-algèbres de $(\mathfrak{H}, ш)$.

- Le système de relations associé à l'intégrale itérée (2) est alors équivalent au fait que l'application $Z : (\mathfrak{H}^0, ш) \to \mathbb{R}$ soit un morphisme d'algèbres.

- En combinant les systèmes de relations en somme itérée et en intégrale itérée, on obtient des *relations de double mélange.* Ce système est équivalent à

$$Z(w_1 * w_2) = Z(w_1 ш w_2),$$

  pour tous $w_1, w_2 \in \mathfrak{H}^0$.

Il est possible d'adjoindre à un tel système des *relations de régularisation* consistant à annuler formellement certaines divergences. On forme ainsi le système de relations de « double mélange et régularisation ». On procède comme suit :

- Il existe deux morphismes d'algèbres ([IKZ, Proposition 1])

$$Z^* : (\mathfrak{H}^1, *) \to \mathbb{R}[T] \text{ et } Z^ш : (\mathfrak{H}^1, ш) \to \mathbb{R}[T]$$

  tels qu'il étendent $Z : \mathfrak{H}^0 \to \mathbb{R}$ et envoient $y_{1,1} = x_1$ vers T.

- Pour $\mathrm{k} = (k_1, \ldots, k_r)$ et $\mathrm{z} = (z_1, \ldots, z_r)$ on note

$$Z^*_{\mathrm{k},\mathrm{z}}(T) = Z^*(x_0^{k_r - 1} x_{z_r} \cdots x_0^{k_1 - 1} x_{z_1 \cdots z_r})$$

  et

$$Z^ш_{\mathrm{k},\mathrm{z}}(T) = Z^ш(x_0^{k_r - 1} x_{z_r} \cdots x_0^{k_1 - 1} x_{z_1 \cdots z_r}).$$

  Ainsi, lorsque $(k_r, z_r) \neq (1, 1)$, on a $Z^*_{\mathrm{k},\mathrm{z}}(T) = Z^ш_{\mathrm{k},\mathrm{z}}(T) = L_{(k_1, \ldots, k_r)}(z_1, \ldots, z_r)$.

- Les relations de régularisation se déduisent de l'identité ([IKZ, Théorème 1.])

$$Z^ш_{\mathrm{k},\mathrm{z}}(T) = \rho(Z^*_{\mathrm{k},\mathrm{z}}(T)), \tag{4}$$

  où $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$ est une application $\mathbb{R}$-linéaire donnée pour $u \in \mathbb{R}$ par

$$\rho\left(\exp(Tu)\right) = \exp\left(\sum_{n \geq 2} \frac{(-1)^n}{n} \zeta(n) u^n\right) \exp(Tu),$$

  où ζ est la fonction zêta de Riemann.

D'après Racinet [Rac], si pour $(k, z) \in \mathbb{N}^* \times \mu_N \diagdown \{(1, 1)\}$, on attribue à $L_k(z)$ le poids k, alors les relations de double mélange et régularisations sont homogènes. On peut dresser la liste des relations de double mélange et régularisation de poids donné pour les petites valeurs du poids :

Poids 1. Un calcul direct permet d'obtenir pour tout entier $k \in]0, N/2[$,

$$L_1(e^{\frac{i2k\pi}{N}}) - L_1(e^{-\frac{i2k\pi}{N}}) = (N - 2k)i\pi.$$

Ainsi pour $z \in \mu_N \diagdown \{1\}$, les $L_1(z) - L_1(z^{-1})$ sont tous $\mathbb{Q}$-colinéaires.

Poids 2. Pour $z_1, z_2 \in \mu_N \diagdown \{1\}$, on obtient en utilisant la somme itérée (1),

$$L_1(z_1)L_1(z_2) = L_{(1,1)}(z_1, z_2) + L_{(1,1)}(z_2, z_1) + L_2(z_1 z_2).$$

Puis, en utilisant l'intégrale itérée (2), on obtient

$$L_1(z_1)L_1(z_2) = L_{(1,1)}(z_1^{-1} z_2, z_1) + L_{(1,1)}(z_1 z_2^{-1}, z_2).$$

On obtient les relations de double mélange

$$L_{(1,1)}(z_1, z_2) + L_{(1,1)}(z_2, z_1) + L_2(z_1 z_2) = L_{(1,1)}(z_1^{-1} z_2, z_1) + L_{(1,1)}(z_1 z_2^{-1}, z_2). \quad (5)$$

Afin d'obtenir les relations de régularisation, utilisons la formule (4) pour $\mathrm{k} = (1, 1)$ et $\mathrm{z} = (z, 1)$ avec $z \in \mu_N \diagdown \{1\}$. On a

$$y_{1,1} * y_{1,z} = y_{1,1} y_{1,z} + y_{1,z} y_{1,z^{-1}} + y_{2,z} \quad \text{et} \quad x_1 ш x_z = x_1 x_z + x_z x_1.$$

Ce qui implique

$$\begin{aligned} Z^*_{\mathrm{k,z}}(T) = Z^*(y_{1,1} y_{1,z}) &= Z^*(y_{1,1}) Z^*(y_{1,z}) - Z^*(y_{1,z} y_{1,z^{-1}}) - Z^*(y_{2,z}) \\ &= T\, L_1(z) - L_{(1,1)}(z^{-2}, z) - L_2(z) \end{aligned}$$

et

$$Z^{ш}_{\mathrm{k,z}}(T) = Z^{ш}(x_1 x_z) = Z^{ш}(x_1) Z^{ш}(x_z) - Z^{ш}(x_z x_1) = T L_1(z) - L_{(1,1)}(z^{-1}, z).$$

L'égalité (4) donne les relations de régularisation

$$L_{(1,1)}(z^{-1}, z) = L_{(1,1)}(z^{-2}, z) + L_2(z). \quad (6)$$

Poids 3 Dans ce cas, nous nous contentons d'utiliser les relations de régularisation (4) pour $\mathrm{k} = (2, 1)$ et $\mathrm{z} = (1, 1)$ afin d'obtenir la formule d'Euler $\zeta(1, 2) = \zeta(3)$:

- On a

$$\begin{aligned} Z^*_{\mathrm{k,z}}(T) = Z^*(y_{1,1} y_{2,1}) &= Z^*(y_{1,1}) Z^*(y_{2,1}) - Z^*(y_{2,1} y_{1,z^{-1}}) - Z^*(y_{3,1}) \\ &= T\, L_2(1) - L_{(1,2)}(1, 1) - L_3(1), \end{aligned}$$

  où la seconde égalité provient du fait que $y_{1,1} * y_{2,1} = y_{1,1} y_{2,1} + y_{2,1} * y_{1,1} + y_{3,1}$.

- Puis,

$$\begin{aligned} Z^{ш}_{\mathrm{k,z}}(T) &= Z^{ш}(x_1x_0x_1) = Z^{ш}(x_1)Z^{ш}(x_0x_1) - Z^{ш}(x_0x_1x_1) - Z^{ш}(x_0x_1x_1) \\ &= TL_2(1) - L_{(1,2)}(1,1) - L_{(1,2)}(1,1) \\ &= TL_2(1) - 2L_{(1,2)}(1,1), \end{aligned}$$

  où la seconde égalité provient du fait que $x_1 ш x_0x_1 = x_1x_0x_1 + x_0x_1x_1 + x_0x_1x_1$.
- Ainsi, l'égalité (4) donne la relation de régularisation

$$L_{(1,2)}(1,1) + L_3(1) = 2L_{(1,2)}(1,1).$$

  C'est-à-dire

$$L_3(1) = L_{(1,2)}(1,1),$$

  qui est la formule annoncée.

Contenu du chapitre 1 : Le torseur de double mélange

Le travail de Racinet [Rac] a contribué significativement à la compréhension des relations de double mélange et régularisation. Il généralise le groupe μ_N à un groupe cyclique fini G et identifie les 1-formes différentielles de (2) à des variables non commutatives x_0 et x_g ($g \in G$) d'une algèbre libre de séries formelles $\mathbf{k}\langle\langle X\rangle\rangle$ sur une $\mathbb{Q}$-algèbre commutative $\mathbf{k}$. Par la suite, il associe à chaque couple (G, ι) où G est un groupe cyclique fini et $\iota : G \to \mathbb{C}^\times$ un morphisme de groupes injectif, un $\mathbb{Q}$-schéma DMR^ι qui, à chaque $\mathbb{Q}$-algèbre commutative $\mathbf{k}$, associe un ensemble $\mathsf{DMR}^\iota(\mathbf{k})$ pouvant être vu comme réunion disjointe d'ensembles $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ pour $\lambda \in \mathbf{k}$ (voir [Rac, Definition 3.2.1]). Le système des relations de double mélange et régularisation sur les MLV est alors encodé dans le fait qu'une série génératrice convenable de ces valeurs (voir [Rac, 2.2.3]) appartient à l'ensemble $\mathsf{DMR}^{\iota_{can}}_{i2\pi}(\mathbb{C})$ où $\iota_{can} : G = \mu_N \hookrightarrow \mathbb{C}^\star$ est l'inclusion canonique.

Par ailleurs, Racinet démontre que pour tout couple (G, ι), l'ensemble $\mathsf{DMR}^\iota_0(\mathbf{k})$, muni du produit $\circledast$ donné en (1.10), est un groupe indépendant du choix du morphisme injectif $\iota : G \to \mathbb{C}^\times$, on le note $\mathsf{DMR}^G_0(\mathbf{k})$. Muni du produit $\circledast$, l'ensemble $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ des éléments diagonaux de $\mathbf{k}\langle\langle X\rangle\rangle$ – pour le coproduit rendant x_0 et x_g ($g \in G$) primitifs – est un groupe (voir Proposition-Définition 1.2.4) contenant $\mathsf{DMR}^G_0(\mathbf{k})$ comme sous-groupe. Le groupe $\mathbf{k}^\times$ agit sur $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ (voir Proposition 1.3.5). Cela nous permet alors de considérer le produit semi-direct $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ dont la loi de groupe est également notée $\circledast$. L'action de $\mathbf{k}^\times$ se restreint au sous-groupe $\mathsf{DMR}^G_0(\mathbf{k})$ (voir Proposition 1.3.16), ce qui permet de construire le sous-groupe produit semi-direct $\mathbf{k}^\times \ltimes \mathsf{DMR}^G_0(\mathbf{k})$. Pour tout $\lambda \in \mathbf{k}$, l'ensemble $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ est un torseur sous l'action de $(\mathsf{DMR}^G_0(\mathbf{k}), \circledast)$ (voir Proposition 1.3.15). Par conséquent, nous avons une structure de torseur sur l'ensemble

$$\mathsf{DMR}^\iota_\times(\mathbf{k}) := \bigsqcup_{\lambda \in \mathbf{k}^\times} \mathsf{DMR}^\iota_\lambda(\mathbf{k})$$

sous l'action de $\mathbf{k}^\times \ltimes \mathsf{DMR}^G_0(\mathbf{k})$ (voir Proposition 1.3.20). Ces résultats motivent l'étude du groupe $(\mathsf{DMR}^G_0(\mathbf{k}), \circledast)$.

Afin de mieux comprendre le groupe $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$, Enriquez et Furusho, dans [EF0], l'ont relié au stabilisateur $\mathsf{Stab}(\hat{\Delta}_\star)(\mathbf{k})$ du coproduit $\hat{\Delta}_\star : \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \to (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\hat{\otimes}2}$ donné dans [Rac, (2.3.1.2)] pour une action de $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ (voir Proposition 1.2.19).

De plus, le travail de Racinet introduit aussi une sous-algèbre $\mathbf{k}\langle\langle Y\rangle\rangle$ de $\mathbf{k}\langle\langle X\rangle\rangle$ engendrée par les mots se terminant en x_g avec $g \in G$. En tant que $\mathbf{k}$-module, $\mathbf{k}\langle\langle Y\rangle\rangle$ s'identifie avec $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ et est munie d'un coproduit $\mathbf{k}\langle\langle Y\rangle\rangle \to \mathbf{k}\langle\langle Y\rangle\rangle^{\hat{\otimes}2}$ compatible avec $\hat{\Delta}_\star$. Pour cette raison, le coproduit précédent possède aussi la même notation dans [Rac]. Toutefois, dans cette thèse, nous allons adopter des notations distinctes pour ces deux coproduits, en notant $\hat{\Delta}_\star^{\mathrm{alg}}$ et $\hat{\Delta}_\star^{\mathrm{mod}}$ les coproduits sur $\mathbf{k}\langle\langle Y\rangle\rangle$ et $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ respectivement.

La situation, détaillée dans §1.1, peut se résumer à l'aide du diagramme de morphismes d'algèbres-modules suivant :

$$\begin{array}{ccc} (\mathbf{k}\langle\langle X\rangle\rangle, \mathbf{k}\langle\langle X\rangle\rangle) & \xrightarrow{(\mathrm{id},\pi_Y)} & (\mathbf{k}\langle\langle X\rangle\rangle, \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0) \\ & & \uparrow{\scriptstyle (\mathrm{incl},\mathrm{id})} \\ & & (\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0) \end{array} \tag{7}$$

Le terme du dessous est munie d'un couple de coproduits compatibles $(\hat{\Delta}_\star^{\mathrm{alg}}, \hat{\Delta}_\star^{\mathrm{mod}})$. Le groupe $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ agit par automorphismes d'algèbres-modules sur les termes horizontaux, la flèche associée étant équivariante. Toutefois, le groupe ne possède pas d'action sur l'objet du dessous en général mais uniquement sur sa partie module; c'est sur cette action qu'est fondée la construction du groupe stabilisateur de [EF0].

Lorsque $G = \{1\}$, il a été démontré dans [EF1, Partie 2, §3] que la sous-algèbre $\mathbf{k}\langle\langle Y\rangle\rangle$ de $\mathbf{k}\langle\langle X\rangle\rangle$ est stable sous l'action de $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ sur $\mathbf{k}\langle\langle X\rangle\rangle$. On peut alors construire le groupe stabilisateur $\mathsf{Stab}(\hat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ de $\hat{\Delta}_\star^{\mathrm{alg}}$ par rapport à l'action de $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ sur $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\hat{\otimes}2})$. Par [EF2, §3.1], on a alors l'inclusion[2] $\mathsf{Stab}(\hat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) \subset \mathsf{Stab}(\hat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$.

Problème I. *Généraliser l'inclusion de stabilisateurs* $\mathsf{Stab}(\hat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) \subset \mathsf{Stab}(\hat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ *à* $G \neq \{1\}$.

Contenu du chapitre 2 : Formalisme de Rham de la théorie des doubles mélanges

Si $G \neq \{1\}$, l'action du groupe $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ sur $\mathbf{k}\langle\langle X\rangle\rangle$ par automorphismes de $\mathbf{k}$-algèbre[3] $\Psi \mapsto \mathrm{Ad}_\Psi \circ \mathrm{aut}_\Psi$ ne se restreint pas à une action sur la sous-algèbre topologique $\mathbf{k}\langle\langle Y\rangle\rangle$. En effet, voici un contre-exemple: puisque $G \neq \{1\}$, soit $g \neq 1$ un élément de G. Posons $\Psi = \exp([x_1, x_0]) \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. On a

$$\mathrm{Ad}_\Psi \circ \mathrm{aut}_\Psi(x_g) = \Psi t_g(\Psi^{-1}) x_g t_g(\Psi)\Psi^{-1}.$$

[2] Plus récemment, Enriquez et Furusho ont démontré dans [EF4] qu'on a, en fait, égalité des deux stabilisateurs.

[3] où, pour $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, l'automorphisme de $\mathbf{k}$-algèbre aut_Ψ est donné dans (1.7).

Posons $u = \Psi t_g(\Psi^{-1})$ et $v = t_g(\Psi)\Psi^{-1}$. On peut aisément vérifier que $u \in \mathbf{k}\langle\langle X\rangle\rangle^\times$ et $v \in \mathbf{k}\langle\langle X\rangle\rangle$. Par conséquent, en utilisant le fait que ([Yad, Lemme 2.16])

$$\forall u \in \mathbf{k}\langle\langle X\rangle\rangle^\times, \forall v \in \mathbf{k}\langle\langle X\rangle\rangle,\ u x_g v \in \mathbf{k}\langle\langle Y\rangle\rangle \iff v \in \mathbf{k}\langle\langle Y\rangle\rangle \tag{8}$$

on obtient que $u x_g v = \Psi t_g(\Psi^{-1}) x_g t_g(\Psi)\Psi^{-1}$ appartient à $\mathbf{k}\langle\langle Y\rangle\rangle$ si et seulement si $v = t_g(\Psi)\Psi^{-1}$ appartient à $\mathbf{k}\langle\langle Y\rangle\rangle$. D'un autre côté, nous avons

$$t_g(\Psi)\Psi^{-1} = \exp([x_g, x_0])\exp(-[x_1, x_0]) = 1 + [x_g - x_1, x_0] + \text{terms of order} > 2.$$

La composante $\mathbf{k}\langle\langle Y\rangle\rangle$ du terme d'ordre 2 est égale à $x_0(x_1 - x_g)$ et la composante $\mathbf{k}\langle\langle X\rangle\rangle x_0$ à $(x_g - x_1)x_0$; cette dernière étant non nulle, $t_g(\Psi)\Psi^{-1}$ n'est, par conséquent, pas dans $\mathbf{k}\langle\langle Y\rangle\rangle$ ce qui implique, par (8) encore, que $\Psi t_g(\Psi^{-1}) x_g t_g(\Psi)\Psi^{-1} \notin \mathbf{k}\langle\langle Y\rangle\rangle$.

Cette situation ne permet pas une généralisation directe du résultat de [EF2]. Par conséquent, résoudre le problème I revient à résoudre le problème suivant

Problème II. *Trouver une algèbre convenable contenant $\mathbf{k}\langle\langle Y\rangle\rangle$ et une action de groupe de $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ sur cette algèbre qui se restreint à une sous-algèbre isomorphe à $\mathbf{k}\langle\langle Y\rangle\rangle$ ayant une action libre de rang 1 sur $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$.*

La généralisation voulue est obtenue l'*algèbre produit croisé* $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ (voir définition 2.1.4), celle-ci contient $\mathbf{k}\langle\langle X\rangle\rangle$. Au chapitre 2, on développe un formalisme parallèle à celui de Racinet à partir de cette algèbre. Ceci constitue le côté « de Rham » de la théorie des doubles mélanges.

Au sein de ce formalisme, l'algèbre produit croisé est identifiée à la $\mathbf{k}$-algèbre topologique $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ définie par une présentation avec générateurs et relations (voir proposition 2.1.9). On construit ensuite une sous-algèbre $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ de $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ isomorphe à l'algèbre $\mathbf{k}\langle\langle Y\rangle\rangle$ (voir proposition 2.1.15) ainsi qu'un module quotient $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ du $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$-module régulier à gauche isomorphe au module $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ (voir proposition-définition 2.1.25. (ii)). L'algèbre $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ est munie d'un coproduit d'algèbre de Hopf $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ et le module $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ est muni d'un coproduit de coalgèbre $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$ compatible avec $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$. Le groupe $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ agit compatiblement sur l'algèbre $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ et sur son module régulier à gauche. Contrairement à la situation avec $\mathbf{k}\langle\langle Y\rangle\rangle \subset \mathbf{k}\langle\langle X\rangle\rangle$, l'action sur l'algèbre $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ se restreint à la sous-algèbre $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$, et l'action sur le $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$-module régulier à gauche induit une action sur le module quotient $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$. On peut résumer ceci à l'aide du diagramme de morphismes d'algèbres-modules suivant :

$$\begin{array}{ccc} (\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{V}}_G^{\mathrm{DR}}) & \xrightarrow{(\mathrm{id}, \widehat{-\cdot 1_{\mathrm{DR}}})} & (\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}) \\ & & \uparrow {\scriptstyle (\mathrm{incl}, \mathrm{id})} \\ & & (\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}) \end{array} \tag{9}$$

Cette situation permet de définir deux stabilisateurs $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})(\mathbf{k})$ et $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k})$ qui sont des sous-groupes de $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. On vérifie alors que $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k})$ est une généralisation du stabilisateur ayant la même notation défini dans [EF2] pour $G = \{1\}$. On a alors une généralisation de [EF2, Theorem 3.1]

Théorème I (Théorème 2.3.5)**.** *On a* $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) \subset \mathsf{Stab}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k})$ *(inclusion de sous-groupes de* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*).*

On relie ensuite un des stabilisateurs ci-dessus au formalisme de Racinet :

Théorème II (Théorème 2.3.9)**.** *On a* $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) = \mathbf{k}^\times \ltimes \mathsf{Stab}(\hat{\Delta}^{\mathrm{mod}}_\star)(\mathbf{k})$ *(égalité de sous-groupes de* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*).*

Puis, au moyen d'isomorphismes adéquats, on définit un stabilisateur $\mathsf{Stab}(\hat{\Delta}^{\mathrm{alg}}_*)(\mathbf{k})$ que l'on exprime purement dans le formalisme de Racinet et on obtient

Théorème III (Théorème 2.3.21)**.** *On a* $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k}) = \mathbf{k}^\times \ltimes \mathsf{Stab}(\hat{\Delta}^{\mathrm{alg}}_*)(\mathbf{k})$ *(égalité de sous-groupes de* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*).*

Contenu du chapitre 3 : Formalisme Betti de la théorie des doubles mélanges

Pour $G = \{1\}$ Enriquez et Furusho ont introduit un sous-schéma $\mathsf{DMR}^{\mathrm{DR,B}}$ de DMR^ι et l'ont identifié à un torseur d'isomorphismes reliant des objets « de Rham » à des objets « Betti ». Dans le dernier chapitre, on note $\mathsf{DMR}^\iota_\times$ le sous-schéma de DMR^ι généralisant $\mathsf{DMR}^{\mathrm{DR,B}}$ pour G général et on démontre un analogue de ce résultat.

Dans un premier temps, décrivons le formalisme « Betti » introduit dans [EF1] pour $G = \{1\}$. Il se base sur l'algèbre filtrée $\mathcal{V}^{\mathrm{B}}$, définie comme l'algèbre de groupe sur $\mathbf{k}$ du groupe libre de rang 2 noté F_2 dont les générateurs sont X_0 et X_1 munie de la filtration induite par son idéal d'augmentation. La complétion $\widehat{\mathcal{V}}^{\mathrm{B}}$ est une $\mathbf{k}$-algèbre topologique qui s'identifie à l'algèbre topologique $\mathbf{k}\langle\langle X\rangle\rangle$ par l'isomorphisme $X_i \mapsto \exp(x_i)$ avec $i \in \{0,1\}$ (voir [EF1, Partie 2, §3.3]). Ensuite, nous avons une algèbre de Hopf $(\widehat{\mathcal{W}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{W},\mathrm{B}})$ où $\widehat{\mathcal{W}}^{\mathrm{B}}$ est la sous-algèbre de $\widehat{\mathcal{V}}^{\mathrm{B}}$ linéairement engendrée par l'unité et l'idéal à gauche engendré par $X_1 - 1$. Elle est présentée en tant qu'algèbre par les générateurs

$$X_1,\, X_1^{-1},\, Y_n^+ = -(X_0-1)^{n-1}X_0(X_1-1) \quad \text{et} \quad Y_n^- = -(X_0^{-1}-1)^{n-1}X_0^{-1}(X_1^{-1}-1)$$

pour $n \in \mathbb{N}^*$, avec la relation $X_1X_1^{-1} = X_{-1}X_1 = 1$. De plus, nous avons un coprduit d'algèbre de Hopf $\widehat{\Delta}^{\mathcal{W},\mathrm{B}} : \widehat{\mathcal{W}}^{\mathrm{B}} \to (\widehat{\mathcal{W}}^{\mathrm{B}})^{\hat{\otimes}2}$ donné par

$$\widehat{\Delta}^{\mathcal{W},\mathrm{B}}(X_1^{\pm 1}) = X_1^{\pm 1} \otimes X_1^{\pm 1} \text{ et pour } n \in \mathbb{N}^*, \widehat{\Delta}^{\mathcal{W},\mathrm{B}}(Y_n^\pm) = Y_n^\pm \otimes 1 + 1 \otimes Y_n^\pm + \sum_{\substack{k,l \in \mathbb{N}^* \\ k+l=n}} Y_k^\pm \otimes Y_l^\pm.$$

Par ailleurs, l'isomorphisme de $\mathbf{k}$-algèbres topologiques $\widehat{\mathcal{V}}^{\mathrm{B}} \to \mathbf{k}\langle\langle X\rangle\rangle$ se restreint en un isomorphisme d'algèbres topologiques $\widehat{\mathcal{W}}^{\mathrm{B}} \to \mathbf{k}\langle\langle Y\rangle\rangle$ (voir [EF1, Partie 2, §3.3]). Enfin, on a une coalgèbre $(\widehat{\mathcal{M}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{M},\mathrm{B}})$ formée d'un module quotient $\widehat{\mathcal{M}}^{\mathrm{B}} = \widehat{\mathcal{V}}^{\mathrm{B}}/\widehat{\mathcal{V}}^{\mathrm{B}}(X_0 - 1)$ isomorphe à $\widehat{\mathcal{W}}^{\mathrm{B}}$, en tant que $\mathbf{k}$-module (voir [EF1, (2.1.1)]) ainsi que d'un coproduit de coalgèbre $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}$ compatible avec le coproduit $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}$. De plus, l'isomorphisme de $\mathbf{k}$-algèbres topologiques $\widehat{\mathcal{V}}^{\mathrm{B}} \to \mathbf{k}\langle\langle X\rangle\rangle$ induit un isomorphisme de $\mathbf{k}$-modules topologiques $\widehat{\mathcal{M}}^{\mathrm{B}} \to \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$.

Problème III. *Pour un groupe cyclique fini* G*, construire des analogues de l'algèbre de Hopf* $(\widehat{\mathcal{W}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{W},\mathrm{B}})$ *et du module-coalgèbre* $(\widehat{\mathcal{M}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{M},\mathrm{B}})$*.*

Soit N l'ordre de G. La solution du problème III introduite au chapitre 3 est basée sur l'algèbre filtrée $\mathcal{V}_N^{\mathrm{B}}$, définie comme l'algèbre de groupe $\mathbf{k}F_2$ munie de la filtration induite par l'idéal $\ker(\mathbf{k}F_2 \to \mathbf{k}\mu_N)$; où $\mathbf{k}F_2 \to \mathbf{k}\mu_N$ est le morphisme d'algèbres induit par le morphisme de groupes $F_2 \to \mu_N$ donné par $X_0 \mapsto e^{\frac{i2\pi}{N}}$ et $X_1 \mapsto 1$. Sa complétion, notée $\widehat{\mathcal{V}}_N^{\mathrm{B}}$, est la limite inverse du système projectif induit par la filtration. C'est une algèbre topologique isomorphe à $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ (voir proposition-définition 3.1.9). Ensuite, on introduit une algèbre filtrée $\mathcal{W}_N^{\mathrm{B}}$ qui est linéairement engendrée par l'unité et l'idéal à gauche engendré par $X_1 - 1$ et dont la filtration est induite par celle de $\mathcal{V}_N^{\mathrm{B}}$. Sa complétion $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ est isomorphe à la $\mathbf{k}$-algèbre topologique $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ (voir proposition-définition 3.1.16). Puis, on introduit le module filtré $\mathcal{M}_N^{\mathrm{B}}$ défini par la donnée du module quotient $\mathbf{k}F_2/\mathbf{k}F_2(X_0 - 1)$ et de la filtration induite par celle de $\mathcal{V}_N^{\mathrm{B}}$. Sa complétion $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ est isomorphe au $\mathbf{k}$-module topologique $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ (voir proposition-définition 3.1.26). On montre alors que, sur $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ et $\widehat{\mathcal{M}}_N^{\mathrm{B}}$, on aboutit respectivement à des structures d'algèbre de Hopf et de coalgèbre compatibles. Cela provient du résultat suivant :

Théorème IV (Théorèmes 3.3.17 et 3.3.19)**.** *Il existe un morphisme de* $\mathbf{k}$*-algèbres topologiques* $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}} : \widehat{\mathcal{W}}_N^{\mathrm{B}} \to (\widehat{\mathcal{W}}_N^{\mathrm{B}})^{\hat{\otimes}2}$ *et un morphisme de* $\mathbf{k}$*-modules topologiques* $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to (\widehat{\mathcal{M}}_N^{\mathrm{B}})^{\hat{\otimes}2}$ *qui munissent respectivement* $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ *et* $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ *d'une structure d'algèbre de Hopf et de coalgèbre compatibles.*

On déduit le résultat suivant :

Théorème V. $\mathsf{DMR}_\times^\iota$ *est contenu dans le torseur d'isomorphismes reliant* $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ *(resp.* $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}$*) à* $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ *(resp.* $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$*).*

Introduction

A *multiple polylogarithm value* or *multiple L-value* (MLV in short) is a complex number defined by the following series

$$L_{(k_1,\ldots,k_r)}(z_1,\ldots,z_r) := \sum_{0<m_1<\cdots<m_r} \frac{z_1^{m_1}\cdots z_r^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}, \tag{1}$$

where $r, k_1, \ldots, k_r$ are positive integers and $z_1, \ldots, z_r$ in μ_N the group of N^{th} roots of unity in $\mathbb{C}$ with N a positive integer. The series (1) converges if and only if $(k_r, z_r) \neq (1,1)$. These values have been defined by Goncharov in [Gon98] and [Gon01a] and studied by many others like Arakawa and Kaneko in [ArKa] and appear as a generalisation of the so called multiple zeta values which in turn generalise the special values of the Riemann zeta function. Among the relations satisfied by the MLVs, our main interest here are the *double shuffle and regularisation* ones. Indeed, it is known that MLVs can be expressed as an iterated integral[4] in the following way (see, for example, [Gon98, Theorem 2.1])

$$L_{(k_1,\ldots,k_r)}(z_1,\ldots,z_r) = \int_0^1 \Omega_0^{k_r-1}\Omega_{z_r}\Omega_0^{k_{r-1}-1}\Omega_{z_{r-1}z_r}\cdots\Omega_0^{k_1-1}\Omega_{z_1\cdots z_r} \tag{2}$$

where $\Omega_0 = \frac{ds}{s}$ and $\Omega_z = \frac{ds}{z^{-1}-s}$ for $z \in \mu_N$. Next, the product of MLVs can be expressed as a $\mathbb{Q}$-linear combination of other MLVs (see [Gon98, §2]).

Example. Using the iterated sum (1) we have for $(k_1, z_1), (k_2, z_2) \in \mathbb{N}^* \times \mu_N \diagdown \{(1,1)\}$

$$\begin{aligned} L_{k_1}(z_1)L_{k_2}(z_2) &= \sum_{m_1,m_2\in\mathbb{N}^*} \frac{z_1^{m_1} z_2^{m_2}}{k_1^{m_1} k_2^{m_2}} = \left(\sum_{0<m_1<m_2} + \sum_{0<m_2<m_1} + \sum_{0<m_1=m_2}\right) \frac{z_1^{m_1} z_2^{m_2}}{k_1^{m_1} k_2^{m_2}} \\ &= L_{(k_1,k_2)}(z_1,z_2) + L_{(k_2,k_1)}(z_2,z_1) + L_{k_1+k_2}(z_1z_2). \end{aligned}$$

[4]Here we use notations of [Kas, Chapter XIX, §11].

Using the iterated integral (2) we have by taking $k_1 = 2$ and $k_2 = 3$

$$
\begin{aligned}
L_2(z_1)L_3(z_2) &= \int_0^1 \Omega_0\Omega_{z_1} \int_0^1 \Omega_0^2\Omega_{z_2} \\
&= \int_0^1 \Omega_0\Omega_{z_1}\Omega_0^2\Omega_{z_2} + 2\int_0^1 \Omega_0^2\Omega_{z_1}\Omega_0\Omega_{z_2} + 3\int_0^1 \Omega_0^3\Omega_{z_1}\Omega_{z_2} \\
&\quad + 3\int_0^1 \Omega_0^3\Omega_{z_2}\Omega_{z_1} + \int_0^1 \Omega_0^2\Omega_{z_2}\Omega_0\Omega_{z_1} \\
&= L_{(3,2)}(z_1^{-1}z_2, z_1) + 2L_{(2,3)}(z_1^{-1}z_2, z_1) + 3L_{(1,4)}(z_1^{-1}z_2, z_1) \\
&\quad + 3L_{(1,4)}(z_1z_2^{-1}, z_2) + L_{(2,3)}(z_1z_2^{-1}, z_2)
\end{aligned}
$$

where the second equality comes from the fact that for any $(n, m) \in \mathbb{N}^2$ and for any complex-valued differential 1-forms $\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_{n+m}$ on an interval $[a, b]$ we have (see, for example, [Kas, Chapter XIX, §11])

$$\int_a^b \omega_1 \cdots \omega_n \int_a^b \omega_{n+1} \cdots \omega_{n+m} = \sum_\sigma \int_a^b \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(n)}\, \omega_{\sigma^{-1}(n+1)} \cdots \omega_{\sigma^{-1}(n+m)},$$

where elements σ are (n, m)-shuffles of the symmetric group $\mathfrak{S}_{n+m}$.

We obtain two systems of relations associated to the iterated sum (1) and to the iterated integral (2) respectively. Combining those two systems we obtain the double shuffle relations and by adjoining regularisation relations obtained by formally annihilating some divergences, one obtains the "double shuffle and regularisation" system of relations.

Contents of Chapter 1 : The double shuffle torsor

Understanding the double shuffle and regularisation relations has been greatly improved thanks to Racinet's work [Rac]. He generalises the group μ_N to a finite cyclic group G and identifies the differential 1-forms in (2) with non-commutative variables x_0 and x_g $(g \in G)$ of a free formal series algebra $\mathbf{k}\langle\langle X\rangle\rangle$ over a commutative $\mathbb{Q}$-algebra $\mathbf{k}$. He then attaches to each pair (G, ι) of a finite cyclic group G and a group injection $\iota : G \to \mathbb{C}^\times$, a $\mathbb{Q}$-scheme DMR^ι which associates to each commutative $\mathbb{Q}$-algebra $\mathbf{k}$, a set $\mathsf{DMR}^\iota(\mathbf{k})$ that can be decomposed as a disjoint union of sets $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ for $\lambda \in \mathbf{k}$ (see [Rac, Definition 3.2.1]). The double shuffle and regularisation relations on MLVs are then encoded in the statement that a suitable generating series of these values (see [Rac, 2.2.3]) belongs to the set $\mathsf{DMR}^{\iota_{can}}_{i2\pi}(\mathbb{C})$ where $\iota_{can} : G = \mu_N \hookrightarrow \mathbb{C}^\star$ is the canonical inclusion.

Racinet also proved that for any pair (G, ι), the set $\mathsf{DMR}^\iota_0(\mathbf{k})$ equipped with the product $\circledast$ given in (1.10) is a group that is independent of the choice of the group embedding $\iota : G \to \mathbb{C}^\times$, we denote it $\mathsf{DMR}^G_0(\mathbf{k})$. The set $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ of grouplike elements of $\mathbf{k}\langle\langle X\rangle\rangle$ – for the coproduct that renders x_0 and x_g $(g \in G)$ primitive – is a group when equipped with the product $\circledast$ (see Proposition-Definition 1.2.4) which contains $\mathsf{DMR}^G_0(\mathbf{k})$ as a subgroup. It is equipped with a group action of $\mathbf{k}^\times$ (see Proposition 1.3.5) which enables us to consider the semidirect product $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ whose group law is denoted $\circledast$ as well. The action of $\mathbf{k}^\times$ restricts to the subgroup

$\mathsf{DMR}_0^G(\mathbf{k})$ (see Proposition 1.3.16). We then obtain the semidirect product subgroup $\mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k})$. Thanks to [Rac, Theorem I], for $\lambda \in \mathbf{k}$, the set $\mathsf{DMR}_\lambda^\iota(\mathbf{k})$ has a torsor structure over the group $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$ (see Proposition 1.3.15). This provides a torsor structure of the set $\mathsf{DMR}_\times^\iota(\mathbf{k}) := \bigsqcup_{\lambda\in\mathbf{k}^\times} \mathsf{DMR}_\lambda^\iota(\mathbf{k})$ over $\mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k})$ (see Proposition 1.3.20). These results motivate the study of the group $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$.

In order to improve the understanding of the group $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$, Enriquez and Furusho, in [EF0], related this group with the stabilizer $\mathsf{Stab}(\hat{\Delta}_\star)(\mathbf{k})$ of the coproduct $\hat{\Delta}_\star : \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \to (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\hat{\otimes}2}$ given in [Rac, (2.3.1.2)] for an action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ (see Proposition 1.2.19).

In addition, Racinet's work also introduced a subalgebra $\mathbf{k}\langle\langle Y\rangle\rangle$ of $\mathbf{k}\langle\langle X\rangle\rangle$ spanned by the words ending with x_g with $g \in G$. It is identified, as a $\mathbf{k}$-module, with $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ and is equipped with a coproduct $\mathbf{k}\langle\langle Y\rangle\rangle \to \mathbf{k}\langle\langle Y\rangle\rangle^{\hat{\otimes}2}$ compatible with $\hat{\Delta}_\star$. For this reason, the former coproduct has also the same notation in [Rac]. However, we will adopt distinct notation for these two coproducts, by denoting respectively $\hat{\Delta}_\star^{\mathrm{alg}}$ and $\hat{\Delta}_\star^{\mathrm{mod}}$ the coproducts on $\mathbf{k}\langle\langle Y\rangle\rangle$ and $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$.

The situation, detailed in §1.1, may be summarised by the diagram of algebra-modules

$$\begin{array}{ccc} (\mathbf{k}\langle\langle X\rangle\rangle, \mathbf{k}\langle\langle X\rangle\rangle) & \xrightarrow{(\mathrm{id},\pi_Y)} & (\mathbf{k}\langle\langle X\rangle\rangle, \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0) \\ & & \uparrow{\scriptstyle (\mathrm{incl},\mathrm{id})} \\ & & (\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0) \end{array} \tag{3}$$

The bottom term has a pair of compatible coproducts $(\hat{\Delta}_\star^{\mathrm{alg}}, \hat{\Delta}_\star^{\mathrm{mod}})$. The group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ acts by automorphisms of algebra-modules on the horizontal terms, the associated arrow being equivariant. However, it does not have an action on the bottom object in general but only on its module part; it is on this action that the stabilizer group construction of [EF0] is based on.

When $G = \{1\}$, it was proved in [EF1, Part 2, §3] that the subalgebra $\mathbf{k}\langle\langle Y\rangle\rangle$ of $\mathbf{k}\langle\langle X\rangle\rangle$ is stable under the action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathbf{k}\langle\langle X\rangle\rangle$. One can therefore construct the stabilizer group $\mathsf{Stab}(\hat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ of $\hat{\Delta}_\star^{\mathrm{alg}}$ with respect to the action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\hat{\otimes}2})$. By [EF2, §3.1], one then has the inclusion[5] $\mathsf{Stab}(\hat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) \subset \mathsf{Stab}(\hat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$.

Problem I. *Generalise the stabilizer inclusion* $\mathsf{Stab}(\hat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) \subset \mathsf{Stab}(\hat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ *to* $G \neq \{1\}$.

Contents of Chapter 2 : De Rham formalism of the double shuffle theory

If $G \neq \{1\}$, the action of the group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathbf{k}\langle\langle X\rangle\rangle$ by $\mathbf{k}$-algebra automorphisms[6] $\Psi \mapsto \mathrm{Ad}_\Psi \circ \mathrm{aut}_\Psi$ does not restrict to an action on the topological subalgebra $\mathbf{k}\langle\langle Y\rangle\rangle$. Indeed, let us provide a counterexample: since $G \neq \{1\}$, let $g \neq 1$ an element of G. Let us set $\Psi = \exp([x_1, x_0]) \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. We have

$$\mathrm{Ad}_\Psi \circ \mathrm{aut}_\Psi(x_g) = \Psi t_g(\Psi^{-1}) x_g t_g(\Psi)\Psi^{-1}.$$

[5] More recently, Enriquez and Furusho showed in [EF4] that the two stabilizers are equal.

[6] where, for $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the $\mathbf{k}$-algebra automorphism aut_Ψ is given in (1.7).

Set $u = \Psi t_g(\Psi^{-1})$ and $v = t_g(\Psi)\Psi^{-1}$. One checks that $u \in \mathbf{k}\langle\langle X\rangle\rangle^\times$ and $v \in \mathbf{k}\langle\langle X\rangle\rangle$. Therefore, one may use the fact that ([Yad, Lemma 2.16])

$$\forall u \in \mathbf{k}\langle\langle X\rangle\rangle^\times, \forall v \in \mathbf{k}\langle\langle X\rangle\rangle,\ ux_gv \in \mathbf{k}\langle\langle Y\rangle\rangle \iff v \in \mathbf{k}\langle\langle Y\rangle\rangle \tag{4}$$

in order to obtain that $ux_gv = \Psi t_g(\Psi^{-1})x_g t_g(\Psi)\Psi^{-1}$ belongs to $\mathbf{k}\langle\langle Y\rangle\rangle$ if and only if $v = t_g(\Psi)\Psi^{-1}$ is in $\mathbf{k}\langle\langle Y\rangle\rangle$. On the other hand, one has

$$t_g(\Psi)\Psi^{-1} = \exp([x_g, x_0])\exp(-[x_1, x_0]) = 1 + [x_g - x_1, x_0] + \text{terms of order} > 2.$$

The order 2 term has $\mathbf{k}\langle\langle Y\rangle\rangle$ component equal to $x_0(x_1 - x_g)$ and $\mathbf{k}\langle\langle X\rangle\rangle x_0$ component equal to $(x_g - x_1)x_0$; the latter being non zero, $t_g(\Psi)\Psi^{-1}$ is, therefore, not in $\mathbf{k}\langle\langle Y\rangle\rangle$ which implies, by (4) again, that $\Psi t_g(\Psi^{-1})x_g t_g(\Psi)\Psi^{-1} \notin \mathbf{k}\langle\langle Y\rangle\rangle$.

This situation forbids a direct generalisation of the result of [EF2]. Therefore, solving Problem I comes down to solve the following

Problem II. *Find a suitable algebra that contains* $\mathbf{k}\langle\langle Y\rangle\rangle$ *and a group action of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on that algebra that restricts to a subalgebra isomorphic to* $\mathbf{k}\langle\langle Y\rangle\rangle$ *with a free rank* 1 *action on* $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$.

The wanted generalisation is obtained by introducing an algebra called the *crossed product algebra* $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ (see Definition 2.1.4) which contains $\mathbf{k}\langle\langle X\rangle\rangle$. In addition, we develop a formalism on it which is parallel to Racinet's as described in Chapter 2. This constitutes the "de Rham" side of the double shuffle theory.

In this framework, the crossed product algebra is identified to a topological $\mathbf{k}$-algebra $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ defined by a presentation with generators and relations (see Proposition 2.1.9). Next, one constructs a subalgebra $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ isomorphic to the algebra $\mathbf{k}\langle\langle Y\rangle\rangle$ (see Proposition 2.1.15) and a quotient module $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ of the left regular $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$-module isomorphic to the module $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ (see Proposition-Definition 2.1.25. (ii)). The algebra $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ is equipped with a Hopf algebra coproduct $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ and the module $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is equipped with a compatible coalgebra coproduct $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$. The group $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ acts compatibly on the algebra $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ and on its regular left module. In contrast to the situation with $\mathbf{k}\langle\langle Y\rangle\rangle \subset \mathbf{k}\langle\langle X\rangle\rangle$, the action on the algebra $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ restricts to the subalgebra $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$, while the action on the left regular $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$-module induces an action of the quotient module $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$. This can be summarised in the following diagram of algebra-modules

$$\begin{array}{ccc} (\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{V}}_G^{\mathrm{DR}}) & \xrightarrow{(\mathrm{id}, \widehat{-\cdot 1_{\mathrm{DR}}})} & (\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}) \\ & & \uparrow {\scriptstyle (\mathrm{incl},\mathrm{id})} \\ & & (\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}) \end{array} \tag{5}$$

This situation allows us to define two stabilizers $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})(\mathbf{k})$ and $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k})$ which are subgroups of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. One shows that $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k})$ is a generalisation of the stabilizer with the same notation defined in [EF2] for $G = \{1\}$. We then have a generalisation of [EF2, Theorem 3.1]

Theorem I (Theorem 2.3.5)**.** *We have* $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) \subset \mathsf{Stab}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k})$ *(inclusion of subgroups of* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*).*

After that, we relate one of the above stabilizers to Racinet's formalism

Theorem II (Theorem 2.3.9)**.** *We have* $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) = \mathbf{k}^\times \ltimes \mathsf{Stab}(\hat{\Delta}^{\mathrm{mod}}_\star)(\mathbf{k})$ *(equality of subgroups of* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*).*

We then define an explicit group $\mathsf{Stab}(\hat{\Delta}^{\mathrm{alg}}_*)(\mathbf{k})$ expressed purely in Racinet's formalism by working out the suitable isomorphisms and obtain

Theorem III (Theorem 2.3.21)**.** *We have* $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k}) = \mathbf{k}^\times \ltimes \mathsf{Stab}(\hat{\Delta}^{\mathrm{alg}}_*)(\mathbf{k})$ *(equality of subgroups of* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*).*

Contents of Chapter 3 : Betti formalism of the double shuffle theory

For $G = \{1\}$ Enriquez and Furusho introduced a subscheme $\mathsf{DMR}^{\mathrm{DR,B}}$ of DMR^ι and proved it to be a torsor of isomorphisms relating "de Rham" and "Betti" objects. In the last chapter, we denote by $\mathsf{DMR}^\iota_\times$ the subscheme of DMR^ι generalizing $\mathsf{DMR}^{\mathrm{DR,B}}$ in the case of a general G and prove an analogue of this result.

Let us first describe the "Betti" formalism introduced in [EF1] for $G = \{1\}$. It is based on the filtered algebra $\mathcal{V}^{\mathrm{B}}$, which denotes the group algebra over $\mathbf{k}$ of the free group of rank 2 denoted F_2 with generators X_0 and X_1 and equipped with the filtration induced by the augmentation ideal. The completion $\widehat{\mathcal{V}}^{\mathrm{B}}$ is a topological $\mathbf{k}$-algebra which is identified with the topological algebra $\mathbf{k}\langle\langle X\rangle\rangle$ by the isomorphism $X_i \mapsto \exp(x_i)$ with $i \in \{0,1\}$ (see [EF1, Part 2, §3.3]). Next, we have a Hopf algebra $(\widehat{\mathcal{W}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{W},\mathrm{B}})$ which consists of a subalgebra $\widehat{\mathcal{W}}^{\mathrm{B}}$ of $\widehat{\mathcal{V}}^{\mathrm{B}}$ linearly generated by $1 \in \widehat{\mathcal{V}}^{\mathrm{B}}$ and the left ideal generated by $X_1 - 1$. It is presented as an algebra with generators X_1, X_1^{-1}, $Y_n^+ = -(X_0-1)^{n-1}X_0(X_1-1)$ and $Y_n^- = -(X_0^{-1}-1)^{n-1}X_0^{-1}(X_1^{-1}-1)$ for $n \in \mathbb{N}^*$, with relation $X_1X_1^{-1} = X_{-1}X_1 = 1$. In addition, we have a Hopf algebra coproduct $\widehat{\Delta}^{\mathcal{W},\mathrm{B}} : \widehat{\mathcal{W}}^{\mathrm{B}} \to (\widehat{\mathcal{W}}^{\mathrm{B}})^{\hat{\otimes}2}$ given by

$$\widehat{\Delta}^{\mathcal{W},\mathrm{B}}(X_1^{\pm1}) = X_1^{\pm1} \otimes X_1^{\pm1} \text{ and for } n \in \mathbb{N}^*, \widehat{\Delta}^{\mathcal{W},\mathrm{B}}(Y_n^\pm) = Y_n^\pm \otimes 1 + 1 \otimes Y_n^\pm + \sum_{\substack{k,l\in\mathbb{N}^* \\ k+l=n}} Y_k^\pm \otimes Y_l^\pm.$$

Moreover, the topological $\mathbf{k}$-algebra isomorphism $\widehat{\mathcal{V}}^{\mathrm{B}} \to \mathbf{k}\langle\langle X\rangle\rangle$ restricts to a topological $\mathbf{k}$-algebra isomorphism $\widehat{\mathcal{W}}^{\mathrm{B}} \to \mathbf{k}\langle\langle Y\rangle\rangle$ (see [EF1, Part 2, §3.3]). Finally, we have a coalgebra $(\widehat{\mathcal{M}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{M},\mathrm{B}})$ which consists of a quotient module $\widehat{\mathcal{M}}^{\mathrm{B}} = \widehat{\mathcal{V}}^{\mathrm{B}}/\widehat{\mathcal{V}}^{\mathrm{B}}(X_0-1)$ isomorphic to $\widehat{\mathcal{W}}^{\mathrm{B}}$, as a $\mathbf{k}$-module (see [EF1, (2.1.1)]) together with a coalgebra coproduct $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}$ compatible with the coproduct $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}$. Moreover, the topological $\mathbf{k}$-algebra isomorphism $\widehat{\mathcal{V}}^{\mathrm{B}} \to \mathbf{k}\langle\langle X\rangle\rangle$ induces a topological $\mathbf{k}$-module isomorphism $\widehat{\mathcal{M}}^{\mathrm{B}} \to \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$.

Problem III. *For a finite cyclic group G, construct analogues of the Hopf algebra* $(\widehat{\mathcal{W}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{W},\mathrm{B}})$ *and of the module-coalgebra* $(\widehat{\mathcal{M}}^{\mathrm{B}}, \widehat{\Delta}^{\mathcal{M},\mathrm{B}})$*.*

Let N be the order of G. The solution of Problem III introduced in Chapter 3 is based on the filtered algebra $\mathcal{V}^{\mathrm{B}}_N$, which denotes the group algebra $\mathbf{k}F_2$ equipped with the filtration induced by the ideal $\ker(\mathbf{k}F_2 \to \mathbf{k}\mu_N)$; where $\mathbf{k}F_2 \to \mathbf{k}\mu_N$ is the algebra morphism induced by the group morphism $F_2 \to \mu_N$ given by $X_0 \mapsto e^{\frac{i2\pi}{N}}$ and $X_1 \mapsto 1$. Its completion is the inverse limit of the projective system induced by the filtration and is denoted $\widehat{\mathcal{V}}_N^{\mathrm{B}}$. It is a topological algebra isomorphic to $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ (see Proposition-Definition 3.1.9). Next, we define a filtered algebra $\mathcal{W}_N^{\mathrm{B}}$ which is linearly generated by $1 \in \mathcal{V}_N^{\mathrm{B}}$ and the left ideal generated by $X_1 - 1$ and whose filtration is induced by that of $\mathcal{V}_N^{\mathrm{B}}$. Its completion $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ is isomorphic to $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ (see Proposition-Definition 3.1.16). We also define a filtered module $\mathcal{M}_N^{\mathrm{B}}$ which consists of the quotient module $\mathbf{k}F_2/\mathbf{k}F_2(X_0 - 1)$ and whose filtration is induced by that of $\mathcal{V}_N^{\mathrm{B}}$. Its completion $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ is isomorphic to $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ (see Proposition-Definition 3.1.26). We then have compatible Hopf algebra and coalgebra structures on $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ respectively thanks to the following result

Theorem IV (Theorems 3.3.17 and 3.3.19)**.** *There exists a topological* $\mathbf{k}$*-algebra morphism* $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}} : \widehat{\mathcal{W}}_N^{\mathrm{B}} \to (\widehat{\mathcal{W}}_N^{\mathrm{B}})^{\hat{\otimes}2}$ *and a topological* $\mathbf{k}$*-module morphism* $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to (\widehat{\mathcal{M}}_N^{\mathrm{B}})^{\hat{\otimes}2}$ *that endows* $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ *and* $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ *respectively with compatible Hopf algebra and coalgebra structures.*

We deduce the following result

Theorem V. $\mathsf{DMR}_{\times}^{\iota}$ *is contained in the torsor of isomorphisms relating* $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ *(resp.* $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}$*) to* $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ *(resp.* $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$*).*

Notation

Throughout this thesis, we consider for a commutative $\mathbb{Q}$-algebra $\mathbf{k}$, a $\mathbf{k}$-algebra A, an element $x \in A$ and a left A-module M the following:

- $\ell_x : M \to M$ to be the $\mathbf{k}$-module endomorphism defined by $m \mapsto xm$ and if x is invertible, then ℓ_x is an automorphism.
- $r_x : A \to A$ to be the $\mathbf{k}$-module endomorphism defined by $a \mapsto ax$ and if x is invertible, then r_x is an automorphism.
- $\mathrm{Ad}_x : A \to A$ to be the $\mathbf{k}$-algebra automorphism defined by $a \mapsto xax^{-1}$ with x being invertible.

1

The double shuffle torsor

Throughout this chapter, let G be a finite abelian group and $\mathbf{k}$ be a commutative $\mathbb{Q}$-algebra. We recall from [Rac] the basic formalism of the double shuffle theory, the main ingredients being presented in §1.1. In §1.2, we introduce the double shuffle group and we recall from [EF0] its stabilizer interpretation. Finally, we define the double shuffle scheme and we recall its torsor structure under the action of the double shuffle group.

1.1 Basic objects of Racinet's formalism

1.1.1 The Hopf algebra $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$

Proposition-Definition 1.1.1. *Let us denote* $X := \{x_0\} \sqcup \{x_g | g \in G\}$.

(i) Let $\mathbf{k}\langle X\rangle$ *be the free noncommutative associative polynomial algebra with unit over the alphabet* X*, including the empty word. It is graded with*

$$\deg(x_0) = \deg(x_g) = 1 \textit{ for } g \in G.$$

(ii) Let $\mathbf{k}\langle\langle X\rangle\rangle$ *be the degree completion of the graded algebra* $\mathbf{k}\langle X\rangle$*. It is a complete graded free noncommutative associative series algebra with unit over* X*.*

Proof. See [Rac, §1.6]. □

Lemma 1.1.2. *The algebra* $\mathbf{k}\langle\langle X\rangle\rangle$ *is endowed with a Hopf algebra structure for the coproduct* $\widehat{\Delta} : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle^{\hat{\otimes}2}$*, which is the unique morphism of topological* $\mathbf{k}$*-algebras given by* $\widehat{\Delta}(x_g) = x_g \otimes 1 + 1 \otimes x_g$*, for any* $g \in G \sqcup \{0\}$.

Proof. See [Rac, §2.2.3]. □

Proposition-Definition 1.1.3. *Let* $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *be the set of grouplike elements of* $\mathbf{k}\langle\langle X\rangle\rangle$ *for the coproduct* $\widehat{\Delta}$ *(see* (1.6)*). It is equipped with a group structure for the product of the algebra* $\mathbf{k}\langle\langle X\rangle\rangle$.

Proof. Immediate. □

Proposition-Definition 1.1.4.

(i) The group G acts on the set X, the permutation t_g corresponding to $g \in G$ being given by $t_g(x_0) = x_0$, $t_g(x_h) = x_{gh}$ for $h \in G$.

(ii) The action t extends to an action by $\mathbf{k}$-algebra automorphisms on $\mathbf{k}\langle X\rangle$ which will also be denoted $g \mapsto t_g$.

Proof. See [Rac, §3.1.1]. □

Proposition 1.1.5.

(i) The group action $t : G \to \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle X\rangle)$ extends to a group action by $\mathbf{k}$-algebra automorphisms on $\mathbf{k}\langle\langle X\rangle\rangle$ which will also be denoted $g \mapsto t_g$.

(ii) The action group $t : G \to \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle X\rangle\rangle)$ restricts to a group action by Hopf algebra automorphisms on $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$.

(iii) For any $g \in G$, the $\mathbf{k}$-algebra automorphism t_g of $\mathbf{k}\langle\langle X\rangle\rangle$ restricts to a group automorphism of $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$.

Proof.

(i) See [Rac, §3.1.1].

(ii) It follows by checking on generators the identity $\widehat{\Delta} \circ t_g = t_g^{\otimes 2} \circ \widehat{\Delta}$ for any $g \in G$.

(iii) It follows from (ii).

□

Lemma 1.1.6. *Each word in X can be uniquely written*

$$\left(x_0^{n_1} x_{g_1} x_0^{n_2} x_{g_2} \cdots x_0^{n_r} x_{g_r} x_0^{n_{r+1}}\right)_{\substack{r, n_1, \dots, n_{r+1} \in \mathbb{N} \\ g_1, \dots, g_r \in G}}.$$

This family forms a $\mathbf{k}$-module basis of $\mathbf{k}\langle X\rangle$.

Proof. See [Rac, §2.2.7]. □

Proposition-Definition 1.1.7.

(i) Let $\mathbf{q}$ be the $\mathbf{k}$-module automorphism of $\mathbf{k}\langle X\rangle$ given by

$$\mathbf{q}(x_0^{n_1-1} x_{g_1} x_0^{n_2-1} x_{g_2} \cdots x_0^{n_r-1} x_{g_r} x_0^{n_{r+1}-1}) = x_0^{n_1-1} x_{g_1} x_0^{n_2-1} x_{g_2 g_1^{-1}} \cdots x_0^{n_r-1} x_{g_r g_{r-1}^{-1}} x_0^{n_{r+1}-1}. \tag{1.1}$$

(ii) The $\mathbf{k}$-module automorphism $\mathbf{q}$ of $\mathbf{k}\langle X\rangle$ extends to a topological $\mathbf{k}$-module automorphism of $\mathbf{k}\langle\langle X\rangle\rangle$ which will be denoted $\mathbf{q}$ as well.

Proof. See [Rac, §2.2.7]. □

1.1.2 The Hopf algebra $(\mathbf{k}\langle\langle Y\rangle\rangle, \widehat{\Delta}_\star^{\mathrm{alg}})$

Proposition-Definition 1.1.8. *For $(n,g) \in \mathbb{N}^* \times G$, set $y_{n,g} := x_0^{n-1}x_g$. Let $Y := \{y_{n,g} | (n,g) \in \mathbb{N}^* \times G\}$.*

(i) Let $\mathbf{k}\langle Y\rangle$ be the free noncommutative associative polynomial algebra with unit over the alphabet Y, including the empty word. It is graded such that for every $(n,g) \in \mathbb{N}^* \times G$, the element $y_{n,g}$ is of degree n.

(ii) Let $\mathbf{k}\langle\langle Y\rangle\rangle$ be the degree completion of the graded algebra $\mathbf{k}\langle Y\rangle$. It is a complete graded free noncommutative associative series algebra with unit over the alphabet Y.

Proof. See [Rac, §2.2.5] and [Rac, §1.6]. □

Lemma 1.1.9. *The algebra $\mathbf{k}\langle\langle Y\rangle\rangle$ is equal to the topological $\mathbf{k}$-subalgebra of $\mathbf{k}\langle\langle X\rangle\rangle$*

$$\mathbf{k} \oplus \bigoplus_{g\in G} \mathbf{k}\langle\langle X\rangle\rangle x_g.$$

Proof. See [Rac, §2.2.5], [Rac, §1.6] and [EF0, §2.2]. □

Proposition-Definition 1.1.10.

(i) Let $\mathbf{q}_Y$ the $\mathbf{k}$-module automorphism of $\mathbf{k}\langle Y\rangle$ given by

$$\mathbf{q}_Y(y_{n_1,g_1}\cdots y_{n_r,g_r}) := y_{n_1,g_1}y_{n_2,g_2g_1^{-1}}\cdots y_{n_r,g_rg_{r-1}^{-1}}. \tag{1.2}$$

(ii) The $\mathbf{k}$-module automorphism $\mathbf{q}_Y$ of $\mathbf{k}\langle Y\rangle$ extends to a topological $\mathbf{k}$-module automorphism of $\mathbf{k}\langle\langle Y\rangle\rangle$ which will be denoted $\mathbf{q}_Y$ as well.

Proof. See [Rac, §2.2.7.]. □

Proposition-Definition 1.1.11. *The algebra $\mathbf{k}\langle\langle Y\rangle\rangle$ is endowed with a Hopf algebra structure for the coproduct $\widehat{\Delta}_\star^{\mathrm{alg}} : \mathbf{k}\langle\langle Y\rangle\rangle \to (\mathbf{k}\langle\langle Y\rangle\rangle)^{\hat{\otimes}2}$, called the* harmonic coproduct*, which is the unique morphism of topological $\mathbf{k}$-algebras given, for any $(n,g) \in \mathbb{N}^* \times G$, by*

$$\widehat{\Delta}_\star^{\mathrm{alg}}(y_{n,g}) = y_{n,g} \otimes 1 + 1 \otimes y_{n,g} + \sum_{\substack{k=1\\ h\in G}}^{n-1} y_{k,h} \otimes y_{n-k,gh^{-1}}. \tag{1.3}$$

Proof. See [Rac, §2.3.1]. □

Proposition 1.1.12.

(i) The group action $t : G \to \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle X\rangle\rangle)$ restricts to a group action by $\mathbf{k}$-algebra automorphisms on $\mathbf{k}\langle\langle Y\rangle\rangle$ which will also be denoted $g \mapsto t_g$.

(ii) The group action $t : G \to \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle)$ restricts to a group action by Hopf algebra automorphisms on $(\mathbf{k}\langle\langle Y\rangle\rangle, \widehat{\Delta}_\star^{\mathrm{alg}})$.

Proof. Immediate. □

1.1.3 The coalgebra $(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \widehat{\Delta}_\star^{\mathrm{mod}})$

Proposition-Definition 1.1.13.

(i) The graded quotient $\mathbf{k}$*-module* $\mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0$ *is a left* $\mathbf{k}\langle Y\rangle$*-module of rank* 1*.*

(ii) The $\mathbf{k}$*-module morphism* $\pi_Y : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0$ *is a surjective map and its restriction to* $\mathbf{k}\langle Y\rangle$ *is a bijective map.*

Proof. See [Rac, §2.2.5] and [EF0, §2.2]. □

Proposition-Definition 1.1.14.

(i) The degree completion of the graded $\mathbf{k}$*-module* $\mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0$ *is identified with the topological* $\mathbf{k}$*-module quotient* $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ *which is a left* $\mathbf{k}\langle\langle Y\rangle\rangle$*-module of rank* 1*.*

(ii) The $\mathbf{k}$*-module morphism* $\pi_Y : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0$ *extends to a topological* $\mathbf{k}$*-module morphism* $\mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ *which will also be denoted* π_Y*. Its restriction to* $\mathbf{k}\langle\langle Y\rangle\rangle$ *is a bijective map.*

Proof. See [Rac, §2.2.5] and [Rac, §1.6]. □

Proposition-Definition 1.1.15.

(i) There exists a unique $\mathbf{k}$*-module automorphism* $\overline{\mathbf{q}}$ *of* $\mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0$ *such that the following diagram*

$$\begin{array}{ccc} \mathbf{k}\langle X\rangle & \xrightarrow{\mathbf{q}} & \mathbf{k}\langle X\rangle \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ \mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0 & \xrightarrow[\overline{\mathbf{q}}]{} & \mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0 \end{array} \tag{1.4}$$

commutes.

(ii) The $\mathbf{k}$*-module automorphism* $\overline{\mathbf{q}}$ *of* $\mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0$ *extends to a topological* $\mathbf{k}$*-module automorphism of* $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ *which will be denonted* $\overline{\mathbf{q}}$ *as well. It is such that the following diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\mathbf{q}} & \mathbf{k}\langle\langle X\rangle\rangle \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\overline{\mathbf{q}}]{} & \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \end{array} \tag{1.5}$$

commutes.

Proof.

(i) It follows from the fact that $\mathbf{q}$ preserves the submodule $\mathbf{k}\langle\langle X\rangle\rangle x_0$.

(ii) It follows from Proposition-Definition 1.1.7.(ii).

□

Proposition-Definition 1.1.16. *The* $\mathbf{k}$*-module* $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ *is endowed with a cocommutative coassociative coalgebra structure for the coproduct*

$$\widehat{\Delta}_\star^{\mathrm{mod}} : \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \to \left(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0\right)^{\hat{\otimes}2},$$

which is the unique morphism of topological $\mathbf{k}$*-modules such that the following diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle Y\rangle\rangle & \xrightarrow{\widehat{\Delta}_\star^{\mathrm{alg}}} & \left(\mathbf{k}\langle\langle Y\rangle\rangle\right)^{\hat{\otimes}2} \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle (\pi_Y)^{\otimes 2}} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\widehat{\Delta}_\star^{\mathrm{mod}}]{} & \left(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0\right)^{\hat{\otimes}2} \end{array}$$

commutes.

Proof. It follows from Proposition-Definitions 1.1.11 and 1.1.14. □

1.2 The double shuffle group

1.2.1 The group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ and its actions

The group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$

Recall that the set of grouplike elements of $\mathbf{k}\langle\langle X\rangle\rangle$ for the coproduct $\widehat{\Delta}$ is

$$\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) = \{\Psi \in \mathbf{k}\langle\langle X\rangle\rangle^\times \mid \widehat{\Delta}(\Psi) = \Psi \otimes \Psi\}. \tag{1.6}$$

Definition 1.2.1 ([EF0, §4.1.3] based on [Rac, §3.1.2])**.** For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, let aut_Ψ be the topological $\mathbf{k}$-algebra automorphism of $\mathbf{k}\langle\langle X\rangle\rangle$ given by

$$x_0 \mapsto x_0 \quad \text{and for } g \in G, x_g \mapsto \mathrm{Ad}_{t_g(\Psi^{-1})}(x_g). \tag{1.7}$$

Definition 1.2.2 ([EF0, (5.15)] based on [Rac, (3.1.2.1)])**.** For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, let S_Ψ to be the topological $\mathbf{k}$-module automorphism of $\mathbf{k}\langle\langle X\rangle\rangle$ given by

$$S_\Psi := \ell_\Psi \circ \mathrm{aut}_\Psi. \tag{1.8}$$

Lemma 1.2.3. *For* $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, the* $\mathbf{k}$*-algebra automorphism* aut_Ψ *is a Hopf algebra automorphism of* $\left(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta}\right)$.

Proof. Both aut_Ψ and $\widehat{\Delta}$ are $\mathbf{k}$-algebra automorphisms. Therefore, thanks to Proposition 1.1.5.(ii), one can check on generators that

$$\widehat{\Delta} \circ \mathrm{aut}_\Psi = (\mathrm{aut}_\Psi)^{\otimes 2} \circ \widehat{\Delta}, \tag{1.9}$$

which is the wanted result. □

Proposition-Definition 1.2.4 ([Rac, Proposition 3.1.6]). *The pair* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *is a group, where for* $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$,

$$\Psi \circledast \Phi := S_\Psi(\Phi). \tag{1.10}$$

A proof of this claim is already available in Racinet's paper, however, somes elements of the proof will be used later on. Thus, we find it useful to rewrite it here. We will then need this result

Lemma 1.2.5. *For* $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, *we have*
(i) $\mathrm{aut}_{\Psi\circledast\Phi} = \mathrm{aut}_\Psi \circ \mathrm{aut}_\Phi$; *(ii)* $S_{\Psi\circledast\Phi} = S_\Psi \circ S_\Phi$.

This, in turn, uses the following technical lemma

Lemma 1.2.6. *For* $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *and* $g \in G$, *we have* $\mathrm{aut}_\Psi \circ t_g = t_g \circ \mathrm{aut}_\Psi$.

Proof. Immediate, it is obtained by checking the identity on generators. □

Proof of Lemma 1.2.5. It is enough to prove the identity (i) on generators. Since for $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ we have $\mathrm{aut}_\Psi(x_0) = x_0$, Identity (i) is immediately true for x_0. Then, for $g \in G$, we have

$$\begin{aligned}\mathrm{aut}_\Psi \circ \mathrm{aut}_\Phi(x_g) &= \mathrm{aut}_\Psi \circ \mathrm{Ad}_{t_g(\Phi^{-1})}(x_g) = \mathrm{Ad}_{\mathrm{aut}_\Psi(t_g(\Phi^{-1}))} \circ \mathrm{aut}_\Psi(x_g) \\ &= \mathrm{Ad}_{\mathrm{aut}_\Psi(t_g(\Phi^{-1}))} \circ \mathrm{Ad}_{t_g(\Psi^{-1})}(x_g) = \mathrm{Ad}_{t_g(\mathrm{aut}_\Psi(\Phi^{-1}))t_g(\Psi^{-1})}(x_g) \\ &= \mathrm{Ad}_{t_g(\mathrm{aut}_\Psi(\Phi^{-1})\Psi^{-1})}(x_g) = \mathrm{Ad}_{t_g((\Psi\circledast\Phi)^{-1})}(x_g) = \mathrm{aut}_{\Psi\circledast\Phi}(x_g),\end{aligned}$$

where the fourth equality is obtained by applying Lemma 1.2.6. This concludes the proof of Identity (i). Finally, by using the latter, we get

$$\begin{aligned}S_\Psi \circ S_\Phi =& \ell_\Psi \circ \mathrm{aut}_\Psi \circ \ell_\Phi \circ \mathrm{aut}_\Phi = \ell_\Psi \circ \ell_{\mathrm{aut}_\Psi(\Phi)} \circ \mathrm{aut}_\Psi \circ \mathrm{aut}_\Phi \\ =& \ell_{\Psi \mathrm{aut}_\Psi(\Phi)} \circ \mathrm{aut}_\Psi \circ \mathrm{aut}_\Phi = \ell_{\Psi\circledast\Phi} \circ \mathrm{aut}_{\Psi\circledast\Phi} = S_{\Psi\circledast\Phi},\end{aligned}$$

thus, establishing Identity (ii). □

Proof of Proposition-Definition 1.2.4. From Lemma 1.2.3, we deduce that $\circledast$ has its image in $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. Next, thanks to Identity (ii) in Lemma 1.2.5, the product $\circledast$ is associative. Indeed, for Ψ, Φ and $\Lambda \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have

$$(\Psi \circledast \Phi) \circledast \Lambda = S_{\Psi\circledast\Phi}(\Lambda) = S_\Psi\left(S_\Phi(\Lambda)\right) = S_\Psi(\Phi \circledast \Lambda) = \Psi \circledast (\Phi \circledast \Lambda).$$

Finally, the other group axioms being easy to check, this proves Proposition 1.2.4. □

Actions of the group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ **by automorphisms**

Corollary 1.2.7.

(i) There is a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on* $\mathbf{k}\langle\langle X\rangle\rangle$ *by topological* $\mathbf{k}$*-algebra automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle X\rangle\rangle),\ \Psi \longmapsto \mathrm{aut}_\Psi. \tag{1.11}$$

(ii) There is a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on* $\mathbf{k}\langle\langle X\rangle\rangle$ *by topological* $\mathbf{k}$*-module automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle),\ \Psi \longmapsto S_\Psi. \tag{1.12}$$

Proof. This result is exactly Lemma 1.2.5. □

Next, we aim to give a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on the topological $\mathbf{k}$-module $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ which is compatible with its action S on $\mathbf{k}\langle\langle X\rangle\rangle$. It is important to notice that this action is not given by compatibility using π_Y but by the following

Proposition-Definition 1.2.8 ([EF0, §5.4])**.** *For* $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, there is a unique topological* $\mathbf{k}$*-module automorphism* S^Y_Ψ *of* $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ *such that the following diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\quad S_\Psi \quad} & \mathbf{k}\langle\langle X\rangle\rangle \\ {\scriptstyle \overline{\mathbf{q}}\circ\pi_Y}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{q}}\circ\pi_Y} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\quad S^Y_\Psi \quad]{} & \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \end{array} \tag{1.13}$$

commutes.

Corollary 1.2.9. *There is a group action of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on* $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ *by topological* $\mathbf{k}$*-module automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \longrightarrow \mathrm{Aut}_{\mathbf{k}\text{-mod}}\left(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0\right),\ \Psi \longmapsto S^Y_\Psi. \tag{1.14}$$

Proof. We have

$$S^Y_\Psi \circ S^Y_\Phi \circ \overline{\mathbf{q}} \circ \pi_Y = S^Y_\Psi \circ \overline{\mathbf{q}} \circ \pi_Y \circ S_\Phi = \overline{\mathbf{q}} \circ \pi_Y \circ S_\Psi \circ S_\Phi = \overline{\mathbf{q}} \circ \pi_Y \circ S_{\Psi\circledast\Phi},$$

and, by uniqueness of the $\mathbf{k}$-module automorphism $S^Y_{\Psi\circledast\Phi}$, we obtain

$$S^Y_\Psi \circ S^Y_\Phi = S^Y_{\Psi\circledast\Phi}.$$

□

The cocycle Γ and twisted actions

Definition 1.2.10. We denote

$$\mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}^{\{\text{words in } x_0,(x_g)_{g\in G}\}}, v \mapsto \big((v|w)\big)_w$$

the map such that $v = \sum_w (v|w)w$ (the empty word is equal to 1).

Definition 1.2.11 ([Rac, (3.2.1.2)])**.** Let $\Gamma : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}[[x]]^\times, \Psi \mapsto \Gamma_\Psi$ the function[1] given by

$$\Gamma_\Psi(x) := \exp\left(\sum_{n\geq 2} \frac{(-1)^{n-1}}{n}(\Psi|x_0^{n-1}x_1)x^n\right). \tag{1.15}$$

[1] This function is related to the classical gamma function as established in [Fu11], page 344 thanks to [Dri90].

Lemma 1.2.12. *For $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have $\Gamma_{\Psi\circledast\Phi} = \Gamma_\Psi\Gamma_\Phi$.*

Proof. Lemma 4.12 in [EF0] says that the map $(-|x_0^{n-1}x_1) : (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to (\mathbf{k}, +)$ is a group morphism, for any $n \in \mathbb{N}^*$. The result is then obtained by straightforward computations. ☐

Definition 1.2.13. For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, let ${}^\Gamma S_\Psi^Y$ be the topological $\mathbf{k}$-module automorphism of $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ given by

$$^\Gamma S_\Psi^Y := \ell_{\Gamma_\Psi^{-1}(x_1)} \circ S_\Psi^Y.$$

Corollary 1.2.14. *There is a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ by topological $\mathbf{k}$-module automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\left(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0\right), \ \Psi \longmapsto {}^\Gamma S_\Psi^Y. \tag{1.16}$$

Proof. Follows from Corollary 1.2.9 and Lemma 1.2.12. ☐

The above automorphism is related to an automorphism introduced in [EF0].

Proposition 1.2.15. *For any $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the $\mathbf{k}$-module automorphism ${}^\Gamma S_\Psi^Y$ is equal to the $\mathbf{k}$-module automorphism $S_{\Theta(\Psi)}^Y$ with $\Theta : (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to ((\mathbf{k}\langle\langle X\rangle\rangle)^\times, \circledast)$ being the group[2] morphism given by ([EF0, Proposition 4.13])*

$$\Theta(\Psi) := \Gamma_\Psi^{-1}(x_1)\Psi\exp(-(\Psi|x_0)x_0). \tag{1.17}$$

Proof. Let $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ and $v \in \mathbf{k}\langle\langle X\rangle\rangle$. First, we have

$$S_{\Theta(\Psi)}(v) = \Theta(\Psi)\mathrm{aut}_{\Theta(\Psi)}(v) = \left(\Gamma_\Psi^{-1}(x_1)\Psi\exp(-(\Psi|x_0)x_0)\right)\mathrm{aut}_{\Theta(\Psi)}(v)$$

Moreover, one can check on generators that

$$\mathrm{aut}_{\Theta(\Psi)} = \mathrm{Ad}_{\exp((\Psi|x_0)x_0)} \circ \mathrm{aut}_\Psi.$$

Therefore, one obtains

$$S_{\Theta(\Psi)}(v) = \Gamma_\Psi^{-1}(x_1)\Psi\mathrm{aut}_\Psi(v)\exp(-(\Psi|x_0)x_0) = \Gamma_\Psi^{-1}(x_1)S_\Psi(v)\exp(-(\Psi|x_0)x_0)$$

Consequently,

$$\begin{aligned} {}^\Gamma S_\Psi^Y\left(\overline{\mathbf{q}} \circ \pi_Y(v)\right) =& \Gamma_\Psi^{-1}(x_1)S_\Psi^Y\Big(\overline{\mathbf{q}} \circ \pi_Y(v)\Big) = \Gamma_\Psi^{-1}(x_1)\Big(\overline{\mathbf{q}} \circ \pi_Y\left(S_\Psi(v)\right)\Big) \\ =& \overline{\mathbf{q}} \circ \pi_Y\left(\Gamma_\Psi^{-1}(x_1)S_\Psi(v)\right) = \overline{\mathbf{q}} \circ \pi_Y\left(S_{\Theta(\Psi)}(v)\right). \end{aligned}$$

This establishes the identity ${}^\Gamma S_\Psi^Y = S_{\Theta(\Psi)}^Y$, thanks to Proposition-Definition 1.2.8. ☐

1.2.2 The group $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$

Proposition-Definition 1.2.16 ([Rac, Definition 3.2.1])**.** *If G is a cyclic group, we define[3] $\mathsf{DMR}_0^G(\mathbf{k})$ to be the set of $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ such that*

[2]The product $\circledast$ extends to a product on $\mathbf{k}\langle\langle X\rangle\rangle^\times$. See [EF0, Lemma 4.1] and [Rac, §3.1.2.].

[3]The notation DMR, used by Racinet, is for "Double Mélange et Régularisation" which is French for "Double Shuffle and Regularisation".

(i) $(\Psi|x_0) = (\Psi|x_1) = 0$; *(iii)* If $|G| \in \{1,2\}$, $(\Psi|x_0x_1) = 0$;

(ii) $\widehat{\Delta}_\star^{\mathrm{mod}}(\Psi_\star) = \Psi_\star \otimes \Psi_\star$; *(iv)* If $|G| \geq 3, \forall g \in G$, $\big(\Psi|x_g - x_{g^{-1}}\big) = 0$;

where $\Psi_\star := \overline{\mathbf{q}} \circ \pi_Y\left(\Gamma_\Psi^{-1}(x_1)\Psi\right) \in \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$. *The pair* $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$ *is a subgroup of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$.

Proof. See [Rac, Theorem I]. □

Proposition-Definition 1.2.17. *There is a group action of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on the* $\mathbf{k}$*-module* $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\Big(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\hat{\otimes}2}\Big)$ *given by*

$$\Psi \cdot D := \left(\left({}^\Gamma S_\Psi^Y\right)^{\otimes 2}\right)^{-1} \circ D \circ {}^\Gamma S_\Psi^Y, \tag{1.18}$$

with $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *and* $D \in \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\Big(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\hat{\otimes}2}\Big)$.

Proof. It follows from Corollary 1.2.14. □

Definition 1.2.18 ([EF0, §5.4])**.** We denote $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k})$ the stabilizer subgroup of the coproduct $\widehat{\Delta}_\star^{\mathrm{mod}} \in \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\Big(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\hat{\otimes}2}\Big)$ for the action (1.18). Namely,

$$\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) := \left\{\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \mid \left({}^\Gamma S_\Psi^Y\right)^{\otimes 2} \circ \widehat{\Delta}_\star^{\mathrm{mod}} = \widehat{\Delta}_\star^{\mathrm{mod}} \circ {}^\Gamma S_\Psi^Y\right\}. \tag{1.19}$$

Proposition 1.2.19. *If* G *is a cylic group, we have*

$$\mathsf{DMR}_0^G(\mathbf{k}) = \{\Psi \in \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) \mid (\Psi|x_0) = (\Psi|x_1) = 0\}. \tag{1.20}$$

Proof. See [EF0, Theorem 1.2]. □

Remark. Since the condition $(\Psi|x_0) = (\Psi|x_1) = 0$ defines a subgroup of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$, Proposition 1.2.19 then identifies $\mathsf{DMR}_0^G(\mathbf{k})$ with the intersection of two subgroups of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$.

1.3 Torsor structure

1.3.1 Action of the group $\mathbf{k}^\times$

Action of $\mathbf{k}^\times$ on $\mathbf{k}\langle\langle X\rangle\rangle$

Definition 1.3.1. Let $\bullet$ be the group action of $\mathbf{k}^\times$ on $\mathbf{k}\langle\langle X\rangle\rangle$ by $\mathbf{k}$-algebra automorphisms given by

$$\mathbf{k}^\times \longrightarrow \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle X\rangle\rangle); \quad \lambda \longmapsto \lambda \bullet - : x_g \mapsto \lambda x_g, \text{ for } g \in G \sqcup \{0\}. \tag{1.21}$$

Lemma 1.3.2. *The action* $\bullet$ *restricts to an action* $\mathbf{k}^\times \to \mathrm{Aut}_{\mathbf{k}\text{-Hopf}}(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$.

Proof. Immediate. □

Lemma 1.3.3. *For any $g \in G$, we have*

$$(\lambda \bullet -) \circ t_g = t_g \circ (\lambda \bullet -).$$

Proof. Since all the morphisms are algebra morphisms, it is enough to check this identity on generators. We have

$$(\lambda \bullet -) \circ t_g(x_0) = \lambda x_0 = \lambda t_g(x_0) = t_g(\lambda x_0) = t_g(\lambda \bullet x_0)$$

and for $h \in G$,

$$(\lambda \bullet -) \circ t_g(x_h) = \lambda x_{gh} = \lambda t_g(x_h) = t_g(\lambda x_h) = t_g(\lambda \bullet x_h).$$

□

Lemma 1.3.4. *For any $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have*

(i) $(\lambda \bullet -) \circ \mathrm{aut}_\Psi = \mathrm{aut}_{\lambda\bullet\Psi} \circ (\lambda \bullet -)$. *(ii)* $(\lambda \bullet -) \circ S_\Psi = S_{\lambda\bullet\Psi} \circ (\lambda \bullet -)$.

Proof. Let $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$.

(i) Since all the morphisms are algebra morphisms, it is enough to check this identity on generators. We have

$$(\lambda \bullet -) \circ \mathrm{aut}_\Psi(x_0) = \lambda \bullet x_0 = \lambda x_0 = \lambda \mathrm{aut}_{\lambda\bullet\Psi}(x_0) = \mathrm{aut}_{\lambda\bullet\Psi}(\lambda x_0) = \mathrm{aut}_{\lambda\bullet\Psi}(\lambda \bullet x_0)$$

and for $g \in G$,

$$\begin{aligned}(\lambda \bullet -) \circ \mathrm{aut}_\Psi(x_g) &= (\lambda \bullet -) \circ \mathrm{Ad}_{t_g(\Psi^{-1})}(x_g) = \mathrm{Ad}_{\lambda\bullet t_g(\Psi^{-1})}(\lambda \bullet x_g) \\ &= \mathrm{Ad}_{t_g(\lambda\bullet\Psi^{-1})}(\lambda \bullet x_g) = \mathrm{aut}_{\lambda\bullet\Psi}(\lambda \bullet x_g),\end{aligned}$$

where the third equality comes from Lemma 1.3.3.

(ii) We have

$$\begin{aligned}(\lambda \bullet -) \circ S_\Psi &= (\lambda \bullet -) \circ \ell_\Psi \circ \mathrm{aut}_\Psi = \ell_{\lambda\bullet\Psi} \circ (\lambda \bullet -) \circ \mathrm{aut}_\Psi \\ &= \ell_{\lambda\bullet\Psi} \circ \mathrm{aut}_{\lambda\bullet\Psi} \circ (\lambda \bullet -) = S_{\lambda\bullet\Psi} \circ (\lambda \bullet -),\end{aligned}$$

where the second equality comes from the fact that $\lambda \bullet - : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle$ is an algebra morphism and the third one from (i).

□

Proposition 1.3.5. *For any $\lambda \in \mathbf{k}^\times$, the map $\lambda \bullet - : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle$ restricts to a group automorphism of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$.*

Proof. Let $\lambda \in \mathbf{k}^\times$. The fact that $\lambda \bullet -$ restricts to a map $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ comes from Proposition 1.3.2. Let $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. We have that

$$\lambda \bullet (\Psi \circledast \Phi) = \lambda \bullet S_\Psi(\Phi) = S_{\lambda\bullet\Psi}(\lambda \bullet \Phi) = (\lambda \bullet \Psi) \circledast (\lambda \bullet \Phi),$$

where the second equality comes from Lemma 1.3.4.(ii). This proves that $\lambda \bullet -$ is a group endomorphism of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$. This endomorphism is injective. Indeed, let $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ such that $\lambda \bullet \Psi = 1$. Since $\lambda \bullet - : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle$ is an algebra isomorphism it immediately follows that $\Psi = 1$.
Now, let $\Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. Since $\lambda \bullet - : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle$ is surjective, there exists $\Psi \in \mathbf{k}\langle\langle X\rangle\rangle$ such that $\lambda \bullet \Psi = \Phi$. Since $\Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we obtain that

$$\widehat{\Delta}(\lambda \bullet \Psi) = (\lambda \bullet \Psi) \otimes (\lambda \bullet \Psi).$$

Moreover, thanks to Lemma 1.3.2, we obtain

$$(\lambda \bullet -)^{\otimes 2} \circ \widehat{\Delta}(\Psi) = \widehat{\Delta}(\lambda \bullet \Psi) = (\lambda \bullet -)^{\otimes 2}(\Psi \otimes \Psi).$$

Finally, by injectivity of $\lambda \bullet - : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle$, we obtain that $\widehat{\Delta}(\Psi) = \Psi \otimes \Psi$ and then $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. This implies that $\lambda \bullet - : \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ is surjective. □

Definition 1.3.6. We denote $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ the semi-direct product of $\mathbf{k}^\times$ and $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ with respect to the action given in Proposition 1.3.5. It consists of the set $\mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ endowed with a group law which will also be denoted $\circledast$ and we have for $(\lambda, \Psi), (\nu, \Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$,

$$(\lambda, \Psi) \circledast (\nu, \Phi) := (\lambda\nu, \Psi \circledast (\lambda \bullet \Phi)). \tag{1.22}$$

A semi-direct product stabilizer

Lemma 1.3.7. *For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have $\Gamma_{\lambda\bullet\Psi} = \lambda \bullet \Gamma_\Psi$.*

Proof. It follows from the identity $(\lambda \bullet \Psi | x_0^{n-1}x_1) = \lambda^n (\Psi | x_0^{n-1}x_1)$ with $n \in \mathbb{N}^*$. □

Lemma 1.3.8. *The action $\bullet$ induces an action of $\mathbf{k}^\times$ on $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$, which will also be denoted $\bullet$.*

Proof. This comes from the fact that for any $\lambda \in \mathbf{k}^\times$, the map $\lambda \bullet -$ preserves the submodule $\mathbf{k}\langle\langle X\rangle\rangle x_0$. □

Proposition-Definition 1.3.9. *There is a group action of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ by topological $\mathbf{k}$-module automorphisms*

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\left(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0\right),\ (\lambda, \Psi) \longmapsto {}^\Gamma S^Y_\Psi \circ (\lambda \bullet -). \tag{1.23}$$

Proof. It follows from Corollary 1.2.14 and Lemma 1.3.7. □

Proposition-Definition 1.3.10. *There is a group action of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on the $\mathbf{k}$-module $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\Big(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\hat{\otimes} 2}\Big)$ by*

$$(\lambda, \Psi) \cdot D := \Big(\big({}^\Gamma S^Y_\Psi \circ (\lambda \bullet -)\big)^{\otimes 2}\Big)^{-1} \circ D \circ {}^\Gamma S^Y_\Psi \circ (\lambda \bullet -), \tag{1.24}$$

with $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ and $D \in \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\hat{\otimes} 2})$.

Proof. It follows from Proposition-Definition 1.3.9. □

Definition 1.3.11. We denote $\mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k})$ the stabilizer subgroup of the coproduct $\widehat{\Delta}_\star^{\mathrm{mod}} \in \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\left(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\hat{\otimes}2}\right)$ for the action (1.24). Namely,

$$\mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) := \left\{ \begin{array}{l} (\lambda, \Psi) \in \mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \mid \\ \left({}^\Gamma S_\Psi^Y \circ (\lambda \bullet -)\right)^{\otimes 2} \circ \widehat{\Delta}_\star^{\mathrm{mod}} = \widehat{\Delta}_\star^{\mathrm{mod}} \circ {}^\Gamma S_\Psi^Y \circ (\lambda \bullet -) \end{array} \right\} \tag{1.25}$$

Proposition 1.3.12. $\mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) = \mathbf{k}^\times \ltimes \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k})$ *(equality of subgroups of* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*).*

It is a consequence of the following general lemma

Lemma 1.3.13. *Let us consider the semidirect product group* $H \ltimes R$*. If* K *is a subgroup of* $H \ltimes R$ *containing* H*, then*

$$K = H \ltimes (K \cap R).$$

Proof of Proposition 1.3.12. We use Lemma 1.3.13 with $H = \mathbf{k}^\times$, $R = \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ and $K = \mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k})$. We have that

$$K \cap R = \mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) \cap \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) = \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}).$$

Since $\widehat{\Delta}_\star^{\mathrm{mod}}$ is compatible with the degree, $\mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k})$ contains $\mathbf{k}^\times$. Therefore, the condition of Lemma 1.3.13 is met and the result then follows. □

1.3.2 The torsor $\mathsf{DMR}^\iota_\times(\mathbf{k})$

Throughout this part, let us suppose that the finite abelian group G is cyclic.

Definition 1.3.14 ([Rac, Definition 3.2.1])**.** Let $\lambda \in \mathbf{k}$ and $\iota : G \to \mathbb{C}^\times$ be a group embedding. We define $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ to be the set of $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ such that

(i) $(\Psi|x_0) = (\Psi|x_1) = 0$;

(ii) $\widehat{\Delta}_\star^{\mathrm{mod}}(\Psi_\star) = \Psi_\star \otimes \Psi_\star$;

(iii) If $|G| \in \{1, 2\}, (\Psi|x_0x_1) = -\frac{\lambda^2}{24}$;

(iv) If $|G| \geq 3$, $\left(\Psi|x_{g_\iota} - x_{g_\iota^{-1}}\right) = \frac{|G|-2}{2}\lambda$;

(v) For $k \in \{1, \ldots, |G|/2\}$, $\left(\Psi|x_{g_\iota^k} - x_{g_\iota^{-k}}\right) = \frac{|G|-2k}{|G|-2}\left(\Psi|x_{g_\iota} - x_{g_\iota^{-1}}\right)$.

where $g_\iota = \iota^{-1}(e^{\frac{i2\pi}{|G|}})$ and $\Psi_\star := \overline{\mathbf{q}} \circ \pi_Y\left(\Gamma_\Psi^{-1}(x_1)\Psi\right) \in \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$.

Remark. $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ is a non-empty set thanks to [Rac, §3.2.3].

Remark. If $|G| \in \{1, 2\}$, the embedding ι is unique; and if $|G| \geq 3$, for $\lambda = 0$, the condition (iv) does not depend of the choice of ι. For this reason, the set $\mathsf{DMR}^\iota_0(\mathbf{k})$ is denoted $\mathsf{DMR}^G_0(\mathbf{k})$ instead (see Proposition-Definition 1.2.16).

Proposition 1.3.15. *Let* $\lambda \in \mathbf{k}$ *and* $\iota : G \to \mathbb{C}^\times$ *be a group embedding. The group* $(\mathsf{DMR}^G_0(\mathbf{k}), \circledast)$ *acts freely and transitively on* $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ *by left multiplication* $\circledast$*.*

Proof. See [Rac, Theorem I]. □

Proposition 1.3.16. *For any $\lambda \in \mathbf{k}^\times$, the group automorphism $\lambda \bullet -$ of $(\mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle), \circledast)$ restricts to a group automorphism of $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$.*

Proof. This follows from Proposition 1.3.5 and immediate computations. □

Definition 1.3.17. We denote $\mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k})$ the semi-direct product of $\mathbf{k}^\times$ and $\mathsf{DMR}_0^G(\mathbf{k})$ with respect to the action given in Proposition 1.3.16. It is a subgroup of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle)$.

Proposition 1.3.18. *We have*

$$\mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k}) = \{(\lambda, \Psi) \in \mathbf{k}^\times \ltimes \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) \,|\, (\Psi|x_0) = (\Psi|x_1) = 0\}. \tag{1.26}$$

Proof. It follows from Proposition 1.2.19. □

Definition 1.3.19. Let $\iota : G \to \mathbb{C}^\times$ be a group embedding. We define

$$\mathsf{DMR}_\times^\iota(\mathbf{k}) := \{(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle) \,|\, \Psi \in \mathsf{DMR}_\lambda^\iota(\mathbf{k})\}. \tag{1.27}$$

Proposition 1.3.20. *Let $\iota : G \to \mathbb{C}^\times$ be a group embedding. The group $\mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k})$ acts freely and transitively on $\mathsf{DMR}_\times^\iota(\mathbf{k})$ by left multiplication $\circledast$.*

Proof. Since the action of the group $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle)$ on the set $\mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle)$ by left multiplication $\circledast$ is free, so is its restriction to the action of $\mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k})$ on $\mathsf{DMR}_\times^\iota(\mathbf{k})$. Let us show the transitivity. Let (λ, Ψ) and $(\nu, \Phi) \in \mathsf{DMR}_\times^\iota(\mathbf{k})$. Set $\mu = \lambda\nu^{-1}$. It follows that $\mu \bullet \Phi \in \mathsf{DMR}_\lambda^\iota(\mathbf{k})$. Thanks to Proposition 1.3.15, the action of the group $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$ on $\mathsf{DMR}_\lambda^\iota(\mathbf{k})$ is transitive, therefore, there exists $\Lambda \in \mathsf{DMR}_0^G(\mathbf{k})$ such that $\Lambda \circledast (\mu \bullet \Phi) = \Psi$. Thus $(\mu, \Lambda) \in \mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k})$ is such that

$$(\mu, \Lambda) \circledast (\nu, \Phi) = (\lambda, \Psi),$$

which proves the transitivity. □

2

The de Rham formalism of the double shuffle theory

Throughout this chapter, let G be a finite abelian group and $\mathbf{k}$ be a commutative $\mathbb{Q}$-algebra. We construct a crossed product version of the double shuffle formalism. The relevant algebras and modules are introduced in §2.1 : (i) an algebra $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ defined by generators and relations, which is then identified with a crossed product algebra involving Racinet's formal series algebra $\mathbf{k}\langle\langle X\rangle\rangle$; (ii) a Hopf algebra $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})$ isomorphic to the Hopf algebra $(\mathbf{k}\langle\langle Y\rangle\rangle, \widehat{\Delta}_\star^{\mathrm{alg}})$, where $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ is a subalgebra of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$; (iii) a coalgebra $(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$ isomorphic to the coalgebra $(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \widehat{\Delta}_\star^{\mathrm{mod}})$, where $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ has a $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$-module structure inducing a free rank one $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$-module structure on it, compatible with the coproducts $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ and $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$. In §2.2, we construct actions of the group $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on these objects by algebra and module automorphisms. This leads us in §2.3 to define the stabilizer groups of the coproducts $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ and $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$ and show in Theorem 2.3.5 that the stabilizer of the latter is included in the stabilizer of the former. Finally, we translate this inclusion of stabilizers into Racinet's formalism as stated in Corollary 2.3.22.

2.1 Algebras and modules

2.1.1 Some results on algebra-modules

Definition 2.1.1.

- We denote $\mathbf{k}$-mod the category of $\mathbf{k}$-modules.
- We denote $\mathbf{k}$-alg the category of $\mathbf{k}$-algebras.
- We denote $\mathbf{k}$-alg-mod the category of pairs (A, M) where A is a $\mathbf{k}$-algebra and M is a left A-module.
- We consider $\mathbf{k}$-alg-mod $\to$ $\mathbf{k}$-alg $\times$ $\mathbf{k}$-mod the forgetful functor.

This forgetful functor induces an inclusion

$$\mathrm{Aut}_{\mathbf{k}\text{-alg-mod}}(A,M) \subset \mathrm{Aut}_{\mathbf{k}\text{-alg}}(A) \times \mathrm{Aut}_{\mathbf{k}\text{-mod}}(M).$$

Definition 2.1.2.

- We denote **k**-coalg the category of coassociative cocommutative **k**-coalgebras.
- We denote **k**-Hopf the category of **k**-Hopf algebras.
- We denote **k**-HAMC the category of pairs $\big((A,\Delta^A),(M,\Delta^M)\big)$ where (A,Δ^A) is a Hopf algebra and (M,Δ^M) is a coassociative cocommutative coalgebra equipped with a left A-module structure such that for $(a,m) \in A \times M$,

$$\Delta^M(am) = \Delta^A(a)\Delta^M(m).$$

- We consider the forgetful functors

$$\mathbf{k}\text{-alg-mod} \leftarrow \mathbf{k}\text{-HAMC} \rightarrow \mathbf{k}\text{-coalg} \times \mathbf{k}\text{-Hopf}$$

  and

$$\begin{array}{ccc} \mathbf{k}\text{-Hopf} & \longrightarrow & \mathbf{k}\text{-alg} \\ \downarrow & & \downarrow \\ \mathbf{k}\text{-coalg} & \longrightarrow & \mathbf{k}\text{-mod} \end{array}$$

The previous forgetful functors induce the following inclusions

$$\begin{array}{cc} \mathrm{Aut}_{\mathbf{k}\text{-alg-mod}}(A,M) \supset & \mathrm{Aut}_{\mathbf{k}\text{-HAMC}}((A,\Delta^A),(M,\Delta^M)) \\ & \cap \\ & \mathrm{Aut}_{\mathbf{k}\text{-Hopf}}(A,\Delta^A) \times \mathrm{Aut}_{\mathbf{k}\text{-coalg}}(M,\Delta^M) \end{array}$$

and

$$\mathrm{Aut}_{\mathbf{k}\text{-coalg}}(M,\Delta^M) \subset \mathrm{Aut}_{\mathbf{k}\text{-mod}}(M) \text{ and } \mathrm{Aut}_{\mathbf{k}\text{-Hopf}}(A,\Delta^A) \subset \mathrm{Aut}_{\mathbf{k}\text{-alg}}(A).$$

Proposition 2.1.3 ([EF3, Lemma 3.20])**.** *Let $\big((A,\Delta^A),(M,\Delta^M)\big)$ be an object in the category* **k**-HAMC*. Let H be a group such that there exists a group morphism $(\varphi,f) : H \to \mathrm{Aut}_{\mathbf{k}\text{-alg-mod}}(A,M)$. We have the following inclusions of subgroups of H*

$$\begin{array}{cc} f^{-1}(\mathrm{Aut}_{\mathbf{k}\text{-coalg}}(M,\Delta^M)) \supset & (\varphi,f)^{-1}(\mathrm{Aut}_{\mathbf{k}\text{-HAMC}}((A,\Delta^A),(M,\Delta^M))) \\ & \cap \\ & \varphi^{-1}(\mathrm{Aut}_{\mathbf{k}\text{-Hopf}}(A,\Delta^A)). \end{array} \tag{2.1}$$

Moreover, if M is a left A-module of rank 1 generated by e such that $\Delta^M(e) \in (A^{\otimes 2})^\times \cdot e^{\otimes 2}$, then

$$f^{-1}(\mathrm{Aut}_{\mathbf{k}\text{-coalg}}(M,\Delta^M)) \subset \varphi^{-1}(\mathrm{Aut}_{\mathbf{k}\text{-Hopf}}(A,\Delta^A)). \tag{2.2}$$

Proof. Let $h \in (\varphi, f)^{-1}(\mathrm{Aut}_{\mathbf{k}\text{-HAMC}}((A, \Delta^A), (M, \Delta^M)))$. This is equivalent to

$$(\varphi_h, f_h) := (\varphi, f)(h) \in \mathrm{Aut}_{\mathbf{k}\text{-HAMC}}((A, \Delta^A), (M, \Delta^M)).$$

Then $\varphi_h = \varphi(h) \in \mathrm{Aut}_{\mathbf{k}\text{-Hopf}}(A, \Delta^A)$ and $f_h = f(h) \in \mathrm{Aut}_{\mathbf{k}\text{-coalg}}(M, \Delta^M)$. Thus, we obtain the inclusions (2.1).
Now, let us assume that M is a left A-module of rank 1 with generator e. Thanks to (2.1), the statement (2.2) can be proved by showing that

$$f^{-1}(\mathrm{Aut}_{\mathbf{k}\text{-coalg}}(M, \Delta^M)) \subset (\varphi, f)^{-1}(\mathrm{Aut}_{\mathbf{k}\text{-HAMC}}((A, \Delta^A), (M, \Delta^M))).$$

Let $h \in f^{-1}(\mathrm{Aut}_{\mathbf{k}\text{-coalg}}(M, \Delta^M))$ such that $(\varphi_h, f_h) \in \mathrm{Aut}_{\mathbf{k}\text{-alg-mod}}(A, M)$ and $a \in A$. We have

$$\begin{aligned} \varphi_h^{\otimes 2}(\Delta^A(a))\, f_h^{\otimes 2}(\Delta^M(e)) &= f_h^{\otimes 2}(\Delta^A(a)\Delta^M(e)) = f_h^{\otimes 2}(\Delta^M(ae)) = \Delta^M(f_h(ae)) \\ &= \Delta^M(\varphi_h(a) f_h(e)) = \Delta^A(\varphi_h(a))\, \Delta^M(f_h(e)) \\ &= \Delta^A(\varphi_h(a))\, f_h^{\otimes 2}(\Delta^M(e)), \end{aligned} \tag{2.3}$$

where the first and fourth equalities come from the fact that (φ_h, f_h) is an automorphism of $\mathbf{k}$-alg-mod, the second and fifth ones from the compatibility of coproducts Δ^A and Δ^M and the third and sixth one from the fact that $f_h \in \mathrm{Aut}_{\mathbf{k}\text{-coalg}}(M, \Delta^M)$. Then, since $\Delta^M(e)$ is a generator of the $A^{\otimes 2}$-module $M^{\otimes 2}$, we have

$$f_h^{\otimes 2}(\Delta^M(e)) \in f_h^{\otimes 2}\left((A^{\otimes 2})^\times e^{\otimes 2}\right) = \varphi_h^{\otimes 2}((A^{\otimes 2})^\times)\, f_h^{\otimes 2}(e^{\otimes 2}) \subset (A^{\otimes 2})^\times\, f_h(e)^{\otimes 2} = (A^{\otimes 2})^\times\, e^{\otimes 2},$$

where the last equality comes from the fact that $f_h(e) \in A^\times e$ and the fact that the map $A \times A \to A \otimes A$ induces a map $A^\times \times A^\times \to (A \otimes A)^\times$. Thus, we obtain from (2.3) that

$$\varphi_h^{\otimes 2}(\Delta^A(a)) = \Delta^A(\varphi_h(a)).$$

□

2.1.2 The algebra $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$

Crossed product algebra

Definition 2.1.4. Let A be a $\mathbf{k}$-algebra such that the group G acts on A by $\mathbf{k}$-algebra automorphisms. Let us denote $G \times A \ni (g, a) \mapsto a^g \in A$ this action. The *crossed product algebra* of the $\mathbf{k}$-algebra A by the group G denoted $A \rtimes G$ is the $\mathbf{k}$-algebra $(A \otimes \mathbf{k}G, *)$ where $*$ is the product given by

$$\sum_{g \in G}(a_g \otimes g) * \sum_{h \in G}(b_h \otimes h) := \sum_{k \in G}\left(\sum_{g,h \in G \mid gh = k} a_g\, b_h^g\right) \otimes k, \tag{2.4}$$

for $a_g, b_g \in A$ with $g \in G$ ([Bou07, Chapter 3, Page 180, Exercise 11]).

Proposition 2.1.5 (Universal property of the crossed product algebra)**.** *For any $\mathbf{k}$-algebra B, there is a natural bijection between the set $\mathrm{Mor}_{\mathbf{k}\text{-alg}}(A \rtimes G, B)$ and the set of pairs $(f, \tau) \in \mathrm{Mor}_{\mathbf{k}\text{-alg}}(A, B) \times \mathrm{Mor}_{\mathrm{grp}}(G, B^\times)$ such that $f(a^g) = \tau(g) f(a) \tau(g)^{-1}$.*

Proof. Indeed, given a **k**-algebra morphism $\beta : A \rtimes G \to B$ we consider:

- The **k**-algebra morphism $f : A \to B$ given for any $a \in A$ by $f(a) = \beta(a \otimes 1)$;
- The group morphism $\tau : G \to B^\times$ given for any $g \in G$ by $\tau(g) = \beta(1 \otimes g)$.

These morphisms verify:

$$\begin{aligned}\tau(g)f(a)\tau(g^{-1}) =&\beta(1 \otimes g)\beta(a \otimes 1)\beta(1 \otimes g^{-1}) = \beta((1 \otimes g) * (a \otimes 1) * (1 \otimes g^{-1}))\\ =&\beta((a^g \otimes g) * (1 \otimes g^{-1})) = \beta(a^g \otimes 1) = f(a^g).\end{aligned}$$

This shows that the map $\beta \mapsto (f, \tau)$ is well-defined. Now let us define a converse map in order to get a bijection. Given any pair (f, τ) of morphisms satisfying the conditions of the proposition, we set $\beta : a \otimes g \mapsto f(a)\tau(g)$ for any $a \otimes g \in A \rtimes G$. This is a **k**-algebra morphism. Indeed, for any $a \otimes g$ and $b \otimes h \in A \rtimes G$

$$\begin{aligned}\beta((a \otimes g) * (b \otimes h)) =&\beta(ab^g \otimes gh) = f(ab^g)\tau(gh) = f(a)f(b^g)\tau(g)\tau(h)\\ =&f(a)\tau(g)f(b)\tau(g)^{-1}\tau(g)\tau(h) = f(a)\tau(g)f(b)\tau(h)\\ =&\beta(a \otimes g)\beta(b \otimes h).\end{aligned}$$

Thus the map $(f, \tau) \to \beta$ is also well-defined. Finally, one can easily check that the composition of the two maps on both sides gives the identity. □

The graded algebra $\mathcal{V}_G^{\mathrm{DR}}$

Definition 2.1.6. Let $\mathcal{V}_G^{\mathrm{DR}}$ be the graded algebra generated by[1] $\{e_0, e_1\} \sqcup G$ where e_0 and e_1 are of degree 1 and elements $g \in G$ are of degree 0 satisfying the relations:

(i) $g \times h = gh$; (ii) $1 = 1_G$; (iii) $g \times e_0 = e_0 \times g$;

for any $g, h \in G$; where "$\times$" is the algebra multiplication[2].

Recall that $g \mapsto t_g$ defines an action of G on $\mathbf{k}\langle X\rangle$ by **k**-algebra automorphisms (see Proposition-Definition 1.1.4. (ii)). We can then consider the crossed product algebra $\mathbf{k}\langle X\rangle \rtimes G$ for this action.

Proposition 2.1.7.

(i) There is a unique **k***-algebra morphism* $\alpha : \mathcal{V}_G^{\mathrm{DR}} \to \mathbf{k}\langle X\rangle \rtimes G$ *such that* $e_0 \mapsto x_0 \otimes 1$, $e_1 \mapsto -x_1 \otimes 1$ *and* $g \mapsto 1 \otimes g$.

(ii) There is a unique **k***-algebra morphism* $\beta : \mathbf{k}\langle X\rangle \rtimes G \to \mathcal{V}_G^{\mathrm{DR}}$ *such that* $x_0 \otimes 1 \mapsto e_0$ *and for* $g \in G$, $x_g \otimes 1 \mapsto -ge_1g^{-1}$ *and* $1 \otimes g \mapsto g$.

(iii) The morphisms α *and* β *given respectively in (i) and (ii) are isomorphisms which are inverse of one another.*

Proof.

[1] The notation e_0 and e_1 is inspired by [EF1] which in turn is inspired by [DT].
[2] which we will no longer denote if there is no risk of ambiguity.

(i) We verify that the images by the morphism α of the generators of $\mathcal{V}_G^{\mathrm{DR}}$ satisfy the relations of $\mathcal{V}_G^{\mathrm{DR}}$.

- For $g, h \in G$, $\alpha(g) * \alpha(h) = (1 \otimes g) * (1 \otimes h) = 1\, t_g(1) \otimes gh = 1 \otimes gh = \alpha(gh)$;
- $\alpha(1_G) = 1 \otimes 1_G = \alpha(1)$;
- For $g \in G, \alpha(g) * \alpha(e_0) = (1 \otimes g) * (x_0 \otimes 1) = 1\, t_g(x_0) \otimes g = x_0 \otimes g$. On the other hand, we have $\alpha(e_0) * \alpha(g) = (x_0 \otimes 1) * (1 \otimes g) = x_0\, t_1(1) \otimes g = x_0 \otimes g$. Thus $\alpha(g) * \alpha(e_0) = \alpha(e_0) * \alpha(g)$.

(ii) First, since for any $g \in G$, the element $-ge_1g^{-1}$ is of degree 1, there is a unique $\mathbf{k}$-algebra morphism $f : \mathbf{k}\langle X\rangle \to \mathcal{V}_G^{\mathrm{DR}}$ such that $x_0 \mapsto e_0$, $x_g \mapsto -ge_1g^{-1}$. Second, there is a unique group morphism $\tau : G \to (\mathcal{V}_G^{\mathrm{DR}})^\times$ given by $g \mapsto g$. Next, for any $g \in G$, the maps $\mathbf{k}\langle X\rangle \to \mathcal{V}_G^{\mathrm{DR}}$ defined by $a \mapsto f(t_g(a))$ and $a \mapsto \tau(g)f(a)\tau(g)^{-1}$ are $\mathbf{k}$-algebra morphisms that are equal by restriction on generators x_h ($h \in \{0\} \sqcup G$) of $\mathbf{k}\langle X\rangle$. Indeed,

$$\tau(g)f(x_0)\tau(g)^{-1} = ge_0g^{-1} = e_0gg^{-1} = e_0 = f(x_0) = f(t_g(x_0)),$$

and for $h \in G$,

$$\tau(g)f(x_h)\tau(g)^{-1} = g(-he_1h^{-1})g^{-1} = -ghe_1(gh)^{-1} = f(x_{gh}) = f(t_g(x_h)).$$

We then have for any $g \in G$ and any $a \in \mathbf{k}\langle X\rangle$, $f(t_g(a)) = \tau(g)f(a)\tau(g)^{-1}$. Finally, according to the universal property of crossed products, the pair (f, τ) gives a unique $\mathbf{k}$-algebra morphism $\beta : \mathbf{k}\langle X\rangle \rtimes G \to \mathcal{V}_G^{\mathrm{DR}}, a \otimes g \mapsto f(a)\tau(g)$ which verifies $\beta(x_0 \otimes 1) = f(x_0)\tau(1) = e_0$, $\beta(x_g \otimes 1) = f(x_g)\tau(1) = -ge_1g^{-1}$ and $\beta(1 \otimes g) = f(1)\tau(g) = g$, for $g \in G$.

(iii) It is enough to show that the compositions of α and β gives the identity. First, since $\beta \circ \alpha : \mathcal{V}_G^{\mathrm{DR}} \to \mathcal{V}_G^{\mathrm{DR}}$, it is enough to compute it on generators. We have $e_0 \mapsto x_0 \otimes 1 \mapsto e_0$, $e_1 \mapsto -x_1 \otimes 1 \mapsto e_1$ and $g \mapsto 1 \otimes g \mapsto g$. Thus $\beta \circ \alpha = \mathrm{id}_{\mathcal{V}_G^{\mathrm{DR}}}$. For the converse, we show that $\alpha \circ \beta \in \mathrm{Mor}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle X\rangle \rtimes G, \mathbf{k}\langle X\rangle \rtimes G)$ and the identity of $\mathbf{k}\langle X\rangle \rtimes G$ have the same image via the bijection of the universal property of crossed products. The image of the identity is the pair

$$f_{\mathrm{id}} : a \mapsto a \otimes 1 \text{ and } \tau_{\mathrm{id}}(g) = 1 \otimes g.$$

Next, let us compute the image of $\alpha \circ \beta$. The $\mathbf{k}$-algebra morphism f is given for any $a \in \mathbf{k}\langle X\rangle$ by

$$f(a) = \alpha \circ \beta(a \otimes 1).$$

Since it is a $\mathbf{k}$-algebra morphism, it is enough to determine it on x_g, $g \in \{0\} \sqcup G$. We have

$$f(x_0) = \alpha \circ \beta(x_0 \otimes 1) = \alpha(e_0) = x_0 \otimes 1,$$

and for $g \in G$,

$$\begin{aligned} f(x_g) =& \alpha \circ \beta(x_g \otimes 1) = \alpha(-ge_1g^{-1}) = -\alpha(g) * \alpha(e_1) * \alpha(g^{-1}) \\ =& -(1 \otimes g) * (-x_1 \otimes 1) * (1 \otimes g^{-1}) = (t_g(x_1) \otimes g) * (1 \otimes g^{-1}) = x_g \otimes 1. \end{aligned}$$

We then deduce that for any $a \in \mathbf{k}\langle X\rangle$, $f(a) = a \otimes 1$. Next, the group morphism $\tau : G \to (\mathbf{k}\langle X\rangle \rtimes G)^\times$ is given for any $g \in G$ by

$$\tau(g) = \alpha \circ \beta(1 \otimes g) = \alpha(g) = 1 \otimes g.$$

Finally, by uniqueness of the images we conclude that $\alpha \circ \beta = \mathrm{id}_{\mathbf{k}\langle X\rangle \rtimes G}$.

□

The complete graded algebra $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$

Definition 2.1.8. Let $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ be the degree completion of the algebra $\mathcal{V}_G^{\mathrm{DR}}$. It is a complete graded topological algebra.

Recall that $g \mapsto t_g$ defines an action of G on $\mathbf{k}\langle\langle X\rangle\rangle$ by $\mathbf{k}$-algebra automorphisms (see Proposition 1.1.5). We can then consider the crossed product algebra $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ for this action. One checks that $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ is the degree completion of $\mathbf{k}\langle X\rangle \rtimes G$. We have

Proposition 2.1.9. *Let $\hat{\beta} : \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ be the degree completion of the $\mathbf{k}$-algebra isomorphism $\beta : \mathbf{k}\langle X\rangle \rtimes G \to \mathcal{V}_G^{\mathrm{DR}}$. It is a topological $\mathbf{k}$-algebra isomorphism.*

Proof. The comes from the fact that the isomorphism $\beta : \mathbf{k}\langle X\rangle \rtimes G \to \mathcal{V}_G^{\mathrm{DR}}$ is homogeneous for the degree. □

2.1.3 The Hopf algebra $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})$

The graded subalgebra $\mathcal{W}_G^{\mathrm{DR}}$

Definition 2.1.10. Let $\mathcal{W}_G^{\mathrm{DR}}$ be the graded $\mathbf{k}$-subalgebra of $\mathcal{V}_G^{\mathrm{DR}}$ given by

$$\mathcal{W}_G^{\mathrm{DR}} := \mathbf{k} \oplus \mathcal{V}_G^{\mathrm{DR}} e_1.$$

Proposition 2.1.11. *The $\mathbf{k}$-subalgebra $\mathcal{W}_G^{\mathrm{DR}}$ is freely generated by the family*

$$Z = \{z_{n,g} := -e_0^{n-1} g e_1 \,|\, (n,g) \in \mathbb{N}^* \times G\},$$

with $\deg(z_{n,g}) = n$.

In order to prove this proposition, we use the following Lemma:

Lemma 2.1.12.

(i) The family

$$\left(e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_r e_1 e_0^{n_{r+1}-1} g_{r+1}\right)_{\substack{r\in\mathbb{N}, n_1,\dots,n_{r+1}\in\mathbb{N}^*,\\ g_1,\dots,g_{r+1}\in G}}$$

is a basis of the $\mathbf{k}$-module $\mathcal{V}_G^{\mathrm{DR}}$.

(ii) The family

$$\{1\} \cup \left(e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_r e_1 e_0^{n_{r+1}-1} g_{r+1} e_1\right)_{\substack{r\in\mathbb{N}, n_1,\dots,n_r,n_{r+1}\in\mathbb{N}^*,\\ g_1,\dots,g_r,g_{r+1}\in G}}$$

is a basis of the $\mathbf{k}$-module $\mathcal{W}_G^{\mathrm{DR}}$.

Proof.

(i) Since the family

$$\left((-1)^r x_0^{n_1-1} x_{g_1} \cdots x_0^{n_r-1} x_{g_1\cdots g_r} x_0^{n_{r+1}-1}\right)_{\substack{r\in\mathbb{N}, n_1,\dots,n_{r+1}\in\mathbb{N}^*,\\ g_1,\dots,g_r\in G}}$$

is a basis of the $\mathbf{k}$-module $\mathbf{k}\langle X\rangle$, it follows that the family

$$\left((-1)^r x_0^{n_1-1} x_{g_1} \cdots x_0^{n_r-1} x_{g_1\cdots g_r} x_0^{n_{r+1}-1} \otimes g_1\cdots g_r g_{r+1}\right)_{\substack{r\in\mathbb{N}, n_1,\dots,n_{r+1}\in\mathbb{N}^*,\\ g_1,\dots,g_{r+1}\in G}}$$

is a basis of the $\mathbf{k}$-module $\mathbf{k}\langle X\rangle \otimes \mathbf{k}G$. Thus, its image by the bijection β defined in Proposition 2.1.7.(ii) is a basis of $\mathcal{V}_G^{\mathrm{DR}}$. Moreover, for $r\in\mathbb{N}$, $n_1,\dots,n_{r+1}\in\mathbb{N}^*$ and $g_1,\dots,g_{r+1}\in G$, we have

$$\begin{aligned}
&x_0^{n_1-1} x_{g_1} \cdots x_0^{n_r-1} x_{g_1\cdots g_r} x_0^{n_{r+1}-1} \otimes g_1\cdots g_r g_{r+1} = \\
&(x_0^{n_1-1}\otimes 1) * (x_{g_1}\otimes 1) * \cdots * (x_0^{n_r-1}\otimes 1) * (x_{g_1\cdots g_r}\otimes 1) * \\
&(x_0^{n_{r+1}-1}\otimes 1) * (1\otimes g_1) * \cdots * (1\otimes g_r) * (1\otimes g_{r+1}).
\end{aligned} \tag{2.5}$$

Then

$$\begin{aligned}
&\beta((-1)^r x_0^{n_1-1} x_{g_1} \cdots x_0^{n_r-1} x_{g_1\cdots g_r} x_0^{n_{r+1}-1} \otimes g_1\cdots g_r g_{r+1}) \\
&= (-1)^r \beta(x_0^{n_1-1}\otimes 1)\beta(x_{g_1}\otimes 1)\cdots\beta(x_0^{n_r-1}\otimes 1)\beta(x_{g_1\cdots g_r}\otimes 1) \\
&\quad \beta(x_0^{n_{r+1}-1}\otimes 1)\beta(1\otimes g_1)\cdots\beta(1\otimes g_r)\beta(1\otimes g_{r+1}) \\
&= e_0^{n_1-1} g_1 e_1 g_1^{-1} \cdots e_0^{n_r-1} g_1\cdots g_r e_1 g_1^{-1}\cdots g_r^{-1} e_0^{n_{r+1}-1} g_1\cdots g_r g_{r+1} \\
&= e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_1^{-1}\cdots g_{r-1}^{-1} g_1\cdots g_{r-1} g_r e_1 e_0^{n_{r+1}-1} g_1^{-1}\cdots g_r^{-1} g_1\cdots g_r g_{r+1} \\
&= e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_r e_1 e_0^{n_{r+1}-1} g_{r+1},
\end{aligned} \tag{2.6}$$

where the first equality comes from (2.5) and the fact that $\beta : \mathbf{k}\langle X\rangle \rtimes G \to \mathcal{V}_G^{\mathrm{DR}}$ is a $\mathbf{k}$-algebra morphism. The second equality is obtained by computing the images of appropriate elements by β. The third equality is a consequence of the equality $ge_0 = e_0 g$ for any $g\in G$ and the last one comes from the fact that the group G is abelian.

(ii) First, $\mathcal{W}_G^{\mathrm{DR}}$ is the image of the $\mathbf{k}$-module morphism $\mathbf{k}\oplus\mathcal{V}_G^{\mathrm{DR}} \to \mathcal{V}_G^{\mathrm{DR}}, (\lambda, v)\mapsto \lambda + ve_1$. Second, according to (i), the family

$$(1,0), \left(0, e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_r e_1 e_0^{n_{r+1}-1} g_{r+1}\right)_{\substack{r\in\mathbb{N}, n_1,\dots,n_r,n_{r+1}\in\mathbb{N}^*,\\ g_1,\dots,g_r,g_{r+1}\in G}}$$

is a basis of the $\mathbf{k}$-module $\mathbf{k}\oplus\mathcal{V}_G^{\mathrm{DR}}$. Moreover, the image of this basis by this $\mathbf{k}$-module morphism is the family

$$\{1\} \cup \left(e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_r e_1 e_0^{n_{r+1}-1} g_{r+1} e_1\right)_{\substack{r\in\mathbb{N}, n_1,\dots,n_r,n_{r+1}\in\mathbb{N}^*,\\ g_1,\dots,g_r,g_{r+1}\in G}}$$

which is free since it is contained in a basis of the target. This implies that this family is a basis of the image of the previous morphism which is $\mathcal{W}_G^{\mathrm{DR}}$.

□

Proof of Proposition 2.1.11. Let $\mathbf{k}\langle Z\rangle$ be the free algebra over Z whose elements $z_{n,g}$ ($(n,g)\in\mathbb{N}^*\times G$) are seen as free variables with $\deg(z_{n,g})=n$. Then there is a unique $\mathbf{k}$-algebra morphism $\mathbf{k}\langle Z\rangle\to\mathcal{W}_G^{\mathrm{DR}}$ given by $z_{n,g}\mapsto -e_0^{n-1}ge_1$. Let us show that it is an isomorphism. The free $\mathbf{k}$-module $\mathbf{k}\langle Z\rangle$ has basis

$$\{1\}\cup(z_{n_1,g_1}\cdots z_{n_{r+1},g_{r+1}})_{\substack{r\in\mathbb{N},n_1,\dots,n_{r+1}\in\mathbb{N}^*\\ g_1,\dots,g_{r+1}\in G}}$$

and, as a $\mathbf{k}$-module, $\mathcal{W}_G^{\mathrm{DR}}$ has basis

$$\{1\}\cup\left(e_0^{n_1-1}g_1e_1\cdots e_0^{n_{r+1}-1}g_{r+1}e_1\right)_{\substack{r\in\mathbb{N},n_1,\dots,n_{r+1}\in\mathbb{N}^*,\\ g_1,\dots,g_{r+1}\in G}}$$

thanks to Lemma 2.1.12.(ii). One computes the image by $z_{n,g}\mapsto -e_0^{n-1}ge_1$ of the former basis and finds it to be equal to the latter basis. Therefore, $z_{n,g}\mapsto -e_0^{n-1}ge_1$ induces a bijection between the two basis - up to appropriate signs - and then a bijection between $\mathbf{k}\langle Z\rangle$ and $\mathcal{W}_G^{\mathrm{DR}}$. Hence, $z_{n,g}\mapsto -e_0^{n-1}ge_1$ is a $\mathbf{k}$-algebra isomorphism between $\mathbf{k}\langle Z\rangle$ and $\mathcal{W}_G^{\mathrm{DR}}$. □

Remark. By abuse of notation, we will identify elements of $\mathcal{W}_G^{\mathrm{DR}}$ with elements of $\mathbf{k}\langle Z\rangle$ by the $\mathbf{k}$-algebra isomorphism $z_{n,g}\mapsto -e_0^{n-1}ge_1$.

Corollary 2.1.13. *There exists a unique* $\mathbf{k}$*-algebra isomorphism* $\varpi:\mathbf{k}\langle Y\rangle\to\mathcal{W}_G^{\mathrm{DR}}$ *given by*

$$y_{n,g}\longmapsto z_{n,g},\qquad \textit{for } (n,g)\in\mathbb{N}^*\times G,$$

Proof. It is a consequence of Proposition 2.1.11. □

The Hopf algebra $(\widehat{\mathcal{W}}_G^{\mathrm{DR}},\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})$

Proposition-Definition 2.1.14. *Let* $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ *be the degree completion of the graded* $\mathbf{k}$*-algebra* $\mathcal{W}_G^{\mathrm{DR}}$*. It is a topological graded* $\mathbf{k}$*-algebra isomorphic to the topological* $\mathbf{k}$*-subalgebra of* $\mathcal{V}_G^{\mathrm{DR}}$ *given by* $\mathbf{k}\oplus\widehat{\mathcal{V}}_G^{\mathrm{DR}}e_1$.

Proof. Immediate. □

Proposition 2.1.15. *The* $\mathbf{k}$*-algebra isomorphism* $\varpi:\mathbf{k}\langle Y\rangle\to\mathcal{W}_G^{\mathrm{DR}}$ *extends to a topological* $\mathbf{k}$*-algebra isomorphism* $\hat{\varpi}:\mathbf{k}\langle\langle Y\rangle\rangle\to\widehat{\mathcal{W}}_G^{\mathrm{DR}}$.

Proof. It follows from Proposition 2.1.11 and then we proceed by degree completion. □

Proposition-Definition 2.1.16. *There exists a unique topological* $\mathbf{k}$*-algebra morphism* $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}:\widehat{\mathcal{W}}_G^{\mathrm{DR}}\to(\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\hat{\otimes}2}$ *such that for any* $(n,g)\in\mathbb{N}^*\times G$

$$\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(z_{n,g})=z_{n,g}\otimes 1+1\otimes z_{n,g}+\sum_{\substack{k=1\\ h\in G}}^{n-1}z_{k,h}\otimes z_{n-k,gh^{-1}}. \tag{2.7}$$

The algebra $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ *is then equipped with a Hopf algebra structure whose coproduct is* $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$*.*

Corollary 2.1.17. *The topological* $\mathbf{k}$*-algebra isomorphism* $\hat{\varpi} : \mathbf{k}\langle\langle Y\rangle\rangle \to \widehat{\mathcal{W}}_G^{\mathrm{DR}}$ *is an isomorphism of Hopf algebras* $(\mathbf{k}\langle\langle Y\rangle\rangle, \widehat{\Delta}_\star^{\mathrm{alg}})$ *and* $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})$*.*

Proof. It follows from Proposition-Definition 2.1.16. □

2.1.4 The coalgebra $(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$

The graded module $\mathcal{M}_G^{\mathrm{DR}}$

Definition 2.1.18. Let $\mathcal{M}_G^{\mathrm{DR}}$ be the graded $\mathbf{k}$-module given by

$$\mathcal{M}_G^{\mathrm{DR}} := \mathcal{V}_G^{\mathrm{DR}} \Big/ \Big(\mathcal{V}_G^{\mathrm{DR}} e_0 + \sum_{g\in G} \mathcal{V}_G^{\mathrm{DR}}(g-1)\Big)$$

Proposition 2.1.19.

(i) The quotient $\mathcal{M}_G^{\mathrm{DR}}$ *is a* $\mathcal{V}_G^{\mathrm{DR}}$*-module and, by restriction, a* $\mathcal{W}_G^{\mathrm{DR}}$*-module.*

(ii) Let 1_{DR} *be the class of* $1 \in \mathcal{V}_G^{\mathrm{DR}}$ *in* $\mathcal{M}_G^{\mathrm{DR}}$*. The map* $-\cdot 1_{\mathrm{DR}} : \mathcal{V}_G^{\mathrm{DR}} \to \mathcal{M}_G^{\mathrm{DR}}$ *is a surjective* $\mathbf{k}$*-module morphism with kernel* $\mathcal{V}_G^{\mathrm{DR}} e_0 + \sum_{g\in G} \mathcal{V}_G^{\mathrm{DR}}(g-1)$*.*

Proof. Immediate. □

Proposition 2.1.20. *There exists a* $\mathbf{k}$*-module isomorphism* $\kappa : \mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0 \to \mathcal{M}_G^{\mathrm{DR}}$ *uniquely determined by the condition that the diagram*

$$\begin{array}{ccc} \mathbf{k}\langle X\rangle & \xrightarrow{\beta\circ(-\otimes 1)} & \mathcal{V}_G^{\mathrm{DR}} \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathrm{DR}}} \\ \mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0 & \xrightarrow[\kappa]{} & \mathcal{M}_G^{\mathrm{DR}} \end{array} \tag{2.8}$$

commutes.

We will prove this proposition by using the following general lemma. In this lemma, for any $\mathbf{k}$-module M and any submodule M', let us denote $\mathrm{can}(M, M') : M \to M/M'$ the canonical projection.

Lemma 2.1.21. *Let* $f : M \to N$ *a* $\mathbf{k}$*-module morphism. Let* M' *a submodule of* M *and* N', N'' *two submodules of* N *such that*

(i) $f(M') \subset N' \subset f(M') + N''$ *and,*

(ii) $\mathrm{can}(N, N'') \circ f : M \to N/N''$ *is a* $\mathbf{k}$*-module isomorphism.*

Then, there is a unique $\mathbf{k}$*-module morphism* $\bar{f} : M/M' \to N/(N' + N'')$ *such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \mathrm{can}(M,M')}\downarrow & & \downarrow{\scriptstyle \mathrm{can}(N,N'+N'')} \\ M/M' & \xrightarrow[\bar{f}]{} & N/(N'+N'') \end{array} \tag{2.9}$$

commutes. Moreover, $\bar{f}$ *is a* $\mathbf{k}$*-module isomorphism.*

Proof. Thanks to ((i)), $f(M') \subset N'$. This implies that $f(M')+N'' \subset N'+N''$. From ((i)) again, we have $N' \subset f(M') + N''$. This implies that $N' + N'' \subset f(M') + N''$. Therefore

$$f(M') + N'' = N' + N''. \tag{2.10}$$

Next, from ((ii)), we have that $\mathrm{can}(N, N'') \circ f : M \to N \to N/N''$ is an isomorphism. One checks that it restricts to an isomorphism from M' to $(f(M')+N'')/N''$. Thanks to equality (2.10), this yields an isomorphism from M' to $(N'+N'')/N''$. This allows us to construct a unique $\mathbf{k}$-module morphism $\tilde{f} : M/M' \to (N/N'')/((N'+N'')/N'')$ such that the lower square of the following diagram

$$\begin{array}{ccc} M' & \xrightarrow{\mathrm{can}(N,N'')\circ f_{|M'}} & (N'+N'')/N'' \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{\mathrm{can}(N,N'')\circ f} & N/N'' \\ {\scriptstyle \mathrm{can}(M,M')}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{can}\left(N/N'',(N'+N'')/N''\right)} \\ M/M' & \xrightarrow[\tilde{f}]{} & (N/N'')/((N'+N'')/N'') \end{array} \tag{2.11}$$

commutes. Moreover, since $\mathrm{can}(N,N'') \circ f : M \to N/N''$ is an isomorphism, so is $\tilde{f} : M/M' \to (N/N'')/((N' + N'')/N'')$. Finally, we construct an isomorphism $\bar{f} : M/M' \to N/(N' + N'')$ by composing $\tilde{f}$ with the inverse map $(N/N'')/((N' + N'')/N'') \simeq N/(N' + N'')$ given by the third isomorphism theorem. Thanks to Diagram (2.11), the isomorphism $\bar{f} : M/M' \to N/(N' + N'')$ is such that Diagram (2.9) commutes. ☐

Proof of Proposition 2.1.20. This is done by application of Lemma 2.1.21 for $M = \mathbf{k}\langle X\rangle$, $N = \mathcal{V}_G^{\mathrm{DR}}$, $M' = \mathbf{k}\langle X\rangle x_0$, $N' = \mathcal{V}_G^{\mathrm{DR}} e_0$, $N'' = \sum_{g\in G} \mathcal{V}_G^{\mathrm{DR}}(g-1)$ and $f = \beta \circ (- \otimes 1)$. It, therefore, suffices to prove that criteria ((i)) and ((ii)) of Lemma 2.1.21 are satisfied.

Criterion (i) $\beta(\mathbf{k}\langle X\rangle x_0 \otimes 1) \subset \mathcal{V}_G^{\mathrm{DR}} e_0 \subset \beta(\mathbf{k}\langle X\rangle x_0 \otimes 1) + \sum_{g\in G} \mathcal{V}_G^{\mathrm{DR}}(g-1)$.
For the first inclusion, we have for any $a \in \mathbf{k}\langle X\rangle$

$$\beta(ax_0 \otimes 1) = \beta(a \otimes 1)\beta(x_0 \otimes 1) = \beta(a \otimes 1)e_0 \in \mathcal{V}_G^{\mathrm{DR}} e_0.$$

Therefore, $\beta(\mathbf{k}\langle X\rangle x_0 \otimes 1) \subset \mathcal{V}_G^{\mathrm{DR}} e_0$.
For the second inclusion, by using the basis of $\mathcal{V}_G^{\mathrm{DR}}$ described in Lemma 2.1.12. (i), we have for $r \in \mathbb{N}$, $n_1, \dots, n_{r+1} \in \mathbb{N}^*$ and $g_1, \dots, g_{r+1} \in G$,

$$\begin{aligned} &\left(e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_r e_1 e_0^{n_{r+1}-1} g_{r+1}\right) e_0 = e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_r e_1 e_0^{n_{r+1}} g_{r+1} \\ &= (-1)^r \beta(x_0^{n_1-1} x_{g_1} \cdots x_0^{n_r-1} x_{g_1\cdots g_r} x_0^{n_{r+1}} \otimes g_1 \cdots g_{r+1}) \\ &= (-1)^r \beta\left(\left(x_0^{n_1-1} x_{g_1} \cdots x_0^{n_r-1} x_{g_1\cdots g_r} x_0^{n_{r+1}-1}\right) x_0 \otimes 1\right) g_1 \cdots g_{r+1} \\ &= (-1)^r \beta\left(\left(x_0^{n_1-1} x_{g_1} \cdots x_0^{n_r-1} x_{g_1\cdots g_r} x_0^{n_{r+1}-1}\right) x_0 \otimes 1\right) \\ &+ (-1)^r \beta\left(\left(x_0^{n_1-1} x_{g_1} \cdots x_0^{n_r-1} x_{g_1\cdots g_r} x_0^{n_{r+1}-1}\right) x_0 \otimes 1\right) (g_1 \cdots g_{r+1} - 1), \end{aligned}$$

where the first equality comes from the relation $ge_0 = e_0 g$ for any $g \in G$; the second one from computation (2.6) and the third one from the fact that $ax_0 \otimes g = (ax_0 \otimes 1) * (1 \otimes g)$ for any $a \in \mathbf{k}\langle X\rangle$ and any $g \in G$. Finally, the last equality shows that we obtain an element of $\beta(\mathbf{k}\langle X\rangle x_0 \otimes 1) + \sum_{g\in G} \mathcal{V}_G^{\mathrm{DR}}(g-1)$, thus proving the claimed inclusion.

Criterion (ii) $\mathrm{can}\Big(\mathcal{V}_G^{\mathrm{DR}}, \sum_{g\in G}\mathcal{V}_G^{\mathrm{DR}}(g-1)\Big) \circ \beta \circ (- \otimes 1) : \mathbf{k}\langle X\rangle \to \mathcal{V}_G^{\mathrm{DR}} \Big/ \sum_{g\in G}\mathcal{V}_G^{\mathrm{DR}}(g-1)$ is an isomorphism.

Let us consider the commutative diagram

$$\begin{array}{ccc} \mathbf{k}\langle X\rangle \otimes \bigoplus_{g\in G} \mathbf{k}G & \longrightarrow & \mathbf{k}\langle X\rangle \otimes \mathbf{k}G \\ \downarrow & & \Vert \\ \bigoplus_{g\in G} (\mathbf{k}\langle X\rangle \otimes \mathbf{k}G) & \longrightarrow & \mathbf{k}\langle X\rangle \otimes \mathbf{k}G \\ {\scriptstyle \bigoplus_{g\in G}\beta}\downarrow & & \downarrow{\scriptstyle \beta} \\ \bigoplus_{g\in G} \mathcal{V}_G^{\mathrm{DR}} & \longrightarrow & \mathcal{V}_G^{\mathrm{DR}} \end{array} \tag{2.12}$$

where the top horizontal arrow is the tensor product of the identity and the **k**-module morphism

$$\bigoplus_{g\in G} \mathbf{k}G \longrightarrow \mathbf{k}G, \quad (h_g)_{g\in G} \longmapsto \sum_{g\in G} h_g(g-1),$$

and the bottom horizontal arrow is the **k**-module morphism

$$\bigoplus_{g\in G} \mathcal{V}_G^{\mathrm{DR}} \longrightarrow \mathcal{V}_G^{\mathrm{DR}}, \quad (v_g)_{g\in G} \longmapsto \sum_{g\in G} v_g(g-1).$$

Since the vertical arrows are isomorphisms, they induce an isomorphism beteween the cokernels of the top and bottom morphisms. We can then extend the above diagram in the following way

$$\begin{array}{ccccc} \mathbf{k}\langle X\rangle \otimes \bigoplus_{g\in G} \mathbf{k}G & \longrightarrow & \mathbf{k}\langle X\rangle \otimes \mathbf{k}G & \longrightarrow & \mathrm{coker}\Big(\mathbf{k}\langle X\rangle \otimes \bigoplus_{g\in G}\mathbf{k}G \to \mathbf{k}\langle X\rangle \otimes \mathbf{k}G\Big) \\ \downarrow & & \Vert & & \\ \bigoplus_{g\in G} (\mathbf{k}\langle X\rangle \otimes \mathbf{k}G) & \longrightarrow & \mathbf{k}\langle X\rangle \otimes \mathbf{k}G & & \downarrow \\ {\scriptstyle \bigoplus_{g\in G}\beta}\downarrow & & \downarrow{\scriptstyle \beta} & & \\ \bigoplus_{g\in G} \mathcal{V}_G^{\mathrm{DR}} & \longrightarrow & \mathcal{V}_G^{\mathrm{DR}} & \longrightarrow & \mathrm{coker}\Big(\bigoplus_{g\in G}\mathcal{V}_G^{\mathrm{DR}} \longrightarrow \mathcal{V}_G^{\mathrm{DR}}\Big) \end{array} \tag{2.13}$$

On the other hand, we have

$$\mathrm{coker}\Big(\bigoplus_{g\in G}\mathcal{V}_G^{\mathrm{DR}} \to \mathcal{V}_G^{\mathrm{DR}}\Big) = \mathcal{V}_G^{\mathrm{DR}} \Big/ \sum_{g\in G}\mathcal{V}_G^{\mathrm{DR}}(g-1).$$

and

$$\operatorname{coker}\Big(\bigoplus_{g\in G}\mathbf{k}G\longrightarrow\mathbf{k}G\Big)=\mathbf{k}G\Big/\sum_{g\in G}\mathbf{k}G(g-1)\simeq\mathbf{k}$$

Therefore

$$\begin{aligned}\operatorname{coker}\Big(\mathbf{k}\langle X\rangle\otimes\textstyle\bigoplus_{g\in G}\mathbf{k}G\to\mathbf{k}\langle X\rangle\otimes\mathbf{k}G\Big)&\simeq\mathbf{k}\langle X\rangle\otimes\operatorname{coker}\Big(\bigoplus_{g\in G}\mathbf{k}G\to\mathbf{k}G\Big)\\&\simeq\mathbf{k}\langle X\rangle\otimes\mathbf{k}\simeq\mathbf{k}\langle X\rangle.\end{aligned}$$

Thus, the isomorphism between cokernels establishes that $\mathbf{k}\langle X\rangle$ is isomorphic to $\mathcal{V}_G^{\mathrm{DR}}\Big/\sum_{g\in G}\mathcal{V}_G^{\mathrm{DR}}(g-1)$. Moreover, thanks to the commutativity of diagram (2.13), this isomorphism is exactly $\operatorname{can}\left(\mathcal{V}_G^{\mathrm{DR}},\sum\limits_{g\in G}\mathcal{V}_G^{\mathrm{DR}}(g-1)\right)\circ\beta\circ(-\otimes 1)$.

□

Corollary 2.1.22.

(i) The following diagram

$$\begin{array}{ccc}\mathbf{k}\langle Y\rangle & \xrightarrow{\quad\varpi\quad} & \mathcal{W}_G^{\mathrm{DR}}\\ {\scriptstyle\pi_Y}\big\downarrow & & \big\downarrow{\scriptstyle -\cdot 1_{\mathrm{DR}}}\\ \mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle\, x_0 & \xrightarrow[\kappa\circ\overline{\mathbf{q}}^{-1}]{} & \mathcal{M}_G^{\mathrm{DR}}\end{array}\tag{2.14}$$

commutes.

(ii) The map $-\cdot 1_{\mathrm{DR}}:\mathcal{W}_G^{\mathrm{DR}}\to\mathcal{M}_G^{\mathrm{DR}}$ *is a* $\mathbf{k}$*-module isomorphism and* $\mathcal{M}_G^{\mathrm{DR}}$ *is free of rank 1 as a* $\mathcal{W}_G^{\mathrm{DR}}$*-module.*

Proof.

(i) One needs to show the equality of two maps from $\mathbf{k}\langle Y\rangle$ to $\mathcal{M}_G^{\mathrm{DR}}$. Since these maps are both $\mathbf{k}$-module morphisms, it is enough to show the equality of the images of the elements of a basis of the source module. Such a basis is

$$(y_{n_1,g_1}\cdots y_{n_rg_r})_{\substack{r\in\mathbb{N},n_1,\dots,n_r\in\mathbb{N}^*,\\ g_1,\dots,g_r\in G}},$$

thanks to [Rac, §2.2.7.]. For $r\in\mathbb{N}$, $n_1,\dots,n_r\in\mathbb{N}^*$ and $g_1,\dots,g_r\in G$ we have

$$-\cdot 1_{\mathrm{DR}}\circ\varpi(y_{n_1,g_1}\cdots y_{n_r,g_r})=z_{n_1,g_1}\cdots z_{n_r,g_r}\cdot 1_{\mathrm{DR}}.$$

On the other hand,

$$\begin{aligned}\kappa\circ\overline{\mathbf{q}}^{-1}\circ\pi_Y(y_{n_1,g_1}\cdots y_{n_r,g_r})&=\kappa(x_0^{n_1-1}x_{g_1}\cdots x_0^{n_1-1}x_{g_1\cdots g_r})\\&=\beta(x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_1\cdots g_r}\otimes 1)\cdot 1_{\mathrm{DR}}\\&=(-1)^r e_0^{n_1-1}g_1e_1\cdots e_0^{n_r-1}g_re_1g_1^{-1}\cdots g_r^{-1}\cdot 1_{\mathrm{DR}}\\&=(-e_0^{n_1-1}g_1e_1)\cdots(-e_0^{n_r-1}g_re_1)\cdot 1_{\mathrm{DR}}\\&=z_{n_1,g_1}\cdots z_{n_r,g_r}\cdot 1_{\mathrm{DR}},\end{aligned}$$

where the first equality comes from [Rac, §2.2.7]; the second one from the commutative diagram (2.8); the third one from computation (2.6) with $n_{r+1} = 1$ and $g_{r+1} = (g_1 \cdots g_r)^{-1}$; and the fourth one from the fact that for any $v \in \mathcal{V}_G^{\mathrm{DR}}$ and any $g \in G$, we have $vg \cdot 1_{\mathrm{DR}} = v \cdot 1_{\mathrm{DR}}$.

(ii) First, the following maps are **k**-module isomorphisms:

- $\varpi : \mathbf{k}\langle Y\rangle \to \mathcal{W}_G^{\mathrm{DR}}$: it sends the basis

$$(y_{n_1,g_1} \cdots y_{n_r,g_r})_{\substack{r\in\mathbb{N}, n_1,\dots,n_r\in\mathbb{N}^*,\\ g_1,\dots,g_r\in G}}$$

  of the **k**-module $\mathbf{k}\langle Y\rangle$ to the basis

$$(z_{n_1,g_1} \cdots z_{n_r,g_r})_{\substack{r\in\mathbb{N}, n_1,\dots,n_r\in\mathbb{N}^*,\\ g_1,\dots,g_r\in G}}$$

  of the **k**-module $\mathcal{W}_G^{\mathrm{DR}}$ (where the latter family is a basis of $\mathcal{W}_G^{\mathrm{DR}}$ thanks to Lemma 2.1.12.(ii)).
- $\pi_Y : \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0$: see [Rac, §2.2.5].
- $\kappa \circ \overline{\mathbf{q}}^{-1} : \mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle x_0 \to \mathcal{M}_G^{\mathrm{DR}}$: see Proposition 2.1.20 and [Rac, §2.2.7].

Next, the diagram (2.14) commutes, thanks to (i). This allows us to conclude that the map $- \cdot 1_{\mathrm{DR}} : \mathcal{W}_G^{\mathrm{DR}} \to \mathcal{M}_G^{\mathrm{DR}}$ is a **k**-module isomorphism and that $\mathcal{M}_G^{\mathrm{DR}}$ is a free $\mathcal{W}_G^{\mathrm{DR}}$-module of rank 1.

□

The coalgebra $(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$

Proposition-Definition 2.1.23. *Let $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ be the degree completion of the graded* **k**-*module $\mathcal{M}_G^{\mathrm{DR}}$. It is a topological graded* **k**-*module isomorphic to the topological graded quotient module $\widehat{\mathcal{V}}_G^{\mathrm{DR}} \Big/ \Big(\widehat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \sum_{g\in G} \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g-1)\Big)$.*

Proof. Immediate. □

Corollary 2.1.24. *The pairs $(\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ and $(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$ are objects in the category* **k**-alg-mod.

Proof. It follows from Proposition-Definition 2.1.23. □

Proposition-Definition 2.1.25.

(i) The surjective **k**-*module morphism $- \cdot 1_{\mathrm{DR}} : \mathcal{V}_G^{\mathrm{DR}} \to \mathcal{M}_G^{\mathrm{DR}}$ extends to a topological graded surjective* **k**-*module morphism $\widehat{- \cdot 1_{\mathrm{DR}}} : \widehat{\mathcal{V}}_G^{\mathrm{DR}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$.*

(ii) The **k**-*module isomorphism $\kappa : \mathbf{k}\langle X\rangle/\mathbf{k}\langle X\rangle \to \mathcal{M}_G^{\mathrm{DR}}$ extends to a topological* **k**-*module isomorphism $\hat{\kappa} : \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$.*

Proof. Immediate. □

Corollary 2.1.26.

(i) The following diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle Y\rangle\rangle & \xrightarrow{\hat{\varpi}} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ {\scriptstyle \pi_Y}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle\, x_0 & \xrightarrow[\hat{\kappa}\circ\overline{\mathbf{q}}^{-1}]{} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{2.15}$$

commutes.

(ii) The map $\widehat{-\cdot 1_{\mathrm{DR}}} : \widehat{\mathcal{W}}_G^{\mathrm{DR}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is a $\mathbf{k}$-module isomorphism and $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is free of rank 1 as a $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$-module.

Proof. If follows from Corollary 2.1.22 and then we proceed by degree completion. □

Proposition-Definition 2.1.27. *There exists a unique topological $\mathbf{k}$-module morphism $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} : \widehat{\mathcal{M}}_G^{\mathrm{DR}} \to (\widehat{\mathcal{M}}_G^{\mathrm{DR}})^{\hat{\otimes}2}$ such that the following diagram*

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}} & (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\hat{\otimes}2} \\ {\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}^{\otimes 2}} \\ \widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow[\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}]{} & (\widehat{\mathcal{M}}_G^{\mathrm{DR}})^{\hat{\otimes}2} \end{array} \tag{2.16}$$

commutes. The module $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is then equipped with a cocommutative coassociative coalgebra structure whose coproduct is $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$.

Proof. It follows from Proposition-Definition 2.1.16 and Corollary 2.1.26.(ii). □

Corollary 2.1.28. *The pair*

$$\left(\left(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}\right), \left(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}\right)\right)$$

is an object of the category $\mathbf{k}$-HAMC.

Proof. It is a consequence of Proposition-Definition 2.1.27. □

Proposition 2.1.29. *The map $\hat{\kappa} \circ \overline{\mathbf{q}}^{-1} : \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is an isomorphism between the coalgebras $(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \widehat{\Delta}_\star^{\mathrm{mod}})$ and $(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$.*

Proof. We have

$$\begin{aligned} \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} \circ \hat{\kappa} \circ \overline{\mathbf{q}}^{-1} \circ \pi_Y &= \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} \circ \widehat{-\cdot 1_{\mathrm{DR}}} \circ \varpi = \widehat{-\cdot 1_{\mathrm{DR}}}^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \circ \varpi \\ &= \widehat{-\cdot 1_{\mathrm{DR}}}^{\otimes 2} \circ \varpi^{\otimes 2} \circ \widehat{\Delta}_\star^{\mathrm{alg}} = (\hat{\kappa} \circ \overline{\mathbf{q}}^{-1})^{\otimes 2} \circ \pi_Y^{\otimes 2} \circ \widehat{\Delta}_\star^{\mathrm{alg}} \\ &= (\hat{\kappa} \circ \overline{\mathbf{q}}^{-1})^{\otimes 2} \circ \widehat{\Delta}_\star^{\mathrm{mod}} \circ \pi_Y, \end{aligned}$$

where the first and the fourth equalities come from the commutativity of Diagram (2.15), the second one from the commutativity of Diagram (2.16), the third one from

Corollary 2.1.17 and the last one from the commutativity of Diagram (DIAG COPROD RAC). Finally, the surjectivity of π_Y implies that

$$\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} \circ \hat{\kappa} \circ \overline{\mathbf{q}}^{-1} = (\hat{\kappa} \circ \overline{\mathbf{q}}^{-1})^{\otimes 2} \circ \widehat{\Delta}_{\star}^{\mathrm{mod}}.$$

□

Corollary 2.1.30. *The pair* $(\hat{\varpi}, \hat{\kappa} \circ \overline{\mathbf{q}}^{-1})$ *is a morphism in the category* $\mathbf{k}$-HAMC.

Proof. It is a consequence of Corollary 2.1.17 and Proposition 2.1.29. □

2.2 Actions of the group $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ by automorphisms

2.2.1 Actions of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ by algebra automorphisms

Proposition-Definition 2.2.1. *Let* $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. *There exists a unique topological* $\mathbf{k}$*-algebra automorphism* $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}$ *of* $\widehat{\mathcal{V}}^{\mathrm{DR}}_G$ *such that*

$$e_0 \mapsto \lambda e_0; \quad e_1 \mapsto \hat{\beta}(\Psi^{-1} \otimes 1)\lambda e_1 \hat{\beta}(\Psi \otimes 1); \quad g \mapsto g, \text{ for } g \in G, \tag{2.17}$$

This automorphism of $\widehat{\mathcal{V}}^{\mathrm{DR}}_G$ *extends the automorphism* $\mathrm{aut}_\Psi \circ (\lambda \bullet -)$ *of* $\mathbf{k}\langle\langle X\rangle\rangle$.

Proof. First, let us verify that the images by the morphism $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}$ of the generators of $\widehat{\mathcal{V}}^{\mathrm{DR}}_G$ satisfy the relations of $\widehat{\mathcal{V}}^{\mathrm{DR}}_G$. Indeed, for $g, h \in G$ we have:

- $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(g) \times \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(h) = g \times h = gh = \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(gh)$;
- $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(1_G) = 1_G = 1 = \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(1)$;
- $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(g) \times \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(e_0) = g \times \lambda e_0 = \lambda e_0 \times g = \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(e_0) \times \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(g)$.

This proves the existence and uniqueness of the algebra endomorphism $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}$. Next, in order to prove that $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}$ is an automorphism, we show that the diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle \rtimes G & \xrightarrow{\hat{\beta}} & \widehat{\mathcal{V}}^{\mathrm{DR}}_G \\ {\scriptstyle \mathrm{aut}_\Psi \circ (\lambda \bullet -) \otimes \mathrm{id}_{\mathbf{k}G}} \downarrow & & \downarrow {\scriptstyle \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}} \\ \mathbf{k}\langle\langle X\rangle\rangle \rtimes G & \xrightarrow[\hat{\beta}]{} & \widehat{\mathcal{V}}^{\mathrm{DR}}_G \end{array} \tag{2.18}$$

commutes, where aut_Ψ is the $\mathbf{k}$-algebra automorphism in (1.7). Since all arrows of Diagram (2.18) are $\mathbf{k}$-algebra morphisms, it is enough to check the commutativity on generators:

- $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \beta(x_0 \otimes 1) = \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(e_0) = \lambda e_0$ and
  $\beta \circ (\mathrm{aut}_\Psi \circ (\lambda \bullet -) \otimes \mathrm{id}_{\mathbf{k}G})(x_0 \otimes 1) = \beta(\mathrm{aut}_\Psi(\lambda x_0) \otimes 1) = \lambda\beta(x_0 \otimes 1) = \lambda e_0$.

- For $g \in G$, $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \beta(1 \otimes g) = \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(g) = g$ and $\beta \circ (\mathrm{aut}_\Psi \circ (\lambda \bullet -) \otimes \mathrm{id}_{\mathbf{k}G})(1 \otimes g) = \beta(1 \otimes g) = g$.

- For $g \in G$, we have that

$$\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \beta(x_g \otimes 1) = \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(-ge_1g^{-1}) = -g\beta(\Psi^{-1} \otimes 1)\lambda e_1 \beta(\Psi \otimes 1)g^{-1}$$

and

$$\begin{aligned}
&\beta \circ (\mathrm{aut}_\Psi \circ (\lambda \bullet -) \otimes \mathrm{id}_{\mathbf{k}G})(x_g \otimes 1) = \beta(\mathrm{aut}_\Psi(\lambda x_g) \otimes 1) = \beta(t_g(\Psi^{-1})\lambda x_g t_g(\Psi) \otimes 1) \\
&= \beta((1 \otimes g) * (\Psi^{-1} \otimes 1) * (1 \otimes g^{-1}) * (\lambda x_g \otimes 1) * (1 \otimes g) * (\Psi \otimes 1) * (1 \otimes g^{-1})) \\
&= \beta(1 \otimes g)\beta(\Psi^{-1} \otimes 1)\beta(1 \otimes g^{-1})\beta(\lambda x_g \otimes 1)\beta(1 \otimes g)\beta(\Psi \otimes 1)\beta(1 \otimes g^{-1}) \\
&= g\beta(\Psi^{-1} \otimes 1)g^{-1}(-g\lambda e_1 g^{-1})g\beta(\Psi \otimes 1)g^{-1} = -g\beta(\Psi^{-1} \otimes 1)\lambda e_1 \beta(\Psi \otimes 1)g^{-1}.
\end{aligned}$$

Therefore, $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}$ is an automorphism thanks to the commutativity of diagram (2.18) and the fact that $\mathrm{aut}_\Psi \circ (\lambda \bullet -) \otimes \mathrm{id}_{\mathbf{k}G} : \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ and $\beta : \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}^{\mathrm{DR}}_G$ are $\mathbf{k}$-algebra isomorphisms.
Finally, the automorphism $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}$ of $\widehat{\mathcal{V}}^{\mathrm{DR}}_G$ extends the automorphism $\mathrm{aut}_\Psi \circ (\lambda \bullet -)$ of $\mathbf{k}\langle\langle X\rangle\rangle$. Indeed, combining Diagram (2.18) with the commutative diagram

$$\begin{array}{ccc}
\mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{-\otimes 1} & \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \\
{\scriptstyle \mathrm{aut}_\Psi \circ (\lambda \bullet -)} \downarrow & & \downarrow {\scriptstyle \mathrm{aut}_\Psi \circ (\lambda \bullet -) \otimes \mathrm{id}_{\mathbf{k}G}} \\
\mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow[-\otimes 1]{} & \mathbf{k}\langle\langle X\rangle\rangle \rtimes G
\end{array}$$

we obtain the following commutative diagram

$$\begin{array}{ccc}
\mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\beta \circ (-\otimes 1)} & \widehat{\mathcal{V}}^{\mathrm{DR}}_G \\
{\scriptstyle \mathrm{aut}_\Psi \circ (\lambda \bullet -)} \downarrow & & \downarrow {\scriptstyle \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}} \\
\mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow[\beta \circ (-\otimes 1)]{} & \widehat{\mathcal{V}}^{\mathrm{DR}}_G
\end{array} \tag{2.19}$$

□

Definition 2.2.2. For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define $\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}$ to be the topological $\mathbf{k}$-algebra automorphism of $\widehat{\mathcal{V}}^{\mathrm{DR}}_G$ given by

$$\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} := \mathrm{Ad}_{\beta(\Psi \otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}. \tag{2.20}$$

Proposition 2.2.3.

(i) There is a group action of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *on* $\widehat{\mathcal{V}}^{\mathrm{DR}}_G$ *by topological* $\mathbf{k}$*-algebra automorphisms*

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{V}}^{\mathrm{DR}}_G),\ (\lambda, \Psi) \longmapsto \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}.$$

(ii) *There is a group action of* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *on* $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ *by topological* $\mathbf{k}$*-algebra automorphisms*

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{V}}_G^{\mathrm{DR}}),\ (\lambda,\Psi) \longmapsto \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}.$$

(iii) *Both group actions induce actions of* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *on* $\mathbf{k}\langle\langle X\rangle\rangle$ *by topological* $\mathbf{k}$*-algebra automorphisms; the former by* $(\lambda,\Psi) \mapsto \mathrm{aut}_\Psi \circ (\lambda \bullet -)$ *and the latter by* $(\lambda,\Psi) \mapsto \mathrm{Ad}_\Psi \circ \mathrm{aut}_\Psi \circ (\lambda \bullet -)$.

Proof.

(i) Let us show that for any $(\lambda,\Psi),(\nu,\Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have

$$\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} = \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\nu,\Phi)}.$$

It suffices to prove this identity on generators. Since for $(\lambda,\Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ we have $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(e_0) = \lambda e_0$ and $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(g) = g$, this is immediately true for e_0 and $g \in G$. Moreover,

$$\begin{aligned}
&\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)\circledast(\nu,\Phi)}(e_1) = \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda\nu,\Psi\circledast(\lambda\bullet\Phi))}(e_1)\\
&= \beta((\Psi \circledast (\lambda\bullet\Phi))^{-1}\otimes 1)\,\lambda\,\nu\,e_1\beta((\Psi\circledast(\lambda\bullet\Phi))\otimes 1)\\
&= \beta(\mathrm{aut}_\Psi((\lambda\bullet\Phi)^{-1})\Psi^{-1}\otimes 1)\,\lambda\,\nu\,e_1\beta(\Psi\mathrm{aut}_\Psi(\lambda\bullet\Phi)\otimes 1)\\
&= \beta(\mathrm{aut}_\Psi(\lambda\bullet\Phi^{-1})\otimes 1)\nu\beta(\Psi^{-1}\otimes 1)\lambda e_1\beta(\Psi\otimes 1)\beta(\mathrm{aut}_\Psi(\lambda\bullet\Phi)\otimes 1)\\
&= \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \beta(\Phi^{-1}\otimes 1)\nu\,\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(e_1)\,\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \beta(\Phi\otimes 1)\\
&= \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}\big(\beta(\Phi^{-1}\otimes 1)\nu e_1\beta(\Phi\otimes 1)\big)\\
&= \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\nu,\Phi)}(e_1),
\end{aligned}$$

where the fifth equality comes from the commutativity of Diagram (2.18).

(ii) For any $(\lambda,\Psi),(\nu,\Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have

$$\begin{aligned}
&\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{V},(1)}_{(\nu,\Phi)} = \mathrm{Ad}_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \mathrm{Ad}_{\beta(\Phi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\nu,\Phi)}\\
&= \mathrm{Ad}_{\beta(\Psi\otimes 1)} \circ \mathrm{Ad}_{\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(\beta(\Phi\otimes 1))} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\nu,\Phi)}\\
&= \mathrm{Ad}_{\beta(\Psi\otimes 1)\beta(\mathrm{aut}_\Psi(\lambda\bullet\Phi)\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\nu,\Phi)}\\
&= \mathrm{Ad}_{\beta((\Psi\circledast(\lambda\bullet\Phi))\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} = \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)\circledast(\nu,\Phi)},
\end{aligned}$$

where the third equality comes from the commutativity of Diagram (2.18) and the fourth one from Identity (i).

(iii) The first part of the claim is a consequence of the fact that for any $(\lambda,\Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the automorphism $\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}$ extends the automorphism $\mathrm{aut}_\Psi \circ (\lambda \bullet -)$ thanks to Proposition-Definition 2.2.1. The same proposition-definition allows us to obtain the following commutative diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\beta\circ(-\otimes 1)} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle \mathrm{Ad}_\Psi\circ\mathrm{aut}_\Psi\circ(\lambda\bullet -)}\downarrow & & \downarrow{\scriptstyle \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}} \\ \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow[\beta\circ(-\otimes 1)]{} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \end{array} \tag{2.21}$$

This implies that the automorphism $\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} = \mathrm{Ad}_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}$ extends the automorphism $\mathrm{Ad}_\Psi \circ \mathrm{aut}_\Psi \circ (\lambda \bullet -)$, thus it proves the second part of the claim.

□

Proposition-Definition 2.2.4. *For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the automorphism $\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} : \widehat{\mathcal{V}}_G^{\mathrm{DR}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ restricts to a topological $\mathbf{k}$-algebra automorphism on the $\mathbf{k}$-subalgebra $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ which will be denoted $\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}$. Moreover, there is a group action of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ by $\mathbf{k}$-algebra automorphisms*

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}}),\ (\lambda,\Psi) \longmapsto \mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}. \tag{2.22}$$

Proof. For $w = k + ve_1 \in \widehat{\mathcal{W}}_G^{\mathrm{DR}}$, we have

$$\begin{aligned} \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(w) =& k + \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(v)\beta(\Psi\otimes 1)\beta(\Psi^{-1}\otimes 1)\lambda e_1 \beta(\Psi\otimes 1)\beta(\Psi^{-1}\otimes 1) \\ =& k + \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(v)\lambda e_1 \in \widehat{\mathcal{W}}_G^{\mathrm{DR}}. \end{aligned}$$

This implies that $\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}$ induces a $\mathbf{k}$-algebra endomorphism of $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$. Moreover, the pullback of this endomorphism under the $\mathbf{k}$-module isomorphism $\mathbf{k} \times \widehat{\mathcal{V}}_G^{\mathrm{DR}} \to \widehat{\mathcal{W}}_G^{\mathrm{DR}}$, $(k, v) \mapsto k + ve_1$ is the pair $(\mathrm{id}, \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)})$, which is a $\mathbf{k}$-module automorphism. This implies that $\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}$ is a $\mathbf{k}$-module automorphism, and therefore a $\mathbf{k}$-algebra automorphism. Thanks to this, the second part of this result can be deduced from Proposition 2.2.3.(ii), by restriction on $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$. □

2.2.2 Actions of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ by module automorphisms

Definition 2.2.5. For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define $\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}$ to be the topological $\mathbf{k}$-module automorphism of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ given by

$$\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} := \ell_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}. \tag{2.23}$$

Remark. Let us notice that, for any $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we also have

$$\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} = \ell_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} = \ell_{\beta(\Psi\otimes 1)} \circ \mathrm{Ad}_{\beta(\Psi^{-1}\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} = r_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}.$$

Proposition 2.2.6. *There is a group action of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ by topological $\mathbf{k}$-module automorphisms given by*

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{V}}_G^{\mathrm{DR}}),\ (\lambda,\Psi) \longmapsto \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}.$$

Proof. For $(\lambda, \Psi), (\nu, \Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have

$$\begin{aligned}
\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\nu,\Phi)} =& \ell_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \ell_{\beta(\Phi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\nu,\Phi)} \\
=& \ell_{\beta(\Psi\otimes 1)} \circ \ell_{\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(\beta(\Phi\otimes 1))} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\nu,\Phi)} \\
=& \ell_{\beta(\Psi\otimes 1)\beta(\mathrm{aut}_\Psi(\lambda\bullet\Phi)\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\nu,\Phi)} \\
=& \ell_{\beta((\Psi\circledast(\lambda\bullet\Phi))\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} = \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)\circledast(\nu,\Phi)},
\end{aligned}$$

where the third equality comes from the commutativity of Diagram (2.18) and the fourth one from Proposition 2.2.3.(i). □

Lemma 2.2.7. *For any $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have*

$$\forall (a, v) \in \widehat{\mathcal{V}}^{\mathrm{DR}}_G \times \widehat{\mathcal{V}}^{\mathrm{DR}}_G,\ \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(av) = \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(a)\,\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(v).$$

Proof. Let $a, b \in \widehat{\mathcal{V}}^{\mathrm{DR}}_G$. We have

$$\begin{aligned}
\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(ab) =& \ell_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(ab) = \ell_{\beta(\Psi\otimes 1)}\left(\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(a)\,\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(b)\right) \\
=& \left(\ell_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(a)\right)\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(b) = \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(a)\,\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(b),
\end{aligned}$$

□

Proposition-Definition 2.2.8. *For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, there is a unique* $\mathbf{k}$*-module automorphism* $\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}$ *of* $\widehat{\mathcal{M}}^{\mathrm{DR}}_G$ *such that the following diagram*

$$\begin{array}{ccc}
\widehat{\mathcal{V}}^{\mathrm{DR}}_G & \xrightarrow{\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}} & \widehat{\mathcal{V}}^{\mathrm{DR}}_G \\
{\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}}\downarrow & & \downarrow{\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}} \\
\widehat{\mathcal{M}}^{\mathrm{DR}}_G & \xrightarrow[\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}]{} & \widehat{\mathcal{M}}^{\mathrm{DR}}_G
\end{array} \tag{2.24}$$

commutes.

Proof. Let us show that the $\mathbf{k}$-module automorphism $\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}$ preserves the submodule $\widehat{\mathcal{V}}^{\mathrm{DR}}_G e_0 + \sum_{g\in G} \widehat{\mathcal{V}}^{\mathrm{DR}}_G (g-1)$. Indeed, for any $a \in \widehat{\mathcal{V}}^{\mathrm{DR}}_G$ and $(v_g)_{g\in G} \in (\widehat{\mathcal{V}}^{\mathrm{DR}}_G)^G$,

$$\begin{aligned}
&\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}\left(ae_0 + \sum_{g\in G} v_g(g-1)\right) = \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(a)\,\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(e_0) + \sum_{g\in G} \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v_g)\mathrm{aut}^{\mathcal{V},(0)}_{(\lambda,\Psi)}(g-1) \\
&= \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(a)e_0 + \sum_{g\in G} \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v_g)(g-1) \in \widehat{\mathcal{V}}^{\mathrm{DR}}_G e_0 + \sum_{g\in G} \widehat{\mathcal{V}}^{\mathrm{DR}}_G (g-1),
\end{aligned}$$

where the first equality comes from Lemma 2.2.7. The result then follows. □

Proposition 2.2.9. *There is a group action of* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *on* $\widehat{\mathcal{M}}^{\mathrm{DR}}_G$ *by topological* $\mathbf{k}$*-module automorphisms*

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\left(\widehat{\mathcal{M}}^{\mathrm{DR}}_G\right),\ (\lambda, \Psi) \longmapsto \mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}. \tag{2.25}$$

Proof. For $(\lambda, \Psi), (\nu, \Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have

$$\begin{aligned}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{M},(10)}_{(\nu,\Phi)} \circ \widehat{- \cdot 1_{\mathrm{DR}}} =& \mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ \widehat{- \cdot 1_{\mathrm{DR}}} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\nu,\Phi)} \\ =& \widehat{- \cdot 1_{\mathrm{DR}}} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\nu,\Phi)} \\ =& \widehat{- \cdot 1_{\mathrm{DR}}} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} = \mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} \circ \widehat{- \cdot 1_{\mathrm{DR}}},\end{aligned}$$

where the first, second and fourth equalities come from Proposition-Definition 2.2.8 and the third one from Proposition 2.2.6. Finally, the surjectivity of $\widehat{- \cdot 1_{\mathrm{DR}}}$ implies that

$$\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{M},(10)}_{(\nu,\Phi)} = \mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)\circledast(\nu,\Phi)}.$$

□

Proposition 2.2.10. *For any $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the following pairs are automorphisms in the category* $\mathbf{k}$-alg-mod*:*

(i) $\left(\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}, \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}\right)$ *is an automorphism of* $(\widehat{\mathcal{V}}^{\mathrm{DR}}_G, \widehat{\mathcal{V}}^{\mathrm{DR}}_G)$.

(ii) $\left(\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}, \mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right)$ *is an automorphism of* $(\widehat{\mathcal{V}}^{\mathrm{DR}}_G, \widehat{\mathcal{M}}^{\mathrm{DR}}_G)$.

(iii) $\left(\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}, \mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right)$ *is an automorphism of* $(\widehat{\mathcal{W}}^{\mathrm{DR}}_G, \widehat{\mathcal{M}}^{\mathrm{DR}}_G)$.

Proof.

(i) Let $(a, v) \in \widehat{\mathcal{V}}^{\mathrm{DR}}_G \times \widehat{\mathcal{V}}^{\mathrm{DR}}_G$. We have

$$\begin{aligned}\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(av) =& r_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(av) = r_{\beta(\Psi\otimes 1)} \left(\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a)\, \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(v)\right) \\ =& \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a) \left(r_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(v)\right) = \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a)\, \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v).\end{aligned}$$

(ii) Let $(a, m) \in \widehat{\mathcal{V}}^{\mathrm{DR}}_G \times \widehat{\mathcal{M}}^{\mathrm{DR}}_G$. There exist $v \in \widehat{\mathcal{V}}^{\mathrm{DR}}_G$ such that $m = v \cdot 1_{\mathrm{DR}}$. We have

$$\begin{aligned}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}(am) =& \mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}(av \cdot 1_{\mathrm{DR}}) = \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(av) \cdot 1_{\mathrm{DR}} \\ =& \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a)\, \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v) \cdot 1_{\mathrm{DR}} = \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a)\, \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v \cdot 1_{\mathrm{DR}}) \\ =& \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a)\, \mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}(m),\end{aligned}$$

where the second and fourth equalities come from Proposition-Definition 2.2.8 and the third one from (i).

(iii) It follows from (ii) thanks to Proposition-Definition 2.2.4.

□

2.2.3 The cocycle Γ and twisted actions

To an element $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, one associates $\Gamma_\Psi \in \mathbf{k}[[x]]^\times$ (see (1.15)). Then $\Gamma_\Psi(-e_1)$ is an invertible element of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$.

Definition 2.2.11. For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define the topological $\mathbf{k}$-algebra automorphism of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$:

$$^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} := \mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)} \circ \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}. \tag{2.26}$$

Proposition-Definition 2.2.12. *For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the topological $\mathbf{k}$-algebra automorphism $^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)} := \mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)} \circ \mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}$ of $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ is such that the following diagram*

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{\ ^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}\ } & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow[\ ^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}\ ]{} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \end{array}$$

commutes.

Proof. Follows from Proposition-Definition 2.2.4 and the fact that $\Gamma_\Psi(-e_1)$ is an invertible element of $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$. □

Proposition 2.2.13.

(i) There is a group action of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ by topological $\mathbf{k}$-algebra automorphisms

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}\left(\widehat{\mathcal{V}}_G^{\mathrm{DR}}\right),\ \Psi \longmapsto {}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}. \tag{2.27}$$

(ii) There is a group action of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ by topological $\mathbf{k}$-algebra automorphisms

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}\left(\widehat{\mathcal{W}}_G^{\mathrm{DR}}\right),\ (\lambda, \Psi) \longmapsto {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}. \tag{2.28}$$

Proof.

(i) It follows from Proposition 2.2.3.(ii), Lemmas 1.2.12 and 1.3.7 and the fact that $\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(e_1) = \lambda e_1$ for any $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$.

(ii) It follows from (i) thanks to Proposition-Definition 2.2.12.

□

Definition 2.2.14. For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define $^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}$ to be the topological $\mathbf{k}$-module automorphism of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ given by

$$^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} := \ell_{\Gamma_\Psi^{-1}(-e_1)} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}. \tag{2.29}$$

Proposition 2.2.15. *There is a group action of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ by topological $\mathbf{k}$-module automorphisms*

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\left(\widehat{\mathcal{V}}_G^{\mathrm{DR}}\right),\ (\lambda, \Psi) \longmapsto {}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}. \tag{2.30}$$

Proof. For any $(\lambda, \Psi), (\nu, \Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have

$$\begin{aligned} {}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} &= \ell_{\Gamma^{-1}_{\Psi\circledast(\lambda\bullet\Phi)}(-e_1)} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} = \ell_{\Gamma^{-1}_{\Psi}(-e_1)(\lambda\bullet\Gamma^{-1}_{\Phi}(-e_1))} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\nu,\Phi)} \\ &= \ell_{\Gamma^{-1}_{\Psi}(-e_1)} \circ \ell_{\lambda\bullet\Gamma^{-1}_{\Phi}(-e_1)} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\nu,\Phi)} \\ &= \ell_{\Gamma^{-1}_{\Psi}(-e_1)} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ \ell_{\Gamma^{-1}_{\Phi}(-e_1)} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\nu,\Phi)} \\ &= {}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ {}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\nu,\Phi)}, \end{aligned}$$

where the second equality uses Lemmas 1.2.12 and 1.3.7 and Proposition 2.2.6 and the fourth equality comes from the fact that for any $v \in \widehat{\mathcal{V}}_G^{\mathrm{DR}}$,

$$\begin{aligned} \ell_{\lambda\bullet\Gamma^{-1}_{\Phi}(-e_1)} \circ \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v) &= \left(\lambda \bullet \Gamma^{-1}_{\Phi}(-e_1)\right) \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v) = \mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(\Gamma^{-1}_{\Phi}(-e_1))\,\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v) \\ &= \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(\Gamma^{-1}_{\Phi}(-e_1)v) = \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ \ell_{\Gamma^{-1}_{\Phi}(-e_1)}(v), \end{aligned}$$

where the second equality uses the fact that $\mathrm{aut}^{\mathcal{V},(1)}_{\Psi}(e_1) = \lambda e_1$ and the third equality comes from Proposition 2.2.10.(i). ☐

Proposition-Definition 2.2.16. *For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the topological $\mathbf{k}$-module automorphism ${}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} := \ell_{\Gamma^{-1}_{\Psi}(-e_1)} \circ \mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}$ of $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is such that the following diagram*

$$\begin{array}{ccc} \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{{}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle\widehat{-\cdot 1_{\mathrm{DR}}}}\big\downarrow & & \big\downarrow{\scriptstyle\widehat{-\cdot 1_{\mathrm{DR}}}} \\ \widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow[{}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}]{} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array}$$

commutes.

Proof. It follows from Proposition-Definition 2.2.8 thanks to the left $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$-module structure of $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$. ☐

Proposition 2.2.17. *There is a group action of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ by topological $\mathbf{k}$-module automorphisms*

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\left(\widehat{\mathcal{M}}_G^{\mathrm{DR}}\right),\ (\lambda, \Psi) \longmapsto {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}. \tag{2.31}$$

Proof. For any $(\lambda, \Psi), (\nu, \Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have

$$\begin{aligned} &{}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} \circ \widehat{-\cdot 1_{\mathrm{DR}}} = \widehat{-\cdot 1_{\mathrm{DR}}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} = \widehat{-\cdot 1_{\mathrm{DR}}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ {}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\nu,\Phi)} \\ &= {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ \widehat{-\cdot 1_{\mathrm{DR}}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\nu,\Phi)} = {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ {}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\nu,\Phi)} \circ \widehat{-\cdot 1_{\mathrm{DR}}}, \end{aligned}$$

where the first, third and fourth equalities comes from Proposition-Definition 2.2.16 and the second one from Proposition 2.2.15. ☐

Proposition 2.2.18. *For any* $(\lambda,\Psi)\in\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, the following pairs are automorphisms in the category* $\mathbf{k}$-alg-mod*:*

(i) $\left({}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)},\,{}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}\right)$ *is an automorphism of* $(\widehat{\mathcal{V}}^{\mathrm{DR}}_G,\widehat{\mathcal{V}}^{\mathrm{DR}}_G)$*.*

(ii) $\left({}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)},\,{}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right)$ *is an automorphism of* $(\widehat{\mathcal{V}}^{\mathrm{DR}}_G,\widehat{\mathcal{M}}^{\mathrm{DR}}_G)$*.*

(iii) $\left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)},\,{}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right)$ *is an automorphism of* $(\widehat{\mathcal{W}}^{\mathrm{DR}}_G,\widehat{\mathcal{M}}^{\mathrm{DR}}_G)$*.*

Proof.

(i) Let $(a,v)\in\widehat{\mathcal{V}}^{\mathrm{DR}}_G\times\widehat{\mathcal{V}}^{\mathrm{DR}}_G$. We have

$$\begin{aligned}{}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(av)=&\ell_{\Gamma_\Psi^{-1}(-e_1)}\circ\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(av)=\ell_{\Gamma_\Psi^{-1}(-e_1)}\left(\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a)\,\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v)\right)\\=&\mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)}\circ\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a)\ \ \ell_{\Gamma_\Psi^{-1}(-e_1)}\circ\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v)\\=&{}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a)\,{}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v).\end{aligned}$$

(ii) Let $(a,m)\in\widehat{\mathcal{V}}^{\mathrm{DR}}_G\times\widehat{\mathcal{M}}^{\mathrm{DR}}_G$. There exist $v\in\widehat{\mathcal{V}}^{\mathrm{DR}}_G$ such that $m=v\cdot 1_{\mathrm{DR}}$. We have

$$\begin{aligned}{}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}(am)=&{}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}(av\cdot 1_{\mathrm{DR}})={}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(av)\cdot 1_{\mathrm{DR}}\\=&{}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a)\,{}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v)\cdot 1_{\mathrm{DR}}\\=&{}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a)\,{}^\Gamma\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}(v\cdot 1_{\mathrm{DR}})={}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(a)\,{}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}(m),\end{aligned}$$

where the second and fourth equalities come from Proposition-Definition 2.2.16 and the third one from (i).

(iii) It follows from (ii) thanks to Proposition-Definition 2.2.12.

□

2.3 The stabilizers $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G)(\mathbf{k})$ and $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k})$

2.3.1 Inclusion of stabilizers

Proposition 2.3.1. *There is a group action of* $\mathbf{k}^\times\ltimes\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *on the* $\mathbf{k}$*-module* $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}\left(\widehat{\mathcal{W}}^{\mathrm{DR}}_G,(\widehat{\mathcal{W}}^{\mathrm{DR}}_G)^{\hat\otimes 2}\right)$ *given by*

$$(\lambda,\Psi)\cdot D^{\mathcal{W}}:=\left(\left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1}\circ D^{\mathcal{W}}\circ{}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)},\tag{2.32}$$

with $(\lambda,\Psi)\in\mathbf{k}^\times\times\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *and* $D^{\mathcal{W}}\in\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}\left(\widehat{\mathcal{W}}^{\mathrm{DR}}_G,(\widehat{\mathcal{W}}^{\mathrm{DR}}_G)^{\hat\otimes 2}\right)$*.*

Proof. It follows from Proposition 2.2.13. □

Definition 2.3.2. We denote $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k})$ the stabilizer subgroup of the coproduct $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \in \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}\left(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\hat{\otimes}2}\right)$ for the action (2.32). Namely,

$$\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k}) := \left\{(\lambda,\Psi) \in \mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \mid \left({}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}\right)^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{W}} = \widehat{\Delta}_G^{\mathcal{W}} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}\right\}. \tag{2.33}$$

Proposition 2.3.3. *There is a group action of* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *on the* $\mathbf{k}$*-module* $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\left(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, (\widehat{\mathcal{M}}_G^{\mathrm{DR}})^{\hat{\otimes}2}\right)$ *given by*

$$(\lambda,\Psi)\cdot D^{\mathcal{M}} := \left(\left({}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1} \circ D^{\mathcal{M}} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}, \tag{2.34}$$

with $(\lambda,\Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *and* $D^{\mathcal{M}} \in \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\left(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, (\widehat{\mathcal{M}}_G^{\mathrm{DR}})^{\hat{\otimes}2}\right)$.

Proof. It follows from Proposition 2.2.17. □

Definition 2.3.4. We denote $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})(\mathbf{k})$ the stabilizer subgroup of the coproduct $\widehat{\Delta}_G^{\mathcal{M}} \in \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}\left(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, (\widehat{\mathcal{M}}_G^{\mathrm{DR}})^{\hat{\otimes}2}\right)$ for the action (2.34). Namely,

$$\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})(\mathbf{k}) := \left\{(\lambda,\Psi) \in \mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \,\middle|\, \left({}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right)^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{M}} = \widehat{\Delta}_G^{\mathcal{M}} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right\}. \tag{2.35}$$

Theorem 2.3.5. $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})(\mathbf{k}) \subset \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k})$ *(as subgroups of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$*).*

Proof. We have

- $\left((\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}), (\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})\right) \in \mathbf{k}$-HAMC thanks to Corollary 2.1.28.
- A group morphism $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathrm{Aut}_{\mathbf{k}\text{-alg-mod}}\left(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}}\right)$ given by the actions (2.32) and (2.34).
- $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is a free $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$-module of rank 1 thanks to Corollary 2.1.26.(ii). It is generated by 1_{DR} and we have that $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}(1_{\mathrm{DR}}) = 1_{\mathrm{DR}}^{\otimes 2}$ is a generator of $(\widehat{\mathcal{M}}_G^{\mathrm{DR}})^{\otimes 2}$ as a $(\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\otimes 2}$-module.

Thus, the result follows from Proposition 2.1.3. □

2.3.2 The stabilizer groups in terms of Racinet's formalism

The stabilizer group $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})$ in Racinet's formalism

Lemma 2.3.6. *For* $(\lambda,\Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, the following diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\hat{\beta}\circ(-\otimes 1)} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle S_\Psi\circ(\lambda\bullet -)}\downarrow & & \downarrow{\scriptstyle \mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)}} \\ \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow[\hat{\beta}\circ(-\otimes 1)]{} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \end{array} \tag{2.36}$$

commutes.

Proof. This is done by composing the bottom of Diagram (2.19) with the following commutative diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\hat{\beta}\circ(-\otimes 1)} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle \ell_\Psi}\downarrow & & \downarrow{\scriptstyle \ell_{\hat{\beta}(\Psi\otimes 1)}} \\ \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow[\hat{\beta}\circ(-\otimes 1)]{} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \end{array}$$

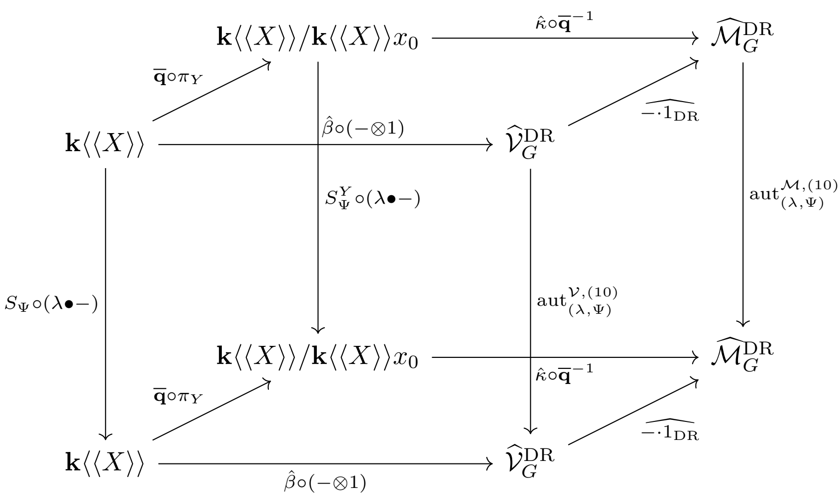
□

Lemma 2.3.7. *For* $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, the following diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow{\hat{\kappa}\circ\overline{\mathbf{q}}^{-1}} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \\ {\scriptstyle S_\Psi^Y\circ(\lambda\bullet -)}\downarrow & & \downarrow{\scriptstyle \mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{M},(10)}} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\hat{\kappa}\circ\overline{\mathbf{q}}^{-1}]{} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{2.37}$$

commutes.

Proof. Let us consider the following cube



The left (resp. right) side commutes by definition of S_Ψ^Y (resp. $\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{M},(10)}$). Then, the upper and lower sides are exactly the same square, which is commutative thanks to Corollary 2.1.26.(i). Finally, Lemma 2.3.6 gives us the commutativity of the front side. This collection of commutativities together with the surjectivity of $\overline{\mathbf{q}}\circ\pi_Y$ implies that the back side of the cube commutes, which is exactly Diagram (2.37). □

Proposition 2.3.8. *For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the following diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow{\hat{\kappa}\circ\overline{\mathbf{q}}^{-1}} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \\ {\scriptstyle \Gamma_{S_\Psi^Y}\circ(\lambda\bullet -)}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma_{\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}}} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\hat{\kappa}\circ\overline{\mathbf{q}}^{-1}]{} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{2.38}$$

commutes.

Remark. It follows from Diagram (2.38) that $\hat{\kappa}\circ\overline{\mathbf{q}}^{-1}$ is an isomorphism between the $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$-modules[3] $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ and $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$.

Proof. This is done by composing the bottom of Diagram (2.37) with the following diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow{\hat{\kappa}\circ\overline{\mathbf{q}}^{-1}} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \\ {\scriptstyle \ell_{\Gamma_\Psi^{-1}(x_1)}}\big\downarrow & & \big\downarrow{\scriptstyle \ell_{\Gamma_\Psi^{-1}(-e_1)}} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\hat{\kappa}\circ\overline{\mathbf{q}}^{-1}]{} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array}$$

The above diagram is commutative because we have

$$\begin{aligned} &\ell_{\Gamma_\Psi^{-1}(-e_1)}\circ\hat{\kappa}\circ\overline{\mathbf{q}}^{-1}\circ\overline{\mathbf{q}}\circ\pi_Y = \ell_{\Gamma_\Psi^{-1}(-e_1)}\circ(\widehat{-\cdot 1_{\mathrm{DR}}})\circ\hat{\beta}\circ(-\otimes 1) \\ &= (\widehat{-\cdot 1_{\mathrm{DR}}})\circ\ell_{\hat{\beta}(\Gamma_\Psi^{-1}(x_1)\otimes 1)}\circ\hat{\beta}\circ(-\otimes 1) = (\widehat{-\cdot 1_{\mathrm{DR}}})\circ\hat{\beta}\circ(-\otimes 1)\circ\ell_{\Gamma_\Psi^{-1}(x_1)} \\ &= \hat{\kappa}\circ\overline{\mathbf{q}}^{-1}\circ\overline{\mathbf{q}}\circ\pi_Y\circ\ell_{\Gamma_\Psi^{-1}(x_1)} = \hat{\kappa}\circ\overline{\mathbf{q}}^{-1}\circ\ell_{\Gamma_\Psi^{-1}(x_1)}\circ\overline{\mathbf{q}}\circ\pi_Y, \end{aligned}$$

where the first and fourth equalities come from Proposition-Definition 2.1.25. (ii); the second one from the fact that $\widehat{-\cdot 1_{\mathrm{DR}}} : \widehat{\mathcal{V}}_G^{\mathrm{DR}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is a $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$-module morphism; the third one from the fact that $\hat{\beta}\circ(-\otimes 1) : \mathbf{k}\langle\langle X\rangle\rangle \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ is a $\mathbf{k}$-algebra morphism and the last one from the fact that $\pi_Y : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ is $\mathbf{k}\langle\langle X\rangle\rangle$-module morphism and that for any $a \in \mathbf{k}\langle\langle X\rangle\rangle$, $\mathbf{q}(x_1 a) = x_1\mathbf{q}(a)$.
Finally, since $\overline{\mathbf{q}}\circ\pi_Y$ is a surjective $\mathbf{k}$-module morphism, it follows that

$$\ell_{\Gamma_\Psi^{-1}(-e_1)}\circ\hat{\kappa}\circ\overline{\mathbf{q}}^{-1} = \hat{\kappa}\circ\overline{\mathbf{q}}^{-1}\circ\ell_{\Gamma_\Psi^{-1}(x_1)},$$

which is the wanted result. □

Theorem 2.3.9. $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k}) = \mathbf{k}^\times \ltimes \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k})$ *(as subgroups of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$).*

Proof. Thanks to Proposition 2.3.8, the map $\hat{\kappa}\circ\overline{\mathbf{q}}^{-1} : \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is a $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$-module isomorphism. Therefore, it induces a $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$-module isomorphism

$$\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, (\widehat{\mathcal{M}}_G^{\mathrm{DR}})^{\otimes 2}) \to \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\otimes 2})$$

[3]For the $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$-module structure of $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ (resp. $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$), see Proposition-Definition 1.3.9 (resp. Proposition 2.2.17).

given by

$$\Delta \mapsto \big((\hat{\kappa} \circ \overline{\mathbf{q}}^{-1})^{\otimes 2}\big)^{-1} \circ \Delta \circ (\hat{\kappa} \circ \overline{\mathbf{q}}^{-1}),$$

where the $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$-module structure on $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}^{\mathrm{DR}}_G, (\widehat{\mathcal{M}}^{\mathrm{DR}}_G)^{\hat{\otimes} 2})$ (resp. $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\otimes 2})$) is defined in (2.34) (resp. (1.24)). Moreover, thanks to Proposition 2.1.29, the coproduct $\widehat{\Delta}^{\mathcal{M}}_G$ is sent to the coproduct $\widehat{\Delta}^{\mathrm{mod}}_\star$ via this isomorphism. Therefore, we obtain the equality $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G)(\mathbf{k}) = \mathsf{Stab}_\bullet(\widehat{\Delta}^{\mathrm{mod}}_\star)(\mathbf{k})$ and thanks to Proposition 1.3.12 this implies the wanted equality of stabilizers. ☐

The stabilizer group $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{W}}_G)$ in Racinet's formalism

Proposition-Definition 2.3.10. *For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we consider the $\mathbf{k}$-algebra automorphism of $\mathbf{k}\langle\langle Y\rangle\rangle$ given by*

$$^\Gamma\mathrm{aut}^Y_\Psi := \hat{\varpi}^{-1} \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(1,\Psi)} \circ \hat{\varpi}. \tag{2.39}$$

Then, there is a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathbf{k}\langle\langle Y\rangle\rangle$ by topological $\mathbf{k}$-algebra automorphisms given by

$$\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle),\ \Psi \longmapsto {}^\Gamma\mathrm{aut}^Y_\Psi. \tag{2.40}$$

Proof. As $\hat{\varpi} : \mathbf{k}\langle\langle Y\rangle\rangle \to \widehat{\mathcal{W}}^{\mathrm{DR}}_G$ is an algebra isomorphism, conjugation by it induces a group isomorphism $\mathrm{Aut}_{\mathbf{k}-\mathrm{alg}}(\widehat{\mathcal{W}}^{\mathrm{DR}}_G) \to \mathrm{Aut}_{\mathbf{k}-\mathrm{alg}}(\mathbf{k}\langle\langle Y\rangle\rangle)$ which, when composed with the group morphism $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathrm{Aut}_{\mathbf{k}-\mathrm{alg}}(\widehat{\mathcal{W}}^{\mathrm{DR}}_G)$, gives rise to a group morphism $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathrm{Aut}_{\mathbf{k}-\mathrm{alg}}(\mathbf{k}\langle\langle Y\rangle\rangle)$. ☐

We aim to give an explicit formulation of the action ${}^\Gamma\mathrm{aut}^Y$ in terms of Racinet's objects. Let us start with the following lemma:

Lemma 2.3.11. *Let $g \in G$. For any $a \in \mathbf{k}\langle\langle X\rangle\rangle$ we have $\hat{\beta}(ax_g \otimes g) = \hat{\varpi} \circ \mathbf{q}_Y(ax_g)$.*

Proof. It is enough to show this on a basis of the $\mathbf{k}$-module $\mathbf{k}\langle\langle X\rangle\rangle$. Let us take the family

$$\Big(x_0^{n_1-1}x_{g_1}x_0^{n_2-1}x_{g_2}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1}\Big)_{\substack{r\in\mathbb{N}, n_1,\dots,n_{r+1}\in\mathbb{N}^*,\\ g_1,\dots,g_r\in G}}$$

as such a basis. For $r \in \mathbb{N}$, $n_1, \dots, n_{r+1} \in \mathbb{N}^*$ and $g_1, \dots, g_r \in G$ we have

$$\begin{aligned} &x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1}x_g \otimes g = \\ &(x_0^{n_1-1}\otimes 1) * (x_{g_1}\otimes 1) * \cdots * (x_0^{n_r-1}\otimes 1) * (x_{g_r}\otimes 1) * \\ &(x_0^{n_{r+1}-1}\otimes 1) * (x_g \otimes 1) * (1 \otimes g). \end{aligned} \tag{2.41}$$

Therefore, we obtain

$$\begin{aligned} &\hat{\beta}(x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1}x_g \otimes g) \\ &= (-1)^{r+1}e_0^{n_1-1}g_1e_1g_1^{-1}e_0^{n_2-1}g_2e_1g_2^{-1}\cdots e_0^{n_{r-1}-1}g_{r-1}e_1g_{r-1}^{-1}e_0^{n_r-1}g_re_1g_r^{-1}e_0^{n_{r+1}-1}ge_1g^{-1}g \\ &= (-1)^{r+1}e_0^{n_1-1}g_1e_1\ e_0^{n_2-1}g_1^{-1}g_2e_1\cdots e_0^{n_{r-1}-1}g_{r-2}^{-1}g_{r-1}e_1\ e_0^{n_r-1}g_{r-1}^{-1}g_re_1\ e_0^{n_{r+1}-1}g_r^{-1}ge_1 \\ &= z_{n_1,g_1}z_{n_2,g_1^{-1}g_2}\cdots z_{n_{r-1},g_{r-2}^{-1}g_{r-1}}z_{n_r,g_{r-1}^{-1}g_r}z_{n_{r+1},g_r^{-1}g}, \end{aligned}$$

where the first equality comes from the computation (2.41) and the fact that $\hat{\beta}$: $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ is a $\mathbf{k}$-algebra morphism and the second equality comes from the fact for any $i \in \{2,\dots,r\}$, $g_i^{-1}e_0 = e_0 g_i^{-1}$. On the other hand,

$$\begin{aligned}
&\hat{\varpi} \circ \mathbf{q}_Y(x_0^{n_1-1}x_{g_1}x_0^{n_2-1}x_{g_2}\cdots x_0^{n_{r-1}-1}x_{g_{r-1}}x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1}x_g) \\
&= \hat{\varpi}(y_{n_1,g_1}y_{n_2,g_1^{-1}g_2}\cdots y_{n_{r-1},g_{r-2}^{-1}g_{r-1}}y_{n_r,g_{r-1}^{-1}g_r}y_{n_{r+1},g_r^{-1}g}) \\
&= z_{n_1,g_1}z_{n_2,g_1^{-1}g_2}\cdots z_{n_{r-1},g_{r-2}^{-1}g_{r-1}}z_{n_r,g_{r-1}^{-1}g_r}z_{n_{r+1},g_r^{-1}g}.
\end{aligned}$$

□

Proposition 2.3.12. *For* $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *and* $(n,g) \in \mathbb{N}^* \times G$ *we have*

$$^{\Gamma}\mathrm{aut}_\Psi^Y(y_{n,g}) = \mathbf{q}_Y\left(\Gamma_\Psi^{-1}(x_1)\Psi x_0^{n-1}t_g\left(\Psi^{-1}\Gamma_\Psi(x_1)\right)x_g\right). \tag{2.42}$$

Proof. Let us start with the following computation

$$\begin{aligned}
&^{\Gamma}\mathrm{aut}_{(1,\Psi)}^{\mathcal{W},(1)}(z_{n,g}) = -\Gamma_\Psi^{-1}(-e_1)\hat{\beta}(\Psi\otimes 1)e_0^{n-1}g\hat{\beta}(\Psi^{-1}\otimes 1)e_1\Gamma_\Psi(-e_1) \\
&= -\Gamma_\Psi^{-1}(-e_1)\hat{\beta}(\Psi\otimes 1)e_0^{n-1}g\hat{\beta}(\Psi^{-1}\otimes 1)\Gamma_\Psi(-e_1)e_1 \\
&= \hat{\beta}((\Gamma_\Psi^{-1}(x_1)\otimes 1)*(\Psi\otimes 1)*(x_0^{n-1}\otimes 1)*(1\otimes g)*(\Psi^{-1}\otimes 1)*(\Gamma_\Psi(x_1)\otimes 1)*(x_1\otimes 1)) \\
&= \hat{\beta}\left(\Gamma_\Psi^{-1}(x_1)\Psi x_0^{n-1}t_g\left(\Psi^{-1}\Gamma_\Psi(x_1)\right)x_g\otimes g\right) \\
&= \hat{\varpi}\circ\mathbf{q}_Y\left(\Gamma_\Psi^{-1}(x_1)\Psi x_0^{n-1}t_g\left(\Psi^{-1}\Gamma_\Psi(x_1)\right)x_g\right),
\end{aligned}$$

where the last equality comes from Lemma 2.3.11. Thanks to this, we have for any $(n,g) \in \mathbb{N}^* \times G$,

$$\begin{aligned}
&^{\Gamma}\mathrm{aut}_\Psi^Y(y_{n,g}) = \hat{\varpi}^{-1}\circ {}^{\Gamma}\mathrm{aut}_{(1,\Psi)}^{\mathcal{W},(1)}\circ\hat{\varpi}(y_{n,g}) = \hat{\varpi}^{-1}\circ {}^{\Gamma}\mathrm{aut}_{(1,\Psi)}^{\mathcal{W},(1)}(z_{n,g}) \\
&= \hat{\varpi}^{-1}\circ\hat{\varpi}\circ\mathbf{q}_Y\left(\Gamma_\Psi^{-1}(x_1)\Psi x_0^{n-1}t_g\left(\Psi^{-1}\Gamma_\Psi(x_1)\right)x_g\right) \\
&= \mathbf{q}_Y\left(\Gamma_\Psi^{-1}(x_1)\Psi x_0^{n-1}t_g\left(\Psi^{-1}\Gamma_\Psi(x_1)\right)x_g\right).
\end{aligned}$$

□

Proposition 2.3.13. *There is a group action of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on the* $\mathbf{k}$*-module* $\mathrm{Mor}_{\mathbf{k}\text{-alg}}\left(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\hat{\otimes}2}\right)$ *given by*

$$\Psi\cdot D := \left(\left({}^{\Gamma}\mathrm{aut}_\Psi^Y\right)^{\otimes 2}\right)^{-1}\circ D\circ {}^{\Gamma}\mathrm{aut}_\Psi^Y, \tag{2.43}$$

with $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *and* $D \in \mathrm{Mor}_{\mathbf{k}\text{-alg}}^{\mathrm{cont}}\left(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\hat{\otimes}2}\right)$.

Proof. It follows from Proposition 2.3.10. □

Definition 2.3.14. We denote $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ the stabilizer subgroup of the coproduct $\widehat{\Delta}_\star^{\mathrm{alg}} \in \mathrm{Mor}_{\mathbf{k}\text{-alg}}^{\mathrm{cont}}\left(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\hat{\otimes}2}\right)$ for the action (2.43). Namely,

$$\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k}) := \left\{\Psi\in\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \mid \left({}^{\Gamma}\mathrm{aut}_\Psi^Y\right)^{\otimes 2}\circ\widehat{\Delta}_\star^{\mathrm{alg}} = \widehat{\Delta}_\star^{\mathrm{alg}}\circ {}^{\Gamma}\mathrm{aut}_\Psi^Y\right\}. \tag{2.44}$$

Lemma 2.3.15. *The action $\bullet$ induces an action of $\mathbf{k}^\times$ on $\mathbf{k}\langle\langle Y\rangle\rangle$, which will also be denoted $\bullet$.*

Proof. This comes from the fact that for any $\lambda \in \mathbf{k}^\times$, the map $\lambda \bullet -$ restricts to $\mathbf{k}\langle\langle Y\rangle\rangle$. □

Lemma 2.3.16. *For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the following diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle Y\rangle\rangle & \xrightarrow{\ {}^\Gamma\mathrm{aut}_\Psi^Y \circ (\lambda\bullet -)\ } & \mathbf{k}\langle\langle Y\rangle\rangle \\ {\scriptstyle\hat{\varpi}}\downarrow & & \downarrow{\scriptstyle\hat{\varpi}} \\ \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow[\ {}^\Gamma\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{W},(1)}\ ]{} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \end{array}$$

commutes.

Proof. Immediate. □

Proposition-Definition 2.3.17. *There is a group action of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on $\mathbf{k}\langle\langle Y\rangle\rangle$ by topological $\mathbf{k}$-module automorphisms*

$$\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \longrightarrow \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle),\ (\lambda,\Psi) \longmapsto {}^\Gamma\mathrm{aut}_\Psi^Y \circ (\lambda \bullet -). \tag{2.45}$$

Proof. It follows from Proposition-Definition 2.3.10 and Lemma 1.3.7. □

Proposition-Definition 2.3.18. *There is a group action of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ on the $\mathbf{k}$-module $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}\left(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\hat{\otimes}2}\right)$ by*

$$(\lambda,\Psi)\cdot D := \left(\left({}^\Gamma\mathrm{aut}_\Psi^Y \circ (\lambda\bullet -)\right)^{\otimes 2}\right)^{-1} \circ D \circ {}^\Gamma\mathrm{aut}_\Psi^Y \circ (\lambda\bullet -), \tag{2.46}$$

with $(\lambda,\Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ and $D \in \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\hat{\otimes}2})$.

Proof. It follows from Proposition-Definition 2.3.17. □

Definition 2.3.19. We denote $\mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ the stabilizer subgroup of the coproduct $\widehat{\Delta}_\star^{\mathrm{alg}} \in \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}\left(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\hat{\otimes}2}\right)$ for the action (2.46). Namely,

$$\mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k}) := \left\{ \begin{array}{l} (\lambda,\Psi) \in \mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \mid \\ \left({}^\Gamma\mathrm{aut}_\Psi^Y \circ (\lambda\bullet -)\right)^{\otimes 2} \circ \widehat{\Delta}_\star^{\mathrm{alg}} = \widehat{\Delta}_\star^{\mathrm{alg}} \circ {}^\Gamma\mathrm{aut}_\Psi^Y \circ (\lambda\bullet -) \end{array} \right\} \tag{2.47}$$

Proposition 2.3.20. $\mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k}) = \mathbf{k}^\times \ltimes \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ *(equality of subgroups of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$).*

Proof. We use Lemma 1.3.13 with $H = \mathbf{k}^\times$, $R = \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ and $K = \mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$. Since $\widehat{\Delta}_\star^{\mathrm{alg}}$ is compatible with degree, then $\mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ contains $\mathbf{k}^\times$. Therefore, the condition of Lemma 1.3.13 is met and

$$K \cap R = \mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k}) \cap \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) = \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k}).$$

The result then follows. □

Theorem 2.3.21. $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k}) = \mathbf{k}^\times \ltimes \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ *(as subgroups of* $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*).*

Proof. Thanks to Lemma 2.3.16, the map $\hat{\varpi} : \mathbf{k}\langle\langle Y\rangle\rangle \to \widehat{\mathcal{W}}_G^{\mathrm{DR}}$ is an isomorphism of $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$-modules. So, it induces a $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$-module isomorphism $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\otimes 2}) \to \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\otimes 2})$ which is given by

$$\Delta \mapsto \left(\hat{\varpi}^{\otimes 2}\right)^{-1} \circ \Delta \circ \hat{\varpi},$$

where the $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$-module structure on the $\mathbf{k}$-module $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\hat{\otimes} 2})$ (resp. $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\otimes 2})$) is defined in (2.32) (resp. (2.46)). Moreover, thanks to Corollary 2.1.17, the coproduct $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ is sent to the coproduct $\widehat{\Delta}_\star^{\mathrm{alg}}$ via this isomorphism. Therefore, we obtain the equality $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k}) = \mathsf{Stab}_\bullet(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ and thanks to Proposition 2.3.20 this implies the wanted equality of stabilizers. □

Corollary 2.3.22. $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) \subset \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ *(as subgroups of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$*).*

Proof. Follows from Theorem 2.3.5 thanks to Theorems 2.3.9 and 2.3.21. □

3

The Betti formalism of the double shuffle theory

Throughout this chapter, let G be a finite cyclic group of order $N \in \mathbb{N}^*$ and $\mathbf{k}$ be a commutative $\mathbb{Q}$-algebra. We construct a Betti version of the double shuffle formalism. The relevant algebras and modules are introduced in §3.1 : (i) an algebra $\widehat{\mathcal{V}}_N^{\mathrm{B}}$ defined as the inverse limit of an algebra $\mathcal{V}_N^{\mathrm{B}}$ endowed with a suitable filtration; (ii) a subalgebra algebra $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ of $\widehat{\mathcal{V}}_N^{\mathrm{B}}$ isomorphic to the topological algebra $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$; (iii) a $\mathbf{k}$-module $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ isomorphic to the $\mathbf{k}$-module $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$, where $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ has a $\widehat{\mathcal{V}}_N^{\mathrm{B}}$-module structure inducing a free rank one $\widehat{\mathcal{W}}_N^{\mathrm{B}}$-module structure on it. In §3.2, we construct comparison isomorphisms between the Betti side and the de Rham side using the actions of the group $\mathbf{k}^\times \ltimes \mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle)$ defined in §2.2 analogue to the ones given in [EF1, §3.3]. This leads us in §3.3 to prove the existence of compatible coproducts $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ and $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}$ which equip $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ with Hopf algebra and coalgebra structures respectively as stated in Theorems 3.3.17 and 3.3.19.

3.1 Algebras and modules

3.1.1 The topological algebra $\widehat{\mathcal{V}}_N^{\mathrm{B}}$

The kernel of a group morphism $F_2 \to \mu_N$

Let F_2 be the free group generated by two elements denoted X_0 and X_1. We consider the group morphism $F_2 \to \mu_N$ given by $X_0 \mapsto \zeta_N$ and $X_1 \mapsto 1$; where $\zeta_N := e^{\frac{i2\pi}{N}}$.

Lemma 3.1.1. *The group* $\ker(F_2 \to \mu_N)$ *is isomorphic to the free group of rank* $N+1$ *denoted* F_{N+1}.

In order to prove this, we use the following result:

Proposition 3.1.2 (Nielsen-Schreier Theorem, see [Ste], Theorem 3)**.** *Let F be a free group on a non-empty set X and let H be a subgroup of F. Let $\sigma : H\diagdown F \to F$ be a section of the canonical projection $F \to H\diagdown F$ such that $T := \sigma(H\diagdown F)$ is stable* *under left prefixation. Then H is freely generated by*

$$\left\{tx(\overline{tx})^{-1} \,|\, (t,x) \in T \times X \text{ and } tx(\overline{tx})^{-1} \neq 1\right\},$$

where for $g \in F$, $\bar{g}$ the image of g under the composition $F \to H\diagdown F \overset{\sigma}{\to} F$.

Proof of Lemma 3.1.1. We apply the Nielsen-Schreier Theorem for $X = \{X_0, X_1\}$, $F = F_2$, $H = \ker(F_2 \to \mu_N)$ and $\sigma : \ker(F_2 \to \mu_N)\diagdown F_2 \simeq \mu_N \to F_2$ where the first map is the isomorphism induced by the surjective morphism $F_2 \to \mu_N$ and the second map given by $e^{i\frac{2n\pi}{N}} \mapsto X_0^n$ for $n \in [\![0, N-1]\!]$. Therefore, we have $T = \{X_0^n, n \in [\![0, N-1]\!]\}$. The theorem then states that $\ker(F_2 \to \mu_N)$ is freely generated by:

- $X_0^n X_0(\overline{X_0^n X_0})^{-1} = X_0^{n+1}(\overline{X_0^{n+1}})^{-1} = \begin{cases} X_0^{n+1}(X_0^{n+1})^{-1} = 1 & \text{if } n \in [\![0, N-2]\!] \\ X_0^N 1^{-1} = X_0^N & \text{if } n = N-1 \end{cases}$
- $X_0^n X_1(\overline{X_0^n X_1})^{-1} = X_0^n X_1(X_0^n)^{-1} = X_0^n X_1 X_0^{-n}$

Finally, $\ker(F_2 \to \mu_N)$ is freely generated by the $N+1$ elements

$$\left\{X_0^N, (X_0^n X_1 X_0^{-n})_{n \in [\![0, N-1]\!]}\right\}.$$

Moreover, if we denote $\left(\widetilde{X}_0, \left(\widetilde{X}_{\zeta_N^n}\right)_{n \in [\![0,N-1]\!]}\right)$ the generators of the free group F_{N+1} of rank $N+1$, one checks that correspondence

$$\widetilde{X}_0 \mapsto X_0^N,\ \widetilde{X}_{\zeta_N^n} \mapsto X_0^n X_1 X_0^{-n} \text{ for } n \in [\![0, N-1]\!]$$

defines a free group isomorphism from F_{N+1} to $\ker(F_2 \to \mu_N)$. ☐

We then obtain the following short exact sequence

$$\{1\} \to F_{N+1} \to F_2 \to \mu_N \to \{1\} \tag{3.1}$$

F_{N+1}-sets and kF_{N+1}-modules

Let $\sigma : \mu_N \to F_2$ be the set-theoretic section of $F_2 \to \mu_N$ given by $e^{\frac{i2n\pi}{N}} \mapsto X_0^n$ for $n \in [\![0, N-1]\!]$. Thanks to the exact sequence (3.1) we obtain a bijection

$$\Sigma : \mu_N \times F_{N+1} \to F_2, \quad (\zeta, x) \mapsto \sigma(\zeta)x; \tag{3.2}$$

where F_{N+1} is seen as $\ker(F_2 \to \mu_N) \subset F_2$ thanks to Lemma 3.1.1.
The set $\mu_N \times F_{N+1}$ is equipped with a right F_{N+1}-set structure by

$$(\zeta, x) * y := (\zeta, xy), \text{ for } (\zeta, x) \in \mu_N \times F_{N+1} \text{ and } y \in F_{N+1}.$$

The group F_2 is also equipped with a right F_{N+1}-set structure given by

$$x * y := xy, \text{ for } x \in F_2 \text{ and } y \in F_{N+1};$$

where F_{N+1} is seen as $\ker(F_2 \to \mu_N) \subset F_2$ thanks to Lemma 3.1.1. One checks that (3.2) upgrades to a right F_{N+1}-set isomorphism.

Now, let us consider the tensor functor

$$\mathbf{k}(-) : \{\text{right } F_{N+1}\text{-sets}\} \longrightarrow \{\text{right } \mathbf{k}F_{N+1}\text{-modules}\}$$

taking X to $\mathbf{k}X$, the set of finitely supported maps $X \to \mathbf{k}$. Applying this functor to the isomorphism of right F_{N+1}-sets (3.2), one obtains the right $\mathbf{k}F_{N+1}$-module isomorphism

$$\mathbf{k}\Sigma : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}F_2,$$

where both the source and the target are equipped with the right $\mathbf{k}F_{N+1}$-module structure given by the right F_{N+1}-set structure on $\mu_N \times F_{N+1}$ and F_2 respectively.

The filtered algebra $\mathcal{V}_N^{\mathrm{B}}$

Let us denote $\mathcal{I} := \ker(\mathbf{k}F_2 \to \mathbf{k}\mu_N)$ where $\mathbf{k}F_2 \to \mathbf{k}\mu_N$ is the $\mathbf{k}$-algebra morphism induced from the group morphism $F_2 \to \mu_N$. Then $\mathcal{I}$ is a two-sided ideal of $\mathbf{k}F_2$.

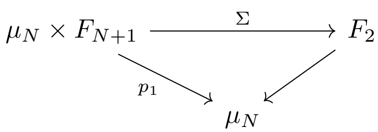
Lemma 3.1.3.

(i) *The* $\mathbf{k}$*-module isomorphism* $\mathbf{k}\Sigma : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}F_2$ *sets up a right* $\mathbf{k}F_{N+1}$*-module isomorphism of the ideal* $\mathcal{I}$ *with the ideal* $\mathbf{k}\mu_N \otimes \ker(\varepsilon)$ *of* $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1}$*, where* $\varepsilon : \mathbf{k}F_{N+1} \to \mathbf{k}$ *is the augmentation morphism.*

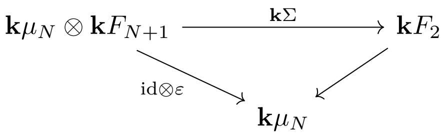
(ii) *The ideal* $\mathcal{I}$ *is linearly generated by* $\sigma(\zeta)(x-1)$ *where* $\zeta \in \mu_N$ *and* $x \in F_{N+1}$*.*

Proof.

(i) The following commutative diagram of F_{N+1}-set morphisms



induces a commutative diagram of $\mathbf{k}F_{N+1}$-module morphisms



One checks that the associated group algebra morphism of the first projection $p_1 : \mu_N \times F_{N+1} \to \mu_N$ is identified with $\mathrm{id} \otimes \varepsilon : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}\mu_N$ thanks to the identification $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \simeq \mathbf{k}(\mu_N \times F_{N+1})$. Therefore, the ideal $\mathcal{I}$ is mapped by the isomorphism $\mathbf{k}\Sigma$ to the ideal $\ker(\mathrm{id} \otimes \varepsilon) = \mathbf{k}\mu_N \otimes \ker(\varepsilon)$.

(ii) Since $\varepsilon : \mathbf{k}F_{N+1} \to \mathbf{k}$ is the augmentation morphism, its kernel is generated by elements $x - 1$ with $x \in F_{N+1}$. Therefore, taking the image of the generators by $\mathbf{k}\Sigma$, we obtain generators of the ideal $\mathcal{I}$ as announced.

$\square$

Proposition-Definition 3.1.4. *Let $\mathcal{V}_N^{\mathrm{B}}$ be the group algebra of F_2 over $\mathbf{k}$ endowed with the filtration*

$$\mathcal{F}^m \mathcal{V}_N^{\mathrm{B}} = \mathcal{I}^m \tag{3.3}$$

for $m \in \mathbb{N}$ where $\mathcal{I}^m$ is the m^{th}-power of the ideal $\mathcal{I}$ with the convention that $\mathcal{I}^0 = \mathcal{V}_N^{\mathrm{B}}$. The filtration $(\mathcal{F}^m \mathcal{V}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ is an algebra filtration.

Proof. Immediate. □

Lemma 3.1.5. *For $m \in \mathbb{N}$*

$$\mathcal{F}^m \mathcal{V}_N^{\mathrm{B}} \simeq \mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0^m, \tag{3.4}$$

where $(\mathbf{k}F_{N+1})_0$ is the augmentation ideal of the group algebra $\mathbf{k}F_{N+1}$.

Proof. If $m = 0$, we have $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \simeq \mathbf{k}(\mu_N \times F_{N+1}) \xrightarrow{\mathbf{k}\Sigma} \mathcal{V}_N^{\mathrm{B}} = \mathcal{F}^0\mathcal{V}_N^{\mathrm{B}}$.
Next, if $m = 1$, we have

$$\mathcal{F}^1\mathcal{V}_N^{\mathrm{B}} = \mathcal{I} \simeq \mathbf{k}\mu_N \otimes \ker(\varepsilon) = \mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0,$$

where the identification is given by Lemma 3.1.3 (i).
Now, let $m \geq 2$. Since $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1}$ is a right $\mathbf{k}F_{N+1}$-module, we have that

$$\mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0^m = (\mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0) \cdot (\mathbf{k}F_{N+1})_0^{m-1}. \tag{3.5}$$

The composition $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \simeq \mathbf{k}(\mu_N \times F_{N+1}) \xrightarrow{\mathbf{k}\Sigma} \mathcal{V}_N^{\mathrm{B}}$ is a right $\mathbf{k}F_{N+1}$-module isomorphism which, combined with the identification $\mathcal{I} \simeq \mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0$ and equality (3.5), gives us

$$\mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0^m \simeq \mathcal{I} \cdot (\mathbf{k}F_{N+1})_0^{m-1},$$

where $(\mathbf{k}F_{N+1})_0^{m-1}$ is seen as a subset of $\mathbf{k}F_{N+1} = \mathbf{k}\ker(F_2 \to \mu_N) \subset \mathbf{k}F_2$. It remains to show that $\mathcal{I} \cdot (\mathbf{k}F_{N+1})_0^{m-1} = \mathcal{I}^m$. First, since $(\mathbf{k}F_{N+1})_0 \subset \mathcal{I}$, we have $\mathcal{I} \cdot (\mathbf{k}F_{N+1})_0^{m-1} \subset \mathcal{I}^m$. Conversely, thanks to Lemma 3.1.3 (ii), $\mathcal{I}^m$ is linearly generated by elements

$$\Pi((\zeta_1, x_1), \dots, (\zeta_m, x_m)) := \sigma(\zeta_1)(x_1 - 1) \cdots \sigma(\zeta_m)(x_m - 1)$$

with $(\zeta_1, x_1), \dots, (\zeta_m, x_m) \in \mu_N \times F_{N+1}$. Moreover, we have that

$$\Pi((\zeta_1, x_1), \dots, (\zeta_m, x_m)) = \sigma(\zeta_1) \cdots \sigma(\zeta_m) \left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}\cdots\sigma(\zeta_2)^{-1}}(x_1) - 1\right)$$
$$\left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}\cdots\sigma(\zeta_3)^{-1}}(x_2) - 1\right) \cdots \left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}}(x_{m-1}) - 1\right) (x_m - 1).$$

Next, since F_{N+1} is a normal subgroup of F_2, we have that

$$\left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}\cdots\sigma(\zeta_3)^{-1}}(x_2) - 1\right) \cdots \left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}}(x_{m-1}) - 1\right) (x_m - 1) \in (\mathbf{k}F_{N+1})_0^{m-1}.$$

In addition, thanks to Lemma 3.1.3 (ii), we have

$$\sigma(\zeta_1) \cdots \sigma(\zeta_m) \left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}\cdots\sigma(\zeta_2)^{-1}}(x_1) - 1\right) \in \mathbf{k}F_2 \cdot (\mathbf{k}F_{N+1})_0.$$

Since $(\mathbf{k}F_{N+1})_0 \subset \mathcal{I}$, it follows that $\mathbf{k}F_2 \cdot (\mathbf{k}F_{N+1})_0 \subset \mathbf{k}F_2 \cdot \mathcal{I}$ and since $\mathcal{I}$ is a two-sided ideal of $\mathbf{k}F_2$, we have $\mathbf{k}F_2 \cdot \mathcal{I} = \mathcal{I}$. Therefore,

$$\sigma(\zeta_1) \cdots \sigma(\zeta_m) \left(\mathrm{Ad}_{\sigma(\zeta_m)^{-1}\cdots\sigma(\zeta_2)^{-1}}(x_1) - 1\right) \in \mathcal{I}.$$

Thus proving the wanted inclusion. □

The elements $X_0^N - 1$ and $X_1 - 1$ belong to $(\mathbf{k}F_{N+1})_0$ therefore to $\mathcal{F}^1\mathcal{V}_N^{\mathrm{B}}$. The element X_0 belongs to $\mathcal{F}^0\mathcal{V}_N^{\mathrm{B}} = \mathcal{V}_N^{\mathrm{B}}$. For $m \in \mathbb{N}$ and $v \in \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}$ we denote $[v]$ the image in $\mathrm{gr}_m(\mathcal{V}_N^{\mathrm{B}})$ of the element v.

Theorem 3.1.6. *For any group embedding $\iota : G \to \mathbb{C}^\times$, there exists a unique graded algebra isomorphism $\rho_\iota^{\mathcal{V}} : \mathcal{V}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$ given by*

$$\begin{aligned} g_\iota &\mapsto [X_0] \in \mathrm{gr}_0(\mathcal{V}_N^{\mathrm{B}}) \\ e_0 &\mapsto [X_0^N - 1] \in \mathrm{gr}_1(\mathcal{V}_N^{\mathrm{B}}) \\ e_1 &\mapsto [X_1 - 1] \in \mathrm{gr}_1(\mathcal{V}_N^{\mathrm{B}}) \end{aligned}$$

where $g_\iota = \iota^{-1}(e^{\frac{i2\pi}{N}})$.

Proof. First, let us show that $\rho_\iota^{\mathcal{V}}$ is a graded algebra morphism. One shows that when the group G is cyclic generated by g_ι, the algebra $\mathcal{V}_G^{\mathrm{DR}}$ is presented with generators e_0, e_1 and g_ι of degrees 1, 1 and 0 respectively; that satisfy the relations

$$g_\iota^N = 1 \text{ and } g_\iota e_0 = e_0 g_\iota.$$

Therefore, it remains to show that their images in $\mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$ have the same degrees and satisfy the same relations. The fact that g_ι resp. e_0, e_1 are mapped to elements of $\mathrm{gr}_0(\mathcal{V}_N^{\mathrm{B}})$ resp. $\mathrm{gr}_1(\mathcal{V}_N^{\mathrm{B}})$ implies the statement on degrees. Since $\rho_\iota^{\mathcal{V}}(g_\iota) = [X_0]$, then $\rho_\iota^{\mathcal{V}}(g_\iota)^N = [X_0^N] \in \mathrm{gr}_0(\mathcal{V}_N^{\mathrm{B}})$ which is the image of X_0^N by the canonical projection $\mathcal{F}^0\mathcal{V}_N^{\mathrm{B}} \to \mathrm{gr}_0(\mathcal{V}_N^{\mathrm{B}})$. In addition, $X_0^N = 1+(X_0^N-1)$ with $X_0^N-1 \in \mathcal{F}^1\mathcal{V}_N^{\mathrm{B}}$. This implies that $[X_0^N] = 1$ in $\mathrm{gr}_0(\mathcal{V}_N^{\mathrm{B}})$. For the second relation, we have $\rho_\iota^{\mathcal{V}}(g_\iota) = [X_0] \in \mathrm{gr}_0(\mathcal{V}_N^{\mathrm{B}})$ and $\rho_\iota^{\mathcal{V}}(e_0) = [X_0^N - 1] \in \mathrm{gr}_1(\mathcal{V}_N^{\mathrm{B}})$. Then

$$\rho_\iota^{\mathcal{V}}(g_\iota)\rho_\iota^{\mathcal{V}}(e_0) = [X_0(X_0^N - 1)] \in \mathrm{gr}_1(\mathcal{V}_N^{\mathrm{B}})$$

and

$$\rho_\iota^{\mathcal{V}}(e_0)\rho_\iota^{\mathcal{V}}(g_\iota) = [(X_0^N - 1)X_0] \in \mathrm{gr}_1(\mathcal{V}_N^{\mathrm{B}})$$

Therefore

$$\rho_\iota^{\mathcal{V}}(g_\iota)\rho_\iota^{\mathcal{V}}(e_0) = \rho_\iota^{\mathcal{V}}(e_0)\rho_\iota^{\mathcal{V}}(g_\iota).$$

Second, let us show that $\rho_\iota^{\mathcal{V}} : \mathcal{V}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$ is a $\mathbf{k}$-algebra isomorphism. Since, it is a $\mathbf{k}$-algebra morphism, it is enough to show that it is a $\mathbf{k}$-module isomorphism. This is done by using the following ingredients:

- The $\mathbf{k}$-module isomorphism

  $$\mathrm{gr}(\mathbf{k}\Sigma) : \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \to \mathbf{k}\mu_N \otimes \mathrm{gr}(\mathbf{k}F_{N+1})$$

  obtained from Lemma 3.1.5.

- The $\mathbf{k}$-algebra isomorphism

  $$\beta : \mathbf{k}\langle X\rangle \rtimes G \to \mathcal{V}_G^{\mathrm{DR}}.$$

  given in Proposition 2.1.7.(ii).

- The right $\mathrm{gr}(\mathbf{k}F_{N+1})$-module structure over the $\mathbf{k}$-module $\mathbf{k}\mu_N\otimes\mathrm{gr}(\mathbf{k}F_{N+1})$ where the action of each element of $\mathrm{gr}(\mathbf{k}F_{N+1})$ is the tensor product of the identity of $\mathbf{k}\mu_N$ with right multiplication by that element of $\mathrm{gr}(\mathbf{k}F_{N+1})$. One checks that $\mathbf{k}\mu_N\otimes\mathrm{gr}(\mathbf{k}F_{N+1})$ is a free right $\mathrm{gr}(\mathbf{k}F_{N+1})$-module with basis $(\zeta_N^l\otimes 1)_{l\in[\![0,N-1]\!]}$.
- The right $\mathbf{k}\langle X\rangle$-module structure on the $\mathbf{k}$-algebra $\mathbf{k}\langle X\rangle\rtimes G$ given by the right multiplication thanks to the fact that $\mathbf{k}\langle X\rangle$ can be seen as a subalgebra of $\mathbf{k}\langle X\rangle\rtimes G$. One checks that $\mathbf{k}\langle X\rangle\rtimes G$ is a free right $\mathbf{k}\langle X\rangle$-module with basis $(1\otimes g_\iota^l)_{l\in[\![0,N-1]\!]}$.
- The $\mathbf{k}$-algebra isomorphism $\mathbf{k}\langle X\rangle\mapsto\mathrm{gr}(\mathbf{k}F_{N+1})$ given by

  $$x_0\mapsto[\widetilde{X}_0-1]\text{ and }x_{g_\iota^n}\mapsto[1-\widetilde{X}_{\zeta_N^n}]\text{ for }n\in[\![0,N-1]\!].$$

  Indeed, thanks to [Qui, Example A2.11], there is a filtered $\mathbb{Q}$-algebra isomorphism $\widehat{\mathbb{Q}F_{N+1}}\to\mathbb{Q}\langle\langle X\rangle\rangle$ such that

  $$\widetilde{X}_0\mapsto\exp(x_0)\text{ and }\widetilde{X}_{\zeta_N^n}\mapsto\exp(x_{g_\iota^n})\text{ for }n\in[\![0,N]\!].$$

  The associated graded algebra morphism is therefore a graded algebra isomorphism $\mathrm{gr}(\mathbb{Q}F_{N+1})\to\mathbb{Q}\langle X\rangle$ such that

  $$[\widetilde{X}_0-1]\mapsto[\exp(x_0)-1]=x_0\text{ and }[\widetilde{X}_{\zeta_N^n}-1]\mapsto[\exp(x_{g_\iota^n})-1]=x_{g_\iota^n}\text{ for }n\in[\![0,N]\!].$$

  Doing the tensor product by $\mathbf{k}$ we obtain an isomorphism $\mathrm{gr}(\mathbf{k}F_{N+1})\to\mathbf{k}\langle X\rangle$ whose inverse is the wanted $\mathbf{k}$-algebra isomorphism.

The composition $\mathrm{gr}(\mathbf{k}\Sigma)^{-1}\circ\rho_\iota^{\mathcal{V}}\circ\beta:\mathbf{k}\langle X\rangle\rtimes G\to\mathbf{k}\mu_N\otimes\mathrm{gr}(\mathbf{k}F_{N+1})$ is a morphism of right modules over the $\mathbf{k}$-algebra isomorphism $\mathbf{k}\langle X\rangle\to\mathrm{gr}(\mathbf{k}F_{N+1})$. Moreover, the restriction to the basis of the source module is given by

$$1\otimes g_\iota^l\mapsto g_\iota^l\mapsto[X_0^l]\mapsto\zeta_N^l\otimes 1,$$

for any $l\in[\![0,N-1]\!]$, thus establishing a bijection with the basis of the target module. It follows that the composed morphism $\mathbf{k}\langle X\rangle\rtimes G\to\mathbf{k}\mu_N\otimes\mathrm{gr}(\mathbf{k}F_{N+1})$ is an isomorphism of right modules over the algebra isomorphism $\mathbf{k}\langle X\rangle\to\mathrm{gr}(\mathbf{k}F_{N+1})$ and, therefore, is an isomorphism of $\mathbf{k}$-modules.
Finally, since $\mathrm{gr}(\mathbf{k}\Sigma):\mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})\to\mathbf{k}\mu_N\otimes\mathrm{gr}(\mathbf{k}F_{N+1})$ and $\beta:\mathbf{k}\langle X\rangle\to\mathcal{V}_G^{\mathrm{DR}}$ are both $\mathbf{k}$-module isomorphisms, it follows that $\rho_\iota^{\mathcal{V}}:\mathcal{V}_G^{\mathrm{DR}}\to\mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$ is also a $\mathbf{k}$-module isomorphism. □

The topological algebra $\widehat{\mathcal{V}}_N^{\mathrm{B}}$

The decreasing filtration $(\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ given in Proposition-Definition 3.1.4 induces an algebra morphism $\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{V}_N^{\mathrm{B}}\to\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}$.

Definition 3.1.7. We denote

$$\widehat{\mathcal{V}}_N^{\mathrm{B}}:=\varprojlim\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}$$

the inverse limit of the system $(\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}},\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{V}_N^{\mathrm{B}}\to\mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}})$.

The algebra $\widehat{\mathcal{V}}^{\mathrm{B}}_N$ is equipped with the filtration $\mathcal{F}^m\widehat{\mathcal{V}}^{\mathrm{B}}_N := \varprojlim \mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{\max(m,l)}\mathcal{V}^{\mathrm{B}}_N$. When equipped with the topology defined by this filtration, $\widehat{\mathcal{V}}^{\mathrm{B}}_N$ is a complete separated topological algebra.
Recall that $\mathbf{k}F_{N+1}$ is a group algebra equipped with a filtration given by the powers of its augmentation ideal. Let us denote $\widehat{\mathbf{k}F_{N+1}}$ the completion of this group algebra with respect to this filtration.

Lemma 3.1.8.

(i) The $\mathbf{k}$*-algebra morphism* $\mathbf{k}\Sigma \circ (1 \otimes -) : \mathbf{k}F_{N+1} \to \mathcal{V}^{\mathrm{B}}_N$ *gives rise to a topological* $\mathbf{k}$*-algebra morphism* $\widehat{\mathbf{k}F_{N+1}} \to \widehat{\mathcal{V}}^{\mathrm{B}}_N$.

(ii) The $\mathbf{k}$*-module morphism* $\mathbf{k}\Sigma : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathcal{V}^{\mathrm{B}}_N$ *gives rise to an isomorphism of topological right* $\widehat{\mathbf{k}F_{N+1}}$*-module* $\widehat{\mathbf{k}\Sigma} : \mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}} \to \widehat{\mathcal{V}}^{\mathrm{B}}_N$.

(iii) The $\mathbf{k}$*-algebra morphism* $\widehat{\mathbf{k}F_{N+1}} \to \widehat{\mathcal{V}}^{\mathrm{B}}_N$ *is injective.*

Proof.

(i) This follows from the fact that the $\mathbf{k}$-algebra morphism $\mathbf{k}\Sigma\circ(1\otimes-) : \mathbf{k}F_{N+1} \to \mathcal{V}^{\mathrm{B}}_N$ is compatible with filtrations, which follows from Lemma 3.1.5.

(ii) This follows from the fact that $\mathbf{k}\Sigma : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathcal{V}^{\mathrm{B}}_N$ is an isomorphism of filtered right module over $\mathbf{k}F_{N+1}$.

(iii) By (i), the topological $\mathbf{k}$-algebra morphism $\widehat{\mathbf{k}F_{N+1}} \to \widehat{\mathcal{V}}^{\mathrm{B}}_N$ is equal to the composition $\widehat{\mathbf{k}\Sigma} \circ (1 \otimes -) : \widehat{\mathbf{k}F_{N+1}} \to \widehat{\mathcal{V}}^{\mathrm{B}}_N$. The map $1 \otimes - : \widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}}$ is trivially injective and $\widehat{\mathbf{k}\Sigma} : \mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}} \to \widehat{\mathcal{V}}^{\mathrm{B}}_N$ is injective by (ii). This implies that their composition is injective, implying the claim.

□

Proposition-Definition 3.1.9. *Let* $\iota : G \to \mathbb{C}^\times$ *be a group embedding. There is a unique topological algebra isomorphism* $\mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}^{\mathrm{B}}_N \to \widehat{\mathcal{V}}^{\mathrm{DR}}_G$ *given by*

$$X_0 \mapsto \exp\left(\frac{1}{N}e_0\right) g_\iota; \quad \textit{and} \quad X_1 \mapsto \exp(e_1),$$

where $g_\iota = \iota^{-1}(e^{\frac{i2\pi}{N}})$*. Moreover, the associated graded* $\mathbf{k}$*-algebra morphism* $\mathrm{gr}\left(\mathrm{iso}^{\mathcal{V},\iota}\right) : \mathrm{gr}(\widehat{\mathcal{V}}^{\mathrm{B}}_N) \to \mathrm{gr}(\widehat{\mathcal{V}}^{\mathrm{DR}}_G) \simeq \mathcal{V}^{\mathrm{DR}}_G$ *is equal to* $\left(\rho^{\mathcal{V}}_\iota\right)^{-1}$.

Proof. Recall that the set $\mathrm{Mor}_{\mathbf{k}-\mathrm{alg}}(\mathbf{k}F_2, \widehat{\mathcal{V}}^{\mathrm{DR}}_G)$ is identified with $\mathrm{Mor}_{\mathrm{grp}}\left(F_2, \left(\widehat{\mathcal{V}}^{\mathrm{DR}}_G\right)^\times\right)$. As a consequence, there is an algebra morphism $\mathcal{V}^{\mathrm{B}}_N \to \widehat{\mathcal{V}}^{\mathrm{DR}}_G$ given by

$$X_0 \mapsto \exp\left(\frac{1}{N}e_0\right) g_\iota \quad \text{and} \quad X_1 \mapsto \exp(e_1)$$

since the images of X_0 and X_1 are invertible. Composing the $\mathbf{k}$-algebra morphism $\mathcal{V}^{\mathrm{B}}_N \to \widehat{\mathcal{V}}^{\mathrm{DR}}_G$ with the $\mathbf{k}$-module isomorphism $\mathbf{k}\Sigma : \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathcal{V}^{\mathrm{B}}_N$ and the inverse of the $\mathbf{k}$-algebra isomorphism $\widehat{\beta} : \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ respectively from the left and from the right, we obtain a $\mathbf{k}$-module morphism

$$\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}\langle\langle X\rangle\rangle \rtimes G. \tag{3.6}$$

One checks that morphism (3.6) is a right module morphism over the $\mathbf{k}$-algebra morphism $\mathbf{k}F_{N+1} \to \mathbf{k}\langle\langle X\rangle\rangle$ given by

$$\widetilde{X}_0 \mapsto \exp(x_0) \text{ and } \widetilde{X}_{\zeta_N^n} \mapsto \exp\left(\frac{n}{N}x_0\right)\exp(-x_{g_\iota^n})\exp\left(-\frac{n}{N}x_0\right), \text{ for } n \in [\![0, N-1]\!].$$

In addition, $(\zeta_N^l \otimes 1)_{l\in[\![0,N-1]\!]}$ and $\left(\exp\left(\frac{l}{N}x_0\right) \otimes g_\iota^l\right)_{l\in[\![0,N-1]\!]}$ are bases of $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1}$ and $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ respectively and the morphism (3.6) induces the following bijection between the bases

$$\zeta_N^l \otimes 1 \mapsto \exp\left(\frac{l}{N}x_0\right) \otimes g_\iota^l, \text{ for } l \in [\![0, N-1]\!]. \tag{3.7}$$

Furthermore, there is a topological $\mathbf{k}$-algebra isomorphism $\widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\langle\langle X\rangle\rangle$ such that the following diagram

$$\begin{array}{ccc} \mathbf{k}F_{N+1} & \longrightarrow & \mathbf{k}\langle\langle X\rangle\rangle \\ & \searrow \quad \nearrow & \\ & \widehat{\mathbf{k}F_{N+1}} & \end{array}$$

commutes, where $\mathbf{k}F_{N+1} \hookrightarrow \widehat{\mathbf{k}F_{N+1}}$ is the canonical $\mathbf{k}$-algebra morphism.
Indeed, such an isomorphism is obtained by composing the topological $\mathbf{k}$-algebra isomorphism $\widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\langle\langle X\rangle\rangle$ obtained from [Qui, Example A2.12] and the topological $\mathbf{k}$-algebra automorphism of $\widehat{\mathbf{k}F_{N+1}}$ given by

$$\widetilde{X}_0 \mapsto \widetilde{X}_0 \text{ and } \widetilde{X}_{\zeta_N^n} \mapsto \mathrm{Ad}_{\exp\left(\frac{n}{N}\log(\widetilde{X}_0)\right)}(\widetilde{X}_{\zeta_N^n}^{-1}) \text{ for } n \in [\![0, N-1]\!].$$

On the other hand, one checks that $\mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}}$ is a free right $\widehat{\mathbf{k}F_{N+1}}$-module with basis $(\zeta_N^l \otimes 1)_{l\in[\![0,N-1]\!]}$ and recall that $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ is a free right $\mathbf{k}\langle\langle X\rangle\rangle$-module with basis $\left(\exp\left(\frac{l}{N}x_0\right) \otimes g_\iota^l\right)_{l\in[\![0,N-1]\!]}$. Therefore, there is a unique module isomorphism $\mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ over the $\mathbf{k}$-algebra isomorphism $\widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\langle\langle X\rangle\rangle$ which extends bijection (3.7) between bases. Therefore, the restriction to the bases of the following diagram

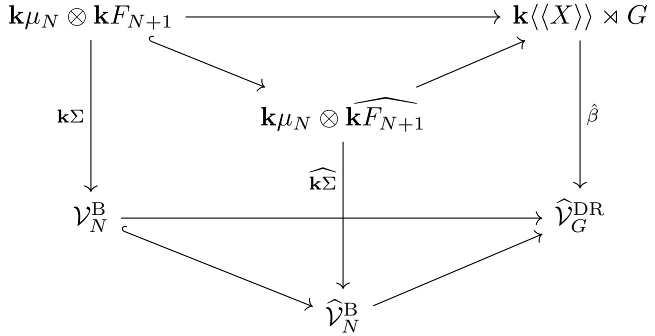
$$\begin{array}{ccc} \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} & \longrightarrow & \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \\ & \searrow \quad \nearrow & \\ & \mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}} & \end{array} \tag{3.8}$$

commutes, where $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}}$ is the tensor product of the identity of $\mathbf{k}\mu_N$ with $\mathbf{k}F_{N+1} \hookrightarrow \widehat{\mathbf{k}F_{N+1}}$. This implies that the diagram commutes.

Next, by composing the $\mathbf{k}$-module isomorphism $\mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}} \to \mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ from the left and from the right with the isomorphisms $\mathbf{k}\Sigma^{-1} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \mathbf{k}\mu_N \otimes \widehat{\mathbf{k}F_{N+1}}$ and $\hat{\beta} : \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_N^{\mathrm{B}}$ respectively, we obtain a $\mathbf{k}$-module isomorphism

$$\mathcal{V}_N^{\mathrm{B}} \longrightarrow \mathcal{V}_G^{\mathrm{DR}}. \tag{3.9}$$

Let us prove that this $\mathbf{k}$-module isomorphism is a $\mathbf{k}$-algebra isomorphism. It is, therefore, enough to show that it is a $\mathbf{k}$-algebra morphism. Let us consider the following prism



The left, right and middle squares commute by definition of $\widehat{\mathbf{k}\Sigma}$, $\widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ and $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ respectively and the upper triangle is Diagram (3.8), so is commutative. Additionally, the arrows going from the upper triangle to the lower triangle are isomorphisms. Therefore, the lower triangle is commutative. The restriction of the topological $\mathbf{k}$-module isomorphism $\widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ to $\mathcal{V}_N^{\mathrm{B}}$ is an algebra morphism, which by the density of $\mathcal{V}_N^{\mathrm{B}}$ in $\widehat{\mathcal{V}}_N^{\mathrm{B}}$ implies that $\widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ to $\mathcal{V}_N^{\mathrm{B}}$ is a topological algebra morphism and therefore a topological algebra isomorphism. Finally, the commutativity of the triangle also implies that the $\mathbf{k}$-algebra isomorphism $\widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ is as announced.

To conclude, let us compute $\mathrm{gr}\left(\mathrm{iso}^{\mathcal{V},\iota}\right)$. The topological $\mathbf{k}$-algebra isomorphism $\mathrm{iso}^{\mathcal{V},\iota}$ induces a graded $\mathbf{k}$-algebra morphism $\mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \to \mathrm{gr}(\widehat{\mathcal{V}}_G^{\mathrm{DR}}) \simeq \mathcal{V}_G^{\mathrm{DR}}$. Its composition with the graded algebra isomorphism $\rho_\iota^{\mathcal{V}} : \mathcal{V}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$ (see Theorem 3.1.6) is the algebra automorphism of $\mathcal{V}_G^{\mathrm{DR}}$ given by

$$\begin{aligned} g_\iota &\mapsto [X_0] \mapsto \left[\exp\left(\frac{1}{N}e_0\right) g_\iota\right] \mapsto g_\iota \\ e_0 &\mapsto [X_0^N - 1] \mapsto [\exp(e_0) - 1] \mapsto e_0 \\ e_1 &\mapsto [X_1 - 1] \mapsto [\exp(e_1) - 1] \mapsto e_1 \end{aligned}$$

which is the identity of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$. Since $\rho_\iota^{\mathcal{V}} : \mathcal{V}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$ is an isomorphism, this implies that $\mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \to \mathcal{V}_G^{\mathrm{DR}}$ is equal to $\left(\rho_\iota^{\mathcal{V}}\right)^{-1}$ to which $\mathrm{gr}\left(\mathrm{iso}^{\mathcal{V},\iota}\right)$ identifies to. □

3.1.2 The topological algebra $\widehat{\mathcal{W}}_N^{\mathrm{B}}$

The filtered algebra $\mathcal{W}_N^{\mathrm{B}}$

Proposition-Definition 3.1.10. *Let us denote*

$$\mathcal{W}_N^{\mathrm{B}} := \mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1). \tag{3.10}$$

It is a subalgebra of $\mathcal{V}_N^{\mathrm{B}}$ endowed with the filtration

$$\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} := \mathcal{W}_N^{\mathrm{B}} \cap \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \tag{3.11}$$

for $m \in \mathbb{N}$. The filtration $(\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ is an algebra filtration.

Proof. Immediate. □

Lemma 3.1.11. *For $m \in \mathbb{N}^*$, we have*

(i) $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cap \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$. *(ii)* $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$.

Proof.

(i) Let $m \in \mathbb{N}^*$. We have

$$\begin{aligned}\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} &= \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cap \big(\mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)\big)\\ &= \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cap \big(\ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}) \cap (\mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1))\big)\\ &= \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cap \mathcal{V}_N^{\mathrm{B}}(X_1 - 1),\end{aligned}$$

where the second equality follows from the inclusion $\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \subset \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k})$ since

$$\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} = \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N)^m \subset \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N) \subset \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}), \tag{3.12}$$

where the last inclusion of (3.12) is a consequence of the fact that $\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}$ is the composition $\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N \to \mathbf{k}$ (the maps with target $\mathbf{k}$ being the augmentation morphisms). The third equality follows from

$$\ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}) \cap (\mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)) = \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$$

which, in turn, follows from the fact that $\ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}) \cap (\mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1))$ is the kernel of the composed map $\mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \subset \mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}$ which is the identity on $\mathbf{k}$ and takes $\mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$ to 0. Its kernel is therefore $\mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$.

(ii) Recall from Lemma 3.1.5 that, for $m \in \mathbb{N}^*$, the $\mathbf{k}$-module isomorphism $\mathbf{k}\Sigma$: $\mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1} \to \mathcal{V}_N^{\mathrm{B}}$ induces an isomorphism

$$\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \simeq \mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0^m, \tag{3.13}$$

where $(\mathbf{k}F_{N+1})_0$ is the augmentation ideal of the group algebra $\mathbf{k}F_{N+1}$. The isomorphism $\mathbf{k}\Sigma$ also induces an isomorphism

$$\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \simeq \mathbf{k}\mu_N \otimes \mathbf{k}F_{N+1}(\widetilde{X}_{\zeta_N^0} - 1).$$

Thanks to Lemma 3.13 (i), this induces the isomorphism

$$\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \simeq \mathbf{k}\mu_N \otimes \Big((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_{\zeta_N^0} - 1)\Big).$$

Next, thanks to [Wei, Proposition 6.2.6], we have a $\mathbf{k}F_{N+1}$-module isomorphism $(\mathbf{k}F_{N+1})^{\oplus(N+1)} \to (\mathbf{k}F_{N+1})_0$. This isomorphism induces the following isomorphisms

$$\mathbf{k}F_{N+1} \oplus \{0\}^{\oplus N} \simeq \mathbf{k}F_{N+1}(X_{\zeta_N^0} - 1) \text{ and } (\mathbf{k}F_{N+1})_0^{m-1})^{\oplus(N+1)} \simeq (\mathbf{k}F_{N+1})_0^m,$$

where for the latter one we use the fact that $(\mathbf{k}F_{N+1})_0^m = (\mathbf{k}F_{N+1})_0^{m-1}(\mathbf{k}F_{N+1})_0$ and the fact that $(\mathbf{k}F_{N+1})_0^{m-1}$ is an ideal of $\mathbf{k}F_{N+1}$.
On the other hand, using the inclusion $(\mathbf{k}F_{N+1})_0^{m-1} \subset \mathbf{k}F_{N+1}$ and the isomorphism $\mathbf{k}F_{N+1} \oplus \{0\}^{\oplus N} \simeq \mathbf{k}F_{N+1}(X_{\zeta_N^0} - 1)$, one obtains

$$(\mathbf{k}F_{N+1})_0^{m-1})^{\oplus(N+1)} \cap \left(\mathbf{k}F_{N+1} \oplus \{0\}^{\oplus N}\right) = (\mathbf{k}F_{N+1})_0^{m-1} \oplus \{0\}^{\oplus N}.$$

Finally, one checks that the isomorphism $(\mathbf{k}F_{N+1})^{\oplus(N+1)} \to (\mathbf{k}F_{N+1})_0$ induces an isomorphism

$$(\mathbf{k}F_{N+1})_0^{m-1} \oplus \{0\}^{\oplus N} \simeq (\mathbf{k}F_{N+1})_0^{m-1}(\widetilde{X}_{\zeta_N^0} - 1)$$

and using (3.13) for m replaced by $m-1$, together with the fact that $\mathbf{k}\Sigma$ intertwines right multiplication by $X_1 - 1$ on $\mathcal{V}_N^{\mathrm{B}}$ with the tensor product of the identity on $\mathbf{k}\mu_N$ with right multiplication by $\widetilde{X}_{\zeta_N^0} - 1$ on $\mathbf{k}F_{N+1}$ implies

$$\mathbf{k}\mu_N \otimes (\mathbf{k}F_{N+1})_0^{m-1}(\widetilde{X}_{\zeta_N^0} - 1) \simeq \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1),$$

thus proving the wanted result.

□

Proposition 3.1.12.

(i) The morphism of graded algebras $\mathrm{gr}(\mathcal{W}_N^{\mathrm{B}}) \to \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$ *induced by the compatibility of the inclusion* $\mathcal{W}_N^{\mathrm{B}} \subset \mathcal{V}_N^{\mathrm{B}}$ *with the filtrations is injective.*

(ii) For any group embedding $\iota : G \to \mathbb{C}^\times$, *there exists a graded algebra isomorphism* $\rho_\iota^{\mathcal{W}} : \mathcal{W}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{W}_N^{\mathrm{B}})$ *such that the following diagram*

$$\begin{array}{ccc} \mathcal{W}_G^{\mathrm{DR}} & \xrightarrow{\rho_\iota^{\mathcal{W}}} & \mathrm{gr}(\mathcal{W}_N^{\mathrm{B}}) \\ \downarrow & & \downarrow \\ \mathcal{V}_G^{\mathrm{DR}} & \xrightarrow[\rho_\iota^{\mathcal{V}}]{} & \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \end{array}$$

commutes.

Proof.

(i) The injectivity of the map $\mathrm{gr}(\mathcal{W}_{\mathrm{B}}^N) \to \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$ follows from the fact that the filtration of $\mathcal{W}_N^{\mathrm{B}}$ is induced by that of $\mathcal{V}_N^{\mathrm{B}}$.

(ii) First, from Lemma 3.1.11 (ii) we have that $\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}} = \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)$. One checks that it is a $\mathbf{k}$-submodule of $\mathcal{V}_N^{\mathrm{B}}$ endowed with the filtration

$$\mathcal{F}^m\left(\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}\right) = \mathcal{F}^{m+1}\mathcal{W}_N^{\mathrm{B}}\ , \text{ for } m \in \mathbb{N}.$$

Next, one checks that the sequence of $\mathbf{k}$-module morphisms

$$\mathcal{V}_N^{\mathrm{B}} \xrightarrow{-\cdot(X_1-1)} \mathcal{F}^1\mathcal{W}_N^{\mathrm{B}} \subset \mathcal{V}_N^{\mathrm{B}}$$

is compatible with the filtrations. More precisely, for $m \in \mathbb{N}$, it restricts as follows

$$\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \to \mathcal{F}^{m+1}\mathcal{W}_N^{\mathrm{B}} \subset \mathcal{F}^{m+1}\mathcal{V}_N^{\mathrm{B}}.$$

The associated graded sequence of maps is given by

$$\mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \xrightarrow{\mathrm{gr}(-\cdot(X_1-1))} \mathrm{gr}(\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}) \hookrightarrow \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}). \tag{3.14}$$

It is such that the following diagram

$$\begin{array}{ccccc} \mathcal{V}_G^{\mathrm{DR}} & & \xrightarrow{\quad -\cdot e_1 \quad} & & \mathcal{V}_G^{\mathrm{DR}} \\ \rho_\iota^{\mathcal{V}}\downarrow & & & & \downarrow\rho_\iota^{\mathcal{V}} \\ \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) & \xrightarrow{\mathrm{gr}(-\cdot(X_1-1))} & \mathrm{gr}(\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}) & \hookrightarrow & \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \end{array}$$

commutes. It then follows from Lemma 3.1.11 (ii) that the map $\mathrm{gr}(-\cdot(X_1-1)) : \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \to \mathrm{gr}(\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}})$ is surjective and then

$$\mathrm{gr}(\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}) \simeq \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})[X_1-1] \simeq \mathcal{V}_G^{\mathrm{DR}}e_1.$$

where the first identification is through the composition (3.14) and the second one is through the graded isomorphism $\rho_\iota^{\mathcal{V}}$. Therefore there exists a graded non-unital algebra isomorphism $\mathrm{gr}(\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}) \to \mathcal{V}_G^{\mathrm{DR}}e_1$ such that the following diagram

$$\begin{array}{ccc} \mathcal{V}_G^{\mathrm{DR}}e_1 & \hookrightarrow & \mathcal{V}_G^{\mathrm{DR}} \\ \downarrow & & \downarrow\rho_\iota^{\mathcal{V}} \\ \mathrm{gr}(\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}) & \hookrightarrow & \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \end{array}$$

commutes. Finally, since $\mathcal{W}_N^{\mathrm{B}} = \mathbf{k} \oplus \mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}$, we have that

$$\mathrm{gr}(\mathcal{W}_N^{\mathrm{B}}) = \mathbf{k} \oplus \mathrm{gr}(\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}).$$

As a consequence, the wanted graded algebra isomorphism

$$\mathrm{gr}(\mathcal{W}_N^{\mathrm{B}}) = \mathbf{k} \oplus \mathrm{gr}(\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}) \xrightarrow{\rho_\iota^{\mathcal{W}}} \mathbf{k} \oplus \mathcal{V}_G^{\mathrm{DR}}e_1 = \mathcal{W}_G^{\mathrm{DR}}$$

is obtained as the direct sum of the identity on $\mathbf{k}$ with the graded non-unital algebra isomorphism $\mathrm{gr}(\mathcal{F}^1\mathcal{W}_N^{\mathrm{B}}) \to \mathcal{V}_G^{\mathrm{DR}}e_1$.

□

The topological algebra $\widehat{\mathcal{W}}_N^{\mathrm{B}}$

The decreasing filtration $(\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ given in (3.11) induces an algebra morphism $\mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{W}_N^{\mathrm{B}} \to \mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}$.

Definition 3.1.13. We denote

$$\widehat{\mathcal{W}}_N^{\mathrm{B}} := \varprojlim \mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}$$

the inverse limit of the projective system $(\mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}, \mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{W}_N^{\mathrm{B}} \to \mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}})$.

The algebra $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ equipped with a filtration

$$\mathcal{F}^m\widehat{\mathcal{W}}_N^{\mathrm{B}} := \lim_{\longleftarrow} \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^{\max(m,l)}\mathcal{W}_N^{\mathrm{B}}$$

and endowed with the topology defined by this filtration is a complete separated topological algebra.

Lemma 3.1.14. *The* $\mathbf{k}$*-algebra inclusion* $\mathcal{W}_N^{\mathrm{B}} \subset \mathcal{V}_N^{\mathrm{B}}$ *gives rise to an injective morphism of topological* $\mathbf{k}$*-algebras* $\widehat{\mathcal{W}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_N^{\mathrm{B}}$.

Proof. This follows from the compatibility of the inclusion $\mathcal{W}_N^{\mathrm{B}} \subset \mathcal{V}_N^{\mathrm{B}}$ with filtrations and the fact that the filtration on $\mathcal{W}_N^{\mathrm{B}}$ is induced by that of $\mathcal{V}_N^{\mathrm{B}}$. ☐

Proposition 3.1.15. *The topological algebra* $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ *is isomorphic to the topological subalgebra* $\mathbf{k} \oplus \widehat{\mathcal{V}}_N^{\mathrm{B}}(X_1 - 1)$ *of* $\widehat{\mathcal{V}}_N^{\mathrm{B}}$.

Proof. This will be done following this program:

Step 1. Construction of the topological $\mathbf{k}$-module $\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}}$.
Let us define a $\mathbf{k}$-submodule $\mathcal{W}_{N,+}^{\mathrm{B}} := \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \subset \mathcal{W}_N^{\mathrm{B}}$. It is equipped with the filtration

$$\mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}} := \mathcal{W}_{N,+}^{\mathrm{B}} \cap \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \text{ , for } m \in \mathbb{N}$$

induced by the inclusion $\mathcal{W}_{N,+}^{\mathrm{B}} \subset \mathcal{W}_N^{\mathrm{B}}$. Denote as follows the associated inverse limit

$$\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} := \lim_{\longleftarrow} \mathcal{W}_{N,+}^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}}.$$

One checks that the $\mathbf{k}$-module inclusion $\mathcal{W}_{N,+}^{\mathrm{B}} \subset \mathcal{W}_N^{\mathrm{B}}$ is compatible with the filtrations, which induces a morphism of topological $\mathbf{k}$-modules $\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} \to \widehat{\mathcal{W}}_N^{\mathrm{B}}$. As the filtration of $\mathcal{W}_{N,+}^{\mathrm{B}}$ is induced by that of $\mathcal{W}_N^{\mathrm{B}}$, this morphism is injective. Thanks to Lemma 3.1.14, we then have a chain of injections

$$\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} \hookrightarrow \widehat{\mathcal{W}}_N^{\mathrm{B}} \hookrightarrow \widehat{\mathcal{V}}_N^{\mathrm{B}}. \tag{3.15}$$

On the other hand, for any $m \in \mathbb{N}^*$, we have

$$\mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}} = \mathcal{W}_{N,+}^{\mathrm{B}} \cap \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \cap \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) = \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1),$$

where the second equality comes from Lemma 3.1.11 (ii). Therefore, for any $m \in \mathbb{N}$,

$$\mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}} = \begin{cases} \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) & \text{if } m = 0 \\ \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) & \text{otherwise} \end{cases} \tag{3.16}$$

Moreover, let us notice that $\mathcal{W}_N^{\mathrm{B}} = \mathbf{k} \oplus \mathcal{W}_{N,+}^{\mathrm{B}}$. Using (3.16) we obtain

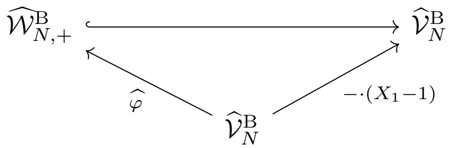
$$\mathcal{F}^0\mathcal{W}_N^{\mathrm{B}} = \mathbf{k} \oplus \mathcal{F}^0\mathcal{W}_{N,+}^{\mathrm{B}};$$
$$\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} = \mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}}, \text{ for } m \in \mathbb{N}^*.$$

These equalities induce the following topological $\mathbf{k}$-algebra isomorphism

$$\widehat{\mathcal{W}}_N^{\mathrm{B}} = \lim_{\longleftarrow} \mathcal{W}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \simeq \mathbf{k} \oplus \lim_{\longleftarrow} \mathcal{W}_{N,+}^{\mathrm{B}}/\mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}} = \mathbf{k} \oplus \widehat{\mathcal{W}}_{N,+}^{\mathrm{B}}. \tag{3.17}$$

where, on the right, the algebra structure is defined by the conditions that $1 \in \mathbf{k}$ is a unit and that the inclusion $\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} \subset \mathbf{k} \oplus \widehat{\mathcal{W}}_{N,+}^{\mathrm{B}}$ is a non-unital algebra morphism.

Step 2. The existence of a topological $\mathbf{k}$-module morphism $\widehat{\varphi} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{W}}_{N,+}^{\mathrm{B}}$ such that the triangle



(3.18)

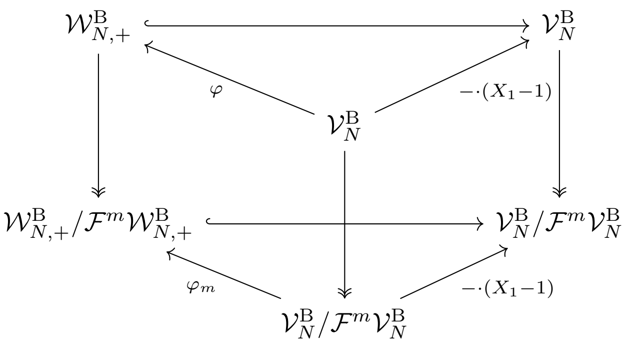
commutes. First, let us consider the $\mathbf{k}$-module morphism $\varphi : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_{N,+}^{\mathrm{B}}$ given by $v \mapsto v(X_1 - 1)$. For any $m \in \mathbb{N}^*$, one has

$$\varphi(\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}) = \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \subset \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) = \mathcal{F}^m\mathcal{W}_{N,+}^{\mathrm{B}},$$

where the first equality follows from the definition of φ, the inclusion follows from decreasing character of $(\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ and the last equality follows from (3.16). One also has

$$\varphi(\mathcal{F}^0\mathcal{V}_N^{\mathrm{B}}) = \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) = \mathcal{F}^0\mathcal{W}_{N,+}^{\mathrm{B}}.$$

This implies that the morphism $\varphi : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_{N,+}^{\mathrm{B}}$ is compatible with filtrations. This induces a $\mathbf{k}$-module morphism $\varphi_m : \mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_N^{\mathrm{B}+}/\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}+}$.
In the following prism



the upper triangle commutes by definition of $\varphi : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_{N,+}^{\mathrm{B}}$ and all the squares commute thanks to the compatibility of the maps $\varphi : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_{N,+}^{\mathrm{B}}$, $- \cdot (X_1 - 1) : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{V}_N^{\mathrm{B}}$ and $\mathcal{W}_{N,+}^{\mathrm{B}} \subset \mathcal{V}_N^{\mathrm{B}}$ with filtrations. Therefore, thanks to the surjectivity of the projection $\mathcal{V}_N^{\mathrm{B}} \to \mathcal{V}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}$, the lower triangle commutes. As a consequence, the morphism $\varphi : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{W}_{N,+}^{\mathrm{B}}$ induces a morphism of topological $\mathbf{k}$-modules $\widehat{\varphi} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{W}}_{N,+}^{\mathrm{B}}$ such that Diagram (3.18) commutes. Finally, the commutativity of the latter diagram implies

$$\begin{aligned}\widehat{\mathcal{V}}_N^{\mathrm{B}}(X_1 - 1) &= \mathrm{Im}\big(- \cdot(X_1 - 1)\big) \\ &= \mathrm{Im}\left(\widehat{\mathcal{V}}_N^{\mathrm{B}} \xrightarrow{\widehat{\varphi}} \widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} \hookrightarrow \widehat{\mathcal{V}}_N^{\mathrm{B}}\right) \subset \mathrm{Im}\left(\widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} \hookrightarrow \widehat{\mathcal{V}}_N^{\mathrm{B}}\right). \qquad (3.19)\end{aligned}$$

Step 3. The existence of a topological $\mathbf{k}$-module morphism $\tilde{\phi} : \widehat{\mathcal{W}}_{N,+}^{\mathrm{B}} \to \widehat{\mathcal{V}}_N^{\mathrm{B}}$ such that the triangle

$$\begin{array}{ccc} \widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} & \hookrightarrow & \widehat{\mathcal{V}}^{\mathrm{B}}_N \\ & {\scriptstyle \tilde{\phi}} \searrow \quad \nearrow {\scriptstyle -\cdot(X_1-1)} & \\ & \widehat{\mathcal{V}}^{\mathrm{B}}_N & \end{array} \tag{3.20}$$

commutes. First, one notices that $\varphi : \mathcal{V}^{\mathrm{B}}_N \to \mathcal{W}^{\mathrm{B}}_{N,+}$ is a surjective **k**-module morphism. It is injective thanks to the integral domain status of the algebra $\mathcal{V}^{\mathrm{B}}_N$. Therefore, the map $\varphi : \mathcal{V}^{\mathrm{B}}_N \to \mathcal{W}^{\mathrm{B}}_{N,+}$ is a **k**-module isomorphism whose inverse will be denoted $\phi : \mathcal{W}^{\mathrm{B}}_{N,+} \to \mathcal{V}^{\mathrm{B}}_N$. Thanks to (3.16), the **k**-module isomorphism $\phi : \mathcal{W}^{\mathrm{B}}_{N,+} \to \mathcal{V}^{\mathrm{B}}_N$ restricts to an isomorphism $\mathcal{F}^m\mathcal{W}^{\mathrm{B}}_{N,+} \to \mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$, for any $m \in \mathbb{N}^*$. This induces a **k**-module isomorphism $\phi_m : \mathcal{W}^{\mathrm{B}}_{N,+}/\mathcal{F}^m\mathcal{W}^{\mathrm{B}}_{N,+} \to \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$, for any $m \in \mathbb{N}^*$ and, via a prism similar to the one of Step 2, one checks that the following triangle

$$\begin{array}{ccc} \mathcal{W}^{\mathrm{B}}_{N,+}/\mathcal{F}^m\mathcal{W}^{\mathrm{B}}_{N,+} & \hookrightarrow & \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N \\ & {\scriptstyle \phi_m} \searrow \quad \nearrow {\scriptstyle -\cdot(X_1-1)} & \\ & \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N & \end{array}$$

commutes where the morphism $-\cdot(X_1-1) : \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N \to \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N$ is well-defined thanks to the inclusion $\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N(X_1-1) \subset \mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N$ being a consequence of (3.16). On the other hand, we have, for any $m \in \mathbb{N}^*$, the following triangle

$$\begin{array}{ccc} & \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N & \\ {\scriptstyle -\cdot(X_1-1)} \nearrow & & \nwarrow {\scriptstyle -\cdot(X_1-1)} \\ \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N & \xleftarrow[\pi_m]{} & \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N \end{array}$$

where $\pi_m : \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N \twoheadrightarrow \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$ is the morphism which associates to the class of an element modulo $\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N$, its class modulo $\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$; this is well-defined and surjective thanks to the inclusion $\mathcal{F}^m\mathcal{V}^{\mathrm{B}}_N \subset \mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$. One then checks that this triangle commutes. By linking the two triangles and doing the inverse limit we obtain the following diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} & \hookrightarrow & \widehat{\mathcal{V}}^{\mathrm{B}}_N \\ {\scriptstyle \widehat{\phi}} \searrow & \nearrow & \nwarrow {\scriptstyle -\cdot(X_1-1)} \\ \varprojlim \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N & \xleftarrow[\widehat{\pi}]{} & \widehat{\mathcal{V}}^{\mathrm{B}}_N \end{array}$$

where $\widehat{\pi} := \varprojlim \pi_m : \widehat{\mathcal{V}}^{\mathrm{B}}_N \to \varprojlim \mathcal{V}^{\mathrm{B}}_N/\mathcal{F}^{m-1}\mathcal{V}^{\mathrm{B}}_N$ is obtained by degree shifting and is therefore a topological **k**-module isomorphism. Let us set $\widetilde{\phi} := \widehat{\pi}^{-1} \circ \widehat{\phi} : \widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} \to \widehat{\mathcal{V}}^{\mathrm{B}}_N$. It is a topological **k**-module morphism such that Diagram (3.20) commutes. Finally, the commutativity of the latter diagram implies

$$\begin{aligned} \mathrm{Im}\left(\widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} \hookrightarrow \widehat{\mathcal{V}}^{\mathrm{B}}_N\right) &= \mathrm{Im}\big(-\cdot(X_1-1) \circ \tilde{\phi}\big) \\ &\subset \mathrm{Im}\big(-\cdot(X_1-1)\big) = \widehat{\mathcal{V}}^{\mathrm{B}}_N(X_1-1). \end{aligned} \tag{3.21}$$

Finally, combining inclusions (3.19) and (3.21) we obtain

$$\widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} \simeq \mathrm{Im}\left(\widehat{\mathcal{W}}^{\mathrm{B}}_{N,+} \hookrightarrow \widehat{\mathcal{V}}^{\mathrm{B}}_{N}\right) = \widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1 - 1).$$

In addition, thanks to 3.17, the topological $\mathbf{k}$-algebras $\mathbf{k} \oplus \widehat{\mathcal{W}}^{\mathrm{B}}_{N,+}$ and $\widehat{\mathcal{W}}^{\mathrm{B}}_{N}$ are isomorphic. One then obtains the isomorphism of topological $\mathbf{k}$-algebras

$$\widehat{\mathcal{W}}^{\mathrm{B}}_{N} \simeq \mathbf{k} \oplus \widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1 - 1).$$

□

Proposition-Definition 3.1.16. *Let $\iota : G \to \mathbb{C}^{\times}$ be a group embedding. There exists a topological algebra isomorphism $\mathrm{iso}^{\mathcal{W},\iota} : \widehat{\mathcal{W}}^{\mathrm{B}}_{N} \to \widehat{\mathcal{W}}^{\mathrm{DR}}_{G}$ such that the following diagram*

$$\begin{array}{ccc} \widehat{\mathcal{W}}^{\mathrm{B}}_{N} & \xrightarrow{\mathrm{iso}^{\mathcal{W},\iota}} & \widehat{\mathcal{W}}^{\mathrm{DR}}_{G} \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{V}}^{\mathrm{B}}_{N} & \xrightarrow[\mathrm{iso}^{\mathcal{V},\iota}]{} & \widehat{\mathcal{V}}^{\mathrm{DR}}_{G} \end{array}$$

commutes.

Proof. We have

$$\mathrm{iso}^{\mathcal{V},\iota}(X_1 - 1) = \exp(e_1) - 1 = ue_1,$$

where $u = f(e_1)$ with $f(x)$ being the invertible formal series $\frac{\exp(x)-1}{x}$. Moreover, since $\mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}^{\mathrm{B}}_{N} \to \widehat{\mathcal{V}}^{\mathrm{DR}}_{G}$ is a $\mathbf{k}$-algebra isomorphism, we obtain

$$\mathrm{iso}^{\mathcal{V},\iota}\left(\widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1 - 1)\right) = \mathrm{iso}^{\mathcal{V},\iota}(\widehat{\mathcal{V}}^{\mathrm{B}}_{N})\ \mathrm{iso}^{\mathcal{V},\iota}(X_1 - 1) = \widehat{\mathcal{V}}^{\mathrm{DR}}_{G} ue_1 = \widehat{\mathcal{V}}^{\mathrm{DR}}_{G} e_1.$$

This implies that $\mathrm{iso}^{\mathcal{V},\iota}_{|\widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1-1)} : \widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1 - 1) \to \widehat{\mathcal{V}}^{\mathrm{DR}}_{G} e_1$ is a surjective $\mathbf{k}$-module morphism which is trivially injective, therefore, is a $\mathbf{k}$-module isomorphism. Taking the direct sum with $\mathbf{k}$, we obtain a $\mathbf{k}$-module isomorphism

$$\mathbf{k} \oplus \widehat{\mathcal{V}}^{\mathrm{B}}_{N}(X_1 - 1) \to \mathbf{k} \oplus \widehat{\mathcal{V}}^{\mathrm{DR}}_{G} e_1,$$

which is a $\mathbf{k}$-algebra isomorphism. Finally, thanks to Lemma 3.1.15, this isomorphism is the wanted $\mathbf{k}$-algebra isomorphism $\mathrm{iso}^{\mathcal{W},\iota} : \widehat{\mathcal{W}}^{\mathrm{B}}_{N} \to \widehat{\mathcal{W}}^{\mathrm{DR}}_{G}$. □

3.1.3 The topological module $\widehat{\mathcal{M}}^{\mathrm{B}}_{N}$

The filtered module $\mathcal{M}^{\mathrm{B}}_{N}$

Proposition-Definition 3.1.17. *The quotient $\mathbf{k}$-module*

$$\mathcal{M}^{\mathrm{B}}_{N} := \mathcal{V}^{\mathrm{B}}_{N} \Big/ \mathcal{V}^{\mathrm{B}}_{N}(X_0 - 1) \tag{3.22}$$

is a $\mathcal{V}^{\mathrm{B}}_{N}$-module. Moreover, if we denote 1_{B} the class of $1 \in \mathcal{V}^{\mathrm{B}}_{N}$ in $\mathcal{M}^{\mathrm{B}}_{N}$, then the canonical projection

$$- \cdot 1_{\mathrm{B}} : \mathcal{V}^{\mathrm{B}}_{N} \to \mathcal{M}^{\mathrm{B}}_{N}$$

is a surjective $\mathcal{V}^{\mathrm{B}}_{N}$-module morphism and its restriction to $\mathcal{W}^{\mathrm{B}}_{N}$ is a $\mathcal{W}^{\mathrm{B}}_{N}$-module isomorphism.

Proof. This follows from the direct sum decomposition

$$\mathcal{V}_N^{\mathrm{B}} = \mathbf{k} \oplus \mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \oplus \mathcal{V}_N^{\mathrm{B}}(X_0 - 1) = \mathcal{W}_N^{\mathrm{B}} \oplus \mathcal{V}_N^{\mathrm{B}}(X_0 - 1)$$

given by [Wei, Proposition 6.2.6]. □

Remark. The statement implies that $(- \cdot 1_{\mathrm{B}})_{|\mathcal{W}_N^{\mathrm{B}}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ is $\mathcal{W}_N^{\mathrm{B}}$-module isomorphism. Therefore $\mathcal{M}_N^{\mathrm{B}}$ to be a free $\mathcal{W}_N^{\mathrm{B}}$-module of rank 1.

Proposition-Definition 3.1.18. *The* $\mathbf{k}$*-module* $\mathcal{M}_N^{\mathrm{B}}$ *is endowed with the decreasing* $\mathbf{k}$*-module filtration given by*

$$\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}} := \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}} \quad \text{for } m \in \mathbb{N}. \tag{3.23}$$

Moreover, the pair $\left(\mathcal{M}_N^{\mathrm{B}}, \left(\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}\right)_{m\in\mathbb{N}}\right)$ *is a filtered module over the filtered algebra* $\left(\mathcal{W}_N^{\mathrm{B}}, \left(\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}}\right)_{m\in\mathbb{N}}\right)$.

Proof. Immediate. □

Lemma 3.1.19.

(i) For any $m \in \mathbb{N}$*, the* $\mathbf{k}$*-module isomorphism* $- \cdot 1_{\mathrm{B}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ *induces a* $\mathbf{k}$*-modules isomorphism* $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \to \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}$.

(ii) For any $m \in \mathbb{N}$*, we have* $\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}} = \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}}$.

Proof.

(i) By definition of $\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}$, the isomorphism $- \cdot 1_{\mathrm{B}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ restricts to a surjective $\mathbf{k}$-module morphism $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \to \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}$. In addition, since $- \cdot 1_{\mathrm{B}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ is injective, so is the restriction $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \to \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}$.

(ii) First, if $m = 0$, the equality follows from Proposition-Definition 3.1.17.
From now on, let $m \in \mathbb{N}^*$. Since $\mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \subset \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}$, we have that

$$\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}} \subset \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}}.$$

Conversely, let us prove the inclusion $\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}} \subset \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}$. This inclusion is equivalent to

$$\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \subset \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} + \mathcal{V}_N^{\mathrm{B}}(X_0 - 1).$$

Since $\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} = \mathcal{I}^m = \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N)^m$ and by Lemma 3.1.11 (i), this inclusion is also equivalent to

$$\mathcal{I}^m \subset \left(\mathcal{I}^m \cap \mathcal{V}_N^{\mathrm{B}}(X_1 - 1)\right) + \mathcal{V}_N^{\mathrm{B}}(X_0 - 1). \tag{3.24}$$

We have

$$\mathcal{I} = \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N) \subset \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}) = \mathcal{V}_N^{\mathrm{B}}(X_0 - 1) + \mathcal{V}_N^{\mathrm{B}}(X_1 - 1),$$

with $\ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k})$ being the augmentation ideal of the group algebra $\mathcal{V}_N^{\mathrm{B}} = \mathbf{k}F_2$ and the last equality being a consequence of [Wei, Proposition 6.2.6].
This implies

$$\begin{aligned}\mathcal{I}^m = \mathcal{I}^{m-1}\mathcal{I} \subset \mathcal{I}^{m-1}\left(\mathcal{V}_N^{\mathrm{B}}(X_1-1) + \mathcal{V}_N^{\mathrm{B}}(X_0-1)\right) \\ \subset \mathcal{I}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1-1) + \mathcal{V}_N^{\mathrm{B}}(X_0-1). \qquad (3.25)\end{aligned}$$

Moreover, $\mathcal{V}_N^{\mathrm{B}}(X_1-1) \subset \ker(\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N)$ since $X_1 - 1 \mapsto 0$ through the map $\mathcal{V}_N^{\mathrm{B}} \to \mathbf{k}\mu_N$. This implies

$$\mathcal{I}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1-1) \subset \mathcal{I}^{m-1}\mathcal{I} = \mathcal{I}^m. \qquad (3.26)$$

On the other hand, we have

$$\mathcal{I}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1-1) \subset \mathcal{V}_N^{\mathrm{B}}(X_1-1). \qquad (3.27)$$

From (3.26) and (3.27) we obtain

$$\mathcal{I}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1-1) \subset \left(\mathcal{I}^m \cap \mathcal{V}_N^{\mathrm{B}}(X_1-1)\right). \qquad (3.28)$$

Finally, from (3.25) and (3.28), we obtain

$$\mathcal{I}^m \subset \left(\mathcal{I}^m \cap \mathcal{V}_N^{\mathrm{B}}(X_1-1)\right) + \mathcal{V}_N^{\mathrm{B}}(X_0-1),$$

which is the wanted inclusion.

□

Corollary 3.1.20.

(i) *The surjective* $\mathbf{k}$*-module morphism* $-\cdot 1_{\mathrm{B}} : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ *induces a surjective graded* $\mathbf{k}$*-module morphism* $\mathrm{gr}(-\cdot 1_{\mathrm{B}}) : \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \to \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$.

(ii) *The* $\mathbf{k}$*-module isomorphism* $-\cdot 1_{\mathrm{B}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ *induces a graded* $\mathbf{k}$*-module isomorphism* $\mathrm{gr}(\mathcal{W}_N^{\mathrm{B}}) \xrightarrow{\mathrm{gr}(-\cdot 1_{\mathrm{B}})} \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$. *It coincides with the* $\mathrm{gr}(\mathcal{W}_N^{\mathrm{B}})$*-module morphism* $\mathrm{gr}(\mathcal{W}_N^{\mathrm{B}}) \to \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ *induced by the action on* $[1_{\mathrm{B}}] \in \mathrm{gr}_0(\mathcal{M}_N^{\mathrm{B}})$.

(iii) *The* $\mathbf{k}$*-module* $\mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ *is a free* $\mathrm{gr}(\mathcal{W}_N^{\mathrm{B}})$*-module of rank* 1.

Proof.

(i) This comes from Lemma 3.1.19 (ii).

(ii) The isomorphism $-\cdot 1_{\mathrm{B}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ is compatible with filtrations by Lemma 3.1.19 (i). Therefore, the morphism $-\cdot 1_{\mathrm{B}}$ induces a graded $\mathbf{k}$-module morphism $\mathrm{gr}(-\cdot 1_{\mathrm{B}}) : \mathrm{gr}(\mathcal{W}_N^{\mathrm{B}}) \to \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$. Moreover, Lemma 3.1.19 (ii) also implies that $-\cdot 1_{\mathrm{B}}$ induces an isomorphism between filtrations. Therefore $\mathrm{gr}(-\cdot 1_{\mathrm{B}})$ is a graded $\mathbf{k}$-module isomorphism. The identification comes from the definitions of the involved maps.

(iii) This is a consequence of (ii).

□

Given a group embedding $\iota : G \to \mathbb{C}^\times$, let us identify $\mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ with $\mathcal{M}_G^{\mathrm{DR}}$. Before we proceed, let us recall that $\mathcal{M}_G^{\mathrm{DR}} = \mathcal{V}_G^{\mathrm{DR}}/\mathcal{V}_G^{\mathrm{DR}}e_0 + \sum_{g\in G}\mathcal{V}_G^{\mathrm{DR}}(g-1)$ (see Definition 2.1.18). Since the group G is cyclic with generator g_ι, the $\mathbf{k}$-submodule $\sum_{g\in G}\mathcal{V}_G^{\mathrm{DR}}(g-1)$ is equal to $\mathcal{V}_G^{\mathrm{DR}}e_0+\mathcal{V}_G^{\mathrm{DR}}(g_\iota-1)$. As a consequence, $\mathcal{M}_G^{\mathrm{DR}} = \mathcal{V}_G^{\mathrm{DR}}/\mathcal{V}_G^{\mathrm{DR}}e_0+\mathcal{V}_G^{\mathrm{DR}}(g_\iota-1)$.

Proposition-Definition 3.1.21. *For any group embedding $\iota : G \to \mathbb{C}^\times$, there exists a graded $\mathbf{k}$-module isomorphism $\rho_\iota^{\mathcal{M}} : \mathcal{M}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ such that the following diagram*

$$\begin{array}{ccc} \mathcal{V}_G^{\mathrm{DR}} & \xrightarrow{\quad\rho_\iota^{\mathcal{V}}\quad} & \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \\ {\scriptstyle -\cdot 1_{\mathrm{DR}}}\downarrow & & \downarrow{\scriptstyle \mathrm{gr}(-\cdot 1_{\mathrm{B}})} \\ \mathcal{M}_G^{\mathrm{DR}} & \xrightarrow[\quad\rho_\iota^{\mathcal{M}}\quad]{} & \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}}) \end{array}$$

commutes.

Proof. Thanks to Corollary 3.1.20 (i), the $\mathbf{k}$-module $\mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ is a graded $\mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$-module and $\mathrm{gr}(-\cdot 1_{\mathrm{B}}) : \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \to \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ is a graded $\mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$-module morphism.
Let us construct a graded module morphism $\rho_\iota^{\mathcal{M}} : \mathcal{M}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ over the graded algebra morphism $\rho_\iota^{\mathcal{V}} : \mathcal{V}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$. We consider the composition

$$\mathcal{V}_G^{\mathrm{DR}} \xrightarrow{\rho_\iota^{\mathcal{V}}} \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}}) \xrightarrow{\mathrm{gr}(-\cdot 1_{\mathrm{B}})} \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}}).$$

This composition sends the $\mathbf{k}$-submodule $\mathcal{V}_G^{\mathrm{DR}}e_0 + \mathcal{V}_G^{\mathrm{DR}}(g_\iota - 1)$ to 0. Indeed, this comes from

$$\begin{aligned} e_0 &\mapsto [X_0^N - 1] &&\mapsto [(X_0^N - 1)\cdot 1_{\mathrm{B}}] = 0 \\ g_\iota - 1 &\mapsto [X_0 - 1] &&\mapsto [(X_0 - 1)\cdot 1_{\mathrm{B}}] = 0 \end{aligned}$$

and the fact that $\mathrm{gr}(-\cdot 1_{\mathrm{B}}) \circ \rho_\iota^{\mathcal{V}}$ is a module morphism over the algebra morphism $\mathcal{V}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$. As a consequence, this composition factorises into a $\mathbf{k}$-module morphism $\rho_\iota^{\mathcal{M}} : \mathcal{M}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ which is a module morphism over the algebra morphism $\rho_\iota^{\mathcal{V}} : \mathcal{V}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$.
Next, let us show that $\rho_\iota^{\mathcal{M}} : \mathcal{M}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ is an isomorphism. Recall from Proposition 3.1.12 that $\rho_\iota^{\mathcal{W}} : \mathcal{W}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{W}_N^{\mathrm{B}})$ is an algebra submorphism of $\rho_\iota^{\mathcal{V}} : \mathcal{V}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{V}_N^{\mathrm{B}})$. As a result, $\rho_\iota^{\mathcal{M}} : \mathcal{M}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ is a module morphism over the algebra isomorphism $\rho_\iota^{\mathcal{W}} : \mathcal{W}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{W}_N^{\mathrm{B}})$. In addition, thanks to Corollaries 2.1.22.(ii) and 3.1.20.(iii), $\mathcal{M}_G^{\mathrm{DR}}$ and $\mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ are both free rank 1 modules over $\mathcal{W}_G^{\mathrm{DR}}$ and $\mathrm{gr}(\mathcal{W}_N^{\mathrm{B}})$ respectively and $\rho_\iota^{\mathcal{M}}$ sends 1_{DR} to $[1_{\mathrm{B}}] \in \mathrm{gr}_0(\mathcal{M}_N^{\mathrm{B}})$ and therefore a basis of the source to a basis of the target. Thus $\rho_\iota^{\mathcal{M}} : \mathcal{M}_G^{\mathrm{DR}} \to \mathrm{gr}(\mathcal{M}_N^{\mathrm{B}})$ is a module isomorphism over $\rho_\iota^{\mathcal{W}}$ and therefore a $\mathbf{k}$-module isomorphism. □

The topological module $\widehat{\mathcal{M}}_N^{\mathrm{B}}$

The decreasing filtration $(\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}})_{m\in\mathbb{N}}$ given in (3.23) induces a **k**-module morphism $\mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{M}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}$.

Definition 3.1.22. We denote

$$\widehat{\mathcal{M}}_N^{\mathrm{B}} := \varprojlim \mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}.$$

the limit of the projective system $(\mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}, \mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^{m+1}\mathcal{M}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}})$.

The **k**-module $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ is a $\widehat{\mathcal{V}}_N^{\mathrm{B}}$-module equipped with the filtration

$$\mathcal{F}^m\widehat{\mathcal{M}}_N^{\mathrm{B}} := \varprojlim \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^{\max(m,l)}\mathcal{M}_N^{\mathrm{B}}, \text{ for } m \in \mathbb{N}^*.$$

When equipped with the topology defined by this filtration, $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ is a complete separated topological **k**-module.

Lemma 3.1.23.

(i) The surjective **k***-module morphism* $-\cdot 1_{\mathrm{B}} : \mathcal{V}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ *induces a topological surjective* **k***-module morphism* $\widehat{-\cdot 1_{\mathrm{B}}} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_N^{\mathrm{B}}$*.*

(ii) The **k***-module isomorphism* $-\cdot 1_{\mathrm{B}} : \mathcal{W}_N^{\mathrm{B}} \to \mathcal{M}_N^{\mathrm{B}}$ *induces a topological* **k***-module isomorphism* $\widehat{-\cdot 1_{\mathrm{B}}} : \widehat{\mathcal{W}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_N^{\mathrm{B}}$*. Therefore, the* **k***-module* $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ *is a free* $\widehat{\mathcal{W}}_N^{\mathrm{B}}$*-module of rank* 1*.*

Proof.

(i) By definition of $\widehat{\mathcal{V}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{M}}_N^{\mathrm{B}}$, this follows from Lemma 3.1.19 (ii).

(ii) By definition of $\widehat{\mathcal{M}}_N^{\mathrm{B}}$, this follows from Lemma 3.1.19 (i).

□

Proposition 3.1.24. *The topological* **k***-module morphism* $\widehat{-\cdot 1_{\mathrm{B}}} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_N^{\mathrm{B}}$ *induces an isomorphism* $\widehat{\mathcal{V}}_N^{\mathrm{B}}/\widehat{\mathcal{V}}_N^{\mathrm{B}}(X_0-1) \to \widehat{\mathcal{M}}_N^{\mathrm{B}}$ *of topological* **k***-modules.*

For the proof, we will need the following lemma

Lemma 3.1.25. *Let* V *be a* **k***-module and* u *be an endomorphism of* V*. Let* $f : V^N \to V^N$ *be the endomorphism given by*

$$(v_0, \dots, v_{N-1}) \mapsto (u(v_{N-1}) - v_0, v_0 - v_1, v_1 - v_2, \dots, v_{N-2} - v_{N-1}).$$

Then we have an isomorphism

$$\operatorname{coker}(f) \simeq \operatorname{coker}(u - \mathrm{id}).$$

Proof of Lemma 3.1.25. Let us consider the **k**-module morphism sum : $V^N \to V$ given by $(v_0, \ldots, v_{N-1}) \mapsto v_0 + \cdots + v_{N-1}$. This morphism sends $\mathrm{Im}(f)$ to $\mathrm{Im}(u - \mathrm{id})$. Therefore, there is a unique **k**-module morphism $V^N/\mathrm{Im}(f) \to V/\mathrm{Im}(u - \mathrm{id})$ such that the diagram

$$\begin{array}{ccc} V^N & \xrightarrow{\quad\text{sum}\quad} & V \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ V^N/\mathrm{Im}(f) & \longrightarrow & V/\mathrm{Im}(u-\mathrm{id}) \end{array}$$

commutes. Let us show that the morphism $V^N/\mathrm{Im}(f) \to V/\mathrm{Im}(u - \mathrm{id})$ is an isomorphism. First, the surjectivity of the morphism sum : $V^N \to V$ implies that the morphism $V^N/\mathrm{Im}(f) \to V/\mathrm{Im}(u - \mathrm{id})$ is surjective as well. Second, let $(w_0, \ldots, w_{N-1}) \in V^N$ such that there exists an element $v \in V$ such that $w_0 + \cdots w_{N-1} = u(v) - v$. The element $(v_0, \ldots, v_{N-1}) \in V^N$ given by

$$v_{N-1} = v, v_{N-2} = w_{N-1} + v, v_{N-3} = w_{N-2} + w_{N-1} + v, \ldots, v_0 = w_1 + \cdots w_{N-1} + v$$

is such that

$$(w_0, \ldots, w_{N-1}) = (u(v_{N-1}) - v_0, v_0 - v_1, \ldots, v_{N-2} - v_{N-1}) \in \mathrm{Im}(f).$$

Thus proving the injectivity of $V^N/\mathrm{Im}(f) \to V/\mathrm{Im}(u - \mathrm{id})$. □

Proof of Proposition 3.1.24. The proof is depicted in the following steps:

Step 1. Construction of the **k**-module isomorphism $P_0 : (\mathbf{k}F_{N+1})^N \to \mathcal{V}_N^{\mathrm{B}}$.
As in §3.1.1, one can define the **k**-module isomorphism $\mathbf{k}F_{N+1} \otimes \mathbf{k}\mu_N \to \mathcal{V}_N^{\mathrm{B}}$ such that for $(x, \zeta) \in F_{N+1} \times \mu_N$

$$x \otimes \zeta \mapsto x\,\sigma(\zeta), \tag{3.29}$$

where F_{N+1} is seen as $\ker(F_2 \to \mu_N) \subset F_2$ thanks to Lemma 3.1.1. Moreover, one checks there is a **k**-module isomorphism $(\mathbf{k}F_{N+1})^N \to \mathbf{k}F_{N+1} \otimes \mathbf{k}\mu_N$ given by

$$(v_0, \ldots, v_{N-1}) \mapsto \sum_{i=0}^{N-1} v_i \otimes \zeta_N^i. \tag{3.30}$$

Therefore, the composition $P_0 : (\mathbf{k}F_{N+1})^N \to \mathbf{k}F_{N+1} \otimes \mathbf{k}\mu_N \to \mathcal{V}_N^{\mathrm{B}}$ is a **k**-module isomorphism and is given by

$$(v_0, \ldots, v_{N-1}) \mapsto v_0 + v_1 X_0 + \cdots + v_{N-1} X_0^{N-1}.$$

Step 2. Identification of $\mathcal{M}_N^{\mathrm{B}}$.
One checks that the endomorphism $f : (\mathbf{k}F_{N+1})^N \to (\mathbf{k}F_{N+1})^N$ given by

$$(v_0, \ldots, v_{N-1}) \mapsto (v_{N-1}\widetilde{X}_0 - v_0, v_0 - v_1, v_1 - v_2, \ldots, v_{N-2} - v_{N-1}), \tag{3.31}$$

is such that the following diagram

$$\begin{array}{ccc} (\mathbf{k}F_{N+1})^N & \xrightarrow{\quad f \quad} & (\mathbf{k}F_{N+1})^N \\ {\scriptstyle P_0}\downarrow & & \downarrow{\scriptstyle P_0} \\ \mathcal{V}_N^{\mathrm{B}} & \xrightarrow[-\cdot(X_0 - 1)]{} & \mathcal{V}_N^{\mathrm{B}} \end{array} \tag{3.32}$$

commutes. This induces a $\mathbf{k}$-module isomorphism $\mathrm{coker}(f) \simeq \mathrm{coker}(- \cdot (X_0 - 1))$. On the other hand, by applying Lemma 3.1.25 with $V = \mathbf{k}F_{N+1}$ and $u = - \cdot \widetilde{X}_0$, we obtain an isomomorphism $\mathrm{coker}(f) \simeq \mathrm{coker}(u - \mathrm{id})$. It then follows that

$$\begin{aligned}\mathcal{M}_N^{\mathrm{B}} = \mathcal{V}_N^{\mathrm{B}}/\mathcal{V}_N^{\mathrm{B}}(X_0 - 1) &= \mathrm{coker}(- \cdot (X_0 - 1)) \\ &\simeq \mathrm{coker}(f) \simeq \mathrm{coker}(u - \mathrm{id}) = \mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1).\end{aligned}$$

Step 3. Compatibility of the isomorphism $\mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1) \to \mathcal{M}_N^{\mathrm{B}}$ with filtrations. Let us show that for any $m \in \mathbb{N}$, we have

$$(\mathbf{k}F_{N+1})_0^m \Big/ \Big((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0 - 1)\Big) \simeq \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}.$$

If $m = 0$, this has been proved in Step 2. From now on, let us assume that $m \in \mathbb{N}^*$. The isomorphism $\mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1) \to \mathcal{M}_N^{\mathrm{B}}$ fits in the following commutative diagram

$$\begin{array}{ccc} \mathbf{k}F_{N+1} & \hookrightarrow & \mathcal{V}_N^{\mathrm{B}} \\ \downarrow & & \downarrow \\ \mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1) & \xrightarrow{\simeq} & \mathcal{M}_N^{\mathrm{B}} \end{array} \tag{3.33}$$

where $\mathbf{k}F_{N+1} \to \mathcal{V}_N^{\mathrm{B}}$ is the group algebra morphism induced by the group morphism $F_{N+1} \simeq \ker(F_2 \to \mu_N) \subset F_2$ obtained in Lemma 3.1.1. This group algebra morphism induces the injection

$$(\mathbf{k}F_{N+1})_0^m \hookrightarrow \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}.$$

Then, thanks to the commutativity of Diagram (3.33), the $\mathbf{k}$-module isomorphism $\mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1) \to \mathcal{M}_N^{\mathrm{B}}$ induces an injection

$$(\mathbf{k}F_{N+1})_0^m \Big/ \Big((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0 - 1)\Big) \hookrightarrow \mathcal{F}^m\mathcal{V}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}} = \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}},$$

where the equality comes from Lemma 3.1.19.(ii). This implies that

$$\mathrm{Im}\Big((\mathbf{k}F_{N+1})_0^m \Big/ \Big((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0 - 1)\Big) \to \mathcal{M}_N^{\mathrm{B}}\Big) \subset \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}}.$$

Conversely, let us show the opposite inclusion. Thanks to Lemma 3.1.11(ii), we have

$$\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}} = \mathcal{F}^m\mathcal{W}_N^{\mathrm{B}} \cdot 1_{\mathrm{B}} = \mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \cdot 1_{\mathrm{B}}.$$

Moreover, we have by definition that $\mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}} = (\mathcal{F}^1\mathcal{V}_N^{\mathrm{B}})^{m-1}$. This implies, thanks to Lemma 3.1.3 (ii) that $\mathcal{F}^{m-1}\mathcal{V}_N^{\mathrm{B}}(X_1 - 1) \cdot 1_{\mathrm{B}}$ is linearly generated by elements

$$\sigma(\zeta_1)(x_1 - 1) \cdots \sigma(\zeta_{m-1})(x_{m-1} - 1)(X_1 - 1) \cdot 1_{\mathrm{B}}$$

with $(\zeta_1, x_1), \dots, (\zeta_{m-1}, x_{m-1}) \in \mu_N \times F_{N+1}$. Additionally, we have that

$$\begin{aligned}&\sigma(\zeta_1)(x_1 - 1) \cdots \sigma(\zeta_{m-1})(x_{m-1} - 1)(X_1 - 1) \cdot 1_{\mathrm{B}} = \\ &\big(\mathrm{Ad}_{\sigma(\zeta_1)}(x_1) - 1\big) \cdots \big(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(x_{m-1}) - 1\big)\big(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(X_1) - 1\big) \\ &\sigma(\zeta_1) \cdots \sigma(\zeta_{m-1}) \cdot 1_{\mathrm{B}} = \\ &\big(\mathrm{Ad}_{\sigma(\zeta_1)}(x_1) - 1\big) \cdots \big(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(x_{m-1}) - 1\big)\big(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(X_1) - 1\big) \cdot 1_{\mathrm{B}}\end{aligned}$$

which belongs to the image of $(\mathbf{k}F_{N+1})_0^m \Big/ \left((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0 - 1)\right)$ by the isomorphism $\mathbf{k}F_{N+1}/\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1) \to \mathcal{M}_N^{\mathrm{B}}$ as

$$\left(\mathrm{Ad}_{\sigma(\zeta_1)}(x_1) - 1\right) \cdots \left(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(x_{m-1}) - 1\right) \left(\mathrm{Ad}_{\sigma(\zeta_1)\cdots\sigma(\zeta_{m-1})}(X_1) - 1\right),$$

seen as an element of $\mathbf{k}F_{N+1}$, belongs to $(\mathbf{k}F_{N+1})_0^m$. This implies that

$$\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}} \subset \mathrm{Im}\left((\mathbf{k}F_{N+1})_0^m \Big/ \left((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0 - 1)\right) \to \mathcal{M}_N^{\mathrm{B}}\right).$$

Therefore, one has equality

$$\mathrm{Im}\left((\mathbf{k}F_{N+1})_0^m \Big/ \left((\mathbf{k}F_{N+1})_0^m \cap \mathbf{k}F_{N+1}(\widetilde{X}_0 - 1)\right) \to \mathcal{M}_N^{\mathrm{B}}\right) = \mathcal{F}^m\mathcal{M}_N^{\mathrm{B}},$$

which establishes the wanted isomorphism.

Step 4. Identification of $\widehat{\mathcal{M}}_N^{\mathrm{B}}$.
Thanks to Step 3, one has for any $m \in \mathbb{N}$

$$\mathcal{M}_N^{\mathrm{B}}/\mathcal{F}^m\mathcal{M}_N^{\mathrm{B}} \simeq \mathbf{k}F_{N+1}\Big/\left(\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1) + (\mathbf{k}F_{N+1})_0^m\right)$$

and, on the other hand, for any $m \in \mathbb{N}^*$,

$$\mathbf{k}F_{N+1}\Big/\left(\mathbf{k}F_{N+1}(\widetilde{X}_0 - 1) + (\mathbf{k}F_{N+1})_0^m\right) \simeq \mathrm{coker}\left(\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m\right),$$

where the morphism

$$\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m \tag{3.34}$$

is induced by the endomorphism $- \cdot (\widetilde{X}_0 - 1)$ of $\mathbf{k}F_{N+1}$. Therefore,

$$\widehat{\mathcal{M}}_N^{\mathrm{B}} \simeq \varprojlim \mathrm{coker}\left(\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^m\right).$$

Step 5. Identification of $\widehat{\mathcal{V}}_N^{\mathrm{B}}/\widehat{\mathcal{V}}_N^{\mathrm{B}}(X_0 - 1)$.
As in Lemma 3.1.5, one can prove that the $\mathbf{k}$-module isomorphism $\mathbf{k}F_{N+1} \otimes \mathbf{k}\mu_N \to \mathcal{V}_N^{\mathrm{B}}$ given in (3.29) allows us to identify $(\mathbf{k}F_{N+1})_0^m \otimes \mathbf{k}\mu_N$ with $\mathcal{F}^m\mathcal{V}_N^{\mathrm{B}}$ for any $m \in \mathbb{N}$. Recall the isomorphism $(\mathbf{k}F_{N+1})^N \to \mathbf{k}F_{N+1} \otimes \mathbf{k}\mu_N$ given in (3.30). One checks that it is compatible with the filtration of $(\mathbf{k}F_{N+1})^N$ given for any $m \in \mathbb{N}$ by $\prod_{i=1}^N (\mathbf{k}F_{N+1})_0^m$. Therefore the isomorphism $P_0 : (\mathbf{k}F_{N+1})^N \to \mathbf{k}F_{N+1} \otimes \mathbf{k}\mu_N \to \mathcal{V}_N^{\mathrm{B}}$ of Step 1 is compatible with filtrations. Therefore, it extends to a topological $\mathbf{k}$-module isomorphism

$$\widehat{P}_0 : (\widehat{\mathbf{k}F_{N+1}})^N \to \widehat{\mathcal{V}}_N^{\mathrm{B}}.$$

On the other hand, the endomorphism $f : (\mathbf{k}F_{N+1})^N \to (\mathbf{k}F_{N+1})^N$ given in (3.31) is compatible with filtrations and then extends to a topological endomorphism $\hat{f} : (\widehat{\mathbf{k}F_{N+1}})^N \to (\widehat{\mathbf{k}F_{N+1}})^N$ and, thanks to Diagram (3.32), it is such that the following diagram

$$\begin{array}{ccc} (\widehat{\mathbf{k}F_{N+1}})^N & \xrightarrow{\hat{f}} & (\widehat{\mathbf{k}F_{N+1}})^N \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ \widehat{\mathcal{V}}_N^{\mathrm{B}} & \xrightarrow[-\cdot(X_0-1)]{} & \widehat{\mathcal{V}}_N^{\mathrm{B}} \end{array}$$

commutes. This induces a $\mathbf{k}$-module isomorphism $\mathrm{coker}(\hat{f}) \simeq \mathrm{coker}(- \cdot (X_0 - 1))$. Similarly to Step 1, by applying Lemma 3.1.25 with $V = \widehat{\mathbf{k}F_{N+1}}$ and $u = - \cdot \widetilde{X}_0$, we obtain an isomomorphism $\mathrm{coker}(\hat{f}) \simeq \mathrm{coker}(u - \mathrm{id})$. It then follows that

$$\begin{aligned}\widehat{\mathcal{V}}_N^{\mathrm{B}}/\widehat{\mathcal{V}}_N^{\mathrm{B}}(X_0 - 1) &= \mathrm{coker}(- \cdot (X_0 - 1)) \\ &\simeq \mathrm{coker}(\hat{f}) \simeq \mathrm{coker}(u - \mathrm{id}) = \widehat{\mathbf{k}F_{N+1}}/\widehat{\mathbf{k}F_{N+1}}(\widetilde{X}_0 - 1).\end{aligned}$$

On the other hand, we have

$$\widehat{\mathbf{k}F_{N+1}}/\widehat{\mathbf{k}F_{N+1}}(\widetilde{X}_0 - 1) \simeq \mathrm{coker}\left(\varprojlim \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \varprojlim \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m}\right),$$

where the morphism $\varprojlim \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \varprojlim \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m}$ is induced by the morphism $\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m}$ given in (3.34).

Step 6. Cokernel of limits and limit of cokernels coincide.
For any $m \in \mathbb{N}^*$, the morphism $\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m}$ induced by the endomorphism $- \cdot (\widetilde{X}_0 - 1)$ of $\mathbf{k}F_{N+1}$ is injective. Indeed, let $x \in \mathbf{k}F_{N+1}$ such that $x(\widetilde{X}_0 - 1) \in (\mathbf{k}F_{N+1})_0^m$. Let l to be the smallest integer such that $x \in (\mathbf{k}F_{N+1})_0^l$. We then have that $l \geq m-1$. Otherwise, since $[x] \in \mathrm{gr}_l(\mathbf{k}F_{N+1})$ and $[\widetilde{X}_0 - 1] \in \mathrm{gr}_1(\mathbf{k}F_{N+1})$, we obtain $[x(\widetilde{X}_0 - 1)] \in \mathrm{gr}_{l+1}(\mathbf{k}F_{N+1})$. Since $l+1 \leq m-1$, then the condition $x(\widetilde{X}_0 - 1) \in (\mathbf{k}F_{N+1})_0^m$ implies that $[x(\widetilde{X}_0 - 1)] = 0$, then since $\mathrm{gr}_{l+1}(\mathbf{k}F_{N+1})$ is an integral domain we would obtain $[x] = 0$, contradicting the minimality of l. Therefore, $l \geq m-1$ and the morphism $\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m}$ is injective. In addition, the image of this morphism is the same as the image of the morphism $\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m}$ induced by the endomorphism $- \cdot (\widetilde{X}_0 - 1)$ of $\mathbf{k}F_{N+1}$. We then have the short exact sequence

$$\begin{aligned}\{0\} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m} \\ \to \mathrm{coker}\big((\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m})\big) \to \{0\}\end{aligned}$$

which, by applying the inverse limit functor, gives us

$$\begin{aligned}&\{0\} \to \varprojlim \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} \to \varprojlim \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m} \to \\ &\varprojlim \mathrm{coker}\big((\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m})\big) \to \varprojlim{}^1\, \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1}.\end{aligned}$$

Since the transition maps of the inverse system $(\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1})_{m\in\mathbb{N}^*}$ are surjective, this implies that $\varprojlim{}^1\, \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m-1} = 0$ (see, for example, [BK72, Propostion IX.2.4]). As a consequence,

$$\begin{aligned}&\varprojlim \mathrm{coker}\big((\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m} \to \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m})\big) \simeq \\ &\mathrm{coker}\big(\varprojlim(\mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m} \to \varprojlim \mathbf{k}F_{N+1}/(\mathbf{k}F_{N+1})_0^{m})\big).\end{aligned}$$

Thanks to Step 4 and Step 5, this proves the wanted result.

□

Proposition-Definition 3.1.26. *Let $\iota : G \to \mathbb{C}^\times$ be a group embedding. There exists an unique topological* $\mathbf{k}$*-module isomorphism* $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ *such that the following diagram*

$$\begin{array}{ccc} \widehat{\mathcal{V}}_N^{\mathrm{B}} & \xrightarrow{\mathrm{iso}^{\mathcal{V},\iota}} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle \widehat{-\cdot 1_{\mathrm{B}}}}\downarrow & & \downarrow{\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}} \\ \widehat{\mathcal{M}}_N^{\mathrm{B}} & \xrightarrow[\mathrm{iso}^{\mathcal{M},\iota}]{} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{3.35}$$

commutes.

Proof. Let us construct a topological module morphism $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ over the topological algebra morphism $\mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$. We consider the composition

$$\widehat{\mathcal{V}}_N^{\mathrm{B}} \xrightarrow{\mathrm{iso}^{\mathcal{V},\iota}} \widehat{\mathcal{V}}_G^{\mathrm{DR}} \xrightarrow{\widehat{-\cdot 1_{\mathrm{DR}}}} \widehat{\mathcal{M}}_G^{\mathrm{DR}}. \tag{3.36}$$

This composition sends the $\mathbf{k}$-submodule $\widehat{\mathcal{V}}_N^{\mathrm{B}}(X_0 - 1)$ to 0. Indeed, this comes from the fact that (3.36) is a module morphism over the algebra morphism $\mathrm{iso}^{\mathcal{V},\iota}$ and the following computation

$$\mathrm{iso}^{\mathcal{V},\iota}(X_0 - 1) = g_\iota \exp\left(\tfrac{1}{N} e_0\right) - 1 = g_\iota \left(\exp\left(\tfrac{1}{N} e_0\right) - 1\right) + (g_\iota - 1) \in \widehat{\mathcal{V}}^{\mathrm{DR}} e_0 + \widehat{\mathcal{V}}^{\mathrm{DR}}(g_\iota - 1)$$

Therefore, thanks to Proposition 3.1.24, the composition (3.36) factorises into a $\mathbf{k}$-module morphism $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ which is a module morphism over the algebra morphism $\mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$.
Next, let us show that $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is an isomorphism. Recall from Proposition 3.1.16 that $\mathrm{iso}^{\mathcal{W},\iota} : \widehat{\mathcal{W}}_N^{\mathrm{B}} \to \widehat{\mathcal{W}}_G^{\mathrm{DR}}$ is an algebra submorphism of $\mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}}$. As a result, $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is a module morphism over the algebra isomorphism $\mathrm{iso}^{\mathcal{W},\iota} : \widehat{\mathcal{W}}_N^{\mathrm{B}} \to \widehat{\mathcal{W}}_G^{\mathrm{DR}}$. In addition, $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ are both free rank 1 modules over $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ respectively and $\mathrm{iso}^{\mathcal{M},\iota}$ sends 1_{B} to 1_{DR} and therefore a basis of the source to a basis of the target. Thus $\mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is a module isomorphism over $\mathrm{iso}^{\mathcal{W},\iota}$ and then a $\mathbf{k}$-module isomorphism. ☐

Remark. Let us notice that we have the following equality of $\mathbf{k}$-submodules of $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$:

$$\widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota \exp(e_0) - 1) = \widehat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota - 1).$$

Indeed, since we have that

$$g_\iota \exp(e_0) - 1 = g_\iota(\exp(e_0) - 1) + (g_\iota - 1),$$

this gives us the inclusion $\widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota \exp(e_0) - 1) \subset \widehat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota - 1)$. Conversely, the inclusion $\widehat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota - 1) \subset \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g_\iota \exp(e_0) - 1)$ follows from

$$e_0 = \frac{e_0}{\exp(N e_0) - 1}\big(1 + g_\iota \exp(e_0) + \cdots + g_\iota^{N-1} \exp((N-1)e_0)\big)(g_\iota \exp(e_0) - 1)$$

and from

$$\begin{aligned} g_\iota - 1 =& \exp(-e_0)(g_\iota \exp(e_0) - 1) + (\exp(-e_0) - 1) \\ =& \exp(-e_0)(g_\iota \exp(e_0) - 1) + \\ & \frac{\exp(-e_0) - 1}{\exp(Ne_0) - 1}\big(1 + g_\iota \exp(e_0) + \cdots + g_\iota^{N-1} \exp((N-1)e_0)\big)(g_\iota \exp(e_0) - 1) \\ =& \Big(\exp(-e_0) + \tfrac{\exp(-e_0)-1}{\exp(Ne_0)-1}\big(1 + g_\iota \exp(e_0) + \cdots + g_\iota^{N-1} \exp((N-1)e_0)\big)\Big)(g_\iota \exp(e_0) - 1). \end{aligned}$$

Proposition 3.1.27. *For any* $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle)$*, the following pairs are isomorphisms in the category* $\mathbf{k}$-alg-mod*:*

(i) $\big(\mathrm{iso}^{\mathcal{V},\iota}, \mathrm{iso}^{\mathcal{M},\iota}\big) : (\widehat{\mathcal{V}}_N^{\mathrm{B}}, \widehat{\mathcal{M}}_N^{\mathrm{B}}) \to (\widehat{\mathcal{V}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$.

(ii) $\big(\mathrm{iso}^{\mathcal{W},\iota}, \mathrm{iso}^{\mathcal{M},\iota}\big) : (\widehat{\mathcal{W}}_N^{\mathrm{B}}, \widehat{\mathcal{M}}_N^{\mathrm{B}}) \to (\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\mathcal{M}}_G^{\mathrm{DR}})$.

Proof.

(i) The fact that $\mathrm{iso}^{\mathcal{V},\iota}$ (resp. $\mathrm{iso}^{\mathcal{M},\iota}$) is a $\mathbf{k}$-algebra (resp $\mathbf{k}$-module) isomorphism follows from Proposition-Definition 3.1.9 (resp. Proposition-Definition 3.1.26). Let $(a, m) \in \widehat{\mathcal{V}}_N^{\mathrm{B}} \times \widehat{\mathcal{M}}_N^{\mathrm{B}}$. There exists $v \in \widehat{\mathcal{V}}_N^{\mathrm{B}}$ such that $m = v \cdot 1_{\mathrm{B}}$. We have

$$\begin{aligned} \mathrm{iso}^{\mathcal{M},\iota}(am) =& \mathrm{iso}^{\mathcal{M},\iota}(av \cdot 1_{\mathrm{B}}) = \mathrm{iso}^{\mathcal{V},\iota}(av) \cdot 1_{\mathrm{DR}} = \mathrm{iso}^{\mathcal{V},\iota}(a)\, \mathrm{iso}^{\mathcal{V},\iota}(v) \cdot 1_{\mathrm{DR}} \\ =& \mathrm{iso}^{\mathcal{V},\iota}(a)\, \mathrm{iso}^{\mathcal{M},\iota}(v \cdot 1_{\mathrm{B}}) = \mathrm{iso}^{\mathcal{V},\iota}(a)\, \mathrm{iso}^{\mathcal{M},\iota}(m), \end{aligned}$$

where the second and fourth equalities come from Proposition-Definition 3.1.26.

(ii) The fact that $\mathrm{iso}^{\mathcal{W},\iota}$ (resp. $\mathrm{iso}^{\mathcal{M},\iota}$) is a $\mathbf{k}$-algebra (resp $\mathbf{k}$-module) isomorphism follows from Proposition-Definition 3.1.16 (resp. Proposition-Definition 3.1.26). One can prove, for any $(w, m) \in \widehat{\mathcal{W}}_N^{\mathrm{B}} \times \widehat{\mathcal{M}}_N^{\mathrm{B}}$, that

$$\mathrm{iso}^{\mathcal{M},\iota}(wm) = \mathrm{iso}^{\mathcal{W},\iota}(w)\mathrm{iso}^{\mathcal{M},\iota}(m)$$

using the argument of (i) and Proposition-Definition 3.1.16.

□

3.2 Comparison isomorphisms

Throughout this section, $\iota : G \to \mathbb{C}^\times$ is a group embedding.

3.2.1 Algebra comparison isomorphisms

Definition 3.2.1. For $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle)$, we denote the composed topological $\mathbf{k}$-algebra isomorphism

$$^{\Gamma}\mathrm{comp}^{\mathcal{V},(1),\iota}_{(\lambda,\Psi)} := {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ \mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}_N^{\mathrm{B}} \to \widehat{\mathcal{V}}_G^{\mathrm{DR}} \tag{3.37}$$

Lemma 3.2.2. *For* $(\lambda, \Psi), (\nu, \Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle)$*, we have*

$$^{\Gamma}\mathrm{comp}^{\mathcal{V},(1),\iota}_{(\lambda,\Psi)\circledast(\nu,\Phi)} = {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{comp}^{\mathcal{V},(1),\iota}_{(\nu,\Phi)}.$$

Proof. We have

$$\begin{aligned}
{}^{\Gamma}\mathrm{comp}^{\mathcal{V},(1),\iota}_{(\lambda,\Psi)\circledast(\nu,\Phi)} &= {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} \circ \mathrm{iso}^{\mathcal{V},\iota} \\
&= {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\nu,\Phi)} \circ \mathrm{iso}^{\mathcal{V},\iota} = {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{comp}^{\mathcal{V},(1),\iota}_{(\nu,\Phi)},
\end{aligned}$$

where the second equality follows from Proposition 2.2.13.(i). □

Proposition-Definition 3.2.3. *For $(\lambda,\Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the topological $\mathbf{k}$-algebra isomorphism ${}^{\Gamma}\mathrm{comp}^{\mathcal{W},(1),\iota}_{(\lambda,\Psi)} := {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)} \circ \mathrm{iso}^{\mathcal{W},\iota} : \widehat{\mathcal{W}}^{\mathrm{B}}_N \to \widehat{\mathcal{W}}^{\mathrm{DR}}_N$ is such that the following diagram*

$$\begin{array}{ccc}
\widehat{\mathcal{W}}^{\mathrm{B}}_N & \xrightarrow{{}^{\Gamma}\mathrm{comp}^{\mathcal{W},(1),\iota}_{(\lambda,\Psi)}} & \widehat{\mathcal{W}}^{\mathrm{DR}}_N \\
\big\downarrow & & \big\downarrow \\
\widehat{\mathcal{V}}^{\mathrm{B}}_N & \xrightarrow[{}^{\Gamma}\mathrm{comp}^{\mathcal{V},(1),\iota}_{(\lambda,\Psi)}]{} & \widehat{\mathcal{V}}^{\mathrm{DR}}_N
\end{array}$$

commutes.

Proof. It follows from Proposition-Definitions 2.2.12 and 3.1.16. □

Lemma 3.2.4. *For $(\lambda,\Psi),(\nu,\Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have*

$${}^{\Gamma}\mathrm{comp}^{\mathcal{W},(1),\iota}_{(\lambda,\Psi)\circledast(\nu,\Phi)} = {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{comp}^{\mathcal{W},(1),\iota}_{(\nu,\Phi)}.$$

Proof. It follows from Lemma 3.2.2. □

3.2.2 Module comparison isomorphisms

Definition 3.2.5. For $(\lambda,\Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we denote the composed topological $\mathbf{k}$-module isomorphism

$${}^{\Gamma}\mathrm{comp}^{\mathcal{V},(10),\iota}_{(\lambda,\Psi)} := {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ \mathrm{iso}^{\mathcal{V},\iota} : \widehat{\mathcal{V}}^{\mathrm{B}}_N \to \widehat{\mathcal{V}}^{\mathrm{DR}}_G. \tag{3.38}$$

Lemma 3.2.6. *For $(\lambda,\Psi),(\nu,\Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have*

$${}^{\Gamma}\mathrm{comp}^{\mathcal{V},(10),\iota}_{(\lambda,\Psi)\circledast(\nu,\Phi)} = {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{comp}^{\mathcal{V},(10),\iota}_{(\nu,\Phi)}. \tag{3.39}$$

Proof. We have

$$\begin{aligned}
{}^{\Gamma}\mathrm{comp}^{\mathcal{V},(10),\iota}_{(\lambda,\Psi)\circledast(\nu,\Phi)} &= {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)\circledast(\nu,\Phi)} \circ \mathrm{iso}^{\mathcal{V},\iota} \\
&= {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(10)}_{(\nu,\Phi)} \circ \mathrm{iso}^{\mathcal{V},\iota} = {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{comp}^{\mathcal{V},(10),\iota}_{(\nu,\Phi)},
\end{aligned}$$

where the second equality follows from Proposition 2.2.15. □

Proposition-Definition 3.2.7. *Let* $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. *The composition* ${}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)} := {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ \mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}^{\mathrm{B}}_{N} \to \widehat{\mathcal{M}}^{\mathrm{DR}}_{G}$ *is a topological* $\mathbf{k}$*-module isomorphism such that the following diagram*

$$\begin{array}{ccc}
\widehat{\mathcal{V}}^{\mathrm{B}}_{N} & \xrightarrow{{}^{\Gamma}\mathrm{comp}^{\mathcal{V},(10),\iota}_{(\lambda,\Psi)}} & \widehat{\mathcal{V}}^{\mathrm{DR}}_{G} \\
{\scriptstyle \widehat{-\cdot 1_{\mathrm{B}}}}\downarrow & & \downarrow{\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}} \\
\widehat{\mathcal{M}}^{\mathrm{B}}_{N} & \xrightarrow[{}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}]{} & \widehat{\mathcal{M}}^{\mathrm{DR}}_{G}
\end{array}$$

commutes.

Proof. It follows from Proposition-Definitions 2.2.16 and 3.1.26. ☐

Lemma 3.2.8. *For* $(\lambda, \Psi), (\nu, \Phi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, *we have*

$${}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)\circledast(\nu,\Phi)} = {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\nu,\Phi)}. \tag{3.40}$$

Proof. We have

$$\begin{aligned}
{}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)\circledast(\nu,\Phi)} \circ \widehat{-\cdot 1_{\mathrm{B}}} &= \widehat{-\cdot 1_{\mathrm{DR}}} \circ {}^{\Gamma}\mathrm{comp}^{\mathcal{V},(10),\iota}_{(\lambda,\Psi)\circledast(\nu,\Phi)} \\
&= \widehat{-\cdot 1_{\mathrm{DR}}} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(10)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{comp}^{\mathcal{V},(10),\iota}_{(\nu,\Phi)} \\
&= {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ \widehat{-\cdot 1_{\mathrm{DR}}} \circ {}^{\Gamma}\mathrm{comp}^{\mathcal{V},(10),\iota}_{(\nu,\Phi)} \\
&= {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ {}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\nu,\Phi)} \circ \widehat{-\cdot 1_{\mathrm{B}}},
\end{aligned}$$

where the first and last equalities come from Proposition-Definition 3.2.7, the second one from Lemma 3.2.6 and the third one from Proposition-Definition 2.2.16. Finally, thanks to the surjectivity of $\widehat{-\cdot 1_{\mathrm{B}}}$, the result follows. ☐

Proposition 3.2.9. *For any* $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, *the following pairs are isomorphisms in the category* $\mathbf{k}$-alg-mod*:*

(i) $\left({}^{\Gamma}\mathrm{comp}^{\mathcal{V},(1),\iota}_{(\lambda,\Psi)}, {}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}\right) : (\widehat{\mathcal{V}}^{\mathrm{B}}_{N}, \widehat{\mathcal{M}}^{\mathrm{B}}_{N}) \to (\widehat{\mathcal{V}}^{\mathrm{DR}}_{G}, \widehat{\mathcal{M}}^{\mathrm{DR}}_{G})$.

(ii) $\left({}^{\Gamma}\mathrm{comp}^{\mathcal{W},(1),\iota}_{(\lambda,\Psi)}, {}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}\right) : (\widehat{\mathcal{W}}^{\mathrm{B}}_{N}, \widehat{\mathcal{M}}^{\mathrm{B}}_{N}) \to (\widehat{\mathcal{W}}^{\mathrm{DR}}_{G}, \widehat{\mathcal{M}}^{\mathrm{DR}}_{G})$.

Proof.

(i) It follows from Proposition 2.2.18.(i) and the fact that $\mathrm{iso}^{\mathcal{V},\iota}$ is an algbera isomorphism.

(ii) It follows from Propositions 2.2.18.(ii) and 3.1.27.(i).

(iii) It follows from Propositions 2.2.18.(iii) and 3.1.27.(ii).

☐

3.3 Betti Coproducts

Throughout this section, let us denote

$$\mathrm{Emb}(G) := \{\iota : G \to \mathbb{C}^\times \mid \iota \text{ is a group embedding}\}. \tag{3.41}$$

3.3.1 Actions of the group $\mathrm{Aut}(G)$

Lemma 3.3.1. *The group* $\mathrm{Aut}(G)$ *acts freely and transitively on* $\mathrm{Emb}(G)$ *by*

$$\phi \cdot \iota := \iota \circ \phi^{-1},$$

for $\phi \in \mathrm{Aut}(G)$ *and* $\iota \in \mathrm{Emb}(G)$*.*

Proof. One knows that for any $\iota \in \mathrm{Emb}(G)$, $\iota(G) = \mu_N$. That gives rise to a group isomorphism $\widetilde{\iota} : G \to \mu_N(\mathbb{C})$. Therefore, for any $\iota_1, \iota_2 \in \mathrm{Emb}(G)$ there is a unique group automorphism $\phi = \widetilde{\iota_2}^{-1} \circ \widetilde{\iota_1}$ of G such that $\iota_1 \circ \phi^{-1} = \iota_2$. □

Proposition-Definition 3.3.2.

(i) The group $\mathrm{Aut}(G)$ *acts on* $\widehat{\mathcal{V}}_G^{\mathrm{DR}}$ *by* $\mathbf{k}$*-algebra automorphisms, the automorphism* $a_\phi^{\mathcal{V}}$ *corresponding to* $\phi \in \mathrm{Aut}(G)$ *being given by*

$$e_0 \mapsto e_0, \quad e_1 \mapsto e_1 \quad \textit{and } g \mapsto \phi(g) \quad \textit{for } g \in G.$$

(ii) The group $\mathrm{Aut}(G)$ *acts on* $\mathbf{k}\langle\langle X\rangle\rangle$ *by* $\mathbf{k}$*-algebra automorphisms, the automorphism* b_ϕ *corresponding to* $\phi \in \mathrm{Aut}(G)$ *being given by*

$$x_0 \mapsto x_0, \quad x_g \mapsto x_{\phi(g)} \textit{ for } g \in G.$$

Proof. Immediate. □

Lemma 3.3.3. *For any* $\phi \in \mathrm{Aut}(G)$*, the following diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{b_\phi} & \mathbf{k}\langle\langle X\rangle\rangle \\ {\scriptstyle \beta\circ(-\otimes 1)}\downarrow & & \downarrow{\scriptstyle \beta\circ(-\otimes 1)} \\ \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow[a_\phi^{\mathcal{V}}]{} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \end{array} \tag{3.42}$$

commutes.

Proof. Immediate. □

Proposition-Definition 3.3.4.

(i) The action $\phi \mapsto a_\phi^{\mathcal{V}}$ *restricts to an action* $a^{\mathcal{W}} : \mathrm{Aut}(G) \to \mathrm{Aut}_{\mathbf{k}\text{-Hopf}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})$*.*

(ii) The action $\phi \mapsto a_\phi^{\mathcal{V}}$ *induces an action* $a^{\mathcal{M}} : \mathrm{Aut}(G) \to \mathrm{Aut}_{\mathbf{k}\text{-coalg}}(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$*.*

Proof.

(i) Let $\phi \in \mathrm{Aut}(G)$. For $(n, g) \in \mathbb{N}^* \times G$ we have

$$a_\phi^{\mathcal{V}}(z_{n,g}) = a_\phi^{\mathcal{V}}(-e_0^{n-1} g e_1) = -e_0^{n-1}\phi(g) e_1 = z_{n,\phi(g)}.$$

Since the algebra $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ is freely generated by the family $(z_{n,g})_{(n,g)\in\mathbb{N}^*\times G}$, it follows that $a_\phi^{\mathcal{V}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}}) \subset \widehat{\mathcal{W}}_G^{\mathrm{DR}}$. Similarly, $(a_\phi^{\mathcal{V}})^{-1}(\widehat{\mathcal{W}}_G^{\mathrm{DR}}) \subset \widehat{\mathcal{W}}_G^{\mathrm{DR}}$. Hence, $a_\phi^{\mathcal{V}}(\widehat{\mathcal{W}}_G^{\mathrm{DR}}) = \widehat{\mathcal{W}}_G^{\mathrm{DR}}$. This implies that $a_\phi^{\mathcal{V}}$ restricts to a $\mathbf{k}$-algebra automorphism of $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ which we denote $a_\phi^{\mathcal{W}}$. Let us show that the following diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{a_\phi^{\mathcal{W}}} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ {\scriptstyle \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}}\downarrow & & \downarrow{\scriptstyle \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}} \\ (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\otimes 2} & \xrightarrow[(a_\phi^{\mathcal{W}})^{\otimes 2}]{} & (\widehat{\mathcal{W}}_G^{\mathrm{DR}})^{\otimes 2} \end{array}$$

commutes. Indeed, for $(n, g) \in \mathbb{N}^* \times G$ we have

$$\begin{aligned} \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \circ a_\phi^{\mathcal{W}}(z_{n,g}) &= \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(z_{n,\phi(g)}) \\ &= z_{n,\phi(g)} \otimes 1 + 1 \otimes z_{n,\phi(g)} + \sum_{\substack{k=1\\ h\in G}}^{n-1} z_{k,h} \otimes z_{n-k,\phi(g)h^{-1}} \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} (a_\phi^{\mathcal{W}})^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}(z_{n,g}) &= (a_\phi^{\mathcal{W}})^{\otimes 2}\left(z_{n,g} \otimes 1 + 1 \otimes z_{n,g} + \sum_{\substack{k=1\\ h\in G}}^{n-1} z_{k,h} \otimes z_{n-k,gh^{-1}} \right) \\ &= z_{n,\phi(g)} \otimes 1 + 1 \otimes z_{n,\phi(g)} + \sum_{\substack{k=1\\ h\in G}}^{n-1} z_{k,\phi(h)} \otimes z_{n-k,\phi(g)\phi(h^{-1})}. \end{aligned}$$

Applying the change of variable $h' = \phi(h)$, we obtain the same quantity as (3.43).

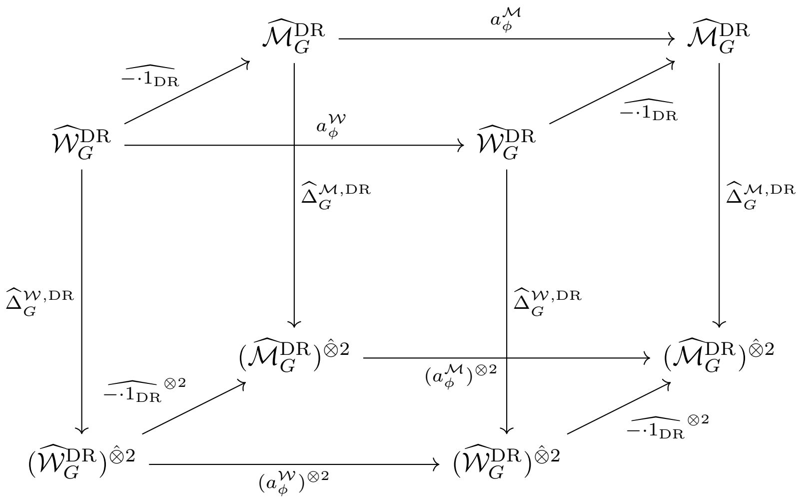
(ii) Let $\phi \in \mathrm{Aut}(G)$. One checks that $a_\phi^{\mathcal{V}}$ preserves the submodule $\widehat{\mathcal{V}}_G^{\mathrm{DR}} e_0 + \sum_{g\in G} \widehat{\mathcal{V}}_G^{\mathrm{DR}}(g - 1)$. It follows that there is a unique $\mathbf{k}$-module automorphism $a_\phi^{\mathcal{M}}$ of $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ such that the following diagram

$$\begin{array}{ccc} \widehat{\mathcal{V}}_G^{\mathrm{DR}} & \xrightarrow{a_\phi^{\mathcal{V}}} & \widehat{\mathcal{V}}_G^{\mathrm{DR}} \\ {\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}}\downarrow & & \downarrow{\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}} \\ \widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow[a_\phi^{\mathcal{M}}]{} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array}$$

commutes. Combined with (i), we obtain the following commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow{a_\phi^{\mathcal{W}}} & \widehat{\mathcal{W}}_G^{\mathrm{DR}} \\ {\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{-\cdot 1_{\mathrm{DR}}}} \\ \widehat{\mathcal{M}}_G^{\mathrm{DR}} & \xrightarrow[a_\phi^{\mathcal{M}}]{} & \widehat{\mathcal{M}}_G^{\mathrm{DR}} \end{array} \tag{3.44}$$

We then have the following cube



The left and the right faces are exactly the same square, which is commutative thanks to Proposition-Definition 2.1.27. The upper side commutes thanks to Diagram (3.44) and the lower side is the tensor square of the upper side so is commutative. Finally, (i) gives us the commutativity of the front side. This collection of commutativities together with the surjectivity of $\widehat{- \cdot 1_{\mathrm{DR}}}$ implies that the back side of the cube commutes, which proves that $a_\phi^{\mathcal{M}}$ is a coalgebra automorphism of $(\widehat{\mathcal{M}}_G^{\mathrm{DR}}, \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})$.

□

Proposition 3.3.5. *Let* $\phi \in \mathrm{Aut}(G)$.

(i) The pair $(a_\phi^{\mathcal{V}}, a_\phi^{\mathcal{M}})$ *is an automorphism in the category* $\mathbf{k}$-alg-mod.

(ii) The pair $(a_\phi^{\mathcal{W}}, a_\phi^{\mathcal{M}})$ *is an automorphism in the category* $\mathbf{k}$-HAMC.

Proof.

(i) Let $(v, m) \in \widehat{\mathcal{V}}_G^{\mathrm{DR}} \times \widehat{\mathcal{M}}_G^{\mathrm{DR}}$. Since $\widehat{- \cdot 1_{\mathrm{DR}}} : \widehat{\mathcal{V}}_G^{\mathrm{DR}} \to \widehat{\mathcal{M}}_G^{\mathrm{DR}}$ is surjective, there exists $v' \in \widehat{\mathcal{V}}_G^{\mathrm{DR}}$ such that $m = v' \cdot 1_{\mathrm{DR}}$. We have

$$\begin{aligned} a_\phi^{\mathcal{M}}(vm) &= a_\phi^{\mathcal{M}}(vv' \cdot 1_{\mathrm{DR}}) = a_\phi^{\mathcal{V}}(vv') \cdot 1_{\mathrm{DR}} \\ &= a_\phi^{\mathcal{V}}(v)\, a_\phi^{\mathcal{V}}(v') \cdot 1_{\mathrm{DR}} = a_\phi^{\mathcal{V}}(v)\, a_\phi^{\mathcal{M}}(m), \end{aligned}$$

where the second and fourth equalities come from Proposition-Definition 3.3.4. (ii).

(ii) It follows from (i) and from Proposition-Definition 3.3.4.

□

Lemma 3.3.6. *Let $\phi \in \mathrm{Aut}(G)$. The map b_ϕ is a Hopf algebra automorphism of $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$.*

Proof. It follows from the equality $\widehat{\Delta}\circ b_\phi = b_\phi^{\otimes 2}\circ\widehat{\Delta}$ which can be checked on generators since all the involved morphisms are algebra morphisms. □

Lemma 3.3.7. *Let $\phi \in \mathrm{Aut}(G)$. We have $b_\phi \circ \mathbf{q} = \mathbf{q} \circ b_\phi$.*

Proof. The family $\left(x_0^{n_1}x_{g_1}x_0^{n_2}x_{g_2}\cdots x_0^{n_r}x_{g_r}x_0^{n_{r+1}}\right)_{\substack{r,n_1,\dots,n_{r+1}\in\mathbb{N}\\ g_1,\dots,g_r\in G}}$ is a basis of the $\mathbf{k}$-module $\mathbf{k}\langle\langle X\rangle\rangle$. It is, therefore, enough to check the identity on this basis. We have

$$\begin{aligned}
&b_\phi\circ\mathbf{q}\left(x_0^{n_1}x_{g_1}x_0^{n_2}x_{g_2}\cdots x_0^{n_r}x_{g_r}x_0^{n_{r+1}}\right) = b_\phi\left(x_0^{n_1-1}x_{g_1}x_0^{n_2-1}x_{g_2g_1^{-1}}\cdots x_0^{n_r-1}x_{g_rg_{r-1}^{-1}}x_0^{n_{r+1}-1}\right)\\
&= x_0^{n_1-1}x_{\phi(g_1)}x_0^{n_2-1}x_{\phi(g_2)\phi(g_1)^{-1}}\cdots x_0^{n_r-1}x_{\phi(g_r)\phi(g_{r-1})^{-1}}x_0^{n_{r+1}-1}\\
&= \mathbf{q}\left(x_0^{n_1-1}x_{\phi(g_1)}x_0^{n_2-1}x_{\phi(g_2)}\cdots x_0^{n_r-1}x_{\phi(g_r)}x_0^{n_{r+1}-1}\right)\\
&= \mathbf{q}\circ b_\phi\left(x_0^{n_1}x_{g_1}x_0^{n_2}x_{g_2}\cdots x_0^{n_r}x_{g_r}x_0^{n_{r+1}}\right).
\end{aligned}$$

□

Proposition-Definition 3.3.8.

(i) The action $\phi\mapsto b_\phi$ restricts to an action $b^Y : \mathrm{Aut}(G)\to\mathrm{Aut}_{\mathbf{k}\text{-Hopf}}(\mathbf{k}\langle\langle Y\rangle\rangle, \widehat{\Delta}_\star^{\mathrm{alg}})$.

(ii) The action $\phi\mapsto b_\phi$ induces an action $\overline{b} : \mathrm{Aut}(G)\to\mathrm{Aut}_{\mathbf{k}\text{-coalg}}(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \widehat{\Delta}_\star^{\mathrm{mod}})$.

Proof.

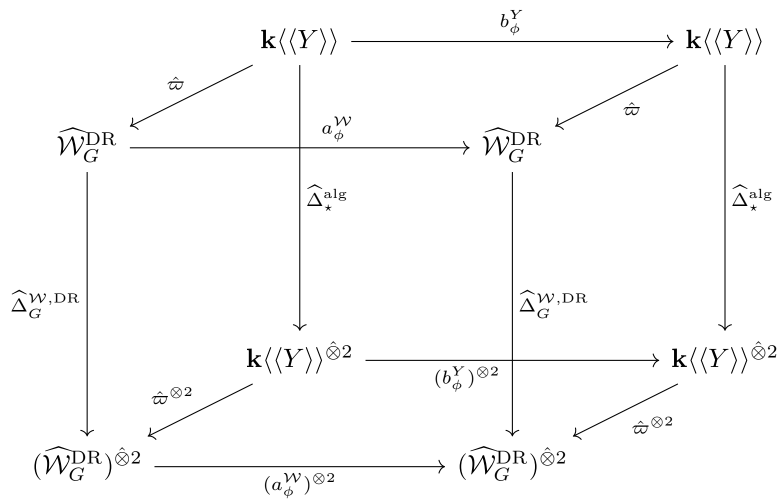
(i) Let $\phi\in\mathrm{Aut}(G)$. For $(n,g)\in\mathbb{N}^*\times G$ we have

$$b_\phi(y_{n,g}) = b_\phi(x_0^{n-1}x_g) = x_0^{n-1}x_{\phi(g)} = y_{n,\phi(g)}.$$

Since the algebra $\mathbf{k}\langle\langle Y\rangle\rangle$ is freely generated by the family $(y_{n,g})_{(n,g)\in\mathbb{N}^*\times G}$, it follows that $b_\phi(\mathbf{k}\langle\langle Y\rangle\rangle)\subset\mathbf{k}\langle\langle Y\rangle\rangle$. Similarly, $(b_\phi)^{-1}(\mathbf{k}\langle\langle Y\rangle\rangle)\subset\mathbf{k}\langle\langle Y\rangle\rangle$. Hence, $b_\phi(\mathbf{k}\langle\langle Y\rangle\rangle)=\mathbf{k}\langle\langle Y\rangle\rangle$. This implies that b_ϕ restricts to a $\mathbf{k}$-algebra automorphism of $\mathbf{k}\langle\langle Y\rangle\rangle$ which we denote b_ϕ^Y. Moreover, thanks Proposition-Definition 3.3.4. (i), this also implies that the following diagram

$$\begin{array}{ccc}
\mathbf{k}\langle\langle Y\rangle\rangle & \xrightarrow{\;b_\phi^Y\;} & \mathbf{k}\langle\langle Y\rangle\rangle\\
{\scriptstyle\hat{\varpi}}\downarrow & & \downarrow{\scriptstyle\hat{\varpi}}\\
\widehat{\mathcal{W}}_G^{\mathrm{DR}} & \xrightarrow[\;a_\phi^{\mathcal{W}}\;]{} & \widehat{\mathcal{W}}_G^{\mathrm{DR}}
\end{array} \tag{3.45}$$

commutes. Next, let us consider the following cube



The left and the right faces are exactly the same square, which is commutative thanks to Corollary 2.1.17. The upper side commutes thanks to Diagram (3.45) and the lower side is the tensor square of the upper side so is commutative. Finally, Proposition-Definition 3.3.4.(i) gives us the commutativity of the front side. This collection of commutativities together with the surjectivity of $\hat{\varpi}$ implies that the back side of the cube commutes, which proves that b_ϕ^Y is a Hopf algebra automorphism of $(\mathbf{k}\langle\langle Y\rangle\rangle, \widehat{\Delta}_\star^{\mathrm{alg}})$.

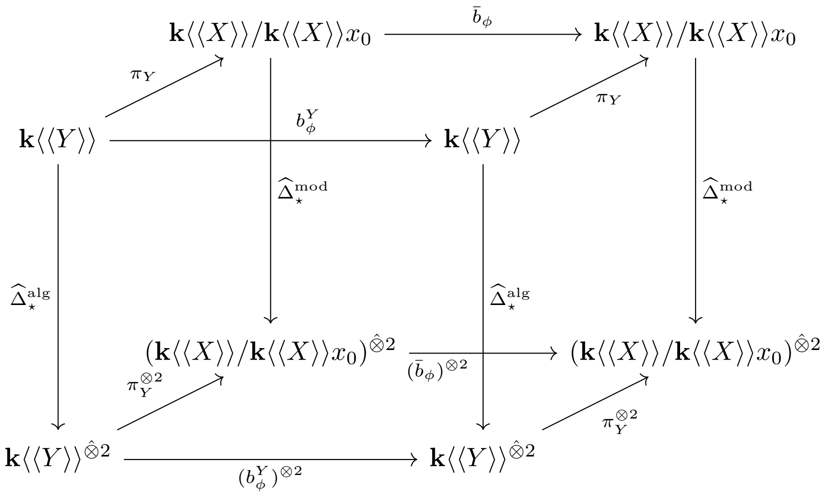
(ii) Let $\phi \in \mathrm{Aut}(G)$. One checks that b_ϕ preserves the submodule $\mathbf{k}\langle\langle X\rangle\rangle x_0$. It follows that there is a unique $\mathbf{k}$-module automorphism $\bar{b}_\phi$ of $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ such that the following diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\ b_\phi\ } & \mathbf{k}\langle\langle X\rangle\rangle \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\ \bar{b}_\phi\ ]{} & \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \end{array} \tag{3.46}$$

commutes. Combined with (i), we obtain the following commutative diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle Y\rangle\rangle & \xrightarrow{\ b_\phi^Y\ } & \mathbf{k}\langle\langle Y\rangle\rangle \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\ \bar{b}_\phi\ ]{} & \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \end{array} \tag{3.47}$$

Next, let us consider the following cube



The left and the right faces are exactly the same square, which is commutative thanks to Proposition-Definition 1.1.16. The upper side commutes thanks to Diagram (3.47) and the lower side is the tensor square of the upper side so is commutative. Finally, (i) gives us the commutativity of the front side. This collection of commutativities together with the surjectivity of π_Y implies that the back side of the cube commutes, which proves that $\bar{b}_\phi$ is a coalgebra automorphism of $(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \widehat{\Delta}_\star^{\mathrm{mod}})$.

□

Proposition 3.3.9. *Let $\phi \in \mathrm{Aut}(G)$. For $\lambda \in \mathbf{k}$ and $\iota \in \mathrm{Emb}(G)$, the automorphism b_ϕ of $\mathbf{k}\langle\langle X\rangle\rangle$ restricts to a bijection $\mathsf{DMR}_\lambda^\iota(\mathbf{k}) \to \mathsf{DMR}_\lambda^{\iota\circ\phi^{-1}}(\mathbf{k})$.*

Proof. It follows from Lemma 3.3.6 that b_ϕ restricts to group automorphism of $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. Next, since $b_\phi(x_0) = x_0$ and $b_\phi(x_1) = x_1$, conditions (i) and (iii) of Definition 1.3.14 are immediate. Moreover, since $g_{\iota\circ\phi^{-1}} = \phi(g_\iota)$, we have for $\Psi \in \mathsf{DMR}_\lambda^\iota$,

$$\left(b_\phi(\Psi)|x_{g_{\iota\circ\phi^{-1}}} - x_{g^{-1}_{\iota\circ\phi^{-1}}}\right) = \left(b_\phi(\Psi)|x_{\phi(g_\iota)} - x_{\phi(g_\iota^{-1})}\right) = \left(b_\phi(\Psi)|b_\phi(x_{g_\iota} - x_{g_\iota^{-1}})\right)$$
$$= \left(\Psi|x_{g_\iota} - x_{g_\iota^{-1}}\right) = \frac{|G|-2}{2}\lambda.$$

Then Identity (iv) of Definition 1.3.14 follows. One checks Identity (v) in a similar way. Finally, we have for $\Psi \in \mathsf{DMR}_\lambda^\iota$,

$$\begin{aligned}(b_\phi(\Psi))_\star &= \overline{\mathbf{q}} \circ \pi_Y\left(\Gamma^{-1}_{b_\phi(\Psi)}(x_1)b_\phi(\Psi)\right) = \overline{\mathbf{q}} \circ \pi_Y\left(\Gamma^{-1}_{\Psi}(x_1)b_\phi(\Psi)\right) \\ &= \pi_Y \circ \mathbf{q}\left(\Gamma^{-1}_{\Psi}(x_1)b_\phi(\Psi)\right) = \pi_Y\left(\Gamma^{-1}_{\Psi}(x_1)\mathbf{q}(b_\phi(\Psi))\right) \\ &= \pi_Y\left(\Gamma^{-1}_{\Psi}(x_1)b_\phi(\mathbf{q}(\Psi))\right) = \pi_Y\left(b_\phi(\Gamma^{-1}_{\Psi}(x_1)\mathbf{q}(\Psi))\right) \\ &= \bar{b}_\phi\left(\pi_Y(\Gamma^{-1}_{\Psi}(x_1)\mathbf{q}(\Psi))\right) = \bar{b}_\phi\left(\pi_Y \circ \mathbf{q}(\Gamma^{-1}_{\Psi}(x_1)\Psi)\right) \\ &= \bar{b}_\phi\left(\overline{\mathbf{q}} \circ \pi_Y(\Gamma^{-1}_{\Psi}(x_1)\Psi)\right) = \bar{b}_\phi(\Psi_\star),\end{aligned}$$

where the second and sixth equalities come from the fact that $\phi(x_1) = x_1$, the third and ninth ones from the commutativity of Diagram (1.5), the fourth and eight ones from the fact that $\mathbf{q} \circ \ell_{\Gamma_\Psi^{-1}(x_1)} = \ell_{\Gamma_\Psi^{-1}(x_1)} \circ \mathbf{q}$, the fifth one from Lemma 3.3.7 and the seventh one from the commutativity of Diagram (3.46). Therefore, thanks to Proposition-Definition 3.3.8.(ii) we obtain that

$$\begin{aligned}\widehat{\Delta}_\star^{\mathrm{mod}}\left((b_\phi(\Psi))_\star\right) = \widehat{\Delta}_\star^{\mathrm{mod}}\left(\bar{b}_\phi(\Psi_\star)\right) &= (\bar{b}_\phi)^{\otimes 2}\left(\widehat{\Delta}_\star^{\mathrm{mod}}(\Psi_\star)\right) \\ &= (\bar{b}_\phi)^{\otimes 2}\left(\Psi_\star^{\otimes 2}\right) = \bar{b}_\phi(\Psi_\star) \otimes \bar{b}_\phi(\Psi_\star),\end{aligned}$$

which implies that condition (ii) of Definition 1.3.14 is verified. This proves that b_ϕ restricts to a map $\mathsf{DMR}_\lambda^\iota \to \mathsf{DMR}_\lambda^{\iota\circ\phi^{-1}}$. Following the same steps, one can prove that b_ϕ^{-1} restricts to a map $\mathsf{DMR}_\lambda^{\iota\circ\phi^{-1}} \to \mathsf{DMR}_\lambda^\iota$ thus showing that the map $b_{\phi|\mathsf{DMR}_\lambda^\iota}$: $\mathsf{DMR}_\lambda^\iota \to \mathsf{DMR}_\lambda^{\iota\circ\phi^{-1}}$ is a bijection. □

3.3.2 The coproducts $\widehat{\Delta}_\iota^{\mathcal{W},\mathrm{DR}}$ and $\widehat{\Delta}_\iota^{\mathcal{M},\mathrm{DR}}$

Lemma 3.3.10. *Let $\phi \in \mathrm{Aut}(G)$ and $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. We have*

$$^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,b_\phi(\Psi))} = a_\phi^{\mathcal{V}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ (a_\phi^{\mathcal{V}})^{-1}. \tag{3.48}$$

Proof. Since all morphisms of Identity (3.48) are algebra morphisms, it is enough to check the equality on generators. We have

$$\begin{aligned}&{}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,b_\phi(\Psi))}(e_0) = \mathrm{Ad}_{\Gamma^{-1}_{b_\phi(\Psi)}(-e_1)\hat{\beta}(b_\phi(\Psi)\otimes 1)}(\lambda e_0) = \mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)a_\phi^{\mathcal{V}}(\hat{\beta}(\Psi\otimes 1))}(\lambda e_0) \\ &= a_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)\hat{\beta}(\Psi\otimes 1)}\big(\lambda (a_\phi^{\mathcal{V}})^{-1}(e_0)\big)\right) = a_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)\hat{\beta}(\Psi\otimes 1)}(\lambda e_0)\right) \\ &= a_\phi^{\mathcal{V}}\left({}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(e_0)\right) = a_\phi^{\mathcal{V}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ (a_\phi^{\mathcal{V}})^{-1}(e_0),\end{aligned}$$

where the second equality comes from the identity $\Gamma_{b_\phi(\Psi)}(-e_1) = \Gamma_\Psi(-e_1)$ and from the commutativity of Diagram (3.42) and the third one from the fact that $a_\phi^{\mathcal{V}}$ is an algebra isomorphism and from the equality $a_\phi^{\mathcal{V}}(\Gamma_\Psi(-e_1)) = \Gamma_\Psi(-e_1)$. Next,

$$\begin{aligned}{}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,b_\phi(\Psi))}(e_1) &= \mathrm{Ad}_{\Gamma^{-1}_{b_\phi(\Psi)}(-e_1)}(\lambda e_1) = \mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)}(\lambda e_1) \\ &= a_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)}\big(\lambda(a_\phi^{\mathcal{V}})^{-1}(e_1)\big)\right) = a_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)}(\lambda e_1)\right) \\ &= a_\phi^{\mathcal{V}}\left({}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}(e_1)\right) = a_\phi^{\mathcal{V}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ (a_\phi^{\mathcal{V}})^{-1}(e_1),\end{aligned}$$

where the second equality comes from the identity $\Gamma_{b_\phi(\Psi)}(-e_1) = \Gamma_\Psi(-e_1)$ and the third one from the fact that $a_\phi^{\mathcal{V}}$ is an algebra isomorphism and from the equality $a_\phi^{\mathcal{V}}(\Gamma_\Psi(-e_1)) = \Gamma_\Psi(-e_1)$. Finally, for $g \in G$,

$$\begin{aligned}&{}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,b_\phi(\Psi))}(g) = \mathrm{Ad}_{\Gamma^{-1}_{b_\phi(\Psi)}(-e_1)\hat{\beta}(b_\phi(\Psi)\otimes 1)}(g) = \mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)a_\phi^{\mathcal{V}}(\hat{\beta}(\Psi\otimes 1))}(g) \\ &= a_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)\hat{\beta}(\Psi\otimes 1)}\big((a_\phi^{\mathcal{V}})^{-1}(g)\big)\right) = a_\phi^{\mathcal{V}}\left(\mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)\hat{\beta}(\Psi\otimes 1)}\big(\phi^{-1}(g)\big)\right) \\ &= a_\phi^{\mathcal{V}}\left({}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)}\big(\phi^{-1}(g)\big)\right) = a_\phi^{\mathcal{V}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{V},(1)}_{(\lambda,\Psi)} \circ (a_\phi^{\mathcal{V}})^{-1}(g),\end{aligned}$$

where the second equality comes from the identity $\Gamma_{b_\phi(\Psi)}(-e_1) = \Gamma_\Psi(-e_1)$ and from the commutativity of Diagram (3.42), the third one from the fact that $a_\phi^{\mathcal{V}}$ is an algebra isomorphism and from the equality $a_\phi^{\mathcal{V}}(\Gamma_\Psi(-e_1)) = \Gamma_\Psi(-e_1)$ and the fourth and sixth ones from the fact that $(a_\phi^{\mathcal{V}})^{-1}(g) = \phi^{-1}(g)$. □

Corollary 3.3.11. *Let* $\phi \in \mathrm{Aut}(G)$ *and* $(\lambda, \Psi) \in \mathbf{k}^\times \times \mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle)$*. We have*

(i) ${}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda, b_\phi(\Psi))} = a_\phi^{\mathcal{W}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)} \circ (a_\phi^{\mathcal{W}})^{-1}$.

(ii) ${}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda, b_\phi(\Psi))} = a_\phi^{\mathcal{M}} \circ {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} \circ (a_\phi^{\mathcal{M}})^{-1}$.

Proof.

(i) It follows from (3.48) thanks to Proposition-Definitions 2.2.12 and 3.3.4. (i).

(ii) It follows from (3.48) thanks to Proposition-Definitions 2.2.16 and 3.3.4. (ii).

□

Proposition-Definition 3.3.12. *Let* $\iota \in \mathrm{Emb}(G)$ *and* $(\lambda, \Psi) \in \mathsf{DMR}^\iota(\mathbf{k})$.

(i) The composition

$$\left(\left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)} : \widehat{\mathcal{W}}^{\mathrm{DR}}_G \to (\widehat{\mathcal{W}}^{\mathrm{DR}}_G)^{\otimes 2}$$

does not depend of the choice of $(\lambda, \Psi) \in \mathsf{DMR}^\iota(\mathbf{k})$*. We denote it* $\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_\iota$.

(ii) The composition

$$\left(\left({}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_G \circ {}^\Gamma\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)} : \widehat{\mathcal{M}}^{\mathrm{DR}}_G \to (\widehat{\mathcal{M}}^{\mathrm{DR}}_G)^{\otimes 2}$$

does not depend of the choice of $(\lambda, \Psi) \in \mathsf{DMR}^\iota(\mathbf{k})$*. We denote it* $\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_\iota$.

Proof. Let $(\lambda, \Psi), (\nu, \Phi) \in \mathsf{DMR}^\iota(\mathbf{k})$. Thanks to Proposition 1.3.20, there exists a unique $(\mu, \Lambda) \in \mathbf{k}^\times \ltimes \mathsf{DMR}^G_0(\mathbf{k})$ such that $(\nu, \Phi) = (\mu, \Lambda) \circledast (\lambda, \Psi)$.

(i) We have

$$\begin{aligned}
&\left(\left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\nu,\Phi)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\nu,\Phi)} \\
&= \left(\left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\mu,\Lambda)\circledast(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\nu,\Phi)} \\
&= \left(\left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1} \circ \left(\left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\mu,\Lambda)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\mu,\Lambda)} \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)} \\
&= \left(\left({}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_G \circ {}^\Gamma\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)},
\end{aligned}$$

where the second equality comes from Proposition 2.2.13. (ii) and the third one from the inclusion

$$\mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k}) \subset \mathbf{k}^\times \ltimes \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) = \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})(\mathbf{k}) \subset \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}})(\mathbf{k}),$$

thanks to Proposition 1.3.18 and Theorems 2.3.9 and 2.3.5.

(ii) We have

$$\begin{aligned}
&\left(\left({}^\Gamma\mathrm{aut}_{(\nu,\Phi)}^{\mathcal{M},(10)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} \circ {}^\Gamma\mathrm{aut}_{(\nu,\Phi)}^{\mathcal{M},(10)} \\
&= \left(\left({}^\Gamma\mathrm{aut}_{(\mu,\Lambda)\circledast(\lambda,\Psi)}^{\mathcal{M},(10)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} \circ {}^\Gamma\mathrm{aut}_{(\nu,\Phi)}^{\mathcal{M},(10)} \\
&= \left(\left({}^\Gamma\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{M},(10)}\right)^{\otimes 2}\right)^{-1} \circ \left(\left({}^\Gamma\mathrm{aut}_{(\mu,\Lambda)}^{\mathcal{M},(10)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} \circ {}^\Gamma\mathrm{aut}_{(\mu,\Lambda)}^{\mathcal{M},(10)} \circ {}^\Gamma\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{M},(10)} \\
&= \left(\left({}^\Gamma\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{M},(10)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}} \circ {}^\Gamma\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{M},(10)},
\end{aligned}$$

where the second equality comes from Proposition 2.2.17 and the third one from the inclusion

$$\mathbf{k}^\times \ltimes \mathsf{DMR}_0^G(\mathbf{k}) \subset \mathbf{k}^\times \ltimes \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) = \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}})(\mathbf{k}),$$

thanks to Proposition 1.3.18 and Theorem 2.3.9.

□

Proposition 3.3.13. *Let $\iota \in \mathrm{Emb}(G)$ and $\phi \in \mathrm{Aut}(G)$. We have*

(i) $\widehat{\Delta}_{\iota\circ\phi^{-1}}^{\mathcal{W},\mathrm{DR}} = (a_\phi^{\mathcal{W}})^{\otimes 2} \circ \widehat{\Delta}_\iota^{\mathcal{W},\mathrm{DR}} \circ (a_\phi^{\mathcal{W}})^{-1}$.

(ii) $\widehat{\Delta}_{\iota\circ\phi^{-1}}^{\mathcal{M},\mathrm{DR}} = (a_\phi^{\mathcal{M}})^{\otimes 2} \circ \widehat{\Delta}_\iota^{\mathcal{M},\mathrm{DR}} \circ (a_\phi^{\mathcal{M}})^{-1}$.

Proof.

(i) Let $(\lambda,\Psi) \in \mathsf{DMR}^{\iota\circ\phi^{-1}}(\mathbf{k})$. From Proposition-Definition 3.3.12. (i), we have

$$\widehat{\Delta}_{\iota\circ\phi^{-1}}^{\mathcal{W},\mathrm{DR}} = \left(\left({}^\Gamma\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{W},(1)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \circ {}^\Gamma\mathrm{aut}_{(\lambda,\Psi)}^{\mathcal{W},(1)}.$$

Next, thanks to Proposition 3.3.8. 3.3.9, there exists a unique $\Phi \in \mathsf{DMR}_\lambda^\iota(\mathbf{k})$ such that $b_\phi(\Phi) = \Psi$. Therefore

$$\begin{aligned}
\widehat{\Delta}_{\iota\circ\phi^{-1}}^{\mathcal{W},\mathrm{DR}} &= \left(\left({}^\Gamma\mathrm{aut}_{(\lambda,b_\phi(\Phi))}^{\mathcal{W},(1)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \circ {}^\Gamma\mathrm{aut}_{(\lambda,b_\phi(\Phi))}^{\mathcal{W},(1)} \\
&= (a_\phi^{\mathcal{W}})^{\otimes 2} \circ \left(\left({}^\Gamma\mathrm{aut}_{(\lambda,\Phi)}^{\mathcal{W},(1)}\right)^{\otimes 2}\right)^{-1} \circ ((a_\phi^{\mathcal{W}})^{\otimes 2})^{-1} \circ \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \circ a_\phi^{\mathcal{W}} \circ {}^\Gamma\mathrm{aut}_{(\lambda,\Phi)}^{\mathcal{W},(1)} \circ (a_\phi^{\mathcal{W}})^{-1} \\
&= (a_\phi^{\mathcal{W}})^{\otimes 2} \circ \left(\left({}^\Gamma\mathrm{aut}_{(\lambda,\Phi)}^{\mathcal{W},(1)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}} \circ {}^\Gamma\mathrm{aut}_{(\lambda,\Phi)}^{\mathcal{W},(1)} \circ (a_\phi^{\mathcal{W}})^{-1} \\
&= (a_\phi^{\mathcal{W}})^{\otimes 2} \circ \widehat{\Delta}_\iota^{\mathcal{W},\mathrm{DR}} \circ (a_\phi^{\mathcal{W}})^{-1},
\end{aligned}$$

where the second equality comes from Corollary 3.3.11. (i), the third one from Proposition-Definition 3.3.4. (i) and the fourth one from Proposition-Definition 3.3.12. (i).

(ii) Let $(\lambda, \Psi) \in \mathsf{DMR}^{\iota\circ\phi^{-1}}(\mathbf{k})$. From Proposition-Definition 3.3.12. (ii), we have

$$\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{\iota\circ\phi^{-1}} = \left(\left({}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}.$$

Next, thanks to Proposition 3.3.8. 3.3.9, there exists a unique $\Phi \in \mathsf{DMR}^{\iota}_{\lambda}(\mathbf{k})$ such that $b_\phi(\Phi) = \Psi$. Therefore

$$\begin{aligned}
\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{\iota\circ\phi^{-1}} &= \left(\left({}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,b_\phi(\Phi))}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,b_\phi(\Phi))} \\
&= (a^{\mathcal{M}}_\phi)^{\otimes 2} \circ \left(\left({}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Phi)}\right)^{\otimes 2}\right)^{-1} \circ ((a^{\mathcal{M}}_\phi)^{\otimes 2})^{-1} \circ \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G} \circ a^{\mathcal{M}}_\phi \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Phi)} \circ (a^{\mathcal{M}}_\phi)^{-1} \\
&= (a^{\mathcal{M}}_\phi)^{\otimes 2} \circ \left(\left({}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Phi)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Phi)} \circ (a^{\mathcal{M}}_\phi)^{-1} \\
&= (a^{\mathcal{M}}_\phi)^{\otimes 2} \circ \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{\iota} \circ (a^{\mathcal{M}}_\phi)^{-1}
\end{aligned}$$

where the second equality comes from Corollary 3.3.11. (ii), the third one from Proposition-Definition 3.3.4. (ii) and the fourth one from Proposition-Definition 3.3.12. (ii).

□

3.3.3 The coproducts $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_N$ and $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_N$

Lemma 3.3.14. *Let $\iota \in \mathrm{Emb}(G)$ and $\phi \in \mathrm{Aut}(G)$. We have*

$$\mathrm{iso}^{\mathcal{V},\iota\circ\phi^{-1}} = a^{\mathcal{V}}_\phi \circ \mathrm{iso}^{\mathcal{V},\iota}. \tag{3.49}$$

Proof. Let us show that these $\mathbf{k}$-algebra morphisms are equal on the generators:

$$\mathrm{iso}^{\mathcal{V},\iota\circ\phi^{-1}}(X_1) = \exp(e_1) = a^{\mathcal{V}}_\phi \circ \mathrm{iso}^{\mathcal{V},\iota}(X_1)$$

and

$$\begin{aligned}
\mathrm{iso}^{\mathcal{V},\iota\circ\phi^{-1}}(X_0) &= \exp\left(\frac{1}{N}e_0\right) g_{\iota\circ\phi^{-1}} = \exp\left(\frac{1}{N}e_0\right)\phi(g_\iota) = a^{\mathcal{V}}_\phi\left(\exp\left(\frac{1}{N}e_0\right)g_\iota\right) \\
&= a^{\mathcal{V}}_\phi \circ \mathrm{iso}^{\mathcal{V},\iota}(X_0)
\end{aligned}$$

□

Corollary 3.3.15. *Let $\iota \in \mathrm{Emb}(G)$ and $\phi \in \mathrm{Aut}(G)$. We have* *(i)* $\mathrm{iso}^{\mathcal{W},\iota\circ\phi^{-1}} = a_\phi^{\mathcal{W}} \circ \mathrm{iso}^{\mathcal{W},\iota}$. *(ii)* $\mathrm{iso}^{\mathcal{M},\iota\circ\phi^{-1}} = a_\phi^{\mathcal{M}} \circ \mathrm{iso}^{\mathcal{M},\iota}$.

Proof.

(i) It follows from (3.49) thanks to Proposition-Definitions 3.1.16 and 3.3.4. (i).

(ii) It follows from (3.49) thanks to Proposition-Definitions 3.1.26 and 3.3.4. (ii).

□

Proposition 3.3.16.

(i) The composition

$$\left(\left(\mathrm{iso}^{\mathcal{W},\iota}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_\iota^{\mathcal{W},\mathrm{DR}} \circ \mathrm{iso}^{\mathcal{W},\iota} : \widehat{\mathcal{W}}_N^{\mathrm{B}} \to (\widehat{\mathcal{W}}_N^{\mathrm{B}})^{\otimes 2}$$

is independent of the choice of $\iota \in \mathrm{Emb}(G)$.

(ii) The composition

$$\left(\left(\mathrm{iso}^{\mathcal{M},\iota}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_\iota^{\mathcal{M},\mathrm{DR}} \circ \mathrm{iso}^{\mathcal{M},\iota} : \widehat{\mathcal{M}}_N^{\mathrm{B}} \to (\widehat{\mathcal{M}}_N^{\mathrm{B}})^{\otimes 2}$$

is independent of the choice of $\iota \in \mathrm{Emb}(G)$.

Proof. Let $\iota_1, \iota_2 \in \mathrm{Emb}(G)$. Thanks to Lemma 3.3.1, there exists a unique $\phi \in \mathrm{Aut}(G)$ such that $\iota_2 = \iota_1 \circ \phi^{-1}$.

(i) We have

$$\begin{aligned}
&\left(\left(\mathrm{iso}^{\mathcal{W},\iota_2}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_{\iota_2}^{\mathcal{W},\mathrm{DR}} \circ \mathrm{iso}^{\mathcal{W},\iota_2} \\
&= \left(\left(\mathrm{iso}^{\mathcal{W},\iota_1\circ\phi^{-1}}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_{\iota_1\circ\phi^{-1}}^{\mathcal{W},\mathrm{DR}} \circ \mathrm{iso}^{\mathcal{W},\iota_1\circ\phi^{-1}} \\
&= \left(\left(a_\phi^{\mathcal{W}} \circ \mathrm{iso}^{\mathcal{W},\iota_1}\right)^{\otimes 2}\right)^{-1} \circ (a_\phi^{\mathcal{W}})^{\otimes 2} \circ \widehat{\Delta}_{\iota_1}^{\mathcal{W},\mathrm{DR}} \circ (a_\phi^{\mathcal{W}})^{-1} \circ a_\phi^{\mathcal{W}} \circ \mathrm{iso}^{\mathcal{W},\iota_1} \\
&= \left(\left(\mathrm{iso}^{\mathcal{W},\iota_1}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_{\iota_1}^{\mathcal{W},\mathrm{DR}} \circ \mathrm{iso}^{\mathcal{W},\iota_1},
\end{aligned}$$

where the second equality comes from Corollary 3.3.15. (i) and from Proposition 3.3.13. (i).

(ii) We have

$$\begin{aligned}
&\left(\left(\mathrm{iso}^{\mathcal{M},\iota_2}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_{\iota_2}^{\mathcal{M},\mathrm{DR}} \circ \mathrm{iso}^{\mathcal{M},\iota_2} \\
&= \left(\left(\mathrm{iso}^{\mathcal{M},\iota_1\circ\phi^{-1}}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_{\iota_1\circ\phi^{-1}}^{\mathcal{M},\mathrm{DR}} \circ \mathrm{iso}^{\mathcal{M},\iota_1\circ\phi^{-1}} \\
&= \left(\left(a_\phi^{\mathcal{M}} \circ \mathrm{iso}^{\mathcal{M},\iota_1}\right)^{\otimes 2}\right)^{-1} \circ (a_\phi^{\mathcal{M}})^{\otimes 2} \circ \widehat{\Delta}_{\iota_1}^{\mathcal{M},\mathrm{DR}} \circ (a_\phi^{\mathcal{M}})^{-1} \circ a_\phi^{\mathcal{M}} \circ \mathrm{iso}^{\mathcal{M},\iota_1} \\
&= \left(\left(\mathrm{iso}^{\mathcal{M},\iota_1}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}_{\iota_1}^{\mathcal{M},\mathrm{DR}} \circ \mathrm{iso}^{\mathcal{M},\iota_1},
\end{aligned}$$

where the second equality comes from Corollary 3.3.15. (ii) and from Proposition 3.3.13. (ii).

□

Theorem 3.3.17.

(i) The composition

$$\left(\left({}^{\Gamma}\mathrm{comp}^{\mathcal{W},(1),\iota}_{(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1}\circ\widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_{G}\circ{}^{\Gamma}\mathrm{comp}^{\mathcal{W},(1),\iota}_{(\lambda,\Psi)}:\widehat{\mathcal{W}}^{\mathrm{B}}_{N}\to(\widehat{\mathcal{W}}^{\mathrm{B}}_{N})^{\otimes 2}$$

is independent of the choice of $(\lambda,\Psi)\in\mathsf{DMR}^{\iota}_{\times}(\mathbf{k})$ *and* $\iota\in\mathrm{Emb}(G)$. *We denote it* $\widehat{\Delta}^{\mathcal{W},\mathrm{B}}_{N}$.

(ii) The composition

$$\left(\left({}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1}\circ\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G}\circ{}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}:\widehat{\mathcal{M}}^{\mathrm{B}}_{N}\to(\widehat{\mathcal{M}}^{\mathrm{B}}_{N})^{\otimes 2}$$

is independent of the choice of $(\lambda,\Psi)\in\mathsf{DMR}^{\iota}_{\times}(\mathbf{k})$ *and* $\iota\in\mathrm{Emb}(G)$. *We denote it* $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_{N}$.

Proof.

(i) Since ${}^{\Gamma}\mathrm{comp}^{\mathcal{W},(1),\iota}_{(\lambda,\Psi)}={}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{(\lambda,\Psi)}\circ\mathrm{iso}^{\mathcal{W},\iota}$, the statement follows from Proposition-Definition 3.3.12. (i) and Proposition 3.3.16. (i).

(ii) Since ${}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}={}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{(\lambda,\Psi)}\circ\mathrm{iso}^{\mathcal{M},\iota}$, the statement follows from Proposition-Definition 3.3.12. (ii) and Proposition 3.3.16. (ii).

□

Proposition 3.3.18. *We have* $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_{N}(1_{\mathrm{B}})=1_{\mathrm{B}}^{\otimes 2}$.

Proof. Thanks to Theorem 3.3.17. (ii), in order to compute $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_{N}(1_{\mathrm{B}})$, let us consider $\iota\in\mathrm{Emb}(G)$ and $(\lambda,\Psi)\in\mathsf{DMR}^{\iota}_{\times}(\mathbf{k})$. We have

$${}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}(1_{\mathrm{B}})={}^{\Gamma}\mathrm{comp}^{\mathcal{V},(10),\iota}_{(\lambda,\Psi)}(1)\cdot 1_{\mathrm{DR}}=\Gamma_{\Psi}^{-1}(-e_1)\hat{\beta}(\Psi\otimes 1)\cdot 1_{\mathrm{DR}}. \tag{3.50}$$

Next,

$$\begin{aligned}
&\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G}\circ{}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}(1_{\mathrm{B}})=\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G}\left(\Gamma_{\Psi}^{-1}(-e_1)\hat{\beta}(\Psi\otimes 1)\cdot 1_{\mathrm{DR}}\right)\\
&=\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G}\left(\hat{\beta}(\Gamma_{\Psi}^{-1}(x_1)\Psi\otimes 1)\cdot 1_{\mathrm{DR}}\right)=\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G}\left(\hat{\kappa}\circ\pi_Y(\Gamma_{\Psi}^{-1}(x_1)\Psi)\right)\\
&=\widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G}\circ\hat{\kappa}\circ\overline{\mathbf{q}}^{-1}\circ\overline{\mathbf{q}}\circ\pi_Y(\Gamma_{\Psi}^{-1}(x_1)\Psi)=(\hat{\kappa}\circ\overline{\mathbf{q}}^{-1})^{\otimes 2}\circ\widehat{\Delta}^{\mathrm{mod}}_{\star}(\Psi_{\star})\\
&=(\hat{\kappa}\circ\overline{\mathbf{q}}^{-1})^{\otimes 2}(\Psi_{\star}^{\otimes 2})=\left(\hat{\kappa}\circ\overline{\mathbf{q}}^{-1}\circ\overline{\mathbf{q}}\circ\pi_Y(\Gamma_{\Psi}^{-1}(x_1)\Psi)\right)^{\otimes 2}\\
&=\left(\hat{\kappa}\circ\pi_Y(\Gamma_{\Psi}^{-1}(x_1)\Psi)\right)^{\otimes 2}=\left(\hat{\beta}(\Gamma_{\Psi}^{-1}(x_1)\Psi\otimes 1)\cdot 1_{\mathrm{DR}}\right)^{\otimes 2}\\
&=\left(\Gamma_{\Psi}^{-1}(-e_1)\hat{\beta}(\Psi\otimes 1)\cdot 1_{\mathrm{DR}}\right)^{\otimes 2}
\end{aligned} \tag{3.51}$$

where the third and the eighth equalities come from Proposition-Definition 2.1.25. (ii), the fifth one from Proposition 2.1.29 and the sixth one from the fact that $\Psi \in \mathsf{DMR}^{\iota}_{\lambda}(\mathbf{k})$. Finally,

$$\begin{aligned}\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_{N}(1_{\mathrm{B}}) &= \left(\left({}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1} \circ \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G} \circ {}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}(1_{\mathrm{B}}) \\ &= \left(\left({}^{\Gamma}\mathrm{comp}^{\mathcal{M},(10),\iota}_{(\lambda,\Psi)}\right)^{\otimes 2}\right)^{-1} \left(\left(\Gamma_{\Psi}^{-1}(-e_1)\hat{\beta}(\Psi \otimes 1) \cdot 1_{\mathrm{DR}}\right)^{\otimes 2}\right) = 1_{\mathrm{B}}^{\otimes 2},\end{aligned}$$

where the first equality is given by definition of $\widehat{\Delta}^{\mathcal{M},\mathrm{B}}_{N}$, the second one is given by computation (3.51) and the last one comes from computation (3.50). □

Theorem 3.3.19.

(i) The pair $(\widehat{\mathcal{W}}^{\mathrm{B}}_{N}, \widehat{\Delta}^{\mathcal{W},\mathrm{B}}_{N})$ is an object in the category **k**-Hopf*.*

(ii) The pair $(\widehat{\mathcal{M}}^{\mathrm{B}}_{N}, \widehat{\Delta}^{\mathcal{M},\mathrm{B}}_{N})$ is an object in the category **k**-coalg*.*

(iii) The pair $\left((\widehat{\mathcal{W}}^{\mathrm{B}}_{N}, \widehat{\Delta}^{\mathcal{W},\mathrm{B}}_{N}), (\widehat{\mathcal{M}}^{\mathrm{B}}_{N}, \widehat{\Delta}^{\mathcal{M},\mathrm{B}}_{N})\right)$ is an object in the category **k**-HAMC*.*

Proof.

(i) It follows from Theorem 3.3.17. (i) and the fact that the pair $(\widehat{\mathcal{W}}^{\mathrm{DR}}_{G}, \widehat{\Delta}^{\mathcal{W},\mathrm{DR}}_{G})$ is an object in the category **k**-Hopf.

(ii) It follows from Theorem 3.3.17. (ii) and the fact that the pair $(\widehat{\mathcal{M}}^{\mathrm{DR}}_{G}, \widehat{\Delta}^{\mathcal{M},\mathrm{DR}}_{G})$ is an object in the category **k**-coalg.

(iii) If follows from Theorem 3.3.17, Proposition 3.2.9. (ii) and Corollary 2.1.28.

□

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Les travaux de Racinet ont permis d'associer à tout groupe cyclique fini G et à toute injection de groupes $\iota : G \to \mathbb{C}^\times$, un $\mathbb{Q}$-schéma DMR^ι décrivant les relations de double mélange et régularisation entre valeurs polylogarithmes multiples aux racines $N^{\text{ième}}$ de l'unité, avec N l'ordre de G. Il a aussi exhibé un $\mathbb{Q}$-schéma en groupes DMR_0^G que les travaux d'Enriquez et Furusho ont permis d'identifier au stabilisateur d'un coproduit intervenant au sein du formalisme de Racinet pour l'action du groupe $\mathcal{G}$ des éléments diagonaux d'une algèbre de Hopf de séries non commutatives munie du produit de Magnus tordu. On reformule les constructions de Racinet en termes de produit croisé. Le coproduit de Racinet peut alors être identifié avec un coproduit $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$ défini sur un module $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ sur une algèbre $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$, munie de son propre coproduit $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$. On construit des actions compatibles d'un produit semi-direct faisant intervenir $\mathcal{G}$ sur $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ et $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$. On aboutit alors à un schéma en groupes stabilisateur contenant DMR_0^G que l'on exprime au sein du formalisme de Racinet.

Par ailleurs, pour $G = \{1\}$, Enriquez et Furusho montrent qu'un sous-schéma $\mathsf{DMR}_\times^\iota$ de DMR^ι est un torseur d'isomorphismes mettant en relation des objets « de Rham » avec des objets « Betti ». Dans la seconde partie de ce travail, on définit les ingrédients principaux pour une généralisation de ce résultat à tout groupe cyclique fini G : on exhibe un module $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ sur une algèbre $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ (N étant l'ordre de G) et on démontre l'existence de deux coproduits compatibles $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ et $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}$ sur $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ et $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ respectivement tels que $\mathsf{DMR}_\times^\iota$ est contenu dans le torseur des isomorphismes reliant $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ (resp. $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}$) à $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ (resp. $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$).

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STRUCTURES ALGEBRIQUES ASSOCIEES AUX RELATIONS DE DOUBLE MELANGE ENTRE VALEURS POLYLOGARITHMES MULTIPES AUX RACINES DE L'UNITE

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Résumé

Les travaux de Racinet ont permis d'associer à tout groupe cyclique fini G et à toute injection de groupes $\iota : G \to \mathbb{C}^\times$ un $\mathbb{Q}$-schéma DMR^ι décrivant les relations de double mélange et régularisation entre valeurs polylogarithmes multiples aux racines $N^{\text{ièmes}}$ de l'unité avec N l'ordre de G. Il a aussi exhibé un schéma en groupes DMR_0^G que les travaux d'Enriquez et Furusho ont permis d'identifier au stabilisateur d'un coproduit intervenant au sein du formalisme de Racinet pour l'action du groupe $\mathcal{G}$ des éléments diagonaux d'une algèbre de Hopf de séries non commutatives munie du produit de Magnus tordu. On reformule les constructions de Racinet en termes de produit croisé. Le coproduit de Racinet peut alors être identifié avec un coproduit $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$ défini sur un module $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ sur une algèbre $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$ munie de son propre coproduit $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$. On construit des actions compatibles d'un produit semi-direct faisant intervenir $\mathcal{G}$ sur $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ et $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$. On aboutit alors à un schéma en groupes stabilisateur contenant DMR_0^G que l'on exprime au sein du formalisme de Racinet. Par ailleurs, pour $G = 1$, Enriquez et Furusho montrent qu'un sous-schéma $\mathrm{DMR}_\times^\iota$ de DMR^ι est un torseur d'isomorphismes mettant en relation des objets de Rham avec des objets Betti. Dans la seconde partie de ce travail, on définit les ingrédients principaux pour une généralisation de ce résultat à tout groupe cyclique fini G : on exhibe un module $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ sur une algèbre $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ (N étant l'odre de G) et on démontre l'existence de deux coproduits $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ et $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}$ sur $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ et $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ respectivement tels que $\mathrm{DMR}_\times^\iota$ est contenu dans le torseur des isomorphismes reliant $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ (resp. $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}$) à $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ (resp. $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$).

Mots clés : Polylogarithmes multiples ; Double mélange ; Coproduit harmonique ; Torseur ; Algèbre de Hopf ; Algèbre produit croisé ; Algèbre filtrée ; de Rham ; Betti.

Résumé en anglais

Racinet attached to each finite cyclic group G and group embedding $\iota : G \to \mathbb{C}^\times$ a $\mathbb{Q}$-scheme DMR^ι which describes the double shuffle and regularization relations between multiple polylogarithm values at N^{th} roots of unity where N is the order of G. He also exhibited a group scheme DMR_0^G, which Enriquez and Furusho identified with the stabilizer of a coproduct element arising in Racinet's formalism with respect to the action of the group $\mathcal{G}$ of grouplike elements of a non-commutative series Hopf algebra, equipped with the twisted Magnus product. We reformulate Racinet's construction in terms of crossed products. Racinet's coproduct can then be identified with a coproduct $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$ defined on a module $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$ over an algebra $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$, which is equipped with its own coproduct $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$. We show that there are compatible group actions of a semidirect product involving $\mathcal{G}$ on $\widehat{\mathcal{M}}_G^{\mathrm{DR}}$and $\widehat{\mathcal{W}}_G^{\mathrm{DR}}$. This yields an explicit stabilizer group scheme containing DMR_0^G, which we also express in the Racinet formalism. Furthermore, for $G = \{1\}$, Enriquez and Furusho showed that a subscheme $\mathrm{DMR}_\times^\iota$ of DMR^ι is a torsor of isomorphisms relating de Rham and Betti objects. In the second part of this work, we define the main ingredients for a generalization of this result to any finite cyclic group G: we exhibit a module $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ over an algebra $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ (where N is the order of G) and we prove the existence of a two compatible coproducts $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ and $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}$ on $\widehat{\mathcal{W}}_N^{\mathrm{B}}$ and $\widehat{\mathcal{M}}_N^{\mathrm{B}}$ respectively such that $\mathrm{DMR}_\times^\iota$ is contained in the torsor of isomorphisms relating $\widehat{\Delta}_N^{\mathcal{W},\mathrm{B}}$ (resp. $\widehat{\Delta}_N^{\mathcal{M},\mathrm{B}}$) to $\widehat{\Delta}_G^{\mathcal{W},\mathrm{DR}}$ (resp. $\widehat{\Delta}_G^{\mathcal{M},\mathrm{DR}}$).

Keywords : Multiple polylogarithm values; Double shuffle ; Harmonic coproduct; Torsor; Hopf algebra; Crossed product algebra; Filtered algebra; de Rham; Betti.