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Antoine FELTZ

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Foncteurs polynomiaux sur les catégories FId

THÈSE dirigée par :

Mme. VESPA Christine	Professeure, université d'Aix-Marseille
M. CHAPOTON Frédéric	Directeur de recherche, université de Strasbourg

RAPPORTEURS :

M. TOUZE Antoine	Professeur, université de Lille
M. ARONE Gregory	Full professor, université de Stockholm

AUTRES MEMBRES DU JURY :

M. DOTSENKO Vladimir	Professeur, université de Strasbourg
M. DJAMENT Aurélien	Chargé de recherche, CNRS

Thèse

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Foncteurs polynomiaux sur les catégories $\mathbf{FI}_d$

Soutenue le 9 février 2024
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Christine Vespa, directrice de thèse
Frédéric Chapoton, co-directeur de thèse
Antoine Touzé, rapporteur
Gregory Arone, rapporteur
Vladimir Dotsenko, examinateur
Aurélien Djament, examinateur

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Foncteurs polynomiaux sur les catégories $\mathbf{FI}_{\mathrm{d}}$

Polynomial functors on the categories $\mathbf{FI}_{\mathrm{d}}$

Feltz Antoine

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L'ennui dans ce monde, c'est que les idiots sont sûrs d'eux et les gens sensés pleins de doutes.

Bertrand Russell

Résumé

Les foncteurs sur la catégorie **FI** des injections entre ensembles finis apparaissent naturellement dans différents contextes. Ils interviennent notamment dans l'étude de la stabilité de familles de représentations des groupes symétriques initiée par Church, Ellenberg et Farb. Djament et Vespa ont montré que la stabilité des représentations s'exprime en termes de polynomialité forte de foncteurs sur **FI**. Ils introduisent également une notion de polynomialité faible mieux adaptée aux phénomènes stables. Ces propriétés polynomiales sont des moyens de mesurer la complexité d'un foncteur et peuvent être considérés comme un analogue des fonctions polynomiales. Il existe des généralisations de la catégorie **FI**, notées $\mathbf{FI}_d$, où l'on rajoute un choix de couleurs parmi d possibles sur le complémentaire de l'image des injections. Les foncteurs sur ces catégories interviennent notamment dans les travaux de Sam et Snowden sur les modules sur les algèbres commutatives tordues libres et dans ceux de Ramos sur l'homologie des espaces de configuration de graphes. Les catégories $\mathbf{FI}_d$ n'ayant pas d'objet initial pour $d > 1$, elles sortent du cadre considéré par Djament et Vespa.

Dans cette thèse on introduit différentes notions (forte et faibles) de foncteurs polynomiaux sur les catégories $\mathbf{FI}_d$ et on étudie leur comportement. On adapte aussi la définition classique de foncteurs polynomiaux (basée sur les effets croisés) au cadre de $\mathbf{FI}_d$, et on montre que les deux définitions obtenues coïncident. Les foncteurs polynomiaux sur $\mathbf{FI}_d$ s'avèrent plus difficiles à étudier que sur **FI**. Par exemple, les projectifs standards sont fortement polynomiaux sur **FI** et on montre que ce n'est plus le cas sur $\mathbf{FI}_d$ pour $d > 1$. On étudie alors différents quotients polynomiaux de ces foncteurs. On amorce également l'étude de la polynomialité des foncteurs considérés par Ramos en calculant explicitement les foncteurs associés aux graphes linéaires. Cependant, la notion forte de foncteurs polynomiaux manque de propriétés essentielles concernant les phénomènes stables. On introduit alors les foncteurs faiblement polynomiaux en considérant le quotient par une sous-catégorie afin de supprimer les foncteurs problématiques. Alors que les foncteurs faiblement polynomiaux de degré 0 sur **FI** sont les foncteurs constants, on donne une description de ceux sur $\mathbf{FI}_d$ qui forment une catégorie plus complexe. On en déduit que l'adaptation directe des méthodes utilisées par Djament et Vespa pour **FI** ne fonctionne pas.

Abstract

The functors over the category **FI** of finite sets and injections appear naturally in various contexts. They intervene especially in the study of representation stability for symmetric groups initiated by Church, Ellenberg and Farb. Djament and Vespa showed that the representation stability is expressed in terms of strong polynomial functors over **FI**. They also introduced a notion of weak polynomiality better suited to stable phenomena. These polynomial properties are ways of measuring the complexity of a functor and can be seen as an analogue of polynomial functions. There are generalizations of the category **FI**, denoted by $\mathbf{FI}_d$, where we add a choice of colours among d possible on the complement of the image of each injection. The functors over these categories intervene in Sam and Snowden's work on modules over the free twisted commutative algebras and in the work of Ramos on the homology of configuration spaces of graphs. Since the categories $\mathbf{FI}_d$ have no initial object for $d > 1$, they fall outside the framework considered by Djament and Vespa.

In this thesis we introduce different notions (strong and weak) of polynomial functors over the categories $\mathbf{FI}_d$ and we study their behaviour. We also adapt the classical definition of polynomial functors (based on cross effects) to the framework of $\mathbf{FI}_d$, and we show that the two definitions obtained coincide. The polynomial functors over $\mathbf{FI}_d$ turn out to be harder to study than over **FI**. For example, the standard projectives are strong polynomial over **FI** and we show that this is no longer the case over $\mathbf{FI}_d$ for $d > 1$. We then study different polynomial quotients of these functors. We also initiate the study of the polynomiality of the functors considered by Ramos by explicitly calculating the functors associated with linear graphs. However, the strong notion of polynomial functors lacks essential properties concerning stable phenomena. We then introduce the weak polynomial functors by considering the quotient by a subcategory in order to eliminate the problematic functors. While the weak polynomial functors of degree 0 over **FI** are the constant functors, we give a description of those over $\mathbf{FI}_d$ which form a more complex category. We deduce that a direct adaptation of the methods used by Djament and Vespa for **FI** does not work.

Contents

Notations

The following is a list of some notations used throughout this thesis:

$\mathbf{R}$	:	A fixed commutative ring
$\mathbb{K}$	:	A fixed field
$\mathbf{Set}$	:	The category of sets
$\mathbf{Ab}$	:	The category of abelian groups
$\mathbf{R}$-$\mathbf{Mod}$	:	The category of R-modules
$\mathcal{C}$-modules	:	The category of functors from a category $\mathcal{C}$ to the category $\mathbf{R}$-$\mathbf{Mod}$
$\mathbb{K}$-$\mathbf{Vect}$	:	The category of $\mathbb{K}$-vector spaces
$\mathbf{Fct}(\mathcal{A},\mathcal{B})$	:	The category of functors from $\mathcal{A}$ to $\mathcal{B}$ if $\mathcal{B}$ is a category and $\mathcal{A}$ a small category
F^*	:	The pre-composition by the functor F
F_*	:	The post-composition by the functor F
S_n	:	The symmetric group on n elements
$x \in \mathcal{A}$	:	An object x of the category $\mathcal{A}$

Introduction (Français)

La science est la croyance en l'ignorance des experts.

Richard Feynman

Des FI-modules aux $\mathbf{FI}_d$-modules

Les **FI**-modules sont les foncteurs de la catégorie **FI** des ensembles finis et des injections (également notée **I** dans [Sch08] et Θ dans [DV19]) vers la catégorie **R-Mod** des **R**-modules (pour **R** un anneau commutatif). Plus généralement, un $\mathcal{C}$-module est un foncteur d'une catégorie $\mathcal{C}$ vers la catégorie **R-Mod**. Les **FI**-modules ont été largement étudiés au cours de la dernière décennie par Church, Ellenberg, Farb, Nagpal, Reinhold et d'autres (voir par exemple [CEF15, CEFN14, CEF14, CE17, CF13, Chu12, CMNR18, Dja16, DV19]). La théorie des **FI**-modules a été introduite dans [CEF15] afin de transformer la notion complexe de stabilité de représentation en un résultat de finitude sur la suite de représentations des groupes symétriques considérée comme un objet unique. Une introduction détaillée à la théorie des **FI**-modules et à la stabilité de représentation peut être trouvée dans [Sam20] mais nous rappelons ici les principes de base. La notation **FI** a été introduite dans [CEF15] en tant qu'acronyme pour la catégorie des ensembles **F**inis (souvent représentés par leur cardinal dans le squelette) et des **I**njections. Un **FI**-module correspond à une famille de représentations linéaires des groupes symétriques avec des conditions de compatibilité données par des applications linéaires, ce qui peut être représenté par le diagramme suivant :

$$\begin{tikzcd}
\mathbf{FI} \arrow[d, "F"'] & 0 \arrow[r, hook] \arrow[loop above, "S_0"] & 1 \arrow[r, hook] \arrow[loop above, "S_1"] & 2 \arrow[r, hook] \arrow[loop above, "S_2"] & \dots \arrow[r, hook] & n \arrow[r, hook] \arrow[loop above, "S_n"] & \dots \\
\mathbf{R\text{-}Mod} & F(0) \arrow[r] \arrow[loop below, "F(S_0)"] & F(1) \arrow[r] \arrow[loop below, "F(S_1)"] & F(2) \arrow[r] \arrow[loop below, "F(S_2)"] & \dots \arrow[r] & F(n) \arrow[r] \arrow[loop below, "F(S_n)"] & \dots
\end{tikzcd}$$

Chaque flèche de ce diagramme représente en fait plusieurs flèches que nous pouvons construire par composition et via l'action des groupes symétriques. Un grand nombre d'exemples concrets de **FI**-modules sont présentés dans [CF13]. D'autres exemples intéressants de **FI**-modules de type fini sont donnés par la cohomologie des groupes de tresses pures dans [Wil18a] et des groupes appelés *pure string motion group* dans [Wil12].

Dans la littérature il existe plusieurs variantes (voir [Sam20] pour une liste détaillée) de la catégorie **FI** : les catégories $\mathbf{FI}_d$ que nous développons dans cette thèse, $\mathbf{FI}_G$ la catégorie des ensembles finis et des couples d'une injection et d'un choix d'un élément du groupe G

pour chaque élément à la source (voir [Ram17b]), $\mathbf{FS}_G$ la catégorie des ensembles finis et des G-surjections pour G un groupe (voir [SS17]), $\mathbf{FI}_W$ pour W certains groupes de Weyl dans [Wil12], **FIM** la catégorie des ensembles finis et des paires d'injection et de couplage parfait sur le complémentaire de l'image (voir [MW19]), ou une version symplectique (voir [Sam20]). Il existe également des variantes pour les représentations des groupes linéaires, comme $\mathbf{VI(R)}$ la catégorie des modules libres de rang fini et des applications linéaires injectives avec inverse à gauche, qui est présentée en détail dans [Wil18a]. Cette catégorie, et sa généralisation $\mathbf{VIC(R)}$ des modules libres de rang fini et des applications linéaires injectives avec un choix de supplémentaire de l'image, ont été introduites sous les noms $\mathbf{S(ab)}$ pour $\mathbf{R} = \mathbb{Z}$ dans [DV19] et $\mathbf{S(R)}$ dans [Dja16].

Dans cette thèse, nous nous concentrerons sur la catégorie $\mathbf{FI}_d$ pour d un entier non nul, introduite par Sam et Snowden dans [SS17], dans laquelle les objets sont toujours les ensembles finis et les morphismes sont les injections colorées. Nous étudions ici les $\mathbf{FI}_d$-modules et nous soulignons en particulier les différences avec les **FI**-modules. Même si nous étudions les foncteurs dont la catégorie but est une catégorie de modules pour plus de clarté, l'essentiel de ce travail reste vrai si nous remplaçons **R-Mod** par une catégorie de Grothendieck générale (voir [Gar01]). Nous récupérons en particulier les **FI**-modules puisque la catégorie $\mathbf{FI}_1$ est isomorphe à la catégorie **FI** (voir Section 2.1). La première différence majeure est que l'unité 0 est un objet initial dans $\mathbf{FI} \cong \mathbf{FI}_1$, mais pas dans $\mathbf{FI}_d$ pour $d > 1$. Nous montrons également dans la Section 2.7 que le foncteur oubli $\mathbf{FI}_d \to \mathbf{FI}$, qui relie les $\mathbf{FI}_d$-modules et les **FI**-modules, possède une famille d'adjoints $\Delta_c : \mathbf{FI} \to \mathbf{FI}_d$ qui ajoutent la couleur c à tous les morphismes de **FI**. Par précomposition, ils permettent de considérer un $\mathbf{FI}_d$-module comme un **FI**-module.

Pour toute catégorie $\mathcal{C}$, une famille d'exemples importants de foncteurs de $\mathcal{C}$ vers **R-Mod** sont les foncteurs projectifs standard. Ces foncteurs fondamentaux apparaissent pour les foncteurs entre les espaces vectoriels $\mathbb{F}_p$ dans [Kuh94], pour $\mathbf{FI}_d$ dans [SS17], et pour $d = 1$ dans [DV19, Dja16, Ves19], ou sous le nom de modules libres dans [CEF15, CEFN14, MW19] ou encore de foncteurs représentables dans [Wil18a]. Ils jouent le rôle des modules libres dans la théorie classique des modules. Nous pouvons déduire beaucoup d'informations sur les $\mathbf{FI}_d$-modules de la structure des foncteurs projectifs standards puisqu'ils forment une famille de générateurs projectifs de $\mathbf{FI}_d$-**Mod** (Proposition 2.2.5).

Les $\mathbf{FI}_d$-modules simples

La catégorie $\mathbf{FI}_d$ est une catégorie EI : *i.e.* une catégorie dont les endomorphismes sont des isomorphismes. Ces catégories et leurs représentations ont été introduites par Dieck dans [Die87] dans le contexte de la K-théorie algébrique, et plus récemment étudiées par Li dans [Li14], en particulier leur propriété de Koszul. Cette propriété nous donne déjà un résultat sur les $\mathbf{FI}_d$-modules simples, c'est-à-dire les $\mathbf{FI}_d$-modules qui n'ont pas de sous-foncteurs propres non nuls. Pour exprimer ce résultat, nous rappelons que les représentations irréductibles du groupe symétrique S_n sur un corps de caractéristique nulle sont indexées par les partitions λ de n. Nous désignons par M^λ la représentation irréductible associée à la partition λ de n, qui est définie comme l'idéal de l'anneau $\mathbb{K}[\mathrm{S}_n]$ engendré par un élément idempotent associé à la partition λ, appelé le symétriseur de Young. Par exemple, la représentation associée à la partition $\lambda = (n)$ est la représentation triviale, celle associée à $\lambda = (1^n)$ est la signature, et celle associée à $\lambda = (n-1, 1)$ est la représentation standard. Nous donnons ensuite dans la Proposition 2.4.3 la description suivante des $\mathbf{FI}_d$-modules simples :

Proposition. *Pour* $\mathbf{R}$ *un corps de caractéristique nulle, les objets simples de la catégorie* $\mathbf{FI}_d$*-**Mod** sont les foncteurs* $(M^\lambda)_k$ *qui envoient un objet* $n \in \mathbf{FI}_d$ *sur* M^λ *si* $n = k$ *et sur zéro sinon, pour* λ *une partition de* k.

Stabilité de représentation

Bien que la catégorie **FI** ait été étudiée dans différents contextes combinatoires, elle a été utilisée pour la première fois dans la cadre de la stabilité de représentation. Cette théorie a été introduite par Church et Farb dans [CF13] pour étudier certaines familles compatibles de représentations de groupes qui admettent une décomposition en irréductibles qui finit par devenir stable. Il s'agit d'une généralisation de la stabilité homologique classique dans le cas où les applications induites en homologie ne deviennent pas des isomorphismes. Une suite de représentations de groupes, tels que les groupes symétriques, est stable en ce sens lorsque les noms des représentations irréductibles (avec une manière appropriée de les indexer) qui apparaissent dans la décomposition finissent par se stabiliser, même si les espaces changent. Des exemples concrets de cette stabilisation sont donnés dans [Sam20] et dans [CF13]. En caractéristique nulle, les représentations irréductibles des groupes symétriques sont indexées par les partitions. Alors la stabilité de représentation pour ces groupes peut être résumée comme suit (voir [CEF15, CEFN14, Far14]) : une famille compatible $(V_n)_n$ de représentations est stable si nous obtenons la décomposition de la représentation V_{n+1} de S_{n+1} en ajoutant une case sur la ligne supérieure des diagrammes associés à la décomposition de la représentation V_n de S_n. Ce processus, ainsi que l'équivalence entre ces deux définitions, est décrit sur des exemples dans [CF13] et [Wil18a, Ex. XXXI].

La théorie des **FI**-modules a été introduite dans [CEF15] pour encoder ce phénomène en un unique objet : en effet, il est prouvé dans [Far14] que, si un **FI**-module est de type fini, alors la famille associée de représentations des groupes symétriques est stable. Notons que la réciproque est vraie pour les foncteurs à valeurs de type fini, et que la preuve est basée sur la propriété noethérienne des **FI**-modules et sur le fait que les familles associées aux générateurs projectifs $P_n^{\mathbf{FI}}$ sont stables comme expliqué dans [Wil18b]. Les exemples concrets de **FI**-modules introduits dans [CF13] et [Wil18b] ont d'abord été considérés comme des représentations stables des groupes symétriques et ont été compris comme étant des **FI**-modules de type fini après, par exemple dans [CEF15]. Un autre exemple intéressant de stabilité de représentation est donné par la cohomologie des *pure string motion group*. Il est traité en détail dans [Wil12] et illustré par un exemple. En pratique, il est généralement plus facile de prouver un résultat de finitude sur un objet que de prouver la stabilité d'une famille entière.

Les résultats centraux sur la stabilité de représentation sont résumés et présentés sur un exemple concret dans [Wil18a, Section 5]. Les principaux outils de ces résultats sont l'étude des représentations apparaissant dans les foncteurs projectifs standard, et les *polynômes des caractères* (voir [Far14, 4.2] pour une définition simple) : il est montré dans [CEF15] et [CMNR18] que les caractères d'un **FI**-module de type fini finissent par être égaux à un polynôme. En particulier, si F est un **FI**-module de type fini sur un corps, alors la dimension des espaces vectoriels $F(n)$ devient polynomiale. Ce résultat, comme beaucoup d'autres concernant les **FI**-modules, a été prouvé pour la première fois dans [CEF15] et dans [Sno13, Theorem 3.1] sur un corps de caractéristique nulle, et a été étendu dans [CEFN14] pour des anneaux plus généraux. De plus, Sam et Snowden ont montré dans [Sno13] et [SS16] que si un **FI**-module est de type fini alors sa série de Hilbert, codant la dimension de ses valeurs, est de la forme $p(t) + e^t q(t)$ où p et q sont des polynômes. Par exemple, les polynômes des caractères de [CEF15] peuvent être récupérés à partir de la fonction polynomiale p de cette série et la fonction polynomiale q peut être récupérée à partir de la cohomologie locale.

Cette théorie a été étendue dans [Ram17a] aux $\mathbf{FI}_d$-modules avec une notion généralisée de stabilité de représentation. Ramos obtient alors le résultat suivant : un $\mathbf{FI}_d$-module F est de type fini si et seulement si l'espace $F(n)$ est de dimension finie pour tout $n \in \mathbb{N}$ et, pour toute partition λ de poids $|\lambda|$ et toute suite d'entiers $n_1 \geq \cdots \geq n_d \geq |\lambda| + \lambda_1$, si $c_{\lambda,n_1,\ldots,n_d}$ désigne la multiplicité de la représentation irréductible associée à la partition $(n_1 - |\lambda|, \ldots, n_d - |\lambda|, \lambda_1, \ldots, \lambda_h)$, alors $c_{\lambda,n_1+l,\ldots,n_d+l}$ est indépendant de l pour l et n suffisamment grands. Ce théorème est une généralisation directe du théorème analogue de [CEF15, CEFN14] pour les $\mathbf{FI}$-modules. Moralement, le dernier point peut être interprété en disant que les représentations irréductibles associées à une partition d'au moins d lignes apparaissent avec une multiplicité qui devient stable dans un $\mathbf{FI}_d$-module de type fini. Ce théorème ne prédit pas le comportement des représentations irréductibles associées à des partitions plus petites, mais le Théorème B de [Ram17a] traite certains de ces cas. Depuis, Sam et Snowden ont défini une série de Hilbert "améliorée" qui encode plus d'informations sur la structure d'un $\mathbf{FI}_d$-module en tant que représentations des groupes symétriques et ils ont prouvé un résultat similaire à celui de la série de Hilbert "classique" ci-dessus pour cette série améliorée, pour $d = 1$ dans [SS16] et pour d général dans [SS17, SS18].

Les foncteurs fortement polynomiaux

Dans une catégorie de foncteurs il existe de très grands foncteurs, souvent incontrôlables, et la propriété polynomiale est un moyen de mesurer la complexité d'un foncteur. Ainsi, les foncteurs polynomiaux doivent être considérés comme un analogue des fonctions polynomiales pour les foncteurs, qui sont plus faciles à comprendre. La notion de foncteur polynomial remonte aux années 1950, lorsque Eilenberg et Mac Lane l'ont introduite dans [EM54] pour les foncteurs entre catégories de modules. Depuis, les foncteurs polynomiaux ont été étudiés pour un large éventail d'applications telles que leur connexion à la théorie des représentations ou à la cohomologie des groupes.

La définition originale d'Eilenberg et Mac Lane a été étendue pour différentes familles de catégories à la source, comme dans [HPV15] au cas où la source est une catégorie monoïdale dont l'unité est un objet nul. Une approche complémentaire dans la généralisation de ces foncteurs polynomiaux consiste à étudier les foncteurs d'une catégorie monoïdale vers une catégorie non abélienne telle que la catégorie des groupes (voir [BP99]). La définition d'Eilenberg et Mac Lane basée sur la notion d'effets croisés est équivalente à la définition basée sur l'endofoncteur différentiel utilisée par Kuhn dans [Kuh94] et Powell dans [Pow98]. Dans [DV19], les auteurs introduisent deux notions de foncteurs polynomiaux à partir d'une catégorie monoïdale symétrique $\mathcal{M}$ dont l'unité est un objet initial vers une catégorie abélienne : la généralisation naïve des foncteurs polynomiaux donne la notion de foncteurs fortement polynomiaux qui ont de mauvaises propriétés comme le fait de ne pas être stables par sous-objet. Cela conduit aux foncteurs faiblement polynomiaux définis en introduisant une catégorie quotient suivant la construction de Gabriel dans [Gab62, pages 366-372]. L'idée de cette catégorie quotient est d'inverser les morphismes dont le noyau et le conoyau sont dans la sous-catégorie en question. Les foncteurs fortement polynomiaux dans ce contexte sont définis en utilisant les endofoncteurs différentiels δ_k, pour $k \in \mathcal{M}$, généralisant celui de [Kuh94] et [Pow98]. Dans [DV19], Djament et Vespa ont également adapté la définition des effets croisés à leur cadre et ont montré que les foncteurs fortement polynomiaux sont égaux à ceux obtenus en utilisant ces effets croisés. La définition utilisant les endofoncteurs différentiels est mieux adaptée à l'étude des comportements stables et a l'avantage d'être récursive, c'est pourquoi nous choisissons de présenter et de généraliser ce point de vue pour les $\mathbf{FI}_d$-modules.

En particulier, la catégorie $\mathbf{FI}$ s'inscrit dans le cadre de Djament et Vespa et nous obtenons la définition suivante des $\mathbf{FI}$-modules fortement polynomiaux en utilisant uniquement l'endofoncteur différentiel δ_1 puisque $1 \in \mathbf{FI}$ est un générateur : le foncteur $F : \mathbf{FI} \to \mathbf{R}\text{-}\mathbf{Mod}$ est fortement polynomial de degré n si nous obtenons le foncteur nul en lui appliquant $n+1$ fois l'endofoncteur δ_1. Ceci est analogue aux polynômes habituels : une fonction $f : \mathbb{R} \to \mathbb{R}$ est polynomiale de degré n si sa $(n+1)$-ième dérivée est nulle. L'endofoncteur δ_1 qui joue le rôle de la dérivée est utilisé dans divers contextes : ceux de Kuhn et Powell sur les foncteurs des $\mathbb{F}_p$-espaces vectoriels vers les $\mathbb{F}_p$-espaces vectoriels ([Kuh94, Pow98]), dans la théorie de la stabilité de représentation ([CEF15, CEFN14, CE17, CMNR18]), dans la définition des foncteurs polynomiaux par Randal-Williams et Wahl dans [RWW17], dans la théorie des algèbres commutatives tordues ([SS12, SS16]) ou dans les travaux de Ramos ([Ram17b, LR18]). Les notions de foncteurs polynomiaux introduites dans [DV19] donnent une autre façon d'exprimer et de comprendre les résultats sur les $\mathbf{FI}$-modules. Par exemple, les foncteurs fortement polynomiaux avec des valeurs de type fini sont les $\mathbf{FI}$-modules de type fini. En utilisant [CEF15], nous déduisons que, sur un corps de caractéristique nulle, la dimension des espaces vectoriels associés à un $\mathbf{FI}$-module polynomial avec des valeurs de dimension finie devient polynomiale. Il existe de nombreux exemples de $\mathbf{FI}$-modules polynomiaux qui apparaissent dans différents contextes. En particulier, un grand nombre des $\mathbf{FI}$-modules présentés dans [CF13] sont fortement polynomiaux. La cohomologie des espaces de configuration sur une variété régulière donne un $\mathbf{FI}$-module fortement polynomial d'un intérêt particulier. Plusieurs $\mathbf{FI}$-modules étudiés par Church, Ellenberg et Farb ont plus de structure : ce sont des $\mathbf{S(ab)}$-modules, où $\mathbf{S(ab)}$ est la catégorie des groupes abéliens et des monomorphismes scindés, correspondant à $\mathbf{VIC}(\mathbb{Z})$ de [Wil18a]. Les $\mathbf{S(ab)}$-modules polynomiaux sont étudiés dans [DV19].

Dans la Section 2.6, nous définissons les foncteurs fortement polynomiaux sur $\mathbf{FI}_d$ de la même manière que sur $\mathbf{FI}$, en utilisant une famille d'endofoncteurs δ_1^c indexés par les d couleurs de $\mathbf{FI}_d$ au lieu d'un seul endofoncteur δ_1 pour les $\mathbf{FI}$-modules. Pour $d=1$, nous retrouvons la définition des foncteurs fortement polynomiaux sur $\mathbf{FI}$ de [DV19] puisque la seule couleur de $\mathbf{FI}_1$ donne l'unique endofoncteur δ_1 de [DV19]. Nous définissons également une notion d'effets croisés pour les $\mathbf{FI}_d$-modules dans la Section 5.4 en introduisant la catégorie cotranche $(0 \downarrow \mathbf{FI}_d)$ (parfois appelée la catégorie au-dessous de 0 comme dans [ML98, P.45]) des paires (k,x) où k est un objet de $\mathbf{FI}_d$ et x un morphisme dans $\mathbf{FI}_d(0,k)$. En effet, nous prouvons dans la Proposition 5.4.4 que la catégorie cotranche $(0 \downarrow \mathbf{FI}_d)$ est une catégorie monoïdale dont l'unité est un objet initial, ce qui nous permet de définir les effets croisés d'un $\mathbf{FI}_d$-module via le foncteur oubli $(0 \downarrow \mathbf{FI}_d) \to \mathbf{FI}_d$ et les travaux de Djament et Vespa dans [DV19]. Nous montrons ensuite dans la Proposition 5.4.12 que les foncteurs polynomiaux définis avec les effets croisés sur $\mathbf{FI}_d$ sont les mêmes que les foncteurs fortement polynomiaux définis avec les endofoncteurs δ_1^c :

Proposition. *Pour $n \in \mathbb{N}$ et F un $\mathbf{FI}_d$-module, F est dans $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ si et seulement si $\mathrm{cr}_{n+1}(F)\big(\ -\ \big)$ est le foncteur nul sur $(0 \downarrow \mathbf{FI}_d)^{\times n+1}$.*

Nous utilisons ensuite cette définition alternative des $\mathbf{FI}_d$-modules fortement polynomiaux pour montrer dans la Proposition 5.4.18 le résultat suivant.

Proposition. *Pour $m, n \in \mathbb{N}$, si $F : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ est fortement polynomial de degré inférieur ou égal à m et si $X : \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$ préserve les épimorphismes et est un foncteur polynomial de degré inférieur ou égal à n, alors la composée $X \circ F : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$ est fortement polynomiale de degré inférieur ou égal à nm.*

Nous utilisons ce résultat pour obtenir dans le Théorème 5.5.4 que le produit tensoriel terme à terme de deux $\mathbf{FI}_d$-modules fortement polynomiaux est fortement polynomial :

Théorème. *Pour $n, m \in \mathbb{N}$ et $F, G : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$, si F est dans $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ et si G est dans $\mathrm{Pol}_m^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, alors leur produit tensoriel $F \otimes G$ est dans $\mathrm{Pol}_{2\max(n,m)}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.*

Cependant, dans ce théorème la borne n'est peut-être pas la meilleure possible. En effet, nous pourrions nous attendre à ce que $F \otimes G$ soit fortement polynomial de degré inférieur ou égal à $n + m$. Par exemple, pour $d = 1$ il est montré dans [Dja16] qu'un $\mathbf{FI}$-module est fortement polynomial de degré inférieur ou égal à n si et seulement s'il est un quotient d'une somme des foncteurs projectifs standards $P_i^{\mathbf{FI}}$ pour $i \leq n$. Cela permet de prouver que, sur $\mathbf{FI}$, le produit tensoriel $F \otimes G$ est polynomial de degré $n + m$ si F est de degré n et G de degré m. Nous prouvons également dans l'annexe A le même résultat dans le cadre étudié par Djament et Vespa dans [DV19], c'est-à-dire les foncteurs sur une catégorie monoïdale symétrique générale dont l'unité est un objet initial :

Théorème. *Soit $\mathcal{M}$ une petite catégorie monoïdale symétrique dont l'unité est un objet initial. Pour $n, m \in \mathbb{N}$ et $F, G : \mathcal{M} \to \mathbf{R}\text{-}\mathbf{Mod}$, si F est dans $\mathrm{Pol}_n^{strong}(\mathcal{M}, \mathbf{R}\text{-}\mathbf{Mod})$ et si G est dans $\mathrm{pol}_m^{strong}(\mathcal{M}, \mathbf{R}\text{-}\mathbf{Mod})$, alors leur produit tensoriel $F \otimes G : \mathcal{M} \to \mathbf{R}\text{-}\mathbf{Mod}$ est dans $\mathrm{Pol}_{2\max(n,m)}^{strong}(\mathcal{M}, \mathbf{R}\text{-}\mathbf{Mod})$.*

Pour $d = 1$, les foncteurs projectifs standard $P_n^{\mathbf{FI}}$ constituent un exemple vraiment important de $\mathbf{FI}$-modules fortement polynomiaux, comme montré dans la [Dja16, Proposition 4.4]. Cela rend l'étude des foncteurs polynomiaux sur $\mathbf{FI}$ beaucoup plus facile. En particulier, cela implique qu'être fortement polynomial (avec des valeurs de type fini) est équivalent à être de type fini pour les $\mathbf{FI}$-modules. Ceci est spécifique à la catégorie $\mathbf{FI}$, dû au fait que les foncteurs projectifs standards sont polynomiaux, et n'est pas vrai en général pour d'autres catégories. Pour les $\mathbf{FI}_d$-modules, ces résultats n'ont aucune raison d'être vrais puisque nous montrons dans le Corollaire 5.2.2 ce qui suit :

Proposition. *Pour $d > 1$, le foncteur projectif standard $P_n^{\mathbf{FI}_d}$ n'est pas fortement polynomial.*

L'exemple des espaces de configuration

Comme expliqué ci-dessus, il existe de nombreux exemples de $\mathbf{FI}$-modules dans la littérature dans une grande variété de domaines. Nous présentons principalement un exemple donné par l'homologie des espaces de configuration d'une variété, qui est entièrement décrit dans [Sam20, Wil19] et [CF13]. Pour M une variété régulière, la cohomologie rationnelle des espaces de configuration de M est un $\mathbf{FI}$-module de type fini ([CEF15, Théorème 6.2.1]), ce qui est presque équivalent à fortement polynomial. De plus, pour M une variété connexe de dimension au moins 2 et vérifiant d'autres hypothèses, il a été montré dans [CMNR18, Theorem A] que $2k$ est une borne supérieure pour le degré polynomial du $\mathbf{FI}$-module $H^i\left(\mathrm{Conf}_{(-)}\left(M\right), \mathbb{K}\right)$.

Les résultats concernant le $\mathbf{FI}$-module $H^i\left(\mathrm{Conf}_{(-)}\left(M\right), \mathbb{K}\right)$ sont prouvés pour une variété de dimension au moins deux. Cette hypothèse est nécessaire pour garantir que les espaces de configuration soient connexes et que les points peuvent se déplacer les uns autour des autres. Mais pour une variété de dimension 1, comme un graphe, il n'y a pas assez d'espace et les points se bloquent les uns les autres dans les espaces de configuration, de sorte que la même approche n'est plus valable. Par exemple, l'espace de configuration d'un graphe linéaire avec une seule arête est homotopiquement équivalent à $n!$ points disjoints. Par conséquent, Ramos a introduit dans [Ram19] l'homologie d'un genre d'espaces de configuration modifiés pour les graphes qui forment un $\mathbf{FI}_d$-module. Dans ces espaces modifiés, appelés *espaces de configuration sink*, nous prenons n points (ordonnés) sur le graphe, comme pour les espaces classiques, mais ils peuvent être distincts deux à deux ou se chevaucher en un sommet du graphe mais pas à l'intérieur d'une arête. Les d sommets du graphe correspondent alors aux d couleurs de $\mathbf{FI}_d$, ce qui donne la structure d'un $\mathbf{FI}_d$-module lorsque nous prenons l'homologie rationnelle de ces espaces topologiques. Cela donne un exemple intéressant de $\mathbf{FI}_d$-module puisque, avant cela, tous les $\mathbf{FI}_d$-modules de la littérature étaient soit libres, soit obtenus à partir de $\mathbf{FI}$-modules via le foncteur oubli. Ramos a prouvé dans [Ram19] que ces $\mathbf{FI}_d$-modules sont de type fini pour tout degré homologique et tout graphe connexe. Dans la Proposition 3.2.8 nous donnons une description explicite de ces foncteurs pour les graphes linéaires :

Proposition. *Pour $\mathcal{G}_d$ le graphe linéaire sur d sommets, le $\mathbf{FI}_d$-module $H_0\left(Conf^{sink}_{(-)}\left(\mathcal{G}_d,[d]\right),\mathbb{Q} \right)$ est le foncteur constant $\mathbb{Q}$, tandis que pour $i \geq 1$ le $\mathbf{FI}_d$-module*

$$H_i\left(Conf^{sink}_{(-)}\left(\mathcal{G}_d,[d]\right),\mathbb{Q} \right)$$

est le foncteur envoyant n sur $\mathbb{Q}^{N(d,i+1)}$ si $n = i+1$ et sur zéro sinon, où

$$N(d,i+1) = \begin{cases} (d-1)^{i+1} - \binom{d-1}{i+1}(i+1)! & si\ d \geq i+2 \\ (d-1)^{i+1} & si\ d \leq i+1. \end{cases}$$

Dans la Proposition 5.1.8 nous déduisons de cette description que ces foncteurs sont fortement polynomiaux et nous donnons leur degré :

Proposition. *Pour $i \in \mathbb{N}$ et $\mathcal{G}_d$ le graphe linéaire sur d sommets, le $\mathbf{FI}_d$-module $H_i\left(Conf^{sink}_{(-)}\left(\mathcal{G}_d,[d]\right),\mathbb{Q} \right)$ est polynomial de degré 0 pour $i = 0$, et de degré $i+1$ pour $i > 1$.*

Les algèbres commutatives tordues

La théorie des algèbres commutatives tordues (ACTs) remonte aux années 1950 et est apparue en topologie algébrique. Elle a été introduite pour étudier différentes structures, telles que des suites d'objets munies d'une action de groupes linéaires ou symétriques. C'est également un analogue de la théorie de l'algèbre commutative adaptée à l'étude des représentations de ces groupes. Par exemple, dans [Bar78] Barratt a défini une algèbre tordue générale et a ajouté une condition pour être une algèbre de Lie tordue ou une algèbre commutative tordue. Comme nous le verrons, les $\mathbf{FI}_d$-modules apparaissent dans ce contexte puisqu'il existe une équivalence de catégories entre les $\mathbf{FI}_d$-modules et les modules sur l'ACT libre sur d générateurs.

Une ACT est un monoïde dans la catégorie monoïdale $\mathbf{Fct}(\mathbf{\Sigma},\mathbb{K}\text{-}\mathbf{Vect})$, où $\mathbf{\Sigma}$ est la catégorie des ensembles finis et des bijections. En considérant plusieurs catégories équivalentes à $\mathbf{Fct}(\mathbf{\Sigma},\mathbb{K}\text{-}\mathbf{Vect})$ nous obtenons différentes définitions équivalentes des ACTs comme expliqué dans [SS12] et [GS10] : il peut s'agir d'un foncteur des espaces vectoriels vers des anneaux commutatifs, ou d'un anneau commutatif muni d'une action du groupe linéaire infini par un morphisme d'algèbre, ou d'un anneau gradué unitaire associatif doté d'une action des groupes symétriques. Dans chaque cas une condition supplémentaire, appelée polynomialité (dans un sens différent de celui des foncteurs polynomiaux que nous étudions ici), est ajoutée pour former une ACT. Parfois, les ACTs sont également traitées comme des objets d'une catégorie abstraite équivalente à n'importe laquelle des catégories précédentes, ce qui conduit à une autre définition équivalente donnée dans [GS10] par l'intermédiaire des opérades. Nous choisissons de considérer les ACTs principalement comme des foncteurs $F : \mathbf{\Sigma} \to \mathbb{K}\text{-}\mathbf{Vect}$, munis d'une loi de multiplication $\nu : F \otimes F \to F$ et d'une unité (Définition 4.1.5). La définition en termes de représentations du groupe linéaire infini $GL(\infty)$, souvent utilisée par Sam et Snowden, est bien décrite dans [SS12] et [DES17]. Ces deux notions utilisant les groupes symétriques ou le groupe linéaire infini sont équivalentes pour $\mathbb{K}$ de caractéristique nulle via la dualité de Schur-Weyl, mais donnent deux notions différentes d'ACTs pour $\mathbb{K}$ de caractéristique positive.

Le premier exemple d'ACT, provenant de [Bar78], est le foncteur envoyant n sur l'espace $\mathbb{K}[\mathrm{S}_n]$ sur lequel le groupe S_n agit par conjugaison et dont la multiplication est donnée par l'inclusion standard de $\mathrm{S}_n \times \mathrm{S}_m$ dans S_{n+m}. Une façon simple de créer d'autres ACTs est de prendre l'algèbre symétrique d'une représentation de $GL(\mathbb{K}^\infty)$. Ces exemples, appelés "ACTs polynomiales" (ce qui n'a rien à voir avec nos foncteurs polynomiaux) sont entièrement décrits dans les différentes définitions équivalentes dans [SS12, Section 8.2.3]. Nous nous concentrons sur les ACTs libres sur d générateurs de degré un $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$, qui ont été largement étudiées, par exemple dans [SS12, SS16, SS19, GS10]. En particulier, Sam et Snowden ont montré dans [SS12] que la catégorie des modules sur cette ACT est équivalente, via le choix d'une base de $\mathbb{K}^d$, à la catégorie des $\mathbf{FI}_d$-modules. Comme mentionné ci-dessus, cela explique comment les $\mathbf{FI}_d$-modules apparaissent dans la théorie des ACTs. Nous donnons la description concrète de l'ACT $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ dans la Définition 4.1.15 et le détail de l'équivalence dans la Section 4.2. Un autre exemple d'ACT est $\mathrm{Sym}(\Lambda^2(\mathbb{K}^\infty))$ qui est étudié dans [SS15]. Par exemple, ils montrent qu'il existe une équivalence similaire à celle de $\mathbf{FI}_d$: les modules de type fini sur cette ACT sont équivalents aux modules de type fini sur la catégorie **FIM** de [MW19] dont les objets sont des ensembles finis et dont les morphismes sont des paires d'injection et de couplage parfait sur le complémentaire de l'image.

Il existe une action naturelle de $GL(\mathbb{K}^d)$ sur les modules sur l'ACT $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ qui agit diagonalement sur les composantes $(\mathbb{K}^d)^{\otimes n}$ de $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ avant d'appliquer la loi de multiplication. Dans la Section 4.3, nous utilisons l'équivalence de catégories de [SS12] pour transformer ceci en une action de $GL(\mathbb{K}^d)$ sur les $\mathbf{FI}_d$-modules. Nous obtenons dans la Proposition 4.3.5 la description concrète suivante :

Proposition. *Soit $\mathcal{B}$ une base de $\mathbb{K}^d$, pour $\varphi \in GL(\mathbb{K}^d)$ et $G \in \mathbf{FI}_d\text{-}\mathbf{Mod}$, le foncteur $\varphi_{\mathcal{B}} \cdot G : \mathbf{FI}_d \to \mathbb{K}\text{-}\mathbf{Vect}$ envoie un objet $n \in \mathbf{FI}_d$ sur $G(n)$ et un morphisme $(f,g) \in \mathbf{FI}_d(n,m)$ sur la somme*

$$\sum_{g' \in \mathbf{FI}_d(0, m \smallsetminus f(n))} \left(\prod_{l \in m \smallsetminus f(n)} m_{g'(l), g(l)} \right) G(f, g').$$

où $(m_{i,j})_{1 \le i,j \le d}$ est la matrice de φ dans la base $\mathcal{B}$ de $\mathbb{K}^d$.

Les foncteurs faiblement polynomiaux

La notion de foncteurs faiblement polynomiaux donne un raffinement de la notion de foncteurs fortement polynomiaux qui est plus intuitive mais manque de propriétés essentielles. En effet, pour une catégorie source qui est une catégorie monoïdale symétrique dont l'unité est un objet nul, les sous-catégories de foncteurs polynomiaux sont épaisses (voir [Dja16] pour le cas général) ce qui permet de regarder les quotients par ces sous-catégories. Cependant, lorsque l'unité est juste un objet initial comme dans **FI**, un sous-foncteur d'un foncteur fortement polynomial peut être de degré plus élevé, ou même ne pas être polynomial. Pour éviter ces phénomènes d'instabilité, Djament et Vespa ont défini une notion de foncteurs faiblement polynomiaux dans [DV19] en supprimant les foncteurs problématiques dans une catégorie quotient. Ils ont montré que la catégorie $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ de ces foncteurs, appelés les foncteurs stablement nuls, est composée des **FI**-modules dont la colimite est nulle. Ces foncteurs stablement nuls correspondent aux modules de torsion sur l'ACT libre sur un générateur de degré 1 étudiés dans [SS16] ou [NSS18], et l'endofoncteur κ qui donne le sous-foncteur maximal d'un **FI**-module dans $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ correspond au foncteur de cohomologie locale noté $\mathrm{H}^0_{\mathfrak{m}}(-)$ dans [SS16, NSS18, CEFN14]. En particulier, les propriétés de leurs foncteurs dérivés à droite $\mathrm{H}^i_{\mathfrak{m}}(-)$ sont étudiées dans [SS16, NSS18] afin de comprendre comment $\mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ est construite à partir des deux morceaux $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ et $\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. De même, le degré polynomial faible pour les **FI**-modules correspond à la notion de degré stable de [CEF15] et [CEFN14] tandis que le degré local précise comment les degrés faibles et forts sont reliés. Il donne moralement le degré polynomial fort modulo le degré polynomial faible et contrôle le rang à partir duquel la famille de représentations associée devient stable.

L'un des principaux objectifs de cette thèse est d'introduire et d'étudier les $\mathbf{FI}_d$-modules faiblement polynomiaux. L'une des différences avec la situation précédente est qu'il existe plusieurs sous-catégories qui peuvent remplacer les foncteurs stablement nuls dans ce cas : les foncteurs globalement stablement nuls $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ et les foncteurs stablement nuls le long de différentes combinaisons de couleurs $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Ces sous-catégories forment un raffinement de la notion des foncteurs stablement nuls introduite dans [DV19] pour **FI**. En effet, pour $d = 1$ il y a une inclusion de l'unique sous-catégorie de foncteurs stablement nuls $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ dans $\mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ mais, pour un d général, ces sous-catégories forment un ensemble partiellement ordonné plus riche pour l'inclusion. Par exemple, pour $d = 2$, l'ensemble partiellement ordonné est le suivant :

$$\begin{tikzcd}
 & \mathcal{SN}_{c_1}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \arrow[dr, hook] & \\
\mathcal{SN}_{c_1,c_2}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \arrow[ur, hook] \arrow[dr, hook] & & \mathcal{SN}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \arrow[r, hook] & \mathbf{Fct}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \\
 & \mathcal{SN}_{c_2}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \arrow[ur, hook] &
\end{tikzcd}$$

Dans la Proposition 6.1.7 et le Corollaire 6.2.5 nous montrons que les sous-catégories $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ et $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ de $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ sont épaisses, c'est-à-dire stables par sous-objet, quotient et extension. Nous pouvons alors considérer la catégorie quotient de $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ par n'importe laquelle de ces sous-catégories en suivant la construction de Gabriel dans [Gab62], et y définir des objets polynomiaux en utilisant les endofoncteurs δ_1^c de $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ qui passent aux quotients. Ceci est possible parce que ces sous-catégories sont stables par colimites et que le foncteur quotient π_d a un adjoint à droite $\mathcal{S}_d$ appelé le foncteur section.

La sous-catégorie $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ des foncteurs globalement stablement nuls est définie dans la Section 6.1 à l'aide d'une famille d'endofoncteurs κ_1^c de $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Ces foncteurs sont définis dans la Section 2.6 d'une manière duale à δ_1^c, et ils s'insèrent tous dans la suite exacte d'endofoncteurs

$$0 \longrightarrow \kappa_1^c \longrightarrow \mathrm{Id} \xrightarrow{i_1^c} \tau_1 \longrightarrow \delta_1^c \longrightarrow 0\ ,$$

où τ_1 est l'endofoncteur décalage $F(-) \mapsto F(-+1)$ et i_1^c une transformation naturelle associée à la couleur c. Nous définissons également une structure d'ensemble partiellement ordonné sur $\mathbb{N}^d$ pour l'ordre du produit et un foncteur $\xi_d : \mathbb{N}^d \to \mathbf{FI}_d$ qui envoie un objet $(n_1, \dots, n_d) \in \mathbb{N}^d$ sur l'objet $n_1 + \cdots + n_d$ de $\mathbf{FI}_d$. Nous montrons ensuite dans la Proposition 6.1.5 qu'il existe une définition équivalente de la catégorie $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ utilisant une colimite filtrée sur $\mathbb{N}^d$:

Proposition. *Soit F un $\mathbf{FI}_d$-module, alors F est dans $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ si et seulement si*

$$\operatorname*{colim}_{\mathbb{N}^d} F \circ \xi_d = 0.$$

Pour $d = 1$, nous retrouvons la description de $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ de [DV19, Proposition 5.7], à savoir que les foncteurs stablement nuls sont ceux dont la colimite est nulle. Rappelons que, par [SS12], la catégorie des $\mathbf{FI}_d$-modules est équivalente à la catégorie des $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$-modules. Dans la Section 6.4, nous donnons une description de $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ en termes de $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$-modules par le biais de cette équivalence. Nous montrons également dans la Proposition 6.4.2 que, pour $d > 1$, la sous-catégorie $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ de $\mathbf{Fct}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ n'est pas stable par l'action de $GL(\mathbb{K}^d)$ définie ci-dessus.

La sous-catégorie $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ de $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ des foncteurs stablement nuls le long des couleurs $c_{i_1}, \dots, c_{i_m}$ est définie dans la Section 6.2 de manière similaire aux foncteurs globalement stablement nuls, mais en utilisant les endofoncteurs κ_1^c pour chaque couleur c dans $\{c_{i_1}, \dots, c_{i_m}\}$. Dans le Corollaire 6.2.4 nous montrons que ces catégories admettent également une définition équivalente, cette fois par l'intermédiaire des foncteurs $\Delta_c^* : \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$:

Proposition. *Un $\mathbf{FI}_d$-module F est dans la sous-catégorie $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ de $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ si et seulement si les foncteurs $\Delta_c^*(F)$ sont dans la sous-catégorie $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ de $\mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ pour toutes les couleurs c dans $\{c_{i_1}, \dots, c_{i_m}\}$.*

Cette définition équivalente nous permet d'utiliser les résultats déjà prouvés pour les foncteurs sur $\mathbf{FI}$, en particulier ceux de Djament et Vespa dans [DV19]. Cependant, nous montrons dans la Section 5.1 que dans le quotient par une sous-catégorie de foncteurs stablement nuls le long des couleurs, les objets polynomiaux sont un peu plus difficiles à définir. Dans ce processus, nous perdons certaines propriétés importantes comme le fait que les endofoncteurs κ_1^c deviennent nuls et que les endofoncteurs δ_1^c deviennent exacts dans le quotient. C'est une première raison pour laquelle nous ne développons que les foncteurs faiblement polynomiaux correspondant à la sous-catégorie globale $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$: celle-ci se comporte mieux avec les endofoncteurs δ_1^c qui constituent un outil crucial pour l'étude des foncteurs polynomiaux.

Dans le Chapitre 7, nous nous concentrons sur la catégorie $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ des foncteurs stables, *i.e.* le quotient par les foncteurs globalement stablement nuls $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, la plus grande de ces sous-catégories, afin d'obtenir une catégorie quotient plus petite qui peut être plus facile à décrire. Bien que les objets de la catégorie quotient $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ soient par définition les foncteurs de $\mathbf{FI}_d$ vers $\mathbf{R}\text{-}\mathbf{Mod}$, il faut les considérer comme des objets abstraits puisque les morphismes dans le quotient sont modifiés par certaines classes d'isomorphismes. Dans la Définition 7.2.1, nous définissons les $\mathbf{FI}_d$-modules faiblement polynomiaux comme les foncteurs sur $\mathbf{FI}_d$ dont l'image dans la catégorie quotient $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ par le foncteur quotient π_d est un objet polynomial (nous identifions parfois F et $\pi_d(F)$ par abus de langage). Avec cette définition, un foncteur fortement polynomial est faiblement polynomial mais la réciproque n'est pas vraie, ce qui justifie la terminologie introduite par Djament et Vespa dans [DV19] pour les $\mathbf{FI}$-modules. Nous désignons par $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ la sous-catégorie pleine de $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ des objets polynomiaux de degré inférieur ou égal à n. Par abus de langage, cela désigne aussi la sous-catégorie pleine de $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ des foncteurs dont l'image par le foncteur quotient π_d est un objet polynomial de degré inférieur ou égal à n. Nous prenons alors $\mathbf{R} = \mathbb{K}$ un corps pour nous assurer que le produit tensoriel est exact et nous montrons dans le Théorème 7.3.6 que le produit tensoriel terme à terme de deux objets polynomiaux de $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ est polynomial.

Théorème. *Soit* $\mathbf{R} = \mathbb{K}$ *un corps, pour* $X \in \mathrm{Pol}_n(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$ *et* $Y \in \mathrm{Pol}_m(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$*, nous avons*

$$X \otimes Y \in \mathrm{Pol}_{n+m}(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect}).$$

Alors que la compréhension des catégories de foncteurs polynomiaux est un problème difficile en général, sauf pour les petites valeurs, le quotient des foncteurs polynomiaux de degré n modulo les foncteurs de degré $n-1$ est bien compris dans plusieurs contextes. En particulier, Djament et Vespa ont décrit ce quotient dans [DV19, Théorème 2.26] pour les objets polynomiaux de $\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ comme nous le rappelons dans le Chapitre 9. Pour $n = 0$, ils obtiennent que les seuls objets de $\mathrm{Pol}_0(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ sont les foncteurs constants.

Dans la Section 7.4, nous décrivons les objets polynomiaux de degré 0 de $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, qui forment une catégorie plus riche que pour $d = 1$. Pour cela, nous introduisons dans la Définition 7.4.8 la catégorie $\mathbf{R}\text{-}\mathbf{Mod}_d$ des $\mathbf{R}$-modules avec $d-1$ automorphismes qui commutent deux à deux. De même, nous introduisons la catégorie des modules sur l'anneau des polynômes commutatifs $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]$ en les $d-1$ variables $x_2, \dots, x_d$ toutes inversibles. Un de nos principaux résultats est alors la description suivante obtenue dans le Théorème 7.4.12 :

Théorème. *Il existe des équivalences de catégories entre la catégorie* $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ *des objets polynomiaux de degré 0 de* $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$*, la catégorie* $\mathbf{R}\text{-}\mathbf{Mod}_d$ *et la catégorie* $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]\text{-}\mathbf{Mod}$*.*

Pour $d = 1$ nous retrouvons que les $\mathbf{FI}$-modules polynomiaux de degré 0 sont les foncteurs constants, mais pour un d général ces foncteurs forment une catégorie plus complexe. Nous prouvons ce théorème en deux étapes : tout d'abord, nous montrons dans la Proposition 7.4.2 que les objets polynomiaux de degré 0 de $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ satisfont une condition abstraite appelée $(POL0)$. Nous utilisons ensuite la catégorie intermédiaire $\underline{\mathbf{FI}_d}$ définie dans la Section 2.5 pour montrer, dans les Propositions 7.4.6 et 7.4.7 que, pour chaque objet F du quotient satisfaisant $(POL0)$, l'image de F par le foncteur section $\mathcal{S}_d$ est complètement déterminée par son image sur les morphismes $c \in \mathbf{FI}_d(0,1)$. Ces images des morphismes $c \in \mathbf{FI}_d(0,1)$ correspondent aux $d-1$ isomorphismes de modules de la catégorie $\mathbf{R}\text{-}\mathbf{Mod}_d$, lorsque nous trivialisons l'action de c_1. Du point de vue des $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]$-modules, les images des morphismes $c_i \in \mathbf{FI}_d(0,1)$ correspondent à l'action des x_i, où x_1 agit par l'identité lorsque nous trivialisons l'action de c_1.

Exemples de quotients polynomiaux des foncteurs $P_n^{\mathbf{FI}_d}$

Le fait que les générateurs projectifs standards $P_n^{\mathbf{FI}_d}$ soient fortement polynomiaux pour $d = 1$ simplifie l'étude des foncteurs polynomiaux sur la catégorie $\mathbf{FI}$. Comme expliqué ci-dessus, ce n'est pas le cas pour $d > 1$. Nous décrivons donc plusieurs quotients des foncteurs $P_n^{\mathbf{FI}_d}$ qui sont polynomiaux. En plus de fournir des exemples concrets, ces quotients peuvent aussi nous donner une meilleure idée de ce à quoi ressemblent les foncteurs polynomiaux sur $\mathbf{FI}_d$. Par exemple, dans la Section 8.1 nous obtenons une famille de quotients du foncteur $P_0^{\mathbf{FI}_d}$ qui sont faiblement polynomiaux de degré 0 en filtrant ses générateurs par le nombre d'occurrences des couleurs. En effet, pour $k_1, \dots, k_d \in \mathbb{N}$, $I \subset \{c_1, \dots, c_d\}$ et $\alpha \in \mathbf{FI}_d(0,k)$ nous notons $\gamma_i(\alpha)$ le nombre d'occurrences de la couleur c_i dans α. Nous disons alors que $\alpha \in \mathbf{FI}_d(0,k)$ satisfait la condition $(P_{I,k_1,\dots,k_d})$ si $\gamma_i(\alpha) \geq k_i$ pour tout $i \in I$ ou s'il existe $j \in \{c_1, \dots, c_d\} \setminus I$ tel que $\gamma_j(\alpha) \geq k_j$. Avec ces notations, nous introduisons dans la Définition 8.1.8 le sous-foncteur $G_{I,k_1,\dots,k_d}$ de $P_0^{\mathbf{FI}_d}$ donné par

$$G_{I,k_1,\dots,k_d}(n) = \left\langle \alpha - X \mid \alpha \in \mathbf{FI}_d(0,n) \text{ that satisfies the condition } (P_{I,k_1,\dots,k_d}) \right\rangle$$

c'est-à-dire le sous-module engendré par les éléments $\alpha - X$, pour $\alpha \in \mathbf{FI}_d(0,n)$ qui satisfait la condition $(P_{I,k_1,\dots,k_d})$ et $X \in \mathbf{FI}_d(0,n)$ un morphisme fixé satisfaisant $(P_{I,k_1,\dots,k_d})$. Nous montrons alors dans la Proposition 8.1.15 ce qui suit :

Proposition. *Pour $k_1,\dots,k_d \in \mathbb{N}$ et $I \subset \{c_1,\dots,c_d\}$, le quotient de $P_0^{\mathbf{FI}_d}$ par son sous-foncteur $G_{I,k_1,\dots,k_d}$ est faiblement polynomial de degré 0.*

De plus, la preuve est basée sur le Lemme 8.1.14 qui montre que ce quotient est égal à un foncteur constant modulo un foncteur stablement nul de $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Cela implique que son image dans le quotient correspond, par l'équivalence donnant la description de $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, à l'objet $(\mathbf{R}, \mathrm{Id}, \dots, \mathrm{Id})$ de $\mathbf{R}\text{-}\mathbf{Mod}_d$ ou au $\mathbf{R}[x_2^{\pm 1},\dots,x_d^{\pm 1}]$-module trivial.

Parallèlement, dans la Section 8.2 nous étudions le quotient du foncteur $P_n^{\mathbf{FI}_d}$ par son sous-foncteur correspondant à l'action des groupes symétriques par post-composition. Ce sous-foncteur, noté F_n dans la Définition 8.2.1, est donné sur les objets par

$$F_n(m) = \langle\, \sigma \circ (f,g) - (f,g) \,|\, (f,g) \in \mathbf{FI}_d(n,m),\ \sigma \in \mathrm{S}_m \rangle\,.$$

Nous montrons que le quotient du foncteur $P_n^{\mathbf{FI}_d}$ par F_n est faiblement polynomial dans le Théorème 8.2.11 :

Théorème. *Pour tout $n \in \mathbb{N}$, le foncteur quotient de $P_n^{\mathbf{FI}_d}$ par F_n est faiblement polynomial de degré 0, où F_n est le sous-foncteur de $P_n^{\mathbf{FI}_d}$ de la Définition 8.2.1.*

Un bon représentant de l'image de ce quotient dans la catégorie $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ pourrait nous aider à le décrire dans la catégorie $\mathbf{R}[x_2^{\pm 1},\dots,x_d^{\pm 1}]\text{–}\mathbf{Mod}$ via l'équivalence donnant la description de $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Cependant, nous expliquons dans la Section 8.2 qu'il n'est pas facile d'en trouver un puisque le passage à la catégorie quotient n'est pas une construction explicite.

Dans la Section 8.3 nous donnons un quotient de $P_n^{\mathbf{FI}_d}$ qui est faiblement polynomial de degré n : pour un morphisme (f,g) dans $\mathbf{FI}_d(n,m)$ la seconde application g correspond à un choix de $m-n$ couleurs. Il existe alors une action du groupe symétrique S_{m-n} permutant ces choix de couleurs, qui donne une action de S_{m-n} sur $P_n^{\mathbf{FI}_d}(m)$. Le sous-foncteur de $P_n^{\mathbf{FI}_d}$ correspondant à cette action des groupes symétriques, noté H_n dans la Définition 8.3.2, est donné sur les objets par

$$H_n(m) = \langle\, (f, \sigma \cdot g) - (f,g) \,|\, (f,g) \in \mathbf{FI}_d(n,m),\ \sigma \in \mathrm{S}_{m-n} \rangle\,.$$

Nous montrons que le quotient du foncteur $P_n^{\mathbf{FI}_d}$ par H_n est faiblement polynomial dans le Théorème 8.3.14 :

Théorème. *Pour tout $n \in \mathbb{N}$, le quotient de $P_n^{\mathbf{FI}_d}$ par H_n est faiblement polynomial de degré n, où H_n est le sous-foncteur de $P_n^{\mathbf{FI}_d}$ de la Définition 8.3.2.*

Nous le prouvons de deux manières : premièrement, nous calculons directement δ_1^c de ce quotient, ce qui est très similaire au calcul de δ_1^c de $P_n^{\mathbf{FI}_d}$ dans la Proposition 5.2.1 mais, puisque nous quotientons par l'action des groupes symétriques sur les couleurs, la composante qui empêche $P_n^{\mathbf{FI}_d}$ d'être polynomial disparait ici. Deuxièmement, nous introduisons dans la Définition 8.2.5 la catégorie $\mathcal{C}_d$ dont les objets sont les entiers et dont les morphismes de n vers m sont les $(m-n)$-uplets de couleurs $(c_{i_1},\dots,c_{i_{m-n}})$ quotientés par l'action de S_{m-n} (ce qui est la même chose que les choix non ordonnés de $m-n$ couleurs). Nous montrons ensuite dans la Proposition 8.2.9 que le quotient de $P_n^{\mathbf{FI}_d}$ par F_n est équivalent au foncteur $P_n^{\mathcal{C}_d}$, et décrivons le quotient de $P_n^{\mathbf{FI}_d}$ par H_n comme un produit tensoriel dans la Proposition 8.3.8 via la formule :

Proposition. *Pour tout $n \in \mathbb{N}$, il existe un isomorphisme naturel*

$$P_n^{\mathbf{FI}_d} / H_n \cong \left(\mathcal{O}^* P_n^{\mathbf{FI}}\right) \otimes P_0^{\mathcal{C}_d}\left((-) - n\right) \circ \Omega \,,$$

où $\mathcal{O}$ est le foncteur oubli $\mathbf{FI}_d \to \mathbf{FI}$ et où $\Omega : \mathbf{FI}_d \to \mathcal{C}_d$ envoie $n \in \mathbf{FI}_d$ sur $n \in \mathcal{C}_d$ et un morphisme $(f,g) \in \mathbf{FI}_d(n,m)$ sur les couleurs de g quotientées par l'action de S_{m-n}.

Ceci explique comment les injections et les couleurs sont mélangées pour former le foncteur $P_n^{\mathbf{FI}_d}$, à l'action des groupes symétriques sur les choix de couleurs près. De plus, comme le foncteur $P_n^{\mathbf{FI}}$ est fortement polynomial de degré n pour $d = 1$, l'image des flèches de $P_n^{\mathbf{FI}}$ vers la somme directe de tous les $P_k^{\mathbf{FI}}$ pour $k \leq i$ est faiblement polynomial de degré i pour tout $i \in \mathbb{N}$. Nous construisons ensuite dans la Proposition 8.4.5 un quotient de $P_n^{\mathbf{FI}_d}$ faiblement polynomial de degré i pour tout $i \in \mathbb{N}$ en utilisant la formule ci-dessus du quotient de $P_n^{\mathbf{FI}_d}$ par H_n.

La construction Cospan

Afin d'étudier les foncteurs polynomiaux sur les catégories monoïdales symétriques dont l'unité est un objet initial, Djament et Vespa ont introduit dans [DV19] un foncteur $\mathcal{M} \to \tilde{\mathcal{M}}$ qui transforme la catégorie $\mathcal{M}$ dont l'unité est un objet initial en la catégorie $\tilde{\mathcal{M}}$ dont l'unité est un objet nul. Cette construction, qui est universelle au sens où elle donne un adjoint au foncteur oubli, ajoute moralement des morphismes "décroissants" des objets de la catégorie vers l'unité tout en préservant les morphismes "croissants" de l'unité vers les objets. Cette construction est équivalente à la construction Cospan$(-)$ de [Ves07] où les foncteurs sur Cospan peuvent être vus comme une généralisation des foncteurs de Mackey. Comme cette construction préserve les foncteurs polynomiaux, elle permet à Djament et Vespa dans [DV19, Théorème 4.8] de transformer l'étude des foncteurs polynomiaux sur une catégorie dont l'unité est un objet initial en l'étude des foncteurs polynomiaux sur une catégorie dont l'unité est un objet nul, qui sont mieux connus.

Ils appliquent ensuite ce résultat à $\mathbf{FI}$ dont l'unité est un objet initial. Cela leur permet de décrire le quotient des objets polynomiaux (dans la catégorie quotient $\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$) de degré inférieur ou égal à n sur $\mathbf{FI}$ par sa sous-catégorie épaisse des foncteurs polynomiaux de degré inférieur ou égal à $n-1$. En effet, les catégories Cospan$(\mathbf{FI})$ et $\tilde{\mathbf{FI}}$ sont équivalentes à la catégorie $\mathbf{FI}\#$ des injections partielles d'ensembles finis de [CEF15]. Ils utilisent ensuite une variante d'un théorème de type Dold-Kan de Pirashvili pour décrire le même quotient pour les foncteurs sur Cospan$(\mathbf{FI})$. Ce théorème de Pirashvili de [Pir00] donne une équivalence de catégories entre les foncteurs sur la catégorie Γ des ensembles finis pointés et les foncteurs sur Ω la catégorie des ensembles finis et des surjections, en utilisant les effets croisés. La variante utilisée dans [DV19], qui est décrite explicitement dans [CEF15, Théorème 4.1.5], donne une équivalence de catégories entre les foncteurs sur la catégorie $\mathbf{FI}\#$ et les foncteurs sur la catégorie $\mathbf{\Sigma}$ des ensembles finis et des bijections, étant donné que $\mathbf{FI}\#$ est une sous-catégorie de Γ et que $\mathbf{\Sigma}$ est une sous-catégorie de Ω. La combinaison de ces deux résultats donne la description suivante dans [DV19, Proposition 5.9] : pour $n \in \mathbb{N}$, il existe une équivalence de catégories

$$\mathrm{Pol}_n(\mathbf{FI}) \big/ \mathrm{Pol}_{n-1}(\mathbf{FI}) \;\cong\; \mathrm{Pol}_n(\mathrm{Cospan}(\mathbf{FI})) \big/ \mathrm{Pol}_{n-1}(\mathrm{Cospan}(\mathbf{FI})) \;\cong\; \mathbf{Fct}\left(\mathbf{\Sigma}_n, \mathbf{R}\text{-}\mathbf{Mod}\right),$$

où $\mathbf{\Sigma}_n$ est la catégorie associée au groupe symétrique S_n. Nous montrons que cette approche ne peut pas être directement généralisée pour décrire les foncteurs polynomiaux sur $\mathbf{FI}_d$.

Dans le Chapitre 9 nous introduisons une généralisation de la construction Cospan pour $\mathbf{FI}_d$ comme suit : les objets de Cospan$(\mathbf{FI}_d)$ sont les mêmes que les objets de $\mathbf{FI}_d$ et les morphismes sont des classes de diagrammes sous une relation d'équivalence. Ces diagrammes sont moralement composés d'une injection et de deux choix de couleurs différents sur des ensembles différents qui interagissent l'un avec l'autre. Ainsi, nous montrons dans la Proposition 9.2.8 que la catégorie $\mathrm{Cospan}(\mathbf{FI}_d)$ est isomorphe à une catégorie combinatoire $\mathbf{FI}_d\#$ dont les morphismes consistent en une injection partiellement définie et deux choix de couleur distincts, l'un sur le complémentaire à la source et l'autre sur le complémentaire au but. De plus, nous montrons dans la Proposition 9.1.6 que chaque morphisme dans $\mathrm{Cospan}(\mathbf{FI}_d)$ admet un diagramme représentatif minimal de la classe, ce qui implique que les morphismes de 0 à n et les morphismes de n à 0 dans $\mathrm{Cospan}(\mathbf{FI}_d)$ sont en bijection avec $\mathbf{FI}_d(0,n)$. Ceci souligne que la catégorie $\mathrm{Cospan}(\mathbf{FI}_d)$ est essentiellement obtenue en conservant les morphismes de 0 à n de $\mathbf{FI}_d$ et en ajoutant de nouveaux morphismes de n à 0 qui leur correspondent.

Nous étudions ensuite les $\mathrm{Cospan}(\mathbf{FI}_d)$-modules comme nous l'avons fait pour les $\mathbf{FI}_d$-modules : dans la Section 9.3 nous définissons les foncteurs polynomiaux sur $\mathrm{Cospan}(\mathbf{FI}_d)$ en utilisant une famille d'endofoncteurs δ_1^c de $\mathrm{Cospan}(\mathbf{FI}_d)$ pour les différentes couleurs. Une différence majeure est que les foncteurs stablement nuls sur $\mathrm{Cospan}(\mathbf{FI}_d)$ sont nuls puisque cette catégorie a un objet nul, ainsi les notions faibles et fortes de foncteurs polynomiaux sur $\mathrm{Cospan}(\mathbf{FI}_d)$ coïncident. Nous obtenons alors dans le Théorème 9.4.9 la description suivante des $\mathrm{Cospan}(\mathbf{FI}_d)$-modules polynomiaux de degré 0 :

Théorème. *Un foncteur* $F \in \mathbf{Fct}(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ *est dans* $\mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ *si et seulement si c'est un foncteur constant. Il existe une équivalence des catégories*

$$\mathrm{Pol}_0(\ \mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod}\ ) \cong \ \mathbf{R}\text{-}\mathbf{Mod}\,.$$

Avec la description des objets polynomiaux de degré 0 de $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ (Théorème 7.4.12), ceci montre que pour un d général la première équivalence de [DV19, Proposition 5.9] présentée ci-dessus échoue déjà pour $n = 0$, c'est-à-dire que le quotient de $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ par $\mathrm{Pol}_{n-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ n'est pas équivalent au même quotient sur $\mathrm{Cospan}(\mathbf{FI}_d)$.

Structure du document

L'organisation du manuscrit est la suivante : dans le premier chapitre nous rappelons la construction et les faits importants concernant le quotient d'une catégorie par une sous-catégorie épaisse. Nous présentons les $\mathbf{FI}_d$-modules dans le Chapitre 2 et nous donnons un aperçu des résultats basiques déjà connus à leur sujet. Nous introduisons également les principaux outils pour leur étude et décrivons les objets simples de cette catégorie. Dans le Chapitre 3, nous présentons l'exemple des espaces de configuration et le décrivons explicitement dans des cas simples. Le Chapitre 4 concerne les algèbres commutatives tordues et leur lien avec les $\mathbf{FI}_d$-modules. Dans le Chapitre 5, nous définissons les foncteurs fortement polynomiaux sur $\mathbf{FI}_d$ et donnons des exemples et des contre-exemples. Nous étendons également les effets croisés à ces foncteurs et montrons que la notion résultante de foncteurs polynomiaux coïncide avec celle utilisant l'endofoncteur différentiel. Le Chapitre 6 est consacré aux différentes notions de foncteurs stablement nuls et à l'ensemble partiellement ordonné qu'elles forment. Dans le Chapitre 7, nous étudions la catégorie quotient $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ et les foncteurs polynomiaux dans ce quotient. En particulier, nous décrivons les objets polynomiaux de degré zéro de $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, qui ne sont pas simplement les foncteurs constants. Dans le Chapitre 8, nous donnons des exemples de quotients polynomiaux des foncteurs projectifs standards. Enfin, dans le dernier chapitre nous introduisons la catégorie $\mathrm{Cospan}(\mathbf{FI}_d)$ et nous montrons que la méthode de [DV19] pour décrire les $\mathbf{FI}$-modules faiblement polynomiaux ne fonctionne pas de la même manière aux $\mathbf{FI}_d$-modules faiblement polynomiaux.

Introduction (English)

Science is the belief in the ignorance of experts.

Richard Feynman

From FI-modules to $\mathbf{FI}_d$-modules

The **FI**-modules are the functors from the category **FI** of finite sets and injections (also denoted by **I** in [Sch08] and Θ in [DV19]) to the category **R-Mod** of **R**-modules (for **R** a commutative ring). More generally, a $\mathcal{C}$-module is a functor from a category $\mathcal{C}$ to the category **R-Mod**. The **FI**-modules have been studied extensively in the last decade by Church, Ellenberg, Farb, Nagpal, Reinhold and others (see for example [CEF15, CEFN14, CEF14, CE17, CF13, Chu12, CMNR18, Dja16, DV19]). The theory of **FI**-modules was introduced in [CEF15] in order to transform the complex notion of representation stability into a finiteness result about the sequence of representations of the symmetric groups viewed as a unique object. A detailed introduction to the theory of **FI**-modules and representation stability can be found in [Sam20] but we recall the basic principles here. The notation **FI** was introduced in [CEF15] as an acronym for the category of **F**inite sets (often represented by their cardinality in the skeleton) and **I**njections. A **FI**-module is a family of linear representations of the symmetric groups together with compatibility conditions given by linear maps, which can be represented by the following diagram:

$$
\begin{tikzcd}
 & S_0 \arrow[d, phantom, "\circlearrowright"] & S_1 \arrow[d, phantom, "\circlearrowright"] & S_2 \arrow[d, phantom, "\circlearrowright"] & & S_n \arrow[d, phantom, "\circlearrowright"] & \\
\mathbf{FI} \arrow[dd, "F"'] & 0 \arrow[r, hook] & 1 \arrow[r, hook] & 2 \arrow[r, hook] & \dots \arrow[r, hook] & n \arrow[r, hook] & \dots \\
 & & & & & & \\
\mathbf{R}\text{-}\mathbf{Mod} & F(0) \arrow[r] & F(1) \arrow[r] & F(2) \arrow[r] & \dots \arrow[r] & F(n) \arrow[r] & \dots \\
 & F(S_0) \arrow[u, phantom, "\circlearrowleft"] & F(S_1) \arrow[u, phantom, "\circlearrowleft"] & F(S_2) \arrow[u, phantom, "\circlearrowleft"] & & F(S_n) \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}
$$

Each arrow in this diagram actually represents many arrows that we can construct by composition with the action of the symmetric groups. A large number of concrete examples of **FI**-modules are presented in [CF13]. Other interesting examples of finitely generated **FI**-modules are given by the cohomology of the pure string motion groups in [Wil12] and the pure braid groups in [Wil18a].

In the literature there are several variants (see [Sam20] for a detailed list) of the category **FI**: the categories $\mathbf{FI}_d$ that we develop in this thesis, $\mathbf{FI}_G$ the category of finite sets and couples of an injection and a choice of an element of the group G for each element at the source (see [Ram17b]), $\mathbf{FS}_G$ the category of finite sets and G-surjections for G a group (see [SS17]), $\mathbf{FI}_W$ for W some Weyl groups in [Wil12], **FIM** the category of finite sets and pairs of injection and perfect matching on the complement of the image (see [MW19]), or a symplectic version (see [Sam20]). There are also variants for representations of linear groups such as $\mathbf{VI}(\mathbf{R})$ the category of free modules of finite rank and injective linear maps with left inverse which is presented in detail in [Wil18a]. This category, and its generalization $\mathbf{VIC}(\mathbf{R})$ of free modules of finite rank and injective linear maps with a choice of direct complement of the image, were introduced under the names $\mathbf{S}(\mathbf{ab})$ for $\mathbf{R} = \mathbb{Z}$ in [DV19] and $\mathbf{S}(\mathbf{R})$ in [Dja16].

In this thesis we will focus on the category $\mathbf{FI}_d$ for d a nonzero integer, introduced by Sam and Snowden in [SS17], in which the objects are still the finite sets and the morphisms are the coloured injections. We study here the $\mathbf{FI}_d$-modules and we emphasize in particular the differences with $\mathbf{FI}$-modules. Even if we study the functors whose target category is a module category for more clarity, most of this work stays true if we replace $\mathbf{R}$-$\mathbf{Mod}$ by a general Grothendieck category (see [Gar01]). We recover in particular the $\mathbf{FI}$-modules since the category $\mathbf{FI}_1$ is isomorphic to the category $\mathbf{FI}$ (see Section 2.1). The first major difference is that the unit 0 is an initial object in $\mathbf{FI} \cong \mathbf{FI}_1$ but not in $\mathbf{FI}_d$ for $d > 1$. We also show in Section 2.7 that the forgetful functor $\mathbf{FI}_d \to \mathbf{FI}$, that connects the $\mathbf{FI}_d$-modules and the $\mathbf{FI}$-modules, has a family of adjoints $\Delta_c : \mathbf{FI} \to \mathbf{FI}_d$ called the colouring functors which add the colour c to all morphisms of $\mathbf{FI}$. By precomposition, they allow us to consider a $\mathbf{FI}_d$-module as a $\mathbf{FI}$-module.

For any category $\mathcal{C}$, a family of important examples of functors from $\mathcal{C}$ to $\mathbf{R}$-$\mathbf{Mod}$ are the standard projective functors. These fundamental functors appear for functors between $\mathbb{F}_p$-vector spaces in [Kuh94], for $\mathbf{FI}_d$ in [SS17], and for $d = 1$ in [DV19, Dja16, Ves19], or under the name of free modules in [CEF15, CEFN14, MW19] or of representable functors in [Wil18a]. They play the role of the free modules in the classical theory of modules. We can deduce a lot of information about the $\mathbf{FI}_d$-modules from the structure of the standard projective functors since they form a family of projective generators of $\mathbf{FI}_d$-$\mathbf{Mod}$ (Proposition 2.2.5).

Simple $\mathbf{FI}_d$-modules

The category $\mathbf{FI}_d$ is an EI-category: *i.e.* a category whose endomorphisms are isomorphisms. These categories and their representations have been introduced by Dieck in [Die87] in the context of algebraic K-theory, and more recently studied by Li in [Li14], in particular their Koszul property. This property already gives us a result about the simple $\mathbf{FI}_d$-modules, that is the $\mathbf{FI}_d$-modules which do not have non-zero proper subfunctors. In order to express this result, we recall that the irreducible representations of the symmetric group S_n over a field of characteristic zero are indexed by the partitions λ of n. We denote by M^λ the irreducible representation associated with the partition λ of n, which is defined as the ideal of the ring $\mathbb{K}[\mathrm{S}_n]$ generated by an idempotent element associated to the partition λ called the Young symmetrizer. For example, the representation associated with the partition $\lambda = (n)$ is the trivial representation, the one associated with $\lambda = (1^n)$ is the sign representation, and the one associated with $\lambda = (n-1,1)$ is the standard representation. We then give in Proposition 2.4.3 the following description of the simple $\mathbf{FI}_d$-modules:

Proposition. *For* $\mathbf{R}$ *a field of characteristic zero, the simple objects of the category* $\mathbf{FI}_d$*-Mod are the functors* $(M^\lambda)_k$ *that sends an object* $n \in \mathbf{FI}_d$ *to* M^λ *if* $n = k$ *and to zero else, for* λ *a partition of* k.

Representation stability

Although the category **FI** has been studied in different combinatorial contexts, it was first used in the frame of representation stability. This theory was introduced by Church and Farb in [CF13] to study some compatible families of representations of groups which admit a decomposition in irreducible that eventually becomes stable. It was thought as a generalization of the classical homological stability in the case where the induced maps in homology do not eventually become isomorphisms. A sequence of representations of groups, such as the symmetric groups, is representation stable when the names of the irreducible representations (with an appropriate way of indexing them) that occur in the decomposition eventually stabilize, even if the spaces change. Concrete examples of this stabilization are given in [Sam20] and in [CF13]. In characteristic zero, the irreducible representations of the symmetric groups are indexed by the partitions. Then the representation stability for these groups can be summarized as follows (see [CEF15, CEFN14, Far14]): a compatible family $(V_n)_n$ of representations is stable if we obtain the decomposition of the representation V_{n+1} of S_{n+1} by adding a box on the top row of the diagrams associated with the decomposition of the representation V_n of S_n. This process, along with the equivalence between these two definitions, is described on examples in [CF13] and [Wil18a, Ex. XXXI].

The theory of **FI**-modules was introduced in [CEF15] to encode this phenomenon in a single object: indeed, it is proven in [Far14] that, if a **FI**-module is finitely generated, then the associated family of representations of the symmetric groups is stable. Note that the converse is true for functors with finitely generated values, and that the proof is based on the noetherian property of **FI**-modules and on the fact that the families associated with the projective generators $P_n^{\mathbf{FI}}$ are stable as explained in [Wil18b]. The concrete examples of **FI**-modules introduced in [CF13] and [Wil18b] were first thought to be stable representations of the symmetric groups and were understood to be finitely generated **FI**-modules after, for example in [CEF15]. Another interesting example of representation stability is given by the cohomology of pure string motion groups. It is treated in detail in [Wil12] and illustrated by an example. In practice, it is generally easier to prove a finiteness result on one object than to prove the stability of an entire family.

The central results on representation stability are summarized and presented on a concrete example in [Wil18a, section 5]. The main tools of these results are the study of the representations appearing in the standard projective functors, and the *character polynomials* (see [Far14, 4.2] for a simple definition): it is shown in [CEF15] and [CMNR18] that the characters of a finitely generated **FI**-module eventually becomes equal to a polynomial. In particular, if F is a finitely generated **FI**-module over a field, then the dimension of the vector spaces $F(n)$ eventually become polynomial. This result, as many others about the **FI**-modules, was first proved in [CEF15] and in [Sno13, Theorem 3.1] over a field of characteristic zero, and was extended in [CEFN14] for more general rings. Moreover, Sam and Snowden showed in [Sno13] and [SS16] that if a **FI**-module is finitely generated then it's Hilbert series, encoding the dimension of its values, is of the form $p(t) + e^t q(t)$ where p and q are polynomials. For example, the character polynomials of [CEF15] can be recovered from the polynomial function p of this series and the polynomial function q can be recovered from the local cohomology.

This theory was extended in [Ram17a] to $\mathbf{FI}_d$-modules with a generalized notion of representation stability. Ramos then got the following result: a $\mathbf{FI}_d$-module F is finitely generated if and only if the space $F(n)$ is finite dimensional for all $n \in \mathbb{N}$ and, for any partition λ of weight $|\lambda|$ and any sequence of integers $n_1 \geq \cdots \geq n_d \geq |\lambda| + \lambda_1$, if $c_{\lambda,n_1,\ldots,n_d}$ denotes the multiplicity of the irreducible representation associated with the padded partition $(n_1 - |\lambda|, \dots n_d - |\lambda|, \lambda_1, \dots \lambda_h)$, then $c_{\lambda, n_1+l, \dots, n_d+l}$ is independent of l for l and n large enough. This theorem is a direct generalization of the analogous theorem of [CEF15, CEFN14] for **FI**-modules. Morally, the last point can be interpreted by saying that the irreducible representations associated with a partition of at least d rows eventually appear with a stable multiplicity in a finitely generated $\mathbf{FI}_d$-module. This theorem does not predict the behavior of the irreducible representations associated with smaller partitions, but the Theorem B from [Ram17a] treats some of these cases. Since then, Sam and Snowden defined an "enhanced" Hilbert series that encodes more information about the structure of a $\mathbf{FI}_d$-module as representations of the symmetric groups and they proved a result similar to the one for the "classical" Hilbert series above for this enhanced series, for $d = 1$ in [SS16] and for a general d in [SS17] and [SS18].

The strong polynomial functors

In a functor category there are very huge functors, often out of control, and the polynomial property is a way of measuring the complexity of a functor. Thus, polynomial functors should be thought of as an analog to polynomial functions for functors, which are easier to understand. The notion of polynomial functors dates back to the 1950s when Eilenberg and Mac Lane introduced it in [EM54] for functors between categories of modules. Since then, polynomial functors have been studied for a wide range of applications such as their connection to representation theory or group cohomology.

The original definition of Eilenberg and Mac Lane has been extended for different families of categories at the source, as in [HPV15] to the case where the source is a monoidal category whose unit is a null object. A complementary approach in the generalization of these polynomial functors is to study functors from a monoidal category to a non-abelian category such as the category of groups (see [BP99]). The definition of Eilenberg and Mac Lane based on the notion of cross effects is equivalent to the definition based on the differential functor as used by Kuhn in [Kuh94] and by Powell in [Pow98]. In [DV19] the authors introduce two notions of polynomial functors from a symmetric monoidal category $\mathcal{M}$ whose unit is an initial object to an abelian category: the naive generalization of polynomial functors gives the notion of strong polynomial functors which have some bad properties like not being closed under subobject. This leads to the weak polynomial functors defined by introducing a quotient category following the construction of Gabriel in [Gab62, pages 366-372]. The idea of this quotient category is to invert the morphisms whose kernel and cokernel are in the subcategory in question. The strong polynomial functors in this context are defined using the differential endofunctors δ_k, for $k \in \mathcal{M}$, generalizing the one from [Kuh94] and [Pow98]. In [DV19], Djament and Vespa also adapted the definition of cross effects to their framework and showed that the strong polynomial functors are equal to the ones obtained by using these cross effects. The definition using the differential endofunctors is better suited for the study of stable behaviour and has the advantage to be recursive, so we choose to mainly present and generalize this point of view for $\mathbf{FI}_d$-modules.

In particular, the category **FI** falls into the framework of Djament and Vespa and we get the following definition of strong polynomial **FI**-modules using only the differential endofunctor δ_1 since $1 \in \mathbf{FI}$ is a generator: the functor $F : \mathbf{FI} \to \mathbf{R}\text{-}\mathbf{Mod}$ is strong polynomial of degree n if we get the zero functor by applying $n + 1$ times the endofunctor δ_1 to it. This is analog to the usual polynomials: a function $f : \mathbb{R} \to \mathbb{R}$ is polynomial of degree n if its $(n + 1)$-th derivative is zero. The endofunctor δ_1 which plays the role of the derivative is used in various contexts: in Kuhn's and Powell's work over functors from $\mathbb{F}_p$-vector spaces to $\mathbb{F}_p$-vector spaces

([Kuh94, Pow98]), in representation stability theory ([CEF15, CEFN14, CE17, CMNR18]), in the definition of polynomial functors by Randal-Williams and Wahl in [RWW17], in the theory of twisted commutative algebras ([SS12, SS16]) or in the work of Ramos ([Ram17b, LR18]). The notions of polynomial functors introduced in [DV19] give an alternative way to express and understand results on **FI**-modules. For example, the strong polynomial functors with finitely generated values are the finitely generated **FI**-modules. Using [CEF15] we then deduce that, over a field of characteristic zero, the dimension of the vector spaces associated with a polynomial **FI**-module with finite dimensional values is eventually polynomial. There are many examples of polynomial **FI**-modules that occur in different contexts. In particular, a large number of the **FI**-modules presented in [CF13] are strong polynomial. The cohomology of configuration spaces over a regular manifold gives a strong polynomial **FI**-module of particular interest. Several **FI**-modules studied by Church, Ellenberg and Farb have more structure: they are $\mathbf{S(ab)}$-modules, were $\mathbf{S(ab)}$ is the category of abelian groups and split monomorphisms, which correspond to $\mathbf{VIC}(\mathbb{Z})$ from [Wil18a]. The polynomial $\mathbf{S(ab)}$-modules are studied in [DV19].

In Section 2.6 we define the strong polynomial functors over $\mathbf{FI}_d$ in a similar way as over **FI**, using a family of endofunctors δ_1^c indexed by the d colours of $\mathbf{FI}_d$ instead of just one endofunctor δ_1 for **FI**-modules. For $d = 1$ we recover the definition of strong polynomial functors over **FI** from [DV19] since the only colour in $\mathbf{FI}_1$ gives the unique endofunctor δ_1 of [DV19]. We also define a notion of cross effects for $\mathbf{FI}_d$-modules in Section 5.4 by introducing the coslice category $(0 \downarrow \mathbf{FI}_d)$ (sometimes called the undercategory under 0 as in [ML98, P.45]) of pairs (k, x) where k is an object of $\mathbf{FI}_d$ and x a morphism in $\mathbf{FI}_d(0, k)$. Indeed, we prove in Proposition 5.4.4 that the coslice category $(0 \downarrow \mathbf{FI}_d)$ is a monoidal category whose unit is an initial object, which allows us to define the cross effects of a $\mathbf{FI}_d$-module via the forgetful functor $(0 \downarrow \mathbf{FI}_d) \to \mathbf{FI}_d$ and the work of Djament and Vespa in [DV19]. We then show in Proposition 5.4.12 that the polynomial functors defined with the cross effects over $\mathbf{FI}_d$ are the same as the strong polynomial functors defined with the endofunctors δ_1^c:

Proposition. *For $n \in \mathbb{N}$ and F a $\mathbf{FI}_d$-module, F is in $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ if and only if $\mathrm{cr}_{n+1}(F)\big(\ -\ \big)$ is the zero functor over $(0 \downarrow \mathbf{FI}_d)^{\times n+1}$.*

We then use this alternative definition of strong polynomial $\mathbf{FI}_d$-modules to show in Proposition 5.4.18 the following result.

Proposition. *For $m, n \in \mathbb{N}$, if $F : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ is a strong polynomial functor of degree less than or equal to m and if $X : \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$ preserves epimorphisms and is a polynomial functor of degree less than or equal to n, then the composite $X \circ F : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$ is a strong polynomial functor of degree less than or equal to nm.*

We use this result to get in Theorem 5.5.4 that the pointwise tensor product of two strong polynomial $\mathbf{FI}_d$-modules is strong polynomial:

Theorem. *For $n, m \in \mathbb{N}$ and $F, G : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$, if F is in $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and if G is in $\mathrm{Pol}_m^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, then their tensor product $F \otimes G$ is in $\mathrm{Pol}_{2\max(n,m)}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.*

However, in this theorem the bound may be not the best possible. Indeed, we could expect for $F \otimes G$ to be strong polynomial of degree less than or equal to $n + m$. For example, for $d = 1$ it is shown in [Dja16] that a **FI**-module is strong polynomial of degree less than or equal to n if and only if it is a quotient of a sum of the standard projective functors $P_i^{\mathbf{FI}}$ for $i \leq n$. This allows us to prove that, over **FI** the tensor product $F \otimes G$ is polynomial of degree $n + m$ if F has degree n and G has degree m. We also prove in Appendix A the same result in the framework studied by

Djament and Vespa in [DV19], that is the functors over a general symmetric monoidal category whose unit is an initial object:

Theorem. *Let $\mathcal{M}$ be a small symmetric monoidal category whose unit is an initial object. For $n, m \in \mathbb{N}$ and $F, G : \mathcal{M} \to$ **R-Mod**, if F is in $\mathrm{Pol}_n^{strong}(\mathcal{M}, \mathbf{R\text{-}Mod})$ and if G is in $\mathrm{Pol}_m^{strong}(\mathcal{M}, \mathbf{R\text{-}Mod})$, then their tensor product $F \otimes G : \mathcal{M} \to$ **R-Mod** is in $\mathrm{Pol}_{2\max(n,m)}^{strong}(\mathcal{M}, \mathbf{R\text{-}Mod})$.*

For $d = 1$, the standard projective functors $P_n^{\mathbf{FI}}$ form a really important example of strong polynomial **FI**-modules, as shown in [Dja16, Proposition 4.4]. This makes the study of polynomial functors over **FI** much easier. In particular, it implies that being strong polynomial (with finitely generated values) is equivalent to being finitely generated for **FI**-modules. This is specific to the category **FI**, due to the fact that the standard projective functors are polynomial, and is not true in general on other categories. For the $\mathbf{FI}_d$-modules these results have no reason to hold since we show in Corollary 5.2.2 the following:

Proposition. *For $d > 1$, the standard projective functor $P_n^{\mathbf{FI}_d}$ is not strong polynomial.*

The example of configuration spaces

As explained above, there are many examples of **FI**-modules in the literature in a wide variety of areas. We mainly present one given by the homology of the configuration spaces of a manifold, which is fully described in [Sam20, Wil19] and [CF13]. For M a regular manifold, the rational cohomology of the configuration spaces of M is a finitely generated **FI**-module ([CEF15, Theorem 6.2.1]), which is almost equivalent to being strong polynomial. Furthermore, for M a connected manifold of dimension at least 2 and under some more assumptions, it was showed in [CMNR18, Theorem A] that $2k$ is an upper bound for the polynomial degree of the **FI**-module $H^i\left(\mathrm{Conf}_{(-)}\left(M\right), \mathbb{K}\right)$.

The results about the **FI**-module $H^i\left(\mathrm{Conf}_{(-)}\left(M\right), \mathbb{K}\right)$ are proved for a manifold of dimension at least two. This hypothesis is necessary to ensure that the configuration spaces are connected and that the points can move around each other. But for a manifold of dimension 1, like a graph, there is not enough space and the points block each other in the configuration spaces, so the same approach is no longer valid. For example, the configuration space of the linear graph with only one edge is homotopy equivalent to $n!$ disjoint points. Therefore, Ramos introduced in [Ram19] the homology of a kind of modified configuration spaces for graphs that form a $\mathbf{FI}_d$-module. In these modified spaces, called the *sink configuration spaces*, we take n (ordered) points on the graph, as for the classical ones, but now they can either be distinct two by two or they can overlap at a vertex of the graph but not within an edge. Then, the d vertices of the graph correspond to the d colours of $\mathbf{FI}_d$ which gives the structure of a $\mathbf{FI}_d$-module when we take the rational homology of these topological spaces. This gives an interesting example of $\mathbf{FI}_d$-module since, before this, all the $\mathbf{FI}_d$-modules in the literature were either free or obtained from **FI**-modules via the forgetful functor. Ramos proved in [Ram19] that these $\mathbf{FI}_d$-modules are finitely generated for every homological degree and every connected graph. In Proposition 3.2.8 we give an explicit description of these functors for the linear graphs:

Proposition. *For $\mathcal{G}_d$ the linear graph on d vertices, the $\mathbf{FI}_d$-module $H_0\left(Conf_{(-)}^{sink}\left(\mathcal{G}_d, [d]\right), \mathbb{Q} \right)$ is the constant functor $\mathbb{Q}$, while for $i \geq 1$ the $\mathbf{FI}_d$-module*

$$H_i\left(Conf_{(-)}^{sink}\left(\mathcal{G}_d, [d]\right), \mathbb{Q} \right)$$

is the functor sending n to $\mathbb{Q}^{N(d,i+1)}$ if $n = i+1$ and zero else, where

$$N(d,i+1) = \begin{cases} (d-1)^{i+1} - \binom{d-1}{i+1}(i+1)! & \text{if } d \geq i+2 \\ (d-1)^{i+1} & \text{if } d \leq i+1. \end{cases}$$

In Proposition 5.1.8 we deduce from this description that these functors are strong polynomial and we give their degree:

Proposition. *For $i \in \mathbb{N}$ and $\mathcal{G}_d$ the linear graph on d vertices, the $\mathbf{FI}_d$-module $H_i\left(Conf^{sink}_{(-)}(\mathcal{G}_d,[d]), \mathbb{Q} \right)$ is polynomial of degree 0 for $i = 0$, and of degree $i+1$ for $i > 1$.*

The twisted commutative algebras

The theory of twisted commutative algebras (TCAs) dates back to the 1950s and appeared in algebraic topology. It was introduced to study different structures, such as sequences of objects endowed with an action of linear or symmetric groups. It is also an analog of the theory of commutative algebra adapted to the study of representations of these groups. For example, in [Bar78] Barratt defined a general twisted algebra and added a condition to be a twisted Lie algebra or a twisted commutative algebra. As we will see, the $\mathbf{FI}_d$-modules appear in this context since there is an equivalence of categories between the $\mathbf{FI}_d$-modules and the modules over the free TCA on d generators.

A TCA is a monoid in the monoidal category $\mathbf{Fct}(\mathbf{\Sigma}, \mathbb{K}\text{-}\mathbf{Vect})$, where $\mathbf{\Sigma}$ is the category of finite sets and bijections. By considering several categories equivalent to $\mathbf{Fct}(\mathbf{\Sigma}, \mathbb{K}\text{-}\mathbf{Vect})$ we get different equivalent definitions of the TCAs, as explained in [SS12] and [GS10]: it can be a functor from vector spaces to commutative rings or a commutative ring endowed with an action of the infinite linear group by an algebra morphism, or an associative unital graded ring endowed with an action of the symmetric groups. In each case there is an additional condition, called polynomiality (in a different sense than the polynomial functors we study here) which is added to form a TCA. Sometimes the TCAs are also treated as objects of an abstract category equivalent to any of the previous ones, which leads to another equivalent definition given in [GS10] via operads. We choose to think of the TCAs mainly as functors $F : \mathbf{\Sigma} \to \mathbb{K}\text{-}\mathbf{Vect}$, endowed with a multiplication law $\nu : F \otimes F \to F$ and a unit law (Definition 4.1.5). The definition in terms of representations of the infinite linear group $GL(\infty)$, often used by Sam and Snowden, is well described in [SS12] and [DES17]. These two notions using the symmetric groups or the infinite linear group are equivalent for $\mathbb{K}$ of characteristic zero via the Schur-Weyl duality, but give two different notions of TCAs for $\mathbb{K}$ of positive characteristic.

The first example of TCA, coming from [Bar78], is the functor sending n to the space $\mathbb{K}[\mathrm{S}_n]$ on which the group S_n acts by conjugation and whose multiplication is given by the standard inclusion of $\mathrm{S}_n \times \mathrm{S}_m$ in S_{n+m}. An easy way to create other TCAs is to take the symmetric algebra of a representation of $GL(\mathbb{K}^\infty)$. These examples, called "polynomial TCAs" (which has nothing to do with our polynomial functors) are fully described in the different equivalent definitions in [SS12, Section 8.2.3]. We focus on the free TCAs on d generators of degree one $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$, which has been studied extensively, for example in [SS12, SS16, SS19, GS10]. In particular, Sam and Snowden showed in [SS12] that the category of modules over this TCA is equivalent, via a choice of a basis of $\mathbb{K}^d$, to the category of $\mathbf{FI}_d$-modules. As mentioned above, this explains how the $\mathbf{FI}_d$-modules appear in the theory of TCAs. We give the concrete description of the TCA $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ in Definition 4.1.15 and the detail of the equivalence in Section 4.2. Another such example of TCA is $\mathrm{Sym}(\Lambda^2(\mathbb{K}^\infty))$ which is studied in [SS15]. For example, they show that there is a equivalence similar to the one of $\mathbf{FI}_d$: the finitely generated modules over this TCA are equivalent to the finitely generated modules over the category **FIM** of [MW19] whose objects are finite sets and whose morphisms are pairs of injection and perfect matching on the complement of the image.

There is a natural action of $GL(\mathbb{K}^d)$ on the modules over the TCA $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ that acts diagonally on the components $(\mathbb{K}^d)^{\otimes n}$ of $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ before applying the multiplication law. In Section 4.3, we use the equivalence of categories from [SS12] to transform this into an action of $GL(\mathbb{K}^d)$ on the $\mathbf{FI}_d$-modules. We obtain in Proposition 4.3.5 the following concrete description:

Proposition. *Let $\mathcal{B}$ be a basis of $\mathbb{K}^d$, for $\varphi \in GL(\mathbb{K}^d)$ and $G \in \mathbf{FI}_d\textbf{-}\boldsymbol{Mod}$, the functor $\varphi_{\mathcal{B}} \cdot G : \mathbf{FI}_d \to \mathbb{K}\textbf{-}\boldsymbol{Vect}$ sends an object $n \in \mathbf{FI}_d$ to $G(n)$ and a morphism $(f,g) \in \mathbf{FI}_d(n,m)$ to the sum*

$$\sum_{g' \in \mathbf{FI}_d(0, m \smallsetminus f(n))} \left(\prod_{l \in m \smallsetminus f(n)} m_{g'(l), g(l)} \right) G(f, g').$$

were $(m_{i,j})_{1 \leq i,j \leq d}$ is the matrix of φ in the basis $\mathcal{B}$ of $\mathbb{K}^d$.

The weak polynomial functors

The notion of weak polynomial functors gives a refinement of the notion of strong polynomial functors which is more intuitive but lacks essential properties. Indeed, for a source category which is a symmetric monoidal category whose unit is a null object, the subcategories of polynomial functors are thick (see [Dja16] for the general case) which allows us to look at the quotients by these subcategories. However, when the unit is just an initial object as in **FI**, a subfunctor of a strong polynomial functor can be of higher degree or even non-polynomial. To avoid these instability phenomena, Djament and Vespa defined a notion of weak polynomial functors in [DV19] by erasing the problematic functors in a quotient category. They showed that the category $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ of these functors, called the stably zero functors, is composed of the **FI**-modules whose colimit is zero. These stably zero functors correspond to the torsion modules over the free TCA over a generator of degree 1 studied in [SS16] or [NSS18], and the endofunctor κ which gives the maximal subfunctor of a **FI**-module in $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ corresponds to the local cohomology functor denoted by $\mathrm{H}^0_{\mathfrak{m}}(-)$ in [SS16, NSS18, CEFN14]. In particular, the properties of their right derived functors $\mathrm{H}^i_{\mathfrak{m}}(-)$ are studied in [SS16, NSS18] in order to understand how $\mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ is constructed from the two pieces $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ and $\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. Similarly, the weak polynomial degree for **FI**-modules corresponds to the notion of stable degree of [CEF15] and [CEFN14] while the local degree precise how the weak and strong degrees are linked. It morally gives the strong polynomial degree modulo the weak polynomial degree and controls the rank from which the associated family of representations becomes stable.

One of the main goal of this thesis is to introduce and study weak polynomial $\mathbf{FI}_d$-modules. One of the differences with the previous situation is that there are several subcategories that can replace the stably zero functors in this case: the globally stably zero functors $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and the functors that are stably zero along different colour combination $\mathcal{SN}_{c_{i_1}, \ldots, c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. These subcategories form a refinement of the notion of stably zero functors introduced in [DV19] for **FI**. Indeed, for $d = 1$ there is an inclusion of the unique subcategory of stably zero functors $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ in $\mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ but, for a general d, these subcategories form a richer poset for the inclusion. For example, for $d = 2$, the poset looks like this:

$$
\begin{tikzcd}
 & \mathcal{SN}_{c_1}(\mathbf{FI}_2, \mathbf{R\text{-}Mod}) \arrow[dr, hook] & & \\
\mathcal{SN}_{c_1,c_2}(\mathbf{FI}_2, \mathbf{R\text{-}Mod}) \arrow[ur, hook] \arrow[dr, hook] & & \mathcal{SN}(\mathbf{FI}_2, \mathbf{R\text{-}Mod}) \arrow[r, hook] & \mathbf{Fct}(\mathbf{FI}_2, \mathbf{R\text{-}Mod}) \\
 & \mathcal{SN}_{c_2}(\mathbf{FI}_2, \mathbf{R\text{-}Mod}) \arrow[ur, hook] & &
\end{tikzcd}
$$

In Proposition 6.1.7 and Corollary 6.2.5 we show that the subcategories $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ and $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ are thick, that is closed under subobject, quotient and extension. Then we can consider the quotient category of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ by any of these subcategories following Gabriel's construction in [Gab62], and define polynomial objects in them using the endofunctors δ_1^c of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ which pass to the quotients. This is possible because these subcategories are closed under colimits and so the quotient functor π_d has a right adjoint $\mathcal{S}_d$ called the section functor.

The subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ of globally stably zero functors is defined in Section 6.1 using a family of endofunctors κ_1^c of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$. These functors are defined in Section 2.6 in a dual way to the δ_1^c, and they all fit in the exact sequence of endofunctors

$$0 \longrightarrow \kappa_1^c \longrightarrow \mathrm{Id} \xrightarrow{i_1^c} \tau_1 \longrightarrow \delta_1^c \longrightarrow 0 \ ,$$

were τ_1 is the shifting endofunctor $F(-) \mapsto F(-+1)$ and i_1^c a natural transformation associated to the colour c. We also define a poset structure on $\mathbb{N}^d$ for the product order and a functor $\xi_d : \mathbb{N}^d \to \mathbf{FI}_d$ that sends an object $(n_1, \dots, n_d) \in \mathbb{N}^d$ to the object $n_1 + \cdots + n_d$ of $\mathbf{FI}_d$. We then show in Proposition 6.1.5 that there is an equivalent definition of the category $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ using a filtered colimit over $\mathbb{N}^d$:

Proposition. *Let F be a $\mathbf{FI}_d$-module, then F is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ if and only if*

$$\operatorname*{colim}_{\mathbb{N}^d} F \circ \xi_d = 0.$$

For $d = 1$ we recover the description of $\mathcal{SN}(\mathbf{FI}, \mathbf{R\text{-}Mod})$ from [DV19, Proposition 5.7], namely that the stably zero functors are those whose colimit is zero. Recall that, by [SS12], the category of $\mathbf{FI}_d$-modules is equivalent to the category of $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$-modules. In Section 6.4 we give a description of $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ in terms of $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$-modules through this equivalence. We also show in Proposition 6.4.2 that, for $d > 1$, the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ is not closed under the action of $GL(\mathbb{K}^d)$ define above.

The subcategory $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ of functors that are stably zero along the colours $c_{i_1}, \dots, c_{i_m}$ is defined in Section 6.2 similarly to the globally stably zero functors, but using the endofunctors κ_1^c for each colour c in $\{c_{i_1}, \dots, c_{i_m}\}$. In Corollary 6.2.4 we show that these categories also admit an equivalent definition, this time via the colouring functors $\Delta_c^* : \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod}) \to \mathbf{Fct}(\mathbf{FI}, \mathbf{R\text{-}Mod})$:

Proposition. *A $\mathbf{FI}_d$-module F is in the subcategory $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ if and only if the functors $\Delta_c^*(F)$ are in the subcategory $\mathcal{SN}(\mathbf{FI}, \mathbf{R\text{-}Mod})$ of $\mathbf{Fct}(\mathbf{FI}, \mathbf{R\text{-}Mod})$ for all colours c in $\{c_{i_1}, \dots, c_{i_m}\}$.*

This equivalent definition allows us to use the results already proved for functors over $\mathbf{FI}$, especially those of Djament and Vespa in [DV19]. However, we show in Section 7.2 that in the quotient by a subcategory of functors that are stably zero along colours the polynomial objects are a bit harder to define. In this process we lose some important properties like the fact that the endofunctors κ_1^c become zero, and the endofunctors δ_1^c become exact in the quotient. This is a first reason why we develop only the weak polynomial functors corresponding to the global subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$: this one behaves better with the endofunctors δ_1^c which are a crucial tool for the study of polynomial functors.

In Chapter 7 we focus on the category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of stable functors, *i.e.* the quotient by the globally stably zero functors $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, the largest of these subcategories, in order to get a smaller quotient category that may be easier to describe. Although the objects of the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ are by definition the functors from $\mathbf{FI}_d$ to $\mathbf{R}\text{-}\mathbf{Mod}$, one should think of them as abstract objects since the morphisms in the quotient are modified by some isomorphisms classes. In Definition 7.2.1, we define the weak polynomial $\mathbf{FI}_d$-modules as the functors over $\mathbf{FI}_d$ whose image in the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by the quotient functor π_d is a polynomial object (we sometimes identify F and $\pi_d(F)$ by an abuse of notation). With this definition, a strong polynomial functor is weak polynomial but the converse is not true, which justifies the terminology introduced by Djament and Vespa in [DV19] for $\mathbf{FI}$-modules. We denote by $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ the full subcategory of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of polynomial objects of degree less than or equal to n. In an abuse of notation, it also denotes the full subcategory of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of functors whose image by the quotient functor π_d is a polynomial object of degree less than or equal to n. We then take $\mathbf{R} = \mathbb{K}$ a field to ensure that the tensor product functor is exact and we show in Theorem 7.3.6 that the pointwise tensor product of two polynomial objects of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is polynomial:

Theorem. *Let $\mathbf{R} = \mathbb{K}$ be a field, for $X \in \mathrm{Pol}_n(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$ and $Y \in \mathrm{Pol}_m(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$, we have $X \otimes Y \in \mathrm{Pol}_{n+m}(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$.*

While the comprehension of the categories of polynomial functors is a hard problem in general, except for small values, the quotient of polynomial functors of degree n modulo the functors of degree $n-1$ is well understood in several contexts. In particular, Djament and Vespa described this quotient in [DV19, Theorem 2.26] for the polynomial objects of $\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$, as we recall in Chapter 9. For $n = 0$, they get that the only objects in $\mathrm{Pol}_0(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ are the constant functors.

In Section 7.4 we describe the polynomial objects of degree 0 of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, which form a richer category than for $d = 1$. For this, we introduce in Definition 7.4.8 the category $\mathbf{R}\text{-}\mathbf{Mod}_d$ of $\mathbf{R}$-modules together with $d-1$ automorphisms which commute two by two. Similarly, we introduce the category of modules over the ring of commutative polynomials $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]$ in the $d-1$ variables $x_2, \dots, x_d$ all invertible. One of our main result is then the following description obtained in Theorem 7.4.12:

Theorem. *There are equivalences of categories between the category $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of polynomial objects of degree 0 of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, the category $\mathbf{R}\text{-}\mathbf{Mod}_d$ and the category $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]\text{–}\mathbf{Mod}$.*

For $d = 1$ we recover that the polynomial $\mathbf{FI}$-modules of degree 0 are the constant functors, but for a general d these functors form a more complex category. We prove this theorem in two steps: first, we show in Proposition 7.4.2 that the polynomial objects of degree 0 of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ satisfy an abstract condition called $(POL0)$. We then use the intermediate category $\underline{\mathbf{FI}_d}$ defined in Section 2.5 to show, in Propositions 7.4.6 and 7.4.7 that, for each object F in the quotient satisfying $(POL0)$, the image of F by the section functor $\mathcal{S}_d$ is completely determined by its image on the morphisms $c \in \mathbf{FI}_d(0,1)$. These images of the morphisms $c \in \mathbf{FI}_d(0,1)$ correspond to the $d-1$ module isomorphisms of the category $\mathbf{R}\text{-}\mathbf{Mod}_d$ when we trivialize the action of c_1. From the point of view of $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]$-modules the images of the morphism $c_i \in \mathbf{FI}_d(0,1)$ correspond to the action of x_i, where x_1 acts by the identity when we trivialize the action of c_1.

Examples of polynomial quotients of the functors $P_n^{\mathbf{FI}_d}$

The fact that the standard projective generators $P_n^{\mathbf{FI}_d}$ are strong polynomial for $d=1$ simplifies the study of polynomial functors over the category $\mathbf{FI}$. As explained above, this is not the case for $d>1$. Therefore we describe several quotients of the functors $P_n^{\mathbf{FI}_d}$ which are polynomial. In addition to providing some concrete examples, these quotients may also give us a better idea of what the polynomial functors on $\mathbf{FI}_d$ look like. For example, in Section 8.1 we obtain a family of quotients of the functor $P_0^{\mathbf{FI}_d}$ which are weak polynomial of degree 0 by filtering its generators by the number of occurrences of the colours. Indeed, for $k_1, \dots, k_d \in \mathbb{N}$, $I \subset \{c_1, \dots, c_d\}$ and $\alpha \in \mathbf{FI}_d(0,k)$ we denote by $\gamma_i(\alpha)$ the number of occurrences of the colour c_i in α. We then say that $\alpha \in \mathbf{FI}_d(0,k)$ satisfies the condition $(P_{I,k_1,\dots,k_d})$ if $\gamma_i(\alpha) \geq k_i$ for all $i \in I$, or there exists $j \in \{c_1, \dots, c_d\} \setminus I$ such that $\gamma_j(\alpha) \geq k_j$. With these notations, we introduce in Definition 8.1.8 the subfunctor $G_{I,k_1,\dots,k_d}$ of $P_0^{\mathbf{FI}_d}$ given by

$$G_{I,k_1,\dots,k_d}(n) = \big\langle \alpha - X \,|\, \alpha \in \mathbf{FI}_d(0,n) \text{ that satisfies the condition } (P_{I,k_1,\dots,k_d}) \big\rangle$$

that is the submodule generated by the elements $\alpha - X$, for $\alpha \in \mathbf{FI}_d(0,n)$ satisfying the condition $(P_{I,k_1,\dots,k_d})$ and $X \in \mathbf{FI}_d(0,n)$ a given morphism in satisfying $(P_{I,k_1,\dots,k_d})$. We then show in Proposition 8.1.15 the following:

Proposition. *For $k_1, \dots, k_d \in \mathbb{N}$ and $I \subset \{c_1, \dots, c_d\}$, the quotient of $P_0^{\mathbf{FI}_d}$ by its subfunctor $G_{I,k_1,\dots,k_d}$ is weak polynomial of degree 0.*

Moreover, the proof is based on the Lemma 8.1.14 which shows that this quotient is equal to a constant functor modulo a stably zero functor of $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. This implies that its image in the quotient corresponds, through the equivalence giving the description of $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, to the object $(\mathbf{R}, \mathrm{Id}, \dots, \mathrm{Id})$ of $\mathbf{R}\text{-}\mathbf{Mod}_d$ or to the trivial $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]$-module.

In parallel, in Section 8.2 we study the quotient of the functor $P_n^{\mathbf{FI}_d}$ by its subfunctor corresponding to the action of the symmetric groups by post-composition. This subfunctor, denoted by F_n in Definition 8.2.1, is given on objects by

$$F_n(m) = \langle\, \sigma \circ (f,g) - (f,g) \,|\, (f,g) \in \mathbf{FI}_d(n,m),\, \sigma \in \mathrm{S}_m \rangle\,.$$

We show that the quotient of the functor $P_n^{\mathbf{FI}_d}$ by F_n is weak polynomial in Theorem 8.2.11:

Theorem. *For all $n \in \mathbb{N}$, the quotient functor of $P_n^{\mathbf{FI}_d}$ by F_n is weak polynomial of degree 0, where F_n is the subfunctor of $P_n^{\mathbf{FI}_d}$ from Definition 8.2.1.*

A nice representative of the image of this quotient in the category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ could help us to describe it in the category $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]$–$\mathbf{Mod}$ through the equivalence giving the description of $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. However, we explain in Section 8.2 that it is not easy to find one since the passage to the quotient category is not an explicit construction.

In Section 8.3 we give a quotient of $P_n^{\mathbf{FI}_d}$ which is weak polynomial of degree n: for a morphism (f,g) in $\mathbf{FI}_d(n,m)$ the second map g corresponds to a choice of $m-n$ colours. So there exists an action of the symmetric group S_{m-n} permuting these colour choices, which gives an action of S_{m-n} on $P_n^{\mathbf{FI}_d}(m)$. The subfunctor of $P_n^{\mathbf{FI}_d}$ corresponding to this action of the symmetric groups, denoted by H_n in Definition 8.3.2, is given on objects by

$$H_n(m) = \langle\, (f, \sigma \cdot g) - (f,g) \,|\, (f,g) \in \mathbf{FI}_d(n,m),\ \sigma \in \mathrm{S}_{m-n} \rangle \,.$$

We show that the quotient of the functor $P_n^{\mathbf{FI}_d}$ by H_n is weak polynomial in Theorem 8.3.14:

Theorem. *For all $n \in \mathbb{N}$, the quotient of $P_n^{\mathbf{FI}_d}$ by H_n is weak polynomial of degree n, where H_n is the subfunctor of $P_n^{\mathbf{FI}_d}$ from Definition 8.3.2.*

We prove this in two ways: first, we directly compute δ_1^c of this quotient, which is very similar to the computation of δ_1^c of $P_n^{\mathbf{FI}_d}$ in Proposition 5.2.1 but, since we take the quotient by the action of the symmetric groups on the colours, the component that prevents $P_n^{\mathbf{FI}_d}$ from being polynomial vanishes here. Second, we introduce in Definition 8.2.5 the category $\mathcal{C}_d$ whose objects are the integers and whose morphisms from n to m are the $(m-n)$-tuple of colours $(c_{i_1}, \ldots, c_{i_{m-n}})$ quotiented by the action of S_{m-n} (which is the same as the unordered choices of $m-n$ colours). We then show in Proposition 8.2.9 that the quotient of $P_n^{\mathbf{FI}_d}$ by F_n is equivalent to the functor $P_n^{\mathcal{C}_d}$, and describe the quotient of $P_n^{\mathbf{FI}_d}$ by H_n as a tensor product in Proposition 8.3.8 via the formula:

Proposition. *For all $n \in \mathbb{N}$, there is a natural isomorphism*

$$P_n^{\mathbf{FI}_d}/H_n \cong \left(\mathcal{O}^* P_n^{\mathbf{FI}} \right) \otimes P_0^{\mathcal{C}_d}\left((-) - n \right) \circ \Omega \,,$$

where $\mathcal{O}$ is the forgetful functor $\mathbf{FI}_d \to \mathbf{FI}$ and $\Omega : \mathbf{FI}_d \to \mathcal{C}_d$ sends $n \in \mathbf{FI}_d$ to $n \in \mathcal{C}_d$ and a morphism $(f,g) \in \mathbf{FI}_d(n,m)$ to the colours of g quotiented by the action of S_{m-n}.

This explains how the injections and the colours are mixed to form the functor $P_n^{\mathbf{FI}_d}$ up to the action of the symmetric groups on the colour choices. Moreover, since the functor $P_n^{\mathbf{FI}}$ is strong polynomial of degree n for $d = 1$, the image of the arrows from $P_n^{\mathbf{FI}}$ to the direct sum of all $P_k^{\mathbf{FI}}$ for $k \leq i$ is weak polynomial of degree i for any $i \in \mathbb{N}$. We then construct in Proposition 8.4.5 a quotient of $P_n^{\mathbf{FI}_d}$ that is weak polynomial of degree i for any $i \in \mathbb{N}$ using the above formula for the quotient of $P_n^{\mathbf{FI}_d}$ by H_n.

The construction Cospan

In order to study the polynomial functors over symmetric monoidal categories whose unit is an initial object, Djament and Vespa introduced in [DV19] a functor $\mathcal{M} \to \tilde{\mathcal{M}}$ which transforms the category $\mathcal{M}$ whose unit is an initial object into the category $\tilde{\mathcal{M}}$ whose unit is a null object. This construction, which is universal in the sense that it gives an adjoint to the forgetful functor, morally adds "decreasing" morphisms from the objects of the category to the unit while preserving the "increasing" morphisms from the unit to the objects. This construction is equivalent to the construction Cospan(−) of [Ves07] where the functors over Cospan can be seen as a generalization of the Mackey functors. Since this construction preserves the polynomial functors, it allows Djament and Vespa in [DV19, Theorem 4.8] to turn the study of polynomial functors over a category whose unit is an initial object into the study of polynomial functors over a category whose unit is a null object, which are better known.

They then apply this result to **FI** whose unit is an initial object. It allows them to describe the quotient of polynomial objects (in the quotient category $\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$) of degree less than or equal to n over **FI** by its thick subcategory of polynomial functors of degree less than or equal to $n-1$. Indeed, the categories $\mathrm{Cospan}(\mathbf{FI})$ and $\tilde{\mathbf{FI}}$ are equivalent to the category $\mathbf{FI}\#$ of partial injections of finite sets of [CEF15]. They then use a variation of a Dold-Kan type theorem of Pirashvili to describe the same quotient for functors over $\mathrm{Cospan}(\mathbf{FI})$. This Pirashvili's theorem from [Pir00] gives an equivalence of categories between the functors over the category Γ of pointed finite sets and the functors over Ω the category of finite sets and surjections, using the cross effects. The variation used in [DV19], which is described explicitly in [CEF15, Theorem 4.1.5], gives an equivalence of categories between the functors over the category $\mathbf{FI}\#$ and the functors over the category $\mathbf{\Sigma}$ of finite sets and bijections, since $\mathbf{FI}\#$ is a subcategory of Γ and $\mathbf{\Sigma}$ of Ω. The combination of these two results give the following description in [DV19, Proposition 5.9]: for $n \in \mathbb{N}$, there is an equivalence of categories

$$\mathrm{Pol}_n(\mathbf{FI}) \big/ \mathrm{Pol}_{n-1}(\mathbf{FI}) \cong \mathrm{Pol}_n(\mathrm{Cospan}(\mathbf{FI})) \big/ \mathrm{Pol}_{n-1}(\mathrm{Cospan}(\mathbf{FI})) \cong \mathbf{Fct}\left(\mathbf{\Sigma}_n, \mathbf{R}\text{-}\mathbf{Mod}\right),$$

where $\mathbf{\Sigma}_n$ is the category associated with the symmetric group S_n. We show that this approach cannot be directly generalized to describe the polynomial functors over $\mathbf{FI}_d$.

In Chapter 9 we introduce a generalization of the construction Cospan for $\mathbf{FI}_d$ as follows: the objects of $\mathrm{Cospan}(\mathbf{FI}_d)$ are the same as the objects of $\mathbf{FI}_d$ and the morphisms are classes of diagrams under an equivalence relation. These diagrams are morally composed of an injection and two different colour choices on different sets which interact with each other. Thus, we show in Proposition 9.2.8 that the category $\mathrm{Cospan}(\mathbf{FI}_d)$ is isomorphic to a combinatorial category $\mathbf{FI}_d\#$ whose morphisms consist of a partial injection and two distinct colour choices, one on the complement at the source and one on the complement at the target. Moreover, we show in Proposition 9.1.6 that each morphism in $\mathrm{Cospan}(\mathbf{FI}_d)$ admits a minimal representative diagram of the class, which implies that both the morphisms from 0 to n and the morphisms from n to 0 in $\mathrm{Cospan}(\mathbf{FI}_d)$ are in bijection with $\mathbf{FI}_d(0,n)$. This emphasizes that the category $\mathrm{Cospan}(\mathbf{FI}_d)$ is essentially obtained by keeping the morphisms from 0 to n of $\mathbf{FI}_d$ and adding new morphisms from n to 0 corresponding to them.

We then study the $\mathrm{Cospan}(\mathbf{FI}_d)$-modules as we did for $\mathbf{FI}_d$-modules: in Section 9.3 we define the polynomial functors on $\mathrm{Cospan}(\mathbf{FI}_d)$ using a family of endofunctors δ_1^c of $\mathrm{Cospan}(\mathbf{FI}_d)$ for the different colours. A major difference is that the stably zero functors over $\mathrm{Cospan}(\mathbf{FI}_d)$ are zero since this category has a null object, so the weak and strong notions of polynomial functors over $\mathrm{Cospan}(\mathbf{FI}_d)$ coincide. We then obtain in Theorem 9.4.9 the following description of the polynomial $\mathrm{Cospan}(\mathbf{FI}_d)$-modules of degree 0:

Theorem. *A functor $F \in \mathbf{Fct}(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ is in $\mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ if and only if it is a constant functor. There is an equivalence of categories*

$$\mathrm{Pol}_0(\ \mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod}\ ) \cong \mathbf{R}\text{-}\mathbf{Mod}\,.$$

Together with the description of the polynomial objects of degree 0 of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ (Theorem 7.4.12), this shows that for a general d the first equivalence of [DV19, Proposition 5.9] presented above already fails for $n = 0$, that is the quotient of $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by $\mathrm{Pol}_{n-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is not equivalent to the same quotient over $\mathrm{Cospan}(\mathbf{FI}_d)$.

Document layout

The organization of the manuscript is the following: in the first chapter we recall the construction and the important facts about the quotient of a category by a thick subcategory. In Chapter

2, we present the $\mathbf{FI}_d$-modules and give an overview of the basic results already known about them. We also introduce the main tools for their study and describe the simple objects of this category. In Chapter 3 we present the example of configuration spaces and describe it explicitly in simple cases. The Chapter 4 concerns the twisted commutative algebras and their connection with the $\mathbf{FI}_d$-modules. In Chapter 5 we define the strong polynomial functors over $\mathbf{FI}_d$ and give examples and counterexamples. We also extend the cross effects to these functors and show that the resulting notion of polynomial functors coincides with the one using the differential endofunctor. Chapter 6 is dedicated to the different notions of stably zero functors and the poset they form. In Chapter 7 we study the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and the polynomial functors in this quotient. In particular, we describe the weak polynomial objects of degree zero of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, which are not just the constant functors. In Chapter 8 we give examples of polynomial quotients of the standard projective functors. Finally, in the last chapter we introduce the category $\mathrm{Cospan}(\mathbf{FI}_d)$, and we show that the method of [DV19] for describing the weak polynomial $\mathbf{FI}$-modules does not work the same for weak polynomial $\mathbf{FI}_d$-modules.

Chapter 1

Recollection on quotient categories

Il n'y a pas besoin de brûler les livres pour détruire une culture, juste de faire en sorte que les gens arrêtent de les lire.

Ray Bradbury

The aim of this section is to recall the construction and some important properties of the quotient of a category by a thick subcategory. Most of these properties are taken from the pages 366-372 of Gabriel's thesis [Gab62] and we refer to it for the proofs of these propositions. In this section $\mathcal{A}$ is an abelian category and $\mathcal{C}$ is a subcategory of $\mathcal{A}$.

1.1 Definition of a quotient category

We start with the construction of the quotient of the category $\mathcal{A}$ by $\mathcal{C}$, when $\mathcal{C}$ is a thick subcategory which is defined below. We will see that this construction depends on the thick hypothesis, so it will be important in the following sections to always check whether the subcategories we are considering are thick or not. The idea of this quotient category is to inverse the morphisms whose kernel and cokernel are in the subcategory $\mathcal{C}$.

Definition 1.1.1. A subcategory $\mathcal{C}$ of $\mathcal{A}$ is *thick* if it is closed under subobjects, quotients and extensions. In other words, $\mathcal{C}$ is thick if, for every short exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ in $\mathcal{A}$ the object N is in $\mathcal{C}$ if and only if both M and P are in $\mathcal{C}$.

Since $\mathcal{A}$ is an abelian category it admits a biproduct denoted by $\coprod$. We then give basic results on the thick subcategories that we use in the following constructions:

Lemma 1.1.2. *For $\mathcal{C}$ a thick subcategory of an abelian category $\mathcal{A}$ and any two objects A and B of $\mathcal{C}$, then $A \coprod B$ is in $\mathcal{C}$ and, if A and B are subobjects of $C \in \mathcal{A}$, then $A + B := \operatorname{Im}(A \coprod B \to C)$ is in $\mathcal{C}$.*

Proof. The first point is a classical result about abelian categories (see for example [ML98]) obtained using the short exact sequence

$$0 \longrightarrow A \longrightarrow A \textstyle\coprod B \longrightarrow B \longrightarrow 0 \ ,$$

and the second point comes from the definition of a thick subcategory since $A + B$ is a quotient of $A \coprod B$. $\square$

In order to define the quotient category we introduce some notations.

Definition 1.1.3. For a thick subcategory $\mathcal{C}$ of $\mathcal{A}$ and any two objects A and B of $\mathcal{A}$ we define

- A *poset*
$$I_{A,B} = \left\{ (A',B') \in \mathcal{A}^2 \;\middle|\; A' \subset A,\, {}^{A}\!/_{A'} \in \mathcal{C},\, B' \subset B,\, B' \in \mathcal{C} \right\},$$
where the order relation is given by: $(A',B') \leq (A'',B'')$ if $A' \supset A''$ and $B' \subset B''$.

- The *category* $\mathcal{I}_{A,B}$ associated with the poset $I_{A,B}$.

- The *functor* $F_{A,B} : \mathcal{I}_{A,B} \to \mathbf{Ab}$ sending an object (A',B') of $\mathcal{I}_{A,B}$ to the group $\mathrm{Hom}_{\mathcal{A}}(A', {}^{B}\!/_{B'})$ and a map $(A',B') \leq (A'',B'')$ in $\mathcal{I}_{A,B}$ to the map
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(A', {}^{B}\!/_{B'}) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A'', {}^{B}\!/_{B''}) \\ f & \mapsto & p^{B'}_{B''} \circ f \circ i^{A'}_{A''}, \end{array}$$
where $i^{A'}_{A''}$ is the inclusion of A'' in A' and $p^{B'}_{B''}$ the projection of ${}^{B}\!/_{B'}$ onto ${}^{B}\!/_{B''}$.

We then want to define the quotient category ${}^{\mathcal{A}}\!/_{\mathcal{C}}$ as the category with the same objects as $\mathcal{A}$ and whose morphisms from A to B are the elements of the colimit of $F_{A,B}$ as in [Gab62, p. 365]. However, using this as a definition of the morphisms, it would be really abstract and not easy to use and the composition would be hard to define. We then use the fact that the category $\mathcal{I}_{A,B}$ is filtered when $\mathcal{C}$ is thick to give a more concrete description of this colimit.

Definition 1.1.4. A category $\mathcal{M}$ is *filtered* if it is not empty, if for every two objects X and Y in $\mathcal{M}$ there exist an object Z and two arrows $f : X \to Z$ and $g : Y \to Z$ in $\mathcal{M}$ and if for every two parallel arrows $f, g : X \to Y$ there exist an object Z and an arrow $h : Y \to Z$ such that $h \circ f = h \circ g$. The colimit of a functor is a filtered colimit if the source category is a filtered category.

Lemma 1.1.5. *For a thick subcategory $\mathcal{C}$ of $\mathcal{A}$ and for any two objects $A, B \in \mathcal{A}$, the category $\mathcal{I}_{A,B}$ is filtered.*

Proof. For (A',B') and (A'',B'') two objects of $\mathcal{I}_{A,B}$, we pose $X = A' \cap A''$ and $Y = B' + B'' := \mathrm{Im}(B' \coprod B'' \to B)$. Since $\mathcal{C}$ is thick (X,Y) is an object of $\mathcal{I}_{A,B}$, *i.e.* an element of the poset $I_{A,B}$. Then we have that $(A',B') \leq (X,Y)$ and $(A'',B'') \leq (X,Y)$ by construction. Furthermore, the element $(A,0)$ is minimal in $I_{A,B}$ so $\mathcal{I}_{A,B}$ is non-empty and, by construction there is one or zero arrow in $\mathcal{I}_{A,B}$ between two objects, so two parallel arrows in $\mathcal{I}_{A,B}$ are equal. □

We now recall a description of filtered colimits over $\mathbf{R}$-modules, which will be useful to describe the colimit of $F_{A,B}$ in a concrete way, even though it is not directly related to quotient categories.

Proposition 1.1.6. *[Bor94, Proposition 2.13.3] For $F : \mathcal{C} \to \mathbf{R}\text{-}\mathbf{Mod}$ a functor, if $\mathcal{C}$ is a small filtered category then the colimit $(M, \mu_C : F(C) \to M)$ of F is given by:*

- *The $\mathbf{R}$-module M is the quotient of the direct sum of all $F(C)$ for $C \in \mathcal{C}$ by the equivalence relation given by: $a \in F(C)$ and $a' \in F(C')$ are equivalent if there exist $C'' \in \mathcal{C}$ and two maps $f \in \mathcal{C}(C, C'')$ and $f' \in \mathcal{C}(C', C'')$ in $\mathcal{C}$ such that $F(f)(a) = F(f')(a')$,*

- *The map of $\mathbf{R}$-modules $\mu_C : F(C) \to M$ is the composition of the inclusion of $F(C)$ in the direct sum and of the quotient map by the equivalence relation that sends an element $a \in F(C)$ to its equivalence class.*

In particular, an element $a \in F(C)$ is in the same equivalence class as zero if and only if there exists an object $C'' \in \mathcal{C}$ and a map $f : C \to C''$ such that $F(f)(a) = 0$.

We deduce the following description of the colimit of the functor $F_{A,B}$ over the category $\mathcal{I}_{A,B}$.

Corollary 1.1.7. *For $A, B \in \mathcal{A}$, there is an isomorphism*

$$\operatorname*{colim}_{\mathcal{I}_{A,B}} F_{A,B} \cong \bigoplus_{(A',B')\in I_{A,B}} \mathrm{Hom}_{\mathcal{A}}\left(A', {}^{B}\!/_{B'}\right) /_{\sim}$$

where $f \in \mathrm{Hom}_{\mathcal{A}}(A', {}^{B}\!/_{B'})$ and $\tilde{f} \in \mathrm{Hom}_{\mathcal{A}}(\tilde{A}', {}^{B}\!/_{\tilde{B}'})$ are equivalent if there exist $X \subset A' \cap \tilde{A}'$ and $Y \supset B' + \tilde{B}'$ such that $(X, Y) \in I_{A,B}$ and $f|_X = \tilde{f}|_X : X \to {}^{B}\!/_{Y}$.

Proof. The Proposition 1.1.6 implies that the colimit of $F_{A,B}$ is equivalent to the direct sum of all $\mathrm{Hom}_{\mathcal{A}}\left(A', {}^{B}\!/_{B'}\right)$ for $(A', B') \in I_{A,B}$ quotient by an equivalence relation. This relation is given by: $f \in \mathrm{Hom}_{\mathcal{A}}(A', {}^{B}\!/_{B'})$ and $\tilde{f} \in \mathrm{Hom}_{\mathcal{A}}(\tilde{A}', {}^{B}\!/_{\tilde{B}'})$ are equivalent if there exist $(X, Y) \in I_{A,B}$, $\varphi \in \mathrm{Hom}_{\mathcal{I}_{A,B}}\left((A', {}^{B}\!/_{B'}), (X, Y) \right)$ and $\tilde{\varphi} \in \mathrm{Hom}_{\mathcal{I}_{A,B}}\left((\tilde{A}', {}^{B}\!/_{\tilde{B}'}), (X, Y) \right)$ such that $F_{A,B}(\varphi)(f) = F_{A,B}(\tilde{\varphi})(\tilde{f})$. The result then follows from the definitions of $I_{A,B}$ and of $F_{A,B}$ on the arrows. □

We can now define the quotient of an abelian category $\mathcal{A}$ by a subcategory $\mathcal{C}$ if it is a thick subcategory.

Definition 1.1.8. The *quotient category* ${}^{\mathcal{A}}\!/_{\mathcal{C}}$ of the abelian category $\mathcal{A}$ by its thick subcategory $\mathcal{C}$ is given by:

- The objects of ${}^{\mathcal{A}}\!/_{\mathcal{C}}$ are the objects of $\mathcal{A}$,
- The morphisms in ${}^{\mathcal{A}}\!/_{\mathcal{C}}$ from A to B are the elements of the direct limit

$$\lim_{(A',B')\in I_{A,B}} \mathrm{Hom}_{\mathcal{A}}(A', {}^{B}\!/_{B'}) = \operatorname*{colim}_{\mathcal{I}_{A,B}} F_{A,B} \cong \bigoplus_{(A',B')\in I_{A,B}} \mathrm{Hom}_{\mathcal{A}}\left(A', {}^{B}\!/_{B'}\right) /_{\sim}$$

  where the equivalence relation is given in Corollary 1.1.7. We denote by $[f]$ the class of a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A', {}^{B}\!/_{B'})$ in this quotient.
- The composition of two morphisms is defined by choosing a representative of the class of each morphism and by composing the (co)-restrictions of them in a natural way:

$$\begin{array}{ccc} \mathrm{Hom}_{A/C}(A, B) \times \mathrm{Hom}_{A/C}(B, C) & \to & \mathrm{Hom}_{A/C}(A, C) \\ ([f], [g]) & \mapsto & [\,\tilde{g} \circ \alpha \circ \tilde{f}\,], \end{array}$$

  where $[\,\tilde{g} \circ \alpha \circ \tilde{f}\,]$ is the class of the composition $\tilde{g} \circ \alpha \circ \tilde{f}$. These last morphisms are defined by: α is the isomorphism ${}^{A' + A''}\!/_{A'} \cong {}^{A''}\!/_{A' \cap A''}$, $\tilde{f}$ is the (co)-restriction $f : f^{-1}({}^{A' + A''}\!/_{A'}) \to {}^{A' + A''}\!/_{A'}$ where $f \in \mathrm{Hom}_{\mathcal{A}}(A', {}^{B}\!/_{B'})$ is a representative of $[f]$ and $\tilde{g} : {}^{A''}\!/_{A' \cap A''} \to {}^{C}\!/_{C' + g(A' \cap A'')}$ is the morphism obtained as $g : A'' \to {}^{C}\!/_{C'}$ passing to the quotient, where $g \in \mathrm{Hom}_{\mathcal{A}}(B'', {}^{C}\!/_{C'})$ is a representative of $[g]$.

Remark 1.1.9. The idea of the composition is to restrict f at the target to the subobject ${}^{A' + A''}\!/_{A'}$ of ${}^{A}\!/_{A'}$ and g at the source to ${}^{A''}\!/_{A' \cap A''}$, so that we can compose them via the isomorphism $\alpha : {}^{A' + A''}\!/_{A'} \cong {}^{A''}\!/_{A' \cap A''}$. The hypothesis that $\mathcal{C}$ is thick is crucial to define the composition in such a way because we use that these objects (or quotients of them) are in $\mathcal{C}$. However, this definition of the composition of two morphisms depends on the choice of two representatives f of $[f]$ and g of $[g]$. We check [Gab62, p.365] that the result does not depend on these choices by making a commutative diagram showing, for $[f'] = [f]$ and $[g'] = [g]$, that $[\,\tilde{g'} \circ \alpha \circ \tilde{f'}\,] = [\,\tilde{g} \circ \alpha \circ \tilde{f}\,]$ when restricted to some $(E, F) \in I_{A,B}$ so that E is small enough to get the information from f and f', and F is large enough to get the information from g and g'.

Proposition 1.1.10. *[Gab62, Proposition 1 p.367] The quotient category $\mathcal{A}/\mathcal{C}$ is abelian.*

We give an example of morphisms in the quotient category which illustrates that this quotient is difficult to understand explicitly, even in simple cases.

Example 1.1.11. For $\mathcal{C}$ a thick subcategory of $\mathcal{A}$ and $A, B \in \mathcal{A}$, if A or B is in $\mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(A, B) = 0.$$

Indeed, if $A \in \mathcal{C}$, for (A', B') in $I_{A,B}$ we have $(0, B') \in I_{A,B}$ since $A/0 = A \in \mathcal{C}$, and by definition of the order on the poset $I_{A,B}$ we have $(A', B') \leq (0, B')$. Then the map $F_{A,B}(A', B') \to \mathrm{colim}\ F_{A,B}$ factors through $F_{A,B}(0, B') = \mathrm{Hom}_{\mathcal{A}}(0, B/B') = 0$. This shows that the maps $F_{A,B}(A', B') \to \mathrm{colim}\ F_{A,B}$ are zero for all $(A', B') \in I_{A,B}$ if $A \in \mathcal{C}$, and so the colimit is zero by minimality. In the case where $B \in \mathcal{C}$, we apply the same reasoning but with $(A', B) \in I_{A,B}$ satisfying $(A', B') \leq (A', B)$ and $F_{A,B}(A', B) = \mathrm{Hom}_{\mathcal{A}}(A', B/B) = 0$.

1.2 The quotient functor

In this section we describe the properties of the canonical quotient functor π from $\mathcal{A}$ to the quotient $\mathcal{A}/\mathcal{C}$.

Definition 1.2.1. The canonical *quotient functor* $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ sends an object A in $\mathcal{A}$ to itself in $\mathcal{A}/\mathcal{C}$ and a morphism f on its class $[f]$ in the colimit for $(A, 0) \in I_{A,B}$, according to Definition 1.1.8.

We now describe some properties of this quotient functor.

Proposition 1.2.2. *[Gab62, Proposition 1 p.367] The quotient functor π is essentially surjective and exact.*

Moreover, the quotient functor is almost a full functor: it is full up to isomorphisms, as explained in the following proposition.

Proposition 1.2.3. *[Gab62, Corollary 1 p.368] For any short exact sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ in $\mathcal{A}/\mathcal{C}$, there is a short exact sequence*

$$0 \longrightarrow M_1 \xrightarrow{f_1} N_1 \xrightarrow{g_1} P_1 \longrightarrow 0$$

in $\mathcal{A}$ such that the induced sequence in the quotient obtained by applying the quotient functor is exact, and there exist isomorphisms $u : M \cong \pi(M_1)$, $v : N \cong \pi(N_1)$, $w : P \cong \pi(P_1)$ in $\mathcal{A}/\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "u"] & N \arrow[r, "g"] \arrow[d, "v"] & P \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & \pi(M_1) \arrow[r, "\pi(f_1)"'] & \pi(N_1) \arrow[r, "\pi(g_1)"'] & \pi(P_1) \arrow[r] & 0 .
\end{tikzcd}$$

We now describe the conditions under which a morphism in $\mathcal{A}$ is sent by the quotient functor π to a monomorphism, an epimorphism or to zero in the quotient. We give a sketch of the proof since it gives another example of how to compute morphism in the quotient category.

Proposition 1.2.4. *[Gab62, Lemma 3 p.366] Let $f : A \to B$ be a morphism in the category $\mathcal{A}$, the morphism $\pi(f)$ in the quotient $\mathcal{A}/_{\mathcal{C}}$ is zero (resp. a monomorphism, an epimorphism) if and only if* $\mathrm{Im}(f)$ *(resp.* $\ker(f)$, $\mathrm{Coker}(f)$*) is in the subcategory $\mathcal{C}$ of $\mathcal{A}$. In particular, if $\pi(A) = 0$ then A is in the subcategory $\mathcal{C}$ of $\mathcal{A}$.*

Proof. We give a sketch of the proof in the first case, the other being similar. If $\mathrm{Im}(f) \in \mathcal{C}$, then the image $\pi(f) = [f]$ of f in the colimit is zero since $(A, \mathrm{Im}(f)) \in I_{A,B}$ so we have a representative of this class in $\mathrm{Hom}_{\mathcal{A}}\left(A, B/_{\mathrm{Im}(f)}\right)$ which is zero. For the converse, if $\pi(f) = [f]$ is zero, we can choose a representative of this class $f' : A' \to B/_{B'}$ which is zero, with $(A', B') \in I_{A,B}$. This implies that $f(A') \subset B'$, and so $f(A') \in \mathcal{C}$ since $B' \in \mathcal{C}$. We have a short exact sequence

$$0 \longrightarrow f(A') \longrightarrow \mathrm{Im}(f) \longrightarrow A/_{A' + \mathrm{Ker}(f)} \longrightarrow 0$$

because $\mathrm{Im}(f) \cong A/_{\mathrm{Ker}(f)}$ and $f(A') \cong A'/_{A' \cap \mathrm{Ker}(f)} \cong A' + \mathrm{Ker}(f)/_{\mathrm{Ker}(f)}$. We then conclude that $\mathrm{Im}(f) \in \mathcal{C}$ using this short exact sequence since $A/_{A' + \mathrm{Ker}(f)}$ is a quotient of $A/_{A'}$ which is in $\mathcal{C}$. The last point of the statement is obtained by applying the functor π to the morphism identity of A. This gives a zero morphism in the quotient, and so the image A of this morphism is in $\mathcal{C}$. □

Finally, we can state a famous result about quotient categories which will not be used in the following.

Theorem 1.2.5 (Gabriel-Popescu Theorem)**.** *If $\mathcal{A}$ is a Grothendieck category with a generator G, then there is an equivalence of categories*

$$\mathcal{A} \cong \textbf{\textit{Mod}}\text{-}R\,/_{\mathcal{C}},$$

where $R = End(G)$ and $\mathcal{C}$ is the thick subcategory corresponding to the kernel of the functor $\mathrm{Hom}(G, -)$.

1.3 The section functor

The quotient functor π makes a link from the category $\mathcal{A}$ to the quotient one $\mathcal{A}/_{\mathcal{C}}$, but it is only in one direction. However, under some hypothesis, this functor has an adjoint, called the section functor, which makes the link in the opposite direction. In this case we say that $\mathcal{C}$ is localizing. When $\mathcal{A}$ is a Grothendieck category, this hypothesis has a concrete description and, since we will only consider Grothendieck categories, we only present this case (see [Gab62, p.377] for a general version). We now describe this adjoint and we give some properties of the adjunction.

Definition 1.3.1. [Gar01, p.7] The category $\mathcal{A}$ is a *Grothendieck category* if it is an AB5 category with a generator. This means that $\mathcal{A}$ is an abelian category with a generator (*i.e.* an object A in $\mathcal{A}$ such that every object is a quotient of a direct sum of copies of A), such that every (possibly infinite) family of objects in $\mathcal{A}$ has a coproduct (direct sum) in $\mathcal{A}$, and every direct limit of short exact sequences is exact (*i.e.* for every family of short exact sequences in $\mathcal{A}$ the induced sequence of direct limits is a short exact sequence).

Example 1.3.2. For any ring R, the categories R-**Mod** and **Mod**-R are Grothendieck categories. Indeed, they are abelian categories generated by R in which one can consider infinite direct sums. The last property can be checked by hand on elements (see [Gar01, p.7]).

We can now give the condition that $\mathcal{C}$ must satisfy so that the quotient functor has an adjoint.

Proposition 1.3.3. *[Gab62, Special case of Propositions 8 and 9 p.377-378] Let $\mathcal{A}$ be a Grothendieck category (Definition 1.3.1), the quotient functor π has a right adjoint*

$$\mathcal{S} : \mathcal{A}/_{\mathcal{C}} \to \mathcal{A}$$

if and only if the subcategory $\mathcal{C}$ is closed under colimits. In this case, the quotient functor commutes with all filtered colimits.

From now on we will assume that the quotient functor π has a right adjoint $\mathcal{S}$. Since we have defined a quotient, we want a statement similar to the usual universal property of the quotient. The following proposition gives this, but only if the functor we want to pass to the quotient is exact.

Proposition 1.3.4. *[Gab62, Corollary 2 p.368] Let F be an exact functor from $\mathcal{A}$ to an abelian category $\mathcal{D}$. If $F(C)$ is zero for all objects C of $\mathcal{C}$, then there exists a unique functor G from $\mathcal{A}/_{\mathcal{C}}$ to $\mathcal{D}$ such that $F = G \circ \pi$.*

More than that, since we need the functor $F : \mathcal{A} \to \mathcal{D}$ to be exact, the resulting functor $G : \mathcal{A}/_{\mathcal{C}} \to \mathcal{D}$ is also exact by using the following corollary which describes when a functor from a quotient is exact:

Corollary 1.3.5. *[Gab62, Corollary 3 p.369] Let G be a functor from $\mathcal{A}/_{\mathcal{C}}$ to an abelian category $\mathcal{D}$, then G is exact if and only if $G \circ \pi$ is exact.*

In particular, as stated above, if G is induced by an exact functor F from $\mathcal{A}$ to $\mathcal{D}$ as in Proposition 1.3.4, then G is also exact. However, the exactness hypothesis on F in Proposition 1.3.4 is a bit restrictive, and at some point we will need a more general version of this proposition. Indeed, if the functor F is not exact but admits derived functors it is sufficient to obtain an induced functor from the quotient as explained in the following proposition. For example, this is typically the case for the **Ext** and **Tor** functors, or for the (co)homology functors.

Proposition 1.3.6. *Let F be a functor from $\mathcal{A}$ to an abelian category $\mathcal{D}$ left (resp. right) exact such that it admits derived functors. If $F(C)$ is zero for all objects C of $\mathcal{C}$, then there exists a unique functor G from $\mathcal{A}/_{\mathcal{C}}$ to $\mathcal{D}$ such that $F = G \circ \pi$.*

Proof. The only time we use the exactness of F in the proof of Proposition 1.3.4 ([Gab62, Corollary 2 p.368]) is when we want to define, for all $A, B \in \mathcal{A}$ and all $(A', B') \in I_{A,B}$, a bijection

$$\psi : \mathrm{Hom}_{\mathcal{D}}\big(F(A), F(B)\big) \cong \mathrm{Hom}_{\mathcal{D}}\big(F(A'), F(B/B')\big).$$

To do this we consider the short exact sequences associated with $i^A_{A'} : A' \to A$ and $p^N_{N/N'} : N \to N/N'$. We get that $F(i^A_{A'}) : F(A') \to F(A)$ is an isomorphism using the fact that $A/A' \in \mathcal{C}$, the hypothesis $F(C) = 0$ for all $C \in \mathcal{C}$, and the exactness of F. The same works for $p^N_{N/N'}$, showing that $F(p)$ is an isomorphism and the two together imply that the morphism ψ is a bijection. This proof still works under the assumption that F admits derived functors since it replace the exactness of F: in the long exact sequence of derived functors the morphism $F(i^A_{A'}) : F(A') \to F(A)$ is between two terms of the type $F(A/A')$ which are zero since $A/A' \in \mathcal{C}$, proving that $F(i^A_{A'}) : F(A') \to F(A)$ is an isomorphism. The same argument works for $p^N_{N/N'} : N \to N/N'$ and the end of the proof is exactly the same. □

Remark 1.3.7. In the last proposition we need $F(C)$ to be zero for all $C \in \mathcal{C}$. For example if $F = H_*(-, \mathbb{K})$ we need for all $C \in \mathcal{C}$ that $H_n(C, \mathbb{K}) = 0$ for all $n \in \mathbb{N}$, and not just $H_0(C, \mathbb{K}) = 0$.

We now give some properties of the adjunction of the quotient functor π and the section functor $\mathcal{S}$, in particular we describe the unit and the co-unit of this adjunction.

Definition 1.3.8. The natural transformations $\eta : \mathrm{Id} \to \mathcal{S} \circ \pi$ and $\varepsilon : \pi \circ \mathcal{S} \to \mathrm{Id}$ such that $(\varepsilon_\pi \circ \pi(\eta) : \pi \to \pi \circ \mathcal{S} \circ \pi \to \pi) = \mathrm{Id}_\pi$ and $(\mathcal{S}(\varepsilon) \circ \eta_{\mathcal{S}} : \mathcal{S} \to \mathcal{S} \circ \pi \circ \mathcal{S} \to \mathcal{S}) = \mathrm{Id}_{\mathcal{S}}$ are respectively the *unit*, and the *co-unit* of the adjunction of $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ and $\mathcal{S} : \mathcal{A}/\mathcal{C} \to \mathcal{A}$.

Applying a general result concerning the unit and the co-unit of an adjunction given in [Bor94, P.115] to the adjunction of the quotient we obtain:

Proposition 1.3.9. *If the unit $\eta : \mathrm{Id} \to \mathcal{S} \circ \pi$ and the co-unit $\varepsilon : \pi \circ \mathcal{S} \to \mathrm{Id}$ of the adjunction of π and $\mathcal{S}$ are isomorphisms, then there is an equivalence of the categories $\mathcal{A}/\mathcal{C} \cong \mathcal{A}$.*

Since we consider the adjunction corresponding to a quotient category, there a some other results about the unit and the co-unit that arise in this case. For the co-unit we have the following.

Proposition 1.3.10. *[Gab62, Proposition 3.a p.371] The co-unit $\varepsilon : \pi \circ \mathcal{S} \to \mathrm{Id}$ of the adjunction of π and $\mathcal{S}$ is always an isomorphism.*

For the unit the description is a bit more complicated and we need the following definition, which is a reformulation of the definition of [Gab62, p.371].

Definition 1.3.11. An object $A \in \mathcal{A}$ is $\mathcal{C}$-closed if $\mathrm{Hom}(H, A)$ and $\mathrm{Ext}^1(H, A)$ are both zero for all $H \in \mathcal{C}$.

Then, the following proposition describes when the unit is an isomorphism.

Proposition 1.3.12. *[Gab62, Corollary p.371] Let A be an object of $\mathcal{A}$, the unit $\eta_A : A \to \mathcal{S} \circ \pi(A)$ of the adjunction of π and $\mathcal{S}$ is an isomorphism if and only if A is $\mathcal{C}$-closed.*

If the unit is not an isomorphism, we can still say something about its kernel and cokernel, as in the following:

Proposition 1.3.13. *[Gab62, Proposition 3.b p.371] For all objects A in $\mathcal{A}$, the kernel and the cokernel of the unit $\eta_A : A \to \mathcal{S} \circ \pi(A)$ of the adjunction of π and $\mathcal{S}$ are in the subcategory $\mathcal{C}$ of $\mathcal{A}$.*

Finally, we gave above some properties of the image of the quotient functor, but there is an important property in the opposite direction about the image of an object of the quotient category by the section functor $\mathcal{S}$:

Lemma 1.3.14. *[Gab62, Lemma 2 p.371] For any object N in the quotient category $\mathcal{A}/\mathcal{C}$, the object $\mathcal{S}(N)$ of $\mathcal{A}$ is $\mathcal{C}$-closed.*

Chapter 2

Functors on the categories $\mathbf{FI}_d$

Le silence éternel de ces espaces infinis m'effraie.

Blaise Pascal

The functors from the category **FI** of finite sets and injections (also denoted by **I** in [Sch08] and by Θ in [DV19]) to **R-Mod** are called **FI**-modules. They have been studied extensively in the last decade by Church, Ellenberg, Farb, Nagpal and some others (see for example [CEF15, CEFN14, CEF14, CE17, CF13, Chu12, CMNR18, Dja16, DV19]), mostly for their link to the theory of representation stability. A complete introduction to these subjects can be found in [Sam20], but we give an outline now: the theory of **FI**-modules was introduced in [CF13] to study the compatible families of representations of the symmetric groups which admit a decomposition in irreducible that eventually becomes stable (in the sense of [Far14]). It is a generalization of the classical homological stability taking into account the action of the symmetric groups. A large family of concrete examples in a wide range of areas are presented in [CF13] and [Wil18b]. Church and Farb then proved that a **FI**-module is finitely generated if and only if it has finitely generated values and the associated family is representation stable. In practice, it is generally easier to prove a finiteness result on one objects than the stability of an entire family, which shows the interest of studying these functors. Replacing the target category **R-Mod** by a more combinatorial category we can also consider the non-abelian categories of **FI**-posets, **FI**-graphs and more generally **FI**-sets with relations (see [RSW20]).

Since then, the category **FI** has been generalized in different directions. The one we are concerned was introduced by Sam and Snowden in [SS17], leading to the categories $\mathbf{FI}_d$ for d a non-zero integer in which the morphisms are coloured injections (see Definition 2.1.2). More precisely, the category $\mathbf{FI}_1$ is isomorphic to the category **FI**. We study here the functors from $\mathbf{FI}_d$ to **R-Mod**, called $\mathbf{FI}_d$-modules, and we emphasize in particular the differences with **FI**-modules. These functors intervene, in particular, in the theory of TCAs and in representation stability. Indeed, the $\mathbf{FI}_d$-modules are equivalent to the modules over the free TCA with d generators of degree 1 (see Chapter 4), and there is a result similar to the one for **FI** from [Ram19]: a $\mathbf{FI}_d$-module is finitely generated if and only if it has finitely generated values and the associated family of representations is stable in a general sense (for large enough partitions). In this Chapter, after recalling the definition of $\mathbf{FI}_d$, we give some examples of $\mathbf{FI}_d$-modules and we describe the simple $\mathbf{FI}_d$-modules. We then define some endofunctors of the category $\mathbf{FI}_d$**-Mod**, which we will use in the following chapters to define the notions of polynomial functors and we study some functors between **FI**-modules and $\mathbf{FI}_d$-modules.

2.1 The categories FI and $\mathbf{FI}_d$

We start with the definition of the category **FI** and its generalization, the category $\mathbf{FI}_d$. We give their first properties and some notations that we will use throughout the manuscript.

Definition 2.1.1. The *category* **FI** has for objects the finite sets and for morphisms the injections between these sets. The composition of morphisms is the usual composition of injections.

The category $\mathbf{FI}_d$ is constructed as the category **FI** to which we add some colours on the morphisms. Explicitly, for $d \in \mathbb{N}^*$ let $C^{(d)}$ be a set of cardinality d whose elements are called colours and are denoted by c_i for $1 \leq i \leq d$.

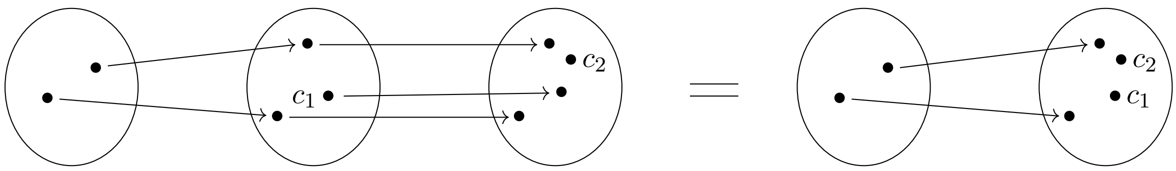
Definition 2.1.2. The *category* $\mathbf{FI}_d$ has for objects the finite sets and for morphisms the injections together with a colour choice in $C^{(d)}$ for each element in the codomain which is not mapped to by any element. In other words, an arrow from X to Y is a pair noted by (f, g) with $f : X \hookrightarrow Y$ an injection and $g : Y \smallsetminus f(X) \to C^{(d)}$ a set map. The composition is given for two composable morphisms (f_1, g_1) and (f_2, g_2), by

$$(f_1 : Y \hookrightarrow Z,\, g_1) \circ (f_2 : X \hookrightarrow Y,\, g_2) = (f_1 \circ f_2 : X \hookrightarrow Z\,, g')$$

where $g' : Z \smallsetminus (f_1 \circ f_2)(X) \to C^{(d)}$ is defined by

$$g'(z) = \begin{cases} g_2(z) & \text{if } \ z \in Z \smallsetminus f_2(Y) \\ g_1\big(f_2^{-1}(z)\big) & \text{if } \ z \in f_2(Y) \smallsetminus f_2 \circ f_1(X). \end{cases}$$

Example 2.1.3. We give an example of the composition of two morphisms with two different colours in $\mathbf{FI}_d$:



Definition 2.1.4. The functor $\odot : \mathbf{FI}_d \times \mathbf{FI}_d \to \mathbf{FI}_d$ is given on objects $X_1, X_2 \in \mathbf{FI}_d$ by the disjoint union $X_1 \odot X_2 = X_1 \sqcup X_2$ and on morphisms by

$$(f_1 : X_1 \hookrightarrow Y_1,\, g_1) \odot (f_2 : X_2 \hookrightarrow Y_2,\, g_2) = (f_1 \sqcup f_2 : X_1 \sqcup X_2 \hookrightarrow Y_1 \sqcup Y_2,\, g_1 + g_2).$$

Lemma 2.1.5. *The functor $\odot$ gives a symmetric monoidal structure on $\mathbf{FI}_d$ with the empty set as the unit.*

The categories $\mathbf{FI}_d$ generalize the category **FI** since for $d = 1$ we have an isomorphism of categories $\mathbf{FI}_1 \cong \mathbf{FI}$. Indeed, the category $\mathbf{FI}_1$ is easy to describe because $C^{(1)} = \{c\}$ so we can define a functor $\Delta_c : \mathbf{FI} \to \mathbf{FI}_1$ by the identity on objects and which sends an injection $f : X \hookrightarrow Y$ to the morphism (f, g), where g is the unique map from $Y \smallsetminus f(X)$ to $C^{(1)}$. This functor gives an isomorphism of categories and its quasi-inverse is a forgetful functor $\mathcal{O} : \mathbf{FI}_1 \to \mathbf{FI}$ which is given by the identity on objects and which sends a morphism (f, g) in $\mathbf{FI}_1$ to the injection f in **FI**. The functors Δ_c and $\mathcal{O}$ can be generalized to get functors between **FI** and $\mathbf{FI}_d$, we define the general forgetful functor now and the general functor Δ_c in Section 2.7.

Definition 2.1.6. The *forgetful functor* $\mathcal{O} : \mathbf{FI}_d \to \mathbf{FI}$ is defined on objects by $\mathcal{O}(X) = X$ and on morphisms by $\mathcal{O}(f, g) = f$.

Lemma 2.1.7. *The forgetful functor $\mathcal{O}$ is monoidal.*

In order to simplify the notation, the set $C^{(d)}$ is simply denoted by C if this dos not leads to misunderstandings. We also consider the skeleton of the category $\mathbf{FI}_d$, which is described in the following.

Notation 2.1.8. The *skeleton* of the category $\mathbf{FI}_d$ has for morphisms the integers, where n corresponds to the class of sets of cardinality n (see also [Wil18a] for $d = 1$). In this skeleton the class of the empty set corresponds to 0 and a morphism from n to m is a couple (f, g) where $f : n \to m$ is an injection and $g : m - n \to C$ is a choice of $m - n$ ordered colours. The monoidal structure $\odot$ on $\mathbf{FI}_d$ corresponds to the addition "+" in the skeleton.

From now on $\mathbf{FI}_d$ will always denote the skeleton of the category of finite sets and coloured injections.

Remark 2.1.9. We use the notation "+" in the skeleton of $\mathbf{FI}_d$ since the cardinality of the disjoint union of two sets is the sum of their cardinality. However, this monoidal structure is not a coproduct, even for $d = 1$. For example the following diagram cannot be completed by any dashed arrow in $\mathbf{FI}$ or $\mathbf{FI}_d$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr, "\mathrm{Id}_1"'] & 2 \arrow[d, dashed] & 1 \arrow[l] \arrow[dl, "\mathrm{Id}_1"] \\ & 1 & \end{tikzcd}$$

In $\mathbf{FI}_d$ the morphisms from 0 to an integer m are really important and will be used a lot. We give there a nice description of these morphisms.

Remark 2.1.10. For $m \in \mathbf{FI}_d$, an element of $\mathbf{FI}_d(0, m)$ corresponds to a choice of m colours in C, so we have a bijection $\mathbf{FI}_d(0, m) \cong C^m$ given by:

$$\begin{array}{ccc} \mathbf{FI}_d(0, m) & \to & C^m \\ \big(0 \hookrightarrow m\,,\, g : m \to C\big) & \mapsto & \big(g(1), \dots, g(m)\big). \end{array}$$

We often denote an element x in $\mathbf{FI}_d(0, m)$ by $x = (c_1, \dots, c_m)$ for some colours $c_1, \dots, c_m \in C$ according to this bijection. In particular, for $c \in C$ we denote by c^m the morphism $(0 \to m, m \to \{c\} \to C)$ in $\mathbf{FI}_d(0, m)$.

The first important difference between $\mathbf{FI}$ and $\mathbf{FI}_d$ is that 0 is an initial object in $\mathbf{FI} \cong \mathbf{FI}_1$, but it is not the case in $\mathbf{FI}_d$ for $d > 1$. Indeed, the set of morphisms

$$\mathbf{FI}_d(0, m) \cong (C^{(d)})^m = \{c_1, \dots, c_d\}^m$$

has d^m elements in general. This gives the existence of a unique morphism from 0 to m in $\mathbf{FI}_1$, but such a morphism is not unique in $\mathbf{FI}_d$ for $d > 1$.

Remark 2.1.11. In the literature they are several variants (see [Sam20] for a detailed list) of the category $\mathbf{FI}$: The categories $\mathbf{FI}_d$, $\mathbf{FI}_W$ for W some Weyl groups in [Wil12], $\mathbf{FS}_G$ the category of finite sets and G-surjections for G a group (see [SS17]), or a symplectic version (see [Sam20]). There are also variants for representations of linear groups presented in [Wil18a], such as $\mathbf{VI(R)}$ the category of free modules of finite rank and injective linear maps with left inverse, and its generalization $\mathbf{VIC(R)}$ of free modules of finite rank and injective linear maps with a choice of direct complement of the image. These categories are particular cases of the category $\mathbf{S}(\mathcal{A})$ introduced by Djament in [Dja12] for $\mathcal{A}$ an abelian category. For $\mathcal{A} = \mathbf{R}$-$\mathbf{Mod}$ this category is denoted by $\mathbf{S(R)}$ in [Dja16] and for $\mathcal{A} = \mathbb{Z}-\mathbf{Mod}$ it is denoted by $\mathbf{S(ab)}$ in [DV19]. They are similar to $\mathbf{FI}$ and $\mathbf{FI}_d$ since the morphisms are given by an injective map coupled with a choice on the complement of the image. Most of these categories are example of the construction $\mathfrak{U}(G)$ from

[RWW17]: for $G = \mathrm{S}_n$ we get the category **FI**, and for $G = GL_n(\mathbf{R})$ we get $\mathbf{S}(\mathbf{R})$. The functors over such categories have been studied, like the category $\mathcal{G}$ in [DV15] for $G = \mathrm{Aut}(F_n)$ and $\mathfrak{U}_\beta$ in [Sou20] for $G = B_n$. Another variant of the category **FI** is the category $\mathbf{FI}_G$ of finite sets and couples of an injection and a choice of an element of the group G for each element at the source. The functors over this category have been studied for example in [Ram17b, LR18] and Sam and Snowden showed that the category of finitely generated $\mathbf{FI}_G$-module is noetherian. This result was extended in [Ram17b] to the notion of degree wise coherent modules using endofunctors similar to the ones we define in Section 2.6. In [LR18] they show that the dimension of the functors over $\mathbf{FI}_G$ eventually becomes polynomial, as for **FI**.

2.2 Functors on the categories $\mathbf{FI}_d$

The main objects that we study in this thesis are the $\mathbf{FI}_d$-modules, which are the functors from $\mathbf{FI}_d$ to the category of modules **R-Mod**. The theory of **FI**-modules was introduced by Church and Farb in [CF13] to encode a large quantity of information about a family of representations of the symmetric groups in one object. Concrete examples of this are given in [CF13] and [Wil18b]. Indeed, it was proven in [CEF15] that if a **FI**-module is finitely generated, then the family of representations of the symmetric groups associated is stable (in the sense of [CEF15, CEFN14, Far14]). The $\mathbf{FI}_d$-modules were then introduced as a generalization of the **FI**-modules, and it was proven in [Ram17a] that, if a $\mathbf{FI}_d$-module is finitely generated, then the family of representations of the symmetric groups associated is stable in a generalized sense (for large enough irreducible representations). In this part we give general results on this category of functors. We start with the definition of this abelian category and we give a family of generating functors which are the standard projective. We will see that we can get a lot of information about the $\mathbf{FI}_d$-modules from the structure of these projective standard functors. We only consider here functors with values in **R**-modules but most of the results admit generalizations for functors with values in a Grothendieck category $\mathcal{A}$ (Definition 1.3.1).

Definition 2.2.1. The category $\mathbf{FI}_d\text{-}\mathbf{Mod} = \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is *the category of functors* from $\mathbf{FI}_d$ to the category of modules **R-Mod**, with natural transformations as morphisms.

Proposition 2.2.2. *The category $\mathbf{FI}_d$-**Mod** $= \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is a Grothendieck category (Definition 1.3.1), in particular it is abelian.*

Proof. The category **R-Mod** is a Grothendieck category and a functor category with values in a Grothendieck category is also a Grothendieck category (see [Gar01]). □

Remark 2.2.3. The structure of a $\mathbf{FI}_d$-module can be represented by a diagram: it is a family of linear representations of the symmetric groups together with compatibility conditions given by linear maps. We give an example for $d = 1$, the diagram for a general d being analogue with more arrows:

$$\begin{tikzcd}
 & S_0 \arrow[d, phantom, "\circlearrowright"] & S_1 \arrow[d, phantom, "\circlearrowright"] & S_2 \arrow[d, phantom, "\circlearrowright"] & & S_n \arrow[d, phantom, "\circlearrowright"] & \\
\mathbf{FI} \arrow[d, "F"'] & 0 \arrow[r, hook] & 1 \arrow[r, hook] & 2 \arrow[r, hook] & \dots \arrow[r, hook] & n \arrow[r, hook] & \dots \\
\mathbf{R}\text{-}\mathbf{Mod} & F(0) \arrow[r] & F(1) \arrow[r] & F(2) \arrow[r] & \dots \arrow[r] & F(n) \arrow[r] & \dots \\
 & F(S_0) \arrow[u, phantom, "\circlearrowleft"] & F(S_1) \arrow[u, phantom, "\circlearrowleft"] & F(S_2) \arrow[u, phantom, "\circlearrowleft"] & & F(S_n) \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

Each arrow in this diagram represents in fact many arrows, however every arrow from i to j in **FI** is obtained from one by composition with an element of the symmetric group S_j.

A family of important examples of $\mathbf{FI}_d$-modules are the standard projective functors. These functors naturally exist in every category of functors with values in an abelian category and we will show that they form a family of generators of $\mathbf{FI}_d$-$\mathbf{Mod}$. These fundamental functors introduced in a different context in [Kuh94] appear for $\mathbf{FI}_d$ in [SS17] and for $d = 1$ in [DV19, Dja16, Ves19], or under the name of free modules in [CEF15, CEFN14, MW19] or representable functors in [Wil18a]. They play the role of the free modules in the classical theory of modules.

Definition 2.2.4. For $n \in \mathbf{FI}_d$, the *standard projective functor* on $\mathbf{FI}_d$ associated with n, denoted by $P_n^{\mathbf{FI}_d} : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$, is given by

$$P_n^{\mathbf{FI}_d} = \mathbf{R}\left[\mathrm{Hom}_{\mathbf{FI}_d}\left(n, -\right)\right],$$

where $\mathbf{R}[-] : \mathbf{Set} \to \mathbf{R}\text{-}\mathbf{Mod}$ is the $\mathbf{R}$-linearization functor (*i.e.* the left adjoint to the forgetful functor $\mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{Set}$). It sends an object $m \in \mathbf{FI}_d$ to the $\mathbf{R}$-module $\mathbf{R}\left[\mathrm{Hom}_{\mathbf{FI}_d}\left(n, m\right)\right]$ and a morphism $(f, g) \in \mathbf{FI}_d(m, k)$ on $\mathbf{R}[(f, g)_*]$, the linearization of the post-composition by (f, g). This functor is sometimes called the *representable functor* on $\mathbf{FI}_d$ associated with n.

We recall that a family $\mathcal{F}$ of objects in a category $\mathcal{C}$ is a generator of $\mathcal{C}$ if for every object C in $\mathcal{C}$ there exists an epimorphism from a direct sum of elements in $\mathcal{F}$ to C. We now show that the projective functors generate the category $\mathbf{FI}_d$-$\mathbf{Mod}$ and that they are projective objects in the category $\mathbf{FI}_d$-$\mathbf{Mod}$, following [Ves19].

Proposition 2.2.5. *The family* $\left(P_n^{\mathbf{FI}_d}\right)_{n\in\mathbb{N}}$ *of standard projective functors forms a set of projective generators of the category of* $\mathbf{FI}_d$*-modules.*

Proof. Let F be an object of $\mathbf{FI}_d$-$\mathbf{Mod}$, the linear Yoneda lemma gives, for all $n \in \mathbf{FI}_d$, a bijection

$$\xi_n : \mathrm{Hom}_{\mathbf{FI}_d\text{-}\mathbf{Mod}}\left(P_n^{\mathbf{FI}_d}, F\right) \xrightarrow{\sim} F(n).$$

Then the natural transformation

$$\bigoplus_{n\in\mathbf{FI}_d}\bigoplus_{x\in F(n)}\left(P_n^{\mathbf{FI}_d} \xrightarrow{\xi_n^{-1}(x)} F \right)$$

is an epimorphism since, for any $n \in \mathbf{FI}_d$ and $x \in F(n)$, the natural transformation $\xi_n^{-1}(x)$ sends the identity of n to the elements $x \in F(n)$. Moreover, every functor $P_n^{\mathbf{FI}_d}$ is a projective object in the category $\mathbf{FI}_d$-$\mathbf{Mod}$. Indeed, for an epimorphism $f : F \to G$ in $\mathbf{FI}_d$ and a natural transformation $g : P_n^{\mathbf{FI}_d} \to G$, the Yoneda lemma gives two natural bijections

$$\mathrm{Hom}_{\mathbf{FI}_d\text{-}\mathbf{Mod}}\left(P_n^{\mathbf{FI}_d} F\right) \cong F(n) \qquad \text{and} \qquad \mathrm{Hom}_{\mathbf{FI}_d\text{-}\mathbf{Mod}}\left(P_n^{\mathbf{FI}_d} G\right) \cong G(n).$$

Since the map $f_n : F(n) \to G(n)$ is surjective, there exist $h : P_n^{\mathbf{FI}_d} \to F$ such that $f \circ h = g$, thus the functor $P_n^{\mathbf{FI}_d}$ is projective. Equivalently, the functor $\mathrm{Hom}(P_n^{\mathbf{FI}_d}, -)$ is equivalent to the evaluation functor $F \mapsto F(n)$ which is exact, so it is also exact. Since the category $\mathbf{FI}_d$-$\mathbf{Mod}$ is abelian, it implies that the functor $P_n^{\mathbf{FI}_d}$ is projective. □

This property allows us to define the notion of finitely generated $\mathbf{FI}_d$-module. Indeed, we have shown that every $\mathbf{FI}_d$-module is a quotient of a direct sum of projective standard functors, and so we say that it is finitely generated if it is a quotient of a finite one.

Definition 2.2.6. A $\mathbf{FI}_d$-module F is finitely generated if there exists an epimorphism

$$\bigoplus_{i=0}^{k}\left(P_i^{\mathbf{FI}_d}\right)^{\oplus c_i} \twoheadrightarrow F$$

from a finite direct sum of standard projective functors to F.

There are equivalent ways to define the finitely generated $\mathbf{FI}_d$-modules: in [CEF15, CEFN14, Ram17a] a **FI**-module F is said finitely generated if and only if there exists a finite set of integers such that every subfunctor of F that coincides with F on this set is equal to F. The equivalence with Definition 2.2.6 is explicitly described in [CEFN14, Proposition 2.3] or in [Ram17a], and from the point of view of TCAs in [SS12, Section 8.3.2]. Sometimes F might be defined as finitely generated if every growing family of subfunctors of F whose union is F is stationary. The equivalence is given, for a large family of categories such as $\mathbf{FI}_d$, in [Dja16, Prop 2.7].

Remark 2.2.7. In recent years it has been proved that several algebraic structures are noetherian (*i.e.* a submodule of a finitely generated module is finitely generated), such as the **FI**-modules (see [CEFN14, SS16]), the **FS**-modules where **FS** is the category of finite sets and surjections also denoted by Ω in [Pir00] (see [SS17]), the $\mathbf{VIC}(R)$-modules (see [PS17]) and many others. The category $\mathbf{FI}_d$ and its ordered version $\mathbf{OI}_d$ appears in [SS17, Section 7.1] where they show that $\mathbf{OI}_d$ is Gröbner and $\mathbf{FI}_d$ quasi-Gröbner (*i.e.* morally there is an essentially surjective functor from the Gröbner category $\mathbf{OI}_d$ to $\mathbf{FI}_d$) and thus that the categories of $\mathbf{FI}_d$-modules are noetherian over any left-noetherian ring $\mathbf{R}$. The idea is to add an order to the category to get a Gröbner category and then use the forgetful functor to transfer the noetherian property from one to the other. This result was first proved in [Sno13, Theorem 2.3] over a field of characteristic zero, then for $d = 1$ in [CEF15] (in characteristic zero) and [CEFN14]. The noetherian property is a crucial tool to prove that a sequence of representation stabilizes since it is equivalent to prove that the **FI**-module associated is finitely generated.

Remark 2.2.8. For $d = 1$, if F is a finitely generated **FI**-module over a field, then the dimension of the vector spaces $F(n)$ eventually becomes polynomial in n. This result was first proved in [Sno13] and in [CEF15] over a field of characteristic zero, then in [CEFN14] and in [SS17] in general. This is false for $d > 1$ since $P_0^{\mathbf{FI}_d}$ is finitely generated but $P_0^{\mathbf{FI}_d}(n) = \mathbf{R}[\mathbf{FI}_d(0,n)] = \mathbf{R}[C^n]$ is of dimension d^n. However, this result admits a generalization for finitely generated $\mathbf{FI}_d$-modules with the notion of Hilbert series introduced by Sam and Snowden. More precisely, they showed in [SS17, Corollary 7.1.7] that if F is a finitely generated $\mathbf{FI}_d$-module, then its Hilbert series $H_F(t) = \sum \dim_{\mathbb{K}} F(n)\, t^n$ is of the form $P(t)/Q(t)$, where $P(t), Q(t)$ are polynomials in $\mathbb{K}[t]$ with $Q(t) = \prod_{j=1}^{d}(1-jt)^{e_j}$ for some $e_j \geq 0$. In particular, this implies that the dimension of $F(n)$ eventually becomes a sum on $1 \leq j \leq d$ of a polynomial multiplied by j^n. For $d = 1$ we recover that the dimension of $F(n)$ is eventually polynomial, but for $d > 1$ it has a more complex expression.

As explained above, the theory of **FI**-modules was introduced to encode the notion of representation stability. It then was proven in [CEF15] that, if a **FI**-module is finitely generated, then the family of representations of the symmetric groups associated is stable (in the sense of [CEF15, CEFN14, Far14]). In order to present the theorem [Ram17a, Theorem A], which is the analogue result for $\mathbf{FI}_d$-modules, we first introduce the padded partitions:

Definition 2.2.9. For $\lambda = (\lambda_1, \ldots, \lambda_h)$ a partition of weight $|\lambda| = \sum \lambda_i$, and $n_1 \geq \cdots \geq n_d \geq |\lambda| + \lambda_1$ d positive integers, the associated d-padded partition is given by $\lambda[n] = (n_1 - |\lambda|, \ldots, n_d - |\lambda|, \lambda_1, \ldots, \lambda_h)$.

As explained in [CF13, Far14, Wil12] this is a way to name the irreducible representations of the symmetric groups such that the name is independent of the index of the symmetric group. Then the representation stability of [CF13] for the symmetric groups can be summarized as follows (see [CEF15, CEFN14, Far14]): a coherent family $(V_n)_n$ of representations is stable if, for each partition λ, the multiplicity of the irreducible representation associated with the 1-padded partition $\lambda[n] = (n - |\lambda|, \lambda_1, \ldots, \lambda_h)$ in V_n is eventually independent of n. For example, the family of spaces $\mathbb{K}^n$, together with the injections of the canonical basis, is stable since each space decomposes into $\mathbb{K}^n = M_{(n)} \oplus M_{(n-1,1)} = M_{(0)}[n] \oplus M_{(1)}[n]$. We now state the analogous theorem for $\mathbf{FI}_d$-modules:

Theorem 2.2.10. *[Ram17a, Theorem A] For $\mathbb{K}$ a field of characteristic 0, a $\mathbf{FI}_d$-module F is finitely generated if and only if the space $F(n)$ is finite dimensional for all $n \in \mathbb{N}$ and, for n large enough:*

- *The intersection of the kernels* $\mathrm{Ker}(F(f,g))$*, for (f,g) the maps starting at n, is zero,*
- *The sum of the spaces $F(\,(f,g) : n \to n+1\,)(F(n))$, for the maps (f,g) in $\mathbf{FI}_d(n,n+1)$, generates $F(n+1)$ as a representation of* S_{n+1}*,*
- *For any partition λ of weight $|\lambda|$ and any integers $n_1 \geq \cdots \geq n_d \geq |\lambda| + \lambda_1$, if $c_{\lambda,n_1,\ldots,n_d}$ is the multiplicity of the irreducible representation associated with the 1-padded partition $\lambda[n] = (n_1 - |\lambda|, \ldots, n_d - |\lambda|, \lambda_1, \ldots \lambda_h)$, then $c_{\lambda,n_1+l,\ldots,n_d+l}$ is independent of l for l large enough.*

This theorem is quite technical, but it is a direct generalization of the analogue theorem of [CEF15, CEFN14] for $\mathbf{FI}$-modules. Morally, one can interpret the last point by saying that the irreducible representation associated with a partition with at least d rows appears eventually with a stable multiplicity in a finitely generated $\mathbf{FI}_d$-module. This theorem does not predict the behavior of the irreducible representations associated with smaller partitions, but the Theorem B in [Ram17a] treats some of these cases.

2.3 First examples of $\mathbf{FI}_d$-modules

In this section we give examples of $\mathbf{FI}_d$-modules. We start with some elementary functors and with a family of functors induced by the tensor product of modules. The first example we can construct is the constant functor. Let $M \in \mathbf{R}\text{-}\mathbf{Mod}$ be an object, we still denote by $M : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ the constant functor which sends any object to M and any morphism to the identity. Since there are only maps in $\mathbf{FI}_d(n,m)$ when $n \leq m$, we can define variations of some of the examples of functor given in [DV19] over a symmetric monoidal category with an initial object. Indeed, we can construct the twisted atomic functor $M_k : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$, which is defined on objects by

$$M_k(n) = \begin{cases} M & \text{if } n = k \\ 0 & \text{else }, \end{cases}$$

and on a morphism $(f,g) \in \mathbf{FI}_d(n,m)$ by $M_k(f,g) = M_k(\sigma)$ if $n = m = k$ and zero else with $\sigma \in \mathrm{S}_n$. When the action of the symmetric group S_n on M given by $M_k(\sigma)$ is trivial, we simply say that M_k is atomic. Note that such a functor cannot be defined over a source category with compatible maps $a \to b \to a$ such that the composition is the identity. We can also consider $M_{\geq k} : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ the subfunctor of the constant functor M defined on objects by

$$M_{\geq k}(n) = \begin{cases} M & \text{if } n \geq k \\ 0 & \text{if } n < k, \end{cases}$$

and on a morphism $(f,g) \in \mathbf{FI}_d(n,m)$ by $M_{\geq k}(f,g) = \mathrm{Id}_M$ if $n, m \geq k$ and zero else. For $d = 1$, this functor is called the truncated module associated to the constant functor in [Wil18a]. We can combine these functors and, for a set $I \subset \mathbb{N}$, we can define the functors

$$\bigoplus_{i \in I} M_i \quad \text{and} \quad \bigoplus_{i \in I} M_{\geq i} \subset \bigoplus_{i \in I} M.$$

Moreover, we can define a functor $M_{<k}$ by the short exact sequence $0 \longrightarrow M_{\geq k} \longrightarrow M \longrightarrow M_{<k} \longrightarrow 0$. We can check that we have

$$M_{<k}(n) = \begin{cases} M & \text{if } n < k \\ 0 & \text{if } n \geq k, \end{cases}$$

and $M_{<k}(f,g) = \mathrm{Id}_M$ if $n, m < k$ and zero else.

Remark 2.3.1. Neither M_k or $M_{<k+1}$ are subfunctors of the constant functor since, in both cases, the image of the spaces $M_k(k) = M$ and $M_{<k+1}(k) = M$ by the maps $M(k \to k+1) = \mathrm{Id}_M$ are equal to M, which is not a subspace of $M_k(k+1) = 0$ or $M_{<k+1}(k+1) = 0$. Thus the category of $\mathbf{FI}_d$-modules is not semisimple since the short exact sequence $0 \longrightarrow M_{\geq k} \longrightarrow M \longrightarrow M_{<k} \longrightarrow 0$ do not split.

Remark 2.3.2. For $d = 1$, the functor $\mathbf{R}_{\geq k}$ corresponds to the image of the functor $P_k^{\mathbf{FI}}$ (see Definition 2.2.4) by the arrow $P_k^{\mathbf{FI}} \to P_0^{\mathbf{FI}}$ given by the unique morphism $0 \to k$ in $\mathbf{FI}$.

We also give a first example of a functor that acts in differently depending on the colours associated with a morphism:

Example 2.3.3. Let $F_{c_1}^{\mathbf{FI}_d} : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ be defined on objects by $F_{c_1}^{\mathbf{FI}_d}(n) = \mathbf{R}$ for all $n \in \mathbf{FI}_d$, and on an arrow (f,g) in $\mathbf{FI}_d(n,m)$ by

$$F_{c_1}^{\mathbf{FI}_d}(f,g) = \begin{cases} 0 & \text{if } g^{-1}(c_1) \neq \varnothing, \\ \mathrm{Id}_{\mathbf{R}} & \text{else.} \end{cases}$$

In other words, $F_{c_1}^{\mathbf{FI}_d}$ sends a morphism to zero if it uses the colour c_1 and to the identity else. It defines a functor since the colour c_1 appears in the composition $(f,g) \circ (f',g')$ if and only if it appears in (f,g) or in (f',g'). One can note that for $d = 1$ this functor is equal to the sum on $i \in \mathbb{N}$ of the atomic functors M_i defined above since it sends every non-bijective morphism to zero and all bijective morphism on the identity.

Another interesting example is given by the tensor product on modules. We will see in Example 5.1.12 that it belongs to a family of strong polynomial functors since the tensor power is a usual polynomial functor over modules.

Example 2.3.4. For $k \in \mathbb{N}$ an integer, let $T_k^{(d)}$ be the $\mathbf{FI}_d$-module defined on objects by

$$T_k^{(d)}(n) = (\mathbb{K}^n)^{\otimes k},$$

and on a morphism $(f,g) \in \mathbf{FI}_d(n,m)$ by the arrow $(\mathbb{K}^n)^{\otimes k} \to (\mathbb{K}^m)^{\otimes k}$ induced by the map that injects $\mathbb{K}^n$ into $\mathbb{K}^m$ along f. For $d = 1$ we get that $T_k^{(1)}$ is the composition of $F : \mathbf{FI} \to \mathbb{K}\text{-}\mathbf{Vect}$ which sends n to $\mathbb{K}^n$, with $T_k : \mathbb{K}\text{-}\mathbf{Vect} \to \mathbb{K}\text{-}\mathbf{Vect}$ which sends V to $V^{\otimes k}$. Since the functor is defined on injections independently of the colour, we also have the composition $T_k^{(d)} = T_k^{(1)} \circ \mathcal{O} = \mathcal{O}^*(T_k^{(1)})$, where $\mathcal{O}$ is the forgetful functor of Definition 2.1.6.

2.4 Simple $\mathbf{FI}_d$-modules

We give a description of the simple objects of the category $\mathbf{FI}_d\text{-}\mathbf{Mod}$ using the fact that $\mathbf{FI}_d$ is an EI-category. The EI-categories and their representations have been introduced among others by Dieck (in [Die87]) in the context of algebraic K-theory and have been studied more recently by Li (in [Li14]), in particular their Koszul property.

Remark 2.4.1. The representation theory of the symmetric groups is well known. A brief summary of the results used in the context of twisted commutative algebras and **FI**-modules can be found in [SS12]. In particular, over a field of characteristic zero, the irreducible representations of the symmetric group S_n are indexed by the partitions λ of n. The irreducible representation associated to λ, denoted by M^λ, is often defined as the ideal of the ring $\mathbb{K}[\mathrm{S}_n]$ generated by an idempotent element associated to the partition λ called the Young symmetrizer. For example, the representation associated with the partition $\lambda = (n)$ is the trivial representation, the one associated with $\lambda = (1^n)$ is the sign representation, and the one associated with $\lambda = (n-1,1)$ is the standard representation.

Definition 2.4.2. An *EI-category* is a category in which every endomorphism is an isomorphism.

The category $\mathbf{FI}_d$ is an EI-category. Indeed, by definition for $n \in \mathbb{N}$ we have

$$\mathbf{FI}_d(n,n) = \{\, (f : n \hookrightarrow n,\, g : n - n \to C) \,\} = \{\, (\sigma \in \mathrm{S}_n,\, 0 \to C) \,\} \cong \mathrm{S}_n\,.$$

Recall that the simple elements of a category of functors $\mathbf{Fct}(\mathcal{C}, \mathbf{R}\text{-}\mathbf{Mod})$ are the functors F which do not have non-zero proper subfunctors. When the source category $\mathcal{C}$ is an EI-category as it is the case here, the simple objects of $\mathbf{Fct}(\mathcal{C}, \mathbf{R}\text{-}\mathbf{Mod})$ can be described as follows:

Proposition 2.4.3. *For $\mathbb{K}$ a field of characteristic zero, the simple objects of the category $\mathbf{Fct}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ are the twisted atomic functors $(M^\lambda)_k$ that sends an object $n \in \mathbf{FI}_d$ to M^λ if $n = k$ and to zero else, for $M^\lambda \in \mathbb{K}\text{-}\mathbf{Vect}$ the irreducible representation of S_k associated with a partition λ of k.*

Proof. First, if a functor $F : \mathbf{FI}_d \to \mathbb{K}\text{-}\mathbf{Vect}$ is non-zero, there exists $k \in \mathbf{FI}_d$ such that $F(k) \neq 0$. Then the twisted atomic functor $F(k)_k$ defined in Section 2.3 is a subfunctor of F which is not zero. If F is not equal to this twisted atomic functor $F(k)_k$, it then admits a proper subfunctor and so it is not simple. Since the category $\mathbf{FI}_d$ is an EI-category, we conclude that a simple element of $\mathbf{Fct}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ is a twisted atomic functor M_k for some $k \in \mathbb{N}$ and some $M \in \mathbf{R}\text{-}\mathbf{Mod}$. Such a functor is given by the vector space M and by the action of the endomorphisms corresponding to the symmetric group S_k. Then the twisted atomic functor M_k is a linear representation of S_k and it is simple as an object of $\mathbf{FI}_d\text{-}\mathbf{Mod}$ if and only if it is irreducible as a representation. □

Remark 2.4.4. The proof of Proposition 2.4.3 remains valid for a general commutative ring $\mathbf{R}$, even if there is no classification of the irreducible representations of S_n on $\mathbf{R}$-modules. Then the simple objects of the category $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ are the twisted atomic functors $(M)_k$, for $M \in \mathbf{R}\text{-}\mathbf{Mod}$ a simple representation of S_k.

2.5 The category $\underline{\mathbf{FI}_d}$

In this section we explain that a functor on $\mathbf{FI}_d$ is completely determined by the image of the morphisms starting from 0 if they are sent to isomorphisms. This property will be used in the following chapters, in particular to describe the polynomial functors of degree 0 on $\mathbf{FI}_d$ in Section 7.4. To prove it we introduce a subcategory $\underline{\mathbf{FI}_d}$ of $\mathbf{FI}_d$ which contains only the morphisms starting from 0 and the symmetric groups. Then we show that, under the condition of sending these morphisms to isomorphisms, the functors on $\mathbf{FI}_d$ correspond to the functors on $\underline{\mathbf{FI}_d}$. First we explain that, under this hypothesis, the order of the colours is not important to define a $\mathbf{FI}_d$-module.

Proposition 2.5.1. *For $F : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$, if there exist an object $k \in \mathbf{FI}_d$ and a colour $c \in C$ such that $F(c^k)$ is an isomorphism, then*

- *For all permutations $\sigma \in \mathrm{S}_k$, the morphism $F(\sigma)$ is the identity,*
- *For all k-tuples of colours $c_{j_1}, \dots, c_{j_k} \in C$ we have the following identity:*

$$F\big(\,(c_{j_1}, \dots, c_{j_k})\,\big) = F\big(\,(c_{j_{\sigma(1)}}, \dots, c_{j_{\sigma(k)}})\,\big).$$

Proof. By definition, the two morphisms c^k and $\sigma \circ c^k$ are equal in $\mathbf{FI}_d$, which give the identity $F(\sigma) \circ F(c^k) = F(c^k)$. Since $F(c^k)$ is an isomorphism by hypothesis, we get the first point. The second point is a consequence using the identity

$$F(\sigma) \circ F\big(\,(c_{j_1}, \dots, c_{j_k})\,\big) = F\big(\,(c_{j_{\sigma(1)}}, \dots, c_{j_{\sigma(k)}})\,\big).$$

□

We now define the subcategory $\underline{\mathbf{FI}_d}$ of $\mathbf{FI}_d$ and we emphasize that a functor F on $\mathbf{FI}_d$ induces canonically a functor $\underline{F}$ on $\underline{\mathbf{FI}_d}$ by restriction.

Definition 2.5.2. The *category* $\underline{\mathbf{FI}_d}$ is the subcategory of $\mathbf{FI}_d$ with the same objects (finite sets) and whose morphisms are given by

$$\underline{\mathbf{FI}_d}(n,m) = \begin{cases} \mathbf{FI}_d(0,m) & \text{if } n = 0 \\ \{\mathrm{S}_m\} & \text{if } n = m \\ \varnothing & \text{else .} \end{cases}$$

The following lemma and proposition explain that, if a functor F sends the morphisms starting from zero to isomorphisms, then it can be re-constructed from its induced functor $\underline{F} : \underline{\mathbf{FI}_d} \to \mathbf{R}\text{-}\mathbf{Mod}$. Morally this states that such a $\mathbf{FI}_d$-module is completely determined by its image on the morphisms starting from 0.

Lemma 2.5.3. *Let F be a $\mathbf{FI}_d$-module, if $F(x)$ is an isomorphism for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$ then, for each $u \in \mathbf{FI}_d(n,m)$, the morphism $F(u)$ is obtained from $\underline{F}$ by the formula:*

$$F(u) = \underline{F}(v) \;\circ\; \big(\underline{F}(c_1^n)\big)^{-1},$$

where $v = u \circ c_1^n \in \mathbf{FI}_d(0,m)$. Moreover, the only relations between the images of morphisms in $\underline{\mathbf{FI}_d}$ by $\underline{F}$ are the one from Proposition 2.5.1.

Proof. For $u \in \mathbf{FI}_d(n, n+m)$ a morphism in $\mathbf{FI}_d$ and $c_1^n \in \mathbf{FI}_d(0,n)$ we have $u \circ c_1^n \in \mathbf{FI}_d(0, n+m)$. By hypothesis, $F(c_1^n)$ and $F(u \circ c_1^n)$ are isomorphisms, so the relation $F(u) \circ F(c_1^n) = F(u \circ c_1^n) = F(v)$ implies the identity

$$F(u) = F(v) \;\circ\; \big(F(c_1^n)\big)^{-1} = \underline{F}(v) \;\circ\; \big(\underline{F}(c_1^n)\big)^{-1}.$$

By Proposition 2.5.1 we get $F(\sigma) = \mathrm{Id}$ and so $\underline{F}(\sigma) = \mathrm{Id}$. Then, for $x, y \in \underline{\mathbf{FI}_d}(0,k)$ we have $\underline{F}(x) = \underline{F}(y)$ when there exist $\sigma \in \mathrm{S}_k$ such that $y = \sigma \circ x$. This gives the conclusion since the only possible compositions in $\underline{\mathbf{FI}_d}$ are of the form $y = \sigma \circ x$ with $x \in \underline{\mathbf{FI}_d}(0,k)$ and $y \in \underline{\mathbf{FI}_d}(0,k)$ and $\sigma \in \mathrm{S}_k$. □

Proposition 2.5.4. *Let $\underline{F} : \underline{\mathbf{FI}_d} \to \mathbf{R}\text{-}\mathbf{Mod}$ be a functor such that the image of all morphisms in $\underline{\mathbf{FI}_d}$ is an isomorphism. This functor can be extended in a unique way in a functor F from $\mathbf{FI}_d$ to $\mathbf{R}\text{-}\mathbf{Mod}$.*

Proof. By hypothesis, $\underline{F}(x)$ is an isomorphism for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$. Then we can define a functor $F \in \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by the formula of Lemma 2.5.3 and by the relations necessary to have a functor. This same lemma proves that F is an extension of $\underline{F}$. □

2.6 Some endofunctors of $\mathbf{FI}_d$-modules

In this section we define some endofunctors of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ which will be used throughout this manuscript, for example to define strong polynomial functors in Section 5.1 or to construct subcategories of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ in Chapter 6. These definitions are inspired by [DV19, Section 2] concerning functors over a symmetric monoidal category where the unit is an initial object. As said in Section 2.1 this is not the case for $\mathbf{FI}_d$ so these definitions are adapted for $\mathbf{FI}_d$-modules. We present here the definitions of these endofunctors and the first general properties about them.

Definition 2.6.1. For $k \in \mathbf{FI}_d$, the *endofunctor*

$$\tau_k : \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$$

is defined by $\tau_k(F) = F((-) + k)$. This means that $\tau_k(F)$ sends an object n to $F(n+k)$ and a morphism (f,g) to $F((f,g) + \mathrm{Id}_k)$. For $x \in \mathbf{FI}_d(0,k)$, the *natural transformation*

$$i_k^x : \mathrm{Id} \to \tau_k$$

is defined on a functor $F \in \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by $i_k^x(F) = F\big(\mathrm{Id}_{(-)} + x\big) : F(-) \to F((-) + k)$.

The main difference with $\mathbf{FI}$-modules is that for $\mathbf{FI}_d$ there is one natural transformation i_k^x for each morphism $x \in \mathbf{FI}_d(0,k)$, while for $\mathbf{FI}$ there is only one natural transformation i_k for k fixed. For example, when $k = 1$ we have $\mathbf{FI}_d(0,1) \cong C$ and so there are d natural transformations i_1^c, one for each $c \in C$.

Definition 2.6.2. For $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$, the *endofunctor* κ_k^x is the kernel of i_k^x, and δ_k^x is its cokernel. Finally, the endofunctor κ is the sum

$$\kappa = \sum_{k \in \mathbf{FI}_d} \sum_{x \in \mathbf{FI}_d(0,k)} \kappa_k^x .$$

For $k = 1$, the endofunctor δ_1^c is called the c-coloured differential endofunctor.

Remark 2.6.3. The endofunctors τ_1 and δ_1 appears under different names in different contexts: in Joyal's work for functors over the category of finite sets and bijections (see [Joy86]), in [Kuh94] for functors from $\mathbb{F}_p$-vector spaces to $\mathbb{F}_p$-vector spaces, in the representation stability theory (see [CEF15, CEFN14, CE17, CMNR18]), in the definition of polynomial functors in [RWW17], in the theory of twisted commutative algebras (see [SS12, SS16]) or in the work of Ramos (see [Ram17b, LR18]). Palmer introduced variations of these endofunctors in [Pal17] for functor over a *category with stabilisers* which encodes the existence of natural transformations like the transformations i_k^x.

Similarly than above, for $d = 1$ there are unique endofunctors δ_1^c and κ_1^c for the only colour $c \in C$, which are denoted by δ_1 and κ_1, and respectively called the differential and evanescent endofunctors in [DV19]. These endofunctors are used to construct both strong and weak polynomial functors over $\mathbf{FI}$. We will use all the endofunctors δ_k^x and κ_k^x for $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$ to define the polynomial functors over $\mathbf{FI}_d$. These endofunctors are arranged in a very important exact sequence:

Lemma 2.6.4. *By definition for* $k \in \mathbf{FI}_d$ *and* $x \in \mathbf{FI}_d(0,k)$ *there is an exact sequence of endofunctors*

$$0 \longrightarrow \kappa_k^x \longrightarrow \mathrm{Id} \xrightarrow{i_k^x} \tau_k \longrightarrow \delta_k^x \longrightarrow 0 \qquad \text{(I)}$$

It is important to note that, on $\mathbf{FI}_d$ there are formulas that associate τ_k with iterations of τ_1 and i_k^x with iterations of i_1^c which are presented in the following proposition, but there is no such formula for δ_k^x or κ_k^x. In particular, δ_k^x is not the composition of k endofunctors δ_1^c.

Proposition 2.6.5. *For $k \in \mathbf{FI}_d$ and $x = (c_1, \dots, c_k) \in \mathbf{FI}_d(0,k)$, there are identities $\tau_k = \tau_1 \circ \cdots \circ \tau_1$ and $i_k^x = \tau_{k-1}(i_1^{c_k}) \circ \dots \circ \tau_1(i_1^{c_2}) \circ i_1^{c_1}$. However, for $d > 1$ and $k \geq 2$, there is no similar isomorphism for δ_k^x or κ_k^x i.e. we do not always have $\delta_k^x \cong \delta_1^{c_k} \circ \cdots \circ \delta_1^{c_1}$ or $\kappa_k^x \cong \kappa_1^{c_k} \circ \cdots \circ \kappa_1^{c_1}$.*

Proof. Since $\mathbf{FI}_d$ is the skeleton of the category of finite sets and coloured injections, τ_k is strictly equal to the composition $\tau_1 \circ \cdots \circ \tau_1$ of τ_1 with itself k times. The relation for i_k^x also follows from the definitions of τ_k and i_k^x. We give a counterexample to prove that there is no similar relation for δ_k^x and κ_k^x if $d > 1$: Let $F = F_{c_1}^{\mathbf{FI}_d} : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ be the functor of Example 2.3.3, we then compute that $\delta_1^{c_1}(F) = \tau_1(F)$, $\delta_1^{c_2}(F) = 0$ and $\delta_2^{(c_1,c_2)}(F) = \tau_2(F)$. This proves that $\delta_1^{c_2} \circ \delta_1^{c_1}(F) = \delta_1^{c_2}(\tau_1(F)) \cong \delta_1^{c_2}(F) = 0$ and $\delta_1^{c_1} \circ \delta_1^{c_2}(F) = \delta_1^{c_1}(0) = 0$, while $\delta_2^{(c_1,c_2)}(F) = \tau_2(F)$ is not zero. Similarly we have $\kappa_1^{c_1}(F) = F$ and $\kappa_1^{c_2}(F) = 0$ which gives $\kappa_1^{c_2} \circ \kappa_1^{c_1}(F) = \kappa_1^{c_1} \circ \kappa_1^{c_2}(F) = 0$, while $\kappa_2^{(c_1,c_2)}(F) = F$. □

Before using these endofunctors to define the polynomial functors in the following chapters, we give some of their basic properties which will be used several times. For $d = 1$ we recover most of [DV19, Proposition 2.4].

Proposition 2.6.6. *For $k, l \in \mathbf{FI}_d$, $x \in \mathbf{FI}_d(0,k)$ and $y \in \mathbf{FI}_d(0,l)$ we have:*

0) For every short exact sequence $0 \to F \to G \to H \to 0$ in $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ there is an exact sequence $0 \longrightarrow \kappa_k^x(F) \longrightarrow \kappa_k^x(G) \longrightarrow \kappa_k^x(H) \longrightarrow \delta_k^x(F) \longrightarrow \delta_k^x(G) \longrightarrow \delta_k^x(H) \longrightarrow 0$.

1) The endofunctors τ_k and τ_l commute up to a natural isomorphism. They also commute with limits and colimits.

2) The endofunctors δ_k^x and δ_l^y commute up to a natural isomorphism. They also commute with colimits.

3) The endofunctors κ_k^x and κ_l^y commute up to a natural isomorphism. They also commute with limits.

4) The inclusion $(\kappa_k^x) \circ (\kappa_k^x) \hookrightarrow (\kappa_k^x)$ gives a natural isomorphism $(\kappa_k^x)^2 \cong (\kappa_k^x)$.

5) The endofunctors τ_l and δ_k^x commute up to a natural isomorphism.

6) The endofunctors τ_l and κ_k^x commute up to a natural isomorphism.

7) There is a natural exact sequence

$$0 \longrightarrow \kappa_l^y \longrightarrow \kappa_{k+l}^{x+y} \longrightarrow \tau_l \circ \kappa_k^x \longrightarrow \delta_l^y \longrightarrow \delta_{k+l}^{x+y} \longrightarrow \tau_l \circ \delta_k^x \longrightarrow 0\ .$$

8) The family of subobjects

$$\left(\kappa_k^x(F) \right)_{k \in \mathbf{FI}_d,\, x \in \mathbf{FI}_d(0,k)}$$

of F forms a filtered set for the inclusion.

9) The endofunctor κ is left exact.

Proof. 0) The endofunctor τ_k is exact by definition so the following diagram has exact rows

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d, "i_k^x(F)"] & G \arrow[r] \arrow[d, "i_k^x(G)"] & H \arrow[r] \arrow[d, "i_k^x(H)"] & 0 \\
0 \arrow[r] & \tau_k(F) \arrow[r] & \tau_k(G) \arrow[r] & \tau_k(H) \arrow[r] & 0
\end{tikzcd}$$

It commutes by naturality of i_k^x, and the snake lemma gives the result.

1) Since $\mathbf{FI}_d$ is the skeleton of the category of finite sets and coloured injections, we the relation $\beta : k + l = l + k$ in $\mathbf{FI}_d$. Then we get $\tau_k \circ \tau_l = \tau_{k+l} = \tau_{l+k} = \tau_l \circ \tau_l$, where the middle equality is given by the natural transformation sending $F : \mathbf{FI}_d \to \textbf{R-Mod}$ to $F(\mathrm{Id}_{(-)} + \beta) : F(-) \to F((-) + k + l)$. Moreover the endofunctors τ_k and τ_l behave well with respect to the universal property of limits and colimits of functors, so they commute with both.

5,6) Applying the exact functor τ_l to the exact sequence (I) from Lemma 2.6.4 on one side, and pre-composing it with τ_l on the other side, we get the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \kappa_k^x \circ \tau_l \arrow[r] \arrow[d, dashed] & \tau_l \arrow[r, "i_k^x \circ \tau_l"] \arrow[d, "\mathrm{id}"'] & \tau_k \circ \tau_l \arrow[r] \arrow[d, "\beta"] & \delta_k^x \circ \tau_l \arrow[r] \arrow[d, dashed, "\gamma"] & 0 \\
0 \arrow[r] & \tau_l \circ \kappa_k^x \arrow[r] & \tau_l \arrow[r, "\tau_l \circ i_k^x"'] & \tau_l \circ \tau_k \arrow[r] & \tau_l \circ \delta_k^x \arrow[r] & 0
\end{tikzcd}$$

By definition of β it commutes in the middle. Using the universal property of kernels for κ_k^x, and of cokernels for δ_k^x, we get the existence of the two dashed morphisms. They are isomorphisms by the five lemma since β is an isomorphism.

2) We again use the exact sequence (I) with l and y but this time we extract the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(s) \xrightarrow{i_l^y|_{\mathrm{Ker}(s)}} \tau_l \xrightarrow{s} \delta_l^y \longrightarrow 0$$

from it, where $\mathrm{Ker}(s)$ is a subfunctor of the identity. Applying the right exact (by point 0) functor δ_k^x to it on one side, and pre-compose it with δ_k^x on the other side, we get another diagram with exact rows

$$\begin{tikzcd}
 & \delta_k^x \circ \mathrm{Ker}(s) \arrow[r, "\delta_k^x \circ i_l^y|_{\mathrm{Ker}(s)}"] \arrow[d, "\mathrm{id}"'] & \delta_k^x \circ \tau_l \arrow[r] \arrow[d, "\gamma"] & \delta_k^x \circ \delta_l^y \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathrm{Ker}(s) \circ \delta_k^x \arrow[r, "i_l^y|_{\mathrm{Ker}(s)} \circ \delta_k^x"'] & \tau_l \circ \delta_k^x \arrow[r] & \delta_l^y \circ \delta_k^x \arrow[r] & 0
\end{tikzcd}$$

since $\mathrm{Ker}(s)$ is a subfunctor of the identity. It commutes in the middle by construction of γ as β passing to the quotient and because $\mathrm{Ker}(s)$ is a subfunctor of the identity. Then the universal property of the cokernel induces the existence of the dashed arrow and it is an isomorphism by the five lemma.

3) It is analogous to the point 2)

4) We again use the exact sequence (I) by pre-composing it with the endofunctor κ_k^x and

by applying the exact endofunctor τ_k to it. We get the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \kappa_k^x \circ \kappa_k^x \arrow[r] & \kappa_k^x \arrow[r, "i_k^x \circ \kappa_k^x"] \arrow[d, "j"'] & \tau_k \circ \kappa_k^x \arrow[r] \arrow[d, "\tau_k(j)"] & \delta_k^x \circ \kappa_k^x \arrow[r] & 0 \\
0 \arrow[r] & \kappa_k^x \arrow[r, "j"'] & \mathrm{id} \arrow[r, "i_k^x"'] \arrow[d] & \tau_k \arrow[r] \arrow[d] & \delta_k^x \arrow[r] & 0 \\
 & & \tau_k & \tau_k \circ \tau_k & &
\end{tikzcd}$$

It commutes in the middle since the transformation i_k^x is natural between id and τ_k. Since the second row is exact we have $\tau_k(j) \circ \left(i_k^x \circ \kappa_k^x\right) = i_k^x \circ j = 0$ but $\tau_k(j)$ is a monomorphism since τ_k is exact, which implies that $i_k^x \circ \kappa_k^x = 0$. By exactness it means that the inclusion $(\kappa_k^x) \circ (\kappa_k^x) \hookrightarrow (\kappa_k^x)$ is an isomorphism.

7) Recall (see [ML98] p.208) that for two composable morphisms $u : a \to b$ and $v : b \to c$ there always exists an exact sequence

$$0 \to \mathrm{Ker}(u) \to \mathrm{Ker}(v \circ u) \to \mathrm{Ker}(v) \to \mathrm{Coker}(u) \to \mathrm{Coker}(v \circ u) \to \mathrm{Coker}(v) \to 0.$$

We use this property for $u = i_l^y : \mathrm{Id} \to \tau_l$ and $v = \tau_l\left(i_k^x\right) : \tau_l \to \tau_l \circ \tau_k$. The endofunctor τ_l commutes with kernels and cokernels as they are limits and colimits, so the result follows from the identity

$$v \circ u = \left(\tau_l\left(i_k^x\right)\right) \circ i_l^y = \left(i_k^x + \mathrm{id}_l\right) \circ i_l^y = i_{k+l}^{x+y}.$$

8) From point 7) we have an inclusion $\kappa_l^y \hookrightarrow \kappa_{k+l}^{x+y}$ and by symmetry we also have $\kappa_k^x \hookrightarrow \kappa_{k+l}^{x+y}$.

9) By the point 8), the colimit κ is a growing filtered colimit so it is exact since **R-Mod** is a Grothendieck category (Definition 1.3.1). Knowing that each κ_k^x is left exact by the point 0), this implies that their colimit κ is also left exact.

□

Unlike what the last proposition suggest, we warn the reader that these endofunctors do not all commute together. In particular, the endofunctors δ_k^x and κ_l^y do not commute, as explained in the following.

Remark 2.6.7. For $k, l \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$, $y \in \mathbf{FI}_d(0,l)$, the endofunctors κ_l^y and δ_k^x of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ do not commute in general, even for $d = 1$. We give there a counterexample for $k = l = 1$ and $x = c \in C$: for $M \in \mathbf{R\text{-}Mod}$, let $M_{\geq k} : \mathbf{FI}_d \to \mathbf{R\text{-}Mod}$ be the functor defined in Section 2.3 as a subfunctor of the constant functor. As explained later in Example 5.1.7, we can compute that $\tau_1(M_{\geq k}) = M_{\geq k-1}$ and that, for any colour c, $\kappa_1^c(M_{\geq k}) = 0$ and $\delta_1^c(M_{\geq k}) = M_{k-1}$, where M_{k-1} is the atomic functor of rank $k-1$. This implies that $\kappa_1^c \circ \delta_1^c(M_{\geq k})$ is the atomic functor of rank $k-1$, while $\delta_1^c \circ \kappa_1^c(M_{\geq k})$ is zero.

2.7 The forgetful and colouring functors

In this section we study the link between the $\mathbf{FI}_d$-modules and the $\mathbf{FI}$-modules. In particular, we present some properties of the forgetful functor $\mathcal{O} : \mathbf{FI}_d \to \mathbf{FI}$ from Definition 2.1.6 and a family of right-adjoints $\Delta_c : \mathbf{FI} \to \mathbf{FI}_d$ for $c \in C$ called the colouring functors. We start this section by showing that the endofunctors of the previous section (Definition 2.6.2) behave well with the precomposition by the forgetful functor.

Proposition 2.7.1. *For all objects $k \in \mathbf{FI}_d$ and all morphism $x \in \mathbf{FI}_d(0,k)$ there are natural isomorphisms :*

$$\begin{array}{rlcl} i) & \mathcal{O}^* \circ \tau_k & = & \tau_k \circ \mathcal{O}^* \\ ii) & \mathcal{O}^* \circ \delta_k & \cong & \delta_k^x \circ \mathcal{O}^* \\ iii) & \mathcal{O}^* \circ \kappa_k & \cong & \kappa_k^x \circ \mathcal{O}^* \end{array} .$$

Proof. For $F \in \mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$, using the fact that the forgetful functor $\mathcal{O}$ is monoidal together with the relations $\mathcal{O}(k) = k$ and $\mathcal{O}(\mathrm{Id}) = \mathrm{Id}$, we have

$$\mathcal{O}^* \circ \tau_k(F) = F(- + k) \circ \mathcal{O} = F(\mathcal{O}(-) + k) = F(\mathcal{O}(- + k)) = \tau_k(F \circ \mathcal{O}) = \tau_k \circ \mathcal{O}^*(F).$$

Moreover, for a natural transformation $\sigma \in \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})(F,G)$, we also have:

$$\mathcal{O}^* \circ \tau_k(\sigma) = \mathcal{O}^*(\sigma_{-+k}) = \sigma_{\mathcal{O}(-)+k} = \sigma_{\mathcal{O}(-+k)} = \tau_k(\sigma_{\mathcal{O}(-)}) = \tau_k \circ \mathcal{O}^*(\sigma).$$

This shows that there is an equality $\mathcal{O}^* \circ \tau_k = \tau_k \circ \mathcal{O}^*$. By pre-composing the exact sequence of endofunctors of $\mathbf{FI}_d$ (I) from Lemma 2.6.4 by the functor $\mathcal{O}^*$ we get the exact sequence

$$0 \longrightarrow \kappa_k^x \circ \mathcal{O}^* \longrightarrow \mathcal{O}^* \xrightarrow{i_k^x \circ \mathcal{O}^*} \tau_k \circ \mathcal{O}^* \longrightarrow \delta_k^x \circ \mathcal{O}^* \longrightarrow 0 \ .$$

By the definition of precomposition on natural transformations, we have for any functor $F \in \mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$:

$$i_k^x \circ \mathcal{O}^*(F) = i_k^x(F \circ \mathcal{O}) = F \circ \mathcal{O}(\mathrm{id} + x) = F\big(\mathrm{id} + (0 \to k)\big) = i_k(F).$$

Next we use the same exact sequence (I), but for $\mathbf{FI} = \mathbf{FI}_1$, and we apply the exact functor $\mathcal{O}^*$ to it, which gives the exact sequence

$$0 \longrightarrow \mathcal{O}^* \circ \kappa_k \longrightarrow \mathcal{O}^* \xrightarrow{\mathcal{O}^*(i_k) = i_k} \mathcal{O}^* \circ \tau_k(F) \longrightarrow \mathcal{O}^* \circ \delta_k(F) \longrightarrow 0 \ .$$

Applying the precomposition functor $\mathcal{O}^*$ to the natural transformation i_k, we have for any functor $F \in \mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$:

$$\mathcal{O}^*\big(i_k\big)(F) = F\big(\ \mathrm{id}_{\mathcal{O}(-)} + (0 \to k)\ \big) = F\big(\ \mathrm{id}_{(-)} + (0 \to k)\ \big) = i_k(F).$$

We then have the following diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}^* \circ \kappa_k \arrow[r] \arrow[d, dashed] & \mathcal{O}^* \arrow[r, "\mathcal{O}^*(i_k) = i_k"] \arrow[d, equal] & \mathcal{O}^* \circ \tau_k \arrow[r] \arrow[d, equal] & \mathcal{O}^* \circ \delta_k \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \kappa_k^x \circ \mathcal{O}^* \arrow[r] & \mathcal{O}^* \arrow[r, "i_k^x \circ \mathcal{O}^* = i_k"'] & \tau_k \circ \mathcal{O}^* \arrow[r] & \delta_k^x \circ \mathcal{O}^* \arrow[r] & 0. \end{tikzcd}$$

It commutes in the middle since $i_k^x \circ \mathcal{O}^* = i_k = \mathcal{O}^*(i_k)$, so the two dashed arrows exist by universal properties and they are isomorphisms by the five lemma. □

We now define a collection of functors from $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ to $\mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ called the colouring functors. These functors add a colour on the morphisms in $\mathbf{FI}$ to get morphisms in $\mathbf{FI}_d$ and, by precomposition they allow us to consider a $\mathbf{FI}_d$-module as a $\mathbf{FI}$-module. We will use this to describe the functors that are stably zero along colours in a concrete way in Section 6.2.

Definition 2.7.2. For $c \in C \cong \mathbf{FI}_d(0,1)$, the *c-colouring functor* $\Delta_c : \mathbf{FI} \to \mathbf{FI}_d$ is the functor given by the identity on objects and on a morphism $f \in \mathbf{FI}(n,m)$ by

$$\Delta_c(f) = (f : n \hookrightarrow m\,, m \smallsetminus \mathrm{Im}(f) \to \{c\} \to C)\,.$$

Let $\Delta_c^* : \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod}) \to \mathbf{Fct}(\mathbf{FI}, \mathbf{R\text{-}Mod})$ denote the precomposition functor defined by $\Delta_c^*(F) = F \circ \Delta_c$ for all functors $F \in \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ and by $\Delta_c(\sigma) = \sigma$ for all natural transformations $\sigma : F \to G$, where σ on the right is seen as a natural transformation between $F \circ \Delta_c$ and $G \circ \Delta_c$.

The functor Δ_c is monoidal since adding the colour c on the arrows does not affect the monoidal structure. By definition we also get $\Delta_c(0 \to 1) = (0 \to 1, c) = c \in \mathbf{FI}_d(0,1)$ and $\Delta_c(i_k) = i_k^{c^k}$, where $i_k : \mathrm{Id} \to \tau_k$ and $i_k^{c^k} : \mathrm{Id} \to \tau_k$ are the natural transformations of Definition 2.6.1 for $\mathbf{FI}$ and $\mathbf{FI}_d$ respectively. Finally, the c-colouring functors are right-inverses of the forgetful functor $\mathcal{O} : \mathbf{FI}_d \to \mathbf{FI}$ as explained in the following.

Proposition 2.7.3. *For all colours $c \in C$ we have the identities:*

$$\mathcal{O} \circ \Delta_c = \mathrm{Id}_{\mathbf{FI}} \quad \textit{and} \quad \Delta_c^* \circ \mathcal{O}^* = \mathrm{Id}_{\mathbf{Fct(FI)}},$$

where $\mathcal{O}$ is the forgetful functor of Definition 2.1.6.

Proof. Following the definitions of $\mathcal{O}$ and Δ_c, it is a direct calculus. □

We now give some properties of the precomposition by colouring functors Δ_c^*, in particular we describe how they behave with endofunctors τ_k, δ_k and κ_k.

Proposition 2.7.4. *For $k \in \mathbf{FI}_d$ and $c \in C$ there are natural isomorphisms:*

$$\begin{array}{rccc} i) & \tau_k \circ \Delta_c^* & \cong & \Delta_c^* \circ \tau_k \\ ii) & \delta_k \circ \Delta_c^* & \cong & \Delta_c^* \circ \delta_k^{c^k} \\ iii) & \kappa_k \circ \Delta_c^* & \cong & \Delta_c^* \circ \kappa_k^{c^k} \end{array}.$$

Proof. For $F \in \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$, the first isomorphism can be checked by hand: since Δ_c is monoidal, we have the following identity

$$\tau_k \circ \Delta_c^*(F) = F \circ \Delta_c(- + k) \cong F\big(\Delta_c(-) + k\big) = F(- + k) \circ \Delta_c(-) = \tau_k(F) \circ \Delta_c(-) = \Delta_c^* \circ \tau_k(F).$$

This isomorphism is natural by definition of the precomposition functors on natural transformations. For the other isomorphisms we take the exact sequence (I) from Lemma 2.6.4 for $k \in \mathbf{FI}_d$ and $x = c^k \in \mathbf{FI}_d(0,k)$ and we apply the exact functor Δ_c^* to it. It gives the exact sequence

$$0 \longrightarrow \Delta_c^* \circ \kappa_k^{c^k}(F) \longrightarrow \Delta_c^*(F) \xrightarrow{\Delta_c^*\left(F(i_k^{c^k})\right)} \Delta_c^* \circ \tau_k(F) \longrightarrow \Delta_c^* \circ \delta_k^{c^k}(F) \longrightarrow 0$$

and by definition of precomposition functor on natural transformations we have $\Delta_c^*(\,F(i_k^{c^k})\,) = F(i_k^{c^k})$. Next we use the exact sequence (I) again, but for $\mathbf{FI} = \mathbf{FI}_1$, and we precompose it with the functor $\Delta_c^*(F)$. This gives the exact sequence

$$0 \longrightarrow \kappa_k \circ \Delta_c^*(F) \longrightarrow \Delta_c^*(F) \xrightarrow{\Delta_c^*(F)\,(i_k)} \tau_k \circ \Delta_c^*(F) \longrightarrow \delta_k^{c^k} \circ \Delta_c^*(F) \longrightarrow 0$$

and by definition we have $\Delta_c^*(F)\,(i_k) = F\circ\Delta_c(i_k) = F(i_k^{c^k})$. We then have the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Delta_c^* \circ \kappa_k^{c^k}(F) \arrow[r] \arrow[d, dashed] & \Delta_c^*(F) \arrow[r, "F\,(i_k^{c^k})"] \arrow[d, equal] & \Delta_c^* \circ \tau_k(F) \arrow[r] \arrow[d, equal] & \Delta_c^* \circ \delta_k^{c^k}(F) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \kappa_k \circ \Delta_c^*(F) \arrow[r] & \Delta_c^*(F) \arrow[r, "F\,(i_k^{c^k})"] & \tau_k \circ \Delta_c^*(F) \arrow[r] & \delta_k \circ \Delta_c^*(F) \arrow[r] & 0.
\end{tikzcd}$$

It commutes in the middle by the previous point, so the two dashed arrows exist by universal properties and they are isomorphisms by the five lemma. $\square$

Chapter 3

Homology of Sink configuration spaces of graphs

Je t'aime sont les mots les plus importants à dire dans une vie.

Albert Dupontel

There are many concrete **FI**-modules that occur in different contexts. Numerous examples are presented in [CF13]. One interesting example, for a regular manifold M, is the cohomology of the configuration spaces of M, which is fully detailed in [Sam20, Wil18a, Wil19] and [CF13]. When M is open, there are a structure of **FI**-module and $\mathbf{FI}^{\mathrm{op}}$-module which are compatible (the first one is given by adding points at infinity on the boundary). This gives a structure of $\mathbf{FI}\#$-modules, where $\mathbf{FI}\#$ is the category of finite sets and partial injections presented in [Wil18a, MW19, MW20], which is equivalent to the category Cospan(**FI**) from [DV19] presented in Chapter 9. Even if there is an extensive literature on the cohomology of the configuration spaces of a manifold, these groups essentially have been studied globally and are known explicitly only in a few cases. The stability theorem from [CEF15] states that, for a non-compact manifold, these **FI**-modules are finitely generated, which can be interpreted in terms of polynomial functor since being finitely generated is almost equivalent to being strong polynomial for **FI**-modules as we will see in Section 5.1.

These results about the **FI**-module $H^i\left(\mathrm{Conf}_{(-)}\left(M\right), \mathbf{R}\right)$ are proved for a manifold M of dimension at least two in order to ensure that the configuration spaces are connected and that the points can move around each other. But for a manifold of dimension 1, like a graph, there is not enough space and the points block each other in the configuration spaces, so the same approach is no longer valid. Therefore, Ramos introduced in [Ram19] the homology of a kind of modified configuration spaces of graphs, called the *sink configuration spaces*, in which we take n (ordered) points on the graph as for the classical ones but in which they can either be distinct two by two or they can overlap at a vertex of the graph but not within an edge. Then, the d vertices of the graph correspond to the d colours of $\mathbf{FI}_d$ which gives the structure of a $\mathbf{FI}_d$-module when we take the homology of these topological spaces. This gives an interesting family of examples of $\mathbf{FI}_d$-modules. Ramos proved in [Ram19] that these $\mathbf{FI}_d$-modules are finitely generated for every homological degree and every connected graph. In Proposition 3.2.8 we give an explicit description of these functors for the linear graphs and we show that they are either twisted atomic or constant functors.

3.1 Cohomology of classical configuration spaces as FI-modules

We start by presenting the **FI**-module $H^i(\mathrm{Conf}_M(-), \mathbf{R})$ of the i-th cohomology of the configuration spaces of a manifold M. We give a concrete example of how it acts on maps and we summarize the results about this **FI**-module.

Definition 3.1.1. For M a manifold the *n-strand configuration space* of M is the topological space

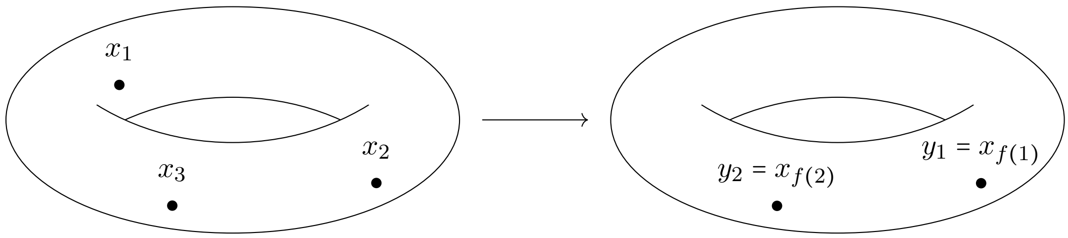
$$\mathrm{Conf}_n\left(M\right) = \left\{(x_1, \ldots, x_n) \in M^n \mid \forall 1 \leq i \neq j \leq n,\, x_i \neq x_j\right\},$$

where the topology is induced from M^n.

In explicit words, we take n (ordered) points on the manifold distinct two by two. These configuration spaces, presented among many others in [CF13, CEF15, MW20, CF13] naturally give a functor from $\mathbf{FI}^{op}$ to **Top** as explained in the following.

Definition 3.1.2. The contravariant *functor* $\mathrm{Conf}_{-}(M) : \mathbf{FI}^{\mathrm{op}} \to \mathbf{Top}$ sends an object $n \in \mathbf{FI}$ to the topological space $\mathrm{Conf}_n\left(M\right)$ and a map $f \in \mathbf{FI}(n, m)$ to the map that sends $(x_1, \ldots, x_m)$ to $(y_1, \ldots, y_n) = (x_{f(1)}, \ldots, x_{f(n)})$.

For example, if M is the torus and f the injection $(0 \to 1) + \mathrm{Id}_2 \in \mathbf{FI}(2, 3)$ which sends 1 to 2 and 2 to 3, it gives an map like the following one:



We can then take the cohomology of these topological spaces, which is contravariant, and with the induced maps in cohomology, this gives a **FI**-module $H^i\left(\mathrm{Conf}_{(-)}\left(M\right), \mathbf{R}\right)$.

Theorem 3.1.3. *For $i \in \mathbb{N}$, if M is a connected oriented manifold of dimension at least 2 with $\dim_{\mathbb{Q}}(H^*(M, \mathbb{Q})) < \infty$, then the* **FI***-module $H^i\left(\mathit{Conf}_{(-)}\left(M\right), \mathbb{Q}\right)$ is finitely generated.*

Proof. It was proved by Church, Ellenberg and Farb in [CEF15, Theorem 6.2.1] . ☐

For example, the hypothesis $\dim_{\mathbb{Q}} H^*(M, \mathbb{Q})) < \infty$ is satisfied is M is compact. This result is illustrated on a concrete example in [Wil19, Section 3.1]. More than that, there is a stronger result of polynomiality, which can be interpreted with Definitions 5.1.1 and 7.2.1 of strong and weak polynomial **FI**-modules (already present in [DV19] for **FI**-modules). More precise boundaries are given in [MW20] since the generation degree corresponds to the strong degree and the presentation degree precise how to the weak and strong degrees are linked.

Theorem 3.1.4. *For $i \in \mathbb{N}$, if M is a connected manifold of dimension at least 2, then the* **FI***-module $H^i\left(\mathit{Conf}_{(-)}\left(M\right), \mathbf{R}\right)$ is strong and weak polynomial of degree less than or equal to $2i$.*

Proof. It is proved in [CEF15, Theorem 1.8] for field of characteristic 0, and in [CMNR18, Application A] in general. ☐

Remark 3.1.5. An additional similar construction can be done if M is a non-compact manifold of dimension at least 2, as explained in [Wil18a] and [Wil19], by sending the injection $\mathrm{Id}_n + (0 \to 1) : n \hookrightarrow n+1$ to a map that adds a new point "at infinity" on the boundary of the rescaled manifold. When $\dim(M) \geq 2$, these maps define a functor over **FI** up to homotopy, where the symmetric group S_n permutes the n points of the n-th configuration space. It has been proven in [CEF15, Theorem 6.4.3] and [MW19, Theorem 3.12] that these **FI**-modules are finitely generated with better bounds than for the compact case which can be interpreted in terms of polynomial functor since being finitely generated is almost equivalent to being strong polynomial for **FI**-modules (see Section 5.1). The proof uses a vanishing result about the spectral sequence associated with the semi-simplicial space called the arc resolution. These arcs connect the points of the configuration space to the boundary of the manifold and are defined when the dimension of M is at least 2. For an open manifold, the structure of **FI**-module and $\mathbf{FI}^{\mathrm{op}}$-module are compatible, giving a structure of **FI**#-modules, where **FI**# is the category of finite sets and partial injections (see in [Wil18a, MW19, MW20]) and is equivalent to the category Cospan(**FI**) from [DV19]. The theorem of [CEF15] which states that these **FI**-modules are finitely generated can be interpreted as follows: if the homological degree is small enough relative to the number of points, the homology of the configuration spaces is spanned by the classes corresponding to the configuration spaces were at least one point is isolated near infinity (*i.e.* near the boundary of the manifold). This was generalized for open manifolds in [MW19] where they showed that, after some point, the homology of the configuration spaces is spanned by the classes were at least one point is stationary at infinity or two points are orbiting around each other near infinity.

Remark 3.1.6. Other interesting examples of finitely generated **FI**-module are given by the cohomology of the pure string motion groups in [Wil12] and the pure braid groups in [Wil18a]. The pure braid groups are equivalent to the fundamental group of the configuration spaces of $\mathbb{C}$, which gives another proof using the finiteness result about the configuration space.

3.2 Homology of a generalized configuration space of graphs as $\mathbf{FI}_d$-modules

The results of the previous section are obtained for a manifold of dimension at least two but for a manifold of dimension 1, like a graph, the same method does not work since the points block each other in the configuration space. For example, as explained in [Wil19] and [Ram19], if G is the linear graph with only one edge then the configuration space is homotopy equivalent to $n!$ disjoints points, while it is always connected when M has higher dimension. In this section we present a variation of this, which makes a $\mathbf{FI}_d$-module for graphs by using the homology of some kind of modified configuration spaces introduced by Ramos in [Ram19]. This gives an interesting non-trivial example of $\mathbf{FI}_d$-module since, before that, all the $\mathbf{FI}_d$-modules in the literature were either free or obtained from **FI**-modules via the forgetful functor. In this section we calculate and give an explicit description of these functors for the linear graphs. We show that they are either twisted atomic or constant functors and we then recover for these examples that these $\mathbf{FI}_d$-modules are finitely generated as proved in [Ram19].

Definition 3.2.1. Let G be a graph with d vertices labeled by $[d] = \{1, \ldots, d\}$, the *n-strand sink configuration space* of G is given by

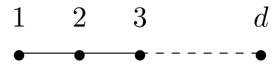
$$\mathrm{Conf}_n^{\mathrm{sink}}\big(G, [d]\big) = \big\{(x_1, \ldots, x_n) \in G^n \mid \forall\, 1 \leq i \neq j \leq n,\ x_i \neq x_j \text{ or } x_i = x_j \in [d]\big\}.$$

In explicit words, as for the classical configuration spaces we take n (ordered) points on the graph, but in these sink configuration spaces they can either be distinct two by two, or they can overlap at a vertex of the graph but not within an edge. These configuration spaces naturally give a functor from $\mathbf{FI}_d$ to **Top**, as explained in the following, since the d vertices of the graph will correspond to the d colours of $\mathbf{FI}_d$.

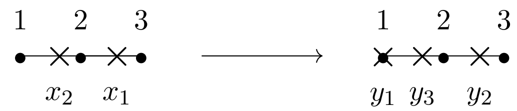
Definition 3.2.2. The covariant *functor* $\mathrm{Conf}^{\mathrm{sink}}_{-}(G,[d]) : \mathbf{FI}_d \to \mathbf{Top}$ sends an object $n \in \mathbf{FI}_d$ to the topological space $\mathrm{Conf}^{\mathrm{sink}}_{n}\big(G,[d]\big)$ and a map $(f,g) \in \mathbf{FI}_d(n,m)$ to the map that sends $(x_1,\dots,x_n)$ to $(y_1,\dots,y_m)$ with

$$y_j = \begin{cases} x_{f^{-1}(j)} & \text{if } j \in \mathrm{Im}(f) \\ g(j) \in [d] & \text{else .} \end{cases}$$

For the rest of this section, let $\mathcal{G}_d$ be the *linear graph* on d vertices:



For $d = 3$, we give an example of an map $(x_1,\dots,x_n) \mapsto (y_1,\dots,y_m)$ from Definition 3.2.2 for $\mathcal{G}_3$ and the injection $c_1 + \mathrm{Id}_2 \in \mathbf{FI}_3(2,3)$ (which sends 1 to 2, 2 to 3 and colours the element 1 with c_1):



We can then consider the homology $H_i\left(\mathrm{Conf}^{\mathrm{sink}}_{(-)}\left(G,[d]\right),\mathbb{Q}\right)$ of these topological spaces and, together with the induced maps in homology, this gives an $\mathbf{FI}_d$-module. We consider the rational homology since it is the main framework studied by Ramos due to its connection with the representation stability. It also allows us to do concrete computations and to use the classification of the irreducible representations of the symmetric groups recalled in Remark 2.4.1. However, most of the following remains true for the homology over a general commutative ring $\mathbf{R}$.

Theorem 3.2.3. *For $i \in \mathbb{N}$ and G a connected graph, the $\mathbf{FI}_d$-module $H_i\left(\mathit{Conf}^{sink}_{(-)}\left(G,[d]\right),\mathbb{Q}\right)$ is finitely generated.*

Proof. It was proved by Ramos in [Ram19, Theorem 4.1]. □

Remark 3.2.4. The case $i = 0$ is simple to describe. Indeed, if G is connected then the space $\mathrm{Conf}^{\mathrm{sink}}_{(-)}\left(G,[d]\right)$ is connected and so $H_0\left(\mathrm{Conf}^{\mathrm{sink}}_{(-)}\left(G,[d]\right),\mathbb{Q}\right)$ is the constant functor $\mathbb{Q}$. For a general graph G, the same argument applies to the connected components of G which implies that $H_0\left(\mathrm{Conf}^{\mathrm{sink}}_{n}\left(G,[d]\right),\mathbb{Q}\right) \cong \mathbb{Q}^{n^c}$, where c is the number of connected components.

In the end of this section, we give an explicit description of this $\mathbf{FI}_d$-module for the linear graphs $\mathcal{G}_d$. To do this we first describe the space $\mathrm{Conf}^{\mathrm{sink}}_{n}\left(\mathcal{G}_2,[2]\right)$ for $d = 2$, then we deduce the general case before computing the homology in Proposition 3.2.8.

Proposition 3.2.5. *For $n \in \mathbb{N}^*$, the space $\mathit{Conf}^{sink}_{n}\left(\mathcal{G}_2,[2]\right)$ is homotopy equivalent to the sphere $\mathbf{S}^{n-1}$ if $n \geq 2$, and to a point if $n = 1$.*

Proof. There is an embedding of $\mathcal{G}_2$ in the subspace $[-1,1]$ of $\mathbb{R}$. This embedding sends $\mathrm{Conf}^{\mathrm{sink}}_{n}\left(\mathcal{G}_2,[2]\right)$ to a subset of the hypercube $\mathcal{C}_n := [-1,1]^n = \{X \in \mathbb{R}^n \mid \parallel X \parallel_\infty \leq 1\}$. We denote by I the image of $\mathrm{Conf}^{\mathrm{sink}}_{n}\left(\mathcal{G}_2,[2]\right)$ by this embedding. We then have the following description

$$I = \mathcal{C}_n \smallsetminus \left\{ (x_1,\dots,x_n) \in [-1,1]^n \mid \exists\, 1 \leq i \neq j \leq n \text{ such that } -1 < x_i = x_j < 1 \right\}.$$

The last inequalities are strict since the points in $\mathrm{Conf}^{\mathrm{sink}}_{n}\left(\mathcal{G}_2,[2]\right)$ can overlap at a vertex, so the boundary $\partial\mathcal{C}_n$ of the cube $\mathcal{C}_n$ is in I. We show now that $\partial\mathcal{C}_n$ is a deformation retract of I.

Indeed, the center of the hypercube $0 \in \mathbb{R}^n$ is not in I so we can define the central retraction (see figure 3.1 for $n = 2$)

$$\begin{array}{rccl} F: & I \times [0,1] & \to & \partial \mathcal{C}_n \\ & (X,t) & \mapsto & tX\frac{1}{\|X\|_\infty} + (1-t)X. \end{array}$$

Then we can check that $F(X,t)$ is a deformation retraction. Finally, the boundary of the hypercube $\partial\mathcal{C}_n$ is homotopy equivalent to the sphere $\mathbf{S}^{n-1}$, which gives the result if $n \geq 2$. The case $n = 1$ is clear. □

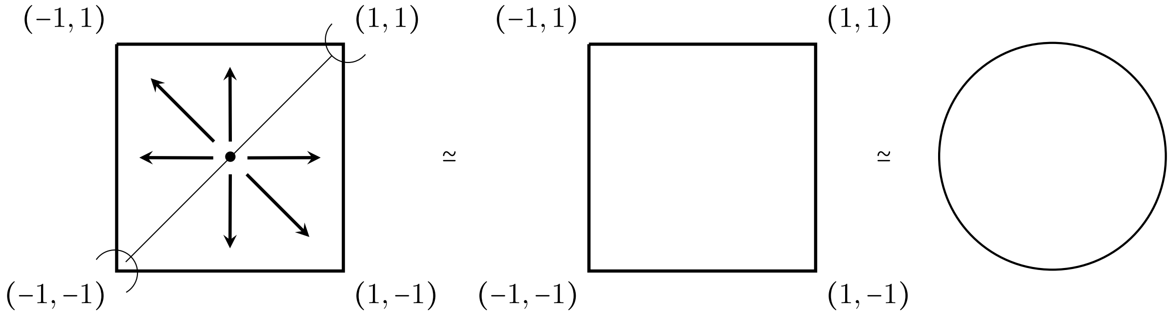


Figure 3.1: The space $\mathrm{Conf}_2^{\mathrm{sink}}\left(G,[2]\right)$ is homotopy equivalent to the sphere $\mathbf{S}^1$.

We now give a similar argument to prove that the space $\mathrm{Conf}_n^{\mathrm{sink}}\left(\mathcal{G}_d,[d]\right)$ for a general d is a bouquet of spheres $\mathbf{S}^{n-1}$.

Proposition 3.2.6. *For $n \in \mathbb{N}^*$ and $\mathcal{G}_d$ the linear graph on d vertices, the space $\mathrm{Conf}_n^{sink}\left(\mathcal{G}_d,[d]\right)$ is homotopy equivalent to the bouquet of $N(d,n)$ spheres $\mathbf{S}^{n-1}$, where*

$$N(d,n) = \begin{cases} (d-1)^n - \binom{d-1}{n} n! & \text{if } d \geq n+1 \\ (d-1)^n & \text{if } d \leq n. \end{cases}$$

Proof. We use the same argument than in the proof of Proposition 3.2.5, but with an embedding of $\mathrm{Conf}_n^{\mathrm{sink}}\left(\mathcal{G}_d,[d]\right)$ in a big hypercube $\mathcal{C}_n \subset \mathbb{R}^n$ given by the embedding of G in the subspace $[0,d-1]$ of $\mathbb{R}$. This big hypercube is composed of $(d-1)^n$ small hypercubes $[i_1-1,i_1] \times \cdots \times [i_n-1,i_n]$ for $1 \leq i_1,\ldots,i_n \leq d-1$. Each small hypercube corresponds to a possible choice of n edges among the $d-1$ in G since $(x_1,\ldots,x_n) \in \mathrm{Conf}_n^{\mathrm{sink}}\left(\mathcal{G}_d,[d]\right)$ is in $[i_1-1,i_1] \times \cdots \times [i_n-1,i_n]$ if and only if x_1 is in the i_1-th edge, x_2 is in the i_2-th edge, ..., x_n is in the i_n-th edge. If we represent a small hypercube by its center, there are two possibilities:

- Either the center has two equal coordinates, and then the center do not belong to $\mathrm{Conf}_n^{\mathrm{sink}}\left(\mathcal{G}_d,[d]\right)$. Since two points are on the same edge, we can use the central retraction as in the proof of Proposition 3.2.5 to show that the subspace of $\mathrm{Conf}_n^{\mathrm{sink}}\left(\mathcal{G}_d,[d]\right)$ included in this hypercube admits the boundary of the small hypercube as a deformation retract.

- Either the center does not have two equal coordinates, and then the whole hypercube is in $\mathrm{Conf}_n^{\mathrm{sink}}\left(\mathcal{G}_d,[d]\right)$ since all the points are on different edges. In this case the subspace of $\mathrm{Conf}_n^{\mathrm{sink}}\left(\mathcal{G}_d,[d]\right)$ corresponding to this hypercube is contractible (it corresponds to the grey squares on the figure 3.2).

The figure 3.2 gives an example of this process for $n = 2$, and $d = 3$. The space $\mathrm{Conf}_n^{\mathrm{sink}}\left(\mathcal{G}_d,[d]\right)$ is then homotopy equivalent to a wedge of boundaries of hypercubes which are homotopy equivalent to the sphere $\mathbf{S}^{n-1}$. The number of spheres in this wedge is given by the number of small hypercubes such that their center has two equal coordinates. This is equal to the number of total hypercubes $(d-1)^n$ minus the number of choices of n different edges among the $d-1$ possible, which is $(d-1)(d-2)\dots(d-n) = \binom{d-1}{n} n!$ if $n \leq d-1$, and zero if $n \geq d$. $\square$

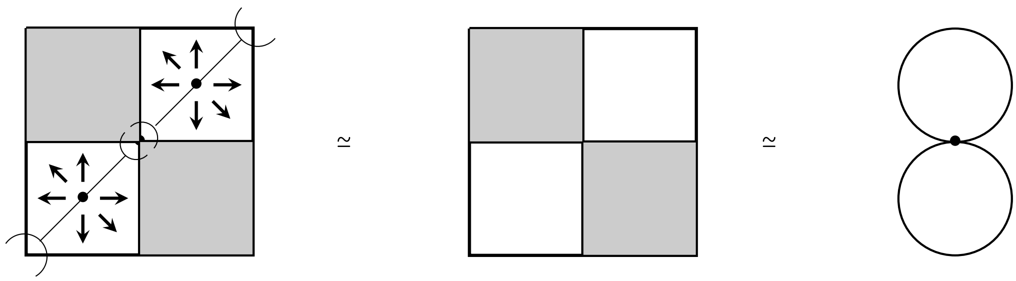


Figure 3.2: The space $\mathrm{Conf}_2^{\mathrm{sink}}\left(G,[3]\right)$ is homotopy equivalent to the wedge of two spheres $\mathbf{S}^1$.

Remark 3.2.7. The process in the proof of Proposition 3.2.6 is similar to the discrete Morse theory arguments used in [Ram19, Section 3.3]. Ramos proved in particular in the corollary 3.25 that, if (G,V,E) is a tree, then $H_i\left(\mathrm{Conf}_n^{\mathrm{sink}}(G,V)\right)$ is torsion free and depends only on i, n and the number of edges $|E|$ in G, and not on the structure of the graph. The proof of this result is based on counting the number of "critical cells" that generates the homology group when we view $\mathrm{Conf}_n^{\mathrm{sink}}(G,V)$ as a CW complex. Then the small hypercubes whose center have two equal coordinates in the proof of Proposition 3.2.6 seems to correspond to the critical cells of the discrete Morse theory, and the other small hypercubes corresponds to collapsible (or redundant) cells. Moreover, the Theorem A in [Ram19] states that, if G is a tree, then $\mathrm{Conf}_n^{\mathrm{sink}}(G,V)$ is homotopy equivalent to a cubical complex and in the proof of [Ram19, Corollary 3.25], it is shown that the group $H_i\left(\mathrm{Conf}_n^{\mathrm{sink}}(G,V)\right)$ is free on the number of critical cells, and so finitely generated. This may indicate that the corresponding $\mathbf{FI}_d$-module has a quite simple description, similar to Proposition 3.2.6, if G is a tree.

Finally, using the description of the space $\mathrm{Conf}_n^{\mathrm{sink}}\left(\mathcal{G}_d,[d]\right)$ from Proposition 3.2.6 we can compute its homology and describe the associated $\mathbf{FI}_d$-module.

Proposition 3.2.8. *For $i \in \mathbb{N}^*$ and $\mathcal{G}_d$ the linear graph on d vertices, the $\mathbf{FI}_d$-module*

$$H_i\left(\mathit{Conf}_{(-)}^{\mathit{sink}}\left(\mathcal{G}_d,[d]\right), \mathbb{Q} \right)$$

is a twisted atomic functor $\left(\mathbb{Q}^{N(d,i+1)}\right)_{i+1}$ of rank $i+1$ defined in Section 2.3, where

$$N(d,i+1) = \begin{cases} (d-1)^{i+1} - \binom{d-1}{i+1}(i+1)! & \text{if } d \geq i+2 \\ (d-1)^{i+1} & \text{if } d \leq i+1. \end{cases}$$

For $i=0$, the $\mathbf{FI}_d$-module $H_0\left(\mathit{Conf}_{(-)}^{\mathit{sink}}\left(\mathcal{G}_d,[d]\right), \mathbb{Q} \right)$ is the constant functor $\mathbb{Q}$.

Proof. By Proposition 3.2.6 the space $\mathrm{Conf}_n^{\mathrm{sink}}\left(\mathcal{G}_d,[d]\right)$ is homotopy equivalent to the bouquet of $N(d,n)$ spheres $\mathbf{S}^{n-1}$. Since $i>0$, we can use the reduced homology and we get the isomorphisms

$$H_i\left(\bigvee_{k=1}^{N(d,n)} \mathbf{S}^{n-1}\right) = \tilde{H}_i\left(\bigvee_{k=1}^{N(d,n)} \mathbf{S}^{n-1}\right) \cong \bigoplus_{k=1}^{N(d,n)} \tilde{H}_i\left(\mathbf{S}^{n-1}\right).$$

Since $\tilde{H}_i\left(\mathbf{S}^{n-1}\right)$ is equal to $\mathbb{Q}$ if $i=n-1$ and zero else, this gives that

$$H_i\left(\mathrm{Conf}_n^{\mathrm{sink}}\left(\mathcal{G}_d,[d]\right)\mathbb{Q}\right) \cong \begin{cases} \mathbb{Q}^{N(d,i+1)} & \text{if } n=i+1 \\ 0 & \text{else .} \end{cases}$$

The case $i=0$ is explained in Remark 3.2.4. $\square$

Remark 3.2.9. In Proposition 3.2.8 we describe the $\mathbf{FI}_d$-module $H_i(\operatorname{Conf}^{\mathrm{sink}}_{(-)}(\mathcal{G}_d, [d]), \mathbb{Q})$ on objects but we do not give its values on morphisms for $i > 0$. It is clear that all the non-bijective morphisms in $\mathbf{FI}_d$ are sent to the zero map by this functor since it is twisted atomic, however there is a non-trivial action of the symmetric group S_{i+1} on $\mathbb{Q}^{N(d,i+1)}$.

Remark 3.2.10. The proof of Proposition 3.2.8 extends to homology over a general commutative ring $\mathbf{R}$, so the result remains true for the $\mathbf{FI}_d$-module $H_i(\operatorname{Conf}^{\mathrm{sink}}_{(-)}(\mathcal{G}_d, [d]), \mathbb{Q})$.

Remark 3.2.11. Ramos also studied the classical unordered configuration spaces of graphs in [Ram20]. He introduced a **FI**-module that sends n to the wedge of a fixed graph and n copies of another fixed graph, and showed this has strong finiteness properties. In particular, this generalizes the fact that homology of the unordered configuration space of a graph is finitely generated.

Chapter 4

Twisted commutative algebras

Ses rêves et désirs, si sages et possibles
sans cri, sans délire, sans inadmissible
Sur dix ou vingt pages de photos banales
bilan sans mystère d'années sans lumière.

Jean-Jacques Goldman

In algebraic topology, the theory of twisted commutative algebras (TCAs) dates back to the 1950s. For example, Barratt defines in [Bar78] a general twisted algebra and adds a condition to be a twisted Lie algebra or a twisted commutative algebra. In this section we explain and exploit the link between the $\mathbf{FI}_d$-modules and the theory of TCAs. Indeed, Sam and Snowden showed in [SS12, SS17] that there is an equivalence of categories between the modules over $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$, the free TCA on d generators, and the $\mathbf{FI}_d$-modules. The modules over these free TCAs have recently been studied in different contexts, such as in [SS12, SS16, SS19] or in [GS10]. They focused on a family of quotient categories given by what they call the determinantal ideals. The TCAs have been used in other contexts, for example, the modules over the TCA $\mathrm{Sym}(\ \mathrm{Sym}^2(\mathbb{C}^\infty)\ )$ are equivalent to the representations of the infinite orthogonal group. In a first section we give some reminders about the different definitions and basic properties of the TCAs. Then we construct explicitly two functors giving the equivalence stated by Sam and Snowden. In a third part, we describe a natural action of the linear group $GL(\mathbb{K}^d)$ on the modules over the TCA $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ and in Proposition 4.3.5 we make explicit the action of $GL(\mathbb{K}^d)$ on the $\mathbf{FI}_d$-modules induced by this action through the equivalence of categories. In this chapter we assume that $\mathbf{R} = \mathbb{K}$ is a field and, in order to use different equivalent definitions of the TCAs, we assume that it is of characteristic zero in the first section.

4.1 A reminder about twisted commutative algebras

A twisted commutative algebra (TCA) is a monoid in an abstract category which is equivalent to several concrete categories, thus there are different equivalent ways to define the TCAs. For example, it can be defined as a functor from vector spaces to commutative rings, or a commutative ring endowed with an action of the infinite linear group, or a graded algebra endowed with an action of the symmetric groups. In each case there is an additional condition, called polynomiality (in a different sense than the polynomial functors we study), which is added to form a TCA and there is a corresponding notion of modules over a TCA. We choose to focus mainly on this last definition, using the others from time to time when it is more relevant. The connection between these definitions is given by the Schur-Weyl duality for a field of characteristic zero, so this is the framework for this chapter. We start with a reminder on the definitions and the basic results on the TCAs. We also introduce examples of TCAs, such as $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$, the free TCA on d generators of degree one.

Definition 4.1.1. The *category* $\mathbf{\Sigma}$ has for objects the finite sets and for morphisms the bijections. The composition of morphisms is the usual composition of bijections.

We now construct a symmetric monoidal category with the functors over $\mathbf{\Sigma}$. First we introduce the following tensor product on these functors, sometimes called the Day convolution as in [MW19]:

Definition 4.1.2. The *functor* $\otimes : \mathbf{Fct}\left(\mathbf{\Sigma}, \mathbb{K}\text{-}\mathbf{Vect}\right)\times\mathbf{Fct}\left(\mathbf{\Sigma}, \mathbb{K}\text{-}\mathbf{Vect}\right)\to\mathbf{Fct}\left(\mathbf{\Sigma}, \mathbb{K}\text{-}\mathbf{Vect}\right)$ is given on objects by

$$(F\otimes G)(S) \coloneqq \bigoplus_{X\sqcup Y=S} F(X)\otimes_{\mathbb{K}} G(Y),$$

where the sum is taken on all the decompositions of S into a disjoint union of two sets X and Y. The functor $(F\otimes G)$ is given on a morphism $\sigma : S\to S$ by the induced map sending the factor $F(X)\otimes G(Y)$ of $(F\otimes G)(S)$ to the factor $F(\sigma(X))\otimes G(\sigma(Y))$.

Lemma 4.1.3. *The category* $\mathbf{Fct}\left(\mathbf{\Sigma}, \mathbb{K}\text{-}\mathbf{Vect}\right)$ *endowed with the functor* $\otimes$*, the functor* $\mathbb{K}_0$ *which sends* $\varnothing\in\mathbf{\Sigma}$ *to* $\mathbb{K}$ *and everything else on* 0*, and with the natural symmetric structure* τ *on* $\otimes$*, gives a symmetric monoidal category.*

Proof. We refer to [SS12, Section 5.1] for a full proof. □

Remark 4.1.4. The category $\mathbf{Fct}\left(\mathbf{\Sigma}, \mathbb{K}\text{-}\mathbf{Vect}\right)$ admits another symmetric structure by introducing signs corresponding to the degree, but it has been shown in [SS12, Section 7.4] that both symmetric categories are equivalent.

As for $\mathbf{FI}_d$, we consider the skeleton of the category $\mathbf{\Sigma}$ given by $\mathbb{N}$, where n corresponds to the class of the sets of cardinality n which is represented by the set $\underline{n}=\{1,\dots,n\}$. We now give the definition of a TCA in the symmetric monoidal category $\mathbf{Fct}\left(\mathbf{\Sigma}, \mathbb{K}\text{-}\mathbf{Vect}\right)$.

Definition 4.1.5. A *twisted commutative algebra* is a commutative monoid in the symmetric monoidal category

$$\left(\mathbf{Fct}\left(\mathbf{\Sigma}, \mathbb{K}\text{-}\mathbf{Vect}\right), \otimes, \mathbb{K}^0, \tau\right).$$

In other words, a *TCA* is a functor $A : \mathbf{\Sigma}\to\mathbb{K}\text{-}\mathbf{Vect}$ together with two laws $\nu : A\otimes A\to A$ and $\varepsilon : \mathbb{K}_0\to A$ such that ν is associative, commutative and admits ε as a unit.

Remark 4.1.6. For $A : \mathbf{\Sigma}\to\mathbb{K}\text{-}\mathbf{Vect}$ a TCA, by taking the sum of the vector spaces $A(n)$ for $n\in\mathbb{N}$ we get an associative graded algebra with an action of S_n on the piece of degree n compatible with the multiplication. Then the algebra

$$\bigoplus_{n\in\mathbb{N}} A(n)$$

is commutative, up to the "twist" which exchanges blocks, as explained in [SS12] or in [GS10].

Example 4.1.7. The first example, already presented in [Bar78] and in [GS10], is the functor sending $n\in\mathbf{\Sigma}$ to $\mathbb{K}[\mathrm{S}_n]$ on which S_n acts by conjugation, while the product is given by the standard inclusion of $\mathrm{S}_n\times\mathrm{S}_m$ in S_{n+m}. This twisted algebra is commutative since the "twist" which exchange the blocks $\mathrm{S}_n\times\mathrm{S}_m$ and $\mathrm{S}_m\times\mathrm{S}_n$ is given by the conjugation by the element of S_{n+m} that exchange the n first integers with the last m.

Remark 4.1.8. The endofunctor τ_1 from Definition 2.6.1 has an equivalent in the context of TCAs. It is the Schur derivative, denoted by $\mathbf{D}$, which is the adjoint of a shift on grading functors. It was used by Joyal in [Joy86] and by Sam and Snowden in [SS12, Section 6.4] and in [SS16, 5.4]. It should not be confused with the endofunctor δ that we call differential. They call it derivative because it verifies the Leibniz rule and a differential equation for the Hilbert series they introduced. To generalize this endofunctor we define a family of endofunctors τ_k for $k \in \mathbb{N}^*$ while, in [SS12], they define a family $\mathbf{D}_\lambda$ for λ a partition to give adjoints of the shift corresponding to the partition λ.

Remark 4.1.9. If $\mathbb{K}$ is a field of characteristic zero, we recall that the irreducible representations of the symmetric groups are indexed by the partitions and that the Littlewood-Richardson rule explains how the tensor product of two such representations is decomposed into irreducible representations.

The Schur-Weyl duality (see [SS12, Section 1]) describes how the space $(\mathbb{K}^n)^{\otimes k}$ is decomposed into irreducible representations of $\mathrm{S}_n \times GL_n(\mathbb{K})$, which are the product of the irreducible representations of S_n and $GL_n(\mathbb{K})$ associated with the same partition. This result is important since it connects the representations of the symmetric and linear groups and gives a concrete way to construct the irreducible representations of $GL_n(\mathbb{K})$ from the representations of S_n. This result is frequently used in the theory of TCAs, as in the following:

Lemma 4.1.10. *For $\mathbb{K}$ a field of characteristic 0, the symmetric monoidal category $\mathbf{Fct}(\mathbf{\Sigma}, \mathbb{K}\text{-}\boldsymbol{Vect})$ is equivalent to the following three other categories:*

- *The category $\boldsymbol{Rep}(S_*)$ of infinite sequences of representations of symmetric groups endowed with the Cauchy product (see Definition 4.1.2).*
- *The category $\boldsymbol{Rep}^{pol}(\mathrm{GL})$ of polynomial representations of the group $\mathrm{GL}(\mathbb{K}^\infty)$, where $\mathbb{K}^\infty$ is the vector space with the basis $e_1, e_2, \dots, e_n, \dots$, and where polynomial means a subquotient of a direct sum of representations of the form $(\mathbb{K}^\infty)^{\otimes k}$.*
- *The full subcategory $\mathcal{S}$ of $\mathbf{Fct}(\mathbb{K}\text{-}\boldsymbol{vect}, \mathbb{K}\text{-}\boldsymbol{Vect})$ of the functors which are isomorphic to direct sums of the Schur functors $(\,(-)^{\otimes n} \otimes M\,)^{\mathrm{S}_n}$, with M a finite dimensional module on the symmetric group S_n.*

Then, TCAs can then be defined as the monoids in any of those categories or even in an abstract equivalent category. These different points of view, the equivalences between the categories and the concrete description of the TCAs in these categories are presented in [SS12, DES17], and with more details in [Fel20]. From the point of view in $\mathbf{Rep}^{\mathrm{pol}}(\mathrm{GL})$, frequently used by Sam and Snowden, a TCA is a commutative, associative, unitary, $\mathbb{K}$-algebra endowed with a compatible polynomial action of $\mathrm{GL}(\mathbb{K}^\infty)$.

Remark 4.1.11. Note that the two notions of TCAs using the action of the symmetric groups or the action of $\mathrm{GL}(\mathbb{K}^\infty)$ are equivalent in characteristic zero via the Schur-Weyl duality, but give two different notions of TCAs in positive characteristic.

Remark 4.1.12. The different definitions of a TCA as a monoid in one of the equivalent categories of Lemma 4.1.10 leads to another definition over the operad *Com*. In general, as explained in [GS10], for any operad $\mathcal{O}$ one can define a twisted algebra over $\mathcal{O}$ as an $\mathcal{O}$-algebra in the symmetric monoidal category $\mathbf{Rep}(S_*)$. Explicitly, it is a graded algebra $A = \bigoplus A(n)$ with an action of S_n on $A(n)$ as in Remark 4.1.6, together with a map $\bigoplus \mathcal{O}_n \otimes_{\mathrm{S}_n} A(n) \to A(n)$ coming from the maps $\mathcal{O}_n \otimes A(i_1) \otimes \cdots \otimes A(i_n) \to A(i_1 + \cdots + i_n)$. To be a twisted algebra, we request in addition that it satisfies the "twist" condition from Remark 4.1.6.

We give right away the definition of a module over a TCA, which is simply a module over the TCA viewed as a monoid in $\mathbf{Fct}\big(\mathbf{\Sigma}\,,\,\mathbb{K}\text{-}\mathbf{Vect}\big)$:

Definition 4.1.13. A *module* over the twisted commutative algebra (A,ν,ε) is a module over the monoid A in the symmetric monoidal category

$$\big(\,\mathbf{Fct}\big(\mathbf{\Sigma}\,,\,\mathbb{K}\text{-}\mathbf{Vect}\big),\otimes,\mathbb{K}^0,\tau\,\big).$$

In other words, a module over the twisted commutative algebra (A,ν,ε) is a functor $F:\mathbf{\Sigma}\to\mathbb{K}\text{-}\mathbf{Vect}$ together with an map $\mu:A\otimes F\to F$ such that

$$\mu\circ(\mathrm{id}\otimes\mu)=\mu\circ(\nu\otimes\mathrm{id}):A\otimes A\otimes F\to F.$$

Example 4.1.14. A simple family of examples of TCAs are the "polynomial TCAs" from [SS12] (in a different sense than the polynomial functors) which are obtained by taking the symmetric algebra of a space. The simplest of them is the symmetric algebra over the space $\mathbb{K}^1$, denoted by $\mathrm{Sym}((\mathbb{K}^1)^{(1)})$ which corresponds, in the category $\mathbf{Fct}(\mathbf{\Sigma},\mathbb{K}\text{-}\mathbf{Vect})$, to the functor sending n to $\mathbb{K}^{\otimes n}$ and, in the category of representations of $GL(\infty)$, to the ring $\mathrm{Sym}(\mathbb{K}^\infty)$. Then we have the symmetric algebras over the space $\mathbb{K}^d$ for $d\geq 1$ which are the free TCAs generated in degree 1 denoted by $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$. They are the most studied ones (see for example [SS12, SS16, SS19]) and are also called the multivariate TCAs.

While the full description of the "polynomial TCAs" in the different equivalent definitions is given in [SS12, 8.2.3], we summarize now the important facts about them. For simplicity we denote $\mathbb{K}^d$ by V in the following.

Definition 4.1.15. The *free twisted commutative algebra* with d generators in degree 1 is the functor $\mathrm{Sym}(V^{(1)})$, which sends an object n of $\mathbf{\Sigma}$ to $V^{\otimes n}$ and a morphism $\sigma\in\mathbf{\Sigma}(n,n)$ to the map which permutes the tensor factors according to σ^{-1}. The multiplication map

$$\mathrm{Sym}(V^{(1)})\otimes\mathrm{Sym}(V^{(1)})\to\mathrm{Sym}(V^{(1)})$$

is the concatenation of the tensor products and the unit is given by $\mathbb{K}_0\cong\mathrm{Sym}(V^{(1)})(0)$.

Remark 4.1.16. The notation "Sym" comes from the equivalent definition of the TCAs in the category $\mathbf{Rep}^{\mathrm{pol}}(\mathrm{GL})$ from Lemma 4.1.10: in this category, the TCA $\mathrm{Sym}(V^{(1)})$ is given by the symmetric algebra $\mathrm{Sym}(\,V\otimes\mathbb{K}^\infty\,)$ over the representation $V\otimes\mathbb{K}^\infty$ of $\mathrm{GL}(\mathbb{K}^\infty)$. As a representation of $GL(\infty)$ it can be identified with the ring $\mathbb{K}[x_{i,j}|1\leq i\leq d, 1\leq j]$, where $x_{i,j}$ corresponds to the tensor product $e_i\otimes\varepsilon_j\in V\otimes\mathbb{K}^\infty$ for $e_1,\dots,e_n$ a basis of V and $\varepsilon_1,\dots,\varepsilon_k,\dots$ a basis of $\mathbb{K}^\infty$. For $d=1$, the TCA $\mathrm{Sym}((\mathbb{K}^1)^{(1)})$ is then identified by Sam and Snowden in [SS16] with the ring $\mathbb{K}[x_1,\dots,x_n,\dots]$. It may also be surprising that a σ^{-1} appears on the arrows instead of a σ in Definition 4.1.15. It follows from this equivalent definition, in which σ permutes the factors $V^{\otimes n}$ according to σ but, as we take the quotient by the action of the symmetric group S_n, we can permute the factors back to get the image expressed in terms of the original factors. This makes the σ^{-1} appear when we pass from this definition to the definition of TCAs in the category $\mathbf{Fct}\big(\mathbf{\Sigma}\,,\,\mathbb{K}\text{-}\mathbf{Vect}\big)$.

In [SS19], Sam and Snowden generalize their work for finitely generated $\mathrm{Sym}((\mathbb{K}^1)^{(1)})$-modules of [SS16] to all $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$-modules. In particular, they introduce and describe the spectrum and ideals of the TCA $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ and focus on a family of ideals called determinantal ideals. The r-th determinantal ideal $\mathfrak{a}_r$, introduced in [SS19] and in [SS12, 8.2.6], is generated by all the $(r+1)\times(r+1)$ minors of the matrix $(x_{i,j})$ with the identification $\mathrm{Sym}(\mathbb{K}^d\otimes\mathbb{K}^\infty)\cong\mathbb{K}[x_{i,j}\,|\,1\leq i\leq d, 1\leq j]$. It also corresponds to the ideal $\bigwedge^{r+1}(\mathbb{K}^d)\otimes\bigwedge^{r+1}(\mathbb{K}^\infty)$

of the ring $\mathrm{Sym}(\mathbb{K}^d \otimes \mathbb{K}^\infty)$ in its decomposition given by the Cauchy formula. They show in [SS19, Theorem 3.3] that the spectrum of the TCA $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ (*i.e.* the set of all prime ideals of $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$) is isomorphic to the total Grassmannian $\mathbf{Gr}(\mathbb{K}^d)$, which is the union of the sets $\mathbf{Gr}_k(\mathbb{K}^d)$ of vector subspaces of $\mathbb{K}^d$ of rank k for $0 \le k \le d$.

Remark 4.1.17. In [SS12] Sam and Snowden define the quotient of the category of modules over a TCA by its full subcategory of modules locally annihilated by a power of a prime ideal of the TCA. In [SS19] they apply this construction to the TCA $\mathrm{Sym}((\mathbb{K}^d)^{(1)}) =: A$ and to the determinantal ideals to define a filtration of subcategories of $\mathrm{Sym}(\mathbb{K}^d \otimes \mathbb{K}^\infty)$-modules: the rank stratification

$$\mathrm{Mod}_{A,\le 0} \subset \mathrm{Mod}_{A,\le 1} \subset \cdots \subset \mathrm{Mod}_{A,\le d} = \mathrm{Mod}_A,$$

where the modules in $\mathrm{Mod}_{A,\le r}$ are locally annihilated by a power of $\mathfrak{a}_r$. They then define $\mathrm{Mod}_{A,r}$ as the quotient of $\mathrm{Mod}_{A,\le r}$ by $\mathrm{Mod}_{A,\le r-1}$, which intuitively corresponds to the part of Mod_A whose support is in $\mathbf{Gr}_r(\mathbb{K}^d)$ within $\mathbf{Gr}(\mathbb{K}^d)$. In particular, they compute the Grothendieck group of $\mathrm{Mod}_{A,r}$ which is free of rank $\binom{d}{r}$ over the ring of symmetric functions. This construction gives, through the equivalence of categories from [SS12] developed in Section 4.2, a family of quotients of $\mathbf{FI}_d$**-Mod** which would be interesting to compare with ours.

We focused on the free TCAs on d generators of degree one, which is the most fundamental example of TCA but there are other interesting examples given by a symmetric algebra $\mathrm{Sym}(\,V\,)$ over a representation V of $\mathrm{GL}(\mathbb{K}^\infty)$. For example, for $V = \Lambda^2(\mathbb{K}^\infty)$ it gives the TCA $\mathrm{Sym}(\,\Lambda^2(\mathbb{K}^\infty)\,)$, or for V the space of symmetric bilinear forms over $\mathbb{K}$ (spanned by the elements $x_{i,j} = e_i e_j$ for $e_1, e_2, \dots$ a basis of $\mathbb{K}^\infty$) this gives the TCA $\mathrm{Sym}(\,\mathrm{Sym}^2(\mathbb{K}^\infty)\,)$ which is equivalent to the algebra $\mathbb{K}[x_{i,j}]$. These two examples have been studied in [NSS16]. Another example is the TCA $\mathrm{Sym}((\mathbb{K}^1)^{(n)})$ generated in degree n, corresponding to the ring $\mathrm{Sym}(\,(\mathbb{K}^\infty)^{\otimes n}\,)$, which is detailed in [SS12, 8.2.4]. The important result about all these TCAs are presented in details in [DES17].

Example 4.1.18. In Section 4.2 we explain that the category of modules over the TCA $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ is equivalent to the category of modules over $\mathbf{FI}_d$, following [SS12]. There is a similar equivalence for the two TCAs $\mathrm{Sym}(\,\mathrm{Sym}^2(\mathbb{C}^\infty)\,)$ and $\mathrm{Sym}(\,\Lambda^2(\mathbb{C}^\infty)\,)$. Indeed, Sam and Snowden showed in [SS15] that the finitely generated modules over these TCAs are equivalent to the finitely generated modules over the upwards Brauer category $\mathbf{B}(\delta)$. This last is equivalent to the category **FIM** from [MW19] whose objects are finite sets, and whose morphisms are pairs of an injection and a perfect matching on the complement of the image (see [NSS16] or [SS17]). Moreover, there is an analog of the Schur-Weyl duality for the infinite orthogonal group which is given by an equivalence of categories between the algebraic representations of $\mathbf{O}(\infty)$ and the functors over the upward Brauer category.

Remark 4.1.19. In the recent years, it has been proven that different algebraic structures similar to the TCAs are noetherian, such as the **FI**-modules and $\mathbf{FI}_d$-modules (see [CEFN14], [SS16, SS19] and [Sno13, Theorem 2.3]), the **FS**-modules (see [SS17]), the $\mathbf{VIC}(R)$-modules (see [PS17]) and many others. Note that in [PS17] and [SS17] it is also shown that the category of $\mathbf{Fct}(\mathbb{F}_q\mathbf{-mod}, \mathbb{F}_d\mathbf{-mod})$ is noetherian, which was known as the Lannes-Schwartz conjecture. They are such results about TCAs, but it is still an open question to find if every finitely generated TCA is noetherian. For now, it was proven that the TCAs generated in degree 1 are noetherian (see [CEF15] and [SS16]), and that the two TCAs generated in degree 2 $\mathrm{Sym}(\,\mathrm{Sym}^2(\mathbb{C}^\infty)\,)$ and $\mathrm{Sym}(\,\Lambda^2(\mathbb{C}^\infty)\,)$ are noetherian (see [NSS16]). In particular, the last theorem implies that the finitely generated **FIM**-modules are noetherian, recalling the result about the **FI**-modules which can be proved using the TCA $\mathrm{Sym}(\mathbb{K}^{(1)})$. Lately, it was shown in [DES17] that the TCA $\mathrm{Sym}(\,\mathrm{Sym}^3(\mathbb{C}^\infty)\,)$ generated in degree 3 is topologically noetherian, which is a weaker notion.

4.2 The equivalence between $\mathbf{FI}_d$-modules and $\mathrm{Sym}(\,(\mathbb{K}^d)^{(1)}\,)$-modules

In this section we give an explicit construction of the equivalence of categories

$$\mathbf{FI}_d\text{-}\mathbf{Mod} \cong \mathrm{Sym}(\,V^{(1)}\,)\text{-}\mathbf{Mod}$$

first stated by Sam and Snowden in [SS12, Section 10.2] for $d = 1$, then proved in [SS17, Proposition 7.2.5]. In particular, we give an explicit construction of two functors $\chi_{\mathcal{B}} : \mathrm{Sym}(V^{(1)})\text{-}\mathbf{Mod} \to \mathbf{FI}_d\text{-}\mathbf{Mod}$ and $\Gamma_{\mathcal{B}} : \mathbf{FI}_d\text{-}\mathbf{Mod} \to \mathrm{Sym}(V^{(1)})\text{-}\mathbf{Mod}$ giving the equivalence. Note that these functors depend on the choice of the basis $\mathcal{B}$ of $V = \mathbb{K}^d$, so we fix one $\mathcal{B} = (e_1, \dots, e_d)$ for this section.

Remark 4.2.1. Since $\mathcal{B} = (e_1, \dots, e_d)$ is a basis of $V = \mathbb{K}^d$, then the elements $e_{i_1} \otimes \cdots \otimes e_{i_n}$ of $V^{\otimes n}$ for $1 \le i_1, \dots, i_n \le d$ form a basis of $V^{\otimes n}$. We denote by $\mathcal{C}$ the set consisting of these elements, and we have a canonical bijection

$$\mathbf{FI}_d(0, n) \cong \mathrm{Hom}_{\mathbf{Set}}(n, d) \cong \mathcal{C}.$$

For $g \in \mathbf{FI}_d(0, n)$ we denote by e_g the basis element $e_{g(1)} \otimes \cdots \otimes e_{g(n)}$ corresponding to g by this bijection. This gives a decomposition

$$V^{\otimes n} \cong \bigoplus_{g \in \mathbf{FI}_d(0,n)} \mathbb{K} \cdot e_g \ .$$

We are now able to define the two functors $\chi_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}}$ which give the equivalence of categories. To give the definition of $\chi_{\mathcal{B}}$ we recall that, for a TCA A, a A-module is a pair (F, μ), where $F : \mathbf{\Sigma} \to \mathbb{K}\text{-}\mathbf{Vect}$ is a functor and $\mu : A \otimes F \to F$ a natural transformation giving the action of A on F.

Definition 4.2.2. The *functor* $\chi_{\mathcal{B}} : \mathrm{Sym}(V^{(1)})\text{-}\mathbf{Mod} \to \mathbf{FI}_d\text{-}\mathbf{Mod}$ sends a $\mathrm{Sym}(V^{(1)})$-module (F, μ) to a $\mathbf{FI}_d$-module $\chi_{\mathcal{B}}(F, \mu) : \mathbf{FI}_d \to \mathbb{K}\text{-}\mathbf{Vect}$ sending the object n to $F(n)$, and the morphism (f, g) of $\mathbf{FI}_d$ to the composition

$$F(n) \xrightarrow{F(f : n \to f(n))} F(f(n)) \xrightarrow{\ \sim\ } \mathbb{K} \cdot e_g \otimes F(f(n)) \xrightarrow{\mu|_{\mathbb{K} \cdot e_g \otimes F(f(n))}} F(m)\,.$$

For $\varepsilon : F \to F'$ a natural transformation in $\mathbf{FI}_d\text{-}\mathbf{Mod}$, the natural transformation $\chi_{\mathcal{B}}(\varepsilon)$ is given for all objects $n \in \mathbf{FI}_d$ by

$$\chi_{\mathcal{B}}(\varepsilon)_n = \varepsilon_n : \chi_{\mathcal{B}}(F)(n) = F(n) \to F'(n) = \chi_{\mathcal{B}}(F')(n).$$

To define the opposite functor $\Gamma_{\mathcal{B}}$ we need to see $\mathbf{\Sigma}$ as a subcategory of $\mathbf{FI}_d$. We then define the functor $\theta : \mathbf{\Sigma} \to \mathbf{FI}_d$, which sends an object n in $\mathbf{\Sigma}$ to n in $\mathbf{FI}_d$, and a morphism $\sigma : n \to n$ in $\mathbf{\Sigma}$ to the morphism $(\sigma, \underline{0} = \varnothing \to d)$ in $\mathbf{FI}_d$.

Definition 4.2.3. The *functor* $\Gamma_{\mathcal{B}} : \mathbf{FI}_d\text{-}\mathbf{Mod} \to \mathrm{Sym}(V^{(1)})\text{-}\mathbf{Mod}$ sends a $\mathbf{FI}_d$-module G to the functor $G \circ \theta : \mathbf{\Sigma} \to \mathbb{K}\text{-}\mathbf{Vect}$ together with the map $\mu : \mathrm{Sym}(V^{(1)}) \otimes (G \circ \theta) \to G \circ \theta$, where for all objects $n \in \mathbf{\Sigma}$, $\mu_n : (\,\mathrm{Sym}(V^{(1)}) \otimes (G \circ \theta)\,)(n) \to G \circ \theta(n)$ is the composition

$$\bigoplus_{i+j=n} V^{\otimes i} \otimes G(j) \xrightarrow{\ \sim\ } \bigoplus_{i+j=n} \bigoplus_{g \in \mathbf{FI}_d(0,i)} \mathbb{K} \cdot e_g \otimes G(j) \xrightarrow{\ \sim\ } \bigoplus_{i+j=n} \bigoplus_{g \in \mathbf{FI}_d(0,i)} G(j) \xrightarrow{\oplus G(j \hookrightarrow n, g)} G(n).$$

For $\varepsilon : (G, \mu) \to (G', \mu')$ a natural transformation in $\mathrm{Sym}(V^{(1)})\text{-}\mathbf{Mod}$, the natural transformation $\Gamma_{\mathcal{B}}(\varepsilon)$ is given for all objects $n \in \mathbf{\Sigma}$ by

$$\Gamma_{\mathcal{B}}(\varepsilon)_n = \varepsilon_n : \Gamma_{\mathcal{B}}(G, \mu)(n) = G(n) \to G'(n) = \Gamma_{\mathcal{B}}(G', \mu')(n).$$

Finally we can state the equivalence of categories shown by Sam and Snowden in [SS17]:

Theorem 4.2.4. *For any basis $\mathcal{B}$ of V, the functors $\chi_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}}$ give an equivalence of categories*

$$\mathbf{FI}_d\text{-}\boldsymbol{Mod} \cong \mathrm{Sym}(\,(\mathbb{K}^d)^{(1)}\,)\text{-}\boldsymbol{Mod}.$$

Proof. This theorem was first stated in [SS12, Section 10.2] for $d = 1$, and proved in [SS17, Proposition 7.2.5]. The proof consist in checking that the functors $\chi_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}}$ are quasi-inverse, but we refer to [Fel20] for more details. □

Remark 4.2.5. For the TCA $\mathrm{Sym}(V^{(1)})$, the representable objects denoted by $\mathbb{K} < n >$ in [SS12] correspond exactly to the projective standard functors $P_n^{\mathbf{FI}_d}$ via the equivalence of Theorem 4.2.4.

4.3 An action of $GL(\mathbb{K}^d)$ on $\mathbf{FI}_d$-modules

In this section we will use the natural action of $GL(V)$ on the $\mathrm{Sym}(V^{(1)})$-modules in the theory of TCAs to get an action of $GL(V)$ on the $\mathbf{FI}_d$-modules. To do this we will use the equivalence of categories given in Theorem 4.2.4 for a fixed basis $\mathcal{B}$ of $V = \mathbb{K}^d$. We start with the definition of this action of $GL(V)$ on the $\mathrm{Sym}(V^{(1)})$-modules, then we will explain how we transpose it into an action on $\mathbf{FI}_d$-modules.

Definition 4.3.1. For $\varphi \in GL(V)$ and $(F,\mu) \in \mathrm{Sym}(V^{(1)})$-**Mod**, the $\mathrm{Sym}(V^{(1)})$-**module** $\varphi \cdot (F,\mu)$ is defined by

$$\varphi \cdot (F,\mu) = (F, \varphi\cdot\mu),$$

where $\varphi\cdot\mu : \mathrm{Sym}(V^{(1)}) \otimes F \to F$ is the natural transformation given on an object $n \in \mathbf{\Sigma}$, by the composition

$$(\varphi\cdot\mu)_n \coloneqq \left(\bigoplus_{i+j=n} V^{\otimes i}\otimes F(j) \xrightarrow{\bigoplus_{i+j=n}\varphi^{\otimes i}\otimes \mathrm{id}} \bigoplus_{i+j=n} V^{\otimes i}\otimes F(j) \xrightarrow{\mu_n} F(n) \right).$$

This gives an invertible endofunctor $\varphi\cdot(-)$ of the category $\mathrm{Sym}(V^{(1)})$-**Mod** sending an object (F,μ) to $\varphi\cdot(F,\mu)$ and a natural transformation σ between (F,μ) and (F',μ') to $\varphi\cdot\sigma = \sigma$ between $(F,\varphi\cdot\mu)$ and $(F',\varphi\cdot\mu')$.

Proposition 4.3.2. *The group $GL(V)$ acts on the category $\mathrm{Sym}(V^{(1)})$-**Mod** by the invertible endofunctor $\varphi\cdot(-)$.*

Proof. The map $\varphi\cdot(-)$ being an extension of the diagonal action of $GL(V)$ on $V^{\otimes n}$, we just need to check that its image on a natural transformation in $\mathrm{Sym}(V^{(1)})$-**Mod** is still a natural transformation in $\mathrm{Sym}(V^{(1)})$-**Mod**. This is true since the following diagram, corresponding to $\sigma : (F,\mu) \to (F',\mu')$, is commutative for all $n \in \mathbf{\Sigma}$:

$$\begin{tikzcd} \bigoplus_{i+j=n} V^{\otimes i}\otimes F(j) \arrow[r, "\bigoplus_{i+j=n}\mathrm{id}\otimes\sigma_j"] \arrow[d, "\mu_n"'] & \bigoplus_{i+j=n} V^{\otimes i}\otimes F'(j) \arrow[d, "\mu'_n"] \\ F(n) \arrow[r, "\sigma_n"'] & F'(n). \end{tikzcd}$$

This means that for all $x \in V^{\otimes i}$ and all $y \in F(j)$ for $i+j=n$, we have the equality $\sigma_n\circ\mu_n(x\otimes y) = \mu'_n(\,x\otimes(\sigma_j(y)\,)$. In particular, for $x = \varphi^{\otimes i}(\tilde{x})$ we have

$$\sigma_n\big(\,\mu_n(\,(\varphi^{\otimes i}(\tilde{x}))\otimes y)\,\big) = \sigma_n\big(\,\mu_n(x\otimes y)\,\big) = \mu'_n\big(\,x\otimes(\sigma_j(y)\,\big) = \mu'_n\big(\,(\varphi^{\otimes i}(\tilde{x}))\otimes(\sigma_j(y)\,\big),$$

which shows that the following diagram, corresponding to $\sigma : (F, \varphi\cdot\mu) \to (F', \varphi\cdot\mu')$, is commutative:

$$\begin{tikzcd}
\bigoplus\limits_{i+j=n} V^{\otimes i}\otimes F(j) \arrow[r, "\bigoplus_{i+j=n}\mathrm{id}\otimes\sigma_j"] \arrow[d, "(\varphi\cdot\mu)_n"'] & \bigoplus\limits_{i+j=n} V^{\otimes i}\otimes F'(j) \arrow[d, "(\varphi\cdot\mu')_n"] \\
F(n) \arrow[r, "\sigma_n"'] & F'(n)
\end{tikzcd}.$$

□

We now use the equivalence of categories to transfer this action of $GL(V)$ from the $\mathrm{Sym}(V^{(1)})$-modules to the $\mathbf{FI}_d$-modules. First, we define this action on $\mathbf{FI}_d$-modules using the equivalence as follows, and then we describe it explicitly in a second step.

Definition 4.3.3. Let $\mathcal{B}$ be a basis of V, for $\varphi \in GL(V)$ the *endofunctor* $\varphi_{\mathcal{B}}\cdot(-)$ of $\mathbf{FI}_d$**-Mod** is given by the composition

$$\mathbf{FI}_d\textbf{-Mod} \xrightarrow{\Gamma_{\mathcal{B}}} \mathrm{Sym}(V^{(1)})\textbf{-Mod} \xrightarrow{\varphi\cdot(-)} \mathrm{Sym}(V^{(1)})\textbf{-Mod} \xrightarrow{\chi_{\mathcal{B}}} \mathbf{FI}_d\textbf{-Mod}\ .$$

To give a more explicit description of this action we need to use the matrix $M = (m_{i,j})_{1\le i,j\le d}$ of φ in the basis $\mathcal{B} = (e_1,\dots,e_d)$ of V. With this notation we can write a formula for $\varphi^{\otimes n}$ which will be useful in the following.

Remark 4.3.4. By definition, for $1 \le k \le d$ we have $\varphi(e_k) = \sum m_{l,k}\, e_l$. Then, for $g \in \mathbf{FI}_d(0,n)$, using the notation $e_g \coloneqq e_{g(1)}\otimes\cdots\otimes e_{g(n)}$ from Remark 4.2.1 and the linearity of the tensor product, we can give the following formula for $\varphi^{\otimes n}(e_g)$:

$$\varphi^{\otimes n}(e_g) = \sum_{l_1,\dots,l_n=1}^{d} m_{l_1,g(1)}\dots m_{l_n,g(n)}\ (e_{i_1}\otimes\cdots\otimes e_{i_n}) = \sum_{g'\in\mathbf{FI}_d(0,n)} m_{g'(1),g(1)}\dots m_{g'(n),g(n)}\ e_{g'}\,,$$

where the last equality is just a relabeling of the sum using the bijection $\mathbf{FI}_d(0,n) \cong \mathrm{Hom}_{\mathbf{Set}}(n,d)$.

Proposition 4.3.5. *Let $\mathcal{B}$ be a basis of V, for $\varphi \in GL(V)$ and $G \in \mathbf{FI}_d$-**Mod**, the functor $\varphi_{\mathcal{B}}\cdot G : \mathbf{FI}_d \to \mathbb{K}$-**Vect** sends an object $n \in \mathbf{FI}_d$ to $G(n)$ and a morphism $(f,g) \in \mathbf{FI}_d(n,m)$ to the sum*

$$\sum_{g'\in\mathbf{FI}_d(0,m\smallsetminus f(n))}\left(\prod_{l\in m\smallsetminus f(n)} m_{g'(l),g(l)}\right) G(f,g').$$

*Moreover, for a natural transformation $\sigma : G \to G'$ in $\mathbf{FI}_d$-**Mod**, the action of $\varphi_{\mathcal{B}}$ is given by*

$$\varphi\cdot\sigma = \big(\sigma_n : \varphi\cdot G(n) = G(n) \to G'(n) = \varphi\cdot G'(n)\big)\,.$$

Proof. First for a natural transformation $\sigma : G \to G'$, by the definitions above, we have

$$\varphi_{\mathcal{B}}\cdot(\sigma) = \chi_{\mathcal{B}}\big(\varphi\cdot(\,\Gamma_{\mathcal{B}}(\sigma)\,)\big) = \chi_{\mathcal{B}}\big(\varphi\cdot(\sigma)\big) = \chi_{\mathcal{B}}(\sigma) = \sigma.$$

Similarly, for a functor G we have $\varphi\cdot(\,\Gamma_{\mathcal{B}}(G)\,) = \varphi\cdot(G\circ\theta,\mu) = (G\circ\theta,\varphi\cdot\mu)$. Then, by definition of $\chi_{\mathcal{B}}$ we get that $\varphi_{\mathcal{B}}\cdot G = \chi_{\mathcal{B}}\,(\varphi\cdot\Gamma_{\mathcal{B}}(g))$ is the functor sending an object $n \in \mathbf{FI}_d$ to $G\circ\theta(n) = G(n)$ and a morphism $(f,g) \in \mathbf{FI}_d(n,m)$ to the composition

$$\begin{array}{ccccccc}
G(n) & \xrightarrow{G(\tilde f,\varnothing)} & G(f(n)) & \xrightarrow{\ \sim\ } & \mathbb{K}\,e_g\otimes G(f(n)) & \xrightarrow{(\varphi\cdot\mu)_m} & G(m) \\
y & \longmapsto & G(\tilde f,\varnothing)(y) & \longmapsto & e_g\otimes G(\tilde f,\varnothing)(y) & \longmapsto & (\varphi\cdot\mu)_m\big(e_g\otimes G(\tilde f,\varnothing)(y)\big),
\end{array}$$

where $\tilde{f}$ is the bijective map $f : n \to f(n)$. However, with Remarks 4.3.4 and 4.2.1 we get that the transformation $(\varphi \cdot \mu)_n$ is given on a basis object $e_g \otimes x$ of $\bigoplus \mathbb{K} \cdot e_g \otimes F(j) = (\mathrm{Sym}(V^{(1)}) \otimes F)(n)$ by

$$(\varphi \cdot \mu)_n(e_g \otimes x) = \mu_n \left(\sum_{g':i \to d} (\prod_{l=1}^{j} m_{g'(l),g(l)}) \, (e_{g'} \otimes x) \right) = \sum_{g':i \to d} (\prod_{l=1}^{j} m_{g'(l),g(l)}) \, G(j \hookrightarrow n, g')(x).$$

This finally give

$$(\varphi \cdot \mu)_m \left(e_g \otimes G(\tilde{f}, \varnothing)(y) \right) = \sum_{g':m \smallsetminus f(n) \to d} \left(\prod_{l=1}^{m \smallsetminus f(n)} m_{g'(l),g(l)} \right) G(f(n) \hookrightarrow m, g') \circ G(\tilde{f}, \varnothing)(y),$$

and we conclude since $(f(n) \hookrightarrow m, g') \circ (\tilde{f}, \varnothing)$ is (f, g') by the definition of composition in $\mathbf{FI}_d$. $\square$

Example 4.3.6. For $d = 1$, the action of $GL(V) = GL(\mathbb{K}) = \mathbb{K}^*$ is simply described. Indeed, for $\varphi \in GL(V)$ the matrix $\mathcal{M}_{\mathcal{B}}(\varphi) = (a)$ is one dimensional, with $a \in \mathbb{K}^*$ and so in the formula of Proposition 4.3.5 the product is a power a^{m-n} of the only possible term a. Also, the sum has only one term since there is only one morphism from 0 to $m \smallsetminus f(n)$ in $\mathbf{FI}$. Then, for $G \in \mathbf{FI}\text{-}\mathbf{Mod}$, the functor $\varphi_{\mathcal{B}} \cdot G$ sends $n \in \mathbf{FI}_d$ to $G(n)$ and $(f, g) \in \mathbf{FI}_d(n, m)$ to $a^{m-n} \cdot (f, g)$ with $a \in \mathbb{K}^*$. Then, the automorphisms of $G(n)$ given by the multiplication by $a^n \in \mathbb{K}^*$ form a natural equivalence between G and $\varphi_{\mathcal{B}} \cdot G$.

Chapter 5

Strong polynomial functors on $\mathbf{FI}_d$

Vous êtes-vous déjà demandé ce qu'il peut bien se passer dans la tête des autres ?

Pete Docter

The polynomial functors were introduced by Eilenberg and Mac Lane in [EM54] in the context of functors from $\mathbf{R}$-modules to $\mathbf{R}$-modules using the notion of cross effects. In such functor categories there are huge functors which are difficult to understand and the polynomial property is as a way to measure their complexity. Indeed, the polynomial functors are easier to understand than the others, thus they should be thought as an analogue of polynomial functions approximating more complex functions. In [DV19] this notion was extended to functors from a symmetric monoidal category with an initial object to the category $\mathbf{R}$-$\mathbf{Mod}$. Djament and Vespa used an equivalent definition of the polynomial functors based on a differential endofunctor δ instead of cross effects. They show in [DV19, 3.3] that these two definitions coincide, however the definition using the differential endofunctor is better suited for the study of stable behavior, so we choose to mainly present and generalize this point of view for $\mathbf{FI}_d$-modules. In Section 5.1 we then introduce and study the strong polynomial $\mathbf{FI}_d$-modules, defined using all the c-coloured differential endofunctors δ_1^c of Definition 2.6.2 to replace the unique endofunctor δ of $\mathbf{FI}$-modules. In particular, for $d = 1$ we recover the definition of strong polynomial $\mathbf{FI}$-modules from [DV19]. After giving examples of polynomial $\mathbf{FI}_d$-modules, we show in Proposition 5.2.2 that the $\mathbf{FI}_d$-modules $P_n^{\mathbf{FI}_d}$ are not strong polynomial for $d > 1$. In a third part, we study the support of $\mathbf{FI}_d$-modules and we explain how it is linked to strong polynomial $\mathbf{FI}_d$-modules. In Section 5.4 we generalize the definition using cross effects and we show that the notion of polynomial functors obtained coincides with the strong polynomial functors, which helps us to prove different kind of results. For example, we prove in Proposition 5.4.18 that the composition $\mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$ of two polynomial functors is polynomial and we use this in Section 5.5 to show that the pointwise tensor product respects strong polynomiality.

5.1 Definition and examples of strong polynomial $\mathbf{FI}_d$-modules

In this section we define the strong polynomial functors from $\mathbf{FI}_d$ to $\mathbf{R}$-$\mathbf{Mod}$ and we give some of their basic properties. For $d = 1$, we recover the definition on $\mathbf{FI}$-modules from [DV19]. We also describe some explicit examples, such as the functors defined in Chapters 2 and 3. First, we define the strong polynomial $\mathbf{FI}_d$-modules using the c -coloured differential endofunctors introduced in Definition 2.6.2.

Definition 5.1.1. The full subcategories of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ of *strong polynomial functors* of degree less than or equal to n, denoted by $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$, are defined by induction. By convention $\mathrm{Pol}_{-1}^{strong}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ is zero and, for $n \in \mathbb{N}$, a $\mathbf{FI}_d$-module F is in $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ if

$$\delta_1^c(F) \in \mathrm{Pol}_{n-1}^{strong}(\mathbf{FI}_d, \mathbf{R\text{-}Mod}) \quad \text{for all } c \in C,$$

where δ_1^c is the c-coloured differential endofunctor from Definition 2.6.2.

Remark 5.1.2. For $d = 1$, since the cardinality of $C = \{c\}$ is 1, we recover the definition of strong polynomial functors over $\mathbf{FI}$ from [DV19] using only one endofunctor $\delta_1 = \delta_1^c$. In particular, the polynomial functors give an alternative way to express and understand results about $\mathbf{FI}$-modules. For example, the strong polynomial functors with finitely generated values are the finitely generated $\mathbf{FI}$-modules. Also, the strong polynomial degree of a $\mathbf{FI}$-module corresponds exactly to its generation degree from [MW19] and [CEFN14] since it is given by the functor denoted by $H_0^{\mathbf{FI}}$, which gives the minimal generators of a $\mathbf{FI}$-module and which corresponds to the cross effect functor on the element 1 of $\mathbf{FI}$. Also, there is a stronger notion of polynomial functors for $\mathbf{FI}$-modules, called *degree-r coefficient systems* in [RWW17], also defined using the endofunctor δ_1, denoted by $D(-)$. The comparison between the degree-r coefficient systems and the strong polynomial functors of [DV19] is given in [Wil18a, 6.2]: morally, the difference is that the degree-r coefficient systems include the stably zero functors of [DV19] in the degree 0, while they are strong polynomial of higher degree (or even not polynomial) according to Djament and Vespa.

We now present the first properties of the strong polynomial functor over $\mathbf{FI}_d$. In particular, we show that in Definition 5.1.1 if we use all the endofunctors δ_k^x for $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$, and not just for $k = 1$, we get an equivalent definition.

Proposition 5.1.3. *For $n \in \mathbb{N}$, $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$, the subcategory $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ is closed under quotient, by extensions, by colimits and by the endofunctors τ_k and δ_k^x.*

Proof. For $c \in C$, by Proposition 2.6.6 the endofunctor δ_1^c commutes (up to isomorphism) with the endofunctors τ_k and δ_k^x and with colimits. We then prove by induction on $n \in \mathbb{N}$ that $\mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ is closed under colimits and by τ_k and δ_k^x. We write the details for the endofunctor τ_k, the other cases being similar. If $F \in \mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ then $\delta_1^c(F)$ is in $\mathrm{Pol}_{n-1}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$. By induction, this subcategory is stable by τ_k, which gives

$$\delta_1^c\,(\,\tau_k(F)\,) \cong \tau_k\,(\,\delta_1^c(F)\,) \in \mathrm{Pol}_{n-1}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod}).$$

Since this is true for all colours $c \in C$, we conclude that $\tau_k(F) \in \mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$. As a special case of the stability by colimits we get that the subcategories $\mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ are closed under quotient. Finally, we show by induction on $n \in \mathbb{N}$ that $\mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ is closed under extension. Let $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ be a short exact sequence in $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ such that F and H are in $\mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$, we want to prove that G is also in $\mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$. For $c \in C$, by Proposition 2.6.6.0) we have an exact sequence

$$0 \longrightarrow \kappa_1^c(F) \longrightarrow \kappa_1^c(G) \longrightarrow \kappa_1^c(H) \longrightarrow \delta_1^c(F) \xrightarrow{f} \delta_1^c(G) \longrightarrow \delta_1^c(H) \longrightarrow 0\,,$$

that we can split to get a short exact sequence

$$0 \longrightarrow \mathrm{Im}(f) \longrightarrow \delta_1^c(G) \longrightarrow \delta_1^c(H) \longrightarrow 0\ .$$

By hypothesis, $\delta_1^c(F)$ and $\delta_1^c(H)$ are in $\mathrm{Pol}_{n-1}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and $\mathrm{Im}(f)$ is also in $\mathrm{Pol}_{n-1}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ since it is a quotient of $\delta_1^c(F)$. Finally, we use induction on the short exact sequence to get that $\delta_1^c(G)$ is in $\mathrm{Pol}_{n-1}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, for all $c \in C$, which means that G is in $\mathrm{Pol}_{n}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. □

In this proposition we showed that the subcategories $\mathrm{Pol}_{n}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ are closed under quotient and by extensions. However, we explain in Remark 5.1.9 that this notion of strong polynomial functors is not completely satisfying since it is not closed under subobjects. We now show that in Definition 5.1.1 we can also use all the endofunctors δ_k^x for $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$ as explained above.

Proposition 5.1.4. *A* $\mathbf{FI}_d$*-module* F *is in* $\mathrm{Pol}_{n}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ *if and only if the functor* $\delta_k^x(F)$ *is in* $\mathrm{Pol}_{n-1}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ *for all* $k \in \mathbf{FI}_d$ *and all* $x \in \mathbf{FI}_d(0,k)$.

Proof. One implication is obvious by taking $k = 1$ and $c \in C = \mathbf{FI}_d(0,1)$, we prove the reverse. Let F be in $\mathrm{Pol}_{n}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and $c, \tilde{c} \in C$ be two colours, we prove that $\delta_2^{(c,\tilde{c})}(F)$ is in $\mathrm{Pol}_{n-1}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. First, we apply the exact sequence of endofunctors of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ from Proposition 2.6.6.7) to F. We get an exact sequence that we can split to get the short exact sequence

$$0 \longrightarrow \mathrm{Im}(f_F) \longrightarrow \delta_2^{(c,\tilde{c})}(F) \longrightarrow \tau_1 \circ \delta_1^c(F) \longrightarrow 0,$$

where f_F is a map from $\delta_1^{\tilde{c}}(F)$ to $\delta_2^{(c,\tilde{c})}(F)$ By Proposition 5.1.3 the subcategory $\mathrm{Pol}_{n-1}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is closed under quotient and by τ_1 so the first and last terms of this short exact sequence are in $\mathrm{Pol}_{n-1}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Since the subcategory $\mathrm{Pol}_{n-1}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is also closed under extensions by Proposition 5.1.3 we then proved that, for any colours $c, \tilde{c} \in C$, the functor $\delta_2^{(c,\tilde{c})}(F)$ is in $\mathrm{Pol}_{n-1}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. By induction, using the exact sequence 2.6.6.7) in a general version, we prove in a similar way that $\delta_k^x(F)$ is in $\mathrm{Pol}_{n-1}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$. □

In the following we give some examples of strong polynomial functors, the first being the functors that are zero after or until some rank.

Lemma 5.1.5. *For* $F \in \mathbf{FI}_d$ ***-Mod*** *and* $k \in \mathbb{N}$*, if* $F(n) = 0$ *for all* $n > k$*, then*

$$F \in \mathrm{Pol}_{k}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}).$$

Proof. We prove this by induction. For $k = 0$ it is clear that, if $F(0)$ is the only non-zero part of F, then $\tau_1(F)$ and $\delta_1^c(F)$ are zero and F is in $\mathrm{Pol}_{0}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Now if $F(n) = 0$ for $n > k$, then $\tau_1(F)(n) = F(n+1) = 0$ for $n > k-1$, and so for any colour $c \in C$, we have $\delta_1^c(F)(n) = 0$ for $n > k-1$. By induction we get that $\delta_1^c(F) \in \mathrm{Pol}_{k-1}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ for any colour $c \in C$, and so $F \in \mathrm{Pol}_{k}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. □

The converse of this result is false (see 5.1.7), but we still have:

Lemma 5.1.6. *For* $F \in \mathbf{FI}_d$ ***-Mod*** *and* $k \in \mathbb{N}$*, if* $F(n) = 0$ *for all* $n < k$ *and* F *is non-zero, then*

$$F \notin \mathrm{Pol}_{k-1}^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}).$$

Proof. We prove this by induction, the case $k = 0$ being empty. For $k \geq 1$, if F is a non-zero $\mathbf{FI}_d$-module such that $F(n) = 0$ for $n < k+1$, then for all $c \in C$, $\delta_1^c(F)(n) = 0$ for $n < k$ since it is a quotient of $\tau_1(F)(n) = F(n+1)$. Since F is non-zero, there exist $m \in \mathbb{N}$ minimal such that $F(m) \neq 0$. Since we consider $k \geq 1$, we have $F(0) = 0$ and so $m \geq 1$. Then $\delta_1^c(F)(m-1)$ is the cokernel of the map $F(m-1) = 0 \to F(m)$, so $\delta_1^c(F)(m-1) = F(m) \neq 0$. We get that $\delta_1^c(F)$ is non-zero and by induction $\delta_1^c(F)$ is not in $\mathrm{Pol}_{k-1}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ for $c \in C$, which gives that F is not in $\mathrm{Pol}_k^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. □

Some interesting examples are the functors defined in Section 2.3:

Example 5.1.7.

1) For $M \in \mathbf{R}\text{-}\mathbf{Mod}$, the constant functor equal to M is strong polynomial of degree 0. Indeed, it sends every morphism to the identity and so we can compute for all $c \in C$:

$$\delta_1^c(M) = \mathrm{Coker}\left(i_1^c(M)\right) = \mathrm{Coker}\left(M(i_1^c)\right) = \mathrm{Coker}\left(\mathrm{Id}\right) = 0.$$

2) For $k \in \mathbb{N}$, the twisted atomic functor $M_k : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ is strong polynomial of degree k. Indeed, $\tau_1(M_k) = M_{k-1}$ and so the natural transformation $i_1^c(M_k) : M_k \to \tau_1(M_k) = M_{k-1}$ is zero since either the source or the target is zero. This gives us that

$$\delta_1^c(M_k) = \tau_1(M_k) = M_{k-1}$$

for all $c \in C$ and by induction we get $M_k \in \mathrm{Pol}_k^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

3) Similarly, we have $\tau_1(M_{\geq k}) = M_{\geq k-1}$ and we can show that, for any colour $c \in C$

$$\delta_1^c(M_{\geq k}) = M_{k-1}$$

proving that $M_{\geq k} \in \mathrm{Pol}_k^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

4) Finally, we can check that the direct sum on $k \in \mathbb{N}$ of the constant functor M is strong polynomial of degree zero, while the functors

$$\bigoplus_{k \in \mathbb{N}} M_k \quad \text{and} \quad \bigoplus_{k \in \mathbb{N}} M_{\geq k}$$

are not strong polynomial since each $M_{\geq k}$ and each M_k has a degree k where k goes to infinity.

As a special case, we retrieve the example of $\mathbf{FI}_d$-module developed in Chapter 3 of the homology of the sink configuration spaces of graphs. In particular, we deduce from Proposition 3.2.8 the following:

Proposition 5.1.8. *For $i \in \mathbb{N}$ and $\mathcal{G}_d$ the linear graph on d vertices, the $\mathbf{FI}_d$-module*

$$H_i\Big(\mathit{Conf}_{(-)}^{\mathit{sink}}\left(\mathcal{G}_d, [d]\right), \mathbf{R} \Big)$$

from Definition 3.2.2 is strong polynomial of degree $i+1$ if $i > 0$ and of degree 0 if $i = 0$.

Proof. By Proposition 3.2.8, for $i > 0$ the $\mathbf{FI}_d$-module $H_i\Big(\mathrm{Conf}_{(-)}^{\mathrm{sink}}\left(\mathcal{G}_d, [d]\right), \mathbb{Q} \Big)$ is twisted atomic of rank $i+1$, so it is strong polynomial of degree $i+1$ by Example 5.1.7.2). For $i = 0$, this $\mathbf{FI}_d$-module is a constant functor by Proposition 3.2.8 so it is strong polynomial of degree 0. This result remains true for the homology over $\mathbf{R}$ since the proof of Proposition 3.2.8 extend to this case. □

In Proposition 5.1.3 we proved that the subcategories $\mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ are closed under quotients, extensions and by the endofunctors τ_k and δ_k^c, but they are not closed under subobjects or by the endofunctors κ_k^x as explained in the following remarks.

Remark 5.1.9. A subfunctor of a strong polynomial functor is not necessarily strong polynomial of lower degree or even strong polynomial at all. For $d = 1$, we can find counterexamples in [DV19, p.362]. For $\mathbf{FI}_d$, we use the variations of these functors defined in Example 2.3: for $M \in \mathbf{R}\text{-}\mathbf{Mod}$, the subfunctor $M_{\geq k}$ of the constant functor M is strong polynomial of degree k, while M is strong polynomial of degree 0. Moreover, the direct sum on $k \in \mathbb{N}$ of the constant functor M is strong polynomial of degree zero, while its subfunctor $\bigoplus M_{\geq k}$ is not strong polynomial at all. These examples emphasize the interest of introducing the notion of weak polynomial functors in Chapter 7.

Remark 5.1.10. As a complement of Proposition 5.1.3, we give a counterexample showing that the subcategory $\mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is not stable by the endofunctors κ_k^x for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$. Indeed, for $M \in \mathbf{R}\text{-}\mathbf{Mod}$ we can consider the quotient of the constant functor M by its subfunctor $M_{\geq k}$ defined in Section 2.3. This quotient is given on objects by

$$M\big/M_{\geq k}(n) = \begin{cases} M & \text{if } n < k \\ 0 & \text{else} \end{cases},$$

and on a morphism $(f,g) \in \mathbf{FI}_d(n,m)$ by the identity if $n, m < k$ and by zero else. We then compute that $\tau_1\left(M\big/M_{\geq k}\right)(n) = M\big/M_{\geq k}(n+1)$ for $n \in \mathbf{FI}_d$. For $c \in C$, as this functor is a quotient of the constant functor M, we deduce that $i_1^c\left(M\big/M_{\geq k}\right) = M\big/M_{\geq k}\left(\ n \xrightarrow{\mathrm{Id}_n + c} n+1\ \right)$ is the identity of M if $n > k-1$ and zero else. This proves that $\delta_1^c\left(M\big/M_{\geq k}\right)$, which is the cokernel of this map, is zero for all $c \in C$, and so

$$M\big/M_{\geq k} \in \mathrm{Pol}_0^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}).$$

On the other hand, these identities also implies that $\kappa_1^c\left(M\big/M_{\geq k}\right) = M_{k-1}$, and we explained in Example 5.1.7 that M_{k-1} is strong polynomial of degree $k-1$, so it is not in $\mathrm{Pol}_0^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ for $k > 1$.

We end this section by showing that the precomposition by the forgetful functor $\mathcal{O} : \mathbf{FI}_d \to \mathbf{FI}$ from Definition 2.1.6 respects the strong polynomiality.

Proposition 5.1.11. *For $n \in \mathbb{N}$, we have the inclusion*

$$\mathcal{O}^*\left(\ \mathrm{Pol}_n^{strong}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})\ \right) \subset \mathrm{Pol}_n^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}).$$

Proof. We prove the result by induction on $n \in \mathbb{N}$, the case $n = 0$ being a special case of the following reasoning. For $F \in \mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$, by Proposition 2.7.1 we have for all colours $c \in C$ the isomorphism

$$\delta_1^c\big(\ \mathcal{O}^*(F)\ \big) = \delta_1^c \circ \mathcal{O}^*(F) \cong \mathcal{O}^* \circ \delta_1(F) = \mathcal{O}^*\left(\delta_1(F)\ \right).$$

By definition of strong polynomial functors over $\mathbf{FI}$ we have $\delta_1(F) \in \mathrm{Pol}_{n-1}^{\mathrm{strong}}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ and by induction we conclude that $\mathcal{O}^*\left(\delta_1^c(F)\ \right) \in \mathrm{Pol}_{n-1}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. □

The Proposition 5.1.11 explains that each strong polynomial functor over $\mathbf{FI}$ provides a strong polynomial functor over $\mathbf{FI}_d$. We give here an example of this process which comes from the Example 2.3.4.

Example 5.1.12. For $k \in \mathbb{N}$, let $T_k^{(d)}$ be the $\mathbf{FI}_d$-module defined in Example 2.3.4 sending n to $(\mathbb{K}^n)^{\otimes k}$. We recall that there is a relation $T_k^{(d)} = \mathcal{O}^*(\ T_k^{(1)}\ )$ and that, for $d = 1$, $T_k^{(1)}$ is the composition of $F : \mathbf{FI} \to \mathbb{K}\text{-}\mathbf{Vect}$, which sends n to $\mathbb{K}^n$, with $T_k : \mathbb{K}\text{-}\mathbf{Vect} \to \mathbb{K}\text{-}\mathbf{Vect}$ which sends V to $V^{\otimes k}$. It is a classical example that T_k is polynomial of degree k in the usual sense (see Definition 5.4.1), and we can compute that F is strong polynomial of degree 1. Indeed, we have

$$\delta_1(F)(n) = \mathrm{Coker}\left(F(\mathrm{Id}_n + (0 \to 1)) \right) = \mathrm{Coker}\left(\mathbb{K}^n \xrightarrow{\mathrm{Id}_{\mathbb{K}^n} \oplus (0 \to \mathbb{K})} \mathbb{K}^{n+1} \right) = \mathbb{K}.$$

Similarly, for any morphism f in $\mathbf{FI}$, by definition we have $\tau_1(F)(f) = F(f + \mathrm{id})$, which means that $\delta_1(F)(f) = \mathrm{Id}_{\mathbb{K}}$, showing that $\delta_1(F)$ is a constant functor. Since $\mathbf{FI}$ has an initial object and T_k preserve epimorphisms, we can use the proposition 3.12 from [DV19] to conclude that $T_k^{(1)} \in \mathrm{Pol}^{\mathrm{strong}}_{k\times 1}(\mathbf{FI}, \mathbb{K}\text{-}\mathbf{Vect})$. Finally, using Proposition 5.1.11 we get

$$T_k^{(d)} = \mathcal{O}^*(T_k^{(1)}) \in \mathrm{Pol}^{\mathrm{strong}}_k(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect}).$$

Remark 5.1.13. This example comes from a factorization $\mathbf{FI}_d \to \mathbf{FI} \to \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$, where the last functor is a polynomial functor in the classical sense (see Definition 5.4.1). The Proposition 5.1.11 together with the proposition 3.12 from [DV19] prove that this kind of composition preserves polynomiality if the last functor preserves epimorphisms. We will show in Proposition 5.4.18 that a direct composition $\mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$ which does not factor by $\mathbf{FI}$ also preserves polynomiality under a similar hypothesis, but with a non optimal bound.

5.2 The standard projective functors

A very important family of examples of strong polynomial $\mathbf{FI}$-modules are the standard projective functors $P_n^{\mathbf{FI}}$ from Definition 2.2.4. The fact that the functors $P_n^{\mathbf{FI}}$ are polynomial simplifies the study of polynomial functors over $\mathbf{FI}$ and leads to important results. In this section we show that the $\mathbf{FI}_d$-modules $P_n^{\mathbf{FI}_d}$ are not strong polynomial when $d > 1$ using an explicit description of $\delta_1^c(P_n^{\mathbf{FI}_d})$, which emphasizes an important difference between $\mathbf{FI}$-modules and $\mathbf{FI}_d$-modules.

Proposition 5.2.1. *For $n \in \mathbf{FI}_d$ and $c \in C$, we have the following relation:*

$$\delta_1^c(P_n^{\mathbf{FI}_d}) \cong (P_{n-1}^{\mathbf{FI}_d})^{\oplus n} \oplus (P_n^{\mathbf{FI}_d})^{\oplus(d-1)}.$$

Proof. By definition $\delta_1^c(P_n^{\mathbf{FI}_d})$ is the cokernel of the morphism $i_1^c(P_n^{\mathbf{FI}_d}) = P_n^{\mathbf{FI}_d}(\mathrm{Id}_{(-)} + c)$ which is given by

$$\mathbf{R}\left[(\mathrm{Id}_{(-)} + c)_*\right] \;:\; P_n^{\mathbf{FI}_d}(-) = \mathbf{R}[\mathbf{FI}_d(n,-)] \longrightarrow \mathbf{R}[\mathbf{FI}_d(n, (-)+1)] = P_n^{\mathbf{FI}_d}((-)+1).$$

For $k \in \mathbf{FI}_d$ with $k \le n-1$ we have $P_n^{\mathbf{FI}_d}(k) = \mathbf{R}[\mathbf{FI}_d(n,k)] = \mathbf{R}[\varnothing] = 0$. For $k \ge n$ the morphism $i_1^c(P_n^{\mathbf{FI}_d})$ sends a basis element $(f,g) \in \mathbf{FI}_d(n,k)$ to the composition $(\mathrm{Id}_k + c) \circ (f,g) : n \to k \to k+1$. Then, the only basis morphisms that do not vanish in its cokernel are the morphisms $(f,g) \in \mathbf{FI}_d(n,k+1)$ such that either the element $k+1$ is in the image of f, or the element $k+1$ is not in the image of f and is coloured with a colour other than c (*i.e.* $g(k+1) \neq c$). Then, we have the isomorphism of $\mathbf{R}$-modules

$$\delta_1^c(P_n^{\mathbf{FI}_d})(k) \cong \mathbf{R}[(f,g) \,|\, k+1 \in \mathrm{Im}(f)] \oplus \mathbf{R}[(f,g) \,|\, k+1 \notin \mathrm{Im}(f),\, g(k+1) \neq c].$$

The generators (f,g) of the first component correspond to all the morphisms in $\mathbf{FI}_d(n \smallsetminus \{j\}, k+1 \smallsetminus \{k+1\})$ for all possible inverse images $1 \le j \le k$ of the element $k+1$. The generators (f,g) of the second component correspond to all the morphisms in $\mathbf{FI}_d(n,k)$ coupled with a colour choice in $C \smallsetminus \{c\}$ for $k+1$. This gives the isomorphism of $\mathbf{R}$-modules

$$\delta_1^c(P_n^{\mathbf{FI}_d})(k) \cong \mathbf{R}[\mathbf{FI}_d(n-1, k)]^{\oplus n} \oplus \mathbf{R}[\mathbf{FI}_d(n,k)]^{\oplus(d-1)}$$

for all $c \in C$. Finally for $(f,g) \in \mathbf{FI}_d(k,l)$, the map $\delta_1^c(P_n^{\mathbf{FI}_d})(f,g)$ is obtained as the map $\tau_1(P_n^{\mathbf{FI}_d})(f,g) = \mathbf{R}\left[\left(\,(f,g) + \mathrm{Id}_1\,\right)_*\right]$ passing to the cokernel. Then, the decomposition is natural since the post-composition by $(\,(f,g) + \mathrm{Id}_1\,)$ preserves the conditions mentioned above. ☐

Corollary 5.2.2. *For $d > 1$, the standard projective functor $P_n^{\mathbf{FI}_d}$ is not strong polynomial of any degree.*

Proof. It follows directly from the relation of Proposition 5.2.1 and from the definition of strong polynomial functors. ☐

Remark 5.2.3. For $d = 1$, the Proposition 5.2.1 gives the relation

$$\delta_1\left(\,P_n^{\mathbf{FI}}\,\right) = \left(\,P_{n-1}^{\mathbf{FI}}\,\right)^{\oplus n}$$

already present in the proof of [Dja16, prop 4.4]. By induction this shows that the functor $P_n^{\mathbf{FI}}$ is strong polynomial of degree n. In particular, this implies that the finitely generated **FI**-modules are the strong polynomial functors with finitely generated values over **FI**, as explained in [DV19]. This result is very specific to the **FI**-modules, due to the fact that the projective standard functors are polynomial. A similar formula is also present in [CEFN14] where the authors show that $\tau_a\left(\,P_n^{\mathbf{FI}}\,\right) \cong \left(\,P_{n-1}^{\mathbf{FI}}\,\right) \oplus Q_a$ with Q_a a direct sum of $P_i^{\mathbf{FI}}$ with $i \leq d-1$. This formula is one of the key points to prove the noetherian property in a general context for **FI**-modules.

5.3 Support of a $\mathbf{FI}_d$-module

For functors over a symmetric monoidal category with an initial object, such as **FI**, the notion of support studied by Djament in [Dja16] is closely related to the notion of strong polynomial functors. Indeed, for **FI**-modules being strong polynomial of degree less than or equal to i is equivalent to being supported by the integers $0,\ldots,i$. This result is specific to the **FI**-modules and is not easily generalized to other categories. In this section, we show that for $\mathbf{FI}_d$-modules this is only an implication and the converse is false. Thus the notion of support has less important applications for $\mathbf{FI}_d$-modules than it has for **FI**-modules, although it is still related to the strong polynomial functors.

Definition 5.3.1. For F a $\mathbf{FI}_d$-module, a *support* of F is a set S of objects of $\mathbf{FI}_d$ such that for any subfunctor $G \subset F$, if $G(n) = F(n)$ for all $n \in S$, then $G = F$. A $\mathbf{FI}_d$-module is said to be *finitely supported* if it admits a support of finite cardinality.

Remark 5.3.2. A support of a $\mathbf{FI}_d$-module F is not unique. Indeed, if S is a support of F, then $S \sqcup \{n\}$ is another support of F for any object n of $\mathbf{FI}_d$ that is not in S.

Example 5.3.3. If a $\mathbf{FI}_d$-module is zero after some rank then it is finitely supported. Conversely, it is not enough to have zero maps after some rank to be finitely supported. For example, the functor $\bigoplus_{k\in\mathbb{N}} M_k$ has no finite support since every support of M_k must contain k.

The following proposition explains how the notion of support of a $\mathbf{FI}_d$-module is related to being generated by the first standard projective functors.

Proposition 5.3.4. *Let F be a $\mathbf{FI}_d$-module and S be a set of objects of $\mathbf{FI}_d$, then S is a support of F if and only if F is a quotient of the direct sum*

$$\bigoplus_{n\in S}\left(\,P_n^{\mathbf{FI}_d}\,\right)^{\oplus k_n},$$

where $k_n \in \mathbb{N}\sqcup\{\infty\}$. In particular, if a $\mathbf{FI}_d$-module is finitely generated, then it is finitely supported.

Proof. This result was proved by Djament in the general context of functors over a small category in the proposition 2.10 and corollary 2.11 from [Dja16]. □

Remark 5.3.5. The direct sum in Proposition 5.3.4 can have an infinite number of terms $P_n^{\mathbf{FI}_d}$. This is why finitely generated implies finitely supported, but the converse is not true. For example $P_0^{\mathbf{FI}_d}$ admits $\{0\}$ as a support, so the direct sum $\bigoplus P_0^{\mathbf{FI}_d}$ with an infinite number of terms also admits $\{0\}$ as a support, and so it is finitely supported even if it is not finitely generated. The Proposition 5.3.4 also shows that a functor is generated in degree $\leq k$, as defined in [Wil18a] for example, if and only if the first k integers form a support of this functor.

Remark 5.3.6. As a consequence of Proposition 5.3.4, the notion of finitely supported is closed under quotient and extension. Indeed, for $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ a short exact sequence of $\mathbf{FI}_d$-modules, if G is a quotient of a direct sum of $P_n^{\mathbf{FI}_d}$ for a finite number of $n \in \mathbb{N}$, then H is also such a quotient since it is a quotient of G. Moreover, if F and H are such quotients the horseshoe lemma implies that G is also such a quotient. However, the notion of finitely supported is not closed under subobjects. Indeed, the direct sum on $k \in \mathbb{N}$ of the constant functor M is supported by $\{0\}$, while its subfunctor $\bigoplus_{k\in\mathbb{N}} M_{\geq k}$ is not finitely supported, since any support of $M_{\geq k}$ must contain k.

We now explain the connection between the support of a $\mathbf{FI}_d$-module and the fact that it is strong polynomial. Explicitly, we show that if $F : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ is in $\mathrm{Pol}_i^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, then the first i integers form a support of F. This is inspired by [Dja16, Proposition 4.1] which gives the same result for functors over a symmetric monoidal category with an initial object, such as $\mathbf{FI}$. We start with the case $i = 0$ and we get the general case by induction.

Lemma 5.3.7. *Let F be a $\mathbf{FI}_d$-module, if F is strong polynomial of degree 0, then $\{0\}$ is a support of F.*

Proof. Let G be a subfunctor of F such that $G(0) = F(0)$, then by hypothesis we have $\delta_1^c(F) = \mathrm{Coker}(\mathrm{Id}_{(-)} + c) = 0$ for all $c \in C$. This shows that $F(\mathrm{Id}_n + c)$ is an epimorphism for all $c \in C$ and all $n \in \mathbb{N}$. We then show by induction that $G(n) = F(n)$ for all $n \in \mathbb{N}$: the case $n = 0$ is true by hypothesis and if $G(n) = F(n)$ we have $F(\mathrm{Id}_n + c)(G(n)) = F(\mathrm{Id}_n + c)(F(n)) = F(n+1)$. Since G is a subfunctor of F, we also have that $F(\mathrm{Id}_n + c)(G(n))$ is a submodule of $G(n+1)$, which shows that $G(n+1) = F(n+1)$. □

Proposition 5.3.8. *Let F be a $\mathbf{FI}_d$-module, if F is strong polynomial of degree less than or equal to i, then $\{0, \ldots, i\}$ is a support of F.*

Proof. We proceed by induction on $i \in \mathbb{N}$, the case $i = 0$ being given by Lemma 5.3.7. For $F \in \mathrm{Pol}_{i+1}^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, let G be a subfunctor of F such that $G(n) = F(n)$ for all $n \in \{0, \ldots, i, i+1\}$. By Proposition 2.6.6.0) we have an exact sequence

$$0 \longrightarrow \kappa_1^c(G) \longrightarrow \kappa_1^c(F) \longrightarrow \kappa_1^c(F/G) \longrightarrow \delta_1^c(G) \xrightarrow{\varphi_1^c} \delta_1^c(F) \longrightarrow \delta_1^c(F/G) \longrightarrow 0$$

for all $c \in C$. Let H_1^c be the subfunctor of $\delta_1^c(F)$ defined by $H_1^c = \mathrm{Im}(\varphi_1^c : \delta_1^c(G) \to \delta_1^c(F))$. We then have $H_1^c(m) = \delta_1^c(m)$ for all $m \in \{0, \ldots, i\}$ since $(\varphi_1^c)_m$ is constructed by the following diagram

$$\begin{tikzcd}
G(m) \arrow[r] \arrow[d] & \tau_1(G)(m) = G(m+1) \arrow[r] \arrow[d] & \delta_1^c(G) \arrow[r] \arrow[d, "(\varphi_1^c)_m"] & 0 \\
F(m) \arrow[r] & \tau_1(F)(m) = F(m+1) \arrow[r] & \delta_1^c(F) \arrow[r] & 0\,,
\end{tikzcd}$$

and the first two vertical maps are epimorphisms for $m \in \{0, \ldots, i\}$ by hypothesis, which implies that the last one is an epimorphism. Since $\delta_1^c(F) \in \mathrm{Pol}_i^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, we have by induction that $\{0, \ldots, i\}$ is a support of $\delta_1^c(F)$, and so $H_1^c = \mathrm{Im}(\ \varphi_1^c : \delta_1^c(G) \to \delta_1^c(F)\ ) = \delta_1^c(F)$. This means that φ_1^c is an epimorphism and, together with the exact sequence above, it gives that $\delta_1^c(F/G) = 0$ for all $c \in C$. We conclude that F/G is in $\mathrm{Pol}_0^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and, by Lemma 5.3.7, that $\{0\}$ is a support of F/G. Finally, $(F/G)(0) = 0$ since $G(0) = F(0)$ by hypothesis, so $F/G = 0$ since $\{0\}$ is a support of F/G, and then $G = F$. □

Remark 5.3.9. The converse of Proposition 5.3.8 is true for **FI**-modules (Proposition 4.4 in [Dja16]) but it is false for $\mathbf{FI}_d$-modules with $d > 1$. Indeed, $P_n^{\mathbf{FI}_d}$ admits $\{0, \ldots, n\}$ for support by Proposition 5.3.4, but it is not strong polynomial if $d > 1$ by Corollary 5.2.2. The fact that the converse of Proposition 5.3.8 is true for **FI**-modules is very specific to the category **FI**. It allows us to describe the strong polynomial functors over **FI** with the notion of support and it comes from the fact that the standard projective functors are polynomial in this case (Remark 5.2.3) which is not often the case over other categories. However, the converse of Lemma 5.3.7, which is the case where $i = 0$, is true for functors over a symmetric monoidal category with an initial object (Remark 2.12 in [Dja16]), but it is particular to 0 since it is the initial object.

5.4 The coslice category $(0 \downarrow \mathbf{FI}_d)$ and cross effects

The original definition of polynomial functors given by Eilenberg and Mac Lane in [EM54] for functors between categories of modules over a ring is based on the notion of cross effects, which as been extended several times. We recall in Definition 5.4.1 the definition of cross effects for functors over monoidal categories whose unit is initial given in [DV19]. In this section we introduce and study the cross effects for $\mathbf{FI}_d$-modules (in Definition 5.4.6) following this generalization which is better adapted for such categories with only increasing morphisms. We then prove in Proposition 5.4.12 that the corresponding polynomial $\mathbf{FI}_d$-modules are exactly the strong polynomial $\mathbf{FI}_d$-modules from Definition 5.1.1. To do this, we introduce a new category which corresponds to the coslice category $(0 \downarrow \mathbf{FI}_d)$ (sometimes also called the undercategory under 0 like in [ML98]page 45) of couples (a, x), where a is an object of $\mathbf{FI}_d$ and x a morphism from 0 to a in $\mathbf{FI}_d$. Then we use this alternative definition to prove that the composition of two polynomial functors is still polynomial in Proposition 5.4.18.

Since its introduction, the definition of polynomial functors based on cross effects has been extended several times, like in [HPV15] to the case where $\mathcal{A}$ is a monoidal category whose unit is a null object. The cross effects are generally defined by the kernel of a morphism where we omit a term of a sum at the target but when the unit is a null object, it is equivalent to use the cokernel of a morphism where we omit a term of a sum at the source (see [DV19] for a proof). In [DV19], Djament and Vespa define the following notion of cross effects for functors over a monoidal category $\mathcal{M}$ whose unit is initial following the definition as a cokernel:

Definition 5.4.1. For $\mathcal{A}$ and $\mathcal{B}$ two monoidal categories whose unit 0 is initial, the n-th *cross effect* of $F : \mathcal{A} \to \mathcal{B}$ is the functor $\mathrm{cr}_n(F) : \mathcal{A}^n \to \mathcal{B}$ given on n objects $a_1, \ldots, a_n$ of $\mathcal{A}$ by

$$\mathrm{cr}_n(F)(a_1, \ldots, a_n) = \mathrm{Coker}\left(\bigoplus_{i=1}^{n} F\Big(\bigoplus_{j \neq i} a_j\Big) \xrightarrow{\bigoplus_{i=1}^{n} F(\sigma_{a_i})} F\Big(\bigoplus_{j=1}^{n} a_j\Big) \right),$$

where σ_{a_i} is given by the unique map $0 \to a_i$ and the identity on the other components. The functor $F : \mathcal{A} \to \mathcal{B}$ is polynomial of degree less than or equal to n if its $(n+1)$-th cross effects $\mathrm{cr}_{n+1}(F)(-, \ldots, -)$ is the zero functor.

When the source is a monoidal category whose unit is a null object, the cross effects functors are exact which implies directly that the categories of (strong) polynomial functors are thick. Djament and Vespa also define a notion of strong polynomial functors over a monoidal category $\mathcal{M}$ whose unit is initial using the endofunctors δ as in Definition 5.1.1. They then show in [DV19, Proposition 3.3] that the two definitions are equivalent: a functor $F : \mathcal{M} \to \mathbf{R}\text{-}\mathbf{Mod}$ is polynomial of degree less than or equal to n if and only if the cross effect $\mathrm{cr}_{n+1}(F)$ is the zero functor. However, since these categories only have an initial object, they lost the exactness of the cross effect functors and so the stability of polynomial functor by subobjects.

We will show in this section that the same thing happens for $\mathbf{FI}_d$-modules. First we introduce the coslice category $(0 \downarrow \mathbf{FI}_d)$ whose unit is initial and then we define the n-th cross effects functor of a $\mathbf{FI}_d$-module through a forgetful functor $\Theta : (0 \downarrow \mathbf{FI}_d) \to \mathbf{FI}_d$ in Definition 5.4.6. This way the n-th cross effect $\mathrm{cr}_n(F)$ of a $\mathbf{FI}_d$-module F is a functor over $(0 \downarrow \mathbf{FI}_d)^n$ and we show in Proposition 5.4.12 that F is in $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ if and only if $F \circ \Theta$ is in $\mathrm{Pol}_n^{strong}(\,(0 \downarrow \mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$.

Definition 5.4.2. The *category* $(0 \downarrow \mathbf{FI}_d)$ has for objects the pairs (a, x) where a is an object of $\mathbf{FI}_d$ and x a morphism in $\mathbf{FI}_d(0, a)$. The morphisms in $(0 \downarrow \mathbf{FI}_d)$ from (a, x) to (b, y) are the morphisms $f \in \mathbf{FI}_d(a, b)$ such that $f \circ x = y$, and the composition comes from $\mathbf{FI}_d$.

Remark 5.4.3. For $d = 1$, the unit 0 of $\mathbf{FI}$ is an initial object so for every $a \in \mathbf{FI}$, there exists a unique morphism from 0 to a. There is then an isomorphism of categories between $(0 \downarrow \mathbf{FI})$ and $\mathbf{FI}$.

Proposition 5.4.4. *The category $(0 \downarrow \mathbf{FI}_d)$ is a symmetric monoidal category and its unit $(0, \mathrm{id}_0)$ is an initial object.*

Proof. The monoidal structure on $(0 \downarrow \mathbf{FI}_d)$ is induced by the monoidal structure on $\mathbf{FI}_d$ (see Lemma 2.1.5) and the unit 0 of $\mathbf{FI}_d$ gives the unit $(0, \mathrm{Id}_0)$ of $(0 \downarrow \mathbf{FI}_d)$. Now for any object (a, x) in $(0 \downarrow \mathbf{FI}_d)$, the only map from $(0, \mathrm{id}_0)$ to (a, x) in $(0 \downarrow \mathbf{FI}_d)$ is x since such a map f must satisfy $f \circ \mathrm{id}_0 = x$. This shows that $(0, \mathrm{id}_0)$ is initial. $\square$

Since $(0 \downarrow \mathbf{FI}_d)$ is a symmetric monoidal category whose unit $(0, \mathrm{Id}_0)$ is an initial object, it falls in the framework of [DV19] and so there is a notion of cross effects for functors over $(0 \downarrow \mathbf{FI}_d)$ recalled in Definition 5.4.1. It gives the following:

Definition 5.4.5. For $G : (0 \downarrow \mathbf{FI}_d) \to \mathbf{R}\text{-}\mathbf{Mod}$ a functor, the n-th *cross effect* of G is the functor $\mathrm{cr}_n(G) : (0 \downarrow \mathbf{FI}_d)^n \to \mathbf{R}\text{-}\mathbf{Mod}$ given on n objects $(a_1, x_1) \dots (a_n, x_n)$ of $(0 \downarrow \mathbf{FI}_d)$ by

$$\mathrm{cr}_n(G)\big(\,(a_1, x_1), \dots, (a_n, x_n)\,\big) = \mathrm{Coker}\left(\bigoplus_{i=1}^{n} G\Big(\sum_{j \neq i} (a_j, x_j)\Big) \xrightarrow{\bigoplus_{i=1}^{n} G(\,\sigma_{(a_i, x_i)}\,)} G\Big(\sum_{j=1}^{n} (a_j, x_j)\Big) \right),$$

where $\sigma_{(a_i, x_i)}$ is given by the unique morphism $x_i : (0, \mathrm{Id}_0) \to (a_i, x_i)$ in $(0 \downarrow \mathbf{FI}_d)$ and the identity on the other components.

We then can define the cross effects of functors over $\mathbf{FI}_d$ using the forgetful functor $\Theta : (0 \downarrow \mathbf{FI}_d) \to \mathbf{FI}_d$, which sends an object (a, x) of $(0 \downarrow \mathbf{FI}_d)$ to $a \in \mathbf{FI}_d$, and an arrow f in $(0 \downarrow \mathbf{FI}_d)$ to itself in $\mathbf{FI}_d$, and the definition over $(0 \downarrow \mathbf{FI}_d)$:

Definition 5.4.6. For F a $\mathbf{FI}_d$-module, $n \in \mathbb{N}^*$ and $(a_1, x_1) \dots (a_n, x_n)$ objects of $(0 \downarrow \mathbf{FI}_d)$, the module $\mathrm{cr}_n(F)\big(\,(a_1, x_1), \dots, (a_n, x_n)\,\big)$ is the n-th cross effect $\mathrm{cr}_n(F \circ \Theta)\big(\,(a_1, x_1), \dots, (a_n, x_n)\,\big)$ of the functor $F \circ \Theta$ over $(0 \downarrow \mathbf{FI}_d)$.

Lemma 5.4.7. *For F a $\mathbf{FI}_d$-module and $n \in \mathbb{N}^*$, the modules $\mathrm{cr}_n(F)\big(\ (a_1,x_1),\dots,(a_n,x_n)\ \big)$ for all objects $(a_1,x_1)\dots(a_n,x_n)$ of $(0\downarrow \mathbf{FI}_d)$ define a functor*

$$\mathrm{cr}_n(F)\big(-,\dots,-\big) : (0\downarrow \mathbf{FI}_d)^n \to \mathbf{R}\text{-}\mathbf{Mod},$$

called the n-th cross effect of F.

Proof. It is a consequence of the fact that $\mathrm{cr}_n(F\circ\Theta)\big(\ -,\dots,-\ \big)$ is a functor over $(0\downarrow \mathbf{FI}_d)^n$ in the definition of cross effects over $(0\downarrow \mathbf{FI}_d)$ whose unit is initial, and that the maps in $(0\downarrow \mathbf{FI}_d)$ are the maps in $\mathbf{FI}_d$ that fits the colours. □

Remark 5.4.8. For $d=1$, we recover the definition of cross effects for $\mathbf{FI}$-modules from [DV19] since $(0\downarrow \mathbf{FI})$ is isomorphic to $\mathbf{FI}$.

We give an explicit description of the cross effects of functors over $\mathbf{FI}_d$, using the category $(0\downarrow \mathbf{FI}_d)$ and the morphisms $\sigma_{a_i}^{x_i} = \Theta(\sigma_{(a_i,x_i)})$ in $\mathbf{FI}_d$ which are similar to the morphisms σ_{a_i} in Definition 5.4.1.

Proposition 5.4.9. *For F a $\mathbf{FI}_d$-module and $n\in\mathbb{N}^*$ the n-th cross effect of F on the objects $(a_1,x_1)\dots(a_n,x_n)$ of $(0\downarrow \mathbf{FI}_d)$ is the $\mathbf{R}$-module*

$$\mathrm{cr}_n(F)\big(\ (a_1,x_1),\dots,(a_n,x_n)\ \big) = \mathrm{Coker}\left(\bigoplus_{i=1}^{n} F\Big(\sum_{j\neq i} a_j\Big) \xrightarrow{\bigoplus_{i=1}^{n} F(\sigma_{a_i}^{x_i})} F\Big(\sum_{j=1}^{n} a_j\Big) \right),$$

where $\sigma_{a_i}^{x_i} = \Theta(\sigma_{(a_i,x_i)})$ is given by the morphism $x_i : 0 \to a_i$ and the identity on the other components.

Proof. For F a $\mathbf{FI}_d$-module $F\circ\Theta$ is a functor from $(0\downarrow \mathbf{FI}_d)$ to $\mathbf{R}$-modules. Then $\mathrm{cr}_n(F\circ\Theta)\big(\ (a_1,x_1),\dots,(a_n,x_n)\ \big)$ is the cokernel of the map $\bigoplus F\circ\Theta(\ \sigma_{(a_i,x_i)}\ )$ and by definition the morphism $\Theta(\sigma_{(a_i,x_i)}) = \sigma_{a_i}^{x_i}$ is given by the morphism $x_i : 0\to a_i$ and the identity on the other components. □

We now give a lemma about the cokernel of cokernel maps that will be used to prove basic properties of the cross effects.

Lemma 5.4.10. *Consider the diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "P_f"] \arrow[d, "h"] & C \arrow[d, dashed, "\overline{h}"] \\
D \arrow[r, "g"] & E \arrow[r, "P_g"] & F
\end{tikzcd}$$

in $\mathbf{R}$-$\mathbf{Mod}$ where $(C,P_f) = \mathrm{Coker}(f)$ and $(F,P_g)=\mathrm{Coker}(g)$. If the left square of the diagram is commutative, there exists a unique $\overline{h} : C\to D$ such that $\overline{h}\circ P_f = P_g\circ h$, and there is an isomorphism

$$\mathrm{Coker}\left(\overline{h}\right) \cong \mathrm{Coker}\Big(\ B\oplus D \xrightarrow{g\oplus h} E\ \Big).$$

Proof. The existence and unicity of $\overline{h}$ is given by the universal property of the cokernel applied to $P_g\circ h$ since $P_g\circ h\circ f = P_g\circ g\circ\alpha = 0$. Since P_f is an epimorphism, the equality $\overline{h}\circ P_f = P_g\circ h$ implies that $\mathrm{Im}\left(\overline{h}\right) = \mathrm{Im}(P_g\circ h)$, which is by definition the image of the map P_g restricted to the image of h. The kernel of the map P_g restricted to the image of h being exactly $\mathrm{Im}(h) \cap \mathrm{Im}(g)$, this gives

$$\mathrm{Im}\left(\overline{h}\right) \cong \mathrm{Im}\left(P_g|_{\mathrm{Im}(h)}\right) \cong \mathrm{Im}(h)/\mathrm{Ker}\left(P_g|_{\mathrm{Im}(h)}\right) \cong \mathrm{Im}(h)/\mathrm{Im}(h) \cap \mathrm{Im}(g).$$

With classical isomorphisms we then get

$$\mathrm{Im}\left(\overline{h}\right) \cong \mathrm{Im}(h) + \mathrm{Im}(g)/\mathrm{Im}(g) = \mathrm{Im}(h \oplus g)/\mathrm{Im}(g).$$

Finally, since F is the cokernel of g, we have

$$\mathrm{Coker}\left(\overline{h}\right) = F/\mathrm{Im}\left(\overline{h}\right) \cong \left(E/\mathrm{Im}(g)\right)\Big/\left(\mathrm{Im}(h \oplus g)/\mathrm{Im}(g)\right) \cong E/\mathrm{Im}(h \oplus g) = \mathrm{Coker}(h \oplus g).$$

□

We now show basic properties of the cross effects of functors over $\mathbf{FI}_d$. In particular, the cross effects satisfy the usual induction relation $\mathrm{cr}_{n+m} = \mathrm{cr}_n(\mathrm{cr}_{m+1}(-))$, where in the second term we use the cross effects of a functor over $(0 \downarrow \mathbf{FI}_d)$.

Proposition 5.4.11. *For F a $\mathbf{FI}_d$-module, $n, m \in \mathbb{N}^*$ and $(a_1, x_1)\ldots(a_n, x_n)$, $(b_1, y_1)\ldots(b_m, y_m)$, (k, x) objects in $(0 \downarrow \mathbf{FI}_d)$,*

1) There is a natural isomorphism

$$\begin{aligned}\mathrm{cr}_{n+m}(F)\big(\,(a_1,x_1),\ldots,(a_n,x_n),(b_1,y_1),\ldots,(b_m,y_m)\,\big) \quad &\cong \\ \mathrm{cr}_n\Big(\,\mathrm{cr}_{m+1}(F)\big(\,-,(b_1,y_1),\ldots,(b_m,y_m)\,\big)\Big)&\big(\,(a_1,x_1)\ldots(a_n,x_n)\,\big).\end{aligned}$$

2) There is a natural isomorphism

$$\mathrm{cr}_{n+1}(F)\big(\,(k,x),(a_1,x_1),\ldots,(a_n,x_n)\,\big) \cong \mathrm{cr}_n\left(\,\delta_k^x(F)\,\right)\big(\,(a_1,x_1),\ldots,(a_n,x_n)\,\big).$$

Proof. 1) It is a formal consequence of the fact that the same properties are true for functors over monoidal categories whose unit is initial such as $(0 \downarrow \mathbf{FI}_d)$ (see [DV19, Proposition 3.2]) since $\mathrm{cr}_n(F)$ is $\mathrm{cr}_n(F \circ \Theta)$ by definition.

2) We consider the following diagram in **R-Mod**:

$$\begin{tikzcd}
\bigoplus_{i=1}^n F\Big(\sum_{j\neq i} a_j\Big) \arrow[r, "\bigoplus_{i=1}^n F\left(\mathrm{Id}_{\sum_{j\neq i} a_j}+x\right)"] \arrow[d, "\bigoplus_{i=1}^n F(\sigma_{a_i}^{x_i})"'] & \bigoplus_{i=1}^n F\Big(\sum_{j\neq i} a_j + k\Big) \arrow[r, two heads] \arrow[d, "\bigoplus_{i=1}^n F(\sigma_{a_i}^{x_i}+\mathrm{Id}_k)"] & \bigoplus_{i=1}^n \delta_k^x(F)\Big(\sum_{j\neq i} a_j\Big) \arrow[d, dashed, "\bigoplus_{i=1}^n \delta_k^x(F)(\sigma_{a_i}^{x_i})"] \\
F\Big(\sum_{j=1}^n a_j\Big) \arrow[r, "F\left(\mathrm{Id}_{\sum a_j}+x\right)"'] & F\Big(\sum_{j=1}^n a_j + k\Big) \arrow[r, two heads] & \delta_k^x(F)\Big(\sum_{j=1}^n a_j\Big)
\end{tikzcd}$$

where the two right horizontal maps are the projection on the cokernels. The left square commutes by naturality of the transformation i_k^x from Definition 2.6.1, which corresponds to the horizontal maps when applied to F and $\bigoplus F$. Also, $\mathrm{cr}_n\left(\delta_k^x(F)\right)\left((a_1,x_1),\dots,(a_n,x_n)\right)$ is exactly the cokernel of the right vertical map $\mathrm{cr}_n(F)\,(\mathrm{Id}+x)$ by Proposition 5.4.9. Then the Lemma 5.4.10 gives an isomorphism between $\mathrm{Coker}\left(\mathrm{cr}_n(F)\,(\mathrm{Id}+x)\right)$ and

$$\mathrm{Coker}\left(\bigoplus_{i=1}^{n} F\Big(\sum_{j\neq i} a_j + k\Big) \oplus F\Big(\sum_{j=1}^{n} a_j\Big) \xrightarrow{\bigoplus_{i=1}^{n} F(\sigma_{a_i}^{x_i}+\mathrm{Id}_k)\oplus F\big(\mathrm{Id}+x\big)} F\Big(\sum_{j=1}^{n} a_j + k\Big)\right).$$

Finally, this last cokernel is exactly $\mathrm{cr}_{n+1}(F)\left((k,x),(a_1,x_1),\dots,(a_n,x_n)\right)$ by Proposition 5.4.9. □

We now prove that the definition of the polynomial functors over $\mathbf{FI}_d$ using the cross effects as in Definition 5.4.1 is equivalent to the definition of strong polynomial functors from Definition 5.1.1 using the endofunctors δ.

Proposition 5.4.12. *Let F be a $\mathbf{FI}_d$-module and $n\in\mathbb{N}$ be an integer, then F is in $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ if and only if $\mathrm{cr}_{n+1}(F)\,(-)$ is the zero functor over $(0\downarrow\mathbf{FI}_d)^{\times(n+1)}$, if and only if $F\circ\Theta$ is in $\mathrm{Pol}_n^{strong}((0\downarrow\mathbf{FI}_d),\mathbf{R\text{-}Mod})$.*

Proof. We prove the first equivalence by induction on $n\in\mathbb{N}$, the second one is given by [DV19, Proposition 3.3] since $(0\downarrow\mathbf{FI}_d)$ is monoidal with an initial object. For $n=0$, the functor $\mathrm{cr}_1(F)$ is zero if and only if the map $F(x)$ is an epimorphism for all $k\in\mathbf{FI}_d$ and all $x\in\mathbf{FI}_d(0,k)$, since $\mathrm{cr}_1(F)(k,x)=\mathrm{Coker}\big(F(x):F(0)\to F(k)\big)$ for any $(k,x)\in(0\downarrow\mathbf{FI}_d)$. In this case, for any $m\in\mathbf{FI}_d$, the map $F\big((\mathrm{id}_m+x)\circ c_1^m\big)=F(\mathrm{id}_m+x)\circ F(c_1^m)$ is an epimorphism, which implies that $F(\mathrm{id}_m+x)$ is an epimorphism. Then $\mathrm{cr}_1(F)=0$ implies that $\delta_k^x(F)(m):=\mathrm{Coker}(F(\mathrm{id}_m+x))=0$ for all $m\in\mathbf{FI}_d$, $k\in\mathbf{FI}_d$ and all $x\in\mathbf{FI}_d(0,k)$, which is equivalent to $F\in\mathrm{Pol}_0^{strong}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ by Proposition 5.1.4. The converse is direct by taking $m=0$ since $\delta_k^x(F)=\mathrm{Coker}(F(x))=0$ implies that $F(x)$ is an epimorphism. For $n\in\mathbb{N}$, by Proposition 5.1.4, F is in $\mathrm{Pol}_{n+1}^{strong}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ if and only if the functor $\delta_k^x(F)$ is in $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ for all $k\in\mathbf{FI}_d$ and all $x\in\mathbf{FI}_d(0,k)$, which is equivalent to $\mathrm{cr}_{n+1}\left(\delta_k^x(F)\right)=0$ by induction. However, by Proposition 5.4.11.2), we have a natural isomorphism

$$\mathrm{cr}_{n+1}\left(\delta_k^x(F)\right)\left(-,\dots,-\right)\cong\mathrm{cr}_{n+2}(F)\left((k,x),-,\dots,-\right).$$

This shows that F is in $\mathrm{Pol}_{n+1}^{strong}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ if and only if $\mathrm{cr}_{n+2}(F)$ is the zero functor. □

A direct consequence is that a $\mathbf{FI}_d$-module F is strong polynomial of degree n if and only if $F\circ\Theta$ is strong polynomial of degree n over $(0\downarrow\mathbf{FI}_d)$ because of Definition 5.4.6. Moreover, if a $\mathbf{FI}_d$-module is strong polynomial, then its cross effects are zero after some rank as in the following:

Corollary 5.4.13. *For F a $\mathbf{FI}_d$-module and $n\in\mathbb{N}$, if F is in $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ then the functors $\mathrm{cr}_k(F)\,(-)$ are the zero functor over $(0\downarrow\mathbf{FI}_d)^{\times k}$ for $k\geq n+1$.*

Proof. It is a consequence of Proposition 5.4.12 together with the induction relation from Proposition 5.4.11. □

Remark 5.4.14. For $d=1$, the generation degree in [MW19] is exactly the strong polynomial degree. Indeed, it is given by the functor denoted by $H_0^{\mathbf{FI}}$, which corresponds to the cross effect functor on the element 1 of $\mathbf{FI}$ and gives the minimal generators of a $\mathbf{FI}$-module.

We end this section by using the cross effects to show that a composition $\mathbf{FI}_d\to\mathbf{R\text{-}Mod}\to\mathbf{R\text{-}Mod}$ of two polynomial functors is polynomial. In this goal we first prove a lemma that will be central in the next proof.

Lemma 5.4.15. *For F a $\mathbf{FI}_d$-module, $m \in \mathbb{N}$, and E a set of $k \geq m$ objects $(a_1, x_1) \dots (a_k, x_k)$ of $(0 \downarrow \mathbf{FI}_d)$ we denote by $\mathcal{P}_m(E)$ the set of the subsets of E of cardinality m and by $\sigma^{x_{E\smallsetminus I}}_{a_{E\smallsetminus I}} : \sum\limits_{i\in I} a_i \to \sum\limits_{i\in E} a_i$ the morphism given by $x_i : 0 \to a_i$ for $(a_i, x_i) \in E \smallsetminus I$ and the identity of a_i for $(a_i, x_i) \in I$. If the functor F is strong polynomial of degree less than or equal to m, then the morphism*

$$\varphi_E = \left(\bigoplus_{I\in\mathcal{P}_m(E)} F\Big(\sum_{i\in I} a_i\Big) \xrightarrow{\bigoplus\limits_{I\in\mathcal{P}_m(E)} F\left(\sigma^{x_{E\smallsetminus I}}_{a_{E\smallsetminus I}}\right)} F\Big(\sum_{i\in E} a_i\Big) \right)$$

is an epimorphism.

Proof. We proceed by induction on $|E| = k \geq m$. For $k = m$, we have $\mathcal{P}_m(E) = \{E\}$ so there is only one term in the sum (for $I = E$) and by definition $\sigma^{x_{E\smallsetminus E}}_{a_{E\smallsetminus E}}$ is the identity so it is an epimorphism. Now if the cardinality of E is $k+1$, we consider the following diagram

$$\begin{tikzcd}
\bigoplus\limits_{l=1}^{k+1} \bigoplus\limits_{I\in\mathcal{P}_m(E\smallsetminus\{(a_l,x_l)\})} F\left(\sum\limits_{(a_i,x_i)\in I} a_i\right) \arrow[rr, "\bigoplus\limits_{l=1}^{k+1} \varphi_{E\smallsetminus\{(a_l,x_l)\}}", "= \bigoplus\limits_{l=1}^{k+1} \bigoplus\limits_{I\in\mathcal{P}_m(E\smallsetminus\{(a_l,x_l)\})} F(\sigma)"'] \arrow[d] & & \bigoplus\limits_{l=1}^{k+1} F\left(\sum\limits_{(a_i,x_i)\in E\smallsetminus\{(a_l,x_l)\}} a_i\right) \arrow[d, "\bigoplus\limits_{l=1}^{k+1} F\left(\sigma^{x_l}_{a_l}\right)"] \\
\bigoplus\limits_{J\in\mathcal{P}_m(E)} F\left(\sum\limits_{(a_j,x_j)\in J} a_j\right) \arrow[rr, "\varphi_E = \bigoplus\limits_{J\in\mathcal{P}_m(E)} F\left(\sigma^{x_{E\smallsetminus J}}_{a_{E\smallsetminus J}}\right)"'] & & F\left(\sum\limits_{(a_i,x_i)\in E} a_i\right).
\end{tikzcd}$$

This diagram commutes because of the relation $F\big(\sigma^{x_{E\smallsetminus J}}_{a_{E\smallsetminus J}}\big) = F\big(\sigma^{x_{E\smallsetminus I}}_{a_{E\smallsetminus I}}\big) \circ F\big(\sigma^{x_{I\smallsetminus J}}_{a_{I\smallsetminus J}}\big)$ for $I = J \sqcup \{(a_l, x_l)\}$. Then by induction each of the maps $\varphi_{E\smallsetminus\{(a_l,x_l)\}}$ is an epimorphism because $|\varphi_E \smallsetminus \{(a_l, x_l)\}| = k$, and so is their sum $\bigoplus \varphi_{E\smallsetminus\{(a_l,x_l)\}})$. Moreover, F is in $\mathrm{Pol}^{strong}_m(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ so the Corollary 5.4.13 implies that the functors $\mathrm{cr}_{k+1}(F)\big(\ -\ \big)$ is the zero functor over $(0 \downarrow \mathbf{FI}_d)^{\times(k+1)}$ because $k+1 \geq m+1$. By Proposition 5.4.9 the module $\mathrm{cr}_{k+1}(F)\big(\,(a_1, x_1), \dots, (a_{k+1}, x_{k+1})\,\big)$ is the cokernel of the right vertical map, which implies that this is an epimorphism. Then, by composition the diagonal of the diagram is an epimorphism, so the bottom map φ_E is also an epimorphism. □

Remark 5.4.16. For $d = 1$, we recover the corollary 3.5 from [DV19] which is used to prove that a composition $\mathbf{FI} \to \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$ of two polynomial functors is polynomial.

Remark 5.4.17. In Lemma 5.4.15 we can replace the set $\mathcal{P}_m(E)$ of the subsets of E of cardinality m by the set $\overline{\mathcal{P}}_m(E)$ of the subsets of E of cardinality less than or equal to m and the result stays true. Indeed, the cokernels of the maps

$$\bigoplus_{I\in\mathcal{P}_m(E)} F\Big(\sum_{i\in I} a_i\Big) \xrightarrow{\bigoplus\limits_{I\in\mathcal{P}_m(E)} F\left(\sigma^{x_{E\smallsetminus I}}_{a_{E\smallsetminus I}}\right)} F\Big(\sum_{i\in E} a_i\Big) \quad \text{and} \quad \bigoplus_{J\in\overline{\mathcal{P}}_m(E)} F\Big(\sum_{j\in J} a_j\Big) \xrightarrow{\bigoplus\limits_{J\in\overline{\mathcal{P}}_m(E)} F\left(\sigma^{x_{E\smallsetminus J}}_{a_{E\smallsetminus J}}\right)} F\Big(\sum_{j\in E} a_j\Big)$$

are equal.It comes from the fact that the new maps on the right ($|J| < m$) all factor by maps that are already present in both sides ($|j| = m$) and so they do not change the cokernel. For example if $|J| = m-1$ and if $(a_l, x_l) \in E \smallsetminus J$ then $|I| = m$ for $I = J \sqcup \{(a_l, x_l)\}$ and we have the relation $F\big(\sigma^{x_{E\smallsetminus J}}_{a_{E\smallsetminus J}}\big) = F\big(\sigma^{x_{E\smallsetminus I}}_{a_{E\smallsetminus I}}\big) \circ F\big(\sigma^{x_{I\smallsetminus J}}_{a_{I\smallsetminus J}}\big)$, which shows that the image of the map $F\big(\sigma^{x_{E\smallsetminus J}}_{a_{E\smallsetminus J}}\big)$ is included in the image of $F\big(\sigma^{x_{E\smallsetminus I}}_{a_{E\smallsetminus I}}\big)$, with $|I| = m$.

We can finally prove that the composition $\mathbf{FI}_d \to \mathbf{R\text{-}Mod} \to \mathbf{R\text{-}Mod}$ of two polynomial functors is polynomial.

Proposition 5.4.18. *For $m, n \in \mathbb{N}$, if $F : \mathbf{FI}_d \to \mathbf{R\text{-}Mod}$ is strong polynomial of degree less than or equal to m and if $X : \mathbf{R\text{-}Mod} \to \mathbf{R\text{-}Mod}$ preserves epimorphisms and is polynomial of degree less than or equal to n (Definition 5.4.1), then the composite $X \circ F : \mathbf{FI}_d \to \mathbf{R\text{-}Mod} \to \mathbf{R\text{-}Mod}$ is strong polynomial of degree less than or equal to nm.*

Proof.
- If $n \neq 0$ and $m \neq 0$: We pose $k = nm + 1$ and we take E a set of k objects $(a_1, x_1), \dots, (a_k, x_k)$ of $(0 \downarrow \mathbf{FI}_d)$. Since $n \neq 0$ we have $k = nm + 1 \geq m$ so we can apply Lemma 5.4.15 to E and $F \in \mathrm{Pol}_m^{strong}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$. Together with Remark 5.4.17 it implies that the morphism

$$\overline{\varphi}_E = \left(\bigoplus_{J \in \overline{\mathcal{P}}_m(E)} F\Big(\sum_{j \in J} a_j\Big) \xrightarrow{\bigoplus_{J \in \overline{\mathcal{P}}_m(E)} F\left(\sigma_{a_{E \setminus J}}^{x_{E \setminus J}}\right)} F\Big(\sum_{j \in E} a_j\Big) \right)$$

is an epimorphism, where $\overline{\mathcal{P}}_m(E)$ is the set of the subsets of E with cardinality less than or equal to m. Since X preserves the epimorphisms we get that $X(\overline{\varphi}_E)$ is an epimorphism. Similarly, since $m \neq 0$ we get that $|\overline{\mathcal{P}}_m(E)| = \sum_{i=0}^{m} \binom{nm+1}{i} \geq n$, and so we can apply the proposition 3.5 of [DV19], which is a version of Lemma 5.4.15 for functors over a symmetric monoidal category with an initial object such as $\mathbf{R\text{-}Mod}$, to X and $E' = \overline{\mathcal{P}}_m(E)$. With the same argument than in Remark 5.4.17 it implies that the morphism

$$\psi = \left(\bigoplus_{J \in \overline{\mathcal{P}}_n\left(\overline{\mathcal{P}}_m(E)\right)} X\Big(\bigoplus_{I \in J} F\Big(\sum_{i \in I} a_i\Big)\Big) \xrightarrow{\bigoplus_{J \in \overline{\mathcal{P}}_n\left(\overline{\mathcal{P}}_m(E)\right)} X\left(\sigma_{a_{\overline{\mathcal{P}}_m(E) \setminus J}}\right)} X\left(\bigoplus_{I \in \overline{\mathcal{P}}_m(E)} F\Big(\sum_{i \in I} a_i\Big)\right) \right)$$

is an epimorphism, where $a_I = F\big(\sum_{i \in I} a_i\big)$. We then consider the following diagram

$$\begin{tikzcd}
\bigoplus_{J \in \overline{\mathcal{P}}_n\left(\overline{\mathcal{P}}_m(E)\right)} X\Big(\bigoplus_{I \in J} F\Big(\sum_{i \in I} a_i\Big)\Big) \arrow[r, "X(\overline{\varphi}_E) \circ \psi"] \arrow[d] & X \circ F\Big(\sum_{i \in E} a_i\Big) \\
\bigoplus_{J \in \overline{\mathcal{P}}_n\left(\overline{\mathcal{P}}_m(E)\right)} X \circ F\Big(\sum_{i \in \sqcup I, I \in J} a_i\Big) \arrow[r] & \bigoplus_{K \in \overline{\mathcal{P}}_{nm}(E)} X \circ F\Big(\sum_{i \in K} a_i\Big) \arrow[u, "\bigoplus_{K \in \overline{\mathcal{P}}_{nm}(E)} X \circ F\left(\sigma_{a_{E \setminus K}}^{x_{E \setminus K}}\right)"']
\end{tikzcd}$$

It commutes since the maps are made with the morphisms σ_a^x and the identities. Then, $X(\overline{\varphi}_E) \circ \psi$ is an epimorphism, and by composition it implies that the right map is an epimorphism. Using the same argument as in Remark 5.4.17, we see that the cokernel of this right map is equal to

$$\mathrm{Coker}\left(\bigoplus_{i=1}^{k} X \circ F\Big(\sum_{j \neq i} a_j\Big) \xrightarrow{\bigoplus_{i=1}^{k} X \circ F\left(\sigma_{a_i}^{x_i}\right)} X \circ F\Big(\sum_{j \in E} a_j\Big) \right),$$

which is $\mathrm{cr}_k(X \circ F)\big((a_1,x_1),\dots,(a_k,x_k)\big)$ by Proposition 5.4.9. We conclude that $\mathrm{cr}_k(X \circ F)\big((a_1,x_1),\dots,(a_k,x_k)\big)$ is zero for all objects $(a_1,x_1),\dots,(a_k,x_k)$ of $(0 \downarrow \mathbf{FI}_d)$, so $\mathrm{cr}_k(X \circ F)\big(-,\dots,-\big)$ is the zero functor. Finally, by Proposition 5.4.12, the functor $X \circ F$ is in $\mathrm{Pol}^{strong}_{nm}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ since $k = nm + 1$.

- If $n = 0$: By the proposition 2.9 of [DV19], if $X : \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$ is polynomial of degree 0, then it is a quotient of a constant functor $B : \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$. Since the precomposition F^* by F is an exact functor, we get that $X \circ F = F^*(X)$ is a quotient of a constant functor $B \circ F = F^*(B)$. Then $B \circ F$ is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and, since it is closed under quotient (Proposition 5.1.3), we get that $X \circ F$ is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

- If $m = 0$: By definition, $F \in \mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ implies that $\delta_1^c(F) = \mathrm{Coker}\, F(\mathrm{Id}_{(-)} + c) = 0$ for all $c \in C$. Then $F(\mathrm{Id}_{(-)} + c)$ is an epimorphism and, since X preserves epimorphisms, we get that $X \circ F(\mathrm{Id}_{(-)} + c) = \delta_1^c(X \circ F)$ is an epimorphism. This implies that $\delta_1^c(X \circ F) = 0$ for all $c \in C$, and so $X \circ F$ is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

□

Remark 5.4.19. The constructions of the cross effects and of the strong polynomial functors were presented for functor $F : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ but it can be extended to the case of functors $F : \mathbf{FI}_d \to \mathcal{A}$ for $\mathcal{A}$ any abelian category. Then the previous result can be extended to $F : \mathbf{FI}_d \to \mathcal{A}$ and $X : \mathcal{A} \to \mathcal{B}$, for $\mathcal{A}$ and $\mathcal{B}$ two abelian categories. Moreover, for $d = 1$, we recover the proposition 3.12 from [DV19] which gives this result for functors over $\mathbf{FI}$.

Remark 5.4.20. The result of Proposition 5.4.18 is generally false if we consider a functor $X : \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$ which does not preserve epimorphisms. We gives a counterexample for $\mathbf{R} = \mathbb{Z}$ that is adapted from [DV19, Remark 3.13]. Let $F_r : \mathbf{FI}_d \to \mathbf{Ab}$ be given on objects by $F_r(n) = \mathbb{Z}$ if $n < r$ and $F_r(n) = \mathbb{Z}/2\mathbb{Z}$ if $n \geq r$, and on morphisms by the identity of $\mathbb{Z}$ or the canonical epimorphism between $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. Then F_r is strong polynomial of degree 0 since $\delta_1^c(F_r) = 0$ for all colours $c \in C$, and $X := \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, -) : \mathbf{Ab} \to \mathbf{Ab}$ is a polynomial functor of degree 1 since it is additive. However, the composition $X \circ F_r$ is the functor $(\mathbb{Z}/2\mathbb{Z})_{\geq r}$ from Example 5.1.7 and so it is strong polynomial of degree r.

5.5 The pointwise tensor product

In this section we present and study the properties of the pointwise tensor product of $\mathbf{FI}_d$-modules. In particular, we show that it preserves strong polynomiality using Proposition 5.4.18. We will see in Section 7.3 that this tensor product pass to the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. We will then extend this result to the weak polynomial degree, when $\mathbf{R}$ is a field, with a more precise bound on the degree by using a simpler argument which require the stability by subobject. We start by defining the pointwise tensor product of two $\mathbf{FI}_d$-modules.

Definition 5.5.1. For F and G two $\mathbf{FI}_d$-modules, their *pointwise tensor product* $F \otimes G \in \mathbf{FI}_d\text{-}\mathbf{Mod}$ is given on an object n by $(F \otimes G)(n) = F(n) \otimes G(n)$ and on a morphism $(f,g) \in \mathbf{FI}_d(n,m)$ by $(F \otimes G)(f,g) = F(f,g) \otimes G(f,g)$.

Remark 5.5.2. This notion of pointwise tensor product should not be confused with the following construction of Sam and Snowden in [SS17]. They define a $\mathbf{FI}_d$-module from d $\mathbf{FI}$-modules $M_1,\dots,M_d$: the $\mathbf{FI}_d$-module $N = M_1 \otimes \cdots \otimes M_d$ is defined on a set S by $N(S) = \bigoplus M_1(S_1) \otimes \cdots \otimes M_d(S_d)$, where the sum is on the decompositions $S = S_1 \sqcup \cdots \sqcup S_d$, and on a morphism $(f,g) \in \mathbf{FI}_d(S,T)$ by the direct sum of the maps

$$M_1(S_1) \otimes \cdots \otimes M_d(S_d) \xrightarrow{\bigotimes_{i=1}^{d} \left(S_i \hookrightarrow S_i \sqcup g^{-1}(c_i)\right)} M_1\big(S_1 \sqcup g^{-1}(c_1)\big) \otimes \cdots \otimes M_d\big(S_d \sqcup g^{-1}(c_d)\big).$$

First we prove that the tensor product of vector spaces is a polynomial functor of degree 2 in the classical sense.

Lemma 5.5.3. *The tensor product* $-\otimes- : \mathbf{R}\text{-}\mathbf{Mod}\times\mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$ *is a polynomial functor of degree 2 in the sense of Definition 5.4.1.*

Proof. Since $\mathbf{R}\text{-}\mathbf{Mod}\times\mathbf{R}\text{-}\mathbf{Mod}$ is an abelian category it is in particular monoidal symmetric category with a null object so it falls in the framework of [DV19]. Then it is enough to prove that $\delta_{X_1}\circ\delta_{X_2}\circ\delta_{X_3}(-\otimes-)$ is the zero functor, while $\delta_{X_1}\circ\delta_{X_2}(-\otimes-)$ is not zero, for all objects $X_1=(M_1,N_1)$, $X_2=(M_2,N_2)$, $X_3=(M_3,N_3)$ of $\mathbf{R}\text{-}\mathbf{Mod}\times\mathbf{R}\text{-}\mathbf{Mod}$. For $U,V\in\mathbf{R}\text{-}\mathbf{Mod}$, the space $\delta_{(M_1,N_1)}(-\otimes-)(U,V)$ is the cokernel of the map $i_{(M_1,N_1)}$ given by

$$\begin{array}{ccc} (-\otimes-)(U,V) & \to & \tau_{(M_1,N_1)}(-\otimes-)(U,V) \\ U\otimes V & \mapsto & (U\oplus M_1)\otimes(V\oplus N_1) \end{array}.$$

By the definition of $i_{(M_1,N_1)}$ in [DV19] the module $U\otimes V$ is sent to $U\otimes V$, and so $\delta_{(M_1,N_1)}(-\otimes-)(U,V)$ is equal to $(U\otimes N_1)\oplus(M_1\otimes V)\oplus(M_1\otimes N_1)$. This decomposition is natural since the map $i_{(M_1,N_1)}$ is natural. Now the space $\delta_{(M_2,N_2)}\circ\delta_{(M_1,N_1)}(-\otimes-)(U,V)$ is the cokernel of the map

$$\begin{array}{ccc} \delta_{(M_1,N_1)}(-\otimes-)(U,V) & \to & \tau_{(M_2,N_2)}\big(\delta_{(M_1,N_1)}(-\otimes-)\big)(U,V) \\ (U\otimes N_1)\oplus(M_1\otimes V)\oplus(M_1\otimes N_1) & \mapsto & \big((U\oplus M_2)\otimes N_1\big)\oplus\big(M_1\otimes(V\oplus N_2)\big)\oplus(M_1\otimes N_1). \end{array}$$

As above, this implies that

$$\delta_{(M_2,N_2)}\circ\delta_{(M_1,N_1)}(-\otimes-)(U,V)=(M_2\otimes N_1)\oplus(M_1\otimes N_2).$$

Again, this is natural and it proves that $\delta_{(M_2,N_2)}\circ\delta_{(M_1,N_1)}(-\otimes-)$ is the constant functor equals to $(M_2\otimes N_1)\oplus(M_1\otimes N_2)$, so it is not zero. Finally, we have that $\delta_{(M_3,N_3)}\circ\delta_{(M_2,N_2)}\circ\delta_{(M_1,N_1)}(-\otimes-)$ is zero since it is given on the object (U,V) by the cokernel of the map

$$\begin{array}{ccc} \delta_{(M_2,N_2)}\circ\delta_{(M_1,N_1)}(-\otimes-)(U,V) & \to & \tau_{(M_3,N_3)}\big(\delta_{(M_2,N_2)}\circ\delta_{(M_1,N_1)}(-\otimes-)\big)(U,V) \\ (M_2\otimes N_1)\oplus(M_1\otimes N_2) & \mapsto & (M_2\otimes N_1)\oplus(M_1\otimes N_2). \end{array}$$

□

We can now prove that the pointwise tensor product respects strong polynomiality by using Proposition 5.4.18.

Theorem 5.5.4. *For* $n,m\in\mathbb{N}$ *and* $F,G:\mathbf{FI}_d\to\mathbf{R}\text{-}\mathbf{Mod}$*, if* F *is in* $\mathrm{Pol}_n^{strong}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod})$ *and if* G *is in* $\mathrm{Pol}_m^{strong}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod})$*, then their tensor product* $F\otimes G$ *is in* $\mathrm{Pol}_{2\max(n,m)}^{strong}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod})$*.*

Proof. We consider the functor (F,G) in $\mathbf{Fct}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod}\times\mathbf{R}\text{-}\mathbf{Mod})$ and we show by induction that (F,G) is polynomial of degree less than or equal to $\max(n,m)$. By hypothesis $F\in\mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod})$, and so we have $\delta_1^c(F)\in\mathrm{Pol}_{n-1}^{\mathrm{strong}}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod})$ for all $c\in C$, and we have the same for G. This implies that $\delta_1^c(F,G)=\big(\delta_1^c(F),\delta_1^c(G)\big)$ is polynomial of degree less than or equal to $\max(n,m)-1=\max(n-1,m-1)$. By induction we get $\delta_1^c(F,G)\in\mathrm{Pol}_{\max(n,m)-1}^{\mathrm{strong}}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod}\times\mathbf{R}\text{-}\mathbf{Mod})$ for all $c\in C$, so $(F,G)\in\mathrm{Pol}_n^{\mathrm{strong}}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod})$. Moreover, we showed Lemma 5.5.3 that $-\otimes-$ is polynomial of degree 2, and the functor $-\otimes-$ preserves epimorphisms since an epimorphism in $\mathbf{R}\text{-}\mathbf{Mod}\times\mathbf{R}\text{-}\mathbf{Mod}$ is a couple (f,g) of epimorphism in $\mathbf{R}\text{-}\mathbf{Mod}$ and then $f\otimes g$ is also an epimorphism. Since $\mathbf{R}\text{-}\mathbf{Mod}\times\mathbf{R}\text{-}\mathbf{Mod}$ is an abelian category, we then conclude by applying a generalization of Proposition 5.4.18 presented in Remark 5.4.19 to the composition

$$\mathbf{FI}_d \xrightarrow{(F,G)} \mathbf{R}\text{-}\mathbf{Mod}\times\mathbf{R}\text{-}\mathbf{Mod} \xrightarrow{-\otimes-} \mathbf{R}\text{-}\mathbf{Mod}.$$

□

Remark 5.5.5. In Appendix A we give a version of Theorem 5.5.4 for the context of symmetric monoidal category whose unit is an initial object studied in [DV19].

Remark 5.5.6. In Theorem 5.5.4, the bound may be not the best possible. Indeed, we could expect for $F \otimes G : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ to be strong polynomial of degree less than or equal to $n+m$. For example, for $d = 1$ the proposition 4.1 from [Dja16] shows that a **FI**-module is strong polynomial of degree less than or equal to n if and only if it is a quotient of a sum of the standard projective functors $P_i^{\mathbf{FI}}$ for $i \leq n$. This implies that, over **FI**, the tensor product $F \otimes G$ is polynomial of degree $n + m$ if F has degree n and G has degree m. One could try to prove a more refined version of Proposition 5.4.18 and use this refinement to get a better bound.

Chapter 6

The poset of stably zero functors

Nous blessons, meurtrissons, polluons et brûlons cette terre, nous sommes les êtres vivants les plus ignobles qui soient.

Hayao Miyazaki

The notion of strong polynomial $\mathbf{FI}_d$-modules introduced in Chapter 5 is not fully satisfactory since it lacks important properties, such as being closed under subobjects (Remark 5.1.9). To solve these problems we want to define a notion of weak polynomial functors inspired by [DV19] for **FI**-modules. In order to define them Djament and Vespa studied the subcategory of $\mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ of functors whose colimit is zero called stably zero functors (Definition 2.10 and Proposition 2.13 in [DV19]) to erase them in a quotient. We do the same for $\mathbf{FI}_d$, but we will see in this section that there are several subcategories that can replace the stably zero functors in the case of $\mathbf{FI}_d$-modules. We first introduce the notion of globally stably zero functors, then we define notions of functors that are stably zero along colours. We end the section by explaining how these notions interact with each other in Section 6.3 and how they interact with the theory of twisted commutative algebras from Chapter 4. In particular, we show that each of these subcategories is thick so we can take the quotient of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by any of them and define a notion of weak polynomial functors for each of these quotients. However, we only develop in Chapter 7 the weak polynomial functors corresponding to the global subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ since it behaves better with the endofunctors δ_k^x that allow us to define polynomial functors.

6.1 The subcategory of globally stably zero functors

We start with the study of the biggest of these subcategories which we will use in Chapter 7 to define a notion of weak polynomial functors in the corresponding quotient of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. It is the subcategory of globally stably zero functors denoted by $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. We will present the other subcategories of stably zero functors, along colours, in the second section.

Definition 6.1.1. The *category* $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is the full subcategory of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ whose objects are the globally stably zero functors, *i.e.* the functors $F : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ such that $\kappa(F) = F$.

Proposition 6.1.2. *Let* $F : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ *be a functor, then* F *is in the subcategory* $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ *if and only if, for every object* $n \in \mathbf{FI}_d$ *and every element* a *in* $F(n)$ *there exist* $k \in \mathbf{FI}_d$ *and* $x \in \mathbf{FI}_d(0, k)$ *such that* $a \in \kappa_k^x(F)(n)$.

Proof. Suppose first that $\kappa(F) = F$, then for all $n \in \mathbf{FI}_d$ and all $a \in F(n)$ we have

$$a \in \kappa(F)(n) = \sum_{k \in \mathbf{FI}_d} \sum_{x \in \mathbf{FI}_d(0,k)} \kappa_k^x(F)(n).$$

By Proposition II.1.14.8) the family of subobjects $(\kappa_k^x(F))$ of F is filtered so there exist $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$ such that $a \in \kappa_k^x(F)(n)$. Conversely, let n be an object of $\mathbf{FI}_d$, if for all $a \in F(n)$ there exist $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$ such that $a \in \kappa_k^x(F)(n)$, then the inclusion $\kappa_k^x(F)(n) \subset \kappa(F)(n)$ implies the inclusion $F(n) \subset \kappa(F)(n)$. Since this is true for all objects $n \in \mathbf{FI}_d$, and since $\kappa(F)$ is a subfunctor of F, this implies the identity $\kappa(F) = F$. □

Remark 6.1.3. Morally, the Proposition 6.1.2 means that a $\mathbf{FI}_d$-module F is globally stably zero if for each $n \in \mathbb{N}$ every element $a \in F(n)$ is sent to zero by some map, of the form $\mathrm{Id}_n + x$ for $x \in \mathbf{FI}_d(0,k)$.

We now give an alternative description of the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ using filtered colimits, which will allow us to prove later that $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ is thick and stable by the endofunctors δ_k^x. To do this, we define a poset structure on $\mathbb{N}^d$ using the product order, which means that for $(n_1, \dots, n_d), (m_1, \dots, m_d) \in \mathbb{N}^d$ we have $(n_1, \dots, n_d) \leq (m_1, \dots, m_d)$ if $n_i \leq m_i$ for all $1 \leq i \leq d$. We then consider the category $\mathbb{N}^d$ associated with this poset which is therefore a filtered category. We now define a functor $\xi_d : \mathbb{N}^d \to \mathbf{FI}_d$ where the i-th component of $\mathbb{N}^d$ corresponds to the i-th colour of $\mathbf{FI}_d$.

Definition 6.1.4. The *functor* $\xi_d : \mathbb{N}^d \to \mathbf{FI}_d$ sends an object $(n_1, \dots, n_d) \in \mathbb{N}^d$ to the object $n_1 + \dots + n_d$ of $\mathbf{FI}_d$ and a morphism $(n_1, \dots, n_d) \leq (m_1, \dots, m_d)$ in $\mathbb{N}^d$ to the morphism

$$\big(\mathrm{Id}_{n_1}, (m_1 \smallsetminus n_1 \to \{c_1\})\big) + \dots + \big(\mathrm{Id}_{n_d}, (m_d \smallsetminus n_d \to \{c_d\})\big)$$

in $\mathbf{FI}_d(n_1 + \dots + n_d\,, m_1 + \dots + m_d)$, which can also be written as

$$\Big(\mathrm{Id}_{n_1+\dots+n_d}\,, \big(\, m_1 \smallsetminus n_1 \xrightarrow{c_1^{m_1-n_1}} C\,\big) + \dots + \big(\, m_d \smallsetminus n_d \xrightarrow{c_d^{m_d-n_d}} C\,\big)\Big).$$

Then the following proposition gives a characterization of the stably zero functors as a colimit using the pre-composition by ξ_d.

Proposition 6.1.5. *Let F be a $\mathbf{FI}_d$-module, then $F \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ if and only if*

$$\operatorname*{colim}_{\mathbb{N}^d} F \circ \xi_d = 0.$$

Remark 6.1.6. For $d = 1$ we recover the propositions 2.13 and 2.14 of [DV19] for the subcategory $\mathcal{SN}(\mathbf{FI}, \mathbf{R\text{-}Mod})$ of $\mathbf{FI}$-modules since the functor $\xi_1 : \mathbb{N} \to \mathbf{FI}$ is the functor ζ from [DV19].

Proof of Proposition 6.1.5. Since the category $\mathbb{N}^d$ is filtered, the colimit of $F \circ \xi_d : \mathbb{N}^d \to \mathbf{R\text{-}Mod}$ is a filtered colimit. Then by Proposition 1.1.6 its elements can be written as the equivalence class of all objects $a \in F \circ \xi_d(n_1, \dots, n_d)$ quotiented by the following equivalence relation: two objects $a \in F \circ \xi_d(n_1, \dots, n_d)$ and $a' \in F \circ \xi_d(n_1', \dots, n_d')$ are equivalent if there exists $(n_1'', \dots, n_d'') \in \mathbb{N}^d$, and two maps $f : (n_1, \dots, n_d) \leq (n_1'', \dots, n_d'')$ and $g : (n_1', \dots, n_d') \leq (n_1'', \dots, n_d'')$ in $\mathbb{N}^d$ such that $F \circ \xi_d(f)(a) = F \circ \xi_d(g)(a')$. In particular, the class of an element $a \in F \circ \xi_d(n_1', \dots, n_d')$ is zero if and only if there exists an object $(m_1, \dots, m_d) \in \mathbb{N}^d$ and a map $f : (n_1, \dots, n_d) \leq (m_1, \dots, m_d)$ such that $F \circ \xi_d(f)(a) = 0$. Now we can prove the equivalence. If $\operatorname{colim} F \circ \xi_d = 0$, for $n \in \mathbf{FI}_d$ and $(n_1, \dots, n_d)$ in $\mathbb{N}^d$ such that $\xi_d(n_1, \dots, n_d) = n_1 + \dots + n_d = n$, then for every element $a \in F(n)$ we can consider the class of $a \in F \circ \xi_d(n_1, \dots, n_d)$ in the colimit of $F \circ \xi_d$. Since this colimit is zero, the class of a is zero which means that there exists an object $(m_1, \dots, m_d) \in \mathbb{N}^d$ and a map

$f : (n_1, \dots, n_d) \le (m_1, \dots, m_d)$ in $\mathbb{N}^d$ such that $F \circ \xi_d(f)(a) = 0$. We now pose $m = m_1 + \dots + m_d$, as well as $k_i = m_i - n_i$ for $1 \le i \le d$ and $k = k_1 + \dots + k_d$. For $x = (k_1 \to \{c_1\}) + \dots + (k_d \to \{c_d\}) \in \mathbf{FI}_d(0,k)$ we can rewrite

$$F \circ \xi_d(f) = F\Big(\big(\mathrm{Id}_{n_1},\, k_1 \to \{c_1\} \big) + \dots + \big(\mathrm{Id}_{n_d},\, k_d \to \{c_d\} \big) \Big) = F(\mathrm{Id}_n + x).$$

Then for all $a \in F(n)$ there exist $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$ such that $F(\mathrm{Id}_n + x)(a) = 0$, so $a \in \kappa_k^x(F)(n) \subset \kappa(F)(n)$. This gives the inclusion $F(n) \subset \kappa(F)(n)$ for all objects $n \in \mathbf{FI}_d$, and since $\kappa(F)$ is a subfunctor of F, this implies the identity $\kappa(F) = F$.

If $F \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ then $\kappa(F) = F$. Let $a \in F \circ \xi_d(n_1, \dots, n_d)$ be a representative of a class $[a \in F \circ \xi_d(n_1, \dots, n_d)]$ in the colimit of $F \circ \xi_d$. For $n = n_1 + \dots + n_d$, by Lemma 6.1.2 there exist $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$ such that $a \in \kappa_k^x(F)(n)$. We then denote by k_i the number of occurrences of c_i in x for $1 \le i \le d$, and by f the map $(n_1, \dots, n_d) \le (n_1 + k_1, \dots, n_d + k_d)$ in $\mathbb{N}^d$. We then have $k_1 + \dots + k_d = k$ and

$$\xi_d(f) = \Big(\mathrm{Id}_{n_1 + \dots + n_d},\, \big(k_1 \longrightarrow \{c_1\} \big) + \dots + \big(k_d \longrightarrow \{c_d\} \big) \Big) \in \mathbf{FI}_d(n, n+k).$$

Then there exits a permutation $\sigma \in S_k$, rearranging the colours as in x, such that $(\mathrm{Id}_n + \sigma) \circ \xi_d(f) = \mathrm{Id}_n + x$. Since $a \in \kappa_k^x(F)(n)$, we have $0 = F(\mathrm{Id}_n + x)(a) = F(\mathrm{Id}_n + \sigma) \circ (F \circ \xi_d(f))(a)$, and so $F \circ \xi_d(f)(a) = 0$ since the map $\mathrm{Id}_n + \sigma$ is bijective. This means that the class $[a \in F \circ \xi_d(n_1, \dots, n_d)]$ in the colimit of $F \circ \xi_d$ is zero. □

Using this description of $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ in terms of a filtered colimit we now state important properties of the stably zero functors.

Proposition 6.1.7. *For F a $\mathbf{FI}_d$-module, $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$, we have*

1) *The subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is thick and closed under colimits.*
2) *The subfunctor $\kappa_k^x(F)$ of F is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.*
3) *The functor $\kappa(F)$ is the biggest subfunctor of F in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.*
4) *The subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is stable by the endofunctor δ_k^x.*

Proof. 1) For $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ a short exact sequence of $\mathbf{FI}_d$-modules, we get another short exact sequence $0 \longrightarrow F \circ \xi_d \longrightarrow G \circ \xi_d \longrightarrow H \circ \xi_d \longrightarrow 0$ in $\mathbf{Fct}(\mathbb{N}^d, \mathbf{R}\text{-}\mathbf{Mod})$ by precomposition with the functor ξ_d. Since $\mathbf{R}\text{-}\mathbf{Mod}$ is a Grothendieck category (Definition 1.3.1) so is $\mathbf{Fct}(\mathbb{N}^d, \mathbf{R}\text{-}\mathbf{Mod})$, which implies that the filtered colimits are exact. By definition $\mathbb{N}^d$ is a filtered category so we get a short exact sequence

$$0 \longrightarrow \operatorname{colim} F \circ \xi_d \longrightarrow \operatorname{colim} G \circ \xi_d \longrightarrow \operatorname{colim} H \circ \xi_d \longrightarrow 0\ .$$

Then, by Proposition 6.1.5, G is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ if and only if $\operatorname{colim} G \circ \xi_d = 0$. This is then equivalent to having $\operatorname{colim} F \circ \xi_d = 0$ and $\operatorname{colim} H \circ \xi_d = 0$, which means that F and H are in the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by Proposition 6.1.5 again. Finally, $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is closed under colimits by Proposition 6.1.5 since colimits commute together.

2) By Proposition 2.6.6 we have an isomorphism $\kappa_k^x \circ \kappa_k^x \cong \kappa_k^x$, so by definition we get

$$\kappa\big(\kappa_k^x(F)\big) = \sum_{l \in \mathbf{FI}_d} \sum_{y \in \mathbf{FI}_d(0,l)} \kappa_l^y \circ \kappa_k^x(F) = \kappa_k^x \circ \kappa_k^x(F) + \sum_{l \in \mathbf{FI}_d} \sum_{y \in \mathbf{FI}_d(0,l), y \ne x} \kappa_l^y \circ \kappa_k^x(F)$$

$$\cong \kappa_k^x(F) + \sum_{l \in \mathbf{FI}_d} \sum_{y \in \mathbf{FI}_d(0,l), y \ne x} \kappa_l^y \circ \kappa_k^x(F).$$

Since $\kappa(\kappa_k^x(F))$ is a subfunctor of $\kappa_k^x(F)$, this shows that $\kappa(\kappa_k^x(F)) = \kappa_k^x(F)$, and so $\kappa_k^x(F)$ is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

3) By definition, $\kappa_k^x(F)$ is a subfunctor of F for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$, so their sum $\kappa(F)$ is also a subfunctor of F by minimality of the sum. By the point 2), each of the $\kappa_k^x(F)$ is in the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ which is closed under colimits by the point 1). This implies that $\kappa(F)$ is also in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. It remains to check that $\kappa(F)$ is the biggest subfunctor of F within $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Let $G \subset F$ be another subfunctor such that $G = \kappa(G)$ and let j denote the inclusion $G \hookrightarrow F$. Then we have a short exact sequence $0 \longrightarrow G \longrightarrow F \longrightarrow \mathrm{Coker}(j) \longrightarrow 0$. By Proposition 2.6.6 the endofunctor κ is left exact, so we get a monomorphism from $G = \kappa(G)$ to $F = \kappa(F)$.

4) Let F be a functor in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, by Proposition 6.1.5 it implies that $\operatorname{colim} F \circ \xi_d = 0$. By Proposition 2.6.6 the endofunctor δ_k^x commutes with colimits so we get that

$$\operatorname{colim} \delta_k^x(F) \circ \xi_d = \delta_k^x(\operatorname{colim} F \circ \xi_d) = \delta_k^x(0) = 0.$$

Then $\delta_k^x(F)$ is in the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by Proposition 6.1.5 again. □

Remark 6.1.8. The point 3) in Proposition 6.1.7 implies in particular that the endofunctor κ is an adjoint of the inclusion functor of $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ in $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

Remark 6.1.9. For $d = 1$, the stably zero functors in $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ correspond exactly to the torsion modules over the free TCA $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ from Definition 4.1.15. Then the endofunctor κ, which gives the maximal subfunctor of a $\mathbf{FI}$-module in $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$, corresponds to the local cohomology functor denoted by $\mathrm{H}^0_{\mathfrak{m}}(-)$ in [SS16] and [NSS18]. In particular, they studied the properties of its right derived functors $\mathrm{H}^i_{\mathfrak{m}}(-)$ in order to understand how $\mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ is constructed from the two pieces $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ and $\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$.

We end this section with some technical results about the $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$-closed objects defined below, which will be used in Chapter 7.

Definition 6.1.10 (Special case of Definition 1.3.11)**.** A $\mathbf{FI}_d$-module F is $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$-closed if, for all $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, both $\mathrm{Hom}(H,F)$ and $\mathrm{Ext}^1(H,F)$ are zero.

Remark 6.1.11. The $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$-closed modules are called saturated from the point of view of TCAs in [SS16] for $d = 1$, and in [SS19, Proposition 4.1] for a general d. In [SS19], the saturation functors denoted by $\Sigma_{>r}$ correspond to the composition $\mathcal{S} \circ \pi$ for the quotient category by $\mathrm{Mod}_{A,\leq r}$. It is shown in [NSS18, Proposition 2.7] that the right derived functors of these functors preserve finitely generated modules and vanish after some rank.

Proposition 6.1.12. *For F a $\mathbf{FI}_d$-module we have*

1) *The subfunctor $\kappa(F)$ is zero if and only if the set of natural transformations* $\mathrm{Hom}(H,F)$ *is reduced to* 0 *for all $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.*

2) *If* $\mathrm{Hom}(H,F) = 0$ *for all $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, then* $\mathrm{Hom}(H, \tau_k(F)) = 0$ *for all $k \in \mathbb{N}$ and all $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.*

3) *If F is $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$-closed, then* $\mathrm{Hom}(H, \delta_k^x(F)) = 0$ *for all $k \in \mathbf{FI}_d$, all $x \in \mathbf{FI}_d(0,k)$ and all $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$*

Proof. 1) If for some H in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ there exists a non-zero natural transformation σ in $\mathrm{Hom}(H, F)$, then its image $\sigma \circ H$ is a non-zero subfunctor of F. We then have the short exact sequence $0 \longrightarrow \mathrm{Ker}(\sigma) \longrightarrow H \xrightarrow{\sigma} \sigma \circ H \longrightarrow 0$ of $\mathbf{FI}_d$-modules. Applying the left exact functor κ to it we get another exact sequence:

$$0 \longrightarrow \kappa\big(\mathrm{Ker}(\sigma)\big) \longrightarrow \kappa(H) \longrightarrow \kappa\big(\sigma \circ H\big)\ .$$

Since H is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ we have $\kappa(H) = H$ and since it is closed under subobjects (Proposition 6.1.7.1) we get that $\mathrm{Ker}(\sigma)$ is also in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Then we have $\kappa(\mathrm{Ker}(\sigma)) = \mathrm{Ker}(\sigma)$ and we can make the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \kappa\big(\mathrm{Ker}(\sigma)\big) \arrow[r] \arrow[d, equal] & \kappa(H) \arrow[r] \arrow[d, equal] & \kappa\big(\sigma \circ H\big) \arrow[d, hook] & \\
0 \arrow[r] & \mathrm{Ker}(\sigma) \arrow[r] & H \arrow[r, "\sigma"] & \sigma \circ H \arrow[r] & 0.
\end{tikzcd}$$

By a careful application of the five lemma we see that the monomorphism $\kappa(\sigma \circ H) \hookrightarrow \sigma \circ H$ is also an epimorphism, so an isomorphism. This means that the image $\sigma \circ H$ of σ is a non-zero subfunctor of F inside $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and by Proposition 6.1.7.3) it implies that $\kappa(F)$ is non-zero. Conversely, by Proposition 6.1.7, the functor $\kappa_k^x(F)$ is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0, k)$. Since this subcategory is closed under colimits by Proposition 6.1.7 it implies that $\kappa(F)$ is also in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Then the inclusion j of $\kappa(F)$ in F is in $\mathrm{Hom}(\kappa(F), F)$ and we showed that $\kappa(F)$ is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Then, by hypothesis we get j is zero and so $\kappa(F) = 0$.

2) By Proposition 2.6.6 the endofunctor τ_k commutes with colimits and with κ_l^y for all $l \in \mathbf{FI}_d$ and all $y \in \mathbf{FI}_d(0, l)$, so it commutes with κ. However, according to the previous point, the hypothesis is equivalent to $\kappa(F) = 0$. We then deduce that $\kappa(\,\tau_k(F)\,) = \tau_k(\,\kappa(F)\,) = 0$ and the result follows by the previous point.

3) In this case, the exact sequence (I) from Lemma 2.6.4 becomes short by hypothesis since the first arrow is in $\mathrm{Hom}(\kappa_k^x(F), F)$, with $\kappa_k^x(F)$ in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. We then get the short exact sequence $0 \longrightarrow F \longrightarrow \tau_k(F) \longrightarrow \delta_k^x(F) \longrightarrow 0$, and for $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ there is a long exact sequence associated with it and with the functor $\mathrm{Hom}(H, -)$:

$$0 \longrightarrow \mathrm{Hom}(H, F) \longrightarrow \mathrm{Hom}(H, \tau_k(F)) \longrightarrow \mathrm{Hom}(H, \delta_k^x(F)) \longrightarrow \mathrm{Ext}^1(H, F) \longrightarrow \dots\ .$$

The first and the fourth terms are zero by hypothesis so, for all $x \in \mathbf{FI}_d(0, k)$, there is an isomorphism $\mathrm{Hom}(H, \tau_k(F)) \simeq \mathrm{Hom}(H, \delta_k^x(F))$. Then, the result is just a consequence of the last point. □

Finally, the precomposition by the colouring functors from Definition 2.7.2 does not preserve the stably zero functors as we will explain in Remark 6.3.3. However, the following proposition explains that it preserves the $\mathcal{SN}$-closed functors.

Proposition 6.1.13. *For $c \in C$ and F a $\mathbf{FI}_d$-module, if F is $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$-closed, then the functor $\Delta_c^*(F)$ is $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$-closed.*

Proof. Let $\eta : \mathrm{Id} \to \mathcal{S}_1 \circ \pi_1$ be the unit of the adjunction of π_1 and $\mathcal{S}_1$. By Proposition 1.3.12 it is enough to prove that the morphism $\eta_{\Delta_c^*(F)} : \Delta_c^*(F) \longrightarrow \mathcal{S}_1 \circ \pi_1 \circ \Delta_c^*(F)$ is an isomorphism. By hypothesis, the set $\mathrm{Hom}(H, F)$ is reduced to zero for all $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and by

Proposition 6.1.12 we have $\kappa(F)=0$. In particular, it gives $\kappa_k^{c^k}(F)=0$ for all $k\in\mathbf{FI}_d$ and by Proposition 2.7.4 we have $\kappa_k\circ\Delta_c^*(F)\cong\Delta_c^*\circ\kappa_k^{c^k}(F)=\Delta_c^*\circ 0=0$. Then by taking the sum on $k\in\mathbf{FI}$ we get $\kappa(\,\Delta_c^*(F)\,)=0$ and finally using 6.1.12 for $\mathbf{FI}=\mathbf{FI}_1$ we have

$$\mathrm{Hom}_{\mathbf{Fct}(\mathbf{FI},\mathbf{R\text{-}Mod})}\left(H,\Delta_c^*(F)\,\right)=0 \text{ for all } H\in\mathcal{SN}(\mathbf{FI},\mathbf{R\text{-}Mod}).$$

But by Proposition 1.3.13, the kernel of the unit η is in $\mathcal{SN}(\mathbf{FI},\mathbf{R\text{-}Mod})$ so the inclusion of $\mathrm{Ker}(\eta_{\Delta_c^*(F)})$ in $\Delta_c^*(F)$ is zero. This shows that the morphism $\eta_{\Delta_c^*(F)}$ is a monomorphism. Now, if N denotes its cokernel we have a short exact sequence

$$0\longrightarrow\Delta_c^*(F)\xrightarrow{\eta_{\Delta_c^*(F)}}\mathcal{S}_1\circ\pi_1\circ\Delta_c^*(F)\longrightarrow N\longrightarrow 0\ .$$

By Lemma 1.3.14 the image of $\mathcal{S}_1$ is $\mathcal{SN}(\mathbf{FI},\mathbf{R\text{-}Mod})$-closed, and Proposition 6.1.12 again we have $\kappa_k\left(\mathcal{S}_1\circ\pi_1\circ\Delta_c^*(F)\right)=0$ for $k\in\mathbf{FI}$. Then the snake lemma gives an exact sequence

$$0\longrightarrow\kappa_k(N)\longrightarrow\delta_k\circ\Delta_c^*(F)\longrightarrow\delta_k\circ\mathcal{S}_1\circ\pi_1\circ\Delta_c^*(F)\longrightarrow\delta_k(N)\longrightarrow 0\ .$$

However, there is a monomorphism $\kappa_k\circ\delta_k\circ\mathcal{S}_1\circ\pi_1\circ\Delta_c^*(F)\hookrightarrow\delta_k\circ\mathcal{S}_1\circ\pi_1\circ\Delta_c^*(F)$, with $\mathcal{S}_1\circ\pi_1\circ\Delta_c^*(F)$ which is $\mathcal{SN}(\mathbf{FI},\mathbf{R\text{-}Mod})$-closed by Lemma 1.3.14. By Proposition 6.1.7 the image of κ_k is in $\mathcal{SN}(\mathbf{FI},\mathbf{R\text{-}Mod})$, so this monomorphism is zero and we have $\kappa_k\circ\delta_k\left(\mathcal{S}_1\circ\pi_1\circ\Delta_c^*(F)\right)=0$. Using this and applying the left exact functor κ_k to the previous exact sequence we get an isomorphism $\kappa_k\circ\kappa_k(N)\cong\kappa_k\circ\delta_k\circ\Delta_c^*(F)$. Using Propositions 2.6.6.4) and 2.7.4 we deduce for all $k\in\mathbf{FI}$ the identity

$$\kappa_k(N)=\kappa_k\circ\kappa_k(N)\cong\kappa_k\circ\delta_k\circ\Delta_c^*(F)\cong\Delta_c^*\circ\kappa_k^{c^k}\circ\delta_k^{c^k}(F).$$

Since F is $\mathcal{SN}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$-closed and since the image of $\kappa_k^{c^k}$ is in $\mathcal{SN}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ the inclusion of $\kappa_k^{c^k}\circ\delta_k^{c^k}(F)$ in $\delta_k^{c^k}(F)$ is the zero map by Proposition 6.1.12.3), implying that the functor $\kappa_k^{c^k}\circ\delta_k^{c^k}(F)$ is zero. We get that $\kappa_k(N)$ is zero for all $k\in\mathbf{FI}$ and, by taking the sum over $k\in\mathbf{FI}$, we get $\kappa(N)=0$. The Proposition 6.1.12.1) implies then that the set $\mathrm{Hom}(H,N)$ is reduced to zero for all $H\in\mathcal{SN}(\mathbf{FI},\mathbf{R\text{-}Mod})$. Since N is the cokernel of the unit η, by Proposition 1.3.13 it is in $\mathcal{SN}(\mathbf{FI},\mathbf{R\text{-}Mod})$ so we can deduce that the map Id_N is zero. This proves that $N=0$, so the monomorphism $\eta_{\Delta_c^*(F)}$ is also an epimorphism and so it is an isomorphism. □

6.2 Functors that are stably zero along colours

We now define the subcategories $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ of $\mathbf{Fct}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ of functors that are stably zero along colours similarly to the globally stably zero functor of the previous section. To do this we use the results already proved for functors over $\mathbf{FI}$, especially those of Djament and Vespa in [DV19], via the colouring functors $\Delta_c^*:\mathbf{Fct}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})\to\mathbf{Fct}(\mathbf{FI},\mathbf{R\text{-}Mod})$ from Definition 2.7.2. We show that each of these subcategories is thick in Corollary 6.2.5, so we can take the quotient of $\mathbf{Fct}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ by each of them and define a notion of weak polynomial functors along colours for any of these quotients.

Definition 6.2.1. Let $c_{i_1},\dots,c_{i_m}\in C$ be distinct colours, the *category* $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ is the full subcategory of $\mathbf{Fct}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})$ of the $\mathbf{FI}_d$-modules F such that, for all colours $c\in\{c_{i_1},\dots,c_{i_m}\}$, we have the identity

$$\sum_{k\in\mathbf{FI}_d}\kappa_k^{c^k}(F)=F.$$

Remark 6.2.2. Morally, it means that a $\mathbf{FI}_d$-module F is stably zero in the colours $c_{i_1}, \dots, c_{i_m} \in C$ if for each $n \in \mathbb{N}$ every element $a \in F(n)$ is sent to zero by some map of the form $\mathrm{Id}_n + c^k$ for every colour c in $\{c_{i_1}, \dots, c_{i_m}\}$.

We can use the precomposition by the colouring functors Δ_c from Definition 2.7.2 to give another description of these subcategories based on the stably zero **FI**-modules. Indeed, we show that a $\mathbf{FI}_d$-module is in the subcategory $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ if and only if the precomposition by the colouring functor is stably zero on **FI** for all the colours in $\{c_{i_1}, \dots, c_{i_m}\}$. This will allow us to use the results about stably zero **FI**-modules to obtain similar properties for stably zero $\mathbf{FI}_d$-modules.

Proposition 6.2.3. *For F a $\mathbf{FI}_d$-module and $c \in C$, the functor F is in the subcategory $\mathcal{SN}_c(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ if and only if the functor $\Delta_c^*(F)$ is in the subcategory $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$.*

Proof. By definition, the functor F is in $\mathcal{SN}_c(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ if and only if the sum on $k \in \mathbf{FI}_d$ of the functors $\kappa_k^{c^k}(F)$ is equal to F. Since every $\kappa_k^{c^k}(F)$ is a subfunctor of F, this is equivalent the equality of **R**-modules

$$\sum_{k \in \mathbf{FI}_d} \kappa_k^{c^k}(F)(n) = F(n)$$

for all $n \in \mathbf{FI}_d$. By definition of Δ_c we have $F(n) = F \circ \Delta_c(n) = \Delta_c^*(F)(n)$ and $\kappa_k^{c^k}(F)(n) = \kappa_k^{c^k}(F) \circ \Delta_c(n) = \Delta_c^* \circ \kappa_k^{c^k}(F)(n) = \kappa_k \circ \Delta_c^*(F)(n)$. This allows us to rewrite the identity as

$$\sum_{k \in \mathbf{FI}} \kappa_k \left(\Delta_c^*(F)\right)(n) = \Delta_c^*(F)(n).$$

Again, each $\kappa_k(\Delta_c^*(F))$ is a subfunctor of $\Delta_c^*(F)$, so this identity (which holds for all objects $n \in \mathbf{FI}$) is equivalent to the equality

$$\sum_{k \in \mathbf{FI}} \kappa_k \left(\Delta_c^*(F)\right) = \Delta_c^*(F),$$

which is the definition of $\Delta_c^*(F)$ being in the subcategory $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. □

Corollary 6.2.4. *For F a $\mathbf{FI}_d$-module and $c_{i_1}, \dots, c_{i_m} \in C$ distinct colours, the functor F is in the subcategory $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ if and only if the functors $\Delta_c^*(F)$ are in the subcategory $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ for all colours $c \in \{c_{i_1}, \dots, c_{i_m}\}$.*

Proof. According to the definition of the categories $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, it follows from Proposition 6.2.3 applied for $c \in \{c_{i_1}, \dots, c_{i_m}\}$. □

An important consequence of this description of the subcategories $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is that they are thick subcategories of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, which allows us to take the quotient by any of them.

Corollary 6.2.5. *For $c_{i_1}, \dots, c_{i_m} \in C$ distinct colours, the subcategory $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is thick.*

Proof. For $c \in \{c_{i_1}, \dots, c_{i_m}\}$ and $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ a short exact sequence in $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, the precomposition functor Δ_c^* being exact, we get the following short exact sequence

$$0 \longrightarrow \Delta_c^*(F) \longrightarrow \Delta_c^*(G) \longrightarrow \Delta_c^*(H) \longrightarrow 0$$

in $\mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. Since the subcategory $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ is thick (see Proposition 6.1.7 for $d = 1$, or [DV19]), the functor $\Delta_c^*(G)$ is in $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ if and only if $\Delta_c^*(F)$ and $\Delta_c^*(H)$ are in $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. Using this for all the colours c in $\{c_{i_1}, \dots, c_{i_m}\}$ we get that $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is thick by Corollary 6.2.4. □

6.3 Poset of stably zero functors

In this section we explain that the subcategories of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of stably zero functors presented in the two previous sections give a refinement of the notion of stably zero functors introduced in [DV19] for $\mathbf{FI}$-modules. Indeed, for $d = 1$ there is an inclusion of the unique subcategory of stably zero functors $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ in $\mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$, but for a general d, these subcategories naturally form a richer poset for the inclusion.

Lemma 6.3.1. *There is a poset of subcategories of* $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ *for the inclusion. It can be represented as follows, where the target of each of these functor categories is* $\mathbf{R}\text{-}\mathbf{Mod}$.

$$\begin{tikzcd}
 & & \cdots \arrow[dr, hook, dashed] & & & \\
 & \mathcal{SN}_{c_2,\cdots,c_d}(\mathbf{FI}_d) \arrow[ur, hook, dashed] \arrow[r, hook, dashed] \arrow[dr, hook, dashed] \arrow[d, dash, dashed] & \cdots \arrow[r, hook, dashed] & \mathcal{SN}_{c_1}(\mathbf{FI}_d) \arrow[dr, hook] \arrow[d, dash, dashed] & & \\
\mathcal{SN}_{c_1,\cdots,c_d}(\mathbf{FI}_d) \arrow[r, hook, dashed] \arrow[ur, hook] \arrow[dr, hook] & \vdots \arrow[r, hook, dashed] \arrow[d, dash, dashed] & \cdots \arrow[r, hook, dashed] \arrow[ur, hook, dashed] \arrow[dr, hook, dashed] & \vdots \arrow[r, hook, dashed] \arrow[d, dash, dashed] & \mathcal{SN}(\mathbf{FI}_d) \arrow[r, hook] & \mathbf{Fct}(\mathbf{FI}_d). \\
 & \mathcal{SN}_{c_1,\cdots,c_{d-1}}(\mathbf{FI}_d) \arrow[r, hook, dashed] \arrow[ur, hook, dashed] \arrow[dr, hook, dashed] & \cdots \arrow[r, hook, dashed] & \mathcal{SN}_{c_d}(\mathbf{FI}_d) \arrow[ur, hook] & & \\
 & & \cdots \arrow[ur, hook, dashed] & & &
\end{tikzcd}$$

Proof. We prove the inclusion $\mathcal{SN}_c(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}) \hookrightarrow \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ for a colour $c \in C$, while the other inclusions of the poset are clear by definition. For $c \in C$ and $F \in \mathcal{SN}_c(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ we have

$$\kappa(F) = \sum_{k \in \mathbf{FI}_d} \sum_{x \in \mathbf{FI}_d(0,k)} \kappa_k^x(F) = \sum_{k \in \mathbf{FI}_d} \kappa_k^{c^k}(F) + \sum_{k \in \mathbf{FI}_d} \sum_{x \in \mathbf{FI}_d(0,k), x \neq c^k} \kappa_k^x(F).$$

However, $F \in \mathcal{SN}_c(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ implies that $\sum_{k \in \mathbf{FI}_d} \kappa_k^{c^k}(F) = F$ and, since $\kappa(F)$ is a subfunctor of F we have

$$\kappa(F) = F + \sum_{k \in \mathbf{FI}_d} \sum_{x \in \mathbf{FI}_d(0,k), x \neq c^k} \kappa_k^x(F) = F.$$

□

We give now some examples of functors in the poset of Lemma 6.3.1 and, in particular, we illustrate that the inclusions of these subcategories are strict.

Example 6.3.2. We illustrate that the inclusions forming the poset are strict for $d = 2$, but the given counterexamples are generalizable for any $\mathbf{FI}_d$. For $d = 2$ the poset is simply the following:

$$\begin{tikzcd}
 & \mathcal{SN}_{c_1}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \arrow[dr, hook] & & \\
\mathcal{SN}_{c_1,c_2}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \arrow[ur, hook] \arrow[dr, hook] & & \mathcal{SN}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \arrow[r, hook] & \mathbf{Fct}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}). \\
 & \mathcal{SN}_{c_2}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \arrow[ur, hook] & &
\end{tikzcd}$$

- The inclusion $\mathcal{SN}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \hookrightarrow \mathbf{Fct}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$ is strict since any constant functor is not stably zero.

- The inclusion $\mathcal{SN}_{c_1,c_2}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \hookrightarrow \mathcal{SN}_{c_1}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$ is strict: if $F_{c_1}^{\mathbf{FI}_2} : \mathbf{FI}_2 \to \mathbf{R}\text{-}\mathbf{Mod}$ is the functor of Example 2.3.3 (sending all objects to $\mathbf{R}$, the maps containing the colour c_1 to zero and the other maps to the identity), then the functor $\Delta_{c_1}^*(F_{c_1}^{\mathbf{FI}_2})$ is the sum of all atomic functors, *i.e.* it sends all objects to $\mathbf{R}$ and all non-bijective morphisms on zero, so it is in the subcategory $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. This implies by Proposition 6.2.3 that $F_{c_1}^{\mathbf{FI}_2}$ is in the subcategory $\mathcal{SN}_{c_1}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$. However $\Delta_{c_2}^*(F_{c_1}^{\mathbf{FI}_2})$ is a constant functor so it is not in $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. By Proposition 6.2.3 it implies that $F_{c_1}^{\mathbf{FI}_2}$ is not in $\mathcal{SN}_{c_2}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$ and even less in $\mathcal{SN}_{c_1,c_2}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$.

- The inclusions $\mathcal{SN}_{c_1}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \hookrightarrow \mathcal{SN}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$ and $\mathcal{SN}_{c_2}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod}) \hookrightarrow \mathcal{SN}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$ are strict: We give an example of a functor in $\mathcal{SN}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$ which is neither in $\mathcal{SN}_{c_1}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$ nor in $\mathcal{SN}_{c_2}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$ using the matrices $A_1, A_2 \in \mathcal{M}_2(\mathbf{R})$ defined by
$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
which verify $A_1A_2 = A_2A_1 = 0$, $A_1^k = A_1$ and $A_2^k = A_2$ for any $k \in \mathbb{N}^*$. We can then define a $\mathbf{FI}_2$-module G by $G(n) = \mathbf{R}^2$ on an object $n \in \mathbf{FI}_2$, and on morphisms $(f,g) \in \mathbf{FI}_2(n,m)$ by
$$G(f,g) = \begin{cases} \mathrm{Id}_{\mathbf{R}} & \text{if } f \text{ is bijective,} \\ A_1 & \text{if the only colour that appears in } g \text{ is the colour } c_1, \\ A_2 & \text{if the only colour that appears in } g \text{ is the colour } c_2, \\ 0 & \text{if both } c_1 \text{ and } c_2 \text{ appears in } g. \end{cases}$$
The functor $\Delta_{c_1}^*(G)$ sends every object of $\mathbf{FI}$ to $\mathbf{R}^2$ and every non-bijective morphism to A_1. This implies that $\kappa(\,\Delta_{c_1}^*(G)\,)(n)$ is the constant functor equals to $\mathrm{Ker}(A_1)$, so G is not in the subcategory $\mathcal{SN}_{c_1}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$. By symmetry, it is not in $\mathcal{SN}_{c_2}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$ either, but we compute
$$\kappa_2^{c_1,c_2}(G) = \ker\Big(G\big(\mathrm{id}_{(-)} + (c_1,c_2)\big) : G(-) \to G(-+2) \Big) = \mathrm{Ker}\Big(\ G(-) \xrightarrow{0} G(-+2)\ \Big) = G.$$
Finally, $\kappa_2^{c_1,c_2}(G) = G$ implies $\kappa(G) = G$, so G is in the subcategory $\mathcal{SN}(\mathbf{FI}_2, \mathbf{R}\text{-}\mathbf{Mod})$.

Remark 6.3.3. The Example 6.3.2 shows that the inclusion of subcategories $\mathcal{SN}_c(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}) \hookrightarrow \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is strict for $c \in C$, so the functor $\pi_1 \circ \Delta_c^*(F)$ is not zero on all functors in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. This implies in particular, in contrast to the endofunctors τ_k and δ_k^x as we will show in Proposition 7.1.6, that the colouring functors do not pass to the quotient by the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Similarly, by Proposition 6.2.3 a $\mathbf{FI}_d$-module F is in the subcategory $\mathcal{SN}_c(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ if and only if the $\mathbf{FI}$-module $\Delta_c^*(F)$ is in the subcategory $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. We conclude that the precomposition by the colouring functors does not preserve the stably zero functors.

Remark 6.3.4. In [SS12] Sam and Snowden define the quotient of the modules over a TCA by its full subcategory of modules locally annihilated by a power of a prime ideal of the TCA. In [SS19] they apply this construction to the modules over the TCA $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ which are equivalent to the $\mathbf{FI}_d$-modules over a field $\mathbf{R} = \mathbb{K}$. They first decompose the category of modules over the TCA $\mathrm{Sym}(\mathbb{K}^d \otimes \mathbb{K}^\infty)$ into two pieces: they define a module over this TCA to be torsion if it is annihilated by a non-zero element of positive degree and they then define the "generic" category $\mathbf{Mod}_A^{\mathrm{gen}}$ as the quotient of the A-modules by the full subcategory of torsion functors. They then study the rank stratification from Remark 4.1.17 defined via the determinant ideals. This gives a filtration of subcategories of the modules over this TCA, which would be interesting to compare with the poset of Lemma 6.3.1, and they decompose the category into the successive quotients of this filtration. They then describe in [SS19] each quotient in the filtration and explain how these pieces come together.

We end this section by showing that the precomposition by the forgetful functor $\mathcal{O} : \mathbf{FI}_d \to \mathbf{FI}$ from Definition 2.1.6 preserves the stably zero functors. More precisely, if a $\mathbf{FI}$-module F is stably zero, we show that the $\mathbf{FI}_d$-module $\mathcal{O}^*(F) = F \circ \mathcal{O}$ is stably zero in every possible way.

Lemma 6.3.5. *For $c_{i_1}, \dots, c_{i_m} \in C$, there are inclusions of categories of functors to* $\mathbf{R}$-$\mathbf{Mod}$*:*

$$\mathcal{O}^* \left(\mathcal{SN}(\mathbf{FI}) \right) \subset \mathcal{SN}_{c_1,\dots,c_d}(\mathbf{FI}_d) \subset \mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d) \subset \mathcal{SN}(\mathbf{FI}_d)\,.$$

Proof. For F in $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ and $c \in C$, by definition $F = \kappa(F) = \sum_{k \in \mathbf{FI}} \kappa_k(F)$. By Proposition 2.7.1 we also have $\mathcal{O}^* \circ \kappa_k(F) \cong \kappa_k^x \circ \mathcal{O}^*(F)$ for all $x \in \mathbf{FI}_d(0,k)$ and all $k \in \mathbb{N}$. This implies that

$$\sum_{k \in \mathbf{FI}_d} \kappa_k^{c^k} \circ \mathcal{O}^*(F) \cong \sum_{k \in \mathbf{FI}_d} \mathcal{O}^* \circ \kappa_k(F) = \mathcal{O}^* \left(\sum_{k \in \mathbf{FI}_d} \kappa_k(F) \right) = \mathcal{O}^*(F).$$

We conclude that $\sum_{k \in \mathbf{FI}_d} \kappa_k^{c^k} \circ \mathcal{O}^*(F) = \mathcal{O}^*(F)$ for all $c \in C$, and so $\mathcal{O}^*(F)$ is in $\mathcal{SN}_{c_1,\dots,c_d}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, the smallest subcategory in the poset. □

6.4 Globally stably zero functors and twisted commutative algebras

For $\mathbf{R} = \mathbb{K}$ a field, there is an equivalence of categories between the category of $\mathbf{FI}_d$-modules and the category of modules over the free TCA $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$ of Definition 4.1.15 given in Theorem 4.2.4. The aim of this section is to study the notion of stably zero functors through this equivalence of categories. We begin with the description of the category equivalent to the notion of globally stably zero functors from the point of view of $\mathrm{Sym}((\mathbb{K}^d)^{(1)})$-modules. We recall that this equivalence depends on a choice of a basis $\mathcal{B}$ of $V = \mathbb{K}^d$, and we use the same notations as in Section 4.2.

Proposition 6.4.1. *The subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$ is equivalent to the full subcategory of $\mathrm{Sym}(V^{(1)})$-$\boldsymbol{Mod}$ having as objects the $\mathrm{Sym}(V^{(1)})$-modules (G, μ) such that for all objects $n \in \mathbf{\Sigma}$ we have the equality*

$$G(n) = \sum_{k \in \mathbf{\Sigma}} \sum_{x \in \mathbf{FI}_d(0,k)} \mathrm{Ker} \left(G(n) \xrightarrow{\Phi_x} \mathbb{K} \cdot e_x \otimes G(n) \xrightarrow{\mu_{n+k}|_{\mathbb{K}\cdot e_x \otimes G(n)}} G(n+k) \right),$$

where Φ_x is the canonical isomorphism $G(n) \cong \mathbb{K} \cdot e_x \otimes G(n)$.

Proof. We prove that the essential image of the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ by the functor $\Gamma_{\mathcal{B}}$ of Theorem 4.2.4's proof consists of the $\mathrm{Sym}(V^{(1)})$-modules (G, μ) which satisfy the condition of the statement. This proves the equivalence since the functor $\Gamma_{\mathcal{B}}$ is full and faithful by Theorem 4.2.4. For (G, μ) a $\mathrm{Sym}(V^{(1)})$-module, $n, k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k) \cong \mathbf{Set}(k,d)$, we can describe $\kappa_k^x(\chi_{\mathcal{B}}(G,\mu))(n)$ by

$$\begin{aligned} \kappa_k^x(\chi_{\mathcal{B}}(G,\mu))(n) &= \mathrm{Ker} \Big(\big(\chi_{\mathcal{B}}(G,\mu)\big) (\mathrm{id}_n + x) \,:\, \chi_{\mathcal{B}}(G,\mu)(n) \longrightarrow \chi_{\mathcal{B}}(G,\mu)(n+k) \Big) \\ &= \mathrm{Ker} \left(G(n) \xrightarrow{G(\mathrm{id}_n)} G(n) \xrightarrow{\Phi_x} \mathbb{K} \cdot e_x \otimes G(n) \xrightarrow{\mu_{n+k}|_{\mathbb{K}\cdot e_x \otimes G(n)}} G(n+k) \right), \end{aligned}$$

where the last equality is just the definition of $\chi_{\mathcal{B}}$. Since $\chi_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}}$ are quasi-inverse of each other, (G, μ) is in the essential image of $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ by $\Gamma_{\mathcal{B}}$ if and only if $\chi_{\mathcal{B}}(G,\mu)$ is in $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ and we conclude using the definition of $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$. □

In Definition 4.3.3 we described an action of $GL(V)$ on $\mathbf{FI}_d$-modules for a fixed basis $\mathcal{B}$ of $V = \mathbb{K}^d$, where $\varphi \in GL(V)$ acts by $\varphi_{\mathcal{B}} \cdot (-)$. For $d = 1$, the subcategory $\mathcal{SN}(\mathbf{FI}, \mathbb{K}\text{-}\mathbf{Vect})$ of $\mathbf{FI}\text{-}\mathbf{Mod}$ (see Definition 6.1.1) is closed under the action of $GL(V)$ since we have shown the following description of this action in Example 4.3.6: for $\varphi \in GL(V)$ and $G \in \mathbf{FI}\text{-}\mathbf{Mod}$ the functor $\varphi_{\mathcal{B}} \cdot G$ sends $n \in \mathbf{FI}_d$ to $G(n)$ and $(f,g) \in \mathbf{FI}_d(n,m)$ to $a \cdot (f,g)$ with $a \in \mathbb{K}^*$. In the following we show that this is not true for $d > 1$ by giving a counterexample. In particular, this implies that the action of $GL(V)$ does not pass to the quotient in an action on the quotient category of $\mathbf{Fct}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ by $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$.

Proposition 6.4.2. *For $d > 1$, the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ is not closed under the action of $GL(V)$ given in Definition 4.3.3.*

Proof. We use the functor $F_{c_1}^{\mathbf{FI}_2}$ from Example 6.3.2 as a counterexample for $d = 2$ which can be generalized for any $d > 1$. We recall that $F_{c_1}^{\mathbf{FI}_2}$ is defined on objects by $F_{c_1}^{\mathbf{FI}_2}(n) = \mathbb{K}$ for all $n \in \mathbf{FI}_2$ and on morphisms by

$$F_{c_1}^{\mathbf{FI}_2}(f,g) = \begin{cases} \mathrm{Id}_{\mathbb{K}} & \text{if } f \text{ is bijective,} \\ \mathrm{Id}_{\mathbb{K}} & \text{if the only colour that appears in } g \text{ is the colour } c_1, \\ 0 & \text{if the colour } c_2 \text{ appears in } g. \end{cases}$$

We showed in Example 6.3.2 that $F_{c_1}^{\mathbf{FI}_2}$ is in the subcategory $\mathcal{SN}(\mathbf{FI}_2, \mathbb{K}\text{-}\mathbf{Vect})$. For $\varphi \in GL(\mathbb{K}^2)$ defined in the basis $\mathcal{B}$ by the matrix

$$\mathcal{M}_{\mathcal{B}}(\varphi) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

by Proposition 4.3.5 we have $(\varphi_{\mathcal{B}} \cdot F_{c_1}^{\mathbf{FI}_2})(n) = \mathbb{K}$ for all objects $n \in \mathbf{FI}_2$ and

$$(\varphi_{\mathcal{B}} \cdot F_{c_1}^{\mathbf{FI}_2})(f,g) = \sum_{g' \in \mathbf{FI}_2(0, m \setminus f(n))} \left(\prod_{l \in m \setminus f(n)} m_{g'(l), g(l)} \right) F_{c_1}^{\mathbf{FI}_2}(f,g')$$

for all morphisms $(f,g) \in \mathbf{FI}_2(n,m)$. Since $F_{c_1}^{\mathbf{FI}_2}(f,g')$ is zero if c_2 appears in g' the only non-zero term in this sum is the one for $g' = (c_1)^{m \setminus f(n)}$. By definition $F_{c_1}^{\mathbf{FI}_2}(f,g')$ is the identity in this case, so we get

$$(\varphi_{\mathcal{B}} \cdot F_{c_1}^{\mathbf{FI}_2})(f,g) = \left(\prod_{l \in m \setminus f(n)} m_{1, g(l)} \right) F_{c_1}^{\mathbf{FI}_2}(f,g') = \left(1^{\#\{c_1 \in g\}} \cdot 1^{\#\{c_2 \in g\}} \right) \cdot \mathrm{Id}_{\mathbb{K}} = \mathrm{Id}_{\mathbb{K}} .$$

Applying this to the morphism $(f,g) = \mathrm{Id}_{(-)} + x$ for $k \in \mathbf{FI}_2$ and $x \in \mathbf{FI}_2(0,k)$, we get that

$$\kappa_k^x(\varphi_{\mathcal{B}} \cdot F_{c_1}^{\mathbf{FI}_2}) = \mathrm{Ker}\left((\varphi_{\mathcal{B}} \cdot F_{c_1}^{\mathbf{FI}_2})(\mathrm{Id}_{(-)} + x) \right) = \mathrm{Ker}(\mathrm{Id}) = 0.$$

Finally, we have $\kappa(\varphi_{\mathcal{B}} \cdot F_{c_1}^{\mathbf{FI}_2}) = \sum \kappa_k^x(\varphi_{\mathcal{B}} \cdot F_{c_1}^{\mathbf{FI}_2}) = 0$, showing that the functor $\varphi_{\mathcal{B}} \cdot F_{c_1}^{\mathbf{FI}_2}$ is not in the subcategory $\mathcal{SN}(\mathbf{FI}_2, \mathbb{K}\text{-}\mathbf{Vect})$ although $F_{c_1}^{\mathbf{FI}_2}$ is in $\mathcal{SN}(\mathbf{FI}_2, \mathbb{K}\text{-}\mathbf{Vect})$. □

Remark 6.4.3. While for $d = 1$ the category of torsion modules over the TCA $\mathrm{Sym}((\mathbb{K}^1)^{(1)})$ presented in Remark 6.3.4 corresponds to $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ via the equivalence of Theorem 4.2.4, Proposition 6.4.2 implies that this is not true for $d > 1$. Indeed, $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is not closed under the action of $GL(V)$, while the category of torsion modules introduced by Sam and Snowden is.

Chapter 7

Weak polynomial functors on $\mathbf{FI}_d$

N'as-tu jamais fait un de ces rêves qui ont l'air plus vrai que la réalité ? Si tu étais incapable de sortir d'un de ces rêves, comment ferais-tu la différence entre le monde réel et le monde des rêves ?

Lana et Lilly Wachowski

For functors over a symmetric monoidal category whose unit is a null object, the polynomial functors are closed under subobjects (see [Dja16]), which simplifies their study. It is because this is not true when the unit is just initial that the weak polynomial functors were introduced in [DV19] to recover this kind of properties. As seen in Remark 5.1.9, even for $d = 1$, the strong polynomial $\mathbf{FI}_d$-modules are not closed under subobject either. The given counterexamples are made of functors which are zero on maps after some rank, which gives rise to unstable phenomena. To avoid this instability we delete the problematic functors, such as the stably zero functors studied in Chapter 6, in a quotient category of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Then we can define polynomial objects in this quotient category and the weak polynomial $\mathbf{FI}_d$-modules as the functors whose image in the quotient is polynomial. This way, the notion of weak polynomial $\mathbf{FI}_d$-modules has all the important properties we want. This idea is inspired by the situation studied in [DV19] for $\mathbf{FI}$-modules but, as seen in Chapter 6, for $\mathbf{FI}_d$ there are several subcategories of stably zero functors that we can consider.

In Section 7.1, we present the quotient by $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, the largest subcategory of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of stably zero functors, to get a smaller quotient category that may be the easiest to describe. In particular, we get a characterization of the simple objects of this quotient in Proposition 7.1.8. We then introduce and study the polynomial objects in this quotient in Section 7.2. We also explain in Remark 7.2.10 that in the quotient by another subcategory of stably zero functors the polynomial objects are a bit harder to define and we lose some important properties like the fact that the endofunctors δ_k^x become exact when they pass to the quotient category. In Section 7.3 we explain that the pointwise tensor product from Definition 5.5.1 passes to the quotient by $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and preserves polynomial objects. Finally, we define the category $\mathbf{R}\text{-}\mathbf{Mod}_d$ whose objects are the tuples $(M, \varphi_2, \ldots, \varphi_d)$, where M is an object of $\mathbf{R}\text{-}\mathbf{Mod}$ and $\varphi_2, \ldots, \varphi_d : M \to M$ are $d-1$ isomorphisms in $\mathbf{R}\text{-}\mathbf{Mod}$ commuting two by two in Section 7.4. We then show that the category of polynomial objects of degree 0 in this quotient is equivalent to $\mathbf{R}\text{-}\mathbf{Mod}_d$.

7.1 The quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$

We showed in Proposition 6.1.7 that the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is thick so, using the construction from Definition 1.1.8, we can define the quotient by this subcategory. In this section we give the definition, the basic results and some more abstract properties of this quotient category of stable functors.

Definition 7.1.1. The *category* $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is the quotient category

$$\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}) = {\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})}\Big/{\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})}$$

where the quotient is described in Definition 1.1.8 and π_d is the canonical quotient functor $\pi_d : \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

Remark 7.1.2. Although the objects of the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ are by definition the functors from $\mathbf{FI}_d$ to $\mathbf{R}\text{-}\mathbf{Mod}$, one should think of them as abstract objects since the morphisms in the quotient are modified by some isomorphism classes, so the objects of the quotient category are only functors up to relations. To make this clear, we often denote by X an object of the quotient and by F a functor in $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. We will say that a functor F is weak polynomial if its image $\pi_d(F)$ in the quotient is a polynomial object, but sometimes we use an abuse of notation and we identify F and $\pi_d(F)$.

Remark 7.1.3. For $d = 1$ and $\mathbf{R} = \mathbb{K}$ a field of characteristic zero, there is an equivalence of categories

$$\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod}) \cong \mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod}),$$

if we consider only finitely generated functors. The proof is done in [SS16, Section 2.5] by defining a $\mathbf{FI}$-module $\mathcal{K} = \mathrm{Sym}(\mathbb{K}^\infty) \otimes \mathrm{Sym}(\mathbb{K}^\infty \otimes \mathbb{K}^\infty)$ from the point of view of TCAs as representations of $GL(\infty)$, which also has a structure of a $\mathbf{FI}^{\mathrm{op}}$-module. They then show that the two functors $\mathrm{Hom}_{\mathbf{FI}}(-, \mathcal{K})$ and $\mathrm{Hom}_{\mathbf{FI}^{op}}(-, \mathcal{K})$, (respectively post and pre) composed with a duality, are quasi-inverse of each other. In [SS19, Section 5], they show that the extremal quotients $\mathrm{Mod}_{A,0}$ and $\mathrm{Mod}_{A,d}$ of the rank stratification of Remark 4.1.17 are equivalent, which generalizes this equivalence for a general d. To prove this, they show that both categories are equivalent to the category of polynomial representations of the group $GL(\infty) \ltimes (\mathbb{K}^\infty \otimes \mathbb{K}^d)$. However, it seems to be false that $\mathrm{Mod}_{A,k}$ and $\mathrm{Mod}_{A,d-k}$ are equivalent for $k \neq 0, d$. The full subcategory $\mathrm{Mod}_{A,0}$ of Mod_A consists, from the point of view of TCAs as representations of $GL(\infty)$, of the modules supported at zero, *i.e.* which are locally annihilated by a power of $\mathbb{K}^d \otimes \mathbb{K}^\infty \subset \mathrm{Sym}(\mathbb{K}^d \otimes \mathbb{K}^\infty)$. With the identification $\mathrm{Sym}(\mathbb{K}^d \otimes \mathbb{K}^\infty) \cong \mathbb{K}[x_{i,j} | 1 \leq i \leq d, 1 \leq j]$ it is the ideal generated by the $x_{i,j}$ with the identification $\mathrm{Sym}(\mathbb{K}^d \otimes \mathbb{K}^\infty) \cong \mathbb{K}[x_{i,j} | 1 \leq i \leq d, 1 \leq j]$, which seems to correspond to the subcategory $\mathcal{SN}_{c_1,\dots,c_d}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ via the equivalence from Theorem 4.2.4 although we have no rigorous proof of this.

Remark 7.1.4. In [SS16, Section 1.3] Sam and Snowden show that, for $d = 1$, the equivalent categories $\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ and $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ can be described as the category of representations of an explicit quiver with relations called $\mathrm{Part}_{\mathrm{HS}}$. The vertices of this quiver are the partitions, there is an order relation on the partitions: $\mu \leq \lambda$ if λ/μ is a "horizontal strip" (called HS), which means that $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all i. There is an arrow in $\mathrm{Part}_{\mathrm{HS}}$ from the partition μ to the partition λ if $\mu \leq \lambda$. For three partitions λ, μ and ν such that $\mu \leq \lambda$ and $\lambda \leq \nu$, the composition of the two maps $\mu \leq \lambda$ and $\lambda \leq \nu$ is equal to the map $\mu \leq \nu$ if $\mu \leq \nu$ and is zero if $\mu \nleq \nu$. This quiver corresponds to the simple elements (see Proposition 2.4.3) of the category and gives the relations between them. The proof of this result uses the functors that give the equivalence $\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod}) \cong \mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ for $d = 1$.

Lemma 7.1.5. *The quotient functor* π_d *is essentially surjective, exact, it commutes with all filtered colimits and has a right adjoint*

$$\mathcal{S}_d : \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$$

called the section functor.

Proof. By Proposition 1.2.2 the quotient functor is always exact and essentially surjective. Since the category **R-Mod** is a Grothendieck category (Definition 1.3.1), the functor category $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is also one. Then Proposition 1.3.3 implies, with Proposition 6.1.7, that $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is a localizing subcategory of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, which means exactly that the quotient functor π_d has a right adjoint. In this case Proposition 1.3.3 implies that it commutes with all filtered colimits. □

We now give some properties of this quotient category inspired by [DV19, section 2] which is similar for **FI**-modules and $\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. We begin with a proposition stating that the endofunctors τ_k and δ_k^x pass to the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, while the endofunctors κ_k^x become all zero in the quotient.

Proposition 7.1.6. *For* $k \in \mathbf{FI}_d$ *and* $x \in \mathbf{FI}_d(0,k)$*, the endofunctors* τ_k *and* δ_k^x *of* $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ *induce two endofunctors* $\tau_k^{\mathbf{St}}$ *and* $(\delta_k^x)^{\mathbf{St}}$ *of* $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ *defined by the relations* $\pi_d \circ \delta_k^x = (\delta_k^x)^{\mathbf{St}} \circ \pi_d$ *and* $\pi_d \circ \tau_k = \tau_k^{\mathbf{St}} \circ \pi_d$*. These endofunctors are exact, they commute to colimits, and there is a short exact sequence of endofunctors of* $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$*:*

$$0 \longrightarrow \mathrm{Id}^{\mathbf{St}} \xrightarrow{(i_k^x)^{\mathbf{St}}} \tau_k^{\mathbf{St}} \longrightarrow (\delta_k^x)^{\mathbf{St}} \longrightarrow 0$$

Proof. For $F \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, by Proposition 2.6.6 the endofunctor τ_k commutes with κ_l^x and colimits, so we have

$$\kappa\big(\tau_k(F)\big) = \sum_{l \in \mathbf{FI}_d} \sum_{x \in \mathbf{FI}_d(0,l)} \kappa_l^x\big(\tau_k(F)\big) = \sum_{l \in \mathbf{FI}_d} \sum_{x \in \mathbf{FI}_d(0,l)} \tau_k\big(\kappa_l^x(F)\big) = \tau_k\big(\kappa(F)\big) = \tau_k(F).$$

This means that $\tau_k(F)$ is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, so the functor $\pi_d \circ \tau_k$ is zero on all objects of the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Moreover, both π_d and τ_k are exact functors, so $\pi_d \circ \tau_k$ is also exact. Then, by Proposition 1.3.4, there exists a unique functor $\tau_k^{\mathbf{St}}$ which satisfies the relation $\pi_d \circ \tau_k = \tau_k^{\mathbf{St}} \circ \pi_d$. By Corollary 1.3.5 we get that it is exact, and it commutes with colimits by construction since π_d and τ_k commute with colimits too (Proposition 2.6.6 and Lemma 7.1.5). Now we do the same for δ_k^x: for every short exact sequence $0 \to F \to G \to H \to 0$ in $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, by applying the exact functor π_d to the exact sequence of Proposition 2.6.6 we get the short exact sequence

$$0 \longrightarrow \pi_d \circ \delta_k^x(F) \longrightarrow \pi_d \circ \delta_k^x(G) \longrightarrow \pi_d \circ \delta_k^x(H) \longrightarrow 0$$

since we have $\pi_d \circ \kappa_k^x = 0$ by Proposition 6.1.7. This means that the functor $\pi_d \circ \delta_k^x$ is exact. Moreover, by Proposition 6.1.7 the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is stable by the endofunctor δ_k^x, which implies that $\pi_d \circ \delta_k^x$ is zero on the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. By Proposition 1.3.4 there exists a unique functor $(\delta_k^x)^{\mathbf{St}}$ which satisfies the relation $\pi_d \circ \delta_k^x = (\delta_k^x)^{\mathbf{St}} \circ \pi_d$. It is also exact and it commutes with colimits with the same arguments as for τ_k. Finally, applying the exact functor π_d to the exact sequence (I) from Lemma 2.6.4 we get the short exact sequence

$$0 \longrightarrow \pi_d(F) \xrightarrow{(i_k^x)^{\mathbf{St}}_{\pi_d(F)}} \tau_k^{\mathbf{St}}(\pi_d(F)) \longrightarrow (\delta_k^x)^{\mathbf{St}}(\pi_d(F)) \longrightarrow 0$$

since $\pi_d \circ \kappa_k^x = 0$ by Proposition 6.1.7. Using Proposition 1.2.3 we get, for all $X \in \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, the existence of the natural short exact sequence of the statement. □

Remark 7.1.7. In [SS16] Sam and Snowden consider the subcategory of finitely generated torsion modules over the TCA $\mathrm{Sym}((\mathbb{K}^1)^{(1)})$ and study the properties of the quotient category of finitely generated $\mathrm{Sym}((\mathbb{K}^1)^{(1)})$-modules by this full subcategory. This quotient is equivalent to the subcategory of $\mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ of finitely generated objects. In particular, they show that the projective finitely generated modules are also injective and that the section functor sends the injective objects of the quotient to projective objects, and that all the finitely generated $\mathbf{FI}$-modules have finite injective dimension.

The following proposition gives a condition that describes the simple objects of the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. It is inspired by [SS16, Proposition 2.2.1] which gives a similar result for finitely generated $\mathbf{FI}$-modules but expressed in terms of modules over the TCA $\mathrm{Sym}((\mathbb{K}^1)^{(1)})$.

Proposition 7.1.8. *For F a $\mathbf{FI}_d$-module, the object $\pi_d(F) \in \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is simple if and only if, for all submodules G of F, either G is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ or F/G is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.*

Proof. If $\pi_d(F)$ is simple, then for all submodules G of F, $\pi_d(G)$ is a subobject of $\pi_d(F)$, since π_d is exact. Then either $\pi_d(G) = 0$, which means that $G \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, or $\pi_d(G) = \pi_d(F)$ which implies that $\pi_d(F/G) = 0$ since π_d is exact, and thus $F/G \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Conversely, for X a subobject of $\pi_d(F)$ we can apply Proposition 1.2.3 to the inclusion of X in $\pi_d(F)$, which gives the existence of $\tilde{F}, G \in \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and of a monomorphism $f : G \to \tilde{F}$ of $\mathbf{FI}_d$-modules and of isomorphisms $\pi_d(G) \cong X$ and $\pi_d(\tilde{F}) \cong \pi_d(F)$, which makes a commutative diagram. In particular, $\pi_d(\tilde{F})$ is simple if and only if $\pi_d(F)$ is simple. Then we consider the image of f which is a submodule of $\tilde{F}$: By hypothesis, either this image is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, either the quotient by this image is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. By Lemma 1.2.4, in the first case we get $\pi_d(f) = 0$ so $\pi_d(G) \cong X = 0$, and in the second case we get $\mathrm{Coker}(f) = 0$ so $\pi_d(f)$ is an epimorphism and $\pi_d(\tilde{F}) = \pi_d(G) \cong X$. □

Remark 7.1.9. In fact, this proof works for any quotient category $\mathcal{A}/_{\mathcal{C}}$, since it uses only the results of Chapter 1. By Proposition 7.1.8 to classify the simple objects of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is equivalent to classify the $\mathbf{FI}_d$-modules F such that for each of its submodule G either G or F/G is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. For $d = 1$, Sam and Snowden used this equivalence to classify explicitly the simple objects of the quotient category for finitely generated $\mathbf{FI}$-modules in [SS16, Section 2.2]. Indeed, they showed that the simple objects of the quotient are indexed by the partitions: the object associated to the partition λ is the image by the section functor π_1 of the direct sum of the simple objects of $\mathbf{FI}\text{-}\mathbf{Mod}$ (see Proposition 2.4.3 for $d = 1$) associated with the partitions of the form (n, λ), with $n \geq \lambda_1$. They then used this result to classify the injective objects of this quotient.

We now use the notion of $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$-closed objects (Definition 6.1.10) and the Proposition 6.1.12 to show that there is a monomorphism from $\delta_k^x \circ \mathcal{S}_d$ to $\mathcal{S}_d \circ (\delta_k^x)^{\mathbf{St}}$. This will be used in Section 7.4 in order to describe the polynomial functors of degree 0.

Proposition 7.1.10. *For all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0, k)$ there is a natural monomorphism*

$$\delta_k^x \circ \mathcal{S}_d \hookrightarrow \mathcal{S}_d \circ (\delta_k^x)^{\mathbf{St}} .$$

Proof. Let $\eta : \mathrm{Id} \to \mathcal{S}_d \circ \pi_d$ be the unit of the adjunction of π_d and $\mathcal{S}_d$, then for $X \in \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, $k \in \mathbb{N}$ and $x \in \mathbf{FI}_d(0, k)$, the object $\eta_{\delta_k^x \circ \mathcal{S}_d(X)}$ is in $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Since the co-unit of the adjunction of π_d and $\mathcal{S}_d$ is always an isomorphism between $\pi_d \circ \mathcal{S}_d$ and Id, we can consider $\tilde{\eta}_X$ the composition

$$\delta_k^x \circ \mathcal{S}_d(X) \xrightarrow{\eta_{\delta_k^x \circ \mathcal{S}_d(X)}} \mathcal{S}_d \circ \pi_d \circ \delta_k^x \circ \mathcal{S}_d(X) \xrightarrow{\sim} \mathcal{S}_d \circ (\delta_k^x)^{\mathbf{St}} \circ \pi_d \circ \mathcal{S}_d(X) \xrightarrow{\sim} \mathcal{S}_d \circ (\delta_k^x)^{\mathbf{St}}(X),$$

where the first isomorphism is given by Proposition 7.1.6 and the second by Proposition 1.3.10. By definition, the kernel of $\tilde{\eta}_X$ is the same as the kernel of $\eta_{\delta_k^x \circ \mathcal{S}_d(X)}$ and, by Proposition 1.3.13 it is in the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ since it is the kernel of the unit of the adjunction of π_d and $\mathcal{S}_d$. Moreover, by Lemma 1.3.14 the functor $\mathcal{S}_d(X)$ is $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$-closed. Then the inclusion j of the kernel $\mathrm{Ker}(\tilde{\eta}_X)$ within $\delta_k^x \circ \mathcal{S}_d(X)$ is in $\mathrm{Hom}\big(\,\mathrm{Ker}(\tilde{\eta}_X),\, \delta_k^x(\mathcal{S}_d(X))\,\big)$, with $\mathrm{Ker}(\tilde{\eta}_X)$ in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ and $\mathcal{S}_d(X)$ which is $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$-closed. By Proposition 6.1.12 we get $j = 0$ and so $\tilde{\eta}_X$ is a monomorphism from $\delta_k^x \circ \mathcal{S}_d(X)$ to $\mathcal{S}_d \circ (\delta_k^x)^{\mathbf{St}}(X)$. It is natural since the co-unit is natural and the two isomorphisms inside $\tilde{\eta}_X$ are also natural. □

Finally we construct an homology functor $h_*^{\mathbf{FI}_d}(-)$ from the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ to the graded category $\mathbf{R\text{-}Mod}_{gr}$ over $\mathbf{R\text{-}Mod}$ as the usual homology functor $H_*(\mathbf{FI}_d, -)$ from the category $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ passing to the quotient. This definition is inspired by Propositions 2.17 and 2.18 of [DV19]. Recall that for $\mathbf{R\text{-}Mod}$ we use the constant functor $\mathbf{R}$ to define the usual homology functor $H_*(\mathbf{FI}_d, -)$ as the functor $\mathrm{Tor}_*^{\mathbf{FI}_d}(\mathbf{R}, -)$.

Proposition 7.1.11. *For F a $\mathbf{FI}_d$-module, if the morphism $i_k^x(F)$ is a split monomorphism for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$, then*

1) *For every functor $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ we have $\mathrm{Ext}^*(H, F) = 0$.*

2) *For every functor $G \in \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$, the morphism*

$$\mathrm{Ext}^*_{\mathbf{Fct}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})}(G, F) \xrightarrow{(\pi_d)^*} \mathrm{Ext}^*_{\mathbf{St}(\mathbf{FI}_d,\mathbf{R\text{-}Mod})}\big(\pi_d(G)\,,\,\pi_d(F)\big)$$

is an isomorphism. In particular, $\pi_d : \mathrm{Hom}(G,F) \to \mathrm{Hom}(\,\pi_d(G), \pi_d(F)\,)$ is an isomorphism.

3) *If $X : \mathbf{FI}_d^{op} \to \mathbf{R\text{-}Mod}$ is a functor such that the morphism $i_k^x(X)$ is a split epimorphism for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$, then for all $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ we have $\mathrm{Tor}_*(X, H) = 0$.*

4) *Let $\mathbf{R\text{-}Mod}_{gr}$ be the graded category over $\mathbf{R\text{-}Mod}$. The homology functor $H_*(\mathbf{FI}_d, -)$: $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod}) \to \mathbf{R\text{-}Mod}_{gr}$ passes to the quotient $\mathbf{St}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$, which means that there exists a unique functor $h_*^{\mathbf{FI}_d}(-)$: $\mathbf{St}(\mathbf{FI}_d, \mathbf{R\text{-}Mod}) \to \mathbf{R\text{-}Mod}_{gr}$ such that $h_*^{\mathbf{FI}_d} \circ \pi_d = H_*(\mathbf{FI}_d, -)$.*

Proof. 1) For H a functor in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$, we first assume that there exist $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$ such that $i_k^x(H) = 0$. For all natural transformations $\sigma : H \to F$, the naturality of i_k^x implies the relation $\tau_k(\sigma) \circ i_k^x(H) = i_k^x(F) \circ \sigma$, so the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}\big(H,F\big) \arrow[rr, "\left(i_k^x(F)\right)_*"] \arrow[dr, "\tau_k"'] & & \mathrm{Hom}\big(H,\tau_k(F)\big) \\
& \mathrm{Hom}\big(\tau_k(H),\tau_k(F)\big) \arrow[ur, "\left(i_k^x(H)\right)^*"'] &
\end{tikzcd}$$

Since we assumed that $i_k^x(H)$ is zero the morphism $\big(i_k^x(F)\big)_*$ is also zero. Moreover $i_k^x(F)$ is a split monomorphism by hypothesis, so $\big(i_k^x(F)\big)_*$ is also an epimorphism. Since $\mathrm{Hom}(-,F)$ and $\mathrm{Hom}(-,\tau_k(F))$ are left exact functors, their derived functors $\mathrm{Ext}^*(-,F)$ and $\mathrm{Ext}^*(-,\tau_k(F))$ are universal δ-functors. So there exist unique morphisms $\big(i_k^x(F)\big)_*^n$: $\mathrm{Ext}^n\big(H,F\big) \longrightarrow \mathrm{Ext}^n\big(H,\tau_k(F)\big)$ extending $\big(i_k^x(F)\big)_*$. By unicity of the extending morphisms all the morphisms

$\left(i_k^x(F)\right)_*^n$ are both split monomorphisms and zero. This implies that, for all $n \in \mathbb{N}$, the object $\mathrm{Ext}^n(H,F)$ is zero proving the statement if there exist $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$ such that $i_k^x(H) = 0$. For H an arbitrary functor in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, we still have

$$H = \kappa(H) = \sum_{k \in \mathbf{FI}_d} \sum_{x \in \mathbf{FI}_d(0,k)} \kappa_k^x(H).$$

By definition, for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$, the morphism $i_k^x(\,\kappa_k^x(H)\,)$ is zero, so from the previous point we get $\mathrm{Ext}^*(\,\kappa_k^x(H)\,,\,F) = 0$. Since the functor $\mathrm{Hom}(-,F)$ commutes with colimits, its derived functors also commute with colimits because they are universal δ-functors. This implies the equality

$$\mathrm{Ext}^*\left(H\,,\,F\right) = \mathrm{Ext}^*\left(\sum_{k \in \mathbf{FI}_d} \sum_{x \in \mathbf{FI}_d(0,k)} \kappa_k^x(H)\,,\,F\right) = \sum_{k \in \mathbf{FI}_d} \sum_{x \in \mathbf{FI}_d(0,k)} \mathrm{Ext}^*\left(\kappa_k^x(H)\,,\,F\right) = 0.$$

2) By Proposition 1.2.3 the functor π_d is full up to inner isomorphisms, so the morphism $\pi_d : \mathrm{Hom}(G,F) \to \mathrm{Hom}(\pi_d(G), \pi_d(F))$ is surjective since we use skew categories (these inner isomorphisms do not count) and we prove that it is also injective. Let $\sigma : G \to F$ be a natural transformation such that $\pi_d(\sigma)$ is zero, by Proposition 1.2.4 its image $\mathrm{Im}(\sigma)$ is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Since the category $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is abelian, the morphism σ splits into $j \circ e$ with $e : G \to \mathrm{Im}(\sigma)$ an epimorphism and $j : \mathrm{Im}(\sigma) \to F$ a monomorphism. Then j is in $\mathrm{Hom}(\mathrm{Im}(\sigma), F)$ and $\mathrm{Im}(\sigma)$ is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. From the previous point we get $j = 0$, so $\sigma = 0$. This means that $\pi_d(\sigma) = 0$ implies $\sigma = 0$ and (since both categories $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ are additives) it means that π_d is injective, so bijective. Now π_d is an exact functor, so $\mathrm{Hom}(-,F)$ and $\mathrm{Hom}(\pi_d(-), \pi_d(F))$ are left exact functors, and their derived functors $\mathrm{Ext}^*(-,F)$ and $\mathrm{Ext}^*(\pi_d(-), \pi_d(F))$ are universal δ-functors. Then there exist unique morphisms

$$(\,\pi_d\,)^* \,:\, \mathrm{Ext}^n_{\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})}\left(G, F\right) \longrightarrow \mathrm{Ext}^n_{\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})}\left(\pi_d(G), \pi_d(F)\right)$$

extending π_d. By unicity of the extending morphisms, all the morphisms $(\pi_d)^*$ are also isomorphisms.

3) This is the dual statement of point 1).

4) The constant functor $\mathbf{R} : \mathbf{FI}_d^{op} \to \mathbf{R}\text{-}\mathbf{Mod}$ satisfies the hypothesis of point 3) since, for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$, the morphism $i_k^x(\mathbf{R}) = \mathbf{R}(\mathrm{Id} + x) = \mathrm{Id}_{\mathbf{R}}$ is a split monomorphism. We then deduce that, for all functors $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, we have the equality $H_*\left(\mathbf{FI}_d, H\right) = \mathrm{Tor}_*^{\mathbf{FI}_d}\left(\mathbf{R}, H\right) = 0$, so the functor $H_*(\mathbf{FI}_d, -)$ is zero on the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Moreover the functor $H_*(\mathbf{FI}_d, -)$ gives a long exact sequence for every short exact sequence in $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. We conclude with Proposition 1.3.6 (with this long exact sequence), and we get the existence of a unique functor $h_*^{\mathbf{FI}_d}(-) : \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{R}\text{-}\mathbf{Mod}_{gr}$ such that $h_*^{\mathbf{FI}_d} \circ \pi_d = H_*(\mathbf{FI}_d, -)$. □

7.2 Generalities on the category $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$

In this section we introduce the weak polynomial functors over $\mathbf{FI}_d$, which are the $\mathbf{FI}_d$-modules that become polynomial objects of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ when passed to the quotient. To define these polynomial objects of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ we use the endofunctors $(\delta_1^c)^{\mathbf{St}}$ of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ from Proposition 7.1.6, for the different colours $c \in C$. After the definition we give the basic properties of these objects. In particular, we show in Proposition 7.2.5 that, unlike to strong polynomial functors, they form a thick subcategory of the quotient $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

Definition 7.2.1. The full subcategories of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of *polynomial objects* of degree less than or equal to n, denoted by $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, are defined by induction. By convention, $\mathrm{Pol}_{-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is zero and, for $n \in \mathbb{N}$, an object X of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is in $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ if

$$(\delta_1^c)^{\mathbf{St}}(X) \in \mathrm{Pol}_{n-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}) \quad \text{for all } c \in C,$$

where $(\delta_1^c)^{\mathbf{St}}$ is the endofunctor from Proposition 7.1.6. A $\mathbf{FI}_d$-module F is a *weak polynomial functor* of degree less than or equal to n if its projection $\pi_d(F)$ is in the subcategory $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

Remark 7.2.2. We say that a functor over $\mathbf{FI}_d$ is weak polynomial if its image in the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by π_d is a polynomial object. By definition of the endofunctors $(\delta_k^x)^{\mathbf{St}}$ we get that a strong polynomial functor is weak polynomial, but the converse is not true as shown in Example 5.1.7. This justifies the terminology introduced by Djament and Vespa in [DV19].

Remark 7.2.3. In an abuse of notation, we sometimes also denote by $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ the full subcategory of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of weak polynomial functors of degree less than or equal to n, *i.e.* the functors F such that $\pi_d(F)$ is in the subcategory $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. In the contrary, we sometimes call by *polynomial functors* the polynomial objects of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, even if they are objects of the quotient category.

One may expect that the image by the section functor of a polynomial object in the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ gives a strong polynomial $\mathbf{FI}_d$-module with the same degree, but this is not always true as shown in the following example.

Example 7.2.4. There exist polynomial objects of degree 1 such that their image by the section functor is not a polynomial $\mathbf{FI}_d$-module of degree 1. We give an example for $\mathbf{R} = \mathbb{Z}$ adapted from [DV19, Example 5.3]. Let $F : \mathbf{FI}_d \to \mathbf{Ab}$ be the kernel of the augmentation map $\mathbb{Z}[-] \to \mathbb{Z}$, where $\mathbb{Z}$ is the constant functor equal to $\mathbb{Z}$ and $\mathbb{Z}[-]$ is the linearization that sends an object n to $\mathbb{Z}^n$ and a map $(f,g) \in \mathbf{FI}_d(n,m)$ to the injection of $\mathbb{Z}^n$ in $\mathbb{Z}^m$ along f. Then we get that $\delta_1^c(F) = \mathbb{Z}_{\geq 1}$ for all $c \in C$ and by the Example 5.1.7 the functor F is strong polynomial of degree 2. It is weak polynomial of degree 1 since we have $\pi_d \circ \delta_1^c(F) = \pi_d(\mathbb{Z}_{\geq 1}) = \pi_d(\mathbb{Z})$ which is weak polynomial of degree 0. We then compute that $\kappa(\mathbb{Z}[-]) = 0$ and $\kappa(\mathbb{Z}[-]) = 0$ so, by Proposition 6.1.12, we get $\mathrm{Hom}(H, \mathbb{Z}[-]) = 0$ and $\mathrm{Hom}(H, \mathbb{Z}) = 0$ for all $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Similarly, since $\mathbb{Z}[-]$ is projective we get $\mathrm{Ext}_1(H, \mathbb{Z}) = 0$ and we deduce from the exact sequence $0 \longrightarrow F \longrightarrow \mathbb{Z}[-] \longrightarrow \mathbb{Z}$ that F is $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$-closed (Definition 6.1.10). Finally, Proposition 1.3.12 gives that $F \cong \mathcal{S}_d \circ \pi_d(F)$, and we showed that $\pi_d(F)$ is polynomial of degree 1 while $\mathcal{S}_d \circ \pi_d(F)$ is strong polynomial of degree 2.

We now prove that the subcategories $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ are thick.

Proposition 7.2.5. *For all $n \in \mathbb{N}$, the subcategory $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is thick, closed under colimits and stable by the endofunctors $(\tau_k)^{\mathbf{St}}$ and $(\delta_k^x)^{\mathbf{St}}$ for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$.*

Proof. The first assumption is proved by induction using Proposition 7.1.6, which implies that all endofunctors $(\delta_1^c)^{\mathbf{St}}$ are exact and commute with colimits. The second assumption is true since τ_k and δ_k^x commute with δ_1^c as endofunctors of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, and it is still true when they pass to the quotient as endofunctors of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by Proposition 7.1.6. □

Using only the endofunctors $(\delta_1^c)^{\mathbf{St}}$ for $c \in C$ in Definition 7.2.1 seems a bit restrictive, but the following lemma shows that, if we use all the endofunctors $(\delta_k^x)^{\mathbf{St}}$ for $k \in \mathbf{FI}_d$ and $x \in \mathbf{FI}_d(0,k)$, we get an equivalent definition.

Lemma 7.2.6. *An object X of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is in $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ if and only if $(\delta_k^x)^{\mathbf{St}}(X)$ is in $\mathrm{Pol}_{n-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$.*

Proof. One implication is obvious by taking $k = 1$ and $c \in C = \mathbf{FI}_d(0,1)$, we prove the converse. For X in $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and $c, \tilde{c} \in C$, we prove that for $(\delta_2^{(c,\tilde{c})})^{\mathbf{St}}(X)$ is in $\mathrm{Pol}_{n-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. By Proposition 7.1.6 we have the identities $\pi_d \circ \tau_1 = \tau_1^{\mathbf{St}} \circ \pi_d$ and $\pi_d \circ \delta_k^x = (\delta_k^x)^{\mathbf{St}} \circ \pi_d$, and by Proposition 6.1.7 we have $\pi_d \circ \kappa_k^x = 0$. Applying the exact functor π_d to the exact sequence of Proposition 2.6.6.7) we get the short exact sequence

$$0 \longrightarrow (\delta_1^{\tilde{c}})^{\mathbf{St}} \circ \pi_d \longrightarrow (\delta_2^{(c,\tilde{c})})^{\mathbf{St}} \circ \pi_d \longrightarrow \tau_1^{\mathbf{St}} \circ (\delta_1^{c})^{\mathbf{St}} \circ \pi_d \longrightarrow 0 \ .$$

The co-unit of the adjunction of π_d and $\mathcal{S}_d$ gives a natural isomorphism $\eta : \pi_d \circ \mathcal{S}_d \simeq \mathrm{Id}$ by Proposition 1.3.10, so applying this exact sequence to the functor $\mathcal{S}_d(X)$, we get the following short exact sequence in $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$:

$$0 \longrightarrow (\delta_1^{\tilde{c}})^{\mathbf{St}}(X) \longrightarrow (\delta_2^{(c,\tilde{c})})^{\mathbf{St}}(X) \longrightarrow \tau_1^{\mathbf{St}} \circ (\delta_1^{c})^{\mathbf{St}}(X) \longrightarrow 0 \ .$$

By Proposition 7.2.5 the subcategory $\mathrm{Pol}_{n-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is thick and stable by τ_1, so the first and last terms of the short exact sequence are in $\mathrm{Pol}_{n-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by hypothesis, and so the middle term $(\delta_2^{(c,\tilde{c})})^{\mathbf{St}}(X)$ is also in $\mathrm{Pol}_{n-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. We then proved that for any colours $c, \tilde{c} \in C$, the functor $(\delta_2^{(c,\tilde{c})})^{\mathbf{St}}(X)$ is in $\mathrm{Pol}_{n-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and we conclude similarly, using the exact sequence of Proposition 2.6.6.7) in a general version, that $(\delta_k^x)^{\mathbf{St}}(X)$ is in $\mathrm{Pol}_{n-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$ by induction. □

Remark 7.2.7. For $d = 1$, the weak polynomial degree corresponds to the notion of stable degree of [CEF15] and [CEFN14] while the local degree precise how the weak and strong degrees are linked. It morally gives the strong polynomial degree modulo the weak polynomial degree and controls the rank from which the representation become stable. For example, in [CMNR18] they use these notions and spectral sequences to obtain representation stability results for two families of $\mathbf{FI}$-modules.

We end this section by showing that the precomposition by the forgetful functor $\mathcal{O} : \mathbf{FI}_d \to \mathbf{FI}$ from Definition 2.1.6 passes to the quotient and respects the polynomiality.

Lemma 7.2.8. *The functor $\mathcal{O}^* : \mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ passes to the quotient and induces a functor $\mathcal{O}^* : \mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ defined by the relation $\mathcal{O}^* \circ \pi_1 = \pi_d \circ \mathcal{O}^*$.*

Proof. By Lemma 6.3.5 we have an inclusion $\mathcal{O}^*\big(\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})\big) \subset \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. This shows that the composition $\pi_d \circ \mathcal{O}^*$ is zero on the subcategory $\mathcal{SN}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ of $\mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. Since the functor $\mathcal{O}^*$ is exact, we can use Proposition 1.3.4, which gives the result. □

We now show that the precomposition by the forgetful functor preserves the polynomiality when passed to the quotient categories.

Proposition 7.2.9. *For $n \in \mathbb{N}$ and $X \in \mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$, if X is in $\mathrm{Pol}_n(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ then $\mathcal{O}^*(X)$ is in $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, where $\mathcal{O}^* : \mathbf{St}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is the functor from Lemma 7.2.8.*

Proof. We prove the result by induction on $n \in \mathbb{N}$, the case $n = 0$ is being a special case of the following reasoning. For $X \in \mathrm{Pol}_n(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ and $c \in C$, we have the isomorphism

$$(\delta_1^c)^{\mathbf{St}} \circ \mathcal{O}^* \circ \pi_1 = (\delta_1^c)^{\mathbf{St}} \circ \pi_d \circ \mathcal{O}^* = \pi_d \circ \delta_1^c \circ \mathcal{O}^* \cong \pi_d \circ \mathcal{O}^* \circ \delta_1 = \mathcal{O}^* \circ \pi_1 \circ \delta_1 = \mathcal{O}^* \circ (\delta_1)^{\mathbf{St}} \circ \pi_1$$

given by Propositions 2.7.1, 7.1.6 and 7.2.8. Since the co-unit always gives an isomorphism from $\pi_1 \circ \mathcal{S}_1$ to the identity by Proposition 1.3.10, we conclude that there is an isomorphism $(\delta_1^c)^{\mathbf{St}} \circ \mathcal{O}^* \cong \mathcal{O}^* \circ (\delta_1)^{\mathbf{St}}$. Finally, since $(\delta_1)^{\mathbf{St}}(X)$ is in $\mathrm{Pol}_{n-1}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$, we get that $(\delta_1^c)^{\mathbf{St}} \circ \mathcal{O}^*(X) \cong \mathcal{O}^* \circ (\delta_1)^{\mathbf{St}}(X)$ is in $\mathrm{Pol}_{n-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by induction. Since this is true for all colours $c \in C$, it implies that $\mathcal{O}^*(X)$ is in $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. □

Finally, we explain in the following remark that the notion of weak polynomial functors corresponding to another quotient category of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by stably zero functors is more complex and have less properties than in $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

Remark 7.2.10. We recall that the composition $\pi_d \circ \delta_k^x$ is an exact functor because of the exact sequence (I) from Lemma 2.6.4 since $\kappa_k^x(F) \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by Proposition 6.1.7. For $c_{i_1}, \dots, c_{i_m} \in C$ some colours, in order to define the polynomial objects in the quotient of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by its thick subcategory $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, we must check that the endofunctors δ_k^x pass to this quotient. However, if π denotes the quotient functor associated with this quotient category, the composition $\pi \circ \delta_k^x$ is not an exact functor. Indeed, in general $\kappa_k^x(F)$ is not in $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$: for example, if $F_{c_1}^{\mathbf{FI}_2}$ is the functor from Example 2.3.3 sending all objects to $\mathbf{R}$, the maps containing the colour c_1 to zero and the others to the identity, then we see that $\kappa_1^{c_1}(F) = F$ is stably zero in c_1 but not in the other colours. We must then adapt Proposition 1.3.4 as we did in Proposition 1.3.6 by replacing the short exact sequence of an exact functor by the exact sequence $0 \longrightarrow \kappa_k^x \longrightarrow \mathrm{Id} \longrightarrow \tau_k \longrightarrow \delta_k^x \longrightarrow 0$ (I) from Lemma 2.6.4. This allows us, as in Proposition 7.1.6, to define the endofunctors $(\kappa_k^x)^{\mathbf{St}_{c_{i_1},\dots,c_{i_m}}}$ and $(\delta_k^x)^{\mathbf{St}_{c_{i_1},\dots,c_{i_m}}}$ of the quotient of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by its thick subcategory $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by the relations

$$\pi \circ \delta_k^x = (\delta_k^x)^{\mathbf{St}_{c_{i_1},\dots,c_{i_m}}} \circ \pi \quad \text{and} \quad \pi \circ \kappa_k^x = (\kappa_k^x)^{\mathbf{St}_{c_{i_1},\dots,c_{i_m}}} \circ \pi.$$

We can then define polynomial objects in this quotient as we did for $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ using these endofunctors $(\delta_k^x)^{\mathbf{St}_{c_{i_1},\dots,c_{i_m}}}$. The problem is that, during this process, we lost the fact that the endofunctors κ_k^x become zero in the quotient. We then also lost the exactness of the endofunctors $(\delta_k^x)^{\mathbf{St}_{c_{i_1},\dots,c_{i_m}}}$, which is fundamental to study the polynomial objects of the quotient. For example these subcategories of polynomial objects in this quotient do not seem to be closed under subobjects.

7.3 The pointwise tensor product

In this section we show that the pointwise tensor product respects polynomial objects of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, as for strong polynomial $\mathbf{FI}_d$-modules (Section 5.5) but with a simpler argument. To do this we introduce a long exact sequence of vector spaces connecting the kernel and the cokernel of the tensor product of two linear maps. We then use it to obtain a natural long exact sequence of functors associated with the tensor product of the two maps $i_k^x(F) : F \to \tau_k(F)$ and $i_l^y(G) : G \to \tau_l(G)$ of Definition 2.6.1. Finally, we use a part of this exact sequence to prove by induction that the tensor product preserves polynomial degree, with more precise bound than for the strong degree. Since this argument requires the stability by subobject, it does not work for strong degree which is why we used the arguments of Section 5.5. In this section we assume that $\mathbf{R} = \mathbb{K}$ is a field but all the statements are true if we consider only flat $\mathbf{R}$-modules and morphisms of $\mathbf{R}$-modules with flat kernels and cokernels. We start by showing that the pointwise tensor product from Definition 5.5.1 passes to the quotient $\mathbf{St}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$.

Lemma 7.3.1. *The pointwise tensor product from Definition 5.5.1 passes to the quotient of* $\mathbf{Fct}(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$ *by the subcategory* $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$*, which gives a functor*

$$\otimes : \mathbf{St}(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect}) \times \mathbf{St}(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect}) \to \mathbf{St}(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect}).$$

Proof. If F or G is in the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$, then so is $F \otimes G$. Indeed, we can compute for all $n \in \mathbf{FI}_d$ that

$$\underset{(n_1,\dots,n_d)\in\mathbb{N}^d}{\operatorname{colim}} (F \otimes G) \circ \xi_d (n_1, \dots, n_d) = \underset{(n_1,\dots,n_d)\in\mathbb{N}^d}{\operatorname{colim}} F \circ \xi_d (n_1, \dots, n_d) \otimes \underset{(n_1,\dots,n_d)\in\mathbb{N}^d}{\operatorname{colim}} G \circ \xi_d (n_1, \dots, n_d)$$

since on vector spaces the tensor product commutes with colimits. By Proposition 6.1.5, if F is in $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ then the colimit of $F \circ \xi_d$ is zero, and so is the colimit of $(F \otimes G) \circ \xi_d$. Using Proposition 6.1.5 again, this implies that $F \otimes G$ is in $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ and it is similar if G is in $\mathcal{SN}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$. Now the functor $\otimes : \mathbb{K}\text{-}\mathbf{Vect} \times \mathbb{K}\text{-}\mathbf{Vect} \to \mathbb{K}\text{-}\mathbf{Vect}$ is exact since every vector space is flat which implies that the pointwise tensor product $\otimes$ of $\mathbf{FI}_d$-modules over $\mathbb{K}$ is also exact. We can post-compose it with the exact functor π_d and, using Proposition 1.3.4 two times, we get an exact tensor functor on the quotient as stated. $\square$

We now introduce a lemma about the tensor product of linear maps, which we will use to construct the long exact sequence of Proposition 7.3.4.

Lemma 7.3.2. *Let $f : X \to Y$ and $f' : X' \to Y'$ be two linear maps between $\mathbb{K}$-vector spaces, the two exact sequences associated with their kernel and cokernel* $0 \longrightarrow K \xrightarrow{i} X \xrightarrow{f} Y \xrightarrow{p} C \longrightarrow 0$ *and* $0 \longrightarrow K' \xrightarrow{i'} X' \xrightarrow{f'} Y' \xrightarrow{p'} C' \longrightarrow 0$ *can be combined to form the following long exact sequence associated with the tensor product $f \otimes f'$:*

$$\begin{tikzcd}
0 \arrow[r] & K \otimes K' \arrow[r, "(\mathrm{Id}_K \otimes i') \oplus (i \otimes \mathrm{Id}_{K'})"] & (K \otimes X') \oplus (X \otimes K') \arrow[r, "(i \otimes \mathrm{Id}_{X'}) - (\mathrm{Id}_X \otimes i')"] & X \otimes X' \arrow[d, "f \otimes f'"] \\
0 & C \otimes C' \arrow[l] & (C \otimes Y') \oplus (Y \otimes C') \arrow[l, "(\mathrm{Id}_C \otimes p') - (p \otimes \mathrm{Id}_{C'})"] & Y \otimes Y' \arrow[l, "(p \otimes \mathrm{Id}_{Y'}) \oplus (\mathrm{Id}_Y \otimes p')"]
\end{tikzcd}$$

Proof. The kernel K of f is a vector subspace so we can choose a complement S such that $X = K \oplus S$. Then the classical isomorphism $X/Ker(f) \cong \mathrm{Im}(f)$ gives an isomorphism $\mathrm{Im}(f) \cong S$. By definition we also have $C = \mathrm{Coker}(f) \cong Y/\mathrm{Im}(f)$, which means that $C \cong Y/S$ and that $Y \cong C \oplus S$ since every short exact sequence of vector spaces splits. Then we have a new exact sequence

$$0 \longrightarrow K \xrightarrow{\mathrm{Id}_K \oplus 0} K \oplus S \xrightarrow{0 \oplus \mathrm{Id}_S} C \oplus S \xrightarrow{\mathrm{Id}_C \oplus 0} C \longrightarrow 0 ,$$

and there is a natural equivalence between this exact sequence and the one with f from the statement since the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "p"] \arrow[d, "\cong"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, "\mathrm{Id}_K \oplus 0"'] & K \oplus S \arrow[r, "0 \oplus \mathrm{Id}_S"'] & C \oplus S \arrow[r, "\mathrm{Id}_C \oplus 0"'] & C \arrow[r] & 0.
\end{tikzcd}$$

The same construction works for f' and we can combine the two commutative diagrams (for f and f') with the tensor product to make the following sequence equivalent to the one of the statement:

$$\begin{tikzcd}
0 \arrow[r] & K\otimes K' \arrow[r, "\Delta_{K\otimes K'}\oplus 0"] & (K\otimes K')^{\oplus 2}\oplus(K\otimes S')\oplus(S\otimes K') \arrow[r, "\nabla_{K\otimes K'}\oplus \mathrm{Id}"] & {\begin{array}{c}(K\otimes K')\oplus(K\otimes S')\\ \oplus(S\otimes K')\oplus(S\otimes S')\end{array}} \arrow[d, "0\oplus \mathrm{Id}_{S\otimes S'}"] \\
0 & C\otimes C' \arrow[l] & (C\otimes C')^{\oplus 2}\oplus(C\otimes S')\oplus(S\otimes C') \arrow[l, "\nabla_{C\otimes C'}\oplus 0"'] & {\begin{array}{c}(C\otimes C')\oplus(C\otimes S')\\ \oplus(S\otimes C')\oplus(S\otimes S')\end{array}} \arrow[l, "\Delta_{C\otimes C'}\oplus \mathrm{Id}\oplus 0"']
\end{tikzcd}$$

where $\Delta_{K\otimes K'} : K\otimes K' \to (K\otimes K')^{\oplus 2}$ is the diagonal map and $\nabla_{K\otimes K'} : (K\otimes K')^{\oplus 2} \to K\otimes K'$ is the identity on the first component and minus the identity on the second one. We can check at each term that this sequence is exact since it consists only of zero and identity maps. This implies that the long sequence of the statement is also exact since they are equivalent. □

Remark 7.3.3. The proof of Lemma 7.3.2 is not canonical since it depends strongly on the choice of the complements S and S' of K and K' in X and X'.

To study the tensor product of polynomial functors we use the exact sequence of vector spaces of Lemma 7.3.2 to induce a similar exact sequence of functors associated with the endofunctors κ_k^x and δ_k^x.

Proposition 7.3.4. *Let* $\mathbf{R} = \mathbb{K}$ *be a field, then for* F, G *in* $\mathbf{FI}_d\text{-}\boldsymbol{Mod} = \mathbf{Fct}(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$ *there is an exact sequence of functors associated with the tensor product* $i_k^x(F)\otimes i_l^y(G)$*:*

$$\begin{tikzcd}
0 \arrow[r] & \kappa_k^x(F)\otimes\kappa_l^y(G) \arrow[r, "\mathrm{Id}\otimes\iota_G\oplus\iota_F\otimes\mathrm{Id}"] & \big(\kappa_k^x(F)\otimes G\big)\oplus\big(F\otimes\kappa_l^y(G)\big) \arrow[r, "\iota_F\otimes\mathrm{Id}-\mathrm{Id}\otimes\iota_G"] & F\otimes G \arrow[d, "i_k^x(F)\otimes i_l^y(G)"'] \\
0 & \delta_k^x(F)\otimes\delta_l^y(G) \arrow[l] & \big(\delta_k^x(F)\otimes\tau_l(G)\big)\oplus\big(\tau_k(F)\otimes\delta_l^y(G)\big) \arrow[l, "\mathrm{Id}\otimes\rho_G-\rho_F\otimes\mathrm{Id}"'] & \tau_k(F)\otimes\tau_l(G) \arrow[l, "\rho_F\otimes\mathrm{Id}\oplus\mathrm{Id}\otimes\rho_G"']
\end{tikzcd}$$

Proof. For $n \in \mathbf{FI}_d$, we apply Lemma 7.3.2 to the two exact sequences of vector spaces

$$0 \longrightarrow \kappa_k^x(F)(n) \longrightarrow F(n) \xrightarrow{i_k^x(F)(n)} \tau_k(F)(n) \longrightarrow \delta_k^x(F)(n) \longrightarrow 0$$

and

$$0 \longrightarrow \kappa_l^y(G)(n) \longrightarrow G(n) \xrightarrow{i_l^y(G)(n)} \tau_l(G)(n) \longrightarrow \delta_l^y(G)(n) \longrightarrow 0\ ,$$

obtained from the exact sequence (I) from Lemma 2.6.4. It implies that, for all $n \in \mathbf{FI}_d$, the long sequence of vector spaces corresponding to the one of the statement evaluated in n is exact. This exact sequence of vector spaces is natural in $n \in \mathbf{FI}_d$ by the definitions of the endofunctors τ_k, δ_k^x, κ_k^x and of the tensor product. □

In the following result we extract a short exact sequence from the long exact sequence of Proposition 7.3.4 that we will use to prove the Theorem 7.3.6.

Corollary 7.3.5. *Let* $\mathbf{R} = \mathbb{K}$ *be a field, for* $F, G \in \mathbf{FI}_d\text{-}\boldsymbol{Mod}$ *there is a natural short exact sequence*

$$0 \longrightarrow \delta_k^x(F\otimes G) \longrightarrow \big(\delta_k^x(F)\otimes\tau_k(G)\big)\oplus\big(\tau_k(F)\otimes\delta_k^x(G)\big) \longrightarrow \delta_k^x(F)\otimes\delta_k^x(G) \longrightarrow 0.$$

Proof. By definition of the pointwise tensor product we get $\tau_k(F\otimes G) = \tau_k(F)\otimes\tau_k(G)$ and $i_k^x(F\otimes G) = i_k^x(F)\otimes i_k^x(G)$. The exact sequence (I) from Lemma 2.6.4 associated with the functor $F\otimes G$ can then be written as

$$0 \longrightarrow \kappa_k^x(F\otimes G) \longrightarrow F\otimes G \xrightarrow[=i_k^x(F)\otimes i_k^x(G)]{i_k^x(F\otimes G)} \tau_k(F)\otimes\tau_k(G) \longrightarrow \delta_k^x(F\otimes G) \longrightarrow 0\ .$$

We then get the short exact sequence of the statement by splitting the long exact sequence of Proposition 7.3.4 for $k = l$ and $x = y$ with the following epi-mono factorization:

$$\tau_k(F)\otimes\tau_k(G) \twoheadrightarrow \delta_k^x(F\otimes G) \hookrightarrow \big(\delta_k^x(F)\otimes\tau_k(G)\big)\oplus\big(\tau_k(F)\otimes\delta_k^x(G)\big)\ ,$$

which holds since $\operatorname{Coker}(\,i_k^x(F)\otimes i_k^x(G)\,) \cong \delta_k^x(F\otimes G)$. □

We finally prove that the pointwise tensor product respects polynomiality. In addition to providing numerous examples of polynomial $\mathbf{FI}_d$-modules, an interesting application of this theorem is to give a second proof (in Theorem 8.3.11) that the quotient of $P_n^{\mathbf{FI}_d}$ that we study in Section 8.3 is weak polynomial of degree n.

Theorem 7.3.6. *For $\mathbf{R} = \mathbb{K}$ be a field, $X \in \mathrm{Pol}_n(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$ and $Y \in \mathrm{Pol}_m(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$, we have $X\otimes Y \in \mathrm{Pol}_{n+m}(\mathbf{FI}_d, \mathbb{K}\text{-}\boldsymbol{Vect})$.*

Proof. We proceed by induction on n, for m fixed. By symmetry we also have the result for n fixed as m varies and the two together give the result for all $n, m \in \mathbb{N}$. For X in $\mathrm{Pol}_{n+1}(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ and Y in $\mathrm{Pol}_m(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$, by Corollary 7.3.5 we have a short exact sequence

$$0 \longrightarrow \delta_k^x(F\otimes G) \longrightarrow \big(\delta_k^x(F)\otimes\tau_k(G)\big)\oplus\big(\tau_k(F)\otimes\delta_k^x(G)\big) \longrightarrow \delta_k^x(F)\otimes\delta_k^x(G) \longrightarrow 0.$$

associated with the $\mathbf{FI}_d$-modules $F = \mathcal{S}_d(X)$ and $G = \mathcal{S}_d(Y)$. As shown in Lemma 7.3.1 we have $\pi_d(F)\otimes\pi_d(G) = \pi_d(F\otimes G)$, so we can apply the exact functor π_d to this short exact sequence. Because of the isomorphisms $\pi_d\circ\delta_k^x = (\delta_k^x)^{\mathbf{St}}\circ\pi_d$, $\pi_d\circ\tau_k = \tau_k^{\mathbf{St}}\circ\pi_d$ and $\pi_d\circ\mathcal{S}_d \cong \mathrm{Id}$ from Propositions 7.1.6 and 1.3.10, we get the following short exact sequence in $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$:

$$0 \to (\delta_k^x)^{\mathbf{St}}(X\otimes Y) \to \begin{matrix}\big((\delta_k^x)^{\mathbf{St}}(X)\otimes(\tau_k)^{\mathbf{St}}(Y)\big)\\ \oplus\big((\tau_k)^{\mathbf{St}}(X)\otimes(\delta_k^x)^{\mathbf{St}}(Y)\big)\end{matrix} \to (\delta_k^x)^{\mathbf{St}}(X)\otimes(\delta_k^x)^{\mathbf{St}}(Y) \to 0$$

By hypothesis, $(\delta_1^c)^{\mathbf{St}}(X)$ is in $\mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ for all $c \in C$. Since the subcategory $\mathrm{Pol}_m(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is stable by $(\tau_1)^{\mathbf{St}}$ and $(\delta_1^c)^{\mathbf{St}}$ by Proposition 7.2.5, we get by induction that both $(\delta_1^c)^{\mathbf{St}}(X)\otimes(\tau_1)^{\mathbf{St}}(Y)$ and $(\tau_1)^{\mathbf{St}}(X)\otimes(\delta_1^c)^{\mathbf{St}}(Y)$ are in $\mathrm{Pol}_{n+m}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Using Proposition 7.2.5 again, this subcategory is closed under subobjects so the short exact sequence above implies that $(\delta_1^c)^{\mathbf{St}}(X\otimes Y) \in \mathrm{Pol}_{n+m}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ for all $c \in C$, showing that $X\otimes Y \in \mathrm{Pol}_{n+m+1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. For $n = 0$, since $(\delta_k^x)^{\mathbf{St}}(X) = 0$, the above short exact sequence gives an isomorphism $(\delta_1^c)^{\mathbf{St}}(X\otimes Y) \cong (\tau_1)^{\mathbf{St}}(X)\otimes(\delta_1^c)^{\mathbf{St}}(Y)$ with $(\tau_1)^{\mathbf{St}}(X) \in \mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and $(\delta_1^c)^{\mathbf{St}}(Y) \in \mathrm{Pol}_{m-1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. We conclude this case by induction on $m \in \mathbb{N}$ to prove that $X\otimes Y \in \mathrm{Pol}_m(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. □

Remark 7.3.7. Since the method used in the proof of Theorem 7.3.6 requires the stability by subobject, it does not work for strong degree. This is why we used a different argument in Section 5.5 to prove that the pointwise tensor product preserves the notion of strong polynomial $\mathbf{FI}_d$-modules.

7.4 Description of $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$

In this section we give a description of the category $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of polynomial objects of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of degree 0. These functors are actually given by an object M of the category $\mathbf{R}\text{-}\mathbf{Mod}$, together with $d-1$ automorphisms of M which commute two by two or by a $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]$-module. More precisely, we define $\mathbf{R}\text{-}\mathbf{Mod}_d$ the category whose objects are the tuples $(M, \varphi_2, \dots, \varphi_d)$, where M is an object of $\mathbf{R}\text{-}\mathbf{Mod}$ and $\varphi_2, \dots, \varphi_d : M \to M$ are $d-1$ isomorphisms in $\mathbf{R}\text{-}\mathbf{Mod}$ commuting two by two and we prove the following in Theorem 7.4.12.

Theorem. *There is an equivalence of categories* $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}) \cong \mathbf{R}\text{-}\mathbf{Mod}_d$ *given by the functor* $\pi_d \circ \Theta_d$*, where* $\Theta_d(M, \varphi_2, \dots, \varphi_d)$ *is the functor that sends all objects* k *in* $\mathbf{FI}_d$ *to* M *and a morphism* $x = (c_{j_1}, \dots, c_{j_k}) \in \mathbf{FI}_d(0,k)$ *to* $\varphi_{j_1} \circ \dots \circ \varphi_{j_k}$*, with* $\varphi_1 = \mathrm{Id}$.

Since $\mathbf{R}\text{-}\mathbf{Mod}_d$ is equivalent to the category of $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]$-modules (see Remark 7.4.9) we also have an equivalence between $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]-\mathbf{Mod}$. For $d = 1$ we recover a special case of [DV19, Theorem 2.26] which says that, for $\mathbf{FI}$, the only objects in $\mathrm{Pol}_0(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ are the constant functors, and that the equivalence is given by $\pi_{\mathcal{M}} \circ c$, where $c : \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{Fct}(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ sends $M \in \mathbf{R}\text{-}\mathbf{Mod}$ to the constant functor M.

We prove the Theorem 7.4.12 in two steps: first we define an abstract condition $(POL0)$ and we show in Proposition 7.4.2 that the polynomial objects of degree 0 are those which satisfy this condition. Then we show that the objects satisfying the condition $(POL0)$ correspond to the objects of $\mathbf{R}\text{-}\mathbf{Mod}_d$.

Definition 7.4.1. An object X of $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ satisfies *the condition* **(POL0)** if, for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$, the morphism

$$i_k^x\big(\,\mathcal{S}_d(X)\,\big) = \mathcal{S}_d(X)\left(\mathrm{Id}_{(-)} + x\right) : \mathcal{S}_d(X)(-) \longrightarrow \tau_k\left(\mathcal{S}_d(X)\right)(-),$$

is an isomorphism, where $\mathcal{S}_d$ is the section functor of Lemma 7.1.5.

Proposition 7.4.2. *An object* X *of* $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ *is in* $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ *if and only if it satisfies the condition* $(POL0)$ *from Definition 7.4.1.*

Proof. If X satisfies the condition $(POL0)$, then for any $c \in C$ the morphism $i_1^c(\mathcal{S}_d(X))$ is an isomorphism, so its cokernel $\delta_c(\mathcal{S}_d(X))$ is zero. Then we have

$$\delta_c^{\mathbf{St}}(X) \cong \delta_c^{\mathbf{St}} \circ \pi_d \circ \mathcal{S}_d(X) \cong \pi_d \circ \delta_c\big(\,\mathcal{S}_d(X)\,\big) \cong \pi_d(0) = 0,$$

where the first equivalence is given by the co-unit $\eta_X : \pi_d \circ \mathcal{S}_d(X) \cong X$ of the adjunction of π_d and $\mathcal{S}_d$ (Proposition 1.3.10) and the second by Proposition 7.1.6. This shows that $\delta_c^{\mathbf{St}}(X) = 0$ for all $c \in C$, so X is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Conversely, if X is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, we show that for any $k \in \mathbf{FI}_d$ and any $x \in \mathbf{FI}_d(0,k)$ the morphism $i_k^x(\mathcal{S}_d(X))$ is a monomorphism and an epimorphism in the abelian category $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, *i.e.* an isomorphism. First, since X is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by Lemma 7.2.6 the functor $(\delta_k^x)^{\mathbf{St}}(X)$ is zero and so we have

$$\delta_k^x\big(\mathcal{S}_d(X)\big) \hookrightarrow \mathcal{S}_d \circ (\delta_k^x)^{\mathbf{St}}(X) = \mathcal{S}_d(0) = 0,$$

where the monomorphism is given by Proposition 7.1.10. This proves that $i_k^x(\mathcal{S}_d(X))$ is an epimorphism. Moreover, by adjunction of π_d and $\mathcal{S}_d$ we have, for all $H \in \mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$:

$$\mathrm{Hom}_{\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})}\left(H, \mathcal{S}_d(X)\right) \cong \mathrm{Hom}_{\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})}\left(\pi_d(H), X\right) = \mathrm{Hom}\left(0, X\right) = 0.$$

By Proposition 6.1.12, this implies that $\kappa(\mathcal{S}_d(X)) = 0$. Since the endofunctor κ is the sum of all the $\kappa_{\tilde{k}}^{\tilde{x}}$, the minimality of the sum implies that all $\kappa_{\tilde{k}}^{\tilde{x}}(\mathcal{S}_d(X))$ are zero. We then have $\kappa_k^x = 0$ and so i_k^x is a monomorphism. $\square$

Remark 7.4.3. The Proposition 7.4.2 and the definitions of strong and weak polynomial $\mathbf{FI}_d$-modules give the following two characterizations for the degree 0:

- A functor $F \in \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ is in $\mathrm{Pol}_0^{strong}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ if and only if the morphism $i_k^x(F)$ is an epimorphism for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$.
- An object $X \in \mathbf{St}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ if and only if the morphism $i_k^x(\mathcal{S}_d(X))$ is an isomorphism for all $k \in \mathbf{FI}_d$ and all $x \in \mathbf{FI}_d(0,k)$.

For $d = 1$, the first point is included in [DV19, Proposition 2.9] and the second in the proof of [DV19, Proposition 2.26]. For $F \in \mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ and $X \in \mathbf{St}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ by applying the exact functor π_d to the exact sequence (I) from Lemma 2.6.4, by Proposition 6.1.7 we have the following:

- If X is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ then $\mathcal{S}_d(X)$ is in $\mathrm{Pol}_0^{strong}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$,
- If F is in $\mathrm{Pol}_0^{strong}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ then $\pi_d(F)$ is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$.

Note that the first point is very specific to the degree 0, while the second one is true in general (see Remark 7.2.2).

We now give an explicit description of the functors satisfying the condition $(POL0)$, which will be used to prove Theorem 7.4.12. First recall that by Definition 7.4.1, if $X \in \mathbf{St}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ satisfies the condition $(POL0)$, then $\mathcal{S}_d(X)(c_1^k)$ is an isomorphism for all $k \in \mathbf{FI}_d$. Using this, the Proposition 2.5.4 and the category $\underline{\mathbf{FI}_d}$ given in Definition 2.5.2 we define a functor H_X isomorphic to $\mathcal{S}_d(X)$ and we give an explicit description of this functor. This equivalence is essential for the proof of Theorem 7.4.12.

Definition 7.4.4. For $X \in \mathbf{St}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ satisfying the condition $(POL0)$, the *functor* $\underline{H_X} : \underline{\mathbf{FI}_d} \to \mathbf{R\text{-}Mod}$ is given on an object $n \in \underline{\mathbf{FI}_d}$ by $\underline{H_X}(n) = \mathcal{S}_d(X)(0)$, and on a morphism $x \in \mathbf{FI}_d(0,k) = \underline{\mathbf{FI}_d}(0,k)$ by

$$\underline{H_X}(x) = \left(\mathcal{S}_d(X)\ (c_1^k)\ \right)^{-1} \circ \mathcal{S}_d(X)\ (x),$$

where c_1 is a fixed colour, and by the identity on $\underline{\mathbf{FI}_d}(k,k) = \mathbf{FI}_d(k,k) = \mathrm{S}_k$.

This functor extends to a unique *functor* $H_X : \mathbf{FI}_d \to \mathbf{R\text{-}Mod}$ by Proposition 2.5.4 since it sends every morphism to isomorphisms. We now explain how the functor H_X is completely determined by its image on the morphisms $c \in \mathbf{FI}_d(0,1) = C$ using the subcategory $\underline{\mathbf{FI}_d}$ of $\mathbf{FI}_d$ from Definition 2.5.2. These images correspond to the $d-1$ isomorphisms of modules of the category $\mathbf{R\text{-}Mod}_d$ as we trivialize the action of c_1.

Proposition 7.4.5. *If $X \in \mathbf{St}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ satisfies the condition $(POL0)$, then $\underline{H_X}$ is determined by the images*

$$\underline{H_X}(c) = \Big(\mathcal{S}_d(X)\ (c_1)\ \Big)^{-1} \circ \mathcal{S}_d(X)\ (c)\ :\ \mathcal{S}_d(X)(0) \to \mathcal{S}_d(X)(0)$$

for $c \in C$ and by the relations $\underline{H_X}(c) \circ \underline{H_X}(c') = \underline{H_X}(c') \circ \underline{H_X}(c)$ for $c, c' \in C$.

Proof. For $x \in \mathbf{FI}_d(0,k)$ and $y \in \mathbf{FI}_d(0,l)$, by applying the functor $\mathcal{S}_d(X)$ to the relation $(\mathrm{Id}_l + x) \circ c_1^l = (c_1^l + \mathrm{Id}_k) \circ x$ in $\mathbf{FI}_d$ we get $\mathcal{S}_d(X) \Big(\mathrm{Id}_l + x \Big) \circ \mathcal{S}_d(X)\,(c_1^l) = \mathcal{S}_d(X) \Big(c_1^l + \mathrm{Id}_k \Big) \circ \mathcal{S}_d(X)\,(x)$. Using this, by definition of $\underline{H_X}$, we get the identities

$$\begin{aligned}
\underline{H_X}(x) \circ \underline{H_X}(y) &= \Big(\mathcal{S}_d(X)\ (c_1^k)\ \Big)^{-1} \circ \mathcal{S}_d(X)\ (x)\ \circ\ \Big(\mathcal{S}_d(X)\ (c_1^l)\ \Big)^{-1} \circ \mathcal{S}_d(X)\ (y) \\
&= \Big(\mathcal{S}_d(X)\ (c_1^k)\ \Big)^{-1} \circ \Big(\mathcal{S}_d(X)\ \Big(c_1^l + \mathrm{Id}_k\ \Big) \Big)^{-1} \circ \mathcal{S}_d(X)\ \Big(\mathrm{Id}_l + x \Big) \circ \mathcal{S}_d(X)\ (y) \\
&= \Big(\mathcal{S}_d(X)\ (c_1^{k+l})\ \Big)^{-1} \circ \mathcal{S}_d(X)\ \Big((y,x) \Big) \\
&= \underline{H_X}\Big((y,x) \Big).
\end{aligned}$$

This proves that for any two morphisms x, y starting from 0 we have the relation $\underline{H_X}(x) \circ \underline{H_X}(y) = \underline{H_X}(y,x)$ and by induction we conclude that $\underline{H_X}$ is determined only by the image $\underline{H_X(c)}$ of the colour morphisms $c \in \mathbf{FI}_d(0,1)$. Finally, since $\mathcal{S}_d(X)(c^k)$ is an isomorphism for all $k \in \mathbf{FI}_d$ and all $c \in C$, we have $\mathcal{S}_d(X)\,(y,x) = \mathcal{S}_d(X)\,(y,x)$ by Proposition 2.5.1. This gives for $c, \tilde{c} \in C$, with the previous relations, the identity

$$\underline{H_X}(c) \circ \underline{H_X}(\tilde{c}) = \underline{H_X}\big(\,(\tilde{c}, c)\,\big) = \underline{H_X}\big(\,(c,\tilde{c})\,\big) = \underline{H_X}(\tilde{c}) \circ \underline{H_X}(c).$$

Finally, any family of commuting isomorphisms $\big(\underline{H_X(c)}\big)_{c \in C}$ of $\mathcal{S}_d(X)(0)$ (*i.e.* satisfying the identities $\underline{H_X}(c) \circ \underline{H_X}(\tilde{c}) = \underline{H_X}(\tilde{c}) \circ \underline{H_X}(c)$ for $c, \tilde{c} \in C$) determines a unique functor $\underline{H_X} : \underline{\mathbf{FI}_d} \to \mathbf{R}\text{-}\mathbf{Mod}$ by the formulas above. □

Proposition 7.4.6. *If $X \in \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ satisfies the condition $(POL0)$, then H_X is determined by the images $H_X(c) = \big(\,\mathcal{S}_d(X)\,(c_1)\,\big)^{-1} \circ \mathcal{S}_d(X)\,(c)\,:\,\mathcal{S}_d(X)(0) \to \mathcal{S}_d(X)(0)$ for $c \in C$, and by the relations $H_X(c) \circ H_X(c') = H_X(c') \circ H_X(c)$ for $c, c' \in C$.*

Proof. It is a consequence of Proposition 7.4.5 and Proposition 2.5.4, which state that a $\mathbf{FI}_d$-module is determined by its underlying functor over $\underline{\mathbf{FI}_d}$. □

We now prove that, for $X \in \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ which satisfies the condition $(POL0)$, the functor $\mathcal{S}_d(X)$ is equivalent to the functor H_X of Definition 7.4.4. This allows us to conclude that the $\mathbf{FI}_d$-module $\mathcal{S}_d(X)$ is determined by its image on the colouring morphism $c \in C$, which is the key point in the proof of the theorem.

Lemma 7.4.7. *For $X \in \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ satisfying the condition $(POL0)$ there is a natural equivalence $\varepsilon : H_X \cong \mathcal{S}_d(X)$ given by $\varepsilon_n = \mathcal{S}_d(X)(c_1^n) : H_X(n) = \mathcal{S}_d(X)(0) \to \mathcal{S}_d(X)(n)$ for $n \in \mathbf{FI}_d$.*

Proof. Let $u \in \mathbf{FI}_d(n, n+m)$ be a general morphism in $\mathbf{FI}_d$, the identity

$$H_X(c_1^n) = \underline{H_X}(c_1^n) = (\mathcal{S}_d(X)\,(c_1^n))^{-1} \circ \mathcal{S}_d(X)\,(c_1^n) = \mathrm{Id}$$

gives that $H_X(u) = H_X(u \circ c_1^n) = \big(\,\mathcal{S}_d(X)\,(c_1^{n+m})\,\big)^{-1} \circ \mathcal{S}_d(X)\,(u \circ c_1^n)$. This implies the naturality of ε:

$$\varepsilon_{n+m} \circ H_X(u) = \mathcal{S}_d(X)\,(c_1^{n+m}) \;\circ\; H_X(u) = \mathcal{S}_d(X)\,(u \circ c_1^n) = \mathcal{S}_d(X)\,(u) \circ \varepsilon_n.$$

Finally, ε is a natural equivalence by definition of the condition $(POL0)$. □

Finally, we define the category $\mathbf{R}\text{-}\mathbf{Mod}_d$ which will be isomorphic to the category $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of polynomial objects of degree 0. The equivalence is given in Theorem 7.4.12.

Definition 7.4.8. The *category* $\mathbf{R}\text{-}\mathbf{Mod}_d$ has for objects the tuple $\big(M, \varphi_2, \dots, \varphi_d\big)$, where M is an object of $\mathbf{R}\text{-}\mathbf{Mod}$ and $\varphi_2, \dots, \varphi_d : M \to M$ are $d-1$ isomorphisms in $\mathbf{R}\text{-}\mathbf{Mod}$ which commute two by two. The morphisms in $\mathbf{R}\text{-}\mathbf{Mod}_d$ from $(M, \varphi_2, \dots, \varphi_d)$ to $(M', \varphi'_2, \dots, \varphi'_d)$ are the morphisms $f : M \to M'$ in $\mathbf{R}\text{-}\mathbf{Mod}$ such that $\varphi'_j \circ f = f \circ \varphi_j$ for all $2 \le j \le d$, and the composition in $\mathbf{R}\text{-}\mathbf{Mod}_d$ comes from $\mathbf{R}\text{-}\mathbf{Mod}$.

Remark 7.4.9. The category $\mathbf{R}\text{-}\mathbf{Mod}_d$ is equivalent to the category $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]\text{–}\mathbf{Mod}$ of modules over the ring $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]$ of commutative polynomials in the $d-1$ variables $x_2, \dots, x_d$, all invertible. The equivalence is given by the functor that sends a $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]$-module M to the tuple $(M, \varphi_2, \dots, \varphi_d)$, where φ_i is given by the action of the variable x_i for $2 \le i \le d$.

We use the subcategory $\underline{\mathbf{FI}_d}$ of $\mathbf{FI}_d$ to define the functor Θ_d which gives the equivalence of categories in Theorem 7.4.12.

Definition 7.4.10. The *functor* $\Theta_d : \mathbf{R}\text{-}\mathbf{Mod}_d \to \mathbf{Fct}\big(\underline{\mathbf{FI}_d}, \mathbf{R}\text{-}\mathbf{Mod}\big)$ is given on an object $(M, \varphi_2, \dots, \varphi_d)$ by $\Theta_d(M, \varphi_2, \dots, \varphi_d)\,(k) = M$, for all $k \in \underline{\mathbf{FI}_d}$, and for $x = (c_{j_1}, \dots, c_{j_k}) \in \underline{\mathbf{FI}_d}(0, k)$ by $\Theta_d(M, \varphi_2, \dots, \varphi_d)\,(c_{j_1}, \dots, c_{j_k}) = \varphi_{j_1} \circ \cdots \circ \varphi_{j_k}$, where φ_1 denotes the identity. The image of a morphism f from $(M, \varphi_2, \dots, \varphi_d)$ to $(M', \varphi'_2, \dots, \varphi'_d)$ by Θ_d is the natural transformation ε defined by $\varepsilon_n = f : M \to M'$.

Remark 7.4.11. Since the images of the morphisms by $\Theta_d(M, \varphi_2, \dots, \varphi_d)$ are all isomorphisms, this definition is extended to obtain a *functor* Θ_d from $\mathbf{R}\text{-}\mathbf{Mod}_d$ to $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ using Proposition 2.5.4.

We end this section with the theorem describing the polynomial functors of degree 0.

Theorem 7.4.12. *There is an equivalence of categories* $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}) \cong \mathbf{R}\text{-}\mathbf{Mod}_d$ *given by the functor* $\pi_d \circ \Theta_d : \mathbf{R}\text{-}\mathbf{Mod}_d \to \mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

Remark 7.4.13. For $d = 1$, since $\mathbf{R}\text{-}\mathbf{Mod}_1 = \mathbf{R}\text{-}\mathbf{Mod}$ we recover the description of $\mathrm{Pol}_0(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$ given by Djament and Vespa in Theorem 2.26 of [DV19]:

$$\mathrm{Pol}_0(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod}) \cong \mathbf{R}\text{-}\mathbf{Mod}\,.$$

Proof of Theorem 7.4.12. First we prove that the essential image of $\pi_d \circ \Theta_d$ is the subcategory $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$: for $(M, \varphi_2, \dots, \varphi_d) \in \mathbf{R}\text{-}\mathbf{Mod}_d$ and $c \in C$, we have

$$\delta_c^{\mathbf{St}}\,\big(\pi_d \circ \Theta_d(M, \varphi_2, \dots, \varphi_d)\big) = \pi_d\Big(\delta_c\big(\Theta_d(M, \varphi_2, \dots, \varphi_d)\big)\Big) = \pi_d(0) = 0,$$

where the first equality is given by Proposition 7.1.6 and the second comes from the fact that $\Theta_d(M, \varphi_2, \dots, \varphi_d)\,(\mathrm{Id} + c)$ is an isomorphism. This shows that the image of the functor $\pi_d \circ \Theta_d$ is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Now if X is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ it satisfies the condition $(POL0)$ by Proposition 7.4.2 and, by Lemma 7.4.7, the functor $\mathcal{S}_d(X)$ is equivalent to the functor H_X which is exactly the image of

$$M_X := \Big(\mathcal{S}_d(X)(0),\, \big(\mathcal{S}_d(X)\,(c_1)\big)^{-1} \circ \mathcal{S}_d(X)\,(c_2)\,, \dots,\, \big(\mathcal{S}_d(X)\,(c_1)\big)^{-1} \circ \mathcal{S}_d(X)\,(c_d)\,\Big)$$

by the functor Θ_d by Proposition 7.4.6. By Proposition 1.3.10 the co-unit of the adjunction of π_d and $\mathcal{S}_d$ is always an isomorphism from $\pi_d \circ \mathcal{S}_d$ to Id and, since π_d is exact, we get the isomorphism

$$X \cong \pi_d \circ \mathcal{S}_d\,(X) \cong \pi_d \circ H_X \cong \pi_d \circ \Theta_d\,(M_X).$$

Finally, we show that the functor $\pi_d \circ \Theta_d$ is full and faithful. The functor Θ_d is faithful since we have $(\,\Theta_d(f)\,)_0 = f$ for any morphism f in $\mathbf{R}\text{-}\mathbf{Mod}_d$. For ε a natural transformation between $\Theta_d(M, \varphi_2, \dots, \varphi_d)$ and $\Theta_d(N, \psi_2, \dots, \psi_d)$, by naturality we have the relations $\varepsilon_1 \circ \varphi_j = \psi_j \circ \varepsilon_0$ and $\varepsilon_n \circ \big(\varphi_{j_n} \circ \cdots \circ \varphi_{j_1}\big) = \big(\psi_{j_n} \circ \cdots \circ \psi_{j_1}\big) \circ \varepsilon_0$ for all $1 \leq j, j_1, \dots, j_n \leq d$ with $\varphi_1 = \psi_1 = \mathrm{Id}$. Using the second relation with $j_1 = \cdots = j_n = 1$ we get $\varepsilon_n = \varepsilon_0$ for all $n \in \mathbf{FI}_d$. The first relations then give $\varepsilon_0 \circ \varphi_j = \psi_j \circ \varepsilon_0$ for all $2 \leq j \leq d$. This means exactly that $\varepsilon = \Theta_d(\varepsilon_0 : M \to N)$, so the functor Θ_d is full and faithful. Moreover, the functor $\Theta_d(N\, \psi_2, \dots, \psi_d)$ sends all morphisms to isomorphisms which are split monomorphisms. Then it satisfies the hypothesis of Proposition 7.1.11 so the functor π_d is an isomorphism on arrows and, with the previous point, we get a natural bijection

$$\begin{aligned}
& \mathrm{Hom}_{\mathbf{R}\text{-}\mathbf{Mod}_d}\,\big(\,(M, \varphi_2, \dots, \varphi_d)\,,\,(N, \psi_2, \dots, \psi_d)\,\big) \\
\cong\ & \mathrm{Hom}_{\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})}\,\big(\,\Theta_d(M, \varphi_2, \dots, \varphi_d)\,,\,\Theta_d(N, \psi_2, \dots, \psi_d)\,\big) \\
\cong\ & \mathrm{Hom}_{\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})}\,\big(\,\pi_d \circ \Theta_d(M, \varphi_2, \dots, \varphi_d)\,,\,\pi_d \circ \Theta_d(N, \psi_2, \dots, \psi_d)\,\big).
\end{aligned}$$

This shows that the functor $\pi_d \circ \Theta_d$ is full and faithful and that its essential image is $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. $\square$

Chapter 8

Weak polynomial quotients of the projective standard functors

Nul n'a le droit en vérité de me blâmer, de me juger
Et je précise que c'est bien la nature qui
Est seule responsable si je suis un homme
Oh, comme ils disent.

Charles Aznavour

By Proposition 5.2.1, the standard projective generators $P_n^{\mathbf{FI}_d}$ from Definition 2.2.4, which form a very important family of $\mathbf{FI}_d$-modules, are not polynomial for $d > 1$. Since the fact that they are polynomial for $d = 1$ simplifies the study of polynomial $\mathbf{FI}$-modules, we give different examples of quotients of the functors $P_n^{\mathbf{FI}_d}$ which are (weak) polynomial. In addition to give some concrete examples, these quotients may give us a better approach of what the polynomial functors on $\mathbf{FI}_d$ look like. The first examples are a family of quotients of the functor $P_0^{\mathbf{FI}_d}$, obtained in Section 8.1 by filtering its generators by the number of occurrences of the colours, which are weak polynomial of degree 0. We also show that the image of these quotients in $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is equal to the image of a constant functor. Then these functors correspond, through the equivalence of categories given in Theorem 7.4.12 giving the description of $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$, to the object $(\mathbf{R}, \mathrm{Id}, \dots, \mathrm{Id})$ of $\mathbf{R}\text{-}\mathbf{Mod}_d$ or to the trivial $\mathbf{R}[x_2^{\pm 1}, \dots, x_d^{\pm 1}]$-module.

In Section 8.2 we show that the quotient of the functor $P_n^{\mathbf{FI}_d}$ by the subfunctor corresponding to the action of the symmetric groups by post-composition is weak polynomial of degree 0. However, we explain that it is not easy to find the corresponding object of $\mathbf{R}\text{-}\mathbf{Mod}_d$ through the equivalence of categories giving the description of $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ since the passage to the quotient category is not explicit. In Section 8.3 we give a quotient of $P_n^{\mathbf{FI}_d}$, which is weak polynomial of degree n, obtained as the quotient by the action of the symmetric groups on the colour choices. To prove this we use a formula from Proposition 8.3.8 which links this quotient of $P_n^{\mathbf{FI}_d}$ with $P_n^{\mathbf{FI}}$ and the functor $P_0^{\mathcal{C}_d}$ over the category $\mathcal{C}_d$, introduced in Definition 8.2.5, whose objects are the integers and whose morphisms from n to m are the unordered choice of $m - n$ colours in C. Finally, in Section 8.4 we construct a quotient of $P_n^{\mathbf{FI}_d}$ that is weak polynomial of degree i for any $i \in \mathbb{N}$ using the above formula and similar quotients of $P_n^{\mathbf{FI}}$.

8.1 Weak polynomial quotients of $P_0^{\mathbf{FI}_d}$

In this section we give examples of quotients of the functor $P_0^{\mathbf{FI}_d}$ which are weak polynomial of degree 0 by filtering its generators by the occurrences of the colours. We begin with a first example where we quotient $P_0^{\mathbf{FI}_d}$ by identifying all its generators.

Definition 8.1.1. For $n \in \mathbf{FI}_d$, the *submodule* $G_0(n)$ of $P_0^{\mathbf{FI}_d}(n)$ is given by

$$G_0(n) = \langle\, \alpha - c_1^n \,|\, \alpha \in \mathbf{FI}_d(0,n) \rangle$$

that is the submodule generated by the elements $\alpha - c_1^n$ for $\alpha \in \mathbf{FI}_d(0,n)$.

Lemma 8.1.2. *The submodules $G_0(n)$ of $P_0^{\mathbf{FI}_d}(n)$ define a subfunctor G_0 of $P_0^{\mathbf{FI}_d}$.*

Proof. For $(f,g) \in \mathbf{FI}_d(n,m)$ and $\alpha - c_1^n$ a generator of $G_0(n)$ we have

$$P_0^{\mathbf{FI}_d}(f,g)\left(\alpha - c_1^n\right) = (f,g) \circ \alpha - (f,g) \circ c_1^n.$$

We then deduce from the equality $(f,g)\circ\alpha - (f,g)\circ c_1^n = (\,(f,g)\circ\alpha - c_1^m\,) - (\,(f,g)\circ c_1^n - c_1^m\,)$ that $P_0^{\mathbf{FI}_d}(f,g)\,(\alpha - c_1^n)$ is in $G_0(m)$. This shows that $P_0^{\mathbf{FI}_d}(f,g)\left(G_0(n)\right)$ is a submodule of $G_0(m)$ and so G_0 define a subfunctor of $P_0^{\mathbf{FI}_d}$. □

We show that this quotient of $P_0^{\mathbf{FI}_d}$ is strong polynomial of degree 0 as it is a constant functor.

Proposition 8.1.3. *The quotient $P_0^{\mathbf{FI}_d}/G_0$ is the constant functor equal to $\mathbf{R}$ and, in particular it is in $\mathrm{Pol}_0^{strong}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.*

Proof. For $n \in \mathbf{FI}_d$, the quotient $P_0^{\mathbf{FI}_d}/G_0\,(n)$ is the module generated by the class $\overline{c_1^n}$ of c_1^n, so it is isomorphic to $\mathbf{R}$. For $(f,g) \in \mathbf{FI}_d(n,m)$ we have $P_0^{\mathbf{FI}_d}/G_0\;(f,g)\;(\overline{c_1^n}) = \overline{(f,g)\circ c_1^n}$, but in the quotient we have $\overline{(f,g)\circ c_1^n} = \overline{c_1^m}$. This shows that $P_0^{\mathbf{FI}_d}/G_0\;(f,g)$ sends the basis element of $P_0^{\mathbf{FI}_d}/G_0\,(n)$ to the basis element of $P_0^{\mathbf{FI}_d}/G_0\,(m)$ and so it is the identity of $\mathbf{R}$. Finally, a constant functor is in $\mathrm{Pol}_0^{\mathrm{strong}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by Example 5.1.7. □

We now generalize this example by identifying only the morphisms with at least i occurrences of c_1 and we show that it gives weak polynomial quotients of $P_0^{\mathbf{FI}_d}$. We first recall that a morphism α in $\mathbf{FI}_d(0,n)$ corresponds to a choice of n colours in $C^{(d)} = \{c_1, \dots, c_d\}$. In the following, we then denote by $\gamma_k(\alpha)$ the number of occurrences of c_k in α, for $1 \le k \le d$.

Definition 8.1.4. For $i \in \mathbb{N}$ and $n \in \mathbf{FI}_d$, the *submodule* $G_i(n)$ of $P_0^{\mathbf{FI}_d}(n)$ is given by

$$G_i(n) = \langle\, \alpha - c_1^n \,|\, \alpha \in \mathbf{FI}_d(0,n),\ \gamma_1(\alpha) \ge i \rangle$$

that is the submodule generated by the elements $\alpha - c_1^n$, for $\alpha \in \mathbf{FI}_d(0,n)$ such that $\gamma_1(\alpha) \ge i$.

Lemma 8.1.5. *The submodules $G_i(n)$ of $P_0^{\mathbf{FI}_d}(n)$ define a subfunctor G_i of $P_0^{\mathbf{FI}_d}$.*

Proof. It is similar to the proof of Lemma 8.1.2. □

We show now that the quotient of $P_0^{\mathbf{FI}_d}$ by its subfunctor G_i is weak polynomial for all $i \in \mathbb{N}$.

Proposition 8.1.6. *For $i \in \mathbb{N}$, the quotient $P_0^{\mathbf{FI}_d}/G_i$ is weak polynomial of degree 0, which means that $\pi_d(P_0^{\mathbf{FI}_d}/G_i) \in \mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.*

Proof. For $n \in \mathbf{FI}_d$, we have by definition

$$P_0^{\mathbf{FI}_d}/G_i\,(n) = \mathbf{R}\,[\alpha \in \mathbf{FI}_d(0,n)] \Big/ \langle\, \alpha - c_1^n \,|\, \alpha \in \mathbf{FI}_d(0,n),\ \gamma_1(\alpha) \geq i \rangle\,.$$

This quotient module is generated by the class $\overline{c_1^n}$ of c_1^n and by the classes $\overline{\alpha}$ for $\alpha \in \mathbf{FI}_d(0,n)$ such that $\gamma_1(\alpha) < i$. For $c \in C$, the module $\delta_1^c(P_0^{\mathbf{FI}_d}/G_i)\,(n)$ is the cokernel of the morphism $P_0^{\mathbf{FI}_d}/G_i\,(\mathrm{Id}_n + c)$. This morphism is obtained as $\mathbf{R}[(\mathrm{Id}_n + c)_*]$ passing to the quotient by G_i. Since $(\mathrm{Id}_n + c)_*$ sends $\overline{c_1^n}$ to $\overline{(c_1^n, c)} = \overline{c_1^{n+1}}$, this gives for $n \geq i$ the formula

$$\delta_1^c(P_0^{\mathbf{FI}_d}/G_i)\,(n) = \langle\, \overline{\alpha} \,|\, \alpha \in \mathbf{FI}_d(0,n+1),\ \gamma_1(\alpha) < i, \text{ and } \alpha \neq (\beta, c) \text{ with } \beta \in \mathbf{FI}_d(0,n) \rangle\,.$$

For $(f,g) \in \mathbf{FI}_d(n,m)$, the map $\delta_1^c(P_0^{\mathbf{FI}_d}/G_i)\,(f,g)$ is obtained as the morphism $\mathbf{R}\left[\overline{(f,g)_*}\right]$ passing to the quotient of $P_0^{\mathbf{FI}_d}$ by G_i, then to the cokernel $\delta_1^c(P_0^{\mathbf{FI}_d}/G_i)$. In particular, for $n \in \mathbf{FI}_d$ and $(f,g) = (\mathrm{Id}_n + c_1^i) \in \mathbf{FI}_d(n, n+i)$ this gives that

$$i_i^{c_1^i}\left(\delta_1^c(P_0^{\mathbf{FI}_d}/G_i)\right)_n = \delta_1^c(P_0^{\mathbf{FI}_d}/G_i)\,(\mathrm{Id}_n + c_1^i) = 0.$$

Then $\kappa_i^{c_1^i} \circ \delta_1^c(P_0^{\mathbf{FI}_d}/G_i)$ is equal to $\delta_1^c(P_0^{\mathbf{FI}_d}/G_i)$ since it is the kernel of this map, and so $\delta_1^c(P_0^{\mathbf{FI}_d}/G_i)$ is in $\mathcal{SN}_{c_1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by Definition 6.2.1. Using Proposition 7.1.6 we get that $\pi_d(P_0^{\mathbf{FI}_d}/G_i) \in \mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. □

These examples are part of a family of weak polynomial quotients of $P_0^{\mathbf{FI}_d}$ obtained by filtering its generators by the number of occurrences of the different colours. Indeed, for d integers $k_1, \ldots, k_d \in \mathbb{N}$ we can define a family of subfunctors indexed by the subsets I of $A = \{1, \ldots, d\}$. After introducing some notation, we define these subfunctors of $P_0^{\mathbf{FI}_d}$, denoted by $G_{I,k_1,\ldots,k_d}$.

Definition 8.1.7. For $k_1, \ldots, k_d \in \mathbb{N}$ and I a subset of $A = \{1, \ldots, d\}$, the morphism $\alpha \in \mathbf{FI}_d(0,k)$ satisfies the condition $(P_{I,k_1,\ldots,k_d})$ if $\gamma_i(\alpha) \geq k_i$ for all $i \in I$, or if there exists $j \in \{c_1, \ldots, c_d\} \smallsetminus I$ such that $\gamma_j(\alpha) \geq k_j$, where $\gamma_i(\alpha)$ denotes the number of occurrences of c_i in α.

Definition 8.1.8. For $k_1, \ldots, k_d \in \mathbb{N}$, $n \in \mathbf{FI}_d$ and a subset $I \subset A$, the *submodule* $G_{I,k_1,\ldots,k_d}(n)$ of $P_0^{\mathbf{FI}_d}(n)$ is given by

$$G_{I,k_1,\ldots,k_d}(n) = \big\langle \alpha - X \,|\, \alpha \in \mathbf{FI}_d(0,n) \text{ that satisfies the condition } (P_{I,k_1,\ldots,k_d}) \big\rangle$$

that is the submodule generated by the elements $\alpha - X$, for $\alpha \in \mathbf{FI}_d(0,n)$ satisfying the condition $(P_{I,k_1,\ldots,k_d})$ from Definition 8.1.7 and $X \in \mathbf{FI}_d(0,n)$ a given morphism in satisfying the condition $(P_{I,k_1,\ldots,k_d})$.

Lemma 8.1.9. *The submodules $G_{I,k_1,\ldots,k_d}(n)$ of $P_0^{\mathbf{FI}_d}(n)$ define a subfunctor $G_{I,k_1,\ldots,k_d}$ of $P_0^{\mathbf{FI}_d}$.*

Proof. This is similar to the proof of Lemma 8.1.2. For any morphism $(f,g) \in \mathbf{FI}_d(n,m)$ the morphism $P_0^{\mathbf{FI}_d}(f,g)$ is the linearization of the post-composition by (f,g) which can only add more colour and increase γ_i. So for any $n \in \mathbb{N}$ we have an inclusion of modules $P_0^{\mathbf{FI}_d}(f,g)\,(\,G_{I,k_1,\ldots,k_d}(n)\,)$ in $G_{I,k_1,\ldots,k_d}(m)$. □

Remark 8.1.10. The subfunctor G_i given in Definition 8.1.4 is a particular case of Definition 8.1.8. In fact, we have $G_{A,i,0,\ldots,0} = G_i$.

In general, the cases $I = \varnothing$ and $I = A$ are easy to describe. Indeed, for $n \in \mathbb{N}$, we have

$$G_{\varnothing,k_1,\dots,k_d}(n) = \langle\, \alpha - X \,|\, \alpha \in \mathbf{FI}_d(0,n),\ \exists j \in A \smallsetminus I \text{ such that } \gamma_j(\alpha) \geq k_j \rangle =: G_{k_1\vee\cdots\vee k_d}(n),$$

and

$$G_{A,k_1,\dots,k_d}(n) = \langle\, \alpha - X \,|\, \alpha \in \mathbf{FI}_d(0,n),\ \forall i \in A\ \gamma_i(\alpha) \geq k_i \rangle =: G_{k_1\wedge\cdots\wedge k_d}(n).$$

We prove in Proposition 8.1.15 that the quotient of $P_0^{\mathbf{FI}_d}$ by the subfunctor $G_{I,k_1,\dots,k_d}$ is weak polynomial of degree 0. To do this, we define a similar family of subfunctors $F_{I,k_1,\dots,k_d}$ of $P_0^{\mathbf{FI}_d}$ and we show that the quotient by these subfunctors are stably zero. Then, we show in Lemma 8.1.14 that the quotient of $P_0^{\mathbf{FI}_d}$ by $G_{I,k_1,\dots,k_d}$ is constant modulo this stably zero quotient.

Definition 8.1.11. For $k_1,\dots,k_d \in \mathbb{N}$, $n \in \mathbf{FI}_d$ and a subset $I \subset A$, the *submodule* $F_{I,k_1,\dots,k_d}(n)$ of $P_0^{\mathbf{FI}_d}(n)$ is given by

$$F_{I,k_1,\dots,k_d}(n) = \big\langle \alpha \in \mathbf{FI}_d(0,n) \,|\, \alpha \text{ that satisfies the condition } (P_{I,k_1,\dots,k_d}) \big\rangle$$

that is the submodule generated by the elements $\alpha \in \mathbf{FI}_d(0,n)$ satisfying the condition $(P_{I,k_1,\dots,k_d})$ from Definition 8.1.7.

Lemma 8.1.12. *The submodules $F_{I,k_1,\dots,k_d}(n)$ of $P_0^{\mathbf{FI}_d}(n)$ define a subfunctor $F_{I,k_1,\dots,k_d}$ of $P_0^{\mathbf{FI}_d}$.*

Proof. It is similar to the proof of Lemma 8.1.9. □

The cases $I = \varnothing$ and $I = A$ are easy to describe. Indeed, for $n \in \mathbb{N}$, we have

$$F_{\varnothing,k_1,\dots,k_d}(n) = \langle\, \alpha \in \mathbf{FI}_d(0,n) \,|\, \exists j \in A \smallsetminus I \text{ such that } \gamma_j(\alpha) \geq k_j \rangle =: F_{k_1\vee\cdots\vee k_d}(n),$$

and

$$F_{A,k_1,\dots,k_d}(n) = \mathbf{R}\,[\, \alpha \in \mathbf{FI}_d(0,n) \,|\, \forall i \in A\ \gamma_i(\alpha) \geq k_i] =: F_{k_1\wedge\cdots\wedge k_d}(n).$$

All the subfunctors $F_{I,k_1,\dots,k_d}$ behave in a similar way so we can consider them all at once, independently of the subset $I \subset A$ and of the d-tuple $(k_1,\dots,k_d)$. Indeed, in all cases the quotient of $P_0^{\mathbf{FI}_d}$ by one of these subfunctors is stably zero as explained in the following proposition.

Proposition 8.1.13. *For $k_1,\dots,k_d \in \mathbb{N}$ and $I \subset A$, the quotient of $P_0^{\mathbf{FI}_d}$ by its subfunctor $F_{I,k_1,\dots,k_d}$ is in the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. In particular, we have* $\pi_d\left(P_0^{\mathbf{FI}_d}/F_{I,k_1,\dots,k_d}\right) = 0$.

Proof. By definition, the functor $P_0^{\mathbf{FI}_d}/F_{I,k_1,\dots,k_d}$ is given on an object $n \in \mathbf{FI}_d$ by the quotient of $\mathbf{R}\,[\alpha \in \mathbf{FI}_d(0,n)]$ by its submodule $F_{I,k_1,\dots,k_d}(n)$ from Definition 8.1.11, which gives

$$P_0^{\mathbf{FI}_d}/F_{I,k_1,\dots,k_d}(n) = \langle \alpha \in \mathbf{FI}_d(0,n) \,|\, \exists i \in I \text{ such that } \gamma_i(\alpha) < k_i, \text{ and } \forall j \in A \smallsetminus I, \gamma_j(\alpha) < k_j \rangle.$$

However, the morphism $P_0^{\mathbf{FI}_d}/F_{I,k_1,\dots,k_d}\ (\mathrm{Id}_{(-)} + (c_1^{k_1},\dots,c_d^{k_d}))$ is zero since it is given by the map $P_0^{\mathbf{FI}_d}(\mathrm{Id}_{(-)} + (c_1^{k_1},\dots,c_d^{k_d}))$, which is the linearization of the map $(\mathrm{Id}_{(-)} + (c_1^{k_1},\dots,c_d^{k_d}))_*$, passing to the quotient. The functor

$$\kappa_{k_1+\cdots+k_d}^{(c_1^{k_1},\dots,c_d^{k_d})}\left(P_0^{\mathbf{FI}_d}/F_{I,k_1,\dots,k_d}\right)$$

being the kernel of this morphism, it is equal to $P_0^{\mathbf{FI}_d}/F_{I,k_1,\dots,k_d}$ itself, showing that the quotient of $P_0^{\mathbf{FI}_d}$ by $F_{I,k_1,\dots,k_d}$ is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. □

The quotients of $P_0^{\mathbf{FI}_d}$ by $G_{I,k_1,\ldots,k_d}$ and by $F_{I,k_1,\ldots,k_d}$ are linked in a short exact sequence by a constant functor as explained in the following.

Lemma 8.1.14. *For $k_1,\ldots,k_d \in \mathbb{N}$ and $I \subset A$, there is a short exact sequence in* $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$*:*

$$0 \longrightarrow \mathbf{R} \longrightarrow P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d} \longrightarrow P_0^{\mathbf{FI}_d}/F_{I,k_1,\ldots,k_d} \longrightarrow 0 \ .$$

Proof. By definition, the functor $P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d}$ is given on an object $n \in \mathbf{FI}_d$ by the quotient of $\mathbf{R}\,[\alpha \in \mathbf{FI}_d(0,n)]$ by its submodule $G_{I,k_1,\ldots,k_d}(n)$ from Definition 8.1.8, which gives

$$P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d}(n) = \left\langle \{\overline{X}\} \sqcup \{\overline{\alpha} \,|\, \exists i \in I \text{ such that } \gamma_i(\alpha) < k_i, \text{ and } \forall j \in A \smallsetminus I, \gamma_j(\alpha) < k_j\} \right\rangle .$$

Then, the quotient of $P_0^{\mathbf{FI}_d}(n)$ by $G_{I,k_1,\ldots,k_d}(n)$ is generated by the class $\overline{X}$ of the fixed morphism X, and by the classes $\overline{\alpha}$ corresponding to the generators of the quotient $P_0^{\mathbf{FI}_d}/F_{I,k_1,\ldots,k_d}(n)$ given in the proof of Proposition 8.1.13. Moreover, both $P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d}$ and $P_0^{\mathbf{FI}_d}/F_{I,k_1,\ldots,k_d}$ send a morphism ϕ in $\mathbf{FI}_d$ to the morphism induced by the post-composition by ϕ passing to the quotient. In particular, $P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d}(\phi)$ sends the class $\overline{X}$ to itself and so the submodules $\langle\overline{X}\rangle$ generated by the class of the fixed morphism X of $P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d}(n)$ give a constant subfunctor of $P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d}$. Then the quotient of $P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d}$ by this constant subfunctor evaluated on $n \in \mathbf{FI}_d$ is generated by the same elements as $P_0^{\mathbf{FI}_d}/F_{I,k_1,\ldots,k_d}$, which gives the short exact sequence of the statement on an object $n \in \mathbf{FI}_d$. It is natural since both $P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d}$ and $P_0^{\mathbf{FI}_d}/F_{I,k_1,\ldots,k_d}$ act on morphism by the post-composition passing to the quotient. □

Finally, we show that the quotient of $P_0^{\mathbf{FI}_d}$ by its subfunctor $G_{I,k_1,\ldots,k_d}$ is weak polynomial of degree 0. To do this, we use the short exact sequence of Lemma 8.1.14 and Proposition 8.1.13.

Proposition 8.1.15. *For $k_1,\ldots,k_d \in \mathbb{N}$ and $I \subset \{c_1,\ldots,c_d\}$, the quotient of $P_0^{\mathbf{FI}_d}$ by its subfunctor $G_{I,k_1,\ldots,k_d}$ is weak polynomial of degree 0. In other words, we have $\pi_d\left(P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d}\right) \in$* $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

Proof. By Lemma 8.1.14 there is a short exact sequence

$$0 \longrightarrow \mathbf{R} \longrightarrow P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d} \longrightarrow P_0^{\mathbf{FI}_d}/F_{I,k_1,\ldots,k_d} \longrightarrow 0 \ ,$$

and by Proposition 8.1.13 the last term is in the subcategory $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. Since the quotient functor π_d is exact, we get an isomorphism $\pi_d(\mathbf{R}) \cong \pi_d(P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d})$. Then for $c \in C$, we have $(\delta_1^c)^{\mathbf{St}} \circ \pi_d\left(P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d}\right) \cong (\delta_1^c)^{\mathbf{St}} \circ \pi_d(\mathbf{R}) \cong \pi_d \circ \delta_1^c(\mathbf{R}) = 0$. Since this is true for all $c \in C$, this shows that $\pi_d(P_0^{\mathbf{FI}_d}/G_{I,k_1,\ldots,k_d})$ is in $\mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. □

Remark 8.1.16. As explained above, the short exact sequence of Lemma 8.1.14 shows that this quotient is equal to a constant functor modulo the stably zero functor $P_0^{\mathbf{FI}_d}/F_{I,k_1,\ldots,k_d}$. This implies that its image in the quotient is equal to $\pi_d(\mathbf{R})$ and so it corresponds through the equivalence of categories given in Theorem 7.4.12 to the object $(\mathbf{R}, \mathrm{Id}, \ldots, \mathrm{Id})$ of $\mathbf{R}\text{-}\mathbf{Mod}_d$, or to the trivial $\mathbf{R}[x_2^{\pm 1}, \ldots, x_d^{\pm 1}]$-module.

We proved that the quotient of $P_0^{\mathbf{FI}_d}$ by the subfunctor $G_{I,k_1,\dots,k_d}$ is weak polynomial of degree 0 for any $k_1,\dots,k_d \in \mathbb{N}$ and $I \subset A$, and that the quotient by the subfunctor $F_{I,k_1,\dots,k_d}$ is stably zero. In the end of this section we look at the strong polynomiality of these quotients. In Proposition 8.1.19 we show that, for $I = \varnothing$ and $k_1,\dots,k_d \in \mathbb{N}^*$, they are strong polynomial because they are constant after some rank. But we explain in Proposition 8.1.18 that, when $|I| \geq 1$, they are not strong polynomial based on the case $I = A$.

Lemma 8.1.17. *For $I = A$, $1 \leq i \leq d$ and the functor $F_{k_1 \wedge \dots \wedge k_d} = F_{A,k_1,\dots,k_d}$ of Definition 8.1.11, we have the relation*

$$\delta_1^{c_i}\left(P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge k_d}\right) \cong \bigoplus_{j=1, j\neq i}^{d} P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge \tilde{k_j}\wedge\dots\wedge k_d},$$

where $\tilde{k_j} = k_j - 1$ if $k_j \geq 1$, and 0 if $k_j = 0$.

Proof. For $n \in \mathbf{FI}_d$, by definition $\delta_1^{c_i}(P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge k_d})(n)$ is the cokernel of the map $P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge k_d}(\mathrm{Id}_n + c_i)$. We can compute

$$\begin{aligned}
\delta_1^{c_i}\left(P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge k_d}\right)(n) &\cong \langle \alpha \in \mathbf{FI}_d(0, n+1) | \, \alpha \neq (\beta, c_i), \exists r \in A \text{ such that } \gamma_r(\alpha) < k_r \rangle \\
&\cong \langle \alpha = (\beta, c) | \, \beta \in \mathbf{FI}_d(0,n), \, c \neq c_i, \, \exists r \in A \text{ such that } \gamma_r(\alpha) < k_r \rangle \\
&\cong \bigoplus_{j=1, j\neq i}^{d} \langle \beta \in \mathbf{FI}_d(0,n) \, | \, \gamma_j(\beta) < k_j - 1 \text{ or } \exists r \in A \smallsetminus j, \, \gamma_r(\beta) < k_r, \rangle
\end{aligned}$$

This shows that $\delta_1^{c_i}\left(P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge k_d}\right)(n)$ is isomorphic to

$$\bigoplus_{j=1, j\neq i}^{d} \left\langle \beta \in \mathbf{FI}_d(0,n) | \, \exists r \in A \, \gamma_r(\beta) < k'_r \right\rangle \cong \bigoplus_{j=1, j\neq i}^{d} P_0^{\mathbf{FI}_d}/F_{k'_1\wedge\dots\wedge k'_d}(n),$$

with $k'_r = k_r - 1$ if $r = j$ and $k'_r = k_r$ else, proving the relation when $k_1,\dots,k_d \geq 1$. This relation is natural in $n \in \mathbb{N}$ since oth $\delta_1^{c_i}\left(P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge k_d}\right)$ and $P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge \tilde{k_j}\wedge\dots\wedge k_d}$ act on morphisms by the post-composition passing to the quotient. The other cases are done in a similar way. □

We now use this lemma to prove that the quotient of $P_0^{\mathbf{FI}_d}$ by the subfunctors $F_{I,k_1,\dots,k_d}$ and $G_{I,k_1,\dots,k_d}$ are not strong polynomial when $|I| \geq 1$.

Proposition 8.1.18. *For $k_1,\dots,k_d \in \mathbb{N}$ all non-zero, $n \in \mathbf{FI}_d$ and a subset $I \subset A$, the quotients of $P_0^{\mathbf{FI}_d}$ by $F_{I,k_1,\dots,k_d}$ and $G_{I,k_1,\dots,k_d}$ are not strong polynomial when the cardinality of I is greater than or equal to 1.*

Proof. For $I = A$, by Lemma 8.1.17 we have the relation for all $n \in \mathbf{FI}_d$ and all $c_i \in C$:

$$\delta_1^{c_i}\left(P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge k_d}\right)(n) \cong \bigoplus_{j=1, j\neq i}^{d} P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge \tilde{k_j}\wedge\dots\wedge k_d}(n),$$

with $\tilde{k_j} = k_j - 1$ if $k_j \geq 1$, and 0 if $k_j = 0$. These relations combined prove that the iterated functors $(\delta_1^{c_1})^i\left(P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge k_d}\right)(n)$ for $i \in \mathbb{N}$ are never zero, and so $P_0^{\mathbf{FI}_d}/F_{k_1\wedge\dots\wedge k_d}$ is not strong polynomial. Using the short exact sequence from Lemma 8.1.14, it implies that $P_0^{\mathbf{FI}_d}/G_{k_1\wedge\dots\wedge k_d}$ is not strong polynomial since these categories are closed under quotients by Proposition 5.1.3. This gives the result when $I = A$, the general case is proved in a similar way because the parts associated with the colours in I take over from the parts associated with the colours in $A \smallsetminus I$ and they prevent the quotient to be strong polynomial. □

While these quotients are not strong polynomial when $|I| \geq 1$, the cases $I = \varnothing$ give polynomial quotients of $P_0^{\mathbf{FI}_d}$ when all the integers $k_1, \ldots, k_d$ are non-zero as explained in the following.

Proposition 8.1.19. *For $k_1, \ldots, k_d \in \mathbb{N}^*$, the quotients of $P_0^{\mathbf{FI}_d}$ by the subfunctors $F_{k_1 \vee \cdots \vee k_d}$ and $G_{k_1 \vee \cdots \vee k_d}$ are strong polynomial of degree $k_1 + \cdots + k_d - d$.*

Proof. For $n \in \mathbf{FI}_d$, the quotient of $P_0^{\mathbf{FI}_d}(n)$ by $F_{k_1 \vee \cdots \vee k_d}(n)$ is generated by the morphisms $\alpha \in \mathbf{FI}_d(0, n)$ such that $\gamma_i(\alpha) < k_i$ for all $1 \leq i \leq d$. Since the integers $k_1, \ldots, k_d$ are non-zero, it implies that $P_0^{\mathbf{FI}_d} / F_{k_1 \vee \cdots \vee k_d}(n)$ is zero for $n > k_1 + \cdots + k_d - d$. Using Lemma 5.1.5 we conclude that $P_0^{\mathbf{FI}_d} / F_{k_1 \vee \cdots \vee k_d}$ is strong polynomial of degree less than or equal to $k_1 + \cdots + k_d - d$. Finally, the short exact sequence from Lemma 8.1.14 implies that $P_0^{\mathbf{FI}_d} / G_{k_1 \wedge \cdots \wedge k_d}$ is also strong polynomial of degree $k_1 + \cdots + k_d - d$ since this category is closed under extension by Proposition 5.1.3 and since the constant functor $\mathbf{R}$ is strong polynomial of degree 0. □

8.2 The quotient of $P_n^{\mathbf{FI}_d}$ by the action of symmetric groups

In this section we define the quotient of the functor $P_n^{\mathbf{FI}_d}$ by the subfunctor, called F_n in Definition 8.2.1, which corresponds to the action of the symmetric groups by post-composition. We show in Theorem 8.2.11 that this quotient of $P_n^{\mathbf{FI}_d}$ is weak polynomial of degree 0. We also show in Proposition 8.2.9 that this quotient of $P_n^{\mathbf{FI}_d}$ is isomorphic to $P_0^{\mathcal{C}_d}$ over the category $\mathcal{C}_d$, introduced in Definition 8.2.5, whose objects are the integers and whose morphisms from n to m are the unordered choice of $m - n$ colours in C. In a second time, we try to find a nice representative of the class of this quotient in the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ for $n = 0$. The objective is to describe the class of $P_0^{\mathbf{FI}_d} / F_0$ in terms of the category $\mathbf{R}\text{-}\mathbf{Mod}_d$ via the equivalence of categories given in Theorem 7.4.12 but this is not always possible since the passage to the quotient category is not an explicit construction.

Definition 8.2.1. For $m \in \mathbf{FI}_d$, we denote by $F_n(m)$ the *submodule* of $P_n^{\mathbf{FI}_d}(m)$ given by

$$F_n(m) = \langle\, \sigma \circ (f, g) - (f, g) \,|\, (f, g) \in \mathbf{FI}_d(n, m),\ \sigma \in \mathrm{S}_m \rangle\,.$$

that is the submodule generated by the elements $\sigma \circ (f, g) - (f, g)$, for $(f, g) \in \mathbf{FI}_d(n, m)$ and $\sigma \in \mathrm{S}_m$.

Lemma 8.2.2. *The submodules $F_n(m)$ of $P_n^{\mathbf{FI}_d}(m)$ define a subfunctor F_n of $P_n^{\mathbf{FI}_d}$.*

Proof. For $(\tilde{f}, \tilde{g}) \in \mathbf{FI}_d(m, l)$ and $\sigma \circ (f, g) - (f, g)$ a generator of $F_n(m)$, we compute

$$P_n^{\mathbf{FI}_d}(\tilde{f}, \tilde{g})\,(\sigma \circ (f, g) - (f, g)) = (\tilde{f}, \tilde{g})_* \big(\sigma \circ (f, g) - (f, g)\big) = (\tilde{f}, \tilde{g}) \circ \sigma \circ (f, g) - (\tilde{f}, \tilde{g}) \circ (f, g).$$

Then there exists $\tilde{\sigma} \in \mathrm{S}_l$ (we can take $\tilde{\sigma}$ which acts as σ on $\mathrm{Im}(\tilde{f}) \cong m$ and is the identity on $l \smallsetminus \mathrm{Im}(\tilde{f})$) such that $\tilde{\sigma} \circ (\tilde{f}, \tilde{g}) \circ (f, g) = (\tilde{f}, \tilde{g}) \circ \sigma \circ (f, g)$. Therefore we have

$$P_n^{\mathbf{FI}_d}(\tilde{f}, \tilde{g})\,(\sigma \circ (f, g) - (f, g)) = \tilde{\sigma} \circ \big((\tilde{f}, \tilde{g}) \circ (f, g) \big) - (\tilde{f}, \tilde{g}) \circ (f, g) \in F_n(l).$$

We then proved on the generators of $F_n(m)$ that we have the inclusion of submodules $P_n^{\mathbf{FI}_d}(\tilde{f}, \tilde{g})\,(\, F_n(m)\,) \subset F_n(l)$. □

For a n-tuple $(c_{i_1}, \ldots, c_{i_n})$ of colours we denote by $\overline{(c_{i_1}, \ldots, c_{i_n})}$ the class of this n-tuple under the action of the symmetric group S_n permuting the positions in the n-tuple. For each class we can choose a representative n-tuple $(c_{j_1}, \ldots, c_{j_n})$ of the class $\overline{(c_{i_1}, \ldots, c_{i_n})}$ such that the colours are in the natural order, *i.e.* such that $1 \leq j_1 \leq \cdots \leq j_n \leq d$. Using this notation we can give a description of the quotient $P_n^{\mathbf{FI}_d} / F_n$ in the following proposition.

Proposition 8.2.3. *The quotient functor $P_n^{\mathbf{FI}_d}/F_n$ sends an object $m \in \mathbf{FI}_d$ to the free $\mathbf{R}$-module generated by the class of $(m-n)$-tuples $\overline{(c_{i_1},\dots,c_{i_{m-n}})}$ under the action of symmetric group S_{m-n}. In other words, we have*

$$P_n^{\mathbf{FI}_d}/F_n\,(m) \cong \mathbf{R}\left[\,\overline{(c_{i_1},\dots,c_{i_{m-n}})}\,|\,1 \le i_1 \le \dots \le i_{m-n} \le d\right].$$

Moreover, it sends a map $(\tilde{f},\tilde{g}) \in \mathbf{FI}_d(m,l)$ to the morphism of $\mathbf{R}$-modules $\mathbf{R}\left[\left(\,\mathrm{Id}_n + \tilde{g}\right)_*\right]$ that sends a basis element $\overline{(c_{i_1},\dots,c_{i_{m-n}})}$ to the element basis $\overline{(c_{i_1},\dots,c_{i_{m-n}},\tilde{g})}$.

Remark 8.2.4. In this proposition we could choose a representative of the class $\overline{(c_{i_1},\dots,c_{i_n},\tilde{g})}$ where the colours are in the natural order to make it more consistent, but it would need more notations for no more information. We give here an example to make it clearer: for $d=5$, $n=1$, $m=3$ and $l=5$ the map $(\,(0\to 2)+\mathrm{Id}_3\,,\,(c_3,c_2)\,) \in \mathbf{FI}_4(3,5)$ sends the basis element $\overline{(c_2,c_4)}$ of $P_1^{\mathbf{FI}_5}/F_1\,(3)$ to the element $\overline{(c_2,c_4,\,c_3,c_2)} = \overline{(c_2,c_2,c_3,c_4)}$ of $P_1^{\mathbf{FI}_5}/F_1\,(5)$.

Proof of Proposition 8.2.3. For $m \in \mathbf{FI}_d$, by definition of F_n we have

$$P_n^{\mathbf{FI}_d}/F_n\,(m) = \mathbf{R}\left[(f,g) \in \mathbf{FI}_d(n,m)\right] \Big/ \langle \sigma \circ (f,g) - (f,g) | (f,g) \in \mathbf{FI}_d(n,m), \sigma \in \mathrm{S}_m \rangle\,.$$

The action of S_m permutes both the injection f and the colours g, so we can choose for each class in the quotient a representative with the injection being the inclusion of the first n elements in m, and with the colours in the natural order. This gives the isomorphisms

$$P_n^{\mathbf{FI}_d}/F_n\,(m) \cong \left\langle \left(\,n \xhookrightarrow{\mathrm{Id}_n + (0\to m-n)} m\;\;,\,\overline{(c_{i_1},\dots,c_{i_{m-n}})}\,\right) | \,1 \le i_1 \le \dots \le i_{m-n} \le d \right\rangle$$

Since these elements are free, it gives an isomorphism of modules

$$P_n^{\mathbf{FI}_d}/F_n\,(m) \cong \mathbf{R}\left[\,\overline{(c_{i_1},\dots,c_{i_{m-n}})}\,|\,1 \le i_1 \le \dots \le i_{m-n} \le d\right].$$

For a map $(\tilde{f},\tilde{g}) \in \mathbf{FI}_d(m,l)$, the morphism $P_n^{\mathbf{FI}_d}/F_n\,(\tilde{f},\tilde{g})$ is induced by $P_n^{\mathbf{FI}_d}(\tilde{f},\tilde{g}) = \mathbf{R}[(\tilde{f},\tilde{g})_*]$ passing to the quotient. Since we take the quotient by the action of S_l, the injection $\tilde{f}$ can be supposed to be the inclusion of the m first elements. Then $P_n^{\mathbf{FI}_d}/F_n\,(\tilde{f},\tilde{g})$ is the morphism of $\mathbf{R}$-modules adding the colours of $\tilde{g}$ on each basis element. □

The description of the quotient of $P_n^{\mathbf{FI}_d}$ by F_n of Proposition 8.2.3 suggests defining a category $\mathcal{C}_d$ corresponding to this functor. More precisely, after giving the definition of this category, we show in Proposition 8.2.9 that the quotient $P_n^{\mathbf{FI}_d}/F_n$ is isomorphic to the functor $P_n^{\mathcal{C}_d}$.

Definition 8.2.5. The *category* $\mathcal{C}_d$ has for objects the integers and its morphisms from n to m are the class of the $(m-n)$-tuples $(c_{i_1},\dots,c_{i_{m-n}})$ of colours in $C^{(d)}$ quotiented by the action of the symmetric group S_{m-n}. In other words, we have

$$\mathcal{C}_d(n,m) = \begin{cases} \{c_1,\dots,c_d\}^{\times(m-n)}/\mathrm{S}_{m-n} = \left\{\overline{(c_{i_1},\dots,c_{i_{m-n}})} | \,1 \le i_1 \le \dots \le i_n \le d\right\} & \text{if } m \ge n \\ \varnothing & \text{if } m < n. \end{cases}$$

The composition is given by the concatenation of two representatives of each class:

$$\overline{(c_{j_1},\dots,c_{j_{k-m}})} \circ \overline{(c_{i_1},\dots,c_{i_{m-n}})} = \overline{(c_{j_1},\dots,c_{j_{k-m}},c_{i_1},\dots,c_{i_{m-n}})}.$$

Remark 8.2.6. As in Remark 8.2.4, in the definition of the composition we could choose a representative of the class of $(c_{j_1},\dots,c_{j_{k-m}},c_{i_1},\dots,c_{i_{m-n}})$ where the colours are in the natural order to make it more consistent, but it would need more notations for no more information. We give here an example to make it clearer: For $d=5$, $n=1$, $m=3$ and $k=5$ we have $\overline{(c_2,c_5)}\circ\overline{(c_1,c_3)}=\overline{(c_2,c_5,c_1,c_3)}=\overline{(c_1,c_2,c_3,c_5)}$.

There is a natural functor between $\mathbf{FI}_d$ and $\mathcal{C}_d$ which we define in the following.

Definition 8.2.7. The *functor* $\Omega:\mathbf{FI}_d\to\mathcal{C}_d$ sends an object n of $\mathbf{FI}_d$ to n in $\mathcal{C}_d$, and a morphism $(f,g)\in\mathbf{FI}_d(n,m)$ to the morphism $\overline{g}=\overline{(c_{i_1},\dots,c_{i_{m-n}})}\in\mathcal{C}_d(n,m)$, where $\overline{g}$ is the class of the colour choice $g=(c_{i_1},\dots,c_{i_{m-n}})$ in the quotient by S_{m-n}.

Remark 8.2.8. The functor $\Omega:\mathbf{FI}_d\to\mathcal{C}_d$ is essentially surjective, full but not faithful for $d>1$ since $\Omega(c_1,c_2)=\overline{(c_1,c_2)}=\Omega(c_2,c_1)$.

Proposition 8.2.9. *For all $n\in\mathbb{N}$, there is a natural isomorphism in $\mathbf{Fct}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod})$:*

$$P_n^{\mathbf{FI}_d}/F_n\cong P_n^{\mathcal{C}_d}\circ\Omega.$$

Proof. For $m\in\mathbf{FI}_d$, by definition of $\mathcal{C}_d$ and by Proposition 8.2.3 we have an isomorphism of $\mathbf{R}$-modules

$$P_n^{\mathbf{FI}_d}/F_n\,(m)\cong\mathbf{R}\left[\overline{(c_{i_1},\dots,c_{i_{m-n}})}\,|\,1\le i_1\le\dots\le i_{m-n}\le d\right]=\mathbf{R}\left[\mathcal{C}_d(n,m)\right]=P_n^{\mathcal{C}_d}\circ\Omega(m).$$

For a morphism $(\tilde f,\tilde g)\in\mathbf{FI}_d(m,l)$, we have by Proposition 8.2.3 that $P_n^{\mathbf{FI}_d}/F_n\,(\tilde f,\tilde g)$ sends the basis element $\overline{(c_{i_1},\dots,c_{i_n})}$ to $\overline{(c_{i_1},\dots,c_{i_n},\tilde g)}$. However, we also have by definition, that $P_n^{\mathcal{C}_d}\circ\Omega(\tilde f,\tilde g)=P_n^{\mathcal{C}_d}(\overline{(\tilde g)})$ sends $\overline{(c_{i_1},\dots,c_{i_n})}$ to $\overline{(\tilde g)}\circ\overline{(c_{i_1},\dots,c_{i_n})}=\overline{(c_{i_1},\dots,c_{i_n},\tilde g)}$. This implies that the diagram

$$\begin{tikzcd}
P_n^{\mathbf{FI}_d}/F_n\,(m) \arrow[r, "\cong"] \arrow[d, "P_n^{\mathbf{FI}_d}/F_n\,(\tilde f,\tilde g)"'] & P_n^{\mathcal{C}_d}\circ\Omega(m) \arrow[d, "P_n^{\mathcal{C}_d}\circ\Omega\,(\tilde f,\tilde g)=P_n^{\mathcal{C}_d}\,\overline{(\tilde g)}"] \\
P_n^{\mathbf{FI}_d}/F_n\,(l) \arrow[r, "\cong"'] & P_n^{\mathcal{C}_d}\circ\Omega(l)
\end{tikzcd}$$

is commutative, showing that the isomorphism $P_n^{\mathbf{FI}_d}/F_n\cong\mathcal{C}_d\circ\Omega$ is natural. □

Using the explicit description of the quotient $P_n^{\mathbf{FI}_d}/F_n$ from Proposition 8.2.3 we compute its image by the endofunctor δ_1^c, for $c\in C$, in the following lemma. We then use this computation to prove in Theorem 8.2.11 that the quotient of $P_n^{\mathbf{FI}_d}$ by F_n is weak polynomial of degree 0.

Lemma 8.2.10. *For $c\in C$, the functor $\delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)$ sends an object $m\in\mathbf{FI}_d$ to the free $\mathbf{R}$-module generated by the class of the $(m-n+1)$-tuples $(c_{i_1},\dots,c_{i_{m-n+1}})$ under the action of symmetric group that does not contain the colour c. In other words, we have*

$$\delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)(m)=\mathbf{R}\left[\overline{(c_{i_1},\dots,c_{i_{m-n}},c_{i_{m-n+1}})}\,|\,c_{i_l}\in C^{(d)}\smallsetminus\{c\},\,1\le i_1\le\dots\le i_{m-n}\le i_{m-n+1}\le d\right].$$

Moreover, this functor sends a map $(\tilde f,\tilde g)\in\mathbf{FI}_d(m,l)$ to zero if c appears in $\tilde g$, and to the morphism of $\mathbf{R}$-modules $\mathbf{R}\left[\left(\mathrm{Id}_{n+1}+\tilde g\right)_*\right]$ if c does not appear in $\tilde g$.

Proof. For $m\in\mathbf{FI}_d$, by definition $\delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)(m)$ is the cokernel of the map $P_n^{\mathbf{FI}_d}/F_n\,(\mathrm{Id}_m+c)$. By construction, this morphism is given by the map $\mathbf{R}[(\mathrm{Id}_m+c)_*]:\mathbf{R}[\mathbf{FI}_d(n,m)]\to\mathbf{R}[\mathbf{FI}_d(n,m+1)]$ passing to the quotient. Using the explicit description of the quotient $P_n^{\mathbf{FI}_d}/F_n$

from Proposition 8.2.3, we get that the morphism $P_n^{\mathbf{FI}_d}/F_n\ (\mathrm{Id}_m + c)$ sends a basis element $\overline{(c_{i_1},\dots,c_{i_{m-n}})}$ to $\overline{(c_{i_1},\dots,c_{i_{m-n}},c)} = \overline{(c_{i_1},\dots,c_k,c,c_{k+1},\dots,c_{i_{m-n}})}$. Then the image of the morphism $P_n^{\mathbf{FI}_d}/F_n\ (\mathrm{Id}_m + c)$ is generated by all the $(m-n)$-tuples of unordered colours where c appears. We then deduce that its cokernel is

$$\delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)(m) = \mathbf{R}\left[\overline{(c_{i_1},\dots,c_{i_{m-n}},c_{i_{m-n+1}})}\,|\,c_{i_j} \in C^{(d)} \smallsetminus \{c\},\ 1 \le i_1 \le \dots \le i_{m-n} \le i_{m-n+1} \le d\right].$$

For $(\tilde f,\tilde g) \in \mathbf{FI}_d(m,l)$, the map $\delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)(\tilde f,\tilde g)$ is induced by the morphism $\tau_1\left(P_n^{\mathbf{FI}_d}/F_n\right)(\tilde f,\tilde g) = P_n^{\mathbf{FI}_d}/F_n\ \left((\tilde f,\tilde g) + \mathrm{Id}_1\right)$ passing to the cokernel. However, by Proposition 8.2.3, this last morphism is the linearization of the map $(\mathrm{Id}_{n+1} + \tilde g)_*$, so its image is in the image of $P_n^{\mathbf{FI}_d}/F_n\ (\mathrm{Id}_n + c)$ if and only if the colour c appears in $\tilde g$. When passing to the cokernel, this gives that $\delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)(\tilde f,\tilde g)$ is 0 if $c \in \tilde g$ and $\mathbf{R}\left[\left(\mathrm{Id}_{n+1} + \tilde g\right)_*\right]$ else. □

Theorem 8.2.11. *For all $n \in \mathbb{N}$, the quotient functor $P_n^{\mathbf{FI}_d}/F_n$ is weak polynomial of degree 0, i.e. we have:*

$$\pi_d\left(P_n^{\mathbf{FI}_d}/F_n\right) \in \mathrm{Pol}_0(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod}).$$

Proof. For $c \in C$ and $m \in \mathbf{FI}_d$, by Lemma 8.2.10 the morphism $\delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)(\mathrm{Id}_m + c)$ is zero. Since $\kappa_1^c \circ \delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)(m)$ is the kernel of this map it is equal to $\delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)(m)$. This equality is natural in $m \in \mathbf{FI}_d$ since $\kappa_1^c \circ \delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)$ is a subfunctor of $\delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)$, proving that the functor $\delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right)$ is in $\mathcal{SN}_c(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. We conclude using Proposition 7.1.6 because, for all $c \in C$, we have

$$(\delta_1^c)^{\mathbf{St}} \circ \pi_d\left(P_n^{\mathbf{FI}_d}/F_n\right) \cong \pi_d \circ \delta_1^c\left(P_n^{\mathbf{FI}_d}/F_n\right) = 0.$$

□

Example 8.2.12. We make this quotient explicit for $n = 0$ and $d = 2$. First we recall that for $n \in \mathbf{FI}_d$, we have $P_0^{\mathbf{FI}_d}(n) = \mathbf{R}\left[\mathbf{FI}_d(0,n)\right] = \mathbf{R}\left[C^n\right]$, so a basis morphism $\alpha \in P_0^{\mathbf{FI}_d}(n)$ corresponds to a choice of n colours in C. Then the post-composition in $\mathbf{FI}_d$ by $\mathrm{S}_n = \mathbf{FI}_d(n,n)$ corresponds to the action of the symmetric group S_n on $P_0^{\mathbf{FI}_d}(n)$ permuting the colours of the generators. Since the subfunctor F_0 of $P_0^{\mathbf{FI}_2}$ correspond to this action, this gives for $d = 2$ and $n \in \mathbf{FI}_2$ the following description:

$$P_0^{\mathbf{FI}_2}/F_0\ (n) \cong \mathbf{R}\left[\overline{(c_{i_1},\dots,c_{i_n})}\,|\,1 \le i_1 \le \dots \le i_n \le 2\right].$$

Moreover, the quotient $P_0^{\mathbf{FI}_2}/F_0$ sends a morphism $(f,g) \in \mathbf{FI}_2(n,m)$ to the linearization of the map $(\mathrm{Id}_n + g)_*$ passing to the quotient. In particular this quotient functor sends a bijective morphism $\sigma \in \mathbf{FI}_2(n,n) = \mathrm{S}_n$ to the identity and the image of a morphism $(f,g) \in \mathbf{FI}_2(n,m)$ is determined only by the colour choice g. We then have an explicit description of the quotient of $P_0^{\mathbf{FI}_2}$ by F_0 as the diagram in Figure 8.2.12. Note that each arrow in the category $\mathbf{FI}_2$ in this diagram actually represents many arrows that we can construct by composition with the action of the symmetric groups. This diagram also represent the functor $P_0^{\mathcal{C}_2}$ since $P_0^{\mathcal{C}_2} \circ \Omega \cong P_0^{\mathbf{FI}_2}/F_0$ by Proposition 8.2.9.

$$\mathbf{FI}_2 \xrightarrow{P_0^{\mathbf{FI}_2}/F_0} \mathbf{R}\text{-}\mathbf{Mod}$$

$$\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[d, shift right, "c_1"'] \arrow[d, shift left, "c_2"] & \mathbf{R}[\mathrm{Id}_0] \arrow[d, dashed, "(\mathrm{Id}_0+c_1)_*"'] \arrow[dr, dashed, "(\mathrm{Id}_0+c_2)_*"] & & & & \\
1 \arrow[r, mapsto] \arrow[d, shift right, "c_1"'] \arrow[d, shift left, "c_2"] & \mathbf{R}[\overline{c_1}] \arrow[r, phantom, "\oplus"] \arrow[d, dashed, "(\mathrm{Id}_1+c_1)_*"'] \arrow[dr, dashed] & \mathbf{R}[\overline{c_2}] \arrow[d, dashed] \arrow[dr, dashed, "(\mathrm{Id}_1+c_2)_*"] & & & \\
2 \arrow[r, mapsto] \arrow[d, shift right, "c_1"'] \arrow[d, shift left, "c_2"] & \mathbf{R}[\overline{(c_1,c_1)}] \arrow[r, phantom, "\oplus"] \arrow[d, dashed, "(\mathrm{Id}_2+c_1)_*"'] \arrow[dr, dashed] & \mathbf{R}[\overline{(c_1,c_2)}] \arrow[r, phantom, "\oplus"] \arrow[d, dashed] \arrow[dr, dashed] & \mathbf{R}[\overline{(c_2,c_2)}] \arrow[d, dashed] \arrow[dr, dashed, "(\mathrm{Id}_2+c_2)_*"] & & \\
3 \arrow[r, mapsto] \arrow[d, shift right, "c_1"'] \arrow[d, shift left, "c_2"] & \mathbf{R}[\overline{(c_1,c_1,c_1)}] \arrow[r, phantom, "\oplus"] \arrow[d, dashed] \arrow[dr, dashed] & \mathbf{R}[\overline{(c_1,c_1,c_2)}] \arrow[r, phantom, "\oplus"] \arrow[d, dashed] \arrow[dr, dashed] & \mathbf{R}[\overline{(c_1,c_2,c_2)}] \arrow[r, phantom, "\oplus"] \arrow[d, dashed] \arrow[dr, dashed] & \mathbf{R}[\overline{(c_2,c_2,c_2)}] \arrow[d, dashed] \arrow[dr, dashed] & \\
\vdots \arrow[r, mapsto] & \cdots & \cdots & \cdots & \cdots & \cdots
\end{tikzcd}$$

Figure 8.1: Explicit description of the quotient of $P_0^{\mathbf{FI}_2}$ by its subfunctor F_0

Remark 8.2.13. We proved in Theorem 8.2.11 that the quotient of $P_n^{\mathbf{FI}_d}$ by its subfunctor F_n of Definition 8.2.1 corresponding to the action of the symmetric groups, is weak polynomial of degree 0. We would like to use Theorem 7.4.12 to describe this quotient in terms of the category $\mathbf{R}\text{-}\mathbf{Mod}_d$, but to do that we need to find a representative of the class $\pi_d\left(P_n^{\mathbf{FI}_d}/F_n\right)$ which is constant on the objects and which sends arrows from 0 to 1 to commutative isomorphisms. However, this is generally difficult to do since passing to the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ is not very explicit.

In the end of this section we give ideas on how to find a nice representative of the class $\pi_d\left(P_n^{\mathbf{FI}_d}/F_n\right)$ as explained in Remark 8.2.13 for $n = 0$. Recall that the quotient $P_0^{\mathbf{FI}_d}/F_0$ sends an object n to the free $\mathbf{R}$-module generated by the class $\overline{(c_{i_1}, \dots, c_{i_n})}$ of the n-tuples of colours quotiented by the action of S_n permuting the colours. For each class we can choose a representative n-tuple $(c_{i_1}, \dots, c_{i_n})$ such that the colours are in the natural order, *i.e.* such that $1 \leq i_1 \leq \dots \leq i_n \leq d$. We start by defining a filtration of subfunctors of the quotient $P_0^{\mathbf{FI}_d}/F_0$ according to the number of occurrences of the colour c_1. We show in Proposition 8.2.18 that each of these subfunctors gives a proper subfunctor of $P_0^{\mathbf{FI}_d}/F_0$, however they are not strictly smaller since they are isomorphic to $P_0^{\mathbf{FI}_d}/F_0$ itself with a shift, as shown in Proposition 8.2.19.

Definition 8.2.14. For $k \in \mathbb{N}$ and $n \in \mathbf{FI}_d$, the *submodules* $L_k(n)$ of $P_0^{\mathbf{FI}_d}/F_0\,(n)$ are generated by the class of the n-tuple $(c_{i_1}, \dots, c_{i_n})$ in the quotient by the action of the symmetric group S_n in which the colour c_1 appears at least k times.

Lemma 8.2.15. *The submodules $L_k(n)$ of $P_0^{\mathbf{FI}_d}/F_0\,(n)$ define a family a subfunctors*

$$\dots \subset L_{k+1} \subset L_k \subset \dots \subset L_2 \subset L_1 \subset L_0 = P_0^{\mathbf{FI}_d}/F_0.$$

Proof. For $(f,g) \in \mathbf{FI}_d(n,m)$, the morphism $P_0^{\mathbf{FI}_d}/F_0\,(f,g)$ is the linearization of the post-composition by (f,g) passing to the quotient by F_0. This morphism can only add more colour in the n-tuples of colours and then only increase the number of occurrences of c_1. When passing to the quotient, this shows that all these subfunctors of $P_0^{\mathbf{FI}_d}/F_0$ are well defined since we have the inclusion of $\mathbf{R}$-modules $P_0^{\mathbf{FI}_d}/F_0\,(f,g)\ (L_k(n)) \subset L_k(m)$. The inclusions follow directly from the definition of L_k. □

Remark 8.2.16. Recall that the quotient of $P_0^{\mathbf{FI}_d}$ by its subfunctor G_i from Definition 8.1.4 is given on an object $n \in \mathbf{FI}_d$ by the free $\mathbf{R}$-module generated by the class $\overline{c_1^n}$ of c_1^n and by the classes $\overline{\alpha}$ for $\alpha \in \mathbf{FI}_d(0,n)$ such that $\gamma_1(\alpha) < i$. Then the quotient of $P_0^{\mathbf{FI}_d}/G_i$ by the action of the symmetric groups by post-composition is equivalent to the quotient of $P_0^{\mathbf{FI}_d}/F_0$ by its subfunctor L_i from Definition 8.2.14.

Remark 8.2.17. We chose to define L_k as the subfunctor where c_1 appears at least k times for simplicity but this is arbitrary. Instead, we could consider any mix of colour and define $L_{k_1,\ldots,k_d}$ the subfunctor generated by the classes of the n-tuple of colour choices in the quotient by the symmetric group in which c_1 appears at least k_1 times, c_2 at least k_2 times, ... and c_d appears at least k_d times. In fact, it appears that every non-zero subfunctor H of $P_0^{\mathbf{FI}_d}/F_0$ is a sum of subfunctors similar to $L_{k_1,\ldots,k_d}$ of $P_0^{\mathbf{FI}_d}$. Indeed, there is a minimal $n \in \mathbb{N}^*$ such that $H(n) \neq 0$. For x non-zero in $H(n)$, H contains a subfunctor similar to $L_{k_1,\ldots,k_d}$ starting at x: it is the subfunctor generated by the classes containing x and in addition at least k_1 times c_1, ... , at least k_d times c_d. Either H is equal to this subfunctor, either it is greater and we can restart the reasoning with the quotient. The process stops in a finite number of steps because each time the number of possible occurrences of the different colours decrease. Then, the subfunctors $L_{k_1,\ldots,k_d}$ generate all the subfunctors of $P_0^{\mathbf{FI}_d}/F_0$. However studying all of them is similar to studying only L_k, but with more complex notations, so we write the details for L_k using only c_1 for more clarity.

We show that the image in $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of each of the subfunctors L_k of $P_0^{\mathbf{FI}_d}/F_0$ gives a representative of the class of $P_0^{\mathbf{FI}_d}/F_0$ itself.

Proposition 8.2.18. *For $k \in \mathbb{N}$, there is a natural isomorphism $\pi_d(L_k) \cong \pi_d\left(P_0^{\mathbf{FI}_d}/F_0\right)$.*

Proof. For $k \in \mathbb{N}$, let K_k denote the quotient of $\left(P_0^{\mathbf{FI}_d}/F_0\right)$ by its subfunctor L_k. By definition the functor $\kappa_k^{(c_1)^k}(K_k)$ is the kernel of the map $K_k\left(\mathrm{Id}_{(-)} + (c_1)^k\right)$ obtained as the morphism $P_0^{\mathbf{FI}_d}/F_0\ (\mathrm{Id}_{(-)} + (c_1)^k)$ passing to the quotient by L_k. By Proposition 8.2.3, this last morphism is the map that sends a basis element $\overline{(c_{i_1},\ldots,c_{i_n})}$ of $P_0^{\mathbf{FI}_d}/F_0\,(n)$ to the element $\overline{((c_1)^k, c_{i_1},\ldots,c_{i_n})}$ of $P_0^{\mathbf{FI}_d}/F_0\,(n+k)$, which is in $L_k(n+k)$ since the colour c_1 appears at least k times. This shows that the image of the map $P_0^{\mathbf{FI}_d}/F_0\left(\mathrm{Id}_{(-)} + (c_1)^k\right)$ is in the subfunctor L_k. When passing to the quotient, it implies that the map $K_k\left(\mathrm{Id}_{(-)} + (c_1)^k\right)$ is zero. We then have $\kappa_k^{(c_1)^k}(K_k) = K_k$, and so K_k is in $\mathcal{SN}_{c_1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. By definition, we have a short exact sequence $0 \longrightarrow L_k \longrightarrow P_0^{\mathbf{FI}_d}/F_0 \longrightarrow K_k \longrightarrow 0$ in $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and, since the functor π_d is exact, we get the wanted natural isomorphism. □

This gives us a family of representatives of the class $\pi_d\left(P_0^{\mathbf{FI}_d}/F_0\right)$, but neither of them is constant on the objects. More than that, we show that each of these subfunctors L_k is in fact isomorphic to a shift of $P_0^{\mathbf{FI}_d}/F_0$ himself.

Proposition 8.2.19. *For all $k \in \mathbb{N}$, there is a natural isomorphism $\tau_k(L_k) \cong P_0^{\mathbf{FI}_d}/F_0$.*

Proof. For $n \in \mathbf{FI}_d$, by definition the $\mathbf{R}$-module $L_k(n+k)$ is generated by the classes $\overline{(c_{i_1}, \ldots, c_{in+k})}$ with $1 \leq i_1 \leq \cdots \leq i_{n+k} \leq d$ such that c_1 appears at least k times. Then we have $i_1 = \cdots = i_k = 1$, so $L_k(n+k)$ is equivalent to the module generated by the classes $\overline{(c_{i_{k+1}}, \ldots, c_{i_{k+n}})}$ with $1 \leq i_{k+1} \leq \cdots \leq i_{n+k} \leq d$. This corresponds to the generators of the module $P_0^{\mathbf{FI}_d}/F_0\,(n)$, so we have the equivalence of $\mathbf{R}$-modules $P_0^{\mathbf{FI}_d}/F_0\,(n) \cong L_k(n+k) = \tau_k\,(L_k)\,(n)$. Since L_n is a subfunctor of $P_0^{\mathbf{FI}_d}/F_0$ we have, for any morphism $(f,g) \in \mathbf{FI}_d(n,m)$, the identity

$$\tau_k(L_k)(f,g) = L_k\left(\,(f,g) + \mathrm{Id}_k\,\right) = P_0^{\mathbf{FI}_d}/F_0\left((f,g) + \mathrm{Id}_k\,\right).$$

By Proposition 8.2.3 this last morphism is induced by $\mathbf{R}\left[\,\left(\,\mathrm{Id}_n + g + \mathrm{Id}_k\,\right)_*\,\right]$ passing to the quotient. In these identities the term "Id_k" corresponds to the k occurrences of c_1 that disappear in the equivalence of modules $P_0^{\mathbf{FI}_d}/F_0\,(n) \cong \tau_k\,(L_k)\,(n)$ above. This shows the naturality of this equivalence since the morphism $\tau_k(L_k)(f,g)$ corresponds then to the morphism $\mathbf{R}\left[\,\left(\,\mathrm{Id}_n + g\right)_*\,\right] = P_0^{\mathbf{FI}_d}/F_0\,(f,g)$. □

Remark 8.2.20. We can prove in the same way that there is a natural isomorphism $\tau_{k_1+\cdots+k_d}\,\left(L_{k_1,\ldots,k_d}\right) \cong P_0^{\mathbf{FI}_d}/F_0$, where $L_{k_1,\ldots,k_d}$ is the subfunctor of $P_0^{\mathbf{FI}_d}/F_0$ described in Remark 8.2.17, where c_1 appears at least k_1 times, ... and c_d appears at least k_d times.

8.3 The quotient of $P_n^{\mathbf{FI}_d}$ by the action of symmetric groups on colours

In this section we define the quotient of the functor $P_n^{\mathbf{FI}_d}$ by the subfunctor, called H_n in Definition 8.3.2, corresponding to the action of the symmetric groups on the colour choices, and we prove in two different ways that:

Theorem. *For all $n \in \mathbb{N}$, the quotient $P_n^{\mathbf{FI}_d}/H_n$ is weak polynomial of degree n, i.e. we have $\pi_d\left(P_n^{\mathbf{FI}_d}/H_n\right) \in \mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.*

First, we prove this theorem if $\mathbf{R} = \mathbb{K}$ is a field in Theorem 8.3.11, using the decomposition

$$P_n^{\mathbf{FI}_d}/H_n \;\cong\; \left(\,\mathcal{O}^* P_n^{\mathbf{FI}}\,\right) \otimes P_0^{\mathcal{C}_d}(\,(-) - n\,) \circ \Omega\,,$$

from Proposition 8.3.8 since the pointwise tensor product respects the polynomial degree. In a second time we prove the theorem in the general case in Theorem 8.3.14 using the direct computation

$$(\delta_1^c)^{\mathbf{St}} \circ \pi_d\left(P_n^{\mathbf{FI}_d}/H_n\right) \cong \pi_d\left(P_{n-1}^{\mathbf{FI}_d}/H_{n-1}\right)^{\oplus n}.$$

from Proposition 8.3.13. In both cases there is a non-trivial stably zero functor that prevents this quotient from being strong polynomial.

Remark 8.3.1. For $(f,g) \in \mathbf{FI}_d(n,m)$ the map $g = (c_{i_1}, \ldots, c_{i_{m-n}})$ corresponds to a choice of $m-n$ colours in C. There is an action of the symmetric group S_{m-n} on these colour choices by permutation, which gives an action on $P_n^{\mathbf{FI}_d}(m)$. For a $m-n$-tuple $(c_{i_1}, \ldots, c_{i_{m-n}})$ of colours we denote by $\overline{(c_{i_1}, \ldots, c_{i_{m-n}})}$ the class of this $m-n$-tuple under this action and, for each class, we can choose a representative $m-n$-tuple $(c_{j_1}, \ldots, c_{j_{m-n}})$ such that the colours are in the natural order, *i.e.* such that $1 \leq j_1 \leq \cdots \leq j_{m-n} \leq d$.

We start with the definition of the subfunctor H_n of $P_n^{\mathbf{FI}_d}$ using an action of S_{m-n} on $P_n^{\mathbf{FI}_d}(m)$.

Definition 8.3.2. For $m \in \mathbf{FI}_d$, the *submodule* $H_n(m)$ of $P_n^{\mathbf{FI}_d}(m)$ is given by

$$H_n(m) = \langle\, (f, \sigma \cdot g) - (f,g) \,|\, (f,g) \in \mathbf{FI}_d(n,m),\ \sigma \in \mathrm{S}_{m-n} \rangle$$

that is the submodule generated by the elements $(f, \sigma \cdot g) - (f,g)$, for $(f,g) \in \mathbf{FI}_d(n,m)$ and $\sigma \in \mathrm{S}_m$ acting on the colours as described in Remark 8.3.1.

Lemma 8.3.3. *The submodules $H_n(m)$ of $P_n^{\mathbf{FI}_d}(m)$ define a subfunctor H_n of $P_n^{\mathbf{FI}_d}$.*

Proof. For $(\tilde{f}, \tilde{g}) \in \mathbf{FI}_d(m,l)$, we have $P_n^{\mathbf{FI}_d}(\tilde{f},\tilde{g})\big((f,\sigma\cdot g) - (f,g)\big) = (\tilde{f},\tilde{g})\circ(f,\sigma\cdot g) - (\tilde{f},\tilde{g})\circ(f,g)$. Then there exists $\tilde{\sigma} \in \mathrm{S}_{l-n}$ (we can take $\tilde{\sigma}$ which acts as σ on $\mathrm{Im}(\tilde{f}) \smallsetminus \mathrm{Im}(\tilde{f}\circ f) \cong m-n$, and is the identity on $l - \mathrm{Im}(\tilde{f}) \cong l-m$) such that $\tilde{\sigma}\cdot(\,(\tilde{f},\tilde{g})\circ(f,g)\,) = (\tilde{f},\tilde{g})\circ(f,\sigma\cdot g)$. We then get

$$P_n^{\mathbf{FI}_d}(\tilde{f},\tilde{g})\big(\,(f,\sigma\cdot g) - (f,g)\,\big) = \tilde{\sigma}\cdot\big(\,(\tilde{f},\tilde{g})\circ(f,g)\,\big) - (\tilde{f},\tilde{g})\circ(f,g) \in H_n(l)$$

showing, on the generators $(f,\sigma\cdot g) - (f,g)$ of $H_n(m)$, that we have the inclusion of $\mathbf{R}$-modules $P_n^{\mathbf{FI}_d}(\tilde{f},\tilde{g})\big(H_n(m)\big) \subset H_n(l)$. □

Remark 8.3.4. For $d = 1$ the subfunctor H_n of $P_n^{\mathbf{FI}_1}$ is zero since the symmetric groups acts on the unique colour choice and so $\sigma\cdot(f,g) = (f,g)$ for all $(f,g) \in \mathbf{FI}_1(n,m)$ and $\sigma \in \mathrm{S}_n$. In particular, Theorem 8.3.11 for $d = 1$ tells that $P_n^{\mathbf{FI}}$ is weak polynomial of degree less than or equal to n. In fact, it is even strong polynomial of degree n as explained in Remark 5.2.3.

We first give a concrete description of the quotient $P_n^{\mathbf{FI}_d}/H_n$ that we will use in both proofs of Theorems 8.3.11 and 8.3.14.

Proposition 8.3.5. *The quotient functor $P_n^{\mathbf{FI}_d}/H_n$ sends an object $m \in \mathbf{FI}_d$ to the free $\mathbf{R}$-module generated by the pairs $(\,f,\ \overline{(c_{i_1},\dots,c_{i_{m-n}})}\,)$, where $f : n \hookrightarrow m$ is an injection and $\overline{(c_{i_1},\dots,c_{i_{m-n}})}$ is a class of a $(m-n)$-tuple of colours under the action of the symmetric group S_{m-n}. In other words, there is an isomorphism of $\mathbf{R}$-modules*

$$P_n^{\mathbf{FI}_d}/H_n\,(m) \cong \mathbf{R}\left[\Big(f,\ \overline{(c_{i_1},\dots,c_{i_{m-n}})}\Big) \,|\, f \in \mathbf{FI}(n,m),\ 1 \le i_1 \le \cdots \le i_{m-n} \le d\right].$$

Moreover, for $(\tilde{f},\tilde{g})$ a morphism in $\mathbf{FI}_d(m,l)$, the image of $(\tilde{f},\tilde{g}) \in \mathbf{FI}_d(m,l)$ by $P_n^{\mathbf{FI}_d}/H_n$ is the morphism of $\mathbf{R}$-modules $\mathbf{R}\left[\,(\tilde{f},\tilde{g})_*\,\right]$ that sends a basis element $(f,\ \overline{(c_{i_1},\dots,c_{i_{m-n}})}\,)$ to the element $(\tilde{f}\circ f,\ \overline{(c_{i_1},\dots,c_{i_{m-n}},\tilde{g})}\,)$.

Remark 8.3.6. As in Remark 8.2.4, in this proposition we could choose a representative of the class $\overline{(c_{i_1},\dots,c_{i_n},\tilde{g})}$ where the colours are in the natural order to make it more consistent but it would need more notations for no more information.

Proof of Proposition 8.3.5. By definition, for $m \in \mathbf{FI}_d$ we have

$$P_n^{\mathbf{FI}_d}/H_n\,(m) = \mathbf{R}\left[(f,g) \in \mathbf{FI}_d(n,m)\right] \Big/ \langle (f,\ \sigma\cdot g) - (f,g) | \,(f,g) \in \mathbf{FI}_d(n,m), \sigma \in \mathrm{S}_{m-n}\rangle\,.$$

Then the action of S_{m-n} permutes the colours of g so we can choose for each class in the quotient a representative with the colours in the natural order as explained in Remark 8.3.1. This gives the formula for $P_n^{\mathbf{FI}_d}/H_n\,(m)$ since these elements are free. For $(\tilde{f},\tilde{g}) \in \mathbf{FI}_d(m,l)$, the morphism $P_n^{\mathbf{FI}_d}/H_n\,(\tilde{f},\tilde{g})$ is induced by $P_n^{\mathbf{FI}_d}(\tilde{f},\tilde{g}) = \mathbf{R}[(\tilde{f},\tilde{g})_*]$ passing to the quotient, which acts on the first component of $(f,\ \overline{(c_{i_1},\dots,c_{i_n})})$ by post-composition by $\tilde{f}$ and on the second component, since we take the quotient by the action of S_{l-m}, it adds the colours of $\tilde{g}$ to the class $\overline{(c_{i_1},\dots,c_{i_{m-n}})}$. □

We now introduce a negative shift on the category $\mathcal{C}_d$ from Definition 8.2.5 that we will use in the following. Recall that this category has for objects the integers and for morphisms from n to m the unordered choices of $m-n$ colours in C, and that there is a natural functor $\Omega : \mathbf{FI}_d \to \mathcal{C}_d$.

Definition 8.3.7. For $F : \mathcal{C}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ and $n \in \mathbb{N}$, the *shifted functor* $F\big((-) - n \big)$ is given on objects by

$$F\big((-) - n \big)(m) = \begin{cases} 0 & \text{if } m < n \\ F(m-n) & \text{if } m \geq n, \end{cases}$$

and on morphisms by $F\big((-)-n \big) = F(-)$ using the natural bijection $\mathcal{C}_d(m,m') \cong \mathcal{C}_d(m-n, m'-n)$.

The first consequence of the description of the quotient $P_n^{\mathbf{FI}_d}/H_n$ in Proposition 8.3.5 is that we can express it as a tensor product of $P_n^{\mathbf{FI}}$ by $P_0^{\mathcal{C}_d}$ shifted using the shift from Definition 8.3.7. This explains how the injections and the colours are mixed to form the functor $P_n^{\mathbf{FI}_d}$ up to the action of the symmetric groups on the colour choices. This relation will allow us to prove in Theorem 8.3.11 that $P_n^{\mathbf{FI}_d}/H_n$ is weak polynomial of degree n using the fact that pointwise tensor product preserves the polynomial degree (see Theorem 7.3.6).

Proposition 8.3.8. *For all $n \in \mathbb{N}$, there is a natural isomorphism*

$$P_n^{\mathbf{FI}_d}/H_n \;\cong\; \big(\mathcal{O}^* P_n^{\mathbf{FI}}\big) \otimes P_0^{\mathcal{C}_d}\big((-) - n \big) \circ \Omega \,,$$

which can equivalently be written as $\tau_n\Big(P_n^{\mathbf{FI}_d}/H_n \Big) \cong \tau_n\big(\mathcal{O}^* P_n^{\mathbf{FI}}\big) \otimes P_0^{\mathbf{FI}_d}/H_0$.

Remark 8.3.9. For $d = 1$, the subfunctor H_n of $P_n^{\mathbf{FI}}$ is zero as explained in Remark 8.3.4. Using this, we can rewrite the formula of Proposition 8.3.8 into a more homogeneous one:

$$P_n^{\mathbf{FI}_d}/H_n \;\cong\; \mathcal{O}^*\Big(P_n^{\mathbf{FI}}/H_n\Big) \otimes P_0^{\mathcal{C}_d}\big((-) - n \big) \circ \Omega \,.$$

Proof of Proposition 8.3.8. For $m \in \mathbf{FI}_d$, using the description of $P_n^{\mathbf{FI}_d}/H_n\,(m)$ from Proposition 8.3.5 for $= 0$ we see that the elements $\overline{x} := \overline{(c_{i_1}, \ldots, c_{i_m})}$ with $1 \leq i_1 \leq \cdots \leq i_m \leq d$ form a set of generators of $P_0^{\mathbf{FI}_d}/H_0\,(m)$ that we denote by $\mathcal{G}_m$. Using the general isomorphism $\mathbf{R}[A \times B] \cong \mathbf{R}[A] \otimes \mathbf{R}[B]$, for any two sets A and B, we can reformulate the isomorphism of Proposition 8.3.5 as:

$$P_n^{\mathbf{FI}_d}/H_n\,(m) \cong \mathbf{R}\left[\, (f,\overline{x}) \mid f \in \mathbf{FI}(n,m),\, \overline{x} \in \mathcal{G}_{m-n}\right] \cong \mathbf{R}\big[\, \mathbf{FI}(n,m) \,\big] \otimes \mathbf{R}\left[\overline{x} \in \mathcal{G}_{m-n}\right].$$

We then have an isomorphism of modules

$$P_n^{\mathbf{FI}_d}/H_n\,(m) \;\cong\; P_n^{\mathbf{FI}}\,(m) \,\otimes P_0^{\mathbf{FI}_d}/H_0\,(m-n) \cong \big(\mathcal{O}^* P_n^{\mathbf{FI}}\big)\,(m) \,\otimes\, P_0^{\mathcal{C}_d}\big((-) - n \big) \circ \Omega\,(m)\,.$$

This isomorphism is natural in $m \in \mathbf{FI}_d$ since, for $(\tilde{f},\tilde{g}) \in \mathbf{FI}_d(m,l)$ the map $P_n^{\mathbf{FI}_d}/H_n\,(\tilde{f},\tilde{g})$ sends a basis element $(f, \overline{(c_{i_1}, \ldots, c_{i_{m-n}})}\,)$ to the element $(\,\tilde{f} \circ f\,, \overline{(c_{i_1}, \ldots, c_{i_{m-n}}, \tilde{g})}\,)$ by Proposition 8.3.5. On the first component of these morphisms $P_n^{\mathbf{FI}_d}/H_n\,(\tilde{f},\tilde{g})$ sends f to $\tilde{f} \circ f$ which corresponds to $\mathcal{O}^*\big(P_n^{\mathbf{FI}}\big)\,(\tilde{f},\tilde{g})\,(f) = P_n^{\mathbf{FI}}(\tilde{f})\,(f) = \tilde{f} \circ f$, while on the second component it sends $\overline{(c_{i_1}, \ldots, c_{i_{m-n}})}$ to $\overline{(c_{i_1}, \ldots, c_{i_{m-n}}, \tilde{g})}$ which is exactly the definition of $P_0^{\mathcal{C}_d}\big((-) - n \big) \circ \Omega(\tilde{f},\tilde{g}) = P_0^{\mathcal{C}_d}(\overline{g})$. This shows the naturality since the isomorphism is given by the separation of the first and the second components. $\square$

The formula of Proposition 8.3.8 for $n = 0$ implies that the quotient $P_n^{\mathbf{FI}_d}/H_n$ is weak polynomial of degree 0 as explained in the following:

Corollary 8.3.10. *There are natural isomorphisms of* $\mathbf{FI}_d$*-modules*

$$P_0^{\mathbf{FI}_d}\big/H_0 \cong P_0^{\mathcal{C}_d} \circ \Omega \cong P_0^{\mathbf{FI}_d}\big/F_0\,,$$

where F_0 *and* H_0 *are the subfunctors of* $P_0^{\mathbf{FI}_d}$ *from Definitions 8.2.1 and 8.3.2. In particular, these functors are weak polynomial of degree 0.*

Proof. For $d = 1$, the functor $P_0^{\mathbf{FI}}$ is a constant functor, and so is $\mathcal{O}^*\left(P_0^{\mathbf{FI}}\right)$ by Proposition 7.2.9. Then the formula of Proposition 8.3.8 applied for $n = 0$ gives the first equivalence since $M^{\mathrm{cst}} \otimes F$ is equivalent to F for any $\mathbf{FI}_d$-module F and any constant $\mathbf{FI}_d$-module M^{cst}. The second equivalence is given by Proposition 8.2.9 and the quotient of $P_0^{\mathbf{FI}_d}$ by F_0 is weak polynomial of degree 0 by Theorem 8.2.11. □

Finally, the relation from Proposition 8.3.8 allows us to prove that the quotient $P_n^{\mathbf{FI}_d}\big/H_n$ is weak polynomial of degree n using the fact that tensor product preserve polynomiality if $\mathbf{R} = \mathbb{K}$ is a field.

Theorem 8.3.11. *If* $\mathbf{R} = \mathbb{K}$ *is a field, for all* $n \in \mathbb{N}$ *the quotient* $P_n^{\mathbf{FI}_d}\big/H_n$ *is weak polynomial of degree less than or equal to* n*, i.e. we have* $\pi_d\left(P_n^{\mathbf{FI}_d}\big/H_n\right) \in \mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$.

Proof. The functor $P_n^{\mathbf{FI}}$ is weak polynomial of degree n on $\mathbf{FI}$ by Lemma 5.2.1 for $d = 1$, and the functor $\mathcal{O}^*$ preserves polynomiality by Proposition 7.2.9. The $\mathbf{FI}_d$-module $P_0^{\mathcal{C}_d} \circ \Omega$ is weak polynomial of degree 0 by Corollary 8.3.10. Then the $\mathbf{FI}_d$-module $P_0^{\mathcal{C}_d}(\ (-) - n\ ) \circ \Omega$ is also weak polynomial of degree 0 since it is the same functor but shifted by n and the weak polynomiality concerns the stable behavior. We then have $\pi_d\left(\mathcal{O}^* P_n^{\mathbf{FI}}\right) \in \mathrm{Pol}_n(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$ and $\pi_d\left(P_0^{\mathcal{C}_d}((-) - n) \circ \Omega\right) \in \mathrm{Pol}_0(\mathbf{FI}_d, \mathbb{K}\text{-}\mathbf{Vect})$. Finally, since $R = \mathbb{K}$ is a field, we conclude that $\pi_d\left(P_n^{\mathbf{FI}_d}\big/H_n\right) \in \mathrm{Pol}_n(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ by Proposition 8.3.8 and Theorem 7.3.6. □

A second consequence of the description of the quotient of $P_n^{\mathbf{FI}_d}$ by the subfunctor H_n from Proposition 8.3.5 is that we can compute explicitly the functor $\delta_1^c\left(P_n^{\mathbf{FI}_d}\big/H_n\right)$. In particular, we describe it with a short exact sequence in Lemma 8.3.12 and we give a description of its image in the quotient category $\mathbf{St}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ in Proposition 8.3.13. The following proposition is very similar to the calculation of $\delta_1^c(P_n^{\mathbf{FI}_d})$ in Proposition 5.2.1. However, the second component of the direct sum in Proposition 5.2.1, which prevents $P_n^{\mathbf{FI}_d}$ from being polynomial, vanishes here since we take the quotient by the action of the symmetric groups on the colours.

Lemma 8.3.12. *For* $c \in C$*, the submodules*

$$I(m) \coloneqq \mathbf{R}\left[\left(f' : n \hookrightarrow m+1,\ \overline{(c_{i_1}, \ldots, c_{i_{m+1-n}})}\right) \middle|\, 1 \le i_1 \le \cdots \le i_{m+1-n} \le d,\ m+1 \in \mathrm{Im}(f')\right]$$

of $\delta_1^c(P_n^{\mathbf{FI}_d}\big/H_n)(m)$*, for* $m \in \mathbf{FI}_d$*, define a subfunctor* I *of* $\delta_1^c(P_n^{\mathbf{FI}_d}\big/H_n)$*, which fits into the following short exact sequence whose last term is in* $\mathcal{SN}_c(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$*:*

$$0 \longrightarrow I \longrightarrow \delta_1^c(P_n^{\mathbf{FI}_d}\big/H_n) \longrightarrow {\delta_1^c(P_n^{\mathbf{FI}_d}\big/H_n)}\big/_{I} \longrightarrow 0\ .$$

Proof. We write the proof for $c = c_1$, the other cases are obtained by symmetry. By Proposition 8.3.5, the module $P_n^{\mathbf{FI}_d}\big/H_n\,(m)$ is generated by the pairs $(\,f,\ \overline{(c_{i_1}, \ldots, c_{i_{m-n}})}\,)$, where $f : n \hookrightarrow m$ is an injection and $\overline{(c_{i_1}, \ldots, c_{i_{m-n}})}$ is a class of a $(m-n)$-tuple of colours under the action of the symmetric group S_{m-n}. For ι_m the injection $m \hookrightarrow m+1$ of m on the first m elements of $m+1$, by definition the functor $\delta_1^{c_1}(P_n^{\mathbf{FI}_d}/H_n)$ is the cokernel of the morphism $P_n^{\mathbf{FI}_d}/H_n\,(\mathrm{Id}_m + c_1)$ which sends a basis element $(f, \overline{(c_{i_1},\dots,c_{i_{m-n}})})$ to the element $(\iota_m \circ f, \overline{(\mathbf{c_1}, c_{i_1},\dots,c_{i_{m-n}})})$ by Proposition 8.3.5. This implies that its cokernel is generated by the elements $(f' : n \hookrightarrow m+1, \overline{(c_{i_1},\dots,c_{i_{m+1-n}})})$ such that $m+1$ is in $\mathrm{Im}(f')$ or that c_1 does not appear, which gives the isomorphism of $\mathbf{R}$-modules

$$\delta_1^{c_1}(P_n^{\mathbf{FI}_d}/H_n)(m) \cong \mathbf{R}\left[\left(f' : n \hookrightarrow m+1, \overline{(c_{i_1},\dots,c_{i_{m+1-n}})}\right) \mid m+1 \in \mathrm{Im}(f') \text{ or } c_{i_k} \in C \smallsetminus \{c_1\}\right].$$

For $(\tilde{f},\tilde{g}) \in \mathbf{FI}_d(m,l)$, by Proposition 8.3.5 the map $\tau_1(P_n^{\mathbf{FI}_d}/H_n)(\tilde{f},\tilde{g}) = P_n^{\mathbf{FI}_d}/H_n\left((\tilde{f},\tilde{g}) + \mathrm{Id}_1\right)$ sends $\left(f', \overline{(c_{i_1},\dots,c_{i_{m+1-n}})}\right)$ to $\left((\tilde{f}+\mathrm{Id}_1)\circ f', \overline{(c_{i_1},\dots,c_{i_{m+1-n}},\tilde{g})}\right)$. The image by $\delta_1^{c_1}(P_n^{\mathbf{FI}_d}/H_n)$ of $(\tilde{f},\tilde{g})$ is induced by this map passing to the quotient, which implies that $\delta_1^{c_1}(P_n^{\mathbf{FI}_d}/H_n)(\tilde{f},\tilde{g})$ sends the basis elements $\left(f' : n \hookrightarrow m+1, \overline{(c_{i_1},\dots,c_{i_{m+1-n}})}\right)$ of $\delta_1^{c_1}(P_n^{\mathbf{FI}_d}/H_n)(m)$ (with either $m+1 \in \mathrm{Im}(f')$ or c_1 that does not appear) to $((\tilde{f}+\mathrm{Id}_1)\circ f' : n \hookrightarrow l+1, \overline{(c_{i_1},\dots,c_{i_{m+1-n}},\tilde{g})})$. In particular, if $m+1 \in \mathrm{Im}(f')$ then $l+1 \in \mathrm{Im}((\tilde{f}+\mathrm{Id}_1)\circ f')$. Then the submodules $I(m)$ of $\delta_1^{c_1}(P_n^{\mathbf{FI}_d}/H_n)(m)$ are stable by the morphisms $\delta_1^{c_1}(P_n^{\mathbf{FI}_d}/H_n)(\tilde{f},\tilde{g})$ for all $(\tilde{f},\tilde{g}) \in \mathbf{FI}_d(m,l)$ so they define a subfunctor $I \subset \delta_1^{c_1}(P_n^{\mathbf{FI}_d}/H_n)$. The quotient of $\delta_1^{c_1}(P_n^{\mathbf{FI}_d}/H_n)$ by this subfunctor I is then generated by the elements $(f' : n \hookrightarrow m+1, \overline{(c_{i_1},\dots,c_{i_{m+1-n}})})$ such that $m+1 \notin \mathrm{Im}(f')$ and c_1 does not appear. The map $\delta_1^{c_1}(P_n^{\mathbf{FI}_d}/H_n)(\mathrm{Id}_m + c_1)$ become zero when passed to the quotient by I since it sends such elements to $\left(\iota_{m+1}\circ f' : n \hookrightarrow m+2, \overline{(\mathbf{c_1}, c_{i_1},\dots,c_{i_{m+1-n}})}\right)$, proving that the quotient of $\delta_1^{c_1}(P_n^{\mathbf{FI}_d}/H_n)$ by I is in $\mathcal{SN}_{c_1}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. ☐

We can now use this short exact sequence to give a formula for $(\delta_1^c)^{\mathbf{St}}$ of $\pi_d(P_n^{\mathbf{FI}_d}/H_n)$ similar to the calculation of $\delta_1^c(P_n^{\mathbf{FI}_d})$ in Proposition 5.2.1.

Proposition 8.3.13. *For all colours $c \in C$, there is a natural isomorphism*

$$(\delta_1^c)^{\mathbf{St}} \circ \pi_d\left(P_n^{\mathbf{FI}_d}/H_n\right) \cong \pi_d\left(P_{n-1}^{\mathbf{FI}_d}/H_{n-1}\right)^{\oplus n}.$$

Proof. Applying the exact functor π_d to the short exact sequence of Lemma 8.3.12, by Proposition 7.1.6 we get the natural isomorphisms

$$(\delta_1^c)^{\mathbf{St}} \circ \pi_d(P_n^{\mathbf{FI}_d}/H_n) \cong \pi_d \circ \delta_1^c(P_n^{\mathbf{FI}_d}/H_n) \cong \pi_d(I)$$

since the last term of the short exact sequence is in $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$. We conclude by showing that there is a natural isomorphism $I \cong \left(P_{n-1}^{\mathbf{FI}_d}/H_{n-1}\right)^{\oplus n}$. Indeed, we have the following isomorphisms

$$\begin{aligned} I(m) &= \mathbf{R}\left[\left(f' : n \hookrightarrow m+1, \overline{(c_{i_1},\dots,c_{i_{m+1-n}})}\right) \mid 1 \le i_1 \le \dots \le i_{m+1-n} \le d,\, m+1 \in \mathrm{Im}(f')\right] \\ &\cong \mathbf{R}\left[\left(f'' : \underline{n} - \{i\} \hookrightarrow m, \overline{(c_{i_1},\dots,c_{i_{m-(n-1)}})}\right) \mid 1 \le i_1 \le \dots \le i_{m+1-n} \le d\right]^{\oplus n} \\ &\cong \left(P_{n-1}^{\mathbf{FI}_d}/H_{n-1}\right)^{\oplus n}(m-1), \end{aligned}$$

where the first is obtained by removing the injection of $i := (f')^{-1}(m+1) \in \underline{n} = \{1,\dots,n\}$ in $m+1$ and where the direct sum on n comes from the choice of i in $\underline{n}$. This isomorphism is natural since we showed in Lemma 8.3.12 that $\delta_1^c(P_n^{\mathbf{FI}_d}/H_n)(\tilde{f},\tilde{g})$ sends the basis element $(f', \overline{(c_{i_1},\dots,c_{i_{m+1-n}})})$ of $\delta_1^c(P_n^{\mathbf{FI}_d}/H_n)(m)$

to $\Big((\tilde{f}+\mathrm{Id}_1)\circ f' : n \hookrightarrow m+1 \hookrightarrow l+1\,,\,\overline{(c_{i_1},\dots,c_{i_{m+1-n}},\tilde{g})}\Big)$ because it comes from $\tau_1(P_n^{\mathbf{FI}_d}/H_n)(\tilde{f},\tilde{g}) = P_n^{\mathbf{FI}_d}/H_n(\,(\tilde{f},\tilde{g}) + \mathrm{Id}_1\,)$. This description also works for the image $I(\tilde{f},\tilde{g})$ since I is a subfunctor of $\delta_1^c(P_n^{\mathbf{FI}_d}/H_n)$. Finally, since the pre-image of $m+1$ by f' is the same as the pre-image of $l+1$ by $(\tilde{f}+\mathrm{Id}_1)\circ f'$, this gives the naturality of the decomposition into the direct sum. □

We then use the computation from Proposition 8.3.13 to prove that the quotient of $P_n^{\mathbf{FI}_d}$ by the subfunctor H_n is weak polynomial of degree n for any ring $\mathbf{R}$, generalizing Theorem 8.3.11.

Theorem 8.3.14. *For all $n\in\mathbb{N}$, the quotient $P_n^{\mathbf{FI}_d}/H_n$ is weak polynomial of degree less than or equal to n, i.e. we have $\pi_d\left(P_n^{\mathbf{FI}_d}/H_n\right)\in \mathrm{Pol}_n(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod})$.*

Proof. We proceed by induction on $n\in\mathbb{N}$ using Proposition 8.3.13 and, for $n=0$, Corollary 8.3.10 which shows that the quotient of $P_0^{\mathbf{FI}_d}$ by H_0 is weak polynomial of degree 0. □

Remark 8.3.15. In both proofs that the quotient of $P_n^{\mathbf{FI}_d}$ by the subfunctor H_n is weak polynomial, there is a non-trivial stably zero functor that appears, which prevents $P_n^{\mathbf{FI}_d}/H_n$ to be strong polynomial: in the first proof (Theorem 8.3.11) we use Corollary 8.3.10 which says that $P_0^{\mathcal{C}_d}\circ\Omega$ is weak polynomial of degree 0 based on Theorem 8.2.11 because $P_0^{\mathcal{C}_d}\circ\Omega\cong P_0^{\mathbf{FI}_d}/F_0$. But this functor is not strong polynomial since the proof of Theorem 8.2.11 use heavily the fact that the functor $\delta_1^c(P_n^{\mathbf{FI}_d}/F_n)$ is in $\mathcal{SN}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod})$ and so that it vanishes in the quotient. In the second proof (Theorem 8.3.14) we use the short exact sequence of Lemma 8.3.12 which gives an isomorphism in the quotient category $\mathbf{St}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod})$ since the last term is in $\mathcal{SN}(\mathbf{FI}_d,\mathbf{R}\text{-}\mathbf{Mod})$.

8.4 Weak polynomial quotients of $P_n^{\mathbf{FI}_d}$ of arbitrary degrees

We explained in Remark 5.2.3 that the functor $P_n^{\mathbf{FI}}$ is strong polynomial of degree n over $\mathbf{FI}$. In this section we present a quotient of $P_n^{\mathbf{FI}}$ that is weak polynomial of degree i for each $i\in\mathbb{N}$, and we explain why this quotient is not strong polynomial of degree less than n. We then use it and the formula from Proposition 8.3.8 to describe a corresponding quotient of $P_n^{\mathbf{FI}_d}$ that is weak polynomial of degree i for each $i\in\mathbb{N}$.

Definition 8.4.1. For $n,i\in\mathbb{N}$, the *functor* $Q_n^i:\mathbf{FI}\to\mathbf{R}\text{-}\mathbf{Mod}$ is given by

$$Q_n^i = \mathrm{Im}\left(P_n^{\mathbf{FI}} \xrightarrow{\bigoplus\limits_{k\le i}\ \bigoplus\limits_{j_k\in\mathbf{FI}(k,n)} j_k^*} \bigoplus_{k\le i}\ \bigoplus_{j_k\in\mathbf{FI}(k,n)} P_k^{\mathbf{FI}} \right),$$

where j_k^* is the precomposition by the injection $j_k\in\mathbf{FI}(k,n)$.

We explain in the following that Q_n^i is a weak polynomial quotient of $P_n^{\mathbf{FI}}$ of degree i and that it is not strong polynomial of degree less than n.

Proposition 8.4.2. *For $n,i\in\mathbb{N}$, the quotient Q_n^i of $P_n^{\mathbf{FI}}$ is strong polynomial of degree n, but not $n-1$.*

Proof. The functor Q_n^i is a quotient of $P_n^{\mathbf{FI}}$ since it is the image of a natural transformation starting from $P_n^{\mathbf{FI}}$ and the $\mathbf{FI}$-module $P_n^{\mathbf{FI}}$ is strong polynomial of degree n by Proposition 5.2.1 for $d=1$. Since the strong polynomial functors are closed under quotient by Proposition 5.1.3 we conclude that Q_n^i is strong polynomial of degree less than or equal to n. However, for $m < n$ we have $P_n^{\mathbf{FI}}(m) = 0$, and so $Q_n^i(m) = 0$ since $Q_n^i(m)$ is the image of the morphism $P_n^{\mathbf{FI}}(m) \to \bigoplus P_k^{\mathbf{FI}}(m)$. Then $Q_n^i(m) = 0$ for $m < n$ and, using Lemma 5.1.6, we conclude that Q_n^i is not strong polynomial of degree $n-1$. □

Proposition 8.4.3. *For $n, i \in \mathbb{N}$, the quotient Q_n^i of $P_n^{\mathbf{FI}}$ is weak polynomial of degree less than or equal to i.*

Proof. The Proposition 5.2.1 implies, for $d = 1$, that the $\mathbf{FI}$-module $P_k^{\mathbf{FI}}$ is weak polynomial of degree k. Then the sum $\bigoplus_{k\leq i} \left(P_k^{\mathbf{FI}}\right)^{\oplus \mathbf{FI}(k,n)}$ is weak polynomial of degree i and Q_n^i is by definition a subfunctor of this sum. We conclude using the fact that the weak polynomial functors are closed under subobjects by Proposition 7.2.5. □

Remark 8.4.4. It is reasonable to believe that Q_n^i is the largest quotient of $P_n^{\mathbf{FI}}$ in $\mathrm{Pol}_i(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$, even if it is not proven yet.

We now use this quotient of $P_n^{\mathbf{FI}}$ to get a quotient of $P_n^{\mathbf{FI}_d}$ that is weak polynomial of degree i for any $i \in \mathbb{N}$ using the formula

$$P_n^{\mathbf{FI}_d} / H_n \cong \left(\mathcal{O}^* P_n^{\mathbf{FI}}\right) \otimes P_0^{\mathcal{C}_d}\left((-) - n\right) \circ \Omega \tag{8.1}$$

from Proposition 8.3.8.

Proposition 8.4.5. *For $n, i \in \mathbb{N}$, the formula $\left(\mathcal{O}^* Q_n^i\right) \otimes P_0^{\mathcal{C}_d}\left((-)-n\right) \circ \Omega$ defines a $\mathbf{FI}_d$-module which is a quotient of $P_n^{\mathbf{FI}_d}$ and is weak polynomial of degree less than or equal to i.*

Proof. By Proposition 8.3.8 the quotient of $P_n^{\mathbf{FI}_d}$ by its subfunctor H_n from Definition 8.3.2 is described by the formula (8.1) above. The $\mathbf{FI}_d$-module $P_0^{\mathcal{C}_d} \circ \Omega$ is weak polynomial of degree 0 by Corollary 8.3.10. Then the $\mathbf{FI}_d$-module $P_0^{\mathcal{C}_d}\left((-)-n\right) \circ \Omega$ is also weak polynomial of degree 0 since it is the same functor but shifted by n and the weak polynomiality concerns the stable behavior. However, in Proposition 8.4.3 we showed that Q_n^i is a quotient of $P_n^{\mathbf{FI}}$ which is weak polynomial of degree less than or equal to i. Since the functor $\mathcal{O}^*$ is exact and preserves the polynomiality by Proposition 7.2.9, we get that $\mathcal{O}^* Q_n^i$ is a quotient of $\mathcal{O}^* P_n^{\mathbf{FI}}$ in $\mathrm{Pol}_i(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. As the pointwise tensor product respects epimorphisms, we get that $\left(\mathcal{O}^* Q_n^i\right) \otimes P_0^{\mathcal{C}_d}\left((-)-n\right) \circ \Omega$ is a quotient of $P_n^{\mathbf{FI}_d} / H_n$, so a quotient of $P_n^{\mathbf{FI}_d}$. Finally, by Proposition 7.3.6, the pointwise tensor product preserves the polynomiality so we conclude that this quotient of $P_n^{\mathbf{FI}_d}$ is in $\mathrm{Pol}_i(\mathbf{FI}, \mathbf{R}\text{-}\mathbf{Mod})$. □

Chapter 9

Functors on the categories $\mathrm{Cospan}(\mathbf{FI}_d)$

Entourez-vous de passionnés, ils instruisent mieux que les livres.

Aienkei

In order to study the polynomial functors over a symmetric monoidal category whose unit is an initial object like **FI**, Djament and Vespa introduce in [DV19, Dja16] the construction $\mathcal{M} \to \tilde{\mathcal{M}}$ which turns the category $\mathcal{M}$ whose unit is an initial object into the category $\tilde{\mathcal{M}}$ whose unit is a null object. This construction is a variation of a construction of Quillen in K-theory, and it morally adds morphisms from the objects of the category to the unit, while preserving the morphisms from the unit to the objects. Since this construction preserves the polynomial properties, it allows us to turn the study of polynomial functors over a category whose unit is an initial object into the study of polynomial functors over a category whose unit is a null object, as shown in [DV19, Theorem 4.8]. The advantage is that the functors on those categories with a null object are better known. In particular, for **FI**-modules, Djament and Vespa show in [DV19, Proposition 5.9] the following equivalences of categories

$$\mathrm{Pol}_n(\mathbf{FI}) \big/ \mathrm{Pol}_{n-1}(\mathbf{FI}) \cong \mathrm{Pol}_n(\mathrm{Cospan}(\mathbf{FI})) \big/ \mathrm{Pol}_{n-1}(\mathrm{Cospan}(\mathbf{FI})) \cong \mathbf{Fct}(\mathbf{\Sigma}_n), \tag{9.1}$$

where $\mathbf{\Sigma}_n$ is the category associated with the symmetric group and $\mathrm{Cospan}(\mathbf{FI})$ is equivalent to $\tilde{\mathbf{FI}}$. This result combines two equivalences of categories: the first describes the quotient of polynomial functors over **FI** as the same quotient of polynomial functors over the intermediate category $\mathrm{Cospan}(\mathbf{FI})$. The second equivalence describes the quotient of polynomial functors over $\mathrm{Cospan}(\mathbf{FI})$ using a variation of a Dold-Kan type theorem of Pirashvili from [CEF15] which gives an equivalence of categories between the functors over the category $\mathbf{FI}\#$ of finite sets and partial injections, and the functors over $\mathbf{\Sigma}$ the category of finite sets and bijections.

In this chapter we show that this approach cannot be directly generalized to describe the polynomial functors over $\mathbf{FI}_d$. Indeed, after introducing a generalization of the construction Cospan for $\mathbf{FI}_d$ and after defining polynomial $\mathrm{Cospan}(\mathbf{FI}_d)$-modules, we show that the polynomial functors of degree 0 over $\mathrm{Cospan}(\mathbf{FI}_d)$ are the constant functors. Together with Theorem 7.4.12, this shows that the first equivalence in (9.1) already fails for $n = 0$. More precisely, in Section 9.1 we introduce the category $\mathrm{Cospan}(\mathbf{FI}_d)$ and we study its properties. In particular, the morphisms in $\mathrm{Cospan}(\mathbf{FI}_d)$ are given by some equivalence classes of diagrams and we show that each class admits a minimal representative. This implies that the morphisms in Cospan($\mathbf{FI}_d$) from 0 to 1 and from 1 to 0 are isomorphic to $\mathbf{FI}_d(0,1)$. In Section 9.2 we introduce a combinatorial category $\mathbf{FI}_d\#$ that is equivalent to Cospan($\mathbf{FI}_d$): it is the category of finite sets and of partial injections coupled with a choice of colours on the complement at the source and the target. Finally, in the following sections we define the polynomial functors over Cospan($\mathbf{FI}_d$), we give their basic properties and we describe the ones of degree 0.

9.1 The categories Cospan($\mathbf{FI}_d$)

In this section we present a generalization of the construction Cospan for the categories $\mathbf{FI}_d$ and we give the main properties of the category Cospan($\mathbf{FI}_d$). This construction can be found for other categories in [Ves07] and is equivalent to the construction $\tilde{} : \mathcal{M}\mathrm{on}_{\mathrm{ini}} \to \mathcal{M}\mathrm{on}_{\mathrm{nul}}$ from [DV19, Dja16]. The idea is to start with $\mathbf{FI}_d$, which has only increasing morphisms (*i.e.* from n to m when $n \leq m$), and to add decreasing morphisms to it. For example, $\mathbf{FI}$ is a monoidal category with an initial object and the construction Cospan turns it into Cospan($\mathbf{FI}$), a monoidal category with a null object. The following definition extends the one from the category $\mathbf{FI}$ to the category $\mathbf{FI}_d$:

Definition 9.1.1. The *category* Cospan($\mathbf{FI}_d$) has the same objects as $\mathbf{FI}_d$. The morphisms in Cospan($\mathbf{FI}_d$) from $n \in \mathbf{FI}_d$ to $m \in \mathbf{FI}_d$ are the classes of diagrams of the form

$$(D) = \left(\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m+\alpha"] \\ n \arrow[r, "f"'] & m+k \end{tikzcd}\right),$$

with $k \in \mathbf{FI}_d$, $\alpha \in \mathbf{FI}_d(0,k)$ and $f \in \mathbf{FI}_d(n, m+k)$ in the quotient by the equivalence relation generated by:

i) All diagrams from 0 to 0 are in relation,

ii) Two diagrams

$$(D) = \left(\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m+\alpha"] \\ n \arrow[r, "f"'] & m+k \end{tikzcd}\right) \quad \text{and} \quad (D') = \left(\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m+\alpha'"] \\ n \arrow[r, "f'"'] & m+k' \end{tikzcd}\right)$$

from n to m are in relation if there exists $\varphi \in \mathbf{FI}_d(k,k')$ such that $(\mathrm{Id}_m+\varphi)\circ f = f'$ and $(\mathrm{Id}_m+\varphi)\circ(\mathrm{Id}_m+\alpha) = \alpha'$.

The class of the diagram (D) in the quotient is denoted by $[D]$. The composition in Cospan($\mathbf{FI}_d$) is given by the relation

$$\left[\begin{tikzcd} & i+0 \arrow[d, "\mathrm{Id}_m+\beta"] \\ m \arrow[r, "g"'] & i+l \end{tikzcd}\right] \circ \left[\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m+\alpha"] \\ n \arrow[r, "f"'] & m+k \end{tikzcd}\right] = \left[\begin{tikzcd} & & i+0+0 \arrow[d, "\mathrm{Id}_i+\beta+\alpha"] \\ n \arrow[r, "f"'] & m+k \arrow[r, "g+\mathrm{Id}_k"'] & i+l+k \end{tikzcd}\right],$$

where we choose a representative diagram of each class.

Remark 9.1.2. A diagram (D) consists of an object $k \in \mathbf{FI}_d$ and two morphisms $\alpha \in \mathbf{FI}_d(0,k)$ and $f \in \mathbf{FI}_d(n, m+k)$. The morphism α starts from zero, so it corresponds to a colour choice on the object k, and the morphism $f = (g,\beta) \in \mathbf{FI}_d(n, m+k)$ consists of an injection g and a colour choice β on the complement of the image. This emphasizes that a morphism in Cospan($\mathbf{FI}_d$) is morally composed of an injection g and colour choices on two sets α and β, which interact with each other.

Remark 9.1.3. The two diagrams $[D]$ and $[D']$ of Definition 9.1.1 from n to m are in relation (along the point ii) if there exists $\varphi \in \mathbf{FI}_d(k,k')$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & & m+0 \arrow[dl, "\mathrm{Id}_m+\alpha"'] \arrow[d, "\mathrm{Id}_m+\alpha'"] \\
n \arrow[r, "f"] \arrow[rr, bend right, "f'"'] & m+k \arrow[r, "\mathrm{Id}_m+\varphi"] & m+k'
\end{tikzcd}$$

The composition in Cospan($\mathbf{FI}_d$) corresponds to the natural way to complete the diagram that links the two diagrams:

$$\begin{tikzcd}
 & & i+0 \arrow[d, "\mathrm{Id}_i+\beta"] \\
 & m+0=m \arrow[r, "g"] \arrow[d, "\mathrm{Id}_m+\alpha"] & i+l \arrow[d, dashed, "\mathrm{Id}_{i+l}+\alpha"] \\
n \arrow[r, "f"'] & m+k \arrow[r, dashed, "g+\mathrm{Id}_k"'] & i+l+k.
\end{tikzcd}$$

Lemma 9.1.4. *The composition in the category* Cospan($\mathbf{FI}_d$) *does not depend on the chosen representative. The category* Cospan($\mathbf{FI}_d$) *is a symmetric monoidal category.*

Proof. For two representatives (D) and (D') of a morphism $[D] \in$ Cospan($\mathbf{FI}_d$)(i,m) and two representatives $(\tilde{D})$ and $(\tilde{D}')$ of a morphism $[\tilde{D}] \in$ Cospan($\mathbf{FI}_d$)(m,n) connected by the commutative diagrams

$$\begin{tikzcd}
 & & i+0 \arrow[dl, "\mathrm{Id}_i+\beta"'] \arrow[d, "\mathrm{Id}_i+\beta'"] \\
m \arrow[r, "g"] \arrow[rr, bend right, "g'"'] & i+l \arrow[r, "\mathrm{Id}_i+\psi"] & i+l'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & & m+0 \arrow[dl, "\mathrm{Id}_m+\alpha"'] \arrow[d, "\mathrm{Id}_m+\alpha'"] \\
n \arrow[r, "f"] \arrow[rr, bend right, "f'"'] & m+k \arrow[r, "\mathrm{Id}_m+\varphi"] & m+k'
\end{tikzcd}$$

the commutative diagram

$$\begin{tikzcd}
 & & i+0 \arrow[d, "\mathrm{Id}_i+\beta"'] \arrow[ddr, bend left, "\mathrm{Id}_i+\beta'"] & \\
 & & i+l \arrow[d, "\mathrm{Id}_i+\mathrm{Id}_l+\alpha"'] \arrow[dr, "\mathrm{Id}_i+\psi"] & \\
n \arrow[r, "f"] \arrow[drr, bend right, "f'"'] & m+k \arrow[r, "g+\mathrm{Id}_k"] \arrow[dr, "\mathrm{Id}_m+\varphi"'] & i+l+k \arrow[dr, "\mathrm{Id}_i+\psi+\varphi"] & i+l' \arrow[d, "\mathrm{Id}_i+\mathrm{Id}_{l'}+\alpha'"] \\
 & & m+k' \arrow[r, "g'+\mathrm{Id}_{k'}"'] & i+l'+k'
\end{tikzcd}$$

shows that the two composition diagrams are related and so their class is the same. For two different representatives (D) and (D') of a morphism $[D]$ from 0 to 0, the two compositions $[D] \circ [\tilde{D}]$ and $[D'] \circ [\tilde{D}]$ give the same result due to the structure of monoidal category. Finally, the structure of symmetric monoidal category directly comes from $\mathbf{FI}_d$. $\square$

Remark 9.1.5. For $d = 1$, the category Cospan($\mathbf{FI}_1$) is equivalent to the category $\tilde{\mathbf{FI}}$ of [DV19, Section 4] whose objects are the ones of $\mathbf{FI}$ and whose morphisms between n and m are the elements of the filtered colimit of the sets $\mathbf{FI}(n, m+a)$ for $a \in \tilde{\mathbf{FI}}$. The equivalence is given by the functor that sends an object k in $\tilde{\mathbf{FI}}$ to k in Cospan($\mathbf{FI}_1$) and a morphism $[f] \in \operatorname{colim}_{a \in \mathbf{FI}} \mathrm{Hom}_{\mathbf{FI}}(n, m+a)$ to the class of diagrams

$$[D] = \left[\begin{tikzcd} & m \arrow[d, "\mathrm{Id}_m + c^a"] \\ n \arrow[r, "f"'] & m+a \end{tikzcd}\right]$$

by choosing a representative $f : n \to m+a$ of the class $[f]$ in the filtered colimit. The value of the functor on a morphism does not depend on the chosen representative by Proposition 1.1.6. This functor is essentially surjective, full and faithful since the equivalence relations defining the morphisms in Cospan($\mathbf{FI}$) correspond exactly to the filtered colimit relations defining the morphisms in $\tilde{\mathbf{FI}}$ by Proposition 1.1.6 again.

We now prove that for every morphism $[D]$ in Cospan($\mathbf{FI}_d$) we can choose a minimal representative diagram (D) of the class $[D]$. This important property will be used in the following and gives a description of the morphisms from 0 to m and from n to 0 in Cospan($\mathbf{FI}_d$).

Proposition 9.1.6. *For each morphism $[D]$ in* Cospan($\mathbf{FI}_d$)(n,m)*, there exists $0 \leq z \leq n$, $\beta \in \mathbf{FI}_d(0,z)$ and $g \in \mathbf{FI}_d(n, m+z)$ such that the diagram (D') defined by these three elements is a minimal representative of the class $[D]$ (in the sense that $z \leq k$ for every $k \in \mathbf{FI}_d$ associated to a representative (D) of the class $[D]$).*

Proof. Let z be the subobject $\mathrm{Im}(f) \cap k$ of k and σ be an isomorphism between k and $z + (k \smallsetminus z)$, the following diagram commutes:

$$\begin{tikzcd} & & m+0 \arrow[dl, "\mathrm{Id}_m + \alpha"'] \arrow[dd, "\mathrm{Id}_m + (\sigma \circ \alpha)"] \\ n \arrow[r, "f"] \arrow[drr, bend right, "(\mathrm{Id}_m + \sigma) \circ f"'] & m+k \arrow[dr, "\mathrm{Id}_m + \sigma"] & \\ & & m+z+(k \smallsetminus z). \end{tikzcd}$$

Since the image of f is in $m+z$, we can define some morphisms $g \in \mathbf{FI}_d(n, m+z)$, $h \in \mathbf{FI}_d(0, k \smallsetminus z)$, $\beta \in \mathbf{FI}_d(0,z)$ and $\gamma \in \mathbf{FI}_d(0, k \smallsetminus z)$ by the relations $(\mathrm{Id}_m + \sigma) \circ f = g + h$ and $\mathrm{Id}_m + (\sigma \circ \alpha) = (\mathrm{Id}_m + \beta) + \gamma$. Since f is an injection we get $0 \leq z \leq n$. By definition, we have the following equality

$$[D] := \left[\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m + \alpha"] \\ n \arrow[r, "f"'] & m+k \end{tikzcd}\right] = \left[\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m + (\sigma \circ \alpha)"'] \\ n \arrow[r, "(\mathrm{Id}_m + \sigma) \circ f"'] & m+z+(k \smallsetminus z) \end{tikzcd}\right].$$

Since there is only one equivalence class in Cospan($\mathbf{FI}_d$)$(0,0)$, so we have

$$[D] = \left[\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m+\beta"] \\ n \arrow[r, "g"'] & m+z \end{tikzcd}\right] + \left[\begin{tikzcd} & 0 \arrow[d, "\gamma"] \\ 0 \arrow[r, "h"'] & k \smallsetminus z \end{tikzcd}\right] = \left[\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m+\beta"] \\ n \arrow[r, "g"'] & m+z \end{tikzcd}\right] =: [D'] .$$

Finally, the diagram (D') is minimal since g restricted to $g^{-1}(z) \to z$ is surjective, *i.e.* bijective. □

Then, the Proposition 9.1.6 gives, in particular, the following description of the morphisms starting and ending at 0 in Cospan($\mathbf{FI}_d$):

Definition 9.1.7. For $n \in \mathbb{N}$ and $x \in \mathbf{FI}_d(0,n)$, we denote by $x \in$ Cospan($\mathbf{FI}_d$) $(0,n)$ and by $\tilde{x} \in$ Cospan($\mathbf{FI}_d$) $(0,n)$ the morphisms given by the classes

$$x = \left[\begin{tikzcd} & n \arrow[d, "\mathrm{Id}_n"] \\ 0 \arrow[r, "x"'] & n \end{tikzcd}\right] \quad \text{and} \quad \tilde{x} = \left[\begin{tikzcd} & 0 \arrow[d, "x"] \\ n \arrow[r, "\mathrm{Id}_n"'] & n \end{tikzcd}\right].$$

Proposition 9.1.8. *For $n,m \in \mathbb{N}$, with the notations of Definition 9.1.7, the set* Cospan($\mathbf{FI}_d$) $(0,m)$ *is the disjoint union of the equivalence classes x for $x \in \mathbf{FI}_d(0,m)$. Similarly, the set* Cospan($\mathbf{FI}_d$) $(n,0)$ *is the disjoint union of the equivalence classes $\tilde{x}$ for $x \in \mathbf{FI}_d(0,n)$. Moreover, for $x \in \mathbf{FI}_d(0,n)$ the composition $\tilde{x} \circ x$ is the identity.*

Proof. For $[D] \in$ Cospan($\mathbf{FI}_d$) $(0,m)$, the Proposition 9.1.6 for $n=0$ gives $z=0$, so $\beta = \mathrm{Id}_0$ and there exists a minimal representative of $[D]$ associated with some $x \in \mathbf{FI}_d(0,m)$ as in Definition 9.1.7. Two such diagrams are never related so we get the description of Cospan($\mathbf{FI}_d$) $(0,m)$. For $[D] \in$ Cospan($\mathbf{FI}_d$) $(n,0)$ the Proposition 9.1.6 for $m=0$ gives a minimal representative of $[D]$ associated with $0 \le z \le n$, $\beta \in \mathbf{FI}_d(0,z)$ and $g \in \mathbf{FI}_d(n,z)$. Since the set $\mathbf{FI}_d(n,z)$ is empty for $z < n$ we get $z = n$ and g is in $\mathbf{FI}_d(n,n) = \mathrm{S}_n$, so it is bijective. We then have the commutative diagram

$$\begin{tikzcd} & & 0+0 \arrow[dl, "\mathrm{Id}_m+\beta"'] \arrow[dd, "\mathrm{Id}_0+g^{-1}\circ\beta"] \\ n \arrow[r, "g"] \arrow[drr, bend right, "\mathrm{Id}_n"'] & 0+n=0+z \arrow[dr, "g^{-1}"] & \\ & & n. \end{tikzcd}$$

which gives the equalities

$$\begin{aligned} \mathrm{Cospan}(\mathbf{FI}_d)\,(n,0) &= \bigcup_{\beta\in\mathbf{FI}_d(0,n)} \bigcup_{g\in\mathrm{S}_n} \left[\begin{tikzcd} & 0+0 \arrow[d, "\mathrm{Id}_0+\beta"] \\ n \arrow[r, "g"'] & 0+n \end{tikzcd}\right] \\ &= \bigcup_{\beta\in\mathbf{FI}_d(0,n)} \bigcup_{g\in\mathrm{S}_n} \left[\begin{tikzcd} & 0 \arrow[d, "g^{-1}\circ\beta"] \\ n \arrow[r, "\mathrm{Id}_n"'] & n \end{tikzcd}\right] \end{aligned}$$

where the last equality is obtained by checking that two diagrams associated with different $x \in \mathbf{FI}_d(0,n)$ are not related. Finally, for $n \in \mathbb{N}$ and $x \in \mathbf{FI}_d(0,n)$, the composition $\tilde{x} \circ x$ is a morphism from 0 to 0 and therefore the identity. □

The Proposition 9.1.8 implies that both Cospan($\mathbf{FI}_d$) $(0,n)$ and Cospan($\mathbf{FI}_d$) $(n,0)$ are in bijection with $\mathbf{FI}_d(0,n)$. This emphasizes that the category Cospan($\mathbf{FI}_d$) is essentially obtained by keeping the morphisms from 0 to n of $\mathbf{FI}_d$ and adding new morphisms from n to 0 corresponding to them. This remark also shows that 0 is a null object in Cospan($\mathbf{FI}_d$) for $d = 1$, as explained in [DV19], but this is not true for $d > 1$.

Remark 9.1.9. As a special case of Proposition 9.1.8, using the notations of Definition 9.1.7, Cospan($\mathbf{FI}_d$) $(0,1)$ is the disjoint union of the equivalence classes c for $c \in C$ and Cospan($\mathbf{FI}_d$) $(1,0)$ is the disjoint union of the equivalence classes $\tilde{c}$ for $c \in C$. In summary, both Cospan($\mathbf{FI}_d$) $(0,1)$ and Cospan($\mathbf{FI}_d$) $(1,0)$ are in bijection with $\mathbf{FI}_d(0,1) = C$ and, for $c \in C$, we have two morphisms $c \in$ Cospan($\mathbf{FI}_d$) $(0,1)$ and $\tilde{c} \in$ Cospan($\mathbf{FI}_d$) $(1,0)$ such that the composition $\tilde{c} \circ c$ is the identity.

In the end of this section, we study the inclusion functor η of $\mathbf{FI}_d$ in Cospan($\mathbf{FI}_d$) and describe an adjoint of this functor, generalizing Proposition 4.7 in [DV19]. This adjunction connects the functors on $\mathbf{FI}_d$ and those on Cospan($\mathbf{FI}_d$) and, in particular, it allows us to create polynomial $\mathbf{FI}_d$-modules from polynomial Cospan($\mathbf{FI}_d$)-modules.

Definition 9.1.10. The *inclusion functor* $\eta : \mathbf{FI}_d \to$ Cospan($\mathbf{FI}_d$) is given on objects by $\eta(n) = n$ and on a morphism $f \in \mathbf{FI}_d(n,m)$ by the class f from Definition 9.1.7.

We describe the adjoint of this functor using the theory of Kan extensions, but first we introduce the slice category $(\eta \downarrow n)$:

Definition 9.1.11. For n in Cospan($\mathbf{FI}_d$), the slice *category* $(\eta \downarrow n)$ has for objects the pairs (t,φ), where $t \in$ Cospan($\mathbf{FI}_d$) and $\phi \in$ Cospan($\mathbf{FI}_d$) (t,n), and for morphisms from (t,ϕ) to (t',ϕ') the maps $f : t \to t'$ in Cospan($\mathbf{FI}_d$) such that $\phi' \circ f = \phi$. The forgetful functor from $(\eta \downarrow n)$ to $\mathbf{FI}_d$ is denoted by ι_n.

Proposition 9.1.12. *The precomposition functor*

$$\eta^* : \mathbf{Fct}(\,\mathrm{Cospan}(\mathbf{FI}_d)\,,\,\mathbf{R}\text{-}\mathbf{Mod}) \longrightarrow \mathbf{Fct}(\,\mathbf{FI}_d\,,\,\mathbf{R}\text{-}\mathbf{Mod})$$

has a left adjoint α which is given on the objects by $\alpha(F)(n) = \operatorname*{colim}_{(t,\phi)\in(\eta\downarrow n)} (F \circ \iota_n)(t,\phi)$.

Proof. Since **R-Mod** is cocomplete, the general theory of Kan extensions (see [ML98, P.236]) gives the existence of a left adjoint to the precomposition functor η^*. It is the functor $\alpha = \mathrm{Lan}_\eta(-)$, given on a functor $F : \mathbf{FI}_d \to \mathbf{R}\text{-}\mathbf{Mod}$ by the functor $\alpha(F) = \mathrm{Lan}_\eta(F) : \mathrm{Cospan}(\mathbf{FI}_d) \to$ **R-Mod**, described on objects by the formula of the statement, and its image on morphisms is obtained by the universal properties of colimits. □

9.2 Equivalence between Cospan($\mathbf{FI}_d$) and $\mathbf{FI}_d\#$

In this section we introduce the combinatorial category $\mathbf{FI}_d\#$ and we prove that it is isomorphic to Cospan($\mathbf{FI}_d$). For example, when $d = 1$ we recover the isomorphism $\tilde{\mathbf{FI}} \cong \mathbf{FI}\#$ of [DV19, Example 4.2] or [Wil19], where $\mathbf{FI}\#$ is the category of partial injections of finite sets from [CEF15] and [Wil18a].

Definition 9.2.1. The *category* $\mathbf{FI}_d\#$ has the same objects as $\mathbf{FI}_d$ and the morphisms in $\mathbf{FI}_d\#$ from n to m are the pairs (f,α) where f is a partially defined morphism in $\mathbf{FI}_d$ - *i.e.* a morphism $f \in \mathbf{FI}_d(l,m)$ for l a subobject of n in $\mathbf{FI}_d$ - and $\alpha \in \mathbf{FI}_d(0, n\setminus l)$ is a colour choice. The composition is given by the relation

$$\left(l \xrightarrow{f} m \ , \ 0 \xrightarrow{\alpha} n\setminus l \right) \circ \left(k \xrightarrow{g} n \ , \ 0 \xrightarrow{\beta} i\setminus k \right) =$$

$$\left(g^{-1}(l) \xrightarrow{g} l \xrightarrow{f} m \ , \ 0 \xrightarrow{\beta+(\alpha|_{\mathrm{Im}(\tilde{g})})} i\setminus k + \mathrm{Im}(\tilde{g}) \xrightarrow{\mathrm{Id}_{i\setminus k}+\tilde{g}^{-1}} (i\setminus k) + (k\setminus g^{-1}(l)) = i\setminus g^{-1}(l) \right)$$

where $\tilde{g}$ is g restricted to $k\setminus g^{-1}(l)$ and co-restricted to its image within $n\setminus l$, so that it is an isomorphism of $k\setminus g^{-1}(l)$.

Remark 9.2.2. As in Remark 9.1.2, a morphism (f,α) in $\mathbf{FI}_d\#(n,m)$ morally consists of an injection and two colour choices, one on the complement at the source and one on the complement at the target, which interact with each other. Indeed, the morphism $f = (g,\beta) \in \mathbf{FI}_d(l,m)$ is itself a pair formed by an injection g from l into m and a colour choice β on the complement of the image of g in m.

We now define two functors between Cospan($\mathbf{FI}_d$) and $\mathbf{FI}_d\#$, and we prove in Theorem 9.2.8 that they are inverse to each other.

Definition 9.2.3. The *functor* $\Gamma : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{FI}_d\#$ is given on objects by $\Gamma(n) = n$ and on morphisms by

$$\Gamma\left(\left[\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m+\alpha"] \\ n \arrow[r, "f"'] & m+k \end{tikzcd}\right]\right) = \left(f^{-1}(m) \xrightarrow{f} m \ , \ 0 \xrightarrow{\alpha|_{\mathrm{Im}(f)\cap k}} \mathrm{Im}(f)\cap k \xrightarrow{\tilde{f}^{-1}} n\setminus f^{-1}(m) \right)$$

where $\tilde{f}$ is f restricted to $n\setminus f^{-1}(m)$ and co-restricted to $\mathrm{Im}(f)\cap k$ so it is an isomorphism.

Definition 9.2.4. The *functor* $\chi : \mathbf{FI}_d\# \longrightarrow \mathrm{Cospan}(\mathbf{FI}_d)$ is given on objects by $\chi(n) = n$ and on morphisms by

$$\chi\left(l \xrightarrow{g} m \ , \ 0 \xrightarrow{\beta} n\setminus l \right) = \left[\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m+\beta"] \\ n = l+(n\setminus l) \arrow[r, "g+\mathrm{Id}_{n\setminus l}"'] & m+(n\setminus l) \end{tikzcd}\right].$$

Lemma 9.2.5. *The functors* $\Gamma : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{FI}_d\#$ *and* $\chi : \mathbf{FI}_d\# \longrightarrow \mathrm{Cospan}(\mathbf{FI}_d)$ *are well defined.*

Proof. By definition of the composition in Cospan($\mathbf{FI}_d$) and $\mathbf{FI}_d\#$, both Γ and χ respect composition and identities. The definition of Γ does not depend on the choice of a representative (D) of the class $[D]$: indeed, for two representatives (D) and (D') of the morphism $[D]$ connected by the commutative diagram

$$\begin{tikzcd} & & m+0 \arrow[dl, "\mathrm{Id}_m+\alpha"'] \arrow[d, "\mathrm{Id}_m+\alpha'"] \\ n \arrow[r, "f"] \arrow[rr, bend right, "g"'] & m+k \arrow[r, "\mathrm{Id}_m+\varphi"] & m+k', \end{tikzcd}$$

as in the proof of Proposition 9.1.6 we can choose k and k' minimal such that $\mathrm{Im}(f) \cap k = k$ and $\mathrm{Im}(g) \cap k' = k'$. The minimality for both k and k' implies that φ is bijective. Then the equality $g = (\mathrm{Id}_m + \varphi) \circ f$ implies that $f^{-1}(m) = g^{-1}(m)$ and that the restrictions of f and g to this subobject are the same. This relation also implies that $\tilde{g} = \varphi \circ \tilde{f}$, and together with the relation $\alpha' = \varphi \circ \alpha$ it gives the relations $\tilde{g}^{-1} \circ \alpha' = (\varphi \circ \tilde{f})^{-1} \circ \varphi \circ \alpha = \tilde{f}^{-1} \circ \varphi^{-1} \circ \varphi \circ \alpha = \tilde{f}^{-1} \circ \alpha$ which shows that $\Gamma([D]) = \Gamma([D'])$. □

Remark 9.2.6. If we choose k minimal as in Proposition 9.1.6, then $\mathrm{Im}(f) \cap k = k$ and the injective morphism f restricted to $n \smallsetminus f^{-1}(m)$ gives an isomorphism $n \smallsetminus f^{-1}(m) \simeq k$. This isomorphism can be used to rewrite the image of the functor Γ on morphisms as follows:

$$\Gamma\left(\left[\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m + \alpha"] \\ n \arrow[r, "f"'] & m+k \end{tikzcd}\right]\right) = \left(f^{-1}(m) \xrightarrow{f} m \ , \ 0 \xrightarrow{\alpha} k \xrightarrow{(f|_{n \smallsetminus f^{-1}(m)})^{-1}} n \smallsetminus f^{-1}(m) \right).$$

In the end of the section we always choose such a minimal k and we use the alternative definition of Γ given in Remark 9.2.6 to simplify the notations.

Remark 9.2.7. We can assume that f restricted to $n \smallsetminus f^{-1}(m)$ is the identity by the following commutative diagram:

$$\begin{tikzcd} & & m+0 \arrow[dl, "\mathrm{Id}_m + \alpha"'] \arrow[d, "\mathrm{Id}_m + ((f|_{n \smallsetminus f^{-1}(m)})^{-1} \circ \alpha)"] \\ f^{-1}(m) + (n \smallsetminus f^{-1}(m)) \arrow[r, "f"] \arrow[rr, bend right, "f + \mathrm{Id}_{n \smallsetminus f^{-1}(m)}"'] & m+k \arrow[r, "\mathrm{Id}_m + (f|_{n \smallsetminus f^{-1}(m)})^{-1}"'] & m + (n \smallsetminus f^{-1}(m)) \end{tikzcd}.$$

This explains why the inverse image of f, restricted to $n \smallsetminus f^{-1}(m)$, appears in the definition of the functor Γ, and illustrates the equivalence below. Indeed, this representative diagram of the class isolates the colour choice on the complement at the source $n \smallsetminus f^{-1}(m)$ and the colour choice on the complement $m \smallsetminus f(n) \cap m$ at the target, which corresponds to a morphism in $\mathbf{FI}_d\#$.

Theorem 9.2.8. *The functors* $\Gamma : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{FI}_d\#$ *and* $\chi : \mathbf{FI}_d\# \longrightarrow \mathrm{Cospan}(\mathbf{FI}_d)$ *from Definitions 9.2.3 and 9.2.4 are inverse of each other. They give an isomorphism of categories*

$$\mathrm{Cospan}(\mathbf{FI}_d) \cong \mathbf{FI}_d\#.$$

Proof. By definition both compositions of Γ and χ are the identity on objects, we prove that this is also the case for morphisms. For $(g : l \to m, \beta : 0 \to n \smallsetminus l)$ a morphism in $\mathbf{FI}_d\#(n,m)$ we have:

$$\Gamma \circ \chi\left(\left(l \xrightarrow{g} m \ , \ 0 \xrightarrow{\beta} n \smallsetminus l \right) \right) = \Gamma\left(\left[\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m + \beta"] \\ n = l + (n \smallsetminus l) \arrow[r, "g + \mathrm{Id}_{n \smallsetminus l}"'] & m + (n \smallsetminus l) \end{tikzcd}\right]\right).$$

Then, the image by Γ of this class is

$$\left((g + \mathrm{Id})^{-1}(m) \xrightarrow{g + \mathrm{Id}_{n \smallsetminus l}} m \ , \ 0 \xrightarrow{\left((g+\mathrm{Id})|_{n \smallsetminus (g+\mathrm{Id})^{-1}(m)}\right)^{-1} \circ \beta} n \smallsetminus l \right) = \left(l \xrightarrow{g} m \ , \ 0 \xrightarrow{\beta} n \smallsetminus l \right),$$

showing the identity $\Gamma \circ \chi = \mathrm{Id}$ on morphisms. For $[D]$ a morphism in Cospan($\mathbf{FI}_d$)(n,m), we use the definitions to get

$$
\begin{aligned}
\chi \circ \Gamma\big(\,[D]\,\big) &:= \chi \circ \Gamma\left(\left[\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m + \alpha"] \\ n \arrow[r, "f"'] & m+k \end{tikzcd}\right]\right) \\
&= \chi\left(f^{-1}(m) \xrightarrow{f|_{f^{-1}(m)}} m \;,\; 0 \xrightarrow{(f|_{n \smallsetminus f^{-1}(m)})^{-1} \circ \alpha} k = n \smallsetminus f^{-1}(m) \right) \\
&= \left[\begin{tikzcd} & m+0 \arrow[d, "\mathrm{Id}_m + \left((f|_{n\smallsetminus f^{-1}(m)})^{-1} \circ \alpha\right)"'] \\ f^{-1}(m) + (n \smallsetminus f^{-1}(m)) \arrow[r, "f|_{f^{-1}(m)} + \mathrm{Id}_{n \smallsetminus f^{-1}(m)}"'] & m + (n \smallsetminus f^{-1}(m)) \end{tikzcd}\right].
\end{aligned}
$$

We denote by $[D']$ this last class of diagrams. Then, since f induces an isomorphism $n \smallsetminus f^{-1}(m) \cong k$, we can make the following commutative diagram:

$$
\begin{tikzcd}
 & & m+0 \arrow[dl, "\mathrm{Id}_m + ((f|_{n\smallsetminus f^{-1}(m)})^{-1} \circ \alpha)"'] \arrow[d, "\mathrm{Id}_m + \alpha"] \\
n = f^{-1}(m) + (n \smallsetminus f^{-1}(m)) \arrow[r, "f|_{f^{-1}(m)} + \mathrm{Id}_{n\smallsetminus f^{-1}(m)}"] \arrow[rr, bend right, "f = f|_{f^{-1}(m)} + f|_{n\smallsetminus f^{-1}(m)}"'] & m + (n \smallsetminus f^{-1}(m)) \arrow[r, "\mathrm{Id}_m + f|_{n\smallsetminus f^{-1}(m)}"'] & m+k,
\end{tikzcd}
$$

which shows that the class $[D']$ is the same as the initial class $[D]$, implying that $\chi \circ \Gamma = \mathrm{Id}$. $\square$

9.3 Polynomial Cospan($\mathbf{FI}_d$)-modules

In this section we study the polynomial functors from the category Cospan($\mathbf{FI}_d$) to the category **R-Mod**, called Cospan($\mathbf{FI}_d$)-modules, as we did for $\mathbf{FI}_d$-modules in the previous chapters. We start by defining the endofunctors τ_k, δ_k^x and κ_k^x on $\mathbf{Fct}$(Cospan($\mathbf{FI}_d$), **R-Mod**) as we did for $\mathbf{Fct}(\mathbf{FI}_d, \mathbf{R\text{-}Mod})$ in Section 2.6 and we define the polynomial functors on Cospan($\mathbf{FI}_d$) using them, as for polynomial $\mathbf{FI}_d$-modules in Section 5.1. One important difference is that all the endofunctors κ_k^x are zero for Cospan($\mathbf{FI}_d$)-modules, as we have shown in Proposition 9.3.4, which simplifies the study of these functors. For example, it implies that the subcategories of stably zero functors over Cospan($\mathbf{FI}_d$) are zero so there is only one notion of polynomial Cospan($\mathbf{FI}_d$)-modules.

Definition 9.3.1. For $k \in$ Cospan($\mathbf{FI}_d$), the *endofunctor* τ_k of $\mathbf{Fct}$(Cospan($\mathbf{FI}_d$) , **R-Mod**) is given on a functor F : Cospan($\mathbf{FI}_d$) $\to$ **R-Mod** by $\tau_k(F) = F(- + k)$ and on a natural transformation $\sigma : F \to F'$ by $\tau_k(\sigma) = \sigma_{(-+k)}$. For $x \in$ Cospan($\mathbf{FI}_d$)$(0,k)$, the *natural transformation* i_k^x : $\mathrm{Id} \to \tau_k$ is given on a functor F : Cospan($\mathbf{FI}_d$) $\to$ **R-Mod** by $i_k^x(F) = F\big(\ \mathrm{Id}_{(-)} + x\ \big) : F(-) \longrightarrow F(- + k)$.

Remark 9.3.2. The transformation i_k^x is natural: for any functor $F : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{R}\text{-}\mathbf{Mod}$ and any morphism $[D]$ in $\mathrm{Cospan}(\mathbf{FI}_d)(n,m)$, using the monoidal structure of $\mathrm{Cospan}(\mathbf{FI}_d)$ and Proposition 9.1.8 we have

$$\begin{aligned}
&\tau_k(F)([D]) \circ F(\mathrm{Id}_n + x)\\
&= F\left(\left[\begin{tikzcd} & m+k \arrow[d, "\mathrm{Id}_{m+k} + \alpha"] \\ n \arrow[r, "(\mathrm{Id}_m + x + \mathrm{Id}_l)\circ f"'] & m+k+l \end{tikzcd}\right]\right) = F\left(\left[\begin{tikzcd} & m+k \arrow[d, "\mathrm{Id}_{m+k} + \alpha"] \\ n \arrow[r, "(f+\mathrm{Id}_k)\circ(\mathrm{Id}_n + x)"'] & m+k+l \end{tikzcd}\right]\right)\\
&= F(\mathrm{Id}_m + x) \circ F([D]).
\end{aligned}$$

We then define the endofunctors δ_k^x and κ_k^x as we did in Definition 2.6.2 for $\mathbf{FI}_d$-modules. However, we immediately show that the endofunctors κ_k^x are all zero over $\mathrm{Cospan}(\mathbf{FI}_d)$.

Definition 9.3.3. For $k \in \mathrm{Cospan}(\mathbf{FI}_d)$ and $x \in \mathrm{Cospan}(\mathbf{FI}_d)(0,k)$, the *endofunctor* κ_k^x of $\mathbf{Fct}(\,\mathrm{Cospan}(\mathbf{FI}_d)\,,\,\mathbf{R}\text{-}\mathbf{Mod}\,)$ is the kernel of the natural transformation i_k^x, and δ_k^x is its cokernel.

Proposition 9.3.4. *For all $k \in \mathrm{Cospan}(\mathbf{FI}_d)$ and all $x \in \mathrm{Cospan}(\mathbf{FI}_d)(0,k)$, the endofunctor κ_k^x is zero.*

Proof. By Proposition 9.1.8 there exists a morphism $\tilde{x}$ in $\mathrm{Cospan}(\mathbf{FI}_d)(k,0)$ such that $\tilde{x} \circ x = \mathrm{Id}_0$. Then, for all functors $F : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{R}\text{-}\mathbf{Mod}$ we have the relation $F(\mathrm{Id}_{(-)} + \tilde{x}) \circ F(\mathrm{Id}_{(-)} + x) = F(\mathrm{Id}_{(-)} + (\tilde{x} \circ x)\,) = \mathrm{Id}_{F(-)}$. This implies that the morphism $i_k^x(F) = F(\mathrm{Id}_{(-)} + x)$ is a monomorphism, so its kernel $\kappa_k^x(F)$ is zero. Since this is natural it shows that κ_k^x is zero as an endofunctor. □

Remark 9.3.5. For $k \in \mathrm{Cospan}(\mathbf{FI}_d)$ and $x \in \mathrm{Cospan}(\mathbf{FI}_d)(0,k)$ there exists a short exact sequence $0 \longrightarrow \kappa_k^x \longrightarrow \mathrm{Id} \xrightarrow{i_k^x} \tau_k \longrightarrow \delta_k^x \longrightarrow 0$.

We now show that the basic properties of these endofunctors that we proved for $\mathbf{FI}_d$-modules remain true for $\mathrm{Cospan}(\mathbf{FI}_d)$-modules. Some of them even become empty since all the endofunctors κ_k^x are zero by Proposition 9.3.4.

Proposition 9.3.6. *For $k,l \in \mathrm{Cospan}(\mathbf{FI}_d)$ and $x \in \mathrm{Cospan}(\mathbf{FI}_d)(0,k)$, $y \in \mathrm{Cospan}(\mathbf{FI}_d)(0,l)$, we have*

0) The endofunctor δ_k^x is exact,

1) The endofunctors τ_k and τ_l commute up to a natural isomorphism. They also commute with limits and colimits.

2) The endofunctors δ_k^x and δ_l^y commute up to a natural isomorphism. They also commute with colimits.

3) The endofunctors τ_l and δ_k^x commute up to a natural isomorphism.

4) There is a natural short exact sequence $0 \longrightarrow \delta_l^y \longrightarrow \delta_{k+l}^{x+y} \longrightarrow \tau_l \circ \delta_k^x \longrightarrow 0$.

Proof. The proof is the same as the proof of Proposition 2.6.6. The only differences are that the exact sequence of points 0) and 4) becomes short since κ_k^x is zero, which implies that all endofunctors δ_k^x are exact and not just right exact. □

Remark 9.3.7. Since all the endofunctors κ_k^x of $\mathbf{Fct}(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ are zero by Proposition 9.3.4, the analogues for Cospan($\mathbf{FI}_d$) of the categories $\mathcal{SN}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ and $\mathcal{SN}_{c_{i_1},\dots,c_{i_m}}(\mathbf{FI}_d, \mathbf{R}\text{-}\mathbf{Mod})$ of Definitions 6.1.1 and 6.2.1 are reduced to zero. Then any corresponding quotient category by one of these subcategories is isomorphic to $\mathbf{Fct}(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ and thus the different notions of weak and strong polynomial functors that we could define for Cospan($\mathbf{FI}_d$)-modules coincide. Therefore, in the following section we present only one notion of polynomial functors over Cospan($\mathbf{FI}_d$).

Finally, we define the polynomial functors on Cospan($\mathbf{FI}_d$), as we did for functors on $\mathbf{FI}_d$ in Section 5.1, using the endofunctors δ_1^c for $c \in C$.

Definition 9.3.8. For $n \in \mathbb{N}$, the full subcategories of $\mathbf{Fct}(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ of *polynomial functors* of degree less than or equal to n, denoted by $\mathrm{Pol}_n(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$, are defined by induction. A Cospan($\mathbf{FI}_d$)-module F is in $\mathrm{Pol}_n(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ if, for all $c \in C$, we have $\delta_1^c(F) \in \mathrm{Pol}_{n-1}(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$, where δ_1^c is the endofunctor of Definition 9.3.3. By convention $\mathrm{Pol}_{-1}(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ is zero.

Remark 9.3.9. For $d = 1$, since $C = \{c\}$ we recover the definition of polynomial functors over the symmetric monoidal category Cospan($\mathbf{FI}$) with a null object of [DV19] using only one endofunctor $\delta_1 = \delta_1^c$.

We end this section with the first properties of the polynomial functor over Cospan($\mathbf{FI}_d$). In particular, we show that this definition using only the endofunctors δ_1^c for $c \in C$ a colour is equivalent to the similar definition using all endofunctors δ_k^x for $k \in \mathrm{Cospan}(\mathbf{FI}_d)$ and $x \in \mathrm{Cospan}(\mathbf{FI}_d(0,k))$.

Proposition 9.3.10. *For all $n \in \mathbb{N}$, the subcategory $\mathrm{Pol}_n(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ is thick, closed under colimits and stable by the endofunctors τ_k and δ_k^x, for $k \in \mathrm{Cospan}(\mathbf{FI}_d)$ and $x \in \mathrm{Cospan}(\mathbf{FI}_d)(0,k)$.*

Proof. The first assumption is proved by induction using Proposition 9.3.6 which implies that all endofunctors δ_1^c are exact. The second assumption is true since τ_k and δ_k^x commute with δ_1^c as endofunctors by Proposition 9.3.6 again. □

Lemma 9.3.11. *A functor $F : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{R}\text{-}\mathbf{Mod}$ is in $\mathrm{Pol}_n(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ if and only if the functor $\delta_k^x(F)$ is in $\mathrm{Pol}_{n-1}(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ for all $k \in \mathrm{Cospan}(\mathbf{FI}_d)$ and all $x \in \mathrm{Cospan}(\mathbf{FI}_d)(0,k)$.*

Proof. One implication is clear, the other is proved by induction using the short exact sequence $0 \longrightarrow \delta_l^y \longrightarrow \delta_{k+l}^{x+y} \longrightarrow \tau_l \circ \delta_k^x \longrightarrow 0$ from Proposition 9.3.6 and the fact that $\mathrm{Pol}_n(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ is thick and stable by the endofunctors τ_k by Proposition 9.3.10. □

9.4 Description of $\mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$

We show that the polynomial functors of degree 0 over Cospan($\mathbf{FI}_d$) are the constant functors, which gives an equivalence of categories between $\mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ and $\mathbf{R}\text{-}\mathbf{Mod}$. We start by adapting the Section 2.5 for Cospan($\mathbf{FI}_d$), which says that a functor on $\mathbf{FI}_d$ is determined by the image of the morphisms starting from 0, if these images are isomorphisms. We show that the same is true for functors over Cospan($\mathbf{FI}_d$) if we keep only the morphisms starting from 0 corresponding to c_1^k for $k \in \mathbb{N}$. We then use the subcategory $\overline{\mathrm{Cospan}(\mathbf{FI}_d)}$ of Cospan($\mathbf{FI}_d$) with only these morphisms to describe the polynomial Cospan($\mathbf{FI}_d$)-modules of degree 0.

Definition 9.4.1. The *subcategory* $\underline{\mathrm{Cospan}(\mathbf{FI}_d)}$ of $\mathrm{Cospan}(\mathbf{FI}_d)$ has the same objects as $\mathrm{Cospan}(\mathbf{FI}_d)$ and the morphisms in $\underline{\mathrm{Cospan}(\mathbf{FI}_d)}$ are the morphisms starting from 0 of the form c_1^k in $\mathrm{Cospan}(\mathbf{FI}_d)$, for $k \in \mathbf{FI}_d$, corresponding to $c_1^k \in \mathbf{FI}_d(0,k)$ via the bijection of Proposition 9.1.8.

We now show that the functors over $\mathrm{Cospan}(\mathbf{FI}_d)$ sending the morphisms c_1^k to isomorphisms correspond to the functors over $\underline{\mathrm{Cospan}(\mathbf{FI}_d)}$.

Lemma 9.4.2. *A functor* $F : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{R}\text{-}\mathbf{Mod}$ *induces a unique functor* $\underline{F} : \underline{\mathrm{Cospan}(\mathbf{FI}_d)} \to \mathbf{R}\text{-}\mathbf{Mod}$. *Moreover, if* $F(c_1^k)$ *is an isomorphism for all* $k \in \mathrm{Cospan}(\mathbf{FI}_d)$*, then for all* $x \in \mathrm{Cospan}(\mathbf{FI}_d)(0,k)$ *we have* $F(x) = F(c_1^k)$ *and, for* $u \in \mathrm{Cospan}(\mathbf{FI}_d)(n,m)$*, the morphism* $F(u)$ *is obtained from* $\underline{F}$ *by the formula:*

$$F(u) = F(u \circ c_1^n) \circ \left(F(c_1^n) \right)^{-1} = \underline{F}(c_1^m) \circ \left(\underline{F}(c_1^n) \right)^{-1}.$$

Proof. For $x \in \mathrm{Cospan}(\mathbf{FI}_d)(0,k)$, let $\tilde{x}$ be the morphism in $\mathrm{Cospan}(\mathbf{FI}_d)(k,0)$ corresponding to x by Proposition 9.1.8, then we have $\tilde{x} \circ x = \mathrm{Id}_0$. Since $\tilde{x} \circ c_1^k$ is in $\mathrm{Cospan}(\mathbf{FI}_d)(0,0)$ we also get by definition $\tilde{x} \circ c_1^k = \mathrm{Id}_0$. Applying F to these relations, since $F(c_1^k)$ is an isomorphism, we get $F(\tilde{x}) \circ F(c_1^k) = F(\tilde{x} \circ c_1^k) = F(\mathrm{Id}_0)$. This gives $F(\tilde{x}) = \left(F(c_1^k)\right)^{-1}$. We then get $\left(F(c_1^k)\right)^{-1} \circ F(x) = F(\tilde{x} \circ x) = \mathrm{Id}_{F(0)}$ and so $F(x) = F(c_1^k)$. The rest of the proof is similar to the proof of Lemma 2.5.3 using that $u \circ c_1^n$ is in $\mathrm{Cospan}(\mathbf{FI}_d)(0,m)$ and that $F(u \circ c_1^n) = F(c_1^m) = \underline{F}(c_1^m)$ because of the relation $F(x) = F(c_1^n)$. □

Corollary 9.4.3. *For* $\underline{F} : \underline{\mathrm{Cospan}(\mathbf{FI}_d)} \to \mathbf{R}\text{-}\mathbf{Mod}$*, if the image of all morphisms in* $\underline{\mathrm{Cospan}(\mathbf{FI}_d)}$ *by* $\underline{F}$ *is an isomorphism, then this functor can be extended to a unique functor* F *from* $\mathrm{Cospan}(\mathbf{FI}_d)$ *to* $\mathbf{R}\text{-}\mathbf{Mod}$.

Proof. By hypothesis, $\underline{F}(c_1^k)$ is an isomorphism for all $k \in \mathrm{Cospan}(\mathbf{FI}_d)$, so we can define a functor $F : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{R}\text{-}\mathbf{Mod}$ by the formula of Lemma 9.4.2. This lemma also proves that F is the unique extension of $\underline{F}$. □

Under the hypothesis that a $\mathrm{Cospan}(\mathbf{FI}_d)$-module sends a morphism c^k to an isomorphism, we show that the actions of the symmetric groups on this functor are trivial.

Proposition 9.4.4. *For* $F : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{R}\text{-}\mathbf{Mod}$*, if there exist* $k \in \mathrm{Cospan}(\mathbf{FI}_d)$ *and* $c \in C$ *such that* $F(c^k)$ *is an isomorphism, then for all permutations* $\sigma \in \mathrm{S}_k = \mathrm{Cospan}(\mathbf{FI}_d)(k,k)$*, the morphism* $F(\sigma)$ *is the identity.*

Proof. It is similar to the proof of Proposition 2.5.1 using the Lemma 9.4.2. □

We finally prove that the polynomial functors of degree 0 over $\mathrm{Cospan}(\mathbf{FI}_d)$ are the constant functors using the category $\underline{\mathrm{Cospan}(\mathbf{FI}_d)}$. We start by giving a concrete condition for functors over $\mathrm{Cospan}(\mathbf{FI}_d)$ to be polynomial of degree 0.

Lemma 9.4.5. *A functor* $F : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{R}\text{-}\mathbf{Mod}$ *is polynomial of degree 0 if and only if the natural transformation* $i_k^x(F) : F \to \tau_k(F)$ *is an equivalence for all* $k \in \mathrm{Cospan}(\mathbf{FI}_d)$ *and all* $x \in \mathrm{Cospan}(\mathbf{FI}_d)(0,k)$.

Proof. By Proposition 9.3.11 F is in $\mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ if and only if the cokernel $\delta_k^k(F)$ of each morphism $i_k^x(F)$ is zero, and by Proposition 9.3.4 its kernel $\kappa_k^x(F)$ is always zero. □

We now define, for each functor $F \in \mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$, an associated functor $H_F : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{R}\text{-}\mathbf{Mod}$ and we show that this functor H_F is constant and isomorphic to F. By Corollary 9.4.3 we can define H_F on the subcategory $\underline{\mathrm{Cospan}(\mathbf{FI}_d)}$ and extend it to a functor over $\mathrm{Cospan}(\mathbf{FI}_d)$ if the morphisms c_1^k are sent to isomorphisms. Moreover, if $F \in \mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ the morphism $i_k^{c_1^k}(F)(0) = F(c_1^k) : F(0) \to F(k) = \tau_k(F)(0)$ is an isomorphism by Lemma 9.4.5, which allows us to give the following definition of H_F:

Definition 9.4.6. For $F \in \mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$, the *functor* $\underline{H_F} : \underline{\mathrm{Cospan}(\mathbf{FI}_d)} \to \mathbf{R}\text{-}\mathbf{Mod}$ is given on objects by $\underline{H_F}(n) = F(0)$, for $n \in \mathrm{Cospan}(\mathbf{FI}_d)$, and on morphisms by $H_F(c_1^k) = \left(F(c_1^k) \right)^{-1} \circ F(c_1^k) = \mathrm{Id}_{F(0)}$ for $k \in \mathrm{Cospan}(\mathbf{FI}_d)$.

Remark 9.4.7. We then consider the *functor* $H_F : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{R}\text{-}\mathbf{Mod}$, which is the one extending $\underline{H_F}$ by Corollary 9.4.3. This definition is similar to Definition 7.4.4 for functors on $\mathbf{FI}_d$ but, for $\mathrm{Cospan}(\mathbf{FI}_d)$, H_F is equal to the constant functor $F(0)^{\mathrm{cst}} : \mathrm{Cospan}(\mathbf{FI}_d) \to \mathbf{R}\text{-}\mathbf{Mod}$.

Lemma 9.4.8. *For $F \in \mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$, there is a natural equivalence $\varepsilon : F(0)^{cst} = H_F \cong F$ given by $\varepsilon_n = F(c_1^n) : H_F(n) = F(0) \to F(n)$.*

Proof. The equality $F(0)^{\mathrm{cst}} = H_F$ follows from Corollary 9.4.3 and from the definition of $\underline{H_F}$. For $u \in \mathrm{Cospan}(\mathbf{FI}_d)(n, m)$, as F is in $\mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$, the morphism $i_k^{c_1^k}(F) = F(\mathrm{Id}_{(-)} + c_1^k)$ is an isomorphism for all $k \in \mathrm{Cospan}(\mathbf{FI}_d)$ by Proposition 9.4.5. Then by Lemma 9.4.2 we have $F(u \circ c_1^n) = F(c_1^m)$ since c_1^m and $u \circ c_1^n$ are in $\mathrm{Cospan}(\mathbf{FI}_d)(0, m)$. This implies the relations

$$\varepsilon_m \circ H_F(u) = \varepsilon_m = F(c_1^m) = F(u \circ c_1^n) = F(u) \circ F(c_1^n) = F(u) \circ \varepsilon_n,$$

which shows that ε is natural. It is an isomorphism by the Lemma 9.4.5 since $F(c_1^k) = i_k^{c_1^k}(F)(0)$ is an isomorphism. □

Finally, we can prove that the polynomial functors of degree 0 over $\mathrm{Cospan}(\mathbf{FI}_d)$ are the constant functors using the Lemma 9.4.8.

Theorem 9.4.9. *There is an equivalence of categories*

$$\mathrm{Pol}_0(\ \mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod}\ ) \cong \mathbf{R}\text{-}\mathbf{Mod}$$

given by the functor $c : \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{Fct}(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ which sends a module $M \in \mathbf{R}\text{-}\mathbf{Mod}$ to the constant functor M^{cst} and a morphism $f \in \mathbf{R}\text{-}\mathbf{Mod}(M, N)$ to the constant natural transformation equal to f.

Proof. By Lemma 9.4.8, if F is in $\mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ then F is equivalent to the constant functor $F(0)^{\mathrm{cst}}$. Conversely, for all $M \in \mathbf{R}\text{-}\mathbf{Mod}$, the constant functor M^{cst} is in $\mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$ since $i_k^x(M^{\mathrm{cst}}) = \mathrm{Id}_{M^{\mathrm{cst}}}$ is a natural equivalence. This proves that the essential image of the functor c is $\mathrm{Pol}_0(\mathrm{Cospan}(\mathbf{FI}_d), \mathbf{R}\text{-}\mathbf{Mod})$. This functor is by definition faithful, and it is full since a natural transformation between two constant functors M^{cst} and N^{cst} is obtained by a morphism in $\mathbf{R}\text{-}\mathbf{Mod}(M, N)$. □

Appendix A: The pointwise tensor product on $\mathcal{M}\mathrm{on}_{\mathrm{ini}}$

This appendix is a complement of [DV19]. In this context $\mathcal{M}$ denotes a small symmetric monoidal category whose unit is an initial object and $\mathcal{M}\mathrm{on}_{\mathrm{ini}}$ denotes the category of these small categories. We prove that the pointwise tensor product of two strong polynomial functors over $\mathcal{M}$ is strong polynomial. Djament and Vespa defined a notion of strong polynomial functors over these categories which is similar to the definition over $\mathbf{FI}_d$. We then use the Proposition 3.12 from [DV19], which corresponds to Proposition 5.4.18 but for $\mathcal{M} \in \mathcal{M}\mathrm{on}_{\mathrm{ini}}$, to prove the following result analogous to Theorem 5.5.4:

Theorem. *For $n, m \in \mathbb{N}$ and $F, G : \mathcal{M} \to \mathbf{R}\text{-}\mathbf{Mod}$, if F is in $\mathrm{Pol}_n^{strong}(\mathcal{M}, \mathbf{R}\text{-}\mathbf{Mod})$ and if G is in $\mathrm{Pol}_m^{strong}(\mathcal{M}, \mathbf{R}\text{-}\mathbf{Mod})$, then their tensor product $F \otimes G : \mathcal{M} \to \mathbf{R}\text{-}\mathbf{Mod}$ is in $\mathrm{Pol}_{2\max(n,m)}^{strong}(\mathcal{M}, \mathbf{R}\text{-}\mathbf{Mod})$.*

Proof. We consider the functor (F, G) in $\mathbf{Fct}(\mathcal{M}, \mathbf{R}\text{-}\mathbf{Mod} \times \mathbf{R}\text{-}\mathbf{Mod})$. We start by proving by induction on $k \in \mathbb{N}$ that $\delta_1^{\circ k}(F, G) = (\,\delta_1^{\circ k}(F), \delta_1^{\circ k}(G)\,)$, where $\delta_1^{\circ k}$ is the composition of δ_1 by itself k times. For $k = 0$ it is by definition, and by induction we have $\delta_1^{\circ k+1}(F, G) = \delta_1(\,\delta_1^{\circ k}(F, G)\,) = \delta_1(\,\delta_1^{\circ k}(F), \delta_1^{\circ k}(G)\,)$. Then, for $a \in \mathcal{M}$, $\delta_1(\,\delta_1^{\circ k}(F), \delta_1^{\circ k}(G)\,)(a)$ is the cokernel of the map

$$\begin{array}{ccc} \big(\,\delta_1^{\circ k}(F), \delta_1^{\circ k}(G)\,\big)(a) & \to & \tau_1\big(\,\delta_1^{\circ k}(F), \delta_1^{\circ k}(G)\,\big)(a) \\ \big(\,\delta_1^{\circ k}(F)(a), \delta_1^{\circ k}(G)(a)\,\big) & \mapsto & \big(\,\delta_1^{\circ k}(F)(a+1), \delta_1^{\circ k}(G)(a+1)\,\big). \end{array}$$

This shows the identity $\delta_1^{\circ k+1}(F, G) = (\,\delta_1^{\circ k+1}(F), \delta_1^{\circ k+1}(G))$ on the objects $a \in \mathcal{M}$ and we can check that it is natural. Then, by hypothesis $F \in \mathrm{Pol}_n^{\mathrm{strong}}(\mathcal{M}, \mathbf{R}\text{-}\mathbf{Mod})$ so $\delta_1^{\circ n+1}(F) = 0$, which implies that $\delta_1^{\circ \max(n,m)+1}(F) = 0$ and we have the same for G. This gives

$$\delta_1^{\circ \max(n,m)+1}(F, G) = \big(\,\delta_1^{\circ \max(n,m)+1}(F), \delta_1^{\circ \max(n,m)+1}(G)\,\big) = (0, 0),$$

showing that (F, G) is polynomial of degree less than or equal to $\max(n, m)$. We conclude by applying the Proposition 3.12 from [DV19] to the composition

$$\mathcal{M} \xrightarrow{(F,G)} \mathbf{R}\text{-}\mathbf{Mod} \times \mathbf{R}\text{-}\mathbf{Mod} \xrightarrow{-\otimes-} \mathbf{R}\text{-}\mathbf{Mod}\ .$$

Indeed, we showed in Lemma 5.5.3 that $-\otimes-$ is polynomial of degree 2, and we proved that (F, G) is polynomial of degree less than or equal to $\max(n, m)$. The functor $-\otimes-$ preserves epimorphisms since an epimorphism in $\mathbf{R}\text{-}\mathbf{Mod} \times \mathbf{R}\text{-}\mathbf{Mod}$ is a couple (f, g) of epimorphism in $\mathbf{R}\text{-}\mathbf{Mod}$ and then $f \otimes g$ is also an epimorphism. $\square$

Remark 9.4.10. In this theorem the bound may not be the best possible. Indeed, we could expect for $F \otimes G : \mathcal{M} \to \mathbf{R}\text{-}\mathbf{Mod}$ to be strong polynomial of degree less than or equal to $n + m$. For example, for $\mathcal{M} = \mathbf{FI}$, the Proposition 4.1 from [Dja16] shows that a $\mathbf{FI}$-module is strong polynomial of degree less than or equal to n if and only if it is a quotient of a sum of the standard projective functors $P_i^{\mathbf{FI}}$ for $i \leq n$. This allows us to prove that, over **FI** the tensor product $F \otimes G$ is polynomial of degree $n+m$ if F has degree n and G has degree m. Similarly, for functors over a symmetric monoidal category where the unit is a null object, this theorem is true with the optimal bound since we have $\deg(F \otimes G) \leq \deg(F) + \deg(G)$. The proof of this result is given in [Ves19, Proposition 2.9] with a direct use of the cross effects. One could try to prove a more refined version of the proposition 3.12 from [DV19] and use this refinement in the proof to get a better bound.

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Les foncteurs sur la catégorie $\mathbf{FI}$ des injections entre ensembles finis apparaissent naturellement dans différents contextes. Ils interviennent notamment dans l'étude de la stabilité de familles de représentations des groupes symétriques initiée par Church, Ellenberg et Farb. Djament et Vespa ont montré que la stabilité des représentations s'exprime en termes de polynomialité forte de foncteurs sur $\mathbf{FI}$. Ils introduisent également une notion de polynomialité faible mieux adaptée aux phénomènes stables. Ces propriétés polynomiales sont des moyens de mesurer la complexité d'un foncteur et peuvent être considérés comme un analogue des fonctions polynomiales. Il existe des généralisations de la catégorie $\mathbf{FI}$, notées $\mathbf{FI}_d$, où l'on rajoute un choix de couleurs parmi d possibles sur le complémentaire de l'image des injections. Les foncteurs sur ces catégories interviennent notamment dans les travaux de Sam et Snowden sur les modules sur les algèbres commutatives tordues libres et dans ceux de Ramos sur l'homologie des espaces de configuration de graphes. Les catégories $\mathbf{FI}_d$ n'ayant pas d'objet initial pour d > 1, elles sortent du cadre considéré par Djament et Vespa.

Dans cette thèse on introduit différentes notions (forte et faibles) de foncteurs polynomiaux sur les catégories $\mathbf{FI}_d$ et on étudie leur comportement. On adapte aussi la définition classique de foncteurs polynomiaux (basée sur les effets croisés) au cadre de $\mathbf{FI}_d$, et on montre que les deux définitions obtenues coïncident. Les foncteurs polynomiaux sur $\mathbf{FI}_d$ s'avèrent plus difficiles à étudier que sur $\mathbf{FI}$. Par exemple, les projectifs standards sont fortement polynomiaux sur $\mathbf{FI}$ et on montre que ce n'est plus le cas sur $\mathbf{FI}_d$ pour d > 1. On étudie alors différents quotients polynomiaux de ces foncteurs. On amorce aussi l'étude de la polynomialité des foncteurs considérés par Ramos en calculant explicitement les foncteurs associés aux graphes linéaires. Cependant, la notion forte de foncteurs polynomiaux manque de propriétés essentielles concernant les phénomènes stables. On introduit alors les foncteurs faiblement polynomiaux en considérant le quotient par une sous-catégorie afin de supprimer les foncteurs problématiques. Alors que les foncteurs faiblement polynomiaux de degré 0 sur $\mathbf{FI}$ sont les foncteurs constants, on donne une description de ceux sur $\mathbf{FI}_d$ qui forment une catégorie plus complexe. On en déduit que l'adaptation directe des méthodes utilisées par Djament et Vespa pour $\mathbf{FI}$ ne fonctionne pas.

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE
UMR 7501
Université de Strasbourg et CNRS
7 Rue René Descartes
67 084 STRASBOURG CEDEX

Tél. 03 68 85 01 29
Fax 03 68 85 03 28

https://irma.math.unistra.fr
irma@math.unistra.fr

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Université de Strasbourg

Antoine FELTZ

Foncteurs polynomiaux sur les catégories FId

École doctorale Mathématiques, sciences de l'information et de l'ingénieur | ED 269 Université de Strasbourg

Résumé

Dans cette thèse on introduit différentes notions (forte et faibles) de foncteurs polynomiaux sur les catégories FId et on étudie leur comportement. On adapte aussi la définition classique de foncteurs polynomiaux (basée sur les effets croisés) au cadre de FId, et on montre que les deux définitions obtenues coïncident. Les foncteurs polynomiaux sur FId s'avèrent plus dificiles à étudier que sur FI. Par exemple, les projectifs standards sont fortement polynomiaux sur FI et on montre que ce n'est plus le cas sur FId pour d > 1. On étudie alors diférents quotients polynomiaux de ces foncteurs. On amorce également l'étude de la polynomialité des foncteurs considérés par Ramos en calculant explicitement les foncteurs associés aux graphes linéaires. Cependant, la notion forte de foncteurs polynomiaux manque de propriétés essentielles concernant les phénomènes stables. On introduit alors les foncteurs faiblement polynomiaux en considérant le quotient par une sous-catégorie afin de supprimer les foncteurs problématiques. Alors que les foncteurs faiblement polynomiaux de degré 0 sur FI sont les foncteurs constants, on donne une description de ceux sur FId qui forment une catégorie plus complexe. On en déduit que l'adaptation directe des méthodes utilisées par Djament et Vespa pour FI ne fonctionne pas.

Mots clés : Foncteurs polynomiaux, catégorie FI des ensembles finis et injections, catégories FId

Résumé en anglais

In this thesis we introduce different notions (strong and weak) of polynomial functors over the categories FId and we study their behaviour. We also adapt the classical definition of polynomial functors (based on cross effects) to the framework of FId, and we show that the two definitions obtained coincide. The polynomial functors over FId turn out to be harder to study than over FI. For example, the standard projectives are strong polynomial over FI and we show that this is no longer the case over FId for d > 1. We then study different polynomial quotients of these functors. We also initiate the study of the polynomiality of the functors considered by Ramos by explicitly calculating the functors associated with linear graphs. However, the strong notion of polynomial functors lacks essential properties concerning stable phenomena. We then introduce the weak polynomial functors by considering the quotient by a subcategory in order to eliminate the problematic functors. While the weak polynomial functors of degree 0 over FI are the constant functors, we give a description of those over FId which form a more complex category. We deduce that a direct adaptation of the methods used by Djament and Vespa for FI does not work.

Keywords : Polynomial functors, category FI of finite sets and injections, categories FId