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Combinatorics, Homotopy, and Embedding of operads
Combinatoire, homotopie et plongements d'opérades

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Contents

Abstract

Algebraic operads are a powerful algebraic tool that can be used to encode some variety of algebras like Lie algebras or the more classical associative algebras. Furthermore, it turns out that algebraic operads are themselves algebraic structures that behave like algebras. Indeed, a well known result of operad theory is that operads are monoids in the category of species relatively to the plethysm. The theory of species is a very fruitful tool of enumerative combinatorics generalizing the notion of generating function, and the plethysm is the analog of the composition in this context. The same way generating functions were a breakthrough in enumerative combinatorics, allowing one to work with a single object, the generating function, instead of having to keep track of several objects, species allow one to work with a single object, however rather than a generating function, it is a functor, thus losing way less information. The fact that algebraic operads are algebras in the category of species opens two paths to explore this theory of algebraic operads. Either a full combinatorial point of view where every object has some very explicit combinatorial description, fully utilizing the underlying category of species. Or one can also choose an algebraic point of view, generalizing homological algebra and homotopical algebra to the operadic context. Of course, both approaches synergies very well together. The combinatorial point of view gives explicit constructions and descriptions of the objects, and tools to work with those combinatorial descriptions. In the other hand, algebraic point of view is full of application in other areas of mathematics, like algebraic topology, rational homotopy, etc... This thesis will focus on interaction between those two points of view, using a rather unusual approach. Indeed, we will use homotopical tools, the operadic Koszul duality to derive combinatorial information on the operads we are studying. We then use those to get combinatorial descriptions allowing use to carry explicit computations. This thesis is divided in three parts. The first part is an introduction to the theory of species. Then we give an introduction to the theory of algebraic operads, and operadic Koszul duality. Finally, we compute some combinatorial descriptions of operads, and apply those to prove a conjecture of Dotsenko on an embedding of the operad encoding the algebraic structure on vector field of weak Frobenius manifolds.

Introduction

In an attempt of the author to write an interesting story about combinatorics and algebra, with the theory of algebraic operads as the main character, this manuscript is divided in three chapters. The first two chapters are mainly introductory with a lot of recollections respectively on the theory of combinatorial species, and on the theory of algebraic operads. The third chapter is the main part of this manuscript, as it is the original work of the author. One of the main goal of this manuscript was to be as self-contained as possible, however the appearance of a lot of category theory and homological algebra made this goal almost impossible to achieve, at least in a reasonable number of pages and amount of time. Background on category theory is assumed to be known by the reader, we refer to the classics: the book of MacLane [63], and the book of Lawvere and Schanuel [54]. We would also like to refer to the amazing book of Riehl [74], and the reader tired of reading can also refer to the video of *The Catsters* [15]. Some elements of homological algebra with be used, however contrary to category theory that will lead our intuition through the chapter, homological algebra, mostly spectral sequences, will be used as a tool to prove some results. We do hope that the appearance of spectral sequences will not scare the reader. Indeed, spectral sequences may seem very cryptic at a first glance, however they are very powerful tools that can be used to prove a lot of results (plus bonus point: they admit a categorical interpretation). We refer to the book of Weibel [79] for an introduction to homological algebra, including spectral sequences. Enough about what is not in this manuscript, let us now present what is in it.

First Chapter: Combinatorial Species The first chapter is all about combinatorics and categories. It is an introduction to the theory of combinatorial species through category theory. Combinatorial species were introduced by Joyal in 1981 [47], and proved to be a very powerful tool in enumerative combinatorics mostly by its ability to translate functional equation to recurrence relation, and conversely. The theory of species can be understood as a "categorification" of the theory of generating functions, meaning that we replace the generating *function* by a "generating *functor*", thus losing way less information. This point of view of categorification will be guiding us through the first chapter.

We will start by exposing a naive categorification of generating functions in the first section. Its first subsection is a recollection on formal power series. To avoid radius of convergence issues, we allow infinite coefficients using the convention that $0 \cdot \infty = 0$, thus giving the following definition for an ordinary generating function, denoted ogf for short:

Definition. An *ogf* is a map $a : \mathbb{N} \to \overline{\mathbb{N}}$, the formal power is denoted $\sum_{n\in\mathbb{N}} a(n)x^n$.

We then define the usual operations on ogf, namely the sum denoted $+$, the product (Cauchy product) denoted $\cdot$, the Hadamard product denoted $\odot$, the composition denoted $\circ$, and the derivative denoted $\,'$. In the next subsection we introduce naive species, that are not yet the species of Joyal, but are a first step towards them.

Definition. A *naive species* is a functor $\mathcal{S} : \mathbb{N} \to \mathrm{Set}$.

We then adapt the usual operations on ogf to the naive species, defining their sum denoted $+$, product (Cauchy product) denoted $\cdot$, Hadamard product denoted $\odot$, plethysm (which corresponds to the composition) denoted $\circ$, and derivative denoted $\,'$. We then end the first section by adapting our definitions to the multi-variate case, and showing an analog of the implicit function theorem on naive species. Here, 2-sort naive species should be understood as "naive species in two variables", meaning that a 2-sort naive species H is a functor $\mathbb{N}^2 \to \mathrm{Set}$.

Theorem. *Let $H[X,Y]$ be a 2-sort naive species in two variables X and Y such that:*

$$H(0,0) = \emptyset \qquad \textit{and} \qquad \frac{\partial H}{\partial Y}(0,0) = \emptyset$$

Then there exists a unique naive species $\mathcal{A}$ such that:

$$\mathcal{A}[X] = H[X, A[X]] \qquad \textit{and} \qquad \mathcal{A}(0) = \emptyset$$

The second section is about the actual theory of species. We start by addressing the issue of the theory of naive species, and we define the combinatorial species in Subsection 1.2.1.

Definition. A *species* is a functor $\mathcal{S} : \mathbb{B} \to \mathrm{Set}$. Where $\mathbb{B}$ is the category of finite sets with the bijections as morphisms.

We then generalize the operations we have defined of naive species to species in the next subsection. Subsection 1.2.3 relate species to actions of the permutation groups $\mathfrak{S}_n$. We define the Schur functor of a species.

Definition. Let $\mathcal{S}$ be a species. The *Schur functor* associated to $\mathcal{S}$ is the functor $\mathcal{F}_{\mathcal{S}} : \mathrm{Set} \to \mathrm{Set}$ such that for any set X, $\mathcal{F}_{\mathcal{S}}(X) = \uplus_{n\in\mathbb{N}} \mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} X^n$.

And we define the Taylor-Joyal expansion of any functor Set $\to$ Set that associates to it a species.

Proposition. *Let us denote by $\widehat{\ }$ the Taylor-Joyal expansion. Let $\mathcal{S}$ be a species. Then we have:*

$$\widehat{\mathcal{F}_{\mathcal{S}}} = \mathcal{S}$$

We then present an original (currently unpublished) work of the author in collaboration with Agugliaro in Section 1.2.4. The main idea is to apply the formalism of the previous subsection to functors $\mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ where $\mathrm{Set}^{\mathrm{op}}$ is the opposite category of Set and to define the Joyal transform of such a functor.

Definition. Let $\mathcal{S}$ be a species. The *categorical L function* associated to $\mathcal{S}$ is the contravariant functor $\mathcal{L}_{\mathcal{S}} : \mathrm{Set} \to \mathrm{Set}$ such that for any set X, $\mathcal{L}_{\mathcal{S}}(X) = \uplus_{n\in\mathbb{N}^*} \mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} \mathrm{Hom}(X, \underline{n})$.

Then, we notice that the diagram of the product is the opposite of the diagram of the sum, allowing us to define "partitions" relatively to the product instead of the sum:

Definition. Let A be a set, and let P and Q be two quotients of A, then we denote $P \veebar Q = A$ when $A \to P \times Q$ is a bijection. We say that $P \veebar Q$ is a *direct product* of A.
Let A be a set and $k \in \mathbb{N}$, a *quotientation of A of length k* is a set $P = \{P_1, \dots, P_k\}$ such that $A = P_1 \veebar \dots \veebar P_k$. We denote by $P \not\ltimes^k A$ the fact that P is a quotientation of A of length k.

These definitions allow us to define the Dirichlet convolution denoted $*$ and the arithmetic plethysm denoted $\square$ of two species. A straightforward generalization of the Taylor-Joyal transform allows us to define the Joyal transform of a functor $\mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$. Moreover, it allows us to associate a Dirichlet series to a such functor, and with well-chosen functors, we can recover usual zeta functions, namely the Riemann zeta function, the Hasse-Weil zeta functions, and the Artin-Mazur zeta functions. We then end the section by a subsection explaining how to generalize the implicit function theorem to species:

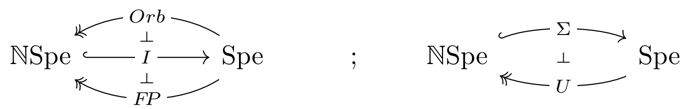
Theorem. *Let $H[X,Y]$ be a 2-sort species in two variables X and Y such that:*

$$H(0,0) = \emptyset \qquad \text{and} \qquad \frac{\partial H}{\partial Y}(0,0) = \emptyset$$

Then there exists a unique species $\mathcal{A}$ such that:

$$\mathcal{A}[X] = H[X, A[X]] \qquad \text{and} \qquad \mathcal{A}(0) = \emptyset$$

The third section takes full advantage of the categorical point of view of the theory of species. In Subsection 1.3.1, we show two very natural ways to embed the category of naive species into the category of species, defining the shuffle species and the ordered species, using the following adjunctions:



Where FP is the fix point functor, I is the inclusion of naive species into species, Orb is the orbit functor, Σ is the symmetrization functor, and U is the forgetful functor. In Subsection 1.3.2, we show how to define species in other categories, defining linear species (that we should not confuse with ordered species because of the inconstancy of the terminology in literature), topological species, and in fact species in any category behaving well-enough.

The last section of the first chapter is about the application of the theory of species to certain tree-like structures, namely the rooted trees, the rooted Greg trees, and the hyperforests (non-empty sets of rooted hypertrees). We define rooted trees and rooted Greg trees in Subsection 1.4.1, and the according species.

Definition. A *rooted tree* is a finite connected graph without cycle with a distinguished vertex called the root.
A *rooted Greg tree* is a rooted tree where the vertices are either black or white such that:

- black vertices are undistinguished, and
- each black vertex has at least two children.

We then adapt those definitions to hypertrees in Subsection 1.4.2.

Definition. A *rooted hypertree* is a finite connected hypergraph without cycle with a distinguished vertex called the root.

And we introduce three new combinatorial objects: First we introduce the tree shapes.

Definition. Let P be a partition of length k of A, a *tree shape on P* is a rooted hypertree structure with k black vertices on A such that:

- the root is black,
- simple edges are only between a black vertex and a white vertex such that the black vertex is bellow (closer to the root),
- hyperedges are only between a white vertex and several black vertices such that the white vertex is bellow,
- for each $p \in P$, we have a black vertex such that p is the set of the white vertices connected to it via simple edges.

We construct the *tree shape of a rooted hypertree* and show that it encodes the data we need to be able to reconstruct a hypertree from its maximal subtrees. Using this insight, we introduce the rooted Greg hypertrees and the reduced rooted Greg hypertrees generalizing both Greg trees and hypertrees in two different ways.

Definition. A *rooted Greg hypertree* is a rooted hypertree where the vertices are either black or white such that:

- black vertices are undistinguished, and
- each black vertex has at least two children.

A *reduced rooted Greg hypertree* is a rooted hypertree where the vertices are either black or white such that:

- black vertices are undistinguished,
- each black vertex has at least two children, and
- each black vertices do not admit any incoming hyperedge (edges connecting strictly more that two vertices).

Second Chapter: Operads The second chapter is about algebraic operads. We start by considering set species (the combinatorial species we have defined in the first chapter), but we very quickly switch to linear species to be able to do actual algebra.

The first section is dedicated to the tree monads. We start by recalling basic definition about monoid and monads. Then each subsection is dedicated to a different tree monad. We start with the symmetric tree monad for species denoted $\mathcal{T}$, then the non-symmetric tree monad for ordered species denoted$\mathcal{T}^{\mathrm{ns}}$, and we end with the shuffle tree monad for shuffle species denoted$\mathcal{T}^{\text{Ш}}$.

In the second section, we define algebraic operads and the main tools we are going to use. Subsection 2.2.1 is dedicated to the definition of the algebraic operads. More precisely, we give three different definitions, and show that they are equivalent.

Definition.

- A *symmetric algebraic operad*, or *operad* for short, is an algebra over the tree monad $\mathcal{T}$.
- Equivalently, an *operad* is a monoid in the category of species according to the plethysm.
- Equivalently, an *operad* is a species $\mathcal{P} \in$ Vect–Spe together with a collection of maps $\circ_i$ and an element $e_i \in \mathcal{P}(\{i\})$ such that $\circ_i : \mathcal{P}(A \sqcup \{i\}) \otimes \mathcal{P}(B) \to \mathcal{P}(A \sqcup B)$ satisfying the sequential composition and parallel composition axioms, and $e_i \in \mathcal{P}(\{i\})$ satisfying the unit axiom.

We take advantage of this subsection to state the point of view of the author on algebraic operads, namely that they are some kind of *algebras* over very combinatorial objects, the linear species. In the second subsection, we show how to give a presentation of an operad by generators and relations. We then give example of operads defined by generators and relations in Subsection 2.2.3. We define the *Three Graces* as they were named by Loday, the operads Ass, Com and Lie. We also define the so called operadic butterfly, and (the nicest of all) the operad PreLie. As we do not yet have the tools to study them, we continue by addressing the issue of canonical representative in operads defined by generators and relations in Subsection 2.2.4. Thus we define operadic rewriting system (ORS) which are slight generalizations of Gröbner bases or PBW bases since they only require a partial order on the set of monomials.

Definition. An *operadic rewriting system* latter denoted *ORS* is a triple $(\mathcal{S}, \mathcal{X}, R)$ where $\mathcal{S}$ is a linear species, $\mathcal{X}$ is a shuffle set species such that $U(\mathcal{S}) = \mathrm{Span}(\mathcal{X})$, and $R = (R_n)_{n\in\mathbb{N}}$ such that R_n is a subset of $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}(n) \times \mathcal{T}(\mathcal{S})(\underline{n})$. We denote by R' the set of *rewritable monomials* which are the monomials admitting at least one rewriting rule in $\mathcal{R}$.

We define the usual monomials order on operads, namely the permutation order, the graded path lexicographic order, and the quantum order. In the next subsection, we define modules on operads, and state the following theorems allowing us to show freeness properties using convergent ORS. Let $(\mathcal{S}, \mathcal{X}, U)$ and $(\mathcal{S} \oplus \mathcal{R}, \mathcal{X} + \mathcal{Y}, U \sqcup V)$ be two convergent ORS admitting associated operads, and let $\mathcal{P}$ and $\mathcal{Q}$ the associated operads. Then we have the following theorem.

Theorem (left freeness version)**.** *[25, Theorem 4] Assume that the root of the rewritable monomial of V are elements of $\mathcal{Y}$. Then $\mathcal{Q}$ is free as left $\mathcal{P}$-module.*

Theorem (right freeness version)**.** *[25, Theorem 4] Assume that the vertices such that each child is a leaf, of the rewritable monomial of V are elements of $\mathcal{Y}$. Then $\mathcal{Q}$ is free as right $\mathcal{P}$-module.*

Theorem. *[31, Theorem 4.1] Let $\mathcal{P}$ be an operad generated by $\mathcal{S}$, and $\mathcal{X}$ a basis of $\mathcal{S}$ satisfying the following conditions:*

- *$\mathcal{P}$ admits a convergent ORS $(\mathcal{S}, \mathcal{X}, U)$ decreasing along the reverse graded path lexicographic ordering such that for each rewritable monomial, the smallest leaf is directly connected to the root.*
- *$\mathcal{P}$ admits a convergent ORS $(\mathcal{S}, \mathcal{X}, V)$ such that each rewritable monomial is a left comb with the smallest leaf and the second-smallest leaf directly connected to the same vertex.*

Then $\mathcal{P}$ has the Nielsen-Schreier property.

The third section is about differential graded operads. In the first subsection, we explain the Koszul sign rule and give example of applications and computations using this rule. In the next subsection, we define the Bar and cobar construction on differential graded operads. Those are key ingredients of operadic homological algebra. We unfortunately do not have the time to go further in this deep and interesting theory. The last subsection is once again an unfortunately short introduction to the operadic twisting. However, the full computation of the operadic twisting of PreLie is given is the last chapter, namely in Subsection 3.1.2.

The last section of the second chapter is an introduction to the operadic Koszul duality. We define the Koszul dual and the Koszul complex of a quadratic operad in the first subsection. In the next subsection, we define the Koszul property, and we show a well known theorem of the operadic Koszul theory for which the author did not find a reference in the literature, namely that an operad admitting a quadratic convergent ORS is Koszul.

Theorem. *The Koszul complex of an operad admitting a quadratic convergent ORS is acyclic.*

In Subsection 2.4.3, we use the tools of operadic Koszul theory to study the operadic butterfly, and answer negatively to a conjecture of Loday. In the penultimate subsection, we quickly discuss the generating series of Koszul operads, and state a conjecture on the generating series of Koszul operads with one generator of arity two.

Conjecture. *Let $\mathcal{P}$ be a Koszul symmetric operad generated by one operation of arity two, then the generating series of $\mathcal{P}$ is differential algebraic of order 1 over $\mathbb{Z}[x]$. Equivalently, $f_{\mathcal{P}}$ and $f'_{\mathcal{P}}$ are algebraically dependent over $\mathbb{Z}[x]$.*

Finally, we end the chapter by a (yet not published) work of the author classifying Koszul set operads with one generator of arity two, proving the conjecture of the previous subsection in the case of set operads.

Theorem. *Let $\mathcal{P}$ be a Koszul set operad over one generator of arity two, then $\mathcal{P}$ is isomorphic to one of the 11 following operads:*

- Mag *the magmatic operad and $f_{\mathcal{P}}(x) = \frac{1}{2}(1 - \sqrt{1-4x})$;*

- NAP *the non-associative permutative operad and* $f_{\mathcal{P}}$ *is the Euler's tree function defined by* $f_{\mathcal{P}}(x) = \sum_{n\in\mathbb{N}^*} \frac{n^{n-1}}{n!} x^n$*;*
- $\mathrm{CMag} \circ \mathrm{ANil}_2$ *which is build from* CMag *and* ANil_2 *with the relation* $[a.b, c] = 0$*, and* $f_{\mathcal{P}}(x) = 1 - \sqrt{1 - 2x - x^2}$*;*
- $\mathrm{ANil}_2 \circ \mathrm{CMag}$ *which is build from* CMag *and* ANil_2 *with the relation* $[a, b].c = 0$*, and* $f_{\mathcal{P}}(x) = 2 - x - 2\sqrt{1 - 2x}$*;*
- CMag#AMag *which is the connected sum of* CMag *and* AMag*, and* $f_{\mathcal{P}}(x) = 2 - x - 2\sqrt{1 - 2x}$*;*
- Ass *the associative operad and* $f_{\mathcal{P}}(x) = \frac{x}{1-x}$*;*
- $\mathrm{CMag\#ANil}_2$ *which is the connected sum of* CMag *and* ANil_2*, and* $f_{\mathcal{P}}(x) = 1 - \sqrt{1 - 2x} + \frac{1}{2}x^2$*;*
- Perm *the permutative operad and* $f_{\mathcal{P}}(x) = x \exp(x)$*;*
- $\mathrm{LieAdm}^!$ *the Koszul dual of the Lie admissible operad and* $f_{\mathcal{P}}(x) = \exp(x) - 1 + \frac{x^2}{2}$*;*
- CMag *the commutative magmatic operad and* $f_{\mathcal{P}}(x) = (1 - \sqrt{1 - 2x})$*;*
- Com *the commutative operad and* $f_{\mathcal{P}}(x) = \exp(x) - 1$*.*

Corollary. *The Hilbert series of a Koszul symmetric set operad generated by one operation of arity two is differential algebraic of order* 1 *over* $\mathbb{Z}[x]$*.*

Third Chapter: Combinatorial interpretations of operads The third chapter, except the first section, is about the original work of the author. In Subsection 3.1.1, we recall the combinatorial interpretation of PreLie as an operadic structure on the species of rooted trees due to Chapoton and Livernet [19].

Theorem. *[19, Theorem 1.9] The underlying species of the operad* PreLie *is the species of rooted trees. Moreover, the operadic structure on the species of rooted trees is given by the insersion of a rooted tree in another rooted tree.*

And in the next subsection, we recall the combinatorial interpretation of the operadic twisting of PreLie due to Dotsenko and Khoroshkin [28].

Proposition. *[30, Subsection 6.7] Let* T *be a twisting rooted tree, then* $d_{\mathrm{Tw}}(T)$ *is given by:*

1. *The sum of all possible ways to split a white vertex of* T *into a white vertex retaining the label and a black vertex above it and to connect the incoming edges to one of the two vertices, up to a sign.*
2. *The sum of all possible ways to split a white vertex of* T *into a white vertex retaining the label and a black vertex bellow it and to connect the incoming edges to one of the two vertices, up to a sign.*
3. *The sum of all possible ways to split a black vertex of* T *into two black vertices and to connect the incoming edges to one of the two vertices, up to a sign.*
4. *The sum of all possible ways to graft an additional black leaf to* T*, taken with a minus sign.*
5. *Grafting* T *on top of a new black root, up to a sign.*

Moreover, many terms cancel due to the signs. In particular, if T *has more than one vertex, all contributions from* 4 *and* 5 *get cancelled by contributions from* 1*,* 2 *and* 3*.*

Which is used to show the following theorem:

Theorem. *[28, Theorem 5.1] The embedding of differential graded operads* $(\mathrm{Lie}, 0) \to \mathrm{TwPreLie}$ *induces an isomorphism in the cohomology.*

In the second section, we present a generalization of this work on rooted Greg trees. Namely, we define the operad Greg which is an operadic structure on the species of rooted Greg trees. We then show its Koszulness, and we relate it to the operadic twisting of PreLie.

Theorem. *The operad* Greg *is generated in arity two and Koszul.*

Theorem. *The embedding of differential graded operads* $\mathrm{Greg}_{-1} \to \mathrm{TwPreLie}$ *induces an isomorphism in the cohomology.*

In the next subsection, we deform this operad using a co-associative co-commutative coalgebra, and we show that the coproduct of several copies of the operad PreLie fibered by the operad Lie can be obtained this way.

Theorem. *The operad* $\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$ *is isomorphic to* $\mathrm{Greg}^{(V,\Delta_{\max})}$ *with:*

$$\Delta_{\max} : e_k \mapsto \sum_{i,j \mid \max(i,j)=k} e_i \otimes e_j$$

In Subsection 3.2.3, we show that the coproduct of $n+1$ operads PreLie fibered by the operad Lie is free over the coproduct of n operads PreLie fibered by the operad Lie as a left and right module, and we explicitly compute the generators in the left module case.

Theorem. *The left* $\bigvee_{\mathrm{Lie}}^{n} \mathrm{PreLie}$*-module* $\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$ *is isomorphic to:*

$$\bigvee_{\mathrm{Lie}}^{n} \mathrm{PreLie} \circ \mathcal{T}\left(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie})\right)$$

with CycLie *the underlying species of the cyclic operad* Lie*.*

The next section is the generalization of the construction on rooted trees and rooted Greg trees to the hyperforests. In Subsection 3.3.1, we generalize the construction of the operad PreLie as an operadic structure on the species of rooted trees to hyperforests, and we show that we get the operad ComPreLie.

Definition. [65, Definition 13] The operad ComPreLie is defined by:

$$\mathcal{T}[x, x.(1\,2), c]/\langle (x \circ_1 x - x \circ_2 x) - (x \circ_1 x - x \circ_2 x).(2\,3), \\ x \circ_1 c - (c \circ_1 x).(2\,3) - c \circ_2 x, c \circ_1 c - c \circ_2 c \rangle$$

where x and c are of arity two, and the action of $\mathfrak{S}_2$ on c is $c.(1\,2) = c$. The three relations should be understood as the pre-Lie identity for x, the Leibniz rule, and the associativity of c.

Theorem. *The underlying species of* ComPreLie *is the species of hyperforests.*

As a side result, we show that it is Koszul.

Theorem. *The operad* ComPreLie *is Koszul.*

We adapt the generalization to the Greg hyperforests in the next subsection, and in the last subsection, we do the same on reduced Greg hyperforest. We then end the chapter by our main result. In Section 3.4, we prove a conjecture of Dotsenko on an embedding of the operad encoding the algebraic structure on vector field of weak Frobenius manifolds. In the first subsection, we recall the definition of the operads FMan encoding the algebraic structure on vector field of weak Frobenius manifolds. Then in the last subsection, we fully use the fact that the operad Greg relate to the operadic twisting of PreLie and we use similar techniques to the operadic twisting to show that FMan embeds in ComPreLie, thus proving the conjecture of Dotsenko.

Theorem. *The operad* FMan *embeds in* ComPreLie*.*

Chapitre 0

Introduction en français

Dans une tentative de l'aut·eur·rice de raconter une histoire intéressante entremêlant combinatoire et algèbre, avec la théorie des opérades algébriques en personnage principal, ce manuscrit est divisé en trois chapitres. Trois chapitres ? Vraiment ? Nous voici pourtant dans le chapitre 0, *L'introduction en français*, alors qu'il y a encore les chapitres 1, 2 et 3 à venir. Il est en effet de coutume en France (et en réalité obligatoire sauf exceptions) d'inclure une partie en français dans un manuscrit de thèse rédigé en anglais. C'est donc ici que nous nous trouvons, au zéroième chapitre de ce manuscrit.

Revenons à notre histoire. Les deux premiers chapitres sont principalement introductifs. En effet, nous rappelons dans le premier chapitre les notions de base de la théorie des espèces combinatoires qui nous servira de base pour le second chapitre. Second chapitre dans lequel nous introduirons la théorie des opérades algébriques, un outil algébrique puissant pour étudier les structures algébriques non-nécessairement associatives. Enfin, le troisième chapitre est le cœur de ce manuscrit, puisqu'il contient les résultats principaux de la thèse de l'aut·eur·rice. Un des buts principaux de ce manuscrit était d'être le plus auto-contenu possible, c'est pourquoi nous avons inclus les deux premiers chapitres, qui bien qu'il contiennent quelques résultats originaux, sont essentiellement des rappels. Cependant, l'apparition de théorie des catégories et d'algèbre homologique a rendu cette tâche quasi-impossible, ou tout du moins pas réalisable compte tenue des contraintes de de temps et de place. Ainsi nous supposerons le lecteur familier avec la théorie des catégories et l'algèbre homologique, et nous ne ferons malheureusement pas de rappels sur ces sujets (à moins qu'il existe par hasard un chapitre en français que l'aut·eur·rice aurait l'obligation d'écrire et qui devrait avoir une taille minimale imposée, au quel cas ce serait l'endroit idéal pour inclure de tels rappels ...).

Dans tous les cas, nous nous référons aux classiques. Pour la théorie des catégories, citons le livre de MacLane [63] ainsi que celui de Lawvere et Schanuel [54]. Nous aimerions également mentionner le fantastique livre de Riehl [74]. On pourra également se retrancher vers les vidéos de *The Catsters* [15] une fois trop assommé par tant de lecture. Des éléments d'algèbre homologique seront utilisés dans ce manuscrit, cependant contrairement à la théorie des catégories qui guidera notre intuition, nous utiliserons l'algèbre homologique, et plus précisément les suites spectrales, comme un outil technique. Nous espérons que l'apparition de ces suites spectrales n'effraiera personne. En effet, bien qu'elles puissent paraître sibylline au premier abord, elles sont en réalité des outils très puissants, qui de plus admettent une interprétation catégorique. Nous revoyons au livre de Weibel [79] pour une introduction à l'algèbre homologique, incluant les suites spectrales.

Préliminaires catégorique Nous commençons par une très brève introduction à la théorie des catégories.

Définition. Une *catégorie* $\mathcal{C}$ est donnée par :

- une classe $\mathrm{Ob}(\mathcal{C})$ d'objets,
- pour tout $X, Y \in \mathrm{Ob}(\mathcal{C})$, un ensemble $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ de flèches, appelés morphismes, de X vers Y,
- pour tout $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, une loi de composition $\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$,
- pour tout $X \in \mathrm{Ob}(\mathcal{C})$, un élément $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$,

tels que :

- pour tout $f : X \to Y$, $g : Y \to Z$ et $h : Z \to W$, on a $h \circ (g \circ f) = (h \circ g) \circ f$,
- pour tout $X, Y \in \mathrm{Ob}(\mathcal{C})$ et $f : X \to Y$, on a $f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f$.

Nous rappelons qu'un monoïde est la donnée d'un ensemble M, d'une loi de composition associative et d'un élément neutre. Cela nous donne notre premier exemple de catégorie :

Exemple. Soit M un monoïde, alors nous notons BM la catégorie avec un seul objet $*$ et $\mathrm{Hom}_{BM}(*, *) = M$, de sorte que la composition est donnée par la loi de composition de M et l'identité est l'élément neutre de M.

Cet exemple permet une première interprétation de la notion de catégorie : une catégorie est un monoïde «ayant plusieurs point base». Cependant, bien que les monoïdes soient des objets fondamentaux en mathématiques (on rappelle que les groupes sont des monoïdes avec des inverses, ou encore que les algèbres associatives unitaires sont des «monoïdes en espaces vectoriels»), cela n'explique pas l'intérêt de la notion de catégorie, ni la raison de son omniprésence en mathématiques. Pour cela, nous devons nous tourner vers une autre interprétation de la notion de catégorie, à savoir que les catégories sont une modélisation très simple et très basique d'une théorie mathématiques. En effet, une catégorie est une collection d'objets, les objets d'intérêt de la théorie, et les flèches sont les applications entre ces objets respectant la structure des objets.

Exemple. La catégorie Set a pour objets les ensembles et pour flèches les applications entre ces ensembles. La composition est la composition d'applications, et l'identité est l'application identité.

On pourrait prendre peur devant le fait que la théorie Set est une «grosse» catégorie, c'est-à-dire une catégorie telle que ses objets forment une classe propre et pas un ensemble, chose qui n'est pas autorisée dans certaine théorie des ensembles comme ZFC. Il est possible d'éviter ces considérations ensemblistes de plusieurs manières, nous nous permettrons de les ignorer entièrement. En effet, il s'agit généralement de considérations purement techniques qui n'apportent pas grand chose à la compréhension de la théorie des catégories, et qui sont bien souvent le reliquat d'un choix plus ou moins judicieux de fondation des mathématiques. (L'aut·eur·rice souhaite en profiter pour rappeler que ZFC n'est pas la panacée dans laquelle toutes les mathématiques doivent être écrites, et que d'autres fondations des mathématiques existent, parfois plus adaptées à certaines théories mathématiques.)
L'un des aspects les plus intéressants de la théorie des catégories est que les (petites) catégories forment une catégorie. Pour cela, nous devons définir les flèches entre catégories, appelées foncteurs.

Définition. Un foncteur $F : \mathcal{C} \to \mathcal{D}$ entre deux catégories $\mathcal{C}$ et $\mathcal{D}$ est donné par :

- une application $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,
- pour tout $X, Y \in \mathrm{Ob}(\mathcal{C})$, une application $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$,

tels que :

- pour tout $X \in \mathrm{Ob}(\mathcal{C})$, on a $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$,
- pour tout $f : X \to Y$, $g : Y \to Z$, on a $F(g \circ f) = F(g) \circ F(f)$.

Les foncteurs donnent une façon de mettre en relation des théories mathématiques différentes.

Définition. Un foncteur $F : \mathcal{C} \to \mathcal{D}$ est *pleinement fidèle* si pour tout $X, Y \in \mathrm{Ob}(\mathcal{C})$, l'application $F_{X,Y}$ est une bijection.

Si un foncteur $F : \mathcal{C} \to \mathcal{D}$ est pleinement fidèle, alors on peut voir $\mathcal{C}$ comme une sous-théorie de $\mathcal{D}$, voulant dire que la théorie encodée par $\mathcal{C}$ est la même que celle de $\mathcal{D}$, mais restreinte à un sous-ensemble des objets de $\mathcal{D}$.

Exemple.

- Soit Mon la catégorie des monoïdes et des morphismes de monoïdes. Soit Grp la catégorie des groupes et des morphismes de groupes. Alors comme un groupe est en particulier un monoïde, il existe un foncteur Grp $\to$ Mon. Ce foncteur est pleinement fidèle, et on peut voir Grp comme une sous-théorie de Mon.
- Soit nuMon la catégorie des monoïdes non-nécessairement unitaires et des morphismes de monoïdes. Alors, comme l'unité est unique dans un monoïde, il existe un foncteur Mon $\to$ nuMon. Ce foncteur n'est pas pleinement fidèle, en effet, soit $(\{0, 1\}, \max)$, alors l'application constante 1 est un morphisme de monoïdes non-nécessairement unitaires, mais n'est pas un morphisme de monoïdes unitaires.

L'aut·eur·rice tient à insister sur le dernier exemple qui montre que même si certaines structures algébriques peuvent être vu comme des propriétés supplémentaires, ces deux points de vue sont en réalité subtilement différents.
On pourrait naïvement définir la notion d'isomorphisme de catégorie de la façon suivante :

Définition. Soit $F : \mathcal{C} \to \mathcal{D}$ un foncteur. On dit que F est un *isomorphisme de catégorie* si :

- $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ est une bijection,
- pour tout $X, Y \in \mathrm{Ob}(\mathcal{C})$, l'application $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ est une bijection.

Cependant, cette notion à le défaut suivant : Supposons que nous travaillons sur, disons, la théorie des groupes et que l'on note Grp la catégorie des groupes. Puis qu'il nous prenne la soudaine et irrépressible envie d'adjoindre un nouvel objet à notre théorie, que l'on appellerait au hasard *Aurélie* et qui soit isomorphe à $\mathbb{Z}/2\mathbb{Z}$. On pourrait noter Grp$'$ cette nouvelle catégorie. Alors, on aurait un foncteur $F : \mathrm{Grp} \to \mathrm{Grp}'$ qui envoie tout groupe sur lui-même, qui ce ne serait pas un isomorphisme de catégorie. Cette situation est absurde, donner un nouveau nom à un objet ne devrait pas changer la théorie, cela montre que la notion d'isomorphisme de catégorie n'est pas la bonne.

Définition. Soit $F : \mathcal{C} \to \mathcal{D}$ un foncteur. On dit que F est une *équivalence de catégories* si :

- F est essentiellement surjectif, c'est-à-dire que pour tout $Y \in \mathrm{Ob}(\mathcal{D})$, il existe $X \in \mathrm{Ob}(\mathcal{C})$ tel que $F(X)$ est isomorphe à Y,
- F est pleinement fidèle.

Cette notion d'équivalence de catégories est la bonne notion pour dire que deux catégories sont «les mêmes». En effet, si $F : \mathcal{C} \to \mathcal{D}$ est une équivalence de catégorie, alors F admet un «quasi-inverse», mais pour définir cette notion nous devons plonger un peu plus loin dans la théorie des catégories.

Définition. Soient $F, G : \mathcal{C} \to \mathcal{D}$ deux foncteurs. Un *transformation naturelle* $\eta : F \to G$ est donnée par un morphisme $\eta_X : F(X) \to G(X)$ pour tout $X \in \mathrm{Ob}(\mathcal{C})$, tel que pour tout $f : X \to Y$, le diagramme suivant commute :

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y) \end{array}$$

Cela veux dire que $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

Nous venons de voir l'un des apports les plus important de la théorie des catégories : les diagrammes commutatifs. Ces diagrammes permettent de représenter graphiquement des égalités entre flèches de façon très compacte. De plus l'on peut remarquer :

Proposition. *Soit* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ *la collection des foncteurs de* $\mathcal{C}$ *dans* $\mathcal{D}$*, alors* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ *est une catégorie, où les objets sont les foncteurs et les flèches sont les transformations naturelles.*

En effet en théorie des catégories, les foncteurs forment une catégorie. On peut maintenant définir la notion de quasi-inverse d'un foncteur.

Définition. Soient $F : \mathcal{C} \to \mathcal{D}$ et $G : \mathcal{D} \to \mathcal{C}$ deux foncteurs. On dit que F et G sont *quasi-inverses* si $F \circ G \simeq \mathrm{id}_{\mathcal{D}}$ et $G \circ F \simeq \mathrm{id}_{\mathcal{C}}$ dans les catégories $\mathrm{Fun}(\mathcal{D},\mathcal{D})$ et $\mathrm{Fun}(\mathcal{C},\mathcal{C})$ respectivement.

Nous remarquons un autre aspect de la théorie des catégories : la bonne notion n'est pas «être égal», mais «être isomorphe». Donnons un exemple pour illustrer cela :

Définition. Soit $\mathcal{C}$ une catégorie, un *objet initial* est un objet $I \in \mathrm{Ob}(\mathcal{C})$ tel que pour tout $X \in \mathrm{Ob}(\mathcal{C})$, il existe un unique morphisme $I \to X$.

Exemple. Dans la catégorie Grp, l'objet initial est le groupe trivial.

On peut alors remarquer les phénomènes suivants :

Proposition. *Soit* $\mathcal{C}$ *une catégorie, alors les objets initiaux de* $\mathcal{C}$ *sont uniques à isomorphisme près.*

En effet, si I et I' sont deux objets initiaux de $\mathcal{C}$, alors il existe un unique morphisme $I \to I'$ et un unique morphisme $I' \to I$. Ces deux morphismes sont inverses l'un de l'autre, et donc I et I' sont isomorphes.

Proposition. *Soit* $F : \mathcal{C} \to \mathcal{D}$ *une équivalence de catégories, alors* $F(I)$ *est initial dans* $\mathcal{D}$ *si et seulement si* I *est initial dans* $\mathcal{C}$*.*

Cela montre que les propriétés purement catégoriques sont conservées par les équivalences de catégories. Cela indique que la notion d'équivalence de catégories est la bonne notion à considérer.

Une dernière notion importante que nous souhaitons introduire est celle d'adjonction. Avant cela remarquons la chose suivante :

Définition. Soit $\mathcal{C}$ une catégorie, notons $\mathcal{C}^{\mathrm{op}}$ la catégorie opposée à $\mathcal{C}$ où les objets sont les mêmes que ceux de $\mathcal{C}$ et où les flèches sont inversées. C'est-à-dire :

- $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$,
- $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$,
- pour $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y)$ et $g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y,Z)$, on a $g \circ_{\mathcal{C}^{\mathrm{op}}} f = f \circ_{\mathcal{C}} g$.

Alors $\mathcal{C}^{\mathrm{op}}$ est effectivement une catégorie.

Nous pouvons maintenant définir la notion d'adjonction.

Définition. Soient $F : \mathcal{C} \to \mathcal{D}$ et $G : \mathcal{D} \to \mathcal{C}$ deux foncteurs. On dit que (F,G) est une *adjonction* avec F l'*adjoint à gauche* de G et G l'*adjoint à droite* de F et l'on note $F \dashv G$ si pour tout $X \in \mathrm{Ob}(\mathcal{C})$ et $Y \in \mathrm{Ob}(\mathcal{D})$, il existe une bijection naturelle :

$$\eta_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \to \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

Dire que η est une *bijection naturelle* signifie que :

- $\eta_{\bullet,Y}$ est un isomorphisme naturel entre les foncteurs $\mathrm{Hom}_{\mathcal{D}}(F(\bullet),Y)$ et $\mathrm{Hom}_{\mathcal{C}}(\bullet,G(Y))$ dans $\mathrm{Fun}(\mathcal{C},\mathrm{Set}^{\mathrm{op}})$,

- $\eta_{X,\bullet}$ est un isomorphisme naturel entre les foncteurs $\mathrm{Hom}_{\mathcal{D}}(F(X),\bullet)$ et $\mathrm{Hom}_{\mathcal{C}}(X,G(\bullet))$ dans $\mathrm{Fun}(\mathcal{D},\mathrm{Set})$.

Les adjonctions sont une notion très importante en théorie des catégories, et sont omniprésentes en mathématiques. Nous nous arrêtons ici pour les préliminaires catégoriques car nous n'avons en effet malheureusement pas le loisir de nous attarder plus longtemps sur ce sujet.

Premier chapitre : Espèces combinatoires. Ce premier chapitre est entièrement consacré à la théorie des catégories et la combinatoire. Il s'agit d'un introduction à la théorie des espèces combinatoires via la théorie des catégories. Les espèces combinatoires ont été introduites par Joyal en 1981 [47], et se sont révélées être un outil extrêmement en combinatoire énumérative, principalement grace à leur capacité à traduire équations fonctionnelles en construction récursive d'objets combinatoires, et vice versa. La théorie des espèces combinatoires peut être comprise comme une ≪catégorification≫ de la théorie des séries génératrices. Cela veut dire que l'on remplace la notion de *fonction* génératrice par un ≪*foncteur* générateur≫, permettant de conserver plus d'informations. C'est ce point de vue catégorique qui nous guidera tout au long du premier chapitre.

Nous commençons par une catégorification naive de la théorie des séries génératrices. Dans la première sous-section, nous faisons des rappels sur les séries formelles. Afin d'éviter les problèmes de convergence, nous nous restreignons à des séries formelles à coefficients dans $\overline{\mathbb{N}}$, en utilisant la convention $0 \cdot \infty = 0$.

Définition. Une *série formelle ordinaire* (ogf) est une application $a : \mathbb{N} \to \overline{\mathbb{N}}$, notée $\sum_{n\geq 0} a_n x^n$.

Nous définissons ensuite les opérations usuelles sur les séries formelles, à savoir l'addition notée $+$, la multiplication (produit de Cauchy) notée $\cdot$, le produit d'Hadamard noté $\odot$, la composition notée $\circ$ et la dérivation notée $\ '$. Dans la sous-section suivante, nous introduisons les espèces naïves, qui ne sont pas pas les espèces combinatoires telles qu'introduites par Joyal, mais qui sont un premier pas vers celles-ci.

Définition. Une *espèce naïve* est un foncteur $F : \mathbb{N} \to \mathrm{Set}$.

Nous adaptons ensuite les opérations sur les séries formelles aux espèces naïves, définissant ainsi leur somme notée $+$, leur produit noté $\cdot$, leur produit d'Hadamard noté $\odot$, leur pléthysme (qui correspond à la composition) notée $\circ$ et leur dérivation notée $\ '$. Nous terminons cette première section par la généralisation des espèces naïves au cas multivarié, montrant ainsi un analogue du théorème des fonctions implicites pour les espèces. Ici, une espèce naïve 2-variée est un foncteur $H : \mathbb{N}^2 \to \mathrm{Set}$.

Théorème. *Soit $H[X,Y]$ une espèce naïve 2-variée dans les variables X et Y telle que :*

$$H(0,0) = \emptyset \qquad \textit{et} \qquad \frac{\partial H}{\partial Y}(0,0) = \emptyset$$

Alors il existe une unique espèce naïve $\mathcal{A}$ telle que :

$$\mathcal{A}[X] = H[X, A[X]] \qquad \textit{et} \qquad \mathcal{A}(0) = \emptyset$$

La seconde sous-section est consacrée à la théorie des espèces combinatoires de Joyal. Nous commençons par aborder les défauts des espèces naïves, et nous définissons les espèces combinatoires dans la sous-section 1.2.1.

Définition. Une *espèce* est un foncteur $\mathcal{S} : \mathbb{B} \to \mathrm{Set}$. Où $\mathbb{B}$ est la catégorie des ensembles finis avec les bijections comme morphismes.

Nous généralisons ensuite les opérations que nous avons définies pour les espèces naïves aux espèces dans la sous-section suivante. La sous-section 1.2.3 relie les espèces aux actions des groupes de permutations $\mathfrak{S}_n$. Nous y définissons également le foncteur de Schur d'une espèce.

Définition. Soit $\mathcal{S}$ une espèce. Le *foncteur de Schur* associé à $\mathcal{S}$ est le foncteur $\mathcal{F}_\mathcal{S}$: Set $\to$ Set tel que pour tout ensemble X, $\mathcal{F}_\mathcal{S}(X) = \uplus_{n\in\mathbb{N}}\mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} X^n$.

Nous définissons le développement de Taylor-Joyal d'un foncteur Set $\to$ Set qui par analogie avec le développement de Taylor d'une fonction, associe à un foncteur une espèce.

Proposition. *Notons* $\widehat{\ }$ *le développement de Taylor-Joyal. Soit* $\mathcal{S}$ *une espèce. Alors nous avons :*

$$\widehat{\mathcal{F}_\mathcal{S}} = \mathcal{S}$$

Nous présentons ensuite un travail original (actuellement non publié) de l'aut·eur·rice en collaboration avec Agugliaro dans la section 1.2.4. L'idée principale est d'appliquer le formalisme de la sous-section précédente à des foncteurs $\mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ où $\mathrm{Set}^{\mathrm{op}}$ est la catégorie opposée de Set et de définir la transformée de Joyal d'un tel foncteur.

Définition. Soit $\mathcal{S}$ une espèce. La *fonction L catégorique* associée à $\mathcal{S}$ est le foncteur contravariant $\mathcal{L}_\mathcal{S}$: Set $\to$ Set tel que pour tout ensemble X, $\mathcal{L}_\mathcal{S}(X) = \uplus_{n\in\mathbb{N}^f r}\mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} \mathrm{Hom}(X, \underline{n})$.

Puis, nous remarquons que le diagramme du produit est l'opposé du diagramme de la somme, nous permettant de définir des "partitions" relativement au produit au lieu de la somme :

Définition. Soit A un ensemble, et soit P et Q deux quotients de A. Nous notons alors $P \underline{\times} Q = A$ lorsque $A \to P \times Q$ est une bijection. Nous disons que $P \underline{\times} Q$ est un *produit direct* de A.
Soit A un ensemble et $k \in \mathbb{N}$, une *quotientation de A de longueur k* est un ensemble $P = \{P_1, \ldots, P_k\}$ tel que $A = P_1 \underline{\times} \ldots \underline{\times} P_k$. Nous notons $P \not\ltimes^k A$ le fait que P est une quotientation de A de longueur k.

Ces définitions nous permettent de définir la convolution de Dirichlet notée $*$ et le pléthysme arithmétique noté $\square$ de deux espèces. Une généralisation directe de la transformée de Taylor-Joyal nous permet de définir la transformée de Joyal d'un foncteur $\mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$. De plus cela nous permet d'associer série de Dirichlet à un tel foncteur, et avec des foncteurs bien choisis, nous pouvons retrouver les fonctions zêta usuelles, à savoir la fonction zêta de Riemann, les fonctions zêta de Hasse-Weil, et les fonctions zêta d'Artin-Mazur. Nous terminons la section par une sous-section expliquant comment généraliser le théorème des fonctions implicites aux espèces :

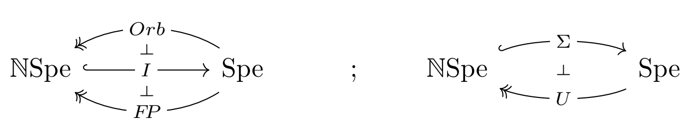
Théorème. *Soit* $H[X, Y]$ *une espèce 2-variée dans les variables* X *et* Y *telle que :*

$$H(0, 0) = \emptyset \qquad et \qquad \frac{\partial H}{\partial Y}(0, 0) = \emptyset$$

Alors il existe une unique espèce $\mathcal{A}$ *telle que :*

$$\mathcal{A}[X] = H[X, A[X]] \qquad et \qquad \mathcal{A}(0) = \emptyset$$

La troisième section tire pleinement parti du point de vue catégorique de la théorie des espèces. Dans la sous-section 1.3.1, nous montrons deux façons très naturelles de plonger la catégorie des espèces naïves dans la catégorie des espèces, définissant les espèces de mélange et les espèces ordonnées, en utilisant les adjonctions suivantes :



Où *FP* est le foncteur des points fixes, I est l'inclusion des espèces naïves dans les espèces, *Orb* est le foncteur orbite, Σ est le foncteur de symétrisation, et U est le foncteur d'oubli. Dans la sous-section 1.3.2, nous montrons comment définir des espèces dans d'autres catégories, définissant les espèces linéaires (qu'il ne faut pas confondre avec les espèces ordonnées à cause de l'inconstance de la terminologie dans la littérature), les espèces topologiques, et en fait les espèces à valeur dans n'importe quelle catégorie se comportant suffisamment bien.

La dernière section du premier chapitre est consacrée à l'application de la théorie des espèces à certaines structures arboricole, à savoir les arbres enracinés, les arbres de Greg enracinés, et les hyperforêts (ensembles non-vides d'hyperarbres enracinés). Nous définissons les arbres enracinés et les arbres de Greg enracinés dans la sous-section 1.4.1, et les espèces correspondantes.

Définition. Un *arbre enraciné* est un graphe fini connexe sans cycle avec un sommet distingué appelé la racine.
Un *arbre de Greg enraciné* est un arbre enraciné où les sommets sont soit noirs soit blancs tel que :

- les sommets noirs sont indistinguables, et
- les sommets noirs ont au moins deux enfants.

Nous adaptons ensuite ces définitions aux hyperarbres dans la sous-section 1.4.2.

Définition. Un *hyperarbre* est un hypergraphe fini connexe sans cycle avec un sommet distingué appelé la racine.

Nous introduisons ensuite trois nouveaux objets combinatoires : tout d'abord les formes d'arbres.

Définition. Soit P une partition de longueur k de A, une *forme d'arbre sur P* est une structure d'hyperarbre avec k sommets noirs sur A telle que :

- la racine est noire,
- les arrêtes simples sont entre un sommet noir et un sommet blanc tel que le sommet noir est en dessous (plus proche de la racine),
- les hyperarrêtes sont entre un sommet blanc et plusieurs sommets noirs tel que le sommet blanc est en dessous,
- pour chaque $p \in P$, il existe un sommet noir tel que p est l'ensemble des sommets blancs connectés à ce sommet via des arrêtes simples.

Nous construisons ensuite la *forme d'arbre d'un hyperarbre enraciné* et montrons qu'elle encode les données nécessaires pour reconstruire un hyperarbre à partir de ses sous-arbres maximaux. En utilisant cette idée, nous introduisons les hyperarbres de Greg enracinés et les hyperarbres de Greg enracinés réduits généralisant à la fois les arbres de Greg et les hyperarbres de deux manières différentes.

Définition. Un *hyperarbre de Greg enraciné* est un hyperarbre enraciné où les sommets sont soit noirs soit blancs tel que :

- les sommets noirs sont indistinguables, et
- les sommets noirs ont au moins deux enfants.

Un *hyperarbre de Greg enraciné réduit* est un hyperarbre de Greg enraciné où les sommets noirs n'admettent pas d'hyperarrête entrante.

Deuxième chapitre : Opérades. Le deuxième chapitre est consacré aux opérades algébriques. Nous commençons par considérer les espèces ensemblistes (les espèces combinatoires que nous avons définies dans le premier chapitre), mais nous passons très rapidement aux espèces linéaires pour pouvoir faire de l'algèbre.

La première section est dédiée aux monades d'arbres. Nous commençons par rappeler les définitions de base sur les monoïdes et les monades. Ensuite, chaque sous-section est dédiée à une monade d'arbres différente. Nous commençons par la monade d'arbres symétriques notée $\mathcal{T}$ pour les espèces, puis la monade d'arbres non-symétriques notée $\mathcal{T}^{\mathrm{ns}}$ pour les espèces ordonnées, et nous terminons par la monade d'arbres de mélange notée $\mathcal{T}^{\text{Ш}}$ pour les espèces de mélange.

La seconde section est dédiée aux opérades algébriques et aux outils principaux que nous allons utiliser. La sous-section 2.2.1 est dédiée à la définition des opérades algébriques. Plus précisément, nous donnons trois définitions différentes, et montrons qu'elles sont équivalentes.

Définition.

- Une *opérade algébrique symétrique*, ou de façon plus concise *opérade*, est une algèbre sur la monade d'arbres $\mathcal{T}$.
- De façon équivalente, une *opérade* est un monoïde dans la catégorie des espèces selon le pléthysme.
- De façon équivalente, une *opérade* est une espèce $\mathcal{P}$ avec des opérations $\circ_i$ et des éléments $e_i \in \mathcal{P}(\{i\})$ tels que $\circ_i : \mathcal{P}(A \sqcup \{i\}) \otimes \mathcal{P}(B) \to \mathcal{P}(A \sqcup B)$, et satisfaisant les axiomes de composition séquentielle, de composition parallèle et d'unité.

Nous profitons de cette sous-section pour donner le point de vue de l'aut·eur·rice sur les opérades algébriques, à savoir qu'elles sont une sorte d'*algèbres* sur des objets très combinatoires, les espèces linéaires. Dans la seconde sous-section, nous montrons comment donner une présentation d'une opérade par générateurs et relations. Nous donnons ensuite des exemples d'opérades définies par générateurs et relations dans la sous-section 2.2.3. Nous définissons les *Trois Grâces* comme elles ont été nommées par Loday, les opérades Ass, Com et Lie. Nous définissons également le papillon opéradique, et (la plus belle de toutes) l'opérade PreLie. Comme nous n'avons pas encore les outils pour les étudier, nous continuons en abordant la question des représentants canoniques dans les opérades définies par générateurs et relations dans la sous-section 2.2.4. Nous définissons les systèmes de réécriture opéradiques (ORS) qui sont des généralisations des bases de Gröbner ou des bases PBW puisqu'ils ne nécessitent qu'un ordre partiel sur l'ensemble des monômes.

Définition. Un *système de réécriture opéradiques* noté *ORS* est un triplet $(\mathcal{S}, \mathcal{X}, R)$ où $\mathcal{S}$ est une espèce linéaire, $\mathcal{X}$ est une espèce de mélange ensembliste tel que $U(\mathcal{S}) = \mathrm{Span}(\mathcal{X})$, et $R = (R_n)_{n\in\mathbb{N}}$ tel que R_n est un sous-ensemble de $\mathcal{T}^{\mathrm{III}}\{\mathcal{X}\}(n) \times \mathcal{T}(\mathcal{S})(\underline{n})$. Nous notons R' l'ensemble des *monômes réécrivables*, c'est-à-dire des monômes admettant au moins une règle de réécriture dans R.

Nous définissons les ordres monomial usuels sur les opérades, à savoir l'ordre par permutation, l'ordre lexicographique gradué sur les chemins, et l'ordre quantique. Dans la sous-section suivante, nous définissons les modules sur les opérades, et nous énonçons les théorèmes suivants nous permettant de montrer des propriétés de liberté en utilisant des ORS convergents. Soient $(\mathcal{S}, \mathcal{X}, U)$ et $(\mathcal{S} \oplus \mathcal{R}, \mathcal{X} + \mathcal{Y}, U \sqcup V)$ deux ORS convergents admettant des opérades associées, notons $\mathcal{P}$ et $\mathcal{Q}$ les opérades associées. Nous avons alors les théorèmes suivants.

Théorème (version libre à gauche)**.** *[25, Théoreme 4] Supposons que les racines des monômes réécrivables de V sont des éléments de $\mathcal{Y}$. Alors $\mathcal{Q}$ est libre en tant que module à gauche sur $\mathcal{P}$.*

Théorème (version libre à droite)**.** *[25, Théoreme 4] Supposons que les sommets tels que chaque enfant est une feuille, des monômes réécrivables de V sont des éléments de $\mathcal{Y}$. Alors $\mathcal{Q}$ est libre en tant que module à droite sur $\mathcal{P}$.*

Théorème. *[31, Théoreme 4.1] Soit $\mathcal{P}$ une opérade engendrée par $\mathcal{S}$, et $\mathcal{X}$ une base de $\mathcal{S}$ satisfaisant les conditions suivantes :*

- *$\mathcal{P}$ admet un ORS convergent $(\mathcal{S}, \mathcal{X}, U)$ descendant le long de l'ordre lexicographique gradué sur les chemins tels que pour chaque monôme réécrivable, la feuille d'étiquette la plus petite est directement connectée à la racine.*
- *$\mathcal{P}$ admet un ORS convergent $(\mathcal{S}, \mathcal{X}, V)$ descendant le long de l'ordre lexicographique gradué sur les chemins tels que pour chaque monôme réécrivable, la feuille d'étiquette la plus petite et la deuxième plus petite sont directement connectées au même sommet.*

Alors $\mathcal{P}$ satisfait la propriété de Nielsen-Schreier.

La troisième section est consacrée aux opérades différentielles graduées. Dans la première sous-section, nous expliquons la règle de signe de Koszul et donnons des exemples d'applications et de calculs utilisant cette règle. Dans la sous-section suivante, nous définissons les constructions Bar et Cobar sur les opérades différentielles graduées. Ce sont des ingrédients clés de l'algèbre homologique opéradique. Nous n'avons malheureusement pas le temps d'aller plus loin dans cette théorie profonde et intéressante. La dernière sous-section est une introduction malheureusement courte à la torsion opéradique. Cependant, le calcul complet de la torsion opéradique de PreLie est donné dans le dernier chapitre, à savoir dans la sous-section 3.1.2.

La dernière section du deuxième chapitre est une introduction à la dualité de Koszul opéradiques. Nous définissons le dual de Koszul et le complexe de Koszul d'une opérade quadratique dans la première sous-section. Dans la sous-section suivante, nous définissons la propriété de Koszul, et nous montrons un théorème bien connu de la théorie de Koszul opéradiques pour lequel l'aut·eur·rice n'a pas trouvé de référence dans la littérature, à savoir qu'une opérade admettant un ORS convergent quadratique est Koszul.

Théorème. *Le complexe de Koszul d'une opérade admettant un ORS convergent quadratique est acyclique.*

Dans la sous-section 2.4.3, nous utilisons les outils de la théorie de Koszul opéradiques pour étudier le papillon opéradique, et répondons négativement à une conjecture de Loday. Dans la sous-section suivante, nous discutons rapidement des séries génératrices des opérades de Koszul, et énonçons une conjecture sur les séries génératrices des opérades Koszul engendrées par une opération d'arité deux.

Conjecture. *Soit $\mathcal{P}$ une opérade symétrique Koszul engendrée par une opération d'arité deux, alors la série génératrice de $\mathcal{P}$ est algébrique différentielle d'ordre 1 sur $\mathbb{Z}[x]$. Autrement dit, $f_{\mathcal{P}}$ et $f'_{\mathcal{P}}$ sont algébriquement dépendants sur $\mathbb{Z}[x]$.*

Enfin, nous terminons le chapitre par un travail original de l'aut·eur·rice classifiant les opérades symétriques ensemblistes Koszul engendrées par une opération d'arité deux, prouvant la conjecture de la sous-section précédente dans le cas des opérades ensemblistes.

Théorème. *Soit $\mathcal{P}$ une opérade symétrique ensembliste Koszul engendrée par une opération d'arité deux, alors $\mathcal{P}$ est isomorphe à l'une des 11 opérades suivantes :*

- Mag *l'opérade magmatique et* $f_{\mathcal{P}}(x) = \frac{1}{2}(1 - \sqrt{1-4x})$;
- NAP *l'opérade permutative non-associative et* $f_{\mathcal{P}}(x) = \sum_{n\in\mathbb{N}^f r} \frac{n^{n-1}}{n!} x^n$;
- $\mathrm{CMag} \circ \mathrm{ANil}_2$ *qui est construite à partir de* CMag *et* ANil_2 *avec la relation* $[a.b, c] = 0$, *et* $f_{\mathcal{P}}(x) = 1 - \sqrt{1-2x-x^2}$;
- $\mathrm{ANil}_2 \circ \mathrm{CMag}$ *qui est construite à partir de* CMag *et* ANil_2 *avec la relation* $[a, b].c = 0$, *et* $f_{\mathcal{P}}(x) = 2 - x - 2\sqrt{1-2x}$;
- CMag#AMag *qui est la somme connexe de* CMag *et* AMag, *et* $f_{\mathcal{P}}(x) = 2 - x - 2\sqrt{1-2x}$;
- Ass *l'opérade associative et* $f_{\mathcal{P}}(x) = \frac{x}{1-x}$;
- $\mathrm{CMag\#ANil}_2$ *qui est la somme connexe de* CMag *et* ANil_2, *et* $f_{\mathcal{P}}(x) = 1 - \sqrt{1-2x} + \frac{1}{2}x^2$;
- Perm *l'opérade permutative et* $f_{\mathcal{P}}(x) = x\exp(x)$;
- $\mathrm{LieAdm}^!$ *le dual de Koszul de l'opérade Lie admissible et* $f_{\mathcal{P}}(x) = \exp(x) - 1 + \frac{x^2}{2}$;
- CMag *l'opérade magmatique commutative et* $f_{\mathcal{P}}(x) = 1 - \sqrt{1-2x}$;

- Com *l'opérade commutative et* $f_{\mathcal{P}}(x) = \exp(x) - 1$.

Corollaire. *La série génératrice d'une opérade symétrique ensembliste Koszul engendrée par une opération d'arité deux est algébrique différentielle d'ordre* 1 *sur* $\mathbb{Z}[x]$*. Autrement dit,* $f_{\mathcal{P}}$ *et* $f'_{\mathcal{P}}$ *sont algébriquement dépendants sur* $\mathbb{Z}[x]$*.*

Troisème chapitre : Interprétations combinatoires des opérades Le troisième chapitre, à l'exception de la première section, est consacré au travail original de l'aut·eur·rice. Dans la sous-section 3.1.1, nous rappelons l'interprétation combinatoire de PreLie en tant que structure opéradiques sur les espèces d'arbres enracinés due à Chapoton et Livernet [19].

Théorème. *[19, Théoreme 1.9] L'espèce sous-jacente de l'opérade* PreLie *est l'espèce des arbres enracinés. De plus, la structure opéradiques sur l'espèce des arbres enracinés est donnée par l'insertion d'un arbre enraciné dans un autre arbre enraciné.*

Et dans la sous-section suivante, nous rappelons l'interprétation combinatoire du twisting opéradiques de PreLie due à Dotsenko et Khoroshkin [28].

Proposition. *[30, Sous-section 6.7] Soit* T *un arbre enraciné tordu, alors* $d_{\mathrm{Tw}}(T)$ *est donné par :*

1. *La somme de toutes les façons possibles de diviser un sommet blanc de* T *en un sommet blanc conservant l'étiquette et un sommet noir au-dessus et de connecter les arêtes entrantes à l'un des deux sommets, au signe près.*

2. *La somme de toutes les façons possibles de diviser un sommet blanc de* T *en un sommet blanc conservant l'étiquette et un sommet noir en dessous et de connecter les arêtes entrantes à l'un des deux sommets, au signe près.*

3. *La somme de toutes les façons possibles de diviser un sommet noir de* T *en deux sommets noirs et de connecter les arêtes entrantes à l'un des deux sommets, au signe près.*

4. *La somme de toutes les façons possibles de greffer une feuille noire supplémentaire à* T*, au signe près.*

5. *Greffer* T *sur une nouvelle racine noire, au signe près.*

De plus, de nombreux termes s'annulent en raison des signes. En particulier, si T *a plus d'un sommet, toutes les contributions de* 4 *et* 5 *sont annulées par les contributions de* 1*,* 2 *et* 3*.*

Qui est utilisé pour montrer le théorème suivant :

Théorème. *[28, Théoreme 5.1] Le plongement d'opérades différentielles graduées* $(\mathrm{Lie}, 0) \to \mathrm{TwPreLie}$ *induit un isomorphisme en cohomologie.*

Dans la deuxième section, nous présentons une généralisation de ce travail sur les arbres de Greg enracinés. À savoir, nous définissons l'opérade Greg qui est une structure opéradiques sur l'espèces d'arbres de Greg enracinés. Nous montrons ensuite sa Koszulité, et nous la relions à la torsion opéradiques de PreLie.

Théorème. *L'opérade* Greg *est engendrée en arité deux et est Koszul.*

Théorème. *Le plongement d'opérades différentielles graduées* $\mathrm{Greg}_{-1} \to \mathrm{TwPreLie}$ *induit un isomorphisme en cohomologie.*

Dans la sous-section suivante, nous déformons cette opérade en utilisant une coalgèbre co-associative co-commutative, et nous montrons que le coproduit de plusieurs copies de l'opérade PreLie fibré par l'opérade Lie peut être obtenu de cette manière.

Théorème. *L'opérade* $\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$ *est isomorphe à* $\mathrm{Greg}^{(V,\Delta_{\max})}$ *avec :*

$$\Delta_{\max} : e_k \mapsto \sum_{i,j \mid \max(i,j)=k} e_i \otimes e_j$$

Dans la sous-section 3.2.3, nous montrons que le coproduit de $n+1$ opérades PreLie fibré par l'opérade Lie est libre sur le coproduit de n opérades PreLie fibré par l'opérade Lie en tant que module à gauche et à droite, et nous calculons explicitement les générateurs dans le cas du module à gauche.

Théorème. *Le* $\bigvee_{\mathrm{Lie}}^{n} \mathrm{PreLie}$*-module à gauche* $\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$ *est isomorphe à :*

$$\bigvee_{\mathrm{Lie}}^{n} \mathrm{PreLie} \circ \mathcal{T}\left(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie})\right)$$

avec CycLie *l'espèce sous-jacente de l'opérade cyclique* Lie*.*

La section suivante est la généralisation de la construction sur les arbres enracinés et les arbres de Greg enracinés aux hyperforêts. Dans la sous-section 3.3.1, nous généralisons la construction de l'opérade PreLie en tant que structure opéradiques sur les espèces d'arbres enracinés aux hyperforêts, et nous montrons que nous obtenons l'opérade ComPreLie.

Définition. [65, Définition 13] L'opérade ComPreLie est définie par :

$$\begin{aligned}\mathcal{T}[x, x.(1\ 2), c]/\langle &(x \circ_1 x - x \circ_2 x) - (x \circ_1 x - x \circ_2 x).(2\ 3),\\ &x \circ_1 c - (c \circ_1 x).(2\ 3) - c \circ_2 x, c \circ_1 c - c \circ_2 c\rangle\end{aligned}$$

où x et c sont d'arité deux, et l'action de $\mathfrak{S}_2$ sur c est $c.(1\ 2) = c$. Les trois relations doivent être comprises comme l'identité pré-Lie pour x, la règle de Leibniz, et l'associativité de c.

Théorème. *L'espèce sous-jacente de* ComPreLie *est l'espèce des hyperforêts.*

Nous montrons ensuite que l'opérade ComPreLie est Koszul.

Théorème. *L'opérade* ComPreLie *est Koszul.*

Nous adaptons ensuite la généralisation aux hyperforêts de Greg dans la sous-section suivante. Enfin dans la dernière sous-section, nous faisons de même pour les hyperforêts de Greg réduits. Nous terminons le chapitre par notre résultat principal. Dans la section 3.4, nous prouvons une conjecture de Dotsenko sur un plongement de l'opérade encodant la structure algébrique sur les champs de vecteurs des variétés de Frobenius faibles. Dans la première sous-section, nous rappelons la définition des opérades FMan encodant la structure algébrique sur les champs de vecteurs des variétés de Frobenius faibles. Puis dans la dernière sous-section, nous utilisons pleinement le fait que l'opérade Greg est liée à la torsion opéradiques de PreLie et nous utilisons des techniques similaires à la torsion opéradiques pour montrer que FMan se plonge dans ComPreLie, prouvant ainsi la conjecture de Dotsenko.

Théorème. *L'opérade* FMan *se plonge dans* ComPreLie*.*

Chapter 1

Combinatorial Species

The theory of combinatorial species was historically introduced by A. Joyal in 1981 [47]. This theory proved itself to very fruitful in enumerative combinatorics, notably because of its ability to transform recursive definitions into differential equations. Indeed, enumerative combinatorics is the area of mathematics dealing with enumeration of certain combinatorial patterns or structures. Questions in this area can often be formulated the following way: "Let $(A_n)_{n\in\mathbb{N}}$ be a sequence a finite set arising from combinatorial considerations, what is the cardinality of A_n?". The first breakthrough in this area can be considered to be the introduction of generating function by A. De Moivre in 1730 [21] allowing to consider one object, the generating function, instead of considering each number separately. The main idea behind the theory of combinatorial species is to replace the generating function by a functor, and thus allowing the functor to carry algebraic structures of A_n that are ignored by the generating function. This point of view allows us to translate some functional equations into combinatorial problems, or to define combinatorial structures by functional equations with functors that we could call "functorial equations". Because of this combinatorial framework, combinatorial species were initially defined to be finite, however, we will occasionally need to consider infinite species in this thesis, and thus we will define non-necessarily finite species. This slight generalization of species was already considered by Joyal in 1986 [46] because of the tight connection between species and Schur functors. Since species will be the basic building block we will use throughout this entire manuscript, we will give an extensive introduction to the theory of combinatorial species in this chapter. We will first define the category of species, then we will define the classical operations on species and the compatibility relations between them. For the sake of precision, we will give proof of the propositions and the theorems we state in this chapter, although they can usually be found in the classical literature on species, see [7] and [1]. The proof we give may seem too formal or abstract, however, first we believe that it is important to give precise definitions of the basic objects we are going to use, and to give precise proofs of their basic properties, and second, we want to emphasize the categorical nature of the theory of combinatorial species. Indeed, in the next chapters, we will use species with value in a category different from the category of sets, see discussion in Subsection 1.3.2. Most of this chapter is a recollection of the theory of combinatorial species, we invite the interested reader to refer to the book of Bergeron, Labelle and Leroux [7], and the book of Aguiar and Mahajan [1] for a more in depth study of this theory.

1.1 Naive categorification of generating functions

One of the point of view we want to emphasize in this manuscript is how considering "natural" categorification of usual mathematical objects can lead to new interesting mathematical objects. The theory of species is a good example of this point of view since it is the categorification of the theory of generating functions. In this section, we start by recalling the theory of generating functions in a quite algebraic point of view since we do not really want to deal with the convergence issues arising with formal power series. We will then "naively" categorify our definition of generating functions to obtain the definition of naive species. The theory of naive species is not the theory we aim for, and it should be seen as a "toy example" of categorification. However, we start with naive species because it is a good warm-up for the theory of species.

1.1.1 Ordinary generating functions

As we stated above, the main idea behind the theory of combinatorial species is to replace a generating function by a functor. Let us first recall the definition of a generating function and then try to naively categorify it. Because we want generating functions to "count something", we will consider formal power series with integer coefficients, however, because it is convenient, we will allow infinite coefficients. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ with $\mathbb{N}$ the set of non-negative integers. We extend the sum on $\mathbb{N}$ to $\overline{\mathbb{N}}$ by setting $n + \infty = \infty$ for each $n \in \mathbb{N}$, and the multiplication by $0 \cdot \infty = 0$ and $n \cdot \infty = \infty$ for each $n \in \overline{\mathbb{N}} \setminus \{0\}$. We use the convention that infinite sums of non-zero elements give ∞. Let x be a formal variable.

Definition 1.1.1.1. An *ordinary generating function*, ogf for short, $f = \sum_{n\in\mathbb{N}} f_n x^n$ is a formal power series such that $f_n \in \overline{\mathbb{N}}$. Since a power series is determined by its coefficients, we can identify the set of ogf with the set of functions $\{\mathbb{N} \to \overline{\mathbb{N}}\}$. Hence, an ogf is uniquely determined by a function $s : \mathbb{N} \to \overline{\mathbb{N}}$. An ogf f is *finite* if f_n is finite for all $n \in \mathbb{N}$. An ogf f is *connected* if $f_0 = 0$, it is *strongly connected* if $f_0 = f_1 = 0$. Let us define the following usual ogf:

- 0 the zero ogf such that all coefficients are 0,
- 1 the ogf such that the first coefficient is 1 and all the other coefficients are 0,
- x the ogf such that the second coefficient is 1 and all the other coefficients are 0,
- e the ogf such that all coefficients are 1.

We can see here why we allow infinite coefficients in the definition of an ogf f. Indeed, we need φ_f to be defined on $\overline{\mathbb{N}}$ and not only on $\mathbb{N}$ in order to avoid the problem of the radius of convergence of the power series. This way, we can compose any two ogf f and g to obtain a new ogf $f \circ g$ without having to impose any condition on the coefficients of f and g. However, we are anticipating a bit here since we have not defined the composition of two ogf yet. Let us recall the natural operations on ogf that we will need to categorify.

Definition 1.1.1.2. Let $f = \sum_{n\in\mathbb{N}} f_n x^n$ and $g = \sum_{n\in\mathbb{N}} g_n x^n$ be two ogf. We define the following operations :

- the *derivation* $f' = \sum_{n\in\mathbb{N}} (n+1) f_{n+1} x^n$,
- the *sum* $f + g = \sum_{n\in\mathbb{N}} (f_n + g_n) x^n$,
- the *product* $f \cdot g = \sum_{n\in\mathbb{N}} \sum_{i=0}^{n} f_i g_{n-i} x^n$,
- the *Hadamard product* $f \odot g = \sum_{n\in\mathbb{N}} f_n g_n x^n$,
- the *composition* $f \circ g = \sum_{n\in\mathbb{N}} f_n g^n$.

All those formulas are straightforward on the level of the coefficients except for the composition. The composition of two power series is a bit more involved, and we will need to use the notion of *composition of an integer* to give a formula on the level of the coefficients. Let us recall the definition of a composition of an integer. The definition we are going to give is slightly different from the usual one since we are allowing 0 to be a part of a composition of an integer.

Definition 1.1.1.3. Let $n, k \in \mathbb{N}$, a *composition* of n of length k is $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k$ such that $\sum_{i=1}^{k} \lambda_i = n$. We denote by $\lambda \vDash^k n$ the fact that λ is a composition of n of length k.

Let us show the following combinatorial lemma on composition that will be useful in the sequel.

Lemma 1.1.1.4. *Let $n, k \in \mathbb{N}$, we denote $\underline{k}$ the set $\{1, \ldots, k\}$. Let us define:*

$$U = \{(m, \gamma, \nu) \mid m \in \mathbb{N}, \gamma \vDash^m n, \nu \vDash^k m\}$$
$$V = \{\left(\lambda, (m_i, \mu^i)_{i\in\underline{k}}\right) \mid \lambda \vDash^k n, \forall i \in \underline{k}, m_i \in \mathbb{N}, \mu^i \vDash^{m_i} \lambda_i\}$$

Then there is a bijection between U and V.

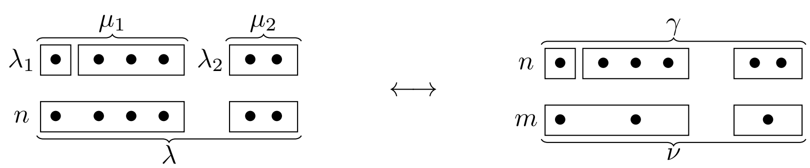
Proof. The idea of the proof is depicted in Figure 1.1. Let us show it in a formal way, so it can be latter categorified. For $(m, \gamma, \nu) \in U$ let us denote $s_i = \sum_{l=1}^{i-1} \nu_l$. We define the following maps:

$$U \to V$$
$$(m, \gamma, \nu) \mapsto \left(\left(\sum_{j=1}^{\nu_i} \gamma_{s_i+j} \right)_{j\in\underline{k}}, \left(\nu_i, (\gamma_{s_i+j})_{j\in\underline{m_i}}\right)_{i\in\underline{k}} \right)$$

$$V \to U$$
$$\left(\lambda, (m_i, \mu^i)_{i\in\underline{k}}\right) \mapsto \left(\sum_{i=1}^{k} m_i, \left(\mu_1^1, \ldots, \mu_{m_1}^1, \ldots, \mu_1^k, \ldots, \mu_{m_k}^k\right), (m_i)_{i\in\underline{k}} \right)$$

One can easily check that those maps are inverse of each other. □

Figure 1.1: Bijection between U and V



Proposition 1.1.1.5. *Let $f = \sum_{n\in\mathbb{N}} f_n x^n$ and $g = \sum_{n\in\mathbb{N}} g_n x^n$ be two ogf, such that g is connected. We have:*

$$f \circ g = \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} f_k \sum_{\lambda\vDash^k n} \prod_{i=1}^{k} g_{\lambda_i} x^n$$

Proof. A straightforward computation is enough to prove this formula:

$$\begin{aligned}
f \circ g &= \sum_{k\in\mathbb{N}} f_k g^k \\
&= \sum_{k\in\mathbb{N}} f_k \left(\sum_{j\in\mathbb{N}} g_j x^j \right)^k \\
&= \sum_{k\in\mathbb{N}} f_k \left(\sum_{(\lambda_1,\ldots,\lambda_k)\in\mathbb{N}^k} \prod_{i=1}^{k} g_{\lambda_i} x^{\lambda_i} \right) \\
&= \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} f_k \sum_{\lambda\vDash^k n} \prod_{i=1}^{k} g_{\lambda_i} x^n
\end{aligned}$$

□

Moreover, we have the following compatibility between the operations defined on ogf, to be precise, we have 19 different relations. To avoid unnecessary parentheses, we use the following order of operations: the composition has the highest priority, then the Hadamard product, then the product, then the sum, and finally the derivative. So for example we have:

$$(f \circ g) \cdot h = f \circ g \cdot h$$

Let us state the 19 different relations:

Proposition 1.1.1.6. *Let f, g and h be three ogf, then the operations $+$, $\cdot$ and $\odot$ are unitary, commutative and associative, and the operation $\circ$ is unitary and associative (we assume g and/or h connected when needed):*

- $f + 0 = f$,
- $f \cdot 1 = f$,
- $f + g = g + f$,
- $f \cdot g = g \cdot f$,
- $(f + g) + h = f + (g + h)$,
- $(f \cdot g) \cdot h = f \cdot (g \cdot h)$,
- $f \odot e = f$,
- $f \circ x = f$,
- $f \odot g = g \odot f$,
- $x \circ f = f$,
- $(f \odot g) \odot h = f \odot (g \odot h)$,
- $(f \circ g) \circ h = f \circ (g \circ h)$,

They also satisfy the following distributivity relations:

- $(f + g) \cdot h = f \cdot h + g \cdot h$,
- $(f + g) \circ h = f \circ h + g \circ h$,
- $(f + g) \odot h = f \odot h + g \odot h$,
- $(f \cdot g) \circ h = f \circ h \cdot g \circ h$,

And they satisfy the following compatibility relations with the derivative:

- $(f + g)' = f' + g'$,
- $(f \cdot g)' = f' \cdot g + f \cdot g'$,
- $(f \circ g)' = f' \circ g \cdot g'$.

Proof. Since the composition is the most involved operation, let us show the associativity of the composition. Let us reuse the notations U and V of Lemma 1.1.1.4 We have:

$$\begin{aligned}
((f \circ g) \circ h)_n &= \sum_{k \in \mathbb{N}} f_k \sum_{\nu \vDash^k m} \prod_{i=1}^{k} g_{\nu_i} \sum_{\gamma \vDash^m n} \prod_{j=1}^{m} h_{\gamma_j} \\
&= \sum_{k \in \mathbb{N}} f_k \sum_{U} \prod_{i=1}^{k} g_{\nu_i} \prod_{j=1}^{m} h_{\gamma_j} \\
&= \sum_{k \in \mathbb{N}} f_k \sum_{V} \prod_{i=1}^{k} g_{m_i} \prod_{j=1}^{m} h_{\mu^i_j} \\
&= \sum_{k \in \mathbb{N}} f_k \sum_{\lambda \vDash^k n} \prod_{i=1}^{k} \sum_{m_i \in \mathbb{N}} g_{m_i} \sum_{\mu^i \vDash^{m_i} \lambda_i} \prod_{j=1}^{m_i} h_{\mu^i_j} \\
&= (f \circ (g \circ h))_n
\end{aligned}$$

Let us show the compatibility relation between the composition and the derivative. Assume that we have shown the compatibility relation between the product and the derivative, then by induction we have $(g^k)' = kg^{k-1}g'$ for any $k \in \mathbb{N}$, and we can compute the n-th coefficient of each side of the equation:

$$\left((g^k)'\right)_n = \left(k(g^{k-1}g')\right)_n$$

$$(n+1) \sum_{\lambda \vDash^k n+1} \prod_{i=1}^{k} g_{\lambda_i} = k \sum_{j=0}^{n} \left(\sum_{\lambda \vDash^{k-1} n-j} \prod_{i=1}^{k-1} g_{\lambda_i} \right) (j+1) g_{j+1}$$

Let us now compute the n-th coefficient of $(f \circ g)'$:

$$\begin{aligned}
(f \circ g)'_n &= (n+1) \sum_{k \in \mathbb{N}} f_k \sum_{\lambda \vDash^k n+1} \prod_{i=1}^{k} g_{\lambda_i} \\
&= \sum_{k \in \mathbb{N}} f_k \left((n+1) \sum_{\lambda \vDash^k n+1} \prod_{i=1}^{k} g_{\lambda_i} \right) \\
&= \sum_{k \in \mathbb{N}} f_k \left(k \sum_{j=0}^{n} \left(\sum_{\lambda \vDash^{k-1} n-j} \prod_{i=1}^{k-1} g_{\lambda_i} \right) (j+1) g_{j+1} \right) \\
&= \sum_{j=0}^{n} \left(\sum_{k \in \mathbb{N}} k f_k \sum_{\lambda \vDash^{k-1} n-j} \prod_{i=1}^{k-1} g_{\lambda_i} \right) (j+1) g_{j+1} \\
&= \sum_{j=0}^{n} \left(\sum_{k \in \mathbb{N}} (k+1) f_{k+1} \sum_{\lambda \vDash^{k} n-j} \prod_{i=1}^{k} g_{\lambda_i} \right) (j+1) g_{j+1} \\
&= (f' \circ g \cdot g')_n
\end{aligned}$$

The other 17 relations are straightforward to prove. We hope that the kind reader will not hold against us the fact that we will not prove them. □

In those proofs, we have only used combinatorial arguments, without having to subtract coefficients nor evaluate the ogf at any point. The 17 relations that we have stated without proving them can be proven the same way, only reasoning on the level of the coefficients, and without any subtraction of coefficients nor evaluation of the ogf at any point. This fact will allow us to categorify the operations on ogf and the compatibility relations between them. Indeed, since we only used sums and products of coefficients, we can replace the coefficients by sets, and the sums and products by disjoint unions and products of sets.

1.1.2 Naive species

Now that we have introduced ogf, we can try to categorify them. We already pointed out that we used $\overline{\mathbb{N}}$ because we wanted the coefficients of the ogf to count something, hence to be the cardinal of some sets. Let us try to directly replace these cardinal of sets by the sets themselves. We will use the category of sets Set and the discrete category $\mathbb{N}$ which is the category with one object for each $n \in \mathbb{N}$ and with only identity morphisms. As we pointed out previously, the fact that we only used sum and products of coefficients in the definitions, the properties, and their proofs, allows us to generalize them to the naive species setting without any problem.

Definition 1.1.2.1. Let $\mathbb{N}$Spe be the functor category $\mathbb{N} \to \text{Set}$. A *naive species* $\mathcal{S}$ is an object of $\mathbb{N}$Spe, it is a functor $\mathcal{S} : \mathbb{N} \to \text{Set}$, the naive species $\mathcal{S}$ is *finite* if each object of its essential image is finite. The naive species $\mathcal{S}$ is *strongly finite* if it is finite, and if we have $N \in \mathbb{N}$ such that $\mathcal{S}(n) = \emptyset$ for each $n > N$. A naive species $\mathcal{S}$ is *connected* if $\mathcal{S}(0) = \emptyset$, it is *strongly connected* if $\mathcal{S}(0) = \mathcal{S}(1) = \emptyset$. A *morphism of naive species* is a morphism in $\mathbb{N}$Spe, it is a natural transformation between two functors $\mathcal{S}, \mathcal{R} : \mathbb{N} \to \text{Set}$. The category $\mathbb{N}$Spe is the *category of naive species*. Let us define the following usual naive species:

- 0 the *trivial species* such that $0(n) = \emptyset$ for each $n \in \mathbb{N}$,
- 1 the *empty set species* such that $1(0) = \{*\}$ and $1(n) = \emptyset$ for each $n \in \mathbb{N}^*$,
- X the *singleton species* such that $X(1) = \{*\}$ and $X(n) = \emptyset$ for each $n \in \mathbb{N} \setminus \{1\}$,
- set the *naive set species* such that $\mathrm{set}(n) = \{*\}$ for each $n \in \mathbb{N}$.

One should not to be afraid of the categorical definition of a naive species. Indeed, a quick inspection of the definition shows that a naive species $\mathcal{S}$ is a very concrete object, it is a sequence of sets $(S_n)_{n\in\mathbb{N}}$, and a morphism of naive species is a sequence of maps $(f_n : S_n \to T_n)_{n\in\mathbb{N}}$. The main idea of giving such a categorical definition is to allow us to modify easily the definition replacing either of the categories Set or $\mathbb{N}$ by another category, and to keep a very similar theory.

Definition 1.1.2.2. Let $\mathcal{S}$ be a naive species. The ogf $f_\mathcal{S}$ associated to $\mathcal{S}$ is the formal power series

$$f_\mathcal{S}(t) = \sum_{n\in\mathbb{N}} |\mathcal{S}(n)| x^n,$$

with $|\mathcal{S}(n)|$ the cardinal (in $\overline{\mathbb{N}}$) of $\mathcal{S}(n)$. Two naive species $\mathcal{S}$ and $\mathcal{R}$ are *equinumerous* if $f_\mathcal{S} = f_\mathcal{R}$.

Proposition 1.1.2.3. *Two isomorphic naive species are equinumerous. The converse is not true in general but is true for finite naive species.*

Proof. It is clear that two isomorphic naive species are equinumerous. Moreover, since finite sets are in bijection if and only if they have the same cardinal (in $\mathbb{N}$), if two finite naive species are equinumerous, then one can find a bijection between the sets of each species, and thus the two species are isomorphic. $\square$

This proposition shows that our categorification of the ogf is probably too naive, since it does not allow us to distinguish between two finite equinumerous naive species. However, let us continue our study of naive species, first as a warm-up, and second because we will need to use the category of naive species at some point. Let us define the following operations on naive species.

Definition 1.1.2.4. Let $\mathcal{S}$ and $\mathcal{R}$ be two naive species. We define the following operations on naive species:

- the *derivative* $\mathcal{S}'$ such that $(\mathcal{S}')(n) = (n+1)\mathcal{S}(n+1) = \mathcal{S}(n+1) \uplus \cdots \uplus \mathcal{S}(n+1)$ for each $n \in \mathbb{N}$,
- the *sum* $\mathcal{S} + \mathcal{R}$ such that $(\mathcal{S} + \mathcal{R})(n) = \mathcal{S}(n) \uplus \mathcal{R}(n)$ for each $n \in \mathbb{N}$,
- the *product* (or Cauchy product) $\mathcal{S} \cdot \mathcal{R}$ such that $(\mathcal{S} \cdot \mathcal{R})(n) = \biguplus_{i=0}^{n} \mathcal{S}(n-i) \times \mathcal{R}(i)$ for each $n \in \mathbb{N}$,
- the *Hadamard product* $\mathcal{S} \odot \mathcal{R}$ such that $(\mathcal{S} \odot \mathcal{R})(n) = \mathcal{S}(n) \times \mathcal{R}(n)$ for each $n \in \mathbb{N}$,
- if $\mathcal{R}$ is connected, the *plethysm* (or composition product) $\mathcal{S} \circ \mathcal{R}$ such that $(\mathcal{S} \circ \mathcal{R})(n) = \biguplus_{k\in\mathbb{N}} \mathcal{S}(k) \times \biguplus_{\lambda \vDash^k n} \prod_{i=1}^{k} \mathcal{R}(\lambda_i)$ for each $n \in \mathbb{N}$.

Here we use the notations $\uplus$ for the disjoint union of sets (the coproduct in Set), $\times$ for the cartesian product of sets, and nA for $A \uplus \cdots \uplus A$ with n copies of A.

Proposition 1.1.2.5. *We have* $f_{\mathcal{S}'} = (f_\mathcal{S})'$. *Same for the sum* $f_{\mathcal{S}+\mathcal{R}} = f_\mathcal{S} + f_\mathcal{R}$, *the product* $f_{\mathcal{S}\cdot\mathcal{R}} = f_\mathcal{S} \cdot f_\mathcal{R}$, *the Hadamard product* $f_{\mathcal{S}\odot\mathcal{R}} = f_\mathcal{S} \odot f_\mathcal{R}$ *and the plethysm* $f_{\mathcal{S}\circ\mathcal{R}} = f_\mathcal{S} \circ f_\mathcal{R}$.

Proof. Since $|A \uplus B| = |A| + |B|$ and $|A \times B| = |A| \cdot |B|$, this follows directly from the definition. $\square$

We also have same 19 compatibility relations between the operations defined on naive species as we had for ogf however, since we are now in the categorical framework, we do not get *relations* but *isomorphisms*. Let us state them:

Proposition 1.1.2.6. *Let $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{U}$ be three naive species, then the operations $+$, $\cdot$ and $\odot$ are unitary symmetric monoidal structures, and the operation $\circ$ is unitary monoidal structure:*

- $\mathcal{S} + 0 \simeq \mathcal{S}$,
- $\mathcal{S} + \mathcal{R} \simeq \mathcal{R} + \mathcal{S}$,
- $(\mathcal{S} + \mathcal{R}) + \mathcal{U} \simeq \mathcal{S} + (\mathcal{R} + \mathcal{U})$,
- $\mathcal{S} \cdot 1 \simeq \mathcal{S}$,
- $\mathcal{S} \cdot \mathcal{R} \simeq \mathcal{R} \cdot \mathcal{S}$,
- $(\mathcal{S} \cdot \mathcal{R}) \cdot \mathcal{U} \simeq \mathcal{S} \cdot (\mathcal{R} \cdot \mathcal{U})$,
- $\mathcal{S} \odot \mathrm{set} \simeq \mathcal{S}$,
- $\mathcal{S} \odot \mathcal{R} \simeq \mathcal{R} \odot \mathcal{S}$,
- $(\mathcal{S} \odot \mathcal{R}) \odot \mathcal{U} \simeq \mathcal{S} \odot (\mathcal{R} \odot \mathcal{U})$,
- $\mathcal{S} \circ X \simeq \mathcal{S}$,
- $X \circ \mathcal{S} \simeq \mathcal{S}$,
- $(\mathcal{S} \circ \mathcal{R}) \circ \mathcal{U} \simeq \mathcal{S} \circ (\mathcal{R} \circ \mathcal{U})$,

We also have the following distributivity morphism:

- $(\mathcal{S} + \mathcal{R}) \cdot \mathcal{U} \simeq \mathcal{S} \cdot \mathcal{U} + \mathcal{R} \cdot \mathcal{U}$,
- $(\mathcal{S} + \mathcal{R}) \odot \mathcal{U} \simeq \mathcal{S} \odot \mathcal{U} + \mathcal{R} \odot \mathcal{U}$,
- $(\mathcal{S} + \mathcal{R}) \circ \mathcal{U} \simeq \mathcal{S} \circ \mathcal{U} + \mathcal{R} \circ \mathcal{U}$,
- $(\mathcal{S} \cdot \mathcal{R}) \circ \mathcal{U} \simeq \mathcal{S} \circ \mathcal{U} \cdot \mathcal{R} \circ \mathcal{U}$,

And the following morphisms when composing with the derivative:

- $(\mathcal{S} + \mathcal{R})' \simeq \mathcal{S}' + \mathcal{R}'$,
- $(\mathcal{S} \cdot \mathcal{R})' \simeq \mathcal{S}' \cdot \mathcal{R} + \mathcal{S} \cdot \mathcal{R}'$,
- $(\mathcal{S} \circ \mathcal{R})' \simeq \mathcal{S}' \circ \mathcal{R} \cdot \mathcal{R}'$.

Moreover, these isomorphisms are coherent with each other, meaning that "any diagram involving only these isomorphisms commutes". Because of this, we will write them as equality in the sequel.

Coherence conditions The situation is slightly more involved than the slogan "any diagram involving only these isomorphisms commutes", indeed, $\tau : A \times A \to A \times A$ the permutation of the two coordinates is almost never the identity. A better setting to state coherence conditions is the following one: Let us understand the coherence conditions between the cartesian product $\times$ and the disjoint union $\uplus$ in the category Set. First let consider the cartesian product $\times$ and the disjoint union $\uplus$ as functors, for example we have $\uplus : \mathrm{Set} \times \mathrm{Set} \to \mathrm{Set}$ where $\mathrm{Set} \times \mathrm{Set}$ is the category of pairs of sets. Then the isomorphisms we impose (associativity of $\times$, distributivity over $\uplus$ ...) are natural isomorphisms between functors, and the coherence conditions state that any two chain of compositions of such natural isomorphisms that have the same source and target should be equal. This is a quite technical definition, and the discerning reader will see ∞-category theory rushing in. Although coherence conditions are very important in category theory, we will leave it as it is, and dodge the appearance of ∞-categories. The author want to apologize to the interested reader, as the author do like ∞-category theory, however, the author already fell down too many rabbit holes while writing this manuscript, and exploring coherence conditions through ∞-category theory would be one too many. Let us go with "any diagram involving only these isomorphisms commutes" and beware that it is not the whole story.

Proof. Replacing the sums by disjoint unions and the cartesian products by products in the proof of Proposition 1.1.1.6 is enough to get the isomorphisms. The compatibility between the isomorphisms is a direct consequence of the fact that the disjoint union and the product are compatible with each other (i.e. that Set is a rig-category, see Definition 1.2.1.3). □

Since sets a closed by products and disjoint unions, we can define a functor Set → Set associated to a naive species in a similar fashion as we can associate a function to a formal power series. Moreover, in this categorical setting, the issue arising from the radius of convergence of formal power series disappears.

Definition 1.1.2.7. Let $\mathcal{S}$ be a naive species. Let $\mathcal{F}_{\mathcal{S}} : \mathrm{Set} \to \mathrm{Set}$ be the Schur functor of $\mathcal{S}$ defined by $\mathcal{F}_{\mathcal{S}}(A) = \biguplus_{n\in\mathbb{N}} \mathcal{S}(\underline{n}) \times A^n$ for each set A, and $\mathcal{F}_{\mathcal{S}}(\varphi) = \biguplus_{n\in\mathbb{N}} \mathrm{id}_{\mathcal{S}(\underline{n})} \times \varphi^n$ for each function $\varphi : A \to B$.

Proposition 1.1.2.8. *Let $\mathcal{S}$ and $\mathcal{R}$ be two naive species. We have $\mathcal{F}_{\mathcal{S}\circ\mathcal{R}} = \mathcal{F}_{\mathcal{S}} \circ \mathcal{F}_{\mathcal{R}}$.*

Proof. We have:

$$\begin{aligned}\mathcal{F}_{\mathcal{S}\circ\mathcal{R}}(A) &= \biguplus_{k\in\mathbb{N}} \mathcal{S}(\underline{k}) \times \left(\biguplus_{j\in\mathbb{N}} \mathcal{R}(\underline{j}) \times A^j\right)^k \\ &= \biguplus_{n\in\mathbb{N}} \biguplus_{k\in\mathbb{N}} \mathcal{S}(\underline{k}) \times \biguplus_{\lambda\vDash^k n} \prod_{i=1}^{k} \mathcal{R}(\underline{\lambda_i}) \times A^n \\ &= (\mathcal{F}_{\mathcal{S}} \circ \mathcal{F}_{\mathcal{R}})(A)\end{aligned}$$

Same for $\mathcal{F}_{\mathcal{S}\circ\mathcal{R}}(\varphi)$ with $\varphi : A \to B$. □

1.1.3 Multi-variate case

We have categorified the ogf, and we have defined the naive species. However, we will need multi-variate case in the sequel. Because there are no major difficulties to go from the univariate and the multi-variate case, we will only discuss the three new operations that appear in the multi-variate case. namely, the partial derivative, the partial composition and the identification of variables which is not as trivial as it may seem. First, let us define the multi-variate ogf. Let us fix the following notations: Let $k \in \mathbb{N}$ and $x_1, \dots, x_k$ be k variables. We denote $x = (x_1, \dots, x_k)$ and for $n = (n_1, \dots, n_k) \in \mathbb{N}^k$ we denote $x^n = x_1^{n_1} \dots x_k^{n_k}$, the same way, if $i = (i_1, \dots, i_k) \in \overline{\mathbb{N}}^k$, we denote $i^n = i_1^{n_1} \dots i_k^{n_k}$. We will denote $y = (y_1, \dots, y_l)$ another tuple of formal variables, and z a single formal variable when needed.

Definition 1.1.3.1. A *k-variate ogf*, k-ogf for short, is a formal power series in k variables with coefficients in $\overline{\mathbb{N}}$. Since a power series is determined by its coefficients, we can identify the set of k-ogf with the set of functions $\{\mathbb{N}^k \to \overline{\mathbb{N}}\}$. Hence, an ogf is uniquely determined by a function $s : \mathbb{N}^k \to \overline{\mathbb{N}}$. A k-ogf f is *finite* if f_n is finite for all $n \in \mathbb{N}^k$. A k-ogf f is *connected* if $f_0 = f_{(0,\dots,0)} = 0$.

The sum, the product and the Hadamard product of k-ogf are defined as for ogf. Let us define the identification of variables, the partial derivative and the partial composition for k-ogf.

Definition 1.1.3.2. Let

$$f(x, y) = f(x_1, \dots, x_k, y_1, \dots, y_l) = \sum_{(n,m)\in\mathbb{N}^{k+l}} f_{(n,m)} x^n y^m$$

be a $(k+l)$-ogf, then we can identify the variables $y_1, \dots, y_l$ to a single variable z and get the $k+1$-ogf $f(x_1, \dots, x_k, y = z)$ defined by:

$$f(x, y = z) = f(x_1, \dots, x_k, z, \dots, z) = \sum_{n\in\mathbb{N}^k} \sum_{m\in\mathbb{N}^l} f_{(n,m)} x^n z^{\sum_{i=1}^{l} m_i}$$

Definition 1.1.3.3. Let $f(x) = f(x_1, \dots, x_k) = \sum_{n\in\mathbb{N}^k} f_n x^n$ be a k-ogf, then we can define the *partial derivative* of f with respect to the variable x_i for $i \in \underline{k}$, and get the k-ogf $\frac{\partial f(x)}{\partial x_i}$ defined by:

$$\frac{\partial f(x)}{\partial x_i} = \sum_{n\in\mathbb{N}^k} n_i f_n x_1^{n_1} \dots x_i^{n_i - 1} \dots x_k^{n_k}$$

Definition 1.1.3.4. Let $f(x) = f(x_1, \dots, x_k) = \sum_{n\in\mathbb{N}^k} f_n x^n$ be a k-ogf, and $g_1, \dots, g_k$ be respectively l_1-ogf up to l_k-ogf, then we can define $f(g)$ the composition of f with g_1 up to g_k. The $(l_1 + \dots + l_k)$-ogf $f(g)$ is defined by:

$$f(g) = f \circ (g_1, \dots, g_k) = \sum_{n\in\mathbb{N}^k} f_n g_1^{n_1} \dots g_k^{n_k}$$

Let $i \in \underline{k}$ and $h = g_i$, then if $g_j = x_j$ for $j \neq i$ then we write $f \circ_i h$ instead of $f(g)$, and we call it the *partial composition* of f with h in i.
Here one needs to be careful when composing ogf. Indeed, to get $(l_1 + \dots + l_k)$ different variables, we need to assume that the sets of variables of the g_i are disjoint. To compose ogf with sets of variables that are not disjoint, we need to rename the variables to make them disjoint, then compose the multi-variate ogf, and finally identify the variables that need to be identified. In the case of ogf, nothing subtle happens, however we rather address this issue here while it is still trivial.

The partial derivative and the partial composition satisfy similar properties as the derivative and the composition for ogf. Let us only recall the chain rule. First we need to show the compatibility relation between the partial derivative and the identification of variables.

Proposition 1.1.3.5. *Let $f(x,y) = \sum_{(n,m)\in\mathbb{N}^{k+l}} f_{(n,m)} x^n y^m$ be a $(k+l)$-ogf and z a formal variable. Then:*

$$\frac{\partial f(x, y = z)}{\partial z} = \sum_{i=1}^{l} \frac{\partial f}{\partial y_i}(x, y = z)$$

Proof. This is a straightforward computation:

$$\begin{aligned}
\frac{\partial f(x, y = z)}{\partial z} &= \sum_{n\in\mathbb{N}^k} \sum_{m\in\mathbb{N}^l} \left(\sum_{i=1}^{l} m_i \right) f_{(n,m)} x^n z^{\left(\sum_{i=1}^{l} m_i\right)-1} \\
&= \sum_{i=1}^{l} \sum_{n\in\mathbb{N}^k} \sum_{m\in\mathbb{N}^l} m_i f_{(n,m)} x^n z^{m_1} \dots z^{m_i-1} \dots z^{m_l} \\
&= \left(\sum_{i=1}^{l} \sum_{n\in\mathbb{N}^k} \sum_{m\in\mathbb{N}^l} m_i f_{(n,m)} x^n y_1^{m_1} \dots y_i^{m_i-1} \dots y_l^{m_l} \right) (x, y = z) \\
&= \sum_{i=1}^{l} \frac{\partial f}{\partial y_i}(x, y = z)
\end{aligned}$$

□

Proposition 1.1.3.6. *Let f be a k-ogf and $g_1, \dots, g_k$ be l_1 up to l_k-ogf. Then:*

$$\frac{\partial(f(g))}{\partial x_i} = \sum_{j=1}^{k} \left(\frac{\partial f}{\partial y_j} \right) (g) \frac{\partial g_j}{\partial x_i}$$

Proof. This is the chain rule which is a more elaborate version of the case with $k = 1$ that we have already proven in Proposition 1.1.1.6. First let us assume that the variables of the g_j are disjoint, and let g_j be the only one where x_i appears. Then from the same argument as in Proposition 1.1.1.6 we have:

$$\frac{\partial(f(g))}{\partial x_i} = \frac{\partial f}{\partial y_j}(g) \frac{\partial g_j}{\partial x_i}$$

If the variables of the g_j are not disjoint, then we can always rename the variables to make them disjoint, then compose the multi-variate ogf, and finally identify the variables that need to be identified. The compatibility between the partial derivative and the identification of variables allow us to conclude. □

We can now define the naive multi-sort species and prove these two propositions. Let $X = (X_1, \dots, X_k)$ be a k-tuple of formal variables, $Y = (Y_1, \dots, Y_l)$ be an l-tuple of formal variables and Z a single formal variable. We use the same notations as for k-ogf, except that variables are capitalized in the case of multi-sort species.

Definition 1.1.3.7. Let us consider $\mathbb{N}^k$ as the category with one object for each $n \in \mathbb{N}^k$ and with only identity morphisms. Let $\mathbb{N}^k$Spe be the functor category $\mathbb{N}^k \to$ Set. A *naive k-sort species* $\mathcal{S}$ is an object of $\mathbb{N}^k$Spe, it is a functor $\mathcal{S} : \mathbb{N}^k \to$ Set, the naive k-sort species $\mathcal{S}$ is *finite* if each object of its essential image is finite. A naive k-sort species $\mathcal{S}$ is *connected* if $\mathcal{S}(0) = \mathcal{S}(0, \dots, 0) = \emptyset$. A *morphism of naive k-sort species* is a morphism in $\mathbb{N}^k$Spe, it is a natural transformation between two functors $\mathcal{S}, \mathcal{R} : \mathbb{N}^k \to$ Set. The category $\mathbb{N}^k$Spe is the *category of naive k-sort species*. If needed, we will specify the formal variables of the naive k-sort species $\mathcal{S}$ in brackets and write $\mathcal{S}[X] = \mathcal{S}[X_1, \dots, X_k]$ in order to be able to identify, derive or compose in a specific variable. One needs to be careful as it is not the same convention as in [7].

Let us define the identification of variables, the partial derivative and the partial composition for naive k-sort species.

Definition 1.1.3.8. Let $\mathcal{S}[X,Y] = \mathcal{S}[X_1, \dots, X_k, Y_1, \dots, Y_l]$ be a $(k+l)$-sort naive species, then we can identify the variables $Y_1, \dots, Y_l$ to a single variable Z and get the $k+1$-sort naive species $\mathcal{S}[X_1, \dots, X_k, Y = Z]$ defined by:

$$\mathcal{S}[X, Y = Z](n, m) = \mathcal{S}[X_1, \dots, X_k, Z, \dots, Z](n, m) = \biguplus_{\lambda \vDash^l m} \mathcal{S}[X, Y](n, \lambda),$$

where $n \in \mathbb{N}^k$ and $m \in \mathbb{N}$.

Definition 1.1.3.9. Let $\mathcal{S}[X] = \mathcal{S}[X_1, \dots, X_k]$ be a naive k-sort species, then we can define the *partial derivative* of $\mathcal{S}$ with respect to the variable X_i for $i \in \underline{k}$, and get the naive k-sort species $\frac{\partial \mathcal{S}[X]}{\partial X_i}$ defined by:

$$\frac{\partial \mathcal{S}[X]}{\partial X_i}(n) = (n_i + 1)\mathcal{S}[X](n_1, \dots, n_i + 1, \dots, n_k)$$

Definition 1.1.3.10. Let $\mathcal{S}[X] = \mathcal{S}[X_1, \dots, X_k]$ be a naive k-sort species, and $\mathcal{R}_1, \dots, \mathcal{R}_k$ be l_1 up to l_k-sort naive species, then we can define the composition of $\mathcal{S}[\mathcal{R}]$ an $(l_1 + \cdots + l_k)$-sort naive species. Let $n \in \mathbb{N}^{l_1} \times \cdots \times \mathbb{N}^{l_k}$, then:

$$\mathcal{S}[\mathcal{R}](n) = \mathcal{S} \circ (\mathcal{R}_1, \dots, \mathcal{R}_k)(n) = \biguplus_{m \in \mathbb{N}^k} \mathcal{S}(m) \times \biguplus_{\lambda \vDash^m n} \prod_{i=1}^{k} \prod_{j=1}^{m_i} \mathcal{R}_i(\lambda_j^{(i)})$$

Here we denote $\lambda \vDash^m n$ to ease the notation, we should have written $(\lambda^{(i)} \vDash^{m_i} n_i)_{i=1}^k$ where n_i is an l_i-tuple and $\lambda^{(i)}$ is a length m_i composition of an l_i-tuple. A length m_i composition of an l_i-tuple is an m_i-tuple of l_i-tuples such that the sum of the l_i-tuples (coefficient by coefficient) is equal to the l_i-tuple n_i.
Let $i \in \underline{k}$ and $\mathcal{U} = \mathcal{R}_i$, then if $\mathcal{R}_j = X_j$ for $j \neq i$ then we write $\mathcal{S} \circ_i \mathcal{U}$ instead of $\mathcal{S}[\mathcal{R}]$, and we call it the *partial composition* of $\mathcal{S}$ with $\mathcal{U}$ in i.
Same as with ogf, one need to be careful when composing. Indeed, to get $(l_1 + \cdots + l_k)$ different variables, we need to assume that the variables of the $\mathcal{R}_i$ are disjoint. When it is not the case, we can always rename the variables to make them disjoint, then compose the multi-sort naive species, and finally identify the variables that need to be identified.

We can now show the compatibility relation between the partial derivative and the identification of variables, and the chain rule for naive k-sort species.

Proposition 1.1.3.11. *Let $\mathcal{S}[X,Y]$ be a $(k+l)$-sort naive species and Z a formal variable. Then:*

$$\frac{\partial \mathcal{S}[X, Y = Z]}{\partial Z} = \sum_{i=1}^{l} \frac{\partial \mathcal{S}}{\partial Y_i}(X, Y = Z)$$

Proof. This is the same straightforward computation as with ogf but on the level of coefficients:

$$\begin{aligned}\frac{\partial \mathcal{S}[X, Y=Z]}{\partial Z}(n,m) &= (m+1)\mathcal{S}[X,Y=Z](n,m+1)\\ &= (m+1)\sum_{\lambda \vDash^l (m+1)} \mathcal{S}[X,Y](n,\lambda)\\ &= \sum_{\lambda \vDash^l (m+1)} \sum_{i=1}^{l} \lambda_i \mathcal{S}[X,Y](n,\lambda)\\ &= \sum_{i=1}^{l} \sum_{\lambda \vDash^l m} (\lambda_i+1)\mathcal{S}[X,Y](n,\lambda_1,\dots,\lambda_i+1,\dots,\lambda_l)\\ &= \sum_{i=1}^{l} \frac{\partial \mathcal{S}}{\partial Y_i}(X, Y=Z)\end{aligned}$$

□

Proposition 1.1.3.12. *Let $\mathcal{S}$ be a naive k-sort species and $R_1,\dots,R_k$ be l_1 up to l_k-sort naive species. Then:*

$$\frac{\partial(\mathcal{S}[\mathcal{R}])}{\partial X_i} = \sum_{j=1}^{k} \left(\frac{\partial \mathcal{S}}{\partial Y_j}\right)[\mathcal{R}]\frac{\partial R_j}{\partial X_i}$$

Proof. Same as with ogf, if we assume that the variables of the g_j are disjoint, and let g_j be the only one where x_i appears. Then from the same argument as in Proposition 1.1.2.6 we have:

$$\frac{\partial(\mathcal{S}[\mathcal{R}])}{\partial X_i} = \frac{\partial \mathcal{S}}{\partial Y_j}[\mathcal{R}]\frac{\partial Y_j}{\partial X_i}$$

If the variables of the R_j are not disjoint, then we can always rename the variables to make them disjoint, then compose the multi-sort naive species, and finally identify the variables that need to be identified. The compatibility between the partial derivative and the identification of variables allow us to conclude. □

The main reason to define multi-sort species is the implicit species theorem. Let us recall the implicit function theorem:

Theorem 1.1.3.13 (Implicit function theorem (fixed point version))**.** *Let h be a continuous differentiable function in two variables x and y defined in a neighborhood of $(0,0)$ such that:*

$$h(0,0) = 0 \qquad \textit{and} \qquad \frac{\partial h}{\partial y}(0,0) = 0$$

Then there exists a neighborhood of 0 and a unique differentiable function a such that:

$$a(x) = h(x, a(x)) \qquad \textit{and} \qquad a(0) = 0$$

The implicit species theorem is an analogous theorem on species

Theorem 1.1.3.14 (Implicit species theorem (naive species version))**.** *Let $H[X,Y]$ be a 2-sort naive species in two variables X and Y such that:*

$$H(0,0) = \emptyset \qquad \textit{and} \qquad \frac{\partial H}{\partial Y}(0,0) = \emptyset$$

Then there exists a unique naive species $\mathcal{A}$ such that:

$$\mathcal{A}[X] = H[X, \mathcal{A}[X]] \qquad \textit{and} \qquad \mathcal{A}(0) = \emptyset$$

Proof. First, let us notice that we have $H[X,Y](0,1) = \emptyset$ because of the second condition. Let us now define the naive species $\mathcal{A}$ by induction. We define $\mathcal{A}(0) = \emptyset$. Then we define $\mathcal{A}(n)$ for $n \in \mathbb{N}$ by induction. Let $n \in \mathbb{N}$, we assume that $\mathcal{A}(m)$ is defined for each $m \in \mathbb{N}$ such that $m < n$. We know that we should have:

$$\begin{aligned}\mathcal{A}(n) &= H[X,\mathcal{A}[X]](n)\\ &= \biguplus_{i\in\mathbb{N}} H[X,\mathcal{A}[Y]](i,n-i)\\ &= \biguplus_{i\in\mathbb{N}}\biguplus_{k\in\mathbb{N}} H[X,Y](i,k) \times \biguplus_{\lambda \vDash^k (n-i)} \prod_{j=1}^{k} \mathcal{A}(\lambda_j)\end{aligned}$$

We may notice that $\mathcal{A}(n)$ only appears when $i = 0$ and $k = 1$, however since $H[X,Y](0,1) = \emptyset$, the term where $\mathcal{A}(n)$ appears cancels. Hence, we can define:

$$\mathcal{A}(n) = \biguplus_{i\in\mathbb{N}}\biguplus_{k\in\mathbb{N}} H[X,Y](i,k) \times \biguplus_{\lambda \vDash^k (n-i)} \prod_{j=1}^{k} \mathcal{A}(\lambda_j)$$

We have defined A by induction, such that $\mathcal{A}(0) = \emptyset$ and $\mathcal{A}(n) = H[X,\mathcal{A}[X]](n)$. Moreover, we can see from the induction that $\mathcal{A}$ is unique. □

This theorem is the main tool that we will be using to define new species. However, naive species are not the objects we will be working with. It is time to explain why we should change our definition of species, and how we will do it.

1.2 Species

We have successfully categorified ogf. However, we have shown that two finite naive species are equal if and only if their ogf are equal. So we have not gained much by categorifying ogf. Another issue is that a lot of interesting power series are not ogf, for example the exponential function. Moreover, when categorifying ogf, we only replaced $\overline{\mathbb{N}}$ by its categorical analog Set and not $\mathbb{N}$, which seems to an arbitrary choice. Naive species are a very natural mathematical object as they are sequences of sets, however, this is not what we were hoping for. To fix this, let us try to find the categorical analog of $\mathbb{N}$, that will hopefully allow us to define a more appealing notion of species. The discerning reader may guess that we will succeed since the theory of species does exist, and they would be right.

1.2.1 Categorical analog of $\mathbb{N}$ and definition of a species

The set $\mathbb{N}$ viewed as a category is a very simple category. Indeed, it is a set, we do not have any morphisms except the identity morphisms. Then, what are the distinctive features of $\mathbb{N}$ that we used? Why did we use $\mathbb{N}$ and not any other set? Well the most naive thing that we can say about integers is that we can add and multiply them together. Let us make this rather trivial observation into a definition.

Definition 1.2.1.1. A (unitary) *rig* is a quintuple $(A, 0, 1, +, \cdot)$ such that $(A, 0, +)$ is a unitary commutative monoid, $(A, 1, \cdot)$ is a unitary monoid, and for any $a, b, c \in A$ we have:

- $(a + b) \cdot c = a \cdot c + b \cdot c$,
- $a \cdot (b + c) = a \cdot b + a \cdot c$,
- $0 \cdot a = a \cdot 0 = 0$.

A rig is *commutative* if the multiplication is commutative.

The rather strange name of this algebraic structure "rig" is a play on words: a rig is a ri*n*g without the *n*egative elements. This definition allows us to characterize $\mathbb{N}$ in a very categorical way, that competes very hard with some other characterizations of $\mathbb{N}$ to be the worse possible definition of $\mathbb{N}$. However, it is quite useful in our context.

Proposition 1.2.1.2. *The set $\mathbb{N}$ (with the usual addition and multiplication) is a commutative rig. Moreover, it is the initial object in the category of commutative rigs.*

Let us categorify the definition of a rig:

Definition 1.2.1.3. A *rig-category* is a quintuple $(\mathcal{C}, 0, 1, \oplus, \otimes)$ such that $(\mathcal{C}, 0, \oplus)$ is a symmetric monoidal category, $(\mathcal{C}, 1, \otimes)$ is a monoidal category, and for any $A, B, C \in \mathcal{C}$ we have:

- $(A \oplus B) \otimes C \simeq A \otimes C \oplus B \otimes C$,
- $A \otimes (B \oplus C) \simeq A \otimes B \oplus A \otimes C$,
- $0 \otimes A \simeq A \otimes 0 \simeq 0$.

Moreover, those isomorphisms should be coherent with each other, meaning that "any diagram involving only these isomorphisms commutes".
A *symmetric rig-category* is a rig-category such that the monoidal structure $(\mathcal{C}, 1, \otimes)$ is symmetric, one need to be careful as it is not a property but an additional structure.

The coherence conditions of a rig-category are quite technical, see [48] and [51]. We will not state them and refer the reader to Paragraph 1.1.2 for a more detailed discussion on coherence conditions. We have already used the notion of rig-category in the previous chapter. Indeed, the category of sets Set is a symmetric rig-category with the disjoint union and the cartesian product. It is from this fact that we were able to show that the compatibility relations stated in Proposition 1.1.2.6 are coherent with each other. We can now guess a categorical analog of $\mathbb{N}$. Let $\mathbb{B}$ be the category of finite sets such that the morphisms are the bijections.

Theorem 1.2.1.4 (Baez' conjecture)**.** *For any symmetric rig-category $\mathcal{C}$, there exists a unique symmetric rig functor $F : \mathbb{B} \to \mathcal{C}$ up to a unique natural isomorphism.*

Informally, this theorem states that the category $\mathbb{B}$ is the 2-initial object in the 2-category of symmetric rig-categories. One can make the remark that we have not defined the notion of 2-category, indeed, the definition of a 2-category is quite technical, and we will not need it in the sequel. Moreover, the 2-initial object is a notion that is not used in the sequel, and we will not need it either. We refer to [33] for a proof of this result. The main point of this theorem is that $\mathbb{B}$ is the categorical analog of $\mathbb{N}$ that we want to consider. With this in mind, we can now define the notion of species.

Definition 1.2.1.5. Let Spe be the functor category $\mathbb{B} \to \text{Set}$. A *species* $\mathcal{S}$ is an object of Spe, it is a functor $\mathcal{S} : \mathbb{B} \to \text{Set}$, the species $\mathcal{S}$ is *finite* if each object of its essential image is finite. The species $\mathcal{S}$ is *strongly finite* if it is finite, and if we have $N \in \mathbb{N}$ such that $\mathcal{S}(\underline{n}) = \emptyset$ for each $n > N$. A species $\mathcal{S}$ is *connected* if $\mathcal{S}(\underline{0}) = \emptyset$, it is *strongly connected* if $\mathcal{S}(\underline{0}) = \mathcal{S}(\underline{1}) = \emptyset$. A *morphism of species* is a morphism in Spe, it is a natural transformation between two functors $\mathcal{S}, \mathcal{R} : \mathbb{B} \to \text{Set}$. The category Spe is the *category of species*. Let us define the following usual naive species:

- 0 the *trivial species* such that $0(A) = \emptyset$ for each $A \in \mathbb{B}$,
- 1 the *empty set species* such that $1(\emptyset) = \{*\}$ and $1(A) = \emptyset$ for each $A \neq \emptyset$,
- X the *singleton species* such that $X(A) = \{*\}$ if $|A| = 1$ and $X(A) = \emptyset$ if $|A| \neq 1$,
- E the *set species* such that $E(A) = \{*\}$ for each $A \in \mathbb{B}$,
- $E_{\geq k}$ the *at least k elements set species* such that $E_{\geq k}(A) = \{*\}$ if $|A| \geq k$ and $E_{\geq k}(A) = \emptyset$ if $|A| < k$,

- $\mathbb{E}$ the *element species* such that $\mathbb{E}(A) = A$ for each $A \in \mathbb{B}$,
- $\mathbb{L}$ the *total order species* such that $\mathbb{L}(A) = \mathrm{Bij}(\underline{n}, A)$ and $\mathbb{L}(\sigma) : f \mapsto f \circ \sigma$ for each $A, B \in \mathbb{B}$ and $\sigma : A \to B$ a bijection,
- $\mathbb{L}_{\geq 1}$ the *total order on non-empty sets species* such that $\mathbb{L}_{\geq 1}(\emptyset) = \emptyset$ and $\mathbb{L}_{\geq 1}(A) = \mathrm{Bij}(\underline{n}, A)$ for $A \neq \emptyset$, with $\mathbb{L}_{\geq 1}(\sigma) : f \mapsto f \circ \sigma$ for each $A, B \in \mathbb{B}$ and $\sigma : A \to B$ a bijection.

The name species come from "species of structure", one can think about a species $\mathcal{S}$ as some structure that can be put on a finite set A, and $\mathcal{S}(A)$ is the set of all possible $\mathcal{S}$-structures on A. Hence, the only "empty set structure" that can be put on a set A is A itself if A is empty, or none if A is non-empty. The only "singleton structure" that can be put on a set A is A itself if A is a singleton, or none if A is not a singleton. Same, the only "set structure" that can be put on a set A is A itself. The "element structure" that can be put on a set A are the elements of A. Finally, the "order structures" (that should be understood as linear order) are the linear orders of A. The zero species 0 is the "impossible structure" such that no finite set can have a 0-structure.

Same as with naive species, let us give a down to earth definition of species and morphism of species.

Definition 1.2.1.6. A *symmetric sequence of sets* is a sequence of sets $(S_n)_{n \in \mathbb{N}}$ together with an action of the symmetric group $\mathfrak{S}_n$ on S_n for each $n \in \mathbb{N}$. A *morphism of symmetric sequences of sets* is a sequence of maps $(f_n)_{n \in \mathbb{N}}$ such that $f_n : S_n \to T_n$ is an $\mathfrak{S}_n$-equivariant map for each $n \in \mathbb{N}$. The category of symmetric sequences of sets is denoted by $\mathfrak{S}_*\mathrm{Set}$.

Proposition 1.2.1.7. *There is an equivalence of categories* $\mathrm{Spe} \simeq \mathfrak{S}_*\mathrm{Set}$.

Proof. Let $\mathbb{S}$ be the groupoid such that the objects are the sets $\underline{n} = \{1, \ldots, n\}$ for $n \in \mathbb{N}$ and $\mathrm{Hom}_{\mathbb{S}}(\underline{n}, \underline{m}) = \mathfrak{S}_n$ if $n = m$ and $\mathrm{Hom}_{\mathbb{S}}(\underline{n}, \underline{m}) = \emptyset$ otherwise. The category $\mathfrak{S}_*\mathrm{Set}$ is exactly the category of functors $\mathbb{S} \to \mathrm{Set}$. Moreover, $\mathbb{S}$ is the skeleton of $\mathbb{B}$ and thus is equivalent to $\mathbb{B}$. Hence, $\mathfrak{S}_*\mathrm{Set}$ is equivalent to Spe, moreover the equivalence is given by the inclusion of $\mathbb{S}$ in $\mathbb{B}$. □

This proposition shows that a species $\mathcal{S}$ is entirely determined by the symmetric sequence $S = (\mathcal{S}(\underline{n}))_{n \in \mathbb{N}}$ of its values on the sets $\underline{n} = \{1, \ldots, n\}$ for $n \in \mathbb{N}$. And moreover, allows us to easily switch between species and symmetric sequences.

1.2.2 Operations on species

We now would like to define the operations on species like we did on naive species, we would not have any issue to do so for the derivative, the sum, the product and the Hadamard product since it clear that the analog of the sum and the product in $\mathbb{N}$ are respectively the disjoint union and the cartesian product in $\mathbb{B}$. However, we have an issue to define the plethysm. Indeed, it is not clear what is the analog of the composition of integers in $\mathbb{B}$. We can start by noticing the following fact about $\mathbb{N}$ and $\mathbb{B}$. The category $\mathbb{B}$ is the maximal sub-groupoid of the category of finite sets with maps as morphisms, and the category $\mathbb{N}$ is the maximal sub-groupoid of the category of finite ordered sets with increasing maps as morphisms. Moreover, the data of $\lambda \vDash^k n$ is exactly the same as the data of an increasing map $f : \underline{k} \to \underline{n}$. This hint the fact that the analog of a composition of integers in $\mathbb{B}$ is a partition of a set.

Definition 1.2.2.1. Let A be a finite set and $k \in \mathbb{N}$. A *partition of A of size k* is a set $P = \{P_1, \ldots, P_k\}$ of subsets of A such that $\bigsqcup_{i=1}^{k} P_i = A$. We denote $P \vdash^k A$ the fact that P is a partition of A of size k.

Definition 1.2.2.2. Let A be a set with n element and $k \in \mathbb{N}$. Let $P \vdash^k A$ and $\lambda \vDash^k n$, we say that P is *of type* λ if we can put a total order on P such that $|P_i| = \lambda_i$. We denote $P \vdash^\lambda A$ the fact that P is of type λ.

Lemma 1.2.2.3. *Let $\lambda \vDash^k n$ and A be a set with n elements, then there are exactly $\frac{n!}{m_\lambda \lambda_1! \dots \lambda_k!}$ non-empty partitions of A of size k of type λ, where:*

$$m_\lambda = \prod_{j=1}^{n} \kappa_j!$$

and κ_j is the number of part of size j in λ.

Proof. Let us put an order on A, we have $n!$ way to do so. Then λ gives us a way to partition A into k parts such that the first λ_1 elements are in the first part, the next λ_2 elements are in the second part, and so on. This give us an ordered partition of A of size k such that each part of the partition is order. To get a partition of A of size k, we need to forget the order of the elements inside each part, this give us a factor of $\lambda_i!$ for each $i \in \underline{k}$. This gives us:

$$\frac{n!}{\lambda_1! \dots \lambda_k!}$$

However, we have overcounted the partitions of A of type λ. Indeed, if we exchange two non-empty parts of the same size in a partition of A of type λ, we get the same partition that we are currently counting twice. Hence, we need to divide by the number of permutation of the non-empty parts of the same size. This gives us a factor of m_λ. The empty parts do not contribute to the overcounting since exchanging two empty parts does not change the order we put on A at the beginning. This gives us the desired result. □

Lemma 1.2.2.4. *Let $P \vdash^k \underline{n}$, then there are exactly $\frac{k!}{m_P}$ composition λ of size k of n such that P is of type λ, where:*

$$m_P = \prod_{j=0}^{n} \kappa_j!$$

and κ_j is the number of part of size j in P.

Proof. To get a composition of n into k parts such that P is of type λ, it suffices to put an order on the parts of P. We have $k!$ ways to do so. However, we have overcounted such compositions. Indeed, if we exchange two parts of the same size in a composition, we get the same composition that we are currently counting twice. Hence, we need to divide by the number of permutation of the parts of the same size. This gives us a factor of m_P. The empty parts does contribute to the overcounting. □

We can now define the usual operations on species: the derivative, the sum, the product, the Hadamard product, and the plethysm of species.

Definition 1.2.2.5. Let $\mathcal{S}$ and $\mathcal{R}$ be two species, let $\sigma : A \to B$ be a bijection between two finite sets A and B. Let us recall that we use the symbol $\uplus$ to denote the disjoint union of sets, that should not be confused with $\sqcup$ which is the usual union of sets in the particular case of disjoint sets. We use the symbol $\times$ to denote the cartesian product of sets, if $I \subseteq A$ we denote $\sigma_{|I}$ the bijection $\sigma_{|I} : I \to \sigma(I)$, and if $P \vdash^k A$ we denote $\sigma^P : P \to \{\sigma(p) \mid p \in P\}$ the bijection induced by σ. Let us define the following operations on species:

- the *derivative* of $\mathcal{S}$ noted $\mathcal{S}'$ by:

$$\mathcal{S}'(A) = \mathcal{S}(A \sqcup \{A\})$$
$$\mathcal{S}'(\sigma) = \mathcal{S}(\sigma \sqcup \{\text{id}\}),$$

- the *sum* $\mathcal{S} + \mathcal{R}$ by:

$$(\mathcal{S} + \mathcal{R})(A) = \mathcal{S}(A) \uplus \mathcal{R}(A)$$
$$(\mathcal{S} + \mathcal{R})(\sigma) = \mathcal{S}(\sigma) \uplus \mathcal{R}(\sigma);$$

- the *product* (or Cauchy product) $\mathcal{S} \cdot \mathcal{R}$ by:

$$(\mathcal{S} \cdot \mathcal{R})(A) = \biguplus_{I \sqcup J = A} \mathcal{S}(I) \times \mathcal{R}(J)$$
$$(\mathcal{S} \cdot \mathcal{R})(\sigma) = \biguplus_{I \sqcup J = A} \mathcal{S}\left(\sigma_{|I}\right) \times \mathcal{R}\left(\sigma_{|J}\right);$$

- the *Hadamard product* $\mathcal{S} \odot \mathcal{R}$ by:

$$(\mathcal{S} \odot \mathcal{R})(A) = \mathcal{S}(A) \times \mathcal{R}(A)$$
$$(\mathcal{S} \odot \mathcal{R})(\sigma) = \mathcal{S}(\sigma) \times \mathcal{R}(\sigma);$$

- if $\mathcal{R}$ is connected, the *plethysm* $\mathcal{S} \circ \mathcal{R}$ by:

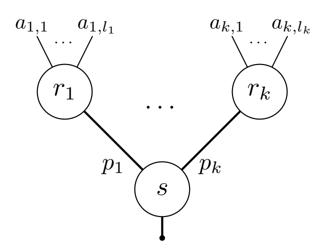
$$(\mathcal{S} \circ \mathcal{R})(A) = \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash^k A} \mathcal{S}(P) \times \prod_{p \in P} \mathcal{R}(p)$$
$$(\mathcal{S} \circ \mathcal{R})(\sigma) = \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash^k A} \mathcal{S}(\sigma^P) \times \prod_{p \in P} \mathcal{R}\left(\sigma_{|p}\right).$$

Since morphisms in $\mathbb{B}$ are not trivial, we need to specify our construction on the level of morphisms.

With these definitions, an $\mathcal{S}'$-structure on a finite set A is an $\mathcal{S}$-structure on $A \sqcup \{*\}$. An $\mathcal{S} + \mathcal{R}$-structure on a finite set A is either an $\mathcal{S}$-structure or an $\mathcal{R}$-structure on A. An $\mathcal{S} \cdot \mathcal{R}$-structure on a finite set A is a pair (I, J) such that $I \sqcup J = A$ with an $\mathcal{S}$-structure on I and an $\mathcal{R}$-structure on J. An $\mathcal{S} \odot \mathcal{R}$-structure on a finite set A is a pair (s, r) such that s is an $\mathcal{S}$-structure on A and r is an $\mathcal{R}$-structure on A. Finally, an $\mathcal{S} \circ \mathcal{R}$-structure on a finite set A is the data of a partition $P \vdash^k A$ for some k with an $\mathcal{S}$-structure on P and an R-structure on each $p \in P$. From the definition of the plethysm, we can see that it bears a tree-like structure. Indeed, let $t \in (\mathcal{S} \circ \mathcal{R})(A)$ then we have P a partition of A, an element $s \in \mathcal{S}(P)$ and a family of elements $(r_p)_{p \in P}$ with $r_p \in \mathcal{R}(p)$ such that:

$$t = (s; (r_p)_{p \in P})$$

If we denote $P = \{p_1, \ldots, p_k\}$ and $p_i = \{a_{i,1}, \ldots, a_{i,l_i}\}$, we can depict t as follows:



Let us define the formal power series associated to a species. The definitions we gave of the operations on species will guide us to define the formal power series associated to a species. Indeed, if $f_{\mathcal{S}}$ is the formal power series associated to a species $\mathcal{S}$, then we would like to have $f_{\mathcal{S} \cdot \mathcal{R}} = f_{\mathcal{S}} \cdot f_{\mathcal{R}}$ for example. Let us compute the cardinal of $(\mathcal{S} \cdot \mathcal{R})(\underline{n})$ in function of the cardinal of $s_i = \mathcal{S}(\underline{i})$ and $r_j = \mathcal{R}(\underline{j})$ for $i, j \leq n$. We have:

$$|(\mathcal{S} \cdot \mathcal{R})(\underline{n})| = \sum_{I \sqcup J = \underline{n}} |\mathcal{S}(I)| \times |\mathcal{R}(J)| = \sum_{i+j=n} \frac{n!}{i!j!} s_i r_j$$

With the appearance of the factorials, we recognize exponential generating functions. Let us define exponential generating functions.

Definition 1.2.2.6. An *exponential generating function*, egf for short, $f = \sum_{n\in\mathbb{N}} \frac{f_n}{n!} x^n$ is a formal power series such that $f_n \in \overline{\mathbb{N}}$. Since a power series is determined by its coefficients, we can identify the set of egf with the set of functions $\{\mathbb{N} \to \overline{\mathbb{N}}\}$. Hence, an egf is uniquely determined by a function $s : \mathbb{N} \to \overline{\mathbb{N}}$. An egf f is *finite* if f_n is finite for all $n \in \mathbb{N}$. An ogf f is *connected* if $f_0 = 0$, it is *strongly connected* if $f_0 = f_1 = 0$. Let us define the following usual ogf:

- 0 the zero egf such that all coefficients are 0,
- 1 the egf such that the first coefficient is 1 and all the other coefficients are 0,
- x the egf such that the second coefficient is 1 and all the other coefficients are 0,
- exp the egf such that all coefficients are 1.

We use the same notation for the ogf 0, 1 and x as for the egf 0, 1 and x since they are the same formal power series.

Let us define the appropriate operations on egf:

Definition 1.2.2.7. Let $f = \sum_{n\in\mathbb{N}} \frac{f_n}{n!} x^n$ and $g = \sum_{n\in\mathbb{N}} \frac{g_n}{n!} x^n$ be two egf. We define the following operations on egf:

- the *derivation* $f' = \sum_{n\in\mathbb{N}} \frac{f_{n+1}}{n!} x^n$,
- the *sum* $f + g = \sum_{n\in\mathbb{N}} \frac{f_n + g_n}{n!} x^n$,
- the *product* $f \cdot g = \sum_{n\in\mathbb{N}} \sum_{i=0}^{n} \frac{f_i g_{n-i}}{n!} x^n$,
- the *exponential Hadamard product* $f \boxdot g = \sum_{n\in\mathbb{N}} \frac{f_n g_n}{n!} x^n$,
- the *composition* $f \circ g = \sum_{n\in\mathbb{N}} \frac{f_n}{n!} g^n$.

One need to be careful as the exponential Hadamard product does not coincide with the Hadamard product of formal power series. The other operations coincide with the operations on formal power series.

We can now define the formal power series associated to a species.

Definition 1.2.2.8. Let $\mathcal{S}$ be a species. The *exponential generating function* of $\mathcal{S}$ is the formal power series:

$$f_{\mathcal{S}}(x) = \sum_{n\in\mathbb{N}} \frac{|\mathcal{S}(\underline{n})|}{n!} x^n$$

Two species $\mathcal{S}$ and $\mathcal{R}$ are *equinumerous* if $f_{\mathcal{S}} = f_{\mathcal{R}}$.

Proposition 1.2.2.9. *Two isomorphic species are equinumerous. The converse is not true in general, even if the species are finite.*

Proof. Since sets in bijection have the same cardinal, two isomorphic species are equinumerous. The converse is not true in general, even if the species are finite. For example, let us consider the species that associate to each set of cardinal n the left action of $\mathfrak{S}_n$ on itself, and the species that associate to each set of cardinal n the action by conjugation of $\mathfrak{S}_n$ on itself. Those two species are equinumerous, however they are not isomorphic. □

We can check that the operations on species we defined are coherent with the operations on formal power series.

Proposition 1.2.2.10. *Let $\mathcal{S}$ and $\mathcal{R}$ be two finite species. Then $f_{\mathcal{S}'} = (f_{\mathcal{S}})'$, same for the sum $f_{\mathcal{S}+\mathcal{R}} = f_{\mathcal{S}} + f_{\mathcal{R}}$, the product $f_{\mathcal{S}\cdot\mathcal{R}} = f_{\mathcal{S}} f_{\mathcal{R}}$, and the Hadamard product $f_{\mathcal{S}\odot\mathcal{R}} = f_{\mathcal{S}} \boxdot f_{\mathcal{R}}$. If $\mathcal{R}$ is connected, we also have $f_{\mathcal{S}\circ\mathcal{R}} = f_{\mathcal{S}} \circ f_{\mathcal{R}}$.*

Proof. All these properties are clear, expect for the plethysm. Let us compute:

$$\begin{aligned}
f_{\mathcal{S}} \circ f_{\mathcal{R}}(x) &= \sum_{k\in\mathbb{N}} \frac{|\mathcal{S}(\underline{k})|}{k!} f_{\mathcal{R}}(x)^k \\
&= \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} \frac{|\mathcal{S}(\underline{k})|}{k!} \sum_{\lambda\vDash^k n} \prod_{i=1}^{k} \frac{|\mathcal{R}(\underline{\lambda_i})|}{\lambda_i!} x^n \\
&= \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} \frac{|\mathcal{S}(\underline{k})|}{k!} \sum_{\lambda\vDash^k n} \frac{m_\lambda \lambda_1! \dots \lambda_k!}{n!} \frac{m_P}{k!} \sum_{P\vdash^\lambda \underline{n}} \prod_{i=1}^{k} \frac{|\mathcal{R}(\underline{\lambda_i})|}{\lambda_i!} x^n \\
&= \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} |\mathcal{S}(\underline{k})| \sum_{\lambda\vDash^k n} \frac{1}{n!} \sum_{P\vdash^\lambda \underline{n}} \prod_{p\in P} |\mathcal{R}(p)| x^n \\
&= \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} |\mathcal{S}(\underline{k})| \sum_{P\vdash^k \underline{n}} \prod_{p\in P} |\mathcal{R}(p)| \frac{x^n}{n!} \\
&= f_{\mathcal{S}\circ\mathcal{R}}(x)
\end{aligned}$$

We can notice that we had to use Lemma 1.2.2.3 and Lemma 1.2.2.4 to go from composition of integers to partition of sets. □

We can now verify the relations we would like to have between the operations on species. We first need a categorified version of Lemma 1.1.1.4.

Lemma 1.2.2.11. *Let $A \in \mathbb{B}$ and $k \in \mathbb{N}$. Let us define:*

$$\begin{aligned}
U &= \{(m, \Lambda, \Gamma) \mid m \in \mathbb{N}, \Lambda \vdash^m A, \Gamma \vdash^k \Lambda\} \\
V &= \{\Big(P, (m_p, Q_p)_{p\in P}\Big) \mid P \vdash^k A, \forall p \in P, m_p \in \mathbb{N}, Q_p \vdash^{m_p} p\}
\end{aligned}$$

Then there is a bijection between U and V.

Proof. A picture of the same kind of the one used in Lemma 1.1.1.4 can be used to understand the situation. Let us write the explicit bijection:

$$\begin{aligned}
U &\to V \\
(m, \Lambda, \Gamma) &\mapsto \left(\bigsqcup_{\gamma\in\Gamma} \left\{ \bigsqcup_{\lambda\in\gamma} \lambda \right\}, (|\gamma|, \gamma)_{\gamma\in\Gamma} \right)
\end{aligned}$$

$$\begin{aligned}
V &\to U \\
(P, (m_p, Q_p)_{p\in P}) &\mapsto \left(\sum_{p\in P} m_p, \bigsqcup_{p\in P} Q_p, \bigsqcup_{p\in P} \{Q_p\} \right)
\end{aligned}$$

We can check that those maps are inverse of each other. □

Let check that the operations on species satisfy the same relations as the analog operations on naive species.

Proposition 1.2.2.12. *Let $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{U}$ be three species, then the operations $+$, $\cdot$ and $\odot$ are unitary symmetric monoidal structures, and the operation $\circ$ is unitary monoidal structure:*

- $\mathcal{S}+0 \simeq \mathcal{S}$,
- $\mathcal{S}+\mathcal{R} \simeq \mathcal{R}+\mathcal{S}$,
- $(\mathcal{S}+\mathcal{R})+\mathcal{U} \simeq \mathcal{S}+(\mathcal{R}+\mathcal{U})$,
- $\mathcal{S}\cdot 1 \simeq \mathcal{S}$,
- $\mathcal{S}\cdot\mathcal{R} \simeq \mathcal{R}\cdot\mathcal{S}$,
- $(\mathcal{S}\cdot\mathcal{R})\cdot\mathcal{U} \simeq \mathcal{S}\cdot(\mathcal{R}\cdot\mathcal{U})$,
- $\mathcal{S}\odot \mathrm{set} \simeq \mathcal{S}$,
- $\mathcal{S}\odot\mathcal{R} \simeq \mathcal{R}\odot\mathcal{S}$,
- $(\mathcal{S}\odot\mathcal{R})\odot\mathcal{U} \simeq \mathcal{S}\odot(\mathcal{R}\odot\mathcal{U})$,
- $\mathcal{S}\circ X \simeq \mathcal{S}$,
- $X\circ\mathcal{S} \simeq \mathcal{S}$,
- $(\mathcal{S}\circ\mathcal{R})\circ\mathcal{U} \simeq \mathcal{S}\circ(\mathcal{R}\circ\mathcal{U})$,

We also have the following distributivity morphism:

- $(\mathcal{S}+\mathcal{R})\cdot\mathcal{U} \simeq \mathcal{S}\cdot\mathcal{U}+\mathcal{R}\cdot\mathcal{U}$,
- $(\mathcal{S}+\mathcal{R})\odot\mathcal{U} \simeq \mathcal{S}\odot\mathcal{U}+\mathcal{R}\odot\mathcal{U}$,
- $(\mathcal{S}+\mathcal{R})\circ\mathcal{U} \simeq \mathcal{S}\circ\mathcal{U}+\mathcal{R}\circ\mathcal{U}$,
- $(\mathcal{S}\cdot\mathcal{R})\circ\mathcal{U} \simeq \mathcal{S}\circ\mathcal{U}\cdot\mathcal{R}\circ\mathcal{U}$,

And the following morphisms when composing with the derivative:

- $(\mathcal{S}+\mathcal{R})' \simeq \mathcal{S}'+\mathcal{R}'$,
- $(\mathcal{S}\cdot\mathcal{R})' \simeq \mathcal{S}'\cdot\mathcal{R}+\mathcal{S}\cdot\mathcal{R}'$,
- $(\mathcal{S}\circ\mathcal{R})' \simeq \mathcal{S}'\circ\mathcal{R}\cdot\mathcal{R}'$.

Moreover, these isomorphisms are coherent with each other, meaning that "any diagram involving only these isomorphisms commutes". Because of this, we will write them as equality in the sequel.

We once again refer to Paragraph 1.1.2 for the discussion on the coherence conditions of the isomorphisms.

Proof. Let us show the associativity of the plethysm, let us reuse the notation U, V and the bijection between them from Lemma 1.2.2.11. We have:

$$\begin{aligned}
((\mathcal{S}\circ\mathcal{R})\circ\mathcal{U})(A) &= \biguplus_{k\in\mathbb{N}}\biguplus_{\Lambda\vdash^k A}\left(\biguplus_{m\in\mathbb{N}}\biguplus_{\Gamma\vdash^m\Lambda}\mathcal{S}(\Gamma)\times\prod_{\gamma\in\Gamma}\mathcal{R}(\gamma)\right)\times\prod_{\lambda\in\Lambda}\mathcal{U}(\lambda)\\
&= \biguplus_{k\in\mathbb{N}}\biguplus_{(m,\Lambda,\Gamma)\in U}\mathcal{S}(\Gamma)\times\prod_{\gamma\in\Gamma}\mathcal{R}(\gamma)\times\prod_{\lambda\in\Lambda}\mathcal{U}(\lambda)\\
&= \biguplus_{k\in\mathbb{N}}\biguplus_{\left(P,(m_p,Q_p)_{p\in P}\right)\in V}\mathcal{S}\left(\bigsqcup_{p\in P}\{Q_p\}\right)\times\prod_{\gamma\in\bigsqcup_{p\in P}\{Q_p\}}\mathcal{R}(\gamma)\times\prod_{\lambda\in\bigsqcup_{p\in P}Q_p}\mathcal{U}(\lambda)\\
&= \biguplus_{k\in\mathbb{N}}\biguplus_{\left(P,(m_p,Q_p)_{p\in P}\right)\in V}\mathcal{S}\left(P\right)\times\prod_{p\in P}\left(\mathcal{R}(Q_p)\times\prod_{q\in Q_p}\mathcal{U}(q)\right)\\
&= \biguplus_{k\in\mathbb{N}}\biguplus_{P\vdash^k A}\biguplus_{(m_p\in\mathbb{N})_{p\in P}}\biguplus_{(Q_p\vdash^{m_p}p)_{p\in P}}\mathcal{S}\left(P\right)\times\prod_{p\in P}\left(\mathcal{R}(Q_p)\times\prod_{q\in Q_p}\mathcal{U}(q)\right)\\
&= \biguplus_{k\in\mathbb{N}}\biguplus_{P\vdash^k A}\mathcal{S}\left(P\right)\times\prod_{p\in P}\left(\biguplus_{m_p\in\mathbb{N}}\biguplus_{Q_p\vdash^{m_p}p}\mathcal{R}(Q_p)\times\prod_{q\in Q_p}\mathcal{U}(q)\right)\\
&= (\mathcal{S}\circ(\mathcal{R}\circ\mathcal{U}))(A)
\end{aligned}$$

We should also check what happens at the level of morphisms. Let $\sigma : A \to B$ be a bijection, we have:

$$\begin{aligned}
((\mathcal{S}\circ\mathcal{R})\circ\mathcal{U})(\sigma) &= \biguplus_{k\in\mathbb{N}}\biguplus_{\Lambda\vdash^k A}\left(\biguplus_{m\in\mathbb{N}}\biguplus_{\Gamma\vdash^m\Lambda}\mathcal{S}\left(\left(\sigma^{\Lambda}\right)^{\Gamma}\right)\times\prod_{\gamma\in\Gamma}\mathcal{R}\left(\left(\sigma^{\Lambda}\right)_{|\gamma}\right)\right)\times\prod_{\lambda\in\Lambda}\mathcal{U}\left(\sigma_{|\lambda}\right)\\
&= \biguplus_{k\in\mathbb{N}}\biguplus_{(m,\Lambda,\Gamma)\in U}\mathcal{S}\left(\left(\sigma^{\Lambda}\right)^{\Gamma}\right)\times\prod_{\gamma\in\Gamma}\mathcal{R}\left(\left(\sigma^{\Lambda}\right)_{|\gamma}\right)\times\prod_{\lambda\in\Lambda}\mathcal{U}\left(\sigma_{|\lambda}\right)\\
&= \biguplus_{k\in\mathbb{N}}\biguplus_{\left(P,(m_p,Q_p)_{p\in P}\right)\in V}\mathcal{S}\left(\left(\sigma^{\bigsqcup_{p\in P}Q_p}\right)^{\bigsqcup_{p\in P}\{Q_p\}}\right)\times\prod_{\gamma\in\bigsqcup_{p\in P}\{Q_p\}}\mathcal{R}\left(\left(\sigma^{\bigsqcup_{p\in P}Q_p}\right)_{|\gamma}\right)\\
&\quad\times\prod_{\lambda\in\bigsqcup_{p\in P}Q_p}\mathcal{U}\left(\sigma_{|\lambda}\right)\\
&= \biguplus_{k\in\mathbb{N}}\biguplus_{\left(P,(m_p,Q_p)_{p\in P}\right)\in V}\mathcal{S}\left(\sigma^{P}\right)\times\prod_{p\in P}\left(\mathcal{R}\left(\left(\sigma_{|p}\right)^{Q_p}\right)\times\prod_{q\in Q_p}\mathcal{U}\left(\left(\sigma_{|p}\right)_{|q}\right)\right)\\
&= \biguplus_{k\in\mathbb{N}}\biguplus_{P\vdash^k A}\biguplus_{(m_p\in\mathbb{N})_{p\in P}}\biguplus_{(Q_p\vdash^{m_p}p)_{p\in P}}\mathcal{S}\left(\sigma^{P}\right)\times\prod_{p\in P}\left(\mathcal{R}\left(\left(\sigma_{|p}\right)^{Q_p}\right)\times\prod_{q\in Q_p}\mathcal{U}\left(\left(\sigma_{|p}\right)_{|q}\right)\right)\\
&= \biguplus_{k\in\mathbb{N}}\biguplus_{P\vdash^k A}\mathcal{S}\left(\sigma^{P}\right)\times\prod_{p\in P}\left(\biguplus_{m_p\in\mathbb{N}}\biguplus_{Q_p\vdash^{m_p}p}\mathcal{R}\left(\left(\sigma_{|p}\right)^{Q_p}\right)\times\prod_{q\in Q_p}\mathcal{U}\left(\left(\sigma_{|p}\right)_{|q}\right)\right)\\
&= (\mathcal{S}\circ(\mathcal{R}\circ\mathcal{U}))(\sigma)
\end{aligned}$$

We used here some relation on sigma in particular that $\left(\sigma_{|p}\right)_{|q}=\sigma_{|p}$, that $\left(\sigma^{\bigsqcup_{p\in P}Q_p}\right)_{|Q_p}=\left(\sigma_{|p}\right)^{Q_p}$ or the more involved:

$$\left(\sigma^{\bigsqcup_{p\in P}Q_p}\right)^{\bigsqcup_{p\in P}\{Q_p\}}=\alpha^{-1}\circ\sigma^{P}\circ\alpha,$$

with $\alpha:\bigsqcup_{p\in P}Q_p\to P$ the bijection that send each Q_p to p.
Since the associativity of the plethysm is the only difficult part of the proof we leave the other relations to the reader. The reader wanting to check the other relations can make use of Propositions 1.1.2.6 and 1.1.1.6 to do so. □

Species also have an additional operation, indeed one can always "point" a structure by adding a marked point to the structure. Let us define this operation.

Definition 1.2.2.13. Let $\mathcal{S}$ be a species, the *pointed* species $\mathcal{S}^\bullet$ is defined by $\mathcal{S}^\bullet=X\cdot\mathcal{S}'$.

If $\mathcal{S}$ is a species, then an $\mathcal{S}^\bullet$-structure on a finite set A is a pair (s,a) such that $a\in A$ and s is an $\mathcal{S}$-structure on A. We can check that $\mathcal{S}^\bullet=X\cdot\mathcal{S}'$.

1.2.3 Action of $\mathfrak{S}_n$ and Schur functors

As we have seen, the data of a species is equivalent to the data of a symmetric sequence. In particular, if $\mathcal{S}$ is a species, then $\mathcal{S}(\underline{n})$ has a natural action of $\mathfrak{S}_n$. In the sequel, the action of $\mathfrak{S}_n$ on $\mathcal{S}(\underline{n})$ will always be denoted as a right action, and the action of $\mathfrak{S}_n$ on $A^n=\mathrm{Hom}(\underline{n},A)$ and on $\mathrm{Hom}(A,\underline{n})$ will both be denoted as a left action. We will use the notation $A\times_{\mathfrak{S}_n}B$ for A a set with a right action of $\mathfrak{S}_n$ and B a set with a left action of $\mathfrak{S}_n$, to denote the quotient of $A\times B$ by the diagonal action of $\mathfrak{S}_n$. Since species are deeply related to symmetric sequences, this explains the terminology Schur functor we used for naive species. Let us define the Schur functor associated to a species.

Definition 1.2.3.1. Let $\mathcal{S}$ be a species, the *Schur functor* associated to $\mathcal{S}$ is the functor $\mathcal{F}_{\mathcal{S}} : \mathrm{Set} \to \mathrm{Set}$ defined by:

$$\mathcal{F}_{\mathcal{S}}(A) = \biguplus_{n\in\mathbb{N}} \mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} A^n$$

$$\mathcal{F}_{\mathcal{S}}(f) = \biguplus_{n\in\mathbb{N}} \mathrm{id}_{\mathcal{S}(\underline{n})} \times_{\mathfrak{S}_n} f^{\times n}$$

A functor $F : \mathrm{Set} \to \mathrm{Set}$ is an *analytic functor* if there is a species $\mathcal{S}$ such that $F \simeq \mathcal{F}_{\mathcal{S}}$.

Let us give an alternative formula for the plethysm of species which does not involve the use of partitions but use group actions. We will use it to show that the plethysm of species corresponds to the composition of the Schur functors.

Proposition 1.2.3.2. *Let $\mathcal{S}$ and $\mathcal{R}$ be two species such that $\mathcal{R}$ is connected and $n \in \mathbb{N}$, we have:*

$$(\mathcal{S} \circ \mathcal{R})(\underline{n}) \simeq \biguplus_{k\in\mathbb{N}} \mathcal{S}(\underline{k}) \times_{\mathfrak{S}_k} \left(\biguplus_{f:\underline{n}\to\underline{k}} \prod_{i=1}^{k} \mathcal{R}(f^{-1}(i)) \right)$$

And the action of $\mathfrak{S}_n$ is given by the pre-composition on f. We will omit the isomorphism in the following and write it as an equality.

Proof. We need to show the following:

$$\biguplus_{P\vdash^k \underline{n}} \mathcal{S}(P) \times \prod_{p\in P} \mathcal{R}(p) \simeq \mathcal{S}(\underline{k}) \times_{\mathfrak{S}_k} \left(\biguplus_{f:\underline{n}\to\underline{k}} \prod_{i=1}^{k} \mathcal{R}(f^{-1}(i)) \right)$$

Here the partition P is unordered, hence we can order P and then quotient by the simply transitive action of $\mathfrak{S}_k$ on the ordered partitions. Since an ordering of P is a bijection $\underline{k} \to P$, we get:

$$\biguplus_{P\vdash^k \underline{n}} \mathcal{S}(P) \times \prod_{p\in P} \mathcal{R}(p) \simeq \biguplus_{P\vdash^k \underline{n}} \left(\biguplus_{\mathrm{Bij}(\underline{k}\to P)} \mathcal{S}(P) \times \prod_{i=1}^{k} \mathcal{R}(p_i) \right) /\mathfrak{S}_k$$

Moreover, by fixing an ordering of P, we have $\mathcal{S}(P) = \mathcal{S}(\underline{k})$ and the action of $\mathfrak{S}_k$ is the diagonal action, hence we get:

$$\biguplus_{P\vdash^k \underline{n}} \mathcal{S}(P) \times \prod_{p\in P} \mathcal{R}(p) \simeq \mathcal{S}(\underline{k}) \times_{\mathfrak{S}_k} \biguplus_{P\vdash^k \underline{n}} \left(\biguplus_{\mathrm{Bij}(\underline{k}\to P)} \prod_{i=1}^{k} \mathcal{R}(p_i) \right)$$

Since the data of an ordered partition of $\underline{n}$ is the same as a map $\underline{n} \to \underline{k}$, we get:

$$\biguplus_{P\vdash^k \underline{n}} \mathcal{S}(P) \times \prod_{p\in P} \mathcal{R}(p) \simeq \mathcal{S}(\underline{k}) \times_{\mathfrak{S}_k} \left(\biguplus_{f:\underline{n}\to\underline{k}} \prod_{i=1}^{k} \mathcal{R}(f^{-1}(i)) \right)$$

□

Proposition 1.2.3.3. *Let $\mathcal{S}$ and $\mathcal{R}$ be two species such that $\mathcal{R}$ is connected, then $\mathcal{F}_{\mathcal{S}\circ\mathcal{R}} \simeq \mathcal{F}_{\mathcal{S}} \circ \mathcal{F}_{\mathcal{R}}$.*

Proof. Let A be a set, we have:

$$\mathcal{F}_{\mathcal{S}} \circ \mathcal{F}_{\mathcal{R}}(A) = \biguplus_{k\in\mathbb{N}} \mathcal{S}(\underline{k}) \times_{\mathfrak{S}_k} \left(\biguplus_{j\in\mathbb{N}} \mathcal{R}(\underline{j}) \times_{\mathfrak{S}_j} A^j \right)^k$$

$$= \biguplus_{n\in\mathbb{N}} \biguplus_{k\in\mathbb{N}} \mathcal{S}(\underline{k}) \times_{\mathfrak{S}_k} \biguplus_{\lambda\vDash^k n} \prod_{i=1}^{k} \left(\mathcal{R}(\underline{\lambda_i}) \times_{\mathfrak{S}_{\lambda_i}} A^{\lambda_i} \right)$$

Inspecting the actions of $\mathfrak{S}_{\lambda_i}$ on $\mathcal{R}(\underline{\lambda_i}) \times A^{\lambda_i}$, we can remark that:

$$\biguplus_{\lambda \vDash^k n} \prod_{i=1}^{k} \left(\mathcal{R}(\underline{\lambda_i}) \times_{\mathfrak{S}_{\lambda_i}} A^{\lambda_i}\right) = \biguplus_{f:\underline{n}\to\underline{k}} \prod_{i=1}^{k} \mathcal{R}(f^{-1}(i)) \times_{\mathfrak{S}_n} A^n$$

Hence, we get:

$$\begin{aligned}
\mathcal{F}_{\mathcal{S}} \circ \mathcal{F}_{\mathcal{R}}(A) &= \biguplus_{n\in\mathbb{N}} \biguplus_{k\in\mathbb{N}} \mathcal{S}(\underline{k}) \times_{\mathfrak{S}_k} \biguplus_{f:\underline{n}\to\underline{k}} \prod_{i=1}^{k} \mathcal{R}(f^{-1}(i)) \times_{\mathfrak{S}_n} A^n \\
&= \biguplus_{n\in\mathbb{N}} \left(\biguplus_{k\in\mathbb{N}} \mathcal{S}(\underline{k}) \times_{\mathfrak{S}_k} \biguplus_{f:\underline{n}\to\underline{k}} \prod_{i=1}^{k} \mathcal{R}(f^{-1}(i)) \right) \times_{\mathfrak{S}_n} A^n \\
&= \biguplus_{n\in\mathbb{N}} (\mathcal{S} \circ \mathcal{R})(\underline{n}) \times_{\mathfrak{S}_n} A^n \\
&= \mathcal{F}_{\mathcal{S}\circ\mathcal{R}}(A)
\end{aligned}$$

The same computation work at the level of morphisms, hence we have $\mathcal{F}_{\mathcal{S}\circ\mathcal{R}} \simeq \mathcal{F}_{\mathcal{S}} \circ \mathcal{F}_{\mathcal{R}}$. □

Remark 1.2.3.4. This proposition allows us to define the plethysm in full generalities, without assuming that one of the species is connected. However, one need to be careful when doing so, as the formula with partitions is not valid in this case.

Definition 1.2.3.5. Let $F, G : \mathrm{Set} \to \mathrm{Set}$ be two functors, we denote by $F + G$ and $F \times G$ the functors defined by:

$$\begin{aligned}
(F+G) &: (f : A \to B) \mapsto (F(f) \uplus G(f) : F(A) \uplus G(A) \to F(B) \uplus G(B)) \\
(F\times G) &: (f : A \to B) \mapsto (F(f) \times G(f) : F(A) \times G(A) \to F(B) \times G(B))
\end{aligned}$$

Proposition 1.2.3.6. *Let $\mathcal{S}$ and $\mathcal{R}$ be two species, then $\mathcal{F}_{\mathcal{S}+\mathcal{R}} \simeq \mathcal{F}_{\mathcal{S}} + \mathcal{F}_{\mathcal{R}}$ and $\mathcal{F}_{\mathcal{S}\cdot\mathcal{R}} \simeq \mathcal{F}_{\mathcal{S}} \times \mathcal{F}_{\mathcal{R}}$.*

Proof. It is straightforward for the sum. Let us do the computation for the product. We have:

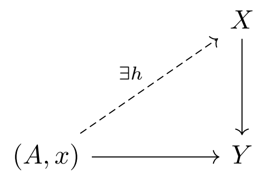
$$\begin{aligned}
(\mathcal{F}_{\mathcal{S}} \times \mathcal{F}_{\mathcal{R}})(A) &= \left(\biguplus_{n\in\mathbb{N}} \mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} A^n \right) \times \left(\biguplus_{n\in\mathbb{N}} \mathcal{R}(\underline{n}) \times_{\mathfrak{S}_n} A^n \right) \\
&= \biguplus_{n\in\mathbb{N}} \biguplus_{i+j=n} \mathcal{S}(\underline{i}) \times_{\mathfrak{S}_i} A^i \times \mathcal{R}(\underline{j}) \times_{\mathfrak{S}_j} A^j \\
&= \biguplus_{n\in\mathbb{N}} \biguplus_{I\sqcup J=\underline{n}} (\mathcal{S}(I) \times \mathcal{R}(J)) \times_{\mathfrak{S}_n} A^n \\
&= \mathcal{F}_{\mathcal{S}\cdot\mathcal{R}}(A)
\end{aligned}$$

The same computation work at the level of morphisms, hence we have $\mathcal{F}_{\mathcal{S}\cdot\mathcal{R}} \simeq \mathcal{F}_{\mathcal{S}} \times \mathcal{F}_{\mathcal{R}}$. □

We can see that the functor $\begin{array}{rccc} \mathcal{F}: & \mathrm{Spe} & \to & \mathrm{Fun}(\mathrm{Set}, \mathrm{Set}) \\ & \mathcal{S} & \mapsto & \mathcal{F}_{\mathcal{S}} \end{array}$ is monoidal. It is quite clear that this functor is not essentially surjective, however one may wonder if it is fully faithful, and what is its essential image. Let us show that it is fully faithful using the technics of [46]. A purely categorical description of the full subcategory of $\mathrm{Fun}(\mathrm{Set}, \mathrm{Set})$ of analytic functors is given in this same article, however, we will not state it here.

Definition 1.2.3.7. Let $F : \mathrm{Set} \to \mathrm{Set}$ be a functor. Let $\mathcal{D}_F$ be the *diagram category of F* defined by: an object of $\mathcal{D}_F$ is a pair (A, x) with A a finite set and $x \in F(A)$ and a morphism from (A, x) to (B, y) is a map $f : A \to B$ such that $F(f)(x) = y$. By definition, such a morphism is said to be *injectif* (resp. *surjectif*, *bijectif*) if the underlying map between the sets is injective (resp. surjective, bijective).

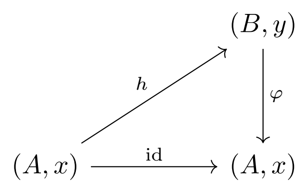
Definition 1.2.3.8. Let $(A,x) \in \mathcal{D}_F$, it is *generic* if for any $X, Y \in \mathcal{D}_F$ and any morphism $(A,x) \to Y$ and $X \to Y$, we have h such that the following triangle commutes:



It means that any maps $(A,x) \to Y$ can be factorized through any map $X \to Y$.

Lemma 1.2.3.9. *Any map to a generic object is surjective. If both the source and the target are generic, then the map is bijective.*

Proof. Let $\varphi : (B,y) \to (A,x)$. If (A,x) is generic, we have h such that the following triangle commutes:

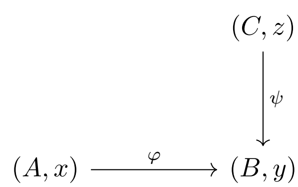


Since id is surjective, we get that φ is surjective. If both (A,x) and (B,y) are generic, then h is surjective since its target is generic, hence φ is bijective. ☐

Definition 1.2.3.10. Let $F : \mathrm{Set} \to \mathrm{Set}$ be a functor. We define its *Taylor-Joyal expansion of* F to be the species $\widehat{F}$ such that $\widehat{F}(A)$ is the set of $x \in F(A)$ such that (A,x) is generic. Since generic objects are stable by isomorphism, we have a well defined species.

Theorem 1.2.3.11. *Let $\mathcal{S}$ be a species, then $\widehat{\mathcal{F}_{\mathcal{S}}} = \mathcal{S}$.*

Proof. First, let $x \in \mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} \mathrm{Bij}(\underline{n}, A)$ and let us show that (A,x) is generic. Let:



Let us construct h such that the triangle commutes. Let $x = [(s,\sigma)]$, $y = [(s_y, f)]$ and $z = [(s_z, g)]$ with $s \in \mathcal{S}(\underline{n})$, $s_y \in \mathcal{S}(\underline{i})$, $s_z \in \mathcal{S}(\underline{j})$, $\sigma \in \mathrm{Bij}(\underline{n}, A)$, $f \in \mathrm{Hom}(\underline{i}, B)$ and $g \in \mathrm{Hom}(\underline{j}, C)$. Since

$$\mathcal{F}_{\mathcal{S}}(\varphi)(x) = \mathcal{F}_{\mathcal{S}}(\psi)(z) = y,$$

we have $\underline{i} = \underline{j} = \underline{n}$ moreover we can assume $s = s_y = s_z$. We know that $(s, \varphi \circ \sigma)$, $(s, \psi \circ g)$ and (s, f) are in the same orbit of $\mathcal{S}(\underline{n}) \times \mathrm{Hom}(\underline{n}, B)$ under the diagonal action of $\mathfrak{S}_n$. Let $\tau \in \mathfrak{S}_n$ such that $\varphi \circ \sigma = \psi \circ g \circ \tau$ and $s.\tau = s$. Finally, let $h = g \circ \tau \circ \sigma^{-1}$, we have:

$$\mathcal{F}_{\mathcal{S}}(h)(x) = [(s, h \circ \sigma)] = [(s, g \circ \tau \circ \sigma^{-1} \circ \sigma)] = [(s, g \circ \tau)] = [(s, g)] = z$$

Moreover, $\psi \circ h = \psi \circ g \circ \tau\sigma^{-1} = \varphi \circ \sigma \circ \sigma^{-1} = \varphi$, hence the triangle commutes. Hence, (A,x) is generic.

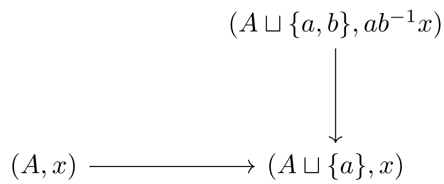
Let us show that these are the only generic objects. Let (D,t) be generic and let us denote $t = [(s,f)]$ with $s \in \mathcal{S}(\underline{n})$ and $f \in \mathrm{Hom}(\underline{n}, D)$. Let $u = [(s, \mathrm{id})] \in \mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} \mathrm{Bij}(\underline{n}, \underline{n})$, we have $f : (\underline{n}, u) \to (D,t)$, moreover $(\underline{n}, u)$ is generic, hence f is bijective.

We have that $\widehat{\mathcal{F}_{\mathcal{S}}}(\underline{n}) = \mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} \mathrm{Bij}(\underline{n}, \underline{n})$, since the left action of $\mathfrak{S}_n$ on $\mathrm{Bij}(\underline{n}, \underline{n})$ is simply transitive, we have $\widehat{\mathcal{F}_{\mathcal{S}}}(\underline{n}) \simeq \mathcal{S}(\underline{n})$ which concludes the proof. ☐

In particular, one may remark that the species $\mathcal{S}$ is entirely determined by its Schur functor $\mathcal{F}_{\mathcal{S}}$. Let us compute the Taylor-Joyal expansion of some classical functors.

Example 1.2.3.12.

- We recall that a monoid is a set together with a binary operation that is associative and has a neutral element. Let Mon be the free monoid functor, we have $\widehat{\mathrm{Mon}} = \mathbb{L}$. Indeed, $\mathrm{Mon}(A) = \biguplus_{n\in\mathbb{N}} A^n$ and we recognize the Schur functor of $\mathbb{L}$. In particular, Mon is an analytic functor.
- Let Grp be the free group functor, we have $\widehat{\mathrm{Grp}} = 0$. Indeed, let (A, x) with $x \in \mathrm{Grp}(A)$. Let us consider $A \sqcup \{a, b\} \to A \sqcup \{a\}$ that is the identity on A and send a and b to a. We have:



  We cannot have a map $(A, x) \to (A \sqcup \{a, b\}, ab^{-1}x)$ that make the diagram commute, hence (A, x) is not generic. In particular, Grp is not an analytic functor.

This example is already quite interesting. It shows that the theory of species allows one to study some algebraic structures such that monoids, but not all of them, since groups escape this scope.

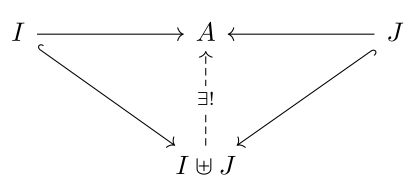
1.2.4 Arithmetic species and categorical L function

This subsection mostly comes from a discussion with Thomas Agugliaro. It will not be used nor developed in the sequel, and thus can be skipped. However, the author thinks that it is interesting nonetheless, and that it would be a shame not to mention it. The goal is to categorify the Dirichlet convolution and to relate species with the Dirichlet convolution to some class of functors. The main idea is that anything that transforms a convolution product into a product is a kind of Fourier transform, in this regard, the Schur functor lets us transform the Cauchy product, that we can see as a kind of convolution product, into the usual cartesian product. Let us try to find the analog of the Schur functor that would transform the Dirichlet convolution into the usual cartesian product. First we need to categorify the Dirichlet convolution. It is already done in [64], however we will have a different approach. Let us recall what is the Dirichlet convolution.

Definition 1.2.4.1. Let $f = \sum_{n\in\mathbb{N}^*} f_n x^n$ and $g = \sum_{n\in\mathbb{N}^*} g_n x^n$ be two formal power series, then the *Dirichlet convolution* of f and g is the formal power series $f * g = \sum_{n\in\mathbb{N}^*} h_n x^n$ defined by:

$$h_n = \sum_{d\times q=n} f_d g_q$$

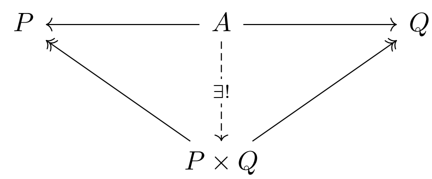
We may remark that the definition of the Dirichlet convolution is very close to the definition of the Cauchy product. Indeed, replacing the product by the sum, we get the Cauchy product. To categorify the Dirichlet convolution, we need to understand the analog of $I \sqcup J = A$ with a cartesian product instead of a union. Let us understand the symbol $\sqcup$ in a categorical way. Let A be a set and let us write the universal property of the coproduct (which is $\uplus$ with our notations):



From this diagram, we can see that we use the symbole $\sqcup$ under two conditions.

- The first condition is that I and J are subsets of A, it is here to ensure that we have a finite number of choices for I and J.
- The second condition is that the dashed arrow is a bijection, the surjectivity ensure that $I \cup J = A$ and the injectivity ensure that $I \cap J = \emptyset$.

Let us write the same diagram for the product:



Definition 1.2.4.2. Let A be a set, and let P and Q be two quotients of A, then we denote $P \boxtimes Q = A$ when $A \to P \times Q$ is a bijection. We say that $P \boxtimes Q$ is a *direct product* of A.

Since the author did not find any reference for this notion, he does not know if it is already known and studied. Let us show the following lemma on direct products which is the main motivation for this notion. It was found by Thomas Agugliaro and the author.

Lemma 1.2.4.3. *Let A be a non-empty finite set and $P \boxtimes Q = A$. Let $p \in P$ and $q \in Q$, then $p \cap q$ is a singleton. Moreover, $|p| = |Q|$ and $|q| = |P|$, in particular each element of P have the same cardinality.*

Proof. Let since $A \to P \times Q$ is a bijection, we have a unique $a \in A$ such that $a \in p \cap q$ hence $p \cap q$ is a singleton, moreover if $(p', q') \neq (p, q)$ then $p' \cap q'$ and $p \cap q$ are disjoint. Hence, $p = \bigsqcup_{q' \in Q} p \cap q'$, in particular $|p| = |Q|$. Idem for $|q| = |P|$. $\square$

With this, we have a very natural candidate for the Dirichlet convolution on species (which coincide with the definition given in [64]):

Definition 1.2.4.4. Let $\mathcal{S}$ and $\mathcal{R}$ be two species, then the *Dirichlet convolution* of $\mathcal{S}$ and $\mathcal{R}$ is the species $\mathcal{S} * \mathcal{R}$ defined by:

$$(\mathcal{S} * \mathcal{R})(A) = \biguplus_{P \boxtimes Q = A} \mathcal{S}(P) \times \mathcal{R}(Q)$$
$$(\mathcal{S} * \mathcal{R})(\sigma) = \biguplus_{P \boxtimes Q = A} \mathcal{S}(\sigma^P) \times \mathcal{R}(\sigma^Q)$$

Now that we have a Dirichlet convolution on species, we can try to find the analog of the Schur functor that would transform the Dirichlet convolution into the usual cartesian product. We saw that reverting some arrows gave us a good notion of Dirichlet convolution, hence we can try to revert some arrows in the definition of the Schur functor.

Definition 1.2.4.5. Let $\mathcal{S}$ be a connected species, the *categorical L function associated to $\mathcal{S}$* is the contravariant functor $\mathcal{L}_\mathcal{S} : \mathrm{Set} \to \mathrm{Set}$ defined by:

$$\mathcal{L}_\mathcal{S}(A) = \biguplus_{n \in \mathbb{N}^*} \mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} \mathrm{Hom}(A, \underline{n})$$
$$\mathcal{L}_\mathcal{S}(f) = \biguplus_{n \in \mathbb{N}^*} \mathrm{id}_{\mathcal{S}(\underline{n})} \times_{\mathfrak{S}_n} f^*$$

A contravariant functor $F : \text{Set} \to \text{Set}$ is a *categorical L function* if there is a connected species $\mathcal{S}$ such that $F \simeq \mathcal{L}_{\mathcal{S}}$.
It is the categorification of the *Dirichlet series associated to* $\mathcal{S}$, defined by:

$$L(\mathcal{S}, s) = \sum_{n \in \mathbb{N}^*} \frac{|\mathcal{S}(\underline{n})|}{n!} n^{-s}$$

Let us show that a connected species $\mathcal{S}$ is entirely determined by its categorical L function $\mathcal{L}_{\mathcal{S}}$. To do so, let us adapt the proof for analytic functors given in [46].

Definition 1.2.4.6. Let $F : \text{Set} \to \text{Set}$ be a contravariant functor. Let $\mathcal{D}_F$ be the *diagram category of* F defined by: an object of $\mathcal{D}_F$ is a pair (A, x) with A a finite set and $x \in F(A)$ and a morphism from (A, x) to (B, y) is a map $f : B \to A$ such that $F(f)(x) = y$. By definition, such a morphism is said to be *injectif* (resp. *surjectif*, *bijectif*) if the underlying map between the sets is injective (resp. surjective, bijective).

Definition 1.2.4.7. Let $(A, x) \in \mathcal{D}_F$, it is *generic* if for any $X, Y \in \mathcal{D}_F$ and any morphism $(A, x) \to Y$ and $X \to Y$, we have h such that the following triangle commutes:

$$\begin{array}{ccc} & & X \\ & \overset{\exists h}{\nearrow} & \downarrow \\ (A,x) & \longrightarrow & Y \end{array}$$

It means that any map $(A, x) \to Y$ can be factorized through any map $X \to Y$.

Lemma 1.2.4.8. *Any map to a generic object is injective. If both the source and the target are generic, then the map is bijective.*

Proof. Let $\varphi : (B, y) \to (A, x)$. If (A, x) is generic, we have h such that the following triangle commutes:

$$\begin{array}{ccc} & & (B,y) \\ & \overset{h}{\nearrow} & \downarrow \varphi \\ (A,x) & \xrightarrow{\text{id}} & (A,x) \end{array}$$

Since id is injective, we get that $\widetilde{\varphi} : A \to B$ is injective. If both (A, x) and (B, y) are generic, then h is injective since its target is generic, hence φ is bijective. □

Definition 1.2.4.9. Let $F : \text{Set} \to \text{Set}$ be a contravariant functor. Let $\widehat{F}$ the *Joyal expansion of* F be the connected species defined by $\widehat{F}(A)$ is the set of $x \in F(A)$ such that (A, x) is generic. Since generic objects are stable by isomorphism, we have a well defined species.

Theorem 1.2.4.10. *Let* $\mathcal{S}$ *be a connected species, then* $\widehat{\mathcal{L}_{\mathcal{S}}} = \mathcal{S}$.

Proof. First, let $x \in \mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} \text{Bij}(A, \underline{n})$ and let us show that (A, x) is generic. Let:

$$\begin{array}{ccc} & & (C,z) \\ & & \downarrow \psi \\ (A,x) & \xrightarrow{\varphi} & (B,y) \end{array}$$

Let us construct h such that the triangle commutes. Let $x = [(s, \sigma)]$, $y = [(s_y, f)]$ and $z = [(s_z, g)]$ with $s \in \mathcal{S}(\underline{n})$, $s_y \in \mathcal{S}(\underline{i})$, $s_z \in \mathcal{S}(\underline{j})$, $\sigma \in \mathrm{Bij}(A, \underline{n})$, $f \in \mathrm{Hom}(B, \underline{i})$ and $g \in \mathrm{Hom}(C, \underline{j})$. Since:

$$\mathcal{L}_{\mathcal{S}}(\widetilde{\varphi})(x) = \mathcal{L}_{\mathcal{S}}(\widetilde{\psi})(y) = z,$$

we have $\underline{i} = \underline{j} = \underline{n}$ moreover we can assume $s = s_y = s_z$. We know that $(s, \widetilde{\varphi} \circ \sigma)$, $(s, \widetilde{\psi} \circ g)$ and (s, f) are in the same orbit of $\mathcal{S}(\underline{n}) \times \mathrm{Hom}(B, \underline{n})$ under the diagonal action of $\mathfrak{S}_n$. Let $\tau \in \mathfrak{S}_n$ such that $\sigma \circ \widetilde{\varphi} = \tau \circ g \circ \widetilde{\psi}$ and $s.\tau = s$. Finally, let $\widetilde{h} = \sigma^{-1} \circ \tau \circ g$, we have:

$$\mathcal{L}_{\mathcal{S}}(\widetilde{h})(x) = [(s, \sigma \circ \widetilde{h})] = [(s, \sigma \circ \sigma^{-1} \circ \tau \circ g)] = [(s, \tau \circ g)] = [(s, g)] = z$$

Moreover, $\widetilde{h} \circ \widetilde{\psi} = \sigma^{-1} \circ \tau \circ g \circ \widetilde{\psi} = \sigma^{-1} \circ \sigma \circ \widetilde{\varphi} = \widetilde{\varphi}$, hence the triangle commutes. Hence, (A, x) is generic.
Let us show that these are the only generic objects. Let (D, t) be generic and let us denote $t = [(s, \widetilde{f})]$ with $s \in \mathcal{S}(\underline{n})$ and $\widetilde{f} \in \mathrm{Hom}(D, \underline{n})$. Let $u = [(s, \mathrm{id})] \in \mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} \mathrm{Bij}(\underline{n}, \underline{n})$, we have $f : (\underline{n}, u) \to (D, t)$, moreover $(\underline{n}, u)$ is generic, hence $\widetilde{f}$ is bijective.
We have that $\widehat{\mathcal{L}_{\mathcal{S}}}(\underline{n}) = \mathcal{S}(\underline{n}) \times_{\mathfrak{S}_n} \mathrm{Bij}(\underline{n}, \underline{n})$, since the action of $\mathfrak{S}_n$ on $\mathrm{Bij}(\underline{n}, \underline{n})$ is simply transitive, we have $\widehat{\mathcal{F}_{\mathcal{S}}}(\underline{n}) \simeq \mathcal{S}(\underline{n})$ which concludes the proof. □

We should check that the transformation we have defined sends the Dirichlet convolution into the usual cartesian product.

Definition 1.2.4.11. Let $F, G : \mathrm{Set} \to \mathrm{Set}$ be two contravariant functors, let $F + G$ and $F \times G$ be the contravariant functors defined by:

$$(F + G) : (f : A \to B) \mapsto (F(f) \uplus G(f) : F(A) \uplus G(A) \to F(B) \uplus G(B))$$
$$(F \times G) : (f : A \to B) \mapsto (F(f) \times G(f) : F(A) \times G(A) \to F(B) \times G(B))$$

Proposition 1.2.4.12. *Let $\mathcal{S}$ and $\mathcal{R}$ be two connected species, then $\mathcal{L}_{\mathcal{S}+\mathcal{R}} = \mathcal{L}_{\mathcal{S}} + \mathcal{L}_{\mathcal{R}}$ and $\mathcal{L}_{\mathcal{S}*\mathcal{R}} = \mathcal{L}_{\mathcal{S}} \times \mathcal{L}_{\mathcal{R}}$.*

Proof. Checking the sum is straightforward. Let us do the computation for the product. We have:

$$\begin{aligned}
(\mathcal{L}_{\mathcal{S}} \times \mathcal{L}_{\mathcal{R}})(A) &= \left(\biguplus_{i \in \mathbb{N}^*} \mathcal{S}(\underline{i}) \times_{\mathfrak{S}_i} \mathrm{Hom}(A, \underline{i}) \right) \times \left(\biguplus_{j \in \mathbb{N}^*} \mathcal{R}(\underline{j}) \times_{\mathfrak{S}_j} \mathrm{Hom}(A, \underline{j}) \right) \\
&= \biguplus_{n \in \mathbb{N}^*} \biguplus_{i \times j = n} \left(\mathcal{S}(\underline{i}) \times \mathrm{Hom}(A, \underline{i}) \times \mathcal{R}(\underline{j}) \times \mathrm{Hom}(A, \underline{j}) \right) / (\mathfrak{S}_i \times \mathfrak{S}_j) \\
&= \biguplus_{n \in \mathbb{N}^*} \biguplus_{I \boxtimes J = \underline{n}} \left(\mathcal{S}(I) \times \mathcal{R}(J) \times \mathrm{Hom}(A, \underline{n}) \right) / \mathfrak{S}_n \\
&= \biguplus_{n \in \mathbb{N}^*} \biguplus_{I \boxtimes J = \underline{n}} \left(\mathcal{S}(I) \times \mathcal{R}(J) \right) \times_{\mathfrak{S}_n} \mathrm{Hom}(A, \underline{n}) \\
&= \mathcal{L}_{\mathcal{S}*\mathcal{R}}(A)
\end{aligned}$$

The computation for the morphisms is the same. □

The same is true for the Dirichlet series:

Proposition 1.2.4.13. *Let $\mathcal{S}$ and $\mathcal{R}$ be two connected species, then $L(\mathcal{S} + \mathcal{R}, s) = L(\mathcal{S}, s) + L(\mathcal{R}, s)$ and $L(\mathcal{S} * \mathcal{R}, s) = L(\mathcal{S}, s) \times L(\mathcal{R}, s)$.*

Proof. Checking the sum is straightforward. Let us do the computation for the product. We need to compute $|(\mathcal{S} * \mathcal{R})(\underline{n})|$:

$$
\begin{aligned}
|(\mathcal{S} * \mathcal{R})(\underline{n})| &= \left| \biguplus_{P \underline{\times} Q = \underline{n}} \mathcal{S}(P) \times \mathcal{R}(Q) \right| \\
&= \left| \biguplus_{d|n} \frac{n!}{d!(n/d)!} \mathcal{S}(\underline{d}) \times \mathcal{R}(\underline{n/d}) \right| \\
&= \sum_{d|n} \frac{n!}{d!(n/d)!} \left|\mathcal{S}(\underline{d})\right| \times \left|\mathcal{R}(\underline{n/d})\right|
\end{aligned}
$$

We have:

$$
\begin{aligned}
L(\mathcal{S} * \mathcal{R}, s) &= \sum_{n \in \mathbb{N}^*} \frac{|(\mathcal{S} * \mathcal{R})(\underline{n})|}{n!} n^{-s} \\
&= \sum_{n \in \mathbb{N}^*} \sum_{d|n} \frac{|\mathcal{S}(\underline{d})|}{d!} \times \frac{\left|\mathcal{R}(\underline{n/d})\right|}{(n/d)!} d^{-s} (n/d)^{-s} \\
&= L(\mathcal{S}, s) \times L(\mathcal{R}, s)
\end{aligned}
$$

□

Since categorical L functions are contravariant, we cannot hope them to be stable under composition. However, we can still generalize the formula of the plethysm by replacing usual partitions by some kind of "multiplicative partitions". Let us do so, we will interpret the construction at the level of categorical L function afterward.

Definition 1.2.4.14. Let A be a set and $k \in \mathbb{N}$, a *quotientation of A of length k* is a set $P = \{P_1, \dots, P_k\}$ such that $A = P_1 \underline{\times} \dots \underline{\times} P_k$, and $\forall i, |P_i| \neq 1$. We denote by $P \not\ltimes^k A$ the fact that P is a quotientation of A of length k.

Proposition 1.2.4.15. *Let $n, k \in \mathbb{N}$, and let $\lambda_1, \dots, \lambda_k \in \mathbb{N}^*$ such that $\lambda_1 \times \dots \times \lambda_k = n$, then the number of $P \not\ltimes^k \underline{n}$ such that we can order P so that $|P_i| = \lambda_i$ is $\frac{n!}{m_\lambda \lambda_1! \dots \lambda_k!}$, with*

$$
m_\lambda = \prod_{j=2}^{n} \kappa_j
$$

where κ_j is the number of part of size j of λ.

Proof. Let us fill each case of a hypercube of size $\lambda_1 \times \dots \times \lambda_k$ with a number between 1 and n such that each number appears exactly once. We have $n!$ ways to do so. The collection of the slides of the hypercube in the direction i is P_i. However each P_i is not ordered, hence, we need to divide by $\lambda_i!$. Moreover, P itself is not ordered hence we can permute parts of the same size, we get:

$$
\frac{n!}{m_\lambda \lambda_1! \dots \lambda_k!}
$$

□

Proposition 1.2.4.16. *Let $P \not\ltimes^k \underline{n}$, the number of $(\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$ such that we can order P so that $|P_i| = \lambda_i$ is $\frac{k!}{m_P}$ where:*

$$
m_P = \prod_{j=1}^{n} \kappa_j!
$$

where κ_j is the number of part of size j of P.

Proof. To get such a $(\lambda_1, \dots, \lambda_k)$, we need to put a order on the parts of P. However, exchanging parts of the same size does not change the $(\lambda_1, \dots, \lambda_k)$, hence we need to divide by $\kappa_j!$ for each j. □

Definition 1.2.4.17. Let $\mathcal{S}$ and $\mathcal{R}$ be two species such that $\mathcal{R}$ is strongly connected, then the *arithmetic plethysm* of $\mathcal{S}$ and $\mathcal{R}$ is the species $\mathcal{S} \,\square\, \mathcal{R}$ defined by:

$$(\mathcal{S} \,\square\, \mathcal{R})(A) = \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash\!\!\!\times^k A} \mathcal{S}(P) \times \prod_{p \in P} \mathcal{R}(p)$$
$$(\mathcal{S} \,\square\, \mathcal{R})(\sigma) = \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash\!\!\!\times^k A} \mathcal{S}(\sigma^{(P)}) \times \prod_{p \in P} \mathcal{R}(\sigma^p)$$

We denote by $\sigma^{(P)}$ the map induced by σ on $P \vdash\!\!\!\times^k A$.

A first inspection of the arithmetic plethysm show that it is not monoidal. In fact the only relation it seems to satisfy is that $X \,\square\, \mathcal{S} = \mathcal{S}$. However, we can still try to understand the arithmetic plethysm at the level of L function.

Proposition 1.2.4.18. *Let $\mathcal{S}$ and $\mathcal{R}$ be two species such that $\mathcal{R}$ is strongly connected, then $L(\mathcal{S} \,\square\, \mathcal{R}, s) = f_{\mathcal{S}} \circ L(\mathcal{R}, s)$.*

Proof. Let us compute $|(\mathcal{S} \,\square\, \mathcal{R})(\underline{n})|$:

$$\begin{aligned}
|(\mathcal{S} \,\square\, \mathcal{R})(\underline{n})| &= |\biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash\!\!\!\times^k \underline{n}} \mathcal{S}(P) \times \prod_{p \in P} \mathcal{R}(p)| \\
&= \left| \biguplus_{k \in \mathbb{N}} \biguplus_{\lambda_1 \times \dots \times \lambda_k = n} \frac{n!}{k! \lambda_1! \dots \lambda_k!} \mathcal{S}(\underline{k}) \times \prod_{i=1}^{k} \mathcal{R}(\underline{\lambda_i}) \right| \\
&= \sum_{k \in \mathbb{N}} \sum_{\lambda_1 \times \dots \times \lambda_k = n} \frac{n!}{k! \lambda_1! \dots \lambda_k!} |\mathcal{S}(\underline{k})| \times \prod_{i=1}^{k} \left|\mathcal{R}(\underline{\lambda_i})\right| \\
&= \sum_{k \in \mathbb{N}} \frac{1}{k!} |\mathcal{S}(\underline{k})| \sum_{\lambda_1 \times \dots \times \lambda_k = n} n! \times \prod_{i=1}^{k} \frac{1}{\lambda_i!} \left|\mathcal{R}(\underline{\lambda_i})\right|
\end{aligned}$$

Hence, we get:

$$\begin{aligned}
L(\mathcal{S} \,\square\, \mathcal{R}, s) &= \sum_{n \in \mathbb{N}^*} \frac{|(\mathcal{S} \,\square\, \mathcal{R})(n)|}{n!} n^{-s} \\
&= \sum_{n \in \mathbb{N}^*} \sum_{k \in \mathbb{N}} \frac{1}{k!} |\mathcal{S}(\underline{k})| \sum_{\lambda_1 \times \dots \times \lambda_k = n} \frac{n!}{n!} \times \prod_{i=1}^{k} \frac{1}{\lambda_i!} \left|\mathcal{R}(\underline{\lambda_i})\right| n^{-s} \\
&= \sum_{n \in \mathbb{N}^*} \sum_{k \in \mathbb{N}} \frac{1}{k!} |\mathcal{S}(\underline{k})| \sum_{\lambda_1 \times \dots \times \lambda_k = n} \prod_{i=1}^{k} \frac{1}{\lambda_i!} \left|\mathcal{R}(\underline{\lambda_i})\right| \lambda_i^{-s} \\
&= \sum_{k \in \mathbb{N}} \frac{1}{k!} |\mathcal{S}(\underline{k})| L(\mathcal{R}, s)^k \\
&= f_{\mathcal{S}} \circ L(\mathcal{R}, s)
\end{aligned}$$

□

Proposition 1.2.4.19. *Let $\mathcal{S}$ and $\mathcal{R}$ be two species such that $\mathcal{R}$ is strongly connected, then $\mathcal{L}_{\mathcal{S} \,\square\, \mathcal{R}} = \mathcal{F}_{\mathcal{S}} \circ \mathcal{L}_{\mathcal{R}}$.*

Proof. Let us do the computation for the objects. We have:

$$\begin{aligned}
\mathcal{F}_{\mathcal{S}}(\mathcal{L}_{\mathcal{R}}(A)) &= \biguplus_{k\in\mathbb{N}} \mathcal{S}(\underline{k}) \times_{\mathfrak{S}_k} \left(\biguplus_{m\in\mathbb{N}^*} \mathcal{R}(\underline{m}) \times_{\mathfrak{S}_m} \mathrm{Hom}(A, \underline{m}) \right) \\
&= \biguplus_{n\in\mathbb{N}} \biguplus_{k\in\mathbb{N}} \biguplus_{\lambda_1\times\dots\lambda_k=n} \left(\mathcal{S}(\underline{k}) \times \prod_{i=1}^{k} \mathcal{R}(\underline{\lambda_i}) \times \mathrm{Hom}(A, \underline{n}) \right) / \left(\mathfrak{S}_k \times \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k} \right) \\
&= \biguplus_{n\in\mathbb{N}} \biguplus_{k\in\mathbb{N}} \biguplus_{P \vdash\!\!\!\times^k \underline{n}} \left(\mathcal{S}(P) \times \prod_{p\in P} \mathcal{R}(P) \times \mathrm{Hom}(A, \underline{n}) \right) / \mathfrak{S}_n \\
&= \biguplus_{n\in\mathbb{N}} \biguplus_{k\in\mathbb{N}} \biguplus_{P \vdash\!\!\!\times^k \underline{n}} \left(\mathcal{S}(P) \times \prod_{p\in P} \mathcal{R}(P) \right) \times_{\mathfrak{S}_n} \mathrm{Hom}(A, \underline{n}) \\
&= \mathcal{L}_{\mathcal{S} \,\square\, \mathcal{R}}(A)
\end{aligned}$$

We allow ourselves not to give all the details since we already did a lot of similar computations. □

Corollary 1.2.4.20. *Let $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{U}$ be three species such that $\mathcal{R}$ and $\mathcal{U}$ are strongly connected species, then $\mathcal{S} \,\square (\mathcal{R} \,\square\, \mathcal{U}) = (\mathcal{S} \circ \mathcal{R}) \,\square\, \mathcal{U}$.*

This fact shows that the category of categorical L functions is stable by left composition with analytic functors. Moreover, the arithmetic plethysm gives a combinatorial interpretation of such composition. With our formalism, it is tempting to construct the species analog of some of the most famous L function, the Hasse-Weil zeta function of an algebraic variety (we recall that the Riemann zeta function is the Hasse-Weil zeta function of the point). Let V be an algebraic variety over $\mathbb{F}_p$, then we would like to define the *point evaluation species* $F_{(V,p)}$ such that $F_{(V,p)}(A)$ would be the "A points of V". The issue is that $V(A)$ is defined only if A is a field. Let A be a finite set such that $|A| = p^n$ since A does not admit any structure of field of characteristic p otherwise. Let us use the following trick to define $F_{(V,p)}(A)$:

$$F_{(V,p)} : A \mapsto \mathrm{Bij}(A, \mathbb{F}_{p^n}) \times_{\mathrm{Aut}(\mathbb{F}_{p^n})} V(\mathbb{F}_{p^n})$$

Let us try to understand this definition: We would like to get the set of "A points of V" which is not defined if A is not a field. Let us put a structure of field on A, an easy way to do so is by considering a bijection $A \to \mathbb{F}_{p^n}$, since $V(\mathbb{F}_{p^n})$ is well defined, such a bijection allows us to define $V(A)$. Then we say that such sets of points are the same if we can go from one to another via an automorphism of $\mathbb{F}_{p^n}$ (hence by the Frobenius). With this, we can copy the definition of the zeta function of the algebraic variety V. Let p be a prime, and $N_n = |V(\mathbb{F}_{p^n})|$. We define $Z_{(V,p)} = E \,\square\, F_{(V,p)}$ the local zeta species of V. Let $\mathbb{P}$ be the set of prime numbers, we define the zeta species of V by:

$$Z_V = \mathop{\Huge *}_{p\in\mathbb{P}} Z_{(V,p)}$$

Let us compute the Dirichlet series of Z_V. First we need to understand the species $F_{(V,p)}$. We have that $F_{(V,p)}(A)$ is non-empty only if $|A| = p^n$ for some $n \in \mathbb{N}^*$, moreover we have:

$$\left|F_{(V,p)}(\underline{p^n})\right| = \left|\mathrm{Bij}(\underline{p^n}, \mathbb{F}_{p^n}) \times_{\mathrm{Aut}(\mathbb{F}_{p^n})} V(\mathbb{F}_{p^n})\right| = \frac{(p^n! N_n)}{n}$$

Then its associated L function is:

$$L(F_{(V,p)}, s) = \sum_{n\in\mathbb{N}^*} \frac{|F_{(V,p)}(\underline{p^n})|}{p^n!} p^{-ns} = \sum_{n\in\mathbb{N}^*} \frac{N_n}{n} p^{-ns}$$

Hence, we have:

$$L(Z_{(V,p)}, s) = \exp(L(F_{(V,p)}, s))$$

Which is exactly the local zeta function of the algebraic variety V. Hence, $L(Z_V, s)$ is exactly the zeta function of V.
We can do the computation for the point. Let F_p be the point evaluation species of the single point, then for A such that $|A| = p^n$ we have:

$$F_p : A \mapsto \mathrm{Bij}(A, \mathbb{F}_{p^n}) \times_{\mathrm{Aut}(\mathbb{F}_{p^n})} \{*\}$$

Then we have:

$$L(F_p) = \sum_{n\in\mathbb{N}^*} \frac{|F_p(\underline{p^n})|}{p^n!} p^{-ns} = \sum_{n\in\mathbb{N}^*} \frac{1}{n} p^{-ns} = \ln\left(\frac{1}{1-p^{-s}}\right)$$

Let us define the *Riemann species* by:

$$Z = \mathop{\LARGE *}_{p\in\mathbb{P}} E \,\square\, F_p$$

Then we have:

$$L(Z, s) = \prod_{p\in\mathbb{P}} \exp\left(\ln\left(\frac{1}{1-p^{-s}}\right)\right) = \zeta(s)$$

We have also an interpretation of the Riemann species using the Artin-Wedderburn theorem:

$$Z(A) = \biguplus_{R \text{ semi-simple finite commutative ring up to isomorphism}} \mathrm{Bij}(A, R)/\mathrm{Aut}(R)$$

And the associated categorical L function is:

$$\mathcal{L}_Z(A) = \biguplus_{R \text{ semi-simple finite commutative ring up to isomorphism}} \mathrm{Hom}(A, R)/\mathrm{Aut}(R)$$

Moreover the computation of the L function of the Riemann species show that $|Z(\underline{n})| = n!$. And we get the contravariant functor:

$$\mathcal{L}_Z : A \mapsto \biguplus_{n\in\mathbb{N}^*} Z(\underline{n}) \times_{\mathfrak{S}_n} \mathrm{Hom}(A, \underline{n})$$

Since the Artin-Wedderburn theorem also works in the context of non-commutative rings, we can also define the *non-commutative Riemann species*. One should not be afraid by the word “non-commutative” since it only means that matrix algebras show up. Let us define the non-commutative Riemann species by:

$$Z_{\mathrm{nc}}(A) = \biguplus_{R \text{ semi-simple finite ring up to isomorphism}} \mathrm{Bij}(A, R)/\mathrm{Aut}(R)$$

Similarly, we have:

$$Z_{\mathrm{nc}} = \mathop{\LARGE *}_{p\in\mathbb{P}} E \,\square\, M_p$$

Where we define M_p on a finite set A such that $|A| = p^n$ by:

$$M_p(A) = \biguplus_{d^2\times\alpha=n} \mathrm{Bij}(A, M_d(\mathbb{F}_{p^\alpha}))/\mathrm{Aut}_{\mathrm{Ring}}(M_d(\mathbb{F}_{p^\alpha}))$$

A computation of the ring automorsphisms of $M_d(\mathbb{F}_{p^\alpha})$ show that:

$$L(Z_{\mathrm{nc}}, s) = \prod_{p\in\mathbb{P}} \prod_{\alpha\in\mathbb{N}^*} \prod_{d\in\mathbb{N}^*} \exp\left(\frac{p^\alpha - 1}{\alpha \prod_{i=1}^{d}(p^{\alpha d} - p^{\alpha(d-i)})} p^{-\alpha d^2 s}\right)$$

Another idea would be to replace rings by groups, and to define the *cyclic group species* and the *commutative group species* by:

$$Z_C(A) = \biguplus_{C \text{ cyclic group up to isomorphism}} \mathrm{Bij}(A, C)/\,\mathrm{Aut}(C)$$

$$Z_G(A) = \biguplus_{G \text{ finite commutative group up to isomorphism}} \mathrm{Bij}(A, G)/\,\mathrm{Aut}(G)$$

The classification of finite commutative groups and the fact that the decomposition of a commutative group in a product of commutative groups of order p^n is canonical show that:

$$Z_C = \mathop{\ast}_{p \in \mathbb{P}} C_p$$

$$Z_G = \mathop{\ast}_{p \in \mathbb{P}} G_p$$

Where we define C_p and G_p on a finite set A such that $|A| = p^n$ by:

$$C_p(A) = \mathrm{Bij}(A, \mathbb{Z}/p^n\mathbb{Z})/\,\mathrm{Aut}(\mathbb{Z}/p^n\mathbb{Z})$$

$$G_p(A) = \biguplus_{G \text{ finite commutative group of order } p^n \text{ up to isomorphism}} \mathrm{Bij}(A, G)/\,\mathrm{Aut}(G)$$

Since the number of automorsphisms of $\mathbb{Z}/n\mathbb{Z}$ is given by $\varphi(n)$ the Euler totient function, we get:

$$L(Z_C, s) = \sum_{n \in \mathbb{N}^*} \frac{1}{\varphi(n)} n^{-s} = \prod_{p \in \mathbb{P}} \left(\frac{p}{(p-1)(p^{s+1}-1)} \right)$$

The author does not know any formula for $L(Z_G, s)$, moreover the fact that the decomposition of a commutative group of order p^n into cyclic groups is not canonical show that $G_p \neq E \,\square\, C_p$. We can compute $|Z(\underline{n})|$, $|Z_{\mathrm{nc}}(\underline{n})|$, $|Z_G(\underline{n})|$ and $|Z_C(\underline{n})|$ for small values of n, and give a link to the according sequence in the OEIS [76].

n	1	2	3	4	5	6	...	16	...	*OEIS*
$\lvert Z(\underline{n})\rvert$	1	2	6	24	120	720	...	20922789888000	...	*A000142*
$\lvert Z_{\mathrm{nc}}(\underline{n})\rvert$	1	2	6	24	120	720	...	24409921536000	...	*A370360*
$\lvert Z_C(\underline{n})\rvert$	1	2	3	12	30	360	...	2615348736000	...	*A034381*
$\lvert Z_G(\underline{n})\rvert$	1	2	3	16	30	360	...	4250979532800	...	*A034382*

The author does not know if these Dirichlet series (except for the function ζ) have already been studied nor if they have any interesting properties. However, the author thinks that this manuscript is an appropriate place to display these quite strange species.

It has been brought to the attention of the author after writing this subsection that the computation of the species associated to the Hasse-Weil zeta functions has been done in the ncatlab [20], see [3]. This leads the author to the following page of the ncatlab: [4], where the same kind of computation are done for the Artin-Mazur zeta function on dynamical systems. A $\mathbb{Z}$-set is the data of a set S equipped with a bijection f. A morphism of $\mathbb{Z}$-set (R, g) to (S, f) is a map $\varphi : R \to S$ such that $\varphi \circ g = f \circ \varphi$. Let (S, f) be a $\mathbb{Z}$-set, a $\mathbb{Z}$-set over (S, f) is a $\mathbb{Z}$-set (R, g) with a morphism $\varphi : (R, g) \to (S, f)$. We can define the Artin-Mazur functor of (S, f) by:

$$\mathcal{J}_{(S,f)} : A \mapsto \biguplus_{(R,g) \text{ finite } \mathbb{Z}\text{-set over } (S,f) \text{ up to isomorphism}} \mathrm{Hom}(A, R)/\,\mathrm{Aut}(R, g)$$

Then it is shown in [4] that $\mathcal{J}_{(S,f)}$ is a categorical L function and that the associated Dirichlet series is the Artin-Mazur zeta function of (S, f).

1.2.5 Implicit species theorem

Same as with naive species, we need to work out the multi-variate case. And same as with naive species, no major difficulties arise. Let us fix the following notations. Let $X = (X_1, \dots, X_k)$ be a k-tuple of formal variables, $Y = (Y_1, \dots, Y_l)$ be an l-tuple of formal variables and Z a single formal variable. We use the same notations as for k-ogf, except that variables are capitalized in the case of multi-sort species.

Definition 1.2.5.1. Let $\mathbb{B}^k$Spe be the functor category $\mathbb{B}^k \to \mathrm{Set}$. A *$k$-sort species* $\mathcal{S}$ is an object of $\mathbb{B}^k$Spe, it is a functor $\mathcal{S} : \mathbb{B}^k \to \mathrm{Set}$, the k-sort species $\mathcal{S}$ is *finite* if each object of its essential image is finite. A k-sort species $\mathcal{S}$ is *connected* if $\mathcal{S}(\underline{0}) = \mathcal{S}(\underline{0}, \dots, \underline{0}) = \emptyset$. A *morphism of k-sort species* is a morphism in $\mathbb{B}^k$Spe, it is a natural transformation between two functors $\mathcal{S}, \mathcal{R} : \mathbb{B}^k \to \mathrm{Set}$. The category $\mathbb{B}^k$Spe is the *category of k-sort species*. If needed, we will specify the formal variables of the k-sort species $\mathcal{S}$ in brackets and write $\mathcal{S}[X] = \mathcal{S}[X_1, \dots, X_k]$ in order to be able to identify, derive or compose in a specific variable. One need to be careful as it is not the same convention as in [7].

Let us define the identification of variables, the partial derivative and the partial composition for k-sort species.

Definition 1.2.5.2. Let $\mathcal{S}[X, Y] = \mathcal{S}[X_1, \dots, X_k, Y_1, \dots, Y_l]$ be a $(k + l)$-sort species, then we can identify the variables $Y_1, \dots, Y_l$ to a single variable Z and get the $k+1$-sort species $\mathcal{S}[X_1, \dots, X_k, Y = Z]$ defined by:

$$\mathcal{S}[X, Y = Z](A, B) = \mathcal{S}[X_1, \dots, X_k, Z, \dots, Z](A, B) = \biguplus_{P \vDash^l B} \mathcal{S}[X, Y](A, P)$$
$$\mathcal{S}[X, Y = Z](\varphi, \psi) = \mathcal{S}[X_1, \dots, X_k, Z, \dots, Z](\varphi, \psi) = \biguplus_{P \vDash^l B} \mathcal{S}[X, Y](\varphi, \psi^P)$$

Here $A \in \mathbb{B}^k$ and $B \in \mathbb{B}$, and φ is a morphism in $\mathbb{B}^k$, and ψ in $\mathbb{B}$. We denote by $P \vDash^l B$ a length l ordered partition of B and ψ^P the morphism in $\mathbb{B}^l$ induced ψ such that the source is P.

Definition 1.2.5.3. Let $\mathcal{S}[X] = \mathcal{S}[X_1, \dots, X_k]$ be a k-sort species, then we can define the partial derivative of $\mathcal{S}$ with respect to the variable X_i for $i \in \underline{k}$, and get the k-sort species $\frac{\partial \mathcal{S}[X]}{\partial X_i}$ defined by:

$$\frac{\partial \mathcal{S}[X]}{\partial X_i}(A) = \mathcal{S}[X](A_1, \dots, A_i \uplus \{*\}, \dots, A_k)$$
$$\frac{\partial \mathcal{S}[X]}{\partial X_i}(\varphi) = \mathcal{S}[X](A\varphi_1, \dots, \varphi_i \uplus \mathrm{id}_{\{*\}}, \dots, \varphi_k)$$

Definition 1.2.5.4. Let $\mathcal{S}[X] = \mathcal{S}[X_1, \dots, X_k]$ be a k-sort species, and $\mathcal{R}_1, \dots, \mathcal{R}_k$ be l_1 up to l_k-sort species, then we can define $\mathcal{S}[\mathcal{R}]$ the composition of $\mathcal{S}$ with $\mathcal{R}_1$ up to $\mathcal{R}_k$. Let $A \in \mathbb{B}^{l_1} \times \cdots \times \mathbb{B}^{l_k}$, then the $(l_1 + \cdots + l_k)$-sort species $\mathcal{S}[\mathcal{R}]$ is defined by:

$$\mathcal{S}[\mathcal{R}](A) = \mathcal{S} \circ (\mathcal{R}_1, \dots, \mathcal{R}_k)(A) = \biguplus_{m \in \mathbb{N}^k} \biguplus_{P \vdash^m A} \mathcal{S}(A) \prod_{i=1}^{k} \prod_{p_i \in P^{(i)}} \mathcal{R}_i(p_i)$$
$$\mathcal{S}[\mathcal{R}](\varphi) = \mathcal{S} \circ (\mathcal{R}_1, \dots, \mathcal{R}_k)(\varphi) = \biguplus_{m \in \mathbb{N}^k} \biguplus_{P \vdash^m A} \mathcal{S}(\varphi) \prod_{i=1}^{k} \prod_{p_i \in P^{(i)}} \mathcal{R}_i((\varphi_i)_{|p_i})$$

Here we denote $P \vdash^m A$ to ease the notation, we should have written $(P^{(i)} \vdash^{m_i} A_i)_{i=1}^k$ where A_i is an l_i-tuple of finite sets and $P^{(i)}$ is a length m_i partition of an l_i-tuple. A length m_i partition of an l_i-tuple is an m_i-tuple of l_i-tuples of sets such that the union of the l_i-tuples is disjoint and equal to the l_i-tuple A_i.

Let $i \in \underline{k}$ and $\mathcal{U} = \mathcal{R}_i$, then if $\mathcal{R}_j = X_j$ for $j \neq i$ then we write $\mathcal{S} \circ_i \mathcal{U}$ instead of $\mathcal{S}[\mathcal{R}]$, and we call it the *partial composition* of $\mathcal{S}$ with $\mathcal{U}$ in i.
One need to be careful when composing. Indeed, to get $(l_1 + \cdots + l_k)$ different variables, we need to assume that the sets of variables of the $\mathcal{R}_i$ are disjoint. When it is not the case, we need to rename the variables to make them disjoint, and to compose the multi-sort species. Finally, we can identify the variables that need to be identified.

We can now show the compatibility relation between the partial derivative and the identification of variables, and the chain rule for k-sort species.

Proposition 1.2.5.5. *Let $\mathcal{S}[X, Y]$ be a $(k + l)$-sort species and Z a formal variable. Then:*

$$\frac{\partial \mathcal{S}[X, Y = Z]}{\partial Z} = \sum_{i=1}^{l} \frac{\partial \mathcal{S}}{\partial Y_i}(X, Y = Z)$$

Proof. Let us compute:

$$\begin{aligned}
\frac{\partial \mathcal{S}[X, Y = Z]}{\partial Z}(A, B) &= \mathcal{S}[X, Y = Z](A, B \uplus \{*\}) \\
&= \biguplus_{P \vDash^l B \uplus \{*\}} \mathcal{S}[X, Y](A, P) \\
&= \biguplus_{P \vDash^l B} \biguplus_{i=1}^{l} \mathcal{S}[X, Y](A, P_1, \ldots, P_i \uplus \{*\}, \ldots, P_l) \\
&= \biguplus_{i=1}^{l} \frac{\partial \mathcal{S}}{\partial Y_i}(X, Y = Z)
\end{aligned}$$

The same computation work at the level of morphisms. $\square$

Proposition 1.2.5.6 (Chain rule)**.** *Let $\mathcal{S}$ be a k-sort species and $R_1, \ldots, R_k$ be l_1 up to l_k-sort species. Then:*

$$\frac{\partial(\mathcal{S}[\mathcal{R}])}{\partial X_i} = \sum_{j=1}^{k} \left(\frac{\partial \mathcal{S}}{\partial Y_j} \right) [\mathcal{R}] \frac{\partial R_j}{\partial X_i}$$

Proof. If we assume that the sets of variables of the $\mathcal{R}_j$ are disjoint, and let $\mathcal{R}_j$ be the only one where X_i appears. Then we have:

$$\frac{\partial(\mathcal{S}[\mathcal{R}])}{\partial X_i} = \frac{\partial \mathcal{S}}{\partial Y_j}[\mathcal{R}] \frac{\partial \mathcal{R}_j}{\partial X_i}$$

If the variables of the $\mathcal{R}_j$ are not disjoint, then in order to compose the multi-sort species, we have renamed the variables to make them disjoint, composed the multi-sort species with disjoint sets of variables, and identified the variables that needed to be identified. Hence, the compatibility between the partial derivative and the identification of variables allow us to conclude. $\square$

Let us state the implicit species theorem.

Theorem 1.2.5.7 (Implicit species theorem)**.** *Let $H[X, Y]$ be a 2-sort species in two variables X and Y such that:*

$$H(\underline{0}, \underline{0}) = \emptyset \qquad \textit{and} \qquad \frac{\partial H}{\partial Y}(\underline{0}, \underline{0}) = \emptyset$$

Then there exists a unique species $\mathcal{A}$ such that:

$$\mathcal{A}[X] = H[X, \mathcal{A}[X]] \qquad \textit{and} \qquad \mathcal{A}(\underline{0}) = \emptyset$$

Proof. It is enough to define $\mathcal{A}(\underline{n})$ and its action of $\mathfrak{S}_n$ for each $n \in \mathbb{N}$ by induction. Let us notice that we have $H[X,Y](\underline{0},\underline{1}) = \emptyset$ because of the second condition. Let us now define the species $\mathcal{A}$ by induction. We define $\mathcal{A}(\underline{0}) = \emptyset$. Then we define $\mathcal{A}(\underline{n})$ for $n \in \mathbb{N}$ by induction. Let $n \in \mathbb{N}$, we assume that $\mathcal{A}(I)$ is defined for each $I \subset \underline{n}$. We know that we should have:

$$\begin{aligned}\mathcal{A}(\underline{n}) &= H[X,\mathcal{A}[X]](\underline{n}) \\ &= \biguplus_{I \sqcup J = \underline{n}} H[X,\mathcal{A}[Y]](I,J) \\ &= \biguplus_{I \sqcup J = \underline{n}} \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash^k J} H[X,Y](I,P) \times \prod_{p \in P} \mathcal{A}(p)\end{aligned}$$

We may notice that $\mathcal{A}(\underline{n})$ only appears when $I = \emptyset$ and $k = 1$, however since $H[X,Y](\underline{0},\underline{1}) = \emptyset$, the term where $\mathcal{A}(\underline{n})$ appears cancels. Hence, we can define:

$$\mathcal{A}(\underline{n}) = \biguplus_{I \sqcup J = \underline{n}} \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash^k J} H[X,Y](I,P) \times \prod_{p \in P} \mathcal{A}(p)$$

We have defined $\mathcal{A}$ by induction, such that $\mathcal{A}(\underline{0}) = \emptyset$ and $\mathcal{A}(\underline{n}) = H[X,\mathcal{A}[X]](\underline{n})$. Let us define the action of $\mathfrak{S}_n$ on $\mathcal{A}(\underline{n})$. Let $\sigma \in \mathfrak{S}_n$, we define:

$$\mathcal{A}(\sigma) = \biguplus_{I \sqcup J = \underline{n}} \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash^k J} H[X,Y](\sigma_{|I}, \sigma^P_{|J}) \times \prod_{p \in P} \mathcal{A}(\sigma_{|p})$$

From the induction, we have that $\mathcal{A}$ is unique. □

We will extensively use this theorem to define various species. Moreover, we can give an interpretation of species constructed this way using the analogy that a sum means a "or", a product means a "and", an X is an "atomic element" or a "root" and $E \circ \mathcal{S}$ means "set of $\mathcal{S}$-structures". As example, let $\mathcal{RT}$ be the species of rooted trees, a rooted tree is a root and a set of rooted trees (which are the children of the root). Hence, we have $\mathcal{RT} = X.E(\mathcal{RT})$. Via the implicit species theorem this is a definition of $\mathcal{RT}$.

1.3 Variation on species

We now have all the constructions we want to work with species. It is now that the very categorical nature of species kicks in. Indeed, we have defined species "valued in Set" however, because everything is categorical, we can replace Set by any category, and as long as the definition makes sense, we will have the same properties. But first, let us investigate a bit on the relation between species and naive species.

1.3.1 Shuffle species and ordered species

As we have defined species and naive species, it is quite natural to ask how they relate to each other. We have the following pair of functors between the discrete category $\mathbb{N}$ and the category $\mathbb{B}$:

$$\mathbb{N} \rightleftarrows \mathbb{B}$$

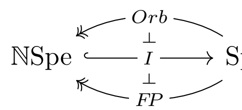
Pre-composition by those functors gives rise to the following pair of functors between the category of naive species and the category of species:

$$\mathbb{N}\mathrm{Spe} \underset{U}{\overset{I}{\rightleftarrows}} \mathrm{Spe}$$

Here, the functor I is given by the pre-composition by $\mathbb{B} \twoheadrightarrow \mathbb{N}$ and its image is the full subcategory of species endowed with a trivial group action. The functor U is given by the pre-composition by $\mathbb{N} \hookrightarrow \mathbb{B}$ and can be seen as the "forgetful functor" that forgets the group action. A naive guess would be that it is an adjunction, however (I, U) does not form an adjunction. Let us compute the adjoints of I and U, starting with I.

Definition 1.3.1.1. Let FP be the *fixed point functor* from Spe to $\mathbb{N}$Spe that send a species $\mathcal{S}$ to the naive species $FP(\mathcal{S})$ such that $FP(\mathcal{S})(n)$ is the set of fixed points of the action of $\mathfrak{S}_n$ on $\mathcal{S}(\underline{n})$. Since morphisms of species send fixed points to fixed points, FP is indeed a functor.
Let Orb be the *orbit functor* from Spe to $\mathbb{N}$Spe that send a species $\mathcal{S}$ to the naive species $Orb(\mathcal{S})$ such that $Orb(\mathcal{S})(n)$ is the set of orbits of the action of $\mathfrak{S}_n$ on $\mathcal{S}(\underline{n})$.

Proposition 1.3.1.2. *The functor FP is right adjoint to I and Orb is left adjoint to I. We have:*



Proof. Let $\mathcal{S}$ be a naive species and $\mathcal{R}$ be a species. Let us show that:

$$\mathrm{Hom}_{\mathbb{N}\mathrm{Spe}}(\mathcal{S}, FP(\mathcal{R})) \simeq \mathrm{Hom}_{\mathrm{Spe}}(I(\mathcal{S}), \mathcal{R})$$

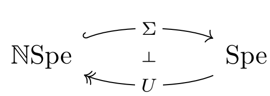
Let $\varphi : I(\mathcal{S}) \to \mathcal{R}$ be a morphism of species. Since the group actions in the species $I(\mathcal{S})$ are trivial, the image of an element of $I(\mathcal{S})$ by φ is a fixed point of $\mathcal{R}$. Hence, φ induces a morphism from $\mathcal{S}$ to $FP(\mathcal{R})$. Moreover, any morphism from $\mathcal{S}$ to $FP(\mathcal{R})$ will give rise to a morphism from $I(\mathcal{S})$ to $\mathcal{R}$ by post-composing with the inclusion of fixed points in $\mathcal{R}$. Hence, we have a bijection between $\mathrm{Hom}_{\mathbb{N}\mathrm{Spe}}(\mathcal{S}, FP(\mathcal{R}))$ and $\mathrm{Hom}_{\mathrm{Spe}}(I(\mathcal{S}), \mathcal{R})$. Those bijections are natural in $\mathcal{S}$ and $\mathcal{R}$, which show that we have an adjunction.
Let us show that $\mathrm{Hom}_{\mathbb{N}\mathrm{Spe}}(Orb(\mathcal{R}), \mathcal{S}) \simeq \mathrm{Hom}_{\mathrm{Spe}}(\mathcal{R}, I(\mathcal{S}))$. Let $\varphi : \mathcal{R} \to I(\mathcal{S})$ be a morphism of species. Let A be a finite set. Since $\varphi_A : \mathcal{R}(A) \to I(\mathcal{S})(A)$ is equivariant, it sends orbits to orbits. Moreover, since the group actions in $I(\mathcal{S})$ are trivial, the orbits are reduced to 1 element, and φ factorize through the projection of $\mathcal{R}(A) \to Orb(\mathcal{R})(A)$. Hence, φ induce a morphism from $Orb(\mathcal{R})$ to $\mathcal{S}$. Moreover, any morphism from $Orb(\mathcal{R})$ to $\mathcal{S}$ will give rise to a morphism from $\mathcal{R}$ to $\mathcal{S}$ by pre-composing with the projection on the orbits of $\mathcal{R}$. Hence, we have a bijection between $\mathrm{Hom}_{\mathbb{N}\mathrm{Spe}}(Orb(\mathcal{R}), \mathcal{S})$ and $\mathrm{Hom}_{\mathrm{Spe}}(\mathcal{R}, I(\mathcal{S}))$. Those bijections are natural in $\mathcal{S}$ and $\mathcal{R}$, which show that we have an adjunction. □

The functor U also admits a left adjoint and a right adjoint. In order to compute the left adjoint, we will use the species of total orders $\mathbb{L}$ introduced in Definition 1.2.1.5. Let use recall that $\mathbb{L}(A) = \mathrm{Bij}(\underline{n}, A)$ and $\mathbb{L}(\sigma) : f \to \sigma \circ f$ for each $A, B \in \mathbb{B}$ and $\sigma : A \to B$ a bijection.

Proposition 1.3.1.3. *The species $\mathbb{L}$ is finite, $f_{\mathbb{L}}(x) = \frac{1}{1-x}$ and the action of $\mathfrak{S}_n$ on $\mathbb{L}(\underline{n})$ is simply transitive. In particular, $Orb(\mathbb{L}) \simeq$ set and $FP(\mathbb{L}) \simeq 1 + X$.*

Proof. It is well known that $|\mathrm{Bij}(\underline{n}, A)| = n!$, hence $\mathbb{L}$ is finite and $f_{\mathbb{L}}(x) = \frac{1}{1-x}$. The action of $\mathfrak{S}_n$ on $\mathbb{L}(\underline{n})$ is simply transitive since it is the action of $\mathfrak{S}_n$ on itself by right multiplication. Hence, $Orb(\mathbb{L}) \simeq$ set and since $\mathfrak{S}_n$ is trivial only if $n = 0$ or $n = 1$, we have $FP(\mathbb{L}) \simeq 1 + X$. □

Proposition 1.3.1.4. *Let Σ be the* symmetrization functor *from* $\mathbb{N}$Spe *to* Spe *defined by $\Sigma(\mathcal{S}) = \mathbb{L} \odot I(\mathcal{S})$. The functor Σ is left adjoint to U. We have:*



Proof. Let us prove that Σ is left adjoint to U. Let $\mathcal{S}$ be a naive species and $\mathcal{R}$ be a species. Since the action of $\mathfrak{S}_n$ is simply transitive on $\mathbb{L}(\underline{n})$, an equivariant map from $\mathbb{L}(\underline{n}) \times \mathcal{S}(\underline{n})$ to $\mathcal{R}(\underline{n})$ is entirely determined by the induced map from the set $\mathcal{S}(\underline{n})$ to the set $\mathcal{R}(\underline{n})$. Moreover, any map from the set $\mathcal{S}(\underline{n})$ to the set $\mathcal{R}(\underline{n})$ give rise to an equivariant map from $\mathbb{L}(\underline{n}) \times \mathcal{S}(\underline{n})$ to $\mathcal{R}(\underline{n})$. Since the data on the sets $\underline{n}$ entirely determine species and morphisms of species, we have a bijection between $\mathrm{Hom}_{\mathrm{Spe}}(\Sigma(\mathcal{S}), \mathcal{R})$ and $\mathrm{Hom}_{\mathbb{N}\mathrm{Spe}}(\mathcal{S}, U(\mathcal{R}))$. Hence, Σ is left adjoint to U. □

Now that we have a better understanding of the situation, we see that we have two very different embedding of $\mathbb{N}$Spe in Spe. Indeed, one can embed $\mathbb{N}$Spe in Spe by the inclusion I and one can embed $\mathbb{N}$Spe in Spe by the functor Σ. Let us compare those two embeddings. Since we have $U \circ I = \mathrm{id}$ and $\Sigma \circ Orb = \mathrm{id}$, we have two main ways of relating $\mathbb{N}$Spe and Spe.

$$\mathbb{N}\mathrm{Spe} \underset{U}{\overset{I}{\rightleftarrows}} \mathrm{Spe} \qquad\qquad \mathbb{N}\mathrm{Spe} \underset{Orb}{\overset{\Sigma}{\rightleftarrows}} \mathrm{Spe}$$

The embedding via I is well suited to study combinatorial structures *with trivial group actions*, however these are not stable by derivative, product nor plethysm as we will see in the following. This embedding is still quite useful when one needs to forget the group actions since $U \circ I = \mathrm{id}$, for example if one needs to impose *a posteriori* a linear order which is the case when defining Gröbner bases. On the other hand, the embedding via Σ is well suited to study combinatorial structures *without group actions*, for example cases when the combinatorial structure depend on an *a priori* linear order. In this case, the embedding via Σ "puts" the linear order inside the combinatorial structure. The main downside of this embedding is that it is not stable by Hadamard product. We may remark that:

- The functor I and U are symmetric monoidal functors with respect to the sum and the Hadamard product. Moreover, they respect the structure morphisms of Proposition 1.2.2.12 which only involve the sum and the Hadamard product.

- The functor Σ and Orb commute with the derivative, are symmetric monoidal functors with respect to the sum and the product and are monoidal with respect to the plethysm. Moreover, they respect the structure morphisms of Proposition 1.2.2.12 which does not involve the Hadamard product. Moreover, Σ is compatible with the generating function, and the Schur functor.

Because those interpretations of the category $\mathbb{N}$Spe are quite different, we will use two different notations for this category.

Definition 1.3.1.5.

- Let us denote by ШSpe the category $\mathbb{N}$Spe when we implicitly compare it to Spe via the functors I and U. Effectively, ШSpe is the category $\mathbb{N}$Spe, however the operations on ШSpe are different from the operations on $\mathbb{N}$Spe. Indeed, the operations on ШSpe are the operations of Spe transported via the functors I and U. Objects of ШSpe are called *shuffle species*, and morphisms of ШSpe are called *morphisms of shuffle species.* We call ШSpe the *category of shuffle species.*

- Let us denote by $\mathbb{L}$Spe the category $\mathbb{N}$Spe when we implicitly compare it to Spe via the functors Σ and Orb. Effectively, $\mathbb{L}$Spe is the category $\mathbb{N}$Spe, however the operations on $\mathbb{L}$Spe are the operations of Spe transported via the functors Σ and Orb. Objects of $\mathbb{L}$Spe are *ordered species*, and morphisms of $\mathbb{L}$Spe are *morphisms of ordered species.* We call $\mathbb{L}$Spe the *category of ordered species.*

One need to be careful with the terminology. Indeed, what we are calling ordered species are also sometimes called linear species in the literature, see [7]. We choose to use the terminology ordered species since we will use the term linear species for a different notion.

The operations on $\mathbb{L}$Spe are exactly the same as the operations on $\mathbb{N}$Spe. However, the operations on ШSpe are different from the operations on $\mathbb{N}$Spe. Indeed, the sum and the Hadamard product are the same, but the derivative, the product and the plethysm are different. Strictly speaking, we have already defined the derivative, the product and the plethysm on ШSpe by:

- $\mathcal{S}' = U(I(\mathcal{S})')$ for a shuffle species $\mathcal{S}$,
- $\mathcal{S} \cdot \mathcal{R} = U(I(\mathcal{S}) \cdot I(\mathcal{R}))$ for two shuffle species $\mathcal{S}$ and $\mathcal{R}$,
- $\mathcal{S} \circ \mathcal{R} = U(I(\mathcal{S}) \circ I(\mathcal{R}))$ for two shuffle species $\mathcal{S}$ and $\mathcal{R}$.

However, it will be way more convenient to define them directly on ШSpe, and to have explicit formulas for them. Let us do this now.

Definition 1.3.1.6. Let $\mathcal{S}$ and $\mathcal{R}$ be two shuffle species. We define the following operations on shuffle species:

- the *shuffle derivative* $\mathcal{S}'$ such that for each $n \in \mathbb{N}$:
$$(\mathcal{S}')(n) = \mathcal{S}(n+1)$$
- the *sum* $\mathcal{S} + \mathcal{R}$ such that for each $n \in \mathbb{N}$:
$$(\mathcal{S} + \mathcal{R})(n) = \mathcal{S}(n) \uplus \mathcal{R}(n)$$
- the *shuffle product* (or shuffle Cauchy product) $\mathcal{S} \cdot_{Ш} \mathcal{R}$ such that for each $n \in \mathbb{N}$:
$$(\mathcal{S} \cdot_{Ш} \mathcal{R})(n) = \biguplus_{i=0}^{n} \binom{n}{i} \mathcal{S}(n-i) \times \mathcal{R}(i)$$
- the *Hadamard product* $\mathcal{S} \odot \mathcal{R}$ such that for each $n \in \mathbb{N}$:
$$(\mathcal{S} \odot \mathcal{R})(n) = \mathcal{S}(n) \times \mathcal{R}(n)$$
- if $\mathcal{R}$ is connected, the *shuffle plethysm* $\mathcal{S} \circ_{Ш} \mathcal{R}$ such that for each $n \in \mathbb{N}$:
$$(\mathcal{S} \circ_{Ш} \mathcal{R})(n) = \biguplus_{k \in \mathbb{N}} \mathcal{S}(k) \biguplus_{\lambda \vDash^k n} \frac{n!}{k! \lambda_1! \dots \lambda_k!} \prod_{i=1}^{k} \mathcal{R}(\lambda_i)$$

Here we use nA for $A \uplus \cdots \uplus A$ with n copies of A.

Because the formulas are quite similar, the shuffle derivative, the sum, the shuffle product, the Hadamard product and the shuffle plethysm satisfy the same relations as the ones of Proposition 1.2.2.12. Since this is the fourth times those 19 relations appear, let not state them again.

We will also need graded species in the sequel, let us define them.

Definition 1.3.1.7. A *k-graded species* $\mathcal{S}$ is a functor $\mathbb{B} \times \mathbb{N}^k \to \text{Set}$. Equivalently, it can be seen as a functor $\mathbb{B} \to \mathbb{N}^k\text{Spe}$. A k-graded species is *finite* if for each $A \in \mathbb{B}$, the k-sort ordered species $\mathcal{S}(A)$ is strongly finite. Species embeds in k-graded species by the functor $\text{Set} \to \mathbb{N}^k\text{Spe}$ such that the k-sort ordered species associated to the finite set A is $A(0) = A$ and $A(n) = \emptyset$ for $n \neq 0$. The *generating function* of a k-graded species $\mathcal{S}$ is the formal power series in $k+1$ variables defined by:

$$f_{\mathcal{S}} = \sum_{n \in \mathbb{N}} f_{\mathcal{S}(\underline{n})} \frac{1}{n!} x^n$$

All our constructions generalize naturally to k-graded species. If $k = 1$, we speak of *graded species*, if $k = 2$, we speak of *bigraded species*. In the sequel, we will only consider graded and bigraded species, and we will denote them $\mathcal{S}_{u,w}$ with u and w two formal variables corresponding to the two grading. We will also use the *graded non-empty set species* $E^w_{\geq 1}$ defined by $E^w_{\geq 1}(A,k) = \{*\}$ if $|A| = k+1$ and $E^w_{\geq 1}(A,k) = \emptyset$ else. Its generating function is:

$$f_{E^w_{\geq 1}}(x,w) = \sum_{n\in\mathbb{N}^*} \frac{1}{n!}x^n w^{n-1} = \frac{1}{w}\left(\exp(wx) - 1\right)$$

Forgetting the grading, is the same as evaluating the formal variable associated at 1.

1.3.2 Species in other categories

In order to do algebra, we may want to consider species in other categories, in particular species which "take values" in the category of vector spaces. Let us do a checklist of the properties of the category Set that we use to define species, and see if we can generalize the definition of species to other categories.

- To define the operations (derivation, sum product, Hadamard product and plethysm) on species (resp. ordered species, and shuffle species), and to show there properties, we heavily used the fact that Set is a symmetric rig-category for the disjoint union $\uplus$ and the cartesian product $\times$. However, as mentioned several times, we did not use any additional structure nor property of Set in this regard. Hence, species (resp. ordered species, and shuffle species) can be defined in any symmetric rig-category $\mathcal{C}$. (One may remark that ordered species can in fact be defined in any rig-category.)

- To define the generating function of a species (resp. ordered species), we used the fact that we have a "good notion of size" in Set. More precisely, we used the fact that we have a functor Set $\to \overline{\mathbb{N}}$ such that $|A \uplus B| = |A| + |B|$ and $|A \times B| = |A| \times |B|$. Since we are working with formal power series, we do not need to assume that the size takes values in $\overline{\mathbb{N}}$, taking values in $\overline{\mathbb{R}}$ is enough (one need to be careful as negative size may create convergence issues). Hence, generating functions of a $\mathcal{C}$-valued species (resp. ordered species) can be defined if $\mathcal{C}$ admits a rig-functor to $\overline{\mathbb{R}}$.

- To define the Schur functor of an ordered species and show its properties, we did not use other structures or properties than being a symmetric rig-category. Hence, the Schur functor of a $\mathcal{C}$-valued ordered species can always be defined and will satisfy the expected properties.

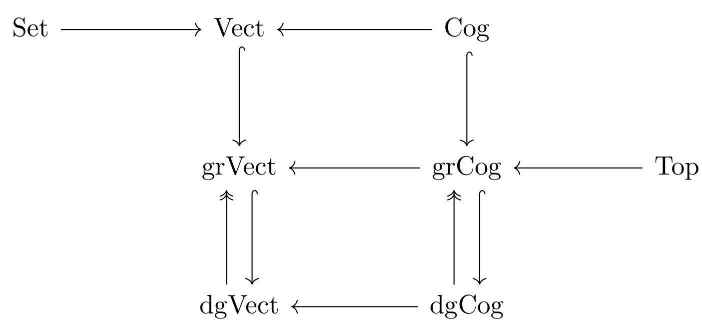
- To define the Schur functor of a species, we used a lot more structure. Indeed, we used the fact that we can define "actions of $\mathfrak{S}_n$ on a set". We can define actions of a group G in any category $\mathcal{C}$ by a functor $BG \to \mathcal{C}$. However, we also used the fact that one could quotient by the action of $\mathfrak{S}_n$ which is not the case in any category. Moreover, we used the fact that action of $\mathfrak{S}_n$ were compatible with the sum and the product, meaning that actions of $\mathfrak{S}_n$ commute with the sum by a structure isomorphism, and that this structure isomorphism is coherent with the structure isomorphism of the rig-category. This condition is way more restrictive than being a symmetric rig-category. However, it is respected by the category of sets, graded sets, vector spaces, graded vector spaces, differential graded vector spaces, co-commutative co-algebras, graded co-commutative co-algebras, differential graded co-commutative co-algebras, topological spaces, and so on...

Let us introduce the following notations:

- Let Vect be the category of vector spaces over a field $\mathbb{K}$ (of characteristic 0), the size of a vector space is its dimension.

- Let grVect be the category of graded vector spaces over a field $\mathbb{K}$, the size of a graded vector space is its Euler characteristic when it is well defined.
- Let dgVect be the category of differential graded vector spaces over a field $\mathbb{K}$, the size of a differential graded vector space is the Euler characteristic of its homology when it exits.
- Let Cog be the category of co-unitary co-commutative co-algebras over a field $\mathbb{K}$, the size of a co-unitary co-commutative co-algebra is its dimension.
- Let grCog be the category of graded co-unitary co-commutative co-algebras over a field $\mathbb{K}$, the size of a graded co-unitary co-commutative co-algebra is its Euler characteristic when it exits.
- Let dgCog be the category of differential graded co-unitary co-commutative co-algebras over a field $\mathbb{K}$, the size of a differential graded co-unitary co-commutative co-algebra is the Euler characteristic of its homology when it exits.
- Let Top be the category of convenient topological spaces (compactly generated Hausdorf for example), we do not define the size in this case.

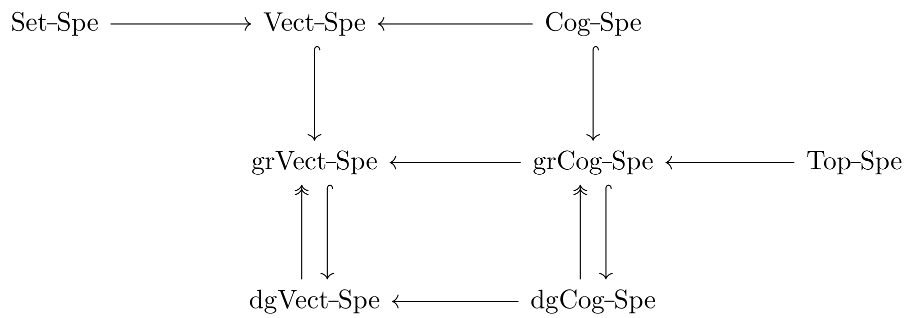
For $\mathcal{C}$ a rig-category, let us denote $\mathcal{C}$–Spe (resp. $\mathcal{C}$–$\mathbb{L}$Spe, and $\mathcal{C}$–ШSpe) the category of $\mathcal{C}$-valued species (resp. ordered species, and shuffle species). We have the following usual functors that are rig-functors respecting the size:



Let us explicitly define these functors:

- The functor Set $\to$ Vect is the free vector space functor, the functor that send a set A to the vector space $\mathrm{Span}_{\mathbb{K}}(A)$.
- The functor Vect $\to$ grVect (resp. Cog $\to$ grCog) is the embedding of Vect in grVect in the 0-th component, this is the functor that sends a vector space V to the graded vector space V concentrated in degree 0.
- The functor grVect $\to$ dgVect (resp. grCog $\to$ dgCog) is the embedding of grVect in dgVect by the trivial differential, this is the functor that sends a graded vector space V to the differential graded vector space V with trivial differential.
- The functor dgVect $\to$ grVect (resp. dgCog $\to$ grCog) is the homology (or the cohomology depending on the context) functor, this is the functor that sends a differential graded vector space V to its homology $H(V)$ (or its cohomology).
- The functor Cog $\to$ Vect (resp. grCog $\to$ grVect, and dgCog $\to$ dgVect) is the forgetful functor that sends a co-unitary co-commutative co-algebra to its underlying vector space.
- The functor Top $\to$ grCog is the homology functor, this is the functor that sends a topological space X to its singular homology $H(X)$. The fact that this functor is a rig-functor is a consequence of the Künneth formula, the co-algebra structure come from the diagonal map $X \to X \times X$. We will give a more detailed explanation in Section 2.3.1.

These functors induce the following functors of species (which respect the operations of species, and their coherence morphisms):



We have the same situation for ordered species and shuffle species. In the following, we will call set species or simply species the Set-valued species, linear species the Vect-valued species, and topological species the Top-valued species. These generalizations can also be done with graded species.

1.4 Rooted trees, rooted Greg trees and hypertrees

We will now apply the theory of species to the combinatorics of rooted trees. No theorem will be stated nor proved in this section however, we will introduce the five species that will play a key role in the last chapter, namely the species of rooted trees, rooted Greg trees, hyperforests, Greg hyperforests, and reduced Greg hyperforests. Let us start with the species related to trees, we will define the species related to hypertrees in the second subsection.

1.4.1 Combinatorics of rooted trees and rooted Greg trees

We will now apply the theory of species to the combinatorics of rooted trees and rooted Greg trees. These are quite usual definitions and results. Let us start by defining graphs, trees, and rooted trees.

Definition 1.4.1.1. A (simple) *graph structure* G on a finite set $\mathcal{V}$ is a pair $(\mathcal{V}, \mathcal{E})$ with $\mathcal{E} \subseteq \mathcal{P}_{=2}(\mathcal{V})$, we use the notation $\mathcal{P}_{=2}(\mathcal{V})$ to denote the set of subsets of $\mathcal{V}$ of cardinal 2. Elements of $\mathcal{V}$ are called the *vertices of* G and elements of $\mathcal{E}$ are the *edges of* G.

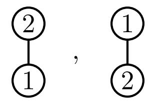
Definition 1.4.1.2. A *path of length* n *in* G is a pair of finite sequences $(v_0, \dots, v_n)$ and $(e_1, \dots, e_n)$ such that $v_0, \dots, v_n \in \mathcal{V}$, $e_1 \dots, e_n \in \mathcal{E}$, $v_i \in e_i$ and $v_i \in e_{i-1}$ for each $i \in \underline{n}$, and $v_i \neq v_j$ for $i \neq j$ except possibly if $(i, j) = (0, n)$ or $(n, 0)$ (meaning that paths do not go twice by the same vertex, except possibly if those are the source and the target). A path is a *cycle* if $v_0 = v_n$ and the e_i are different from each other.

Definition 1.4.1.3. A graph structure is *connected* if there exists a path between any two vertices. A *tree structure* on $\mathcal{V}$ is a connected graph structure on $\mathcal{V}$ without cycles. A *rooted tree structure* on $\mathcal{V}$ is a tree structure on $\mathcal{V}$ with a distinguished vertex called the *root*. A *forest structure* on $\mathcal{V}$ is partition of $\mathcal{V}$ into disjoint sets together with a rooted tree structure on each set, equivalently it is a graph structure on $\mathcal{V}$ without cycles together with a distinguished vertex in each connected component. Forests are assumed to be non-empty.

Definition 1.4.1.4. Let $(\mathcal{V}_1, \mathcal{E}_1)$ and $(\mathcal{V}_2, \mathcal{E}_2)$ be two graph structures. A *graph isomorphism* from $(\mathcal{V}_1, \mathcal{E}_1)$ to $(\mathcal{V}_2, \mathcal{E}_2)$ is a bijection $\sigma : \mathcal{V}_1 \to \mathcal{V}_2$ such that for each $v_1, v_2 \in \mathcal{V}_1$, $\{v_1, v_2\} \in \mathcal{E}_1$ if and only if $\{\sigma(v_1), \sigma(v_2)\} \in \mathcal{E}_2$. A *rooted tree isomorphism* is a graph isomorphism that preserves the root. A *forest isomorphism* is a bijection respecting the partitions of the set of vertices, and such that the induced bijection on each part is a rooted tree isomorphism.

The set of rooted tree structures of $\underline{2}$ is depicted in Figure 1.2. Rooted trees will always be depicted with the root at the bottom (just like in real life).

Figure 1.2: The set of rooted tree structures on the set $\underline{2}$.



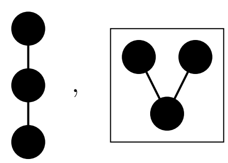
Some notations in rooted trees Let T be a rooted tree. Any vertex v of T admits a unique path to the root of T. If v is the root then it does not admit a parent. If v is not the root, the second vertex of this path w is called the *parent* of v and v is called a *child* of w. We also call child of w the tree above v rooted at v. With this convention, the children of w form a possibly empty set rooted trees. The edge $\{v, w\}$ is the *outgoing edge* of v and is an *incoming edge* of w. (We go down the tree.)

These definitions of graphs, trees and forests can instantly be translated into species, however, with these definitions, vertices of graphs, trees, and forests are always bijectively labeled, meaning that the vertices can be distinguished from each other. We will also need to consider unlabeled graphs, trees, and forests. Let us define them.

Definition 1.4.1.5. Let us consider the set of graph (resp. tree, rooted tree, and forest) structures on $\underline{n}$, we have an action of $\mathfrak{S}_n$ on this set given by the permutation of the vertices. An *unlabeled graph (resp. tree, rooted tree, and forest) with n vertices* is an orbit of the action of $\mathfrak{S}_n$ on the set of graph (resp. tree, rooted tree, and forest) structures on $\underline{n}$. Such an unlabeled graph (resp. tree, rooted tree, and forest) is *asymmetric* if the orbit contain exactly $n!$ elements.

The set of unlabeled rooted trees with 3 vertices is depicted in Figure 1.3. The non-asymmetric one is depicted inside a box. Indistinguishable vertices will always be depicted in black.

Figure 1.3: The set of unlabeled rooted trees with 3 vertices.

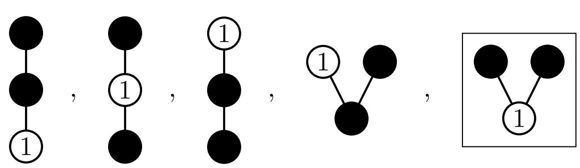


One may want to consider graphs where some vertices are distinguishable vertices, and others are indistinguishable vertices. Let us define them.

Definition 1.4.1.6. A *graph (resp. tree, rooted tree, and forest) structure with k black vertices* on the finite set $\mathcal{V}$ is an orbit of the action of $\mathfrak{S}_k$ on the set of graph (resp. tree, rooted tree, and forest) structures on the finite set $\mathcal{V} \uplus \underline{k}$. Such a structure is *asymmetric* if the orbit contains exactly $k!$ elements.

The set of rooted trees structure with 2 black vertices on $\underline{1}$ is depicted in Figure 1.4. All of these are asymmetric except the one in a box.

Figure 1.4: The set of rooted trees with 2 black vertices on the set $\underline{1}$.



In the sequel, we will refer to the distinguishable vertices as the white vertices, and to the indistinguishable vertices as the black vertices, moreover the set of black vertices a tree τ will always be denoted $BV(\tau)$. Let us now define the relevant species.

Definition 1.4.1.7. Let $\mathcal{RT}$ be the *rooted trees species* such that $\mathcal{RT}(A)$ is the set of rooted tree structures on A.

Proposition 1.4.1.8. *The species $\mathcal{RT}$ is the unique species satisfying the following functional equation:*

$$\mathcal{RT} = X \cdot E \circ \mathcal{RT}$$

Proof. Let $H[X,Y] = X \cdot E[Y]$, we may remark that H satisfies the conditions of the implicit species theorem. Hence, there is a unique species satisfying the functional equation. Let us show that $\mathcal{RT}$ satisfies the functional equation. Let A be a finite set, let us show that $\mathcal{RT}(A) = (X \cdot E \circ \mathcal{RT})(A)$. Let T be a rooted tree structure on A, let r be the root of T, and let $\{r, r_1\}, \ldots \{r, r_k\}$ be the edges of T containing r. Let T_i be the maximal connected subgraph of T containing r_i and not containing r. We have that T_i is a tree, let us root T_i in r_i. Then the data of $(r, \{T_1, \ldots, T_k\})$ is the same as the data of T. Indeed, connecting r to each root of the T_i recovers T, and removing the edges containing r recovers $(r, \{T_1, \ldots, T_k\})$. Hence, we have a bijection between the set of rooted tree structures on A and the set of pairs $(r, \{T_1, \ldots, T_k\})$ where $r \in A$ and $T_1, \ldots, T_k$ are rooted tree structures on the block of a partition of $A \setminus \{r\}$. □

An immediate consequence of this definition, is a formula for the generating function of the species of rooted trees. Let us denote by $\text{rev}_x(f)$ the composition reversed of f in the variable x, that is $\text{rev}_x(f) \circ_x f = f \circ_x \text{rev}_x(f) = \text{id}$.

Proposition 1.4.1.9. *We have:*

$$f_{\mathcal{RT}}(x) = \text{rev}_x\left(x \exp(-x)\right)$$

Proof. From the definition of $\mathcal{RT}$, we have:

$$f_{\mathcal{RT}}(x) = x \exp f_{\mathcal{RT}}(x)$$

Hence:

$$f_{\mathcal{RT}} \cdot \exp(-f_{\mathcal{RT}}) = x$$

□

A Lagrange inversion formula let us compute the coefficients of the series $f_{\mathcal{RT}}(x)$, we have that

$$|\mathcal{RT}(n)| = n^{n-1},$$

which is known as the Cayley formula. This is the sequence A000169 of the OEIS, see [76].

Another example of species that we will use is the species of twisting rooted trees, the name comes from the operadic twisting that we will define in Subsection 2.3.3.

Definition 1.4.1.10. A *black rooted tree structure* on A is a rooted tree structure with k black vertices on A for some $k \in \mathbb{N}$. We define $\mathcal{BRT}$ the *black rooted trees species* such that $\mathcal{BRT}(A)$ is the set of black rooted tree structures on A. The *weight* of a black rooted tree is the number of black vertices. We define $\mathcal{TRT}$ the *twisting rooted trees species* as the subspecies of $\mathcal{BRT}$ of asymmetric black rooted trees.

One may notice that $\mathcal{BRT}$ is graded by the weight. Let u be a formal variable encoding the weight, we denote by $\mathcal{BRT}_u$ the species $\mathcal{BRT}$ graded by the weight. Same for the subspecies $\mathcal{TRT}_u$. One may notice that $\mathcal{BRT}_0 = \mathcal{TRT}_0 = \mathcal{RT}$.

Remark 1.4.1.11. We would like to write:

$$\mathcal{BRT}_u = X \cdot E \circ \mathcal{BRT}_u + u \cdot E \circ \mathcal{BRT}_u$$

Since the root of a black rooted tree is either white or black, and that the contribution from the white root would be $X \cdot E \circ \mathcal{BRT}_u$, and the contribution from the black root would be $u \cdot E \circ \mathcal{BRT}_u$. However, $\mathcal{BRT}_u$ is not connected, and although we stated in Remark 1.2.3.4 that we could define the plethysm in this case, we did not. Moreover, we may notice that $H[X,Y] = X \cdot E[Y] + uE[Y]$ does not satisfies the conditions of the implicit species theorem.

This species is quite useful despite the fact that it is not finite. However, we cannot apply the implicit species theorem. Let us try to solve this issue by tweaking the equation a bit. Let $H[X,Y]$ defined by:

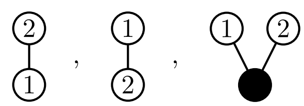
$$H[X,Y] = X \cdot E[Y] + uE_{\geq 2}[Y]$$

We remark that H satisfies the conditions of the implicit species theorem. Let us understand the species it defined. It is the species of rooted Greg trees. Rooted Greg trees were initially introduced by Flight in [34] to encode and solve a problem from textual criticism which was first stated by Greg in [40] and studied by Maas in [62].

Definition 1.4.1.12. A *rooted Greg tree structure* on A is a rooted tree structure with k black vertices on A for some $k \in \mathbb{N}$ such that each black vertex has at least two children. We define $\mathcal{G}$ the *rooted Greg trees species* such that $\mathcal{G}(A)$ is the set of rooted Greg tree structures on A.

Once again this species is graded by the weight (the number of black vertices). We denote by $\mathcal{G}_u$ the species $\mathcal{G}$ graded by the weight. We denote $\mathcal{G}_k$ the species of rooted Greg tree of weight k. One may notice that $\mathcal{G}_0 = \mathcal{RT}$. One may notice that the condition that each black vertex has at least two children is exactly the condition that the species $E_{\geq 2}$ in the functional equation imposes, moreover one may check that it implies that each rooted Greg tree is asymmetric. Hence, $\mathcal{G}_u$ is a subspecies of $\mathcal{TRT}_u$. The set $\mathcal{G}(\underline{2})$ is depicted in Figure 1.5.

Figure 1.5: The set $\mathcal{G}(\underline{2})$.



Proposition 1.4.1.13. *The species of rooted Greg trees $\mathcal{G}_u$ is the unique graded species verifying:*

$$\mathcal{G}_u = X \cdot E \circ \mathcal{G}_u + uE_{\geq 2} \circ \mathcal{G}_u$$

Proof. Let $H[X,Y] = X \cdot E[Y] + uE_{\geq 2}[Y]$, we may remark that H satisfies the conditions of the implicit species theorem. Hence, there is a unique species satisfying the functional equation. The fact that $\mathcal{G}_u$ satisfies the functional equation is a direct consequence of the definition of $\mathcal{G}_u$. □

Proposition 1.4.1.14. *We have:*

$$f_{\mathcal{G}}(x,u) = \mathrm{rev}_x\left((x + ux + u)\exp(-x) - u\right)$$

Proof. From the definition of $\mathcal{G}_u$, we have:

$$f_{\mathcal{G}}(x,u) = x\exp(f_{\mathcal{G}}(x,u)) + u\exp(f_{\mathcal{G}}(x,u)) - uf_{\mathcal{G}}(x,u) - u$$

Hence:

$$(f_{\mathcal{G}}(x,u) - uf_{\mathcal{G}}(x,u) - u)\exp(f_{\mathcal{G}}(x,u)) - u = x$$

□

Remark 1.4.1.15. We can recover the recursive formula enumerating the rooted Greg trees from [45, Proposition 2.1] by resolving a differential equation. We have $h(x,u) = ((u+1)x + u)\exp(-x) - u$. Hence:

- $\frac{\partial h}{\partial x} = -((u+1)x - 1)\exp(-x)$,
- $\frac{\partial h}{\partial u} = (x+1)\exp(-x) - 1$.

Hence:

$$(u+2)h + \frac{\partial h}{\partial x} - (u+1)^2\frac{\partial h}{\partial u} = 1$$

Let f be such that $h(f(x,u),u) = h \circ (f, \mathrm{id}) = x$. We have that $\frac{\partial h}{\partial x} \circ (f, \mathrm{id}).\frac{\partial f}{\partial x} = 1$ and $\frac{\partial h}{\partial x} \circ (f, \mathrm{id}).\frac{\partial f}{\partial u} + \frac{\partial h}{\partial u} \circ (f, \mathrm{id}) = 0$, hence:

$$((u+2)x-1)\frac{\partial f}{\partial x} + (u+1)^2\frac{\partial f}{\partial u} = -1$$

Let $f(x,u) = \sum \frac{g_k(u)}{k!}x^k$ with g_k polynomials in u, we get the following recursive relation:

- $g_0(u) = 0$,
- $g_1(u) = 1$, and
- $g_{k+1}(u) = (u+2)kg_k(u) + (u+1)^2 g_k'(u)$.

Once again the Lagrange inversion formula (or the recursive formula we just get) allows us to compute the coefficients of the series $f_{\mathcal{G}}(x,u)$ which give the sequence A005264 and the non-graded version, the coefficients of the series $f_{\mathcal{G}}(x,1)$ which give the sequence A048160.

Definition 1.4.1.16. Since $\mathcal{G}_u$ is a subspecies of $\mathcal{TRT}_u$, we can define the *rooted non-Greg trees species* $\mathcal{NG}_u$ as the species such that $\mathcal{TRT}_u = \mathcal{G}_u + \mathcal{NG}_u$.

A generalization of rooted Greg trees that we will need in the sequel are rooted Greg trees with k different kind of black vertices. Let us define them.

Definition 1.4.1.17. A *rooted k-Greg tree structure* on A is a rooted Greg tree structure together with a labeling of the black vertices on $\underline{k}$. A labeling of the black vertices on $\underline{k}$ is a map $B \to \underline{k}$ where B is the set of black vertices. It is important to notice that this labeling is a priori not a bijection, meaning that some black vertices may have the same label. We define $\mathcal{G}_u^{(k)}$ the *rooted k-Greg tree species* such that $\mathcal{G}_u^{(k)}(A)$ is the set of rooted k-Greg tree structures on A.

One may notice that $\mathcal{G}_u^{(k)}$ is a subspecies of $\mathcal{G}_u^{(k+1)}$, that $\mathcal{G}_u^{(0)} = \mathcal{RT} \subseteq \mathcal{G}_u = \mathcal{G}_u^{(1)}$, and finally $\mathcal{G}_0^{(k)} = \mathcal{RT}$.

Proposition 1.4.1.18. *The species of rooted Greg trees $\mathcal{G}_u^{(k)}$ is the unique graded species verifying:*

$$\mathcal{G}_u^{(k)} = X \cdot E \circ \mathcal{G} + kuE_{\geq 2} \circ \mathcal{G}_u^{(k)}$$

Moreover:

$$f_{\mathcal{G}^{(k)}}(x,u) = \mathrm{rev}_x\left((x + kux + ku)\exp(-x) - ku\right)$$

Proof. The proofs are similar to the ones of $\mathcal{G}_u$. □

Remark 1.4.1.19. A strange phenomenon in combinatorics is that sometimes evaluating a generating function at -1 gives a meaningful answer although, we are technicality trying to understand our construction on a (non existing) set of with -1 element which does not mean anything. We have:

$$f_{\mathcal{G}}(x,-1) = -\ln(1-x)$$

We may notice that all the coefficients of this series are positive indeed those are $(n-1)!$, so it appears to enumerate something. The naive combinatorial interpretation would be that it enumerates the number of rooted Greg trees with -1 kind of black vertices, which is not meaningful at all. A conceptual explanation will be given in Section 3.2.

To summarize, we have defined a sequence of species $(\mathcal{G}^{(k)})_{k\in\mathbb{N}}$ starting by the rooted trees and the rooted Greg trees. We have computed their generation function, and an evaluation in -1 of the generating function hint that an interesting phenomenon is hidden behind this sequence.

1.4.2 Generalization to hypertrees

Let us now generalize what we have done to hypertrees. Hypertrees are a generalization of rooted trees where the edges can have more than two vertices. Same as trees are connected graphs without cycles, hypertrees are connected hypergraphs without cycles once the correct definitions are given. Hypergraphs were introduced by Berge in [6] to generalize graphs. Let us define them.

Definition 1.4.2.1. A (simple) *hypergraph structure* H on a finite set $\mathcal{V}$ is a pair $(\mathcal{V}, \mathcal{E})$ with $\mathcal{E} \subseteq \mathcal{P}_{\geq 2}(\mathcal{V})$, we use the notation $\mathcal{P}_{\geq 2}(\mathcal{V})$ to denote the set of subsets of $\mathcal{V}$ of cardinal at least 2. Elements of $\mathcal{V}$ are called the *vertices of* H and elements of $\mathcal{E}$ are the *edges of* H. The *hyper-weight* of an edge is its number of elements minus 2. Edges of hyper-weight 0 are *simple edge*, and edges of positive weight are the *hyperedges*.

Definition 1.4.2.2. A *path of length* n *in* H is a pair of finite sequences $(v_0, \ldots, v_n)$ and $(e_1, \ldots, e_n)$ such that $v_1, \ldots, v_n \in \mathcal{V}$, $e_0, \ldots, e_n \in \mathcal{E}$, $v_i \in e_i$ and $v_i \in e_{i-1}$ for each $i \in \underline{n}$, and $v_i \neq v_j$ for $i \neq j$ except possibly if $(i, j) = (0, n)$ or $(n, 0)$ (meaning that paths do not go twice by the same vertex, except possibly if those are the source and the target). A path is a *cycle* if $v_0 = v_n$ and the e_i are different from each other.

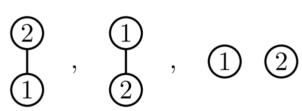
An important fact to notice is that if the intersection of two edges contains more than one vertex, then it create a cycle.

Definition 1.4.2.3. A hypergraph structure is *connected* if there exists a path between any two vertices. A *hypertree structure* on $\mathcal{V}$ is a connected hypergraph structure on $\mathcal{V}$ without cycles. A *rooted hypertree structure* on $\mathcal{V}$ is a tree structure on $\mathcal{V}$ with a distinguished vertex called the *root*. The *hyper-weight of a hypertree* is the sum of the hyper-weight of its edges. A *hyperforest structure* on $\mathcal{V}$ is a partition of $\mathcal{V}$ into disjoint sets together with a rooted hypertree structure on each set, equivalently it is a hypergraph structure on $\mathcal{V}$ without cycles together with a distinguished vertex in each connected component. The *hyper-weight of a hyperforest* is the number of hypertrees it contains minus 1 plus the hyper-weight of each of these hypertrees. Hyperforests are assumed to be non-empty.

Definition 1.4.2.4. Let $(\mathcal{V}_1, \mathcal{E}_1)$ and $(\mathcal{V}_2, \mathcal{E}_2)$ be two hypergraph structures. A *hypergraph isomorphism* from $(\mathcal{V}_1, \mathcal{E}_1)$ to $(\mathcal{V}_2, \mathcal{E}_2)$ is a bijection $\sigma : \mathcal{V}_1 \to \mathcal{V}_2$ such that for each $v_1, \ldots, v_k \in \mathcal{V}_1$, $\{v_1, \ldots, v_k\} \in \mathcal{E}_1$ if and only if $\{\sigma(v_1), \ldots, \sigma(v_k)\} \in \mathcal{E}_2$. A *rooted hypertree isomorphism* is a hypergraph isomorphism that preserves the root. A *hyperforest isomorphism* is a bijection respecting the partitions of the set of vertices, and such that the induced bijection on each part is a rooted hypertree isomorphism.

The set of hyperforest structures of $\underline{2}$ is depicted in Figure 1.6. Hypertrees will always be depicted with the root at the bottom (just like in real life, if hypertrees existed).

Figure 1.6: The set of hyperforest structures on the set $\underline{2}$.



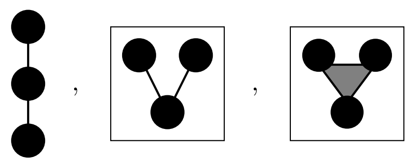
Some notations in rooted hypertrees Let T be a rooted hypertree. Any vertex v of T admits a unique minimal path to the root of T. If v is the root then it does not admit a parent. If v is not the root, the second vertex of this path w is called the *parent* of v and v is called a *child* of w. We also call child of w the hypertree above v rooted at v, with this convention, the children of the vertex w form a possibly empty set of hyperforests. The edge containing v and w is the *outgoing edge* of v and is an *incoming edge* of w. (We go down the hypertree.)

These definitions of hypergraphs, hypertrees, and hyperforests can instantly be translated into species, however, with these definitions, vertices of hypergraphs, hypertrees, and hyperforests are always bijectively labeled, meaning that the vertices can be distinguished from each other. We will also need to consider unlabeled hypergraphs, hypertrees, and hyperforests. Let us define them.

Definition 1.4.2.5. Let us consider the set of hypergraph (resp. hypertree, rooted hypertree, and hyperforest) structures on $\underline{n}$, we have an action of $\mathfrak{S}_n$ on this set given by the permutation of the vertices. An *unlabeled hypergraph (resp. hypertree, rooted hypertree, and hyperforest) with n vertices* is an orbit of the action of $\mathfrak{S}_n$ on the set of hypergraph (resp. hypertree, rooted hypertree, and hyperforest) structures on $\underline{n}$. Such an unlabeled hypergraph (resp. hypertree, rooted hypertree, and hyperforest) is *asymmetric* if the orbit contains exactly $n!$ elements.

The set of unlabeled rooted hypertrees with 3 vertices is depicted in Figure 1.7. The non-asymmetric ones are depicted in a box. Indistinguishable vertices will always be depicted in black. The hyperedges are depicted in gray.

Figure 1.7: The set of unlabeled rooted hypertrees with 3 vertices.



One may want to consider hypergraph where some vertices are distinguishable vertices, and others are indistinguishable vertices. Let us define them.

Definition 1.4.2.6. A *hypergraph (resp. hypertree, rooted hypertree, and hyperforest) structure with k black vertices* on the finite set $\mathcal{V}$ is an orbit of the action of $\mathfrak{S}_k$ on the set of hypergraph (resp. hypertree, rooted hypertree, and hyperforest) structures on the finite set $\mathcal{V} \uplus \underline{k}$. Such a structure is *asymmetric* if the orbit contain exactly $n!$ elements.

In the sequel, we will refer to the distinguishable vertices as the white vertices, and to the indistinguishable vertices as the black vertices, moreover the set of black vertices a hypertree τ will always be denoted $BV(\tau)$. Let us now define the relevant species.

Definition 1.4.2.7. Let $\mathcal{HT}$ be the *rooted hypertrees species* such that $\mathcal{HT}(A)$ is the set of rooted hypertree structures on A. Let $\mathcal{HF}$ be the *hyperforests species* such that $\mathcal{HF}(A)$ is the set of hyperforest structures on A. These species are graded by the hyper-weight. Let w be a formal variable encoding the hyper-weight, we denote $\mathcal{HT}_w$ the species $\mathcal{HT}$ graded by the hyper-weight. We denote by $\mathcal{HT}_k$ the species of rooted hypertree of hyper-weight k. Same for the species $\mathcal{HF}_w$.

Proposition 1.4.2.8. *The species $\mathcal{HT}$ is the unique species satisfying the following functional equation:*

$$\mathcal{HT}_w = X \cdot E \circ E^w_{\geq 1} \circ (\mathcal{HT}_w)$$

The species $\mathcal{HF}$ is the unique species satisfying the following functional equation:

$$\mathcal{HF}_w = E^w_{\geq 1} \circ (X \cdot E \circ \mathcal{HF}_w)$$

Moreover, $\mathcal{HF}_w = E^w_{\geq 1} \circ \mathcal{HT}_w$.

Proof. It is clear from the definition that:

$$\mathcal{HF}_w = E^w_{\geq 1} \circ \mathcal{HT}_w$$

Moreover, let $H_1[X,Y] = X \cdot E \circ E^w_{\geq 1}[Y]$ and $H_2[X,Y] = E^w_{\geq 1} \circ (X \cdot E[Y])$, we may notice that H_1 and H_2 satisfy the conditions of the implicit species theorem. Hence, for each equation, there is a unique species satisfying it. Let us show that we have:

$$\mathcal{HT}_w = X \cdot E \circ \mathcal{HF}_w$$

Let A be a finite set, and T be a rooted hypertree structure on A, let r be the root of T, and let $\{r\} \sqcup R_1, \dots \{r\} \sqcup R_k$ be the edges of T containing r. For $v \in R_i$, let $T_{i,v}$ be the maximal connected sub-hypergraph of T containing v and not containing r. We have that $T_{i,v}$ is a hypertree, let us mark v as the root of $T_{i,v}$. Then $F_i = \{T_{i,v} \mid v \in R_i\}$ is a hyperforest. Then the data of $(r, \{F_1, \dots, F_k\})$ is the same as the data of T. Indeed, connecting r to each root of the F_i recovers T, and removing the edges containing r recovers $(r, \{F_1, \dots, F_k\})$. Hence, we have a bijection between the set of rooted hypertree structures on A and the set of pairs $(r, \{F_1, \dots, F_k\})$ where $r \in A$ and $F_1, \dots, F_k$ are hyperforest structures on the block of a partition of $A \setminus \{r\}$.
To get the grading, it suffices to notice T and $(r, \{F_1, \dots, F_k\})$ have the same hyper-weight, where the hyper-weight of $(r, \{F_1, \dots, F_k\})$ is the sum of the hyper-weight of the F_i. □

Proposition 1.4.2.9. *We have:*

$$f_{\mathcal{HT}}(x,w) = \operatorname{rev}_x \left(x \exp \left(\frac{1}{w} \left(1 - \exp(wx)\right) \right) \right)$$

$$f_{\mathcal{HF}}(x,w) = \operatorname{rev}_x \left(\frac{1}{w} \ln\left(1 + wx\right) \exp(-x) \right)$$

Proof. From the definition of $\mathcal{HT}$, we have:

$$f_{\mathcal{HT}}(x,w) = x \exp \left(\frac{1}{w} \left(\exp\left(f_{\mathcal{HT}}(x,w)\right) - 1 \right) \right)$$

Hence:

$$f_{\mathcal{HT}}(x,w) \exp \left(\frac{1}{w} \left(1 - \exp\left(f_{\mathcal{HT}}(x,w)\right)\right) \right) = x$$

From the definition of $\mathcal{HF}$, we have:

$$f_{\mathcal{HF}}(x,w) = \frac{1}{w} \left(\exp\left(wx \exp\left(f_{\mathcal{HT}}(x,w)\right)\right) - 1 \right)$$

Hence:

$$\frac{1}{w} \ln\left(1 + w f_{\mathcal{HF}}(x,w)\right) \exp\left(-f_{\mathcal{HF}}(x,w)\right) = x$$

□

This is not the first time, and will not be the last, that the Lagrange inversion formula allows us to compute the coefficients of the series $f_{\mathcal{HT}}(x,w)$ and $f_{\mathcal{HF}}(x,w)$. The sequence A210586 is the sequence of the coefficients of $f_{\mathcal{HT}}(x,w)$ and the sequence A035051 is non-graded version, hence the sequence of the coefficients of $f_{\mathcal{HT}}(x,1)$. Same for $f_{\mathcal{HT}}(x,w)$, where its sequence of coefficients is A364709 and its non-graded version is A052888. The sequences of the OEIS may need to be slightly shifted to match the sequences of the coefficients of the above-mentioned series.

Let us introduce the notion of *tree shape*, defined by the author in [53], which allows us to decompose rooted hypertrees into rooted trees, and to reconstruct rooted hypertrees from rooted trees.

Definition 1.4.2.10. Let P be a partition of length k of A, a *tree shape on* P is a rooted hypertree structure with k black vertices on A such that:

- the root is black,

- simple edges are only between a black vertex and a white vertex such that the black vertex is bellow (closer to the root),
- hyperedges are only between a white vertex and several black vertices such that the white vertex is bellow,
- for each $p \in P$, we have a black vertex such that p is the set of the white vertices connected to it via simple edges.

A *forest shape on P* is a partition Q of P together with a tree shape on each block of the partition Q. Let $\mathrm{TS}_w(P)$ be the set of tree shape on P, and $\mathrm{FS}_w(P)$ be the set of forest shape on P. As before, w is the formal variable encoding the hyper-weight and the hyper-weight of a tree shape is the hyper-weight of its underlying rooted hypertree, same for the hyper-weight of a forest shape which is the hyper-weight of its underlying hyperforest.

Definition 1.4.2.11. Let T be a rooted hypertree structure on A. We denote by G be the maximal subgraph of T. Since G is a subgraph, it only keeps simple edges and all the hyperedges are removed. Since T has no cycles, G has no cycles either, hence G is a non-empty set of trees. Moreover, since T is rooted and connected, each tree τ of G admits a unique vertex v with the shortest path to the root, and we can mark v as the root of τ. Thus G is a forest. The forest G is the *forest of maximal subtrees of T*. Let S be the *tree shape of T*, it is the hypertree obtained by collapsing each maximal subtree of T into a corolla rooted at a black vertex with all the white vertices of the maximal subtree as leaves.
Let φ be the *decomposition map* such that $\varphi(T) = (P, S, G)$ with P the partition of the set of vertices of T corresponding to the connected components of G, S the tree shape of T, and G the forest of maximal subtrees of T.

Definition 1.4.2.12. The shape of a hyperforest F called *forest shape of F* is the set of the tree shapes of rooted hypertrees of F. We extend φ, the *decomposition map*, to the hyperforest by $\varphi(F) = (P, S, G)$ with P the partitions of the set of vertices of F corresponding to the connected components of G, S the forest shape of F, and G the union of the forests of maximal subtrees of each hypertrees of F.

Proposition 1.4.2.13. *Let A be a finite set. The decomposition map φ gives a bijection between $\mathcal{HT}(A)$ and the set of triples (P, S, G) such that P is a partition of A, S is a tree shape on P, and G is a forest such that for each $p \in P$ we have a tree of G with p as its set of vertices. Same with $\mathcal{HF}(A)$ and the set of triples (P, S, G) such that P is a partition of A, S is a forest shape on P, and G is a forest such that for each $p \in P$ we have a tree of G with p as its set of vertices.*

Proof. Since the forest shape of a hyperforest is the set of the tree shapes of its elements, it is enough to prove it for rooted hypertrees. Let T be a rooted hypertree structure on A, by definition the map φ send T to the triple (P, S, G) such that P is a partition of A, S is a tree shape and G is a forest such that for each $p \in P$ we have a tree of G with p as its set of vertices. Let rebuild a rooted hypertree from the data (P, S, G). Let $\psi(P, S, G)$ be the hypertree obtained by replacing corolla of S by the tree of G with the same set of vertices. We have that $\varphi \circ \psi$ and $\psi \circ \varphi$ are identity maps. $\square$

Corollary 1.4.2.14. *We have:*

$$\mathcal{HT}_w(A) = \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash^k A} \mathrm{TS}_w(P) \times \prod_{p \in P} \mathcal{RT}(p)$$

$$\mathcal{HF}_w(A) = \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash^k A} \mathrm{FS}_w(P) \times \prod_{p \in P} \mathcal{RT}(p)$$

Remark 1.4.2.15. One need to be careful since neither TS nor FS are species. We could define the species:

$$\mathcal{TS}_w : A \mapsto \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash^k A} \mathrm{TS}_w(P)$$

$$\mathcal{FS}_w : A \mapsto \biguplus_{k\in\mathbb{N}} \biguplus_{P\vdash^k A} \mathrm{FS}_w(P)$$

However, we have $\mathcal{HT}_w \neq \mathcal{TS}_w \circ \mathcal{RT}$ and $\mathcal{HF}_w \neq \mathcal{FS}_w \circ \mathcal{RT}$. Indeed, when computing $\mathcal{TS}_w \circ \mathcal{RT}$ or $\mathcal{FS}_w \circ \mathcal{RT}$, we lose the information of the partition. Hence, we do not get $\mathcal{HT}_w$ nor $\mathcal{HF}_w$. The sequence of cardinals of $\mathcal{TS}$ is A367752 and the sequence of cardinals of $\mathcal{FS}$ is A367753.

Since both the species of rooted hypertrees and the species rooted Greg trees generalize the species of rooted trees, it could be interesting to define a species that generalize both the species of rooted hypertrees and the species of rooted Greg trees. We have two quite natural ways of doing so. Either we consider rooted hypertrees with black and white vertices such that each black vertex has at least two incoming edges, or we replace $\mathcal{RT}$ by $\mathcal{G}_w$ in the above corollary. Let us start by the first option, and define *Greg hypertrees* which were introduced by the author in [53].

Definition 1.4.2.16. A *Greg hypertree structure* on A is a rooted hypertree with black vertices on A such that each black vertex has at least two children. A *Greg hyperforest structure* on A is a partition of A together with a Greg hypertree structure on each block of the partition. Let $\mathcal{GH}_{u,w}$ be the *Greg hypertrees species* such that $\mathcal{GH}_{u,w}(A)$ is the set of Greg hypertrees structures on A. Let $\mathcal{GF}_{u,w}$ be the *Greg hyperforests species* such that $\mathcal{GF}_{u,w}(A)$ is the set of Greg hyperforests structures on A. Those species are graded by the hyper-weight and the weight. Those species are bigraded by u the number of black vertices and w the hyper-weight.

Once again, we may notice from the definition that Greg hypertrees and Greg hyperforests are asymmetric.

Proposition 1.4.2.17. *The species of Greg hypertrees $\mathcal{GH}_{u,w}$ is the unique graded species verifying:*

$$\mathcal{GH}_{u,w} = X \cdot E \circ E^w_{\geq 1} \circ \mathcal{GH}_{u,w} + uE_{\geq 2} \circ E^w_{\geq 1} \circ \mathcal{GH}_{u,w}$$

The species of Greg hyperforests $\mathcal{GF}_{u,w}$ is the unique graded species verifying:

$$\mathcal{GF}_{u,w} = E^w_{\geq 1} \circ (X \cdot E \circ \mathcal{GF}_{u,w} + uE_{\geq 2} \circ \mathcal{GF}_{u,w})$$

Moreover, we have $\mathcal{GF}_{u,w} = E^w_{\geq 1} \circ \mathcal{GH}_{u,w}$.

Proof. It is clear from the definition that:

$$\mathcal{GF}_{u,w} = E^w_{\geq 1} \circ \mathcal{GH}_{u,w}$$

Moreover, let $H_1[X,Y] = X \cdot E \circ E^w_{\geq 1}[Y] + uE \circ E^w_{\geq 1}[Y]$ and $H_2[X,Y] = E^w_{\geq 1} \circ (X \cdot E[Y] + uE_{\geq 2}[Y])$, we may notice that H_1 and H_2 satisfy the conditions of the implicit species theorem. Hence, for each equation, there is a unique species satisfying it. Let us show that we have:

$$\mathcal{GH}_{u,w} = X \cdot E \circ E^w_{\geq 1} \circ \mathcal{GH}_{u,w} + uE_{\geq 2} \circ E^w_{\geq 1} \circ \mathcal{GH}_{u,w}$$

It suffice to notice that the root is either white or black, and that if it is black it has at least two children. The rest of the proof is the same as with rooted hypertrees. □

Proposition 1.4.2.18. *We have:*

$$f_{\mathcal{GH}}(x,u,w) = \mathrm{rev}_x\left(\left(x + \frac{u}{w}\left(\exp(wx)-1\right) + u - u\exp\left(\frac{1}{w}\left(\exp(wx)-1\right)\right)\right)\exp\left(-\frac{1}{w}\left(\exp(wx)-1\right)\right)\right)$$

$$f_{\mathcal{GF}}(x,u,w) = \mathrm{rev}_x\left(\left(\frac{1}{w}\ln\left(1+wx\right) + ux + u - u\exp(x)\right)\exp\left(-x\right)\right)$$

Proof. This is a direct consequence of the functional equations. □

We can once again use the Lagrange inversion formula to compute the coefficients of the series $f_{\mathcal{GH}}(x,u,w)$ and $f_{\mathcal{GF}}(x,u,w)$. The sequence of coefficients of $f_{\mathcal{GF}}(x,1,1)$ is A364816, the graded versions are also in the OEIS, $f_{\mathcal{GF}}(x,1,w)$ is A370949, and $f_{\mathcal{GF}}(x,u,1)$ is A370948.

The species of Greg hyperforest and rooted Greg hypertrees will be used in the sequel, however, the description with forest shapes and tree shapes does not work for those species. Let us define a subspecies of the species of Greg hypertrees that will admit such a description. Let us introduce the *reduced Greg hypertrees*.

Definition 1.4.2.19. A Greg hypertree is *reduced* if black vertices have no incoming hyperedges. A Greg hyperforest is *reduced* if each of its elements is reduced. Let $\mathcal{RGH}_{u,w}$ be the *reduced Greg hypertrees species* such that $\mathcal{RGH}_{u,w}(A)$ is the set of reduced Greg hypertrees structures on A. Let $\mathcal{RGF}_{u,w}$ be the *reduced Greg hyperforests species* such that $\mathcal{RGF}_{u,w}(A)$ is the set of reduced Greg hyperforests structures on A. Those species are bigraded by u the number of black vertices and w the hyper-weight.

Proposition 1.4.2.20. *The species of reduced Greg hypertrees $\mathcal{RGH}_{u,w}$ is the unique graded species verifying:*

$$\mathcal{RGH}_{u,v} = X \cdot E \circ E^{w}_{\geq 1} \circ \mathcal{RGH}_{u,w} + uE_{\geq 2} \circ \mathcal{RGH}_{u,w}$$

Moreover, we have $\mathcal{RGF}_{u,w} = E^{w}_{\geq 1} \circ \mathcal{RGH}_{u,w}$.

Proof. It is clear from the definition that:

$$\mathcal{RGF}_{u,w} = E^{w}_{\geq 1} \circ \mathcal{RGH}_{u,w}$$

Moreover, let $H[X,Y] = X \cdot E \circ E^{w}_{\geq 1}[Y] + uE_{\geq 2}[Y]$, we may notice that H satisfies the conditions of the implicit species theorem. Hence, the equation admit a unique solution. Let us show that we have:

$$\mathcal{RGH}_{u,v} = X \cdot E \circ \mathcal{RGF}_{u,w} + uE_{\geq 2} \circ \mathcal{RGH}_{u,w}$$

Let us notice that the root is either white or black. If the root is white, then the same argument as for rooted hypertrees applies, and we get:

$$X \cdot E \circ \mathcal{RGF}_{u,w}$$

If the root is black, then it has at least two children, and each edge is a simple edge. Hence, the same argument as for rooted trees applies, and we get:

$$uE_{\geq 2} \circ \mathcal{RGH}_{u,w}$$

Adding the two contributions gives the result. □

Proposition 1.4.2.21. *We have:*

$$f_{\mathcal{RGH}}(x,u,w) = \mathrm{rev}_x \left((x + u + ux - u\exp(x)) \exp\left(-\frac{1}{w} \left(\exp(wx) - 1\right) \right) \right)$$

$$f_{\mathcal{RGF}}(x,u,w) = \mathrm{rev}_x \left(\left(\frac{1}{w} \ln(1+wx) + u + \frac{u}{w} \ln(1+wx) - u \exp\left(\frac{1}{w} \ln(1+wx) \right) \right) \exp\left(-x\right) \right)$$

Proof. This is a direct consequence of the functional equations. □

We can once again use the Lagrange inversion formula to compute the coefficients of the series $f_{\mathcal{RGH}}(x,u,w)$ and $f_{\mathcal{RGF}}(x,u,w)$.

Proposition 1.4.2.22. *We have:*

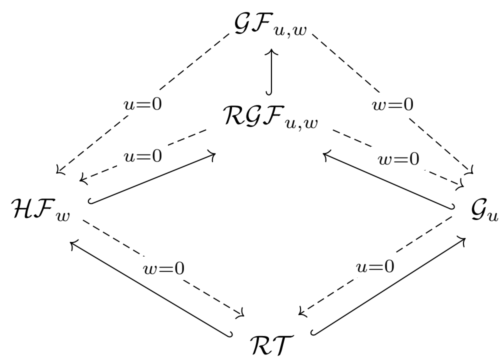
$$\mathcal{RGH}_{u,w}(A) = \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash^k A} \mathrm{TS}_w(P) \times \prod_{p \in P} \mathcal{G}_u(p)$$

$$\mathcal{RGF}_{u,w}(A) = \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash^k A} \mathrm{FS}_w(P) \times \prod_{p \in P} \mathcal{G}_u(p)$$

Proof. The proof is the same as for rooted hypertrees and hyperforests, we only need to check that the maximal subtrees of a reduced Greg hypertree are rooted Greg trees, which is the case since the black vertices have no incoming hyperedges and have at least two children. □

Remark 1.4.2.23. As with hypertrees, one needs to be careful since $\mathcal{RGH}_w \neq \mathcal{TS}_w \circ \mathcal{G}_u$ and $\mathcal{RGF}_w \neq \mathcal{FS}_w \circ \mathcal{G}_u$. Indeed, we would lose the information of the partition.

To summarize, we have defined the following generalization of rooted trees:



Chapter 2

Operads

We defined and studied species in the first chapter to lay down a nice mathematical fundament to the theory of operads that we are going to study in this chapter. This choice of introducing species before operads is not arbitrary, as it allows the author to emphasize their point of view: an operad should be understood as an algebra in the category of species. However, this point of view is quite anachronic. Indeed, the theory of combinatorial species was founded by Joyal in 1981, see [47], and the first instance of the word operad with a formal definition was in 1972 by May, see [67]. The notion of operads did not emerge out of nowhere, a lot of proto-examples (or straight example using other terminologies) can be found. We can think about the article of Artamonov in 1969, see [2], where *clones of multilinear operations* are literally *symmetric algebraic operads*. Before that in 1968, the article [9] of Boardman and Vogt where *category of operators in standard form* are introduced, and are in fact *PROP completion of symmetric algebraic operads* (that we are not going to define). One can even go back to 1898 with the work of Whitehead, see [80] where the *complete algebraic systems* are *quadratic operads generated by one operation of arity two* in disguise. We refer to the book of Markl, Shnider and Stasheff [61] for a more detail history of operads. We can also refer the article of Dotsenko [26] were some historical elements are discussed.
The operads we are interested in are the *symmetric algebraic operads*, these are algebra in the category of linear species. We once again point out that this terminology is not standard since there is no standard terminology in the literature. What we call linear species are species that take values in the category of vector spaces, this agree with the terminology in [60], however this is in contradiction with the terminology in [7] where what we call ordered species is called linear species, and what we call linear species is called tensor species. To define operads, we will start by defining the tree monads (which are secretly the free operad functors). One need to be very careful since tree monads have very few in common with the trees we defined at the end of the first chapter. In the first section, we will define the tree monads on set species, the species we have defined in the first chapter. After this section, we will switch to linear species as explain in Subsection 1.3.2. We will define symmetric algebraic operads, non-symmetric algebraic operads and shuffle algebraic operads, that we will respectively call operads, ns operads and shuffle operads for short. We will extend our definitions to the differential graded case, where the only additional difficulty is the appearance of Koszul signs. Then, since operads are algebras in disguise, we will quickly introduce some tools of homological operad theory which really is homological algebra on operads. The main tool we are introducing is the Koszul theory for operads, which is a generalization of the Koszul duality for associative algebras introduced by Priddy in 1970, see [72], and then extended to algebraic operads in 1994 by Ginzburg and Kapranov [39], and Getzler and Jones [37]. We will conclude by a classification of Koszul set operads generated by one operation of arity two that the author did in order to check a conjecture on the generating function of Koszul operads.

2.1 The tree monads

First let us recall the definition of a monoidal object and a monad. Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. A *monoid object* in $\mathcal{C}$ is a triple (M, γ, ε) with $M \in \mathcal{C}$, and with $\gamma : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ and $\varepsilon : \mathrm{id} \to \mathcal{M}$ morphisms of $\mathcal{C}$ such that γ is associative and ε is its unit, meaning that the following diagrams commute:

$$\begin{array}{ccc} (M \otimes M) \otimes M & = & M \otimes (M \otimes M) \xrightarrow{\mathrm{id} \otimes \gamma} M \otimes M \\ \downarrow{\scriptstyle \gamma \otimes \mathrm{id}} & & \downarrow{\scriptstyle \gamma} \\ M \otimes M & \xrightarrow{\gamma} & M \end{array} \qquad \text{(associativity diagram)}$$

$$\begin{array}{ccccc} 1 \otimes M & \xrightarrow{\varepsilon \otimes \mathrm{id}} & M \otimes M & \xleftarrow{\mathrm{id} \otimes \varepsilon} & M \otimes 1 \\ & \searrow{\scriptstyle =} & \downarrow{\scriptstyle \gamma} & \swarrow{\scriptstyle =} & \\ & & M & & \end{array} \qquad \text{(unit diagram)}$$

A *co-monoid object* in $\mathcal{C}$ is a a monoid object in the opposite category $\mathcal{C}^{op}$, meaning that it is a triple (M, Δ, η) with the above diagrams reversed. A *semi-monoid object* is a "non-necessarily unital" monoid object, meaning that it is a pair (M, γ) such that γ is associative. Semi-monoid are often called "semi-groups", however the author find this terminology misleading, and rather use the term "semi-monoid" to avoid confusion. A *monoid morphism* $f : (M, \gamma, \varepsilon) \to (N, \delta, \nu)$ is a morphism of $\mathcal{C}$ respecting the multiplication and the unit, meaning that the following diagram commute:

$$\begin{array}{ccc} M \otimes M & \xrightarrow{f \otimes f} & N \otimes N \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\ M & \xrightarrow{f} & N \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{\varepsilon} & M \\ & \searrow{\scriptstyle \nu} & \downarrow{\scriptstyle f} \\ & & N \end{array}$$

A *semi-monoid morphism* is a morphism of $\mathcal{C}$ respecting the multiplication, in particular any monoid morphism is a semi-monoid morphism, but the converse is false in general. A *monad* $(\mathcal{M}, \mu, \eta)$ on $\mathcal{C}$ is a monoid object in the endofunctor category $(\mathrm{End}(\mathcal{C}), \circ, \mathrm{id})$ of $\mathcal{C}$. An *algebra over* $\mathcal{M}$ is a pair (A, α) with $A \in \mathcal{C}$ and $\alpha : \mathcal{M}(A) \to A$ a morphism of $\mathcal{C}$ such that the following diagrams commute:

$$\begin{array}{ccc} (\mathcal{M} \circ \mathcal{M})(A) & \xrightarrow{\mathcal{M}(\alpha)} & \mathcal{M}(A) \\ \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \alpha} \\ \mathcal{M}(A) & \xrightarrow{\alpha} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathcal{M}(A) \\ & \searrow{\scriptstyle \mathrm{id}_A} & \downarrow{\scriptstyle \alpha} \\ & & A \end{array}$$

A *morphism of $\mathcal{M}$-algebras* $f : (A, \alpha) \to (B, \beta)$ is a morphism of $\mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}(A) & \xrightarrow{\mathcal{M}(f)} & \mathcal{M}(B) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{f} & B \end{array}$$

The category of $\mathcal{M}$-algebras and there morphisms is the *Eilenberg-Moore category of* $\mathcal{M}$ and is usually denoted $\mathcal{C}^{\mathcal{M}}$, however since in our context algebras over a monad will literally be algebras, we will denote it $\mathcal{M}$–Alg. Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a morphism of monads, it induces a functor:

$$\begin{aligned} \varphi_* : \mathcal{M}\text{–Alg} &\to \mathcal{N}\text{–Alg} \\ (A, \alpha) &\mapsto (A, \alpha \circ \varphi_A) \end{aligned}$$

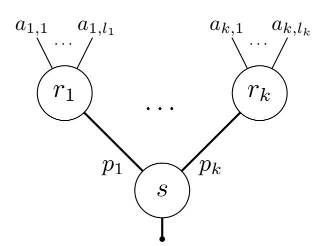
2.1.1 The symmetric tree monad

First let us recall the tree-like structure of the plethysm of species. Let $\mathcal{X}$ and $\mathcal{Y}$ be two species. We have that:

$$(\mathcal{X} \circ \mathcal{Y})(A) = \biguplus_{k \in \mathbb{N}} \biguplus_{P \vdash^k A} \mathcal{X}(P) \times \prod_{p \in P} \mathcal{Y}(p)$$

Hence, an element of $(\mathcal{X} \circ \mathcal{Y})(A)$ is a pair $(s, (r_p)_{p \in P})$ with $s \in \mathcal{X}(P)$ and $r_p \in \mathcal{Y}(p)$. We depict such an element by a tree-like structure, as depicted in Figure 2.1 with $P = \{p_1, \dots, p_k\}$ and $p_i = \{a_{i,1}, \dots, a_{i,l_i}\}$.

Figure 2.1: An element of $(\mathcal{X} \circ \mathcal{Y})(A)$



One need to be very careful with those tree-like representations of elements of $(\mathcal{X} \circ \mathcal{Y})(A)$ since they are very different from the species we have defined in Section 1.4.

Let us inductively define the *tree monad* $\mathcal{T}$ on the category of species.

Definition 2.1.1.1. Let $\mathcal{X}$ a connected species. We define the *tree monad* $\mathcal{T}$ by the following induction:

- $\mathcal{T}_{\leq 0}\{\mathcal{X}\} = X$,
- $\mathcal{T}_{\leq n+1}\{\mathcal{X}\} = \mathcal{X} \circ \mathcal{T}_{\leq n}\{\mathcal{X}\} + X$,
- $\mathcal{T}\{\mathcal{X}\} = \bigcup_{n \in \mathbb{N}} \mathcal{T}_{\leq n}\{\mathcal{X}\}$.

Similarly, we define the *reduced tree monad* $\overline{\mathcal{T}}$ by the following induction:

- $\overline{\mathcal{T}}_{\leq 0}\{\mathcal{X}\} = \emptyset$,
- $\overline{\mathcal{T}}_{\leq n+1}\{\mathcal{X}\} = \mathcal{X} \circ (\overline{\mathcal{T}}_{\leq n}\{\mathcal{X}\} + X)$,
- $\overline{\mathcal{T}}\{\mathcal{X}\} = \bigcup_{n \in \mathbb{N}} \overline{\mathcal{T}}_{\leq n}\{\mathcal{X}\}$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of species, we define $\mathcal{T}(f) : \mathcal{T}\{\mathcal{X}\} \to \mathcal{T}\{\mathcal{Y}\}$ and $\overline{\mathcal{T}}(f)$ by the same inductions. From these definitions, we have $\mathcal{T}\{\mathcal{X}\} = \overline{\mathcal{T}}\{\mathcal{X}\} + X$ and $\overline{\mathcal{T}}\{\mathcal{X}\} = \mathcal{X} \circ \mathcal{T}\{\mathcal{X}\}$.

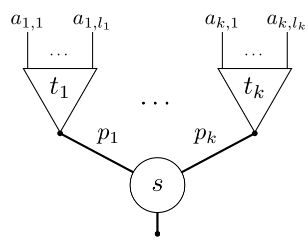
It is not clear from the definitions why we call these monads the tree monads. Let us depict an element of $\mathcal{T}\{\mathcal{X}\}$. Let A a finite set, $P = \{p_1, \dots, p_k\}$ a partition of A and let us denote $p_i = \{a_{i,1}, \dots, a_{i,l_i}\}$, let $s \in \mathcal{X}(P)$ and $t_i \in \mathcal{T}_{\leq n}\{\mathcal{X}\}(p_i)$. Let us depict an element $T \in \mathcal{T}_{\leq n+1}\{\mathcal{X}\}(A)$ in Figure 2.2. Let us explain our notations and the structure of T.

- The circles are the internal vertices.

- The triangles are subtrees.
- Leaves are bijectively labeled by elements of A, meaning that we have a bijection between the set of leaves and A.
- Each edge is labeled by a subset of A which is the set labels of leaves of the subtree it is incident to. For example, in the figure the edge between the internal vertex labeled by s and the subtree t_1 is labeled by the set $p_1 = \{a_{1,1}, \dots, a_{1,l_1}\}$.
- Each internal vertex is labeled by an element $v \in \mathcal{X}(\{q_1, \dots, q_k\})$ with $q_1, \dots, q_k$ the labels of the incoming edges of this vertex.

In Figure 2.2, we depict the element $T = (P, s; t_1, \dots, t_k) \in \mathcal{X} \circ \mathcal{T}_{\leq n}\{\mathcal{X}\}(A) \subseteq \mathcal{T}_{\leq n+1}\{\mathcal{X}\}(A)$.

Figure 2.2: An element of $\mathcal{T}_{\leq n+1}\{\mathcal{X}\}(A)$



We can now understand the inductive definition of the tree monad. The elements of $\mathcal{T}_{\leq n+1}\{\mathcal{X}\}$ are trees of *length* at most $n+1$. A tree of length 0 is reduced to a leaf, hence $\mathcal{T}_{\leq 0}\{\mathcal{X}\} = X$. A tree of length at most $n+1$ is either a root with subtrees of length at most n or a leaf. Hence, $\mathcal{T}_{\leq n+1}\{\mathcal{X}\} = \mathcal{X} \circ \mathcal{T}_{\leq n}\{\mathcal{X}\} + X$. The reduced tree monad is the same, but we do not allow the root to be a leaf, hence $\overline{\mathcal{T}}_{\leq 0}\{\mathcal{X}\} = \emptyset$. An important fact to notice is that $\mathcal{T}_{\leq n}\{\mathcal{X}\}$ naturally gives us a filtration of $\mathcal{T}\{\mathcal{X}\}$ but not a grading. We denote by $\mathcal{T}_l\{\mathcal{X}\}$ the associated grading which corresponds to considering the length of the longest path. We can also define the *weight* of a tree $T \in \mathcal{T}\{\mathcal{X}\}(A)$ as the cardinality of the set of internal vertices of T. Hence, the species $\mathcal{T}\{\mathcal{X}\}$ is bi-graded by the length and the weight we denote by $\mathcal{T}_{l,w}\{\mathcal{X}\}$ this bi-graded species. The species $\overline{\mathcal{T}}\{\mathcal{X}\}$ is also bi-graded by the length and the weight, however, since we would like the grading to start at 0, and that we do not have the trivial tree (of length and weight 0) in $\overline{\mathcal{T}}\{\mathcal{X}\}$, the weight of a tree $T \in \overline{\mathcal{T}}\{\mathcal{X}\}(A)$ is the weight of the according tree in $\mathcal{T}\{\mathcal{X}\}(A)$ minus 1, same for the length. Hence, we have:

$$\mathcal{T}_{l,w}\{\mathcal{X}\} = X + \overline{\mathcal{T}}_{l+1,w+1}\{\mathcal{X}\} = X + lw\overline{\mathcal{T}}_{l,w}\{\mathcal{X}\}$$

Proposition 2.1.1.2. *The tree monad is a monad.*

Proof. First we need to ensure that $\mathcal{T}$ is well-defined. To do so, we need to check that $\mathcal{T}_{\leq n}\{\mathcal{X}\} \subseteq \mathcal{T}_{\leq n+1}\{\mathcal{X}\}$. This is clear for $n = 0$. Assume that $\mathcal{T}_{\leq n}\{\mathcal{X}\} \subseteq \mathcal{T}_{\leq n+1}\{\mathcal{X}\}$. Then we have:

$$\mathcal{T}_{\leq n+1}\{\mathcal{X}\} = \mathcal{X} \circ \mathcal{T}_{\leq n}\{\mathcal{X}\} + X \subseteq \mathcal{X} \circ \mathcal{T}_{\leq n+1}\{\mathcal{X}\} + X = \mathcal{T}_{\leq n+2}\{\mathcal{X}\}$$

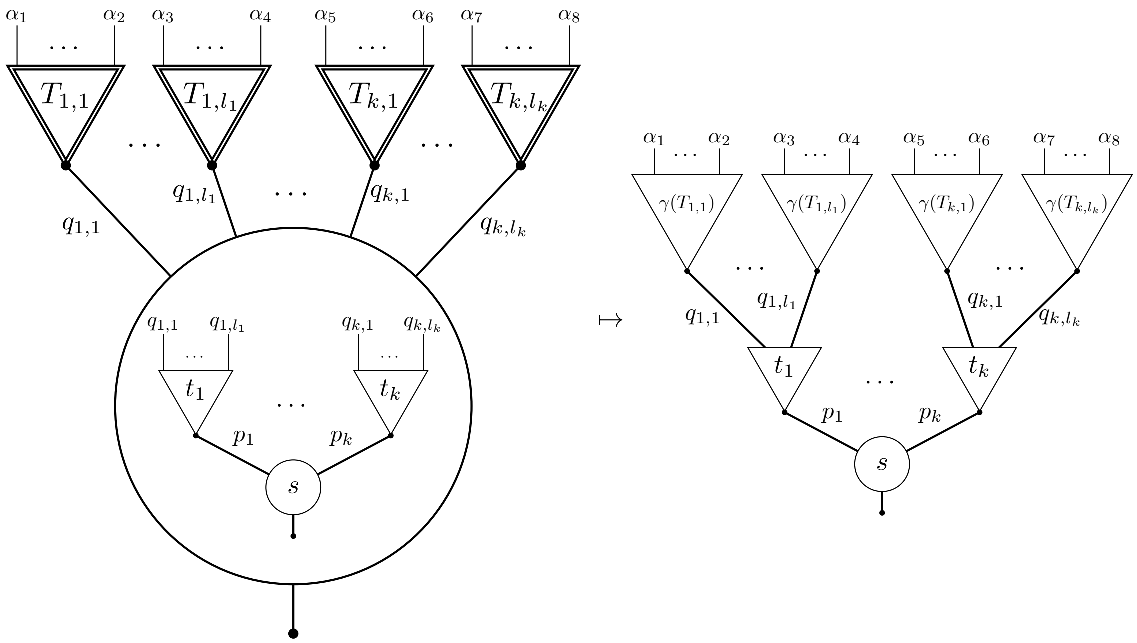
Moreover, one can check that $\mathcal{T}$ is a functor from the category of connected species to itself. From the inclusion $\mathcal{X} \circ \mathcal{T}_{\leq n}\{\mathcal{X}\} \subseteq \mathcal{T}_{\leq n+1}\{\mathcal{X}\}$, we have:

$$\mathcal{T}\{\mathcal{X}\} = \mathcal{X} \circ \mathcal{T}\{\mathcal{X}\} + X$$

To show that $\mathcal{T}$ is a monad, we need to define $\gamma : \mathcal{T} \circ \mathcal{T} \to \mathcal{T}$ and $\varepsilon : \mathrm{id} \to \mathcal{T}$. The morphism ε is straightforward to define:

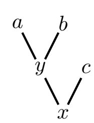
$$\mathcal{X} \subseteq \mathcal{X} \circ \mathcal{X} \circ \mathcal{T}\{\mathcal{X}\} + \mathcal{X} + X \subseteq \mathcal{X} \circ (\mathcal{X} \circ \mathcal{T}\{\mathcal{X}\} + X) + X = \mathcal{X} \circ \mathcal{T}\{\mathcal{X}\} + X = \mathcal{T}\{\mathcal{X}\}$$

To describe it more explicitly, we have that $\varepsilon_{\mathcal{X}}$ send s to the tree with one internal vertex labeled by s.
Let us define $\gamma_{\mathcal{X}} : \mathcal{T}(\mathcal{T}\{\mathcal{X}\}) \to \mathcal{T}\{\mathcal{X}\}$. It should send a tree labeled by trees labeled by $\mathcal{X}$ to a tree labeled by $\mathcal{X}$. The idea is to replace each internal vertex by the tree labeling it. Let us depict on an example the inductive definition of $\gamma_{\mathcal{X}}$:

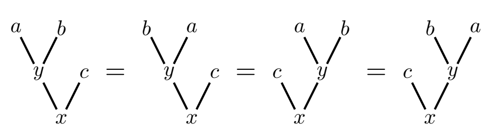


It is quite clear from this kind of picture that γ associative and unital with respect to ε. Explicit formulas for γ and ε can be written down, but they are quite cumbersome and do not bring any insight.
Same for the reduced tree monad which is a sub-monad of the tree monad. We have the same situation for the ns and shuffle tree monads. □

We will no longer write the labels of internal edges since they are entirely determined by the labels of the leaves. Here is an example of the trees we will draw from now on:



An issue we can notice is that we do not have a canonical way to depict an element of $\mathcal{T}\{\mathcal{X}\}$ with trees. Indeed, the following trees depict the same elements of $\mathcal{T}\{\mathcal{X}\}$:



We will see that this issue does not appear in the non-symmetric case, and we will use this fact to make to have canonical representation of each element in the shuffle case.

2.1.2 The non-symmetric tree monad

Let us copy-past (literally) the previous section and adapt it to the non-symmetric case. Let us inductively define the *non-symmetric tree monad* $\mathcal{T}^{\mathrm{ns}}$, later *ns tree monad* for short, on the category of ordered species by the same induction as the tree monad.

Definition 2.1.2.1. Let $\mathcal{X}$ a connected ordered species. We define the *ns tree monad* $\mathcal{T}$ by the following induction:

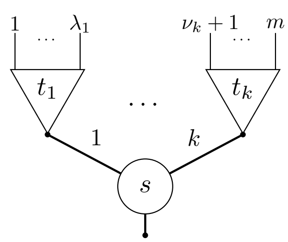
- $\mathcal{T}^{\mathrm{ns}}_{\leq 0}\{\mathcal{X}\} = X$,
- $\mathcal{T}^{\mathrm{ns}}_{\leq n+1}\{\mathcal{X}\} = \mathcal{X} \circ \mathcal{T}^{\mathrm{ns}}_{\leq n}\{\mathcal{X}\} + X$,
- $\mathcal{T}^{\mathrm{ns}}\{\mathcal{X}\} = \bigcup_{n\in\mathbb{N}} \mathcal{T}^{\mathrm{ns}}_{\leq n}\{\mathcal{X}\}$.

Similarly, we define the *reduced ns tree monad* $\overline{\mathcal{T}}$ by the following induction:

- $\overline{\mathcal{T}}^{\mathrm{ns}}_{\leq 0}\{\mathcal{X}\} = \emptyset$,
- $\overline{\mathcal{T}}^{\mathrm{ns}}_{\leq n+1}\{\mathcal{X}\} = \mathcal{X} \circ (\overline{\mathcal{T}}^{\mathrm{ns}}_{\leq n}\{\mathcal{X}\} + X)$,
- $\overline{\mathcal{T}}^{\mathrm{ns}}\{\mathcal{X}\} = \bigcup_{n\in\mathbb{N}} \overline{\mathcal{T}}^{\mathrm{ns}}_{\leq n}\{\mathcal{X}\}$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of ordered species, we define $\mathcal{T}^{\mathrm{ns}}(f) : \mathcal{T}^{\mathrm{ns}}\{\mathcal{X}\} \to \mathcal{T}^{\mathrm{ns}}\{\mathcal{Y}\}$ and $\overline{\mathcal{T}}^{\mathrm{ns}}(f)$ by the same inductions. From these definitions, we have $\mathcal{T}^{\mathrm{ns}}\{\mathcal{X}\} = \overline{\mathcal{T}}^{\mathrm{ns}}\{\mathcal{X}\} + X$ and $\overline{\mathcal{T}}^{\mathrm{ns}}\{\mathcal{X}\} = \mathcal{X} \circ \mathcal{T}^{\mathrm{ns}}\{\mathcal{X}\}$.

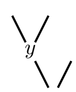
Let us depict element of $\mathcal{T}^{\mathrm{ns}}\{\mathcal{X}\}$ by tree-like structures and point out the differences with the tree monad. Let $m \in \mathbb{N}$ and λ a composition of m of length k, we denote $\nu_i = \sum_{j=1}^{i-1} \lambda_i$. Let $s \in \mathcal{X}(k)$ and $t_i \in \mathcal{T}^{\mathrm{ns}}_{\leq n}\{\mathcal{X}\}(\lambda_i)$. We depict the element of $\mathcal{T}^{\mathrm{ns}}_{\leq n+1}\{\mathcal{X}\}(m)$ as follows:



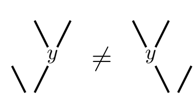
Proposition 2.1.2.2. *The ns tree monad is a monad.*

Proof. The same discussion as for the tree monad applies here. □

We will no longer write the labels of internal edges nor the leaves since they are in increasing order. Hence, in the non-symmetric case, we have planar trees such that each vertex is labeled by an element of $\mathcal{X}(k)$ with k the number of incoming edges of this vertex. Here is an example of the trees we will draw from now on with $x, y \in \mathcal{X}(2)$:



However, since we work with planar trees, one need to be careful since the order of the incoming edges of each vertex is important. For example, the following trees are different:



This allows us to have a canonical representation for each element of $\mathcal{T}^{\mathrm{ns}}\{\mathcal{X}\}$, and we will use this idea of considering planar trees to make sure we have canonical representation of each element in the shuffle case. We can also understand why we could not have canonical representations in the symmetric case, indeed the symmetric group action on the leaves cannot be compatible with the planar structure of the trees.

We have a fully faithful functor $\Sigma : \mathbb{L}\mathrm{Spe} \to \mathrm{Spe}$ the symmetrization functor. Furthermore, we know it commutes with the plethysm, hence we have:

$$\mathcal{T}\{\Sigma(\mathcal{X})\} = \Sigma(\mathcal{T}^{\mathrm{ns}}\{\mathcal{X}\})$$

This allows us to see the non-symmetric case as a particular case of the symmetric case. Hence, everything we will do for the symmetric case will be valid for the non-symmetric case, however some tools of the non-symmetric case will not be available in the symmetric case.

2.1.3 The shuffle tree monad

We can also define the shuffle tree monad for shuffle species. The trees that are going to appear are going to be a “mix” between the trees of the symmetric case and the non-symmetric case. Let us inductively define the *shuffle tree monad* $\mathcal{T}^{\text{Ш}}$ on the category of shuffle species by the same induction as the tree monad.

Definition 2.1.3.1. Let $\mathcal{X}$ a connected shuffle species. We define the *shuffle tree monad* $\mathcal{T}^{\text{Ш}}$ by the following induction:

- $\mathcal{T}^{\text{Ш}}_{\leq 0}\{\mathcal{X}\} = X$,
- $\mathcal{T}^{\text{Ш}}_{\leq n+1}\{\mathcal{X}\} = \mathcal{X} \circ_{\text{Ш}} \mathcal{T}^{\text{Ш}}_{\leq n}\{\mathcal{X}\} + X$,
- $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\} = \bigcup_{n\in\mathbb{N}} \mathcal{T}^{\text{Ш}}_{\leq n}\{\mathcal{X}\}$.

Similarly, we define the *reduced ns tree monad* $\overline{\mathcal{T}}$ by the following induction:

- $\overline{\mathcal{T}}^{\text{Ш}}_{\leq 0}\{\mathcal{X}\} = \emptyset$,
- $\overline{\mathcal{T}}^{\text{Ш}}_{\leq n+1}\{\mathcal{X}\} = \mathcal{X} \circ_{\text{Ш}} (\overline{\mathcal{T}}^{\text{Ш}}_{\leq n}\{\mathcal{X}\} + X)$,
- $\overline{\mathcal{T}}^{\text{Ш}}\{\mathcal{X}\} = \bigcup_{n\in\mathbb{N}} \overline{\mathcal{T}}^{\text{Ш}}_{\leq n}\{\mathcal{X}\}$.

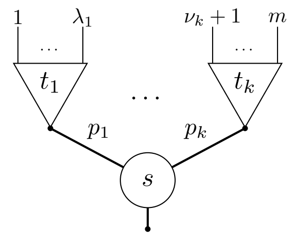
If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of ordered species, we define $\mathcal{T}^{\text{Ш}}(f) : \mathcal{T}^{\text{Ш}}\{\mathcal{X}\} \to \mathcal{T}^{\mathrm{ns}}\{\mathcal{Y}\}$ and $\overline{\mathcal{T}}^{\text{Ш}}(f)$ by the same inductions. From these definitions, we have $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\} = \overline{\mathcal{T}}^{\text{Ш}}\{\mathcal{X}\} + X$ and $\overline{\mathcal{T}}^{\text{Ш}}\{\mathcal{X}\} = \mathcal{X} \circ_{\text{Ш}} \mathcal{T}^{\text{Ш}}\{\mathcal{X}\}$.

Let us recall that the shuffle plethysm is the image of the plethysm under the forgetful functor from the category of species to the category of shuffle species. In particular, for $\mathcal{X}$ and $\mathcal{Y}$ to shuffle species, we have:

$$\begin{aligned}(\mathcal{X} \circ_{\text{Ш}} \mathcal{Y})(n) &= \biguplus_{k\in\mathbb{N}} \biguplus_{P\vdash^k \underline{n}} \mathcal{X}(k) \times \prod_{p\in P} \mathcal{Y}(|p|) \\ &= \biguplus_{k\in\mathbb{N}} \biguplus_{\lambda\vDash^k \underline{n}} \frac{n!}{k!\lambda_1!\dots\lambda_k!}\mathcal{X}(k) \times \prod_{i=1}^{k} \mathcal{Y}(\lambda_i)\end{aligned}$$

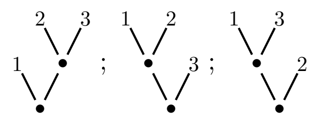
From this fact, we see that we can depict elements of $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}$ by tree-like structures, same as in the symmetric case but with integers labeling the leaves. Let $m \in \mathbb{N}$, $P = \{p_1, \dots, p_k\}$ a partition of $\underline{m}$ and λ the composition of m given by $\lambda_i = |p_i|$. We denote by ν_i the sum $\sum_{j=1}^{i-1} \lambda_j$. Figure 2.3 depicts an element of $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}(m)$ with those notations. To get a canonical representation of elements of $\mathcal{T}^{\mathrm{ns}}\{\mathcal{X}\}$ by tree-like structures, we need to define the notion of shuffle tree.

Figure 2.3: An element of $\mathcal{T}^{\text{III}}\{\mathcal{X}\}(A)$

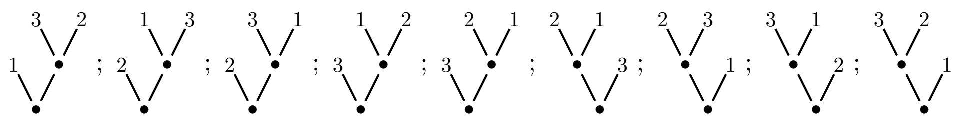


Definition 2.1.3.2. A *shuffle tree* is a planar rooted tree such that the leaves are bijectively labeled by the set $\{1, \dots, n\}$ with n the number of leaves, such that:
When we label each edge by the smallest label of the leaves above it, the labels of the incoming edges of each vertex are in increasing order.

Example 2.1.3.3. The following planar trees are shuffle trees:



The following planar trees are not shuffle trees:

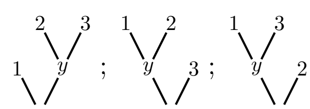


All the example we give are binary planar trees. Shuffle trees do not have to be binary, however for all the explicit computations we will do, only binary trees will appear.

Proposition 2.1.3.4. *The shuffle tree monad is a monad.*

Proof. The same discussion as for the tree monad applies here. □

We will depict elements of $\mathcal{T}^{\text{III}}\{\mathcal{X}\}$ by shuffle trees such that each internal vertex is labeled by elements of $\mathcal{X}(k)$ with k the number of incoming edges of this vertex. Here some examples with $x, y \in \mathcal{X}(2)$:



2.2 Algebraic operads

With the definition of the tree monads, we are ready to give definitions of algebraic operads. The last point to clarify is the category in which we are going to work. We would like to work with vector spaces over a field of characteristic 0 to do algebra. Hence, our goal is to work in the linear case, with linear species. The functor Span : Set → Vect induces a functor:

$$\text{Span} : \text{Spe} \to \text{Vect–Spe}$$

Same in the ns and shuffle case. We can define our tree monads in the linear case by the same induction as in the set theoretical case. Since the Span functor is compatible with the operations of the category of species, it commutes with the tree monads. By a slight abuse of notation, we denote by $\mathcal{T}$: Vect–Spe $\to$ Vect–Spe the linear tree monad. Same in the ns and shuffle case. To avoid confusion, we will denote by $\mathcal{T}\{\mathcal{X}\}$ the tree monad applied on $\mathcal{X}$ a set species, by $\mathcal{T}(\mathcal{S})$ applied on $\mathcal{S}$ a linear species, and by $\mathcal{T}[\mathcal{X}]$ applied on $\mathrm{Span}(\mathcal{X})$. We may notice that:

$$\mathcal{T}[\mathcal{X}] = \mathrm{Span}(\mathcal{T}\{\mathcal{X}\}) = \mathcal{T}(\mathrm{Span}(\mathcal{X}))$$

Proposition 2.2.0.1. *We have the following:*

- *Let $\mathcal{S}$ be an ordered linear species and $\mathcal{X}$ a basis of $\mathcal{S}$ as a vector space. Then $\mathcal{X}$ is an ordered set species. We have that the planar trees such that the internal vertices are labeled by elements of $\mathcal{X}$ constitute a basis of $\mathcal{T}^{\mathrm{ns}}(\mathcal{S})$ as a vector space. Namely:*

$$\mathcal{T}^{\mathrm{ns}}(\mathcal{S}) = \mathrm{Span}(\mathcal{T}^{\mathrm{ns}}\{\mathcal{X}\})$$

- *Let $\mathcal{S}$ be a shuffle linear species and $\mathcal{X}$ a basis of $\mathcal{S}$ as a vector space. Then $\mathcal{X}$ is a shuffle set species. We have that the shuffle trees such that the internal vertices are labeled by elements of $\mathcal{X}$ constitute a basis of $\mathcal{T}^{\text{Ш}}(\mathcal{S})$ as a vector space. Namely:*

$$\mathcal{T}^{\text{Ш}}(\mathcal{S}) = \mathrm{Span}(\mathcal{T}^{\text{Ш}}\{\mathcal{X}\})$$

- *Let $\mathcal{S}$ be a linear species and $\mathcal{X}$ a basis of $\mathcal{S}$ as a vector space. Then $\mathcal{X}$ is a shuffle set species. Indeed, it is not possible to guaranty that the basis is compatible with the group action. We have that the shuffle trees such that the internal vertices are labeled by elements of $\mathcal{X}$ constitute a basis of $\mathcal{T}(\mathcal{S})$ as a vector space. Namely:*

$$U(\mathcal{T}(\mathcal{S})) = \mathrm{Span}(\mathcal{T}^{\text{Ш}}\{\mathcal{X}\})$$

The case we are interested in is the last one. However, we see that it is the most involved one. It is exactly because of this technicality that we needed to introduce shuffle species.

Proof. The easy case is the ordered case. Let $\mathcal{S}$ be an ordered linear species and $\mathcal{X}$ a basis of $\mathcal{S}$ as a vector space. We have that $\mathcal{X}$ is an ordered set species such that $\mathrm{Span}(\mathcal{X}) = \mathcal{S}$. Hence, $\mathrm{Span}(\mathcal{T}^{\mathrm{ns}}\{\mathcal{X}\}) = \mathcal{T}^{\mathrm{ns}}(\mathcal{S})$, and the discussion of Subsection 2.1.2 explicitly describes the basis. The shuffle case is similar with shuffle trees instead of planar trees.
The linear case is slightly more tricky as the basis given by the shuffle trees is not compatible with the symmetric group action. However, by the definition of the linear shuffle species, we have that $U(\mathcal{T}(\mathcal{S})) = \mathcal{T}^{\text{Ш}}(U(\mathcal{S}))$ for $\mathcal{S}$ a linear species and U : Vect–Spe $\to$ Vect–ШSpe the forgetful functor. Hence, $\mathcal{T}^{\text{Ш}}(U(\mathcal{S}))$ is the underlying vector space of the linear species $\mathcal{T}(\mathcal{S})$. The same discussion as in the shuffle case concludes the proof. $\square$

From now on all the species will be assumed to be linear species. Same for ordered and shuffle species that will be assumed to be respectively linear ordered species and linear shuffle species.

2.2.1 Definitions of algebraic operads

We are now ready to give the definitions of algebraic operads. We will give three equivalent definitions of operads. The first one is through the tree monad which ease the study of the free operads, and allows us to do computations by working on trees in this case. The second one is through the plethysm which shows that an operad is literally a monoid in the category of species relatively to the plethysm. The last one is a partial definition which allows us to see an operad as a collection of abstract multilinear maps closed under composition.

Definition 2.2.1.1 ($\mathcal{T}$-algebra definition of an operad)**.** A *symmetric algebraic operad*, or *operad* for short, is an algebra over the tree monad $\mathcal{T}$.

Definition 2.2.1.2 (Monoidal definition of an operad)**.** An *operad* is a monoid in the category of species according to the plethysm.

Definition 2.2.1.3 (Partial definition of an operad)**.** An *operad* is a species $\mathcal{P} \in$ Vect–Spe together with a collection of maps $\circ_i$ and an element $e_i \in \mathcal{P}(\{i\})$ such that $\circ_i : \mathcal{P}(A \sqcup \{i\}) \otimes \mathcal{P}(B) \to \mathcal{P}(A \sqcup B)$ such that the following diagrams commute:

$$\begin{array}{ccccc} \mathcal{P}(\{j\}) \otimes \mathcal{P}(A \sqcup \{i\}) & \xrightarrow{\circ_j} & \mathcal{P}(A \sqcup \{i\}) & \xleftarrow{\circ_i} & \mathcal{P}(A \sqcup \{i\}) \otimes \mathcal{P}(\{i\}) \\ & {\scriptstyle e_j \otimes \mathrm{id}} \nwarrow & \Vert & \nearrow {\scriptstyle \mathrm{id} \otimes e_i} & \\ & & \mathcal{P}(A \sqcup \{i\}) & & \end{array}$$

(identity)

$$\begin{array}{ccc} \mathcal{P}(A \sqcup \{i\}) \otimes (\mathcal{P}(B \sqcup \{j\}) \otimes \mathcal{P}(C)) & \xrightarrow{\mathrm{id} \otimes \circ_j} & \mathcal{P}(A \sqcup \{i\}) \otimes \mathcal{P}(B \sqcup C) \\ \Vert & & \\ (\mathcal{P}(A \sqcup \{i\}) \otimes \mathcal{P}(B \sqcup \{j\})) \otimes \mathcal{P}(C) & & \downarrow {\scriptstyle \circ_i} \\ \downarrow {\scriptstyle \circ_i \otimes \mathrm{id}} & & \\ \mathcal{P}(A \sqcup B \sqcup \{j\}) \otimes \mathcal{P}(C) & \xrightarrow{\circ_j} & \mathcal{P}(A \sqcup B \sqcup C) \end{array}$$

(sequential composition)

$$\begin{array}{ccc} (\mathcal{P}(A \sqcup \{i, j\}) \otimes \mathcal{P}(B)) \otimes \mathcal{P}(C) & \xrightarrow{\circ_j \otimes \mathrm{id}} & \mathcal{P}(A \sqcup B \sqcup \{i\}) \otimes \mathcal{P}(C) \\ \Vert & & \\ (\mathcal{P}(A \sqcup \{i, j\}) \otimes \mathcal{P}(C)) \otimes \mathcal{P}(B) & & \downarrow {\scriptstyle \circ_i} \\ \downarrow {\scriptstyle \circ_i \otimes \mathrm{id}} & & \\ \mathcal{P}(A \sqcup C \sqcup \{j\}) \otimes \mathcal{P}(B) & \xrightarrow{\circ_j} & \mathcal{P}(A \sqcup B \sqcup C) \end{array}$$

(parallel composition)

$$\begin{array}{ccc} \mathcal{P}(A \sqcup \{i\}) \otimes \mathcal{P}(B) & \xrightarrow{\mathcal{P}(\sigma) \otimes \mathcal{P}(\tau)} & \mathcal{P}(C \sqcup \{\sigma(i)\}) \otimes \mathcal{P}(D) \\ \downarrow {\scriptstyle \circ_i} & & \downarrow {\scriptstyle \circ_{\sigma(i)}} \\ \mathcal{P}(A \sqcup B) & \xrightarrow{\mathcal{P}((\sigma \setminus \{i\}) \sqcup \tau)} & \mathcal{P}(C \sqcup D) \end{array}$$

(compatibility with the species structure)

These diagrams may seem cryptic at first, but they are very natural. One needs to see a partial operad as the algebraic structure encoding partial composition of multivariate maps with elements of $\mathcal{P}(A)$ being multivariate maps such that the input are labeled by the set A. Indeed, the first diagram is the unitary condition stating that composing with the identity in the input i does not change the map, and that composing f in the input of the identity gives f once again.

$$f \circ_i \mathrm{id} = f \text{ and } \mathrm{id} \circ_j f = f$$

The second diagram is the associativity of the composition stating that composing a map h in the input j of a map g and then composing the result $g \circ_j h$ in the input i of f is the same as composing g in the input i of f and then composing h in the input j of the result $g \circ_i f$.

$$f \circ_i (g \circ_j h) = (f \circ_i g) \circ_j h$$

The third diagram is the sequential axiom stating that the inputs of f are independent hence composing g in the input i of f and then composing h in the input j of the result $f \circ_i g$ is the same as composing h in the input j of f and then composing g in the input i of the result $f \circ_j h$.

$$(f \circ_i g) \circ_j h = (f \circ_j h) \circ_i g$$

The last diagram is the fact that relabeling the inputs of the maps does not change the result of the composition. Hence, with σ a relabeling of the inputs of f and τ of the inputs of g, we have:

$$(f.\sigma) \circ_{\sigma(i)} (g.\tau) = (f \circ_i g).((\sigma \setminus \{i\}) \sqcup \tau)$$

Proposition 2.2.1.4. *The three definitions of an operad are equivalent.*

Proof. Let $\mathcal{P}$ be an operad defined by partial compositions, let us define a monoid structure on $\mathcal{P}$. We need to define a map $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ and a map $\varepsilon : X \to \mathcal{P}$. Let us define γ as follows:

$$\gamma_A : (s; r_1, \dots, r_n) \mapsto (\dots (s \circ_{p_1} r_1) \circ_{p_2} r_2 \dots) \circ_{p_n} r_n$$

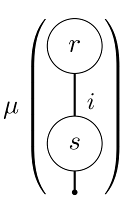
Where A is a finite set, $P = \{p_1, \dots, p_n\}$ is a partition of A, $s \in \mathcal{P}(P)$, $r_i \in \mathcal{P}(p_i)$, and thus $(s; r_1, \dots, r_n) \in (\mathcal{P} \circ \mathcal{P})(A)$. Let us define ε by $\varepsilon_{\{*\}} : * \mapsto e_*$, where e_* is the element of $\mathcal{P}(\{*\})$ given by the partial operad structure. The parallel composition diagram ensures that γ is well-defined. The compatibility with the species structure diagram ensures that γ and ε are morphisms of species. The sequential composition diagram and unitary diagram ensure respectively that γ is associative, and that ε is its unit. Hence, $\mathcal{P}$ is a monoid in the category of species according to the plethysm.
Let $\mathcal{P}$ an operad defined by the monoidal definition, let us inductively define a $\mathcal{T}$-algebra structure on $\mathcal{P}$:

- Let us define $\mu_0 : \mathcal{T}_{\leq 0}(\mathcal{P}) \to \mathcal{P}$ by $\mu_0 = \varepsilon$.

- Assume that we have defined $\mu_n : \mathcal{T}_{\leq n}(\mathcal{P}) \to \mathcal{P}$, let us define $\mu_{n+1} : \mathcal{T}_{\leq n+1}(\mathcal{P}) \to \mathcal{P}$ by:

$$\mu_{n+1} = \gamma(\mathrm{id} \circ \mu_n) + \varepsilon : \mathcal{P} \circ \mathcal{T}_{\leq n}(\mathcal{P}) + X \to \mathcal{P}$$

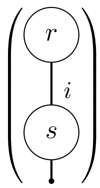
The fact that γ is associative and unital with respect to ε ensures that μ is a $\mathcal{T}$-algebra structure on $\mathcal{P}$.
Let $\mathcal{P}$ an operad defined by the $\mathcal{T}$-algebra definition, let us define a partial operad structure on $\mathcal{P}$. Let us define the partial operad structure on $\mathcal{P}$ by:

$\circ_i : (s, r) \mapsto \mu$ 

Where we omitted the leaves for readability. Let us define $e_i = \mu(*)$. The fact that μ is a $\mathcal{T}$-algebra structure on $\mathcal{P}$ ensures that the partial compositions verify the required axioms. Hence, $\mathcal{P}$ is a partial operad.

To summarize, if $\mathcal{P}$ is an operad defined by $\mathcal{T}$-algebra definition, then we can get the partial compositions by setting:

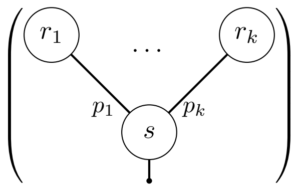
$$\circ_i : (s, r) \mapsto \mu$$



(Leaves are omitted for readability.)

Moreover, we can get the monoidal structure by setting:

$$\gamma : (s; r_1, \ldots, r_n) \mapsto \mu$$



(Leaves are omitted for readability.)

Then, if we only have the partial compositions, we can recover the monoid structure γ by setting:

$$\gamma : (s; r_1, \ldots, r_n) \mapsto (\ldots (s \circ_{p_1} r_1) \circ_{p_2} r_2 \ldots) \circ_{p_n} r_n$$

And if we only have the monoidal structure, we can recover the $\mathcal{T}$-algebra definition by inductively defining a $\mathcal{T}$-algebra structure on $\mathcal{P}$ as follows:

$$\mu = \gamma(\mathrm{id} \circ \mu) + \varepsilon : \mathcal{P} \circ \mathcal{T}_n(\mathcal{P}) + X \to \mathcal{P}$$

Hence, the three definitions are equivalent. The full data are given by the $\mathcal{T}$-algebra structure. The data of the partial compositions are enough to recover the monoidal structure, and the data of the monoidal structure allows us to inductively recover the $\mathcal{T}$-algebra structure. □

Definition 2.2.1.5. A *morphism of operads* is:

- a $\mathcal{T}$-algebra morphism; or
- a monoid morphism; or
- a morphism of species that respects the partial compositions and the unit.

Those three definitions are equivalent.

The equivalence of the three definitions of an operad allows us to have three interpretations of the notion of operad. The partial definition tells us that an operad behave like a collection of multivariate maps stable by partial compositions. The monoid definition allows us to understand an operad as a combinatorial object, a species, together with an algebraic structure, a monoidal structure. Finally, the $\mathcal{T}$-algebra definition provide a way to represent elements of operads as trees and to do actual computations with them.

Let us give the analogous definitions for the ns and shuffle case.

Definition 2.2.1.6. Similarly to the symmetric case, we have three equivalent definitions for a *non-symmetric algebraic operad*, or *ns operad* for short:

- A *ns operad* is an algebra over the ns tree monad $\mathcal{T}^{\mathrm{ns}}$.
- A *ns operad* is a monoid in the category $(\mathrm{Vect}\text{-}\mathbb{L}\mathrm{Spe}, \circ, X)$ of ordered species according to the plethysm.

- A *ns operad* is an ordered species $\mathcal{P} \in \text{Vect–}\mathbb{L}\text{Spe}$ together with an element $e \in \mathcal{P}(1)$ and a collection of maps $\circ_i : \mathcal{P}(n+1) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m)$ satisfying the unitary, sequential composition, and parallel composition axioms, meaning that the following diagrams commute:

$$\begin{array}{ccccc} \mathcal{P}(1)\otimes\mathcal{P}(n) & \xrightarrow{\circ_1} & \mathcal{P}(n) & \xleftarrow{\circ_i} & \mathcal{P}(n)\otimes\mathcal{P}(1) \\ & \nwarrow^{e\otimes \mathrm{id}} & \| & \nearrow^{\mathrm{id}\otimes e} & \\ & & \mathcal{P}(n) & & \end{array} \tag{ns identity}$$

$$\begin{array}{ccc} \mathcal{P}(n+1)\otimes(\mathcal{P}(m+1)\otimes\mathcal{P}(k)) & \xrightarrow{\mathrm{id}\otimes\circ_j} & \mathcal{P}(n+1)\otimes\mathcal{P}(m+k) \\ \| & & \\ (\mathcal{P}(n+1)\otimes\mathcal{P}(m+1))\otimes\mathcal{P}(k) & & \downarrow \circ_i \\ \downarrow \circ_i\otimes\mathrm{id} & & \\ \mathcal{P}(n+m+1)\otimes\mathcal{P}(k) & \xrightarrow{\circ_{j+i-1}} & \mathcal{P}(n+m+k) \end{array} \tag{ns sequential composition}$$

$$\begin{array}{ccc} (\mathcal{P}(n+2)\otimes\mathcal{P}(m))\otimes\mathcal{P}(k) & \xrightarrow{\circ_j\otimes\mathrm{id}} & \mathcal{P}(n+m+1)\otimes\mathcal{P}(k) \\ \| & & \\ (\mathcal{P}(n+2)\otimes\mathcal{P}(k))\otimes\mathcal{P}(m) & & \downarrow \circ_{i'} \\ \downarrow \circ_i\otimes\mathrm{id} & & \\ \mathcal{P}(n+k+1)\otimes\mathcal{P}(m) & \xrightarrow{\circ_{j'}} & \mathcal{P}(n+m+k) \end{array} \tag{ns parallel composition}$$

With $i' = i$ if $i < j$ and $i' = i + k - 1$ if $i > j$, same with $j' = j$ if $j < i$ and $j' = j + m - 1$ if $j > i$.

The equivalence of these three definitions is a direct consequence of the discussion we had in the symmetric case. One may remark that indexing the partial compositions over the integers $\{1, \ldots, n\}$ rather than any finite set creates some technicalities. Indeed, one need to renumber the inputs of the maps when composing them, which is the reason why $j + i - 1$ appears in Diagram ns sequential composition.

Definition 2.2.1.7. Similarly to the symmetric case, we have three equivalent definitions for a *shuffle algebraic operad*, or *shuffle operad* for short:

- A *shuffle operad* is an algebra over the shuffle tree monad $\mathcal{T}^{\text{ns}}$.

- A *shuffle operad* is a monoid in the category $(\text{Vect–Ш}\text{Spe}, \circ_{\text{Ш}}, X)$ of shuffle species according to the shuffle plethysm.

- A *shuffle operad* is a shuffle species $\mathcal{P} \in \text{Vect–Ш}\text{Spe}$ together with an element $e \in \mathcal{P}(1)$ and a collection of maps $\circ_{i,I} : \mathcal{P}(n+1) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m)$ with $I \subseteq \underline{n+m}$ such that $|I| = m$ and $i = \min(I)$, satisfying the unitary, sequential composition, and parallel composition axioms, meaning that the following diagrams commute:

$$\begin{array}{ccccc} \mathcal{P}(1)\otimes\mathcal{P}(n) & \xrightarrow{\circ_{1,\{1\}}} & \mathcal{P}(n) & \xleftarrow{\circ_{i,\{i\}}} & \mathcal{P}(n)\otimes\mathcal{P}(1) \\ & \nwarrow^{e\otimes \mathrm{id}} & \| & \nearrow^{\mathrm{id}\otimes e} & \\ & & \mathcal{P}(n) & & \end{array}$$

(shuffle identity)

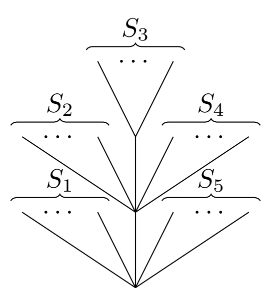
$$\begin{array}{ccc} \mathcal{P}(n+1)\otimes(\mathcal{P}(m+1)\otimes\mathcal{P}(k)) & \xrightarrow{\mathrm{id}\otimes\circ_{j,J}} & \mathcal{P}(n+1)\otimes\mathcal{P}(m+k) \\ \| & & \\ (\mathcal{P}(n+1)\otimes\mathcal{P}(m+1))\otimes\mathcal{P}(k) & & \downarrow{\scriptstyle \circ_{i',I'}} \\ \downarrow{\scriptstyle \circ_{i,I}\otimes\mathrm{id}} & & \\ \mathcal{P}(n+m+1)\otimes\mathcal{P}(k) & \xrightarrow{\circ_{j',J'}} & \mathcal{P}(n+m+k) \end{array}$$

(shuffle sequential composition)

$$\begin{array}{ccc} (\mathcal{P}(n+2)\otimes\mathcal{P}(m))\otimes\mathcal{P}(k) & \xrightarrow{\circ_{b,B}\otimes\mathrm{id}} & \mathcal{P}(n+m+1)\otimes\mathcal{P}(k) \\ \| & & \\ (\mathcal{P}(n+2)\otimes\mathcal{P}(k))\otimes\mathcal{P}(m) & & \downarrow{\scriptstyle \circ_{a',A'}} \\ \downarrow{\scriptstyle \circ_{a,A}\otimes\mathrm{id}} & & \\ \mathcal{P}(n+k+1)\otimes\mathcal{P}(m) & \xrightarrow{\circ_{b',B'}} & \mathcal{P}(n+m+k) \end{array}$$

(shuffle parallel composition)

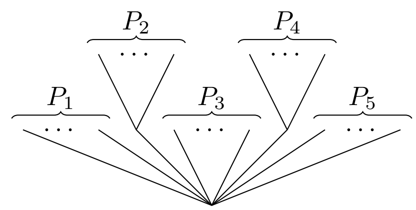
Like in the ns case, some renumbering appears in the parallel and sequential compositions. For the sake of completeness, let us explicitly describe the renumbering. For the sequential composition, let us consider the following shuffle tree:



Then I, J, I', J' are defined as follows:

- $I' = S_2 \sqcup S_3 \sqcup S_4$;
- $J' = S_3$;
- $I = S_2 \sqcup \{j'\} \sqcup \{s - j_s + 1 \mid s \in S_4\}$ with $j_s = |\{r \in S_3 \mid r \leq s\}|$;
- $J = \{s - i' - i_s + 1 \mid s \in S_3\}$ with $i_s = |\{r \in S_5 \mid r \leq s\}|$.

We may notice that either (I, J') or (I', J) are enough to recover the full data of the shuffle tree. For the parallel composition, let us consider the following shuffle tree:



Then A, B, A', B' are defined as follows:

- $A' = P_2$;
- $B' = P_4$;
- $A = \{p - b_p \mid p \in P_2\}$ with $b_p = |\{q \in P_4 \mid q \leq p\}|$;
- $B = \{p - a_p + 1 \mid p \in P_4\}$ with $a_p = |\{q \in P_2 \mid q \leq p\}|$.

Once again, we may notice that either (A, B') or (A', B) are enough to recover the full data of the shuffle tree. The renumbering have become much more painful in the shuffle case, because of those technicalities, we will allow ourselves to index the inputs over finite sets even in the ns or shuffle case, thus avoiding the need to renumber the inputs of the maps when composing them. Once again, the equivalence of these three definitions is a direct consequence of the discussion we had in the symmetric case.

First let us define our first example of operad, the trivial operad.

Definition 2.2.1.8. The *trivial operad* is the operad $\mathcal{I} = (X, 0, \mathrm{id})$ with the monoid definition. We have that $\mathcal{I} = \mathcal{T}(0)$, hence $\mathcal{I}$ is the initial object of the category of operads.

The trivial operad is the initial object of the category of operads, it is the simplest operad. It is the operad with only one element, the identity. It also allows us to define the notion of *augmented* operad, which is quite important since all the concrete example of operads we are going to give are actually augmented operads.

Definition 2.2.1.9. An *augmented operad* is an operad $(\mathcal{P}, \gamma, \varepsilon)$ together with an augmentation map $\eta : \mathcal{P} \to \mathcal{I}$ such that $\eta \circ \varepsilon = \mathrm{id}$. The *augmentation map* is the map η, and the *augmentation ideal* is the kernel of η.

Definition 2.2.1.10. Let $\mathcal{P}$ and $\mathcal{Q}$ be two augmented operads and $\overline{\mathcal{P}}$ and $\overline{\mathcal{Q}}$ their augmentation ideal. The *connected sum* of $\mathcal{P}$ and $\mathcal{Q}$ is the operad $\mathcal{I} \oplus \overline{\mathcal{P}} \oplus \overline{\mathcal{Q}}$ such that the partial composition of an element of $\overline{\mathcal{P}}$ with an element of $\overline{\mathcal{Q}}$ is zero, and similarly the partial composition of an element of $\overline{\mathcal{Q}}$ with an element of $\overline{\mathcal{P}}$ is zero. It means that: $f \circ_i g = g \circ_j f = 0$ for all $f \in \overline{\mathcal{P}}$ and $g \in \overline{\mathcal{Q}}$.

To see the link between operads and algebraic structures, let us define our second and less trivial example of operad, the endofunctor operad, and see how it is related to the notion of algebra over a monad.

Definition 2.2.1.11. Let V be a vector space, the endomorphism operad of V is the operad End_V such that $\mathrm{End}_V(A) = \mathrm{Hom}(V^{\otimes A}, V)$ with the composition of endomorphisms.

Definition 2.2.1.12. Let $\mathcal{P}$ be an operad, let $\mathcal{F}_{\mathcal{P}}$ be the Schur functor associated to the underlying species of $\mathcal{P}$. Since the plethysm of operads corresponds to the composition of endomorphisms, $\mathcal{F}_{\mathcal{P}}$ is a monad of Vect. A $\mathcal{P}$*-algebra* is an algebra over the monad $\mathcal{F}_{\mathcal{P}}$.

Proposition 2.2.1.13. *The data of a $\mathcal{P}$-algebra structure on V is equivalent to the data of a morphism of operads $\mathcal{P} \to \mathrm{End}_V$.*

Proof. A $\mathcal{P}$-algebra structure on V is a morphism $\mu : \mathcal{F}_{\mathcal{P}}(V) \to V$ compatible with the monadic structure of $\mathcal{F}_{\mathcal{P}}$. From the definition of $\mathcal{F}_{\mathcal{P}}$, it is a family of morphisms $\mu_n : \mathcal{P}(\underline{n}) \otimes_{\mathfrak{S}_n} V^{\otimes n} \to V$. Moreover we have:

$$\mathrm{Hom}\left(\mathcal{P}(\underline{n}) \otimes_{\mathfrak{S}_n} V^{\otimes n}, V\right) = \mathrm{Hom}^{\mathfrak{S}_n}\left(\mathcal{P}(\underline{n}), \mathrm{Hom}\left(V^{\otimes n}, V\right)\right)$$

Hence, we have a morphism of species $\mathcal{P} \to \mathrm{End}_V$. Moreover, the fact that μ is compatible with the monad structure of $\mathcal{F}_{\mathcal{P}}$ ensures that the morphism of species is a morphism of operads.
Conversely, a morphism of operads $\mathcal{P} \to \mathrm{End}_V$ give rise to a $\mathcal{P}$-algebra structure on V. □

2.2.2 Presentation by generators and relations

We have a nice definition of operads and the promise that they can encode algebraic objects. However, we do not know (yet) how to construct useful operads. A very general way to construct mathematical objects is by generators and relations. Let us see how to do that for operads. To construct a mathematical object by generator and relations, one usually need two ingredients: a notion of free object and a notion of quotient. The $\mathcal{T}$-algebra definition already provides us with a notion of free object, the free operad on a species $\mathcal{S}$ is $\mathcal{T}(\mathcal{S})$. Let us see how to define a notion of quotient for operads.

Definition 2.2.2.1. Let $(\mathcal{P}, \gamma, \varepsilon)$ be an operad (resp. a ns operad, a shuffle operad) given by the monoid definition, a *left ideal* of $\mathcal{P}$ is a subobject L of $\mathcal{P}$ such that $\gamma(\mathcal{P} \circ L) \subset L$. A *right ideal* of $\mathcal{P}$ is a subobject R of $\mathcal{P}$ such that $\gamma(R \circ \mathcal{P}) \subset R$.

These definitions of left and right ideals may seem equivalent to each other, however, they are not. Indeed, a technical issue in the world of operads is that the plethysm is left linear but not right linear. Let us define the infinitesimal composition as a linearization of the plethysm. We recall that the plethysm is defined by:

- $(\mathcal{S} \circ \mathcal{R})(A) = \bigoplus_{k\in\mathbb{N}} \bigoplus_{P \vdash^k A} \mathcal{S}(P) \otimes \bigotimes_{p\in P} R(p)$ for species;
- $(\mathcal{S} \circ \mathcal{R})(n) = \bigoplus_{k\in\mathbb{N}} \bigoplus_{\lambda \vDash^k n} \mathcal{S}(k) \otimes \bigotimes_{i=1}^{k} R(\lambda_i)$ for ordered species;
- $(\mathcal{S} \circ_{\text{Ш}} \mathcal{R})(n) = \bigoplus_{k\in\mathbb{N}} \bigoplus_{\lambda \vDash^k n} \frac{n!}{k!\lambda_1! \dots \lambda_k!} \mathcal{S}(k) \otimes \bigotimes_{i=1}^{k} R(\lambda_i)$ for shuffle species.

With those formulas, we clearly say the plethysm is not right linear.

Definition 2.2.2.2. Let us define the *infinitesimal composition* of $\mathcal{S}$ with $\mathcal{R}$ relatively to $\mathcal{U}$ by:

- $(\mathcal{S} \circ' (\mathcal{U}, \mathcal{R}))(A) = \bigoplus_{k\in\mathbb{N}} \bigoplus_{P \vdash^k A} \mathcal{S}(P) \otimes \bigoplus_{p\in P} R(p) \otimes \bigotimes_{q\neq p} \mathcal{U}(q)$ for species;
- $(\mathcal{S} \circ' (\mathcal{U}, \mathcal{R}))(n) = \bigoplus_{k\in\mathbb{N}} \bigoplus_{\lambda \vDash^k n} \mathcal{S}(k) \otimes \bigoplus_{i=1}^{k} R(\lambda_i) \otimes \bigotimes_{j\neq i} \mathcal{U}(\lambda_j)$ for ordered species;
- $(\mathcal{S} \circ'_{\text{Ш}} (\mathcal{U}, \mathcal{R}))(n) = \bigoplus_{k\in\mathbb{N}} \bigoplus_{\lambda \vDash^k n} \frac{n!}{k!\lambda_1! \dots \lambda_k!} \mathcal{S}(k) \otimes \bigoplus_{i=1}^{k} R(\lambda_i) \otimes \bigotimes_{j\neq i} \mathcal{U}(\lambda_j)$ for shuffle species.

We will denote $\mathcal{S} \circ' \mathcal{R}$ for $\mathcal{S} \circ' (\mathcal{I}, \mathcal{R})$.

We have clearly fixed the linearity issue. Indeed, we have: $\mathcal{S} \circ' (\mathcal{U}, \mathcal{R}_1 + \mathcal{R}_2) = \mathcal{S} \circ' (\mathcal{U}, \mathcal{R}_1) + \mathcal{S} \circ' (\mathcal{U}, \mathcal{R}_2)$. However, we added a new issue: one need to choose a third species $\mathcal{U}$ to define the infinitesimal composition. This is not a big issue, as we will have a canonical choice for $\mathcal{U}$. Another slight issue is that one need to be careful that $\mathcal{S} \circ' (\mathcal{R}, \mathcal{R}) \neq \mathcal{S} \circ \mathcal{R}$ in general. Indeed, a lot of elements of $\mathcal{S} \circ \mathcal{R}$ are present multiple times in $\mathcal{S} \circ' (\mathcal{R}, \mathcal{R})$ since we have chosen a preferred copy of $\mathcal{R}$. However, we still get a canonical morphism $\mathcal{S} \circ' (\mathcal{R}, \mathcal{R}) \to \mathcal{S} \circ \mathcal{R}$

Definition 2.2.2.3. Let $\mathcal{P}$ be an operad (resp. a ns operad, a shuffle operad), a *left infinitesimal ideal* of $\mathcal{P}$ is a subobject L of $\mathcal{P}$ such that $\gamma(\mathcal{P} \circ' L) \subset L$. An *operadic ideal* of $\mathcal{P}$ is a subobject $\mathcal{S}$ such that $\mathcal{S}$ is a left infinitesimal ideal and a right ideal.

Proposition 2.2.2.4. *Let $\varphi : \mathcal{P} \to \mathcal{Q}$ be a morphism of operads. The kernel of φ is an operadic ideal of $\mathcal{P}$.*

Proof. Let us check this property using partial compositions. Let $f \in \mathcal{P}$ such that $\varphi(f) = 0$. Then for all $g \in \mathcal{P}$ we have:

$$\varphi(f \circ_i g) = \varphi(f) \circ_i \varphi(g) = 0$$

Hence, $\ker(\varphi)$ is a right ideal of $\mathcal{P}$. Moreover, we have:

$$\varphi(g \circ_i f) = \varphi(g) \circ_i \varphi(f) = 0$$

Hence, $\ker(\varphi)$ is a left infinitesimal ideal of $\mathcal{P}$. Same in the ns and shuffle cases. □

Definition 2.2.2.5. Let $\mathcal{P}$ be an operad (resp. a ns operad, a shuffle operad) and $\mathcal{S}$ be a subspecies of $\mathcal{P}$. The quotient of $\mathcal{P}$ by $\mathcal{S}$ is the species $\mathcal{P}/\mathcal{S}$ such that $(\mathcal{P}/\mathcal{S})(A) = \mathcal{P}(A)/\mathcal{S}(A)$ for all A.

Proposition 2.2.2.6. *Let $\mathcal{P}$ be an operad (resp. a ns operad, a shuffle operad) and $\mathcal{S}$ a subspecies of $\mathcal{P}$. The operadic structure of $\mathcal{P}$ induces an operadic structure on $\mathcal{P}/\mathcal{S}$ if and only if $\mathcal{S}$ is an operadic ideal of $\mathcal{P}$.*

Proof. If $\mathcal{P}$ induces an operadic structure on $\mathcal{P}/\mathcal{S}$, then $\pi : \mathcal{P} \to \mathcal{P}/\mathcal{S}$ is a morphism of operads. Hence, $\mathcal{S}$ is an operadic ideal of $\mathcal{P}$. Conversely, if $\mathcal{S}$ is an operadic ideal of $\mathcal{P}$, let $f_1, f_2 \in \mathcal{P}$ and $g_1, g_2 \in \mathcal{P}$ such that $f_1 - f_2 \in \mathcal{S}$ and $g_1 - g_2 \in \mathcal{S}$. Then:

$$f_1 \circ_i g_1 - f_1 \circ_i (g_1 - g_2) - (f_1 - f_2) \circ_i g_1 = f_2 \circ_i g_2$$

Since $\mathcal{S}$ is an operadic ideal, we have:

$$\pi(f_1 \circ_i g_1) = \pi(f_2 \circ_i g_2)$$

Hence, the partial compositions are well-defined on $\mathcal{P}/\mathcal{S}$. Same in the ns and shuffle cases. □

The last ingredient we are missing to define an operad by generator and relator is the operadic ideal generated by a subset.

Definition 2.2.2.7. Let $\mathcal{P}$ be an operad (resp. a ns operad, a shuffle operad) and R a subset of $\mathcal{P}$. The *operadic ideal generated by* R is the smallest operadic ideal of $\mathcal{P}$ containing R. More precisely, it is the intersection of all operadic ideals of $\mathcal{P}$ containing R which is an operadic ideal of $\mathcal{P}$ since operadic ideals are stable by intersection.

Definition 2.2.2.8. Let $\mathcal{S}$ be a species (resp. an ordered species, a shuffle species) and $R = \{r_1, \ldots, r_k\}$ a set of elements of $\mathcal{T}(\mathcal{S})$. The *operad (resp. the ns operad, the shuffle operad) generated by $\mathcal{S}$ with relations* R denoted by $\mathcal{T}(\mathcal{S})/\langle R\rangle$ or $\mathcal{T}(\mathcal{S})/\langle r_1, \ldots, r_k\rangle$ is the quotient of $\mathcal{T}(\mathcal{S})$ by the operadic ideal generated by R. In the case of set species, we introduce the following two notations:

- If $\mathcal{X}$ is a set species, and $\mathcal{R}$ an equivalence relation on $\mathcal{T}\{\mathcal{X}\}$, then we denote by $\mathcal{T}\{\mathcal{X}\}/\mathcal{R}$ the set operad generated by $\mathcal{X}$ with relation $\mathcal{R}$.

- If $\mathcal{X}$ is a set species, we denote $\mathcal{T}[\mathcal{X}] = \mathcal{T}(\mathrm{Span}(\mathcal{X}))$. If $R \subseteq \mathcal{T}[\mathcal{X}]$, then we denote by $\mathcal{T}[\mathcal{X}]/\langle R\rangle$ the algebraic operad generated by $\mathrm{Span}(\mathcal{X})$ with relations R.

With those definitions, we have a way to construct operads by generators and relations, and we can finally define the coproduct of operads.

Definition 2.2.2.9. Let $\mathcal{P}$ and $\mathcal{Q}$ be two operads. The *coproduct* of $\mathcal{P}$ and $\mathcal{Q}$ is the operad $\mathcal{P} \vee \mathcal{Q}$ satisfying the universal property of the coproduct in the category of operads. More explicitly, if $\mathcal{P} = \mathcal{T}(\mathcal{S}_1)/\mathcal{R}_2$ and $\mathcal{Q} = \mathcal{T}(\mathcal{S}_2)/\mathcal{R}_2$, then $\mathcal{P} \vee \mathcal{Q} = \mathcal{T}(\mathcal{S}_1 \oplus \mathcal{S}_2)/\langle \mathcal{R}_1 \oplus \mathcal{R}_2\rangle$. If we have a third operad $\mathcal{U}$ such that:

$$\mathcal{P} \xleftarrow{\ \varphi\ } \mathcal{U} \xrightarrow{\ \psi\ } \mathcal{Q}$$

Then the *fibered coproduct* of $\mathcal{P}$ and $\mathcal{Q}$ over $\mathcal{U}$ denoted $\mathcal{P} \vee_{\mathcal{U}} \mathcal{Q}$ is:

$$\mathcal{P} \vee_{\mathcal{U}} \mathcal{Q} = (\mathcal{P} \vee \mathcal{Q})/\langle \varphi(x) - \psi(x) \mid x \in \mathcal{U}\rangle$$

We would like to draw the attention of the reader towards the last notation $\mathcal{T}[\mathcal{X}]/\langle R\rangle$ since this is the notation we are going to use in the rest of the document.

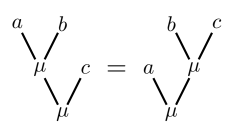
2.2.3 The Three Graces and the operadic butterfly

We can now define our favorite operads, the Three Graces. The Three Graces are the three operads Ass, Com, and Lie respectively encoding the associative, commutative associative, and Lie algebras. They were named the Three Graces by Loday since these are the three most important example of operads. Let us start with the operad Ass of associative algebras. We specify "associative" algebras because most algebra we are going to consider are not associative at all. Hence, the reader should keep in mind that we use the term algebra for a general algebraic structure, in particular algebra are always assumed to be non-necessarily associative algebra.

Definition 2.2.3.1. The *associative operad* denoted Ass is the operad generated by one generator of arity two without symmetry μ and the relations $\{\mu \circ_1 \mu - \mu \circ_2 \mu\}$.

$$\mathrm{Ass} = \mathcal{T}[\mu, \mu.(1\ 2)]/\langle \mu \circ_1 \mu - \mu \circ_2 \mu\rangle$$

One can "draw" the relation $\mu \circ_1 \mu = \mu \circ_2 \mu$ as follows:



If interpret μ a product (a bilinear map) of a vector space V and $a, b, c \in V$, we get $(ab)c = a(bc)$ and we recognize the associativity relation. Formally, we have the following result:

Proposition 2.2.3.2. *The category* Ass–Alg *is the category of (non-necessarily unital) associative algebras.*

Proof. Let (A, m) be an associative algebra, then by the universal property of the free operad, there exists a unique morphism of operads such that:

$$\begin{aligned} \mathcal{T}[\mu, \mu.(1\ 2)] &\to \mathrm{End}_A \\ \mu &\mapsto m \end{aligned}$$

Moreover since m is associative, it factorizes through Ass. Hence, we have a morphism of operads $\mathrm{Ass} \to \mathrm{End}_A$. Conversely, let $\varphi : \mathrm{Ass} \to \mathrm{End}_A$ be a morphism of operads, then let $m = \varphi(\mu)$. We have that (A, m) is an associative algebra. $\square$

This basic example is interesting in many aspects. First it shows that operads can be used to encode algebraic structures, indeed replacing the associativity relation by let say the Jacobi identity, we would get the operad of Lie algebras. Moreover, the operad Ass have a lot of interesting properties:

Proposition 2.2.3.3. *The underlying species of* Ass *is the species* $\mathbb{L}_{\geq 1}$ *of total order on non-empty sets.*

Proof. Since Ass-algebras are exactly the associative algebras, the Schur functor $\mathcal{F}_{\text{Ass}}$ is the free associative algebra. Hence, we have:

$$\mathcal{F}_{\text{Ass}}(V) = \bigoplus_{n\geq 1} V^{\otimes n}$$

We recognize the Schur functor of the species $\mathbb{L}_{\geq 1}$. Hence, $\text{Ass} = \mathbb{L}_{\geq 1}$ as species. □

Similarly, we can define the operad Com of (non-necessarily unital) commutative associative algebras. The only modification we need to do is to make the product commutative.

Definition 2.2.3.4. The *commutative operad* denoted Com is the operad generated by one symmetric generator c of arity two and the relations $\{c \circ_1 c - c \circ_2 c\}$.

$$\text{Com} = \mathcal{T}[c]/\langle c \circ_1 c - c \circ_2 c\rangle$$

Such that the action of $\mathfrak{S}_2$ is given by $c.(1\ 2) = c$.

We can see from this example that the data of the group action on the generators is fundamental to define the operad. Indeed, the operad Ass and Com are quite different, however they only differ by the action of $\mathfrak{S}_2$ on the generator.

Proposition 2.2.3.5. *The category* Com–Alg *is the category of commutative associative algebras.*

Proof. Let (A, m) be a commutative associative algebra, then by the universal property of the free operad, there exists a unique morphism of operads such that:

$$\begin{aligned} \mathcal{T}[c] &\to \text{End}_A \\ c &\mapsto m \end{aligned}$$

Moreover since m is associative, it factorizes through Com. Hence, we have a morphism of operads $\text{Com} \to \text{End}_A$. Conversely, let $\varphi : \text{Com} \to \text{End}_A$ be a morphism of operads, then let $m = \varphi(c)$. We have that (A, m) is a commutative associative algebra. □

Proposition 2.2.3.6. *The underlying species of* Com *is the species* $E_{\geq 1}$ *of non-empty sets.*

Proof. Since Com-algebras are exactly the commutative associative algebras, the Schur functor $\mathcal{F}_{\text{Com}}$ is the free commutative associative algebra. Hence, we have:

$$\mathcal{F}_{\text{Com}}(V) = \bigoplus_{n\geq 1} V^{\otimes n}/\mathfrak{S}_n$$

We recognize the Schur functor of the species $E_{\geq 1}$. Hence, $\text{Com} = E_{\geq 1}$ as species. □

Let us define the last of the Three Graces, the operad Lie of Lie algebras. The Lie operad is a bit more complicated than the two previous ones, since we need to encode the Jacobi identity.

Definition 2.2.3.7. The *Lie operad* denoted Lie is the operad generated by one skew-symmetric generator ℓ of arity two and the relations $\{\ell \circ_1 \ell + (\ell \circ_1 \ell).(1\ 2\ 3) + (\ell \circ_1 \ell).(1\ 3\ 2)\}$.

$$\text{Lie} = \mathcal{T}[\ell]/\langle \ell \circ_1 \ell + (\ell \circ_1 \ell).(1\ 2\ 3) + (\ell \circ_1 \ell).(1\ 3\ 2)\rangle$$

Such that the action of $\mathfrak{S}_2$ is given by $\ell.(1\ 2) = -\ell$.

Proposition 2.2.3.8. *The category* Lie–Alg *is the category of Lie algebras.*

Proof. Let $(\mathfrak{g}, [.\,,.])$ be a Lie algebra, then by the universal property of the free operad, there exists a unique morphism of operads such that:

$$\begin{aligned} \mathcal{T}[\ell] &\to \mathrm{End}_{\mathfrak{g}} \\ \ell &\mapsto [.\,,.] \end{aligned}$$

Moreover since $[.\,,.]$ satisfies the Jacobi identity, it factorizes through Lie. Hence, we have a morphism of operads $\mathrm{Lie} \to \mathrm{End}_{\mathfrak{g}}$. Conversely, let $\varphi : \mathrm{Lie} \to \mathrm{End}_{\mathfrak{g}}$ be a morphism of operads, then let $[.\,,.] = \varphi(\ell)$. We have that $(\mathfrak{g}, [.\,,.])$ is a Lie algebra. □

These three operads are related the following way:

$$\mathrm{Lie} \to \mathrm{Ass} \twoheadrightarrow \mathrm{Com}$$

The way to understand this is that any associative algebra is a Lie algebra since the commutator is a Lie bracket, and any commutative associative algebra is an associative algebra. The morphisms are given by:

$$\mathrm{Lie} \xrightarrow{\ \ell \mapsto \mu - \mu.(1\ 2)\ } \mathrm{Ass} \xrightarrow{\ \mu \mapsto c\ } \mathrm{Com}$$

It was remarked that it is possible to extend this diagram. Let us define four other operads:

Definition 2.2.3.9. Let us define the following operads:

- The *diassociative operad* denoted Diass:

  $$\mathrm{Diass} = \mathcal{T}[a, b, a.(1\ 2), b.(1\ 2)]/\langle a \circ_2 a - a \circ_1 a, a \circ_2 a - a \circ_2 b, a \circ_1 b - b \circ_2 a, b \circ_1 a - b \circ_2 b, b \circ_1 a - b \circ_1 b\rangle$$

- The *dendriform operad* denoted Dend:

  $$\mathrm{Dend} = \mathcal{T}[x, y, x.(1\ 2), y.(1\ 2)]/\langle x \circ_1 x - x \circ_2 x - x \circ_2 y, x \circ_1 y - y \circ_2 x, y \circ_1 x + y \circ_1 y - y \circ_2 y\rangle$$

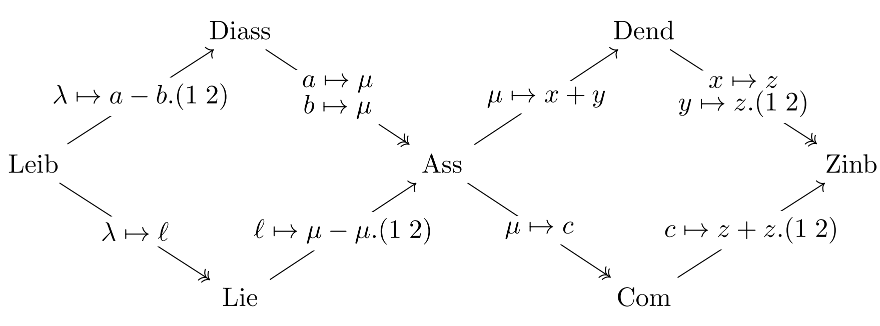
- The *Leibniz operad* denoted Leib:

  $$\mathrm{Leib} = \mathcal{T}[\lambda, \lambda.(1\ 2)]/\langle \lambda \circ_1 \lambda + (\lambda \circ_1 \lambda).(1\ 2\ 3) + (\lambda \circ_1 \lambda).(1\ 3\ 2)\rangle$$

- And the *Zinbiel operad* denoted Zinb:

  $$\mathrm{Zinb} = \mathcal{T}[z, z.(1\ 2)]/\langle z \circ_1 z - z \circ_2 z - (z \circ_2 z).(2\ 3)\rangle$$

These operads were respectively introduced in [59] for the diassociative and dendriform operads, in [56] for the Leibniz operad, and in [58] for the Zinbiel operad. We then have the following commutative diagram:



Let us define the true favorite operad of the author, the operad PreLie encoding pre-Lie algebras which was first introduced in [19].

Definition 2.2.3.10. The *pre-Lie operad* denoted PreLie is the operad generated by one generator of arity two without symmetry r and the relations $\{r \circ_1 r - r \circ_2 r = (r \circ_1 r - r \circ_2 r).(1\ 2)\}$.

$$\mathrm{PreLie} = \mathcal{T}[r, r.(1\ 2)]/\langle (r \circ_1 r - r \circ_2 r) - (r \circ_1 r - r \circ_2 r).(1\ 2)\rangle$$

The way to understand this definition is that pre-Lie algebras (which are algebras over PreLie) are a generalization of associative algebras, indeed pre-Lie algebras satisfy a weakened version of associativity: instead of having the associator that vanishes, we have that the associator is right symmetric. The reason PreLie is interesting is that we still have:

$$\mathrm{Lie} \to \mathrm{PreLie}$$

With the morphism $\ell \mapsto r - r.(1\ 2)$. Moreover, we have the following fact:

Proposition 2.2.3.11. *Let $\mathcal{P}$ be an operad. Let $\star$ be defined by $\star = \sum \circ_i$ with $\circ_i$ the partial compositions. Then $(\mathcal{P}, \star)$ is a pre-Lie algebra.*

Proof. Let $f, g, h \in \mathcal{P}$ and let us compute $(f \star g) \star h$. We have that:

$$(f \star g) \star h = \sum_{(i,j) \in Seq} (f \circ_i g) \circ_j h + \sum_{(i,j) \in Par} (f \circ_i g) \circ_j h$$

With Seq the set of indices (i, j) such that the composition is sequential and Par the set of indices (i, j) such that the composition is parallel. Using the sequential composition axiom we have that:

$$\begin{aligned}(f \star g) \star h &= \sum_{(i,j) \in Seq} f \circ_i (g \circ_j h) + \sum_{(i,j) \in Par} (f \circ_i g) \circ_j h \\ &= f \star (g \star h) + \sum_{(i,j) \in Par} (f \circ_i g) \circ_j h\end{aligned}$$

Hence, we get:

$$(f \star g) \star h - f \star (g \star h) = \sum_{(i,j) \in Par} (f \circ_i g) \circ_j h$$

Using the parallel composition axiom we get the intended result. ☐

The property that any operad is a pre-Lie algebra is quite interesting, and is part of the reason why the author is so fond of pre-Lie algebras. We will see more about the pre-Lie operad in the next chapter.

2.2.4 Operadic rewriting systems

We now know how to define operads by generators and relations. However, a usual issue with this kind of definition is that it is not easy to compute with it. Indeed, we have no canonical way to represent an element of a quotient. To solve this issue, we will be using rewriting systems. We will first recall the basic definitions of abstract rewriting systems (ARS). We will then adapt this notion to vector spaces to get linear rewriting systems (LRS), this terminology is not standard and once again the reader should be careful as in the literature LRS is used to denote a different notion. In our case we use the terminology LRS to denote a rewriting system on vector spaces compatible with the structure. Finally, we will define operadic rewriting systems (ORS) which are rewriting systems on operads. The adapted context to these constructions are in fact *polygraph*, see [41], however the author do not wish to introduce this general context, and we will adapt the definitions and proofs to the operadic case. What we are going to achieve with operadic rewriting systems is usually done either with PBW basis, see [43], or with Gröbner basis, see [28] and [10]. However, since the author tends to prefer rewriting systems, we will use this approach.

Definition 2.2.4.1. An *abstract rewriting system*, denoted ARS, is a pair $(A, \to)$ where A is a set and $\to$ is a binary relation on A, called the *reduction relation*. We denote by $a \to b$ a *rewriting step* which is a couple (a, b) in relation by $\to$. We denote by $\overset{*}{\to}$ the reflexive and transitive closure of $\to$, meaning that $a \overset{*}{\to} b$ if we have a sequence:

$$a = a_0 \to a_1 \to \ldots \to a_n = b$$

We will denote $\leftarrow$ the opposite relation of $\to$, meaning that $a \leftarrow b$ if $b \to a$.

Definition 2.2.4.2. Let $(A, \to)$ be an ARS, and $a, b \in A$. We say that a and b are *joinable* if there exists $c \in A$ such that $a \overset{*}{\to} c \overset{*}{\leftarrow} b$. Denote by $a \downarrow b$ the fact that a and b are joinable.

Definition 2.2.4.3. The ARS $(A, \to)$ is *confluent* if for all $a, b, c \in A$ such that $a \overset{*}{\leftarrow} c \overset{*}{\to} b$, we have $a \downarrow b$. It is *locally confluent* if for all $a, b, c \in A$ such that $a \leftarrow c \to b$ we have $a \downarrow b$.

Definition 2.2.4.4. The ARS $(A, \to)$ is *terminating* if there is no infinite sequence $a_0 \to a_1 \to a_2 \to \ldots$. It is *convergent* if it is confluent and terminating.

Theorem 2.2.4.5 (Diamond Lemma)**.** *Let $(A, \to)$ be an ARS. If $(A, \to)$ is locally confluent and terminating, then it is convergent.*

The first proof of this theorem was given in [68]. However, we refer to [8] for a more concise and understandable proof.

Definition 2.2.4.6. Let $(A, \to)$ be an ARS. An element $a \in A$ is *reducible* if there exists $b \in A$ such that $a \to b$. It is *irreducible* otherwise. If $a \overset{*}{\to} b$ and b is irreducible, we say that b is a *normal form* of a.

Theorem 2.2.4.7. *Let $(A, \to)$ be a convergent ARS. Then each $a \in A$ admits a unique normal form.*

Proof. Since $(A, \to)$ is terminating, we have that each $a \in A$ admits a normal form. Since $(A, \to)$ is confluent, we have that this normal form is unique. □

This theorem is the reason why we are interested in rewriting systems. Indeed, it gives us a canonical way to represent an element of a quotient. Before adapting this notion to vector spaces, let us relate termination and wellness of partial order.

Definition 2.2.4.8. Let A be a set. Let us recall that a *partial order* on A is a binary relation $\geq$ on A which is:

- *reflexive*: for all $a \in A$, $a \geq a$;
- *transitive*: for all $a, b, c \in A$, if $a \geq b$ and $b \geq c$ then $a \geq c$;
- *antisymmetric*: for all $a, b \in A$, if $a \geq b$ and $b \geq a$ then $a = b$.

Let us denote $a \geq b$ and $a \neq b$ by $a > b$. A *well partial order*, denoted *wpo*, is a partial order such that there is no infinite sequence:

$$a_0 > a_1 > a_2 > \ldots$$

It can be seen as a Noetherian property for partial orders.

Theorem 2.2.4.9. *Let $(A, \to)$ be an ARS. Then $(A, \to)$ is terminating if and only if we have a wpo $\geq$ on A such that $\to$ is strictly decreasing for $\geq$, meaning that for all $a, b \in A$ such that $a \to b$, we have $a > b$.*

Proof. Let us first prove that if $(A, \to)$ is strictly decreasing along $\geq$ a wpo, then it is terminating. Let us suppose that $(A, \to)$ is not terminating, then there is an infinite sequence $a_0 \to a_1 \to a_2 \to \dots$. Hence, we get $a_0 > a_1 > a_2 > \dots$, which is a contradiction since $\geq$ is a wpo.
Let us now prove that if $(A, \to)$ is terminating, then we have a wpo $\geq$ on A such that $\to$ is strictly decreasing for $\geq$. Let us show that $\overset{*}{\to}$ is a wpo. It is reflexive and transitive, moreover $\to$ is strictly decreasing for $\overset{*}{\to}$ by definition. Let us show that $\overset{*}{\to}$ is antisymmetric. Let us suppose that $a \overset{*}{\to} b$ and $b \overset{*}{\to} a$ for $a \neq b$. Then we have $a \to a_1 \to \dots \to a_n \to b$ and $b \to b_1 \to \dots \to b_m \to a$. Hence, we can construct:

$$a \to a_1 \to \dots \to a_n \to b \to b_1 \to \dots \to b_m \to a \to a_1 \to \dots a_n \to b \to b_1 \to \dots \to b_m \to a \to \dots$$

Which is a contradiction with the fact that $(A, \to)$ is terminating. Hence, $\overset{*}{\to}$ is antisymmetric. Let us show that $\overset{*}{\to}$ is a wpo. Let us suppose that there is an infinite sequence $a_0 \overset{*}{\to} a_1 \overset{*}{\to} a_2 \overset{*}{\to} \dots$ with $a_i \neq a_{i+1}$. Then we have:

$$a_0 \to a_0^{(1)} \to \dots \to a_0^{(n_0)} \to a_1 \to a_1^{(1)} \to \dots \to a_1^{(n_1)} \to a_2 \to \dots$$

This is a contradiction with the fact that $(A, \to)$ is terminating. Hence, $\overset{*}{\to}$ is a wpo. □

However, we are working with algebraic operads, operads on vector spaces, not on sets. Let us adapt the notion of rewriting system to linear algebra. We recall we are not using standard definition for LRS.

Definition 2.2.4.10. A *linear rewriting system*, denoted *LRS*, is a pair (M, R) where M is a set usually referred as the set of monomials, and R is a subset of $M \times \mathrm{Span}(M)$. Elements of R are called *rewriting rules* and are denoted by $m \to v$ instead of (m, v). If $v = \lambda_{i_0} m_{i_0} + \sum \lambda_i m_i$, we denote by $v \to w$ the fact that $m_{i_0} \to v_{i_0}$ and $w = \lambda_{i_0} v_{i_0} + sum \lambda_i m_i$. With those notations, we have that $(\mathrm{Span}(M), \to)$ is an ARS.

From the fact that we can get an ARS from an LRS, we can get a notion of confluence, termination, and convergence for LRS. The useful part is that the ARS given by an LRS is compatible with the vector space structure of $\mathrm{Span}(M)$.

Definition 2.2.4.11. Let (M, R) be an LRS. Let us write $m \succ m_i$ if we have $m \to \sum \lambda_i m_i \in R$ with $\lambda_i \neq 0$. Let $\geq$ be the reflexive and transitive closure of $\succ$.

Lemma 2.2.4.12. *Let (M, R) be a terminating LRS. Then (M, R) induces a complete transfinite filtration $F_\alpha M$ on the set M such that $\succ$ is decreasing along this filtration.*

Proof. First, let us define a transfinite filtration on $\mathrm{Span}(M)$. Let $G_0\mathrm{Span}(M)$ be the set of irreducible elements of V. Let $G_\alpha\mathrm{Span}(M)$ be the set of elements of $\mathrm{Span}(M)$ such that for all $v \in G_\alpha\mathrm{Span}(M)$ and $v \to w$, we have $w \in G_\beta\mathrm{Span}(M)$ with $\alpha > \beta$. One need to be careful since $G_\alpha\mathrm{Span}(M)$ and $F_\alpha\mathrm{Span}(M) = \bigcup_{\beta<\alpha} G_\alpha\mathrm{Span}(M)$ are a priori not sub vector spaces of $\mathrm{Span}(M)$. Let ν be the smallest ordinal such that $G_\nu\mathrm{Span}(M) = \emptyset$. Such an ordinal exists, indeed since the $G_\alpha\mathrm{Span}(M)$ are disjoints, let γ be an ordinal of cardinality greater than $\mathrm{Span}(M)$, we have $\beta < \gamma$ such that $G_\beta\mathrm{Span}(M) = \emptyset$. We have $F_\nu\mathrm{Span}(M) = \mathrm{Span}(M)$, indeed let $v \notin F_\nu\mathrm{Span}(M)$, then $v \notin F_{\nu+1}\mathrm{Span}(M)$ hence we have $v_1 \notin F_\nu\mathrm{Span}(M)$ such that $v \to v_1$. By induction we can construct:

$$v \to v_1 \to v_2 \to \dots$$

Which lead to a contradiction since (M, R) is terminating. Let $F_\alpha M = (F_\alpha\mathrm{Span}(M)) \cap M$. This is a complete transfinite filtration of M. From its definition, $\succ$ is decreasing along this filtration. □

Theorem 2.2.4.13. *Let (M, R) be an LRS. Then it is terminating if and only $\geq$ is a wpo on M. In particular, it is terminating if and only if it is terminating on M.*

Proof. Assume that $\geq$ is a wpo on M and let us show that (M, R) is terminating. Let us suppose that (M, R) is not terminating, then there is an infinite sequence

$$m \to v_1 \to v_2 \to v_3 \to \dots$$

Hence, we get:

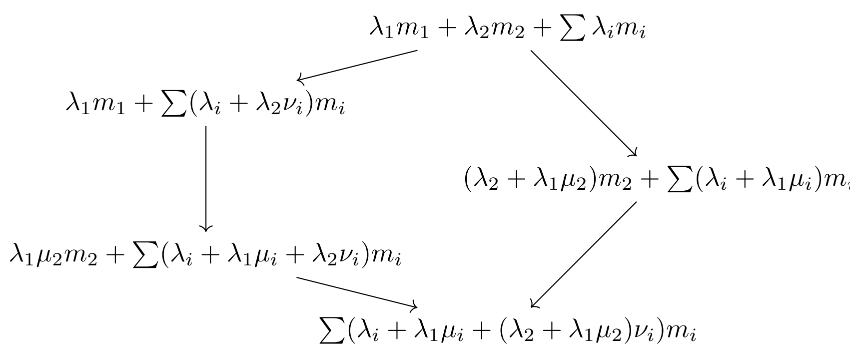
$$m_0 > m_1 > m_2 > m_3 > \dots$$

This is a contradiction since $\geq$ is a wpo.
Assume that (M, R) is terminating. By the previous lemma, we have a complete transfinite filtration $F_\alpha M$ of M such that $\succ$ is decreasing along this filtration. By definition, $\geq$ is reflexive and transitive, and for each rewriting rule $m \to \sum \lambda_i m_i$, if $\lambda_i \neq 0$ then we have $m > m_i$. In particular, we have that $m > m_i$ imply that $m \in F_\alpha M$ and $m_i \in F_\beta M$ with $\alpha > \beta$. Hence, $\geq$ is a wpo on M. ☐

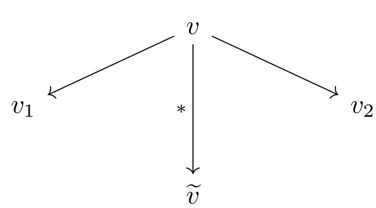
The proof is quite technical, however the idea is quite simple: if the LRS is terminating, then each rewriting rule is decreasing for a well-chosen wpo on M.

Lemma 2.2.4.14. *Let (M, R) be a terminating LRS. Then it is confluent if and only if it is confluent on M.*

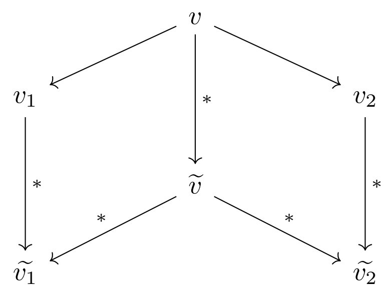
Proof. It is clear that if (M, R) is confluent, then it is confluent on M. Let us show the converse. Assume that (M, R) is confluent on M. Let us show that (M, R) admits unique normal forms. Let $v = \lambda_1 m_1 + \lambda_2 m_2 + \sum \lambda_i m_i$ and let us rewrite m_1 and m_2. By Theorem 2.2.4.13, we have that either there is no occurrence of m_1 in the rewriting rules of m_2 or there is no occurrence of m_2 in the rewriting rules of m_1. Up to reordering the basis, we can assume that m_1 is not in the rewriting rules of m_2. Let us write $m_1 \to \sum \mu_i m_i$ and $m_2 \to \sum \nu_i m_i$, we have:



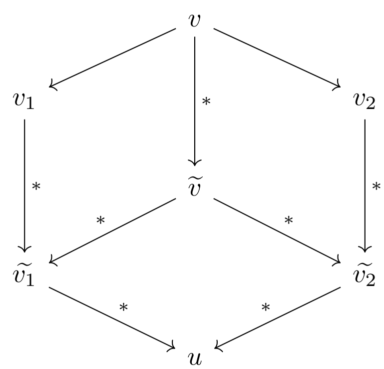
Hence, (M, R) is locally confluent in v if the two rewriting rules rewrite different monomials. Let us rewrite the same monomial m of v by two different rewriting rules to get: $v_1 \leftarrow v \to v_2$. By Theorem 2.2.4.13, we have a wpo on the monomials. For each monomial lower than m, let us chose a rewriting rule, and let us rewrite v by those rules to get $v \overset{*}{\to} \widetilde{v}$ such that the only monomial lower than m in $\widetilde{v}$ are irreducible. By the previous computation we see that the order does not matter. We have:



By the previous computation, we can complete it and get:



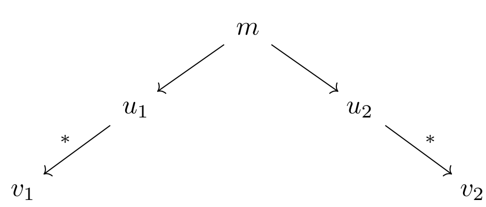
The other monomials of $\widetilde{v}$ does not interfere with the rewriting of m, indeed they are either greater than m and we do not rewrite them, or they are irreducible and we can not rewrite them. Hence, since (M, R) is confluent in m we have:



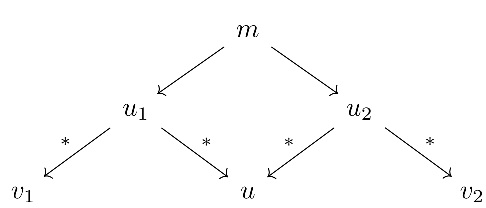
Hence, (M, R) is locally confluent. By the diamond lemma, we have that (M, R) is confluent. □

Theorem 2.2.4.15 (Linear Diamond Lemma)**.** *Let (M, R) be a terminating LRS. Then it is convergent if and only if it is locally confluent on M.*

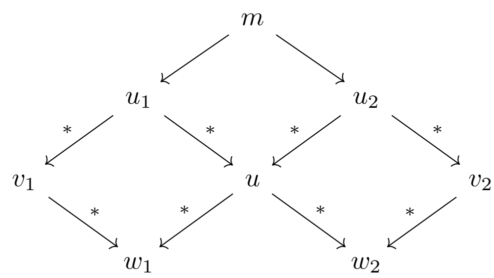
Proof. It is clear that if (M, R) is convergent, then it is locally convergent on M. Let us show the converse. Assume that (M, R) is locally convergent on M. By theorem 2.2.4.13, we have a complete ordinal filtration on M, let us denote it $F_\alpha M$. Let us do an ordinal induction. It is clear that (M, R) is convergent once restricted to $\mathrm{Span}(F_1 M)$ since $F_1 M$ is the set of irreducible monomials. Let α be an ordinal, and assume that (M, R) is convergent once restricted to $\mathrm{Span}(F_\beta M)$, with $\beta < \alpha$. Let $m \in F_\alpha M$, let us show that (M, R) is convergent in m. Let $v_1 \overset{*}{\leftarrow} m \overset{*}{\rightarrow} v_2$, assuming $v_1 \neq m$ and $v_2 \neq m$, we have:



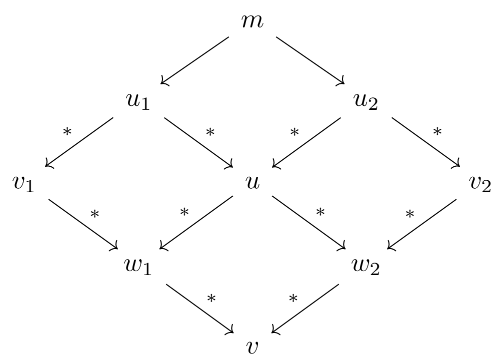
We know that (M, R) is locally convergent in m, hence:



By the induction hypothesis, we have that (M, R) is convergent in u_1 and u_2. Hence, we have:



By induction hypothesis, we have that (M, R) is convergent in u. We get:



Hence, (M, R) is convergent in m. Hence, (M, R) is convergent on the monomials of $\mathrm{Span}(F_\alpha M)$. By the previous lemma we have that (M, R) is on $F_\alpha M$. By induction, we have that (M, R) is convergent. □

One need to be a bit careful with the last theorem, indeed the terminating hypothesis is crucial. Let us show an example of a non-terminating LRS which is confluent on M but not on $\mathrm{Span}(M)$.

Example 2.2.4.16. Let V be the vector space with basis $M = \{a, b, c, d\}$ and let $R = \{a \to c - b, b \to d - a\}$. It is quite clear that (M, R) is confluent on M. However, we have $c \leftarrow a + b \to d$, so it is not confluent on V.

We still do not have exactly what we want. Indeed, we want to work with operads, hence we would like to have a notion of rewriting system for operads.

Definition 2.2.4.17. An *operadic rewriting system*, denoted *ORS*, is a triple $(\mathcal{S}, \mathcal{X}, R)$ where $\mathcal{S}$ is a linear species, $\mathcal{X}$ is a shuffle set species such that $U(\mathcal{S}) = \mathrm{Span}(\mathcal{X})$, and $R = (R_n)_{n \in \mathbb{N}}$ such that R_n is a subset of $\mathcal{T}^{\mathrm{III}}\{\mathcal{X}\}(n) \times \mathcal{T}(\mathcal{S})(\underline{n})$. We denote by R' the set of *rewritable monomials* which are the monomials admitting at least one rewriting rule in R.

We now would like to get an LRS compatible with the shuffle operadic structure of $\mathcal{T}^{\mathrm{III}}(U(\mathcal{S}))$ from an ORS $(\mathcal{S}, \mathcal{X}, R)$. The key idea is to define a notion of divisibility in $\mathcal{T}^{\mathrm{III}}\{\mathcal{X}\}$.

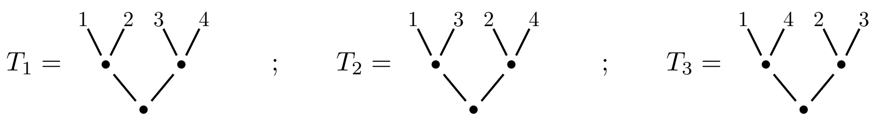
Definition 2.2.4.18. Let $D \in \mathcal{T}^{\mathrm{III}}\{\mathcal{X}\}(k)$ and $T \in \mathcal{T}^{\mathrm{III}}\{\mathcal{X}\}(n)$ with $\mathcal{X}$ a shuffle set species. D is a *divisor* of T if we have T_0 and $T_1, \ldots, T_k$ such that:

$$T = T_0 \circ_{i,I} \left(\gamma_{I_1,\ldots,I_k}(D; T_1, \ldots, T_k)\right)$$

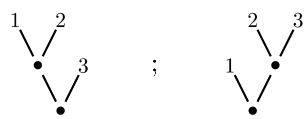
With $\circ_{i,I}$ given by the shuffle operad structure of $\mathcal{T}^{\mathrm{III}}\{\mathcal{X}\}$ via the partial compositions, and $\gamma_{I_1,\ldots,I_k}$ given by the shuffle operad structure of $\mathcal{T}^{\mathrm{III}}\{\mathcal{X}\}$ via the monoid definition. This definition means that D divides T if we can get T by composing some T_i in the leaves of D and then composing the result in a leaf of T_0.

To grasp the intuition behind this definition, let us give a more visual definition of divisibility. An element of $\mathcal{T}^{Ш}\{\mathcal{X}\}(n)$ is a shuffle tree with n leaves such that internal vertices are labeled by elements of $\mathcal{X}$. Hence, we can think of T as a triple (τ, σ, f) with τ a planar tree, σ the bijective labeling of the leaves respecting the shuffle tree condition, and f the labeling of the internal vertices. Then D is also a triple (ν, ρ, g). Let τ' be a sub-tree of τ, the labeling σ' of the leaves of τ' induced by σ is defined by: σ' of l a leaf of τ' is the minimal label of leaves of τ which are above the vertex of τ corresponding to l. Then D divides T if we can find a sub-tree τ' of τ such that the labeling f restricted to the internal vertices of τ' is equal to g, and the labeling σ' induced by σ on the leaves of τ' defined by σ' is equal to $\rho \circ g$ with g an increasing function. Let us show some examples and non-examples of divisibility.

Example 2.2.4.19. For simplicity, let us assume that $\mathcal{X}$ contains only one element of arity two, and let us omit the labeling of the internal vertices. Let us consider the following elements of $\mathcal{T}^{Ш}\{\mathcal{X}\}(4)$ and find their divisor in $\mathcal{T}^{Ш}\{\mathcal{X}\}(3)$:



Then the divisor of T_1 in $\mathcal{T}^{Ш}\{\mathcal{X}\}(3)$ are:



The divisor of T_2 and T_3 are the same, and are:

Definition 2.2.4.20. Let $D \in \mathcal{T}^{Ш}\{\mathcal{X}\}(k)$ and $T \in \mathcal{T}^{Ш}\{\mathcal{X}\}(n)$ with $\mathcal{X}$ a shuffle set species, such that D divides T. We have:

$$T = T_0 \circ_{i,I} (\gamma_{I_1,\dots,I_k}(D; T_1, \dots, T_k))$$

Let $S \in \mathcal{T}^{Ш}\{\mathcal{X}\}(k)$. The *substitution* of D by S in T is the element of $\mathcal{T}^{Ш}\{\mathcal{X}\}(n)$ defined by:

$$\underset{T,D}{\overset{(T_0,I);(T_1,I_1),\dots,(T_k,I_k)}{\square}}(S) = T_0 \circ_{i,I} (\gamma_{I_1,\dots,I_k}(S; T_1, \dots, T_k))$$

Formally this depends on the choice of $T_0, I; T_1, I_1, \dots, T_k, I_k$, however since it makes the notation heavy, we will instead write:

$$\square_{T,D}(S)$$

We still need to remember that this depends on the choice of $T_0, I; T_1, I_1, \dots, T_k$, and that we implicitly fixed such a choice when using this notation.

Definition 2.2.4.21. Let $(\mathcal{S}, \mathcal{X}, R)$ be an ORS. Let $m \in \mathcal{T}^{Ш}\{\mathcal{X}\}$, we denote $m \to t$ if there exists D divisor of m and $(D, r) \in R$ such that $r = \sum \lambda_i r_i$ and $t = \sum \lambda_i \square_{m,D}(r_i)$. We denote by $R_\square$ the set of couples (m, t) such that $m \to t$. Then $(\mathcal{T}^{Ш}\{\mathcal{X}\}, R_\square)$ is an LRS.

Since we have an LRS, we also have a ARS associated to any ORS. Moreover, from our constructions the ARS associated to an ORS $(\mathcal{S}, \mathcal{X}, R)$ is compatible with the shuffle operadic structure of $\mathcal{T}^{Ш}(U(\mathcal{S}))$. Hence, we have a notion of confluence, termination, and convergence for ORS.

We have defined the formalism of ORS, in order to define operads by generators and relations, and to get canonical basis of those operads. Let us define an operad via an ORS.

Definition 2.2.4.22. Let $(\mathcal{S},\mathcal{X},R)$ be an ORS. We recall that we have a canonical identification: $\mathrm{Span}\left(\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}\right) = U\left(\mathcal{T}(\mathcal{S})\right)$. Let us define $\mathcal{R}^{\text{Ш}} = \langle m - r \mid (m,r) \in R\rangle \subseteq \mathcal{T}^{\text{Ш}}(U(\mathcal{S}))$ the shuffle operadic ideal generated by $m-r$ such that $(m,r)\in R$. Then the *shuffle operad* defined by $(\mathcal{S},\mathcal{X},R)$ is the operad $\mathcal{P}^{\text{Ш}} = \mathcal{T}^{\text{Ш}}(U(\mathcal{S}))/\mathcal{R}^{\text{Ш}}$. If we have $\mathcal{R} \subseteq \mathcal{T}(\mathcal{S})$ such that $U(\mathcal{R}) = \mathcal{R}^{\text{Ш}}$, then we can define the *operad* defined by $(\mathcal{S},\mathcal{X},R)$ by $\mathcal{P} = \mathcal{T}(\mathcal{S})/\mathcal{R}$.

We can see in this definition that we need both the data of the species $\mathcal{S}$ and the shuffle set species $\mathcal{X}$. Indeed, we need the data of the shuffle set species $\mathcal{X}$ in order to have a basis of $\mathcal{T}(\mathcal{S})$, and we need the data of the species $\mathcal{S}$ in order to get the action of the permutation groups.

Proposition 2.2.4.23. *Let $(\mathcal{S},\mathcal{X},R)$ be a convergent ORS. Then the operad $\mathcal{P}$ defined by $(\mathcal{S},\mathcal{X},R)$ admits a canonical basis.*

Proof. It suffices to compute the normal forms of $(\mathcal{S},\mathcal{X},R)$. They constitute a basis of $\mathcal{P}$. □

Definition 2.2.4.24. Let us define the notion of critical monomial. Let $(\mathcal{S},\mathcal{X},R)$ be an ORS. A monomial $m \in \mathcal{T}^{\text{Ш}}\{\mathcal{X}\}$ is *critical* if there exists t_1 and t_2 two shuffle trees such that we have (t_1,r_1) and (t_2,r_2) in R, and:

$$m = T_0 \circ_{i,I} \left(\gamma_{I_1,\ldots,I_k}(v;T_1,\ldots,T_k)\right)$$

With:

$$t_1 = T_0 \circ_{i',I'} v \qquad \text{and} \qquad t_2 = \gamma_{I_1,\ldots,I_k}(v;T_1,\ldots,T_k)$$

The shuffle tree v is called an *overlap* of t_1 and t_2.

The intuitive idea behind the notion of critical monomial is that it is a small common multiple of two monomials t_1 and t_2 appearing in the rewriting rules. The critical monomials are the smaller monomials which can be rewritten in two different ways, and where those two ways overlap.

Theorem 2.2.4.25 (Operadic Diamond Lemma)**.** *Let $(\mathcal{S},\mathcal{X},R)$ be a terminating ORS. Then $(\mathcal{S},\mathcal{X},R)$ is confluent if and only if it is locally confluent on the critical monomials.*

Proof. It is clear that if $(\mathcal{S},\mathcal{X},R)$ is confluent, then it is locally confluent on the critical monomials. Let us show the converse. Assume that $(\mathcal{S},\mathcal{X},R)$ is locally confluent on the critical monomials. Let $m \in \mathcal{T}^{\text{Ш}}\{\mathcal{X}\}$, let us show that $(\mathcal{S},\mathcal{X},R)$ is locally confluent in m. Let $t_1 \leftarrow m \rightarrow t_2$. Let us denote by $d_1 \rightarrow r_1$ and $d_2 \rightarrow r_2$ the rewriting rules we used. If we have an overlap in the division of m by d_1 and d_2, then we have a critical monomial c of d_1 and d_2 such that c divide m, and the local confluence of $(\mathcal{S},\mathcal{X},R)$ in c conclude. If we do not have an overlap, then the two rewriting rules commutes, and we have the local confluence of $(\mathcal{S},\mathcal{X},R)$ in m. Theorem 2.2.4.15 allows us to conclude. □

From this theorem, we can see that the local confluence of an ORS can be checked on the critical monomials. In the case of a finite number of rewriting rules, we have a finite number of critical monomials, hence we can check the local confluence of an ORS by checking it on a finite number of cases, which can be done by hand or by a computer. The last issue is to check the termination of an ORS. It is in general not possible to reduce the termination of an ORS to the termination of a finite number of cases. However, we can use the good old trick of having a well order. Let us define the notion of monomial partial order.

Definition 2.2.4.26. Let $\mathcal{X}$ be a shuffle set species. A *monomial partial order* Ξ on $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}$ is a collection of wpo Ξ_n on $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}(n)$ which is compatible with the shuffle operad structure of $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}$, meaning that for $T_1 \succ_{\Xi_n} T_2$, we have:

$$T_1 \circ_{i,I} S \succ_{\Xi_m} T_2 \circ_{i,I} S \qquad \text{and} \qquad S \circ_{j,J} T_1 \succ_{\Xi_m} S \circ_{j,J} T_2$$

Definition 2.2.4.27. Let $(\mathcal{S},\mathcal{X},R)$ be an ORS. Let us write $m \succ_n m_i$ if we have $m \rightarrow \sum \lambda_i m_i \in R_n$ with $\lambda_i \neq 0$. Let Ξ_n be the reflexive and transitive closure of $\succ_n$, we denote $\Xi = (\Xi_n)_{n\in\mathbb{N}}$.

Theorem 2.2.4.28. *Let $(\mathcal{S}, \mathcal{X}, R)$ be an ORS. Then it is terminating if and only Ξ is a monomial partial order. In particular, it is terminating if and only if it is terminating on $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}$.*

Proof. This is a direct corollary of Theorem 2.2.4.15. The fact that Ξ is compatible with the shuffle operad structure is direct consequence of its definition. □

This theorem allows us to reduce the termination of an ORS to finding a monomial partial order such that the rewriting rules are decreasing. Any terminating ORS give such a monomial partial order, however we would like to go the other way around. We would like some "classical" monomial partial orders such that we can check the termination of an ORS by checking that the rewriting rules are decreasing for those monomial partial orders. However, because they are more convenient than partial orders, let us work with preorders:

Definition 2.2.4.29. We recall that a *preorder* on a set A is a reflexive and transitive relation. meaning that:

- For all $a \in A$, we have $a \leq a$.
- For all $a, b, c \in A$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

Let $a, b \in A$, we denote by $a \sim b$ the fact that $a \leq b$ and $b \leq a$, and we denote by $a < b$ the fact that $a \leq b$ and $a \nsim b$. A *well preorder* is a preorder such that each non-empty subset has a minimal element. A *total preorder* is a preorder such that for all $a, b \in A$, we have $a \leq b$ or $b \leq a$.

Definition 2.2.4.30. Let M be a monoid. A *monoidal preorder* is a preorder which is compatible with the monoid structure of $\text{Mon}(\mathcal{X})$, meaning that for $u \leq v$, we have $u \cdot w \leq v \cdot w$ and $w \cdot u \leq w \cdot v$.

Let us give some important example and non-example of monoidal orders.

Example 2.2.4.31. Let $\mathcal{X} = \{a, b\}$ and let $a \leq b$. The lexicographic order on $\text{Mon}(\mathcal{X})$ is not a monoidal preorder. Indeed, we have $a \leq aa$ but $ab \geq aab$. However, the graded lexicographic order is a monoidal preorder.

Example 2.2.4.32. Let us consider $\text{Q} = \text{Mon}(\{x, y, q\})/\langle xq - qx, yq - qy, yx - xyq\rangle$ the so-called *quantum monoid*. It is quite clear that any element of Q admits a unique representation of the form $x^i y^j q^k$ with $i, j, k \in \mathbb{N}$. Let us put the following order on Q: we write $x^i y^j q^k \geq x^{i'} y^{j'} q^{k'}$ if

- $i < i'$, or
- $i = i'$ and $j > j'$, or
- $i = i'$, $j = j'$ and $k > k'$.

One may notice the condition $i < i'$ which seems to be the wrong way, however it is the correct way. We have:

$$x^i y^j q^k x^{i'} y^{j'} q^{k'} = x^{i+i'} y^{j+j'} q^{k+k'+ji'},$$

and we can check that it is a monoidal preorder using this formula.

Definition 2.2.4.33. Let $\mathcal{X}$ be a shuffle set species. A *monomial preorder* Ξ on $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}$ is a collection of total well preorders Ξ_n on $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}(n)$ which is compatible with the shuffle operad structure of $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}$, meaning that for $T_1 \succ_{\Xi_n} T_2$, we have:

$$T_1 \circ_{i,I} S \succ_{\Xi_m} T_2 \circ_{i,I} S \qquad \text{and} \qquad S \circ_{j,J} T_1 \succ_{\Xi_m} S \circ_{j,J} T_2$$

We are interested in monomial partial orders, not in monomial preorders. However, we can always get a partial order from a preorder the following way:

Proposition 2.2.4.34. *Let Ξ be a preorder, then Ξ' defined by $a \leq_{\Xi'} b$ if $a <_{\Xi} b$ or $a = b$. Then Ξ' is a partial order. Moreover:*

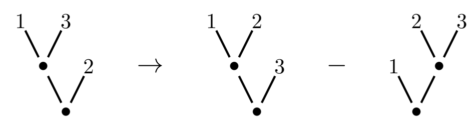
- *if Ξ is a well preorder, then Ξ' is a well order,*
- *if Ξ is a monoidal preorder, then Ξ' is a monoidal partial order, and*
- *if Ξ is a monomial preorder, then Ξ' is a monomial partial order.*

This allows us to get a partial order from a preorder. In particular, an ARS which decreases along a preorder will also decrease along the associated partial order. Since they are more convenient, we will mostly work with monomial preorders, i.e. total well preorders which are compatible with the shuffle operad structure.

Let us give some examples of monomial preorders.

Definition 2.2.4.35. Let $T \in \mathcal{T}^{\text{III}}\{\mathcal{X}\}(n)$. The *leaf permutation* of T is the sequence of integers $(i_1, \ldots, i_n)$ such that i_j is the label of the j-th leaves of T from left to right. The *permutation* order is the order on T induced by the lexicographic order on the leaf permutations. The *reverse permutation* order is the order on T induced by the reverse lexicographic order on the leaf permutations.

It is quite clear that the permutation and reverse permutation orders are monomial preorders. However, these monomial preorders are quite "weak" as most shuffle trees are equivalent. They are not useless as they already provide a way of checking the termination of ORS of the Lie operad and of the Com operad. Let us consider the operad Lie, it is generated by one element of arity two, so we will omit the labeling by this element in the shuffle trees. We can see the Jacobi identity as the following rewriting rule:



There is only one critical monomial for this rewriting rule. Its local confluence is checked in Figure 4.5 of the appendix. Counting the normal forms for this ORS allows us to show the following non-trivial fact:

Proposition 2.2.4.36. *We have that* $\dim(\text{Lie}(n)) = (n-1)!$.

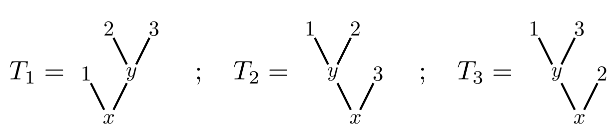
Proof. Since we have a convergent ORS, we have a canonical basis of Lie. It suffices to count the normal forms of the ORS. The only rewritable monomial is the left comb with leaves $(1,3,2)$, let d be this monomial. We have 2 normal forms in arity 3. Let us assume that we have $(n-1)!$ in arity n, and show that we have $n!$ normal forms in arity $n+1$. To get a normal form in arity $n+1$ from a normal form in arity n, we need to know where we composed the Lie bracket carrying the leaf $n+1$. We cannot have composed it in a left input since it would be divisible by d in this case. Hence, either we have composed it in a right input or it is the root. We have $n-1$ choices for composing is a right input and 1 for the root. Hence, we have $n \cdot (n-1)! = n!$ normal forms in arity $n+1$. □

Let us construct more "powerful" monomial preorders, meaning that they allow us to compare more shuffle trees.

Definition 2.2.4.37. Let $T \in \mathcal{T}^{\text{III}}\{\mathcal{X}\}(n)$. The *path sequence* of T is the sequence $(v_1, \ldots, v_n)$ defined the following way: Let us consider the unique path from the root of T to the leaf i, let v_i be the sequence of the labels of the internal vertices of this path.

Let us compute some path sequences.

Example 2.2.4.38. Let us consider the following shuffle trees, and compute their path sequences.



Then the path sequence of T_1 is (x, xy, xy), the path sequence of T_2 is (xy, xy, x) and the path sequence of T_3 is (xy, x, xy).

The path sequences are quite useful to define monomial preorders. Indeed, they allow us to define monomial preorders using monoidal preorders via the so called path extension:

Definition 2.2.4.39. Let $T \in \mathcal{T}^{\text{Ш}}\{\mathcal{X}\}(n)$, and let θ be a preorder on $\text{Mon}(\mathcal{X})$. The *path extension* of θ is the sequence of order on $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}(n)$ induced by θ on the path sequences of the shuffle trees. Let T_1 and T_2 be two shuffle trees of $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}(n)$ such that the path sequences of T_1 and T_2 are $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$, we have $T_1 \succ_\theta T_2$ if and only if we have i such that $v_i \succ_\theta w_i$, and for any $j < i$ we have $v_j = w_j$. The *reverse path extension* of θ is the reverse order of the path extension of θ.

Proposition 2.2.4.40. *If θ is a monoidal preorder, then the path extension of θ is a monomial partial order.*
Respectively, if the reversion of θ is a monoidal preorder, then the reverse path extension of θ is a monomial partial order.

Proof. Let T_1 and T_2 be two shuffle trees of $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}(n)$ such that the path sequences of T_1 and T_2 are $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$. Let S be a shuffle tree of $\mathcal{T}^{\text{Ш}}\{\mathcal{X}\}(k)$ such that its path sequence is $(u_1, \dots, u_k)$. Let us compute the path sequence of $T_1 \circ_{i,I} S$ with $I = \{i = i_1, i_2, \dots, i_k\}$ in increasing order. We get:

$$(v_1, \dots, v_i u_1, v_{i_1+1}, \dots, v_i u_2, v_{i_2}, \dots, v_i u_3, v_{i_3-1}, \dots, v_i u_k, v_{i_k-k+2}, \dots, v_n)$$

Same for $T_2 \circ_{i,I} S$, we get:

$$(w_1, \dots, w_i u_1, w_{i_1+1}, \dots, w_i u_2, w_{i_2}, \dots, w_i u_3, w_{i_3-1}, \dots, w_i u_k, w_{i_k-k+2}, \dots, w_n)$$

Hence, if we have $T_1 \succ_\theta T_2$, we have j such that $v_j \succ_\theta w_j$, and for any $l < j$ we have $v_l = w_l$. Hence, by the explicit computation of the path sequence of $T_1 \circ_{i,I} S$ and $T_2 \circ_{i,I} S$, we have $T_1 \circ_{i,I} S \succ_\theta T_2 \circ_{i,I} S$. The same computation concludes for $S \circ_{i,I} T_1 \succ_\theta S \circ_{i,I} T_2$. □

Let us give three key example of such path extension.

Example 2.2.4.41. Let $\mathcal{X} = \{x_1, \dots, x_n\}$ and let us consider a function $\psi : \mathcal{X} \to \mathbb{N}$. We extend ψ to $\text{Mon}(\mathcal{X})$ by $\psi(uv) = \psi(u) + \psi(v)$. Then the weight order relatively to ψ on $\text{Mon}(\mathcal{X})$ is a monoidal preorder. Hence, the path extension of weight order is a monomial preorder. This monomial preorder is called the *weight order*, it depends on the function ψ.

Example 2.2.4.42. Let $\mathcal{X} = \{x_1, \dots, x_n\}$ and let us put the order $x_1 \leq x_2 \leq \dots \leq x_n$. Then the graded lexicographic order on $\text{Mon}(\mathcal{X})$ is a monoidal preorder. Hence, the path extension of the graded lexicographic order is a monomial preorder. This monomial preorder is called the *graded path lexicographic order*. Similarly, the *reversed graded path lexicographic order* is a monomial preorder.

Example 2.2.4.43. Let $\mathcal{X} = \{x_1, \dots, x_n, y_1, \dots, y_k\}$. Let us consider the morphism of monoid $\text{Mon}(\mathcal{X}) \to \text{Q}$ defined by $x_i \mapsto x$ and $y_i \mapsto y$ with Q for Example 2.2.4.32. The monoidal preorder on Q induces a monoidal preorder on $\text{Mon}(\mathcal{X})$. The path extension of this preorder is a monomial preorder. This monomial preorder is called the *quantum order on $\mathcal{X}$*. One may notice that the quantum order on $\mathcal{X}$ depends on the choice of the morphism of monoid $\text{Mon}(\mathcal{X}) \to \text{Q}$, hence it depends on the choice of the x-like and y-like elements of $\mathcal{X}$.

We have defined the most classical monomial preorders. However, we can do more, we can combine them together.

Definition 2.2.4.44. Let Ξ and Θ be two total preorders. The *concatenated order* $\Xi.\Theta$ is the preorder defined by $a \leq_{\Xi.\Theta} b$ if:

- $a <_\Xi b$, or
- $a \sim_\Xi b$ and $a \leq_\Theta b$.

We can directly see from this definition that the reverse of the concatenated order is the concatenation of the reverse orders. Moreover, we have the following lemma.

Lemma 2.2.4.45. *If Ξ and Θ are monomial preorder, then $\Xi.\Theta$ is a monomial preorder.*

Proof. The verification is direct from the definition. □

This lemma allows us to construct more monomial preorders by concatenating the classical monomial preorders. It would have been crucial if we were working with Gröbner bases or PBW bases since we would have needed to find total orders, and that this lemma allows us to refine preorders more and more by concatenating them together to get total orders. It is still useful in our case since it allows us to construct new monomial preorders from old ones.

2.2.5 Application of ORS to freeness properties

Let us give some applications of ORS to show freeness properties of operads, namely we will recall the main theorem of [25] and [31]. These theorems were originally stated with Gröbner basis, however we can restate them with ORS. First, we need to define those freeness properties. Let us define modules over operads.

Definition 2.2.5.1. Let $\mathcal{P}$ be an operad.

- A *left module* L over $\mathcal{P}$ is a species L with a morphism $\mathcal{P} \circ L \to L$ such that the following commutes:

$$\begin{CD} \mathcal{P} \circ \mathcal{P} \circ L @>>> \mathcal{P} \circ L \\ @VVV @VVV \\ \mathcal{P} \circ L @>>> L \end{CD}$$

- A *right module* R over $\mathcal{P}$ is a species R with a morphism $R \circ \mathcal{P} \to R$ such that the following commutes:

$$\begin{CD} R \circ \mathcal{P} \circ \mathcal{P} @>>> R \circ \mathcal{P} \\ @VVV @VVV \\ R \circ \mathcal{P} @>>> R \end{CD}$$

- A *bimodule* M over $\mathcal{P}$ is a left and right module over $\mathcal{P}$ such that the two structures commute, meaning that the following diagram commutes:

$$\begin{CD} \mathcal{P} \circ M \circ \mathcal{P} @>>> M \circ \mathcal{P} \\ @VVV @VVV \\ \mathcal{P} \circ M @>>> M \end{CD}$$

These are the usual definition of left, right, and bimodule. However, since $\circ$ is not symmetric, left and right modules behave quite differently in general. Let $\mathcal{Q}$ be an operad such that we have a morphism of operads $\mathcal{P} \to \mathcal{Q}$, then this morphism induces a canonical structure of a left and a right module on $\mathcal{Q}$ over $\mathcal{P}$. (It in fact induces a structure of bimodule over $\mathcal{P}$.)

Definition 2.2.5.2. Let $\mathcal{P}$ be an operad. A left module L over $\mathcal{P}$ is *free* if we have a species $\mathcal{X}$ such that $L \simeq \mathcal{P} \circ \mathcal{X}$ and the structure of left module is given by the operadic structure of $\mathcal{P}$, namely:

$$\mathcal{P} \circ L \simeq \mathcal{P} \circ \mathcal{P} \circ \mathcal{X} \to \mathcal{P} \circ \mathcal{X} \simeq L$$

Same for a right module.

Let $(\mathcal{S}, \mathcal{X}, U)$ and $(\mathcal{S} \oplus \mathcal{R}, \mathcal{X} + \mathcal{Y}, U \sqcup V)$ be two convergent ORS admitting associated operads, and let $\mathcal{P}$ and $\mathcal{Q}$ the associated operads. Then we have the following theorem.

Theorem 2.2.5.3 (left freeness version)**.** *[25, Theorem 4] Assume that the root of the rewritable monomial of V are elements of $\mathcal{Y}$. Then $\mathcal{Q}$ is free as left $\mathcal{P}$-module.*

Theorem 2.2.5.4 (right freeness version)**.** *[25, Theorem 4] Assume that the vertices such that each child is a leaf, of the rewritable monomial of V are elements of $\mathcal{Y}$. Then $\mathcal{Q}$ is free as right $\mathcal{P}$-module.*

Using well-chosen ORS, these theorems allow us to show several interesting result. For example it is shown in [25] that the operad PreLie is free as a left module and as a right module over the operad Lie. One could also show that Ass is free as a right module over Lie. An interesting fact to notice is that Ass is not free as a left module over Lie.

Proposition 2.2.5.5. *The operad* Ass *is not free as a left module over the operad* Lie*.*

Proof. We know the dimensions of Ass and of Lie. Indeed, for $n \geq 1$, we have that $\dim(\mathrm{Ass}(n)) = n!$ and $\dim(\mathrm{Lie}(n)) = (n-1)!$. Hence we have that $f_{\mathrm{Ass}}(x) = \frac{x}{1-x}$ and $f_{\mathrm{Lie}}(x) = -\ln(1-x)$. Assume that we have a species $\mathcal{X}$ such that $\mathrm{Ass} = \mathrm{Lie} \circ \mathcal{X}$ then we have $f_{\mathrm{Ass}} = f_{\mathrm{Lie}} \circ f_{\mathcal{X}}$. Hence we can compute $f_{\mathcal{X}}$ and we get:

$$f_{\mathcal{X}}(x) = 1 - \exp\left(\frac{x}{x-1}\right)$$

We can compute the first few terms of the series expansion of $f_{\mathcal{X}}$ and we get:

$$f_{\mathcal{X}}(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \mathcal{O}(x^5)$$

We get a negative dimension in arity 4, hence Ass is not free as a left module over Lie. □

Let us now define the Nielsen-Schreier property.

Definition 2.2.5.6. An operad $\mathcal{P}$ has the *Nielsen-Schreier property* if any subalgebra of a free $\mathcal{P}$-algebra is free.

Let us recall the following theorem:

Theorem 2.2.5.7. *[31, Theorem 4.1] Let $\mathcal{P}$ be an operad generated by $\mathcal{S}$, and $\mathcal{X}$ a basis of $\mathcal{S}$ satisfying the following conditions:*

- *$\mathcal{P}$ admits a convergent ORS $(\mathcal{S}, \mathcal{X}, U)$ decreasing along the reverse graded path lexicographic ordering such that for each rewritable monomial, the smallest leaf is directly connected to the root.*
- *$\mathcal{P}$ admits a convergent ORS $(\mathcal{S}, \mathcal{X}, V)$ such that each rewritable monomial is a left comb with the smallest leaf and the second-smallest leaf directly connected to the same vertex.*

Then $\mathcal{P}$ has the Nielsen-Schreier property.

We could once again use the operad Lie as an example. Indeed, it is shown in [31] that Lie has the Nielsen-Schreier property.

2.3 Differential graded operads

We are ready to generalize everything we have done to the differential graded setting. From a categorical point a view, we have nothing to do. We just write down our nice categorical definitions using commutating diagrams and everything will work smoothly. However, once we want to do computations, we need to be careful. We will hit the usual technical difficulty of computation in homological algebra: the random appearance of signs coming from nowhere. Except those signs are not random, and they do not come from nowhere, they come from the Koszul sign rule. Let us recall it, explain it, and do some computations with it.

2.3.1 The Koszul sign rule

The *Koszul sign rule* is not a formal math theorem, and is more like a general principle. Let us recall it. "When two symbols a and b of homological (or cohomological) degree $|a|$ and $|b|$ are exchanged, a sign $(-1)^{|a||b|}$ should appear". One of the motto of homological algebra is that "any sign comes from the Koszul sign rule". Let us give some examples of the Koszul sign rule (and play the game of finding the Koszul sign rule behind usual formulas).

Tensor product of chain complexes Let us consider two chain complexes $(C^\bullet, \partial_C)$ and $(D^\bullet, \partial_D)$. The tensor product of the two chain complexes should be a chain complex $(C \otimes D, \partial)$ such that ∂ is a differential defined from ∂_C and ∂_D. The most natural way of defining ∂ would be:

$$\partial = \partial_C \otimes \mathrm{id} + \mathrm{id} \otimes \partial_D$$

Let us compute it a naive way on an element $a \otimes b$:

$$\partial(a \otimes b) = \partial_C(a) \otimes b + a \otimes \partial_D(b)$$

Then we have:

$$\partial^2(a \otimes b) = \partial_C^2(a) \otimes b + 2\partial_C(a) \otimes \partial_D(b) + a \otimes \partial_D^2(b) = 2\partial_C(a) \otimes \partial_D(b)$$

It fails to be a differential, since it does not square to zero in general. To fix it, we need to add a sign:

$$\partial(a \otimes b) = \partial_C(a) \otimes b + (-1)^{|a|} a \otimes \partial_D(b)$$

And we now have a differential. This sign comes from the fact that from $(\mathrm{id} \otimes \partial_D)(a \otimes b)$ to $a \otimes \partial_D(b)$ the symbols a and ∂_D have been exchanged, hence we need to add a sign $(-1)^{|a||\partial_B|}$. Since ∂_D is a differential, we have $|\partial_D| = 1$ (or -1 depending on whether we are using the homological or cohomological convention). Hence, we get the sign $(-1)^{|a|}$.

Braiding map of chain complexes Let us consider two chain complexes $C = (C^\bullet, \partial_C)$ and $D = (D^\bullet, \partial_D)$. We have defined the tensor product of C and D as $C \otimes D = (C^\bullet \otimes D^\bullet, \partial)$. However, since we have $C^\bullet \otimes D^\bullet \simeq D^\bullet \otimes C^\bullet$ via a trivial isomorphism, we should have $C \otimes D \simeq D \otimes C$. Let us naively define a braiding map $\tau : C \otimes D \to D \otimes C$ by $\tau(a \otimes b) = b \otimes a$. We have:

$$\begin{array}{ccc}
a \otimes b & \overset{\partial}{\longmapsto} & \partial_C(a) \otimes b + (-1)^{|a|} a \otimes \partial_D(b) \\
\downarrow \tau & & \downarrow \tau \\
 & & (-1)^{|a|} \partial_D(b) \otimes a + b \otimes \partial_C(a) \\
 & & \neq \\
b \otimes a & \overset{\partial}{\longmapsto} & \partial_D(b) \otimes a + (-1)^{|b|} b \otimes \partial_C(a)
\end{array}$$

We see that τ is not a chain map. To fix it, we need to add a sign:

$$\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$$

With this sign, we have that τ is a chain map.

Internal Hom **of chain complexes** Let us consider two chain complexes $C = (C^\bullet, \partial_C)$ and $D = (D^\bullet, \partial_D)$. Let us define the chain complex $(\mathrm{Hom}(C, D), d)$ the following way:

- $\mathrm{Hom}(C, D)^n$ is the set of function f of degree n, meaning that $f(C^i) \subseteq D^{i+n}$,
- $d = (\partial_D)_* - (\partial_C)^*$ meaning that d is the difference between the post-composition by ∂_D and the pre-composition by ∂_C. We take the difference since we would like to have $df = 0$ for f a degree 0 map that is a chain map.

One may notice the appearance of the Koszul sign rule, indeed when computing $(\partial_C)^* \circ f$, we should get $f \circ \partial_C$, however we exchange the symbols f and ∂_C, hence we need to add a sign $(-1)^{|f||\partial_C|}$. Hence, we have:

$$d(f) = \partial_D \circ f - (-1)^{|f|} f \circ \partial_C$$

Künneth formula Let us consider two chain complexes C and D. We have the Künneth formula:

$$H(C \otimes D) \simeq H(C) \otimes H(D)$$

Moreover if X and Y are topological spaces, then we have the Künneth formula in homology:

$$H_*(X \times Y; \mathbb{R}) \simeq H_*(X; \mathbb{R}) \otimes H_*(Y; \mathbb{R})$$

Moreover, since $X \times Y \simeq Y \times X$, we have a map:

$$H_*(X; \mathbb{R}) \otimes H_*(Y; \mathbb{R}) \simeq H_*(X \times Y; \mathbb{R}) \simeq H_*(Y \times X; \mathbb{R}) \simeq H_*(Y; \mathbb{R}) \otimes H_*(X; \mathbb{R})$$

Since the Künneth map is explicit, we can compute the map $H_*(X; \mathbb{R}) \otimes H_*(Y; \mathbb{R}) \to H_*(Y; \mathbb{R}) \otimes H_*(X; \mathbb{R})$. One may check that we exactly get τ the braiding map, ensuring that the homology with real coefficients (or more generally coefficient in a field) is a monoidal functor. This allows us to show the following fact: the homology of a topological space is a co-commutative co-monoid. Indeed, any topological space is a co-commutative co-monoid in the category of topological spaces, it is clear that the diagonal map $\Delta : X \to X \times X$ and the trivial map $\eta : X \to \{*\}$ give a structure of co-commutative co-monoid to X. Hence, we have that $H_*(X; \mathbb{R})$ is a co-commutative co-monoid in the category of graded vector spaces. However, the co-monoid structure is no longer trivial at the level of homology, indeed this co-monoid structure is exactly the dual of the cup product in the cohomology.

Orientation of $\mathbb{R}^n$ Let us consider the manifold $\mathbb{R}^n$. Since $\mathbb{R}^n$ is orientable, the data of an orientation of $\mathbb{R}^n$ is the same as the data of a local orientation of $\mathbb{R}^n$ around 0. A local orientation of $\mathbb{R}^n$ around $p \in \mathbb{R}^n$ is a choice of generator of the homology group $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{p\}; \mathbb{Z})$, which is isomorphic to $\mathbb{Z}$. Hence, an orientation of $\mathbb{R}^n$ is a choice of generator of $\mathbb{Z}$ as a group, so it is ± 1. Let us consider $\mathbb{R}^n$ and $\mathbb{R}^m$ and try to get an orientation of $\mathbb{R}^{n+m}$ from an orientation of $\mathbb{R}^n$ and of $\mathbb{R}^m$. Using the relative version of the Künneth formula in homology, we have the following exact sequence:

$$\begin{aligned} 0 \to \bigoplus_{i=0}^{n+m} &(H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \otimes H_{n+m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; \mathbb{Z})) \\ &\to H_{n+m}(\mathbb{R}^n \times \mathbb{R}^m, (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^m \cup \mathbb{R}^n \times (\mathbb{R}^m \setminus \{0\}); \mathbb{Z}) \\ &\to \bigoplus_{i=0}^{n+m-1} \mathrm{Tor}\left(H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}), H_{n+m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; \mathbb{Z})\right) \to 0 \end{aligned}$$

Hence, we get:

$$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \otimes H_m(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; \mathbb{Z}) \simeq H_{n+m}(\mathbb{R}^n \times \mathbb{R}^m, (\mathbb{R}^n \times \mathbb{R}^m) \setminus \{0\}; \mathbb{Z})$$

Since the tensor product is symmetric, we get:

$$H_{m+n}(\mathbb{R}^m \times \mathbb{R}^n, (\mathbb{R}^m \times \mathbb{R}^n) \setminus \{0\}; \mathbb{Z}) \simeq H_{n+m}(\mathbb{R}^n \times \mathbb{R}^m, (\mathbb{R}^n \times \mathbb{R}^m) \setminus \{0\}; \mathbb{Z})$$

This result is quite obvious since we know that these two homology groups are isomorphic to $\mathbb{Z}$. However, the non-trivial part is the actual isomorphism. Indeed, one may check that the isomorphism is $(-1)^{nm}$. Hence, the fact that the orientation of $\mathbb{R}^{n+m}$ given by the orientation of $\mathbb{R}^n \times \mathbb{R}^m$ is $(-1)^{nm}$ times the orientation given by $\mathbb{R}^m \times \mathbb{R}^n$ can be seen as an instance of the Koszul sign rule.

Dimension and Euler characteristic Let V be a vector space. The dimension of V is the cardinal of a basis of V, however this definition is not "categorical enough" to easily generalize it. Another definition of the dimension of V is to say that it is the trace of the identity. In this case, finding a categorical definition of the trace would be enough to define the dimension in a categorical way. We have:

$$\begin{array}{ccccc} \mathbb{K} & \overset{\text{eval}}{\leftarrow} & V^\vee \otimes V & \overset{\xi}{\rightarrow} & \text{End}(V) \\ f(v) & \leftarrow\!\shortmid & f \otimes v & \mapsto & (x \mapsto f(x)v) \end{array}$$

We remark that if φ is in the image of ξ, we have that $\text{eval}(\xi^{-1}(\varphi))$ is exactly the trace of φ. Let us show where the Koszul sign rule appears in this construction. We have defined ξ such that it is the unique function satisfying::

$$\text{eval}(\xi(f \otimes v) \otimes x) = \text{eval}(f \otimes x)v$$

Taking the Koszul sign rule seriously would lead us to:

$$\text{eval}(\xi(f \otimes v) \otimes x) = (-1)^{|x||v|}\, \text{eval}(f \otimes x)v$$

In the case of vector spaces, then nothing changes since the degree of a vector is 0. However, in the case of chain complexes, we get:

$$\begin{array}{ccccc} \mathbb{K} & \overset{\text{eval}}{\leftarrow} & C^\vee \otimes C & \overset{\xi}{\rightarrow} & \text{End}(C) \\ f(v) & \leftarrow\!\shortmid & f \otimes v & \mapsto & (x \mapsto (-1)^{|x||v|} f(x)v) \end{array}$$

We remark that if id is in the image of ξ then $\text{eval}(\xi^{-1}(\text{id}))$ exactly gives the Euler characteristic of C. This gives an interpretation of the $(-1)^n$ in the Euler characteristic as an instance of the Koszul sign rule.

Principle of inclusion and exclusion Let us consider A and B two finite sets, and let us show (probably in the worst possible way) that:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

More generally, let $(A_i)_{i \in \{1,\dots,n\}}$ be a family of finite sets, let us show that:

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{i=1}^n (-1)^{i-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \left| \bigcap_{j=1}^k A_{i_j} \right|$$

We have:

$$\bigcap_i A_i \; \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \; \biguplus_j \bigcap_{i \neq j} A_i \; \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \; \cdots \; \Rrightarrow \; \biguplus_{i>j} A_i \cap A_j \; \rightrightarrows \; \biguplus_i A_i$$

With the j-th arrow being the inclusion from $\bigcap_l A_{i_l} \to \bigcap_{l|l \neq j} A_{i_l}$ with $i_1 < \cdots < i_k$. We may notice that this is a semi-simplicial set, let us consider its geometric realization $\mathcal{R}$. We have a bijection between $\pi_0(\mathcal{R})$ and $A_1 \cup \cdots \cup A_n$, indeed, two point of $A_1 \uplus \cdots \uplus A_n$ are in the same connected component if and only if they are connected by an edge, if and only they have the same image in

$A_1 \cup \cdots \cup A_n$. Let us show that each connected component of $\mathcal{R}$ is contractible. Let $a \in A_1 \cup \cdots \cup A_n$ and let us consider $\mathcal{R}_a$ the connected component associated to a. We have that $\mathcal{R}_a$ is a simplex since its vertices are in bijection with the indices i such that $a \in A_i$, its edges are in bijection with the indices i, j such that $a \in A_i \cap A_j$, and so on. Hence, $\mathcal{R}_a$ is contractible. Hence, the homology of its chain complex is concentrated in degree 0. Moreover, its chain complex C is:

$$\mathbb{R}^{\bigcap_i A_i} \xrightarrow{d} \mathbb{R}^{\biguplus_j \bigcap_{i \neq j} A_i} \xrightarrow{d} \ldots \xrightarrow{d} \mathbb{R}^{\biguplus_{i>j} A_i \cap A_j} \xrightarrow{d} \mathbb{R}^{\biguplus_i A_i}$$

Hence, we have that the Euler characteristic of C is the cardinal of $A_1 \cup \cdots \cup A_n$, we have:

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left| \bigcap_{j=1}^{k} A_{i_j} \right|$$

Since we just saw that the signs in the Euler characteristic are instances of the Koszul sign rule, so are the signs in the principle of inclusion and exclusion.

We could continue the game and express the signature of a permutation or the Legendre symbol as an instance of the Koszul sign rule. However, let us stop here and go to the main topic of this section: differential graded operads, and more precisely where does the Koszul sign rule appears in the definition of a differential graded operad. Because we defined operads in a quite categorical way, either as algebra over the tree monad, as monoids in the category of species, or with the partial compositions, we do not need to change those definitions to get differential graded operads.

Definition 2.3.1.1. A *symmetric differential graded operad*, denoted *dg operad*, is either:

- An algebra over the tree monad in the dg species,
- A monoid in the category of dg species relatively to the plethysm,
- A dg species with partial compositions satisfying the same axioms as in Definition 2.2.1.3, namely the identity, the parallel composition, the sequential composition, and the compatibility with the species structure.

These three definitions are equivalent. Same for *shuffle dg operads* and *ns dg operads*.

To understand where the Koszul sign appears, let us do two very basic computations, one for the sequential composition and one for the parallel composition. Let us consider a dg operad $\mathcal{P}$ and let $f, g, h \in \mathcal{P}$ then:

- For the sequential composition we should relate $(f \circ_i g) \circ_j h$ and $f \circ_i (g \circ_j h)$. Since “nothing is exchanged”, we get:
$$(f \circ_i g) \circ_j h = f \circ_i (g \circ_j h)$$
- For the parallel composition we should relate $(f \circ_a g) \circ_b h$ and $(f \circ_b h) \circ_a g$. Since g and h are exchanged, we get:
$$(f \circ_a g) \circ_b h = (-1)^{|g||h|}(f \circ_b h) \circ_a g$$

Another quite important place where the Koszul sign rule appears is in the definition of the endomorphism operad. Let $C = (C^\bullet, \partial)$ be a chain complex, then let us recall that $\mathrm{Hom}(C^{\otimes k}, C)$ is a chain complex such that:

- $\mathrm{Hom}(C^{\otimes k}, C)_n$ is the set of functions f degree n, meaning that $f = (f_i)_{i \in \mathbb{Z}}$ and $f_i : \left(C^{\otimes k}\right)^i \to C^{n+i}$,
- and $d(f) = \partial_C \circ f - (-1)^{|f|} \sum_j f \circ_j \partial_C$.

Then we define End_C the endomorphism operad of C as the operad such that $\mathrm{End}_C(n) = \mathrm{Hom}(C^{\otimes n}, C)$ and the composition is the composition of functions. We have that End_C is a dg operad. Let us show some computations in End_C to see where the Koszul sign rule appears. First, let us understand the action of $\mathfrak{S}_n$ on $\mathrm{End}_C(n)$. Let $\sigma \in \mathfrak{S}_n$ such that $\sigma = (i\ i+1)$, let $f \in \mathrm{End}_C(n)$, and $x_1, \dots, x_n \in C$. Then $(f.\sigma)(x_1 \otimes \cdots \otimes x_n)$ should give $f(x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n)$. However, we once again need to exchange some symbols, here we exchange x_i and x_{i+1}, hence we need to add a sign $(-1)^{|x_i||x_{i+1}|}$. Hence, we have:

$$(f.\sigma)(x_1 \otimes \cdots \otimes x_n) = (-1)^{|x_i||x_{i+1}|} f(x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n)$$

Since we can write any permutation as a product of $(i\ i+1)$, we can compute the sign that appears for any permutation. We may notice that if all the x_i are of odd degree, the sign is exactly the sign of the permutation. If all of the x_i are of even degree, then no sign appears.
Let us now compute the sign that appears when composing two elements of End_C. Let $f \in \mathrm{End}_C(n)$, $g \in \mathrm{End}_C(m)$, and $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in C$, and $y_1, \dots, y_m \in C$. Let us compute

$$(f \circ_i g)(x_1 \otimes \cdots \otimes x_{i-1} \otimes y_1 \otimes \cdots \otimes y_m \otimes x_{i+1} \otimes \cdots \otimes x_n \otimes)$$

It should give:

$$f(x_1 \otimes \cdots \otimes x_{i-1} \otimes g(y_1 \otimes \cdots \otimes y_m) \otimes x_{i+1} \otimes \cdots \otimes x_n)$$

However, we see that a lot of symbols are exchanged, hence we get:

$$\begin{aligned}(f \circ_i g)(x_1 \otimes \cdots \otimes x_{i-1} \otimes y_1 \otimes \cdots \otimes y_m \otimes x_{i+1} \otimes \cdots \otimes x_n \otimes) = \\ (-1)^s f(x_1 \otimes \cdots \otimes x_{i-1} \otimes g(y_1 \otimes \cdots \otimes y_m) \otimes x_{i+1} \otimes \cdots \otimes x_n)\end{aligned}$$

Where we denote:

$$s = \sum_{j=1}^{i-1} |x_j||g|$$

As we can see quite intricate signs appear. One cannot avoid those signs since they are needed for the theory to be coherent as we saw in the first two paragraphs of this section. However, the Koszul sign rule allows us to postpone the appearance of these signs to the very end, when we need to actually apply the function to some elements. This is the main reason why we use the Koszul sign rule: signs are unavoidable, but let us try to avoid them for as long as we can, and postpone their appearance to the very end, when we need to actually evaluate functions.

2.3.2 Bar and cobar constructions

Now that we have dg operads, let us consider operads as they are: algebra in a category of combinatorial object, the species; and let us do some "operadic homological algebra". A powerful and unavoidable tool of homological algebra is the bar and cobar constructions. Let us adapt them to the operadic context. Before doing so, we need to define the notion of a cooperad. Since we understood what operads are, defining cooperads should be a formality. As cooperad is "an operad with the arrows the other way around". Let us give the formal definition.

Definition 2.3.2.1. Let $\mathcal{T}^c$ be the *tree comonad*. Let us recall that we have a basis for $\mathcal{T}$ given by the (unlabeled) shuffle trees. Since at number of leaves fixed and at number of internal vertices fixed, there are a finite number of (unlabeled) shuffle trees, we have an isomorphism between $\mathcal{T}$ and $\mathcal{T}^\vee$ its linear dual. We define $\mathcal{T}^c$ to be the functor $\mathcal{T}^\vee$ together with the comonad structure induced on $\mathcal{T}^\vee$ by the monad structure of $\mathcal{T}$. Same for the *shuffle tree comonad* and the *ns tree comonad*.

This definition may be a bit too fast. To understand it, and to work out the comonad structure of $\mathcal{T}^c$, one need to go back to the discussion about the monad structure of $\mathcal{T}$ and flip the arrows. Long story short: $\mathcal{T} \circ \mathcal{T}$ is encoded by the shuffle trees such that the labels of internal vertices are shuffle trees. The monad structure corresponds to "erasing the circles around the shuffle trees labeling vertices" and making them actual subtrees, see Figure 2.2. The comonad structure corresponds to the reversed process: "drawing circles around subtrees and making them labels of internal vertices", since there are no canonical subtrees, one need to do the sum over all possible choices of collection of subtrees.

Definition 2.3.2.2. An *algebraic symmetric cooperad*, denoted *cooperad*, is either:

- A coalgebra over the tree comonad,
- A comonoid in the category of species relatively to the plethysm,
- A species with partial co-compositions satisfying the same axioms as in Definition 2.2.1.3 with the arrows reversed, namely the co-identity, the parallel co-composition, the sequential co-composition, and the compatibility with the species structure.

These three definitions are equivalent. Same for *shuffle cooperads* and *ns cooperads*.

It is clear from this definition that the linear dual of a finite operad is a cooperad, and that reciprocally the linear dual of a finite cooperad is an operad. (Here we need to assume finiteness to have that $(\mathcal{T}^c(\mathcal{S}))^\vee \simeq \mathcal{T}(\mathcal{S}^\vee)$. The finiteness hypothesis is necessary when one takes the linear dual of an operad, however it can be dropped when one takes the linear dual of a cooperad.) Let us now define the suspension of an operad. One need to be quite careful since we have two possible suspension, the "classical suspension" and the "operadic suspension". The classical suspension is the suspension of the underlying chain complex, which does not respect the operadic structure, while the operadic suspension is the suspension of the operad.

Definition 2.3.2.3. Let $\mathcal{P}$ be a dg operad. The *classical suspension* of $\mathcal{P}$ is the dg species $\mathrm{s}_+\mathcal{P}$ defined by $\mathrm{s}_+\mathcal{P}(n) = \mathrm{s}_+\mathbb{K} \otimes \mathcal{P}(n)$. Similarly, the *classical desuspension* of $\mathcal{P}$ is the dg species $\mathrm{s}_-\mathcal{P}$ defined by $\mathrm{s}_-\mathcal{P}(n) = \mathrm{s}_-\mathbb{K} \otimes \mathcal{P}(n)$. We denote by $\mathrm{s}_+f = \mathrm{s}_+ \otimes f \in \mathrm{s}_+\mathcal{P}$ and $\mathrm{s}_-f = \mathrm{s}_- \otimes f \in \mathrm{s}_-\mathcal{P}$.

Let us point out the Koszul signs that appear. Let us compute $(\mathrm{s}_+f).(1\ 2)$:

$$\begin{aligned}(\mathrm{s}_+f).(1\ 2) &= (\mathrm{s}_+ \otimes f).(1\ 2)\\ &= \mathrm{s}_+ \otimes (f.(1\ 2))\\ &= \mathrm{s}_+(f.(1\ 2))\end{aligned}$$

The differential $d_\mathcal{P}$ induce a differential $\mathrm{id} \otimes d_\mathcal{P}$ on $\mathrm{s}_+\mathcal{P}$ that we will also denote $d_\mathcal{P}$. by the Koszul sign rule, we have:

$$d_\mathcal{P}(\mathrm{s}_+f) = -\mathrm{s}_+(d_\mathcal{P}f)$$

Let us denote $\gamma_{\mathrm{s}_+} : \mathrm{s}_+\mathbb{K} \otimes \mathrm{s}_+\mathbb{K} \to \mathrm{s}_+\mathbb{K}$ such that $\gamma_{\mathrm{s}_+}(\mathrm{s}_+, \mathrm{s}_+) = \mathrm{s}_+$. Let us try to define the partial compositions in $\mathrm{s}_+\mathcal{P}$ by $\bullet_i = \gamma_{\mathrm{s}_+} \otimes \circ_i$, and compute $\bullet_i\left((\mathrm{s}_+f) \otimes (\mathrm{s}_+g)\right)$:

$$\begin{aligned}\bullet_i\left((\mathrm{s}_+f) \otimes (\mathrm{s}_+g)\right) &= \bullet_i\left((\mathrm{s}_+ \otimes f) \otimes (\mathrm{s}_+ \otimes g)\right)\\ &= (-1)^{|f|}\, \bullet_i\left((\mathrm{s}_+ \otimes \mathrm{s}_+) \otimes (f \otimes g)\right)\\ &= (-1)^{|f|}\gamma_{\mathrm{s}_+}(\mathrm{s}_+ \otimes \mathrm{s}_+) \otimes f \circ_i g\\ &= (-1)^{|f|}\mathrm{s}_+(f \circ_i g)\end{aligned}$$

Finally, let us show why we do not get an operadic structure with the classical suspension. We have:

$$\begin{aligned}\bullet_i\left((\mathrm{s}_+f) \otimes \left(\bullet_j\left((\mathrm{s}_+g) \otimes (\mathrm{s}_+h)\right)\right)\right) &= (-1)^{|f|+|g|}(\mathrm{s}_+(f \circ_i g \circ_j h))\\ &= (-1)^{|f|}(-1)^{|f|+|f|+|g|}(\mathrm{s}_+f \circ_i g \circ_j h)\\ &= (-1)^{|f|}\, \bullet_j\left(\left(\bullet_i\left((\mathrm{s}_+f) \otimes (\mathrm{s}_+g)\right)\right) \otimes (\mathrm{s}_+h)\right)\end{aligned}$$

Indeed, in $s_+\mathcal{P}$, the operations $\bullet_i$ are of degree -1, hence the parallel composition and sequential composition axioms are satisfied *up to a sign.* Since the parallel composition and sequential composition axioms are not satisfied on the nose, but only up a sign, we do not get an operad. Let us now define the operadic suspension and desuspension.

Definition 2.3.2.4. Let us define the *suspension operad* $ß_+\mathrm{Com}$. We saw that $\mathrm{Com} = \mathrm{End}_{\mathbb{K}}$, hence let us define $ß_+\mathrm{Com} = \mathrm{End}_{s_-\mathbb{K}}$. We have that $ß_+\mathrm{Com}(n) = \mathrm{Hom}(s_-\mathbb{K}^{\otimes n}, s_-\mathbb{K})$, hence $ß_+\mathrm{Com}(n)$ is of dimension 1 and is generated by $ß_+^{(n)}$ of degree $n-1$ defined by:

$$ß_+^{(n)} : s_- \otimes \cdots \otimes s_- \mapsto s_-$$

We have that $ß_+\mathrm{Com}$ is the commutative associative operad on one generator of degree 1. By a slight abuse of notation, we will denote by $ß_+$ instead of $ß_+^{(n)}$. We define the *desuspension operad* $ß_-\mathrm{Com}$ as the operad $\mathrm{End}_{s_+\mathbb{K}}$, it is the commutative associative operad on one generator of degree -1.

We use here the notation $ß_+\mathrm{Com}$ for the suspension operad. This is not a classical notation, however s, σ, $\mathcal{S}$, and Σ are already used. The author chose to use an Eszett, ß, to denote the suspension operad since the other *s*-like letters are not available.

Definition 2.3.2.5. Let $\mathcal{P}$ be a dg operad. The *operadic suspension* of $\mathcal{P}$ is the operad $ß_+\mathcal{P} = ß_+\mathrm{Com} \odot \mathcal{P}$. Similarly, the *operadic desuspension* of $\mathcal{P}$ is the operad $ß_-\mathcal{P} = ß_-\mathrm{Com} \odot \mathcal{P}$. We denote by $ß_+ f = ß_+ \otimes f \in ß_+\mathcal{P}$ and $ß_- f = ß_- \otimes f \in ß_-\mathcal{P}$.

Let us point out the Koszul signs that appear. Let us compute $ß_+ f.(1\ 2)$:

$$\begin{aligned} ß_+ f.(1\ 2) &= (ß_+ \otimes f).(1\ 2) \\ &= (ß_+.(1\ 2)) \otimes (f.(1\ 2)) \\ &= (-1)ß_+(f.(1\ 2)) \end{aligned}$$

Let us compute $ß_+ f \circ_i ß_+ g$ with k the arity of g:

$$\begin{aligned} ß_+ f \circ_i ß_+ g &= (ß_+ \otimes f) \circ_i (ß_+ \otimes g) \\ &= (-1)^{(k-1)|f|}(ß_+ \circ_i ß_+) \otimes (f \circ_i g) \\ &= (-1)^{(k-1)|f|}ß_+(f \circ_i g) \end{aligned}$$

We can see from these computations that the *classical suspension* and the *operadic suspension* are quite different, and that one need to be careful not to mix them up. To understand these two suspensions, let us consider the following example. Let $\mathcal{P}$ be a dg operad, and let C be a $\mathcal{P}$-algebra, meaning that C is a chain complex and we have a degree k morphism of dg operads $\mathcal{P} \to \mathrm{End}_C$. Let us suspend C with the classical suspension. We get s_+C, we still have an operad End_{s_+C}, however the linear map $\mathcal{P} \to \mathrm{End}_{s_+C}$ is no longer of constant degree, indeed $\mathcal{P}(n) \to \mathrm{End}_{s_+C}(n)$ is of degree $k+1-n$. However, if we do the operadic desuspension of $\mathcal{P}$, we get $ß_-\mathcal{P}$ and we get back our degree k morphism $ß_-\mathcal{P} \to \mathrm{End}_{s_+C}$. Hence, the operadic suspension and desuspension are the correct way to suspend and desuspend an operad, while the classical suspension and desuspension are the correct way to suspend and desuspend a chain complex.

The bar construction Let $\mathcal{P} = \mathcal{I} + \overline{\mathcal{P}}$ be an augmented operad with $\overline{\mathcal{P}}$ its augmentation ideal. Let us denote by $(\mathbf{B}(\mathcal{P}), d_1)$ the dg cooperad $\mathcal{T}^c\left(s_+\overline{\mathcal{P}}\right)$. The cooperad $\mathbf{B}(\mathcal{P})$ will be the underlying cooperad of the bar construction, we need to define the differential of the bar construction. The partial compositions of $\mathcal{P}$ induce a map:

$$\begin{aligned} s_+\overline{\mathcal{P}} \circ' s_+\overline{\mathcal{P}} &\to s_+\overline{\mathcal{P}} \\ s_+ f \circ_i s_+ g &\mapsto (\gamma_{s_+} \otimes \circ_i)(s_+ f \otimes s_+ g) \end{aligned}$$

We have computed the Koszul sign that appear in the example after Definition 2.3.2.3. One can extend this map on $\mathcal{T}^c\left(\mathrm{s}_+\overline{\mathcal{P}}\right)$ by a map that we denote d_2:

$$d_2 : \mathcal{T}^c\left(\mathrm{s}_+\overline{\mathcal{P}}\right) \to \mathcal{T}^c\left(\mathrm{s}_+\overline{\mathcal{P}}\right)$$

Proposition 2.3.2.6. *The map d_2 is a differential on $\mathbf{B}(\mathcal{P})$.*

Proof. This follows from the parallel composition and sequential composition axioms of $\mathcal{P}$, and the Koszul sign rule. Let us compute d_2^2 on an element of $\mathbf{B}(\mathcal{P})$ as an example. Let:

$$T = (\mathrm{s}_+ f \circ_i \mathrm{s}_+ g) \circ_j \mathrm{s}_+ h = \mathrm{s}_+ f \circ_i (\mathrm{s}_+ g \circ_j \mathrm{s}_+ h)$$

Let us compute $d_2^2(T)$. We have:

$$\begin{aligned} d_2(T) &= (d_2(\mathrm{s}_+ f \circ_i \mathrm{s}_+ g)) \circ_j \mathrm{s}_+ h + (-1)^{|\mathrm{s}_+ f|} \mathrm{s}_+ f \circ_i d_2(\mathrm{s}_+ g \circ_j \mathrm{s}_+ h) \\ &= (-1)^{|f|} \mathrm{s}_+ (f \circ_i g) \circ_j \mathrm{s}_+ h + (-1)^{|f|+1} (-1)^{|g|} \mathrm{s}_+ f \circ_i \mathrm{s}_+ (g \circ_j h) \\ &= (-1)^{|f|} \mathrm{s}_+ (f \circ_i g) \circ_j \mathrm{s}_+ h + (-1)^{|f|+|g|+1} \mathrm{s}_+ f \circ_i \mathrm{s}_+ (g \circ_j h) \end{aligned}$$

Hence:

$$\begin{aligned} d_2^2(T) &= (-1)^{|f|} d_2\left(\mathrm{s}_+ (f \circ_i g) \circ_j \mathrm{s}_+ h\right) + (-1)^{|f|+|g|+1} d_2\left(\mathrm{s}_+ f \circ_i \mathrm{s}_+ (g \circ_j h)\right) \\ &= (-1)^{|f|} (-1)^{|f|+|g|} \mathrm{s}_+ (f \circ_i g \circ_j h) + (-1)^{|f|+|g|+1} (-1)^{|f|} \mathrm{s}_+ (f \circ_i g \circ_j h) \\ &= \mathrm{s}_+ (f \circ_i g \circ_j h) - \mathrm{s}_+ (f \circ_i g \circ_j h) \\ &= 0 \end{aligned}$$

□

Proposition 2.3.2.7. *Let us recall that d_1 is the differential on $\mathbf{B}(\mathcal{P})$ induced by the differential of $\mathcal{P}$. Then d_1 and d_2 anticommute.*

Proof. This directly follows from the fact that $d_{\mathcal{P}}$ commutes with the partial compositions of $\mathcal{P}$, and from the Koszul sign rule. Let us compute $d_1 d_2$ and $d_2 d_1$ on an element of $\mathbf{B}(\mathcal{P})$ as an example. Let: $T = \mathrm{s}_+ f \circ_i \mathrm{s}_+ g$ Then we have:

$$\begin{aligned} d_1 d_2(T) &= d_1\left((-1)^{|f|} \mathrm{s}_+ (f \circ_i g)\right) \\ &= -(-1)^{|f|} \mathrm{s}_+ d_{\mathcal{P}} (f \circ_i g) \\ &= -\left((-1)^{|f|} \mathrm{s}_+ \left(d_{\mathcal{P}}(f) \circ_i g\right) + \mathrm{s}_+ \left(f \circ_i d_{\mathcal{P}}(g)\right)\right) \end{aligned}$$

And:

$$\begin{aligned} d_2 d_1(T) &= d_2\left(d_1(\mathrm{s}_+ f) \circ_i \mathrm{s}_+ g + (-1)^{|\mathrm{s}_+ f|} \mathrm{s}_+ f \circ_i d_1(\mathrm{s}_+ g)\right) \\ &= -d_2\left(\mathrm{s}_+ d_{\mathcal{P}}(f) \circ_i \mathrm{s}_+ g + (-1)^{|f|+1} \mathrm{s}_+ f \circ_i \mathrm{s}_+ d_{\mathcal{P}}(g)\right) \\ &= -\left((-1)^{|d_{\mathcal{P}}(f)|} \mathrm{s}_+ \left(d_{\mathcal{P}}(f) \circ_i g\right) + (-1)^{|f|+1} (-1)^{|f|} \mathrm{s}_+ \left(f \circ_i d_{\mathcal{P}}(g)\right)\right) \\ &= (-1)^{|f|} \mathrm{s}_+ \left(d_{\mathcal{P}}(f) \circ_i g\right) + \mathrm{s}_+ \left(f \circ_i d_{\mathcal{P}}(g)\right) \end{aligned}$$

□

Since the proofs that $d_2^2 = 0$ and that d_1 and d_2 anticommute are purely technical, we will not give them here, and we hope that the examples of computations we gave are enough to convince the reader that he could write down the complete proof.
We define the *bar construction of* $\mathcal{P}$ to be the dg cooperad $(\mathbf{B}(\mathcal{P}), d)$, with $d = d_1 + d_2$. From the fact that $\mathcal{T}^c = X + \overline{\mathcal{T}}^c$, one may notice that $(\mathbf{B}(\mathcal{P}), d)$ is co-augmented. Let us now define the cobar construction.

The cobar construction Let us define the cobar construction. Since the cobar construction is the same as the bar construction, with the arrows reversed, we will allow ourselves to be a bit sketchy, and not to show the computations. Let $\mathcal{C} = \mathcal{I} + \overline{\mathcal{C}}$ a co-augmented cooperad. Let us denote by $(\mathbf{\Omega}(\mathcal{C}), d_1)$ the dg operad $\mathcal{T}\left(\mathrm{s}_+\overline{\mathcal{C}}\right)$. The operad $\mathbf{\Omega}(\mathcal{C})$ will be the underlying operad of the cobar construction, we need to define the differential of the cobar construction. The partial co-compositions of $\mathcal{C}$ induce a map:

$$\begin{aligned} \mathrm{s}_+\overline{\mathcal{C}} &\to \mathrm{s}_+\overline{\mathcal{C}} \circ' \mathrm{s}_+\overline{\mathcal{C}} \\ \mathrm{s}_+ t &\mapsto \textstyle\sum_{f \circ_i g = t} (-1)^{|f|} \mathrm{s}_+ f \circ_i \mathrm{s}_+ g \end{aligned}$$

One can extend this map on $\mathcal{T}\left(\mathrm{s}_+\overline{\mathcal{C}}\right)$ by a map that we denote d_2:

$$d_2 : \mathcal{T}\left(\mathrm{s}_+\overline{\mathcal{C}}\right) \to \mathcal{T}\left(\mathrm{s}_+\overline{\mathcal{C}}\right)$$

The map d_2 is a differential on $\mathbf{\Omega}(\mathcal{C})$. Moreover, d_2 and d_1, the differential induced by the differential of $\mathcal{C}$, anticommute. Hence, we define the *cobar construction of $\mathcal{C}$cobar construction* to be the dg operad $(\mathbf{\Omega}(\mathcal{C}), d)$, with $d = d_1 + d_2$. From the fact that $\mathcal{T} = X + \overline{\mathcal{T}}$, one may notice that $(\mathbf{\Omega}(\mathcal{C}), d)$ is augmented.

The bar and cobar constructions are in fact functors. More over they are related by the following theorem:

Theorem 2.3.2.8. *The functors* $\mathbf{B}$ *and* $\mathbf{\Omega}$ *respectively of the bar and cobar construction are adjoint functors. We have:*

$$\{\textit{aug. dg operads}\} \underset{\mathbf{\Omega}}{\overset{\mathbf{B}}{\rightleftarrows}} \{\textit{coaug. dg cooperads}\} \quad (\perp)$$

We refer to [60], Theorem 6.5.10 page 214 for a proof of this theorem.

We will unfortunately not have the time to go further in the study of the bar and cobar constructions. However, the bar/cobar constructions are a very powerful tool in homological algebra, and we refer to [60] for a more detailed study of the bar and cobar constructions, and how they allow us to relate the next section, operadic Koszul duality, to resolutions and minimal models of operads. More specifically, we refer the interested reader to the Rosetta Stone Theorem 10.1.22 page 356 of [60] which relates homotopy algebra, bar/cobar constructions, Koszul duality, and twisting morphisms.

2.3.3 The operadic twisting

Let us quickly define the operadic twisting. We will not explicitly need it, however we will use construction that are quite similar to the operadic twisting. Although, we will go a bit on the definition, Subsection 3.1.2 is entirely dedicated to an example of operadic twisting, the operadic twisting of the operad of pre-Lie algebras. We refer the reader interested in a more in depth study of the operadic twisting to [22] and the book [30].

We will use the cohomological convention for the degree, hence a differential is a map a degree 1 that squares to zero. Let us describe the operadic twisting. Let $\mathfrak{g}$ be a differential graded Lie algebra, a Maurer-Cartan element of $\mathfrak{g}$ is a degree 1 element $\alpha \in \mathfrak{g}$ such that $d\alpha + \frac{1}{2}[\alpha\,, \alpha] = 0$. This condition ensures that the map $d^\alpha = d + [\alpha\,, \cdot]$ is a differential on $\mathfrak{g}$.

Definition 2.3.3.1. Let $\mathcal{P}$ be an operad. The pre-Lie algebra $(\mathfrak{g}, \star)$ associated to $\mathcal{P}$ is the "weighted" vector space $\mathfrak{g} = \bigoplus_{n\in\mathbb{N}} \mathcal{P}(\underline{n})$ (here we use the word "weighted" instead of "graded" to emphasize the fact that this is not the cohomological grading) with the product $\star$ defined by:

$$\mu \star \nu = \sum_{i=1}^{n} \mu \circ_i \nu$$

In particular, this is a Lie algebra. If the operad is differential graded, $(\mathfrak{g}, \star)$ is a differential graded pre-Lie algebra. The Maurer-Cartan equation can be written as:

$$d\mu + \mu \star \mu = 0$$

This imposes that μ is an arity 1, degree 1 element, hence $\mu \star \mu = \mu \circ_1 \mu$. Such an element is called an *operadic Maurer-Cartan element*.

Definition 2.3.3.2. Let $\mathcal{P}$ be an operad, and $\varphi : \mathrm{Lie} \to \mathcal{P}$ a morphism of operads from the operad Lie to $\mathcal{P}$ and $\tilde{l}$ the image of l in $\mathcal{P}$. The operadic twisting of $\mathcal{P}$ by φ is the differential graded operad $(\mathrm{Tw}\mathcal{P}, d_{\mathrm{Tw}})$ defined by as follows:

- Let α be a formal Maurer-Cartan element, α is an arity 0, degree 1 operation symbol.
- Let $\mathrm{Tw}\mathcal{P} = \mathcal{P}\hat{\vee}\alpha$ be the operad $\mathcal{P}$ extended by the operation symbol α without any relation. The symbol $\vee$ denotes the coproduct in the category of operads. We use the notation $\hat{\vee}$ since we need complete to the operad $\mathcal{P} \vee \alpha$ because of the appearance of potentially infinite sums in the general theory developed in [22] and in the book [30]. In our case, this technicality is not relevant.
- The differential d_{MC} is defined by: $d_{\mathrm{MC}}(\alpha) = -\frac{1}{2}\tilde{l}(\alpha, \alpha)$ which is $-\frac{1}{2}(\tilde{l} \circ_1 \alpha) \circ_2 \alpha$ when written with the partial compositions, and for any $p \in \mathcal{P}$, $d_{\mathrm{MC}}(p) = 0$. It extends to the whole operad $\mathrm{Tw}\mathcal{P}$ by compatibility with the composition. With this differential, any $(\mathrm{Tw}\mathcal{P}, d_{\mathrm{MC}})$-algebra is a graded differential $\mathcal{P}$-algebra, with a marked Maurer-Cartan element which is the image of α.
- Let μ be an operadic Maurer-Cartan element of $(\mathrm{Tw}\mathcal{P}, d_{\mathrm{MC}})$. The operadic Maurer-Cartan equation ensure that the map $d_{\mathrm{MC}} + \mu \star \cdot - \cdot \star \mu$ is a differential on $\mathrm{Tw}\mathcal{P}$. The differential d_{Tw} is defined by $d_{\mathrm{Tw}} = d_{\mathrm{MC}} + \mu \star \cdot - \cdot \star \mu$ with $\mu = \tilde{l}(\alpha, \cdot)$ which is an operadic Maurer-Cartan element.

2.4 Operadic Koszul theory

Koszul duality was originally introduced for associative algebras by Priddy in 1970, see [72]. It was then extended to algebraic operads in 1994 by Ginzburg and Kapranov [39], and Getzler and Jones [37]. The theory of Koszul duality for operads is a vast and deep theory, and we will only scratch the surface of it. The main strength of Koszul duality is its link with homotopical algebra as for example the duality between Lie algebras and commutative algebras in rational homotopy theory. This is even more the case in operadic Koszul duality, where Koszul duality and the Koszul property are related to minimal models of operads, allowing to study homotopical properties of algebra over an operad. We refer the interested to the book [60], as we unfortunately will not have the time to do the link between Koszul duality and homotopical algebra. We will still introduce the main concepts of operadic Koszul duality, give some basic results, and give a nice criterion for the Koszul property of an operad. However, we are going to use the operadic Koszul theory in a rather unusual way. Indeed, we are going to use it to compute arity-wise dimension of operads. This is a very specific use of the operadic Koszul theory, and we will only introduce and prove the relevant results for this specific use. The author would like to point out that the operadic Koszul theory is a very interesting theory going much further than what we will present here, and that using it to compute arity-wise dimension of operads is a quite strange use of it since it is usually used to study homotopical properties of algebras. The author believes that there is something somewhat funny in using such a deep theory to derive something as elementary as equality between integers. We will start by defining the Koszul dual of a quadratic operad, and the Koszul complex. We will then give criterions for the Koszul property of an operad, a positive criterion sufficient but not necessary for an operad to be Koszul, and a negative criterion necessary but not sufficient for and operad to be Koszul. Those criterions will be enough to prove or disprove Koszulness of all operads we will encounter in this manuscript, however, they are by no mean sufficient to answer the question in the general case. Moreover, we should point out that other criterions to prove or disprove Koszulness of operads does exist. We will apply our criterion to the operadic butterfly, disproving a conjecture Loday made in [57]. Finally, we will discuss the generating function of Koszul operads, and classify Koszul set operads on one generator of arity 2 in order to prove a conjecture of the author.

2.4.1 Koszul duality of quadratic operads

Let us introduce the Koszul theory for quadratic operads. It does exist in a more general context, however we will only need the Koszul theory in the quadratic case. Let us define the Koszul dual of quadratic operads. First we need to define the notion of quadratic operad.

Definition 2.4.1.1. An operad $\mathcal{P}$ is *quadratic* if we have $\mathcal{P} = \mathcal{T}(\mathcal{S})/\langle R\rangle$ with $R \in \mathcal{S} \circ' \mathcal{S}$.

Definition 2.4.1.2. Let $\mathcal{S}$ be a species. The *cofree cooperad* generated by $\mathcal{S}$ is the cooperad $\mathcal{T}^c(\mathcal{S})$. Let $R \subseteq \mathcal{S} \circ' \mathcal{S}$. The *cooperad cogenerated by $\mathcal{S}$ with corelators* R is the maximal sub-cooperad $\mathcal{C}(\mathcal{S}, R)$ of $\mathcal{T}^c(\mathcal{S})$ such that the following composition is zero:

$$\mathcal{C}(\mathcal{S}, R) \hookrightarrow \mathcal{T}^c(\mathcal{S}) \twoheadrightarrow \mathcal{S} \circ' \mathcal{S}/\mathcal{R}$$

We remark that with this definition, the corelators are always quadratic. We could have defined the cooperad cogenerated by $\mathcal{S}$ with corelators R in a more general setting, however we will not need this generality since the only use of this definition is in the following definition. Let us define Koszul dual cooperad.

Definition 2.4.1.3. Let $\mathcal{P} = \mathcal{T}(\mathcal{S})/\mathcal{R}$ be a quadratic operad. The *Koszul dual cooperad* of $\mathcal{P}$ is the cooperad

$$\mathcal{P}^{\text{i}} = \mathcal{C}(\mathrm{s}_+\mathcal{S}, \mathrm{s}_+^2\mathcal{R})$$

Definition 2.4.1.4. Let $\mathcal{P} = \mathcal{T}(\mathcal{S})/\mathcal{R}$ be a quadratic operad. The *Koszul dual operad* is $\mathcal{P}^!$ which is $ß_+ \left(\mathcal{P}^{\text{i}}\right)^\vee$ where $\vee$ denotes the arity-wise linear dual. We denote $\mathcal{S}_* = ß_+ \left(\mathrm{s}_+\mathcal{S}\right)^\vee$. If $\mathcal{X}$ is a basis of $\mathcal{S}$ and $\mathcal{S}$ is a finite species, we denote by $\mathcal{X}_*$ the according basis in $\mathcal{S}_*$.

Let us give the explicit construction of the Koszul dual of a finite quadratic operad. Let $\mathcal{P} = \mathcal{T}(\mathcal{S})/\mathcal{R}$. First, let us chase the action of the symmetric group through the construction. Let $\mathcal{X}$ be a basis of $\mathcal{S}$, let $f \in \mathcal{X}$ and $\sigma \in \mathfrak{S}_n$. We denote:

$$f.\sigma = g$$

Then by definition, in $\mathrm{s}_+\mathcal{S}$, we have that:

$$\mathrm{s}_+f.\sigma = \mathrm{s}_+g$$

Since $\mathcal{X}$ is a basis of $\mathcal{S}$, we have $(\mathrm{s}_+\mathcal{X})^\vee$ a basis of $(\mathrm{s}_+\mathcal{S})^\vee$. We get:

$$(\mathrm{s}_+f)^\vee.\sigma = ((\mathrm{s}_+f).\sigma)^\vee = \mathrm{s}_+g^\vee$$

Finally, we have a basis $ß_+(\mathrm{s}_+\mathcal{X})^\vee$ of $ß_+(\mathrm{s}_+\mathcal{S})^\vee$. Let us denote them $\mathcal{X}_*$ and $\mathcal{S}_*$. We have:

$$ß_+(\mathrm{s}_+f)^\vee.\sigma = \left(ß_+ \otimes (\mathrm{s}_+f)^\vee\right).\sigma = (ß_+.\sigma) \otimes ((\mathrm{s}_+f)^\vee.\sigma) = \mathrm{sgn}(\sigma)ß_+(\mathrm{s}_+g)^\vee$$

We denote by $\mathrm{sgn}(\sigma)$ the sign of the permutation σ. Hence, we have that $\mathcal{S}_* \simeq \mathcal{S} \otimes \mathrm{sgn}$. One need to be careful since this isomorphism is not canonical, it depends on the choice of a basis of $\mathcal{S}$. This isomorphism gives a non-degenerate pairing between $\mathcal{S}$ and $\mathcal{S}_*$, hence we get a non-degenerate pairing between $\mathcal{S}_* \circ_1 \mathcal{S}_* = \mathcal{S}_* \otimes \mathcal{S}_*$ and $\mathcal{S} \circ_1 \mathcal{S} = \mathcal{S} \otimes \mathcal{S}$. It gives a pairing between $\mathcal{S} \circ' \mathcal{S}$ and $\mathcal{S}_* \circ' \mathcal{S}_*$ since $\mathcal{S} \circ_1 \mathcal{S}$ generates $\mathcal{S} \circ' \mathcal{S}$ through the action of the symmetric groups, moreover this pairing is non-degenerate if $\mathcal{S}$ is finite. We can compute it the following way, if $f, g, h, k \in \mathcal{S}$, we have:

$$\langle (f_* \circ_1 g_*).\sigma, (h \circ_1 k).\sigma\rangle = \mathrm{sgn}(\sigma)(-1)^{|g_*||h|}f_*(h)g_*(k)$$

Since $\mathcal{S} \circ' \mathcal{S}$ is generated by $\mathcal{S} \circ_1 \mathcal{S}$ under the action of $\mathfrak{S}_n$, this formula gives the pairing on $\mathcal{S} \circ' \mathcal{S}$.

Theorem 2.4.1.5. *Let $\mathcal{P}$ be a finite quadratic operad generated by $\mathcal{S}$. Then, we have that $\mathcal{P}^! = \mathcal{T}(\mathcal{S}_*)/\mathcal{R}_*^{\perp}$ where $\mathcal{R}_*^{\perp}$ is the orthogonal to $\mathcal{R}$ in $\mathcal{S}_* \circ' \mathcal{S}_*$ for the pairing defined above.*

Proof. By construction, we know that $\mathcal{P}^!$ is generated by $\mathcal{S}_*$. Moreover, from the definition of $\mathcal{P}^{¡}$ we know that $f \in \mathrm{s}_+\mathcal{S} \circ' \mathrm{s}_+\mathcal{S}$ is in $\mathcal{P}^{¡}$ is and only if the corresponding element in $\mathcal{S} \circ \mathcal{S}$ is in $\mathcal{R}$. Since we take the linear dual we have the image of $\mathcal{R}_*^{\perp}$ is zero in $\mathcal{P}^!$. Hence, we have $\mathcal{T}(\mathcal{S}_*)/\mathcal{R}_*^{\perp} \twoheadrightarrow \mathcal{P}^!$. By maximality of $\mathcal{P}^{¡}$ which is defined by cogenerators and corelators, we have that it is an isomorphism. □

Proposition 2.4.1.6. *Let $\mathcal{P}$ be a finitely generated quadratic operad. Then we have that $(\mathcal{P}^!)^! = \mathcal{P}$.*

Proof. The finiteness hypothesis ensure that $\mathcal{S} = (\mathcal{S}^\vee)^\vee$ with $\mathcal{S}$ a species generating $\mathcal{P}$. The proof directly follows from this fact. □

The Koszul complex Let $\mathcal{P} = \mathcal{T}(\mathcal{S})/\mathcal{R}$ be a quadratic dg operad. Let us define the *Koszul complex* of $\mathcal{P}$ to be $\mathcal{P}^{¡} \circ \mathcal{P}$. We need to define the differential of the Koszul complex using the operadic structure of $\mathcal{P}$. First, let us denote by d_1 the map $d_{\mathcal{P}^{¡}} \circ \mathrm{id}_{\mathcal{P}} + \mathrm{id}_{\mathcal{P}^{¡}} \circ d_{\mathcal{P}}$, one need to be careful since this map is not a composition of maps, but the map induced by the plethysm. We need to denote the "second part" of the differential. We know that $\mathcal{P}^{¡}$ is a sub-cooperad of $\mathcal{T}(\mathrm{s}_+\mathcal{S})$ by definition. We have a map:

$$\mathcal{T}^c(\mathrm{s}_+\mathcal{S}) \to \mathcal{T}^c(\mathcal{T}^c(\mathrm{s}_+\mathcal{S})) \to \mathcal{T}^c(\mathrm{s}_+\mathcal{S}) \circ' \mathrm{s}_+\mathcal{S}$$

Indeed, the map $\mathcal{T}^c(\mathrm{s}_+\mathcal{S}) \to \mathcal{T}^c(\mathcal{T}^c(\mathrm{s}_+\mathcal{S}))$ is induced by the comonad structure of $\mathcal{T}^c$, and second map is the projection on $\mathcal{T}^c(\mathrm{s}_+\mathcal{S}) \circ' \mathcal{S} \subseteq \mathcal{T}^c(\mathcal{T}^c(\mathrm{s}_+\mathcal{S}))$. This induces a map on $\mathcal{P}^{¡}$:

$$\Delta' : \mathcal{P}^{¡} \to \mathcal{P}^{¡} \circ' \mathrm{s}_+\mathcal{S}$$

Hence, we have:

$$\mathcal{P}^{¡} \circ \mathcal{P} \to (\mathcal{P}^{¡} \circ' \mathrm{s}_+\mathcal{S}) \circ \mathcal{P} \to \mathcal{P}^{¡} \circ' (\mathcal{I} \circ \mathcal{P}, \mathrm{s}_+\mathcal{S} \circ \mathcal{P}) \to \mathcal{P}^{¡} \circ' (\mathcal{P}, \mathcal{S} \circ \mathcal{P}) \to \mathcal{P}^{¡} \circ' (\mathcal{P}, \mathcal{P}) \to \mathcal{P}^{¡} \circ \mathcal{P}$$

This is a map of degree -1, since the only map of non-zero degree used to define it is $\mathrm{s}_+\mathcal{S} \to \mathcal{S}$. We need to check that it square to zero. To do so, let us explicitly describe it. An element T of $\mathcal{P}^{¡} \circ \mathcal{P}$ is:

$$T = \gamma(C; P_1, \ldots, P_n)$$

With $C \in \mathcal{P}^{¡} \subseteq \mathcal{T}^c(\mathrm{s}_+\mathcal{S})$, and $P_i \in \mathcal{P}$. First, let us describe d_2 if C is a shuffle tree. We can understand T as follows: The bottom part of T is C, the *cooperadic part* of T, and we have $P_1, \ldots, P_n$ element of $\mathcal{P}$ grafted to the leaves of C, the *operadic part* of T. Then, $d_2(T)$ is the sum over all the possible ways to switch an internal vertex of C from the cooperadic part to the operadic part, let us point out that from the construction, only internal vertices of C such that all its children are leaves can be switched. Then when applying d_2 twice, we get the sum over all the possible ways to switch two internal vertices of C from the cooperadic part to the operadic part. We can see that we switched two vertices v_1 and v_2 that were either "in parallel", meaning that the vertices are not one on top of the other, or "in series", meaning that the vertices are one on top of the other. In the first case, switching s_+v_1 before s_+v_2 give the opposite sign of switching s_+v_2 before s_+v_1 because of the Koszul sign rule, and the two contributions cancel out. In the second case, it means that $\mathrm{s}_+v_2 \circ_i \mathrm{s}_+v_2$ viewed as a shuffle tree in $\mathcal{T}^c(\mathrm{s}_+\mathcal{S})$ is in $\mathcal{P}^{¡}$, and by construction, it means that $\pi(v_2 \circ_i v_2) = 0 \in \mathcal{P}$, when $v_2 \circ_i v_2$ viewed as a shuffle tree in $\mathcal{T}(\mathcal{S})$. Hence, we get 0. We cannot assume that C is a shuffle tree since $\mathcal{P}^{¡}$ is a sub-cooperad of $\mathcal{T}^c(\mathrm{s}_+\mathcal{S})$, however this proof still work when considering linear combinations. Hence, $d_2 = 0$. Because $d_{\mathcal{P}^{¡}}$ and $d_{\mathcal{P}}$ are compatible with the composition, and because of the Koszul sign rule, we have that d_1 and d_2 anticommute. Let $d_\kappa = d_1 + d_2$, we have that $d_\kappa^2 = 0$.

Definition 2.4.1.7. Let $\mathcal{P}$ be a quadratic dg operad. The *Koszul complex* of $\mathcal{P}$ is the chain complex $(\mathcal{P}^{¡} \circ \mathcal{P}, d_\kappa)$. If $\mathcal{P}$ is not differential graded, then $d_\kappa = d_2$ in the above discussion.

Definition 2.4.1.8. Let $\mathcal{P}$ be a quadratic operad. The operad $\mathcal{P}$ is *Koszul* if $H(\mathcal{P}^{¡} \circ \mathcal{P}, d_\kappa) = \mathbb{K}$. By a sight abuse of notation, we will say that the Koszul complex of $\mathcal{P}$ is acyclic.

Proposition 2.4.1.9. *Let $\mathcal{P}$ be a finitely generated quadratic operad. Then $\mathcal{P}$ is Koszul if and only if $\mathcal{P}^!$ is Koszul.*

Proof. The Koszul complex splits according to the number of generators. The Koszul complex of $\mathcal{P}$ is acyclic if and only its linear dual (arity-wise and number of generator wise) is acyclic. Hence, we have that $\mathcal{P}$ is Koszul if and only if $\mathcal{P}^!$ is Koszul. □

From the definition of the Koszul complex, we can prove the following theorem.

Theorem 2.4.1.10. *Let $\mathcal{P}$ be a quadratic finite operad. Let $\chi_{\mathcal{P}^{¡}\circ\mathcal{P}}(x, y) = \sum \frac{\chi_{(\mathcal{P}^{¡}\circ\mathcal{P})(n)}(y)}{n!} x^n$ be the generating series of the Euler characteristic of the Koszul complex of $\mathcal{P}$ graded by the number of generators. Then we have:*

$$f_{\mathcal{P}^!}(f_{\mathcal{P}}(x, y), -y) = \chi_{\mathcal{P}^{¡}\circ\mathcal{P}}(x, y)$$

Proof. Let y_1 be the formal variable associated to the number of *operadic* generators, and y_2 the formal variable associated to the number of *cooperadic* generators. It is quite clear from the definition of the Koszul complex that its generating series bi-graded by the number of operadic and cooperadic generators $f_{\mathcal{P}^{¡}\circ\mathcal{P}}(x, y_1, y_2)$ is the composition of the generating series of $\mathcal{P}^{¡}$ and $\mathcal{P}$. Hence, we have:

$$f_{\mathcal{P}^{¡}\circ\mathcal{P}}(x, y_1, y_2) = f_{\mathcal{P}^{¡}}(f_{\mathcal{P}}(x, y_1), y_2)$$

Since the degree is the number of cooperadic generators, we have that the Euler characteristic of the Koszul complex of $\mathcal{P}$ bi-graded by the number of operadic and cooperadic generators is:

$$\chi_{\mathcal{P}^{¡}\circ\mathcal{P}}(x, y_1, y_2) = f_{\mathcal{P}^{¡}\circ\mathcal{P}}(x, y_1, -y_2)$$

Since the total number of generators is the sum of the number of operadic and cooperadic generators, we have:

$$\chi_{\mathcal{P}^{¡}\circ\mathcal{P}}(x, y) = f_{\mathcal{P}^{¡}}(f_{\mathcal{P}}(x, y), -y)$$

Since $f_{\mathcal{P}^{¡}} = f_{\mathcal{P}^!}$, we get the result. □

This lead to a nice criterion on generating series to check if an operad is Koszul.

Corollary 2.4.1.11 (Ginzburg-Kapranov criterion for Koszulness)**.** *Let $\mathcal{P}$ be a quadratic finite operad. If $\mathcal{P}$ is Koszul then:*

$$f_{\mathcal{P}^!}(f_{\mathcal{P}}(x, y), -y) = x$$

If $\mathcal{P}$ is generated in arity two, we can drop the grading by the number of generators which is redundant with the arity and we get:

$$-f_{\mathcal{P}^!}(-f_{\mathcal{P}}(x)) = x$$

This lead to another criterion on generating series to check if an operad is Koszul:

Corollary 2.4.1.12. *Let $\mathcal{P}$ be a quadratic finite operad generated in arity two. If $\mathcal{P}$ is Koszul then each coefficient of* $\mathrm{rev}(f_{\mathcal{P}}(x))$ *is non-negative.*

This last criterion is particularly useful since it is quite easy to check. Indeed, the n-th coefficient of $\mathrm{rev}(f_{\mathcal{P}}(x))$ is determined by the n first coefficient of $f_{\mathcal{P}}(x)$, hence one can compute the n-th coefficient of $\mathrm{rev}(f_{\mathcal{P}}(x))$ by a finite number of operations. Since we have a software to compute arity-wise dimension of operads from a presentation by generators and relations, we can use the Lagrange inversion formula to check this criterion. Its is quite nice since we do not need to compute the Koszul dual of $\mathcal{P}$ to check if it is Koszul.

2.4.2 Koszulness and convergent quadratic ORS

Let us show a particularly useful example of Koszulness result. Namely, that an operad defined via a quadratic convergent ORS is Koszul. This will be our main tool to prove Koszulness of operads in this manuscript. This is a slight generalization of the PBW case, see [43], and the Gröbner basis case, see [28] and [10], since we only require a partial monomial ordering instead of a total monomial ordering. It was long known that this generalization was possible, however the author could not find a reference for this result. We will give a proof of this result in this subsection.

Definition 2.4.2.1. An operad is *monomial* if it is the quotient of a free operad by a two-sided ideal generated by monomials. It is *quadratic monomial* if the ideal is generated by monomials involving exactly 2 generators. Same for a cooperad, shuffle operads and shuffle cooperads.

Let us show that quadratic monomial operads are Koszul. To do so, we need to show that the Koszul complex is acyclic. Let $\mathcal{P}$ be a quadratic monomial operad, given by $\mathcal{P} = \mathcal{T}\{\mathcal{X}\}/\langle M\rangle$ with $\mathcal{X}$ the set of generators and M the set of monomials generating the operadic ideal of relation of $\mathcal{P}$. Since $\mathcal{P}$ is monomial, we have a preferred basis for $\mathcal{P}$ given by the set of monomials in $\mathcal{T}\{\mathcal{X}\}$ that are not divisible by any $m \in M$. Same for the Koszul dual cooperad $\mathcal{P}^{\text{i}}$ for which the preferred basis is given by the set of monomials in $\mathcal{T}\{\mathrm{s}_+\mathcal{X}\}$ that are not divisible by any $m \in M_*$. Hence, the Koszul complex of $\mathcal{P}$ has a preferred basis given by the shuffle trees of the form:

$$\gamma_{I_1,\ldots,I_n}(C; P_1, \ldots, P_n)$$

where C is an element of the preferred basis of $\mathcal{P}^{\text{i}}$ and $P_1, \ldots, P_n$ are elements of the preferred basis of $\mathcal{P}$. Hence, one can get a shuffle tree of $\mathcal{T}\{\mathcal{X}\}$ through the bijection $\mathrm{s}_+\mathcal{X} \to \mathcal{X}$.

Proposition 2.4.2.2. *The differential d_κ of the Koszul complex of $\mathcal{P}$ split according to the underlying shuffle tree of $\mathcal{T}\{\mathcal{X}\}$ obtain by the bijection $\mathrm{s}_+\mathcal{X} \to \mathcal{X}$.*

Proof. Let us differentiate a shuffle tree T of the Koszul complex of $\mathcal{P}$. We can write $T = \gamma_{I_1,\ldots,I_n}(C; P_1, \ldots, P_n)$. Let us denote v_i the internal vertices of C such that all their children are leaves, and $\mathrm{s}_+x_i \in \mathrm{s}_+\mathcal{X}$ their labels. Then, $d_\kappa(\gamma_{I_1,\ldots,I_n}(C; P_1, \ldots, P_n))$ is, up to a Koszul sign, the sum over i of the same shuffle tree with s_+x_i flipped to x_i. Hence, if one "forget the s_+" in the labels, the underlying shuffle tree did not change, more formally, the underlying shuffle tree of $\mathcal{T}\{\mathcal{X}\}$ obtain by the bijection $\mathrm{s}_+\mathcal{X} \to \mathcal{X}$ did not change after applying d_κ. □

Lemma 2.4.2.3. *The sub-complex (A, d_κ) of the Koszul complex of $\mathcal{P}$ spanned by the shuffle trees of height 2 is acyclic, meaning that the cohomology of (A, d_κ) is zero.*

Proof. Let us consider a shuffle tree of height 2 given by

$$\gamma_{I_1,\ldots,I_m}(f; \mathrm{id}, \ldots, \mathrm{id}, g_1, \mathrm{id}, \ldots, \mathrm{id}, g_2, \ldots, g_{n-1}, \mathrm{id}, \ldots, \mathrm{id}, g_n, \mathrm{id}, \ldots, \mathrm{id})$$

where $f, g_1, \ldots, g_n \in \mathcal{X}$. By a slight abuse of notation, let us denote it:

$$\gamma(f; g_1, \ldots, g_n)$$

Then let us construct the following simplex: The vertex i is:

$$\gamma(\mathrm{s}_+f; g_1, \ldots, g_{i-1}, \mathrm{s}_+g_i, g_{i+1}, \ldots, g_n)$$

The edge linking i and j is:

$$\gamma(\mathrm{s}_+f; g_1, \ldots, g_{i-1}, \mathrm{s}_+g_i, g_{i+1}, \ldots, g_{j-1}, \mathrm{s}_+g_j, g_{j+1}, \ldots, g_n)$$

And so on. We may notice that the Koszul complex restricted to the shuffle tree $\gamma(f; g_1, \ldots, g_n)$ (under the bijection $\mathrm{s}_+\mathcal{X} \to \mathcal{X}$) is the reduced chain complex of the simplex we just constructed, with $\gamma(\mathrm{s}_+f; g_1, \ldots, g_n)$ for the empty set. We may in fact only have a facet of the simplex, indeed if

$s_+ f \circ_i s_+ g_i \in s_+^2 M$, then the simplex is degenerate as the i vertex is 0. If $s_+ f \circ_i s_+ g_i \in s_+ M$ for all i, then it is the only case where $\gamma(f; g_1, \ldots, g_n)$ is not zero in the Koszul complex, hence in this case we get the chain complex:

$$0 \to \gamma(s_+ f; g_1, \ldots, g_n) \to \gamma(f; g_1, \ldots, g_n) \to 0$$

In all these cases, we get acyclic complexes since simplex are contractible. Hence, the Koszul complex restricted to the shuffle trees of height 2 is acyclic. □

Theorem 2.4.2.4. *The Koszul complex of a quadratic monomial operad is acyclic. The same is true for quadratic monomial shuffle operads.*

Proof. Let us use the previous lemma and a spectral sequence argument. Since the shuffle tree are a basis of the tree monad, we can assume that $\mathcal{P}$ is a shuffle operad without loss of generality. Let $\mathcal{P} = \mathcal{T}^{\text{III}}\{\mathcal{X}\}/M$ and let us prove that $(\mathcal{P}^{\text{i}} \circ \mathcal{P}, d_\kappa)$ is acyclic. Let T be a shuffle tree of the Koszul complex of $\mathcal{P}$. Let S be the underlying shuffle tree under the identification $s_+\mathcal{X} \to \mathcal{X}$. The minimal representative of S denoted $S_{\min}$ is the shuffle tree of the Koszul complex of $\mathcal{P}$ such that its underlying shuffle tree under the identification $s_+\mathcal{X} \to \mathcal{X}$ is S and such that it has the minimal number of internal vertices labeled by elements of $\mathcal{X}$. Let v be (one of) the highest vertices labeled by elements of $s_+\mathcal{X}$ in $S_{\min}$, there is at least one vertex labeled by elements of $s_+\mathcal{X}$ in $S_{\min}$ since the root is always labeled by $s_+\mathcal{X}$ in $S_{\min}$. Let us define t as either the maximal subtree of T of height 2 such that the parent of v is its root, or as the maximal subtree of T of height 2 such that v is its root if v is the root of T. Let us write T as:

$$T = R \circ_{i,I} (\gamma I_1, \ldots, I_n(t; P_1, \ldots, P_n))$$

We have $P_1, \ldots, P_n$ are labeled over $\mathcal{X}$, and t is of height 2. We define the degree of T as the number of internal vertices labeled by elements of $\mathcal{X}$ in R. The differential d_κ is increasing along the degree since it turns elements of $\mathcal{X}$ into elements of $s_+\mathcal{X}$. Let us consider the decreasing filtration induced by the degree, and the associated spectral sequence. We have that d_0 is exactly the restriction of d_κ on t. Hence, by the previous lemma, we have that the spectral sequence abuts at the first page. The differential d_κ splits accordingly to the number of internal vertices of the shuffle trees. Since the degree is bounded by the number of internal vertices, at number of internal vertices fixed, the filtration is bounded. By [79][Classical Convergence Theorem of spectral sequence 5.5.1], at number of internal vertices fixed we have that the spectral sequence converges. Hence, the Koszul complex of $\mathcal{P}$ is acyclic. □

This theorem is quite useful and already has some non-trivial consequences. In particular, the criterion given by this theorem is sufficient to classify Koszul set operads on one generator of arity two, see 2.4.5. However, it can be seen as the base case of a more general theorem. Indeed, any operad admitting a presentation by a quadratic convergent ORS is Koszul. This is a quite powerful result, and one of the main computational tools to show that an operad is Koszul. Let us show this theorem, using our favorite tool: the spectral sequence. To do so, we need a filtration. Let $(\mathcal{S}, \mathcal{X}, R)$ be a quadratic convergent ORS admitting an associated operad, and $\mathcal{P}$ its associated operad. We recall that $(\mathcal{S}, \mathcal{X}, R)$ induces a transfinite filtration F_α on $\mathcal{T}\{\mathcal{S}\}$.

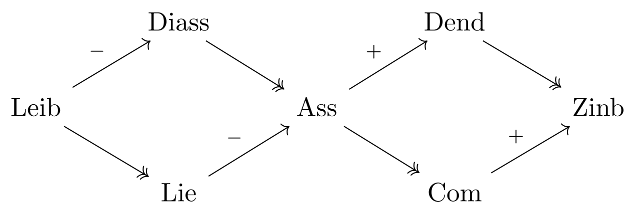
Theorem 2.4.2.5. *The Koszul complex of an operad admitting a quadratic convergent ORS is acyclic.*

Proof. Since $(\mathcal{S}, \mathcal{X}, R)$ is a quadratic convergent ORS, we have canonical representations of the elements of $\mathcal{P}$ by shuffle trees of $\mathcal{T}^{\text{III}}\{\mathcal{X}\}$. By definition of $\mathcal{P}^{\text{i}}$ we have $\mathcal{P}^{\text{i}} \subseteq \mathcal{T}^{\text{III}}\{s_+\mathcal{X}\}$. Since the Koszul complex is given by $\mathcal{P}^{\text{i}} \circ \mathcal{P}$, we have canonical representations of the elements of the Koszul complex by shuffle trees of $\mathcal{T}^{\text{III}}\{s_+\mathcal{X}\} \circ \mathcal{T}^{\text{III}}\{\mathcal{X}\}$. Moreover, any shuffle tree of $\mathcal{T}^{\text{III}}\{s_+\mathcal{X}\} \circ \mathcal{T}^{\text{III}}\{\mathcal{X}\}$ can be seen as a shuffle tree of $\mathcal{T}^{\text{III}}\{\mathcal{X}\}$ by "forgetting the s_+" in the labels. Hence, we have transfinite filtration on the shuffle trees of $\mathcal{T}^{\text{III}}\{s_+\mathcal{X}\} \circ \mathcal{T}^{\text{III}}\{\mathcal{X}\}$. The differential d_κ is decreasing along this transfinite filtration since we may be forced to rewrite the part of shuffle tree of $\mathcal{T}^{\text{III}}\{\mathrm{s}_+\mathcal{X}\} \circ \mathcal{T}^{\text{III}}\{\mathcal{X}\}$ which is in $\mathcal{T}^{\text{III}}\{\mathcal{X}\}$ after applying d_κ. Let us consider the associated transfinite spectral sequence. We have that d_0 is exactly the differential d'_κ of the Koszul complex of $\mathcal{P}' = \mathcal{T}^{\text{III}}\{\mathcal{X}\}/\langle R' \rangle$ with R' the rewritable monomials of $(\mathcal{S}, \mathcal{X}, R)$. Hence, by the previous theorem, d_0 is acyclic. Hence, the spectral sequence abuts at the first page, hence the Koszul complex of $\mathcal{P}$ is acyclic. □

One should not be afraid of the appearance of transfinite spectral sequence, since in most cases the filtration is bounded. We refer to [79] for a detailed study of spectral sequence (and for a very nice introduction to homological algebra). We refer more specifically to [44] and [73] for the case of transfinite spectral sequence.

2.4.3 Koszulness in the operadic butterfly

Let us apply the previous theorem to the operadic butterfly. We recall that the operadic butterfly is the following diagram of operad:



Let us recall the definition of the operads in the butterfly diagram. This will allow us to fix some notation for the generators of these operads. For $\mathcal{X}$ a set species, we denote $\mathcal{T}[\mathcal{X}] = \mathcal{T}(\mathrm{Span}(\mathcal{X}))$ to ease the notations.

$$\mathrm{Diass} = \mathcal{T}[a, b, a.(1\ 2), b.(1\ 2)]/\langle a \circ_2 a - a \circ_1 a, a \circ_2 a - a \circ_2 b, a \circ_1 b - b \circ_2 a, b \circ_1 a - b \circ_2 b, b \circ_1 a - b \circ_1 b \rangle$$

$$\mathrm{Dend} = \mathcal{T}[x, y, x.(1\ 2), y.(1\ 2)]/\langle x \circ_1 x - x \circ_2 x - x \circ_2 y, x \circ_1 y - y \circ_2 x, y \circ_1 x + y \circ_1 y - y \circ_2 y \rangle$$

$$\mathrm{Leib} = \mathcal{T}[\lambda, \lambda.(1\ 2)]/\langle \lambda \circ_1 \lambda + (\lambda \circ_1 \lambda).(1\ 2\ 3) + (\lambda \circ_1 \lambda).(1\ 3\ 2) \rangle$$

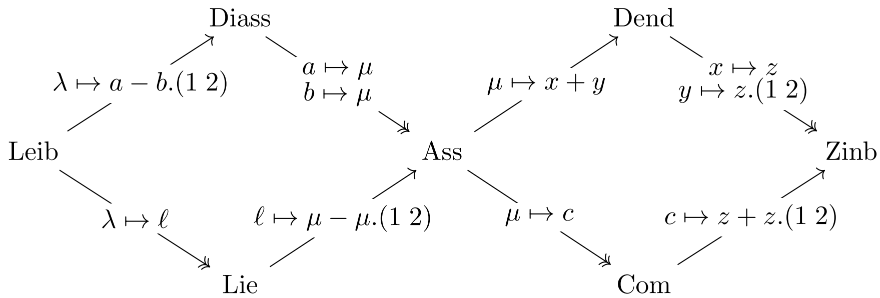
$$\mathrm{Ass} = \mathcal{T}[\mu, \mu.(1\ 2)]/\langle \mu \circ_1 \mu - \mu \circ_2 \mu \rangle$$

$$\mathrm{Zinb} = \mathcal{T}[z, z.(1\ 2)]/\langle z \circ_1 z - z \circ_2 z - (z \circ_2 z).(2\ 3) \rangle$$

$$\mathrm{Lie} = \mathcal{T}[\ell]/\langle \ell \circ_1 \ell + (\ell \circ_1 \ell).(1\ 2\ 3) + (\ell \circ_1 \ell).(1\ 3\ 2) \rangle$$

$$\mathrm{Com} = \mathcal{T}[c]/\langle c \circ_1 c - c \circ_2 c \rangle$$

From these definitions, one can wonder how Ass is different from Com, or Leib from Lie. This is because our description is not complete, we need to specify the action of $\mathfrak{S}_2$ on the generators. Let us do it. The generators a, b, x, y, λ, μ and z have no symmetries. For example, in the case of Diass, it means that $(a, b, a.(1\ 2), b.(1\ 2))$ is a basis of Diass(2) as a vector space. The generator ℓ of Lie is antisymmetric so $\ell.(1\ 2) = -\ell$, and the generator c of Com is symmetric so $c.(1\ 2) = c$. This is the difference between Ass and Com, and Leib and Lie. The attentive reader may have noticed that $\{\ell\}$ is not a set species since it is not stable under the action of $\mathfrak{S}_2$, however $\mathrm{Span}(\{\ell\})$ is indeed a species so the notation $\mathcal{T}[\ell]$ is still licit. We can specify the 8 maps of the operadic butterfly. It is enough to specify the image of the generators. We let the reader check that the following maps are well-defined, and that they make the diagram commute.

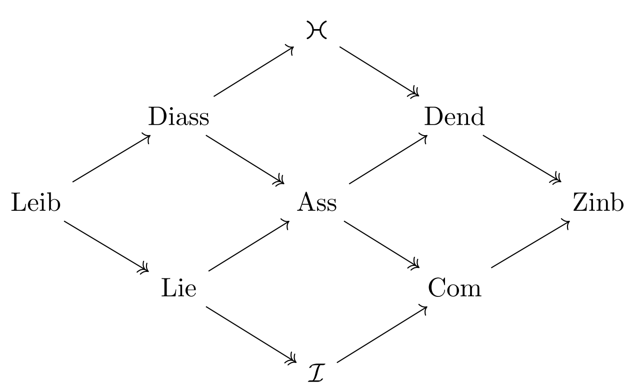


A well known result in operad theory is the following. With our theorem on Koszulness of operads admitting a quadratic convergent ORS, we could find convergent quadratic ORS for the operads in the butterfly. We let the reader find such a convergent quadratic ORS for the operads in the butterfly. The tricky part is to find the monomial partial ordering. After that the software developed in [27] allows us to skip the tedious verification by hand of the confluence of the rewriting system. We will not give the solution here, but we will give the result:

Theorem 2.4.3.1. *The operads* Diass*,* Dend*,* Leib*,* Ass*,* Zinb*,* Lie *and* Com *are Koszul. Moreover, we have:*

- $\mathrm{Diass}^! = \mathrm{Dend}$*,*
- $\mathrm{Leib}^! = \mathrm{Zinb}$*,*
- $\mathrm{Ass}^! = \mathrm{Ass}$*, and*
- $\mathrm{Lie}^! = \mathrm{Com}$*.*

From this theorem, we can see that the left/right symmetry of the operadic butterfly corresponds to the Koszul duality. This fact led Loday to ask the following question: "Is there a Koszul operad $\asymp$ such that $\asymp^! = \asymp$ which completes the diagram?" This question stayed open for quite a long time, and we will see that the answer is no. However, we first need to make the question more precise, what does "completing the diagram" mean? We want an operad $\asymp$ such that:



Where $\asymp$ has four non-symmetric generators α, β, γ and δ. The morphism $\mathrm{Diass} \to \asymp$ is given by:

$$a \mapsto \alpha + \gamma$$
$$b \mapsto \beta + \delta$$

And finally, the morphism $\bowtie \to \mathrm{Dend}$ is given by:

$$\begin{aligned} \alpha &\mapsto x \\ \beta &\mapsto x \\ \gamma &\mapsto y \\ \delta &\mapsto y \end{aligned}$$

Let us define the following 15 relations:

$$\begin{array}{lll} \alpha \circ_1 \alpha = \alpha \circ_2 \alpha + \alpha \circ_2 \beta, & \alpha \circ_1 \beta = \beta \circ_2 \alpha, & \beta \circ_2 \alpha + \beta \circ_2 \beta = \beta \circ_1 \beta, \\ \alpha \circ_1 \alpha = \alpha \circ_2 \gamma + \alpha \circ_2 \delta, & \alpha \circ_1 \beta = \beta \circ_2 \gamma, & \beta \circ_2 \alpha + \beta \circ_2 \beta = \beta \circ_1 \delta, \\ \alpha \circ_1 \gamma = \gamma \circ_2 \alpha + \gamma \circ_2 \beta, & \alpha \circ_1 \delta = \delta \circ_2 \alpha, & \beta \circ_2 \gamma + \beta \circ_2 \delta = \delta \circ_1 \beta, \\ \gamma \circ_1 \alpha = \gamma \circ_2 \gamma + \gamma \circ_2 \delta, & \gamma \circ_1 \beta = \delta \circ_2 \gamma, & \delta \circ_2 \alpha + \delta \circ_2 \beta = \delta \circ_1 \delta, \\ \gamma \circ_1 \gamma = \gamma \circ_2 \gamma + \gamma \circ_2 \delta, & \gamma \circ_1 \delta = \delta \circ_2 \gamma, & \delta \circ_2 \gamma + \delta \circ_2 \delta = \delta \circ_1 \delta \end{array} \tag{1-15}$$

And let define two relations:

$$\delta \circ_1 \gamma - \delta \circ_1 \alpha = +\alpha \circ_2 \beta - \alpha \circ_2 \delta \tag{16+}$$

$$\delta \circ_1 \gamma - \delta \circ_1 \alpha = -\alpha \circ_2 \beta + \alpha \circ_2 \delta \tag{16-}$$

Let us define the operad $\bowtie^+$ as the quotient:

$$\mathcal{T}[\alpha, \beta, \gamma, \delta, \alpha.(1\ 2), \beta.(1\ 2), \gamma.(1\ 2), \delta.(1\ 2)]/R^+,$$

where R^+ is the operadic ideal generated by the relations (1-15) and (16+). Similarly, let us define the operad $\bowtie^-$ as the quotient:

$$\mathcal{T}[\alpha, \beta, \gamma, \delta, \alpha.(1\ 2), \beta.(1\ 2), \gamma.(1\ 2), \delta.(1\ 2)]/R^-,$$

where R^- is the operadic ideal generated by the relations (1-15) and (16-). The study done in [57]

Theorem 2.4.3.2. *Let $\bowtie$ be an operad such that $\bowtie^! = \bowtie$ and $\bowtie$ completes the butterfly diagram. Then $\bowtie = \bowtie^+$ or $\bowtie = \bowtie^-$. Moreover, if $\bowtie$ is Koszul, $\frac{\dim(\bowtie(n))}{n!} = 4^{n-1}$.*

However, the question if either $\bowtie^+$ or $\bowtie^-$ are Koszul stayed open for quite a long time, almost 20 years. We now have software that can compute arity-wise dimension of operads from a presentation. Let us do it.

Proposition 2.4.3.3 (Solution to Exercice 3.12 [10])**.** *We have $\frac{\dim(\bowtie^+(n))}{n!} = 56 \neq 64$ and $\frac{\dim(\bowtie^-(n))}{n!} = 58 \neq 64$. In particular neither $\bowtie^+$ nor $\bowtie^-$ are Koszul.*

Proof. We used the software [27] to compute the arity-wise dimension of $\bowtie^+$ and $\bowtie^-$. We get:

n	1	2	3	4	5	6	7	...
$\frac{\dim(\bowtie^+(n))}{n!}$	1	4	16	56	210	792	3003	...
$\frac{\dim(\bowtie^-(n))}{n!}$	1	4	16	58	211	793	3004	...

□

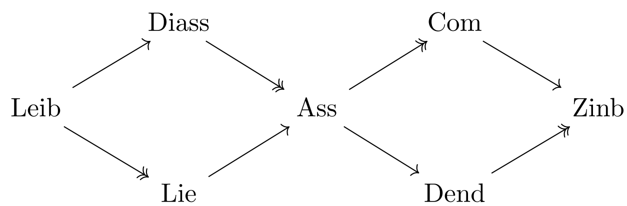
These computations are quite surprising. Indeed, the first terms on the sequence of arity-wise dimensions of $\bowtie^+$ seem to link to A001791. This lead the author to a new conjecture on the arity-wise dimension of $\bowtie^+$.

Conjecture 2.4.3.4. *Let $n \neq 3$. Up to n=7, we have:*

$$\frac{\dim(\asymp^+(n))}{n!} = \binom{2n}{n-1}$$

Does this formula hold for $n > 7$?

The fact that one cannot complete the operadic butterfly with a Koszul operad hints that we did not look at operadic butterfly the right way. Indeed, we may equivalently write the operadic butterfly the following way:



In this case, it is the central symmetry that corresponds to the Koszul duality. Let us recall the definition of the following operad:

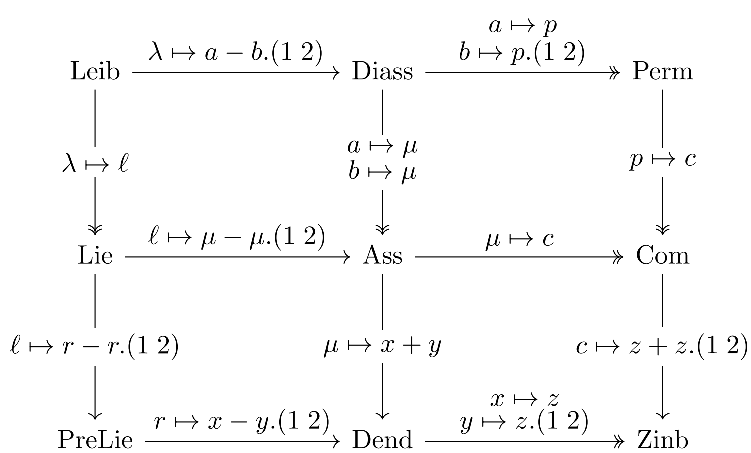
$$\mathrm{PreLie} = \mathcal{T}[r, r.(1\ 2)]/\langle r \circ_1 r - r \circ_2 r = (r \circ_1 r - r \circ_2 r).(1\ 2)\rangle$$

One can compute its Koszul dual:

$$\mathrm{Perm} = \mathcal{T}[p, p.(1\ 2)]/\langle p \circ_1 p = p \circ_2 p, p \circ_1 p = (p \circ_2 p).(1\ 2)\rangle$$

Then we have:

Theorem 2.4.3.5. *The operads* Perm *and* PreLie *are Koszul. Moreover, we have the following commutative diagram:*



This way of completing the operadic butterfly is quite interesting. It was done in [17], and we refer to this paper for a proof of this theorem.

2.4.4 Generating function of Koszul operads

We already saw that Koszulness of an operad imposes some conditions on the generating function of the operad. There is in fact a quite famous conjecture of Polishchuk and Positselski [71] on the generating function of Koszul algebra let us cite it:

Conjecture 2.4.4.1 (Polishchuk, Positselski)**.** *Let A be a finitely generated Koszul algebra, then the Hilbert series of A is a rational function.*

This conjecture can be extended to operads with the insight of Khoroshkin and Piontkovski [49]:

Conjecture 2.4.4.2 (Khoroshkin, Piontkovski)**.**

- *Let $\mathcal{O}$ be a finitely generated Koszul non-symmetric operad, then the Hilbert series of $\mathcal{O}$ is an algebraic function.*
- *Let $\mathcal{P}$ be a finitely generated Koszul symmetric operad, then the Hilbert series of $\mathcal{P}$ is a differential algebraic function.*

It states that the Hilbert series of any finitely generated Koszul symmetric operad should satisfy a non-trivial differential algebraic equation over $\mathbb{Z}[t]$. It can be experimentally checked on examples appearing in the literature, however most of the examples are operads generated in arity two, and are often generated by one operation. We have already introduced the Koszul operads Ass, Com, Lie, Leib, Zinb, PreLie and Perm which are all generated by one operation of arity two. In order to be the most exhaustive possible, we looked through Operadia[77] which is an under construction database of operads and their property inspired by the article of Zinbiel [81]. One can find at least 17 Koszul operads generated by one operation of arity two in [77], at the time this is written down. A summary table of those operads and their Hilbert series is given in Table 4.1 of the appendix, the sequence of arity-wise dimensions of the operads are given in the OEIS, they might be shifted by 1 or have a non-zero first term.

One may remark that all their Hilbert series satisfy differential algebraic identities. Moreover, those identities are of *order* 1 meaning that only f and f' appear and no higher differential of f. It leads us to the following conjecture:

Conjecture 2.4.4.3. *Let $\mathcal{P}$ a Koszul symmetric operad generated by one operation of arity two, then the Hilbert series of $\mathcal{P}$ is differential algebraic of order 1 over $\mathbb{Z}[x]$. Equivalently, $f_{\mathcal{P}}$ and $f'_{\mathcal{P}}$ are algebraically dependent over $\mathbb{Z}[x]$.*

Remark 2.4.4.4. One cannot expect this conjecture to be true for binary finitely generated Koszul symmetric operads since one can define the operad Com ∘ Com by generators and relations:

$$\mathrm{Com} \circ \mathrm{Com} = \mathcal{F}(c_1, c_2)/\langle c_1 \circ_1 c_1 - c_1 \circ_2 c_1,\ c_2 \circ_1 c_2 - c_2 \circ_2 c_2,\ c_1 \circ_1 c_2 \rangle$$

where the action of $\mathfrak{S}_2$ on $\{c_1, c_2\}$ is given by $c_i.(1\ 2) = c_i$. One can show that this operad is Koszul, moreover its Hilbert series is $\exp(\exp(x) - 1) - 1$ which is not algebraically dependent with its differential over $\mathbb{Z}[x]$.
More generally this construction allows us to build a Koszul operad $\mathcal{P}$ with n generators such that its Hilbert series is not algebraically dependent with its first $n-1$ differentials over $\mathbb{Z}[x]$.

A very pedestrian way to prove this conjecture would be to classify all Koszul symmetric operads generated by one operation of arity two, and to compute the Hilbert series of each of them. For the sake of simplicity, we will restrict ourselves to the case of set operads and prove the conjecture in this particular case.

2.4.5 Classification of Koszul set operads generated by one operation of arity two

To check the last subsection conjecture, let us try to classify Koszul operads generated by one operation of arity two. To ease this classification and to reduce the number of cases, we will restrict ourselves to the case of set operads. Let us note KSetOp_1 for set operads generated by one operation of arity two without symmetries. We denote by Mag the so called magmatic operad which is the free operad generated by one operation of arity two without symmetries. The following lemma allows us to reduce the study of KSetOp_1 to the study of a finite number of operads.

Lemma 2.4.5.1. *Let $\mathcal{P}$ be a* KSetOp_1 *operad, then $\mathcal{P}$ is a quotient of* Mag *by an equivariant equivalence relation on the monomials of* Mag(3).

Proof. Since $\mathcal{P}$ is generated by one operation of arity two, it is a quotient of Mag, moreover since $\mathcal{P}$ is Koszul, $\mathcal{P}$ is quadratic, hence the relations are quadratic. Moreover, since $\mathcal{P}$ is a set operad, the relations are given by an equivariant equivalence relation on the monomials. □

This lemma allows us to reduce the study to a finite number of cases. However, in each case, we will need to show if it is Koszul or not. Our main tool to show Koszulness will be Theorem 2.4.2.4. Our main tool to show non-Koszulness will be Corollary 2.4.1.12. Using the software SageMath[78] to compute reverse power series, we can state the following proposition:

Proposition 2.4.5.2. *Let $\mathcal{P}$ be an operad such that its generating series is one of the power series of Table 4.2 of the appendix. Then $\mathcal{P}$ is not Koszul.*

Proof. For each power series of Table 4.2, we computed the reverse power series and found some negative coefficients. This implies that the operad is not Koszul. Table 4.2 was assembled after the actual classification and contains most of the power series of the non-Koszul operads we had to check to establish the classification. □

Some KSetOp_1 operads are well known:

- The magmatic operad on one generator without symmetries,
- The non-associative permutative operad,
- The associative operad,
- The permutative operad,
- The Koszul dual of the Lie admissible operad (which is indeed a set operad when considering the non-symmetric generator).

We have already introduce most of them. Let us introduce the last ones. The non-associative permutative operad denote NAP is the operad defined by:

$$\mathrm{NAP} = \mathcal{T}[x, x.(1\ 2)]/\langle x \circ_1 x = (x \circ_1 x).(2\ 3)\rangle$$

The Koszul dual of the Lie admissible operad denoted $\mathrm{LieAdm}^!$ is the operad defined by:

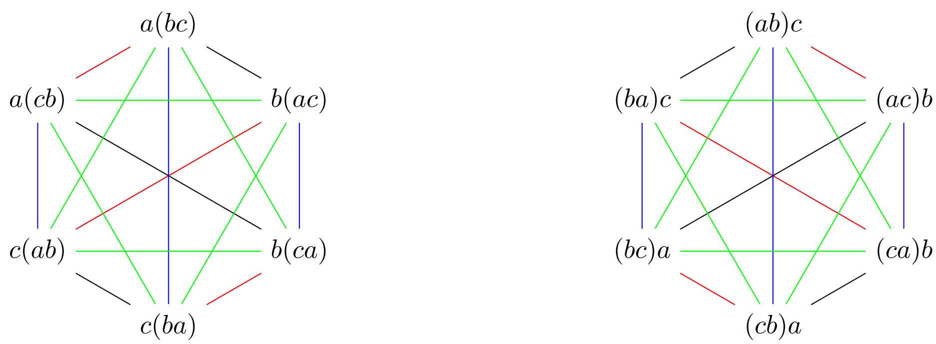
$$\mathrm{NAP} = \mathcal{T}[x, x.(1\ 2)]/\langle x \circ_1 x = (x \circ_1 x).(1\ 2) = (x \circ_1 x).(2\ 3) = x \circ_2 x = (x \circ_2 x).(1\ 2) = (x \circ_2 x).(2\ 3)\rangle$$

Using convergent quadratic ORS, One can check that those are indeed Koszul.

Let us define CMag and AMag respectively the commutative magmatic operad and the anti-commutative magmatic operad. They are defined by $\mathrm{CMag} = \mathcal{T}[c]$ with $c.(1\ 2) = c$ and $\mathrm{AMag} = \mathcal{T}[l]$ with $l.(1\ 2) = -l$. We also define CNil_2 and ANil_2 the commutative nilpotent operad of order 2 and the anti-commutative nilpotent operad of order 2. They are defined by $\mathrm{CNil}_2 = \mathrm{CMag}/\mathrm{CMag}(3)$ and $\mathrm{ANil}_2 = \mathrm{AMag}/\mathrm{AMag}(3)$, this means that in CNil_2 and ANil_2 any non-trivial partial composition gives 0. It is clear from Theorem 2.4.2.4 that these four operads are Koszul. Moreover, we have $\mathrm{ANil}_2 = \mathrm{CMag}^!$ and $\mathrm{CNil}_2 = \mathrm{AMag}^!$.

The operads AMag, CNil_2, and ANil_2 are clearly not set operads. However, they will be the building blocks of the four new Koszul set operads that we will introduce to complete the classification.

The magmatic operad We denote by Mag the magmatic operad, and let x be its generator without symmetry. The operad Mag admits a non-trivial automorphism given by $x \mapsto x.(1\ 2)$, let us call it the *reverse automorphism*. The operad Mag is Koszul and is in fact the first example of KSetOp_1. By Lemma 2.4.5.1, any KSetOp_1 operad is a quotient of Mag by an equivariant equivalence relation on the monomials of Mag(3). Let us study the action of $\mathfrak{S}_3$ on the set of monomials of Mag(3). They can be represented by all possible ways to parenthesize the product of three elements a, b and c with a, b and c in any order. Let us represent it this way:



Black edges represent the action of the transposition (1 2), red edges represent the action of the transposition (2 3), blue edges represent the action of the transposition (1 3), green edges represent the action of the 3-cycles.

As we can see, the action of $\mathfrak{S}_3$ on the monomials of Mag(3) have 2 orbits let us call them the left and right orbits. They are exchanged by the reverse automorphism of Mag.

Proposition 2.4.5.3. *Let $\mathcal{R}$ be an equivariant equivalence relation on the monomials of* Mag(3)*, then $\mathcal{R}$ satisfies exactly one of the following property:*

1. *all equivalence classes are either subsets of the left orbit or subsets of the right orbit;*
2. *all equivalence classes contain elements of both orbits.*

In the first case, the relation is entirely determined by the class of $(ab)c$ and the class of $a(bc)$. In the second case, the class of $(ab)c$ is enough to determine the relation.

The first case can be refined in four sub-cases:

Proposition 2.4.5.4. *Let $\mathcal{R}$ be an equivariant equivalence relation on the monomials of* Mag(3) *satisfying Property 1 of Proposition 2.4.5.3, then $\mathcal{R}$ satisfies exactly one of the following property:*

- *1.1 the relation is trivial (equivalence classes are singletons);*
- *1.2 equivalence classes of the left orbit are reduced to singletons, and equivalence classes of the right are not;*
- *1.3 equivalence classes of the right orbit are reduced to singletons, and equivalence classes of the left are not;*
- *1.4 no equivalence class is reduced to a singleton.*

The first sub-case gives rise to the operad Mag. The second and third sub-cases are equivalent by the reverse automorphism. The last sub-case is the same as giving one equivalence relation on the left orbit and another one on the right orbit.

The second case in Proposition 2.4.5.3 can be refined in two sub-cases:

Proposition 2.4.5.5. *Let $\mathcal{R}$ an equivariant equivalence relation on the monomials of* Mag(3) *satisfying Property 2 of Proposition 2.4.5.3, then $\mathcal{R}$ satisfies exactly one of the following property:*

2.1 all equivalence classes contain exactly two elements;

2.2 all equivalence classes contain strictly more than two elements (namely 4, 6 *or* 12 *elements).*

We start by studying Sub-case 1.2. We then study Sub-case 1.4 in Paragraph 2.4.5. Sub-case 2.1 is studied in Paragraph 2.4.5 and Sub-case 2.2 in Paragraph 2.4.5. Since Sub-case 1.1 is trivial and Sub-cases 1.2 and 1.3 are equivalent, all the possible cases are explicitly considered.
This approach leads us to consider 72 operads, however some of them are isomorphic. Removing isomorphic operads allows us to reduce this number to 39 operads to check.

Equivariant relations on the right orbit Let us first study Sub-case 1.2. By equivariance, a relation of Sub-case 1.2 is entirely determined by the class of $(ab)c$. Moreover, the equivalence class of $(ab)c$ contains either 2, 3 or 6 elements. We have a priori 5 possibilities to relate two elements of the right orbit:

- $(ab)c \sim (ac)b$;
- $(ab)c \sim (ba)c$;
- $(ab)c \sim (bc)a$;
- $(ab)c \sim (ca)b$;
- $(ab)c \sim (cb)a$.

However, since the relation is equivariant, the relation generated by $(ab)c \sim (ca)b$ is the same as the relation generated by $(ab)c \sim (bc)a$. Moreover, this relation is the only one such that the equivalence class of $(ab)c$ contains 3 elements.
We have only one possibility for a relation such that the equivalence class of $(ab)c$ contains 6 elements:

$$(ab)c \sim (cb)a \sim (ca)b \sim (ba)c \sim (ac)b \sim (bc)a$$

Hence, we have 5 possibles relations which gives rise to 5 operads:

- $\mathcal{P}_1$ with $(ab)c = (ac)b$;
- $\mathcal{P}_2$ with $(ab)c = (ba)c$;
- $\mathcal{P}_3$ with $(ab)c = (cb)a$;
- $\mathcal{P}_4$ with $(ab)c = (bc)a = (ca)b$;
- $\mathcal{P}_5$ with $(ab)c = (cb)a = (ca)b = (ba)c = (ac)b = (bc)a$.

We recognize that $\mathcal{P}_1$ is the (right) non-associative permutative operad, hence is Koszul.

Proposition 2.4.5.6. *The operads $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_5$ are not Koszul.*

Proof. Let us compute the first dimensions of the operads $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_5$ using the Haskell calculator [27]:

n	$\mathcal{P}_2(n)$	$\mathcal{P}_3(n)$	$\mathcal{P}_4(n)$	$\mathcal{P}_5(n)$
1	1	1	1	1
2	2	2	2	2
3	9	9	8	7
4	60	60	40	29
5	525	520	210	146

We recognize the generating series 1, 2, 3 and 4 of Table 4.2, thus $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_5$ are not Koszul. □

The sub-case 1.2 has been considered, since the sub-case 1.3 is equivalent by the reverse automorphism, we only need to consider the sub-case 1.4 to finish the study of the case 1.

Compatibility between the relations on the left and right orbits Let us study Sub-case 1.4. We need to study the compatibility between the relations on the left and right orbits. By the previous section, we have 5 relations involving only the right orbit :

- $\mathrm{RR}_1 = \{(ab)c = (ac)b\}$,
- $\mathrm{RR}_2 = \{(ab)c = (ba)c\}$,
- $\mathrm{RR}_3 = \{(ab)c = (cb)a\}$,
- $\mathrm{RR}_4 = \{(ab)c = (bc)a = (ca)b\}$,
- $\mathrm{RR}_5 = \{(ab)c = (cb)a = (ca)b = (ba)c = (ac)b = (bc)a\}$,

And their symmetric by the reverse automorphism:

- $\mathrm{LL}_1 = \{a(bc) = b(ac)\}$,
- $\mathrm{LL}_2 = \{a(bc) = a(cb)\}$,
- $\mathrm{LL}_3 = \{a(bc) = c(ba)\}$,
- $\mathrm{LL}_4 = \{a(bc) = b(ca) = c(ab)\}$,
- $\mathrm{LL}_5 = \{a(bc) = b(ca) = c(ab) = a(cb) = b(ac) = c(ba)\}$,

We have 25 possibilities to combine the relations RR_i and LL_j, however by the automorphism of the magmatic operad, we only need to study those with $i \leq j$, and thus we have 15 possibilities.

A naive guess would be that since RR_1 gives rise to the right non-associative permutative operad, and LL_1 gives rise to the left non-associative permutative operad, the operad with both relations would be Koszul. We will show that it is not the case.

Let $\mathcal{P}_{i;j} = \mathrm{Mag}/\langle \mathrm{RR}_i; \mathrm{LL}_j \rangle$.

Proposition 2.4.5.7. *The operads $\mathcal{P}_{1;1}$, $\mathcal{P}_{1;2}$, $\mathcal{P}_{1;3}$, $\mathcal{P}_{1;4}$ and $\mathcal{P}_{1;5}$ are not Koszul.*

Proof. Let us compute the first dimensions of the operads $\mathcal{P}_{1;1}$, $\mathcal{P}_{1;2}$, $\mathcal{P}_{1;3}$, $\mathcal{P}_{1;4}$ and $\mathcal{P}_{1;5}$ using the Haskell calculator [27]:

n	$\mathcal{P}_{1;1}(n)$	$\mathcal{P}_{1;2}(n)$	$\mathcal{P}_{1;3}(n)$	$\mathcal{P}_{1;4}(n)$	$\mathcal{P}_{1;5}(n)$
1	1	1	1	1	1
2	2	2	2	2	2
3	6	6	6	5	4
4	14	20	14	6	5
5	30	75	30	10	6
6	$*$	312	$*$	18	7
7	$*$	$*$	$*$	$*$	8
8	$*$	$*$	$*$	$*$	9

We recognize the generating series 7, 8, 7, 10 and 14 of Table 4.2 of the appendix, thus $\mathcal{P}_{1;1}$, $\mathcal{P}_{1;2}$, $\mathcal{P}_{1;3}$, $\mathcal{P}_{1;4}$ and $\mathcal{P}_{1;5}$ are not Koszul. □

Proposition 2.4.5.8. *The operads $\mathcal{P}_{2;3}$, $\mathcal{P}_{2;4}$, $\mathcal{P}_{2;5}$, $\mathcal{P}_{3;4}$, $\mathcal{P}_{3;5}$ and $\mathcal{P}_{4;4}$ are not Koszul.*

Proof. The proof is the same.
Let us compute the first dimensions of the operads $\mathcal{P}_{2;3}$, $\mathcal{P}_{2;4}$, $\mathcal{P}_{2;5}$, $\mathcal{P}_{3;4}$, $\mathcal{P}_{3;5}$ and $\mathcal{P}_{4;4}$ using the Haskell calculator [27]:

n	$\mathcal{P}_{2;3}(n)$	$\mathcal{P}_{2;4}(n)$	$\mathcal{P}_{2;5}(n)$	$\mathcal{P}_{3;4}(n)$	$\mathcal{P}_{3;5}(n)$	$\mathcal{P}_{4;4}(n)$
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	6	5	4	5	4	4
4	14	8	5	2	2	2
5	21	18	6	2	1	2
6	$*$	55	7	$*$	1	2
7	$*$	$*$	8	$*$	1	2
8	$*$	$*$	9	$*$	$*$	$*$

We recognize the generating series 9, 11, 14, 12, 15 and 13 of Table 4.2 of the appendix, thus $\mathcal{P}_{2;3}$, $\mathcal{P}_{2;4}$, $\mathcal{P}_{2;5}$, $\mathcal{P}_{3;4}$, $\mathcal{P}_{3;5}$ and $\mathcal{P}_{4;4}$ are not Koszul. □

Proposition 2.4.5.9. *The operad $\mathcal{P}_{4;5}$ is not Koszul.*

Proof. The idea of the proof is the same but more dimensions are needed, indeed the obstruction appears in dimension 15. Let us compute the first dimensions of the operad $\mathcal{P}_{4;5}$ using the Haskell calculator [27] which give a (non-quadratic) convergent ORS. We get:

$$(1, 2, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \dots)$$

This is the generating series 17 of Table 4.2 of the appendix, thus $\mathcal{P}_{4;5}$ is not Koszul. □

Proposition 2.4.5.10. *The operad $\mathcal{P}_{3;3}$ is not Koszul.*

Proof. The method consisting of finding negative coefficients in $\mathrm{rev}(-f_{\mathcal{P}}(-t))$ does not seem to work in this case. However, one can remark that this operad is self-dual (in the sens of Koszul duality). We recall that x is the generator without symmetries of this operad, let $y = x.(1\ 2)$. The relations of $\mathcal{P}_{3;3}$ are:

$$x \circ_1 x - y \circ_2 y \qquad ; \qquad y \circ_1 y - x \circ_2 x$$

And those relations are the same as the one of the operad $\mathcal{P}^!_{3;3}$ with $x \mapsto x_*$.
Let us compute the first dimensions of the operad $\mathcal{P}_{3;3}$ using the Haskell calculator [27]. We get $(1, 2, 6, 20, 60, 182, 546, ?, ?, \dots)$. Thus, we can compute:

$$f_{\mathcal{P}_{3;3}}(-f_{\mathcal{P}_{3;3}}(-t)) = t - \frac{7}{12}t^7 + \mathcal{O}(t^8)$$

Which show that $\mathcal{P}_{3;3}$ is not Koszul. □

Proposition 2.4.5.11. *The operad $\mathcal{P}_{5;5}$ is not Koszul.*

Proof. Once again the generating series method does not seem to work in this case. Let us polarize the relations of $\mathcal{P}_{5;5}$ by defining $[a, b] = ab - ba$ and $a.b = ab + ba$ and rewriting the relations of $\mathcal{P}_{5;5}$ using those:

- $(a.b).c = a.(b.c)$,
- $[a.b, c] = -[a, b.c]$,
- $[a, b].c = 0$,
- $[[a, b], c] = 0$.

From this presentation we can see that $\mathcal{P}_{5;5}$ is graded by $[\cdot, \cdot]$ and the dimensions can be easily computed. We get $(1, 1 + u, 1 + u, 1, 1, 1, \dots)$ where u is the generator of the grading. This case look very much like [11, Proposition 3.6] (however this is not the same operad). We have that:

$$f_{\mathcal{P}_{5;5}} = \exp(t) - 1 + \frac{u}{2}t^2 + \frac{u}{6}t^3$$

The power series criterion can be used on this series and the first negative term is

$$\frac{-1983044460002323872u^2}{20!}t^{20}$$

which is the exact same term as in [11, Proposition 3.6] since the power series are the same. Thus, $\mathcal{P}_{5;5}$ is not Koszul. □

Theorem 2.4.5.12. *The operad $\mathcal{P}_{2;2}$ is Koszul and self-dual.*

Proof. Let us polarize the relations of $\mathcal{P}_{2;2}$ by defining $[a,b] = ab - ba$ and $a.b = ab + ba$ and rewriting the relations of $\mathcal{P}_{2;2}$ using those:

- $[a,[b,c]] = 0$,
- $[a,b].c = 0$,

By Theorem 2.4.2.4, $\mathcal{P}_{2;2}$ is Koszul. Moreover, its generating series can be computed and is $2 - t - \sqrt{1-2t}$. Computation of its Koszul dual show that it is self-dual. □

Let us denote by $\mathrm{ANil}_2 \circ \mathrm{CMag}$ the operad $\mathcal{P}_{2;2}$. This notation comes from distributive laws of operads, that we did not define in this manuscript. It was first introduced in [5], we refer the reader to [60] for its application to operads. With this theorem, we have finished the study of Case 1. We will now study Case 2, starting with Sub-case 2.1.

Equivariant relations relating exactly one term of the left and one of right orbits As in Paragraph 2.4.5, knowing the equivalence class of $(ab)c$ is enough to determine the relation. Thus, we have 6 possibilities:

- $(ab)c \sim a(bc)$,
- $(ab)c \sim a(cb)$,
- $(ab)c \sim b(ac)$,
- $(ab)c \sim b(ca)$,
- $(ab)c \sim c(ab)$,
- $(ab)c \sim c(ba)$.

Moreover, none of those relations are equivalent. We get the 6 following operads:

- $\mathcal{P}_6$ with $(ab)c = a(bc)$,
- $\mathcal{P}_7$ with $(ab)c = a(cb)$,
- $\mathcal{P}_8$ with $(ab)c = b(ac)$,
- $\mathcal{P}_9$ with $(ab)c = b(ca)$,
- $\mathcal{P}_{10}$ with $(ab)c = c(ab)$,
- $\mathcal{P}_{11}$ with $(ab)c = c(ba)$.

We recognize that $\mathcal{P}_6$ is the associative operad, hence is Koszul.

Proposition 2.4.5.13. *The operads $\mathcal{P}_7$, $\mathcal{P}_8$ and $\mathcal{P}_9$ are not Koszul.*

Proof. Let us compute the first dimensions of the operads $\mathcal{P}_7$, $\mathcal{P}_8$ and $\mathcal{P}_9$ using the Haskell calculator [27]:

n	$\mathcal{P}_7(n)$	$\mathcal{P}_8(n)$	$\mathcal{P}_9(n)$
1	1	1	1
2	2	2	2
3	6	6	6
4	12	12	12
5	20	20	1

We recognize the generating series 5, 5 and 6 of Table 4.2 of the appendix, thus $\mathcal{P}_7$, $\mathcal{P}_8$ and $\mathcal{P}_9$ are not Koszul. ☐

Theorem 2.4.5.14. *The operad $\mathcal{P}_{10}$ is Koszul and is self-dual.*

Proof. Let us polarize the relations of $\mathcal{P}_{10}$ by defining $[a,b] = ab - ba$ and $a.b = ab + ba$ and rewriting the relations of $\mathcal{P}_{10}$ using those:

- $[a,[b,c]] = 0$,
- $[a,b.c] = 0$,

By Theorem 2.4.2.4, $\mathcal{P}_{10}$ is Koszul. Moreover, its generating series can be computed and is $1 - \sqrt{1-2t-t^2}$. Computation of its Koszul dual show that it is self-dual. ☐

We will denote by $\mathrm{CMag} \circ \mathrm{ANil}_2$ the operad $\mathcal{P}_{10}$.

Theorem 2.4.5.15. *The operad $\mathcal{P}_{11}$ is Koszul.*

Proof. Let us polarize the relations of $\mathcal{P}_{11}$ by defining $[a,b] = ab - ba$ and $a.b = ab + ba$ and rewriting the relations of $\mathcal{P}_{11}$ using those:

- $a.[b,c] = 0$,
- $[a,b.c] = 0$,

We get the connected sum of the magmatic operad over a symmetric generator and the magmatic operad over a skew-symmetric generator. By Theorem 2.4.2.4, it is Koszul. Moreover, its generating series can be computed and is $2 - t - \sqrt{1-2t}$. ☐

The operad $\mathcal{P}_{11}$ will be denoted by CMag#AMag since it is the connected sum of CMag and AMag. This concludes the study of Sub-case 2.1. The last sub-case to study is Sub-case 2.2.

Equivariant relations mixing several terms of the left and right orbits Let us study Sub-case 2.2. Because the relations are equivariant, the same number of elements of the left and right orbits must appear in each class of the equivalence relation, either 2, 3 or 6. Let us name the relations of the previous section:

- $\mathrm{RL}_6 = \{(ab)c = a(bc)\}$,
- $\mathrm{RL}_7 = \{(ab)c = a(cb)\}$,
- $\mathrm{RL}_8 = \{(ab)c = b(ac)\}$,
- $\mathrm{RL}_9 = \{(ab)c = b(ca)\}$,
- $\mathrm{RL}_{10} = \{(ab)c = c(ab)\}$,
- $\mathrm{RL}_{11} = \{(ab)c = c(ba)\}$.

Any relation of Sub-case 2.2 is a combination of a relation of Sub-case 2.1 and a relation of Sub-case 1.2.

Let define the operads $\mathcal{P}_{i;j} = \mathrm{Mag}/\langle \mathrm{RR}_i; \mathrm{RL}_j\rangle$. We get a total of 30 operads, however some of them are isomorphic.

Proposition 2.4.5.16. *The couples of operads* $(\mathcal{P}_{1;6}, \mathcal{P}_{1;7})$, $(\mathcal{P}_{1;8}, \mathcal{P}_{1;10})$, $(\mathcal{P}_{1;9}, \mathcal{P}_{1;11})$, $(\mathcal{P}_{2;6}, \mathcal{P}_{2;8})$, $(\mathcal{P}_{2;7}, \mathcal{P}_{2;9})$, $(\mathcal{P}_{2;10}, \mathcal{P}_{2;11})$, $(\mathcal{P}_{3;6}, \mathcal{P}_{3;11})$, $(\mathcal{P}_{3;7}, \mathcal{P}_{3;10})$ *and* $(\mathcal{P}_{3;8}, \mathcal{P}_{3;9})$ *are couples of isomorphic operads.*

Proof. Let us compute the class of $(ab)c$ in $\mathcal{P}_{1;6}$. We get: $(ab)c = (ac)b$ by definition of RR_1 and $(ab)c = a(bc)$ by definition of RL_6. Moreover, by equivariance, RL_6 implies $(ac)b = a(cb)$ and thus the class of $(ab)c$ is

$$\{(ab)c, (ac)b, a(bc), a(cb)\}$$

Which is the same as in $\mathcal{P}_{1;7}$.
The proof is the same for the other couples. □

With the exact same method, one can prove the following result:

Proposition 2.4.5.17. *The triples of operads* $(\mathcal{P}_{4;6}, \mathcal{P}_{4;9}, \mathcal{P}_{4;10})$ *and* $(\mathcal{P}_{4;7}, \mathcal{P}_{4;8}, \mathcal{P}_{4;11})$ *are triples of isomorphic operads.*

Theorem 2.4.5.18. *The operads* $\mathcal{P}_{5;6}$, $\mathcal{P}_{5;7}$, $\mathcal{P}_{5;8}$, $\mathcal{P}_{5;9}$, $\mathcal{P}_{5;10}$ *and* $\mathcal{P}_{5;11}$ *are all isomorphic to the Koszul dual of the Lie admissible operad and thus Koszul.*

Proof. It is clear in all six cases that the class of $(ab)c$ is the entire set $\mathrm{Mag}(3)$. Thus, we recognize the Koszul dual of the Lie admissible operad which is Koszul. □

Proposition 2.4.5.19. *The operads* $\mathcal{P}_{1;8}$, $\mathcal{P}_{1;9}$, $\mathcal{P}_{2;7}$, $\mathcal{P}_{3;6}$, $\mathcal{P}_{3;7}$ *and* $\mathcal{P}_{3;8}$ *are not Koszul.*

Proof. Although those operads are not proved to be isomorphic, they are all share the same dimensions which are $(1, 2, 3, 1, 1, 1, \dots)$ this can be computed using the Haskell calculator [27] which give a (non-quadratic) convergent ORS. This is the series 16 of Table 4.2 of the appendix and thus those operads are not Koszul. □

Proposition 2.4.5.20. *The operad* $\mathcal{P}_{4;6}$ *is not Koszul.*

Proof. Let us polarize the relations. We get:

- $[a, [b, c]] = 0$,
- $[a, b].c = a.[b, c]$,
- $[a, b.c] = 0$,
- $(a.b).c = a.(b.c)$

We recognize the operad of [11, Proposition 3.6] which is not Koszul. □

Proposition 2.4.5.21. *The operad* $\mathcal{P}_{4;7}$ *is not Koszul.*

Proof. Let us polarize the relations. We get:

- $[a, [b, c]] = -[[a, b], c]$,
- $[a, b].c = 0$,
- $[a, b.c] = 0$,
- $(a.b).c = a.(b.c)$.

From this presentation, we can remark that $\mathcal{P}_{4;7}$ is graded by $[\cdot,\cdot]$ and the dimensions can be easily computed. We get $(1, 1+u, 1+u^2, 1, 1, 1, \dots)$ where u is the generator of the grading. We have that:

$$f_{\mathcal{P}_{4;7}} = \exp(t) - 1 + \frac{u}{2}t^2 + \frac{u^2}{6}t^3$$

The power series criterion can be used on this series and the first negative term is $\frac{-35u^5}{6!}t^6$ which show that this operad is not Koszul. ☐

Theorem 2.4.5.22. *The operad $\mathcal{P}_{2;10}$ is Koszul.*

Proof. Let us polarize the relations. We get:

- $[[a,b],c] = 0$,
- $[a,b].c = 0$,
- $[a,b.c] = 0$.

One may recognize the connected sum of CMag and ANil_2. By Theorem 2.4.2.4, $\mathcal{P}_{2;10}$ is Koszul. Moreover, its generating series can be computed and is $1 - \sqrt{1-2t} + \frac{1}{2}t^2$. ☐

The operad $\mathcal{P}_{2;10}$ will be denoted by $\mathrm{CMag}\#\mathrm{ANil}_2$ since it is the connected sum of CMag and ANil_2. The two last operads to check are $\mathcal{P}_{1;6}$ and $\mathcal{P}_{2;6}$. One can remark that they are in fact isomorphic to the permutative operad and thus are Koszul.

Now that all the possible cases have been studied, we can state the main result:

Theorem 2.4.5.23. *Let $\mathcal{P}$ a KSetOp_1, then $\mathcal{P}$ is isomorphic to one of the 9 following operads:*

- Mag *the magmatic operad;*
- NAP *the non-associative permutative operad;*
- $\mathrm{CMag} \circ \mathrm{ANil}_2$ *which is build from* CMag *and* ANil_2 *with the relation* $[a.b,c] = 0$*;*
- $\mathrm{ANil}_2 \circ \mathrm{CMag}$ *which is build from* CMag *and* ANil_2 *with the relation* $[a,b].c = 0$*;*
- CMag#AMag *which is the connected sum of* CMag *and* AMag*;*
- Ass *the associative operad;*
- $\mathrm{CMag}\#\mathrm{ANil}_2$ *which is the connected sum of* CMag *and* ANil_2*;*
- Perm *the permutative operad;*
- $\mathrm{LieAdm}^!$ *the Koszul dual of the Lie admissible operad.*

Moreover, only Ass, $\mathrm{CMag} \circ \mathrm{ANil}_2$ *and* $\mathrm{ANil}_2 \circ \mathrm{CMag}$ *are self-dual. And only* NAP *and* Perm *do not inherit the reverse automorphism from* Mag.

Proof. By Propositions 2.4.5.3, We know that it is enough to study equivariant equivalence relations on the monomials of Mag(3) satisfying either Case 1 or Case 2. Moreover, Proposition 2.4.5.4 refine Case 1 into Sub-cases 1.1, 1.2, 1.3 and 1.4.

- Sub-case 1.1 is trivial and correspond to the magmatic operad Mag.
- Sub-case 1.2 is studied in Section 2.4.5. It gives rise to 5 operads, however only 1 is Koszul, the operad $\mathcal{P}_1$ which is isomorphic to NAP.
- Sub-case 1.3 is equivalent to Sub-case 1.2 by the reverse automorphism.

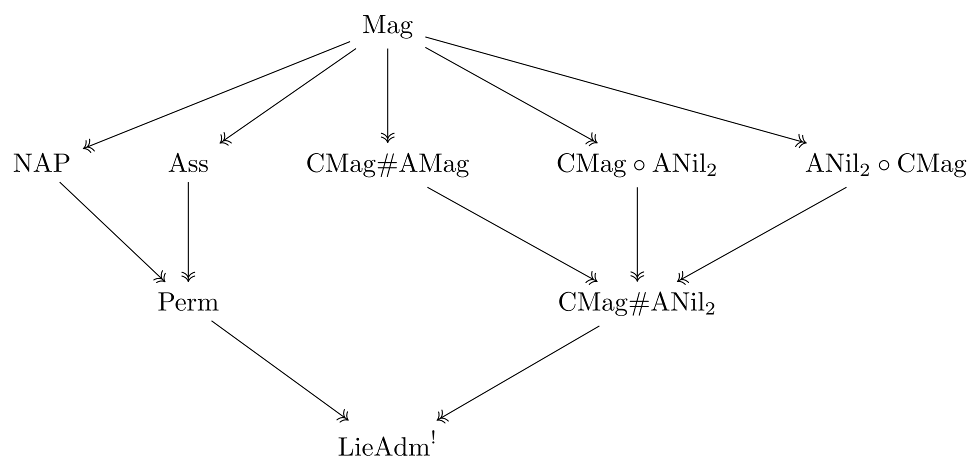
- Sub-case 1.4 is studied in Section 2.4.5. It gives rise to 25 operads, the reverse automorphism allows us to reduce this number to 15 operads. Among those 15 operads, only 1 is Koszul, the operad $\mathcal{P}_{2;2}$ that we denote $\mathrm{ANil}_2 \circ \mathrm{CMag}$.

Proposition 2.4.5.5 refine Case 2 into Sub-cases 2.1 and 2.2.

- Sub-case 2.1 is studied in Section 2.4.5. It gives rise to 6 operads, 3 of them are Koszul, the operads $\mathcal{P}_6$ (isomorphic to Ass), $\mathcal{P}_{10}$ and $\mathcal{P}_{11}$ that we respectively denote $\mathrm{CMag} \circ \mathrm{ANil}_2$ and CMag#AMag.
- Sub-case 2.2 is studied in Section 2.4.5. It gives rise to 30 operads, however some of them are isomorphic, we can reduce this number to 12 operads. Among those 12 operads, 3 of them are Koszul, the operads $\mathcal{P}_{1;6}$ (isomorphic to Perm), $\mathcal{P}_{5;6}$ (isomorphic to $\mathrm{LieAdm}^!$) and the new operad $\mathcal{P}_{2;10}$ that we denote $\mathrm{CMag\#ANil}_2$.

All the cases have been exhausted and we have found 9 Koszul operads. ☐

Moreover, we have the following poset of quotient of operads:



Corollary 2.4.5.24. *Let $\mathcal{P}$ a Koszul set operad over one generator of arity two, then $\mathcal{P}$ is isomorphic to one of the* 11 *following operads:*

- Mag *the magmatic operad and* $f_{\mathcal{P}}(x) = \frac{1}{2}(1 - \sqrt{1-4x})$;
- NAP *the non-associative permutative operad and* $f_{\mathcal{P}}$ *is the Euler's tree function defined by* $f_{\mathcal{P}}(x) = \sum_{n\in\mathbb{N}^*} \frac{n^{n-1}}{n!} x^n$;
- $\mathrm{CMag} \circ \mathrm{ANil}_2$ *which is build from* CMag *and* ANil_2 *with the relation* $[a.b, c] = 0$, *and* $f_{\mathcal{P}}(x) = 1 - \sqrt{1-2x-x^2}$;
- $\mathrm{ANil}_2 \circ \mathrm{CMag}$ *which is build from* CMag *and* ANil_2 *with the relation* $[a, b].c = 0$, *and* $f_{\mathcal{P}}(x) = 2 - x - 2\sqrt{1-2x}$;
- CMag#AMag *which is the connected sum of* CMag *and* AMag, *and* $f_{\mathcal{P}}(x) = 2 - x - 2\sqrt{1-2x}$;
- Ass *the associative operad and* $f_{\mathcal{P}}(x) = \frac{x}{1-x}$;
- $\mathrm{CMag\#ANil}_2$ *which is the connected sum of* CMag *and* ANil_2, *and* $f_{\mathcal{P}}(x) = 1 - \sqrt{1-2x} + \frac{1}{2}x^2$;
- Perm *the permutative operad and* $f_{\mathcal{P}}(x) = x\exp(x)$;
- $\mathrm{LieAdm}^!$ *the Koszul dual of the Lie admissible operad and* $f_{\mathcal{P}}(x) = \exp(x) - 1 + \frac{x^2}{2}$;

- CMag *the commutative magmatic operad and* $f_{\mathcal{P}}(x) = (1 - \sqrt{1-2x})$*;*
- Com *the commutative operad and* $f_{\mathcal{P}}(x) = \exp(x) - 1$*.*

All those operads satisfies Conjecture 2.4.4.3.

Corollary 2.4.5.25. *The Hilbert series of a Koszul symmetric set operad generated by one operation of arity two is differential algebraic of order* 1 *over* $\mathbb{Z}[x]$*.*

Chapter 3

Combinatorial interpretations of operads

We have introduced the species in the first chapter, and the operads in the second chapter as algebra in the category of species. In this chapter, we will focus on the combinatorial interpretation of some operads. Indeed, understanding the underlying species of an operad defined by generators and relations is a highly non-trivial, same as it is not trivial to find the exhaustive set of relations of an operad when it is not defined by generators and relations. We will study the species introduced in Section 1.4, and put operadic structures on them. We will then relate those newly defined operads to various construction of the operads PreLie and Lie, and ultimately we will use those results to prove a conjecture of Dotsenko on the operad FMan, see [23]. First we will recall the construction of Chapoton and Livernet of an operadic structure on the species of rooted trees, recall the amazing theorem stating that this operad is isomorphic to PreLie, see [19]. We will then use this combinatorial interpretation of PreLie to give a very precise description of TwPreLie the operadic twisting of PreLie, as it is done in [30].

In the second section, we display the results of the first article of the author [52]. We relate the combinatorial description of PreLie to the rooted Greg trees for two main reasons. First, because vector spaces with two pre-Lie product sharing the same Lie bracket do appear in geometry with the notion of Joyce structure, see [13]. This is not strange at all since any flat torsion-free connection gives a pre-Lie structure on the space of vector fields of a smooth manifold, see [14] for an overview of the appearance of pre-Lie algebras in geometry. Hence, a vector spaces with two pre-Lie product sharing the same Lie bracket appears naturally when considering the space of vector fields of a smooth manifold with two flat torsion-free connections. Then when computing the first dimension of the operad encoding this structure, we get the same numbers as when enumerating the rooted Greg trees. The second reason is because of the apparent link between the operadic twisting of PreLie and rooted Greg trees that we make explicit in Theorem 3.2.1.20. We conclude the section by the explicit computation of the generator of the coproduct of several copies of PreLie fibered over Lie, thus generalizing a conjecture of Chapoton [18] that was proven by Dotsenko [24].

In the third section, we display the generalization of the first two sections to hypertrees that was made in the second article of the author [53]. We relate hyperforests to the operad ComPreLie first define in [65] by Mansuy. Then using the insight that rooted Greg trees relates to the operadic twisting of PreLie, we relate the operadic twisting of ComPreLie to Greg hyperforests. We conclude the section by adapting the construction to reduced Greg hyperforests, since they also generalize both rooted Greg trees and hyperforests.

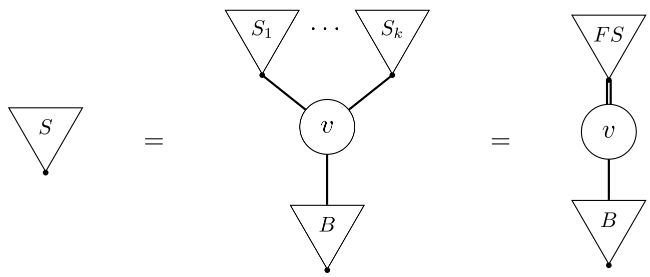
In the last section, we finally prove the main theorem of [53]. First we introduce the operad FMan defined by Hertling and Manin in [42], and state the conjecture of Dotsenko [23] that FMan admits an embedding in ComPreLie. We then use all the tools we have introduced so far to prove this conjecture.

3.1 Combinatorics of the PreLie operad

Let us relate the combinatorics of PreLie to rooted trees. To do so, we follow the construction of [19] of an operadic structure on the rooted trees species. We then use this construction to give a precise combinatorial description of the operadic twisting of PreLie.

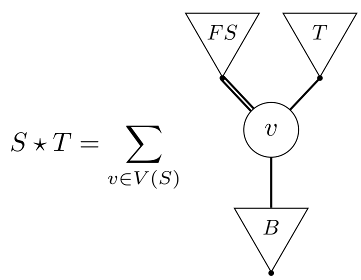
3.1.1 Rooted trees and insertions

First, let us define the insertion product in the species of rooted trees. For S a rooted tree, v a vertex of S, the forest $FS = \{S_1, \ldots, S_k\}$ the forest of the children of v and B the rooted tree below v. Let us introduce the following notation:

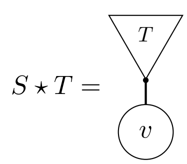


We use circles to represent vertices, triangles to represent trees or forests and double edges to represent that each tree of the forest FS is grafted to v.

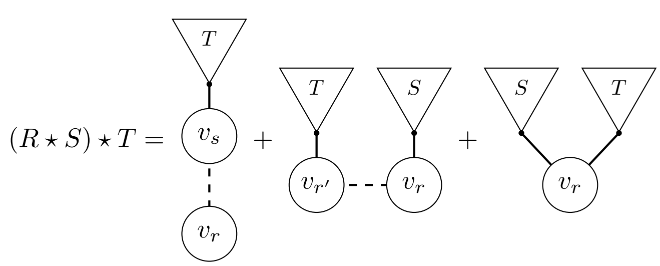
Definition 3.1.1.1. Let S and T be two rooted trees labeled over disjoint sets and $V(S)$ the set of vertices of S, let $S \star T$ be the *fall product* of T over S defined by:



For the sake of readability, let us omit the sums, the tree B and the forest FS:



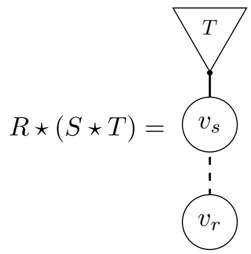
Example 3.1.1.2. Let compute $(R \star S) \star T$ to grasp the definition of the fall product. Let v_r and $v_{r'}$ be generic vertices of R and v_s a generic vertex of S.



Here S falls on the vertex v_r of R, then either T falls on a vertex of S, or T falls on another vertex $v_{r'}$ of R, or T falls on the vertex v_r of R.
We denote by a dotted line the path between two vertices which may contain several edges. The dotted line is vertical if one vertex is above the other. The dotted line is horizontal if the two vertices can be either one above the other, or not.

Proposition 3.1.1.3. *The fall product is pre-Lie.*

Proof. We have computed $(R \star S) \star T$, moreover we have:



Hence, $(R \star S) \star T - R \star (S \star T)$ is symmetric in S and T. Hence, the fall product is pre-Lie. □

Remark 3.1.1.4. The fall product allows us to graft a rooted tree T over another rooted tree S on all possible vertices of S. However, as we can see in the above computation, a naive composition of the fall product is not enough to make several trees fall on the same tree. Indeed, making several trees (two trees T and S for example) fall on the same tree R should not depend on the order in which we make them fall. However, when computing $(R \star S) \star T$, we have that S fall on R, but T can either fall on S or on R, hence this is not symmetric in S and T. The solution is to use the symmetric brace products.

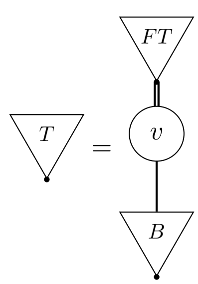
The symmetric brace products were first introduced by Lada and Markl [50] and the following formula to get the symmetric brace products from a pre-Lie product was given by Oudom and Guin [69]:

- $Br(S) = S$
- $Br(S; T) = S \star T$
- $Br(S; T_1, \ldots, T_{n+1}) = Br(S; T_1, \ldots, T_n) \star T_{n+1} - \sum_{i=1}^n Br(S; T_1, \ldots, T_i \star T_{n+1}, \ldots, T_n)$

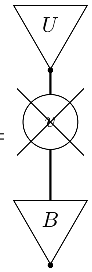
The symmetric brace product $Br(S; T_1, \ldots, T_n)$ is the sum of all possible ways to graft the trees $T_1, \ldots, T_n$ on vertices of S. It is symmetric in the T_i's. For $FT = \{T_1, \ldots, T_n\}$, we will write $Br(S; FT)$.

Let us recall that an operad structure on a species can be given by a collection of operations $\circ_i$ of arity 2, the partial compositions satisfying the sequential and parallel composition axioms. Let us define these operations on rooted trees using the symmetric brace products.

Definition 3.1.1.5. Let T and S be two rooted trees, let i be a label of a vertex of T and v this vertex.



Let us consider $U = Br(S; FT)$, then the partial composition $\circ_i$ is defined by:

$T \circ_i S =$ 

The children of v fall on S, then the result of this symmetric brace product is grafted on v, and finally the vertex v is removed. This is exactly the insertion of S in T at the vertex v, with all the children of v falling on S. Although we do not describe the construction the same way as [19], the two constructions are the same. Hence, those partial compositions satisfy the sequential and parallel composition axioms.

Remark 3.1.1.6. When defining partial compositions on a species P, we define

$$\circ_i : \mathcal{P}(A \sqcup \{i\}) \otimes \mathcal{P}(B) \to \mathcal{P}(A \sqcup B),$$

where A and B are disjoint sets. It is equivalent to the definition with

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n + m - 1)$$

which involves the compatibility with the actions of the symmetric groups and also involves some renumbering.

Let us recall the following result:

Theorem 3.1.1.7. *[19, Theorem 1.9] Let $\mathcal{RT}$ be the species of rooted trees. The operad $(\mathcal{RT}, \{\circ_i\})$ with the partial compositions defined as above is isomorphic to the operad* PreLie*. Moreover, the isomorphism is*  *$\mapsto x$ with x the generator of* PreLie*.*

3.1.2 Operadic twisting of PreLie

Let us now explicitly describe the operadic twisting of PreLie by the morphism $\varphi : \mathrm{Lie} \to \mathrm{PreLie}$ given by $\varphi(l) = x - x.(1\ 2)$, using the combinatorial description of PreLie. Let α be a formal Maurer-Cartan element, α is an arity 0, degree 1 operation symbol, let us denote it by a black vertex. Then, the underlying species of PreLie$\hat{\vee}\alpha$ is the species twisting rooted trees, see Definition 1.4.1.10. As example, the element

$$((x \circ_1 x - x \circ_2 x) \circ_2 \alpha) \circ_2 \alpha$$

would be represented by the rooted tree with the white vertex as the root having two black children. However, computations show that this element is equal to its opposite, hence is zero. This species contain the species of rooted Greg trees as a subspecies, however, it is infinite dimensional in each arity. The differential d_{MC} is defined by $d_{\mathrm{MC}}(\alpha) = -\frac{1}{2}\tilde{l}(\alpha, \alpha)$, hence, $d_{\mathrm{MC}}(\alpha) = -\frac{1}{2}(x(\alpha, \alpha) - x.(1\ 2)(\alpha, \alpha)) = -x(\alpha, \alpha)$ by the Koszul sign rule. The vector space of arity 1 degree 1 elements is spanned by $x \circ_1 \alpha$ and $x \circ_2 \alpha$, let us compute their differential:

$$d_{\mathrm{MC}}(x \circ_1 \alpha) = x \circ_1 d_{\mathrm{MC}}(\alpha) = -x \circ_1 ((x \circ_1 \alpha) \circ_2 \alpha)$$

$$d_{\mathrm{MC}}(x \circ_2 \alpha) = x \circ_2 d_{\mathrm{MC}}(\alpha) = -x \circ_2 ((x \circ_1 \alpha) \circ_2 \alpha)$$

Those computations can be represented using rooted trees, see Figure 3.1, however one need to be careful with the order in which the black vertices are "filled", indeed $(x \circ_1 \alpha) \circ_2 \alpha$ and $(x \circ_2 \alpha) \circ_1 \alpha$

Figure 3.1: Example of computation of d_{MC} on some trees.

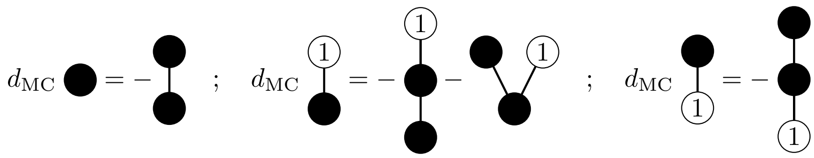
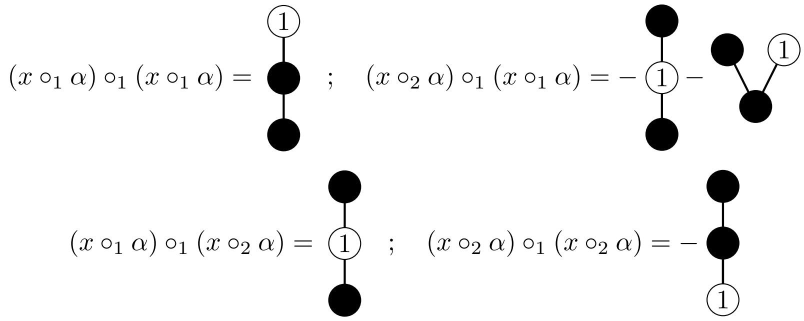


Figure 3.2: Composition of the arity 1 degree 1 elements.



are represented by the same rooted tree but have opposite signs, we use the convention bottom to top and left to right.

Let us find a Maurer-Cartan element. We need to compute $(x \circ_i \alpha) \circ_1 (x \circ_j \alpha)$ with $i, j \in \{1, 2\}$, see Figure 3.2. Since an operadic Maurer-Cartan element is a degree one arity one element, this shows that the unique operadic Maurer-Cartan element (up to multiplication by a scalar) is $\mu = (x \circ_1 \alpha) - (x \circ_2 \alpha)$. This allows to describe the differential d_{Tw} on the rooted trees:

Proposition 3.1.2.1. *[30, Subsection 6.7] Let T be a twisting rooted tree, then $d_{\mathrm{Tw}}(T)$ is the sum of:*

1. *All possible ways to split a white vertex of T into a white vertex retaining the label and a black vertex above it and to connect the incoming edges to one of the two vertices, up to an explicitly computable sign.*
2. *All possible ways to split a white vertex of T into a white vertex retaining the label and a black vertex below it and to connect the incoming edges to one of the two vertices, up to an explicitly computable sign.*
3. *All possible ways to split a black vertex of T into two black vertices and to connect the incoming edges to one of the two vertices, up to an explicitly computable sign.*
4. *All possible ways to graft an additional black leaf to T, up to an explicitly computable sign.*
5. *And the tree obtain by grafting T on top of a new black root, up to an explicitly computable sign.*

Moreover, many terms cancel due to the signs. In particular, if T has more than one vertex, all contributions from 4 and 5 get cancelled by contributions from 1, 2 and 3.

Remark 3.1.2.2. The signs in the previous proposition depend on the order in which the black vertices are "filled". In this description, we assume that the newly created black vertex is filled first. The signs created when changing the ordering can be computed using the Koszul sign rule. Assume that T has k black vertices and n white vertices. To explicitly compute $d_{\mathrm{Tw}}(T)$, we may notice that:

$$T = \gamma(U; \varepsilon_1, \ldots, \varepsilon_{n+k}),$$

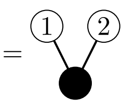
where exactly k of the ε_i are α, and the other n are the identity. One need to be careful with the order in which the ε_i are composed, indeed, this way of writing T is not unique, and the signs in the previous proposition depend on the order in which the ε_i are composed. However, once such a decomposition is chosen, one can compute $d_{\mathrm{Tw}}(T)$ using the formula:

$$d_{\mathrm{Tw}}(T) = d_{\mathrm{MC}}(T) + [\mu, T]$$

The Koszul sign appears during this computation.

A direct computation show that:

Proposition 3.1.2.3. *Let us denote $g = -d_{\mathrm{Tw}}(x)$ then we have that:*

$g =$ 

Moreover, x and g satisfies the relation:

$$(x \circ_1 g - (g \circ_1 x).(2\ 3) - g \circ_2 x) - (x \circ_1 g - (g \circ_1 x).(2\ 3) - g \circ_2 x).(2\ 3)$$

Theorem 3.1.2.4. *[29, Theorem 5.1] The embedding of differential graded operads* $(\mathrm{Lie}, 0) \to \mathrm{TwPreLie}$ *induces an isomorphism in the cohomology.*

3.2 Coproducts of the PreLie operad over the Lie operad

Smooth manifold with two flat torsion-free connections appear in the literature with the notion of Joyce structure, see [13]. The algebraic structure given by such two flat torsion-free connections is exactly the algebraic structure encoded by the operad $\mathrm{PreLie} \vee_{\mathrm{Lie}} \mathrm{PreLie}$ which is the coproduct of two copies of the PreLie fibered over Lie. Computations show that dimensions of the low arity components of $\bigvee^2_{\mathrm{Lie}} \mathrm{PreLie}$ coincide with the number of rooted Greg trees A005264 in [76]. This leads to the natural following questions:

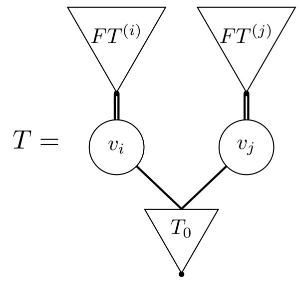
- Are $\bigvee^2_{\mathrm{Lie}} \mathrm{PreLie}$ and the species of rooted Greg trees *equinumerous*?
- Are they *isomorphic as species*?
- Are they *isomorphic with their extra algebraic structure*?

3.2.1 Rooted Greg trees and the Greg operad

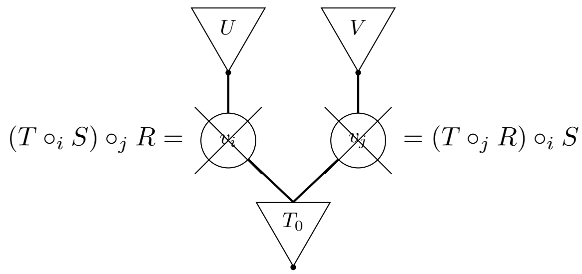
First let us put an operadic structure on the rooted Greg trees species. Let us naively generalize the construction described in the last section to the rooted Greg trees. Definition 3.1.1.1 of the fall product is the same where we allow rooted Greg trees to fall on either black or white vertices, it is straightforward to check that it is a pre-Lie product on $\mathcal{G}$. Definition 3.1.1.5 of partial compositions is also the same, however one can only compose in the white vertices. One may want to check that it satisfies the sequential and parallel composition axioms.

Proposition 3.2.1.1. *The partial compositions on $\mathcal{G}$ satisfy the sequential and parallel composition axioms.*

Proof. Parallel composition axiom: Let us compute $(T \circ_i S) \circ_j R$ in the case where i and j are the labels v_i and v_j which are white vertex of T. Let us write T the following way:

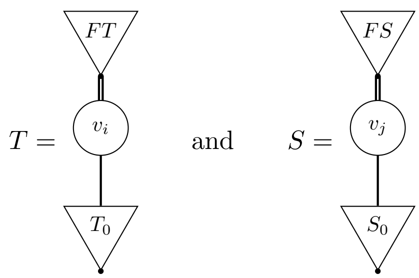


Applying the definition of the partial composition, we get:



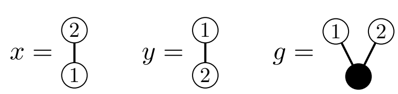
with $U = Br(S; FT^{(i)})$ and $V = Br(R; FT^{(j)})$. The parallel axiom is verified.

Sequential composition axiom: Let us compute $T \circ_i (S \circ_j R)$ in the case where i is the label of v_i a white vertex of T and j is the label of v_j a white vertex of S. Let us write:



With those notations for T and S, the computation of $T \circ_i S$ gives Subfigure 3.3a, with the additional notations $FS = (S_1, \dots, S_k)$, $FU = (U_1, \dots, U_k)$, $U_l = Br(S_l; FT_l)$ and $FT = \bigsqcup_{l=0}^{k+1} FT_l$ for every possible such decomposition of FT in (possibly empty) subforest. From this, the computation of $(T \circ_i S) \circ_j R$ gives Subfigure 3.3b, with the additional notation $V = Br(R; FU \cup FT_{k+1})$. Finally, the computation of $T \circ_i (S \circ_i jR)$ gives Subfigure 3.3c, with $W = Br(Br(R; FS); \widetilde{FT_1})$ and $U_0 = Br(S_0; FT_0)$, and $FT = FT_0 \sqcup \widetilde{FT_1}$ for every possible such decomposition of FT in (possibly empty) subforest. To conclude, one needs to remark that $V = W$. Indeed, in the definition of W, the forest FS falls on R, then $\widetilde{FT_1}$ falls on the result; and in the definition of V, some trees of $\widetilde{FT_1}$ fall on some trees of FS and the resulting forest falls on R. Those two operations give the same result. □

Definition 3.2.1.2. Let Greg be the operad $(\mathcal{G}, \{\circ_i\})$. Let us note



We know from the previous section that x satisfies the pre-Lie relation. Let us introduce the following notation:

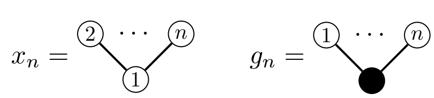
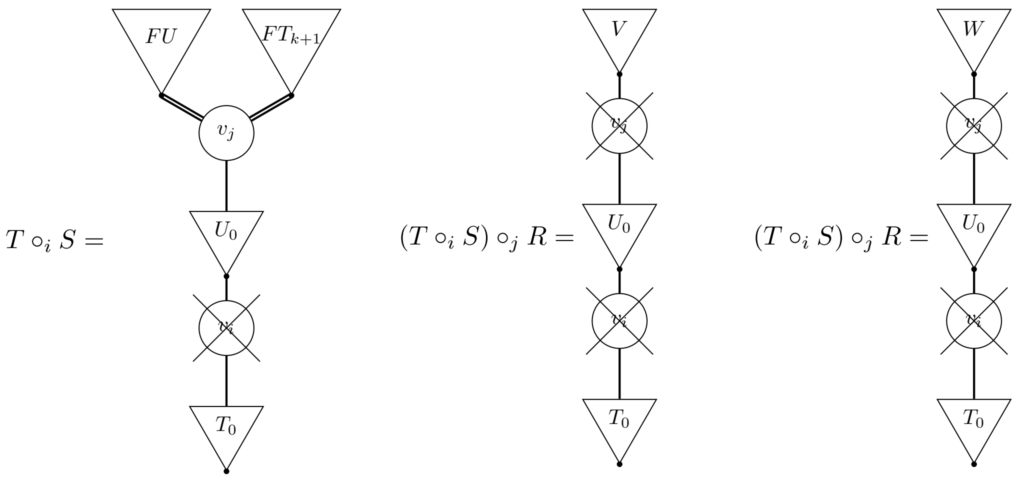
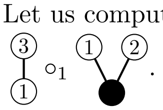


Figure 3.3: Computation of several composition of rooted Greg trees

(a) Computation of $T \circ_i S$. (b) Computation of $(T \circ_i S) \circ_j R$. (c) Computation of $T \circ_i (S \circ_j R)$.



Example 3.2.1.3. Let us compute $x \circ_1 g$. Because of the renumbering that we have been ignoring, we have to compute



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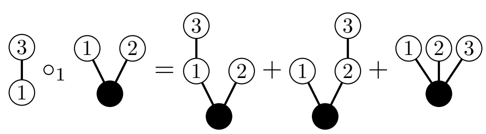
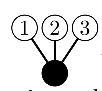


Hence, we have $x \circ_1 g - (g \circ_1 x).(2\ 3) - g \circ_2 x =$



. One may recognize that the left-hand side of the equation is the Leibniz rule. Moreover, since the right-hand side is symmetric, we have $x \circ_1 g - (g \circ_1 x).(2\ 3) - g \circ_2 x = (x \circ_1 g - (g \circ_1 x).(2\ 3) - g \circ_2 x).(2\ 3)$. Let us call this relation the Greg relation.

Remark 3.2.1.4. The element g_3 encodes the failure to verify the Leibniz's rule in the operad Greg, the same as the element x_3 encodes the failure to verify the associativity relation.

Proposition 3.2.1.5. *The operad* Greg *is generated in arity two.*

Proof. Let $\mathcal{P}(x, g)$ be the suboperad of Greg generated by x and g. We have to show that $\mathcal{P}(x, g) =$ Greg. Let us prove it by induction on the arity.

- *Base case:* By definition $\mathcal{P}(x, g)(2) = \mathrm{Greg}(2)$.
- *Induction step:* Let $n \geq 2$ and suppose that $\mathcal{P}(x, g)(k) = \mathrm{Greg}(k)$ for all $k \leq n$. We have to show that $\mathcal{P}(x, g)(n+1) = \mathrm{Greg}(n+1)$. Computing $x \circ_1 x_n$ and $x \circ_1 g_n$ shows that $x_{n+1} \in \mathcal{P}(x, g)(n+1)$ and $g_{n+1} \in \mathcal{P}(x, g)(n+1)$. Since we can obtain any rooted Greg trees by inductively composing corollas in the leaves of smaller trees, we have $\mathrm{Greg}(n + 1) = \mathcal{P}(x, g)(n + 1)$.

By induction, $\mathcal{P}(x, g) =$ Greg. □

We want to prove that Greg is quadratic to get a quadratic presentation. In order to do so, we will introduce a quadratic operad Greg′, show that Greg′ is Koszul and use the information given on its dimensions of components to show that Greg′ is isomorphic to Greg.

Definition 3.2.1.6. Let Greg' be the quotient of the free operad generated by $\tilde{x}$ and $\tilde{g}$ such that $\tilde{x}$ has no symmetry and $\tilde{g}.(1\ 2) = \tilde{g}$, by the following relations:

- $(\tilde{x} \circ_1 \tilde{x} - \tilde{x} \circ_2 \tilde{x}) - (\tilde{x} \circ_1 \tilde{x} - \tilde{x} \circ_2 \tilde{x}).(2\ 3)$
- $(\tilde{x} \circ_1 \tilde{g} - (\tilde{g} \circ_1 \tilde{x}).(2\ 3) - \tilde{g} \circ_2 \tilde{x}) - (\tilde{x} \circ_1 \tilde{g} - (\tilde{g} \circ_1 \tilde{x}).(2\ 3) - \tilde{g} \circ_2 \tilde{x}).(2\ 3)$

The first relation is the pre-Lie relation and the second one is the Greg relation. Since Greg is generated in arity two and those relations are satisfied in Greg, we have a surjective morphism of operads $\mathrm{Greg}' \twoheadrightarrow \mathrm{Greg}$.

Remark 3.2.1.7. From this definition and using the formalism of shuffle operads, it would already be possible to prove that Greg' is Koszul. However, the dimensions of components of its Koszul dual are much simpler, hence we will compute and work with the Koszul dual of Greg'.

Definition 3.2.1.8. Let $(\mathrm{Greg}')^!$ be the quotient of the free operad generated by x_* and g_* such that x_* has no symmetry and $g_*.(1\ 2) = -g_*$, by the following relations:

$$x_* \circ_1 x_* - x_* \circ_2 x_* \quad ; \quad x_* \circ_1 x_* - (x_* \circ_1 x_*).(2\ 3) \quad ; \quad x_* \circ_1 g_* - g_* \circ_2 x_*$$

$$x_* \circ_1 g_* + (x_* \circ_1 g_*).(1\ 2\ 3) + (x_* \circ_1 g_*).(1\ 3\ 2) \quad ; \quad x_* \circ_2 g_* \quad ; \quad g_* \circ_1 g_*$$

The two first relation are associativity and permutativity. The third one can be see as a Leibniz relation if we add the fifth one to it. The fourth one can be seen as some kind of Chasles relation (see the next definition). The two others are some nilpotency relations.

Proposition 3.2.1.9. *The operad* $(\mathrm{Greg}')^!$ *is the Koszul dual of* Greg'*.*

Proof. We have already detailed the explicit computation of the Koszul dual in Subsection 2.4.1. To check if it is indeed the dual, we need to show we get $\mathcal{R}^\perp$ using notation of Subsection 2.4.1. We can easily check that the relations we are giving are in $\mathcal{R}^\perp$. To show that we have the whole $\mathcal{R}^\perp$, we need to compute the dimension. We have:

$$\dim\left(\mathrm{Span}\left(\{\tilde{x}, \tilde{g}\}\right) \circ' \mathrm{Span}\left(\{\tilde{x}, \tilde{g}\}\right)\right) = 3 \times 3^2 = 27$$

The space of relation of Greg' is of dimension $3 + 2 = 5$, indeed the first relation is a pre-Lie relation and its orbit contain 3 elements, and the second relation contain 2 elements in its orbit and they are linearly independent. The space of relation of $(\mathrm{Greg}')^!$ is of dimension $(6+3)+(6+1+3)+3 = 22$. This can be computed by explicitly writing down the orbits of each relations. □

Let us give an example of $(\mathrm{Greg}')^!$-algebra that allows us to show that $\dim((\mathrm{Greg}')^!(4)) \geq 7$.

Definition 3.2.1.10. Let χ be a finite alphabet and $\mathbf{W}(\chi)$ be the linear span of finite words on χ with the following extra decorations: either one letter is pointed with a dot, or there is an arrow from one letter to another. Let $W(\chi)$ be the quotient of $\mathbf{W}(\chi)$ by the following relations, letters commute with each other (the dot its letter, and the arrow follow the two letters it links), reverting the arrow changes the sign and the Chasles relation holds:

$$\overset{\curvearrowright}{ab}cv = \overset{\curvearrowright}{cb}av + \overset{\curvearrowright}{ac}bv$$

for any $a, b, c \in \chi$ and v a finite word. Because the letters commute, we can write the elements of $W(\chi)$ with the pointed letter (or arrowed letters) at the start. Let the $\textcircled{x}$ and $\textcircled{g}$ products on $W(\chi)$ defined by:

- $\dot{a}v \,\textcircled{x}\, \dot{b}w = \dot{a}vbw$
- $\overset{\curvearrowright}{ab}v \,\textcircled{x}\, \dot{c}w = \overset{\curvearrowright}{ab}vcw$

- $\dot{a}v\textcircled{g}\dot{b}w = \overset{\frown}{ab}vw$

All other cases give 0.

Proposition 3.2.1.11. *The algebra* $(W(\chi), \textcircled{x}, \textcircled{g})$ *is a* $(\mathrm{Greg}')^!$*-algebra generated by* χ*.*

Proof. Indeed, $\dot{a}v = \dot{a}\textcircled{x}w$ with w the word v with a dot on a letter (let us say the first one for example) and

$$\overset{\frown}{ab}v = (\dot{a}\textcircled{g}\dot{b})\textcircled{x}w$$

so $(W(\chi), \textcircled{x}, \textcircled{g})$ is generated by χ under the operations $\textcircled{x}$ and $\textcircled{g}$. The product $\textcircled{g}$ is skew-symmetric, indeed

$$\dot{a}\textcircled{g}\dot{b} = \overset{\frown}{ab} = \overset{\frown}{ba} = -\overset{\frown}{ba} = -\dot{b}\textcircled{g}\dot{a}$$

The 6 relations of $(\mathrm{Greg}')^!$ are easily checked. □

Proposition 3.2.1.12. *We have* $\dim((\mathrm{Greg}')^!(4)) \geq 7$*.*

Proof. Let $A = W(\{a, b, c, d\})$. Let us compute the dimension of $\mathrm{Mult}(A(4))$ the multilinear part of $A(4)$. Let us write words of $\mathrm{Mult}(A(4))$ such that the arrow always starts from a and go to the second letter, and aside from that, the letters are in the lexicographic order. The rewriting rule $\overset{\frown}{\beta\gamma}\alpha \mapsto \overset{\frown}{\alpha\gamma}\beta - \overset{\frown}{\alpha\beta}\gamma$ is confluent. Indeed:

$$\overset{\frown}{cd}ab = \overset{\frown}{bd}ac - \overset{\frown}{bc}ad = (\overset{\frown}{ad}bc - \overset{\frown}{ab}cd) - (\overset{\frown}{ac}bd - \overset{\frown}{ab}cd) = \overset{\frown}{ad}bc - \overset{\frown}{ac}bd$$

Hence, $\mathrm{Mult}(A(4))$ is spanned by the words $\dot{a}bcd$, $a\dot{b}cd$, $ab\dot{c}d$, $abc\dot{d}$, $\overset{\frown}{a\dot{b}}cd$, $\overset{\frown}{ac}bd$ and $\overset{\frown}{ad}bc$. Since A is a $(\mathrm{Greg}')^!$-algebra on 4 generators, $\dim((\mathrm{Greg}')^!(4)) \geq \dim(\mathrm{Mult}(A(4))) = 7$. □

Remark 3.2.1.13. The algebra $(W(\chi), \textcircled{x}, \textcircled{g})$ is in fact the free $(\mathrm{Greg}')^!$-algebra generated by χ.

Let us use the formalism of shuffle operads to write down a convergent ORS of $(\mathrm{Greg}')^!$. Writing the relations of $(\mathrm{Greg}')^!$ using shuffle trees is a good exercise to familiarize ourselves with shuffle trees, and to be careful not to confuse them with the species of rooted trees or rooted Greg trees. Since the actions of the symmetric groups are disposed of when working with shuffle operads, let us note $y_* = x_*.(1\ 2)$. The result of the computation is displayed in Figure 4.6 of the appendix.

From those computations, the only missing ingredient to get a terminating ORS is a monomial order. We will consider the three following orders to get terminating rewriting systems. We refer to Definitions 2.2.4.35 and 2.2.4.39, and Examples 2.2.4.41 and 2.2.4.42 for the definitions of the orders. One should understand the juxtaposition of the orders as their concatenation see Definition 2.2.4.44.

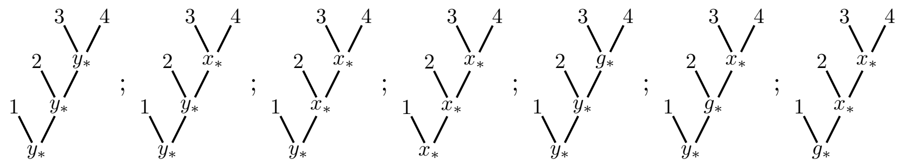
- The ORS associated to the graded path lexicographic permutation order with $x_* > y_* > g_*$ gives the rewriting system displayed in Figure 4.7 of the appendix.
- The ORS associated to the permutation reverse graded path lexicographic order with $g_* > y_* > x_*$ gives the rewriting system displayed in Figure 4.8 of the appendix.
- The ORS associated to the reverse graded path lexicographic permutation order with $x_* > y_* > g_*$ gives the rewriting system displayed in Figure 4.9 of the appendix.

Proposition 3.2.1.14. *The ORS displayed in Figures 4.7, 4.7 and 4.7 of the appendix are convergent.*

Proof. To prove this fact one has to choose either checking the confluence of the critical monomials (more than 250 cases to check for the rewriting system displayed in Figure 4.7 of the appendix), or noticing that there are 7 normal forms in arity 4 for each rewriting system and because $\dim((\mathrm{Greg}')^!(4)) \geq 7$, no new relation can appear. Since we have a monomial order and the rewriting rules are quadratic in an operad generated in arity two, checking arity 4 is enough. □

Proposition 3.2.1.15. *We have that* $\dim((\mathrm{Greg}')^!(n)) = 2n-1$ *for all* $n \geq 1$*, hence its exponential generating series is* $(2t-1)\exp(t)+1$.

Proof. It suffices to count the normal forms of a rewriting system for example the one displayed in Figure 4.7 of the appendix. Let $n \geq 2$ and count the number of normal forms in arity n. Those are right combs with at most one g_*, with all the x_* above the g_* and the y_*, and all the y_* below the g_* and the x_*. Let us depict them in arity 4:



Hence, the normal forms are determined by the number of occurrences x_* and g_*, and have either zero or one occurrence g_*. If there is no g_*, then one can have from 0 to $n-1$ occurrences of x_*. If there is one g_*, then one can have from 0 to $n-2$ occurrences of x_*. Hence, the number of normal forms in arity n is $2n-1$. □

Theorem 3.2.1.16. *The operad* Greg′ *is Koszul.*

Proof. We have a quadratic convergent ORS for $(\mathrm{Greg}')^!$, hence $(\mathrm{Greg}')^!$ is Koszul, hence Greg′ is Koszul. □

Theorem 3.2.1.17. *The exponential generating series of the operad* Greg′ *is the inverse under composition of* $(2t+1)\exp(-t)-1$. *Hence,* Greg′ *is isomorphic to* Greg.

Proof. We know that a Koszul operad $\mathcal{P}$ satisfies $f_{\mathcal{P}}(f_{\mathcal{P}^!}(-t)) = -t$. Hence, the exponential generating series of the operad Greg′ is the inverse under composition of $(2t+1)\exp(-t)-1$. Since we have a surjective morphism from Greg′ to Greg, and they have the same exponential generating series, the morphism is an isomorphism. □

Corollary 3.2.1.18. *The operad* Greg *is generated in arity two and Koszul.*

Definition 3.2.1.19. Let us define the differential graded operad Greg_{-1} as the operad Greg such that g is of degree 1 and with the differential d such that $d(x) = d(y) = g$. We may notice that:

$$\begin{aligned}
&d((x \circ_1 x - x \circ_2 x) - (x \circ_1 x - x \circ_2 x).(2\,3)) \\
&\quad = (g \circ_1 x - g \circ_2 x) - (g \circ_1 x - g \circ_2 x).(2\,3) + (x \circ_1 g - x \circ_2 g) - (x \circ_1 g - x \circ_2 g).(2\,3) \\
&\qquad = (g \circ_1 x - g \circ_2 x) - (g \circ_1 x - g \circ_2 x).(2\,3) + (x \circ_1 g) - (x \circ_1 g).(2\,3) \\
&\qquad\quad = (x \circ_1 g - (g \circ_1 x).(2\,3) - g \circ_2 x) - (x \circ_1 g - (g \circ_1 x).(2\,3) - g \circ_2 x).(2\,3)
\end{aligned}$$

In particular, we have that d is indeed well defined on Greg_{-1}.

Proposition 3.1.2.3 states that we have a morphism of differential graded operads $\mathrm{Greg}_{-1} \to$ TwPreLie, moreover since rooted Greg trees have no non-trivial tree automorphisms, this morphism is injective. Hence, using Theorem 3.1.2.4, we get the following theorem.

Theorem 3.2.1.20. *The embedding of differential graded operads* $(\mathrm{Lie}, 0) \to \mathrm{Greg}_{-1}$ *induces an isomorphism in the cohomology.*

Proof. From Proposition 3.1.2.1, we have that the differential of TwPreLie splits on rooted Greg trees and rooted non-Greg trees. We have $\mathrm{TwPreLie} = \mathrm{Greg}_{-1} \oplus \mathcal{NG}$ with some differential on $\mathcal{NG}$ hence, $H_*(\mathrm{TwPreLie}) = H_*\left(\mathrm{Greg}_{-1}\right) \oplus H_*(\mathcal{NG})$. Since the cohomology of TwPreLie is generated by the image of l the generator of Lie and that it is in Greg_{-1}, we have that $\mathcal{NG}$ is acyclic and that the cohomology of Greg_{-1} is Lie. □

This theorem give us a nice interpretation of the fact that $f_{\mathcal{G}}(x,-1) = -\ln(1-x)$, indeed the exponential generating series of Lie is $-\ln(1-x)$ which is the cohomology of Greg_{-1}.

3.2.2 Deformations of Greg parametrized by coalgebras

We have studied the operad Greg, however we have not yet relate this operad to $\bigvee^2_{\mathrm{Lie}}\mathrm{PreLie}$ which is another notation for $\mathrm{PreLie}\vee_{\mathrm{Lie}}\mathrm{PreLie}$. In fact, we shall now establish a much more general result about the operad $\bigvee^{n+1}_{\mathrm{Lie}}\mathrm{PreLie}$.

Proposition 3.2.2.1. *The operad $\bigvee^{n+1}_{\mathrm{Lie}}\mathrm{PreLie}$ is isomorphic to the operad $\mathcal{T}[x,c_1,\dots,c_n]/\langle\mathcal{R}\rangle$, with x without symmetries, $c_k.(1\ 2)=c_k$ and $\mathcal{R}$ the relations:*

$$(x\circ_1 x - x\circ_2 x)-(x\circ_1 x - x\circ_2 x).(2\ 3) \qquad \text{(pre-Lie)}$$

$$(x\circ_1 c_k-(c_k\circ_1 x).(2\ 3)-c_k\circ_2 x)-(x\circ_1 c_k-(c_k\circ_1 x).(2\ 3)-c_k\circ_2 x).(2\ 3) + \sum_{i,j\mid\max(i,j)=k}(c_i\circ_1 c_j-(c_i\circ_1 c_j).(2\ 3)) \qquad \text{(diff pre-Lie)}$$

Proof. We already know the following presentation: $\bigvee^{n+1}_{\mathrm{Lie}}\mathrm{PreLie}\simeq\mathcal{T}[x_1,\dots,x_{n+1}]/\langle\mathcal{R}'\rangle$ with x_k without symmetries and $\mathcal{R}'$ the relations:

$$(x_k\circ_1 x_k - x_k\circ_2 x_k)-(x_k\circ_1 x_k - x_k\circ_2 x_k).(2\ 3) \qquad \text{(pre-Lie k)}$$

$$(x_k-x_{k+1})-(x_k-x_{k+1}).(1\ 2) \qquad \text{(share)}$$

Let $c_k=x_{k+1}-x_k$, then $c_k=c_k.(1\ 2)$ is equivalent to Relation (share). Let $x=x_1$. We have that $x_{k+1}=x+\sum_{i=1}^k c_i$. Hence, $\mathcal{R}'$ is equivalent to:

$$(x\circ_1 x - x\circ_2 x)-(x\circ_1 x - x\circ_2 x).(2\ 3)+ \sum_{i=1}^{k}((x\circ_1 c_i - x\circ_2 c_i)-(x\circ_1 c_i - x\circ_2 c_i).(2\ 3)+(c_i\circ_1 x-c_i\circ_2 x)-(c_i\circ_1 x - c_i\circ_2 x).(2\ 3))+ \sum_{i=1}^{k}\sum_{j=1}^{k}((c_i\circ_1 c_j-c_i\circ_2 c_j)-(c_i\circ_1 c_j-c_i\circ_2 c_j).(2\ 3)),$$

which is equal to:

$$(x\circ_1 x - x\circ_2 x)-(x\circ_1 x - x\circ_2 x).(2\ 3)+ \sum_{i=1}^{k}(x\circ_1 c_i-(x\circ_1 c_i).(2\ 3)+(c_i\circ_1 x-c_i\circ_2 x)-(c_i\circ_1 x-c_i\circ_2 x).(2\ 3))+ \sum_{i=1}^{k}\sum_{j=1}^{k}(c_i\circ_1 c_j-(c_i\circ_1 c_j).(2\ 3))$$

Finally, if we subtract consecutive relations; we obtain

$$x\circ_1 c_k-(x\circ_1 c_k).(2\ 3)+(c_i\circ_1 x-c_k\circ_2 x)-(c_k\circ_1 x-c_k\circ_2 x).(2\ 3)+ \sum_{i,j\mid\max(i,j)=k}(c_i\circ_1 c_j-(c_i\circ_1 c_j).(2\ 3)),$$

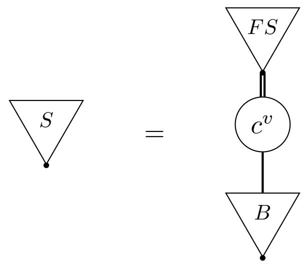
which is the intended relation. □

This quadratic presentation very much look like the presentation of the operad Greg. The operad $\bigvee^2_{\mathrm{Lie}}\mathrm{PreLie}$ is not isomorphic to Greg, however one may wonder if the operad $\bigvee^2_{\mathrm{Lie}}\mathrm{PreLie}$ is a deformation of Greg. We shall now show that it is indeed the case.

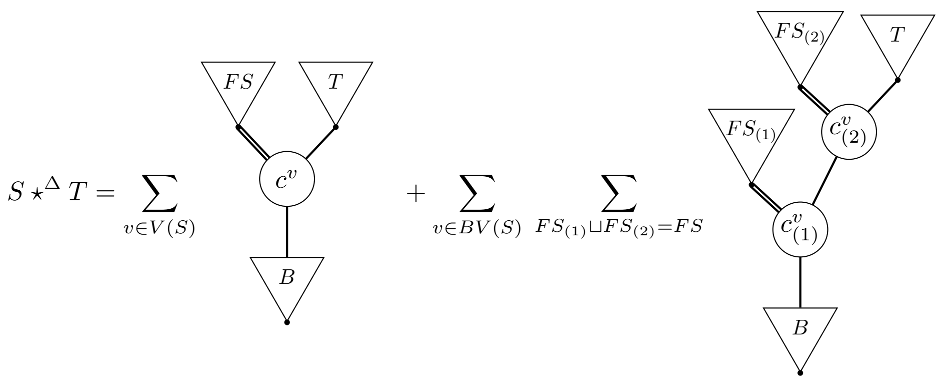
Deformation via a coalgebra Let $C = (V, \Delta)$ with V be a vector space of finite dimension n and Δ a co-associative co-commutative co-multiplication on V.

Definition 3.2.2.2. The vector space of *rooted Greg trees over* V is $\mathcal{G}_k^V(m) = \bigotimes_{\tau \in G_k(m)} V^{\otimes BV(\tau)}$. It has a basis of rooted Greg trees whose black vertices are labeled by a basis of V. Let $\mathcal{G}^V(m) = \bigoplus_k \mathcal{G}_k^V(m)$ and $\mathcal{G}^V = \bigoplus_m \mathcal{G}^V(m)$.

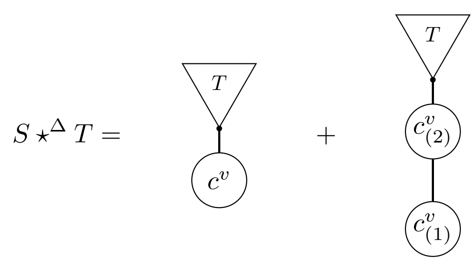
Definition 3.2.2.3. Let us define the deformed fall product $\star^\Delta$ on $\mathcal{G}^V$. Let S and T be two rooted Greg trees over V. For v a vertex of S, the forest $FS = \{S_1, \dots, S_k\}$ the forest of the children of v, B the rooted tree below v and c^v the label of v. Let us write:



For v a black vertex and c^v its label, let us write $c^v_{(1)} \otimes c^v_{(2)} = \Delta(c^v)$ using the Sweedler notation. Let us define the product $\star^\Delta$ by:



For the sake of readability, let us write:



Proposition 3.2.2.4. *The deformed fall product* $\star^\Delta$ *is pre-Lie.*

Proof. The proof is the tedious computation of $(R \star^\Delta S) \star^\Delta T - R \star^\Delta (S \star^\Delta T)$. The computation of $(R \star^\Delta S) \star^\Delta T$ is written down in Figure 4.10; r, r' and s are labels of vertices of R and S respectively. The boxed terms are the terms of $R \star^\Delta (S \star^\Delta T)$. Using the co-associativity and co-commutativity of Δ, we get that $(R \star^\Delta S) \star^\Delta T - R \star^\Delta (S \star^\Delta T)$ is symmetric in S and T. Hence, the deformed fall product $\star^\Delta$ is pre-Lie. □

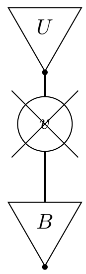
Remark 3.2.2.5. One may notice that co-commutativity is stronger than the needed condition, indeed the weaker needed condition is that $r_{(1)} \otimes r_{(2)} \otimes r_{(3)} = r_{(1)} \otimes r_{(3)} \otimes r_{(2)}$ using Sweedler notation, which is known as the co-permutativity property. Moreover, one may notice that if C admits a co-unity, then the co-permutativity property is equivalent to the co-commutativity property.

The symmetric brace product Br^{Δ} associated to $\star^{\Delta}$ is also defined by the following formula:

$$Br^{\Delta}(S; T_1, \dots, T_{n+1}) = Br^{\Delta}(S; T_1, \dots, T_n) \star^{\Delta} T_{n+1} - \sum_{i=1}^{n} Br^{\Delta}(S; T_1, \dots, T_i \star^{\Delta} T_{n+1}, \dots, T_n)$$

Same as Br, Br^{Δ} is symmetric in the T_i's.

Definition 3.2.2.6. The partial compositions $\circ_i^{\Delta}$ are defined the same way as in Definition 3.1.1.5 by:

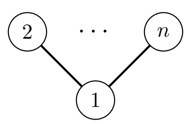
$T \circ_i^{\Delta} S =$ 

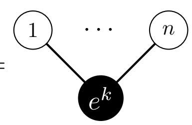
with $U = Br^{\Delta}(S; FT)$.

Proposition 3.2.2.7. *The partial compositions $\circ_i^{\Delta}$ satisfy the sequential composition and parallel composition axioms.*

The proof is the same as the one of Proposition 3.2.1.1.

Definition 3.2.2.8. Let us denote $C = (V, \Delta)$, and let Greg^C be the operad $(\mathcal{G}^V, \{\circ_i^{\Delta}\})$. Let $(e^1, \dots, e^n)$ be a basis of V and:

$x_n =$ 

$g_n^k =$ 

Let $x = x_2$ and $g^k = g_2^k$.

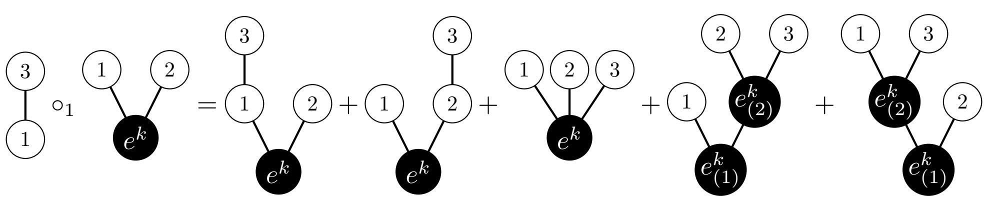
Proposition 3.2.2.9. *The operad* Greg^C *is generated in arity two and satisfy the following relations:*

$$(x \circ_1 x - x \circ_2 x) - (x \circ_1 x - x \circ_2 x).(2\ 3) \qquad \text{(pre-Lie)}$$

$$(x \circ_1 g^k - (g^k \circ_1 x).(2\ 3) - g^k \circ_2 x) - (x \circ_1 g^k - (g^k \circ_1 x).(2\ 3) - g^k \circ_2 x).(2\ 3) + (g^k_{(1)} \circ_1 g^k_{(2)} - (g^k_{(1)} \circ_1 g^k_{(2)}).(2\ 3)) \qquad \text{(greg } \Delta)$$

With $g^k_{(1)}$ and $g^k_{(2)}$ defined by Δ under the identification of V with the linear span of the generators g^k.

Proof. Let us compute $x \circ_1 g^k$:



This shows that the operad Greg^C satisfies Relation greg Δ.
Let $\mathcal{P}(x, g^1, \dots, g^n)$ be the suboperad of Greg^C generated by x and $g^1, \dots, g^n$. We have to show that $\mathcal{P}(x, g^1, \dots, g^n) = \mathrm{Greg}^C$. Let us prove it by induction on the arity.

- *Base case:* By definition $\mathcal{P}(x, g^1, \dots, g^n)(2) = \mathrm{Greg}^C(2)$.
- *Induction step:* Let $m \geq 2$ and suppose that $\mathcal{P}(x, g^1, \dots, g^n)(k) = \mathrm{Greg}^C(k)$ for all $k \leq m$. We have to show that $\mathcal{P}(x, g^1, \dots, g^n)(m+1) = \mathrm{Greg}^C(m+1)$. Computing $x \circ_1 x_m$ and $x \circ_1 g^k_m$ shows that $x_{m+1} \in \mathcal{P}(x, g^1, \dots, g^n)(m+1)$ and $g^k_{m+1} \in \mathcal{P}(x, g^1, \dots, g^n)(m+1)$. Since we can obtain any rooted Greg trees by inductively composing corollas in the leaves of smaller trees, we have $\mathrm{Greg}^C(m+1) = \mathcal{P}(x, g^1, \dots, g^n)(m+1)$.

By induction, $\mathcal{P}(x, g^1, \dots, g^n) = \mathrm{Greg}^C$. □

Let us use the same strategy as in the previous sections to prove that the operad Greg^C is Koszul.

Definition 3.2.2.10. Let $\mathcal{G}^C$ the operad defined by generators and relations as follows: $\tilde{x}$ is a generator without symmetries, $\tilde{g}^k$ are symmetric generators such that:

$$(\tilde{x} \circ_1 \tilde{x} - \tilde{x} \circ_2 \tilde{x}) - (\tilde{x} \circ_1 \tilde{x} - \tilde{x} \circ_2 \tilde{x}).(2\ 3)$$

$$(\tilde{x} \circ_1 \tilde{g}^k - (\tilde{g}^k \circ_1 \tilde{x}).(2\ 3) - \tilde{g}^k \circ_2 \tilde{x}) - (\tilde{x} \circ_1 \tilde{g}^k - (\tilde{g}^k \circ_1 \tilde{x}).(2\ 3) - \tilde{g}^k \circ_2 \tilde{x}).(2\ 3) \\ + (\tilde{g}^k_{(1)} \circ_1 \tilde{g}^k_{(2)} - (\tilde{g}^k_{(1)} \circ_1 \tilde{g}^k_{(2)}).(2\ 3))$$

With $\tilde{g}^k_{(1)}$ and $\tilde{g}^k_{(2)}$ defined by Δ under the identification of V with the linear span of the generators $\tilde{g}^k$.

Let $C^* = (V^*, \mu)$ the linear dual of C. This is a commutative algebra of dimension n.

Definition 3.2.2.11. Let $(\mathcal{G}^C)^!$ the operad defined by generators and relations as follows: x_* is a generator without symmetries, g^k_* are skew-symmetric generators such that:

$$x_* \circ_1 x_* - x_* \circ_2 x_* \quad ; \quad x_* \circ_1 x_* - (x_* \circ_1 x_*).(2\ 3) \quad ; \quad x_* \circ_1 g^k_* - g^k_* \circ_2 x_*$$

$$x_* \circ_1 g^k_* + (x_* \circ_1 g^k_*).(1\ 2\ 3) + (x_* \circ_1 g^k_*).(1\ 3\ 2) \quad ; \quad x_* \circ_2 g^k_* \quad ; \quad g^i_* \circ_1 g^j_* - x \circ_1 g^{i.j}_*$$

With the notation $g^{i.j}_* = \mu(g^i_*, g^j_*)$, one should be careful since $i.j$ in not the product of i and j; this is just a way to keep notation more compact.

Let us generalize the construction of $W(\chi)$ of Definition 3.2.1.10 to get an example of $(\mathcal{G}^C)^!$-algebra that allows us to show that $\dim((\mathcal{G}^C)^!(4)) \geq 4 + 3n$.

Definition 3.2.2.12. Let χ be a finite alphabet and $\mathbf{W}_C(\chi)$ be the linear span of finite words on χ with the following extra decorations: either one letter is pointed with a dot or there is an arrow from one letter to another, the arrow is linearly labeled by V^*.

Let us write $\overset{i}{\curvearrowright}$ instead of $\overset{e^*_i}{\curvearrowright}$ and $\overset{i.j}{\curvearrowright}$ instead of $\overset{\mu(e^*_i, e^*_j)}{\curvearrowright}$.

Let $W_C(\chi)$ be the quotient of $\mathbf{W}_C(\chi)$ by the following relations, letters commute with each other (the dot follows its letter, and the arrow follow the two letter it links), reverting the arrow changes the sign and the Chasles relation holds:

$$\overset{\overset{i}{\curvearrowright}}{ab}cv = \overset{\overset{i}{\curvearrowright}}{cb}av + \overset{\overset{i}{\curvearrowright}}{ac}bv$$

for any $a, b, c \in \chi$ and v a finite word. Because the letters commute, we can write the elements of $W(\chi)$ with the pointed letter (or arrowed letters) at the start. Let the $\textcircled{x}$ and $\textcircled{g}_i$ products on $W(\chi)$ defined by:

- $\dot{a}v\textcircled{x}\dot{b}w = \dot{a}vbw$
- $\overset{\overset{i}{\curvearrowright}}{ab}v\textcircled{x}\dot{c}w = \overset{\overset{i}{\curvearrowright}}{ab}vcw$

- $\dot{a}v\textcircled{g}_i\dot{b}w = \overset{\overset{i}{\curvearrowright}}{ab}vw$
- $\overset{\overset{i}{\curvearrowright}}{ab}v\textcircled{g}_j\dot{c}w = \overset{\overset{i.j}{\curvearrowright}}{ab}vcw$
- $\dot{a}w\textcircled{g}_j\overset{\overset{i}{\curvearrowright}}{bc}v = -\overset{\overset{i.j}{\curvearrowright}}{bc}vaw$

All other cases give 0.

Proposition 3.2.2.13. *The algebra $(W_C(\chi), \textcircled{x}, \{\textcircled{g}_i\})$ is a $(\mathcal{G}^C)^!$-algebra generated by χ.*

Proof. Indeed, $\dot{a}v = \dot{a}\textcircled{x}w$ with w the word v with a dot on a letter (let us say the first one for example) and

$$\overset{\overset{i}{\curvearrowright}}{ab}v = (\dot{a}\textcircled{g}_i\dot{b})\textcircled{x}w$$

so $(W(\chi), \textcircled{x}, \{\textcircled{g}_i\})$ it is generated by χ under the operations $\textcircled{x}$ and $\textcircled{g}_i$. The products $\textcircled{g}_i$ are skew-symmetric since

$$\dot{a}\textcircled{g}_i\dot{b} = \overset{\overset{i}{\curvearrowright}}{ab} = \overset{\overset{i}{\curvearrowleft}}{ba} = -\overset{\overset{i}{\curvearrowright}}{ba} = -\dot{b}\textcircled{g}_i\dot{a}$$

The relations of $(\mathcal{G}^C)^!$ are easily checked. □

Proposition 3.2.2.14. *We have* $\dim((\mathcal{G}^C)^!(4)) \geq 4 + 3n$.

Proof. Let $A = W(\{a, b, c, d\})$. Let us compute the dimension of $\mathrm{Mult}(A(4))$ the multilinear part of $A(4)$. Let us write words of $\mathrm{Mult}(A(4))$ such that the arrow always starts from a and go to the second letter, and aside from that, the letters are in the lexicographic order. The rewriting rule $\overset{\overset{i}{\curvearrowright}}{\beta\gamma}\alpha \mapsto \overset{\overset{i}{\curvearrowright}}{\alpha\gamma}\beta - \overset{\overset{i}{\curvearrowright}}{\alpha\beta}\gamma$ is confluent. Indeed:

$$\overset{\overset{i}{\curvearrowright}}{cd}ab = \overset{\overset{i}{\curvearrowright}}{bd}ac - \overset{\overset{i}{\curvearrowright}}{bc}ad = (\overset{\overset{i}{\curvearrowright}}{ad}bc - \overset{\overset{i}{\curvearrowright}}{ab}cd) - (\overset{\overset{i}{\curvearrowright}}{ac}bd - \overset{\overset{i}{\curvearrowright}}{ab}cd) = \overset{\overset{i}{\curvearrowright}}{ad}bc - \overset{\overset{i}{\curvearrowright}}{ac}bd$$

Hence, $\mathrm{Mult}(A(4))$ is spanned by the words $\dot{a}bcd$, $a\dot{b}cd$, $ab\dot{c}d$, $abc\dot{d}$, $\overset{\overset{i}{\curvearrowright}}{ab}cd$, $\overset{\overset{i}{\curvearrowright}}{ac}bd$ and $\overset{\overset{i}{\curvearrowright}}{ad}bc$. Since A is a $(\mathcal{G}^C)^!$-algebra on 4 generators, $\dim((\mathcal{G}^C)^!(4)) \geq \dim(\mathrm{Mult}(A(4))) = 4 + 3n$. □

Remark 3.2.2.15. The algebra $(W(\chi), \textcircled{x}, \{\textcircled{g}_i\})$ is in fact the free $(\mathcal{G}^C)^!$-algebra generated by χ.

Let us consider the three following monomials orders. We refer to Definitions 2.2.4.35 and 2.2.4.39, and Examples 2.2.4.41 and 2.2.4.42 for the definitions of the orders. One should understand the juxtaposition of the orders as their concatenation see Definition 2.2.4.44:

- the rewriting system associated to the graded path lexicographic permutation order with $x_* > y_* > g_*$ gives the rewriting system displayed in Figure 4.11 of the appendix;
- the rewriting system associated to the weighted permutation reverse graded path lexicographic order with $g_* > y_* > x_*$ and g_* of degree 1 gives the rewriting system displayed in Figure 4.12 of the appendix;
- and the rewriting system associated to the reverse graded path lexicographic permutation order with $x_* > y_* > g_*$ gives the rewriting system displayed in Figure 4.13 of the appendix.

Proposition 3.2.2.16. *The rewriting systems displayed in Figures 4.7, 4.12 and 4.13 of the appendix are convergent.*

Proof. As in Proposition 3.2.1.14, noticing that there are $4+3n$ normal forms in arity 4 for each rewriting system and that $\dim((\mathcal{G}^C)^!(4)) \geq 4+3n$ is enough. Since we have a monomial order and the rewriting rules are quadratic in an operad generated in arity two, checking arity 4 is enough. ☐

Proposition 3.2.2.17. *We have that* $\dim((\mathcal{G}^C)^!(m)) = (n+1)m - n$ *for all* $m \geq 1$*, hence its exponential generating series is* $((n+1)t - n)\exp(t) + n$*.*

Proof. It suffices to count the normal forms of a rewriting system for example the one displayed in Figure 4.11 of the appendix. Let $m \geq 2$ and count the number of normal forms in arity m. Those are right combs with at most one g_*^k, with all the x_* above the g_*^k and the y_*, and all the y_* below the g_*^k and the x_*. Hence, the normal forms are determined by the number of occurrences x_* and g_*^k, and have either zero or one occurrence g_*^k. If there is no g_*^k, then one can have from 0 to $m-1$ occurrences of x_*. If there is one g_*^k, then one can have from 0 to $m-2$ occurrences of x_* and n choice for the g_*^k that appears. Hence, the number of normal forms in arity m is $m + n(m-1) = (n+1)m - n$. ☐

Theorem 3.2.2.18. *The operad* $\mathcal{G}^C$ *is Koszul.*

Proof. We have a quadratic convergent ORS for $(\mathcal{G}^C)^!$, hence $(\mathcal{G}^C)^!$ is Koszul, hence $\mathcal{G}^C$ is Koszul. ☐

Proposition 3.2.2.19. *The exponential generating series of* Greg^C *verifies:*

$$f_{\mathrm{Greg}^C} = t\exp(f_{\mathrm{Greg}^C}) + n(\exp(f_{\mathrm{Greg}^C}) - f_{\mathrm{Greg}^C} - 1)$$

Proof. An inspection of the species $\mathcal{G}^V$ which is the species of rooted Greg trees such that the black vertices are labeled by $\{e_1, \ldots, e_n\}$ shows that:

$$\mathcal{G}^V = X \cdot E(\mathcal{G}^V) + nE_{\geq 2}(\mathcal{G}^V)$$

With the usual notation of species, X is the singleton species, E is the species of sets and $E_{\geq 2}$ is the species of sets with at least two elements. The above equation means that a rooted Greg tree is either a white vertex and a set of rooted Greg trees connected to it, or a black vertex labeled by e_k (so n possibilities) and a set of at least 2 rooted Greg trees connected to it. Since $\mathcal{G}^V$ is the underlying species of Greg^C, we have that:

$$f_{\mathrm{Greg}^C} = t\exp(f_{\mathrm{Greg}^C}) + n(\exp(f_{\mathrm{Greg}^C}) - f_{\mathrm{Greg}^C} - 1)$$

☐

Remark 3.2.2.20. We can recover the recursive formula enumerating the rooted Greg trees from [45, Proposition 2.1] by resolving a differential equation. Let $h(t,z) = ((z+1)t + z)\exp(-t) - z$. Hence:

- $\frac{\partial h}{\partial t}(t,z) = -((z+1)t - 1)\exp(-t)$,
- $\frac{\partial h}{\partial z}(t,z) = (t+1)\exp(-t) - 1$.

Hence:

$$(z+2)h(t,z) + \frac{\partial h}{\partial t}(t,z) - (z+1)^2\frac{\partial h}{\partial z}(t,z) = 1$$

Let f be such that $h(f(t,z),z) = h \circ (f, \mathrm{id}) = t$. We have that $\frac{\partial h}{\partial t} \circ (f,\mathrm{id}).\frac{\partial f}{\partial t} = 1$ and $\frac{\partial h}{\partial t} \circ (f,\mathrm{id}).\frac{\partial f}{\partial z} + \frac{\partial h}{\partial z} \circ (f,\mathrm{id}) = 0$, hence:

$$((z+2)t - 1)\frac{\partial f}{\partial t} + (z+1)^2\frac{\partial f}{\partial z} = -1$$

Let $f(t,z) = \sum \frac{g_k(z)}{k!}t^k$ with g_k polynomials in z, we get the following recursive relation:

- $g_1(z) = 1$
- $g_{k+1}(z) = (z+2)kg_k(z) + (z+1)^2 g_k'(z)$

Theorem 3.2.2.21. *The operad* $\mathcal{G}^C$ *is isomorphic to* Greg^C*.*

Proof. We know that $\dim((\mathcal{G}^C)^!(m)) = (n+1)m - n$, hence its exponential generating series is $f_{(\mathcal{G}^C)^!} = ((n+1)t - n)\exp(t) + n$. Let $h(t,n) = -f_{(\mathcal{G}^C)^!}(-t) = ((n+1)t + n)\exp(-t) - n$, since $\mathcal{G}^C$ is Koszul, we know that $h(f_{\mathcal{G}^C}(t,n), n) = t$. Hence:

$$t = ((n+1)f_{\mathcal{G}^C} + n)\exp(-f_{\mathcal{G}^C}) - n$$

Hence:

$$f_{\mathcal{G}^C} = t\exp(f_{\mathcal{G}^C}) + n(\exp(f_{\mathcal{G}^C}) - f_{\mathcal{G}^C} - 1)$$

Which shows that $f_{\mathcal{G}^C} = f_{\mathrm{Greg}^C}$. Since we have a surjective morphism from $\mathcal{G}^C$ to Greg^C and equality of dimensions of components, we have that $\mathcal{G}^C$ is isomorphic to Greg^C. □

Corollary 3.2.2.22. *The operad* Greg^C *is generated in arity two and Koszul.*

Definition 3.2.2.23. Let Greg_n be the operad $\mathrm{Greg}^{(V,0)}$, with 0 the trivial co-multiplication on V.

Corollary 3.2.2.24. *The operad* $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ *is isomorphic to* $\mathrm{Greg}^{(V,\Delta_{\max})}$ *with:*

$$\Delta_{\max} : e_k \mapsto \sum_{i,j \mid \max(i,j)=k} e_i \otimes e_j$$

Moreover, $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ *is filtered by the grading of the rooted Greg trees by the number of black vertices. The associated graded operad is* Greg_n*.*

Corollary 3.2.2.25. *The operad* $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ *is Koszul.*

Remark 3.2.2.26. This fact is not a direct consequence of the definition of $\bigvee_{\mathrm{Lie}}^{m+1}\mathrm{PreLie}$ as a coproduct. Indeed, the fiber coproduct of two Koszul operads $\mathcal{P}$ and $\mathcal{Q}$ over a Koszul operad $\mathcal{R}$ is not necessarily Koszul. Take for instance the operads $\bigvee_{\mathrm{Lie}}^{2}\mathrm{Ass}$ and $\bigvee_{\mathrm{Lie}}^{2}\mathrm{Pois}$ which are not Koszul, it can be checked by comparing the exponential generating series of those operads and of their Koszul dual. Worst, freeness as right $\mathcal{R}$-modules of $\mathcal{P}$ and $\mathcal{Q}$ does not solve the issue as shown by the example $\bigvee_{\mathrm{Lie}}^{2}\mathrm{Pois}$. It seems that freeness as left modules solve this issue. Indeed, for instance the operads $\bigvee_{\mathrm{Com}}^{2}\mathrm{Pois}$ and $\bigvee_{\mathrm{Com}}^{2}\mathrm{Zinb}$ are Koszul. However, the author does not know how to prove that left freeness ensure that Koszulness is preserved. Left and right freeness are defined at the very beginning of the next section.

3.2.3 Freeness and explicit computation of the generators

We have seen that Greg^C is Koszul using quadratic convergent ORS. However, one quadratic convergent ORS was enough to show this fact. Three different quadratic convergent ORS were computed with particular normal forms. Indeed, the goal was to apply the freeness theorems of Subsection 2.2.5. Let C be a co-associative co-commutative coalgebra and C' a sub-coalgebra of C.

Theorem 3.2.3.1. *The operad* Greg^C *is free as left and as a right* $\mathrm{Greg}^{C'}$*-module. (And not as a bimodule.)*

Proof. By reversing the order, one can go from an ORS of an operad to an ORS of its Koszul dual, this exchanges the rewritable monomials and the normal forms, and reverse the monomial partial order. Hence, the ORS displayed in Figure 4.12 of the appendix witness the left freeness and the ORS displayed in Figure 4.13 of the appendix witness the right freeness. □

Remark 3.2.3.2. When we have $\mathcal{P} \to \mathcal{Q}$ a morphism of operads, we can never expect $\mathcal{Q}$ to be free as a $\mathcal{P}$-bimodule. Indeed, $\mathcal{P}$ itself is not free as a $\mathcal{P}$-bimodule.

Corollary 3.2.3.3. *The operad* $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ *is free as left and as a right* $\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}$*-module.*

Theorem 3.2.3.4. *The operad* Greg^C *has the Nielsen-Schreier property.*

Proof. The ORS displayed in Figure 4.11 of the appendix witness the first condition and the ORS displayed in Figure 4.12 of the appendix witness the second condition. □

Corollary 3.2.3.5. *The operad* $\bigvee_{\mathrm{Lie}}^n \mathrm{PreLie}$ *has the Nielsen-Schreier property.*

Let us compute the explicit generators of $\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$ as a left $\bigvee_{\mathrm{Lie}}^n \mathrm{PreLie}$-module. To do so, let us mimic the proof of [24].

Cyclic operad A structure of *cyclic operad* on an operad $\mathcal{P}$ is given by an action of $\mathfrak{S}_{n+1}$ on $\mathcal{P}(n)$ compatible with the operadic structure. This is equivalently given by an action of $\tau = (1, 2, \dots, n+1)$ on $\mathcal{P}(n)$ verifying:

- $\tau(\mu \circ_i \nu) = \tau(\mu) \circ_{i+1} \nu$ for $i < m$ with m the arity of μ;
- $\tau(\mu \circ_m \nu) = \tau(\nu) \circ_1 \tau(\mu)$.

It is known that Lie is a cyclic operad, see [38], CycLie is the species underlying this cyclic operad so as vector space we have $\mathrm{Lie}(k) = \mathrm{CycLie}(k+1)$. In the particular case of CycLie, the action of τ is given by $\tau(l) = l$. Let us describe CycLie a bit more explicitly. We have that CycLie is a right Lie-module, moreover it is generated by r commutatif in arity 2 and satisfies the following relation:

$$r \circ_1 l = r \circ_2 l$$

This relation should be understood the following way: r correspond to id in Lie. The action of $\mathfrak{S}_2$ on id is trivial. Since $\tau(l) = l$ we have that $\tau(l \circ_2 \mathrm{id}) = \tau(\mathrm{id}) \circ_1 \tau(l) = l$, where l correspond to $r \circ_2 l$ in CycLie since r correspond to id in Lie, and $\tau(l \circ_2 \mathrm{id})$ correspond to $(r \circ_2 l).(1\ 2\ 3)$ in CycLie. In particular, we have that CycLie is the right Lie-module generated by r and satisfying the relation $r \circ_1 l = r \circ_2 l$.
Let us introduce the following notation, x is the generator of PreLie without symmetries, $x = \mu + l$ with μ symmetric and l skew-symmetric. Then l is the generator of the suboperad Lie of PreLie.

We want to prove that $\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie} \simeq \bigvee_{\mathrm{Lie}}^n \mathrm{PreLie} \circ \mathcal{T}(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie}))$ with $\mathcal{T}$ the free operad functor, $\overline{\mathcal{T}}$ the reduced free operad functor such that $\mathcal{T}(\mathcal{X}) = \overline{\mathcal{T}}(\mathcal{X}) \oplus \mathcal{X}$ and $\overline{\mathcal{T}}^{(n)}$ the n-th iteration of $\overline{\mathcal{T}}$. We will do so by induction on n. The initialization is exactly the main theorem of [24].

Theorem 3.2.3.6. *[24, Theorem 1] Let* $\mathcal{Y}$ *be the subspecies of* PreLie *such that* $y \in \mathcal{Y}$ *if and only if* $y = (\mu \circ_2 a) \circ_1 b$ *with* $a, b \in \mathrm{Lie}$*. Then:*

- $\mathcal{Y}$ *is isomorphic to* CycLie *as species;*
- *Let* $\mathcal{P}(\mathcal{Y})$ *the suboperad of* PreLie *generated by* $\mathcal{Y}$*, then* $\mathcal{P}(\mathcal{Y})$ *is free;*
- *The left* Lie*-submodule of* PreLie *generated by* $\mathcal{P}(\mathcal{Y})$ *is free and coincide with* PreLie*.*

Let us use the exact same technic as the one used in [24] to explicitly compute the generators of $\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$ as a left $\bigvee_{\mathrm{Lie}}^n \mathrm{PreLie}$-module. The idea is the following, we introduce some explicit generators of $\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$ as a left Lie-module and as a left $\bigvee_{\mathrm{Lie}}^n \mathrm{PreLie}$-module. We then define a bunch of surjective morphisms of species involving those generators. We then compute the dimensions of the species involved to show that the morphisms are isomorphisms. Finally, we conclude that the generators we introduced freely generate $\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$.

Notations We fix the inclusions $\bigvee_{\mathrm{Lie}}^{k}\mathrm{PreLie}\to\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ given by x is send to x, c_1 is send to c_1 up to c_k is send to c_k. We denote by $\mathcal{Y}_0$ the species $\mathcal{Y}$ of [24, Theorem 1], which is defined as the subspecies PreLie generated by the $(\mu\circ_2 a)\circ_1 b$ with $a,b\in\mathrm{Lie}$. We define $\mathcal{Y}_n$ to be the subspecies of $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ generated by the $(c_n\circ_2 a)\circ_1 b$ such that $a,b\in\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}$. We denote $\mathcal{X}_0=\mathcal{Y}_0$, and we define $\mathcal{X}_n$ be the subspecies of $\mathcal{Y}_n$ generated by the $(c_n\circ_2 a)\circ_1 b$ with $a,b\in\mathrm{Lie}$ which is different from $\mathcal{Y}$ when $n\neq 0$. Let $\mathcal{P}(\mathcal{Y}_n)$ be the suboperad of $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ generated by $\mathcal{Y}_n$. And let $\mathcal{Z}_n$ be the species inductively defined by $\mathcal{Z}_0=\mathcal{T}(\mathcal{Y}_0)$ and $\mathcal{Z}_{n+1}=\mathcal{Z}_n\circ\mathcal{T}(\mathcal{Y}_{n+1})$.
We can notice the following facts: By fixing the morphisms $\bigvee_{\mathrm{Lie}}^{k}\mathrm{PreLie}\to\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$, we have that $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ is a left $\bigvee_{\mathrm{Lie}}^{k}$-module. Moreover, by definition, $\mathcal{Y}_n$ is a right $\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}$-module which implies that it is a right Lie-module. Hence, $\mathcal{T}(\mathcal{Y}_n)$ is a right Lie-module, hence $\mathcal{Z}_n$ is also a right Lie-module. The species $\mathcal{X}_n$ is also a right Lie-module, moreover it is generated by c_n as a right Lie-module.
The goals are to show that: $\mathcal{X}_n$ is isomorphic to CycLie, that $\mathcal{Y}_n$ is isomorphic to $\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie})$, that $\mathcal{P}(\mathcal{Y}_n)$ is free, and that $\mathcal{Z}_n$ is isomorphic to:

$$\mathcal{T}(\mathrm{CycLie})\circ\mathcal{T}(\overline{\mathcal{T}}(\mathrm{CycLie}))\circ\cdots\circ\mathcal{T}(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie}))$$

Let us define the surjective morphisms that we will need.

Lemma 3.2.3.7. *We have a surjective morphism of* Lie-*bimodule from* $\mathrm{Lie}\circ\mathcal{Z}_n$ *to* $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$.

Proof. Let $k\leq n+1$. Since $\mathcal{Y}_k$ is a subspecies of $\bigvee_{\mathrm{Lie}}^{k}\mathrm{PreLie}\subseteq\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$, we have a morphism of species from $\mathcal{T}(\mathcal{Y}_k)$ to $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$. Hence, we have a morphism from $\mathcal{Z}_n$ to $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$. Since Lie is a suboperad of $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$, we have a morphism of left Lie-module from $\mathrm{Lie}\circ\mathcal{T}(\mathcal{Y}_k)$ to $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$. This is a morphism of right Lie-module since it is compatible with the composition by l. Moreover, this morphism is surjective since $l\in\mathrm{Lie}$, $\mu\in\mathcal{Y}_0$, $c_1\in\mathcal{Y}_1$, ..., $c_n\in\mathcal{Y}_n$ which imply that $l,\mu,c_1,\dots,c_n\in\mathrm{Lie}\circ\mathcal{Z}_n$ and since those are the generators of $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$, we have that the morphism $\mathrm{Lie}\circ\mathcal{Z}_n\to\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ is surjective. □

Let us define the following filtration on $\mathrm{Lie}\circ\mathcal{Z}_n$:

Definition 3.2.3.8. Let us define the weight of an element of $\mathcal{T}(\mathcal{Y}_k)$ as the usual weight in free operad, which is the number of generators needed in the composition. Then we define inductively the weight of an element $\gamma(z,f_1,\dots,f_k)$ of $\mathcal{Z}_n$ with $z\in\mathcal{Z}_{n-1}$ and $f_i\in\mathcal{T}(\mathcal{Y}_n)$ as the total sum of the weight of those elements. For an element $\alpha=\gamma(l,z_1,\dots,z_r)$ of $\mathrm{Lie}\circ\mathcal{Z}_n$ such that $z_i\in\mathcal{Z}_n$ of weight w_i and $l\in\mathrm{Lie}$ of arity r, let $w=r+\sum w_i$ be the weight of α. We define the filtration by $\alpha\in F^w(\mathrm{Lie}\circ\mathcal{Z}_n)$ with w the weight of α.

Proposition 3.2.3.9. *This filtration is compatible with the* Lie-*bimodule structure. (It is in fact a filtration by infinitesimal* Lie-*bimodule.)*

Proof. Indeed, we have that $l(F^p(\mathrm{Lie}\circ\mathcal{Z}_n),F^q(\mathrm{Lie}\circ\mathcal{Z}_n))\subseteq F^{p+q}(\mathrm{Lie}\circ\mathcal{Z}_n)$ and $F^p(\mathrm{Lie}\circ\mathcal{Z}_n)\circ\mathrm{Lie}\subseteq F^p(\mathrm{Lie}\circ\mathcal{Z}_n)$. □

Hence, this filtration induces a filtration on $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ by the surjective morphism of Lie-bimodule of the previous lemma.

Lemma 3.2.3.10. *We have a surjective morphism of species from* CycLie *to* $\mathcal{X}_n$.

Proof. Let us compute Relation (diff pre-Lie) with $x=\mu+l$. We get:

$$\begin{aligned}(l\circ_1 c_n-(c_n\circ_1 l).(2\,3)-c_n\circ_2 l)-(l\circ_1 c_n-(c_n\circ_1 l).(2\,3)-c_n\circ_2 l).(2\,3)\\ (\mu\circ_1 c_n-(c_n\circ_1 \mu).(2\,3)-c_n\circ_2 \mu)-(\mu\circ_1 c_n-(c_n\circ_1 \mu).(2\,3)-c_n\circ_2 \mu).(2\,3)\\ +\sum_{i,j\mid\max(i,j)=n}(c_i\circ_1 c_j-(c_i\circ_1 c_j).(2\,3))\end{aligned}$$

Let us rewrite it a bit:

$$2 \times (c_n \circ_2 l) + (c_n \circ_1 l).(2\ 3) - (c_n \circ_1 l) = l \circ_1 c_n - (l \circ_1 c_n).(2\ 3) + (\mu \circ_1 c_n - (c_n \circ_1 \mu).(2\ 3)) - (\mu \circ_1 c_n - (c_n \circ_1 \mu).(2\ 3)).(2\ 3) + \sum_{i,j \mid \max(i,j)=n} (c_i \circ_1 c_j - (c_i \circ_1 c_j).(2\ 3))$$

Let us point out that element of the left-hand side are in $F^2 \bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$ and elements of the right-hand side are in $F^3 \bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$. Indeed, at the left-hand side, we have composition of the identity (arity 1) with elements of $\mathcal{T}(\mathcal{Y}_n)$ having exactly one occurrence of an element of $\{\mu, c_1, \ldots, c_n\}$, hence degree 2. At the right-hand side, we have either composition of l (arity 2) with elements of $\mathcal{T}(\mathcal{Y}_n)$ having exactly one occurrence of an element of $\{\mu, c_1, \ldots, c_n\}$, hence degree 3; or composition the identity (arity 1) with elements of $\mathcal{T}(\mathcal{Y}_n)$ having exactly two occurrences of an element of $\{\mu, c_1, \ldots, c_n\}$, hence degree 3.
Let us consider $\mathrm{gr}_F \mathcal{X}_n$ the graded species associated to the restriction of the filtration F of $\mathcal{X}_n$. As species we have that $\mathrm{gr}_F \mathcal{X}_n$ is isomorphic to $\mathcal{X}_n$. Moreover, in $\mathrm{gr}_F \mathcal{X}_n$, the above relation gives:

$$2 \times (c_n \circ_2 l) + (c_n \circ_1 l).(2\ 3) - (c_n \circ_1 l) = 0$$

Let us denote r this relation and compute $\frac{1}{3}(r + r.(1\ 3))$:

$$\frac{1}{3}(2 \times (c_n \circ_2 l) + (c_n \circ_1 l).(2\ 3) - (c_n \circ_1 l) + 2 \times (c_n \circ_2 l).(1\ 3) + (c_n \circ_1 l).(1\ 2\ 3) - (c_n \circ_1 l).(1\ 3)) = 0$$

We get:

$$(c_n \circ_2 l) = (c_n \circ_1 l) \tag{cyc}$$

This relation allows us to define a morphism of species from CycLie to $\mathrm{gr}_F \mathcal{X}_n$ by sending $\tilde{\mathrm{id}} \mapsto c_n$ and $\tilde{l} \mapsto (c_n \circ_2 l)$, with $\tilde{\mathrm{id}}$ and $\tilde{l}$ the identity and the Lie bracket of CycLie. Indeed, we have an action of $\tau = (1\ 2\ 3)$ on $c_n \circ_2 l$ and $(c_n \circ_2 l).\tau = (c_n \circ_1 l)$, which gives $(c_n \circ_2 l)$ by the relation above, hence $\tau(\tilde{l}) = \tilde{l}$. This morphism is surjective since $\mathrm{gr}_F \mathcal{X}_n$ is a right Lie-module generated by c_n. Hence, we have a surjective morphism of species from CycLie to $\mathcal{X}_n$. □

Lemma 3.2.3.11. *We have a surjective morphism of species from $\mathcal{X}_n \circ \mathcal{Z}_{n-1}$ to Y_n.*

Proof. Let $\gamma(c_n, y_1, \ldots, y_k)$ a monomial element of $\mathcal{Y}_n$, since $y_i \in \bigvee_{\mathrm{Lie}}^n \mathrm{PreLie}$ and $\mathrm{Lie} \circ \mathcal{Z}_{n-1}$ surjects on $\bigvee_{\mathrm{Lie}}^n \mathrm{PreLie}$, we have l_i such that:

$$y_i = \gamma(l_i, \alpha_{(i,1)}, \ldots, \alpha_{(i,r_i)})$$

with $l_i \in \mathrm{Lie}$ and $\alpha_{(i,j)} \in \mathcal{Z}_{n-1}$. Let $\beta = \gamma(c_n, l_1, \ldots, l_k)$, we have $\beta \in \mathcal{X}_n$, hence $\gamma(c_n, y_1, \ldots, y_k)$ is in the image of $\mathcal{X}_n \circ \mathcal{Z}_{n-1}$. □

Let us summarize the morphisms of species we have:

$$\bigvee_{\mathrm{Lie}}^{n} \mathrm{PreLie} \circ \mathcal{T}(\mathrm{CycLie} \circ \mathcal{Z}_{n-1}) \twoheadrightarrow \bigvee_{\mathrm{Lie}}^{n} \mathrm{PreLie} \circ \mathcal{T}(\mathcal{X}_n \circ \mathcal{Z}_{n-1}) \twoheadrightarrow \bigvee_{\mathrm{Lie}}^{n} \mathrm{PreLie} \circ \mathcal{T}(\mathcal{Y}_n) \twoheadrightarrow \bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$$

One last ingredient is needed: the equality of dimensions of the components to show that those morphisms are in fact isomorphisms.

Proposition 3.2.3.12. *Let S a species, $f_S(t)$ its exponential generating series. Then*

$$f_{\mathcal{T}(\overline{\mathcal{T}}^{(n)}(S))}(t) = \frac{\mathrm{rev}_t(t - (n+1) f_S(t)) - t}{n+1} + t$$

where rev_t is the inverse of the composition in the argument t and $f_{\mathcal{T}(\overline{\mathcal{T}}^{(n)}(S))}$ the exponential generating series of $\mathcal{T}(\overline{\mathcal{T}}^n(S))$.

Proof. For a species S with exponential generating series $f_S(t)$, the exponential generating series $f_{\mathcal{T}(S)}(t,z)$ of $\mathcal{T}(S)$ is given by $f_{\mathcal{T}(S)}(t,z)$ which is the inverse of $t - zf_S(t)$ for the composition in the argument t, hence we have $f_{\mathcal{T}(S)}(t,z) = \mathrm{rev}_t(t - zf_S(t))$. Hence, the exponential generating series of $\overline{\mathcal{T}}(S)$ is $f_{\overline{\mathcal{T}}(S)}(t,z) = f_{\mathcal{T}(S)}(t,z) - t = \mathrm{rev}_t(t - zf_S(t)) - t$. The exponential generating series $f_{\mathcal{T}(S)}(t,z)$ and $f_{\overline{\mathcal{T}}(S)}(t,z)$ have two arguments, the first one, t, count the arity of the elements and the second one, z, count the number of generators of the elements in the free operad. Since z count the number of generators of the elements in the free operad, dividing by z allows us to count the number of compositions of generators. Hence, the exponential generating series of $\overline{\mathcal{T}}^{(n)}(S)(t)$ is $\frac{\mathrm{rev}_t(t-nf_S(t))-t}{n}$. Finally, we get:

$$f_{\mathcal{T}(\overline{\mathcal{T}}^{(n)}(S))}(t) = \frac{\mathrm{rev}_t(t-(n+1)f_S(t)) - t}{n+1} + t$$

□

Lemma 3.2.3.13. *The exponential generating series of* $\bigvee_{\mathrm{Lie}}^{n} \mathrm{PreLie} \circ \mathcal{T}(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie}))$ *is equal to the exponential generating series of* $\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$.

Proof. Let us compute the exponential generating series of $\mathcal{T}(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie}))$. The exponential generating series of CycLie is well known to be $(1-t)\ln(1-t)+t$, indeed its dimensions are $(n-2)!$. Hence, the exponential generating series of $\mathcal{T}(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie}))$ is

$$f_{\mathcal{T}(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie}))}(t) = \frac{\mathrm{rev}_t(t-(n+1)(1-t)\ln(1-t)-(n+1)t) - t}{n+1} + t$$

We have already computed the exponential generating series of $\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}$ in Proposition 3.2.2.19 which is

$$f_{\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}}(t) = \mathrm{rev}_t((nt+t+n)\exp(-t) - n)$$

And the exponential generating series of $\bigvee_{\mathrm{Lie}}^{n} \mathrm{PreLie}$ is

$$f_{\bigvee_{\mathrm{Lie}}^{n} \mathrm{PreLie}}(t) = \mathrm{rev}_t((nt+n-1)\exp(-t) - n + 1)$$

Let us show that $f_{\bigvee_{\mathrm{Lie}}^{n+1} \mathrm{PreLie}}(t) = (f_{\bigvee_{\mathrm{Lie}}^{n} \mathrm{PreLie}} \circ f_{\mathcal{T}(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie}))})(t)$. Let

$$f(t) = (nt+t+n)\exp(-t) - n \qquad g(t) = (nt+n-1)\exp(-t) - n + 1$$

$$h(t) = t - (n+1)(1-t)\ln(1-t) - (n+1)t$$

We want to show that:

$$\mathrm{rev}_t(f)(t) = (\mathrm{rev}_t(g) \circ (\frac{\mathrm{rev}_t(h) - t}{n+1} + t))(t)$$

It suffices to show that $h((n+1)g - nf) = f$. Let us compute:

$$\begin{aligned}(n+1)g(t) - nf(t) \quad &= (n+1)((nt+n-1)\exp(-t) - n + 1) - n((nt+t+n)\exp(-t) - n)\\ &= ((n+1)nt\exp(-t) + (n+1)(n-1)\exp(-t) - (n+1)(n-1)) - \\ &\quad (n(n+1)t\exp(-t) - n^2\exp(-t) + n^2)\\ &= -\exp(-t) + 1\end{aligned}$$

Hence:

$$h((n+1)g - nf) = -\exp(-t) + 1 - (n+1)\exp(-t)\ln(\exp(-t)) - (n+1)(-\exp(-t)+1) = f$$

which concludes the proof. □

We can state and prove the generalization of the previous theorem:

Theorem 3.2.3.14. *We have:*

1. *The species* $\mathcal{X}_n$ *is isomorphic to* CycLie *as a species.*
2. *The species* $\mathcal{Y}_n$ *is isomorphic to* $\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie})$ *as species;*
3. *The suboperad* $\mathcal{P}(\mathcal{Y}_n)$ *of* $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ *generated by* $\mathcal{Y}_n$ *is free;*
4. *The left* $\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}$*-submodule of* $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ *generated by* $\mathcal{P}(\mathcal{Y}_n)$ *is free and coincide with the* $\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}$*-module* $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$.
5. *The species* $\mathcal{Z}_n$ *is isomorphic to* $\mathcal{T}(\mathrm{CycLie})\circ\cdots\circ\mathcal{T}(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie}))$ *as species;*
6. *The left* Lie*-submodule of* $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ *generated by* $\mathcal{Z}_n$ *is free and coincide with the* Lie*-module* $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$.

Proof. Let us prove this theorem by induction on n. The base case is the theorem of [24]. From the previous lemmas we have

$$\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}\circ\mathcal{T}(\mathrm{CycLie}\circ\mathcal{Z}_{n-1})\twoheadrightarrow\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}\circ\mathcal{T}(\mathcal{X}_n\circ\mathcal{Z}_{n-1})\twoheadrightarrow$$
$$\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}\circ\mathcal{T}(\mathcal{Y}_n)\twoheadrightarrow\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}\circ\mathcal{P}(\mathcal{Y}_n)\twoheadrightarrow\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$$

By item (5), we have $\mathcal{Z}_{n-1}\simeq\mathcal{T}(\mathrm{CycLie})\circ\cdots\circ\mathcal{T}(\overline{\mathcal{T}}^{(n-1)}(\mathrm{CycLie}))$, hence

$$\mathrm{CycLie}\circ\mathcal{Z}_{n-1}\simeq\mathrm{CycLie}\circ\mathcal{T}(\mathrm{CycLie})\circ\cdots\circ\mathcal{T}(\overline{\mathcal{T}}^{(n-1)}(\mathrm{CycLie}))\simeq\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie})$$

Those surjective morphisms are isomorphisms by equality of dimensions. This shows that:

1. The species $\mathcal{X}_n$ is isomorphic to CycLie;
2. The species $\mathcal{Y}_n$ is isomorphic to $\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie})$;
3. The species $\mathcal{P}(\mathcal{Y}_n)$ is isomorphic to $\mathcal{T}(\mathcal{Y}_n)$;
4. And the left $\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}$-module $\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}\circ\mathcal{P}(\mathcal{Y}_n)$ is isomorphic to $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ as left $\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}$-module.

Moreover, since $\mathcal{Z}_n=\mathcal{Z}_{n-1}\circ\mathcal{T}(\mathcal{Y}_n)$ we have that $\mathcal{Z}_n$ is isomorphic to:

$$\mathcal{T}(\mathrm{CycLie})\circ\cdots\circ\mathcal{T}(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie}))$$

as a species. Since $\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}\circ\mathcal{P}(\mathcal{Y}_n)$ is isomorphic to $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ as left $\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}$-module, they are isomorphic as left Lie-module. Hence, $\mathrm{Lie}\circ\mathcal{Z}_n$ is isomorphic to $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ as left Lie-module, in particular the left Lie-submodule generated by $\mathcal{Z}_n$ is free and coincide with $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$. □

Remark 3.2.3.15. This proof can be adapted to show that $\mathrm{Greg}_n\simeq\mathrm{Greg}_{n-1}\circ\mathcal{T}(\overline{\mathcal{T}}^{(n)}(\mathrm{CycLie}))$.

The operad $\bigvee_{\mathrm{Lie}}^{n+1}\mathrm{PreLie}$ is also free as a right $\bigvee_{\mathrm{Lie}}^{n}\mathrm{PreLie}$-module. It could be interesting to compute explicit generator in this case.

3.3 Generalization to hypertrees

We saw in Section 1.4 that hyperforests were a generalization of rooted trees, and that Greg hyperforests and reduced Greg hyperforests were two possible ways to generalize both hyperforests and rooted Greg trees at the same time. Let us do the constructions of the above section for hypertrees, generalizing [19, Theorem 1.9] to the operad ComPreLie that we will introduce in this section.

3.3.1 Hyperforests and the operad ComPreLie

Let us now give a description *à la* Chapoton-Livernet of the operad ComPreLie. The operad ComPreLie was first introduced in [65], it is defined by the following presentation:

$$\mathcal{T}[x,y,c]/\langle (x\circ_1 x - x\circ_2 x) - (x\circ_1 x - x\circ_2 x).(2\ 3),\\ x\circ_1 c - (c\circ_1 x).(2\ 3) - c\circ_2 x,\ c\circ_1 c - c\circ_2 c\rangle$$

where x, y and c are operations of arity two, and the action of $\mathfrak{S}_2$ on x, y and c is given by $x.(1\ 2) = y$ and $c.(1\ 2) = c$. The first relation is the pre-Lie relation for x, the second one is a Leibniz rule and the third is the associativity of c. Its Koszul dual, the operad $\mathrm{ComPreLie}^!$, is defined by the following presentation:

$$\mathcal{T}[x_*,y_*,c_*]/\langle x_*\circ_1 x_* - x_*\circ_2 x_*,\ x_*\circ_1 x_* - (x_*\circ_2 x_*).(2\ 3),\\ x_*\circ_2 c_*,\ x_*\circ_1 c_* - (c_*\circ_1 x_*).(2\ 3),\ c_*\circ_1 c_* - c_*\circ_2 c_* - c_*\circ_1 c_*.(2\ 3)\rangle$$

where x_*, y_* and c_* are operations of arity two, and the action of $\mathfrak{S}_2$ on x_*, y_* and c_* is given by $x_*.(1\ 2) = y_*$ and $c_*.(1\ 2) = -c_*$. In order to compute arity-wise dimensions of $\mathrm{ComPreLie}^!$, let us introduce the following $\mathrm{ComPreLie}^!$-algebra admitting an explicit description:

Definition 3.3.1.1. Let X a finite set, $\mathrm{Lie}(X)$ the free Lie algebra generated by X and $\mathrm{uCom}(X)$ the free unitary commutative associative algebra generated by X. Let $LC(X) = \mathrm{Lie}(X)\otimes \mathrm{uCom}(X)$. For $a_1\otimes a_2$ and $b_1\otimes b_2$ in $LC(X)$, let us define two operations of arity two $[\cdot,\cdot]$ and $\cdot.\cdot$ by:

- $(a_1\otimes a_2).(b_1\otimes b_2) = a_1\otimes (a_2.b_1.b_2)$ if $b_1\in \mathrm{Vect}(X)$;
- $(a_1\otimes a_2).(b_1\otimes b_2) = 0$ if $b_1\notin \mathrm{Vect}(X)$;
- $[(a_1\otimes a_2),(b_1\otimes b_2)] = [a_1,b_1]\otimes (a_2.b_2)$.

One can check that $LC(X)$ is a $(\mathrm{ComPreLie})^!$-algebra with $[\cdot,\cdot]$ the image of c_* and $\cdot.\cdot$ the image of x_*. Moreover, it is generated as a $(\mathrm{ComPreLie})^!$-algebra by the elements of the form $a\otimes 1$ with $a\in X$ and 1 the unit of $\mathrm{uCom}(X)$.

Definition 3.3.1.2. Let u_n be the sequence of logarithmic numbers, see Sequence A002104 in the OEIS [76]. This sequence is defined by:

$$\sum_{n\geq 1}\frac{u_n}{n!}t^n = -\log(1-t)\exp(t)$$

Proposition 3.3.1.3. *We have that* $u_n = \dim(\mathrm{Mult}(LC(\{a_1,\dots,a_n\})))$ *with* Mult *the multilinear part, in particular* $u_4 = 24$.

Proof. Since $LC(\{a_1,\dots,a_n\}) = \mathrm{Lie}(\{a_1,\dots,a_n\})\otimes \mathrm{uCom}(\{a_1,\dots,a_n\})$, we have that:

$$\mathrm{Mult}(LC(\{a_1,\dots,a_n\})) = \bigoplus_{I\sqcup J=\{a_1,\dots,a_n\}}\mathrm{Mult}(\mathrm{Lie}(I))\otimes \mathrm{Mult}(\mathrm{uCom}(J))$$

Hence, $n\to \mathrm{Mult}(LC(\{a_1,\dots,a_n\}))$ give rise to a species which is the Cauchy product of Lie and uCom. Hence, its exponential generating series is $-\log(1-t)\exp(t)$. □

The operad $\mathrm{ComPreLie}^!$ admits a terminating quadratic rewriting system displayed in the appendix Figure 4.14 of the appendix, it has 19 rules. This rewriting system is obtained using the quantum permutation graded path lexicographic order with x_* and y_* of x-type and c_* of y-type. We refer to Definitions 2.2.4.43 and 2.2.4.39, and Examples 2.2.4.41, 2.2.4.35 and 2.2.4.42 for the definitions of the orders. One should understand the juxtaposition of the orders as their concatenation see Definition 2.2.4.44. Moreover, this rewriting system has the following property:

Proposition 3.3.1.4. *The sequence of numbers of normal form of the rewriting system displayed in Figure 4.14 of the appendix is the sequence of logarithmic numbers u_n.*

Proof. Since $\mathrm{ComPreLie}^!$ is graded by the number of occurrences of c_*, it is clear that the suboperad generated by c_* is the Lie operad. The rewriting system displayed in Figure 4.14 of the appendix restricted to c_* is the rewriting system associated to the permutation graded path lexicographic order, which is known to be a Gröbner basis (in particular this is a convergent ORS) for the Lie operad, see [10, Example 5.6.1.1]. In particular, normal forms of $\mathrm{Lie}(n)$ are in bijection with a basis of $\mathrm{Lie}(\{a_1, \dots, a_n\})$.
An analogous observation shows that the rewriting system displayed in Figure 4.14 of the appendix restricted to x_* and y_* is a convergent ORS of the operad Perm.
Moreover, a normal form is given by a pair (a, b) with a a normal form of Lie and b a normal form of Perm, with a composed in the non-symmetric input of b. This allows us to build a bijection between normal forms of $\mathrm{ComPreLie}^!(n)$ and $\mathrm{Mult}(LC(\{a_1, \dots, a_n\}))$. Hence, the number of normal forms of $\mathrm{ComPreLie}^!(n)$ is the number of multilinear elements of $LC(\{a_1, \dots, a_n\})$. This concludes the proof. □

Theorem 3.3.1.5. *The operad* $\mathrm{ComPreLie}^!$ *is Koszul. Moreover, its Hilbert series is given by:*

$$f_{\mathrm{ComPreLie}^!}(t) = -\log(1-t)\exp(t)$$

Proof. The inequality $\dim(\mathrm{ComPreLie}^!(4)) \geq 24$ and the fact that the rewriting system admit 24 normal forms in arity 4 ensures that the ORS is convergent. Hence, $\mathrm{ComPreLie}^!$ is Koszul. □

Corollary 3.3.1.6. *The operad* ComPreLie *is Koszul. Moreover, its Hilbert series is given by:*

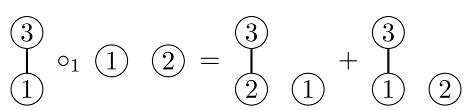
$$f_{\mathrm{ComPreLie}}(t) = \mathrm{rev}(\log(1+t)\exp(-t))$$

with rev *the compositional inverse in t of a series.*

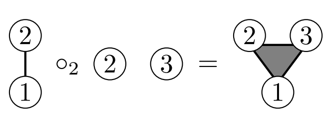
With this result, we know the dimensions of the operad ComPreLie, indeed it is Sequence A052888 of the OEIS [76]. Let us now give a combinatorial description of the operad ComPreLie.

Definition 3.3.1.7. Let S and T be two hyperforests and i be a label of a vertex v_i of S. Let B be the tree below the vertex v_i in S, and $C = \{C_1, \dots, C_n\}$ be the set of forests of children of v_i in S such that each rooted hypertrees that are grafted at v_i by the same edge are in the same forest. The *insertion* of T in S at the vertex i denoted $S \circ_i T$ is the formal sum of all possible way to graft the set of hyperforests $C_1, \dots, C_n$ on vertices of T such that each rooted hypertrees of C_j are grafted at T by the same edge, and then grafting the result on the parent of v_i in B. If T is a forest this creates a unique hyperedge that connects all its rooted hypertrees to the parent of v_i.

Let us compute the following examples:



and:



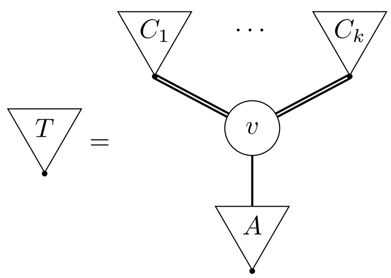
As one can remark, a hyperedge is created in the second example.

Proposition 3.3.1.8. *The insertions satisfy the parallel and sequential axioms. Hence, they give a structure of operad on the species of hyperforests.*

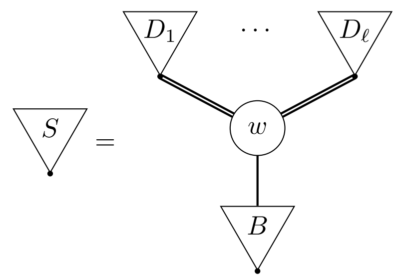
Proof. It is clear that the parallel axiom is verified since we are inserting hyperforests in different vertices. The proof of the sequential axiom is the computation shown in Figure 3.4 with the convention that double edges means that all the rooted hypertrees of the forest are grafted at the same vertex via the same edge. Indices are omitted for readability. □

Figure 3.4: The sequential axiom for $\mathcal{HF}$

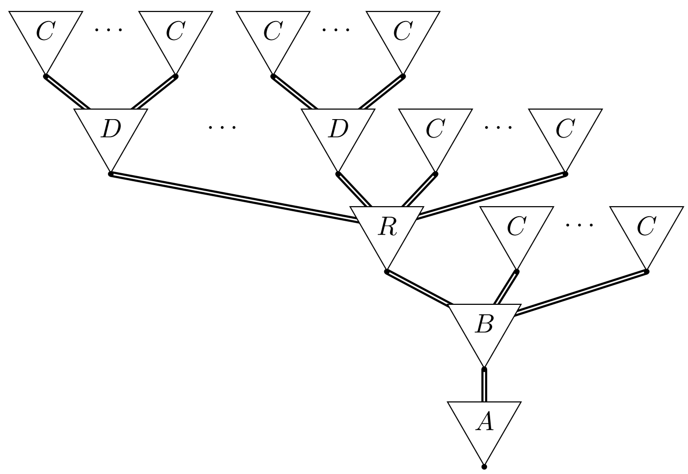
(a) Notation for T



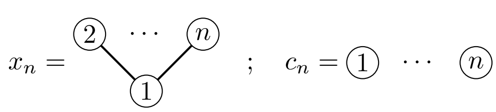
(b) Notation for S



(c) Computation of $T \circ_v S \circ_w R$



Let us denote:



As in the operad PreLie, the elements x_n are the symmetric braces, see [50].

Proposition 3.3.1.9. *The operad $\mathcal{HF}$ is generated by arity 2 elements. It means that $\mathcal{HF}$ is generated by x_2 and c_2.*

Proof. Let $\mathcal{P}$ be the suboperad of $\mathcal{HF}$ generated by $\mathcal{HF}(2)$. Let us prove inductively that $\mathcal{P} = \mathcal{HF}$:

- Initial case: $\mathcal{P}(2) = \mathcal{HF}(2)$ by definition.
- Induction step: If $T = x_n$, then T is a rooted tree, and thus in the suboperad generated by x_2 since PreLie is generated by arity 2 elements. If $T = c_n$, then $T = (\dots(c_2 \circ_1 c_2) \circ_1 \dots) \circ_1 c_2$. Else, T can be obtained by inductively composing copies of x_i and of c_j at the leaves.

□

Theorem 3.3.1.10. *The operad* $(\mathcal{HF}, \{\circ_i\})$ *is isomorphic to the operad* ComPreLie*. Moreover, the morphism is given by* $x_2 \mapsto x$ *and* $c_2 \mapsto c$*.*

Proof. The example of computation show that $\mathcal{HF}$ satisfies the relations of ComPreLie. Since, $\mathcal{HF}$ is generated by arity 2 elements, we have a surjective morphism ComPreLie $\to \mathcal{HF}$. The equality of the Hilbert series show that this morphism is bijective. □

Remark 3.3.1.11. A combinatorial interpretation of the operad ComPreLie was already given in [35] using partitioned trees. This theorem shows in particular that the species of hyperforests is isomorphic to the species of partitioned trees. However, the author finds the description of ComPreLie as the species of hyperforests more convenient to carry out computations.

3.3.2 From rooted hypertrees to the Greg hypertrees

Now that we have a combinatorial description of the operad ComPreLie, we want an analogue of the operad Greg in this context. Let us define the ComGreg operad by the following presentation:

$$\begin{aligned}\mathcal{T}[x,y,c,g]/\langle &(x\circ_1 x - x\circ_2 x) - (x\circ_1 x - x\circ_2 x).(2\ 3),\\ &x\circ_1 c - (c\circ_1 x).(2\ 3) - c\circ_2 x,\ c\circ_1 c - c\circ_2 c,\\ &(x\circ_1 g - (g\circ_1 x).(2\ 3) - g\circ_2 x) - (x\circ_1 g - (g\circ_1 x).(2\ 3) - g\circ_2 x)\,.(2\ 3)\rangle\end{aligned}$$

with x, y, c and g operations of arity two, and the action of $\mathfrak{S}_2$ on x, y, c and g is given by $x.(1\ 2) = y$, $c.(1\ 2) = c$ and $g.(1\ 2) = g$. The first relation is the pre-Lie relation for x, the second one is a Leibniz rule, the third one is theassociativity of c and the last one is the Greg relation. Its Koszul dual, the operad ComGreg$^!$ is defined by the following presentation:

$$\begin{aligned}\mathcal{T}[x_*,y_*,c_*,g_*]/\langle &x_*\circ_1 x_* - x_*\circ_2 x_*,\ x_*\circ_1 x_* - (x_*\circ_1 x_*).(2\ 3),\\ &x_*\circ_2 c_*,\ c_*\circ_2 x_* - x_*\circ_1 c_*,\ c_*\circ_1 c_* - (c_*\circ_1 c_*).(2\ 3) - c_*\circ_2 c_*,\\ &x_*\circ_1 g_* - g_*\circ_2 x_*,\ x_*\circ_2 g_*,\ c_*\circ_1 g_*,\ g_*\circ_1 c_*,\ g_*\circ_1 g_*\rangle\end{aligned}$$

Remark 3.3.2.1. Let us denote $\vee$ the coproduct of operads, and for $\mathcal{P}$ an operad, $\vee_{\mathcal{P}}$ the fibered coproduct of operads over $\mathcal{P}$. One may remark that ComGreg = ComPreLie $\vee_{\text{PreLie}}$ Greg. This is not enough to show that ComGreg is Koszul, however as we have worked out a description of the free ComPreLie$^!$-algebras and Greg$^!$-algebras in the previous section, we can guess a description of the free ComGreg$^!$-algebras, and show that ComGreg$^!$ is Koszul.

Definition 3.3.2.2. Let X be a finite alphabet. Let $\mathbf{Ar}(X)$ be the linear span of finite words on X with the following extra decoration: there is an arrow from one letter to another. Let $Ar(X)$ be the quotient of $\mathbf{Ar}(X)$ by the following relations: letters commute with each other (the arrow follows the letters), reverting the arrow change the sign and

$$\overset{\curvearrowright}{ab}cv = \overset{\curvearrowright}{cb}av + \overset{\curvearrowright}{ac}bv$$

for any $a, b, c \in X$ and v a finite word. Because the letters commute, we can write the elements of $Ar(X)$ with the arrow going from the first letter to the second one.
Let $LCA(X) = LC(X) \oplus Ar(X)$ and let us define $\cdot.\cdot$, $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ on $LCA(X)$ by:

- $\{a\otimes v, b\otimes w\} = \overset{\curvearrowright}{ab}vw$ for $a, b \in X$,
- $(\overset{\curvearrowright}{ab}v).(c\otimes w) = \overset{\curvearrowright}{ab}vcw$ for $c \in X$,
- and $\cdot.\cdot$ and $[\cdot,\cdot]$ are the same as in $LC(X)$.

All other cases give 0.

Proposition 3.3.2.3. *The algebra* $(LCA(X), \cdot\cdot, [\cdot,\cdot], \{\cdot,\cdot\})$ *is a* $(\mathrm{ComGreg})^!$*-algebra generated by* X.

Proof. Direct computations show that $LCA(X)$ is a $(\mathrm{ComGreg})^!$-algebra. Moreover, it is generated by X since $LC(X)$ is generated by X and $\overset{\frown}{ab}v = \{a \otimes v, b \otimes \varepsilon\}$. ☐

Proposition 3.3.2.4. *We have* $\dim(\mathrm{Mult}(LCA(\{a,b,c,d\}))) = 27$.

Proof. We have:

$$\mathrm{Mult}(LCA(\{a,b,c,d\})) = \mathrm{Mult}(LC(\{a,b,c,d\})) \oplus \mathrm{Mult}(Ar(\{a,b,c,d\}))$$

We already know the dimension of $\mathrm{Mult}(LC(\{a,b,c,d\}))$, and it is not difficult to check that $\mathrm{Mult}(Ar(\{a,b,c,d\}))$ is of dimension 3. Hence, $\mathrm{Mult}(LCA(\{a,b,c,d\}))$ is of dimension 27. ☐

The rewriting system of $\mathrm{ComGreg}^!$ is displayed in Figures 4.14 and 4.15 of the appendix with the rules not involving g_* in the first one and the ones involving g_* in the second one, it has 38 rules. This rewriting system is obtained using the quantum permutation graded path lexicographic order with x_* and y_* of x-type, and c_* and g_* of y-type. We refer to Definitions 2.2.4.35 and 2.2.4.39, and Examples 2.2.4.41 and 2.2.4.42 for the definitions of the orders. One should understand the juxtaposition of the orders as their concatenation see Definition 2.2.4.44. Moreover, this rewriting system has the following property:

Proposition 3.3.2.5. *The exponential generating function of the number of normal forms of the rewriting system displayed in Figures 4.14 and 4.15 of the appendix is given by:*

$$f = -\ln(1-t)\exp(t) + t\exp(t) - exp(t) + 1$$

In particular it has 27 *normal forms in arity* 4.

Proof. One may remark that c_* and g_* cannot appear at the same time in a normal form. Hence, either g_* appears or not. If g_* does not appear, then we have a normal form of $\mathrm{ComPreLie}^!$. If g_* appears, then we have a left comb with only g_* and x_* appearing, and only one occurrence of g_* at the top of the left comb. Hence, a normal form with g_* appearing is entirely determined by the label of the second leave of g_*, hence we have $n-1$ such normal form in arity n. Computation of the exponential generating series show that it is:

$$-\ln(1-t)\exp(t) + t\exp(t) - exp(t) + 1$$

☐

Theorem 3.3.2.6. *The operad* $\mathrm{ComGreg}^!$ *is Koszul.*

Proof. The inequality $\dim(\mathrm{ComGreg}^!(4)) \geq 27$ and the fact that the rewriting system admit 27 normal forms in arity 4 ensure that the ORS is convergent. Hence, $\mathrm{ComGreg}^!$ is Koszul. ☐

Corollary 3.3.2.7. *The operad* ComGreg *is Koszul. Moreover, its Hilbert series is given by:*

$$f_{\mathrm{ComGreg}}(t) = \mathrm{rev}(\ln(1+t)\exp(-t) + t\exp(-t) + \exp(-t) - 1)$$

with rev *the compositional inverse in* t *of a series.*

Now that we know that the arity-wise dimensions of the operad ComGreg, we can describe the underlying species.

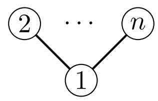
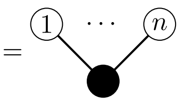
Same as in the previous subsection, one can define insertions and show that they define an operad structure on $\mathcal{GF}$.

Definition 3.3.2.8. Let S and T be two Greg hyperforests and i be a label of a vertex v_i of S. Let B be the tree below the vertex v_i in S, and $C = \{C_1, \ldots, C_n\}$ be the set of forests of children of v_i in S such that each rooted Greg hypertrees that are grafted at v_i by the same edge are in the same forest. The *insertion* of T in S at the vertex i denoted $S \circ_i T$ is the formal sum of all possible way to graft the set of Greg hyperforests $C_1, \ldots, C_n$ on black or white vertices of T such that each rooted Greg hypertrees of C_j are grafted at T by the same edge, and then grafting the result on the parent of v_i in B. If T is a forest it creates a unique hyperedge that connects all its rooted Greg hypertrees to the parent of v_i.

The same computations show that:

Proposition 3.3.2.9. *The insertions satisfy the parallel and sequential axioms. Hence, they give a structure of an operad on the species of Greg hyperforests.*

Let us denote:

$x_n =$  ; $c_n =$  ; $g_n =$ 

Proposition 3.3.2.10. *The operad $\mathcal{GF}$ is generated by arity 2 elements.*

Proof. Let $\mathcal{P}$ the suboperad of $\mathcal{GF}$ generated by $\mathcal{GF}(2)$, let us prove by induction on the arity that $\mathcal{P} = \mathcal{GF}$.

- Base case: by definition $\mathcal{P}(2) = \mathcal{GF}(2)$.
- Induction step: let $T \in \mathcal{GF}(n)$, if $T = x_n$ or g_n then $T \in \mathcal{P}$ since Greg is generated by arity 2 elements. If $T = c_n$ then $T \in \mathcal{P}$ since $\mathcal{HF}$ is generated by arity 2 elements. Else, T can be obtained by inductively composing copies of x_i, c_j and g_k at the leaves.

□

Theorem 3.3.2.11. *The operad $\mathcal{GF}$ is isomorphic to* ComGreg.

Proof. Computations show that the relations of ComGreg are satisfied in the operad $\mathcal{GF}$. Hence, we have a morphism ComGreg $\to \mathcal{GF}$. Since $\mathcal{GF}$ is generated by arity 2 elements, the morphism is surjective. Moreover, we have $f_{\mathcal{GF}}(t,1,1) = f_{\text{ComGreg}}(t)$. The equality of the Hilbert series shows that this morphism is bijective. □

3.3.3 Reduced Greg hypertrees

As we have seen in Theorem 3.2.1.20, the link between the operad Greg_{-1} and the operadic twisting of PreLie depicted in Proposition 3.1.2.3 allowed us to prove that $H^*(\text{Greg}_{-1}) = \text{Lie}$ which is the suboperad of PreLie generated by the Lie bracket. To use the same idea for the operad ComPreLie, we would need to define a differential d on ComGreg such that $d(x) = g$ and $d(c) = 0$. However, such a differential would not be compatible with the operad structure since we would have:

$$d(0) = d(x \circ_1 c - c \circ_2 x - (c \circ_1 x).(2\ 3)) = g \circ_1 c - c \circ_2 g - (c \circ_1 g).(2\ 3) \neq 0$$

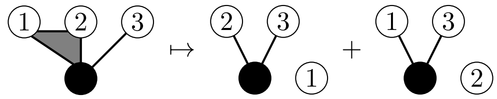
In order to fix this issue, we will need *reduced* version of the operad ComGreg which will not be Koszul, but on which such a differential can be defined.

Definition 3.3.3.1. Let us define the *reduced* ComGreg operad RedComGreg by

$$\text{RedComGreg} = \text{ComGreg}/\langle g \circ_1 c - c \circ_2 g - (c \circ_1 g).(2\ 3)\rangle$$

Let us describe the underlying species of RedComGreg as a subspecies of $\mathcal{GF}$.

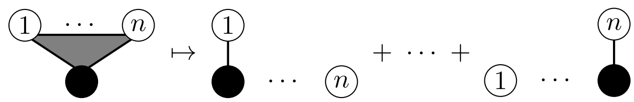
Let us study the rewriting rule $g \circ_1 c \mapsto c \circ_2 g + (c \circ_1 g).(2\ 3)$ at the level of the Greg hyperforests. It may be written the following way:



The hyperedge above the black vertex is no longer present in the right-hand side. This lead to the definition of the following species:

Definition 3.3.3.2. The *height* of a Greg hyperforests is the sum over all hyperedges of their hypertree weight times the number of white vertices in the path from this hyperedge to the root.

Proposition 3.3.3.3. *The following rewriting system on $\mathcal{GF}$ is convergent:*



For readability, other edges of the black vertex are omitted in the picture, however they are present and stay connected to the black vertex.

Proof. To be fair, we did not write down all the rewriting rules of this LRS. Indeed, we should have one rewriting rule for each incoming hyperedge of each black vertex of each Greg hyperforest. Moreover, if the black vertex is not a root, then the hyperforest depicted in the right-hand side of the rewriting rule should be grafted on the parent of the black vertex by an hyperedge.
Now that we have more precisely state the rewriting system, let us remark that those rewriting rules strictly decrease the height of the Greg hyperforests. Hence, it terminates. Let us apply consecutively two rewriting rules. We let the reader do the computation and notice that the result does not depend on the order of the rewriting rules. □

Corollary 3.3.3.4. *The species underlying the operad* RedComGreg *is $\mathcal{RGF}$.*

Proof. The rewriting system of the previous proposition gives us a projection of $\mathcal{GF}$ on $\mathcal{RGF}$, by applying the rewriting system in any order. Moreover, all those rewriting rules are consequences of the rule $g \circ_1 c \mapsto c \circ_2 g + (c \circ_1 g).(2\ 3)$. Hence, $\mathcal{RGF}$ is the operad $\mathcal{GF}$ quotiented by the relation $g \circ_1 c \mapsto c \circ_2 g + (c \circ_1 g).(2\ 3)$, which is the definition of RedComGreg. □

Let u be a formal variable encoding the hypertree weight grading and v be a formal variable encoding the Greg weight grading. Let us denote $f_{\mathcal{RGF}}(t,u,v)$ the exponential generating series of $\mathcal{RGF}$ according to these grading. It means that $f_{\mathcal{RGF}}(t,u,v) = \sum a_{i,j,k} \frac{t^i u^j v^k}{i!}$ where $a_{i,j,k}$ is the number of reduced Greg hyperforests with i white vertices of hypertree weight j, and Greg weight k.

Proposition 3.3.3.5. *The exponential generating series of $\mathcal{RGF}$ is given by:*

$$f_{\mathcal{RGF}}(t,u,v) = \operatorname{rev}\left(\left((v+1)\frac{\ln(1+ut)}{u} + v - v\exp\left(\frac{\ln(1+ut)}{u}\right)\right)\exp(-t)\right)$$

In particular, $f_{\mathcal{RGF}}(t,1,-1)$ is the series $\sum_{n\geq 1} n^{n-1}\frac{t^n}{n!}$.

Proof. Let us inspect the species $\mathcal{RGF}$ and $\mathcal{G}$ of reduced rooted Greg hypertrees. We have:

$$\mathcal{RGF} = \frac{1}{u}E_{\geq 1}(u.\mathcal{G})$$

and:

$$\mathcal{G} = X.E(\mathcal{RGF}) + vE_{\geq 2}(\mathcal{G})$$

Hence, we have:

$$f_{\mathcal{G}} = \frac{\ln(1+u.tf_{\mathcal{RGF}})}{u}$$

and:

$$\frac{\ln(1+u.tf_{\mathcal{RGF}})}{u} = t\exp(f_{\mathcal{RGF}}) + v\exp\left(\frac{\ln(1+u.tf_{\mathcal{RGF}})}{u}\right) - v\frac{\ln(1+u.tf_{\mathcal{RGF}})}{u} - v$$

We get that:

$$f_{\mathcal{RGF}}(t,u,v) = \operatorname{rev}\left(\left((v+1)\frac{\ln(1+ut)}{u} + v - v\exp\left(\frac{\ln(1+ut)}{u}\right)\right)\exp(-t)\right)$$

□

Remark 3.3.3.6. The computation of the composition reverse of this exponential generating series show that RedComGreg is not Koszul.

We can now define the analogue of Greg_{-1} for the operad ComPreLie. We will compute its cohomology in the next section to show the main theorem.

Definition 3.3.3.7. Let dgComGreg be the differential graded operad such that the underlying operad is RedComGreg with x, y and c in degree 0 and g in degree 1, and the differential is given by $d(x) = g$ and $d(c) = 0$. Computing d on the relations defining RedComGreg show that d is well defined. The underlying species of this operad is $\mathcal{RGF}$, hence d is also defined on $\mathcal{RGF}$. Let $F^p\mathcal{RGF}$ be the subspecies of $\mathcal{RGF}$ of reduced Greg hyperforests of height less or equal to p. The differential d respect the filtration by the height.

3.4 Application to the FMan operad

3.4.1 The operad FMan

The operad FMan is the operad encoding the algebraic structure on the vector fields of a Frobenius manifold. It is conjectured in [23] that FMan is isomorphic to the suboperad of ComPreLie generated by $x - y$ and c. In this section, we will prove this conjecture. First, let us state presentation of the operad FMan by generators and relations from [42]. The operad FMan admit the following presentation:

$$\begin{aligned}\mathcal{T}[l,c]/\langle l\circ_1 l - l\circ_2 l - (l\circ_1 l).(2\ 3),\ c\circ_1 c - c\circ_2 c,&\\ (l\circ_1 c)\circ_3 c - (c\circ_1 l)\circ_1 c - ((c\circ_1 l)\circ_1 c).(3\ 4) - (c\circ_2 l)\circ_3 c - ((c\circ_2 l)\circ_3 c).(1\ 2)+&\\ ((c\circ_1 c)\circ_3 l).(2\ 3) + ((c\circ_1 c)\circ_3 l).(1\ 3) - ((c\circ_1 c)\circ_3 l).(1\ 4) - ((c\circ_1 c)\circ_3 l).(2\ 4)\rangle,&\end{aligned}$$

where the action of $\mathfrak{S}_2$ on l and c is given by $l.(1\ 2) = -l$ and $c.(1\ 2) = c$. The relations defining FMan are the Jacobi relation of the Lie bracket l, the associativity relation of the commutative product c and the so-called Hertling-Manin relation which is cubical. The Hertling-Manin relation can be understood the following way: Let $LR = l\circ_2 c - c\circ_1 l - (c\circ_2 l).(1\ 2)$ be the failure to satisfy the Leibniz rule. Then LR satisfy the Leibniz rule in its first input, meaning that:

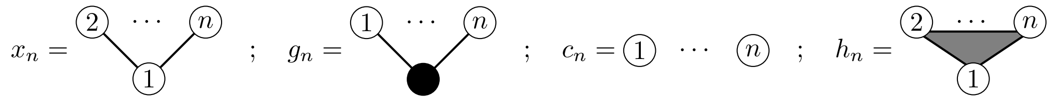
$$LR\circ_1 c - c\circ_2 LR - (c\circ_1 LR).(2\ 4\ 3) = 0$$

Since this relation is cubical, FMan is not quadratic, hence escapes the scope of the Koszul duality theory. However, this operad is closely related to the operad PreLie, indeed from [23], we know that FMan is the graded operad associated to the filtration of PreLie by the embedding of Lie into PreLie. In particular, the arity-wise dimensions of FMan are the same as the arity-wise dimensions of PreLie, which are given by the sequence n^{n-1}.

3.4.2 Proof of a conjecture of Dotsenko

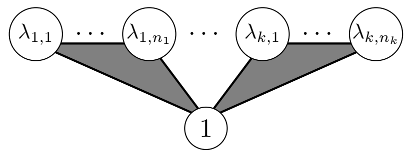
In order to prove that we have an embedding of FMan into ComPreLie, we will compute the cohomology of dgComGreg and show that it is FMan. In order to do so, we will show that the cohomology of dgComGreg is concentrated in degree 0, then since we know the arity-wise Euler characteristic of dgComGreg, we know the arity-wise dimension of the cohomology of dgComGreg. Moreover, since those are the same as the arity-wise dimension of FMan, we will have an isomorphism between FMan and the cohomology of dgComGreg, thus showing the embedding of FMan into ComPreLie.

Let us give a description of the differential d on dgComGreg similar to the description of the differential of TwPreLie given in Proposition 3.1.2.1. To do so, let us denote:



Moreover, for $P = \{\{\lambda_{1,1}, \ldots, \lambda_{1,n_1}\}, \ldots, \{\lambda_{k,1}, \ldots, \lambda_{k,n_k}\}\}$ a partition of $\{2, \ldots, n\}$, let us denote:

$p_P =$



Then any reduced rooted Greg hypertree which is a corolla is g_n or p_P for some n or P up to a permutation of the labels.

Definition 3.4.2.1. We recall that the complex $(\mathcal{RGF}, d)$ is filtered by the height. Let us denote by $(\mathrm{gr}_h\mathcal{RGF}, d_0)$ the associated graded complex. We have a canonical isomorphism $\mathcal{RGF}$ and $\mathrm{gr}_h\mathcal{RGF}$.

Proposition 3.4.2.2. *Let T be a reduced Greg hyperforests, i the label of a leaf and C a corolla, then $d_0(T \circ_i C) = d_0(T) \circ_i C + (-1)^{|T|} T \circ_i d_0(C)$.*

Proof. Let us denote *lht* for "lower height terms", meaning reduced Greg hyperforests of lower height. We have:

$$\begin{aligned} d_0(T \circ_i C) &= d(T \circ_i C) + lht \\ &= d(T) \circ_i C + (-1)^{|T|} T \circ_i d(C) + lht \\ &= d_0(T) \circ_i C + (-1)^{|T|} T \circ_i d_0(C) + lht \end{aligned}$$

Moreover, since we compose a single reduced rooted Greg hypertree in a leaf of T, no rewriting are involved in the composition. Hence, $d_0(T) \circ_i C$ and $T \circ_i d_0(C)$ have the same height. Hence:

$$d_0(T \circ_i C) = d_0(T) \circ_i C + (-1)^{|T|} T \circ_i d_0(C)$$

□

Lemma 3.4.2.3. *The differential d_0 on $\mathrm{gr}_h\mathcal{RGF}$ admits a description similar to Proposition 3.1.2.1. The image of a reduced Greg hyperforests T is obtained as the sum of six terms:*

1. *The sum of all possible ways to split a white vertex of T into a white vertex retaining the label and a black vertex above it and to connect the incoming edges to one of the two vertices (hyperedges cannot be grafted on the black vertex), up to a sign.*
2. *The sum of all possible ways to split a white vertex of T into a white vertex retaining the label and a black vertex below it and to connect the incoming edges to one of the two vertices (hyperedges cannot be grafted on the black vertex), up to a sign.*

3. *The sum of all possible ways to split a black vertex of T into two black vertices and to connect the incoming edges to one of the two vertices, up to a sign.*

4. *The sum over all the white vertex directly above a hyperedge to graft this white vertex on top of a new black vertex, and to put this new black vertex in the hyperedge in place of the white vertex, up to a sign.*

5. *The sum of all possible ways to graft an additional black leaf to T, up to a sign.*

6. *The sum of all possible ways graft a tree of T on top of a new black root, up to a sign.*

In this description, we forbid the grafting of rooted hyperedges on black vertices to ensure that the result is a reduced Greg hyperforests. Some black vertex that are created have zero or one child, however, those terms cancel out in the differential, and we are left with a sum of reduced Greg hypertrees.

Proof. Let us denote d_{descr} the map described in the proposition. Let us prove that $d_{\mathrm{descr}} = d_0$. In order to do so, let us prove that $d_{\mathrm{descr}}(C) = d_0(C)$ for C a corolla, and then that $d_{\mathrm{descr}}(T \circ_i C) = d_{\mathrm{descr}}(T) \circ_i C + (-1)^{|T|} T \circ_i d_{\mathrm{descr}}(C)$ for i a label of a leaf of T and C a corolla.
Let us first prove that $d_{\mathrm{descr}}(C) = d_0(C)$ for C a corolla. Let C be a corolla, if $C = x_n$ or g_n (up to a permutation) then $d_{\mathrm{descr}}(C) = d(C) = d_0(C)$ from Proposition 3.1.2.1. Else, we have $C = p_P$ for some partition P, which allows us to write C as a composition the following way:

$$C = (\dots(x_k \circ_{i_1} c_{k_1}) \circ_{i_2} \dots) \circ_{i_s} c_{k_s}$$

Let us compute $d_0(C)$:

$$\begin{aligned} d_0(C) &= d(C) + lht \\ &= d((\dots(x_k \circ_{i_1} c_{k_1}) \circ_{i_2} \dots) \circ_{i_s} c_{k_s}) + lht \\ &= ((\dots(d(x_k) \circ_{i_1} c_{k_1}) \circ_{i_2} \dots) \circ_{i_s} c_{k_s}) + lht \\ &= ((\dots(d_{\mathrm{descr}}(x_k) \circ_{i_1} c_{k_1}) \circ_{i_2} \dots) \circ_{i_s} c_{k_s}) + lht \end{aligned}$$

Let T be a reduced rooted Greg hypertree appearing in $d_{\mathrm{descr}}(x_k)$. To conclude, we need to know of when $((\dots(T \circ_{i_1} c_{k_1}) \circ_{i_2} \dots) \circ_{i_s} c_{k_s})$ has the same height as C. This is the case if and only no rewriting is involved in the compositions, hence if and only if each i_j is the label of a leaf which is the child of a white vertex. This is exactly the condition that "hyperedges cannot be grafted on the black vertex" in the terms from (1) and (2). Moreover, since all the new vertices coming from the c_{k_j} are leaves connected by hyperedges, the terms from (1) compensate with the terms from (5), and the terms from (2) compensate with the terms from (4). Hence, we have that $d_{\mathrm{descr}}(C) = d_0(C)$.
Let us show that $d_{\mathrm{descr}}(T \circ_i C) = d_{\mathrm{descr}}(T) \circ_i C + (-1)^{|T|} T \circ_i d_{\mathrm{descr}}(C)$ for i a label of a leaf of T and C a corolla. Let us assume that T is not the identity since the result is obvious if T is the identity. The vertex v labeled i is a white leaf which is not the root, hence the contributions of v in the sum come from (1) and (5) which compensate, and from (2) which create a new black vertex below it. The only thing that changes for the vertices of C, once composed in T, is that the root of C will no longer be a root, hence the contribution from (6) will no longer appear. However, the contribution of v is exactly the missing contribution of C that no longer appears once composed in T, hence:

$$d_{\mathrm{descr}}(T \circ_i C) = d_{\mathrm{descr}}(T) \circ_i C + (-1)^{|T|} T \circ_i d_{\mathrm{descr}}(C)$$

The sign $(-1)^{|T|}$ comes from the order in which we fill the black vertices, see Remark 3.1.2.2.
Since any reduced Greg hyperforests can be obtained by inductively composing corollas in leaves, we have that $d_{\mathrm{descr}} = d_0$. □

Now that we have this description, let us compute the cohomology of dgComGreg using the Künneth formula and the fact that the cohomology of Greg_{-1} is Lie.

Proposition 3.4.2.4. *We have the following isomorphism of chain complexes, where* $\mathrm{TS}(P)$ *is the linear span of the tree shapes on* P *a partition as defined in Definition 1.4.2.10:*

$$(\mathcal{RGF}(n), d_0) \simeq \bigoplus_k \bigoplus_{P \vdash^k \underline{n}} \left(\left(\bigotimes_{p \in P} \mathcal{G}(p), d \right) \right) \otimes \mathrm{TS}(P)$$

Proof. Let T be a forest of reduced rooted Greg trees, and $\varphi(T) = (S, M_1, \ldots, M_k)$. Then from the description of the differential d_0 on $\mathcal{RGF}$, we have:

$$d_0(T) = \sum_{i=1}^{k} \pm \varphi^{-1}(S, M_1, \ldots, d_0(M_i), \ldots, M_k)$$

This proves the isomorphism of chain complexes. □

Remark 3.4.2.5. Let $\lambda \vdash n$ and let $\mathrm{TS}(\lambda)$ the direct sum of $\mathrm{TS}(P)$ for P a partition of $\underline{n}$ in parts of size λ_i. We have a right action of the group $\mathfrak{S}_\lambda = \prod \mathfrak{S}_{\lambda_i}$ on $\bigotimes_i \mathcal{G}(\lambda_i)$, a left action of $\mathfrak{S}_\lambda$, and a right action of $\mathfrak{S}_n$ on $\mathrm{TS}(\lambda)$. The isomorphism of chain complexes is compatible with those actions, meaning that we have the following isomorphism of $\mathfrak{S}_n$-modules:

$$\mathcal{RGF}(n) \simeq \bigoplus_{\lambda \vdash n} \left(\bigotimes_i \mathcal{G}(\lambda_i) \right) \otimes_{\mathfrak{S}_\lambda} \mathrm{TS}(\lambda)$$

We can finally apply the Künneth formula to show that the cohomology of dgComGreg is concentrated in degree 0.

Theorem 3.4.2.6. *The cohomology of the operad* dgComGreg *is concentrated in degree* 0.

Proof. From the previous proposition, we have that:

$$(\mathcal{RGF}(n), d_0) \simeq \bigoplus_k \bigoplus_{P \vdash^k \underline{n}} \left(\bigotimes_{p \in P} (\mathcal{G}(p), d) \right) \otimes \mathrm{TS}(P)$$

Hence, we have:

$$H^*(\mathcal{RGF}(n), d_0) \simeq \bigoplus_k \bigoplus_{P \vdash^k \underline{n}} \left(\bigotimes_{p \in P} H^*(\mathcal{G}(p), d) \right) \otimes \mathrm{TS}(P)$$

From [30, Theorem 5.1], we have that $H^*(\mathcal{G}(p), d)$ is concentrated in degree 0. Hence, we have that $H^*(\mathcal{RGF}(n), d_0)$ is concentrated in degree 0. The spectral sequence associated to the filtration by the height abuts at the first page, hence the cohomology of $(\mathcal{RGF}, d)$ is concentrated in degree 0. □

Remark 3.4.2.7. From this proof, we can get the following description of the cohomology of $(\mathcal{RGF}, d)$:

$$H^*(\mathcal{RGF}(n), d) \simeq \bigoplus_k \bigoplus_{P \vdash^k \underline{n}} \left(\bigotimes_{p \in P} \mathrm{Lie}(p) \right) \otimes \mathrm{TS}(P)$$

This description could allow us to get a recursive formula for the dimension of $H^*(\mathcal{RGF}(n), d_0)$ if the dimensions of $\mathrm{TS}(P)$ were known.

Corollary 3.4.2.8. *The morphism* FMan $\to$ ComPreLie *is injective.*

Proof. Since $H^*(\mathrm{dgComGreg}) = H^0(\mathrm{dgComGreg})$, and using the description of $H^*(\mathrm{dgComGreg})$ with tree shapes, we have that $H^0(\mathrm{dgComGreg})$ is the suboperad of ComPreLie generated by $x - y$ and c. Hence, we have a surjective morphism FMan $\to H^0(\mathrm{dgComGreg})$. We know that $\dim(\mathrm{FMan}(n)) = n^{n-1}$ from [24], and we have computed the Euler characteristic of $\mathcal{RGF}$ in Proposition 3.3.3.5. Since the dimensions are the same, the morphism FMan $\to H^0(\mathrm{dgComGreg})$ is an isomorphism. Hence, φ is injective. □

Corollary 3.4.2.9. *Let u be the additional grading of* FMan *by the number of commutative product. The Hilbert series of* FMan *is given by:*

$$f_{\mathrm{FMan}}(t,u) = f_{\mathcal{RGF}}(t,u,-1) = \mathrm{rev}\left(\left(\exp\left(\frac{\ln(1+ut)}{u}\right)-1\right)\exp(-t)\right)$$

Remark 3.4.2.10. Moreover, from Remarks 3.4.2.5 and 3.4.2.7, and using the same notations, we have the following isomorphism of $\mathfrak{S}_n$-modules:

$$\mathrm{FMan}(n) \simeq \bigoplus_{\lambda\vdash n}\left(\bigotimes_i \mathrm{Lie}(\lambda_i)\right)\otimes_{\mathfrak{S}_\lambda}\mathrm{TS}(\lambda)$$

Remark 3.4.2.11. Theorem 3.3 from [66] gives a description of the subspace $\mathrm{Lie}(V) \subseteq \mathrm{PreLie}(V)$ using constructions similar to the operadic twisting of PreLie. This description can be understood as a consequence of [30, Theorem 5.1]. It can be generalized to give description of the subspace $\mathrm{FMan}(V) \subseteq \mathrm{ComPreLie}(V)$ using Theorem 3.4.2.8.

Appendix

Figure 4.5: Local confluence of the critical monomial of Lie.

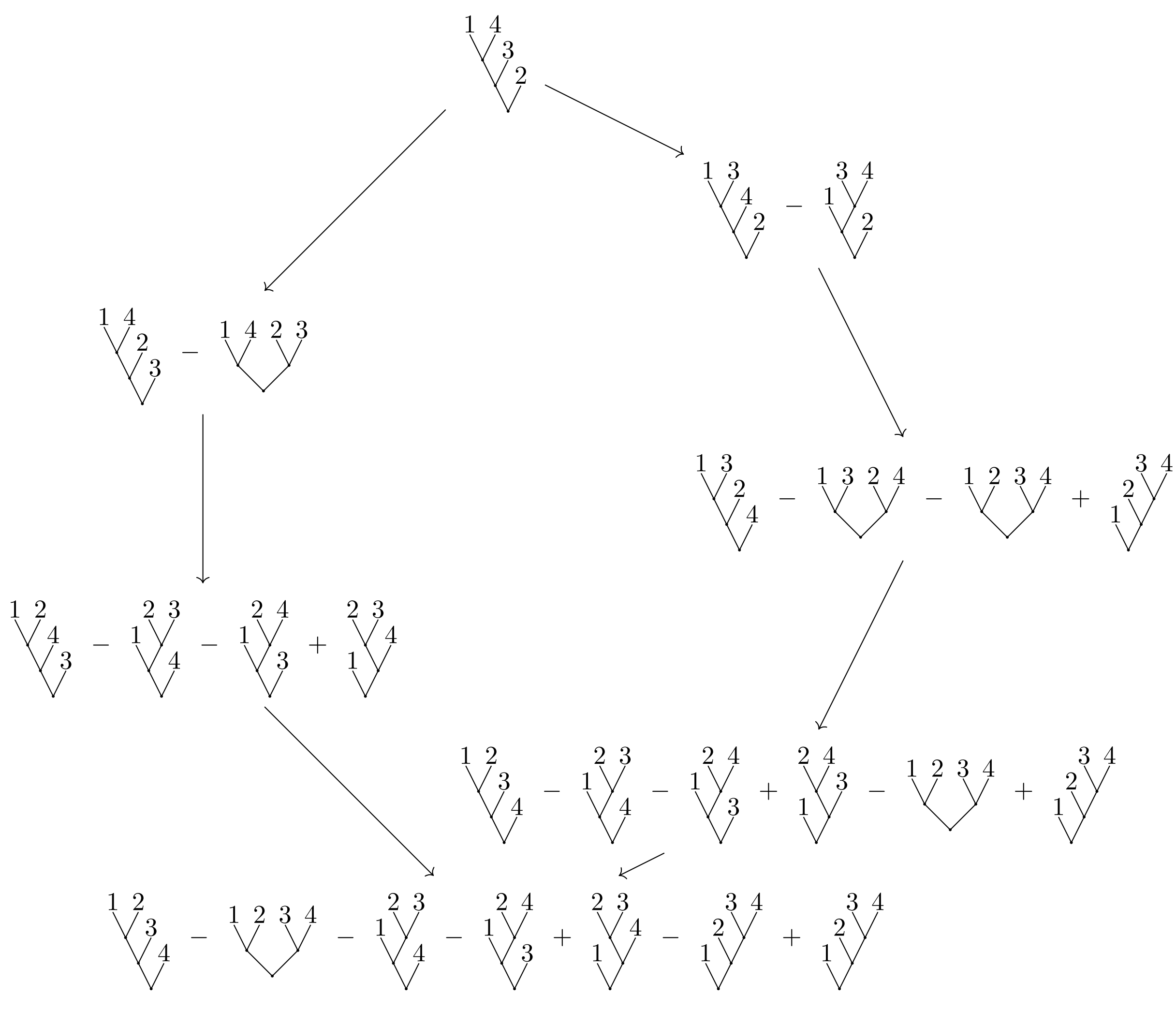


Table 4.1: Summary table of binary mono-generated Koszul operads in Operadia.

Operad	Entry in the OEIS	Hilbert series	Equation	Reference
Lie	A000142	$-\ln(1-x)$	$(1-x)f'-1$	
Com	A000012	$\exp(x)-1$	$f-f'+1$	
Ass	A000142	$\frac{x}{1-x}$	$(1-x)f-x$	[70]
Pois	—— ” ——	—— ” ——	—— ” ——	[36]
Leib	—— ” ——	—— ” ——	—— ” ——	[56]
Zinb	—— ” ——	—— ” ——	—— ” ——	[58]
LeftNil	—— ” ——	—— ” ——	—— ” ——	[12]
PreLie	A000169	$\mathrm{rev}(x\exp(x))$	$xff'-xf'+f$	[19]
NAP	—— ” ——	—— ” ——	—— ” ——	[55]
Perm	A000027	$x\exp(x)$	$(1+x)f-xf'$	[17]
$\mathrm{NAP}^!$	—— ” ——	—— ” ——	—— ” ——	[55]
Alia	A220433	$\mathrm{rev}\left(-x+x^2-\frac{x^3}{6}\right)$	$f-f^2+\frac{f^3}{6}-x$	[32]
LeftAlia	—— ” ——	—— ” ——	—— ” ——	[32]
$\mathrm{Alia}^!$	None	$x+x^2+\frac{x^3}{6}$	$f-\left(x+x^2+\frac{x^3}{6}\right)$	
LieAdm	A337017	$\mathrm{rev}\left(1-\frac{t^2}{2}-\exp(-x)\right)$	$f'\left(\frac{f^2}{2}+f+x-1\right)+1$	[75]
$\mathrm{LieAdm}^!$	A294619	$\frac{x^2}{2}+\exp(x)-1$	$f-f'-\left(\frac{x^2}{2}-2x-1\right)$	
Bess	A001515	$\exp(1-\sqrt{1-2x})-1$	$(1-2x)(f'-f-1)^2-(f+1)^2$	[16]

Table 4.2: Generating series appearing in the classification of KSetOp_1 and the first negative term of their reverse.

	$f_{\mathcal{P}}(t)$	first negative term in $\mathrm{rev}_t(-f_{\mathcal{P}}(-t))$
1	$t + \frac{2}{2!}t^2 + \frac{9}{3!}t^3 + \frac{60}{4!}t^4 + \frac{525}{5!}t^5 + \mathcal{O}(t^6)$	$\frac{-15}{5!}t^5$
2	$t + \frac{2}{2!}t^2 + \frac{9}{3!}t^3 + \frac{60}{4!}t^4 + \frac{520}{5!}t^5 + \mathcal{O}(t^6)$	$\frac{-10}{5!}t^5$
3	$t + \frac{2}{2!}t^2 + \frac{8}{3!}t^3 + \frac{40}{4!}t^4 + \frac{210}{5!}t^5 + \mathcal{O}(t^6)$	$\frac{-50}{5!}t^5$
4	$t + \frac{2}{2!}t^2 + \frac{7}{3!}t^3 + \frac{29}{4!}t^4 + \frac{146}{5!}t^5 + \mathcal{O}(t^6)$	$\frac{-46}{5!}t^5$
5	$t + \frac{2}{2!}t^2 + \frac{6}{3!}t^3 + \frac{12}{4!}t^4 + \frac{20}{5!}t^5 + \mathcal{O}(t^6)$	$\frac{-140}{5!}t^5$
6	$t + \frac{2}{2!}t^2 + \frac{6}{3!}t^3 + \frac{12}{4!}t^4 + \frac{1}{5!}t^5 + \mathcal{O}(t^6)$	$\frac{-121}{5!}t^5$
7	$t + \frac{2}{2!}t^2 + \frac{6}{3!}t^3 + \frac{14}{4!}t^4 + \frac{30}{5!}t^5 + \mathcal{O}(t^6)$	$\frac{-90}{5!}t^5$
8	$t + \frac{2}{2!}t^2 + \frac{6}{3!}t^3 + \frac{20}{4!}t^4 + \frac{75}{5!}t^5 + \frac{312}{6!}t^6 + \mathcal{O}(t^7)$	$\frac{-318}{6!}t^6$
9	$t + \frac{2}{2!}t^2 + \frac{6}{3!}t^3 + \frac{14}{4!}t^4 + \frac{21}{5!}t^5 + \mathcal{O}(t^6)$	$\frac{-81}{5!}t^5$
10	$t + \frac{2}{2!}t^2 + \frac{5}{3!}t^3 + \frac{6}{4!}t^4 + \frac{10}{5!}t^5 + \frac{18}{6!}t^6 + \mathcal{O}(t^7)$	$\frac{-2572}{6!}t^6$
11	$t + \frac{2}{2!}t^2 + \frac{5}{3!}t^3 + \frac{8}{4!}t^4 + \frac{18}{5!}t^5 + \frac{55}{6!}t^6 + \mathcal{O}(t^7)$	$\frac{-1541}{6!}t^6$
12	$t + \frac{2}{2!}t^2 + \frac{5}{3!}t^3 + \frac{2}{4!}t^4 + \frac{2}{5!}t^5 + \mathcal{O}(t^6)$	$\frac{-112}{5!}t^5$
13	$t + \frac{2}{2!}t^2 + \frac{4}{3!}t^3 + \frac{2}{4!}t^4 + \frac{2}{5!}t^5 + \frac{2}{6!}t^6 + \frac{2}{7!}t^7 + \mathcal{O}(t^8)$	$\frac{-26238}{7!}t^7$
14	$t + \frac{2}{2!}t^2 + \frac{4}{3!}t^3 + \frac{5}{4!}t^4 + \frac{6}{5!}t^5 + \frac{7}{6!}t^6 + \frac{8}{7!}t^7 + \frac{9}{8!}t^8 + \mathcal{O}(t^9)$	$\frac{-95669}{8!}t^8$
15	$t + \frac{2}{2!}t^2 + \frac{4}{3!}t^3 + \frac{2}{4!}t^4 + \frac{1}{5!}t^5 + \frac{1}{6!}t^6 + \frac{1}{7!}t^7 + \mathcal{O}(t^8)$	$\frac{-29093}{7!}t^7$
16	$t + \frac{2}{2!}t^2 + \frac{3}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \frac{1}{6!}t^6 + \frac{1}{7!}t^7 + \frac{1}{8!}t^8 + \frac{1}{9!}t^9 + \frac{1}{10!}t^{10} + \frac{1}{11!}t^{11} + \mathcal{O}(t^{12})$	$\frac{-802543633}{11!}t^{11}$
17	$t + \frac{2}{2!}t^2 + \frac{3}{3!}t^3 + \frac{2}{4!}t^4 + \frac{2}{5!}t^5 + \frac{2}{6!}t^6 + \frac{2}{7!}t^7 + \frac{2}{8!}t^8 + \frac{2}{9!}t^9 + \frac{2}{10!}t^{10} + \frac{2}{11!}t^{11} + \frac{2}{12!}t^{12} + \frac{2}{13!}t^{13} + \frac{2}{14!}t^{14} + \frac{2}{15!}t^{15} + \mathcal{O}(t^{16})$	$\frac{-1080639958361062}{15!}t^{15}$

Figure 4.6: The 25 relations of the shuffle operad $(\mathrm{Greg}')^{!}$.

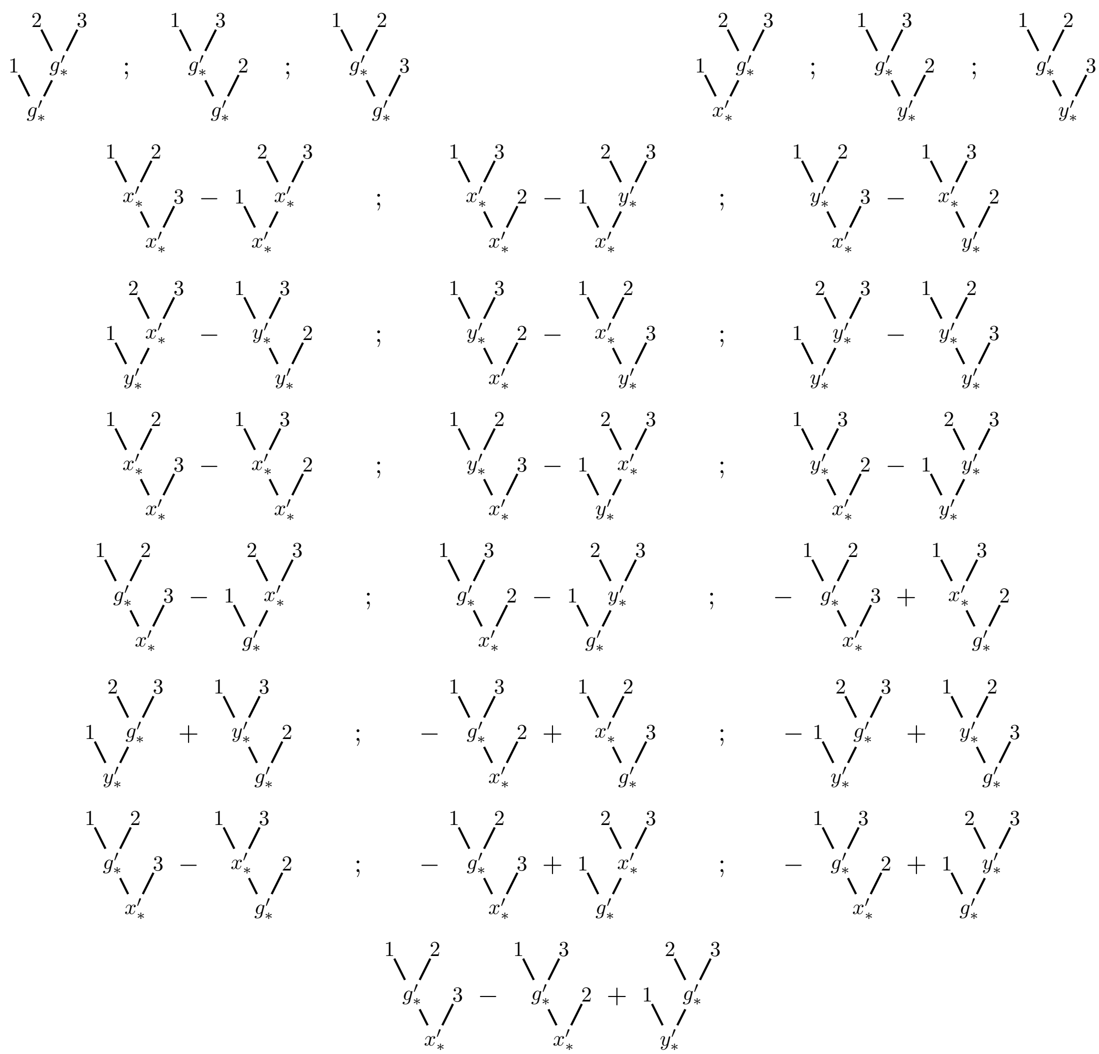


Figure 4.7: The “dlp” rewriting system for $(\mathrm{Greg}')^!$.

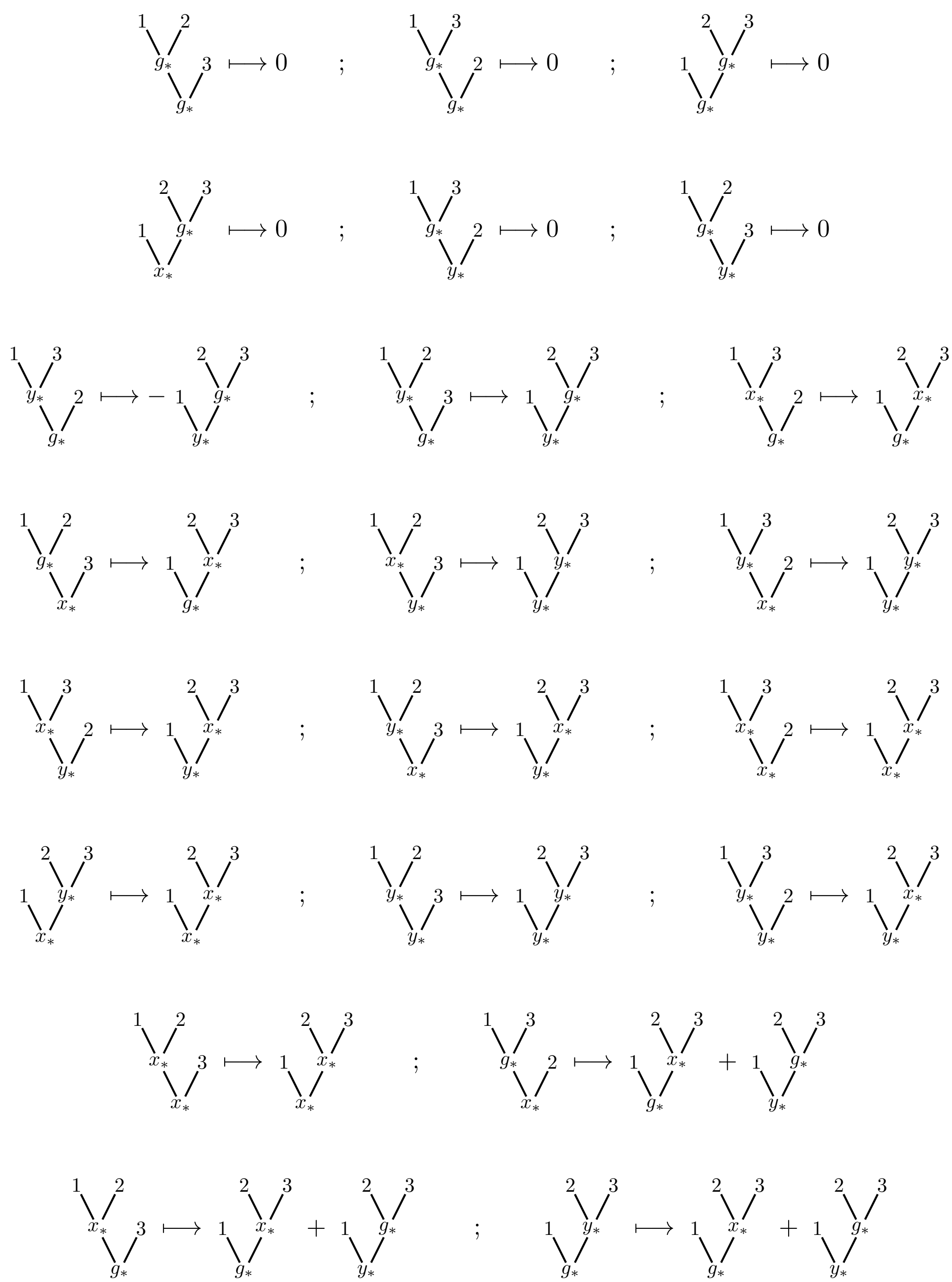


Figure 4.8: The "prdl" rewriting system for $(\mathrm{Greg}')^!$.

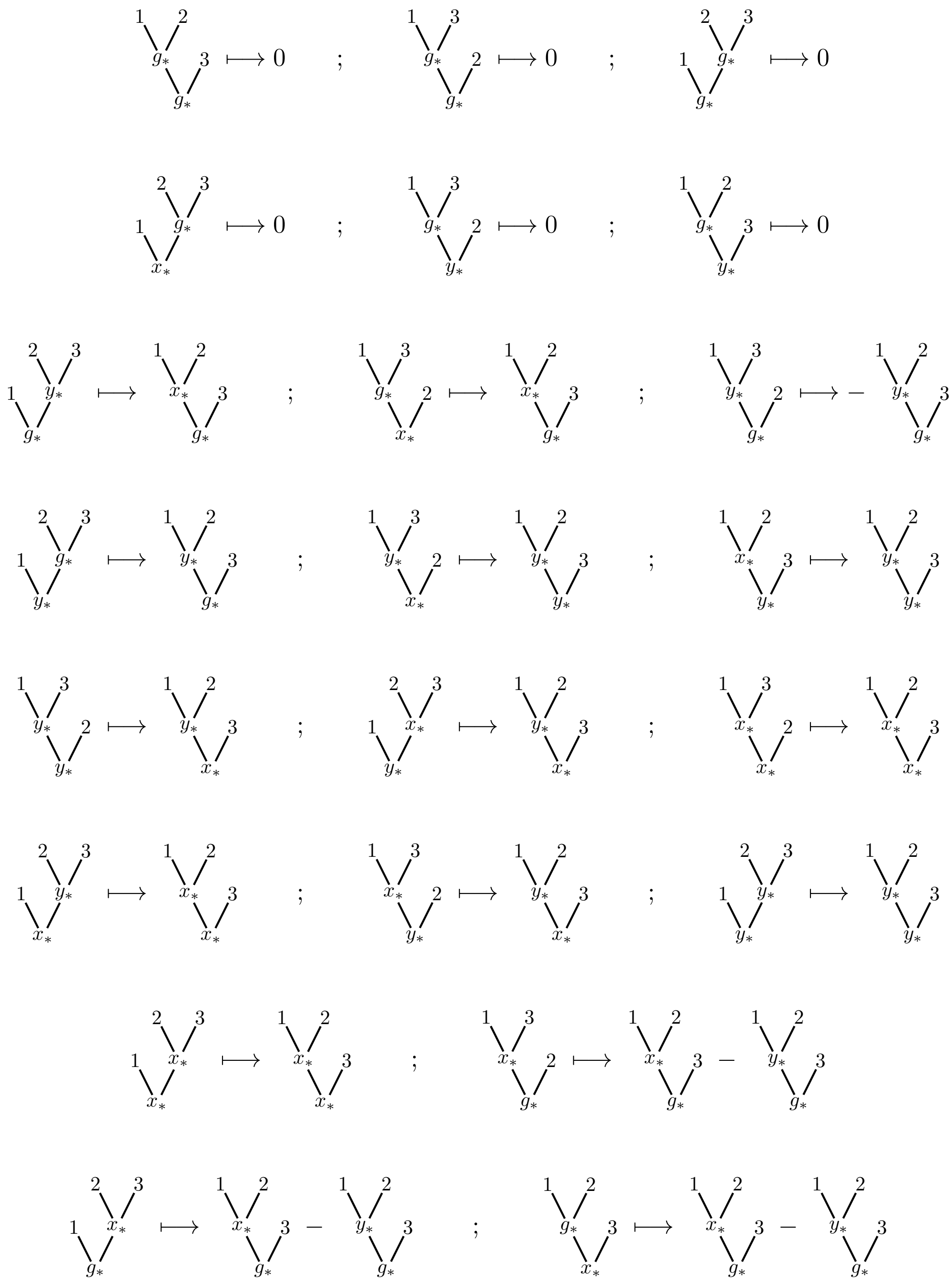


Figure 4.9: The "rdlp" rewriting system for $(\mathrm{Greg}')^!$.

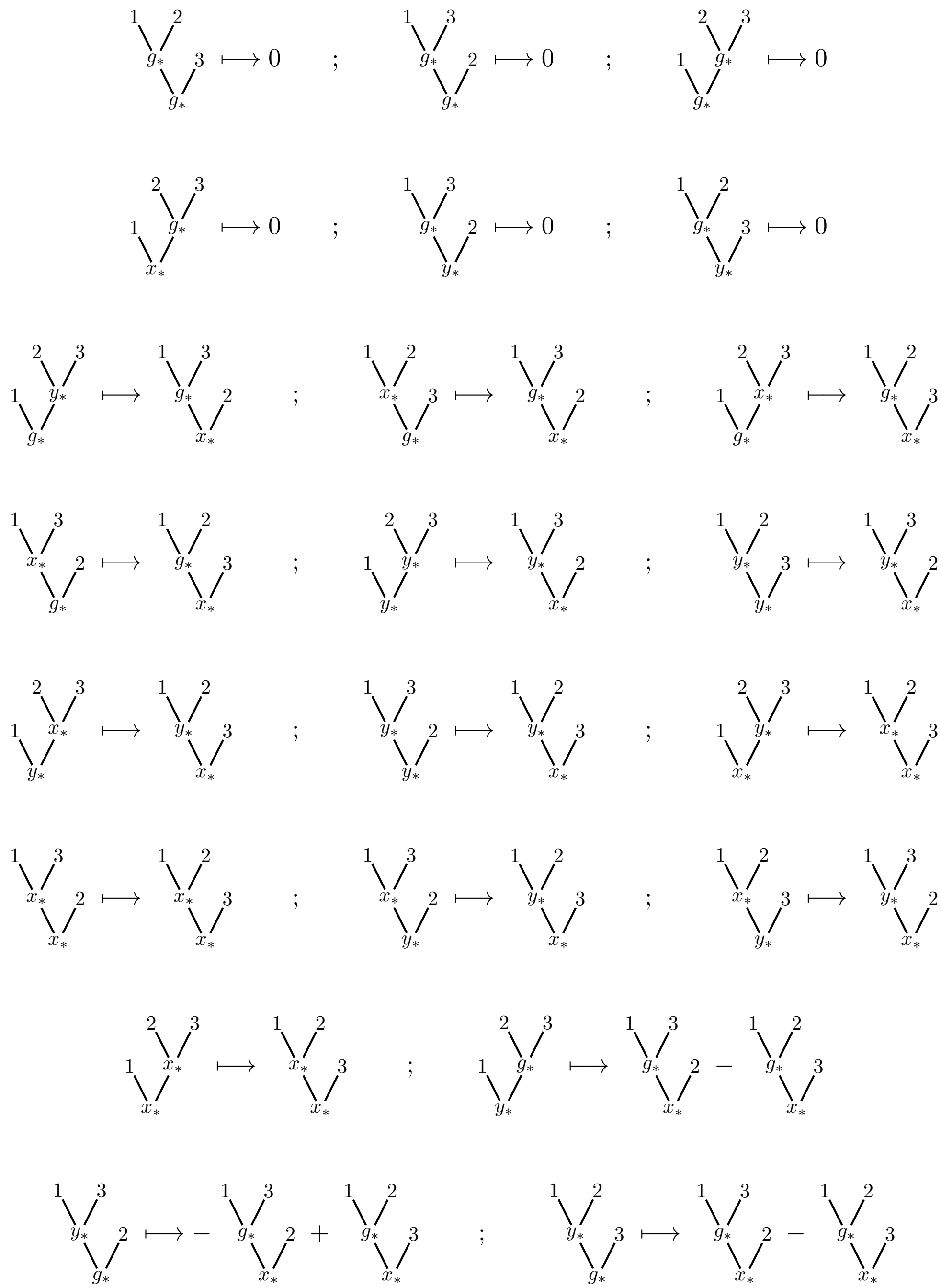


Figure 4.10: The computation of $(R \star^{\Delta} S) \star^{\Delta} T$.

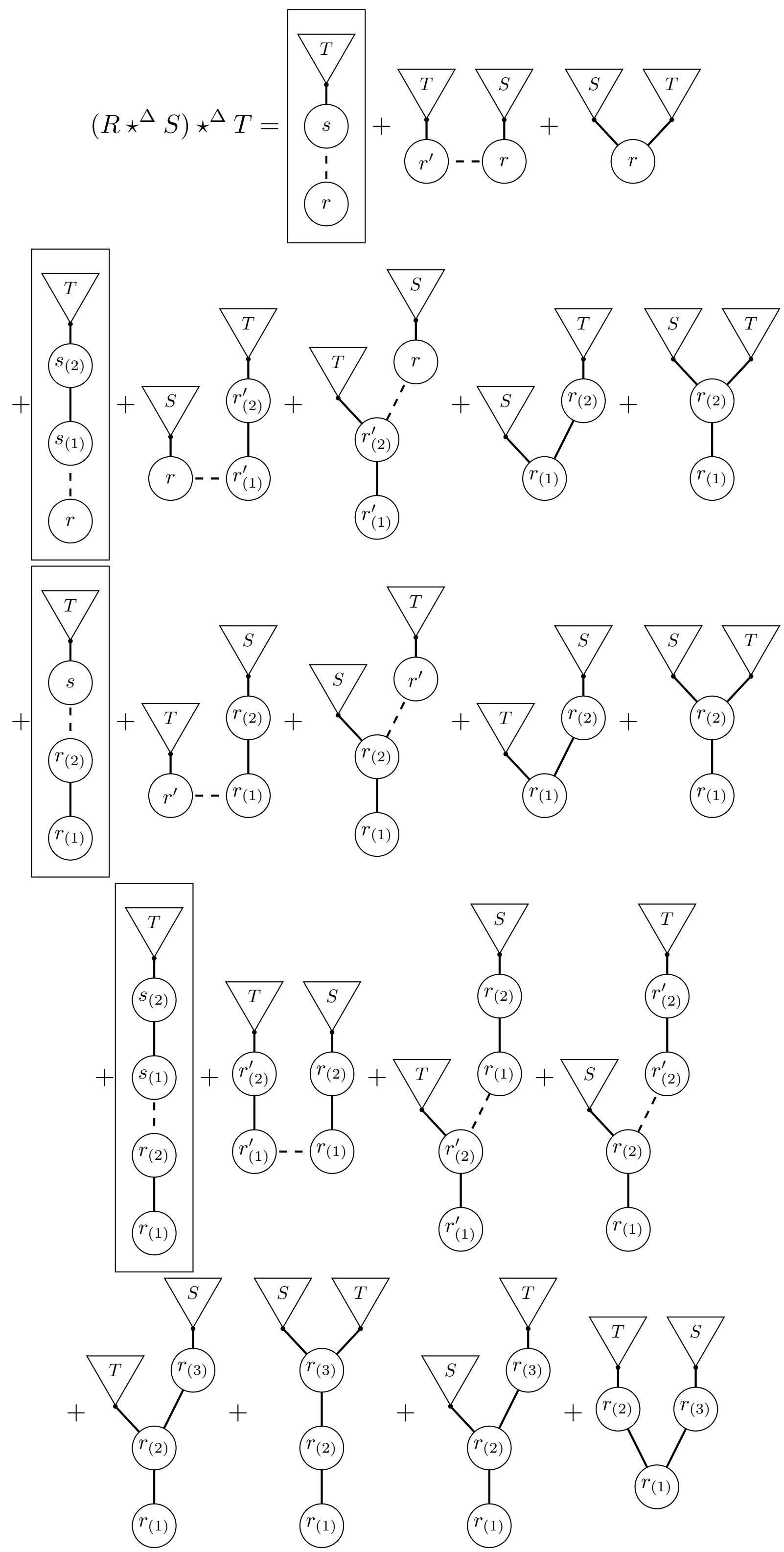


Figure 4.11: The “gplp” rewriting system for $(\mathcal{G}^C)^!$.

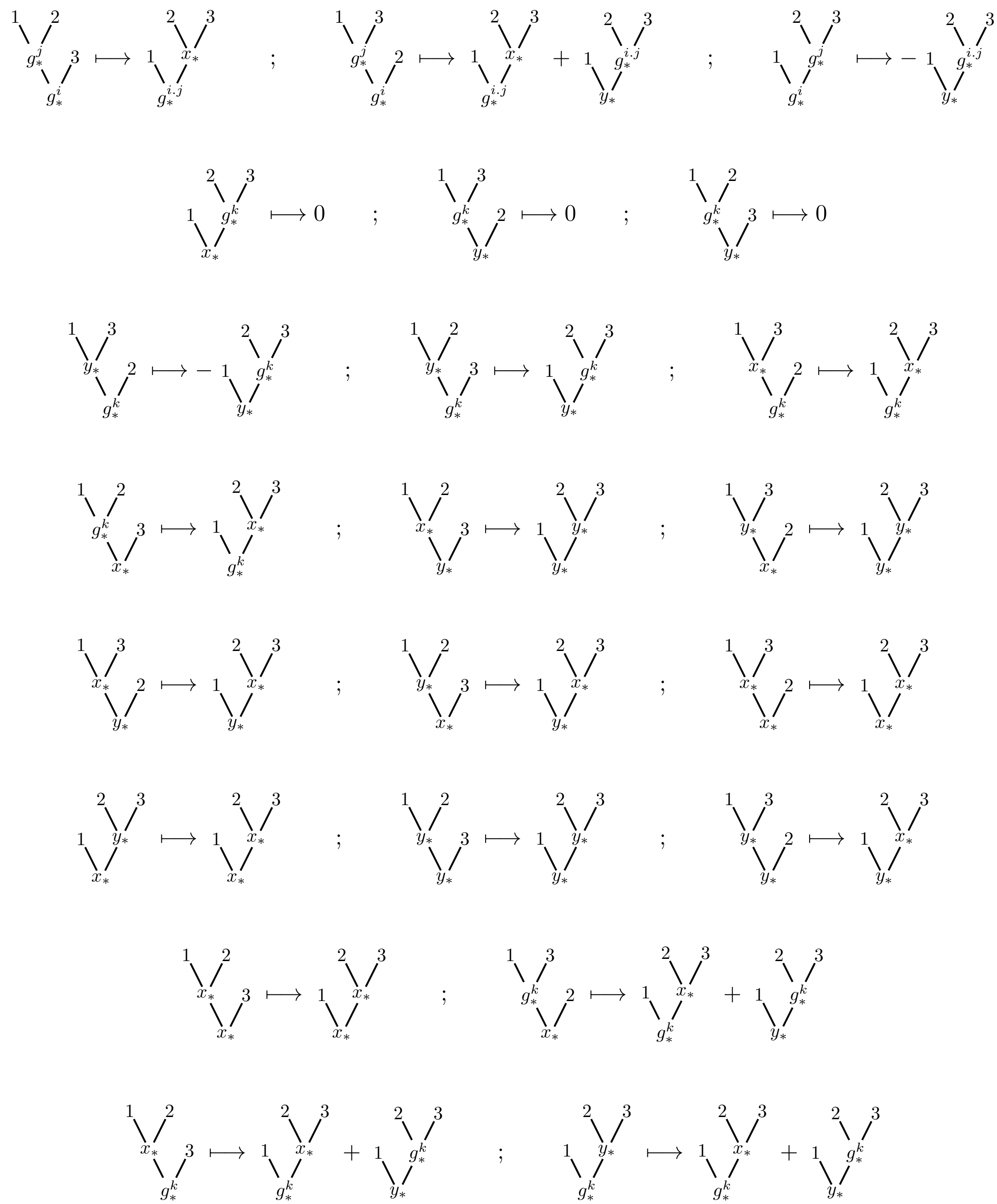


Figure 4.12: The "wprgpl" rewriting system for $(\mathcal{G}^C)^!$.

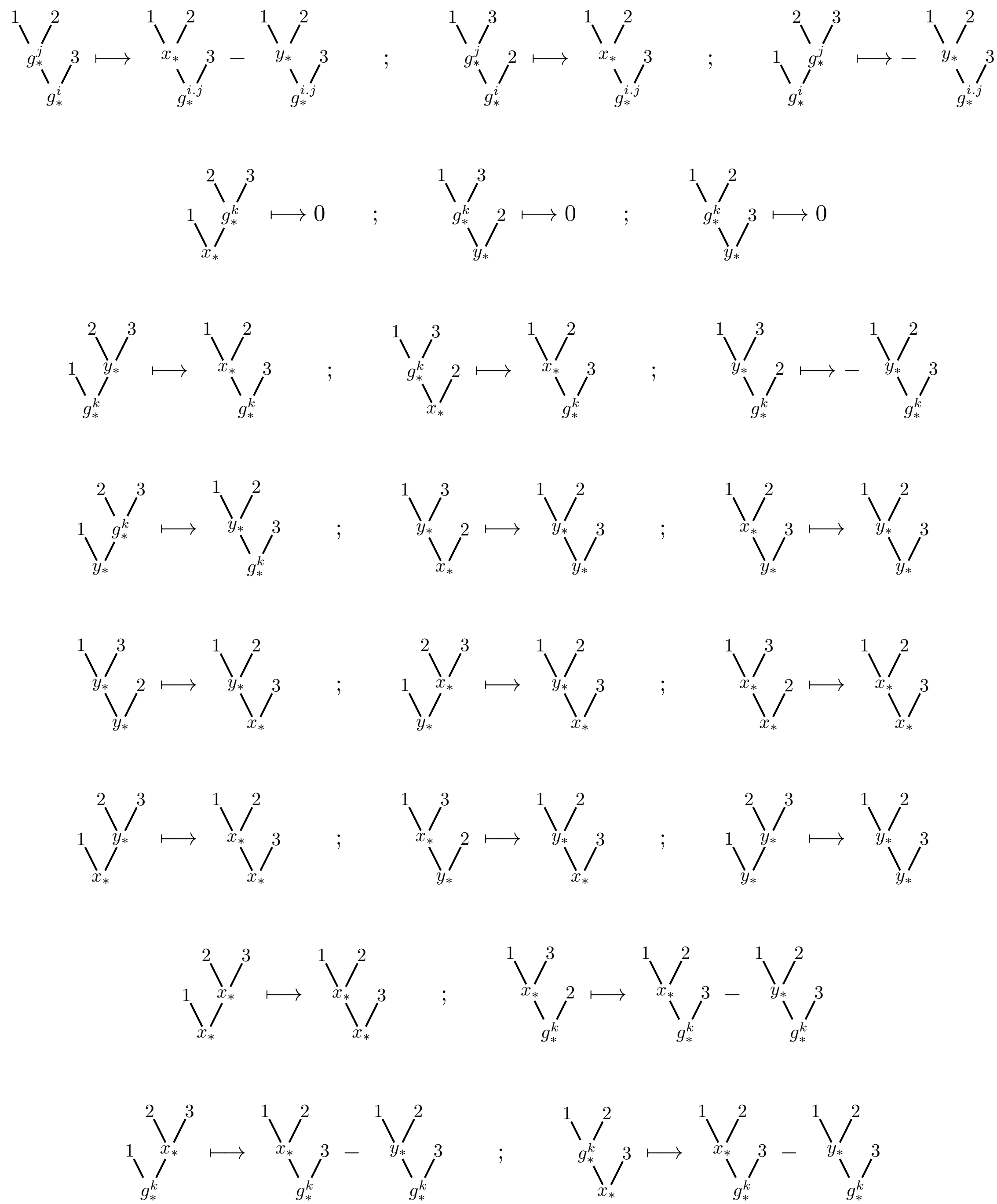


Figure 4.13: The “rgplp” rewriting system for $(\mathcal{G}^C)^!$.

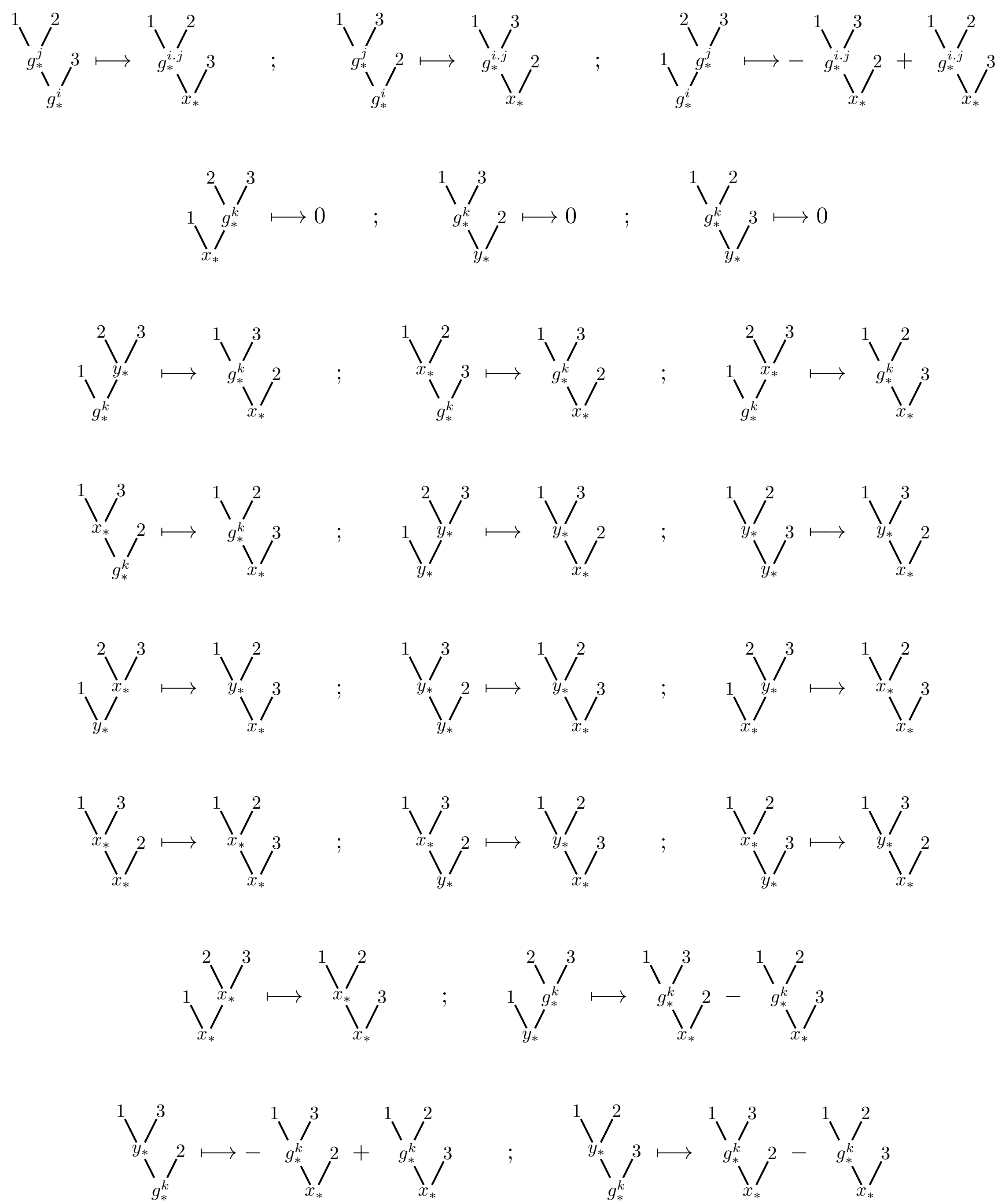


Figure 4.14: The “qpgpl” rewriting system for $\mathrm{ComPreLie}^{!}$.

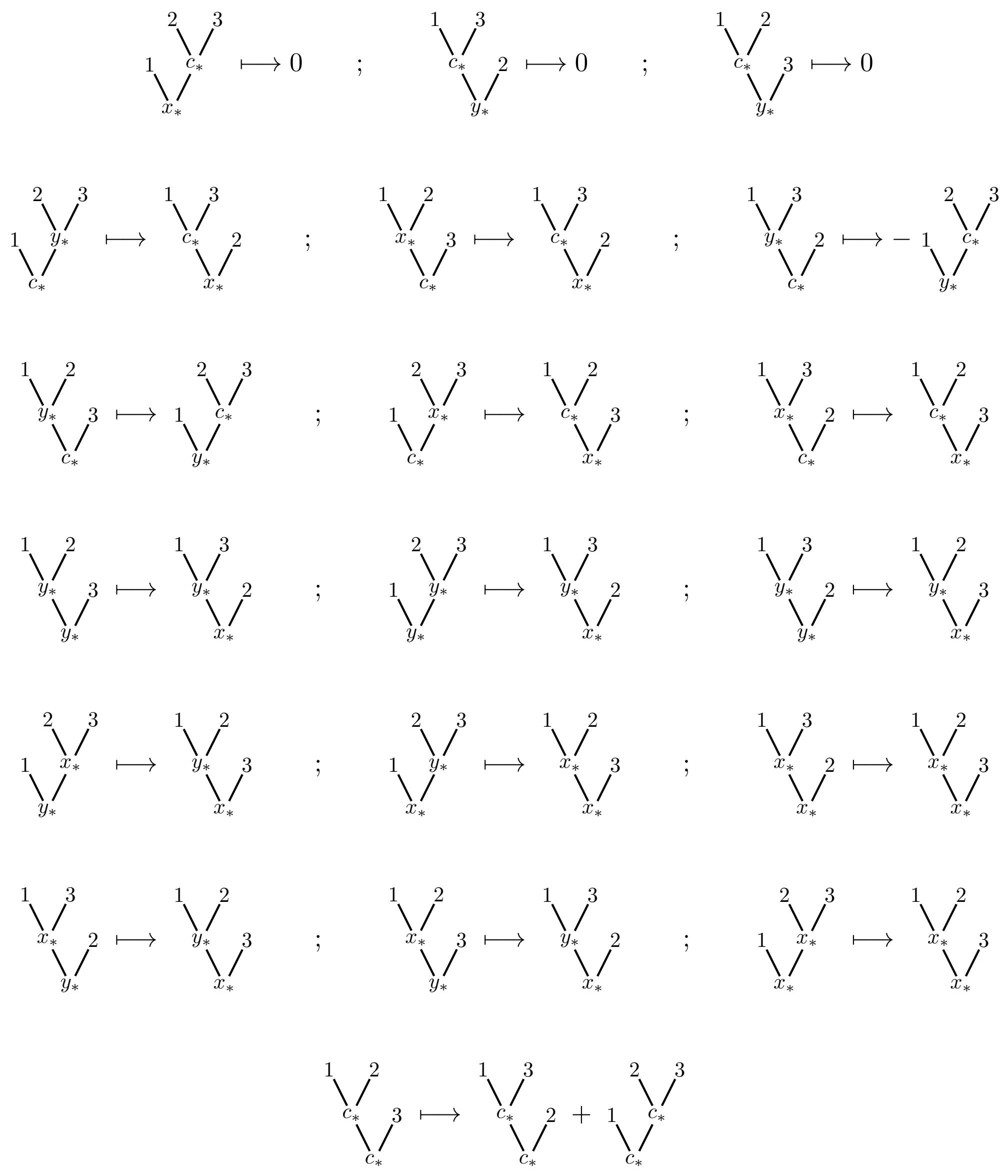
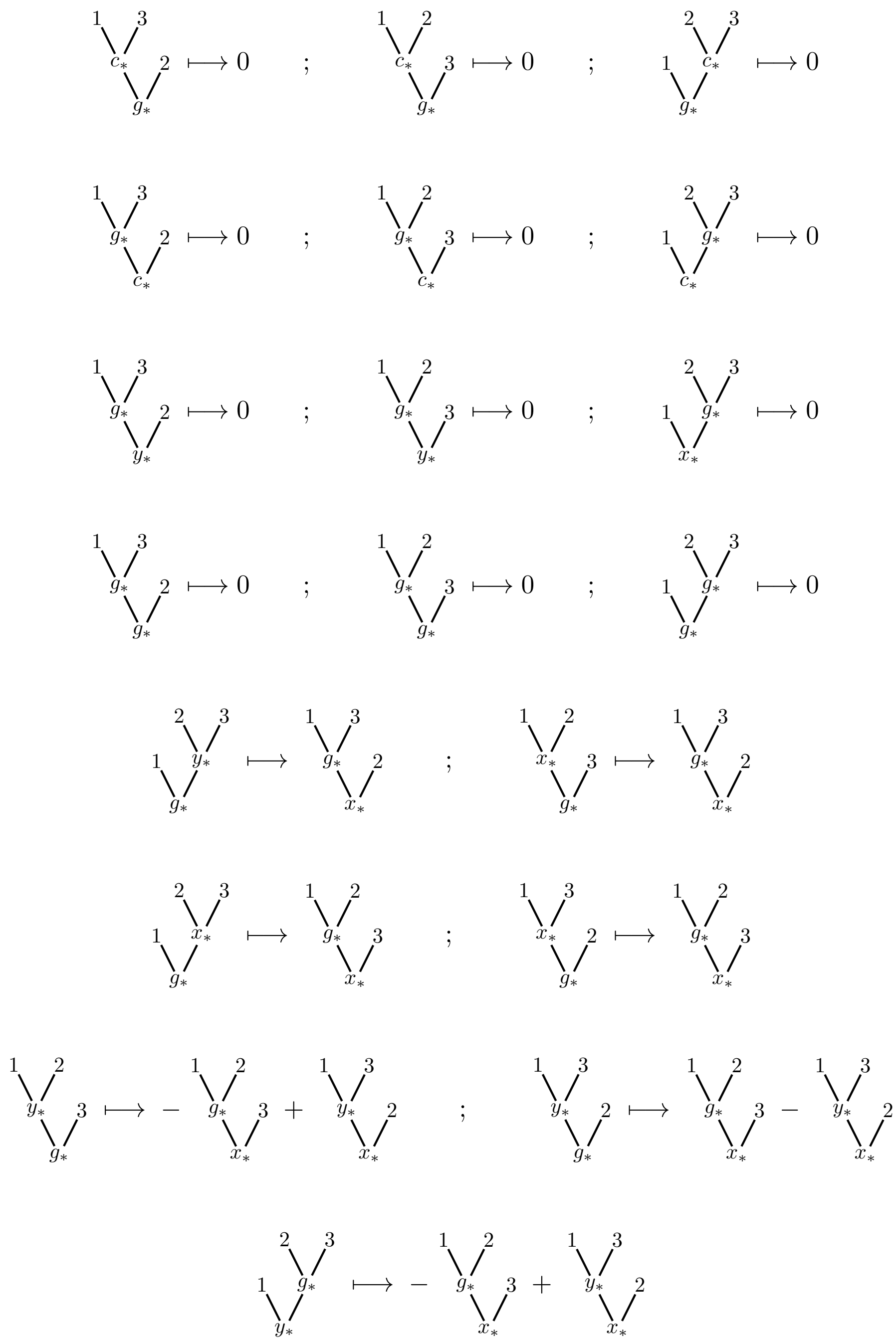


Figure 4.15: The fragment of the "qpgpl" rewriting system for ComGreg$^!$ involving g_*.



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Combinatoire, homotopie et plongement d’opérades

ÉCOLE DOCTORALE MSII

Résumé :

Les opérades algébriques sont un outil algébrique permettant d’encoder certaines variétés d’algèbres non-nécessairement associatives, comme les algèbres de Lie ou les algèbres pre-Lie. De plus, les opérades algébriques peuvent elles-mêmes être vues comme des algèbres dans une catégorie bien choisie. Cette remarque permet l’étude des opérades via les puissants outils de l’algèbre homologique. En parallèle, la catégorie monoïdale telle que les objets en monoïde de cette catégorie sont les opérades est la catégorie des espèces combinatoires munie du pléthysme. Cela permet d’adopter un point de vue très combinatoire sur les opérades donnant ainsi des descriptions très explicites des objets considérés. Ces deux approches synergisent très bien ensemble et cette thèse se concentrera sur l’interaction entre ces deux points de vue. En effet, nous utiliserons des outils homotopiques tels que la dualité de Koszul opéradique pour obtenir des informations combinatoires sur les opérades que nous étudions. Nous utilisons ensuite celles-ci pour obtenir des descriptions combinatoires permettant d’effectuer des calculs explicites. Cette thèse est divisée en trois parties. La première partie est une introduction à la théorie des espèces. Ensuite, nous donnons une introduction à la théorie des opérades algébriques et à la dualité de Koszul opéradique. Enfin, nous calculons certaines descriptions combinatoires d’opérades, et les appliquons pour prouver une conjecture de Dotsenko sur un plongement de l’opérade encodant la structure algébrique sur le champ de vecteurs des variétés de Frobenius.

Combinatoire, Opérade, Espèce combinatoire, Dualité de Koszul, Algèbre de Lie, Algèbre pre-Lie, Arbres enracinés, Arbres de Greg, Hyperarbres.

Résumé en anglais :

Algebraic operads are an algebraic tool for encoding some varieties of algebras, not necessarily associative, such as Lie algebras or pre-Lie algebras. Moreover, algebraic operads can themselves be viewed as algebras in a well-chosen category. This observation allows the study of operads using the powerful tools of homological algebra. Simultaneously, the monoidal category where the monoid objects are operads is the category of combinatorial species equipped with plethysm. This enables a very combinatorial perspective on operads, providing explicit descriptions of the considered objects. These two approaches synergize well together, and this thesis will focus on the interaction between these two viewpoints. Indeed, we will use homotopical tools such as operadic Koszul duality to obtain combinatorial information on the operads we study. We then use this information to derive combinatorial descriptions that allow for explicit computations. This thesis is divided into three parts. The first part is an introduction to the theory of species. Next, we provide an introduction to the theory of algebraic operads and operadic Koszul duality. Finally, we compute descriptions of operads and apply them to prove a conjecture by Dotsenko on embedding the operad encoding the algebraic structure on the vector field of Frobenius manifolds.

Combinatorics, Operad, Combinatorial species, Koszul duality, Lie algebra, Pre-Lie algebra, Rooted trees, Greg trees, Hyper-trees.