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Asymptotic invariants of flows in dimension 3

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Abstract

The aim of this thesis is to prove new results on asymptotic invariants of flows in dimension 3. These new invariants come from invariants used in knot theory, which can be generalised to vector fields following Arnol'd's method.
We first define the bridge number of vector fields and show some results on its continuity, its relation to the asymptotic bridge number and to two other asymptotic invariants, the helicity and the trunkunness of vector fields.
We then prove the existence of the asymptotic genus for right-handed vector fields preserving an ergodic volume. We show that in this case the asymptotic genus is equal to half the helicity.

Résumé

L'objectif de cette thèse est de démontrer de nouveaux résultats sur les invariants asymptotiques de flots en dimension 3. Ces nouveaux invariants proviennent d'invariants utilisés en théorie des nœuds qui peuvent être généralisés aux champs de vecteurs en suivant la méthode d'Arnol'd.
Nous définissons dans un premier temps le «nombre de ponts» des champs de vecteurs et démontrons plusieurs résultats au sujet de sa continuité, sa relation avec le «nombre de ponts» asymptotique et avec deux autres invariants asymptotiques, l'hélicité et le tronc des champs de vecteurs.
Nous prouvons ensuite l'existence du genre asymptotique pour les champs de vecteurs dextrogyres préservant un volume ergodique. Nous montrons que ce genre asymptotique est égal à la moitié de l'hélicité dans ce cas.

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Introduction (en français)

Contexte

L'objet de cette thèse est d'introduire deux nouveaux invariants asymptotiques pour les champs de vecteurs lisses non-singuliers préservant une mesure sur $\mathbb{S}^3$, le second invariant étant défini uniquement pour la classe des champs de vecteurs dextrogyres préservant un volume ergodique. Pour expliquer l'intérêt de la recherche d'invariants de champs de vecteurs préservant une mesure, commençons par un résumé historique. Considérons les équations d'Euler (1755) dans $\mathbb{R}^3$, qui décrivent le champ de vitesses v_t d'un fluide parfait - non-visqueux et incompressible - à partir de l'application des lois de la mécanique de Newton à des volumes infinitésimaux :

$$\begin{cases} \nabla \cdot v_t &= 0 \\ \frac{\partial v_t}{\partial t} + (v_t \cdot \nabla) v_t + \nabla p &= \vec{0} \end{cases}$$

La première équation exprime la conservation de la masse du fluide, tandis que la seconde représente la conservation de la quantité de mouvement. Ici p désigne la pression appliquée au fluide et $(v_t \cdot \nabla)v_t$ la dérivée directionnelle de v_t. En 1858, Helmholtz remarqua une propriété particulière de ces équations [Hel58] : il démontra que la circulation du champ de vitesses v_t le long d'une courbe fermée est conservée au cours du temps. Au niveau infinitésimal, cela signifie que le champ de vorticité $\omega_t = \nabla \times v_t$ est transporté par le champ de vitesses v_t.

Cette observation a d'importantes conséquences : puisque ϕ_X^t, une hypothétique solution du système, est un difféomorphisme préservant le volume pour tout temps t, toute propriété du champ de rotationnel ω_t qui est préservée par difféomorphisme préservant le volume constitue un invariant indépendant du temps du champ de vitesses v_t, et par conséquent, du système original. Parmi ces propriétés possibles, la présence d'une orbite périodique de ω_t représentant un type de nœud spécifique prend une importance particulière. Bien que l'identification d'orbites périodiques isolées puisse être une tâche difficile, un voisinage tubulaire d'un nœud peut également être préservé par le flot du champ de vecteurs, ce qui conduit à la notion de *tube invariant noué*. Cette notion est à l'origine de la théorie des atomes de Thomson [Tho69] et a motivé la fondation de la théorie des nœuds par Tait [Tai77].

Le premier - et le plus simple - invariant de champs de vecteurs découvert est l'hélicité. Il a été introduit dans les années 60 par Woltjer [Wol58], Moreau [Mor61] et Moffatt [Mof69]. Par souci de simplicité, nous donnons sa définition pour un champ de vecteurs statique X préservant un volume Ω sur $\mathbb{S}^3$, bien qu'elle puisse être définie dans n'importe quelle sphère d'homologie rationnelle.

Définition. *L'hélicité de* (X, Ω) *est donnée par*

$$Hel(X, \Omega) = \int_{\mathbb{S}^3} \alpha_X \wedge d\alpha_X$$

où $d\alpha_X = i_X \Omega$.

Comme cette définition repose uniquement sur des notions de calcul différentiel, l'hélicité est facile à calculer ou à approximer. Moffatt a esquissé le lien avec la théorie des nœuds un peu plus tard en calculant l'hélicité de tubes invariants noués [Mof69] et Arnol'd l'a approfondi comme suit [Arn73]. Soit p un point de $\mathbb{S}^3$ et appelons $k_X(p,t)$ la courbe fermée construite en commençant au point p, en suivant l'orbite de p pendant un temps t et en bouclant par un segment géodésique de longueur bornée entre p et $\phi_X^t(p)$. Soit $\text{link}(k_X(p_1,t_1), k_X(p_2,t_2))$ le nombre d'enlacements - le *linking number* - de deux nœuds obtenus de cette façon. Arnol'd [Arn73] et Vogel [Vog03] ont prouvé le théorème suivant :

Théorème (Arnol'd-Vogel)**.** *Soit* X *un champ de vecteurs sur* $\mathbb{S}^3$ *préservant une mesure* μ *telle que* μ *n'est concentrée sur aucune orbite périodique. Alors pour* μ*-presque toute paire de points* (p_1, p_2)*, la limite*

$$lk_X(p_1,p_2) := \lim_{t_1,t_2 \to \infty} \frac{1}{t_1 t_2} link(k_X(p_1,t_1), k_X(p_2,t_2))$$

existe. Si de plus μ *est un volume et si* X *est ergodique par rapport à* μ*, alors pour presque tous* p_1*,* p_2 *cette limite vaut* $\frac{1}{\mu(\mathbb{S}^3)^2} Hel(X, \mu)$.

L'hélicité peut donc être interprétée comme un nombre d'enlacements asymptotique moyen de deux orbites du champ de vecteurs.

Remarquons que si I est votre invariant de nœuds (ou d'entrelacs) préféré, et si pour presque tous $p_1, ..., p_i$ l'invariant $I(k_X(p_1,t_1), ..., k_X(p_i,t_i))$ a un comportement asymptotique de la forme $I_\infty(p_1, ..., p_i) \times t_1^{n_1} ... t_i^{n_i}$ et si la fonction $(p_1, ..., p_i) \mapsto I_\infty(p_1, ..., p_i)$ est intégrable par rapport à la mesure μ, alors son intégrale sur $(\mathbb{S}^3)^i$ est un invariant de (X, μ) par difféomorphisme préservant μ. On peut donc naturellement se demander si en remplaçant le nombre d'enlacements par un autre invariant d'entrelacs ou de nœuds, et en utilisant les mêmes méthodes, on obtiendrait un autre invariant asymptotique. En suivant la méthode d'Arnol'd, Freedman et He ont construit le nombre de croisement asymptotique [FH91], tandis que Dehornoy et Rechtman ont construit la *trunkenness* des champs de vecteurs comme une généralisation du tronc d'un nœud [DR17]. Bien qu'il ne provienne pas d'un invariant de nœuds, il faut également mentionner l'invariant asymptotique de Ruelle construit par Gambaudo et Ghys [GG97]. Ces trois exemples d'invariants ne sont pas fonction de l'hélicité, même sur des champs de vecteurs ergodiques, et sont donc considérés comme des invariants totalement indépendants. En revanche, si l'on considère les ω-signatures des nœuds [GG01], les invariants de selle linéaires [Baa11], et les invariants de type fini de Vassiliev [BM12], les invariants obtenus par la méthode d'Arnol'd sont fonction de l'hélicité au moins pour les champs de vecteurs ergodiques, ce qui peut réduire leur intérêt. Le début d'une explication à ce phénomène d'ubiquité de l'hélicité est venu de deux équipes différentes en 2016

[EPSTdL16],[Kud16]. Elles ont montré que tout invariant suffisamment régulier - dans le sens où sa dérivée de Fréchet est donnée par l'intégrale d'un noyau continu - est une fonction de l'hélicité. Mais comme aucun des invariants mentionnés ci-dessus n'a cette régularité, il pourrait y avoir une explication encore plus forte à l'omniprésence de l'hélicité.

Je présenterai d'abord les résultats concernant le nombre de ponts des champs de vecteurs, qui a été défini à l'aide de la méthode d'Arnol'd.

Nombre de ponts d'un champ de vecteurs

Au cours de cette thèse, je me suis d'abord intéressée au nombre de ponts des nœuds, noté $b_{\text{nœuds}}$ dans la suite, qui est un invariant de même nature que le tronc des nœuds à partir duquel Dehornoy et Rechtman ont construit la *trunkenness* dans [DR17].

Considérons un champ de vecteurs lisse non-singulier X sur $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ et supposons que X préserve une mesure de probabilité μ. Notons ϕ_X^t le flot de X au temps t. Nous appelons *fonction hauteur* sur $\mathbb{S}^3$ une fonction obtenue en précomposant la fonction hauteur standard de $\mathbb{S}^3$, dont les niveaux sont des sphères centrées en l'origine, par un difféomorphisme préservant l'orientation de $\mathbb{S}^3$ - une définition précise sera donnée dans la section 2.2.1. Pour une fonction hauteur h et $t \in]0,1[$, soit $T_X\left(h^{-1}(t)\right)$ l'ensemble (fermé) des points où X est tangent au niveau $h^{-1}(t)$.

Définition. *Pour une fonction hauteur h donnée, le nombre de ponts du champ de vecteurs X pour h est défini par*

$$B_h(X,\mu) = \frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mu\left(\bigcup_{0<t<1}\phi_X^{[0,\epsilon]}\left(T_X\left(h^{-1}(t)\right)\right)\right).$$

On définit le nombre de ponts de (X,μ) par

$$B(X,\mu) = \inf_{h \text{ fonction hauteur}} \frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mu\left(\bigcup_{0<t<1}\phi_X^{[0,\epsilon]}\left(T_X\left(h^{-1}(t)\right)\right)\right).$$

Dans cette thèse, nous prouvons certaines propriétés de cette dernière définition. La première est qu'il s'agit bien d'un invariant, et qu'il est d'ordre un :

Théorème A. *Soit f un $\mathcal{C}^1$-difféomorphisme de $\mathbb{S}^3$, et X_1 et X_2 deux champs de vecteurs préservant respectivement les mesures de probabilité μ_1 et μ_2, telles que $\mu_1 = f^*\mu_2$ et $f\circ\phi_{X_1}^t = \phi_{X_2}^t \circ f$ pour tout $t\in\mathbb{R}$. Alors*

$$B(X_1,\mu_1) = B(X_2,\mu_2).$$

De plus, pour $\lambda\in\mathbb{R}_+^*$, $B(\lambda X_1,\mu_1) = \lambda\, B(X_1,\mu_1)$.

Notons que le nombre de ponts est un invariant de classe $\mathcal{C}^1$ alors que la *trunkenness* est un invariant de classe $\mathcal{C}^0$. Cela vient du fait que nous devons considérer les points de tangence du champ de vecteurs aux niveaux de la fonction hauteur pour calculer le nombre de ponts, et ceux-ci requièrent plus de régularité que la condition de transversalité requise pour la *trunkenness* [DR17]. Une propriété cruciale pour un invariant est sa continuité en le champ de vecteurs et la mesure. Dans notre cas, nous avons le résultat suivant :

Théorème B. *Soit $(X_n, \mu_n)_{n\in\mathbb{N}}$ une suite de champs de vecteurs préservant une mesure de probabilité. Supposons que $(X_n)_{n\in\mathbb{N}}$ converge vers un champ de vecteurs X en topologie $\mathcal{C}^0$, que $(\mu_n)_{n\in\mathbb{N}}$ converge vers une mesure μ faiblement-$*$ et que $\lim_{n\to\infty} B(X_n, \mu_n) = L \in \mathbb{R} \cup \{\infty\}$. Dans ce cas*

$$\lim_{n\to\infty} B(X_n, \mu_n) \leqslant B(X, \mu)\,.$$

Le théorème B donne plus d'options pour calculer le nombre de ponts des champs de vecteurs, comme nous le montrerons dans un exemple avec le cas des *flots de Seifert* dans la section 2.4. De plus, le théorème B implique que le nombre de ponts $B(X, \mu)$ peut être obtenu comme une limite asymptotique lorsque X est ergodique pour un volume μ. Ainsi, nous définissons un nombre de ponts asymptotique :

Théorème C. *Soit (X, μ) un champ de vecteurs sur $\mathbb{S}^3$ préservant un volume ergodique μ. Soit x un point récurrent pour le flot de X et générique pour μ. Alors :*

$$\lim_{T\to\infty} \frac{1}{T} b_{n\oe uds}\,(k_X(x, T)) = B(X, \mu)\,.$$

De même que pour l'hélicité, le théorème C fournit une interprétation du nombre de ponts comme la moyenne du nombre de ponts asymptotique de ses orbites, en les considérant comme des nœuds arbitrairement longs.

Comme nous l'avons expliqué plus haut, un point clé lorsqu'on définit un nouvel invariant est de savoir s'il est indépendant de l'hélicité et des autres invariants déjà connus. Dans le cas du nombre de ponts, nous disposons de ce qui suit :

Théorème D. *Pour une mesure ergodique μ, il n'existe pas de fonction telle que $B(X, \mu) = f(Hel(X, \mu))$.*

Le tronc - l'invariant de nœuds dont dérive la *trunkenness* - et le nombre de ponts sont liés en tant qu'invariants de nœuds : pour un nœud k nous avons $\mathrm{Tronc}(k) \leqslant 2b_{\text{nœuds}}(k)$ en général, avec une égalité pour les nœuds méridionalement petits, c'est-à-dire les nœuds k dont l'extérieur ne contient pas de surface essentielle avec un bord méridional [Oza10]. C'est pourquoi nous abordons également la question d'une relation entre la *trunkenness* et le nombre de ponts des champs de vecteurs. Nous montrerons dans un exemple qu'ils sont indépendants.

Genre des champs de vecteurs dextrogyres

Dans la deuxième partie de cette thèse, j'ai défini un genre asymptotique pour les champs de vecteurs dextrogyres préservant un volume ergodique sur $\mathbb{S}^3$. Étant donné un nœud k dans $\mathbb{S}^3$, il est possible de construire une surface de Seifert, c'est-à-dire une surface plongée orientée dont le bord est k, et de calculer son genre.

Nous appelons *genre de k* le genre minimal que l'on peut obtenir par ce procédé. C'est un invariant de nœuds. Il y a déjà eu plusieurs tentatives pour définir un genre asymptotique à partir de la méthode d'Arnol'd précédemment évoquée. Par exemple les travaux antérieurs de Dehornoy [Deh15] suggèrent que l'ordre de cet invariant est 2, et Dehornoy et Rechtman [DR22] ont prouvé le théorème suivant,

Théorème. *Soit M une trois-sphère d'homologie entière, X un champ de vecteurs dextrogyre sur M et μ une mesure invariante par le flot de X. Si $(\gamma_n)_{n\in\mathbb{N}}$ est une suite d'orbites périodiques dont les longueurs $(t_n)_{n\in\mathbb{N}}$ tendent vers l'infini et telle que $\left(\frac{1}{t_n}\gamma_n\right)_{n\in\mathbb{N}}$ tend vers μ faiblement-$*$, alors la suite $\left(\frac{1}{t_n^2}genus(\gamma_n)\right)_{n\in\mathbb{N}}$ tend vers la moitié de l'hélicité de (X,μ).*

Ce théorème laisse espérer que l'on puisse définir le genre asymptotique pour cette classe particulière de champs de vecteurs. Les champs de vecteurs dextrogyres ont été introduits par Ghys dans [Ghy09] et seront présentés dans le chapitre 1. Très informellement, un champ de vecteurs dextrogyre satisfait que toute paire d'orbites suffisamment longues est positivement enlacée. Bien que cela puisse sembler restrictif, cette classe de champs de vecteurs a la propriété dynamique intéressante que toute collection d'orbites périodiques est le bord d'une surface transverse au flot et intersectant toutes ses orbites en temps fini, c'est-à-dire une section de Birkhoff. Le champ de vecteurs de Hopf, et plus généralement les champs de vecteurs de Seifert sur $\mathbb{S}^3$ sont des champs de vecteurs dextrogyres [Ghy09]. Un autre exemple est donné par l'attracteur de Lorenz. Récemment, Florio et Hryniewicz ont démontré que le flot géodésique d'une 3-sphère est dextrogyre si la courbure est pincée entre deux constantes [FH23].

La stratégie pour construire le genre asymptotique est de considérer un très long arc d'orbite d'un point récurrent pour le flot et générique pour la mesure, et de le fermer artificiellement par une perturbation $\mathcal{C}^1$-petite du champ de vecteurs pour obtenir le nœud $k(x,t_n)$. Nous devons alors montrer que le champ perturbé reste dextrogyre. Ensuite, grâce aux travaux de Dehornoy et Rechtman [DR22], il est possible de calculer le genre de cette orbite fermée particulière, et nous obtenons le théorème :

Théorème E. *Soit X un champ de vecteurs dextrogyre préservant un volume ergodique μ sur $\mathbb{S}^3$. Soit x un point récurrent pour le flot de X et générique pour μ. Alors :*

$$\lim_{n\to\infty}\frac{1}{{t_n}^2}g(k(x,t_n)) = \frac{1}{2}Hel(X,\mu)\,.$$

Bien qu'elle ne soit pas totalement complète, je présente également une tentative de preuve que le genre asymptotique des champs de vecteurs dextrogyres est majoré par la moitié de l'hélicité. À mes yeux, cela pourrait être intéressant car cela utilise des méthodes complètement différentes et fournit une formule générale pour borner le genre de deux nœuds positivement enlacés.

Organisation de la thèse

L'organisation de la thèse est la suivante. Le chapitre 1 est considéré comme un chapitre préalable dans lequel nous expliquons le cas particulier de l'hélicité parmi les invariants de champs de vecteurs dans la section 1.1, et nous présentons des définitions et des résultats concernant les champs de vecteurs dextrogyres dans la section 1.2. Enfin nous présentons quelques exemples à garder à l'esprit pour le reste du texte dans la section 1.3. Dans le chapitre 2, nous expliquons la construction du nombre de ponts des champs de vecteurs et nous prouvons les théorèmes A à D. Le chapitre 3 présente les deux constructions du genre asymptotique dans les sections 3.1 et 3.2. Ce dernier chapitre s'appuie sur le matériel présenté dans les sections 1.1 et 1.2 et peut être lu indépendamment du chapitre 2.

Introduction

Context

The purpose of this thesis is to introduce two new asymptotic invariants for smooth measure preserving non-singular vector fields on $\mathbb{S}^3$, one of them on the specific class of right-handed vector fields preserving an ergodic volume. To explain the interest of the search for invariants of measure preserving vector fields, let us begin with a historical summary. Let us consider the equations of Euler (1755) in $\mathbb{R}^3$, which describe the velocity field v_t of an ideal fluid - characterized by non-viscosity and incompressibility - from the application of Newton's laws of mechanics to infinitesimal volumes:

$$\begin{cases} \nabla \cdot v_t &= 0 \\ \frac{\partial v_t}{\partial t} + (v_t \cdot \nabla) v_t + \nabla p &= \vec{0} \end{cases}$$

The first equation expresses the conservation of the fluid's mass, while the second represents the conservation of momentum, with p standing for the pressure applied to the fluid and $(v_t \cdot \nabla) v_t$ denoting the directional derivative of v_t. In 1858, Helmholtz found a particular property of these equations [Hel58]: he demonstrated that the circulation of the velocity field v_t along a closed curve is preserved over time. At the infinitesimal level, this means that the vorticity field $\omega_t = \nabla \times v_t$ is carried by the velocity field v_t, resulting in the concept of it being *frozen in*.

This observation has significant implications: since ϕ_X^t, the hypothetical solution to the equation, is a volume-preserving diffeomorphism for all times t, any property of the vorticity field ω_t that is preserved under volume-preserving diffeomorphisms constitutes a time-independent invariant of the velocity field v_t, and consequently, of the original system. Among these possible properties, the presence of a periodic orbit of a specific knot type for ω_t is particularly interesting. Although identifying isolated periodic orbits might be a difficult task, a tubular neighbourhood of a knot might also be preserved by the flow of the vector field, leading to the notion of a *knotted invariant tube*. This was at the origin of Thomson's theory of atoms [Tho69] and motivated the foundation of knot theory by Tait [Tai77].

The first - and simplest - invariant of vector fields discovered was helicity. It was introduced in the 60s by Woltjer [Wol58], Moreau [Mor61] and Moffatt [Mof69]. For simplicity we state its definition for a vector field X preserving a volume Ω on $\mathbb{S}^3$, though it can be defined in any rational homology sphere.

Definition. *$Hel(X, \Omega) = \int_{\mathbb{S}^3} \alpha_X \wedge d\alpha_X$ is the helicity of X, where $d\alpha_X = i_X \Omega$.*

Because this definition relies only on differential calculus, helicity is easy to compute or to approximate. Moffatt sketched the connection with knot theory a little later when computing the helicity of vector fields in knotted tubes and Arnol'd deepened it as follows [AK21]. Let us denote by $k_X(p,t)$ a loop starting at the point p, following the orbit of p during a time t and closed by a geodesic segment of bounded length. Denote by $\mathrm{link}(k_X(p_1,t_1), k_X(p_2,t_2))$ the linking number of two knots obtained in this way. Arnol'd [Arn73] and Vogel [Vog03] proved the following theorem:

Theorem (Arnold-Vogel)**.** *Let X be a vector field on $\mathbb{S}^3$ preserving a measure μ so that μ does not charge any periodic orbit. Then for μ-almost pair of points (p_1, p_2), the limit*

$$lk_X(p_1,p_2) := \lim_{t_1,t_2\to\infty} \frac{1}{t_1 t_2} link(k_X(p_1,t_1), k_X(p_2,t_2))$$

exists. Moreover if μ is a volume and X is ergodic with respect to μ, then for almost every p_1, p_2 the limit equals $\frac{1}{\mu(\mathbb{S}^3)^2} Hel(X,\mu)$.

Thanks to this result, helicity can be interpreted as an average linking number of two orbits of the vector field.

Remark that if I is your favorite knot (or link) invariant, and if for almost every $p_1,...,p_i$ the invariant $I(k_X(p_1,t_1),...,k_X(p_i,t_i))$ has an asymptotic behavior of the form $I_\infty(p_1,...,p_i) \times t_1^{n_1}...t_i^{n_i}$ and the function $(p_1,...,p_i) \mapsto I_\infty(p_1,...,p_i)$ is integrable with respect to the measure μ, then its integral on $(\mathbb{S}^3)^i$ is an invariant of (X,μ) under μ-preserving diffeomorphisms. Thus one naturally wonders if replacing the linking number with another link or knot invariant, and using the same methods, would produce another asymptotic invariant. Following Arnol'd's method, Freedman and He constructed the asymptotic crossing number [FH91], while Dehornoy and Rechtman constructed the trunkenness of vector fields as a generalisation of the trunk of a knot [DR17]. Although it does not come from a knot invariant, we should also mention the asymptotic Ruelle invariant constructed by Gambaudo and Ghys [GG97]. These three examples of invariants are not proportional to helicity, even on ergodic vector fields, and thus are totally independent new invariants. On the other hand, when considering the ω-signatures of knots [GG01], linear saddle invariants [Baa11], and Vassiliev's finite type invariants [BM12], the invariants obtained with Arnol'd's method are function of helicity at least for ergodic vector fields, which might reduce their interest. The beginning of an explanation to this phenomenon of ubiquity of the helicity came from two different teams in 2016 [EPSTdL16],[Kud16]. They showed that any invariant which is regular enough - in the sense that its Fréchet derivative is given by the integral of a continuous kernel - is a function of helicity. But since none of the above mentioned invariants has this regularity, there might be an even stronger explanation to the omnipresence of helicity.

I will first present the results about the bridge number of vector fields, which was defined using the method of Arnol'd.

Bridge number of vector fields

During this thesis I was first interested in the bridge number of knots, denoted b_{knots} in what follows, which is an invariant of the same nature as the trunk of knots from which Dehornoy and Rechtman constructed the trunkenness in [DR17].

We consider a smooth non-singular vector field X on $\mathbb{S}^3$ and we suppose that X preserves a probability measure μ. We denote ϕ_X^t the flow of X at the time t. We call *height function* on $\mathbb{S}^3$ a function obtained by pre-composing the standard height function of $\mathbb{S}^3$, whose level sets are spheres centered in the origin in the standard stereographic projection, by a $\mathcal{C}^1$-orientation preserving diffeomorphism - a precise definition will be given in Section 2.2.1. For h a height function and $t \in]0,1[$, we denote $T_X\left(h^{-1}(t)\right)$ the (closed) set of points where X is tangent to the level set $h^{-1}(t)$.

Definition. *For h a height function, the bridge number of the vector field X for h is defined by*

$$B_h(X,\mu) = \frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mu\left(\bigcup_{0<t<1}\phi_X^{[0,\epsilon]}\left(T_X\left(h^{-1}(t)\right)\right)\right).$$

We define the bridge number of (X,μ) by

$$B(X,\mu) = \inf_{h \text{ height function}}\frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mu\left(\bigcup_{0<t<1}\phi_X^{[0,\epsilon]}\left(T_X\left(h^{-1}(t)\right)\right)\right).$$

In this thesis we prove some properties of this last definition. The first one is that it is indeed an invariant, and its order is one:

Theorem A. *Let f be a $\mathcal{C}^1$-diffeomorphism of $\mathbb{S}^3$, and X_1 and X_2 two vector fields that preserve respectively the probability measures μ_1 and μ_2, and so that $\mu_1 = f^*\mu_2$ and $f\circ\phi_{X_1}^t = \phi_{X_2}^t\circ f$ for all $t\in\mathbb{R}$. Then*

$$B(X_1,\mu_1) = B(X_2,\mu_2).$$

Moreover, for $\lambda\in\mathbb{R}_+^*$, $B(\lambda X_1,\mu_1) = \lambda\, B(X_1,\mu_1)$.

Note that the bridge number is a $\mathcal{C}^1$-invariant whereas the trunkenness is a $\mathcal{C}^0$-invariant. This comes from the fact that we have to consider the tangency points of the vector field to the level sets of the height function to compute the bridge number, and this requires more regularity than the transversality condition required for the trunkenness [DR17]. A crucial property for an invariant is its continuity in the vector field and the measure. In our case we have the following result:

Theorem B. *Let $(X_n,\mu_n)_{n\in\mathbb{N}}$ be a sequence of vector fields preserving a probability measure on $\mathbb{S}^3$. Suppose that $(X_n)_{n\in\mathbb{N}}$ converges to a vector field X in the $\mathcal{C}^0$-topology and that $(\mu_n)_{n\in\mathbb{N}}$ converges to μ weakly-$*$, and that $\lim_{n\to\infty} B(X_n,\mu_n) = L \in \mathbb{R}\cup\{\infty\}$. Then*

$$\lim_{n\to\infty} B(X_n,\mu_n) \leqslant B(X,\mu).$$

Theorem B gives more options to compute the bridge number of vector fields, as we will show in an example with the case of *Seifert flows* in Section 2.4. Also, Theorem B implies that the bridge $B(X,\mu)$ can be obtained as an asymptotic limit when μ is an ergodic volume. In this direction, we define an asymptotic bridge number:

Theorem C. *Let (X,μ) be a volume preserving vector field on $\mathbb{S}^3$ and suppose that X is ergodic with respect to μ. Let x be a recurrent point for the flow of X and generic for μ. Then:*

$$\lim_{T\to\infty} \frac{1}{T} b_{knots}\left(k_X(x,T)\right) = B(X,\mu)\,.$$

As it was the case for helicity, Theorem C provides us with an interpretation of the bridge number as the average of the asymptotic bridge number of its orbits, seen as arbitrarily long knots.

As we explained above, a key point when one defines a new invariant is to know whether it is independent from helicity and other known invariants. We have the following:

Theorem D. *For μ an ergodic measure, there is no function so that $B(X,\mu) = f(Hel(X,\mu))$.*

The trunkenness and the bridge number are related as knot invariants: for a knot k we have $\mathrm{Trunk}(k) \leqslant 2b_{\mathrm{knots}}(k)$ in general, with an equality for meridionally small knots, that is to say knots so that the exterior of the knot does not contain an essential surface with meridional boundary [Oza10]. This is why we also address the question of a relation between the trunkenness and the bridge number of vector fields. In Section 2.4 we show in an example that they are independent.

Genus of right-handed vector fields

In the second part of this thesis I define an asymptotic genus for smooth non-singular right-handed vector fields preserving an ergodic volume on $\mathbb{S}^3$. Given a knot k in $\mathbb{S}^3$, it is possible to construct a Seifert surface, i.e. an oriented embedded surface whose boundary is k, and to compute its genus. We call *genus of k* the minimal genus that we can obtain with this process. It is a knot invariant. There where already several attempts to define an asymptotic genus by mean of the method of Arnol'd. For instance previous works from Dehornoy [Deh15] suggest that the order of this invariant is 2, and Dehornoy and Rechtman [DR22] proved the following theorem:

Theorem. *Let M be a 3-manifold that is an integer homology sphere, X a non-singular right-handed vector field on M and μ an X-invariant measure. If $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence of periodic orbits whose lengths $(t_n)_{n\in\mathbb{N}}$ tend to infinity and such that $\left(\frac{1}{t_n}\gamma_n\right)_{n\in\mathbb{N}}$ tends to μ in the weak-$*$ sense, then the sequence $\left(\frac{1}{t_n^2} genus(\gamma_n)\right)_{n\in\mathbb{N}}$ tends to half the helicity of (X,μ).*

This result gives hope that one could define the asymptotic genus for this particular class of vector fields. Right-handed vector fields were introduced by Ghys in [Ghy09] and will be presented in Chapter 1. Roughly speaking, a right-handed vector field satisfies that any pair of long enough orbits are positively linked. Although this may seem restrictive, this class of vector fields has the interesting dynamical property that any collection of periodic orbits bounds a surface transverse to the flow and intersecting all of its orbits in bounded time, that is to say a Birkhoff section. The Hopf vector field, and more generally Seifert vector fields on $\mathbb{S}^3$ are right-handed [Ghy09]. Another example is given by the Lorenz attractor. Recently, Hryniewicz and Florio demonstrated that the geodesic flow of a 3-sphere is right-handed if the curvature is pinched between two constants [FH23].

The strategy to construct the asymptotic genus is to consider a very long arc of orbit of a recurrent point generic for the measure and to artificially close it with a $\mathcal{C}^1$-perturbation of the vector field to obtain the knot $k(x, t_n)$. We have to show that the perturbed field remains right-handed. Then thanks to the work of Dehornoy and Rechtman [DR22] it is possible to compute the genus of this particular closed orbit, and we obtain the following:

Theorem E. *Let X be a smooth right-handed non-singular vector field X preserving a smooth ergodic volume μ on $\mathbb{S}^3$. Let x be a recurrent point for the flow of X and generic for μ. Then*

$$\lim_{n\to\infty} \frac{1}{{t_n}^2} g(k(x, t_n)) = \frac{1}{2} Hel(X, \mu)\,.$$

Although it is not fully complete, I also present an attempt of a proof that the asymptotic genus of right-handed vector fields is bounded by half of the helicity. To my eyes this might be interesting as it uses completely different methods and provides us with a general formula to bound the genus of two positively linked knots.

Organization of the thesis

The organization of the thesis is as follows. Chapter 1 is thought as a prerequisite chapter where we explain the special case of helicity among vector fields invariants in Section 1.1, present definitions and results concerning right-handed vector fields in Section 1.2 and finally present some examples to have in mind for the rest of the text in Section 1.3. In Chapter 2 we explain the construction of the bridge number of vector fields and we prove Theorems A to D. Chapter 3 presents the two constructions of the asymptotic genus in Sections 3.1 and 3.2. This last chapter relies on the material presented in Sections 1.1 and 1.2 and can be read appart from Chapter 2.

Chapter 1

Prerequisites and examples

In this chapter we introduce the main ideas and notions that will be used in the thesis. At first it is necessary to understand the idea of Arnol'd and to examine the special case of helicity, which we do in Section 1.1. Then in Section 1.2 we present the particular class of right-handed vector fields, for which we will define the asymptotic genus in Chapter 3. Lastly in Section 1.3 we present a class of examples of flows for which it is possible to compute the trunkenness and the bridge number.

1.1 Vector fields invariants and helicity

1.1.1 Helicity

For simplicity and because we are interested in asymptotic invariants of vector fields in $\mathbb{S}^3$, we will be working in $\mathbb{S}^3$, though helicity exists for vector fields in homology spheres of any nature, and can also be extended to submanifolds of $\mathbb{S}^3$ that have a boundary, under the condition that the vector field remains tangent to the boundary. So let X be a smooth measure-preserving vector field on $\mathbb{S}^3$, preserving a volume form μ. One can define a 2-form β_X using the vector field X by mean of the formula $\beta_X = i_X\mu$. Because X preserves the volume, the Lie derivative of μ along X is zero and Cartan's formula implies that β_X is closed, thus exact since $H^2(\mathbb{S}^3)$ is trivial. Hence there exists a 1-form α_X - a *potential form* of X - such that $d\alpha = \beta_X$. Of course it is not unique since other potential forms can be obtained by adding an exact form, but one can show that the integral

$$\int_{\mathbb{S}^3} \alpha_X \wedge \beta_X$$

is independent of the choice of the potential form α_X. This leads to the definition of helicity:

Definition 1.1. *The helicity $Hel(X, \mu)$ of (X, μ) is given by*

$$Hel(X, \mu) := \int_{\mathbb{S}^3} \alpha_X \wedge d\alpha_X$$

where α_X is a potential form of X.

It is important to remark that by definition, helicity is significantly influenced by the choice of the invariant volume μ. Different choices of invariant volumes lead to distinct helicity values.

Proposition 1.2. *The helicity of (X, μ) is invariant under the action of μ-preserving diffeomorphisms.*

As Pierre Dehornoy remarks in [Deh15], while the above definition is succinct, it might seem enigmatic. Following his summary text about asymptotic invariants, we present an alternative understanding of helicity in the particular case of $\mathbb{R}^3$ endowed with an auxiliary metric, denoted g. In this case μ is the volume associated to the metric g. The volume-preservation of X writes $\nabla \cdot X = 0$ in this case and this implies that X is the rotational of some vector potential w, that is to say $\nabla \times w = X$. Then one can check that the definition of helicity boils down to $\mathrm{Hel}(X, \mu) = \int w \cdot X d\mu$. But on $\mathbb{R}^3$, we have a fundamental example of a vector potential given by the Biot-Savard formula:

$$w(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus \{x\}} \frac{X(y) \times (x-y)}{\|x-y\|^3} dy\,.$$

Using this potential in the definition of helicity and the relation $x \cdot (y \times z) = \det(x, y, z)$, one finally gets:

$$\mathrm{Hel}(X, \mu) = \frac{1}{4\pi} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta} \frac{\det(X(x), X(y), x-y)}{\|x-y\|^3} dxdy$$

where Δ is the diagonal set $\{(x, x) | x \in \mathbb{R}^3\}$. Before we continue by presenting the method of Arnol'd, we need to introduce the linking number and the linking forms.

1.1.2 Linking forms and system of short paths

Linking number and linking forms The linking number stands out as the most straightforward invariant for 2-component links. Given two separate knots k_1 and k_2 in $\mathbb{R}^3$, their linking number, denoted as $\mathrm{link}(k_1, k_2)$, can be described using multiple equivalent definitions [Rol76]:

- the number of signed crossings of a projection of the knots k_1, k_2 on a plane;
- the algebraic intersection number of k_1 with a Seifert surface for k_2 - and conversely;
- the degree of the Gauss map defined on $\mathbb{S}^1 \times \mathbb{S}^1$ and given by $(t_1, t_2) \mapsto \frac{\gamma_1(t_1)-\gamma_2(t_2)}{\|\gamma_1(t_1)-\gamma_2(t_2)\|}$ where γ_1 and γ_2 are any parametrizations of the knots k_1, k_2;
- the Gauss integral (note the similarity with the previous computation for helicity in $\mathbb{R}^3$):

$$\frac{1}{4\pi} \int\int_{\mathbb{S}^1 \times \mathbb{S}^1} \frac{\det\left(\dot{\gamma}_1(t_1), \dot{\gamma}_2(t_2), \gamma_2(t_2) - \gamma_1(t_1)\right)}{\|\gamma_2(t_2) - \gamma_1(t_1)\|^3} dt_1 dt_2$$

An important remark is that the above definitions of the linking number also work in $\mathbb{S}^3$: the point is to perturb the knots so that they do not pass through the point at infinity (in a given stereographic projection), and then to compute the linking number of their stereographic projections. Actually, another way to compute the linking number is given by the integration of a Gauss linking form on $\mathbb{S}^3 \times \mathbb{S}^3$.

Definition 1.3. *A Gauss linking form on $\mathbb{S}^3$ is a double form $\mathcal{L}$ on $\mathbb{S}^3 \times \mathbb{S}^3$ such that for any two disjoint closed oriented one-dimensional submanifolds γ_1, γ_2 of $\mathbb{S}^3$, the equality:*

$$\mathit{link}(\gamma_1, \gamma_2) = \int_{\gamma_1} \int_{\gamma_2} \mathcal{L}$$

holds.

Example. According to the previous definitions of the linking number and the example at the end of Section 1.1.1, if x, $y \in \mathbb{R}^3$, $V \in T_x\mathbb{R}^3$ and $W \in T_y\mathbb{R}^3$ the 2-form given by

$$\mathcal{L}_{\mathbb{R}^3} = \frac{1}{4\pi} \frac{V \cdot (W \times (x - y))}{\|x - y\|^3}$$

is a Gauss linking form on $\mathbb{R}^3$.

In his article [Vog03], Vogel proved using the theory of Poisson equation on Riemannian manifolds, that Gauss linking form exist on any closed oriented three-dimensional manifold M having the real cohomology of a three-sphere, that is to say $H^1(M, \mathbb{R}) = H^2(M, \mathbb{R}) = 0$. We will not present the detail of this construction but we will list some of its properties that will be useful in Chapter 3.

Let us consider the projections:

$$\pi_L,\ \pi_R : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3$$

The bundle of double forms over $\mathbb{S}^3 \times \mathbb{S}^3$ is the tensor product of the pullbacks of the bundle $\Lambda^*(T^*\mathbb{S}^3)$ of differential forms by these two projections:

$$\pi_L^* \left(\Lambda^*(T^*\mathbb{S}^3)\right) \otimes \pi_R^* \left(\Lambda^*(T^*\mathbb{S}^3)\right)$$

Let g be a Riemannian metric on $\mathbb{S}^3$. There are left and right exterior derivative operators d_L and d_R which act on double forms, and moreover g induces the left and right Hodge star operators $*_L$ and $*_R$. Denote by $\mathcal{G}$ the Green form of the Hodge Laplacian Δ associated with g. This form is constructed explicitly with help of the distance function in Chapter V of [dR84]. $\mathcal{G}$ is an integrable double form, smooth outside the diagonal and satisfying the pointwise bound:

$$\|\mathcal{G}_{p,q}\|_\infty = O(\mathrm{dist}(p, q)^{-1})$$

where $\|\alpha_{p,q}\|_\infty$ is the maximum of the absolute value of the coefficients of α in an orthogonal basis, evaluated in the point $(p, q) \in (\mathbb{S}^3 \times \mathbb{S}^3) \setminus \Delta$, Δ being the diagonal of $\mathbb{S}^3 \times \mathbb{S}^3$. Then a Gauss linking form is the double form defined by

$$\mathcal{L} := *_R d_R \mathcal{G}\,.$$

It is immediate that $\mathcal{L}$ is integrable and smooth outside of the diagonal in $\mathbb{S}^3 \times \mathbb{S}^3$, and satisfies the pointwise bound:

Lemma 1.4.
$$\|\mathcal{L}_{p,q}\|_\infty = O(dist(p,q)^{-2})\,.$$

The other property of the Gauss linking form we need in this thesis is the boundedness of its integral on pairs of short geodesics. Let g be a Riemannian metric on $\mathbb{S}^3$ and denote by $r_{inj}(g)$ the injectivity radius of g.

Lemma 1.5. *Fix $r_0 := r_{inj}(g)/100$. Then there exists a constant $C(g) > 0$ depending only on g so that for any pair of distinct geodesics γ_1, γ_2 of length less than r_0 and intersecting at most one time,*

$$\left| \int_{\gamma_1 \times \gamma_2} \mathcal{L} \right| < C(g)\,.$$

As noted Prasad in [Pra22], Lemma 1.5 is stated in the proof of [[Vog03], Theorem 5], and is proven more precisely in the article of Prasad [[Pra22], Lemma 2.3].

System of short paths. We explained in the introduction that there is a suitable way to close up the arc of trajectories of a flow in order to get a knot. This can be achieved using a *system of short paths*, introduced by Arnol'd [AK21] and refined by Vogel [Vog03]. Here we present briefly the definition and properties of a *system of short paths*. The following definition comes directly from [Vog03].

Definition 1.6. *A set $\mathcal{S}$ of paths on $\mathbb{S}^3$ is a system of short paths if it has the following properties:*

1. *For any two points p, $q \in \mathbb{S}^3$ there is a unique path $\sigma(p,q) \in \mathcal{S}$ starting at p and ending at q.*
2. *Each path in $\mathcal{S}$ is piecewise differentiable.*
3. *The paths depend continuously on their endpoints almost everywhere and the following limits exist in the L^1-sense:*

$$\lim_{T,S\to\infty} \frac{1}{TS} \int_{\phi_X^{[0,T]}(x)} \int_{\sigma(\phi_X^S(y),y)} |\mathcal{L}| = 0 \tag{1.1}$$
$$\lim_{T,S\to\infty} \frac{1}{TS} \int_{\sigma(\phi_X^T(x),x)} \int_{\phi_X^{[0,S]}(y)} |\mathcal{L}| = 0 \tag{1.2}$$
$$\lim_{T,S\to\infty} \frac{1}{TS} \int_{\sigma(\phi_X^T(x),x)} \int_{\sigma(\phi_X^S(y),y)} |\mathcal{L}| = 0 \tag{1.3}$$

4. *The sets*

$$I_{X,\mathcal{S}} = \{(x,y) \in \mathbb{S}^3 \times \mathbb{S}^3 | \phi_X^{[0,T]}(x) \cap \sigma(\phi_X^S(y),y) \neq \emptyset\}$$
$$I_{\mathcal{S},Y} = \{(x,y) \in \mathbb{S}^3 \times \mathbb{S}^3 | \sigma(\phi_X^T(x),x) \cap \phi_X^{[0,S]}(y) \neq \emptyset\}$$
$$I_{\mathcal{S},\mathcal{S}} = \{(x,y) \in \mathbb{S}^3 \times \mathbb{S}^3 | \sigma(\phi_X^T(x),x) \cap \sigma(\phi_X^S(y),y) \neq \emptyset\}$$

have measure zero at any given time T, respectively S.

Since this definition is technical, we explain the aim of such requirements. Point 2 ensures that integrating along a short path is possible. Thanks to the continuity condition in point 3, the integrals of the linking form on artificially closed orbits are measurable functions on $\mathbb{S}^3 \times \mathbb{S}^3$ and conditions (1.1) to (1.3) ensure that the short paths do not count for the asymptotic linking number. Then point 4 avoids having short paths intersecting either one another or the flow lines too often.

Vogel showed that a subset of the set of geodesics is a system of short paths:

Theorem 1.7 (Vogel, [Vog03])**.** *Let $\mathcal{S}$ be the set consisting of a geodesic of minimal length having starting point p and ending point q for any p, $q \in \mathbb{S}^3$. Then $\mathcal{S}$ is a system of short paths.*

Now we can use this result to relate helicity to the asymptotic linking number.

1.1.3 Connection between helicity and the linking number

In his work about helicity, Moffatt had already figured out that helicity could be interpreted as an average linking number. To refine Moffatt's idea and to get around the fact that the orbits of the flow are generally not closed curves, Arnol'd introduced a technique to transform open segments of orbits into closed loops [Arn73]. Although the initial definition lacked some precision to achieve the result, Vogel presented later the enhancement of the system of short paths that we presented in the precedent section.

Definition 1.8. *We define the knot $k_X(p,t)$ as the concatenation of the segment of orbit $\phi_X^{[0,T]}(p)$ with the short path $\sigma(\phi_X^T(p), p)$ in the set $\mathcal{S}$ given by Theorem 1.7.*

According to the definition of the system of short paths $\mathcal{S}$, this is a well-defined knot for almost every positive time T.

Theorem 1.9 (Arnol'd-Vogel [Arn73] [Vog03])**.** *Let X be a vector field on $\mathbb{S}^3$ preserving a measure μ not charging any periodic orbit. Then for μ-almost every pair of points p_1, p_2, the limit*

$$lk_X(p_1,p_2) := \lim_{t_1,t_2\to\infty} \frac{1}{t_1 t_2} link(k_X(p_1,t_1), k_X(p_2,t_2))$$

exists. Moreover, if μ is a volume form and if X is ergodic with respect to μ, then for almost every p_1, p_2 the limit equals $\frac{1}{\mu(\mathbb{S}^3)^2} Hel(X,\mu)$.

The proof of this theorem is an application of Birkhoff's ergodic theorem. First we know that the linking number of the knots $k_X(p_1,t_1)$, $k_X(p_2,t_2)$ is given by the integral of the Gauss linking form along the knots. Because of the choice of the closing short paths and the fact that we are considering arbitrary long pieces of orbits and dividing by the product of times, only the following integral may have a (strictly) positive limit:

$$\frac{1}{t_1 t_2}\int_0^{t_1}\int_0^{t_2} \frac{\det\left(X(\phi_X^{s_1}(p_1)), X(\phi_X^{s_2}(p_2)), \phi_X^{s_2}(p_2)-\phi_X^{s_1}(p_1)\right)}{\|\phi_X^{s_2}(p_2)-\phi_X^{s_1}(p_1)\|^3} ds_1 ds_2\,,$$

and this integral is a time average. One has then to check that the function $(x,y) \mapsto \frac{\det(X(x),Y(y),y-x)}{\|y-x\|^3}$ is integrable on $\mathbb{S}^3 \times \mathbb{S}^3 \setminus \Delta$ before concluding.

An example of computation of the helicity will be presented in Section 1.3.

1.2 Right-handed vector fields

Right-handed vector fields are a special class of non-singular vector fields on closed, oriented rational homology three-spheres introduced by Étienne Ghys [Ghy09]. To say it shortly, non-singular vector field X is right-handed if any pair of long pieces of distinct recurrent trajectories of X, after being closed up in an appropriate way, form a pair of embedded knots with positive linking number. One example is the vector field generating the Hopf fibration on $\mathbb{S}^3$, where any pair of orbits forms a Hopf link with linking number 1. A motivation to the study of these vector fields is given by the following theorem of Ghys:

Theorem 1.10 ([Ghy09])**.** *Let X be a right-handed vector field in $\mathbb{S}^3$. Then any finite collection of periodic orbits is a fibered link. More precisely, any finite collection of periodic orbits is the binding of some Birkhoff section.*

Here we present the definition of right-handedness as explained by Anna Florio and Umberto Hryniewicz in [FH23], focusing on the case of (non-singular) vector fields X on $\mathbb{S}^3$ as it will be our preoccupation later on this thesis. The advantage of this equivalent definition compared to Ghys' original definition, is that we do not need to deal with the existence of linking forms. We also present briefly Ghys' definition, in order to state a theorem from Ghys that we need in Section 3.1.

1.2.1 Transverse rotation numbers

Let X be a smooth vector field on $\mathbb{S}^3$ and denote its flow by ϕ_X^t. Let γ be a non-constant periodic orbit of ϕ_X^t of primitive period $T > 0$. We think of γ as a map $\gamma : \mathbb{R}/T\mathbb{Z} \mapsto \mathbb{S}^3$. On a small tubular neighbourhood N of γ, consider the *tubular coordinates* $(t, z = x + iy) \in \mathbb{R}/T\mathbb{Z} \times \mathbb{C}$ such that $dt \wedge dx \wedge dy > 0$ and $\phi_X^t(\gamma(0)) = (t, 0)$. For every $\theta_0 \in \mathbb{R}$, let $t \mapsto \theta(t)$ be the continuous real valued function defined by $\theta(0) = \theta_0$ and

$$D\phi_X^t(0,0) \cdot (0, e^{i\theta_0}) \in \mathbb{R}(1,0) + \mathbb{R}_+(0, e^{i\theta(t)})\,.$$

For a 1-form $y \in H^1(N \setminus \gamma, \mathbb{R})$ homologous to $pdt + qd\theta$ we define the *transverse rotation number of γ with respect to y*:

$$\rho^y(\gamma) = \frac{T}{2\pi}\left(p + q \lim_{t\to+\infty} \frac{\theta(t)}{t}\right)\,.$$

As Hryniewicz showed in [Hry20], Section 2, this number $\rho^y(\gamma)$ does not depend on the choice of tubular coordinates nor on the initial condition θ_0. Consider any oriented Seifert surface S spanned by γ, with the orientation of the boundary of S, ∂S, being consistent with the orientation of γ given by the flow ϕ_X^t. Denote by $S^* \in H^1(\mathbb{S}^3 \setminus \gamma; \mathbb{Z})$ the class dual to S. Since we are working in $\mathbb{S}^3$, S^* is independent of S. Actually $\langle S^*, \beta\rangle = \text{link}(\gamma, \beta)$ for any oriented loop β in $\mathbb{S}^3 \setminus \gamma$. One can think of it as a class in $H^1(N \setminus \gamma; \mathbb{Z})$ after restricting to $N \setminus \gamma$.

Definition 1.11. *Let y be the cohomology class dual to some (thus any since we are in a homology 3-sphere) oriented Seifert surface for γ. We say that $\rho^y(\gamma)$ is the transverse rotation number of γ in a Seifert framing.*

In Section 3.1.4 we will explain a more geometric interpretation of this quantity in terms of a Seifert framing. For now on, let us pursue with a definition of right-handedness.

1.2.2 Right-handedness according to Florio and Hryniewicz

Now we suppose in addition to the above hypothesis that X is non-singular. Let $\mathcal{P}$ be the set of ϕ_X^t-invariant Borel probability measures, $\mathcal{R}$ the set of recurrent points, and R the following measurable set:

$$R = \left\{(x, y) \in \mathcal{R} \times \mathcal{R} | \phi_X^{\mathbb{R}}(x) \cap \phi_X^{\mathbb{R}}(y) = \emptyset\right\} .$$

Let μ_1 and μ_2 be two ergodic probability measures in $\mathcal{P}$, and denote by $\mu_1 \times \mu_2$ the product measure. There are two cases to consider:

1. $(\mu_1 \times \mu_2)(R) = 1$.
2. $(\mu_1 \times \mu_2)(R) = 0$ and $\mathrm{supp}(\mu_1) \cup \mathrm{supp}(\mu_2) \subset \gamma$ for some periodic orbit γ.

One needs to treat each case separately. Let g be an auxiliary Riemannian metric on $\mathbb{S}^3$.

Case 1. Choose $(p, q) \in R$ and denote by $\mathcal{S}(p, q)$ the set of ordered pairs of sequences $\left((T_n)_{n\in\mathbb{N}}, (S_n)_{n\in\mathbb{N}}\right)$ such that when n goes to infinity $(T_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ tend to infinity, $\phi_X^{T_n}(p) \to p$ and $\phi_X^{S_n}(q) \to q$. For an n large enough denote α_n (resp. β_n) the shortest geodesic arc from $\phi_X^{T_n}(p)$ to p (resp. $\phi_X^{S_n}(q)$ to q). In order to get two closed loops $k(T_n, p)$ and $k(S_n, q)$ not intersecting each other, consider $\mathcal{C}^1$-small perturbations $\tilde{\alpha}$, $\tilde{\beta}$ of α_n and β_n, fixing the extremities. We set:

$$\mathrm{link}_-\left(\phi_X^{[0,T_n]}(p), \phi_X^{[0,S_n]}(q)\right) = \liminf_{\tilde{\alpha}\xrightarrow{\mathcal{C}^1}\alpha_n, \tilde{\beta}\xrightarrow{\mathcal{C}^1}\beta_n} \mathrm{link}\left(k(T_n, p), k(S_n, q)\right) ,$$

and

$$l(p, q) = \inf_{\left((T_n)_{n\in\mathbb{N}}, (S_n)_{n\in\mathbb{N}}\right)\in\mathcal{S}(p,q)} \frac{1}{T_n S_n} \mathrm{link}_-\left(\phi_X^{[0,T_n]}(p), \phi_X^{[0,S_n]}(q)\right) .$$

In this case, μ_1 and μ_2 are said to be positively linked if $l(p, q) > 0$ for $\mu_1 \times \mu_2$-almost all points (p, q).

Case 2. μ_1 and μ_2 are said to be positively linked if the transverse rotation number $\rho^y(\gamma)$ of the periodic orbit γ containing the supports of μ_1, μ_2 computed in a Seifert framing is strictly positive.

Definition 1.12. *We say that the vector field X is right-handed if all pairs of ergodic measures in $\mathcal{P}$ are positively linked.*

As we said earlier, the advantage of this definition is that it avoids dealing with details on the existence of linking forms. In [Ghy09], Ghys defines right-handedness in a different way that we are now going to sketch.

1.2.3 Right-handedness according to Ghys

As we have seen in Section 1.1, there is a suitable way to close the arc trajectory from a point $p \in \mathbb{S}^3$ to $\phi_X^t(p)$ in order to get a knot. Using this fact, Ghys defines the linking number of two ergodic (for the flow of X) probability measures μ_1 and μ_2 as follows [Ghy09]. Let p_1 and p_2 be two points which are generic respectively for μ_1 and μ_2, and t_1, t_2 two large times. Connecting the endpoints of the arcs of orbit $\phi_X^{[0,t_1]}(p_1)$ and $\phi_X^{[0,t_2]}(p_2)$, one obtains a link $k(p_1,t_1) \cup k(p_2,t_2)$ in $\mathbb{S}^3$. Adapting the proof of Arnol'd, Ghys proved that if μ_1 and μ_2 are not concentrated on the same periodic orbit, the limit of linking numbers

$$\mathrm{lk}(\mu_1,\mu_2) = \lim_{t_1,t_2\to\infty} \frac{1}{t_1 t_2} \mathrm{link}\left(k(p_1,t_1), k(p_2,t_2)\right)$$

exists $\mu_1 \times \mu_2$-almost everywhere and is independent from the choice of (p_1,p_2). If μ_1 and μ_2 are distributed on the same periodic orbit, there is also a way to define the self-linking number without using a preferred trivialization of the normal bundle. In our situation, Ghys explains that one can define some kind of self-linking number of a periodic orbit going through a point p, by considering the asymptotic linking number of the orbits of two sequences of different points p_1^n, p_2^n converging to the point p.

Using the ergodic decomposition theorem and the previous definition of linking number of measures, one can define a bilinear form $\mathrm{lk}(\mu_1,\mu_2)$ on the set $\mathcal{P}$ of invariant probability measures for X. Ghys proves that this bilinear extension is possible in a continuous way and finally defines the fundamental linking form on the compact convex set $\mathcal{P}$:

$$\mathrm{lk} : \mathcal{P} \times \mathcal{P} \mapsto \mathbb{R}\,.$$

We end up with the following definition of a right-handed vector field:

Definition 1.13. *A non-singular vector field X on $\mathbb{S}^3$ is right-handed if the quadratic linking form is positive on the convex set $\mathcal{P}$ of invariant probability measures.*

We can now state Ghys' Theorem which is an analogue to the Schwarzman–Fried–Sullivan Theorem:

Theorem 1.14. *Let X be a non-singular vector field on $\mathbb{S}^3$, generating a flow ϕ_X^t. Choose some Gauss linking form Ω. The following conditions are equivalent:*

1. *X is right-handed, i.e., the quadratic linking form is positive on the convex set $\mathcal{P}$.*

2. *There is some $T > 0$ such that for every pair of points p_1, p_2 on different orbits, the integral $\int_0^T \int_0^T \Omega_{\phi_X^{t_1}(p_1),\phi_X^{t_2}(p_2)}(X_{\phi_X^{t_1}}(p_1), X_{\phi_X^{t_2}}(p_2))dt_1 dt_2$ is positive.*

3. *There is some Gauss linking form $\overline{\Omega}$ which is pointwise positive on X, i.e. for every distinct points p_1, p_2, one has $\overline{\Omega}_{p_1,p_2}(X(p_1), X(p_2)) > 0$.*

1.3 Examples

1.3.1 Trunk of knots and trunkenness

Before we look at some examples of vector fields that are relevant for this thesis, I would like to introduce the trunkenness of vector fields, an invariant which was defined by Pierre Dehornoy and Ana Rechtman in [DR17]. The objective is twofold: first to present an invariant very close to the bridge number of knots, that we will generalize to vector fields in Chapter 2, and also to have a second invariant to compute when we will look at examples. Let us start with the trunk of knots defined by Ozawa in [Oza10]. Define the standard height function on $\mathbb{R}^3$ by $h_z : (x, y, z) \mapsto z$.

Definition 1.15. *A height function h on $\mathbb{R}^3$ is a function of the form*

$$h(x, y, z) = h_z(\psi(x, y, z))$$

where ψ is a smooth orientation-preserving diffeomorphism of $\mathbb{R}^3$. In particular, the level sets of a height function on $\mathbb{R}^3$ are topologically smooth planes.

Let k be a knot in $\mathbb{R}^3$, and h a height function which is Morse with respect to k. Ozawa [Oza10] defines the trunk of k for h by

$$\mathrm{tk}_h(k) := \max_{t\in\mathbb{R}} \#\{k \cap h^{-1}(t)\}\,.$$

Definition 1.16 ([Oza10])**.** *The trunk of the knot k is given by*

$$tk(k) := \inf_{h \text{ Morse height function}} tk_h(k) = \inf_{h \text{ Morse height function}} \max_{t\in\mathbb{R}} \#\{k \cap h^{-1}(t)\}\,.$$

Examples.

- The trunk of the unknot is 2, and a knot is trivial if and only if its trunk is 2.
- The trunk of the trefoil knot is 4.
- Since a torus knot $T(p, q)$ can be realized as the closure of a braid with q strands, and using symmetry of the knot, one has $\mathrm{tk}(k) \leqslant \min\{p, q\}$. Actually, Ozawa proved that this is an equality [Oza10].

In order to generalize this invariant, Dehornoy and Rechtman proposed the following definition. Define a height function h on $\mathbb{S}^3$ to be a function whose level sets are 2-dimensional spheres and with only two singular points. Let X be a vector field in $\mathbb{S}^3$ preserving a measure μ. The trunkenness of (X, μ) for h is given by

$$\mathrm{tks}_h(X, \mu) := \max_{t\in[0,1]} \mathrm{Flux}\left(X, \mu, h^{-1}(t)\right)$$

where $\mathrm{Flux}\,(X, \mu, h^{-1}(t))$ is the geometric flux of (X, μ) through the level set $h^{-1}(t)$. Roughly speaking, it represents the instantaneous measure of the points crossing the level set $h^{-1}(t)$.

Definition 1.17 (Dehornoy-Rechtman [DR17]). *The trunkenness of* (X, μ) *is*

$$Tks(X,\mu) := \inf_{h \text{ height function}} tks_h(X,\mu) = \inf_{h \text{ height function}} \max_{t\in[0,1]} Flux\Big(X,\mu,h^{-1}(t)\Big)$$

In their article [DR17], Dehornoy and Rechtman prove that the trunkenness is an order one invariant under measure preserving homeomorphisms. It is continuous in the sense that if the sequence of vector fields $(X_n)_{n\in\mathbb{N}}$ tends to X in the $\mathcal{C}^0$-topology and preserves the sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ converging to μ weakly-$*$, the trunkenness of (X_n, μ_n) converges to the trunkenness of (X, μ). In particular, the asymptotic trunk is well-defined:

Theorem 1.18 (Dehornoy-Rechtman, [DR17]). *If* X *is ergodic with respect to* μ*, for* μ*-almost every point* p*,*

$$\lim_{t\to\infty} \frac{1}{t} tk\left(k_X(p,t)\right)$$

exists and is equal to $Tks(X, \mu)$*.*

The trunk of knots behaves well under connected sum. Davies and Zupan proved the following [DZ17]:

Proposition 1.19 (Davies-Zupan, [DZ17]). *For any two knots* k_1*,* k_2 *in* $\mathbb{R}^3$*,*

$$tk(k_1\#k_2) = \max\{tk(k_1), tk(k_2)\}\,.$$

The trunk is also related to another knot invariant, the *bridge number* of knots. There are several ways to define the bridge number, here to be consistent we choose to fix the knot and vary the height function. Let k be a knot and h be a height function so that $h_{|k}$ is a Morse function, which means that the function $h_{|k}$ has finitely many extrema.

Definition 1.20. *Let* $b_h(k)$ *be the number of maxima (or minima) of* $h_{|k}$*. The bridge number of* k *is then defined by:*

$$b(k) = \min_{h \text{ height function}} b_h(k)\,.$$

In general, we have the relation $\mathrm{tk}(k) \leqslant 2b(k)$. Ozawa [Oza10] proved that for the class of meridionally small knots, equality holds. A knot k is called meridionally small if there exists no essential surface in its exterior $E(k)$ with meridional boundary.

Theorem 1.21 (Ozawa [Oza10]). *If a knot* k *is meridionally small, then* $tk(k) = 2b(k)$*.*

Because of this relation, we asked if the trunkenness of vector fields could be related to the bridge number of vector fields that we define in this thesis. We show in an example in Section 2.4 that it is not the case and that these invariants are independent.

1.3.2 Seifert flows

The Seifert flow of parameters $(\alpha, \beta) \in \mathbb{R}_+^*$ is the flow preserving the Haar measure Ω_{Haar} on $\mathbb{S}^3$ and defined by :

$$\phi^t_{(\alpha,\beta)}(z_1, z_2) = (z_1 \exp 2i\pi\alpha t, z_2 \exp 2i\pi\beta t)\,,$$

where $(z_i)_{i=1,2}$ are complex coordinates on $\mathbb{S}^3 \subset \mathbb{C}^2$. This flow has two particular orbits given by $z_i = 0$ that are closed and unknotted for any choice of parameters. Moreover the flow preserves the torii $\|\frac{z_2}{z_1}\| = $ constant. If α/β is rational and equal to p/q with p and q two coprime integers, then every orbit is periodic and forms a torus knot of type $T(p, q)$ - except the two previously mentioned. This is why for these flows, it is easy to compute vector field invariants using their asymptotic definition.

Helicity of Seifert flows. In the case where (α, β) takes on the rational values (p, q) with the condition that $p \wedge q = 1$, all the orbits are periodic of period 1 and the linking number between any pair of orbits is pq with the exception of the two specific unknotted orbits that have linking number 1. Hence the asymptotic linking number is also pq and using Arnol'd's Theorem 1.9, the helicity is pq. In the general case, when $\alpha/\beta \in \mathbb{R} \setminus \mathbb{Q}$, one can approximate the vector field $X_{\alpha,\beta}$ with a sequence of Seifert flows with rational slope, for which the helicity is the product of the parameters. Using the continuity of helicity, one eventually has

$$\mathrm{Hel}(X_{\alpha,\beta}, \Omega_{Haar}) = \alpha\beta\,.$$

for arbitrary parameters α, $\beta \in \mathbb{R}$.

Trunkenness of Seifert flows.

Proposition 1.22 (Dehornoy-Rechtman, [DR17])**.** *The trunkenness of* $(X_{\alpha,\beta}, \Omega_{Haar})$ *is* $\min(\alpha, \beta)$.

We briefly explain the proof of this result. It stands in two steps. First, using the definition of the trunkenness in terms of geometric fluxes, one shows that $\mathrm{Tks}(X_{\alpha,\beta}, \Omega_{Haar}) \leqslant 2\beta$ by exhibiting a particular height function. Here we have to consider the standard height function h_0 on $\mathbb{S}^3$, whose level sets are 2-spheres centered in the origin of the standard stereographic projection, and more precisely its particular level set $h_0^{-1}(1/2)$ which contains the special orbit $z_2 = 0$ and is the only sphere that intersects all the orbits of the flow.

The second step is to prove the converse inequality $\mathrm{Tks}(X_{\alpha,\beta}, \Omega_{Haar}) \geqslant 2\min(\alpha, \beta)$. This can be achieved by approximating the vector field $X_{\alpha,\beta}$ with a sequence of vector fields with rational slope $(X_{p_n/r_n, q_n/r_n})_{n\in\mathbb{N}}$, where p_n, q_n and r_n are integer numbers (think of the decimal expansions of α and β). By Theorem 1.18,

$$\mathrm{Tks}\left(X_{\alpha,\beta}, \Omega_{Haar}\right) = \lim_{n\to\infty} \mathrm{Tks}\left(X_{p_n/r_r, q_n/r_n}, \Omega_{Haar}\right)\,.$$

Using the fact that the trunkenness is an order one invariant, it is enough to prove $\mathrm{Tks}(X_{p,q}, \Omega_{Haar}) = 2\min(p, q)$ for p, q two coprime natural numbers. We have to

compute the right-hand side of the above equality.
As we remarked, the flow is particularly simple in this case. One can choose a sequence $(K_n)_{n\in\mathbb{N}}$ of collections of n periodic orbits so that μ_n, the *invariant measure supported by* K_n, tends to Ω_{Haar}. This sequence of measures is defined as follows. Let X_n be a vector field tangent to the link K_n in every point. This induces a flow ϕ_n^t along the link. The invariant measure supported by K_n is given by:

$$\mu_n(A) := \frac{1}{n}\mathrm{Leb}\left(\cup_{i=1}^n \{t \in [0,1], \phi_n^t(x_i) \in A\}\right)$$

for all $A \subset \mathbb{S}^3$, where the x_i belong to the i-th component of K_n. Since the period of each component is 1 and we have n of them, μ_n is of total mass 1 and tends to Ω_{Haar} when n goes to infinity by choice of the collection K_n. But the orbits that constitute K_n are of course torus knots of type (p,q) so the trunkenness of K_n is $n\min(p,q)$ by a theorem of Zupan about cable links [Zup12]. Thus we have:

$$\mathrm{Tks}(X_{p,q}, \Omega_{Haar}) = \lim_{n\to\infty} \mathrm{Tks}(X_{p,q}, \mu_n) = \lim_{n\to\infty} \frac{1}{n} \times n\min(p,q) = \min(p,q)\,.$$

We will show in Chapter 2 that the bridge number of this flow is the same as the trunkenness - because torus knots are meridionally small - and in Chapter 3 that the asymptotic genus of this flow is half of the helicity.

Chapter 2

The bridge number of vector fields

In this chapter we are interested in the bridge number of knots, which is an invariant of the same nature as the trunk of knots defined by Schubert [Sch54] from which Dehornoy and Rechtman defined the trunkenness of vector fields in [DR17]. In Section 2.1 we present the definition of the bridge number of knots and some needed results of Milnor relating the bridge number of knots to the curvature of closed curves. In Section 2.2 we introduce a definition for the bridge number of a vector field X preserving a probability measure μ on $\mathbb{S}^3$ and we show that it is an order one invariant by $\mathcal{C}^1$-diffeomorphisms preserving μ (Theorem A). Then we prove a regularity result for this new invariant (Theorem B) in Section 2.2 and use it to define an asymptotic bridge number (Theorem C) in Section 2.3. Finally we investigate the relations between the bridge number of vector fields, the trunkenness and the helicity in Section 2.4.

2.1 Bridge number, crookedness and curvature of a curve

First we define what is a *height function* in $\mathbb{S}^3$: it is the precomposition of the standard height function:

$$\begin{array}{rccl} h_0: & \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\} & \to & [0,1] \\ & (x,y,z) & \mapsto & 1 - \frac{1}{1+x^2+y^2+z^2}, \end{array}$$

with a $\mathcal{C}^1$-orientation-preserving diffeomorphism of $\mathbb{S}^3$. Let K be a knot and $h : \mathbb{S}^3 \to \mathbb{R}$ be a height function. We denote $\mathcal{K}$ the set of all embeddings k of $\mathbb{S}^1$ into $\mathbb{S}^3$ that are isotopic to K and so that $h_{|k}$ is a Morse function.

Definition 2.1. *Let $b_h(k)$ be the number of maxima (or minima) of $h_{|k}$. The bridge number of K is then defined by:*

$$b(K) = \min_{k \in \mathcal{K}} b_h(k)\,.$$

In this definition we choose to set the height function h and then minimize the bridge number over the embeddings of the knot for which h is Morse. Another point of view (which we used in Chapter 1) is to fix an embedding of the knot and then minimize the bridge number over the height functions that are Morse for this knot:

$$b(K) = \min_{h \text{ height function}} b_h(k)\,.$$

We will use this last definition in preference to the previous one when defining a bridge number for vector fields.

The bridge number is almost additive under the connected sum of knots. This result was proven by Schubert and later by Schultens [Sch03] and it will be used in Section 2.4:

Proposition 2.2. *The quantity* $b-1$ *is additive under the connected sum of knots: if* K_1 *and* K_2 *are two knots,*

$$b(K_1 \sharp K_2) = b(K_1) + b(K_2) - 1\,.$$

Relation with the curvature. In the case of knots, the bridge number happens to coincide with the *crookedness of a curve*, a notion which was introduced by Milnor [Mil50] in the 50s.

Let K be a knot, k an embedding of $\mathbb{S}^1$ into $\mathbb{R}^n$ that is isotopic to K and $u \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ a unit vector. We are considering k as a periodic parametrized curve $\gamma(t)$. We denote $\mu(k,u)$ the number of maxima of the function $t \mapsto u \cdot \gamma(t)$ during one period, and we call *crookedness of* k the quantity $\mu(k) := \min_{u\in\mathbb{S}^{n-1}}\{\mu(k,u)\}$. Then the *crookedness of* K is defined by

$$\mu(K) := \min_{k \text{ embedding isotopic to } K} \mu(k)\,.$$

In his article Milnor relates the crookedness of a closed curve with its curvature. Here we state his result in our particular setting : $\mathbb{S}^3$ embedded in $\mathbb{R}^4$. The point is that the crookedness of K is exactly the bridge number of K. The difference with the definition that we chose before is that for the crookedness, one fixes a height function and then isotopes the knot, while our first definition fixes the embedding and then changes the height function. Now let k be an oriented knot seen as the support of a curve C of class $\mathcal{C}^2$, parametrized by arc length by $\gamma : \mathbb{S}^1 \to \mathbb{R}^4$ and with total length l, and set $\kappa(k) = \kappa(C) = \int_0^l |\gamma''(s)|\, ds$ the total curvature of the curve. Then the following holds:

Theorem 2.3 (Milnor, [Mil50])**.** *The integral* $\int_{\mathbb{S}^2} \mu(C,u)\, dS$ *exists and*

$$\int_{\mathbb{S}^2} \mu(C,u)\, dS = \frac{Vol(\mathbb{S}^2)}{2\pi} \times \kappa(C)\,.$$

This allows Milnor to obtain an upper bound of the crookedness by the curvature, and thus also an upper bound of the bridge number. In our case because $\mu(C,u) \geqslant \mu(k)$ we have from the precedent theorem:

Corollary 2.4. $\kappa(k) \geqslant 2\pi b(k)$.

This will be useful in Section 2.3 to show an upper bound on the asymptotic bridge number.

2.2 Bridge number of vector fields

2.2.1 Definition and proof of Theorem A

We consider a smooth non-singular vector field X on $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ and we suppose that X preserves a probability measure μ. We denote ϕ_X^t the flow of X at the time t. For h a height function and $t \in [0,1]$, we note $T_X\left(h^{-1}(t)\right)$ the (closed) set of points where X is tangent to the level set $h^{-1}(t)$.

The problem that we face now is to mimic counting the local extrema of a height function on a closed curve when the curve is replaced with a vector field. The first remark is that when considering a vector field X, the local extrema correspond to the tangency points of X to the level sets of the height function. The second step is to find a way to count these tangency points. To see this let us consider a knot k of isotopy class $\mathcal{K}$ in $\mathbb{R}^3$. On k we define X_k a unitary vector field tangent to k. Since k is a knot, the flow $\phi_{X_k}^t$ is T-periodic for some fixed $T > 0$. It allows a measure to be defined in the following way. For A a mesurable set and any $p \in k$,

$$\mu_k(A) = \mathrm{Leb}\{t \in [0,T],\, \phi_{X_k}^t(p) \in A\}\,.$$

Of course this measure does not see the tangency points of X_k to the level S of a Morse height function. But it can detect short arcs of orbit $\phi_{X_k}^{[0,\epsilon]}(p)$. In particular if ϵ is small enough, $\mu_k(\phi_{X_k}^{[0,\epsilon]}(k \cap S)) = \epsilon \cdot \mathrm{Card}\{k \cap S\}$, that is to say

$$\mathrm{Card}\{k \cap S\} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mu_k(\phi_{X_k}^{[0,\epsilon]}(k \cap S))\,.$$

Now the general case is more complicated, because an arbitrary measure may not detect a 1-dimensional subset in general, for instance if μ is a volume. Thus in order to detect the tangency points of X to the level sets using the measure μ we have to consider the union of all the tangency points to all the level sets of h, push them by the flow and then compute their measure. We have the following definition:

Definition 2.5. *For a given height function h, we define the bridge number of the vector field X preserving a probability measure μ for h by*

$$B_h(X,\mu) = \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mu \left(\bigcup_{0<t<1} \phi_X^{[0,\epsilon]} \left(T_X \left(h^{-1}(t) \right) \right) \right)\,.$$

Note that when μ charges only one orbit of the flow, this formula counts morally the number of local extrema of the height function h restricted to this orbit, divided by two, and this is just the bridge number of the orbit. As we said in the introduction, we can then define the bridge number of (X,μ):

Definition 2.6. *The bridge number of (X,μ) is given by*

$$B(X,\mu) = \inf_{h \text{ height function}} \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mu \left(\bigcup_{0<t<1} \phi_X^{[0,\epsilon]} \left(T_X \left(h^{-1}(t) \right) \right) \right)\,.$$

From this definition, we deduce that if for a given height function h, $B_h(X,\mu) < +\infty$, then $\mu(T_X) = 0$, where T_X denotes the union of the tangency points of X to the level sets of h. Indeed, if this is not the case, then pushing T_X by a small time ϵ just makes its measure bigger, and dividing by ϵ makes $\frac{1}{\epsilon}\mu(T_X^\epsilon)$ go to infinity when ϵ tends to zero.

Another important remark about this definition is that given (X, μ) as above, it is not immediate that there exists a height function h so that $B_h(X,\mu) < \infty$. Indeed, if $B_h(X,\mu) = \infty$ for a given height function h, it means that the μ-measure of the union of tangency points of X to the level sets of h is (strictly) positive. Let us show that we can always find a height function so that the μ-measure of the tangency points of X to the level sets of this function is zero. We begin with the following lemma.

Lemma 2.7. *Let X be a vector field in $\mathbb{S}^3$. Let $A \subset \mathbb{S}^3$ and suppose that there exists a fixed $\eta > 0$ so that any two intersections of A with the same orbit of the flow of X are separated by a time greater than η. Then for all invariant measures μ and all $\epsilon > 0$ so that $\epsilon < \eta$, we have*

$$\mu(A^\epsilon) := \mu\left(\phi_X^{[0,\epsilon]}(A)\right) = f \times \epsilon\,,$$

where f is a constant depending on A and μ and which we call the geometric flux of the flow through A.

Proof. Choose an invariant measure μ and let x be a generic point for μ and recurrent for the flow of X. It induces an ergodic measure defined by

$$\mu_x(B) = \lim_{T\to\infty} \frac{1}{T}\mathrm{Leb}\left(\{t \in [0,T],\, \phi_X^t(x) \in B\}\right)$$

for $B \subset \mathbb{S}^3$ a measurable set. Then

$$\begin{aligned}\mu_x(A^\epsilon) &= \lim_{T\to\infty} \frac{1}{T}\mathrm{Leb}\left(\{t \in [0,T],\, \phi_X^t(x) \in A^\epsilon\}\right)\\ &= \lim_{T\to\infty} \frac{1}{T}\epsilon \cdot \sharp\{A \cap \phi_X^{[0,T]}(x)\}\,.\end{aligned}$$

But by hypothesis the intersection points of A with the orbit of x are isolated along the orbit and there exists a fixed small time $\eta > 0$ which separates two intersections. Thus

$$\sharp\{A \cap \phi_X^{[0,T]}(x)\} \leqslant \frac{T}{\eta}\,,$$

so finally $\mu_x(A^\epsilon) = \epsilon \times f$ where f is a constant depending on A and x. Since by the ergodic decomposition theorem ([Mn87], Section II.6) any invariant measure μ is a linear combination of such ergodic measures μ_x, we are done.

□

Proposition 2.8. *Let (X,μ) be as in Definition 2.5. Let h be a height function so that $B_h(X,\mu) = \infty$. Then there exists a height function $\tilde{h}$ so that $B_{\tilde{h}}(X,\mu) < \infty$.*

Proof. Let us suppose that $B_h(X,\mu) = \infty$ for a given height function h, meaning that the μ-measure of the union of tangency points of X to the level sets of h is strictly positive. Then at least one of these two options occurs:

1. The union of tangency points of X to the level sets of h contains an open set $\mathcal{U}$ in the support of μ which has positive measure.

2. There exists $t_0 \in]0,1[$ so that the level set $h^{-1}(t_0)$ contains a piece of orbit which is in the support of μ and has positive measure.

By regularity of X and h, one cannot avoid the union of the tangency points of h to be a set of dimension 2. However, for I a closed time interval and $x \in \mathbb{S}^3$, we can avoid having pieces of orbits $\phi_X^I(x)$ as tangency points of a given level set, which is the issue in the above two cases. We are going to construct a perturbation $\tilde{h}$ of h so that in the support of μ, the tangency points of $\tilde{h}$ are discrete along any orbit.

Let us set $T_X(h) := \bigcup_{0<t<1} T_X\left(h^{-1}(t)\right)$ and we will omit the reference to h when it is clear which height function we are referring to. So our aim is to change the level sets of h in a neighbourhood $\mathcal{U}$ of the set of tangency points $T_X(h)$, in order to get a height function for which the pair (X,μ) has finite bridge number.

Since $\overline{\mathcal{U}}$ is closed in $\mathbb{S}^3$, thus compact, we can cover this set with a finite number of flowboxes $(\mathcal{F}_i)_{(1\leqslant i\leqslant n)}$. We want to gradually change the level sets of h in a neighbourhood of $\mathcal{U}$, with perturbations located inside the $\mathcal{F}_i$. To do this we need another family of flowboxes $\mathcal{G}_i$ so that each $\overline{\mathcal{G}_i}$ is contained in $\mathcal{F}_i$ and the family $(\mathcal{G}_i)_{1\leqslant i\leqslant n}$ is a covering of $\overline{\mathcal{U}}$. This can be made because the family $(\mathcal{F}_i)_{1\leqslant i\leqslant n}$ is a covering, so shrinking a little the generating disks of these flowboxes does not change this property.

Consider the flowbox $\mathcal{F}_1$. There is a family of level sets of h that are problematic, i.e. tangent to X along segments of orbits in $\mathcal{F}_1$. In this case we can perform a $\mathcal{C}^1$-perturbation of the level sets of h in $\mathcal{G}_1$ so that the new tangency points of X inside $\mathcal{G}_1$ form 2-dimensional sets that are transverse to X. The idea of the perturbation is to allow the level sets to make $\mathcal{C}^1$ little waves instead of being flat (and tangent). As there is a finite number of oscillations in $\mathcal{G}_1$, one can find a small time η_1 which separates two tangency points in the same orbit. Figure 2.1 illustrates this modification. Let us call $\tilde{h}$ the perturbed function.

Note that since we only modify the level sets inside of $\mathcal{G}_1$, the obtained level sets glue well with the other level sets outside $\mathcal{F}_1$ since the level sets in $\mathcal{F}_1 \setminus \mathcal{G}_1$ are unchanged. This also means that we do not add tangency points with this perturbation, we can only remove some of them. Indeed after perturbation we have $\mu(T_X \cap \mathcal{G}_1) = 0$. We iterate this process on the n flowboxes $\mathcal{F}_i$ to finally get $\mu(\mathcal{U}) = 0$. For each flowbox $\mathcal{G}_i$ we have a small positive constant η_i and we have a finite number of flowboxes, so with $\eta = \min_{i=1,\dots,n} \eta_i$ the set $T_X(\tilde{h})$ satisfies the hypothesis of Lemma 2.7.

Finally after a finite number of changes, we obtain a height function $\tilde{h}$ so that $\mu\left(\bigcup_{0<t<1} T_X\left(\tilde{h}^{-1}(t)\right)\right) = 0$ and the set $\bigcup_{0<t<1} T_X\left(\tilde{h}^{-1}(t)\right)$ satisfies the hypothesis of

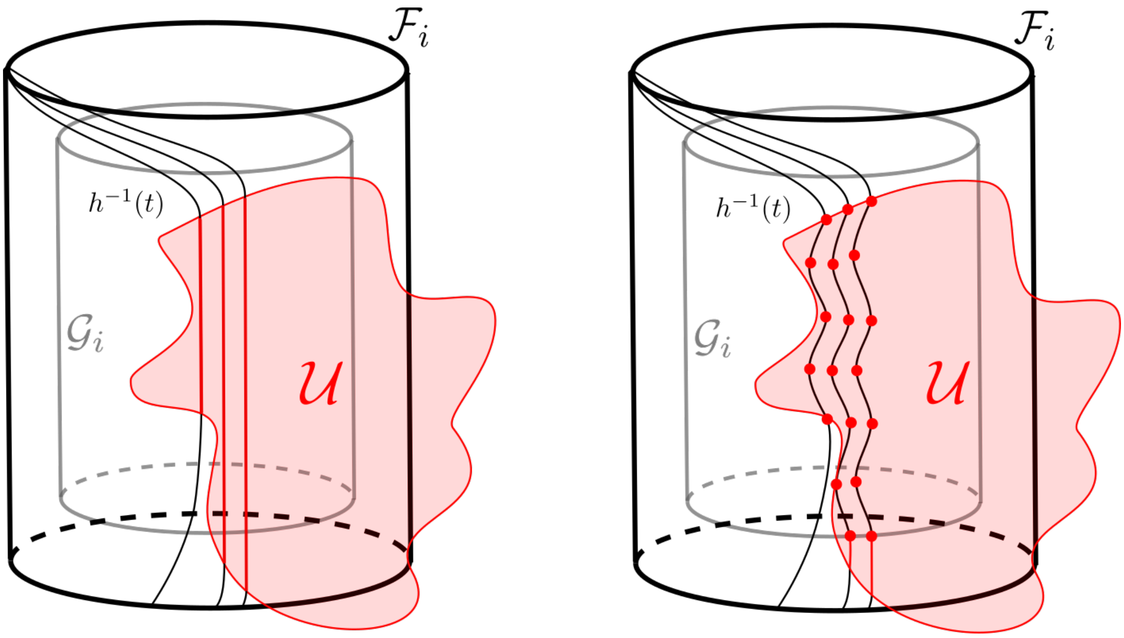

Figure 2.1: Changing the level sets of h to obtain a 2-dimensional set of tangency points.

Lemma 2.7, and thus

$$\begin{aligned} B_{\tilde{h}}(X,\mu) &= \frac{1}{2}\lim_{\epsilon\to\infty}\frac{1}{\epsilon}\mu\left(\bigcup_{0<t<1}\phi_X^{[0,\epsilon]}\left(T_X\left(\tilde{h}^{-1}(t)\right)\right)\right) \\ &= \frac{1}{2}\lim_{\epsilon\to\infty}\frac{1}{\epsilon}a_{\tilde{h}}\times\epsilon \\ &= \frac{a_{\tilde{h}}}{2} < +\infty\,. \end{aligned}$$

□

Proposition 2.8 and Lemma 2.7 and their proof call for a definition of a class of height functions which could be usable to compute the bridge number of a given vector field X:

Definition 2.9. *Let X be a vector field on $\mathbb{S}^3$. We say that a height function h is* good for X *if $T_X(h)$ does not contain any orbit segment of strictly positive length.*

Given any height function, we can change its level sets as explained in the Proposition 2.8 so that it becomes an $\tilde{h}$ that is *good for X* and $B_{\tilde{h}}(X,\mu) \leqslant B_h(X,\mu)$. Moreover from the proof of Proposition 2.8, any good function h for X admits an $\epsilon_h > 0$ so that any two points of $T_X(h)$ that belong to the same orbit are at distance at least ϵ_h. Then for any $\epsilon < \epsilon_h$, we have that $\mu(T_X^{\epsilon}) = a_h \times \epsilon$, where a_h is sometimes called the geometric flux through T_X, as we have seen in Section 1.3. It is a flux that does not take the orientations into account and that counts everything positively. This a_h exists because μ is an invariant measure, so it can be desintegrated if one thinks of the flow lines as a foliation, as it is explained in Lemma 2.7. In this case, a_h is the bridge number of (X,μ) with respect to the good height function h.

Now we prove Theorem A that asserts that this definition is invariant under measure-preserving $\mathcal{C}^1$-diffeomorphisms.

Proof of Theorem A Let X_1 and X_2 be two vector fields that preserve respectively the probability measures μ_1 and μ_2, f a $\mathcal{C}^1$-diffeomorphism of $\mathbb{S}^3$ so that $\mu_1 = f^*\mu_2$ and $f \circ \phi^t_{X_1} = \phi^t_{X_2} \circ f$ for all $t \in \mathbb{R}$. Suppose by contradiction that $0 < \delta = B(X_1, \mu) - B(X_2, \mu)$. Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of differentiable height functions so that

$$\lim_{n\to\infty} B_{H_n}(X_2, \mu_2) = B(X_2, \mu_2)\,.$$

For all n we define the height function $h_n = H_n \circ f$ and we call $T_{X_1}(h_n^{-1}(t))$ the points of the level set $h_n^{-1}(t)$ where X_1 is tangent. We have:

$$T_{X_1}(h_n^{-1}(t)) = T_{X_1}(f^{-1} \circ H_n^{-1}(t)) = T_{f^*X_2}(f^{-1} \circ H_n^{-1}(t)) = f^{-1}\left(T_{X_2}(H_n^{-1}(t))\right)\,.$$

So, since f conjugates the flows and $\mu_1 = f^*\mu_2$:

$$\begin{aligned}
\lim_{\epsilon\to 0} \frac{1}{\epsilon}\mu_1\left(\bigcup_{0<t<1} \phi^{[0,\epsilon]}_{X_1}\left(T_{X_1}\left(h_n^{-1}(t)\right)\right)\right) &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\mu_1\left(\bigcup_{0<t<1} \phi^{[0,\epsilon]}_{X_1}\left(f^{-1}\left(T_{X_2}\left(H_n^{-1}(t)\right)\right)\right)\right)\\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\mu_1\left(f^{-1}\left(\bigcup_{0<t<1} \phi^{[0,\epsilon]}_{X_2}\left(\left(T_{X_2}\left(H_n^{-1}(t)\right)\right)\right)\right)\right)\\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\mu_2\left(\bigcup_{0<t<1} \phi^{[0,\epsilon]}_{X_2}\left(T_{X_2}\left(H_n^{-1}(t)\right)\right)\right),
\end{aligned}$$

thus $B_{h_n}(X_1, \mu_1) = B_{H_n}(X_2, \mu_2)$. Let now $N \in \mathbb{N}$ be so that for all $n > N$, $B_{H_n}(X_2, \mu_2) - B(X_2, \mu_2) < \frac{\delta}{2}$. For any $n > N$ we have that

$$\begin{aligned}
0 < \delta = B(X_1, \mu_1) - B(X_2, \mu_2) = &\quad B(X_1, \mu_1) - B_{h_n}(X_1, \mu_1) + B_{H_n}(X_2, \mu_2) - B(X_2, \mu_2)\\
< &\quad 0 + \frac{\delta}{2},
\end{aligned}$$

since $B(X_1, \mu_1) \leqslant B_{h_n}(X_1, \mu_1)$ for all n. This gives a contradiction, so $B(X_2, \mu_2) = B(X_1, \mu_1)$.

It remains to prove that the order of B is one. Let $\lambda \in \mathbb{R}^*_+$ and h be a height function and consider the vector field λX_1. It has the same tangency points to any level set of h as X_1, and we also have $\phi^{[0,\epsilon]}_{\lambda X_1} = \phi^{[0,\lambda\epsilon]}_{X_1}$, so that we have:

$$\begin{aligned}
B_h(\lambda X_1, \mu_1) &= \frac{1}{2}\lim_{\epsilon\to 0} \frac{1}{\epsilon}\mu_1\left(\bigcup_{0<t<1} \phi^{[0,\epsilon]}_{\lambda X_1}\left(T_{\lambda X_1}\left(h^{-1}(t)\right)\right)\right)\\
&= \frac{\lambda}{2}\lim_{\epsilon\to 0} \frac{1}{\lambda\epsilon}\mu_1\left(\bigcup_{0<t<1} \phi^{[0,\lambda\epsilon]}_{X_1}\left(T_{X_1}\left(h^{-1}(t)\right)\right)\right)\\
&= \lambda B_h(X_1, \mu_1)\,.
\end{aligned}$$

Since this holds for all height functions h, the bridge number is an order one invariant.

2.2.2 Proof of Theorem B

In this section we prove Theorem B. The proof is by contradiction. Remember that X is a smooth non-singular vector field on $\mathbb{S}^3$ preserving a probability measure μ, and we have a sequence $(X_n, \mu_n)_{n\in\mathbb{N}}$ so that $(X_n)_{n\in\mathbb{N}}$ tends to X in the $\mathcal{C}^1$-topology and $(\mu_n)_{n\in\mathbb{N}}$ tends to μ weakly-$*$.

Let us begin the proof with some general results. Fix a Riemannian metric g on $\mathbb{S}^3$ so that $\|X\| = 1$ everywhere. To exploit weak-$*$ convergence of $(\mu_n)_{n\in\mathbb{N}}$ to μ, we are going to use the theory of currents. The current associated to (X, μ) is given for every differential 1-form α on $\mathbb{S}^3$ by

$$C_{(X,\mu)}(\alpha) = \int \alpha(X) d\mu = \int \alpha(Y) d\nu$$

where Y is the unitary vector field $\frac{X}{\|X\|}$ and $\nu = \|X\|\mu$; observe that in our case $X = Y$ and $\mu = \nu$, but we can define by analogy Y_n and ν_n as well. The mass of a current is then given by

$$\mathbf{M}\left(C_{(X,\mu)}\right) = \sup_{\alpha,\,\|\alpha\|\leqslant 1} \left(\int \alpha(X)d\mu\right) = \nu(\mathbb{S}^3)\,.$$

If the vector fields X_n tend to X in the $\mathcal{C}^0-$topology and $(\mu_n)_{n\in\mathbb{N}}$ tends to μ weakly-$*$, the currents $C_{(X_n,\mu_n)}$ tend to $C_{(X,\mu)}$ in mass topology, so

$$\mathbf{M}\left(C_{(X,\mu)} - C_{(X_n,\mu_n)}\right) = \Delta_n \to_{n\to\infty} 0\,.$$

Lemma 2.10. *For any measurable set $A \subset \mathbb{S}^3$, $|\nu(A) - \nu_n(A)| \leqslant \delta_n$, where δ_n is a positive decreasing sequence converging to zero, and independent from A.*

Proof. Let α be the dual form to Y and α_n be the dual form to Y_n with respect to the metric g. Then $\|\alpha\| = 1$, $\|\alpha_n\| = 1$, $\alpha(Y) = \alpha_n(Y_n) = 1$ and since Y and Y_n are $\mathcal{C}^0$-close,

$$0 \leqslant \alpha(Y_n) = \langle Y, Y_n\rangle = \alpha_n(Y) \leqslant 1$$

and there exists a positive sequence $(\epsilon_n)_{n\in\mathbb{N}}$ so that $\langle Y, Y_n\rangle \geqslant 1-\epsilon_n$ tends to 1 when n goes to infinity. Then

$$\begin{aligned}
\Delta_n &= \mathbf{M}\left(C_{(X,\mu)} - C_{(X_n,\mu_n)}\right)\\
&\geqslant \frac{1}{2}\left(\int_A \alpha(Y)d\nu - \int_A \alpha(Y_n)d\nu_n + \int_A \alpha_n(Y)d\nu - \int_A \alpha_n(Y_n)d\nu_n\right)\\
&= \frac{1}{2}\left(\nu(A) - \nu_n(A) - \int_A \alpha(Y_n)d\nu_n + \int_A \alpha_n(Y)d\nu\right)\\
&\geqslant \frac{1}{2}\left(\nu(A) - \nu_n(A) - \int_A d\nu_n + \int_A (1-\epsilon_n)d\nu\right)\\
&= \frac{1}{2}\left((2-\epsilon_n)\nu(A) - 2\nu_n(A)\right)\,.
\end{aligned}$$

Finally

$$\frac{2}{2-\epsilon_n}\Delta_n \geqslant \nu(A) - \frac{2}{2-\epsilon_n}\nu_n(A)\,.$$

We can then write

$$\begin{aligned}\nu(A)-\nu_n(A) &= \nu(A)-\frac{2}{2-\epsilon_n}\nu_n(A)+\left(\frac{2}{2-\epsilon_n}-1\right)\nu_n(A)\\ &\leqslant \frac{2}{2-\epsilon_n}\Delta_n+\frac{\epsilon_n}{2-\epsilon_n}\nu_n(\mathbb{S}^3)\\ &=: \delta_n\end{aligned}$$

and δ_n is independent from A and tends to 0 as n goes to infinity. Using the 1−forms $-\alpha$ and $-\alpha_n$, the same computation gives

$$\begin{aligned}\nu(A)-\nu_n(A) &= \nu_n(A)-\frac{2}{2-\epsilon_n}\nu(A)+\left(\frac{2}{2-\epsilon_n}-1\right)\nu(A)\\ &\leqslant \frac{2}{2-\epsilon_n}\Delta_n+\frac{\epsilon_n}{2-\epsilon_n}\nu(\mathbb{S}^3)\\ &= \delta_n\,,\end{aligned}$$

thus

$$\Big|\nu(A)-\nu_n(A)\Big| \leqslant \delta_n\,.$$

□

We continue by considering the following sets, which we reintroduce for greater clarity:

$$T_{X_n} := \bigcup_{0<t<1} T_{X_n}\left(h^{-1}(t)\right) \qquad \text{and} \qquad T_X := \bigcup_{0<t<1} T_X\left(h^{-1}(t)\right)\,,$$

and denote them by $T_X(h)$ and $T_{X_n}(h)$ when we need to specify the height function.

Lemma 2.11. *Let h be a good height function for X. Then $\lim_{n\to\infty}\mu_n(T_{X_n})=\mu(T_X)$.*

Proof. Let θ_n be the positive angle between X and X_n. Then $\cos(\theta_n)=\langle Y,Y_n\rangle \geqslant 1-\epsilon_n$ for the positive sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to 0. By compactness of $\mathbb{S}^3$, the continuous function which maps any point $p\in\mathbb{S}^3$ to the angle $\theta_n(p)$ is strictly bounded by a constant d_n which tends to 0 as n goes to infinity. Let us extract this sequence $(d_{n_k})_{k\in\mathbb{N}}$ so that it is strictly decreasing to 0. For $n_k<n<n_{k+1}$, set $d_n=d_{n_k}$. Let W_n be the set of the points of each level set of h for which the tangent space of the level set makes an angle strictly less than d_n with X. These W_n have the following properties:

- $W_n \supseteq W_{n+1}$ for any n and $\bigcap_{n\in\mathbb{N}} W_n = T_X$;
- $\mu(W_n)\leqslant\mu(\mathbb{S}^3)$, hence it is finite. Then

$$0=\mu(T_X)=\mu\left(\bigcap_{n\in\mathbb{N}} W_n\right)=\lim_{n\to\infty}\mu(W_n)$$

- $T_{X_n}\subset W_n$ for all $n\in\mathbb{N}$.

By the additivity of the measure μ, we have:

$$0 \leqslant \mu(T_{X_n}) - \mu(T_X) \leqslant \mu(T_{X_n} \setminus T_X) \leqslant \mu(W_n \setminus T_X)\,.$$

But

$$\lim_{n\to\infty} \mu(W_n \setminus T_X) = \mu\left(\bigcap_{n\in\mathbb{N}} (W_n \setminus T_X)\right) = \mu\left(\left(\bigcap_{n\in\mathbb{N}} W_n\right) \setminus T_X\right) = 0\,.$$

Hence $\mu(T_{X_n})$ tends to $\mu(T_X) = \nu(T_X)$ when n tends to infinity. Thus, $\nu(T_{X_n})$ tends to $\nu(T_X)$ as well.

By Lemma 2.10 we have for all $n \in \mathbb{N}$:

$$\left|\nu(T_{X_n}) - \nu_n(T_{X_n})\right| \leqslant \delta_n\,,$$

which implies for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ so that for all $n > N_\epsilon$:

$$\nu(T_X) - \delta_n - \epsilon \leqslant \nu_n(T_{X_n}) \leqslant \delta_n + \nu(T_X) + \epsilon\,.$$

Thus when n tends to infinity, $\nu_n(T_{X_n})$ tends to $\nu(T_X) = \mu(T_X)$.

Now we want to prove that $\mu_n(T_{X_n})$ converges to $\mu(T_X)$. Let $M_n = \max_{\mathbb{S}^3} \|X_n\|^{-1}$ and $m_n = \min_{\mathbb{S}^3} \|X_n\|^{-1}$. By the convergence of X_n to X, given $\epsilon > 0$ there exists N'_ϵ so that for all $n > N'_\epsilon$, $1-\epsilon \leqslant M_n \leqslant 1+\epsilon$ and $1-\epsilon \leqslant m_n \leqslant 1+\epsilon$. Moreover, there exists K so that $|\mu(T_X)-\nu_n(T_{X_n})| < \epsilon$ for all $n > K_\epsilon$. Hence for all $n > \max(N'_\epsilon, K_\epsilon)$, we can write

$$(1-\epsilon)\nu_n(T_{X_n}) \leqslant m_n\nu_n(T_{X_n}) \leqslant \mu_n(T_{X_n}) \leqslant M_n\nu_n(T_{X_n}) \leqslant (1+\epsilon)\nu_n(T_{X_n})$$

from which we deduce

$$\epsilon(-\mu(T_X) - \epsilon) < -\epsilon\nu_n(T_{X_n}) \leqslant \mu_n(T_{X_n}) - \nu_n(T_{X_n}) \leqslant \epsilon\nu_n(T_{X_n}) < \epsilon(\epsilon + \mu(T_X))\,.$$

Thus $\mu_n(T_{X_n}) - \nu_n(T_{X_n})$ tends to 0 when n goes to infinity, implying that $\mu_n(T_{X_n})$ converges to $\mu(T_X)$. □

We now push the sets with the flow of X (resp. X_n). For a small positive time ϵ let us consider the sets:

$$T_X^\epsilon := \phi_X^{[0,\epsilon]}(T_X) \qquad \text{and} \qquad T_{X_n}^\epsilon := \phi_{X_n}^{[0,\epsilon]}(T_{X_n})\,.$$

Recall that W_n is the set of the points of each level set of h for which the tangent space of the level set makes an angle strictly less than d_n with X. We have seen that $T_{X_m} \subseteq W_m \subset W_n$ for all n, $m \in \mathbb{N}$ so that $m > n$.

Fix a small $\epsilon > 0$ that we might take smaller later in the proof. Set

$$W_n^\epsilon = \left(\bigcup_{m\geqslant n} T_{X_m}^\epsilon\right) \cup \phi_X^{[0,\epsilon]}(W_n)\,.$$

These sets have the following properties:

- $W_n^\epsilon \supset W_{n+1}^\epsilon$;
- $T_X^\epsilon \subset W_n^\epsilon$ and $T_{X_m}^\epsilon \subset W_n^\epsilon$ for all n and $m \geqslant n$;
- $\bigcap_{n\in\mathbb{N}} W_n^\epsilon = T_X^\epsilon$. Let us prove the two inclusions:
  - Because of the properties of $(W_n)_{n\in\mathbb{N}}$, $T_X^\epsilon \subset \phi_X^{[0,\epsilon]}(W_n)$ for all n and thus $T_X^\epsilon \subset W_n^\epsilon$ for all n, so that $T_X^\epsilon \subset \bigcap_{n\in\mathbb{N}} W_n^\epsilon$.
  - Let us show the converse inclusion. Let $x \in \left(\bigcap_{n\in\mathbb{N}} W_n^\epsilon\right) \setminus T_X^\epsilon$. Then

$$x \in \left(\bigcap_{n\in\mathbb{N}} \left(\bigcup_{m>n} T_{X_m}^\epsilon\right)\right) \setminus T_X^\epsilon$$

and $x \notin T_X$. Then for all $n \in \mathbb{N}$, there exists $y_n \in T_{X_n}$ so that the orbit of X_n going through y_n goes through x and $\phi_{X_n}^{t_n}(y_n) = x$ with $0 \leqslant t_n \leqslant \epsilon$. By compactness of $\mathbb{S}^3$, $(y_n)_{n\in\mathbb{N}}$ converges to a point y up to extraction of a subsequence and there exists $t = \lim_{n\to\infty} t_n \in [0,\epsilon]$ so that $\phi_X^t(x) = y$. Thus x belongs to T_X^ϵ and this contradicts the choice of x. So $\bigcap_{n\in\mathbb{N}} W_n^\epsilon \subset T_X^\epsilon$.

Let us suppose that $\bigcap_{n\in\mathbb{N}} T_{X_n}^\epsilon \not\subset T_X^\epsilon$. If x belongs to $\in \bigcap_{n\in\mathbb{N}} T_{X_n}^\epsilon \setminus T_X^\epsilon$, there exist two sequences $(p_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ so that $p_n \in T_{X_n}$, $t_n \in [0,\epsilon]$ and $p_n = \phi_{X_n}^{t_n}(x)$. Since the T_{X_n} are closed and decreasing for the inclusion, $(p_n)_{n\in\mathbb{N}}$ converges to $p \in T_X$ up to extraction of a subsequence and $(t_n)_{n\in\mathbb{N}}$ converges to $t \in [0,\epsilon]$. In the end $x \in T_X^\epsilon$ and this is a contradiction. So $\bigcap_{n\in\mathbb{N}} T_{X_n}^\epsilon \subset T_X^\epsilon$ and finally $\bigcap_{n\in\mathbb{N}} W_n^\epsilon = T_X^\epsilon$.

Now we can use these results to prove the theorem by contradiction. Suppose that $L > B(X,\mu)$. Then there exists h good for X such that $a_h < L$. Fix $\eta > 1$ such that $\eta a_h < L$. There exist $N(\eta)$ so that for all $n > N(\eta)$, $\eta a_h < B(X_n, \mu_n)$. Fix $\epsilon < \epsilon_h/2$. Then

$$0 \leqslant a_h \epsilon = \mu(T_X^\epsilon) = \mu\left(\bigcap_{n\in\mathbb{N}} W_n^\epsilon\right) = \lim_{n\to\infty} \mu(W_n^\epsilon)\,.$$

So there exists $N(\epsilon) > 0$ such that for all $n > N(\epsilon)$, we have $\mu(W_n^\epsilon) < \eta a_h \epsilon$ and thus $\mu(T_{X_n}^\epsilon) < \eta a_h \epsilon$.

As in the proof of Lemma 2.11, given $\delta > 0$, there exists N'_δ so that

$$|1 - M_n| \leqslant \delta \qquad \text{and} \qquad |1 - m_n| \leqslant \delta$$

for all $n > N'_\delta$. Then

$$\nu(T_{X_n}^\epsilon) - M_n \nu_n(T_{X_n}^\epsilon) \leqslant \mu(T_{X_n}^\epsilon) - \mu_n(T_{X_n}^\epsilon) \leqslant \nu(T_{X_n}^\epsilon) - m_n \nu_n(T_{X_n}^\epsilon)\,.$$

The lower bound can be bounded from below by

$$\begin{aligned}
\nu(T^{\epsilon}_{X_n}) - M_n\nu_n(T^{\epsilon}_{X_n}) &= M_n\left(\nu(T^{\epsilon}_{X_n}) - \nu_n(T^{\epsilon}_{X_n})\right) + (1-M_n)\nu(T^{\epsilon}_{X_n}) \\
&\geqslant -M_n\delta_n + (1-M_n)\nu(T^{\epsilon}_{X_n}) \\
&> -(\delta+1)\delta_n - \delta\nu(T^{\epsilon}_{X_n}) \\
&= -\delta(\delta_n + \nu(T^{\epsilon}_{X_n})) - \delta_n
\end{aligned}$$

for every $n > N'_\delta$. Since $\nu(T^{\epsilon}_{X_n}) = \mu(T^{\epsilon}_{X_n})$ is bounded, by taking n big, the above quantity is arbitrarily close to 0. Proceeding the same way, the upper bound is bounded by

$$\begin{aligned}
\nu(T^{\epsilon}_{X_n}) - m_n\nu_n(T^{\epsilon}_{X_n}) &= m_n\left(\nu(T^{\epsilon}_{X_n}) - \nu_n(T^{\epsilon}_{X_n})\right) + (1-m_n)\nu(T^{\epsilon}_{X_n}) \\
&< (\delta+1)\delta_n + \delta\nu(T^{\epsilon}_{X_n}) \\
&= \delta(\delta_n + \nu(T^{\epsilon}_{X_n})) + \delta_n
\end{aligned}$$

which can be taken arbitrarily close to 0. So there is an $N(\epsilon,\eta)$ so that for all $n > N(\eta,\epsilon)$,

$$\mu_n(T^{\epsilon}_{X_n}) < \eta a_h \epsilon\,.$$

The aim of what comes next is to find a good height function h_n for X_n, so that $B_{h_n}(X_n,\mu_n) < \eta a_h$. By definition, h has the following properties:

- The set $T_X(h)$ does not contain any connected segment of orbit (for X) of strictly positive length;
- If $x, y \in T_X$ are such that there exists $t > 0$ with $\phi^t_X(x) = y$ and $\phi^s_X(x) \notin T_X$ for all $s \in]0,t[$, then $t > \epsilon_h$.

Let $\delta < \epsilon_h$ and n large enough so that W_n satisfies the following conditions:

- The set W_n does not contain any connected segment of orbit (for X_n) of length greater than $\delta/2$;
- If $x, y \in T_{X_n}$ are such that there exists $t > 0$ with $\phi^t_{X_n}(x) = y$ and $\phi^s_{X_n}(x) \notin T_{X_n}$ for all $s \in]0,t[$, then $t > \delta/2$.

This is possible because $T_{X_{n_k}} \subset W_{n_k}$ and $\cap_k W_{n_k} = T_X$ which satisfies the above properties. Indeed since h is good for X, T_X does not contain any orbit segment of strictly positive length and two points of T_X on the same orbit are at a distance bigger than $\epsilon_h > \delta$. Moreover T_X does not contain any piece of orbit of strictly positive length for X_n.

For each such n - given by the above conditions on W_n, let us choose a finite covering of $(\mathbb{S}^3, X_n)$ with flowboxes $(\mathcal{F}_i)_{1\leqslant i\leqslant M}$ whose lengths are at least 2δ and so that any connected orbit segment of length δ is contained in a flowbox.

Fact: For each such n, we can construct a height function h_n which is good for X_n and so that $\epsilon_{h_n} > \delta/4$. Moreover, $T_{X_n}(h_n) \subset T_{X_n}(h)$.

Proof: We want to use the above covering to obtain a height function h_n good for X_n and so that $T_{X_n}(h_n) \subset T_{X_n}(h)$. To do this we need another family of

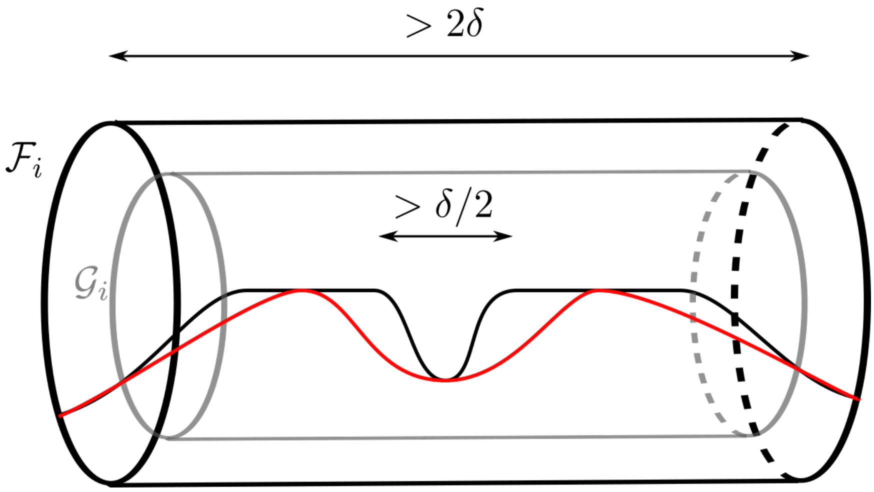

Figure 2.2: Changing the level sets of h inside a flowbox $\mathcal{G}_i$ to make it good for X_n. There can be several "plateaux" in the same flowbox, the important point is that there are all at least $\delta/2$ apart, which ensures that we have enough space to perform a modification.

flowboxes $(\mathcal{G}_i)_{1\leqslant i\leqslant M}$ so that each $\overline{\mathcal{G}_i}$ is contained in $\mathcal{F}_i$ and the family $(\mathcal{G}_i)_{1\leqslant i\leqslant M}$ is a covering of T_{X_n}. This can be made because the family $(\mathcal{F}_i)_{1\leqslant i\leqslant M}$ is a covering, so shrinking a little the generating disks of these flowboxes does not change this property. In each flowbox $\mathcal{G}_i$, the level sets of h have a certain number of "plateaux", that is to say points where a piece of orbit of strictly positive length is tangent, and these"plateaux" are apart one from each other from at least $\delta/2$ by choice of W_n. Inside these flowboxes, we can change the "plateaux" into"bump", with a $\mathcal{C}^1$-perturbation as pictured in Figure 2.2.

Since the "plateaux" are apart from at least $\delta/2$, by choosing the middle of the "plateau" as top of the "bump", we ensure that two tangency points on the same orbit are apart from at least $\delta/2$, thus a fortiori $\epsilon_{h_k} > \delta/4$.

Let $\epsilon < \delta/4 < \epsilon_h$. Recall that there exists $N(\epsilon,\eta) > 0$ so that for all $n > N(\epsilon,\eta)$ we have $\mu_n(T^\epsilon_{X_n}) < \eta a_h \epsilon$. Let $n > \max(N(\epsilon,\eta), N(\eta))$, then

$$a_{h_n}\epsilon = \mu_n(T^\epsilon_{X_n}(h_n)) \leq \mu_n(T^\epsilon_{X_n}(h)) < \eta a_h \epsilon.$$

Thus $a_{h_n} < \eta a_h$ and

$$B(X_n,\mu_n) \leq a_{h_n} < \eta a_h < B(X_n,\mu_n),$$

a contradiction. So $L \leq B(X,\mu)$.

If $L = \infty$, we can fix $A > B(X,\mu)$ and find $N > 0$ so that for all $n > N$ we have $B(X_n,\mu_n) > A$. Then repeating the above proof with A instead of L gives again a contradiction. So Theorem B holds.

2.3 Asymptotic bridge number

Let X be a smooth non-singular vector field on $\mathbb{S}^3$ which preserves an ergodic volume μ. Following the work of Arnol'd, the aim of this section is to prove that the asymptotic bridge number of a vector field is well defined (Theorem C) for this setting and that it coincides with the bridge number of a vector field that we defined in the previous section.

2.3.1 Asymptotic bridge number and curvature

Before proving Theorem C, we present an interesting relation between the asymptotic bridge number and the mean curvature of a knot. In order to use the relation between bridge number and curvature that was established by Milnor, we need to set a Riemannian metric g on $\mathbb{S}^3$ induced by $\mathbb{R}^4$. We denote ϕ_X^T the flow of X at the time $T \in \mathbb{R}$.

System of short paths. As we have already seen with the asymptotic linking number, an obvious difference between the bridge number of knots and the one we want to define is that we are working with pieces of trajectories which are generally not closed. So we need a system of short paths $\mathcal{S}$ that allows us to close the trajectories, which has been defined by Thomas Vogel in [Vog03] and which we have presented in Section 1.1.2. We recall some of its properties that we will need in this section:

- For each pair of points p, $q \in \mathbb{S}^3$, there exists an unique short path from p to q; each short path is piecewise differentiable.
- The short paths depend continuously on their extremities almost everywhere.
- For almost each point p and for all time T, the short path from p to $\phi_X^T(p)$ does not intersect the piece of trajectory $\phi_X^{[0,T]}(p)$, that is to say that we can close pieces of orbits with short paths and obtain simple closed curves.

With this system and the results of Milnor (Corollary 2.4), we can prove an upper bound on the asymptotic bridge number when it exists. We begin with a construction.

Construction

- For $x \in \mathbb{S}^3$ and $T \in \mathbb{R}$, we follow the trajectory of x during the time T and then close it with the right short path α in $\mathcal{S}$ given by Theorem 1.7. Then we do a little trick : we take a $\mathcal{C}^2$ perturbation $\tilde{\alpha}$ of α so that $\tilde{\alpha}$ has the same properties as α and the curve $k(x,T) := \phi_X^{[0,T]}(x) \cup \tilde{\alpha}$ is of class $\mathcal{C}^2$. Such a construction is possible for almost every x.
- We note $b(x,T)$ the bridge number of the knot $k(x,T)$.
- Following Milnor, we also define $\kappa(x,T) = \kappa(k(x,T)) = \int_0^{T+l(\tilde{\alpha})} \mid \gamma''(s) \mid ds$ where γ is the parametrized by arc length curve of support $k(x,T)$ and $l(\tilde{\alpha})$ is the length of $\tilde{\alpha}$. We set:

$$\kappa(x) = \lim_{T\to\infty} \frac{1}{T}\kappa(x,T)\,.$$

The above limit exists because the curve γ is of class $\mathcal{C}^2$ on $\mathbb{S}^3$ which is compact. Moreover the contribution of the perturbed geodesic to the curvature divided by T tends to 0 when the length of the piece of trajectory goes to infinity, so we have:

$$\begin{aligned}\kappa(x) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \mid \gamma''(s)\mid ds + \lim_{T\to\infty}\frac{1}{T}\int_T^{T+l(\tilde{\alpha})} \mid \gamma''(s)\mid ds \\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \mid \gamma''(s)\mid ds\end{aligned}$$

and the limit $\kappa(x)$ can be interpreted as the average curvature of the positive orbit of x.

Using Corollary 2.4, we know that for almost every $x \in \mathbb{S}^3$ and for all $T > 0$ we have $\kappa(x,T) \geqslant 2\pi\, b(x,T)$, so

$$\kappa(x) = \lim_{T\to\infty}\frac{1}{T}\kappa(x,T) \geqslant 2\pi \lim_{T\to\infty}\frac{1}{T}b(x,T) =: 2\pi\, b_\infty(x)\,,$$

when the limit exists for the right-hand side of the inequality. Now if we consider a sequence $(T_n)_{n\in\mathbb{N}}$ of times converging to infinity, the sequence $(\frac{1}{T_n}b(x,T_n))_{n\in\mathbb{N}}$ is bounded between 0 and $2\kappa(x)$ and converges up to extraction of a subsequence. So the mean curvature bounds the asymptotic bridge number.

2.3.2 Proof of Theorem C

Recall that for this result X is ergodic with respect to the volume μ. Fix a Riemannian metric and let x be a generic point in $\mathbb{S}^3$ for μ. We consider a small open disk-like section D of radius R transverse to X around x and push it with the flow of X on the time interval $]-t,t[$ for a fixed small $t > 0$ so as to get a flowbox $\mathcal{G}$. We call $(t_n)_{n\in\mathbb{N}}$ a sequence of return times of the point x to D, associated to the points $(x_n)_{n\in\mathbb{N}} = (\phi_X^{t_n}(x))_{n\in\mathbb{N}}$ so that $d(x,x_n)$ tends monotonically to 0 when n goes to infinity. Now fix $n \gg 0$. We want to close the arc of orbit between x and x_n with a $\mathcal{C}^1$-perturbation of X in a smaller flowbox $\mathcal{F}_n$ that we are going to define. Consider a radius r_n so that x_n is the first return of x in the transverse open disk centered on x and of radius r_n, and $r_n \ll R$. Now we push:

$$\mathcal{F}_n := \phi_X^{[-t/100,0]}(D(x,r_n))\,.$$

Let us abusively denote x_n the intersection point of the arc of orbit $\phi_X^{[0,t_n]}(x)$ with $\phi_X^{-t/100}(D(x,r_n))$. In this flowbox $\mathcal{F}_n$, we can close the arc of orbit $\phi_X^{[0,t_n-t/100]}(x)$ with a $\mathcal{C}^1$-perturbation of X in $\mathcal{F}_n$ - this is possible with the $\mathcal{C}^1$-closing lemma of Pugh and Robinson [PR83]. We denote $k_X(x,t_n)$ the obtained knot. For each return time t_n, we have a perturbed vector field X_n so that $X_n = X$ outside $\mathcal{F}_n$ and X_n is $\mathcal{C}^1$-close to X in $\mathcal{F}_n$.

As we have seen previously in the beginning of Section 2.2.1, we can associate a Dirac linear measure to the knot $k_X(x,t_n)$ as follows : for a point $x \in \mathbb{S}^3$, $t > 0$, $\mu_{x,t}(A) := \mathrm{Leb}(\{s \in [0,t], \phi_X^s(x) \in A\})$. We consider the normalized measure $\frac{1}{t}\mu_{x,t}$. Since X is ergodic for μ, for μ-almost all $x \in \mathbb{S}^3$ and for all $(t_n)_{n\in\mathbb{N}}$ so that $(t_n)_{n\in\mathbb{N}}$

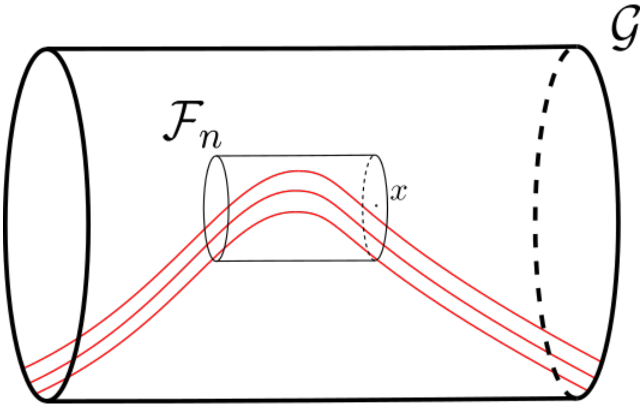

Figure 2.3: Possible position of the level sets of h_n in $\mathcal{G}$ and $\mathcal{F}_n$

tends to infinity, $\frac{1}{t_n}\mu_{x,t_n}$ converges to μ weakly-$*$. Now we denote μ_n the normalized Dirac linear measure supported by the periodic orbit (for X_n) $k_X(x,t_n)$, which is invariant by X_n. As X_n tends to X in the $\mathcal{C}^1$-topology and μ_n tends to μ weakly-$*$, by Theorem B the sequence $B(X_n,\mu_n)$ admits a limit, that is less or equal to $B(X,\mu)$.

Take h_n so that $B_{h_n}(X_n,\mu_n) = B(X_n,\mu_n)$. Such a height function exists since the support of μ_n is a knot in $\mathbb{S}^3$.

Lemma 2.12. *Let $n \in \mathbb{N}$. Let $\mathcal{F}_n$ be the flowbox where $X \neq X_n$. We can choose h_n so that*

- $\mu(T_X(h_n)) = 0$*;*
- $T_{X_n}(h_n) \cap \mathcal{F}_n = \emptyset$*.*

Proof. From the choice of h_n, we can already have that $\mu_n(T_X(h_n)) = 0$. Indeed, since $X = X_n$ outside of $\mathcal{F}_n$ and h_n is a good height function for X_n, in an open tubular neighbourhood $\mathcal{N}$ of $k_X(x,t_n)$, the level sets of h_n satisfy that $\mu_n(T_X(h_n)) = 0$. Thus by Proposition 2.8 we can modify the level sets of h_n in $\mathbb{S}^3 \setminus \mathcal{N}$ so that the tangency points to X are isolated along the orbits, which implies that $\mu(T_X(h_n)) = 0$. Note that since $\mathbb{S}^3 \setminus \mathcal{N}$ is not in the support of μ_n, anything could have happened with the tangency points there although h_n is a good function for $X_n = X$. We have then to look at what happens inside $\mathcal{F}_n$. If X_n does not have tangency points for h_n in $\mathcal{F}_n$, we are done. If not, since the flowboxes are arbitrarily small and h_n is chosen to minimize the bridge, one can suppose that the tangency points look like in Figure 2.3.

Then we can pull away the problematic points by modifying the level sets in $\mathcal{G}$, so that the level sets of h_n are transverse both to X_n and X in $\mathcal{F}_n$, as pictured in Figure 2.4

This concludes the proof. □

Observe that $\mu_n(T_{X_n}(h_n)) = \mu_n(T_X(h_n))$, by the definition of μ_n, and for $\epsilon > 0$ small we have $\mu_n(T^\epsilon_{X_n}(h_n)) = \mu_n(T^\epsilon_X(h_n))$ too. Thus

$$\mu(T^\epsilon_X(h_n)) - \mu_n(T^\epsilon_{X_n}(h_n)) = \mu(T^\epsilon_X(h_n)) - \mu_n(T^\epsilon_X(h_n))\,.$$

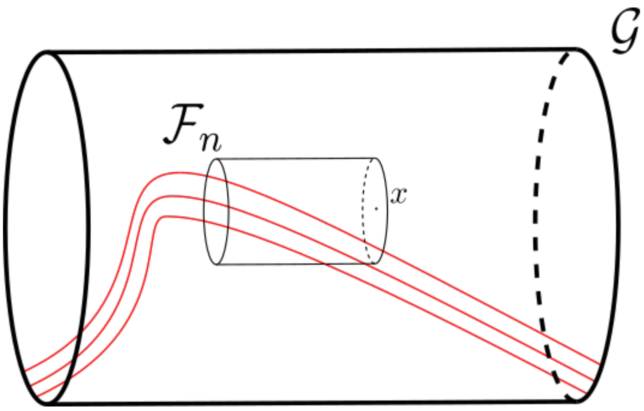

Figure 2.4: Modification of the level sets of h_n in $\mathcal{G}$ so that there are no tangency points in $\mathcal{F}_n$

Since $\mu(T_X(h_n)) = 0$, $\mu(T_X^\epsilon(h_n)) = a_n \cdot \epsilon$, and also $\mu_n(T_{X_n}^\epsilon(h_n)) = b_n \cdot \epsilon$, for some a_n and $b_n \geqslant 0$. Observe that $b_n = B_{h_n}(X_n, \mu_n)$ and $a_n = B_{h_n}(X, \mu)$, and by Theorem B $b_n \leqslant a_n$ for n big enough. Given a positive sequence $(\epsilon_k)_{k\in\mathbb{N}}$ converging to 0, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ converging to infinity so that

$$\delta_{n_k} < \epsilon_k^2 , \qquad \left|\frac{1 - m_{n_k}}{m_{n_k}}\right| < \epsilon_k^2 , \qquad \left|\frac{1 - M_{n_k}}{M_{n_k}}\right| < \epsilon_k^2 ,$$

where δ_{n_k} is given by Lemma 2.10 and we recall that $M_{n_k} = \max_{\mathbb{S}^3} \|X_{n_k}\|^{-1}$ and $m_{n_k} = \min_{\mathbb{S}^3} \|X_{n_k}\|^{-1}$. At the same time,

$$\frac{b_n \cdot \epsilon}{M_n} \leqslant \nu_n\left(T_{X_n}^\epsilon(h_n)\right) \leqslant \frac{1}{m_n}\mu_n\left(T_{X_n}^\epsilon(h_n)\right) = \frac{b_n \cdot \epsilon}{m_n}$$

where the last equality holds for ϵ small enough. Then

$$\begin{aligned} -\delta_{n_k} + (1 - M_{n_k})\nu_{n_k}\left(T_{X_{n_k}}(h_{n_k})\right) &\leqslant \mu\left(T_X^{\epsilon_k}(h_{n_k})\right) - \mu_{n_k}\left(T_{X_{n_k}}^{\epsilon_k}(h_{n_k})\right) \\ -\epsilon_k^2 + \tfrac{1-M_{n_k}}{M_{n_k}} b_{n_k}\epsilon_k &< \mu\left(T_X^{\epsilon_k}(h_{n_k})\right) - \mu_{n_k}\left(T_{X_{n_k}}^{\epsilon_k}(h_{n_k})\right) , \end{aligned}$$

and

$$\begin{aligned} \mu\left(T_X^{\epsilon_k}(h_{n_k})\right) - \mu_{n_k}\left(T_{X_{n_k}}^{\epsilon_k}(h_{n_k})\right) &\leqslant \delta_{n_k} + (1 - m_{n_k})\nu_{n_k}\left(T_{X_{n_k}}^{\epsilon_k}(h_{n_k})\right) \\ \mu\left(T_X^{\epsilon_k}(h_{n_k})\right) - \mu_{n_k}\left(T_{X_{n_k}}^{\epsilon_k}(h_{n_k})\right) &< \epsilon_k^2 + \tfrac{1-m_{n_k}}{m_{n_k}} b_{n_k}\epsilon_k . \end{aligned}$$

Now we need to remember that for any $n \in \mathbb{N}$, $b_n = B(X_n, \mu_n)$ and by Theorem B up to extraction of a subsequence,

$$\lim_{n\to\infty} b_n \leqslant B(X, \mu) .$$

Thus $b_{n_k}\epsilon_k$ tends to zero when k tends to infinity, and for k sufficiently big we can assume that

$$-2\epsilon_k^2 \;< -\epsilon_k^2(1 + b_{n_k}\epsilon_k) \;< \mu\left(T_X^{\epsilon_k}(h_{n_k})\right) - \mu_{n_k}\left(T_{X_{n_k}}^{\epsilon_k}(h_{n_k})\right) \;< \epsilon_k^2(1 + b_{n_k}\epsilon_k) \;< 2\epsilon_k^2 .$$

Hence when k tends to infinity,

$$\lim_{k\to\infty} \mu\left(T_X^{\epsilon_k}(h_{n_k})\right) = \lim_{k\to\infty} \mu_{n_k}\left(T_{X_{n_k}}^{\epsilon_k}(h_{n_k})\right)$$

and

$$\lim_{k\to\infty} \frac{1}{\epsilon_k}\mu_{n_k}\left(T_{X_{n_k}}^{\epsilon_k}(h_{n_k})\right) = \lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty} B(X_{n_k}, \mu_{n_k}) = L\,.$$

But $\mu\left(T_X^{\epsilon_k}(h_{n_k})\right) \leqslant a_{n_k}\epsilon_k$ - with an inequality because ϵ_k might not be small enough. This implies that

$$a = \lim_{k\to\infty} a_{n_k} \leqslant L\,.$$

If $a < L$, there exists K big enough so that for all $k > K$, $a_{n_k} < L$. Then by definition of the bridge number of vector fields,

$$B_{h_{n_k}}(X,\mu) < L \leqslant B(X,\mu)$$

which is a contradiction. If not,

$$\lim_{k\to\infty} a_{n_k} = L \geqslant B(X,\mu)$$

and we are done.

Corollary 2.13. *Let X be a smooth non-singular vector field on $\mathbb{S}^3$ preserving an ergodic volume μ. Let x be a recurrent point generic for μ. Then*

$$B(X,\mu) \leqslant \frac{\kappa(x)}{2\pi}$$

where $\kappa(x)$ is the limit defined in Section 2.3.1.

Proof. Let x be a recurrent point generic for μ. Then from Section 2.3.1, we have:

$$\kappa(x) \geqslant 2\pi b_\infty(x),$$

and since (X,μ) is ergodic by Theorem C we know that $b_\infty(x) = B(X,\mu)$, so the result follows.

□

2.4 Connection with other invariants

2.4.1 Independance of helicity

As we said in the introduction, it happens sometimes that a new invariant turns out to be a function of the well-known helicity. The aim of this section is to show that it is not the case for the bridge number of vector fields.

Here we restrict to vector fields X preserving a volume form Ω on $\mathbb{S}^3$. We recall from Chapter 1 that the helicity of (X,Ω) is defined by

$$Hel(X,\Omega) = \int_{\mathbb{S}^3} \alpha \wedge d\alpha\,,$$

where α is any potential 1-form for $i_X\Omega$, that is to say $d\alpha = i_X\Omega$. To prove Theorem D we want to show an example of a vector field for which the bridge number of vector fields is not a function of helicity. So we first compute the bridge number in the case of Seifert flows, a class of flows that we presented in Section 1.3. We start by giving an upper bound and then we show that this number is also the infimum.

We consider the standard height function on $\mathbb{S}^3$. The tangency points of the flow to the surface level sets (spheres centered on $0^3_{\mathbb{R}}$) are exactly the points of the "plane" P (actually a sphere) made of the two flat disks bounded by the particular orbit $t \mapsto (\exp 2i\pi\alpha t, 0)$. For $x \in P$, the first return time (in P) is $\frac{1}{2\beta}$ and this time multiplied by the area of P is equal to the volume of $\mathbb{S}^3$, that is to say 1. So the bridge number of this flow is bounded by $\frac{1}{2} \times 2\beta = \beta$. Now if we change the stereographic projection of $\mathbb{S}^3$, the same argument shows that $B(X_{\alpha,\beta}, \Omega_{Haar}) \leqslant \alpha$. Thus we can conclude:

$$B(X_{\alpha,\beta}, \Omega_{Haar}) \leqslant \min(\alpha, \beta)\,.$$

In order to prove the converse inequality, we approximate $X_{\alpha,\beta}$ in the C^∞-topology by a Ω_{Haar}-preserving sequence $(X_{p_n/r_n, q_n/r_n})_{n\in\mathbb{N}}$, with p_n, q_n, r_n three sequences of integer numbers so that p_n/r_n and q_n/r_n are decimal expansions of α and β. Then by Theorem B, we have:

$$B(X_{\alpha,\beta}, \Omega_{Haar}) \geqslant \lim_{n\to\infty} B(X_{p_n/r_n, q_n/r_n}, \Omega_{Haar})\,.$$

We have to compute the right-hand side of this inequality. Fortunately since the bridge number is an invariant of order 1, it is enough to prove that $B(X_{p,q}, \Omega_{Haar}) = \min(p,q)$ where p, q are two coprime positive integers.
The proof is the same as the computation of the trunkenness of Seifert flows presented in Section 1.3. One can choose a sequence $(K_n)_{n\in\mathbb{N}}$ of collections of periodic orbits so that its induced normalized invariant measures μ_n tends to Ω_{Haar}. These orbits are torus knots of type (p,q) so the bridge number of K_n is $n\min(p,q)$ since the bridge number of torus knots is $\min(p,q)$ and we have n copies of them. The period of each component of K_n is 1, so that its total length is n and we have:

$$B(X_{p,q}, \Omega_{Haar}) \geqslant \lim_{n\to\infty} B(X_{p,q}, \mu_n) = \lim_{n\to\infty} \frac{1}{n} \times n\min(p,q) = \min(p,q)\,.$$

Finally we have

$$\min(\alpha,\beta) \geqslant B(X_{\alpha,\beta}, \Omega_{Haar}) \geqslant \lim_{n\to\infty} \min(p_n/r_n, q_n/r_n) = \min(\alpha,\beta)\,,$$

so that the bridge number of a Seifert flow of parameters (α,β) is $\min(\alpha,\beta)$.

Proof of Theorem D. For a Seifert flow on $\mathbb{S}^3$ with the standard Haar measure, we have $\mathrm{Hel}(X_{\alpha,\beta}, \Omega_{Haar}) = \alpha\beta$ and $B(X_{\alpha,\beta}, \Omega_{Haar}) = \min(\alpha,\beta)$. Since there is no function f so that $\min(\alpha,\beta) = f(\alpha\beta)$, the helicity and the bridge number of vector fields are independent in this case. But as we already said, the Seifert flows are not ergodic with respect to Ω_{Haar}.

Let (α, β) so that α/β is irrational. Let x be a point on an invariant torus. The measure μ_x associated with x is ergodic and supported precisely on this invariant torus, and we have $\text{Hel}(X, \mu_x) = \alpha\beta$ while $B(X, \mu_x) = \min(\alpha, \beta)$. This shows that the bridge number is not a function of helicity in this particular ergodic case. □

But since the trunkenness is two times the bridge number in the case of torus knots, we also need to check if the bridge number and the trunkenness are unrelated in general, which we do in the next section.

2.4.2 Relation with the trunkenness of vector fields

As we have seen in the previous section and in Chapter 1, the bridge number of knots is strongly related to the trunk, at least for meridionally small knots [Oza10]. Thus one could ask: is the bridge number of vector fields independent from the trunkenness of the vector fields?
We can answer that these two invariants are independent by showing a construction. Let us consider two Seifert flows of parameters $(3, 2)$ on two copies of $\mathbb{S}^3$ that we consider as $\mathbb{R}^3 \cup \{\infty\}$.

Fix a projection of $\mathbb{S}^3$ to $\mathbb{R}^3$ so that in any invariant torus, the torus knots (the orbits) make 3 turns meridionally and 2 turns longitudinally. Choose also a metric g which induced distance will be denoted d. Then fix an invariant solid torus and choose a small open disk $D = D(x_0, r)$ transverse to the flow that does not contain the torus axis. The disk D can be made small enough so that any orbit in the solid torus has at most one intersection with it. It is because any orbit crosses any longitudinal section of the solid torus in two opposite points. Fix a positive $\epsilon \ll r$ and consider the set of points:

$$\mathcal{L}(x_0, r) := \bigcup_{x \in D(x_0, r)} \phi_X^{]-\epsilon(1-\frac{d(x,x_0)}{r})^2, \epsilon(1-\frac{d(x,x_0)}{r})^2[}(x)$$

which is some kind of open lens of radius r centered on x_0. Choose this same particular lens $\mathcal{L}(x_0, r)$ in the two copies of $\mathbb{S}^3$.

Now perform the connected sum of the flows along these two lenses. We obtain a flow that preserves the volume since the fluxes across each sections are identical.The new flow X has two types of orbits:

- The ones that do not intersect D and that are $T(3, 2)$ torus knots;
- and the ones that intersect D and are a connected sum of two $T(3, 2)$ torus knots.

Now we need to specify the invariant measure that we consider for this new flow. Let $p \in \mathcal{L}(x_0, r)$ and let us choose the measure supported by the periodic orbit of p, which is thus ergodic with respect to X. Because of this choice, the values of $B(X, \mu)$ and $\text{Tk}(X, \mu)$ are the ones of the knot invariants. Thus the bridge number of the orbit of p is 3 because of the almost additivity of the bridge by Proposition 2.2. In the same time, by Proposition 1.19, the trunk of a connected sum is the maxima of their trunks [DZ17], thus the trunk of the orbit of p is 2. So the trunkenness and the bridge number of vector fields are independent.

Chapter 3

The asymptotic genus

Given a knot k in $\mathbb{S}^3$, it is possible to construct a Seifert surface, i.e. an oriented embedded surface whose boundary is k, and to compute its genus. We call *genus of* k the minimal genus that we can obtain with this process. It is a knot invariant. If we consider a vector field X on $\mathbb{S}^3$, the trajectories of the flow of X are generally not closed curves in $\mathbb{S}^3$. Thus it would make sense to try to define an asymptotic genus with Arnol'd's method, just like the asymptotic linking number [AK21] or the trunkenness [DR17]. In this chapter we explain two attempts to define an asymptotic genus for right-handed vector fields, a class of vector fields that we have presented in Chapter 1.

The first strategy is to consider a very long arc of orbit of a recurrent point x and to artificially close it after a return time t_n with a perturbation of the vector field to obtain the knot $k(x, t_n)$. Then we show that the perturbed vector field remains right-handed, and finally we compute the genus of this particular closed orbit to obtain the following:

Theorem E. *Let X be a smooth right-handed non singular vector field X preserving a smooth ergodic volume μ on $\mathbb{S}^3$. Let x be a recurrent point for the flow of X and generic for μ. Then*

$$\lim_{n\to\infty} \frac{1}{{t_n}^2} g(k(x, t_n)) = \frac{1}{2} Hel(X, \mu)\,.$$

Another idea to define an asymptotic genus, which uses completely different methods, is to use an iterated construction that is presented in Section 3.2. We choose a recurrent point and transform its arcs of orbits between two successive returns to a flowbox into knots, using a short path to close them. These knots may be linked, and we find a formula to bound the genus of this link. Then we present a sketch of proof from which we could obtain that half of the helicity is an upper bound for the genus of an orbit.

We start by explaining the first strategy in Section 3.1. The attempt of second strategy is presented in Section 3.2.

3.1 Proof of Theorem E

3.1.1 Outline of the proof

In this section we are going to show Theorem E. As it is technical, we start by explaining the general idea of the proof together with some needed results. For the detailed proof see Section 3.1.2.

We consider a smooth right-handed, non-singular vector field X preserving a smooth ergodic volume μ on $\mathbb{S}^3$. Choose x a recurrent point for the flow ϕ_X^t of X generic for μ in $\mathbb{S}^3$ and S a disk transverse to the vector field containing x. Let x_N be the N-th return of the orbit of X to S, then $x_N = \phi_X^{T_N}(x)$. Set $\phi_X^{[0,T_N]}(x)$ to be the orbit segment between x and x_N. As we will explain below, we slightly modify the vector field X in a flowbox around S in order to obtain a knot $k(x, T_N)$. More precisely we want to close the arc of orbit $\phi_X^{[0,T_N]}(x)$ with a $\mathcal{C}^1$-perturbation of X preserving the volume μ. To do this we need to find a "good" flowbox around x. Lemma 3.1, together with some technical assumptions, allows to choose a suitable flowbox.

Lemma 3.1. *Let X be a right-handed vector field in $\mathbb{S}^3$. Then there exists two constants $T_r > 0$ and $C_r > 0$ such that for any pair of recurrent points (p, q) of $\mathbb{S}^3$,*

$$\frac{1}{ts} link(k(p,t), k(q,s)) \geqslant C_r$$

for all times t, $s \geqslant T_r$. We say that the vector field X is (T_r, C_r)-right-handed.

The first step is to use this lemma to construct a flowbox $\mathcal{F}_N$ around x so that the return time of any recurrent point in $\mathcal{F}_N$ is at least T_r. Secondly, we consider x_N the N-th return of x to $\mathcal{F}_N$ with $N \gg 0$ being fixed. Then using the $\mathcal{C}^1$-closing lemma of Pugh and Robinson [PR83], we can perturb locally X in X_N inside $\mathcal{F}_N$ in order to close the arc of orbit $\phi_X^{[0,T_N]}(x)$ while preserving the volume μ. Note that $X_N = X$ outside $\mathcal{F}_N$ and that these two vector fields are $\mathcal{C}^1$-close in $\mathcal{F}_N$. Even if this perturbation is local for the vector fields, it is changing much more than the prescribed orbit between x_N and x since any orbit entering $\mathcal{F}_N$ might be affected and re-branched on another (different) orbit.

After performing the perturbation we have to show that the perturbed vector field X_N is still right-handed. To prove that the linking of two orbits stays positive, we cut the integral of the linking form into pieces depending whether we integrate or not on an orbit segment passing through $\mathcal{F}_N$ or close to $\mathcal{F}_N$. Then we use the technical bounds on the linking form presented in Chapter 1 to bound some terms of the sum and estimate others, and finally we obtain a positive asymptotic. In order to conclude, we then show that periodic orbits created by the perturbation have positive self-linking.

As the linking of any two long enough pieces of orbit of X_N is positive this implies that the piece of orbit $\phi_X^{[0,T_N]}(x)$, the one which we artificially closed, is linked positively to any other orbit. Then by a theorem of Ghys [Ghy09], this orbit is a fibered knot and thus binds a Birkhoff section. But by a known result (see

Theorem 4.1.10 in [Kaw96]), any Seifert surface S_N for this orbit and transverse to the flow has minimal genus. Dehornoy and Rechtman computed the genus of such a surface in [DR22] using its self-linking number : one has $g(S_N) = \frac{Slk^{\zeta_{X_N}}(\gamma)+1}{2}$ where ζ_{X_N} is a vector field transverse to X_N. So using the fact that the asymptotic self-linking number is equal to helicity (Arnol'd [Arn73]), we obtain Theorem E.

3.1.2 Proof of theorem E

In all this section, X is a smooth non-singular right-handed vector field preserving a smooth ergodic volume μ on $\mathbb{S}^3$. Fix x a recurrent point for the flow of X and generic for μ.

3.1.3 Choice of a good neighbourhood

We start by proving Lemma 3.1.

Proof. Suppose it is false. Then there exist $(T_n)_{n\in\mathbb{N}}$ a monotonically increasing time sequence, diverging to $+\infty$, and $(C_n)_{n\in\mathbb{N}}$ a positive monotonically decreasing sequence converging to 0, and a sequence $((p_n, q_n))_{n\in\mathbb{N}}$ of pairs of recurrent points and $(S_n)_{n\in\mathbb{N}}$, $(R_n)_{n\in\mathbb{N}}$ two sequences of return times so that $S_n > T_n$, $R_n > T_n$ and such that we have for all n large enough:

$$\frac{1}{S_n R_n}\text{link}(k(p_n, S_n), k(q_n, R_n)) < C_n\,.$$

The limit of this quantity should be strictly positive according to Ghys' Theorem 1.14 and we have a contradiction. In the case where (X, μ) is ergodic, we give another argument to prove Lemma 3.1. With the ergodic assumption, the above quantity

$$\frac{1}{S_n R_n}\text{link}(k(p_n, S_n), k(q_n, R_n))$$

tends to a constant when n tends to infinity, and if this constant is zero or negative, it is a contradiction to the definition of right-handedness according to Hryniewicz-Florio [FH23] which we presented in Chapter 1, see Section 1.2.2. Thus the limit is strictly positive and this concludes the proof.

□

As we want to change the vector field X in a small neighbourhood of a point x without changing too much the linking number of the orbits of the flow, we will also be needing Lemma 1.5 and Lemma 1.4 from Chapter 1 to bound the contribution to the linking number of the perturbed parts of the orbits. Recall that on our setup in $\mathbb{S}^3$, given a Riemannian metric g there is a construction of a Gauss linking form $\mathcal{L}$ on $\mathbb{S}^3 \times \mathbb{S}^3$ given by Vogel [Vog03]. In this case Lemma 1.4 ensures that there exists a constant C_l depending on g so that we have the punctual bound:

$$\|\mathcal{L}_{p,q}\|_\infty \leqslant C_l \text{dist}(p, q)^{-2}\,.$$

Choice of a flowbox around x. Here we explain how to choose the flowbox in which we will perform a perturbation to close the orbit of x. Suppose that X is right-handed of constants (T_r, C_r), meaning that if we follow two orbits during a time greater than T_r then their normalized linking number is at least C_r as in Lemma 3.1.
Let g be a Riemannian metric of injectivity radius r_{inj}. As in Lemma 1.5 set $r_0 = r_{inj}/100$, so that we have an isometric embedding i_1 of the Euclidian open ball of radius $r_0/2$ in $\mathbb{R}^3$ into $\mathbb{S}^3$ with $i_1(0_{\mathbb{R}^3}) = x$ and so that the image of the disk $\{(u,v,w) \in \mathbb{R}^3 | u^2+v^2 < (r_0/2)^2, w = 0\}$ is a disk transverse to the flow of X, which we denote $i_1\left(D_{r_0/2}(x)\right)$. Similarly, the image of the ball of radius $r_0/2$ will be denoted $i_1\left(B_{r_0/2}(x)\right)$. We also ask that the flow lines of X are just the image of straight vertical lines in $\mathbb{R}^3$ and that geodesics are parametrized by arc-length in the image of the embedding. Since X is recurrent there exists an open neighbourhood $\mathcal{W} \subset i_1\left(D_{r_0/8}(x)\right)$ of x so that the first return time to $\mathcal{W}$ is greater than a constant $T_0 > T_r + r_0/2$. We also have to ask T_0 to be greater than the following constants, up to shrinking $\mathcal{W}$:

$$T_+ := \frac{r_0}{8} + \frac{3G}{4C_r} + \frac{1}{2}\sqrt{\left(\frac{r_0}{4} + \frac{3G}{2C_r}\right)^2 + 64\frac{C_l}{C_r}}$$
$$T'_+ := \frac{1}{2}\left(\frac{r_0}{2} + \frac{2G}{C_r} + \sqrt{8\frac{C(g)}{C_r} + \frac{2G}{C_r}\left(r_0 + \frac{2G}{C_r}\right)}\right)$$

where G is the geometric constant

$$G := \frac{4}{r_0}\left(\frac{C(g)}{\sqrt{3}} + 8C_l\right)$$

and $C(g)$ is the constant from Lemma 1.5 and depends only on g. This restriction will arise from the computations of Section 3.1.4. The idea is that T_0 must be large enough.

Suppose that we have $T_0 > \max\left\{T_r + r_0/2, T_+, T'_+\right\}$. We want to define a flow-box around x to close very long pieces of orbit, and thus we need to avoid having short periodic orbits, that is to say periodic orbits of period less than T_0, in a neighbourhood of x. Suppose by contradiction that there exist short periodic orbits arbitrarily close to x. Then in $\mathcal{W}$, we can find a sequence $(y_n)_{n\in\mathbb{N}}$ converging to x and so that the return time of any y_n is less or equal to T_0. By continuity of the flow, this implies that x comes back to $\overline{\mathcal{W}}$ in a time $T_0 + \epsilon$, for $\epsilon > 0$ small, so because of the definition of $\mathcal{W}$, x comes back in $\overline{\mathcal{W}} \setminus \mathcal{W}$. Let $\mathcal{W}'$ be an open set around x strictly contained in $\mathcal{W}$ and so that $\left(\overline{\mathcal{W}} \setminus \mathcal{W}\right) \cap \mathcal{W}' = \emptyset$. The same reasoning applied to $\mathcal{W}'$ tells us that x comes back to $\overline{\mathcal{W}'}$, which is strictly contained in $\mathcal{W}$, in a time $T_0 + \epsilon'$, with $\epsilon' > 0$ small, and this is a contradiction. So there exists a small neighbourhood of x which does not contain short periodic orbits.

Now by regularity of the $\mathcal{C}^\infty$-diffeomorphism $\phi_X^{T_0}$, there exist U a connected open set around x in $\mathcal{W}$ so that any recurrent point has a first return time to U greater than T_0. Consider a sequence $(x_n)_{n\in\mathbb{N}}$ of returns of x to U so that $d(x_n, x)$ is monotonically decreasing to zero when n goes to infinity. Fix $N \gg 0$. There exists

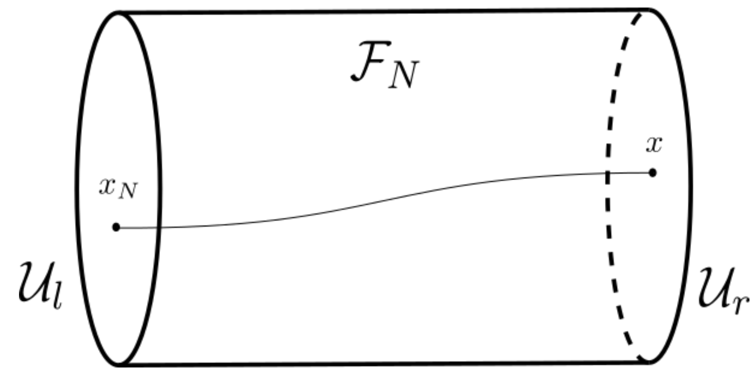

Figure 3.1: Closing the orbit in the flowbox $\mathcal{F}_N$

a connected open set $\mathcal{U}_N \subset U$, containing x and x_N but not the precedent returns of x. This is the set from which we will define $\mathcal{F}_N$. Let us fix $0 < \epsilon < r_0/4$. We define the flowbox $\mathcal{F}_N$ to be $\phi_X^{[-\epsilon,0]}(\mathcal{U}_N)$. In order to make this choice of neighbourhood more understandable, let us list the properties of $\mathcal{F}_N$:

- The point x belongs to the exit region of $\mathcal{F}_N$;
- Any recurrent point in $\mathcal{F}_N$ will come back in a time greater than $T_0 > T_r$, and we will prove later that this ensures that it will link positively with long enough orbits;
- In $\mathcal{F}_N$ the flow lines of X are just straight, parallel lines;
- Thanks to the inclusion $\mathcal{F}_N \subset i_1\left(B_{r_0/2}(0)\right)$, the geodesics are just straight lines, parametrized by arc-length, and $\mathcal{F}_N$ is small enough to use Lemma 1.5 for geodesics (or perturbations of geodesics) in it.

To simplify the notation we denote $\mathcal{U}_l$ and $\mathcal{U}_r$ the left and right transverse regions of the boundary of the flowbox $\mathcal{F}_N$, and we suppose that the flow goes from left to right.

We are ready to modify the orbit of x so as to close it. Abusing notation, consider x_N in $\mathcal{U}_l$. Then the $\mathcal{C}^1$-closing Lemma for volume preserving vector fields of Pugh and Robinson [PR83] states that we can construct a vector field X_N that is $\mathcal{C}^1$-close to X inside $\mathcal{F}_N$, so that X_N preserves the ergodic volume μ and it has a prescribed arc of orbit from x_N to x in $\mathcal{F}_N$. The vector field X_N coincides with X on the boundary of $\mathcal{F}_N$ and hence we set X_N to be X outside $\mathcal{F}_N$.

We are going to show that the perturbation is small enough to ensure that the perturbed vector field X_N is right-handed.

3.1.4 The perturbation X_N is right-handed

As we said in Section 3.1.1, we want to cut the integral of the linking form in different parts and bound them. In order to do this we use Lemma 1.5 and Lemma 1.4 from Chapter 1. Lemma 1.4 is a punctual bound on the linking form which we will be using when we integrate the linking form on arcs of orbits that are far one from the other. If the arcs of orbit are too close one from each other, then we use Lemma 1.5, which was already helpful to define the flowbox $\mathcal{F}_N$. Note that because of the smoothness of the linking form outside the diagonal of $\mathbb{S}^3 \times \mathbb{S}^3$, Lemma 1.5 stays true for $\mathcal{C}^1$-perturbations of geodesics. This fact will be useful when considering short arcs of orbits in $i_1\left(B_{r_0/2}(0)\right)$, and also for the short paths closing long arcs of orbits.

Consider a pair of recurrent points (p, q) of $\mathbb{S}^3$ on distinct orbits, i.e. $\mathcal{O}(p) \cap \mathcal{O}(q) = \emptyset$. If none of the orbits of p and q enters $\mathcal{F}_N$, then their asymptotic linking number is positive since $X_N = X$ outside $\mathcal{F}_N$ and X is right-handed. Thus we have to examine the two following cases:

1. The orbit of p enters $\mathcal{F}_N$ while the orbit of q does not (up to changing p in q and conversely);

2. Both of the orbits enter $\mathcal{F}_N$.

In addition to these two cases, one has to consider that changing X in X_N may have created new periodic orbits, that are necessary going through $\mathcal{F}_N$. Actually there is at least one, which is the new orbit of x. According to the definition of right-handedness (see Chapter 1) one has to verify that these new orbits have positive self-linking number. This will be made in a third case to complete the proof.

The proof of the second case contains somehow the proof of the first one, but for the sake of comprehension we will detail both of them. We begin with the first case.

Proof of the first case : $\mathcal{O}(q) \cap \mathcal{F}_N = \emptyset$.

Since we are interested in very long pieces of orbits, we can suppose that p belongs to the “exit region” $\mathcal{U}_r$ of $\mathcal{F}_N$. Indeed, otherwise the piece of orbit that is before the first intersection of the orbit of p with $\mathcal{F}_N$ would have a bounded contribution to the linking with the orbit of q, which would disappear as the times tend to infinity. Since we would consider the linking for arbitrary big times, the bounded part can be neglected.

Notations and setup. We modify the metric g so as to have an isometric embedding i_2 of the Euclidean ball $B_{r_0/2}(0)$ around q, and more precisely so that $i_2(0_{\mathbb{R}^3}) = q$. Note that we already have an embedding i_1 around the point x. It is exaclty the same procedure as in Section 3.1.3 when the flowbox $\mathcal{F}_N$ was defined. To sum up, there is one isometric embedding centered at x and containing $\mathcal{F}_N$, which means it also contains p and its arcs of orbits through $\mathcal{F}_N$, and another isometric embedding centered at q. Thus in the neighbourhood of both p and q, the orbits are images of straight lines (and geodesics), any other geodesic is the image of a straight line, all geodesics are parametrized by arc-length and we are going to consider the successive return times to $\mathcal{U}_r$ for p and to $i_2(D_{r_0/8}(0))$ for q. Let $(R_k)_{k\in\mathbb{N}}$ and $(S_k)_{k\in\mathbb{N}}$ be two return time sequences for respectively p and q to the exit region of $\mathcal{F}_N$.

In this first case we decided that the orbit of q would not enter the flowbox $\mathcal{F}_N$, but it is possible that it goes very close to it. As far as the orbit of p is concerned, it can also go through $i_2\left(D_{r_0/8}(0)\right)$. In this case it is not possible to use the punctual

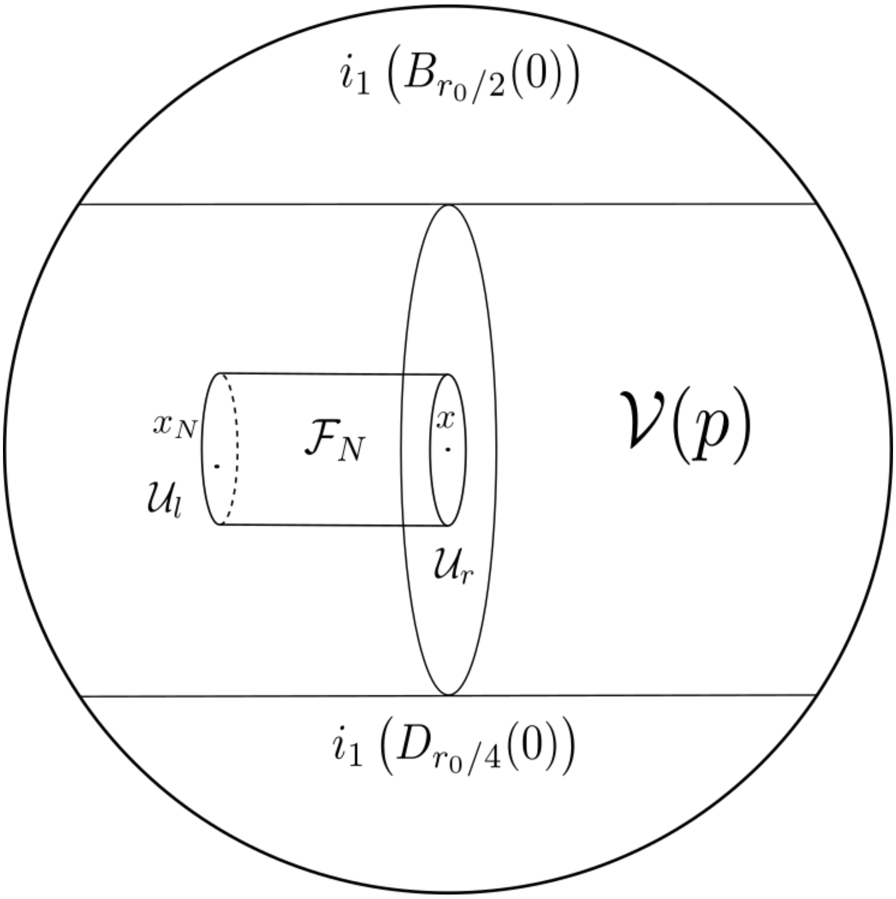

Figure 3.2: The configuration of the different neighbourhoods around x

bound from Lemma 1.4 on the linking form, and we need to define some additional neighbourhoods of p and q. Set:

$$\begin{aligned}\mathcal{V}(p) :=\ & i_1\left(B_{r_0/2}(0)\right) \cap \phi_X^{]-r_0/2,r_0/2[}\left(i_1\left(D_{r_0/4}(0)\right)\right)\\ \mathcal{V}(q) :=\ & i_2\left(B_{r_0/2}(0)\right) \cap \phi_X^{]-r_0/2,r_0/2[}\left(i_2\left(D_{r_0/4}(0)\right)\right).\end{aligned}$$

These are two open flowboxes around p and q respectively, and $\mathcal{F}_N \subset \mathcal{V}(p)$. Note that any piece of orbit that goes through $\mathcal{V}(p)$ or $\mathcal{V}(q)$ is a geodesic of length at least $\frac{r_0\sqrt{3}}{2}$ and at most r_0, and thus since it is parametrized by arc-length it stays in the neighbourhood during a time which is at least $\frac{r_0\sqrt{3}}{2}$ and at most r_0.

From now on the flow of X_N will be denoted by ϕ_N. Fix $K \gg 0$. Let α_K (respectively β_K) be a geodesic from $\phi_N^{R_K}(p)$ to p (resp. $\phi_N^{S_K}(q)$ to q). It is a straight line in the image of the embedding $i_1\left(D_{r_0/8}(0)\right)$ (resp. $i_2\left(D_{r_0/8}(0)\right)$) because of the isometry. Fix α a path $\mathcal{C}^1$-close to α_K (resp. β a path $\mathcal{C}^1$-close to β_K) so that the knots

$$k_\alpha(p, K) = \phi_N^{[0,R_k]}(p) \cup \alpha \quad \text{and} \quad k_\beta(q, K) = \phi_N^{[0,S_k]}(q) \cup \beta$$

are respectively $\mathcal{C}^1$-close to:

$$\phi_N^{[0,R_k]}(p) \cup \alpha_K \quad \text{and} \quad \phi_N^{[0,S_k]}(q) \cup \beta_K$$

and so that these two knots $k_\alpha(p, K)$ and $k_\beta(q, K)$ do not intersect.

Estimation of the linking integral. According to the definition of the asymptotic linking number, the linking number of the two knots $k_\alpha(p, K)$ and $k_\beta(q, K)$ divided by the time product $R_K S_K$ must be close to their asymptotic linking number if K is chosen large enough. We are now going to split the integral of the linking form into different pieces, then bound some of them and estimate the others. In our first case, the linking number of $k_\alpha(p, K)$ and $k_\beta(q, K)$ is equal to:

$$\begin{aligned}
\operatorname{link}\left(k_\alpha(p,K), k_\beta(q,K)\right) &= \int_{k_\alpha(p,K)\times k_\beta(q,K)} \mathcal{L} \\
&= \underbrace{\int_{\phi_N^{[0,R_K]}(p)\times\phi_N^{[0,S_K]}(q)} \mathcal{L}}_{(A)} + \underbrace{\int_{\alpha\times\beta} \mathcal{L}}_{(B)} \\
&+ \underbrace{\int_{\alpha\times\phi_N^{[0,S_K]}(q)} \mathcal{L}}_{(C)} + \underbrace{\int_{\phi_N^{[0,R_K]}(p)\times\beta} \mathcal{L}}_{(D)} .
\end{aligned}$$

We begin with the terms (B), (C) and (D). We are going to deduce the following bounds:

$$\left|\int_{\alpha\times\beta} \mathcal{L}\right| \leqslant 16 C_l \tag{3.1}$$

$$\left|\int_{\alpha\times\phi_N^{[0,S_K]}(q)} \mathcal{L}\right| \leqslant \frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}} + 8C_l\right) S_K \tag{3.2}$$

$$\left|\int_{\phi_N^{[0,R_K]}(p)\times\beta} \mathcal{L}\right| \leqslant \frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}} + 8C_l\right) R_K \tag{3.3}$$

We start by explaining how to bound (B). Here the two paths α and β are $\mathcal{C}^1$-perturbations of geodesics far away one from each other because p belongs to $\mathcal{F}_N$ and q belongs to $i_2\left(D_{r_0/8}(0)\right)$. So we can use the punctual bound on the linking form from Lemma 1.4 to bound the integral, and:

$$\left|\int_{\alpha\times\beta} \mathcal{L}\right| \leqslant \left(\frac{r_0}{2}\right)^2 C_l \operatorname{Dist}\left(\mathcal{F}_N, i_2\left(D_{r_0/8}(0)\right)\right)^{-2} .$$

Since $i_2\left(D_{r_0/8}(0)\right)$ does not intersect $i_1\left(B_{r_0/2}(0)\right)$, the distance from $\mathcal{F}_N$ to $i_2\left(D_{r_0/8}(0)\right)$ is bounded from below by $\frac{r_0}{8}$ and thus

$$\left|\int_{\alpha\times\beta} \mathcal{L}\right| \leqslant 16 C_l \, .$$

The terms (C) and (D) are symmetric and they are bounded for the same reason. Thus we will just explain how to bound (D). The path β lies in $i_2(D_{r_0/8}(0))$ and the arc of orbit $\phi_N^{[0,R_K]}(p)$ lies somewhere in $\mathbb{S}^3$, possibly intersecting $\mathcal{V}(q)$ a finite number of times m during certain periods of time t_i, with $\sum_{i=1}^m t_i < R_K$. Consider a hypothetical arc c of $\phi_N^{[0,R_K]}(p)$ in $\mathcal{V}(q)$. Because of the isometric embedding, c is a geodesic, while β is $\mathcal{C}^1$-close to the short geodesic β_K lying in $i_2(D_{r_0/8}(0))$ and joining $\phi_N^{S_K}(q)$ to q. Thus by Lemma 1.5 the linking of the two short arcs is uniformly bounded by the constant $C(g)$. Thus

$$\Big| \int_{\left(\phi_N^{[0,R_K]}(p)\cap\mathcal{V}(p)\right)\times\beta} \mathcal{L} \Big| \leqslant m \times C(g)$$

and we know that $mr_0\frac{\sqrt{3}}{2} \leqslant \sum_{i=1}^m t_i < R_K$, so

$$\Big| \int_{\left(\phi_N^{[0,R_K]}(p)\cap\mathcal{V}(p)\right)\times\beta} \mathcal{L} \Big| \leqslant R_K \frac{2}{r_0\sqrt{3}} \times C(g)\,.$$

Now consider the parts of $\phi_N^{[0,R_K]}(p)$ that are not in $\mathcal{V}(q)$. By the punctual bound of Lemma 1.4 the linking of these pieces of orbit with β is bounded:

$$\Big| \int_{\left(\phi_N^{[0,R_K]}(p)\cap\mathcal{V}(p)^c\right)\times\beta} \mathcal{L} \Big| \leqslant C_l \left(\frac{1}{r_0/8}\right)^2 \times \frac{r_0}{4} \times \left(R_K - \sum_{i=1}^m t_i\right)\,,$$

because β is of length (thus time) at most $r_0/4$ since q and $\phi_N^{S_K}(q)$ both belong to $i_2(D_{r_0/8}(0))$ and the distance between β and $\mathcal{V}(q)^c$ is at least $r_0/8$. Rearranging the factors and bounding $R_K - \sum_{i=1}^m t_i$ with R_K, we have:

$$\Big| \int_{\left(\phi_N^{[0,R_K]}(p)\cap\mathcal{V}(p)^c\right)\times\beta} \mathcal{L} \Big| \leqslant C_l \frac{16}{r_0} R_K\,.$$

Finally we conclude :

$$\Big| \int_{\phi_N^{[0,R_K]}(p)\times\beta} \mathcal{L} \Big| \leqslant \frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}} + 8C_l\right) R_K\,.$$

The same argument yields the precedent bound (3) for (C). Now we have to consider the term (A) in the above equation. We are going to cut it in different parts, depending whether the orbit of p passes inside or outside $\mathcal{F}_N$, as follows:

$$\int_{\phi_N^{[0,R_K]}(p)\times\phi_N^{[0,S_K]}(q)} \mathcal{L} = \sum_{i=0}^{K-1} \int_{\phi_N^{[R_i,R_i+r_{i+1}]}(p)\times\phi_N^{[0,S_K]}(q)} \mathcal{L} \tag{3.4}$$

$$+ \sum_{i=1}^{K} \int_{\phi_N^{[R_i+r_{i+1},R_{i+1}]}(p)\times\phi_N^{[0,S_K]}(q)} \mathcal{L} \tag{3.5}$$

Here the r_{i+1} are the times between $\phi^{R_i}(p) \in \mathcal{U}_r$ and the next return to the flowbox, which belongs to $\mathcal{U}_l$.What happens here - and it will be the same in the second case - is that the first term in the right-hand side of the equation will be positive because of the assumption that X is right-handed. At the same time, the second term will have a finite contribution to the linking that can be made small enough not to interfere with the positivity of the total integral.

For the same reasons as the precedent term (C) - i.e. the punctual bound on the linking form and Lemma 1.5 when necessary - we have the following bound for all $1 \leqslant i \leqslant K$:

$$\Big| \int_{\phi_N^{[R_i+r_{i+1},R_{i+1}]}(p)\times\phi_N^{[0,S_K]}(q)} \mathcal{L} \Big| \leqslant C_l \left(\frac{1}{r_0/8}\right)^2 \times \frac{r_0}{4} \times S_K + \frac{2}{r_0\sqrt{3}} C(g) \times S_K \qquad (3.6)$$

Thus we obtain the following bound on the sum:

$$\Big| \sum_{i=1}^{K} \int_{\phi_N^{[R_i+r_{i+1},R_{i+1}]}(p)\times\phi_N^{[0,S_K]}(q)} \mathcal{L} \Big| \leqslant \frac{2}{r_0} \left(\frac{C(g)}{\sqrt{3}} + 8C_l\right) K \times S_K \,.$$

Before explaining how, we need to bound from below the first term (3.4). Now we use the fact that X is right-handed and that all the r_i are larger than the time constant T_r fixed in Lemma 3.1, by construction. Since for all $0 \leqslant i \leqslant K-1$, we have:

$$\frac{1}{r_{i+1}S_K} \int_{\phi_N^{[R_i,R_i+r_{i+1}]}(p)\times\phi_N^{[0,S_K]}(q)} \mathcal{L} \geqslant C_r \,,$$

then

$$\int_{\phi_N^{[R_i,R_i+r_{i+1}]}(p)\times\phi_N^{[0,S_K]}(q)} \mathcal{L} \geqslant C_r \times r_{i+1} S_K$$

and finally since all terms are positive and $r_{i+1} \geqslant R_{i+1} - R_i - \frac{r_0}{4}$ because of the choice of $\epsilon < r_0/4$:

$$\sum_{i=0}^{K-1} \int_{\phi_N^{[R_i,R_i+r_{i+1}]}(p)\times\phi_N^{[0,S_K]}(q)} \mathcal{L} \geqslant C_r \times (R_K - K\frac{r_0}{4}) S_K \,.$$

Putting everything together we have:

$$\int_{\phi_N^{[0,R_K]}(p)\times\phi_N^{[0,S_K]}(q)} \mathcal{L} \geqslant C_r (R_K - K\frac{r_0}{4}) S_K - \frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}} + 8C_l\right) K S_K \,.$$

Dividing by the product of times $R_K S_K$,

$$\frac{1}{R_K S_K} \int_{\phi_N^{[0,R_K]}(p)\times\phi_N^{[0,S_K]}(q)} \mathcal{L} \geqslant C_r \left(1 - \frac{K r_0}{4R_K}\right) - \frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}} + 8C_l\right) \frac{K}{R_K} \,.$$

Now if we consider the whole linking number, we have:

$$\begin{aligned} \frac{1}{R_K S_K} \mathrm{link}\,(k_\alpha(p,K), k_\beta(q,K)) \geqslant{}& C_r \left(1 - \frac{K r_0}{4R_K}\right) - \frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}} + 8C_l\right) \frac{K}{R_K} \\ &- \frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}} + 8C_l\right)\left(\frac{1}{R_K} + \frac{1}{S_K}\right) - \frac{16 C_l}{R_K S_K} \,. \end{aligned}$$

From the choice of the return time R_K, we know that $R_K \geqslant KT_0$, so $-\frac{K}{R_K} \geqslant -\frac{1}{T_0}$ and *a fortiori* $R_K > T_0$ - respectively $S_K > T_0$ - implies $-\frac{1}{R_K} > -\frac{1}{T_0}$ - resp. $-\frac{1}{S_K} > -\frac{1}{T_0}$. Finally we obtain:

$$\frac{1}{R_K S_K}\text{link}\left(k_\alpha(p,K),k_\beta(q,K)\right) \geqslant \underbrace{C_r\left(1-\frac{r_0}{4T_0}\right)-\frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}}+8C_l\right)\frac{3}{T_0}-\frac{16C_l}{T_0^2}}_{C_N}\,.$$

The right-hand side of the equation is positive if and only if

$$T_0^2 - T_0\left(\frac{r_0}{4}+\frac{3G}{2C_r}\right)-16\frac{C_l}{C_r} > 0\,,$$

where we set $G := \frac{4}{r_0}\left(\frac{C(g)}{\sqrt{3}}+8C_l\right)$. The discriminant of this polynomial is positive:

$$\Delta := \left(\frac{r_0}{4}+\frac{3G}{2C_r}\right)^2 + 64\frac{C_l}{C_r}\,,$$

thus for T_0 bigger than

$$T_+ := \frac{r_0}{8}+\frac{3G}{4C_r}+\frac{1}{2}\sqrt{\left(\frac{r_0}{4}+\frac{3G}{2C_r}\right)^2+64\frac{C_l}{C_r}}$$

the above constant C_N is positive. This condition holds because of our choice for T_0 in the beginning, see Section 3.1.3. Gathering things together we have:

$$\lim_{K\to\infty}\frac{1}{R_K S_K}\text{Link}\left(k_\alpha(p,K),k_\beta(q,K)\right) \geqslant C_N > 0\,,$$

and this concludes the proof of the first case.

Proof of the second case : $\mathcal{O}(q)\cap\mathcal{F}_N \neq \emptyset$

We start with a similar setting as for the first case : now p and q both belong to $\mathcal{U}_r$, and we have the neighbourhood $\mathcal{V}=\mathcal{V}(p)$ of $\mathcal{F}_N$. As before we start with the decomposition:

$$\begin{aligned}\text{link}\left(k_\alpha(p,K),k_\beta(q,K)\right) &= \int_{k_\alpha(p,K)\times k_\beta(q,K)}\mathcal{L}\\ &= \underbrace{\int_{\phi_N^{[0,R_K]}(p)\times\phi_N^{[0,S_K]}(q)}\mathcal{L}}_{(A')}+\underbrace{\int_{\alpha\times\beta}\mathcal{L}}_{(B')}\\ &\quad+\underbrace{\int_{\alpha\times\phi_N^{[0,S_K]}(q)}\mathcal{L}}_{(C')}+\underbrace{\int_{\phi_N^{[0,R_K]}(p)\times\beta}\mathcal{L}}_{(D')}\,.\end{aligned}$$

This time, the term (B') is bounded by Lemma 1.5, and (C') and (D') are bounded for the same reasons as (D) of first case. We have the following bounds:

$$\Big|\int_{\alpha\times\beta}\mathcal{L}\Big| \leqslant C(g)$$
$$\Big|\int_{\alpha\times\phi_N^{[0,S_K]}(q)}\mathcal{L}\Big| \leqslant \frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}}+8C_l\right)S_K$$
$$\Big|\int_{\phi_N^{[0,R_K]}(p)\times\beta}\mathcal{L}\Big| \leqslant \frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}}+8C_l\right)R_K\,.$$

We need to consider the term (A'). With the precedent notations for R_i and r_i extended to the return times S_j and s_j for the orbit of q, the term (A') decomposes in four parts:

$$\int_{\phi_N^{[0,R_K]}(p)\times\phi_N^{[0,S_K]}(q)}\mathcal{L} = \sum_{i=0}^{K-1}\sum_{j=0}^{K-1}\int_{\phi_N^{[R_i,R_i+r_{i+1}]}(p)\times\phi_N^{[S_j,S_j+s_{j+1}]}(q)}\mathcal{L} \tag{3.7}$$
$$+\sum_{i=1}^{K}\sum_{j=0}^{K-1}\int_{\phi_N^{[R_i+r_{i+1},R_{i+1}]}(p)\times\phi_N^{[S_j,S_j+s_{j+1}]}(q)}\mathcal{L} \tag{3.8}$$
$$+\sum_{j=1}^{K}\sum_{i=0}^{K-1}\int_{\phi_N^{[R_i,R_i+r_{i+1}]}(p)\times\phi_N^{[S_j+s_{j+1},S_{j+1}]}(q)}\mathcal{L} \tag{3.9}$$
$$+\sum_{i=1}^{K}\sum_{j=1}^{K}\int_{\phi_N^{[R_i+r_{i+1},R_{i+1}]}(p)\times\phi_N^{[S_j+s_{j+1},S_{j+1}]}(q)}\mathcal{L} \tag{3.10}$$

The absolute value of Equation (3.10) is bounded by $K^2C(g)$ thanks to Lemma 1.5, since the short arcs of orbit in $\mathcal{F}_N$ are geodesics. The terms (3.8) and (3.9) are again symmetric, and for (3.8) we have from our previous efforts (see Equation 3.6):

$$\Big|\int_{\phi_N^{[R_i+r_{i+1},R_{i+1}]}(p)\times\phi_N^{[S_j,S_j+s_{j+1}]}(q)}\mathcal{L}\Big| \leqslant C_l\left(\frac{1}{r_0/8}\right)^2\frac{r_0}{4}s_{j+1}+\frac{2}{r_0\sqrt{3}}C(g)s_{j+1}\,.$$

Thus since $\sum_{j=0}^{K-1}s_{j+1}<S_K$ we obtain the following bound on the sum:

$$\Big|\sum_{i=1}^{K}\sum_{j=0}^{K-1}\int_{\phi_N^{[R_i+r_{i+1},R_{i+1}]}(p)\times\phi_N^{[S_j,S_j+s_{j+1}]}(q)}\mathcal{L}\Big| \leqslant \frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}}+8C_l\right)KS_K\,,$$

and the symmetric one for (3.9):

$$\Big|\sum_{j=1}^{K}\sum_{i=0}^{K-1}\int_{\phi_N^{[R_i,R_i+r_{i+1}]}(p)\times\phi_N^{[S_j+s_{j+1},S_{j+1}]}(q)}\mathcal{L}\Big| \leqslant \frac{2}{r_0}\left(\frac{C(g)}{\sqrt{3}}+8C_l\right)KR_K\,,$$

As far as (3.7) is concerned, we have since that X is (T_r,C_r)-right-handed and thus (T_0,C_r)-right-handed:

$$\int_{\phi_N^{[R_i,R_i+r_{i+1}]}(p)\times\phi_N^{[S_j,S_j+s_{j+1}]}(q)}\mathcal{L} \geqslant r_{i+1}s_{j+1}\times C_r\,,$$

so

$$\sum_{i=0}^{K-1}\sum_{j=0}^{K-1}\int_{\phi_N^{[R_i+r_{i+1},R_{i+1}]}(p)\times\phi_N^{[S_j,S_j+s_{j+1}]}(q)}\mathcal{L}\geqslant C_r\left(R_K-K\frac{r_0}{4}\right)\left(S_K-K\frac{r_0}{4}\right).$$

Now just like in the first case, we use the bounds that we have found to bound from below the term (A'):

$$(A')\geqslant C_r\left(R_K-K\frac{r_0}{4}\right)\left(S_K-K\frac{r_0}{4}\right)-\frac{G}{2}K(R_K+S_K)-K^2\times C(g)$$

where we set G to be the constant, already presented in Section 3.1.3:

$$G:=\frac{4}{r_0}\left(\frac{C(g)}{\sqrt{3}}+8C_l\right).$$

So the whole linking number is bounded by:

$$\begin{aligned}\operatorname{link}\left(k_\alpha(p,K),k_\beta(q,K)\right)\geqslant C_r&\left(R_K-K\frac{r_0}{4}\right)\left(S_K-K\frac{r_0}{4}\right)\\&-\frac{G}{2}K(R_K+S_K)-K^2C(g)\\&-\frac{G}{2}(R_K+S_K)-C(g)\,.\end{aligned}$$

Dividing by the time product R_KS_K and then using the fact $-\frac{K}{R_K}\geqslant-\frac{1}{T_0}$ and $-\frac{K}{S_K}\geqslant-\frac{1}{T_0}$, we have

$$\begin{aligned}\frac{1}{R_KS_K}\operatorname{link}\left(k_\alpha(p,K),k_\beta(q,K)\right)&\geqslant C_r\left(1-\frac{Kr_0}{4R_K}\right)\left(1-\frac{Kr_0}{4S_K}\right)\\&\quad-\frac{G}{2}\left(\frac{K+1}{R_K}+\frac{K+1}{S_K}\right)-\frac{K^2+1}{R_KS_K}C(g)\\&\geqslant\underbrace{C_r\left(1-\frac{r_0}{4T_0}\right)^2-\frac{2G}{T_0}-\frac{2C(g)}{T_0^2}}_{C'_N}\end{aligned}$$

The asymptotic linking number of p and q will be positive if

$$C_r\left(1-\frac{r_0}{4T_0}\right)^2-\frac{2G}{T_0}-\frac{2C(g)}{(T_0)^2}>0\,,$$

that is to say

$$\left(T_0-\frac{r_0}{4}\right)^2-2T_0\frac{G}{C_r}-\frac{2C(g)}{C_r}>0\,.$$

We find the polynomial in T_0:

$$T_0^2-T_0\left(\frac{r_0}{2}+\frac{2G}{C_r}\right)+\left(\frac{r_0}{4}\right)^2-\frac{2C(g)}{C_r}>0$$

whose discriminant is positive:

$$\begin{aligned}\Delta &= \left(\frac{r_0}{2} + \frac{2G}{C_r}\right)^2 - 4\left(\left(\frac{r_0}{4}\right)^2 - \frac{2C(g)}{C_r}\right) \\ &= 8\frac{C(g)}{C_r} + \frac{2G}{C_r}\left(r_0 + \frac{2G}{C_r}\right) .\end{aligned}$$

This is why for T_0 greater than

$$T'_+ = \frac{1}{2}\left(\frac{r_0}{2} + \frac{2G}{C_r} + \sqrt{8\frac{C(g)}{C_r} + \frac{2G}{C_r}\left(r_0 + \frac{2G}{C_r}\right)}\right) ,$$

the linking number will be positive. Gathering everything we finally have:

$$\lim_{K\to\infty} \frac{1}{R_K S_K} \operatorname{link}\left(k_\alpha(p, K), k_\beta(q, K)\right) \geqslant C'_N > 0 ,$$

and the orbits of p and q have a positive asymptotic linking number in the second case.

Proof of the third case

Suppose that changing X into X_N added new periodic orbits for the flow of $\widetilde{X}$. Actually there is at least the orbit of x to consider, the one that we closed with the perturbation. There might be others, and the following proof applies to any of these newly created periodic orbits.

Lemma 3.2. *Let γ be a periodic orbit for the flow ϕ_N of X_N and passing through $\mathcal{F}_N$. Then γ has a (strictly) positive self-linking number.*

Proof. We are going to use a geometric interpretation of Hryniewicz's formula [FH23] (see Chapter 1) for the self-linking number. First we recall the definition of the self-linking number in our specific setting. Consider a tubular neighbourhood $\mathcal{N}$ of γ with complex coordinate $z = (r, \theta)$ in the transverse disks and the time t as third coordinate (the longitude), so that $\phi_N^t(\gamma(0)) = (t, 0)$ in these coordinates. Set p to be an intersection of γ with $\mathcal{U}_r$ and $(R_i)_{i=1,\dots,K}$ the successive return times of the orbit of p to $\mathcal{U}_r$, with $R_K = T$ being the period of γ. Note that the orbit of p may cross the flowbox $\mathcal{F}_N$ several times before closing, although this does not happen for the specific orbit of x that we have closed with the perturbation X_N. Recall that for a choice $\theta_0 \in \mathbb{R}$, we have the continuous function given by

$$D\phi_N^t(0,0) \cdot (0, e^{i\theta_0}) \in \mathbb{R}(1,0) + \mathbb{R}_+(0, e^{i\theta_N(t)})$$

and $\theta_N(0) = \theta_0$. If $\lambda \in \mathrm{H}_1(\mathcal{N} \setminus \gamma, \mathbb{R})$ is a one-form of the form $\lambda = pdz + qdt$, which also represents a Seifert surface S for γ, the self-linking number of γ is given by the formula

$$\rho^\lambda(\gamma) = \frac{T}{2\pi}\left(p + q \lim_{t\to\infty} \frac{\theta_N(t)}{t}\right) ,$$

where we set T to be the period of γ. Now let $\mathcal{S}$ be a Seifert surface for γ, and choose a trivialization of the tubular neighbourhood $\mathcal{N}$ which has $\mathcal{S}$ as the zero section. Then $\lambda = d\theta$ is a cohomology class dual to $\mathcal{S}$. We have to consider the two following quantities:

- the transverse rotation number of γ, $\rho^\lambda(\gamma) = \frac{T}{2\pi} \lim_{t\to\infty} \frac{\theta_N(t)}{t}$;
- the number $n_t^{\mathcal{S}}(p)$ of oriented intersections of the arc of orbit $\mathcal{O}\left(x, \phi_N^t(p)\right)$ with $\mathcal{S}$, that is to say the number of times that this arc of orbit crosses the zero section.

In the same time, consider a sequence of points $(p_n)_{n\in\mathbb{N}} \subset \mathcal{U}_r$ converging to p. We call S_n the K-th return time of the point p_n to $\mathcal{F}_N$. From the study of case 2, we have since $T > T_0$ and $S_n > T_0$:

$$\frac{1}{TS_n}\text{link}\left(\phi_N^{[0,T]}(p), \phi_N^{[0,S_n]}(p_n)\right) > C'_N$$

and according to the definition of the linking number for knots, this linking number is exactly $n_{S_n}^{\mathcal{S}}(p_n)$, thus

$$n_{S_n}^{\mathcal{S}}(p_n) > C'_N T S_n \,.$$

In the other hand, $\theta_N(S_n) \geqslant 2\pi(n_{S_n}^{\mathcal{S}}(p_n) - 1)$ since the arc of orbit $\mathcal{O}\left(x, \phi_N^t(p)\right)$ crosses the zero section at least $n_{S_n}^{\mathcal{S}}(p_n)$ times and between two consecutive intersections the function θ_N has to increase by 2π since intersections are oriented. For this reason we have

$$\theta_N(T) \geqslant 2\pi \lim_{n\to\infty} n_{S_n}^{\mathcal{S}}(p_n) - 1 > 2\pi \lim_{n\to\infty} TS_nC'_N - 1 \geqslant 2\pi T^2 C'_N - 1\,.$$

The same argument starting with multiples kT of the period shows that $\theta_N(kT) \geqslant 2\pi(kT)^2C'_N - 1$. By iteration of this process, we finally obtain a monotonically increasing sequence $(kT)_{k\in\mathbb{N}^*}$ of time values for which the quantity $\frac{T}{2\pi}\frac{\theta_N(kT)}{kT}$ is greater than $kT^2C'_N - 1$. As a consequence the limit is positive and $Slk(\gamma) > 0$.

□

So this concludes the third case and the μ-preserving $\mathcal{C}^1$-perturbation X_N of X is still right-handed.

3.1.5 Genus of the prescribed orbit

Since X_N is right-handed, a Ghys' Theorem 1.10 from [Ghy09] states that $k(x, T_N)$ bounds a surface S_N which is a Birkhoff section for X_N. But by Theorem 4.1.10 in [Kaw96], any Seifert surface S_N for this orbit and transverse to the flow has minimal genus, so S_N has minimal genus. In our case, Dehornoy and Rechtman [DR22] proved that

$$g(S_N) = \frac{1 + Slk^{\zeta_X}(k(x, T_N))}{2}\,.$$

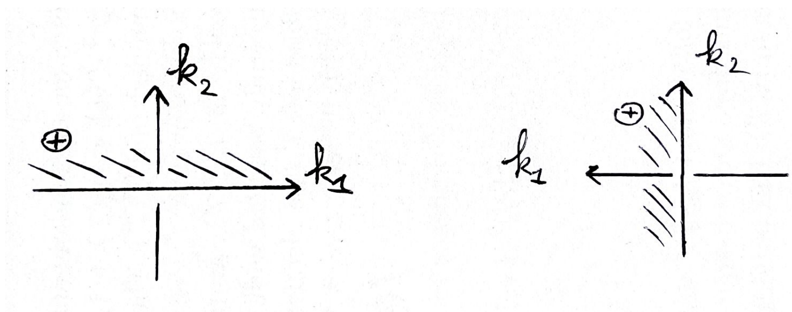

Figure 3.3: The linking number is symmetric: on the left, k_2 intersects positively a Seifert surface for k_1. Looking at this crossing from the back of the picture, k_1 intersects positively a Seifert surface for k_2.

Using the fact that $\lim_{N\to\infty} \frac{Slk^{\zeta_X}(k(x,T_N))}{{T_N}^2} = \mathrm{Hel}(X,\mu)$ where ζ_X is a vector field everywhere transverse to X along the closed orbit $k(x,T_N)$ [Arn73] we can conclude:

$$\lim_{N\to\infty} \frac{1}{{T_N}^2} g(k(x,T_N)) = \lim_{N\to\infty} \frac{1+Slk^{\xi}(k(x,T_N))}{2{T_N}^2} = \frac{1}{2}\mathrm{Hel}(X,\mu)\,.$$

3.2 Genus of a two components link

3.2.1 Notation and definitions

Let k_1 and k_2 be two oriented knots in $\mathbb{S}^3$ that are disjoint and may be linked one with another. For $i = 1, 2$, we denote by $g(k_i)$ the genus and by $\chi(k_i)$ the Euler characteristic of a co-oriented Seifert surface for k_i that minimizes the genus.

Let $k_1 \cup k_2$ be the link formed by the two knots. We need to define a notion of crossing number which suits the cases we are interested in. First we have to say that since the finality of this work is to deal with flows of vector fields, we will not be interested in isotopies of a link. Thus we have chosen a definition of the crossing number which is coherent with the asymptotic crossing number as defined by Freedman and He in [FH91]. We call crossing number of this link the least number (over all projections) of crossings between k_1 and k_2, divided by two.

In this section we assume that all crossings (between k_1 and k_2) are positive, meaning that in each double point in a projection, the upper oriented strand has to rotate anticlockwise to align with the lower oriented strand. Alternatively, there exists S_1 a co-oriented Seifert surface for k_1 that minimizes the genus such that all the intersection points between S_1 and k_2 are positive. Under this assumption $\mathrm{link}(k_1,k_2) = |\mathrm{link}(k_1,k_2)| = \mathrm{cross}(k_1,k_2)$. Note that this is a symmetric condition. Indeed, consider a positive crossing, for example an above strand from k_1 crossing horizontally to the right and an under strand from k_2 crossing vertically in the upper direction. Then if we look from the back of the picture (see Figure 3.3), where the vertical strand from k_2 is above the other, and consider a Seifert surface S_2 for k_2, it will be co-oriented by the boundary and thus k_1 will intersect S_2 positively. Since this happens for each positive crossing, we have $\mathrm{link}(k_1,k_2) = \mathrm{link}(k_2,k_1)$ with our previous definition.

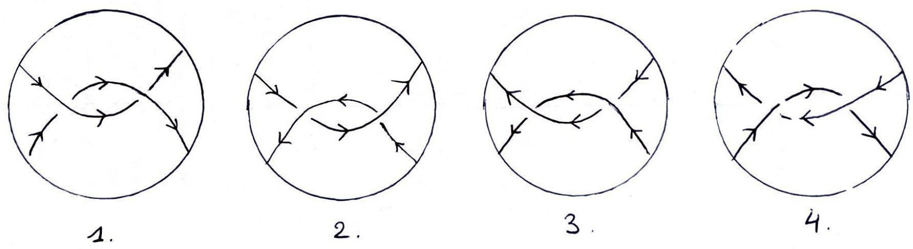

Figure 3.4: Four positions of knots that can happen; positions 1 and 3 are mirror images, as well as 2 and 4.

Proposition 3.3. *Let k_1 and k_2 form a link with positive crossings and $n = link(k_1, k_2)$. Then modulo isotopy, there is a position of the link $k_1 \cup k_2$ such that there exist disjoint embedded 2-spheres F, F_1, $F_2, \ldots, F_n$ such that*

- *$F \cap k_1 = \emptyset$ and the intersection of $F \cap k_2$ is transverse.*
- *If $n > 0$, $F \cap k_2$ has exactly $n = link(k_1, k_2)$ pairs of points, that is $2n$ points. If $n = 0$, $F \cap k_2 = \emptyset$ and F separates k_1 and k_2, that is to say that k_1 and k_2 belong to distinct components of $\mathbb{S}^3 \setminus F$.*

  We define the interior of F as the component of $\mathbb{S}^3 \setminus F$ that contains k_1. Define also the interior of F_i as the component of $\mathbb{S}^3 \setminus F_i$ that is contained in the interior of F.
- *For $i = 1, 2, \ldots, n$ the intersection of F_i with the two knots is transverse and F_i is contained in the interior of F. Moreover, $F_i \cap k_j$ has exactly 2 points, for $j = 1, 2$ and in the interior of F_i the knots are in positions presented in Figure 3.4.*

Proof. Choose a projection where the link has the least crossing number. We can imagine that the two components k_1 and k_2 of the link are lying on a plane or on a table. Now lift k_1 in a plane above k_2 while k_2 stays on the table. In this new position we have a number $n = \text{link}(k_1, k_2)$ of pairs of linking strands in the space between the two knots. We choose a disc between the two planes where (most of) k_1 and k_2 are located, and close it above k_1 to get a 2-sphere F. Then F satisfies that $k_1 \cap F = \emptyset$, and that $k_2 \cap F = n$. In particular, if k_1 and k_2 are unlinked, by definition we can find F so that k_1 and k_2 are in distinct components of $\mathbb{S}^3 \setminus F$.

In this new position, we can put little spheres F_i around each crossing. These crossings are isolated enough so that the F_i do not intersect each other and do not intersect F, and because of the positive condition for the crossings, only the above configurations can occur in the spheres F_i. □

3.2.2 The Fried surface for $k_1 \cup k_2$.

In this section, we describe the construction of a Seifert surface S_F for the link $k_1 \cup k_2$ from two genus minimizing Seifert surfaces S_1 and S_2 for k_1 and k_2 respectively. We

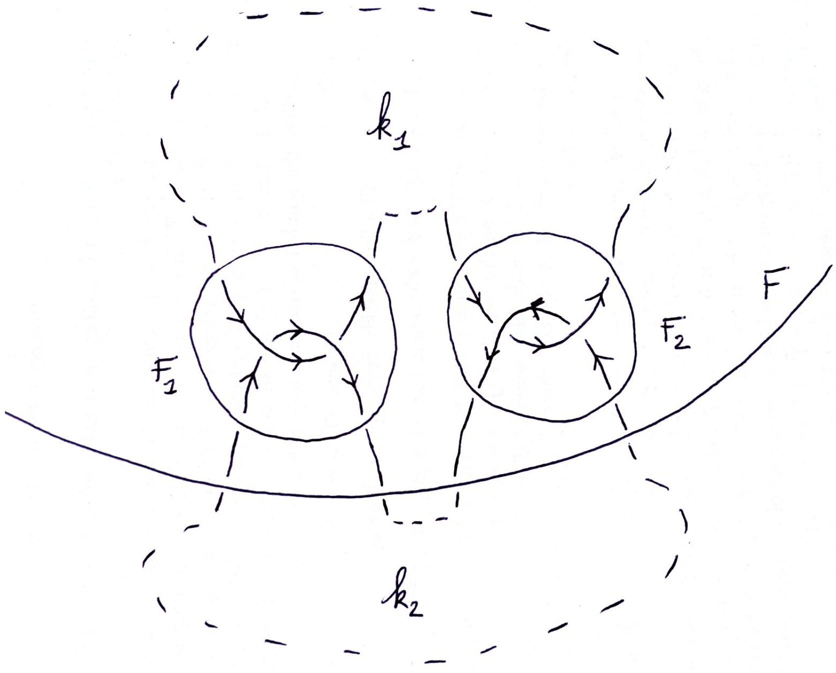

Figure 3.5: Lifting k_1 while k_2 stays on the lower plane.

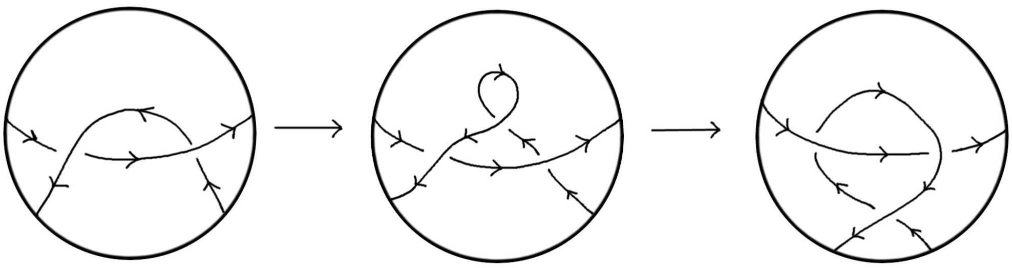

Figure 3.6: Transforming case 2 into case 1 with two Reidemeister moves.

will show that S_F is oriented, and thus its genus is an upper bound for the genus of the link. In particular, if $\text{link}(k_1, k_2) = 0$ then S_F is just the disjoint union of S_1 and S_2 and hence the genus is just the sum, and $\chi(S_F) = \chi(S_1) + \chi(S_2)$.

First let us consider the link $k_1 \cup k_2$ in the position described in Proposition 3.3. In each of the spheres F_i, we have one of the four diagrams described in Figure 3.4.

We keep the cases 1 and 3 and we slightly modify the cases 2 and 4, according to the following moves, as illustrated in Figure 3.6:

- From case 2, first make a loop with the upper part of k_2, i.e. a Reidemeister move of type 1;
- Then move the bottom strands to the opposite side : the left one goes to the right and conversely - it is a Reidemeister move of type 3.

Note that this adds a crossing to k_2 in the exterior of F_i for each ball where a modification occurs, but this will not be a problem. In the same manner, case 4

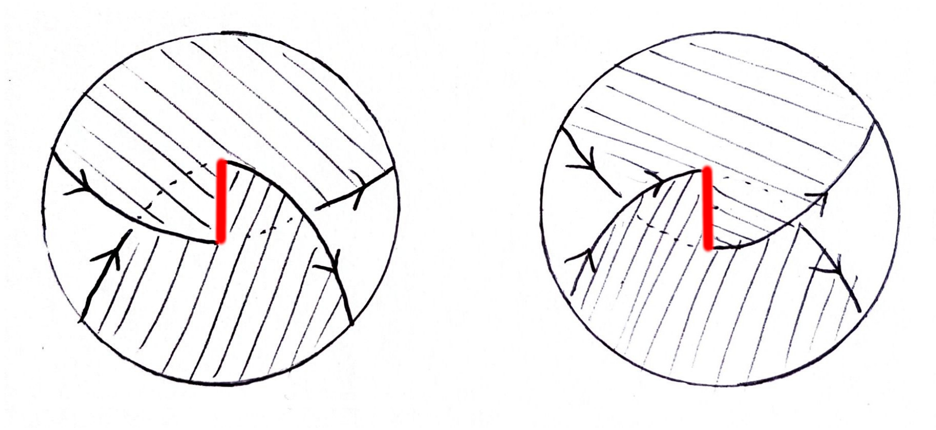

Figure 3.7: Intersections of the surfaces S_1 and S_2 in the cases 1 and 2 of Figure 3.4

turns into case 3. Now let us consider two genus minimizing Seifert surfaces S_1 and S_2 for k_1 and k_2 (separately). In the balls F_i, we can choose these surfaces so that they look as in Figure 3.7. Here is a way to do this. Consider the knots in the same configuration as in the proof of Proposition 3.3, with k_2 lying on a plane and k_1 lifted above. Now we unlink temporarily k_1 and k_2, keeping track of the small parts of strands $(c_i^1)_{i\in\{1,\dots n\}}$ and $(c_i^2)_{i\in\{1,\dots n\}}$ that where inside the F_i. Then consider two Seifert surfaces S_1 and S_2 for k_1 and k_2. Since k_1 and k_2 are now unlinked, we can suppose that F separates S_1 and S_2 up to slightly moving the Seifert surfaces. Now pull and extend the pieces of strands $(c_i^1)_{i\in\{1,\dots n\}}$ and $(c_i^2)_{i\in\{1,\dots n\}}$, avoiding any self-intersection of S_1 and S_2, in order to link k_1 and k_2 again as they were in the beginning. After these movements, $F\cap S_1=\emptyset$ and $F\cap S_2$ consists of n segments. In particular if B_i denotes the closure of the connected component of $\mathbb{S}^3\setminus F_i$ containing the i-th linking between k_1 and k_2, there is an unique intersection segment (marked in red) in each ball B_i.

We are going to remove the self-intersections of this surface using an idea of Fried [Fri83]. To do this, let us denote $S_F^i=(S_1\cup S_2)\cap B_i$ and consider a vector field R on B_i positively transverse to the interior of S_F^i and tangent to the boundary ∂S_F^i, that we note abusively $k_1\cup k_2$. On each boundary k_i, S_F^i extends to the unit normal bundle Σ_{k_i} into a collection of immersed curves transverse to the extension of R and in general position. Now we blow up B_i along ∂S_F^i to get the compact manifold $B_{\partial S}$ bounded by two cylinders. By the precedent remark, S_F^i extends to an immersed compact surface that we still denote S_F^i on $B_{\partial S}$. Moreover this extension is transverse to the extension of R along $\partial B_{\partial S}$ and every self-intersection is transverse, even in $\partial B_{\partial S}$.

By choice of R, S_F^i has a transversal given by R and thus we can resolve its self-intersections consistently to get an embedded surface $\tilde{S}_F^i$ positively transverse to R in $B_{\partial S}$, as pictured in Figure 3.8. Each boundary piece of torus of $B_{\partial S}$ has well-defined meridians which corresponds to the unit bundle of points in ∂S_F^i. Thus we can isotope the resolved boundary in $\partial B_{\partial S}$ with an isotopy transverse to R, so that it becomes transverse to the foliation by meridians as pictured in Figure 3.9. This isotopy can be extended to an isotopy of S_F^i transverse to R and supported in a small neighbourhood of $\partial B_{\partial S}$. After blow-down, the isotoped surface yields an

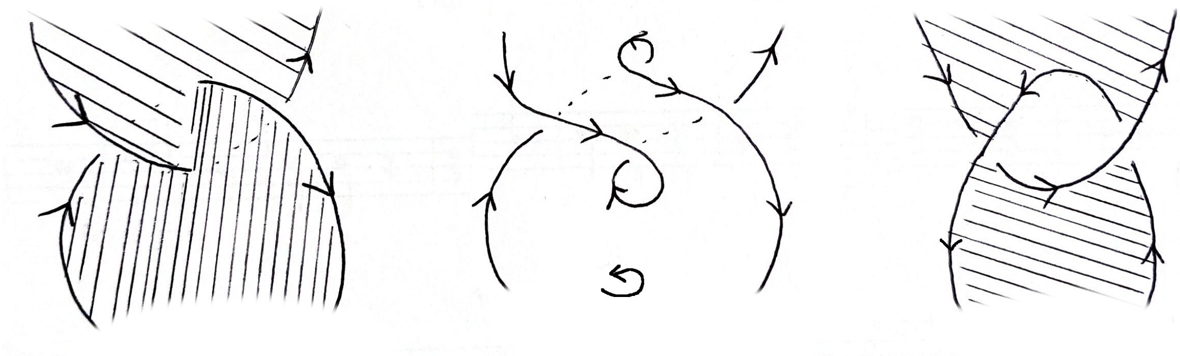

Figure 3.8: Desingularization of the intersection after Fried

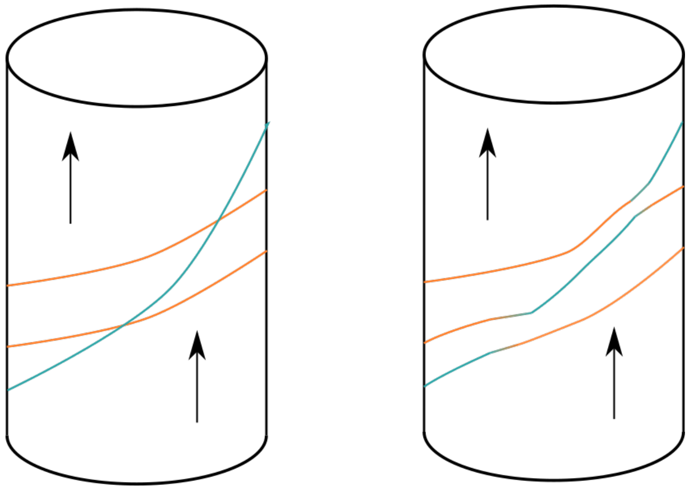

Figure 3.9: Resolution of the double points of ∂S_F^i transversally to R.

immersed surface $\tilde{S}_F^i$ in B_i. This surface has no self-intersection and has $(k_1 \cup k_2) \cap B_i$ as a boundary inside B_i.

This operation acts as if we added two half-twisted strips between S_1 and S_2 in each sphere F_i. To see this, look at the picture of the desingularized disks in Figure 3.8, and make half a turn with the lower disk.

Now we can perform this process in each of the cases 1 and 3 as they appear in the spheres F_i. The orientations are locally consistent and, since S_1 and S_2 are oriented surfaces, we obtain a globally oriented surface that we call the Fried surface, denoted by S_F. This surface is a Seifert surface for the link $k_1 \cup k_2$. As we have seen while explaining the Fried desingularization, S_F is obtained by adding locally two twisted strips per crossing between S_1 and S_2.

Proposition 3.4. *The Euler characteristic of S_F is given by the following formula*

$$\chi(S_F) = \chi(S_1) + \chi(S_2) - 2link(k_1, k_2).$$

Proof. This follows from the precedent construction, since the number of crossing is equal to the number of linking by hypothesis. □

Corollary 3.5. *If $link(k_1, k_2) \geq 1$, then $g(S_F) = g(k_1) + g(k_2) + link(k_1, k_2) - 1$. If $link(k_1, k_2) = 0$ then $g(S_F) = g(k_1) + g(k_2)$.*

Proof. If $\text{link}(k_1, k_2) = 0$, the Fried surface S_F has two connected components which are Seifert surfaces for k_1 and k_2, so the result follows.
If $\text{link}(k_1, k_2) = 1$, the linking implies that S_F is connected. Since the Euler characteristic of an oriented surface equals two minus twice the genus minus the number of boundary components we have that

$$2 - 2g(S_F) - 2 = 2 - 2g(k_1) - 1 + 2 - 2g(k_2) - 1 - 2\text{link}(k_1, k_2),$$

and the result follows immediately. □

3.2.3 How good is this formula for $g(k_1 \cup k_2)$?

The idea is to study how good the formulas in Proposition 3.4 and Corollary 3.5 are for the Euler characteristic of a genus minimizing Seifert surface S for $k_1 \cup k_2$. We show that it is accurate if $\text{link}(k_1, k_2) \leqslant 1$, but not for $\text{link}(k_1, k_2) \geqslant 2$.

Case link$(k_1, k_2) = 0$.
Since the two knots are unlinked, by the assumption of positiveness of the crossings we can apply Proposition 3.3. We consider hence the 2-sphere F and the surface S. We can now assume that F and S are in general position with respect to each other, thus the intersection between them defines a finite collection of circles $\mathcal{C}$ on F that are two by two disjoint, because of the assumption $\text{link}(k_1, k_2) = 0$.

If S is the disjoint union of two surfaces S_1' and S_2' with boundary k_1 and k_2 respectively, we obtain that

$$g(S_1) + g(S_2) \leq g(S_1') + g(S_2') = g(S) \leq g(S_F) = g(S_1) + g(S_2),$$

implying that S_1' and S_2' are also genus minimizing for k_1 and k_2 respectively.

If S is connected, then the cardinality of $\mathcal{C}$ is (strictly) positive. Let $C_0 \in \mathcal{C}$ be such that one of the components of $F \setminus C_0$ is empty, meaning that it does not contain any other circle of $\mathcal{C}$. We now cut S along C_0 and paste two disks along the two boundaries that are created. Let S_c be the obtained surface, that might be disconnected. Observe that $\partial S_c = k_1 \cup k_2$.

Lemma 3.6. $g(S_c) = g(S)$.

Proof. If C_0 is an essential curve for S, then cutting along C_0 induces a lost of genus because the first homology group of S is getting smaller. Then S would not be a genus minimizing surface for $k_1 \cup k_2$. Thus C_0 is not an essential curve and the cut and paste operation does not change the genus of the surface. □

In particular, after performing this operation, S_c is disconnected and we have two possibilities:

1. One component (or a union of components) has the link $k_1 \cup k_2$ as boundary and the only other component is a 2-sphere. Then we throw away the 2-sphere and we denote abusively S_c the lasting component.

2. S was the connected sum of two surfaces S_c^1 and S_c^2 with respective boundaries k_1 and k_2, which we separated with the cut and paste operation. We keep the notation $S_c = S_c^1 \cup S_c^2$.

Observe that in both cases, F is in general position with respect to S_c and the intersection of these two surfaces is along a finite collection of circles that has one less circle than $\mathcal{C}$. We call this operation *erasing an empty circle*.

We can hence repeat the above procedure to obtain a new surface S_{2c} that intersects F along a collection of circles with one less circle. We continue the process until finding a surface S' such that $S' \cap F = \emptyset$, $\partial S' = k_1 \cup k_2$ and $g(S') = g(S)$. This implies that $S' = S_1' \cup S_2'$ where S_1' has k_1 (respectively k_2) as a boundary.Then

$$g(S_1) + g(S_2) \leq g(S_1') + g(S_2') = g(S) \leq g(S_F) = g(S_1) + g(S_2).$$

This implies that the formulas are the best possible for this case.

Case link$(k_1, k_2) = 1$**.**

Proposition 3.3 implies that we have two spheres F and F_1 such that F contains k_1 and intersects k_2 in two points, and F_1 is a small ball around the crossing located in the same connected component of $\mathbb{S}^3 \setminus F$ as k_1. As before, let S be a genus minimizing surface for the link $k_1 \cup k_2$. We place F so that it is in general position with respect to S. In this case $S \cap F$ consists of a finite collection of two by two disjoint circles $\mathcal{C}$ and a segment $\mathcal{S}$ whose endpoints are the two points in $F \cap k_2$. For the intersection $F_1 \cap S$ we have two possibilities to consider:

1. Up to erasing irrelevant intersections that we could erase, $F_1 \cap S$ consists of two segments $(s_i)_{i=1,2}$, and each s_i links the two intersections of F_1 with k_i;

2. $F_1 \cap S$ consists of two segments s_{12} and s_{21}, with s_{12} linking the endpoint (according to the orientation) of the strand of k_1 with the start-point of the strand of k_2 and conversely for s_{21}. This is necessary because the surface S is oriented, and so is S_{F_1}, the part of S which is inside F_1.

In both cases, we orient the intersection segments of S with F_1 to make them coherent with the orientations of the strands of k_1 and k_2 inside F_1. In case 1, we have a link made of two unknotted components, positively linked. The genus of this link is zero and so $g(S_{F_1}) = 0$ because otherwise S would not be a genus minimizing surface. Thus S_{F_1} is an annulus as described in Figure 3.10.

In case two, the link that appears in F_1 after orienting the intersection segments of S with F_1 is the unknot, since its trunk number is 2 when we consider a height function whose level sets are vertical planes. Thus by the same argument as above $g(S_{F_1}) = 0$ and this time S_{F_1} is a disk as described in Figure 3.10.

Now that we know how S looks like in F_1, we modify S to show that the formula is optimal. We start with case one. Observe that $\mathcal{S}$ is disjoint from every circle in $\mathcal{C}$. So for each circle in $\mathcal{C}$, $\mathcal{S}$ is in one of the components of its complement. We can thus erase one by one the circles in $\mathcal{C}$ as before and obtain a surface S' such that its boundary is $k_1 \cup k_2$, its intersection with F is the segment and $g(S') = g(S)$.

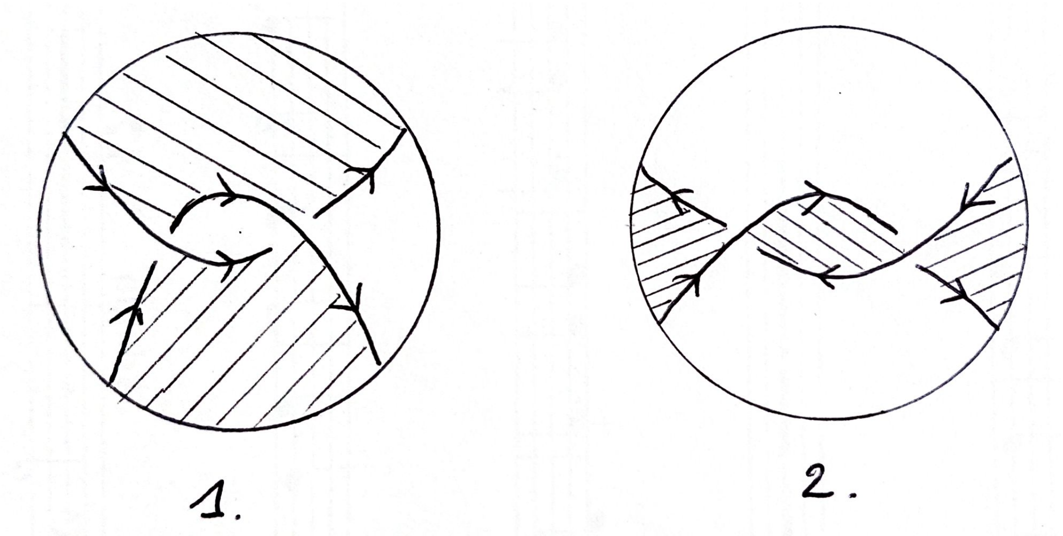

Figure 3.10: Two possible configurations in F_1

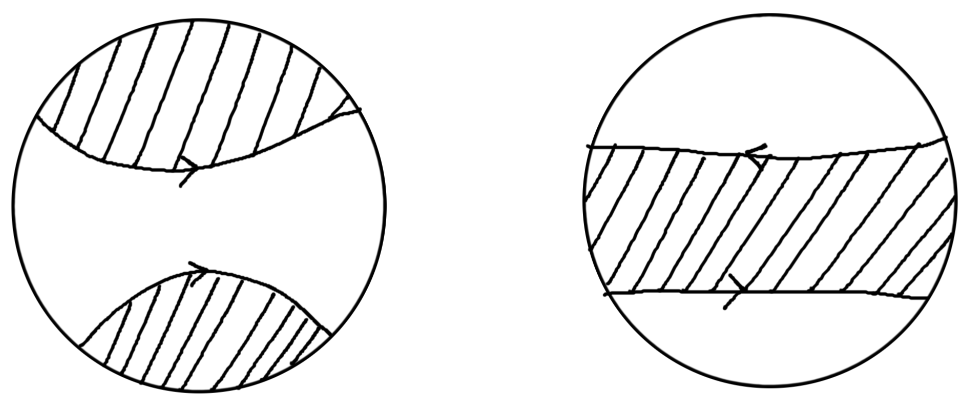

Figure 3.11: Unlinking the two possible configurations in F_1

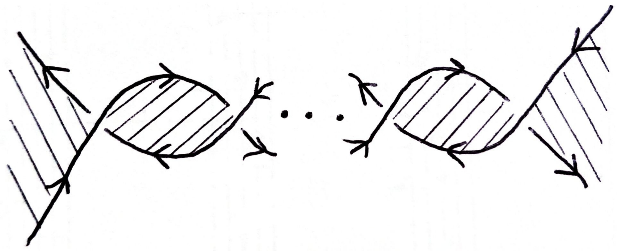

Figure 3.12: n linking number in a row

Because we throw away the obtained 2-sphere when we erase the empty circles, S' is connected.

Now cut the two half-twisted bands of the annulus inside F_1 to obtain two disks, as pictured in Figure 3.11. This disconnects S' and we obtain S'_2 a Seifert surface for k_2 and S'_1 another surface which stays in F. Since S' is the connected sum of these two surfaces, we have

$$g(S) = g(S') = g(S'_1) + g(S'_2) \geqslant g(k_2) + g(k_1)\,.$$

In another hand if we consider the Fried Surface for $k_1 \cup k_2$, we know by Corollary 3.5 that

$$g(k_1) + g(k_2) = g(S_F) \geqslant g(S)\,,$$

thus S_F is minimizing and the formula is optimal in this case.

Now we deal with the second case. First, as before erase one by one the circles in $\mathcal{C}$ to obtain a surface S' as in the first case. Then cut the twisted band in F_1 and replace it with a straight band, as pictured in Figure 3.11. This operation preserves the genus and $g(S'') = g(S)$. Note that after this operation, the knots k_1 and k_2 are not linked anymore and S'' is a Seifert surface for $k_1 \cup k_2$. Then we can use the process described in the case $\mathrm{link}(k_1, k_2) = 0$ to get two Seifert surfaces S_1 and S_2 for k_1 and k_2 so that $g(S'') = g(S_1) + g(S_2)$. Since S was genus minimizing for the link $k_1 \cup k_2$, we have

$$g(S_F) = g(k_1) + g(k_2) \geqslant g(S) = g(S'') = g(S_1) + g(S_2) \geqslant g(k_1) + g(k_2)\,,$$

thus S_F is minimizing for $k_1 \cup k_2$ when $\mathrm{link}(k_1, k_2) = 1$.

What happens from link$(k_1, k_2) \geqslant 2$? In this case, the main obstruction to prevent the accuracy of our formula is that several linkings can happen *in a row*, as illustrated in Figure 3.12.

Suppose that a minimal surface S looks like in Figure 3.12 in the neighbourhood of the linkings, and that there is no other linking. Then we could replace the n-twisted band with a straight one to unlink k_1 and k_2, getting a new surface S' with the same genus as S. Then S' is also minimizing for $k_1 \sqcup k_2$, and from the study of the case $\mathrm{Link}(k_1, k_2) = 0$ above,

$$g(S') = g(k_1) + g(k_2) < g(S_F) = g(k_1) + g(k_2) + n - 1$$

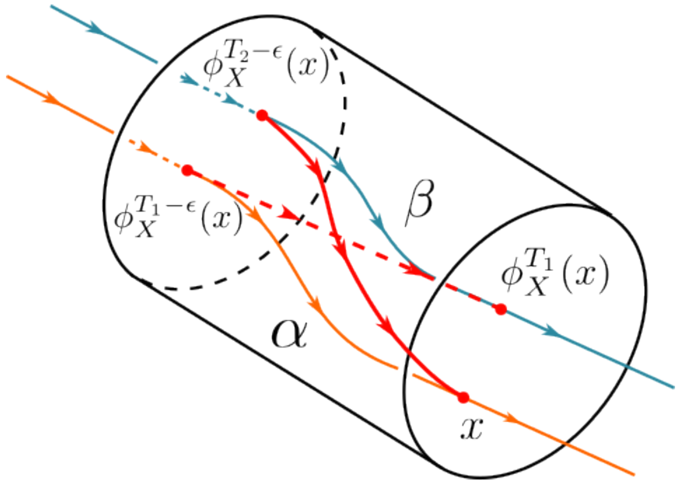

Figure 3.13: Closing pieces of orbits in a flowbox.

so the Fried surface is highly non-optimal for this class of examples. As far as we know, there is no way to distinguish between two linkings that happen in a row and two linkings appart one from another.

3.2.4 A bound on the asymptotic genus of an orbit

In the previous section we obtained a bound on the genus of a two component link with positive crossings depending on the genera of the two knots. We want to use this bound to estimate the genus of a very long pieces of orbit for the flow ϕ_X^t of a right-handed vector field X preserving an ergodic volume μ.

Setting. Consider a recurrent point x, D a small disk-like section around x and call $(T_n)_{n\in\mathbb{N}}$ the sequence of the successive return times of x in D with $T_0 := 0$, and set $t_n = T_n - T_{n-1}$ for any $n \in \mathbb{N} \setminus \{0\}$. According to Lemma 3.1 we can shrink D so that $t_n > T_r$ for all $n \in \mathbb{N}$. Set $\epsilon \ll 1$ and let F be the flowbox $\phi_X^{[-\epsilon,0]}(D)$. For each return time $T_n > 0$, we close the piece of orbit $\phi_X^{[T_{n-1},T_n-\epsilon]}(x)$ in F with a short geodesic path in the set $\mathcal{S}$ given by Theorem 1.7, so that we get a knot $k(\phi_X^{T_{n-1}}(x), t_n)$. In the same idea, $k(x, T_n)$ is the knot obtained by closing the piece of orbit $\phi^{[0,T_n-\epsilon]}(x)$ with a short $\mathcal{C}^1$-path in F. We denote $\mathrm{link}(T_{i-1}, t_i)$ the linking number of the knots $k(x, T_{i-1})$ and $k(\phi_X^{T_{i-1}}(x), t_i)$.

For $N \in \mathbb{N}$, we want to split the knot $k(x, T_N)$ constructed as above into a kind of sum of the knots $k\left(\phi_X^{T_k}(x), t_{k+1}\right)$, with the sum being made in the flowbox. To explain this consider the knots $k(x, t_1)$ and $k(\phi_X^{T_1}, t_2)$ obtained by following respectively the arc of orbits $\phi_X^{[0,t_1-\epsilon]}(x)$ and $\phi_X^{[0,t_2-\epsilon]}(\phi_X^{T_1}(x))$ and closing them with the appropriate short paths from the set $\mathcal{S}$ given by Theorem 1.7. From these two knots we can construct $k(x, T_2)$ by replacing the two closing arc (in dotted lines in Figure 3.13) by the orbit segment $\phi_X^{[T_1-\epsilon,T_1]}(x)$ (in dotted red) and the segment in $\mathcal{S}$ joining $\phi_X^{T_2-\epsilon}(x)$ to x (in red). Figure 3.13 illustrates this operation. We can then iterate this process to decompose the knots $k(x, T_n)$ into a *sum by the flow* of the knots $k(\phi_X^{T_{n-1}}(x), t_n)$ and use the precedent computation on the genus of two components

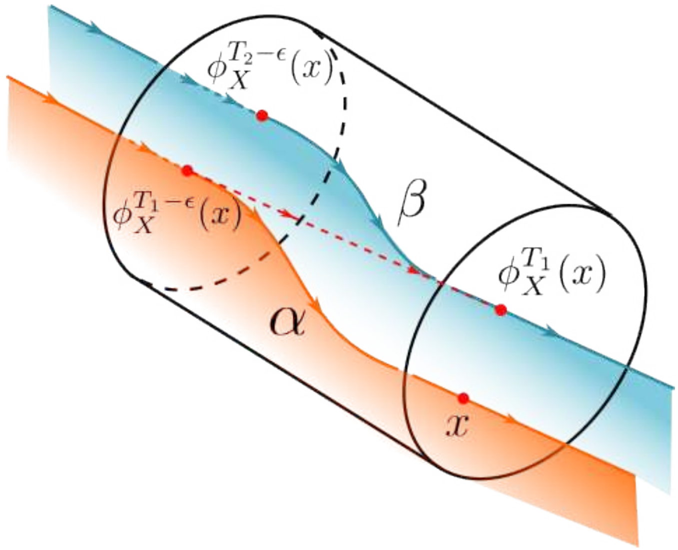

Figure 3.14: The Fried surface in the neighbourhood of the flowbox F

positive links to bound the asymptotic genus.

The first thing to see is that the link $k(\phi_X^{T_{i-1}}(x), t_i) \cup k(x, T_{i-1})$ has only positive crossings. If we see each link as an orbit of a $\mathcal{C}^1$-perturbation of X, which is right-handed, then because right-handedness is $\mathcal{C}^1$-open (Ghys [Ghy09]) these two orbits bound a Birkhoff section and we know that all the crossings are positive.

Then we need to understand what happens with the genus when we connect the two knots of the two-component link $k(\phi_X^{T_{i-1}}(x), t_i) \cup k(x, T_{i-1})$. Let us consider $k_1 = \phi_X^{[0,T_1-\epsilon]}(x) \cup \alpha$ and $k_2 = \phi_X^{[T_1,T_2-\epsilon]}(x) \cup \beta$ where α (resp. β) is the short geodesic path connecting $\phi_X^{T_1-\epsilon}(x)$ to x (resp. $\phi_X^{T_2-\epsilon}(x)$ to $\phi_X^{T_1}(x)$) in F. Let us consider the Fried surface S_F for the link $k_1 \cup k_2$, made out of two genus minimizing surfaces for k_1 and k_2. In F, because of the orientation of the boundaries, S_F looks like in Figure 3.14.

Thus to join the boundaries consistently with the orientations - in order to obtain a Seifert surface for the knot $k(x, T_2)$, we only have to attach a positively half-twisted band to the boundaries α and β in F, as illustrated in Figure 3.15.

Let S be the surface made with S_F plus the above half-twisted band. Since $\chi(S) = \chi(S_F) - 1$, we have:

$$2 - 2g(S) - 1 = 2 - 2g(S_F) - 2 - 1\,,$$

thus $g(S) = g(S_F) + 1 = g(k_1) + g(k_2) + \mathrm{link}(k_1, k_2)$.

Iterating this formula by cutting the knot $k(x, T_n)$ into pieces at each return time to F, we have:

$$\begin{aligned}\frac{1}{T_n^2} g\left(k(x,T_n)\right) &\leqslant \frac{1}{T_n^2}\Big(g\left(k(x,T_{n-1})\right) + g\left(k(\phi_X^{T_{n-1}}(x), t_n)\right) + \mathrm{link}(T_{n-1}, t_n)\Big) \\ &\leqslant \underbrace{\sum_{i=1}^{n} \frac{g\left(k(\phi_X^{T_{i-1}}(x), t_i)\right)}{T_n^2}}_{(A)} + \underbrace{\sum_{i=1}^{n} \frac{\mathrm{link}(T_{i-1}, t_i)}{T_n^2}}_{(B)}\,.\end{aligned}$$

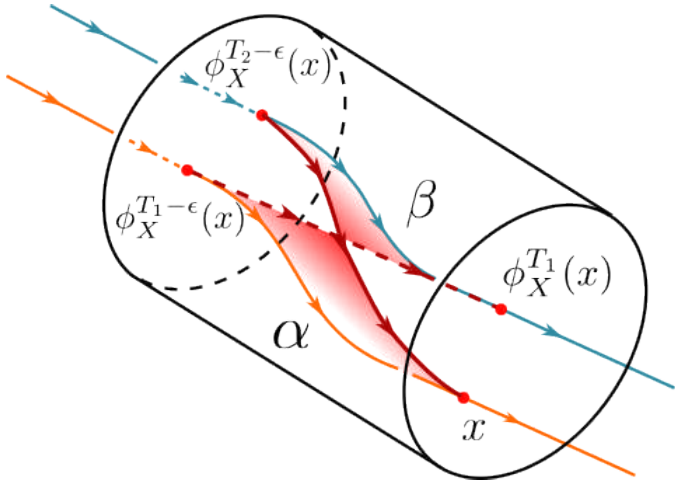

Figure 3.15: Connecting the boundaries α and β with a half-twisted band inside F.

The sequence of time interval $(t_n)_{n\in\mathbb{N}}$ between two successive returns of x in D is bounded. This implies that the knots $k(\phi_X^{T_{i-1}}(x), t_i)$ are tame and thus have a finite genus bounded by some g_{max}, and also that $\frac{n}{T_n}$ is bounded. Therefore part (A) in the above equation is bounded by $\frac{C}{T_n}$ for some positive constant C and thus tends to zero as n goes to infinity.

Now we need estimate the term (B), and this is where the proof becomes sketchy and unfinished. Since X is right-handed, Ghys' Theorem 1.14 ensure that there exists a Gauss linking form Ω which is pointwise positive. By definition, the sum of $\text{link}(T_{i-1}, t_i)$ is the sum of the integrals of this linking form Ω on the collection of knots $k(x, T_{i-1}) \times k(x, t_i)$, which is, up to throwing away the contribution of the short closing paths which is negligible if the return times are big enough (i.e. if D is chosen small enough), the integral of Ω on the collection of curves $\{\phi^t(x) \times \phi^s(x), (t, s) \in \cup_{i=1}^n [0, T_{i-1} - \epsilon] \times [T_{i-1}, T_i - \epsilon]\}$. Thus the term (B) is bounded by the integral $I(T_n)$ of this particular linking form Ω on the set $\{\phi^t(x) \times \phi^s(x), t \in [0, T_n - \epsilon]$ and $s \in [t+\epsilon, T_n - \epsilon]\}$. The problem to estimate this integral is that we are considering only one orbit instead of two distinct orbits. Indeed, we usually have:

$$\text{link}(\gamma_1, \gamma_2) = \lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \Omega_{\gamma_1(t_1), \gamma_2(t_2)} \left(X\left(\gamma_1(t_1)\right), X\left(\gamma_2(t_2)\right)\right) dt_1 dt_2$$

and using the symmetry of the linking number we could conclude that the term (B) is bounded by half of the asymptotic linking number of the two orbits, which is the helicity here because X is ergodic with respect to μ. But here we are considering points on the same orbit, and we have to avoid the diagonal of $\mathbb{S}^3 \times \mathbb{S}^3$ on which Ω has a pole of order 2. A solution to compute the integral $I(T_n)$ would be to take inspiration from Ghys' definition of the self-linking number without a preferred framing (see Section 1.2.3) and approximate x with two sequences (p_k), (q_k) of points converging to x, and then compute the integral

$$\int_0^{T_n-\epsilon} \int_{t+\epsilon}^{T_n-\epsilon} \Omega_{\phi_X^t(p_k), \phi_X^s(q_k)} \left(X\left(\phi_X^t(p_k)\right), X\left(\phi_X^s(q_k)\right)\right) dt_1 dt_2$$

but it is yet to be done.

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